Approaches to Set Optimization Without Convexity Assumptions

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Contents

Notations — Symbols — Abbreviations

F . . . . . . notation for a set-valued mapping
$\mathbb{R}$ . . . . set of real numbers
$\overline{\mathbb{R}}$ . . . . . . set of extended real numbers, i.e., $\mathbb{R} \cup \{\pm\infty\}$
$\mathbb{N}$ . . . . set of non-negative integers
$\mathbb{R}_+$ . . . . . . set of non-negative real numbers
$\mathbb{R}^p_+$ . . . . nonnegative orthant in $\mathbb{R}^p$
$(a, b),]a, b[, (a, b],]a, b], [a, b), [a, b[, [a, b]$ . . . intervals in $\overline{\mathbb{R}}$
X . . . . real linear space
Y . . . . real linear (topological) space
Y^* . . . . . . topological dual space of a linear topological space Y
Y' . . . . algebraic dual space of a linear space Y
D . . . . subset in Y
$\operatorname{co} D$ . . . . convex hull of a set D
$\operatorname{cl} D$ . . . . topological closure of a set D
$\operatorname{int} D$ . . . . topological interior of a set D
$\operatorname{cor} D$ . . . . core of a set D
$\operatorname{vcl} D$ . . . . vector closure of a set D
$\operatorname{vcl}_k D$ . . . . k-vector closure of a set D
$\mathcal{P}(Y)$ . . . . power set of Y without the empty set
$\overline{\mathcal{P}}(Y)$ . . . . power set of Y
D^* . . . . positive dual of a set D
D' . . . . . algebraic dual cone of the cone D
$D^{\#}_{Y'}$ . . . . algebraic quasi-interior of D'
$\preceq$ . . . . binary relation among sets / set relation
$\mathcal{A}$ . . . . family of sets
$\mathcal{A}_{min}$ . . . . set of all minimal elements of $\mathcal{A}$
$\leq$ . . . order relation in a real linear space Y
$\leq_C$ . . . order relation in a real linear space Y induced by a cone $C \subseteq Y$
w.r.t. . . . . with respect to

Chapter 1

Introduction

1.1 Motivation

Set optimization is a modern, dynamic field that subsumes scalar and vector optimization, and therefore provides an important extension in optimization theory. Due to a large number of applications, such as duality principles in vector optimization, gap functions for vector variational inequalities, inverse problems for partial differential equations and variational inequalities, fuzzy optimization, image processing, optimal control problems with differential inclusions, viability theory, medical image registration or in mathematical economics, set optimization has recently expanded as a distinct branch of applied mathematics. As a result, set optimization became a bridge between different areas in optimization.

For an introduction, let us briefly describe how set optimization arises from uncertain multiobjective problems. Many optimization problems are faced with conflicting goals which have to be minimized simultaneously. Such problem structures lead to *multi-objective optimization* programs, where different conflicting functions are optimized in parallel, meaning at the same time. Almost any real-world application of mathematics has conflictive multiple criteria; see, for example, the problem of choosing a portfolio in financial mathematics (compare [82]). Optimal elements of a feasible set are then defined by the concept of *Pareto optimality* (see, for example, [21]). If one expands this concept even further (for instance to infinite dimensional spaces), it is possible to define optimality in more general settings. Then one arrives at *vector optimization*, compare, for example, [5, 53].

Moreover, most complex multi-objective problems arising in Operational Research are contaminated with uncertain data. The reasons for this can be diverse, and include, among others, rounding errors or numerical inaccuracy, errors in measurements, incomplete information or broad estimations leading to contaminated data. For instance, in traffic optimization, uncertain weather conditions, construction works, or traffic jams can highly influence the computed optimal solutions of a train schedule or shortest path problem (compare, for example, [36]). Several examples for uncertain programming can be found in medicine. For instance, in intensity-modulated radiation therapy, Eichfelder and Pilecka [27, 28] explain that for safety purposes one might prefer to do necessary calculations of the optimal radiation dose based on several data sets. Portfolio optimization is subject to uncertainty on account of unreliable predictions, political decisions influencing the markets, etc. Moreover, network flow and network design problems are also heavily faced with uncertainty (see, for instance, [72]).

Uncertainty here means that some parameters are not known. Instead, only an estimated value or a set of possible values can be determined. As inaccurate data can have severe impacts on the model and therefore on the computed solution, it is important to take such uncertainty into account when modeling an optimization problem.

If uncertainty is included in the optimization model, one is left with not only one objective function value, but possibly a whole set of values. This leads to a set optimization problem, where the objective map is set-valued. This non-probabilistic approach gained recognition since the fundamental paper by Ehrgott et al. [22], who introduced robust solutions for uncertain multiobjective optimization problems, and has since been studied intensively, see, for example, [50, 51].

For instance, several diverse concepts of robustness for dealing with uncertainties in vector optimization can be described using approaches from set-valued optimization (see [51]). The concept of interval arithmetics for computations with strict error bounds [83] is also a special case of dealing with set-valued mappings. An interesting application of set optimization in welfare economics is given in [84]. We refer to [59] for a recent introduction to set optimization and its applications.

An important part of set optimization includes comparing sets by means of set relations, which are binary relations among sets. There is a variety of set relations based on convex cones known in the literature (for an overview, see [59, Chapter 2.6.2]), and several authors have discussed which set relations are appropriate for certain applications (compare [51]).

This work is concerned with dealing with *set optimization problems*, i.e., the problem of minimizing a set-valued mapping over a set of feasible elements. In particular, we will introduce and examine more general set relations, where the involved sets do not necessarily have to be convex. This necessarily includes the definition of the corresponding solution concepts as well. So far, in the literature convexity of the involved sets plays a crucial role, for instance, when representing set relations by means of linear functionals. In this thesis, we will show that it is possible to characterize set relations without any convexity assumptions through nonlinear functionals. In addition, we provide efficient algorithms for solving set optimization problems, where our approach is two-fold: When dealing with continuous problems, we derive a derivative-free descent method. When only a finite number of sets is known, we propose some extensions of so-called Jahn-Graef-Younes methods. Both algorithm types make use of the aforementioned characterization of set relations by means of nonlinear functionals. We furthermore study various examples that confirm that the use of the nonlinear functionals is applicable.

Moreover, we study set optimization problems with a variable domination structure. Variable domination structures for vector optimization problems play a key role in medicine, for example, in medical image registration [25]. We will show in this thesis that, when allowing uncertain parameters, this problem structure leads to a set-valued optimization problem w.r.t. a variable domination structure, i.e., the set that defines the ordering varies among the variables.

Furthermore, we derive notions for approximate solutions of set optimization problems, and apply some of our results to uncertain programming.

Below we describe the content of this thesis in more detail.

The present chapter contains a short description of the results derived in this thesis, and Chapter 1.2 covers notation, some preliminary results and the problem formulation.

The set relations that we present in Chapter 2 involve sets that describe the domination structure; in contrast to traditional approaches found in the literature (see [76, 77]), they do not need to be convex or cones. This is a novel approach and shall be motivated in this thesis. We then characterize these new generalized set relations by means of a scalarizing functional that is well known from vector optimization. Our generalized set relations are broader than the ones found in the literature, and the assumptions for their representation by means of the scalarizing functional are more general.

The easy structure of the nonlinear scalarizing functional allows for a convenient computation to check whether two sets fulfill the considered new set relations. This will enable us to derive efficient algorithms for solving set optimization problems, which shall constitute a significant part of this thesis.

In addition to deriving descent methods for computing minimal solutions of set optimization problems (see Chapter 2.3.1), we will also present methods for obtaining minimal elements of a family of finitely many sets (compare Chapter 2.3.2). The condition of dealing with a finite number of sets is not a difficult restriction, as most set optimization problems, even if given in a continuous framework, need to be handled in a discrete manner concerning computations. Therefore, given a finite discrete family of sets, in Chapter 2.3.2, we propose several methods that sort out non-minimal elements and determine all minimal elements of the family of sets. An according approach for *approximate* minimal solutions of set optimization problems is presented in Chapter 4.4. Numerical tests justify that our approaches are useful and the numerical effort is drastically reduced.

Chapter 3 is concerned with set relations based on a variable domination structure and their corresponding set optimization problems. Variable domination structures enable the decision-maker to include specific information into the data while modeling the problem.

From the theory of optimization, it is well-known that minimal solutions do not always exist and one needs to consider approximations thereof. In accordance with this knowledge, in Chapter 4 we introduce notions of approximate minimality in set-valued optimization. Theoretical investigations as well as algorithmic findings are presented in that chapter.

Finally, Chapter 5 presents unified concepts to uncertain programming problems based on three approaches, namely, the vector-valued approach, set-valued approach and a nonlinear scalarization approach. In particular, using the set-valued approach by means of the techniques derived in this thesis shows that it is possible to handle a number of concepts from uncertain scalar programming.

The results presented in this work have been selected from the the following publications:

- J. Chen, E. Köbis, M. A. Köbis and J.-C. Yao: A New Set Order Relation in Set Optimization. Journal of Nonlinear and Convex Analysis 18(4), 637–649, 2017.
- C. Gutiérrez, L. Huerga, E. Köbis and Chr. Tammer: Approximate Solutions of Set-Valued Optimization Problems Using Set-Criteria. Applied Analysis and Optimization 1(3), 501–519, 2017.
- N. Hebestreit and E. Köbis: Representation of Set Relations in Real Linear Spaces. Journal of Nonlinear and Convex Analysis 19(2), 287-296, 2018.
- K. Klamroth, E. Köbis, A. Schöbel and Chr. Tammer: A Unified Approach to Uncertain Optimization. European Journal of Operational Research 260(2), 403–420, 2017.
- E. Köbis: Variable Ordering Structures in Set Optimization. Journal of Nonlinear and Convex Analysis 18(9), 1571–1589, 2017.
- E. Köbis, D. Kuroiwa and Chr. Tammer: Generalized Set Order Relations and Their Numerical Treatment. Applied Analysis and Optimization 1(1), 45–65, 2017.
- E. Köbis and M. A. Köbis: Treatment of Set Order Relations by Means of a Nonlinear Scalarization Functional: A Full Characterization. Optimization 65(10), 1805–1827, 2016.

Most of the results presented in this thesis have been gained in collaboration with co-authors. The following sections were solely obtained by the author:

- Section 2.1, Section 2.2, Section 2.3.2, Section 2.4.1, Section 2.4.2, Section 2.5;
- Chapter 3;
- Section 4.3.1, Section 4.4;
- Section 5.1.2, Section 5.1.3, Section 5.2, Section 5.3, Section 5.4.

1.2 Preliminaries and Problem Formulation

Throughout this work, unless stated otherwise, we consider a set-valued optimization problem in the following setting: Let X be a real linear space, Y a real linear *topological* space or Y a real linear space, and let a set-valued mapping $F\colon X \rightrightarrows Y$ (the objective map that is to be minimized) and a set relation $\preceq$, which is a binary relation among sets, be given. $\overline{\mathcal{P}}(Y)$ is the *power set* of Y. By $\mathcal{P}(Y)$, we denote the *power set* of Y without the empty set, i.e., $\mathcal{P}(Y) := \{A \subseteq Y \mid A \text{ is nonempty}\}$.

For two elements A, B of $\mathcal{P}(Y)$, we denote the sum of sets by

$$A + B := \{a + b \mid a \in A,\ b \in B\}.$$

The set $C \subseteq Y$ is a *cone* if for all $c \in C$ and $\lambda \geq 0$, $\lambda c \in C$ holds true. The cone C is *convex* if $C + C \subseteq C$. We say that a set C is *proper* (or *nontrivial*) if $C \neq \{0\}$ and $C \neq Y$. The cone C is *pointed* of $C \cap (-C) = \{0\}$ holds. We call the cone C *reproducing* if $C - C = Y$.

Below we give some properties of a cone.

Remark 1.2.1 (See [5, Section 1.1]). *(a) A cone C may or may not be convex.*

(b) A cone C may be open, closed or neither open nor closed.

(c) A set C is a convex cone if it is both convex as well as a cone.

(d) If C_1 and C_2 are convex cones, then $C_1 \cap C_2$ and $C_1 + C_2$ are also convex cones.

(e) If C is a cone, then the convex hull of C, $\operatorname{co} C$ is a convex cone.

(f) If C_1 and C_2 are convex cones, then $C_1 + C_2 = \operatorname{co}(C_1 \cup C_2)$.

The following figures illustrate the notion of a (convex) cone.

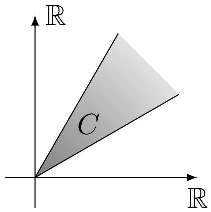

Figure 1.1: A proper, pointed, convex cone C in $\mathbb{R}^2$.

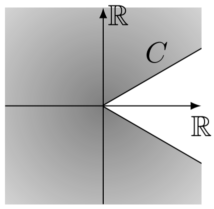

Figure 1.2: A cone C in $\mathbb{R}^2$ which is not convex.

In the following, we collect a few examples of cones.

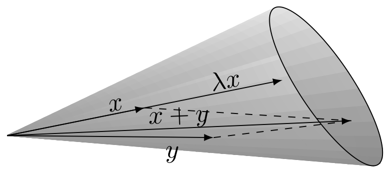

Figure 1.3: A convex cone in $\mathbb{R}^3$.

Example 1.2.2 ([59, Example 2.1.10]). *1. Let*

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0 \ \forall\, i \in \{1, \ldots, n\}\} \tag{1.1}$$

be the nonnegative orthant in $\mathbb{R}^n$. Obviously, $\mathbb{R}^n_+$ is a cone in the linear space $\mathbb{R}^n$, which is convex, proper, reproducing and pointed. We call $\mathbb{R}^n_+$ the natural ordering cone *in $\mathbb{R}^n$.*

2. Let $C[0,1]$ be the linear space of all real functions defined and continuous on the interval $[0,1] \subset \mathbb{R}$. Addition and multiplication by scalars are defined, as usual, by

$$(x+y)(t) = x(t) + y(t), \quad (\lambda x)(t) = \lambda x(t) \quad \forall\, t \in [0,1]$$

for $x, y \in C[0,1]$ and $\lambda \in \mathbb{R}$. Then

$$C_+[0,1] := \{x \in C[0,1] \mid x(t) \geq 0 \ \forall\, t \in [0,1]\} \tag{1.2}$$

is a convex, nontrivial, pointed, and reproducing cone in $C[0,1]$. Note that the set

$$Q := \{x \in C_+[0,1] \mid x \text{ is nondecreasing}\} \tag{1.3}$$

is also a convex, nontrivial, and pointed cone in the space $C[0,1]$, but it is not reproducing in general: $Q - Q$ is the proper linear subspace of all functions with bounded variation of $C[0,1]$.

3. Consider the set $C \subset \mathbb{R}^n$ defined by

$$\begin{aligned} C := \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \mid x_1 > 0, \ \textit{or} \\ x_1 = 0, \ x_2 > 0, \ \textit{or} \\ \ldots \\ x_1 = \cdots = x_{n-1} = 0, \ x_n > 0, \ \textit{or} \\ x = 0\}. \end{aligned}$$

Then the cone C is convex, proper, reproducing and pointed.

If Y is a real linear *topological* space, then Y^* denotes the *topological dual space* of Y. The *topological interior* of $F \subset Y$ will be denoted by $\operatorname{int} F$ and its *topological closure* by $\operatorname{cl} F$.

The *positive dual set* of a set F is given by $F^* := \{\ell \in Y^* \mid \forall\ y \in F : \ \ell(y) \geq 0\}$ and the nonnegative orthant of $\mathbb{R}^p$ is denoted by $\mathbb{R}^p_+$. A nonempty set $F \subset Y$ is called K-proper if $F + K \neq Y$. If there is no confusion, for some $y \in Y$, we write y instead of $\{y\}$ in the single-valued case.

1.2.1 Binary Relations

In this section, our objective is to study some useful order relations. We begin by recalling that given a nonempty set M, by $M \times M$ we represent the set of ordered pairs of elements of M, that is,

$$M \times M := \{(x_1, x_2) \mid x_1, x_2 \in M\}.$$

The following definition gives the notion of an order relation.

Definition 1.2.3. *Let M be a nonempty set and let $\mathcal{R}$ be a nonempty subset of $M \times M$. Then $\mathcal{R}$ is called an **order relation** (or a **binary relation**) on M and the pair $(M, \mathcal{R})$ is called a set M with **order relation** $\mathcal{R}$. The containment $(x_1, x_2) \in \mathcal{R}$ will be denoted by $x_1 \mathcal{R} x_2$. The order relation $\mathcal{R}$ is called:*

(a) ***reflexive*** *if for every $x \in M$, we have $x \mathcal{R} x$;*

(b) ***transitive*** *if for all $x_1, x_2, x_3 \in M$, the relations $x_1 \mathcal{R} x_2$ and $x_2 \mathcal{R} x_3$ imply that $x_1 \mathcal{R} x_3$;*

(c) ***antisymmetric*** *if for all $x_1, x_2 \in M$, the relations $x_1 \mathcal{R} x_2$ and $x_2 \mathcal{R} x_1$ imply that $x_1 = x_2$.*

*Moreover, an order relation $\mathcal{R}$ is called a **preorder** on M if $\mathcal{R}$ is transitive and a **partial order** on M if $\mathcal{R}$ is reflexive, transitive, and antisymmetric. In both cases, the containment $(x_1, x_2) \in \mathcal{R}$ is denoted by $x_1 \leq_{\mathcal{R}} x_2$, or simply by $x_1 \leq x_2$ if there is no risk of confusion. The binary relation $\mathcal{R}$ is called a **linear** or **total order** if $\mathcal{R}$ is a partial order and any two elements of M are **comparable**, that is*

(d) *for all $x_1, x_2 \in M$ either $x_1 \leq_{\mathcal{R}} x_2$ or $x_2 \leq_{\mathcal{R}} x_1$.*

*Furthermore, if each nonempty subset M' of M has a first element x' (meaning that $x' \in M'$ and $x' \leq_{\mathcal{R}} x \ \forall x \in M'$), then M is called **well-ordered**.*

We recall Zermelo's theorem: For every nonempty set M there exists a partial order $\mathcal{R}$ on M such that $(M, \mathcal{R})$ is well-ordered.

An illustrative example of a relation is $\Delta_M := \{(x, x) \mid x \in M\}$, which is reflexive, transitive, and antisymmetric, but it satisfies (d) only when M is a singleton.

We recall that the **inverse** of the relation $\mathcal{R} \subset M \times M$ is the relation

$$\mathcal{R}^{-1} := \{(x_1, x_2) \in M \times M \mid (x_2, x_1) \in \mathcal{R}\},$$

and if $\mathcal{S}$ is a relation on M, then the **composition** of $\mathcal{R}$ and $\mathcal{S}$ is the relation

$$\mathcal{S} \circ \mathcal{R} := \{(x_1, x_3) \mid \exists\, x_2 \in M \mid (x_1, x_2) \in \mathcal{R},\ (x_2, x_3) \in \mathcal{S}\}.$$

Using these two notations, the conditions (a), (b), (c), and (d) are equivalent to $\Delta_M \subset \mathcal{R}$, $\mathcal{R} \circ \mathcal{R} \subset \mathcal{R}$, $\mathcal{R} \cap \mathcal{R}^{-1} \subset \Delta_M$ and $\mathcal{R} \cup \mathcal{R}^{-1} = M \times M$, respectively.

In the following result, we characterize the relations between order relations and cones:

Theorem 1.2.4. *Let Y be a linear space and let C be a cone in Y. Then the relation*

$$\mathcal{R}_C := \{(x_1, x_2) \in Y \times Y \mid x_2 - x_1 \in C\} \tag{1.4}$$

is reflexive and satisfies

$$\forall\, x_1, x_2 \in Y,\ \forall\, \lambda \in \mathbb{R}\ :\ x_1 \mathcal{R} x_2,\ 0 \le \lambda \Rightarrow \lambda x_1 \mathcal{R} \lambda x_2 \tag{1.5}$$

and

$$\forall\, x_1, x_2, x \in Y\ :\ x_1 \mathcal{R} x_2 \Rightarrow (x_1 + x) \mathcal{R} (x_2 + x). \tag{1.6}$$

Moreover, C is convex if and only if $\mathcal{R}_C$ is transitive, and, respectively, C is pointed if and only if $\mathcal{R}_C$ is antisymmetric. Conversely, if $\mathcal{R}$ is a reflexive relation on X satisfying (1.5) and (1.6), then $C := \{x \in X \mid 0\mathcal{R}x\}$ is a cone and $\mathcal{R} = \mathcal{R}_C$.

Proof. See [37, Theorem 2.1.13]. □

The above result shows that when $\emptyset \neq C \subset X$, the relation $\mathcal{R}_C$ defined by (1.4) is a reflexive preorder iff C is a convex cone, and $\mathcal{R}_C$ is a partial order iff C is a pointed convex cone.

Let Y be a linear topological space, partially ordered by a proper pointed convex closed cone $C \subset Y$. Denote this order by " $\le_C$ ". Its ordering relation is described by

$$y_1 \le_C y_2 \quad \text{if and only if} \quad y_2 - y_1 \in C \quad \text{for all } y_1, y_2 \in Y. \tag{1.7}$$

In the sequel, we omit the subscript C if no confusion occurs.

Definition 1.2.5. *Let $\mathcal{R}$ be an order relation on the nonempty set M and let $M_0 \subset M$ be nonempty. An element $x_0 \in M_0$ is called a **maximal (minimal) element** of M_0 **relative to** $\mathcal{R}$ if for every $x \in M_0$,*

$$x_0 \mathcal{R} x \Rightarrow x \mathcal{R} x_0 \qquad (x \mathcal{R} x_0 \Rightarrow x_0 \mathcal{R} x). \tag{1.8}$$

The collection of all maximal (minimal) elements of M_0 with respect to (w.r.t. for short) $\mathcal{R}$ is denoted by $\mathbf{Max}(M_0, \mathcal{R})$ *(*$\mathbf{Min}(M_0, \mathcal{R})$*).*

Note that x_0 is a maximal element of M_0 w.r.t. $\mathcal{R}$ if and only if x_0 is a minimal element of M_0 w.r.t. $\mathcal{R}^{-1}$, and hence $\mathbf{Max}(M_0, \mathcal{R}) = \mathbf{Min}(M_0, \mathcal{R}^{-1})$.

Remark 1.2.6 ([59, Remark 2.1.3]). **1.** *If the order relation $\mathcal{R}$ in Definition 1.2.5 is antisymmetric, then $x_0 \in M_0$ is maximal (minimal) if and only if for every $x \in M_0$*

$$x_0 \mathcal{R} x \Rightarrow x = x_0 \quad (\, x \mathcal{R} x_0 \Rightarrow x_0 = x). \tag{1.9}$$

2. *If $\mathcal{R}$ is an order relation on M and $\emptyset \neq M_0 \subset M$, then $\mathcal{R}_0 := \mathcal{R} \cap (M_0 \times M_0)$ is an order relation on M_0. In such a situation, the set M_0 will always be endowed with the order structure $\mathcal{R}_0$ if not stated explicitly otherwise. If $\mathcal{R}$ is a preorder (partial order, linear order) on M, then $\mathcal{R}_0$ is a preorder (partial order, linear order) on M_0. Therefore, x_0 is a maximal (minimal) element of M_0 relative to $\mathcal{R}$ iff x_0 is a maximal (minimal) element of M_0 relative to $\mathcal{R}_0$.*

In the following, we give some examples to illustrate the above notions.

Example 1.2.7 ([59, Example 2.1.4]). **(1)** *Assume that X is a nonempty set and $M := \mathcal{P}(X)$ represents the collection of subsets of X. Then the order relation $\mathcal{R} := \{(A, B) \in M \times M \mid A \subset B\}$ is a partial order on M. However, if X contains at least two elements, then $\mathcal{R}$ is not a linear order.*

(2) *Assume that $\mathbb{N}$ is the set of non-negative integers and*

$$\mathcal{R}_{\mathbb{N}} := \{(n_1, n_2) \in \mathbb{N} \times \mathbb{N} \mid \exists \, p \in \mathbb{N} \; : \; n_2 = n_1 + p\}.$$

Then $\mathbb{N}$ is well-ordered by $\mathcal{R}_{\mathbb{N}}$. Note that $\mathcal{R}_{\mathbb{N}}$ defines the usual order relation on $\mathbb{N}$, and $n_1 \mathcal{R}_{\mathbb{N}} n_2$ will always be denoted by $n_1 \leq n_2$ or, equivalently, $n_2 \geq n_1$.

(3) *Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+ := [0, \infty[$ be the set of non-negative real numbers. The usual order relation on $\mathbb{R}$ is defined by*

$$\mathcal{R}_1 := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} \mid \exists \, y \in \mathbb{R}_+ \; : \; x_2 = x_1 + y\}.$$

Then $\mathcal{R}_1$ is a linear order on $\mathbb{R}$, but $\mathbb{R}$ is not well-ordered by $\mathcal{R}_1$. In the following, the fact $x_1 \mathcal{R}_1 x_2$ will always be denoted by $x_1 \leq x_2$ or, equivalently, $x_2 \geq x_1$.

(4) *Given $n \in \mathbb{N}$, $n \geq 2$, we consider the binary relation $\mathcal{R}_n$ on $\mathbb{R}^n$ defined by*

$$\mathcal{R}_n := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid \forall \, i \in \{1, \ldots, n\} \; : \; x_i \leq y_i\},$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Then $\mathcal{R}_n$ is a partial order on $\mathbb{R}^n$, but $\mathcal{R}_n$ is not a linear order. For example, the elements e_1 and e_2 are not comparable w.r.t. $\mathcal{R}_n$, where $e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$. As usual, by e_i we denote the vector whose entries are all 0 except the ith one, which is 1.

Remark 1.2.8 ([59, Remark 2.1.5]). *Every well-ordered subset W of $\mathbb{R}$ (equipped with its usual partial order defined above) is at most countable. Indeed, every element $y \in W$, except the greatest element w of W (provided that it exists), has a successor $s(y) \in W$. Clearly, if $y, y' \in W$, $y < y'$, then $s(y) \leq y'$. Therefore, fixing $q_y \in \mathbb{Q}$ such that $y < q_y < s(y)$ for $y \in W \setminus \{w\}$, we get an injective function from $W \setminus \{w\}$ into $\mathbb{Q}$, and so W is at most countable.*

We emphasize that even when $\mathcal{R}$ is a partial order on M, a nonempty subset M_0 of M may have zero, one, or several maximal elements, but if $\mathcal{R}$ is a linear order, then every subset has at most one maximal (minimal) element.

Definition 1.2.9. *Let $\emptyset \neq M_0 \subset M$ and let $\mathcal{R}$ be an order relation on M. Then:*

1. *M_0 is **lower (upper) bounded** (w.r.t. $\mathcal{R}$) if there exists $a \in M$ such that $a\mathcal{R}x$ ($x\mathcal{R}a$) for every $x \in M_0$. In this case, the element a is called a **lower (upper) bound** of M_0 (w.r.t. $\mathcal{R}$).*

2. *If, moreover, $\mathcal{R}$ is a partial order, we say that $a \in M$ is the **infimum (supremum)** of M_0 if a is a lower (upper) bound of M_0 and for any lower (upper) bound a' of M_0 we have that $a'\mathcal{R}a$ ($a\mathcal{R}a'$).*

In set-valued optimization, the existence of maximal elements w.r.t. order relations is an important problem. For this, the following **Zorn's lemma** plays a crucial role.

Lemma 1.2.10 (Zorn). *Let $(M, \leq)$ be a reflexively preordered set. If every nonempty totally ordered subset of M is upper bounded, then M has maximal elements.*

1.2.2 Problem Formulation: Minimal Elements of a Family of Sets and Minimal Solutions of Set Optimization Problems

When studying optimization problems with a set-valued objective map, one is usually looking for feasible elements that satisfy some kind of optimality notion. One possibility for such a definition is the following one (see Definition 1.2.5).

Definition 1.2.11 (Minimal Elements of a Family of Sets). *Let $\mathcal{A}$ be a family of nonempty subsets of Y and let a set relation $\preceq$ on $\mathcal{P}(Y)$ be given. $\overline{A} \in \mathcal{A}$ is called a **minimal element** of $\mathcal{A}$ w.r.t. $\preceq$ if*

$$A \preceq \overline{A},\ A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq A\,.$$

The set of all minimal elements of $\mathcal{A}$ w.r.t. $\preceq$ is denoted by $\mathcal{A}_{min}$.

Note that if the elements of $\mathcal{A}$ are single-valued, $D \subset Y$ is a convex cone, and $\leq_D := \preceq$, where $\leq_D$ is defined by $A_1 \leq_D A_2 :\Longleftrightarrow A_1 \in A_2 - D$ (see (1.7)), then Definition 1.2.11 reduces to the standard notion of minimality in vector optimization (compare, for example, [53, Definition 4.1]).

Moreover, we are looking for **minimal elements** w.r.t. the set relation $\preceq$ in the sense of Definition 1.2.11 of the problem

$$\min_{x \in S} F(x)\,. \tag{1.10}$$

Definition 1.2.12 (Minimal Solutions of Problem (1.10))**.** *We say that $\bar{x} \in S$ is a **minimal solution** of* (1.10) *w.r.t. $\preceq$ if $F(\bar{x})$ is a minimal element of the family of sets $F(x)$, $x \in \mathbb{R}^n$ w.r.t. $\preceq$. The family of sets $F(x)$, $x \in S$, is denoted by $\mathcal{A}$.*

Chapter 2

Generalized Set Relations

In this chapter, we formulate generalizations of existing set relations that are useful for applications in uncertain programming. In addition to motivating the necessity of these new relations, we will give equivalent characterizations of these set relations by means of a well-known scalarization functional in Section 2.2. Moreover, our intention is to study set-valued optimization problems with these general set relations and to derive corresponding algorithms in order to determine (approximate) solutions (see Section 2.3). In addition, we will propose a new set relation which is able to act as a weighting function between two important set relations and therefore balances out possible gaps that can occur in modeling set optimization problems (see Section 2.4). Finally, in Section 2.5 we will no longer assume that the objective space is a priori equipped with a topology. We will then show characterizations of set relations in real linear spaces as an extension of Section 2.2.

2.1 Preparatory Work

Throughout this chapter, unless stated otherwise, let Y be a linear topological space. Note that in Chapter 2.5, we will drop the *topology* assumption on Y and consider linear spaces. We assume that $D \in \mathcal{P}(Y)$ is a closed proper set satisfying the inclusion

$$D + [0, +\infty) \cdot k \subseteq D \tag{2.1}$$

for some $k \in Y \setminus \{0\}$. In $\mathbb{R}^2$, a set D (that is not necessarily a cone) satisfying (2.1) for $k = (1, 1)$ is, for instance, the set $\mathbb{R}^2_+ - \{(0, 1)\}$. If the relation (2.1) is fulfilled, the functional $z^{D,k} \colon Y \to \mathbb{R} \cup \{\pm\infty\} =: \overline{\mathbb{R}}$ defined by

$$z^{D,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - D\} \tag{2.2}$$

is well-defined. We call $z^{D,k}$ *nonlinear scalarizing functional*, as it plays an important role in scalarization methods for obtaining efficient solutions of a vector-valued optimization problem. It can be shown that for a given vector $k \in Y \setminus \{0\}$ and by a variation of the set D satisfying the property (2.1), all efficient elements of a vector optimization problem without any convexity assumptions can be found. The functional $z^{D,k}$ was used to obtain a separation theorem for not necessarily onvex sets, see Gerstewitz [33], Gerstewitz and Iwanow [34] and also Gerth and Weidner [35]. Additionally, numerous applications of $z^{D,k}$ are known in the literature, for instance, coherent risk measures in financial mathematics (see [47]) and uncertain optimization (in particular, in robustness theory, compare [60]). Many properties of $z^{D,k}$ can be found in [35, 37, 34, 104]. It is interesting to notice that the construction of $z^{D,k}$ was mentioned by Krasnoselskiĭ [75] (see Rubinov [97]) in the context of operator theory.

Definition 2.1.1. *Let Y be a linear space and $\widetilde{D} \in \mathcal{P}(Y)$. A functional $z\colon Y \to \mathbb{R} \cup \{\pm\infty\}$ is $\widetilde{D}$-**monotone** if*

$$y_1,\ y_2 \in Y:\ y_1 \in y_2 - \widetilde{D} \Rightarrow z(y_1) \leq z(y_2).$$

Important properties of the functional $z^{D,k}$ which will be used in this work are given in the following theorem.

Theorem 2.1.2 ([37, Theorem 2.3.1])**.** *Let Y be a linear topological space, $D \in \mathcal{P}(Y)$ a closed proper set, $\widetilde{D} \in \mathcal{P}(Y)$ and let $k \in Y \setminus \{0\}$ be such that (2.1) is satisfied. Then the following properties hold for $z = z^{D,k}$:*

(a) *z is lower semi-continuous.*

(b) *(i) z is convex $\Longleftrightarrow$ D is convex,*

(ii) $[\forall\ y \in Y,\ \forall\ \ r > 0:\ z(ry) = rz(y)] \Longleftrightarrow D$ is a cone.

(c) *z is proper $\Longleftrightarrow$ D does not contain lines parallel to k, i.e., $\forall\ y \in Y\ \exists\ r \in \mathbb{R}:\ y + rk \notin D$.*

(d) *z is $\widetilde{D}$-monotone $\Longleftrightarrow D + \widetilde{D} \subseteq D$.*

(e) *z is subadditive $\Longleftrightarrow D + D \subseteq D$.*

(f) *$\forall\ y \in Y,\ \forall\ r \in \mathbb{R}:\ z(y) \leq r \Longleftrightarrow y \in rk - D$.*

(g) *$\forall\ y \in Y,\ \forall\ r \in \mathbb{R}:\ z(y + rk) = z(y) + r$.*

(h) *z is finite-valued $\Longleftrightarrow$ D does not contain lines parallel to k and $\mathbb{R}k - D = Y$.*

(i) *Let furthermore $D + (0, +\infty) \cdot k \subseteq \operatorname{int} D$. Then z is continuous.*

The following examples illustrates the choice concerning the set D and the vector k in the formulation of the functional $z^{D,k}$.

Example 2.1.3 (Compare [68, Example 2.3])**.** *(a) Pascoletti, Serafini [90] use a special optimization problem related to the functional $z^{D,k}$ in the special case $Y = \mathbb{R}^n$.*

Given a function $f : \Omega \to \mathbb{R}^n$, where $\Omega \subseteq \mathbb{R}^m$, a closed convex cone $D \subset \mathbb{R}^n$ with nonempty interior, parameters $a \in \mathbb{R}^n$, $r \in \operatorname{int} D$, they propose the problem

$$\begin{aligned} &\min t \\ &s.t.\ \ x \in \Omega \\ &\qquad f(x) \in a + tr - D \\ &\qquad t \in \mathbb{R}. \end{aligned}$$

(b) *Many well known concepts of proper efficiency (compare [59, Chapter 2.4]) also fit into the general approach of the nonlinear scalarizing concept with the functional $z^{D,k}$. Since many of them are based on a certain kind of generalized linear scalarization, they are endowed with a polyhedral structure: In [105], Weidner characterizes properly efficient elements in the sense of Geoffrion by solutions of the auxiliary problem*

$$\min_{y \in \mathbb{R}^n} \max_{i=1,\dots,n} (\langle v_i, y\rangle - \nu_i)$$

with $v_i \in \operatorname{int} \mathbb{R}^n_+$, $\sum_{j=1}^n v_i^j = 1$, $\nu_i \in \mathbb{R}$, $i = 1, \dots, n$. Without effort, we can verify that these auxiliary problems coincide with the problem $\min_{y \in \mathbb{R}^n} z^{D,k}$ for $D := \{y \in \mathbb{R}^n \ :\ \forall\, i = 1, \dots, n : \langle v_i, y\rangle - \nu_i \geq 0\}$ and $k := (1, \dots, 1)^T \in \mathbb{R}^n$.

(c) *Kaliszewski [58] characterizes efficiency in vector optimization with respect to polyhedral cones by some inconsistency assertions. He uses a polyhedral cone D given by*

$$D := \{y \in \mathbb{R}^n :\ \langle -b_i, y\rangle \geq 0,\ i = 1, ..., m\}$$

with $b_i \in \mathbb{R}^n$, $i = 1, \dots, m$. The inconsistency notions he uses can equivalently be represented by means of the functional $z^{D,k}$, as was shown by Tammer and Winkler in [102].

Most of the set relations to be defined in Section 2.2 rely on set inclusions, where the set D is attached pointwise to the considered sets $A, B \in \mathcal{P}(Y)$. In that spirit, the following result relates $A - D$ by means of the functional $z^{D,k}$.

Lemma 2.1.4 ([67, Corollary 2.13])**.** *Let $D \subset Y$ be a closed proper set in Y, $k \in Y \setminus \{0\}$ such that* (2.1) *is fulfilled, $A \in \mathcal{P}(Y)$, and let $\widetilde{D} \subseteq Y$ such that $0 \in \widetilde{D}$ and $D + \widetilde{D} \subseteq D$. Then it holds*

$$\sup_{a \in A} z^{D,k}(a) = \sup_{y \in A - \widetilde{D}} z^{D,k}(y)\,.$$

Proof. Under the given assumptions, $z^{D,k}$ is $\widetilde{D}$-monotone. Because $A - \widetilde{D} \subseteq A - \widetilde{D}$, it holds

$$\forall\, y \in A - \widetilde{D},\ \exists\, a \in A :\ z^{D,k}(y) \leq z^{D,k}(a)\,,$$

hence, $\sup_{y \in A - \widetilde{D}} z^{D,k}(y) \leq \sup_{a \in A} z^{D,k}(a)$. The converse inequality follows directly from the definition of the supremum and $A \subseteq A - \widetilde{D}$, as $0 \in \widetilde{D}$. □

Remark 2.1.5 ([67, Remark 2.14]). *Note that the assumption $0 \in \widetilde{D}$ in Lemma 2.1.4 is only necessary for the validity of the inequality*

$$\sup_{a \in A} z^{D,k}(a) \leq \sup_{y \in A - \widetilde{D}} z^{D,k}(y)\,. \tag{2.3}$$

For the inequality

$$\sup_{y \in A - \widetilde{D}} z^{D,k}(y) \leq \sup_{a \in A} z^{D,k}(a)\,,$$

$0 \in \widetilde{D}$ is not required. It is important to mention that the assumption $0 \in \widetilde{D}$ in Lemma 2.1.4 cannot be dropped in order for (2.3) *to hold true. Consider, for example, the selection $D = \mathbb{R}^2_+$, $\widetilde{D} = \mathbb{R}^2_+ + d$, where $d \in \operatorname{int} \mathbb{R}^2_+$, $A = \{(0,0)\}$ and $k = (1,1)$. Then, clearly $0 \notin \widetilde{D}$ and $\sup_{a \in A} z^{D,k}(a) = 0 \not\leq \sup_{y \in A - \widetilde{D}} z^{D,k}(y) < 0$.*

When D is a proper closed convex cone, we immediately obtain the following result from Lemma 2.1.4.

Corollary 2.1.6 ([66, Corollary 2.2]). *Let $C \subset Y$ be a proper closed convex cone and $k \in C \setminus \{0\}$. For two sets A, $B \in \mathcal{P}(Y)$ it holds*

$$\sup_{b \in B} z^{C,k}(b) = \sup_{y \in B - C} z^{C,k}(y)\,,$$
$$\inf_{a \in A} z^{C,k}(a) = \inf_{y \in A + C} z^{C,k}(y)\,.$$

Let $C \subset Y$ be a proper closed convex cone. Notice that it is possible that the function value $z^{C,k}(b)$ may be $+\infty$, if there is no $t \in \mathbb{R}$ with $b \in tk - C$, and by convention $\inf \emptyset = +\infty$. This can be the case if $k \in \operatorname{bd} C$. Then the relations in Corollary 2.1.6 hold true, because $\sup_{b \in B} z^{C,k}(b) = \sup_{y \in B - C} z^{C,k}(y) = +\infty$, as $0 \in C$. Similarly, for instance if C is a halfspace and $k \in \operatorname{bd} C$, then function values $-\infty$ are possible. In that case, $z^{C,k}$ is not proper, because C contains lines parallel to k (see Theorem 2.1.2 (c)). For example, consider the halfspace $C = \{y = (y_1, y_2)^T \in \mathbb{R}^2 \mid y_2 \geq 0\}$, $k = (0, -1)^T \in \operatorname{bd} C$, $b_1 = (-1, -1)^T$, $b_2 = (1, 1)^T$ and $B = \{b_1, b_2\}$. Then $z^{C,k}(b_1) = -\infty$, $z^{C,k}(b_2) = \inf \emptyset = +\infty$. Because $0 \in C$, we get that $\sup_{b \in B} z^{C,k}(b) = \sup_{y \in B - C} z^{C,k}(y) = +\infty$ and $\inf_{b \in B} z^{C,k}(b) = \inf_{y \in B - C} z^{C,k}(y) = -\infty$. If $k \in \operatorname{int} C$, the functional is finite-valued, see [37, Corollary 2.3.5.]. Throughout this work, unless stated otherwise, we will assume that $k \in Y \setminus \{0\}$, such that $z^{C,k}$ is an extended real-valued functional, i.e., function values $\pm\infty$ are possible.

2.2 Generalized Set Relations and Representation by Means of a Scalarizing Functional

In this chapter, we formulate generalizations of known set relations. In addition to describing the need for these new relations, our intention is to study set-valued optimization problems with these general set relations and, afterwards, to derive corresponding algorithms in Section 2.3. The findings presented in this chapter are based on [66, 67, 68].

In the following definition, we introduce a generalized set relation w.r.t. a nonempty subset D of Y, which is not assumed to be convex or a cone. The following set relation generalizes the upper set less relation by Kuroiwa [76, 77], where the involved set D is a convex cone.

Definition 2.2.1 (Generalized Upper Set Less Relation, [67, Definition 2.1])**.** *Let $D \in \mathcal{P}(Y)$. The **generalized upper set less relation** $\preceq_D^u$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by*

$$A \preceq_D^u B :\Longleftrightarrow A \subseteq B - D,$$

which is equivalent to

$$\forall\, a \in A,\ \exists\, b \in B:\ a \in b - D.$$

Figure 2.1 illustrates this definition.

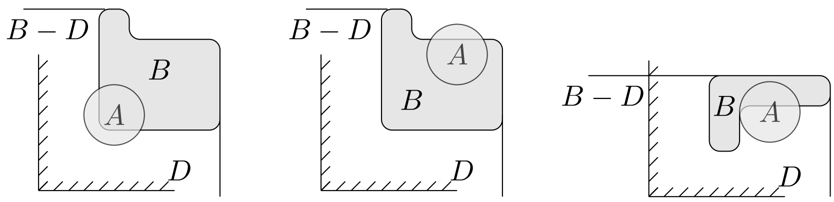

Figure 2.1: Illustration of the relations $A \preceq_D^u B$ in the first image, $A \not\preceq_D^u B$ in the second illustration, and $A \preceq_D^u B$ in the right picture.

Remark 2.2.2. *Notice that $\preceq_D^u$ is transitive if $D + D \subseteq D$. If D is a cone, then $D + D \subseteq D$ implies that D is convex. If, for instance, $D = \mathbb{R}^2_+ \setminus \{0\}$, then $D + D \subseteq D$ is fulfilled, but D is not a cone. Moreover, $\preceq_D^u$ is reflexive if $0 \in D$. Therefore, $\preceq_D^u$ is a preorder if $D + D \subseteq D$ and $0 \in D$.*

In the following remark, we note what kind of set relations are comprised by the generalized upper set less relation $\preceq_D^u$.

Remark 2.2.3 ([67, Remark 2.2])**.** *Let $Y = \mathbb{R}^q$ for $q \in \mathbb{N}$, $q > 0$, and $A, B \in \mathcal{P}(\mathbb{R}^q)$. If $D = \mathbb{R}^q_+$, then the relation $\preceq_D^u$ has been used to model robust solutions of uncertain multiobjective optimization problems (compare [22]). If $D = \{0\}$, then the relation $\preceq_D^u$ describes set-inclusions. In case $D = \{d\}$, where $d \in \mathbb{R}^q$, the relation $\preceq_D^u$ can be used to judge whether one set A is a relocation of another set B (compare Figure 2.2).*

The following result gives a necessary condition for the generalized upper set less relation to hold by means of the nonlinear scalarizing functional $z^{D,k}$.

Theorem 2.2.4 ([67, Theorem 2.10])**.** *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, $k \in Y \setminus \{0\}$ such that (2.1) is fulfilled, $\widetilde{D} \subseteq Y$ such that $D + \widetilde{D} \subseteq D$, and $A, B \in \mathcal{P}(Y)$. Then it holds*

$$A \subseteq B - \widetilde{D} \implies \sup_{a \in A} z^{D,k}(a) \leq \sup_{b \in B} z^{D,k}(b).$$

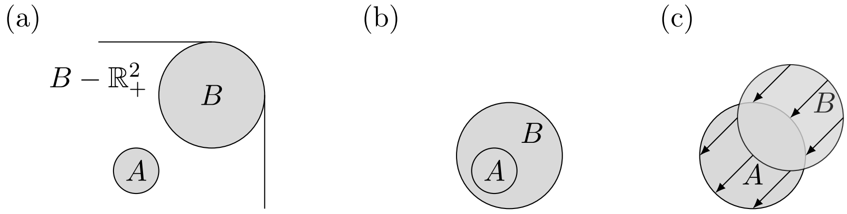

Figure 2.2: The relation $\preceq^u_D$ includes, among others, (a) worst-case-oriented set comparisons, (b) set-inclusion and (c) relocations of a set.

Proof. Choose an arbitrary vector $k \in Y \setminus \{0\}$ such that (2.1) is satisfied, and let $A \subseteq B - \widetilde{D}$. Then, we have

$$\forall\, a \in A,\ \exists\, b \in B:\ a \in b - \widetilde{D}\,.$$

The monotonicity property of the functional $z^{D,k}$ (compare Theorem 2.1.2 (d)) yields

$$\forall\, a \in A,\ \exists\, b \in B:\ z^{D,k}(a) \leq z^{D,k}(b)\,.$$

Therefore, we conclude with the stated inequality. □

The example below illustrates Theorem 2.2.4 for the case that $D = \widetilde{D}$ coincides with the natural ordering cone in $\mathbb{R}^2$ and verifies that the inverse implication of the assertion in Theorem 2.2.4 is generally not satisfied, even if the underlying sets are convex (and even singletons).

Example 2.2.5 ([66, Example 3.2])**.** *Let $Y := \mathbb{R}^2$, $a := (-1/4, -1/4)^T$, $A := \{a\}$, $\overline{a} := (3/4, 3/4)^T$, $\overline{A} := \{\overline{a}\}$, $B := \{(s, 1-s)^T \mid s \in [0,1]\}$, $k := (k_1, k_2)^T$, $k_1, k_2 > 0$, and consider the natural ordering cone $D = \widetilde{D} = C = \mathbb{R}^2_+$, see Figure 2.3. It holds*

$$\sup_{b \in B} z^{C,k}(b) = \sup_{s \in [0,1]} \max\left\{\frac{s}{k_1}, \frac{1-s}{k_2}\right\} = \max\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} > 0$$

and

$$\sup_{\widetilde{a} \in A} z^{C,k}(\widetilde{a}) = z^{C,k}(a) = -\frac{1}{4}\min\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} < 0 < \sup_{b \in B} z^{C,k}(b)$$

corresponding to $A \subseteq B - C$ as well as

$$\sup_{\widetilde{a} \in \overline{A}} z^{C,k}(\widetilde{a}) = z^{C,k}(\overline{a}) = \frac{3}{4}\max\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} < \sup_{b \in B} z^{C,k}(b)$$

but clearly

$$\overline{A} \not\subseteq B - C\,.$$

The following result, shown in [67], gives an equivalent representation for $A \preceq^u_D B$.

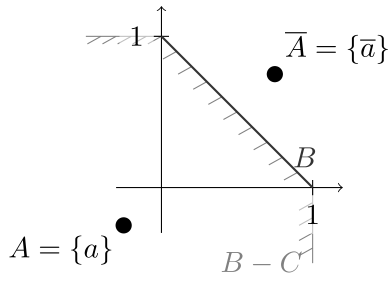

Figure 2.3: Illustration of Example 2.2.5.

Theorem 2.2.6 ([67, Theorem 2.8]). *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1). *For two sets $A, B \in \mathcal{P}(Y)$, the following implication holds:*

$$A \subseteq B - D \quad \Longrightarrow \quad \sup_{a \in A} \inf_{b \in B} z^{D,k}(a-b) \leq 0\,.$$

Assume on the other hand, that there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{b \in B} z^{D,k_0}(a-b)$ is attained for all $a \in A$, then the converse is also true, i.e.,*

$$\sup_{a \in A} \inf_{b \in B} z^{D,k_0}(a-b) \leq 0 \quad \Longrightarrow \quad A \subseteq B - D\,.$$

Proof. Let $A \subseteq B - D$. This means

$$\forall\, a \in A,\quad \exists\, b \in B:\ a \in b - D \quad \Longrightarrow \quad \forall\, a \in A,\quad \exists\, b \in B:\ a - b \in -D\,.$$

Because of Theorem 2.1.2 (f) with $r = 0$ and $y = a - b$, we have

$$\forall\, a \in A,\ \exists\, b \in B:\ z^{D,k}(a-b) \leq 0\,,$$

and this implies

$$\sup_{a \in A} \inf_{b \in B} z^{D,k}(a-b) \leq 0\,.$$

Conversely, let $k_0 \in Y \setminus \{0\}$ be given such that for all $a \in A$ the infimum $\inf_{b \in B} z^{D,k_0}(a-b)$ is attained. Let

$$\sup_{a \in A} \inf_{b \in B} z^{D,k_0}(a-b) \leq 0\,, \tag{2.4}$$

but assume that $A \not\subseteq B - D$. Thus, there exists some $\bar{a} \in A$ with $\bar{a} \notin B - D$. So for all $b \in B$ it holds $\bar{a} - b \notin -D$ and with Theorem 2.1.2 (f) with $r = 0$ and $y = \bar{a} - b$, we obtain

$$\exists\, \bar{a} \in A,\ \forall\, b \in B : z^{D,k_0}(\bar{a}-b) > 0 \quad \Longrightarrow \quad \exists\, \bar{a} \in A:\ \inf_{b \in B} z^{D,k_0}(\bar{a}-b) > 0\,.$$

Because the last infimum is attained by assumption, one concludes that

$$\sup_{\bar{a} \in A} \inf_{b \in B} z^{D,k_0}(\bar{a}-b) > 0\,,$$

in contradiction to the inequality (2.4). $\square$

The following example illustrates the statements in Theorems 2.2.4 and 2.2.6 and verifies again that the inverse implication of the assertion in Theorem 2.2.4 is generally not fulfilled.

Example 2.2.7 ([67, Example 2.11]). *Let* $Y := \mathbb{R}^2$, $A := \{(0,0)\} =: \{a\}$, $\bar{A} := \{(3/4, 7/4)^T\} =: \{\bar{a}\}$, $B := \{(s, 1-s)^T \mid s \in [0,1]\}$, $k := (k_1, k_2)^T$, k_1, $k_2 > 0$, $\widetilde{D} = \mathbb{R}^2_+$, $D = \mathbb{R}^2_+ - \{(0,1)\}$, *see Figure 2.4. So we have* $A \subseteq B - D$. *It holds for* $y \in \mathbb{R}^2$

$$\begin{aligned} z^{D,k}(y) &= \inf\{t \in \mathbb{R} | y \in tk - D\} \\ &= \inf\left\{t \in \mathbb{R} | (y_1, y_2) - (0,1) \in t(k_1, k_2) - \mathbb{R}^2_+\right\} \\ &= \max\left\{\frac{y_1}{k_1}, \frac{y_2 - 1}{k_2}\right\}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \sup_{a \in A} \inf_{b \in B} z^{D,k}(a - b) &= \inf_{b \in B} z^{D,k}(a - b) \\ &= \inf_{b \in B} \max\left\{\frac{a_1 - b_1}{k_1}, \frac{a_2 - b_2 - 1}{k_2}\right\} \\ &= \inf_{b \in B} \max\left\{\frac{-b_1}{k_1}, \frac{-b_2 - 1}{k_2}\right\} \\ &= \min_{s \in [0,1]} \max\left\{\frac{-s}{k_1}, \frac{s - 2}{k_2}\right\} \leq 0\,. \end{aligned}$$

Moreover, it holds $\bar{A} \not\subseteq B - D$, *and it can be shown that* $\sup_{\bar{a} \in \bar{A}} \inf_{b \in B} z^{D,k}(\bar{a} - b) = \min_{s \in [0,1]} \max\left\{\frac{3/4 - s}{k_1}, \frac{s - 1/4}{k_2}\right\} > 0$, *in accordance with Theorem 2.2.6. Furthermore, it holds* $A \subseteq B - \widetilde{D}$, *and* $D + \widetilde{D} \subseteq D$. *So, the assumptions in Theorem 2.2.4 are satisfied. Thus, we obtain*

$$\sup_{a \in A} z^{D,k}(a) = \max\left\{0, \frac{-1}{k_2}\right\} = 0\,,$$

and

$$\sup_{b \in B} z^{D,k}(b) = \sup_{s \in [0,1]} \max\left\{\frac{s}{k_1}, \frac{(1-s) - 1}{k_2}\right\} = \sup_{s \in [0,1]} \max\left\{\frac{s}{k_1}, \frac{-s}{k_2}\right\} = \frac{1}{k_1}\,.$$

Therefore, we obtain $\sup_{a \in A} z^{D,k}(a) \leq \sup_{b \in B} z^{D,k}(b)$. *Furthermore, we have*

$$\sup_{\bar{a} \in \bar{A}} z^{D,k}(\bar{a}) = \max\left\{\frac{3}{4k_1}, \frac{3}{4k_2}\right\},$$

and for $k = (k_1, k_2) = (1,1)$ *we obtain*

$$\sup_{\bar{a} \in \bar{A}} z^{D,k}(\bar{a}) = \frac{3}{4} < 1 = \sup_{b \in B} z^{D,k}(b)\,,$$

but $\bar{A} \not\subseteq B - \widetilde{D}$. *This shows that the converse implication of the assertion in Theorem 2.2.4 is not fulfilled.*

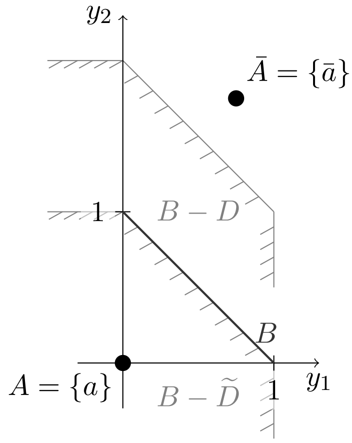

Figure 2.4: Illustration of Example 2.2.7.

Remark 2.2.8. *The generalized upper set less relation $\preceq^u_D$ can be used for the treatment of set optimization problems to compare sets. Theorems 2.2.4 and 2.2.6 are useful to decide whether two sets fulfill the relation $\preceq^u_D$ in a numerical manner. Furthermore, these results even give a quantification by means of the extremal points of the functional values $z^{D,k}(a-b)$. Therefore, Theorems 2.2.4 and 2.2.6 can be used to derive algorithms, for instance an iterative pattern search where in each iteration the minimal function value is determined to specify the locally best search direction (i.e., a so-called descent method, compare [55]). Such algorithm types are very useful for solving set optimization problems and will be presented in Chapter 2.3.*

As our goal in this work is to study different extensions of several known set relations (and their corresponding representation by means of the functional $z^{D,k}$), we continue with introducing the following extension of the lower set less relation by Kuroiwa [76, 77].

Definition 2.2.9 (Generalized Lower Set Less Relation, [68, Definition 3.5])**.** *Let $D \in \mathcal{P}(Y)$. The **generalized lower set less relation** $\preceq^l_D$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by*

$$A \preceq^l_D B :\Longleftrightarrow B \subseteq A + D,$$

which is equivalent to

$$\forall\ b \in B,\ \exists\ a \in A:\ b \in a + D.$$

Remark 2.2.10 ([68, Remark 3.6])**.** *Notice that $\preceq^l_D$ is transitive if $D + D \subseteq D$ and it is reflexive if $0 \in D$. Therefore, $\preceq^l_D$ is a preorder if $D + D \subseteq D$ and $0 \in D$.*

If the set D in Definition 2.2.9 is replaced by a convex cone $C \subset Y$, then this definition coincides with the definition of the lower set less relation introduced by Kuroiwa [76, 77], and $B \subseteq A + C$ can be replaced by

$$\forall\ b \in B,\ \exists\ a \in A:\ a \leq_C b\,,$$

where $\leq_C$ relates to the order relation induced by the convex cone C, thus, $a \leq_C b$ means that $a \in b - C$.

The following theorem gives a first insight into the relations between the generalized lower set less relation and the functional $z^{D,k}$.

Theorem 2.2.11 ([68, Theorem 2.7]). *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, $k \in Y \setminus \{0\}$ such that* (2.1) *is fulfilled, let $\widetilde{D} \subseteq Y$ such that $D + \widetilde{D} \subseteq D$, and $A, B \in \mathcal{P}(Y)$. Then it holds*

$$B \subseteq A + \widetilde{D} \implies \inf_{a \in A} z^{D,k}(a) \leq \inf_{b \in B} z^{D,k}(b).$$

Proof. Choose an arbitrary vector $k \in Y \setminus \{0\}$ such that (2.1) is satisfied, and let $B \subseteq A + \widetilde{D}$. Then, we have

$$\forall\, b \in B,\ \exists\, a \in A:\ b \in a + \widetilde{D}\,.$$

The monotonicity property of the functional $z^{D,k}$ (compare Theorem 2.1.2 (d)) yields

$$\forall\, b \in B,\ \exists\, a \in A:\ z^{D,k}(a) \leq z^{D,k}(b)\,.$$

Therefore, we conclude with the stated inequality. □

A consequence of Theorem 2.2.11 is comprised in the following corollary, which was proven in [66, Theorem 3.15].

Corollary 2.2.12 ([68, Corollary 3.8]). *Let $D \in \mathcal{P}(Y)$ be a closed proper convex cone in Y, $k \in Y \setminus \{0\}$, $A, B \in \mathcal{P}(Y)$. Then it holds*

$$B \subseteq A + D \implies \inf_{a \in A} z^{D,k}(a) \leq \inf_{b \in B} z^{D,k}(b).$$

We derive the following result in correspondence with Theorem 2.2.6 for the generalized lower set less relation.

Theorem 2.2.13 ([68, Theorem 3.9]). *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, the following implication holds:*

$$B \subseteq A + D \quad \implies \quad \sup_{b \in B} \inf_{a \in A} z^{D,k}(a - b) \leq 0\,.$$

On the other hand, assume that there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{a \in A} z^{D,k_0}(a - b)$ is attained for all $b \in B$, then*

$$\sup_{b \in B} \inf_{a \in A} z^{D,k_0}(a - b) \leq 0 \quad \implies \quad B \subseteq A + D\,.$$

Proof. Let $B \subseteq A + D$. This means

$$\forall\, b \in B,\ \ \exists\, a \in A:\ b \in a + D \quad \implies \quad \forall\, b \in B,\ \ \exists\, a \in A:\ a - b \in -D\,.$$

Because of Theorem 2.1.2 (f) with $r = 0$ and $y = a - b$, we have

$$\forall\, b \in B,\ \exists\, a \in A:\ z^{D,k}(a - b) \leq 0\,,$$

and this implies
$$\sup_{b\in B}\ \inf_{a\in A} z^{D,k}(a-b) \le 0\,.$$
Conversely, let $k_0 \in Y\backslash\{0\}$ be given such that for all $b \in B$ the infimum $\inf_{a\in A} z^{D,k_0}(a-b)$ is attained. Let
$$\sup_{b\in B}\inf_{a\in A} z^{D,k_0}(a-b) \le 0\,.$$
That means
$$\forall\ b \in B :\ \inf_{a\in A} z^{D,k_0}(a-b) \le 0\,.$$
Because for all $b \in B$ the infimum $\inf_{a\in A} z^{D,k_0}(a-b)$ is attained, we obtain
$$\forall\ b \in B\ \exists\ \overline{a} \in A :\ z^{D,k_0}(\overline{a}-b) = \inf_{a\in A} z^{D,k_0}(a-b) \le 0\,.$$
By Theorem 2.1.2 (f) with $r = 0$ and $y = a-b$, we conclude with
$$\forall\ b \in B\ \exists\ \overline{a} \in A :\ \overline{a} - b \in -D,$$
thus $B \subseteq A + D$. □

Example 2.2.14 (Weighted Sum Scalarization, [66, Corollary 3.12])**.** *Let $Y := \mathbb{R}^m$, a vector $w := (w_1, \ldots, w_m)^T \in \mathbb{R}^m$ with $w_i \ge 0$, $i = 1, \ldots, m$, $D := \{y \in \mathbb{R}^m \mid w^T y \ge 0\}$ (note that D is a convex cone, but D is not pointed) and $k := (k_1, \ldots, k_m)^T \in \operatorname{int} D$ be given. Then we have for $A, B \in \mathcal{P}(\mathbb{R}^m)$, $a \in A$ and $b \in B$:*
$$\begin{aligned}
z^{D,k}(a-b) &= \inf\{t \in \mathbb{R} \mid a-b \in tk - D\} \\
&= \inf\{t \in \mathbb{R} \mid w(a-b) \le w^T tk\} \\
&= \inf\{t \in \mathbb{R} \mid w(a-b) \le t \cdot w^T k\} \\
&\overset{k\in \operatorname{int} D}{=} \inf\{t \in \mathbb{R} \mid \frac{1}{w^T k} \cdot \sum_{i=1}^m w_i(a_i - b_i) \le t\} \\
&= \frac{1}{w^T k} \cdot \sum_{i=1}^m w_i(a_i - b_i)\,.
\end{aligned}$$
This leads to
$$\begin{aligned}
\sup_{a\in A}\inf_{b\in B} z^{D,k}(a-b) &= \sup_{a\in A}\inf_{b\in B} \frac{1}{w(k)} \cdot \sum_{i=1}^m w_i(a_i - b_i) \\
&= \sup_{a\in A} \frac{1}{w^T k} \cdot \sum_{i=1}^m w_i a_i - \sup_{b\in B} \frac{1}{w^T k} \cdot \sum_{i=1}^m w_i b_i \\
&= \frac{1}{w^T k} \cdot \left(\sup_{a\in A} \sum_{i=1}^m w_i a_i - \sup_{b\in B} \sum_{i=1}^m w_i b_i \right).
\end{aligned}$$

Hence, with the above definitions of D and k and weights $w_i > 0$, $i = 1, \dots, m$, we obtain due to Theorems 2.2.6 and 2.2.13

$$\begin{aligned} A \subseteq B - D \quad &\iff \quad \forall\ k \in \operatorname{int} D : \ \frac{1}{w^T k} \sup_{a\in A} \sum_{i=1}^m w_i a_i \le \frac{1}{w^T k} \sup_{b \in B} \sum_{i=1}^m w_i b_i \\ &\iff \quad \sup_{a\in A} \sum_{i=1}^m w_i a_i \le \sup_{b \in B} \sum_{i=1}^m w_i b_i \quad \textit{and, in analogy,} \\ B \subseteq A + D \quad &\iff \quad \forall\ k \in \operatorname{int} D : \ \frac{1}{w^T k} \inf_{a\in A} \sum_{i=1}^m w_i a_i \le \frac{1}{w^T k} \inf_{b \in B} \sum_{i=1}^m w_i b_i \\ &\iff \quad \inf_{a\in A} \sum_{i=1}^m w_i a_i \le \inf_{b \in B} \sum_{i=1}^m w_i b_i. \end{aligned}$$

Note that we only considered $k \in \operatorname{int} D$ here in order to exclude division by zero for this rather algorithmic example. Moreover, the attainment of the infima and suprema, respectively, is implicitly required.

Example 2.2.15 (Natural Ordering [66, Corollary 3.13])**.** *Let again $Y := \mathbb{R}^m$, $D := \mathbb{R}^m_+$ and $k := (k_1, \dots, k_m)^T \in \operatorname{int} C$. Then we have*

$$z^{D,k}(a-b) = \sup_{i=1,\dots,m} \frac{(a-b)_i}{k_i}.$$

Hence, with the above definitions of D and k, the assertions in Theorems 2.2.6 and 2.2.13 lead to

$$\begin{aligned} A \subseteq B - D \quad &\iff \quad \forall\ k \in \operatorname{int} D : \ \sup_{a\in A} \inf_{b\in B} \max_{i=1,\dots,m} \frac{(a-b)_i}{k_i} \le 0, \\ B \subseteq A + D \quad &\iff \quad \forall\ k \in \operatorname{int} D : \ \sup_{b\in B} \inf_{a\in A} \max_{i=1,\dots,m} \frac{(a-b)_i}{k_i} \le 0. \end{aligned}$$

Example 2.2.16 (Polyhedral Cones [66, Corollary 3.14])**.** *More generally, if $Y = \mathbb{R}^m$ and the cone D is given by $D := \{y \in \mathbb{R}^m \mid (Wy)_i \ge 0$ for all $i = 1, \dots, l\}$ for a given matrix $W \in \mathbb{R}^{l,m}$, $w_{ij} \ge 0$ for all $i = 1, \dots, l$, $j = 1, \dots, m$, where every row of the matrix W cannot be the zero vector, the value of the nonlinear scalarizing functional $z^{D,k}(y)$ can be obtained by*

$$z^{D,k}(y) = \max_{i=1,\dots,l} \frac{(Wy)_i}{(Wk)_i}.$$

Note that $k \in \operatorname{int} D$ implies $(Wk)_i \neq 0$ for all $i = 1, \dots, l$, such that this value is well defined and also that Examples 2.2.14 and 2.2.15 are special cases with $l = 1$ and $W = I_m$ (identity matrix), respectively.

In the following definition, we extend the notion of the set less relation (see Young [111] and Nishnianidze [85]).

Definition 2.2.17 (Generalized Set Less Relation, [68, Definition 3.10])**.** *Let $D \in \mathcal{P}(Y)$. The* ***generalized set less relation*** $\preceq^s_D$ *is defined for two sets $A, B \in \mathcal{P}(Y)$ by*

$$A \preceq^s_D B :\Longleftrightarrow A \preceq^u_D B \text{ and } A \preceq^l_D B.$$

The next result follows directly from Theorems 2.2.6 and 2.2.13.

Corollary 2.2.18. *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, we have*

$$A \preceq^s_D B \quad \Longrightarrow \quad \sup_{a \in A} \inf_{b \in B} z^{D,k}(a-b) \leq 0 \text{ and } \sup_{b \in B} \inf_{a \in A} z^{D,k}(a-b) \leq 0.$$

If, on the other hand, there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{b \in B} z^{D,k_0}(a-b)$ is attained for all $a \in A$, and if there exists $k_1 \in Y \setminus \{0\}$ satisfying* (2.1) *such that $\inf_{a \in A} z^{D,k_1}(a-b)$ is attained for all $b \in B$, then*

$$A \preceq^s_D B \quad \Longleftarrow \quad \sup_{a \in A} \inf_{b \in B} z^{D,k_0}(a-b) \leq 0 \text{ and } \sup_{b \in B} \inf_{a \in A} z^{D,k_1}(a-b) \leq 0.$$

The following definition is an extension of the certainly less relation (see Jahn, Ha [56], Eichfelder, Jahn [26]).

Definition 2.2.19 (Generalized Certainly Less Relation, [68, Definition 3.12])**.** *Let $D \in \mathcal{P}(Y)$. The* ***generalized certainly less relation*** $\preceq^c_D$ *is defined for two sets $A, B \in \mathcal{P}(Y)$ by*

$$A \preceq^c_D B :\Longleftrightarrow (A = B) \text{ or } (\forall\, a \in A,\ \forall\, b \in B:\ a \in b - D)\,.$$

Figure 2.5 illustrates Definition 2.2.19.

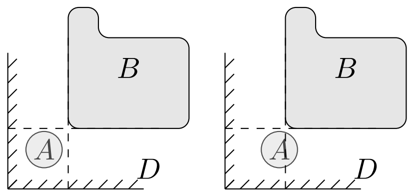

Figure 2.5: Illustration of the relations $A \preceq^c_D B$ in the first image and $A \not\preceq^c_D B$ in the second picture.

The following result does not require any attainment property. We omit its proof, as it is similar to that of Theorem 2.2.13.

Theorem 2.2.20 ([68, Theorem 3.13])**.** *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying* (2.1)*. For two sets $A, B \in \mathcal{P}(Y)$, the following equivalence holds:*

$$\forall\, a \in A,\ \forall\, b \in B:\ a \in b - D \quad \Longleftrightarrow \quad \sup_{(a,b) \in A \times B} z^{D,k}(a-b) \leq 0\,.$$

Applying Theorem 2.2.20 to the definition of the generalized certainly less relation, we obtain the following result.

Corollary 2.2.21 ([68, Corollary 3.14]). *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, $k \in Y \setminus \{0\}$ such that (2.1) is fulfilled, $A, B \in \mathcal{P}(Y)$. Then we have the following equivalence for the generalized certainly less relation:*

$$A \preceq_D^c B \quad \Longleftrightarrow \quad (A = B) \ \textit{or} \ \left(\sup_{(a,b)\in A\times B} z^{D,k}(a-b) \leq 0 \right).$$

Example 2.2.22. *Note that adding the possibility that A equals B in Corollary 2.2.21 is necessary in order to get the classification $A \preceq_D^c B$. If, for example, $A = B \subset \mathbb{R}^2$ is the unit ball in $\mathbb{R}^2$ and the natural ordering is considered by $D = C = \mathbb{R}^2_+$, we have*

$$\forall\ k \in \operatorname{int} C \quad \exists\ a, b \in A : \ z^{C,k}(a-b) > 0,$$

but clearly $A \preceq_C^c B$, see Figure 2.6. If C is pointed (that is, $C \cap (-C) = \{0\}$), note that

$$\begin{aligned}
\sup_{a,a'\in A} z^{C,k}(a-a') \leq 0 \quad &\Longleftrightarrow \quad \forall a, a' \in A : \ z^{C,k}(a-a') \leq 0 \\
&\Longleftrightarrow \quad \forall a, a' \in A : \ a - a' \in -C \\
&\Longleftrightarrow \quad A - A \subseteq -C \\
&\Longleftrightarrow \quad A - A \subseteq (-C) \cap C = \{0\} \\
&\Longleftrightarrow \quad A \textit{ is a singleton.}
\end{aligned}$$

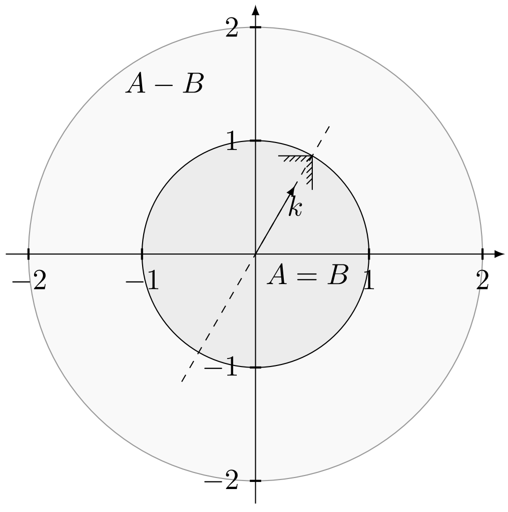

Figure 2.6: $\sup_{(a,b)\in A\times B} z^{C,k}(a-b) > 0$ for $A = B$ in Example 2.2.22.

Remark 2.2.23. *Notice that it is remarkable that the result in Corollary 2.2.21 holds true for arbitrary $k \in Y \setminus \{0\}$ fulfilling (2.1). Therefore, we conclude that $A \preceq^c_D B$ is equivalent to*

$$(A = B) \ \text{or} \ \left(\forall\, k \in Y \setminus \{0\} \text{ satisfying (2.1)}: \sup_{(a,b)\in A\times B} z^{D,k}(a-b) \le 0 \right).$$

The next definition is a more general form of the possibly less relation (see [17, 56]).

Definition 2.2.24 (Generalized Possibly Less Relation, [68, Definition 3.15])**.** *Let $D \in \mathcal{P}(Y)$. The **generalized possibly less relation** $\preceq^p_D$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by*

$$A \preceq^p_D B :\Longleftrightarrow \exists\, a \in A,\ \exists\, b \in B:\ a \in b - D.$$

The following result shows that the nonlinear scalarizing functional $z^{D,k}$ is useful for the characterization of the generalized possibly less relation.

Theorem 2.2.25 ([68, Theorem 3.16])**.** *Let $D \in \mathcal{P}(Y)$ be a closed proper set in Y, and $k \in Y \setminus \{0\}$ satisfying (2.1). For two sets $A, B \in \mathcal{P}(Y)$, the following implication holds:*

$$\exists\, a \in A,\ \exists\, b \in B:\ a \in b - D \quad \Longrightarrow \quad \inf_{(a,b)\in A\times B} z^{D,k}(a-b) \le 0..$$

If there exists $k_0 \in Y \setminus \{0\}$ satisfying (2.1) such that $\inf_{(a,b)\in A\times B} z^{D,k_0}(a-b)$ is attained, we have:

$$\inf_{(a,b)\in A\times B} z^{D,k_0}(a-b) \le 0 \quad \Longrightarrow \quad \exists\, a \in A, \exists\, b \in B:\ a \in b - D.$$

Remark 2.2.26 ([68, Remark 3.17])**.** *Of course, many other set relations can be found in the literature. Some of them can be generalized in the way we conducted so far. For example, the minmax less relation and the minmax certainly less relation, given in Jahn, Ha [56] can be generalized and expressed via the nonlinear scalarizing functional $z^{D,k}$. Moreover, in Kuroiwa et al. [79] the following set relations are presented (with D being a proper closed convex cone):*

$$A \preceq^{(ii)} B :\Longleftrightarrow \exists\, a \in A:\ \forall\, b \in B,\ a \in b - D$$

and

$$A \preceq^{(iv)} B :\Longleftrightarrow \exists\, b \in B:\ \forall\, a \in A,\ a \in b - D.$$

Under appropriate attainment properties and if D and $k \in Y \setminus \{0\}$ satisfy (2.1), these relations are concerned with

$$\inf_{a\in A} \sup_{b\in B} z^{D,k}(a-b) \le 0 \quad \text{and} \quad \inf_{b\in B} \sup_{a\in A} z^{D,k}(a-b) \le 0.$$

However, we will not pursue them any further, as they are similar to $\preceq^u_D$ as well as $\preceq^l_D$, and coincide by simply interchanging the infima and suprema.

2.3 Algorithms for Determining Minimal Elements

This section is concerned with presenting several algorithms for finding minimal elements (minimal solutions, respectively) of a family of sets with respect to the generalized set relations that we introduced and characterized in Section 2.2. The results presented in this section rely mainly on [66] and [68].

2.3.1 A descent method

In the literature, there already exist some algorithms for solving set-valued optimization problems based on descent methods. For example, Jahn [55] proposes a descent method that generates approximations of minimal elements of set-valued optimization problems under convexity assumptions on the considered sets. In [55], the set less relation is characterized by means of linear functionals. More recently, in [66], the authors propose a similar descent method for obtaining approximations of minimal elements of set-valued optimization problems. In [66], several set relations are characterized by the nonlinear scalarizing functional $z^{D,k}$, where D is assumed to be a proper convex cone. Since the nonlinear functional $z^{D,k}$ is used in [66], no convexity assumptions on the considered outcome sets $F(x)$ are needed. Note that the approaches in [55, 66] all rely on set relations where the involved domination structure is given by cones, whereas in this section, we consider arbitrary nonemtpy sets $D \subset Y$.

Here we consider the set-valued optimization problem (1.10) with $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, thus, we have the following setting: The objective map is $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a set relation $\preceq$ is given. In this section, we are looking for approximations of **minimal solutions** w.r.t. the relation $\preceq$ in the sense of Definition 1.2.12 of the problem

$$\min_{x \in \mathbb{R}^n} F(x)\,.$$

The results in Section 2.2 provide us with a possibility to decide whether two sets fulfill the set relation or not in a numerical manner and even give a quantification by means of the extremal points of the functional values $z^{D,k}(a-b)$, $z^{D,k}(b-a)$, respectively. So a natural way of constructing an algorithm for solving problem (1.10) is an iterative pattern search where in each iteration the minimal function value is determined to specify the locally best search direction. For this reason we refer to Algorithm 2.3.2 below as a *descent method*, cf. [55].

For the following algorithm it is very important to have an easy way to calculate the functional $z^{D,k}$. With this aim, in the following example, we consider a special structure of the set D in the definition of $z^{D,k}$ to exemplarily show how the functional $z^{D,k}$ can be computed numerically. In order to study such a special structure, we introduce a set A_γ in the following way (see Tammer, Winkler [102]):

Example 2.3.1 (Compare [66]). *Let γ be a norm on $\mathbb{R}^m$ which is characterized by its (closed) unit ball*

$$B_\gamma := \{y \in \mathbb{R}^m \mid \ \gamma(y) \leq 1\}.$$

A norm γ is called a **block norm**, *if its unit ball B_γ is polyhedral (a polytope). Let $\bar{y} \in \mathbb{R}^m$. The* **reflection set** *of $\bar{y}$ is defined by*

$$R(\bar{y}) := \{y \in \mathbb{R}^m \mid \ |y_i| = |\bar{y}_i| \ \ \forall\, i = 1, ..., m\}.$$

A norm γ is called **absolute**, *if $\gamma(y) = \gamma(\bar{y})$ for all $y \in R(\bar{y})$. A block norm γ is called* **oblique**, *if γ is absolute and satisfies $(y - \mathbb{R}^m_+) \cap \mathbb{R}^m_+ \cap \operatorname{bd} B_\gamma = \{y\}$ for all $y \in \mathbb{R}^m_+ \cap \operatorname{bd} B_\gamma$.*

Let γ be a block norm with unit ball B_γ, given for $a^i \in \mathbb{R}^m$, $\alpha_i \in \mathbb{R}$, $i = 1, ..., n$, by

$$B_\gamma = \{y \in \mathbb{R}^m \mid \ \langle a^i, y\rangle \le \alpha_i, \ i = 1, ..., n\}.$$

The number of halfspaces that define the ball B_γ coincides with the dimension of the decision space of problem (1.10). *Using a^i from this formula for B_γ, we define a set $A_\gamma \subset \mathbb{R}^m$ by*

$$A_\gamma := \{y \in \mathbb{R}^m \mid \ \langle a^i, y\rangle \le \alpha_i, \ i \in I\} \tag{2.5}$$

with the index set

$$I := \{i \in \{1, ..., n\} \mid \ \{y \in \mathbb{R}^m : \ \langle a^i, y\rangle = \alpha_i\} \cap B_\gamma \cap \operatorname{int} \mathbb{R}^m_+ \neq \emptyset\}.$$

The set I is exactly the set of indices $i = 1, ..., n$ for which the hyperplanes $\langle a^i, y\rangle = \alpha_i$ are active in the positive orthant.

Let γ be an absolute block norm with unit ball B_γ and the corresponding set A_γ defined as in (2.5), let vectors $k \in \operatorname{int} \mathbb{R}^m_+$ and $w \in \mathbb{R}^m$ be given. We define a functional $z^{A_\gamma + w, k} : \mathbb{R}^m \to \mathbb{R}$ by

$$z^{A_\gamma + w, k}(y) = \inf\{\tau \in \mathbb{R} \mid \ y \in \tau k + A_\gamma + w\}, \qquad y \in \mathbb{R}^m. \tag{2.6}$$

The functional $z^{A_\gamma + w, k}$ depends on the norm γ and the parameters k and w.

Let γ be an oblique block norm with unit ball B_γ and the corresponding set A_γ; let $k \in \operatorname{int} \mathbb{R}^m_+$ and $w \in \mathbb{R}^m$ be arbitrary. Then the functional $z^{A_\gamma + w, k}$ defined by formula (2.6) is strictly $\mathbb{R}^m_+$-monotone.

For given $y \in \mathbb{R}^m$, we can calculate the value $z^{A_\gamma + w, k}(y)$ by the following formula (see Tammer, Winkler [102]):

Let γ be an absolute (oblique) block norm with unit ball B_γ and the corresponding set A_γ defined as in (2.5), let vectors $k \in \operatorname{int} \mathbb{R}^m_+$ and $w \in \mathbb{R}^m$ be given. We consider the functional $z^{A_\gamma + w, k} : \mathbb{R}^m \to \mathbb{R}$ defined by (2.6). Then $z^{A_\gamma + w, k}$ is a finite-valued, continuous, convex, $\mathbb{R}^m_+$-monotone (strictly $\mathbb{R}^m_+$-monotone) functional with

$$z^{A_\gamma + w, k}(y) = \max_{i \in I} \frac{\langle a^i, y\rangle - \langle a^i, w\rangle - \alpha_i}{\langle a^i, k\rangle}. \tag{2.7}$$

With the formula (2.7) it is very easy to compute the objective function values $z^{D,k}(a - b)$ in the following algorithm.

The following algorithm calculates an approximation of a minimal solution of the set-valued problem (1.10), where $\preceq$ is assumed to be a preorder. It is presented in [66] for the case that D is a convex cone, and given here more generally.

Algorithm 2.3.2. *(A descent method for finding an approximation of a minimal solution of the set-valued problem* (1.10)*)*

Input: *$F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, set D, preorder $\preceq$, starting point $x^0 \in \mathbb{R}^n$, a set K of vectors $k_0^i \in D \setminus \{0\}$ to determine the required attainment property, maximal number i_{max} of iterations, number of search directions n_s, maximal number j_{max} of iterations for the determination of the step size, initial step size h_0 and minimum step size $h_{\min}$*

% initialization
$i := 0$, $h := h_0$
choose n_s points $\widetilde{x}^1, \widetilde{x}^2, \ldots, \widetilde{x}^{n_s}$ on the unit sphere around $0_{\mathbb{R}^n}$

% iteration loop
while $i \leq i_{max}$ ***do***
    check $F(x^i + h\widetilde{x}^j) \preceq F(x^i)$ for every $j \in \{1, \ldots, n_s\}$ by evaluating the extremal term (e. g. $\sup_{a\in A} \inf_{b\in B} z^{D,k_0^i}(a-b)$ for $A = F(x^i + h\widetilde{x}^j)$ and $B = F(x^i)$, when $\preceq = \preceq_D^u$ for some $k_0^i \in K$ fulfilling the required attainment property). Choose the index $n_0 := j$ with the smallest function value extremal_{term}.
    if $\text{extremal}_{term} \leq 0$ ***then***
        $x^{i+1} := x^i + h\widetilde{x}^{n_0}$ *% new iteration point*
        $j := 1$
        while $F(x^i + (j+1)h\widetilde{x}^{n_0}) \preceq F(x^i + jh\widetilde{x}^{n_0})$ *and* $j \leq j_{max}$ ***do***
            $j := j + 1$
            $x^{i+1} := x^{i+1} + h\widetilde{x}^{n_0}$ *% new iteration point*
        end while
    else
        $h := h/2$
        if $h \leq h_{\min}$ ***then***
            STOP $x := x^i$
        end if
    end if
    $i := i + 1$
end while

Output: *An approximation x of a minimal solution of the set-valued problem* (1.10) *w.r.t. $\preceq$.*

For one given starting point x^0, Algorithm 2.3.2 approximates one minimal solution of problem (1.10). To find more than one approximation of minimal solutions, one needs to vary the input parameters, such as choosing a different starting point $x^0 \in \mathbb{R}^n$, or modifying the vector $k^0 \in D\setminus\{0\}$ (which should fulfill the required attainment property). Determining efficient ways to ensure that all minimal solutions are well-approximated will be the topic of future research.

We emphasize that for Algorithm 2.3.2, we do not need any convexity assumptions on the considered sets. So in the following numerical example we turn our attention to a set-valued map with nonconvex images.

Example 2.3.3. *Let $\Delta_t := 2\pi/40$ and $\mathcal{T} := \{j \cdot \Delta_t,\, j = 0, \ldots, 40\}$. We define the set valued mapping $F\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by*

$$F(x) := \left\{ \begin{pmatrix} x_1^2 + x_2^2 \cdot \sin(2t) \\ x_2^2 + x_1^2 \cdot \cos(3t) \end{pmatrix} \;\middle|\; t \in \mathcal{T} \right\}$$

where $x = (x_1, x_2)^T$. The example is chosen such that the unique minimizer is attained at $x = 0_{\mathbb{R}^2}$.

We apply Algorithm 2.3.2 to the problem with starting point $x^0 := (6,5)^T$ using the natural ordering cone $D := \mathbb{R}^2_+$ and the upper set less relation $\preceq^u_D$. Initial and minimal step lengths $h_0 := 2.5$ and $h_{\min} := 10^{-4}$ have been used.

For this discrete example the attainment property is trivially fulfilled such that any $k \in D \setminus \{0\}$ can be used in order to get the equivalences in Theorem 2.2.6. For the numerical example presented here $k^0 := \frac{1}{2}(\sqrt{2}, \sqrt{2})^T$ and $n_s := 5$ search directions were chosen.

Numerical results are depicted in Figure 2.7. On the diagrams to the left the iterates $x^i \in \mathbb{R}^2$ are shown with their corresponding image sets in the right diagrams. For this setup the algorithm performed 35 *main iterations and the objective function F is evaluated* 240 *times which is the appropriate measure of computational effort for realistic problems.*

For the chosen minimal step length $h_{\min}$ the algorithm terminates at $x^{35} \approx 10^{-5} \cdot (-3.894, 3.991)^T$ which is clearly within a ball of radius $h_{\min}$ around the actual minimum.

Example 2.3.4 ([66, Example 4.7])**.** *As a second example, we propose a set-valued extension $F\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ of the linear-quadratic objective function*

$$f(x_1, x_2) = \begin{pmatrix} x_1^2 + x_2^2 \\ 2(x_1 + x_2) \end{pmatrix}.$$

To this end, the values are clustered on a circle around f similar to the previous example:

$$F(x_1, x_2) = \left\{ f(x_1, x_2) + \frac{1}{4} \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} | t \in \bar{\mathcal{T}} \right\}, \quad \left(\bar{\mathcal{T}} = \left\{ \frac{2\pi}{14} \cdot i | i = 0, 1, \ldots, 13 \right\} \right).$$

Since functions with a similar form as $f(x_1, x_2)$ form the basis of the Markowitz stock model [82], this may—apart from the rather simple mathematical structure—be regarded as a representative example for a large class of real-world applications. In this example we do not consider convex objective sets only to simplify the reasoning that the optimal solutions are aligned along the line $x_1 = x_2$, $x_1 \leq 0$ with objective values forming discretized circles around the Pareto front $\{(f_1, f_2) : f_2 \leq 0, f_1 = f_2^2/8\}$, where we considered again the upper set less order relation and the natural ordering cone $D = \mathbb{R}^2_+$. The results for algorithmic parameters $k^0 = (1,1)$, $i_{max} = 40$, $j_{max} = 15$, $n_s = 16$ (equally distributed search directions) are displayed in Figures 2.8 and 2.9 for the arguments and objective values, respecively. Initial and minimal step sizes $h_0 = 1.1$ and $h_{\min} = 10^{-4}$ have been chosen and a series of 20 *different starting points. It can clearly be seen that the algo-*

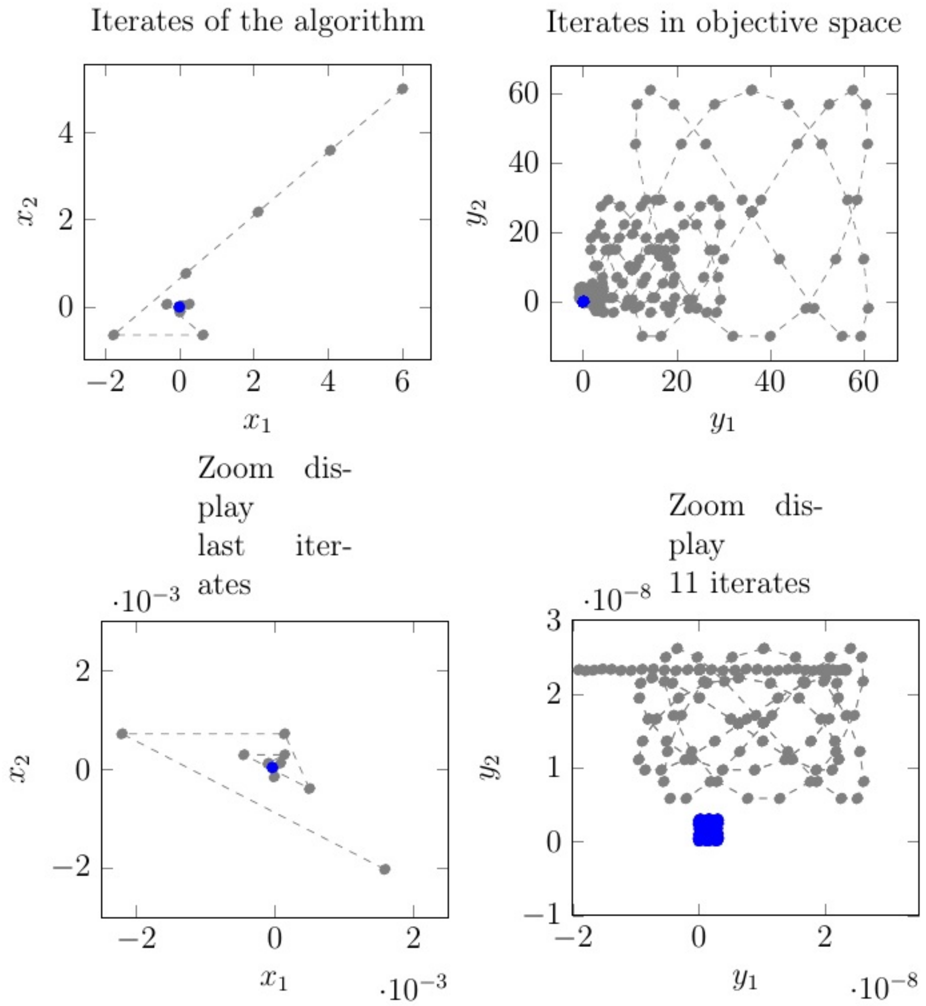

Figure 2.7: Numerical results for Example 2.3.3

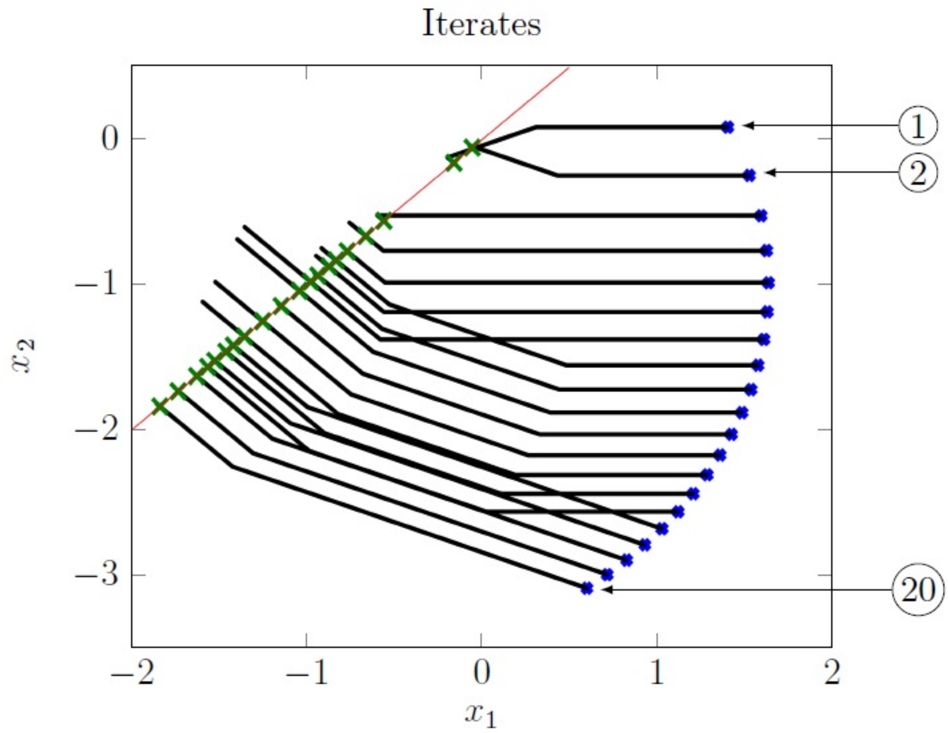

Figure 2.8: Iterates for Example 2.3.4: Argument space

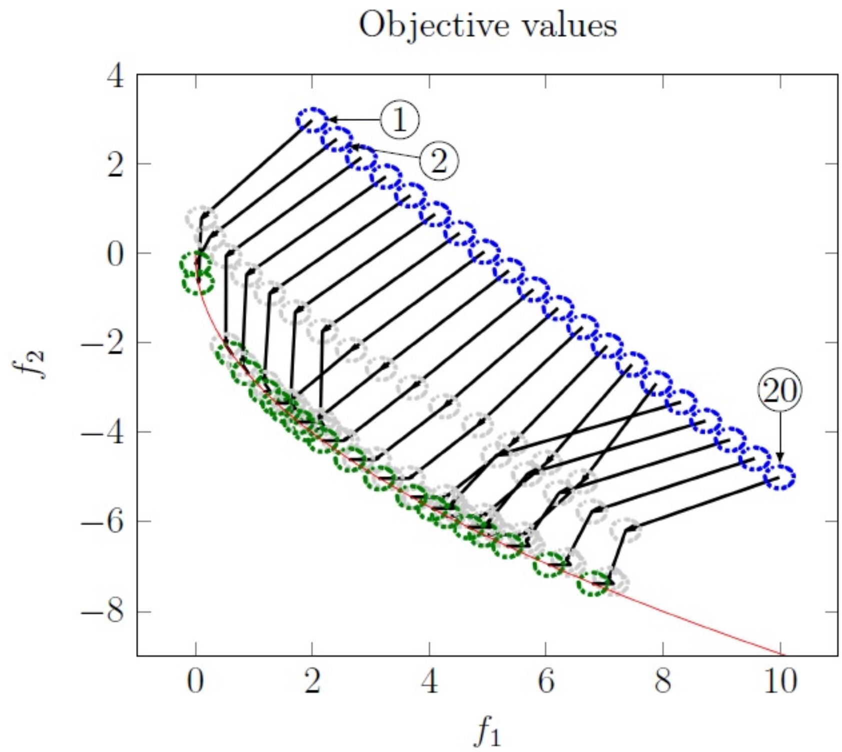

Figure 2.9: Iterates for Example 2.3.4: Objective space

rithm robustly approximates different minimal solutions for varying starting points. For better visualization we indicate the Pareto front of f and its respective argument values in the plots as well. To judge the accuracy and efficiency of the method we also added some performance statistics in Figure 2.10. It is verified that the algorithm approximates the minimal elements sufficiently well, i. e. with errors smaller than the minimal step length. The average error in argument values for all 20 *experiments was* $4.92 \cdot 10^{-5}$ *while in the*

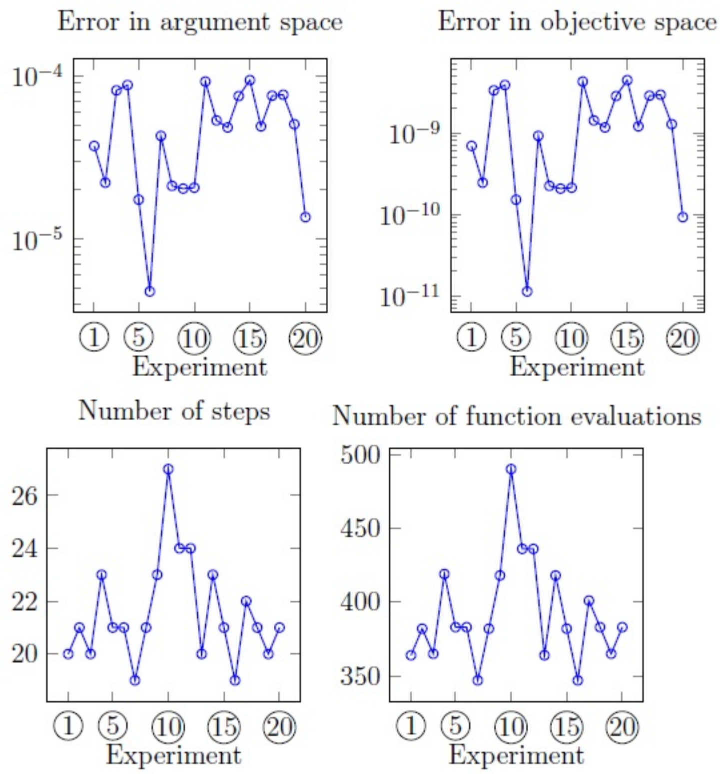

Figure 2.10: Performance statistics for Example 2.3.4

objective space (calculated as distance of the center points from the Pareto front) it was on average even $1.61 \cdot 10^{-9}$. *The lower two plots in Figure 2.10 show that also regarding efficiency the method performed satisfyingly. The average number of steps was found to be* 21.55 *with an average number of function evaluations of* 392.4.

2.3.2 Jahn-Graef-Younes Methods

In this section, our aim is to present algorithms for computing all minimal elements of a nonempty finite family of sets $\mathcal{A} \subseteq \mathcal{P}(Y)$ with respect to a set relation $\preceq$ defined on the power set $\mathcal{P}(Y)$ of a real linear space Y. These algorithms are inspired from two methods originally conceived for vector optimization problems:

(i) *Graef-Younes method*, proposed by Younes [110] and formulated algorithmically by Graef, as mentioned by Jahn [53, Sec. 12.4];

(ii) *Jahn-Graef-Younes method*, also called *Graef-Younes method with backward iteration*, proposed by Jahn [52, 53], Jahn and Rathje [57], and reformulated in a more general setting by Eichfelder [24];

Our approach in this section is two-fold: First, we extend the well-known Jahn-Graef-Younes method from vector to set optimization. The Jahn-Graef-Younes method in vector optimization selects minimal elements of a set of **finitely** many elements. Its advantage is that this method reduces the numerical effort by excluding elements which cannot be minimal for a given set. Here we extend this method to the set-valued case in order to obtain minimal elements of a family of **finitely** many sets. We propose several extensions of the Jahn-Graef-Younes method under different assumptions on the generalized set relations introduced in Section 2.2.

Secondly, when the involved sets are compared by means of any of those proposed set relations, we use the results from Section 2.2 to evaluate $A \preceq B$ by using the nonlinear scalarizing functional $z^{D,k}$. The results presented in this chapter can be found in [68].

When the family of sets $\mathcal{A}$ is given by a large number of elements, it may take a long time to compare the sets pairwise according to Definition 1.2.11. We propose a method that significantly reduces the number of comparisons of sets. Reducing the numerical effort is especially useful if each comparison is rather expensive. The following algorithm filters out elements of a family of sets which cannot be minimal. This procedure extends the Jahn-Graef-Younes method which is given in the dissertation by Younes [110], Jahn and Rathje [57] (compare also Jahn [53, Section 12.4]) for minimal elements in the vector-valued case, where $Y = \mathbb{R}^n$. Eichfelder [25] formulated corresponding algorithms for vector-valued problems with a variable ordering structure. We extend the idea of such a method to set optimization problems, where we assume that a family of finitely many sets $\mathcal{A}$ is given and minimal elements of $\mathcal{A}$ are to be identified.

Algorithm 2.3.5 ([68, Algorithm 4.2])**.** *(Jahn-Graef-Younes method for sorting out non-minimal elements of a family of finitely many sets)*

Input: $\mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{R}^n$, *set relation* $\preceq$
% initialization
$\mathcal{T} := \{A_1\}$,
% iteration loop
for $j = 2 : 1 : m$ ***do***
    if $(A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A)$ ***then***
        $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
    end if
end for
Output: $\mathcal{T}$

Algorithm 2.3.5 is a reduction method which sorts out sets that cannot be minimal. Moreover, it is a self learning method which becomes better and better in each step.

In the if-statement of Algorithm 2.3.5, each element is compared only with elements that have been considered so far (which belong to the set $\mathcal{T}$), so it is not necessary to compare all elements with each other pairwise, which can reduce the computation time of determining minimal elements significantly. Notice that the conditions $A \preceq A_j$ and $A_j \preceq A$ in the if-statement in Algorithm 2.3.5 can be evaluated by means of computing the nonlinear scalarizing functional $z^{D,k}$ (compare Theorems 2.2.6, 2.2.13, 2.2.25 and Corollaries 2.2.18 and 2.2.21 for representations of different set relations by means of $z^{D,k}$). This will be done on page 38. Below we show that all minimal elements of the family of sets $\mathcal{A}$ are contained in the output set $\mathcal{T}$ generated by Algorithm 2.3.5.

Theorem 2.3.6 ([68, Theorem 4.3]). *1. Algorithm 2.3.5 is well-defined.*

2. Algorithm 2.3.5 generates a nonempty set $\mathcal{T} \subseteq \mathcal{A}$.

3. Every minimal element of $\mathcal{A}$ also belongs to the set $\mathcal{T}$ generated by Algorithm 2.3.5.

Proof. As 1. and 2. are obvious, we only prove part 3. Let A_j be a minimal element of $\mathcal{A}$, but assume that $A_j \notin \mathcal{T}$. Clearly $j \neq 1$. As A_j is a minimal element of $\mathcal{A}$, we have

$$A \preceq A_j, \ A \in \mathcal{A} \implies A_j \preceq A.$$

Since $\mathcal{T} \subseteq \mathcal{A}$, we have

$$A \preceq A_j, \ A \in \mathcal{T} \implies A_j \preceq A.$$

But then the condition in the if-statement is fulfilled and A_j is added to $\mathcal{T}$, which is a contradiction to our assumption. $\square$

As mentioned before, the conditions $A \preceq A_j$ and $A_j \preceq A$ in the if-statement in Algorithm 2.3.5 shall be evaluated by means of the nonlinear scalarizing functional $z^{D,k}$ for all introduced set relations. In order to prepare this evaluation, we first consider the following attainment properties:

Assumption 2.3.7 (Attainment Property). *(u) Assume that there exist $k_0^u,\ k_1^u \in Y \setminus \{0\}$ satisfying* (2.1) *such that $\inf_{\overline{a} \in A_j} z^{D,k_0^u}(a - \overline{a})$ is attained for all $a \in A$ and $\inf_{a \in A} z^{D,k_1^u}(\overline{a} - a)$ is attained for all $\overline{a} \in A_j$.*

(l) Assume that there exist $k_0^l,\ k_1^l \in Y \setminus \{0\}$ satisfying (2.1) *such that $\inf_{a \in A} z^{D,k_0^l}(a - \overline{a})$ is attained for all $\overline{a} \in A_j$ and $\inf_{\overline{a} \in A_j} z^{D,k_1^l}(\overline{a} - a)$ is attained for all $a \in A$.*

(s) Assume that there exist $k_0^s,\ k_1^s,\ k_2^s,\ k_3^s \in Y \setminus \{0\}$ satisfying the inclusion (2.1) *such that $\inf_{\overline{a} \in A_j} z^{D,k_0^s}(a - \overline{a})$ is attained for all $a \in A$, $\inf_{a \in A} z^{D,k_1^s}(\overline{a} - a)$ is attained for all $\overline{a} \in A_j$, $\inf_{a \in A} z^{D,k_2^s}(a - \overline{a})$ is attained for all $\overline{a} \in A_j$ and $\inf_{\overline{a} \in A_j} z^{D,k_3^s}(\overline{a} - a)$ is attained for all $a \in A$.*

(p) Assume that there exist $k_0^p, k_1^p \in Y \setminus \{0\}$ satisfying the inclusion (2.1) *such that $\inf_{(a,\overline{a}) \in A \times A_j} z^{D,k_0^p}(a - \overline{a})$ and $\inf_{(a,\overline{a}) \in A \times A_j} z^{D,k_1^p}(\overline{a} - a)$ are attained.*

Remark 2.3.8 ([68, Remark 4.5]). *The attainment properties above are important for the representation of the introduced generalized set relations by means of the nonlinear scalarizing functional* $z^{D,k}$ *(compare Theorems 2.2.6, 2.2.13, 2.2.25 and Corollary 2.2.18). Sufficient conditions ensuring the existence of solutions of corresponding optimization problems (extremal principles) are given in the literature. The well-known Theorem of Weierstrass says that a lower semi-continuous function on a nonempty compact set has a minimum. An extension of the Theorem of Weierstrass is given by Zeidler [113, Proposition 9.13]: A proper lower semi-continuous and quasi-convex function on a nonempty closed bounded convex subset of a reflexive Banach space has a minimum. Taking into account that the functional* z^{D,k_0} *is lower semi-continuous and convex if* $D \subset Y$ *is a proper closed convex cone and* $k_0 \in D \setminus \{0\}$ *(compare Theorem 2.1.2), we get that the attainment property for* $\inf_{a \in A} z^{D,k_0}(a - b)$ *(with* $b \in B$ *fixed) is fulfilled if* A *is a nonempty closed bounded convex subset of a reflexive Banach space and* D *is a proper closed convex cone.*

In the following, we will give an implementation of the implication $A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A$ in Algorithm 2.3.5 in order to show how we are using the results from Section 2.2 for deriving the algorithm. Especially in Step 5 of the following implementation of Algorithm 2.3.5 it can be seen that the results concerning the scalarizing functional $z^{D,k}$ are important for computing minimal elements of the set $\mathcal{A}$. In the following, we assume that the set relation used in Algorithm 2.3.5 is given by $\preceq^t_D$, where t is replaced by u, l, s, c, p for the generalized *upper set less relation* $\preceq^u_D$, *lower set less relation* $\preceq^l_D$, *set less relation* $\preceq^s_D$, *certainly set less relation* $\preceq^c_D$ or *possibly set less relation* $\preceq^p_D$, respectively.

We use the following implications (see Theorems 2.2.6, 2.2.13, 2.2.25 and Corollaries 2.2.18 and 2.2.21) in our implementation of Algorithm 2.3.5 (note that these are equivalent to $A \preceq A_j \implies A_j \preceq A$ under appropriate attainment properties):

$$\sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_0^u}(a-\overline{a})\leq 0 \quad\Longrightarrow\quad \sup_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_1^u}(\overline{a}-a)\leq 0 \tag{I_u}$$

$$\sup_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_0^l}(a-\overline{a})\leq 0 \quad\Longrightarrow\quad \sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_1^l}(\overline{a}-a)\leq 0 \tag{I_l}$$

$$\begin{cases} \sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_0^s}(a-\overline{a})\leq 0 \wedge \sup_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_2^s}(a-\overline{a})\leq 0 \\ \Longrightarrow\quad \sup_{\overline{a}\in \overline{A}}\inf_{a\in A} z^{D,k_1^s}(\overline{a}-a)\leq 0 \wedge \sup_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_3^s}(\overline{a}-a)\leq 0 \end{cases} \tag{I_s}$$

$$\begin{cases} (A=A_j) \ \vee\ \sup_{a\in A}\sup_{\overline{a}\in A_j} z^{D,k}(a-\overline{a})\leq 0 \\ \Longrightarrow\ (A=A_j) \ \vee\ \sup_{\overline{a}\in A_j}\sup_{a\in A} z^{D,k}(\overline{a}-a)\leq 0 \end{cases} \tag{I_c}$$

$$\inf_{a\in A}\inf_{\overline{a}\in A_j} z^{D,k_0^p}(a-\overline{a})\leq 0 \quad\Longrightarrow\quad \inf_{\overline{a}\in A_j}\inf_{a\in A} z^{D,k_1^p}(\overline{a}-a)\leq 0 \tag{I_p}$$

The following implementation of Algorithm 2.3.5 checks whether the implication $A \preceq A_j,\ A\in\mathcal{T} \implies A_j \preceq A$ in the **if**-statement in Algorithm 2.3.5 is fulfilled for some input A_j, given $\mathcal{T}$, and $t\in\{u,l,s,c,p\}$ for $\preceq_D^t:=\preceq$ which was chosen in the input of Algorithm 2.3.5. If this implication is satisfied for all $A\in\mathcal{T}$, then the set A_j is added to the family of sets $\mathcal{T}$. Then the **for**-loop in Algorithm 2.3.5 continues with $j:=j+1$. If this implication is not fulfilled for some $A\in\mathcal{T}$, then the **for**-loop in Algorithm 2.3.5 continues with $j:=j+1$, but the set A_j is not added to the family of sets $\mathcal{T}$. Note that the set $K:=\{k\in Y\setminus\{0\}\mid D+[0,+\infty)\cdot k\subseteq D\}$, which is necessary for the definition of the functional $z^{D,k}$, as $k\in K$, should be determined at the beginning of Algorithm 2.3.5. Furthermore, notice that the set D and $t\in\{u,l,s,c,p\}$ were already chosen in the input of Algorithm 2.3.5.

Realization the implication $A\preceq_D^t A_j,\ A\in\mathcal{T} \implies A_j\preceq_D^t A$ in Algorithm 2.3.5:

Input: $\mathcal{T}$ and j

Step 1: Set $\widetilde{\mathcal{T}}:=\mathcal{T}$. Go to Step 2.

Step 2: If $\widetilde{\mathcal{T}}=\emptyset$, then the implication $A\preceq_D^t A_j,\ A\in\mathcal{T} \implies A_j\preceq_D^t A$ holds and STOP. Otherwise, go to Step 3.

Step 3: Choose $A\in\widetilde{\mathcal{T}}$. Set $\widetilde{\mathcal{T}}:=\widetilde{\mathcal{T}}\setminus\{A\}$. Go to Step 4.

Step 4: When $t\in\{u,l,p,s\}$, choose $k_r^t\in K$ ($r=0,1$ if $t\in\{u,l,p\}$, $r=0,1,2,3$ if $t=s$) such that Assumption 4.3 (t) is fulfilled. When $t=c$, choose $k\in K$. Go to Step 5.

Step 5: If the implication (I_t) is true, then go to Step 2. Otherwise, the implication does not hold and STOP.

Remark 2.3.9 ([68, Remark 4.6])**.** *The above implementation of the implication $A\preceq_D^t A_j,\ A\in\mathcal{T} \implies A_j\preceq_D^t A$ Algorithm 2.3.5 is especially easy for the generalized certainly* *less relation $\preceq_D^c$ (when $t = c$), as no attainment property needs to be fulfilled for this particular set relation (compare Theorem 2.2.20).*

Example 2.3.10 ([68, Example 4.7])**.** *Let $D := \mathbb{R}^2_+$ and $\preceq := \preceq_D^c$. We have randomly computed 1,000 sets, for easy comparison each set is a ball of radius one in $\mathbb{R}^2$. Out of those 1,000 sets, a total number of 93 are minimal w.r.t. to $\preceq$. Algorithm 2.3.5 generates 103 sets in $\mathcal{T}$, which is already a reduction of 897 sets. In Figure 2.11 the elements of the set $\mathcal{T}$ are the filled circles.*

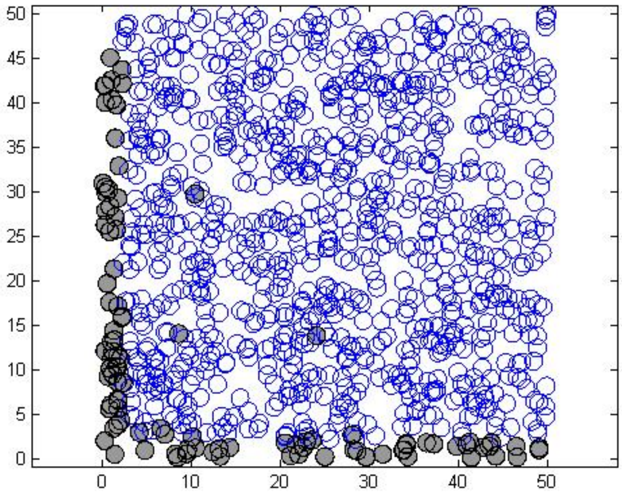

Figure 2.11: A randomly generated family of sets. The filled circles belong to the set $\mathcal{T}$ generated by Algorithm 2.3.5.

Remark 2.3.11. *Notice that the set relation $\preceq$ does not need to be transitive in Algorithm 2.3.5, in contrast to descent methods (see Jahn [55]), which rely on the transitivity of the considered set relation.*

Example 2.3.12 ([68, Example 4.9])**.** *Let $D := \mathbb{R}^2_+$, $\preceq := \preceq_D^c$, $A_1 := B_1(3,3)$, $A_2 := B_1(5,5)$, $A_3 := B_1(0,0)$ (where $B_1(y_1, y_2)$ denotes the closed ball of radius one around the point $(y_1, y_2) \in \mathbb{R}^2$). Let the family of sets be given by these balls, i.e., $\mathcal{A} := \{A_1, A_2, A_3\}$. The only minimal element of $\mathcal{A}$ w.r.t. $\preceq$ is $A_3 = B_1(0,0)$. Algorithm 2.3.5 generates the set $\mathcal{T} := \{A_1, A_3\}$.*

In the following, we will apply the for-loop in Algorithm 2.3.5 backwards. This will lead to Algorithm 2.3.15 presented on page 40, which determines all minimal elements of a family of sets under an external stability assumption on the set of minimal elements $\mathcal{A}_\preceq$, when the set relation is antisymmetric. For example, the generalized certainly less relation $\preceq_D^c$ is antisymmetric if D is a **pointed** cone (see Proposition 2.3.22). We use the following notion of external stability of the set $\mathcal{A}_{min}$, i.e., the set of all minimal elements of $\mathcal{A}$ w.r.t. the set relation $\preceq$.

Definition 2.3.13. *If for all non-minimal elements $A \in \mathcal{A} \setminus \mathcal{A}_{min}$ there exists a minimal element $\overline{A} \in \mathcal{A}_{min}$ with $\overline{A} \preceq A$, then $\mathcal{A}_{min}$ is called **externally stable**.*

Remark 2.3.14 ([40, Remark 2.2])**.** *It is well-known that every nonempty finite subset of a general preordered set is externally stable (see, e.g., Podinovskiĭ and Nogin [92, p. 21]). Thus, whenever $\mathcal{A}$ is nonempty and finite and $\preceq$ is a preorder, the set $\mathcal{A}_{min}$ is externally stable.*

Algorithm 2.3.15 ([68, Algorithm 4.11])**.** *(Jahn-Graef-Younes method with backward iteration for finding minimal elements of a family of finitely many sets, where $\mathcal{A}_{min}$ is externally stable)*

Input: $\mathcal{A} := \{A_1, \dots, A_m\} \subset \mathbb{R}^n$, *antisymmetric set relation* $\preceq$
% initialization
$\mathcal{T} := \{A_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ ***do***
    if $(A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A)$ ***then***
        $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
    end if
end for
$\{A_1, \dots, A_p\} := \mathcal{T}$
$\mathcal{U} := \{A_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ ***do***
    if $(A \preceq A_j,\ A \in \mathcal{U} \implies A_j \preceq A)$ ***then***
        $\mathcal{U} := \mathcal{U} \cup \{A_j\}$
    end if
end for
Output: $\mathcal{U}$

Theorem 2.3.16 ([68, Theorem 4.12])**.** *Let the set relation $\preceq$ be antisymmetric and the set of minimal elements $\mathcal{A}_{min}$ be nonempty and externally stable. Then the output $\mathcal{U}$ of Algorithm 2.3.15 consists of exactly all minimal elements of the family of sets $\mathcal{A}$.*

Proof. Let $\mathcal{U} := \{A_1, \dots, A_q\}$. By assertion 3 of Theorem 2.3.6, we know that all minimal elements of $\mathcal{A}$ are contained in $\mathcal{T}$ as well as in $\mathcal{U}$. Now we prove that every element of $\mathcal{U}$ is also a minimal element of the set $\mathcal{A}$. Let $A_j \in \mathcal{U}$ be arbitrarily chosen. By the forward iteration of Algorithm 2.3.15, we obtain

$$\forall\ i < j\ (i \geq 1):\ A_i \preceq A_j \Longrightarrow A_j \preceq A_i.$$

The backward iteration of Algorithm 2.3.15 yields

$$\forall\ i > j\ (i \leq q):\ A_i \preceq A_j \Longrightarrow A_j \preceq A_i.$$

This means that

$$\forall\ i \neq j\ (1 \leq i \leq q):\ A_i \preceq A_j \Longrightarrow A_j \preceq A_i. \tag{2.8}$$

(2.8) implies that

$$\forall\ A_i \in \mathcal{U} \setminus \{A_j\}:\ A_i \preceq A_j \quad \Longrightarrow \quad A_j \preceq A_i.$$

Then A_j is a minimal element of $\mathcal{U}$. Now suppose that A_j is not a minimal element in $\mathcal{A}$, then $A_j \notin \mathcal{A}_{min}$. Then, as $\mathcal{A}_{min}$ was assumed to be externally stable, there exists a minimal element A in $\mathcal{A}_{min}$ (especially, $A \neq A_j$) with the property $A \preceq A_j$. Since A is a minimal element in $\mathcal{A}$, Theorem 2.3.6, 3. implies that $A \in \mathcal{U}$. Therefore, by (2.8), $A_j \preceq A$, as A_j is minimal in $\mathcal{U}$ and $A \in \mathcal{U}$. By the antisymmetry of the set relation $\preceq$, we obtain $A = A_j$, a contradiction. □

It is again possible to formulate an implementation the implication $A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A$ of Algorithm 2.3.15. This can be performed for the first for-loop analogously to the process on page 38, and for the second for-loop simply by replacing $\mathcal{T}$ by $\mathcal{U}$ and changing $j := j + 1$ to $j := j - 1$.

Example 2.3.17 ([68, Example 4.13]). *We return to Example 2.3.12. The backward iteration in Algorithm 2.3.15 generates the set $\mathcal{U} = \{A_3\}$, which is exactly the minimal element of $\mathcal{A}$ w.r.t. $\preceq$.*

Example 2.3.18 ([68, Theorem 4.14]). *The minimal elements of the randomly generated family of sets of Example 2.3.10 are illustrated as dark filled circles in Figure 2.12. The remaining elements which are lighter belong to the set $\mathcal{T}$, but not to $\mathcal{U}$.*

In the following, we give a sufficient condition for the set of minimal elements $\mathcal{A}_{min}$ to be externally stable (see also Remark 2.3.14).

Lemma 2.3.19 ([68, Lemma 4.15]). *Let a family $\mathcal{A}$ of finitely many nonempty subsets of Y be given and let the set relation $\preceq$ be transitive and antisymmetric. Assume that the set of minimal elements w.r.t. $\preceq$, denoted as $\mathcal{A}_{min}$, is nonempty. Then $\mathcal{A}_{min}$ is externally stable.*

Proof. Let some $A \in \mathcal{A}$, and A is assumed to be not minimal w.r.t. $\preceq$. Then there exists some $A_1 \in \mathcal{A}$ such that $A_1 \preceq A$ and $A \not\preceq A_1$. If $A_1 \in \mathcal{A}_{min}$, then there is nothing to show. If $A_1 \notin \mathcal{A}_{min}$, then there exists some $A_2 \in \mathcal{A}$ with $A_2 \preceq A_1$ and $A_1 \not\preceq A_2$. As $\preceq$ is transitive, we get $A_2 \preceq A$. As $\mathcal{A}$ consists of finitely many elements and $\preceq$ is antisymmetric, this procedure stops with a minimal element. □

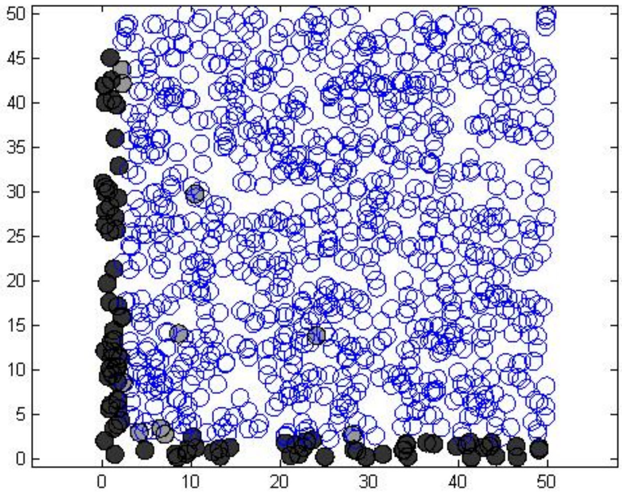

Figure 2.12: A randomly generated family of sets. The minimal elements w.r.t. $\preceq_D^c$ are dark, the lighter sets belong to the set $\mathcal{T}$ generated by Algorithm 2.3.15.

Remark 2.3.20 ([68, Remark 4.16])**.** *Let us briefly explain the difference between our extension of the Jahn-Graef-Younes-Algorithm to set optimization to the originally introduced version by Younes (compare [53, Section 12.4]) in vector optimization. Let $Y = \mathbb{R}^n$ with the ordering $\leq_C$ induced by a closed convex cone C. The if-statement in the original Jahn-Graef-Younes-Algorithm in vector optimization reads*

$$\textit{for all } y \in \mathcal{T} \setminus \{\overline{y}\} : \; y \not\leq_C y_j,$$

and transferring this notion to our set optimization setting would yield the condition

$$\textit{for all } A \in \mathcal{T} \setminus \{A_j\} : \; A \not\preceq A_j.$$

However, then the set $\mathcal{T}$ generated by Algorithm 2.3.5 would possibly not contain all minimal elements. The reason for this is the following: We work with the minimality notion given in Definition 1.2.11:

$$A \preceq \overline{A}, \; A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq A\,. \tag{2.9}$$

However, the implication (2.9) *does not imply*

$$\forall \; A \in \mathcal{A} \setminus \{\overline{A}\} : \; A \not\preceq \overline{A}, \tag{2.10}$$

unless $\preceq$ is antisymmetric. We note that (2.10) *always implies* (2.9)*, even if $\preceq$ is not antisymmetric. We exemplarily illustrate this with a small example in vector optimization. Let $a = (a^1, a^2) \in \mathbb{R}^2$ be given, $C := \{y \in \mathbb{R}^2 \mid a^T y \geq 0\}$, $\mathcal{A} = \{y \in \mathbb{R}^2 \mid a^T y = 0\}$ and $\overline{A} \in \mathcal{A}$ arbitrarily given. The binary relation $\leq_C := \preceq$ is defined as $y_1 \leq_C y_2 \; :\Longleftrightarrow$*

$y_1 \in y_2 - C$. Then all elements in $\mathcal{A}$ are minimal w.r.t. $\preceq$. Then (2.9) *is satisfied for all $A = y \in \mathcal{A}$. However, we have for all $y_1, y_2 \in \mathcal{A}$ the relation $y_1 \leq_C y_2$. Therefore,* (2.10) *does not hold true for any $A = y \in \mathcal{A}$. The reason, of course, is that the cone C is a halfspace and therefore not pointed, hence the binary relation $\leq_C$ is not antisymmetric.*

The above remark also relates to the following proposition:

Proposition 2.3.21 ([68, Proposition 4.17])**.** *We consider the statements*

$$\nexists\ A \in \mathcal{A} \setminus \{\overline{A}\}:\ A \preceq \overline{A} \tag{2.11}$$

and

$$A \preceq \overline{A},\ A \in \mathcal{A} \quad\Longrightarrow\quad \overline{A} \preceq A. \tag{2.12}$$

Then we have (2.11) $\Longrightarrow$ (2.12)*. Conversely, if $\preceq$ is antisymmetric, then* (2.12) *implies* (2.11)*.*

Proof. Let (2.11) be true, and suppose that (2.12) is not fulfilled. Then there is some $A \in \mathcal{A} \setminus \{\overline{A}\}$ such that $A \preceq \overline{A}$, but $\overline{A} \not\preceq A$. Because of (2.11), we obtain $\overline{A} = A$, a contradiction.

Conversely, let (2.12) be fulfilled, but suppose that (2.11) does not hold. Then there exists some $A \in \mathcal{A} \setminus \{\overline{A}\}$ with the property $A \preceq \overline{A}$. By (2.12), we get $\overline{A} \preceq A$. As $\preceq$ was assumed to be antisymmetric, this yields $\overline{A} = A$, a contradiction. $\square$

By the above results, it is possible to replace the if-condition in Algorithms 2.3.5 and 2.3.15 by $A \not\preceq A_j$ for all $A \in \mathcal{T}$ (and $A \not\preceq A_j$ for all $A \in \mathcal{U}$ in the backwards-iteration of Algorithm 2.3.15) under the assumption that the set relation $\preceq$ is antisymmetric. However, among our introduced generalized set relations, only the generalized certainly less relation $\preceq^c_D$ is antisymmetric if D is a pointed cone (i.e., the cone D fulfills $D \cap (-D) = \{0\}$). One possibility to overcome this issue is to use different notions of minimality, as it has been done in Köbis and Le [69]. In [69], *strict*, *strong* and *ideal* minimal solutions have been introduced and numerical methods based on Jahn-Graef-Younes algorithms have been presented and analyzed.

Notions similar to antisymmetry, that are fulfilled by $\preceq^u_D$, $\preceq^l_D$ and $\preceq^s_D$, are summarized below (see [59, Chapter 2.6.2]).

Proposition 2.3.22 ([68, Proposition 4.18])**.**

1. *If D is a convex cone, then $A \preceq^u_D B$ and $B \preceq^u_D A$ imply that $A - D = B - D$.*
2. *If D is a convex cone, then $A \preceq^l_D B$ and $B \preceq^l_D A$ imply that $A + D = B + D$.*
3. *If D is a convex cone, then $A \preceq^s_D B$ and $B \preceq^s_D A$ imply that $A - D = B - D$ and $A + D = B + D$.*
4. *If D is a pointed cone, then the generalized certainly set relation $\preceq^c_D$ is antisymmetric. Moreover, $A \preceq^c_D B$ and $B \preceq^c_D A$ imply that the set $A = B$ is single-valued.*

Proof. The first three assertion are obvious. Concerning the last statement, the assertions $a - b \in -D$ for all $a \in A$ and for all $b \in B$ and $a - b \in D$ for all $a \in A$ and for all $b \in B$ imply that $a = b$ all $a \in A$ and for all $b \in B$. □

Though $\preceq_D^u$, $\preceq_D^l$ and $\preceq_D^s$ are not antisymmetric in $\mathcal{A}$, we can use Algorithm 2.3.15 effectively to some antisymmetric subfamily of $\mathcal{A}$. This analysis is presented in [68] along with an algorithm to create such an antisymmetric subfamily.

Finally, we propose the following algorithm that does not rely on antisymmetry or external stability of the set relation $\preceq$. The idea stems from Eichfelder [25, Algorithm 1], who gave a similar numerical procedure for finding minimal elements in vector optimization with a variable domination structure. In the following algorithm, a third for-loop is added which compares the elements that were obtained in the set $\mathcal{U}$ by Algorithm 2.3.15 with all remaining elements in $\mathcal{A} \setminus \mathcal{U}$.

Algorithm 2.3.23 ([68, Algorithm 4.21])**.** *(Jahn-Graef-Younes method with backward iteration for finding minimal elements of a family of finitely many sets)*

Input: $\mathcal{A} := \{A_1, \ldots, A_m\} \subset \mathbb{R}^n$, *set relation* $\preceq$
% initialization
$\mathcal{T} := \{A_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ ***do***
    if $(A \preceq A_j,\ A \in \mathcal{T} \implies A_j \preceq A)$ ***then***
        $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
    end if
end for
$\{A_1, \ldots, A_p\} := \mathcal{T}$
$\mathcal{U} := \{A_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ ***do***
    if $(A \preceq A_j,\ A \in \mathcal{U} \implies A_j \preceq A)$ ***then***
        $\mathcal{U} := \mathcal{U} \cup \{A_j\}$
    end if
end for
$\{A_1, \ldots, A_q\} := \mathcal{U}$
$\mathcal{V} := \emptyset$
% final comparison
for $j = 1 : 1 : q$ ***do***
    if $(A \preceq A_j,\ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq A)$ ***then***
        $\mathcal{V} := \mathcal{V} \cup \{A_j\}$
    end if
end for
Output: $\mathcal{V}$

Theorem 2.3.24 ([68, Theorem 4.22]). *Algorithm 2.3.23 consists of exactly all minimal elements of the family of sets $\mathcal{A}$.*

Proof. Let A_j be an arbitrary element in $\mathcal{V}$. Then $A_j \in \mathcal{U}$, as $\mathcal{V} \subseteq \mathcal{U}$, and

$$A \preceq A_j,\ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq A.$$

Suppose that A_j is not minimal in $\mathcal{A}$. Then there exists some $A \in \mathcal{A}$ such that $A \preceq A_j$ and $A_j \not\preceq A$. If $A \notin \mathcal{U}$, then this is a contradiction. If $A \in \mathcal{U}$, then A is also minimal in $\mathcal{U}$ (compare the proof of Theorem 2.3.16). Since $A_j \in \mathcal{U}$, and A_j is also minimal in $\mathcal{U}$, we obtain from $A \preceq A_j$ that $A_j \preceq A$, a contradiction.

Conversely, let A_j be minimal in $\mathcal{A}$. Then we get

$$A \preceq A_j,\ A \in \mathcal{A} \implies A_j \preceq A.$$

Now suppose that $A_j \notin \mathcal{V}$. Then there exists some $A \in \mathcal{A} \setminus \mathcal{U}$ with $A \preceq A_j$ and $A_j \not\preceq A$. As A_j is minimal in $\mathcal{A}$, we get $A_j \preceq A$, a contradiction. □

Remark 2.3.25 ([68, Remark 4.23]). *Note that it is again possible to evaluate the implication*

$$A \preceq A_j,\ A \in \mathcal{T}\ (\mathcal{U},\ \mathcal{A} \setminus \mathcal{U},\ \textit{resp.}) \implies A_j \preceq A$$

in Algorithm 2.3.23 by means of the nonlinear scalarizing functional $z^{D,k}$. This can be done analogously to the proposed process on page 38, but we refrain from repeating it here due to its similarities.

Example 2.3.26 ([68, Example 4.24]). *Let $D := \mathbb{R}^2_+$ and $\preceq := \preceq^u_D$. We use the same family of randomly computed sets from Example 2.3.10. Out of the considered 1.000 sets, a total number of 5 are minimal w.r.t. to $\preceq$. Algorithm 2.3.23 first generates 18 sets in $\mathcal{T}$, which is already a huge reduction, and finally collects all minimal elements within the set $\mathcal{U}$, which coincides with $\mathcal{V}$. In Figure 2.13 the minimal elements are darkly filled, while the lighter sets are those elements that are not minimal, but belong to the set $\mathcal{T}$. Of course, in our case the set of minimal elements is externally stable because of the unified structure of the sets.*

Example 2.3.27 ([68, Example 4.25]). *Let $D := \mathbb{R}^2_+$, $\preceq := \preceq^p_D$, $A_1 := \{(0,0)\}$, $A_2 := \{(1,1),(2,-1)\}$, $A_3 := \{(3,-0.5)\}$. The family of sets is given as $\mathcal{A} := \{A_1, A_2, A_3\}$. The only minimal element of $\mathcal{A}$ w.r.t. $\preceq$ is $A_1 = \{0,0\}$. Algorithm 2.3.5 generates the sets $\mathcal{T} := \{A_1, A_3\}$ and $\mathcal{U} = \{A_3, A_1\}$. A final comparison then yields $\mathcal{V} = \{A_1\}$.*

Remark 2.3.28 ([68, Example 4.26]). *A finite family of sets $\mathcal{A}$ can also be computed by an appropriate discretization of the outcome sets of the considered (continuous) set optimization problem.*

Note that our generalizations of Jahn-Graef-Younes methods have recently been extended to obtain different solution concepts, namely, to strict and strong solutions of set-valued optimization problems (see [69]).

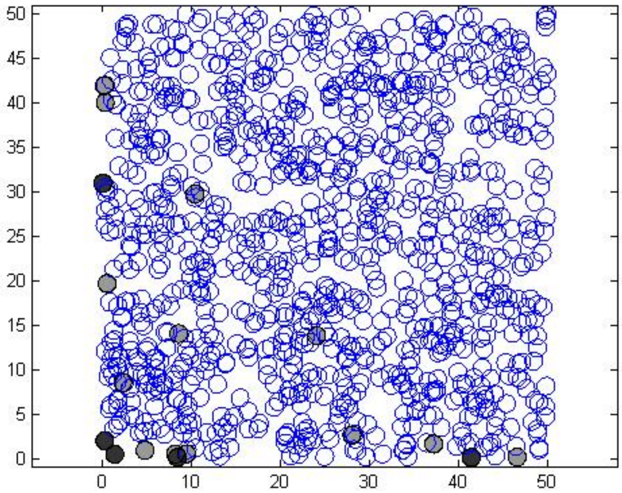

Figure 2.13: The lightly filled circles belong to the set $\mathcal{T}$ generated by Algorithm 2.3.23 and the darkly filled circles are the elements which are minimal w.r.t. $\preceq^u_D$ (see Example 2.3.26).

2.4 A New Set Relation in Set Optimization

In this section, it is our goal to introduce a new set relation that can be regarded as a "weighting" between two prominent set relations, namely the relations $\preceq^u_D$ and $\preceq^l_D$. We discuss its properties, formulate a set optimization problem by means of this new set relation, give an existence theorem and propose a numerical method for obtaining approximations of minimal elements. The results in this section rely mainly on [16].

2.4.1 Formulation of the New Set Relation and its Properties

So far, we have recalled and extended different set relations from the literature in order to compare sets in abstract spaces. Among those, the set relations $\preceq^u_D$ and $\preceq^l_D$ play a crucial role in applications involving uncertainty (see Chapter 5). The upper set less relation $\preceq^u_D$ is widely used to model *robust solutions* of uncertain vector optimization problems, whereas the lower set less relation $\preceq^l_D$ can be utilized in order to obtain solutions that work well in the best-case scenario (so-called *optimistic solutions*). However, these relations are somewhat counterparts of each other and do not reflect well the attitude of a decision maker who is looking for a compromise. In this section, we will introduce a new set relation that resolves this issue. The new relation, that is based on the characterizations given in Theorems 2.2.6 and 2.2.13, will involve both the upper as well as the lower set relation as special cases, and allows to alternate between the two in a continuous manner.

Throughout this section, we assume that the attainment properties in Theorems 2.2.6

and 2.2.13 are satisfied. For this reason we need some further assumptions on the sets $A, B \in \mathcal{P}(Y)$ (see Assumption 2.4.1 below).

Assumption 2.4.1. *Let Y be a real quasicompact topological linear space, $D \in \mathcal{P}(Y)$ a closed proper convex cone with nonempty interior, $k \in \operatorname{int} D$, and assume that $A, B \in \mathcal{P}(Y)$ are closed and bounded.*

Now we are ready to present a new set relation.

Definition 2.4.2 ([16, Definition 2.5]). *Let Assumption 2.4.1 be satisfied, and let $\lambda \in [0,1]$. The **weighted set relation** $\preceq_D^\lambda$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by*

$$A \preceq_D^\lambda B :\Longleftrightarrow \quad \lambda g^u(A,B) + (1-\lambda) g^l(A,B) \leq 0,$$

where

$$g^u(A,B) := \sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b),$$
$$g^l(A,B) := \sup_{b\in B} \inf_{a\in A} z^{D,k}(a-b).$$

Remark 2.4.3 ([16, Remark 2.6]). *We assume that D has nonempty interior in Assumption 2.4.1, because we need this property in order to show that $\preceq_D^\lambda$ is reflexive (see Proposition 2.4.6, (iii), below). This assumption facilitates the presumption of the attainment properties in Theorems 2.2.6 and 2.2.13: According to Theorem 2.1.2, the functional $z^{D,k}$ (with $k \in \operatorname{int} D$ and D a proper closed convex cone) is continuous. Therefore, by the well-known Theorem of Weierstrass, we get that the attainment properties in Theorems 2.2.6 and 2.2.13 are fulfilled if the involved sets are nonempty closed bounded subset of a real quasicompact topological linear space Y. If Assumption 2.4.1 is fulfilled, then we can assume that for any $k \in \operatorname{int} D$, $\inf_{b\in B} z^{D,k}(a-b)$ and $\inf_{a\in A} z^{D,k}(a-b)$ are attained for each $a \in A$, $b \in B$, respectively.*

Remark 2.4.4 ([16, Remark 2.7]). *Obviously, if $\lambda = 1$, then $A \preceq_D^\lambda B$ if and only if $A \preceq_D^u B$. For $\lambda = 0$, we have $A \preceq_D^\lambda B$ if and only if $A \preceq_D^l B$. If $A \preceq_D^u B$ and $A \preceq_D^l B$ hold, then $A \preceq_D^\lambda B$ is true for all $\lambda \in [0,1]$. The converse is not true, and this is exactly the intention of introducing $\preceq_D^\lambda$: The parameter λ serves as a weight factor which indicates the importance of either of the two relations $\preceq_D^u$ and $\preceq_D^l$. The relation which is more important should be associated with a higher weight factor. For instance, if $g^u(A,B) \leq 0$ and $g^l(A,B) > 0$, then, for large enough λ, $A \preceq_D^\lambda B$ can hold and then $A \preceq_D^u B$ "outweighs" the effects of $A \not\preceq_D^l B$.*

Proposition 2.4.5 ([16, Proposition 2.8]). *Let Assumption 2.4.1 be satisfied. Then $A \not\subseteq A - \operatorname{int}D$ and $A \not\subseteq A + \operatorname{int}D$.*

Proof. Suppose that $A \subseteq A - \operatorname{int}D$. Then $A - D \subseteq A - \operatorname{int}D - D \subseteq A - \operatorname{int}D$. Since A is nonempty, closed and bounded, $\emptyset \neq A - D \neq Y$ and $\emptyset \neq A - \operatorname{int}D \neq Y$. It is worth noting that $A - D$ is a closed set and $A - \operatorname{int}D$ is an open set. Taking into account that $A - \operatorname{int}D \subseteq A - D$, we have $A - D = A - \operatorname{int}D$. This implies that $A - D = \emptyset$ (or, Y) and $A - \operatorname{int}D = \emptyset$ (or, Y), which is a contradiction. The case $A \not\subseteq A + \operatorname{int}D$ can be derived similarly. $\square$

Proposition 2.4.6 ([16, Proposition 2.9])**.** *Let Assumption 2.4.1 be satisfied and, additionally, let $C \in \mathcal{P}(Y)$ be closed and bounded. Then the following assertions hold:*

(i) *We have*

$$g^u(A,C) \leq g^u(A,B) + g^u(B,C),$$
$$g^l(A,C) \leq g^l(A,B) + g^l(B,C).$$

(ii) $g^u(\alpha A, \alpha B) = \alpha g^u(A,B)$ *and* $g^l(\alpha A, \alpha B) = \alpha g^l(A,B)$ *for any* $\alpha \geq 0$.

(iii) *For any* $\lambda \in [0,1]$ *the relation* $\preceq_D^\lambda$ *is reflexive and transitive. Hence,* $\preceq_D^\lambda$ *is a preorder.*

(iv) *The relation* $\preceq_D^\lambda$ *is compatible with nonnegative scalar multiplication, i.e., for given* $A, B \in \mathcal{P}(Y)$ *and any* $\alpha \geq 0$*, we have*

$$A \preceq_D^\lambda B \implies \alpha A \preceq_D^\lambda \alpha B.$$

Proof. (i) Choose $\bar{b} \in B$ such that $\sup_{a\in A} z^{D,k}(a-\bar{b}) := \sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b)$. Such a $\bar{b}$ always exists according to Assumption 2.4.1 (see also Remark 2.4.3). Then

$$\begin{aligned}
g^u(A,C) &= \sup_{a\in A} \inf_{c\in C} z^{D,k}(a-c) = \sup_{a\in A} \inf_{c\in C} z^{D,k}(a-\bar{b}+\bar{b}-c)\\
&\leq \sup_{a\in A} \inf_{c\in C} \left(z^{D,k}(a-\bar{b}) + z^{D,k}(\bar{b}-c) \right) \quad (\text{as } z^{D,k} \text{ is subadditive})\\
&= \sup_{a\in A} z^{D,k}(a-\bar{b}) + \inf_{c\in C} z^{D,k}(\bar{b}-c)\\
&= \sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b) + \inf_{c\in C} z^{D,k}(\bar{b}-c)\\
&\leq \sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b) + \sup_{b\in B} \inf_{c\in C} z^{D,k}(b-c).
\end{aligned}$$

The triangle inequality for g^l follows a similar pattern, so the proof is omitted here.

(ii) We have for any $\alpha > 0$,

$$\begin{aligned}
g^u(\alpha A, \alpha B) &= \sup_{a\in \alpha A} \inf_{b\in \alpha B} z^{D,k}(a-b)\\
&= \sup_{\frac{a}{\alpha}\in A} \inf_{\frac{b}{\alpha}\in B} z^{D,k}(a-b)\\
&= \sup_{\tilde{a}\in A} \inf_{\tilde{b}\in B} z^{D,k}(\tilde{a}\alpha - \tilde{b}\alpha) \quad (\text{with } \tilde{a} := \frac{a}{\alpha},\ \tilde{b} := \frac{b}{\alpha})\\
&= \sup_{\tilde{a}\in A} \inf_{\tilde{b}\in B} \alpha z^{D,k}(\tilde{a}-\tilde{b}) \quad (\text{as } z^{D,k} \text{ is positive homogeneous})\\
&= \alpha \sup_{\tilde{a}\in A} \inf_{\tilde{b}\in B} z^{D,k}(\tilde{a}-\tilde{b}) = \alpha g^u(A,B).
\end{aligned}$$

For $\alpha = 0$, $g^u(0,0) = 0$ is obvious. The proof for g^l is similar and left out.

(iii) Here we show that, for arbitrary sets $A \in \mathcal{P}(Y)$, $g^u(A, A) = 0$. We already know that $g^u(A, A) \leq 0$, as $A \subseteq A - D$ is fulfilled, and according to Theorem 2.2.6, this is equivalent to $g^u(A, A) \leq 0$. Suppose now that $g^u(A, A) < 0$. Then we have

$$\sup_{a \in A} \inf_{\overline{a} \in A} z^{D,k}(a - \overline{a}) < 0.$$

This means that for all $a \in A$, $\inf_{\overline{a} \in A} z^{D,k}(a - \overline{a}) < 0$. Due to Assumption 2.4.1, for all $a \in A$ there exists some $\tilde{a} \in A$ such that $z^{D,k}(a - \tilde{a}) < 0$. It follows from Theorem 2.1.2 (iv) that $a - \tilde{a} \in -\mathrm{int}D$, and so, $A \subseteq A - \mathrm{int}D$, which contradicts the fact that $A \not\subseteq A - \mathrm{int}\ D$.

Note that $A \subseteq A + D$, as $0 \in D$. Again, from Theorem 2.2.13, it yields that $g^l(A, A) \leq 0$. The case $g^l(A, A) = 0$ can be proven similarly, leading to the contradiction that $A \subseteq A + \mathrm{int}D$ when assuming $g^l(A, A) < 0$.

Therefore, $g^u(A, A) = g^l(A, A) = 0$, which implies $A \preceq_D^\lambda A$, and $\preceq_D^\lambda$ is reflexive.

Now we show that $\preceq_D^\lambda$ is transitive for arbitrary $\lambda \in [0, 1]$. Let the sets A, B, C be given according to Assumption 2.4.1, and let $A \preceq_D^\lambda B$ and $B \preceq_D^\lambda C$. Then

$$\lambda g^u(A, B) + (1 - \lambda) g^l(A, B) \leq 0,$$
$$\lambda g^u(B, C) + (1 - \lambda) g^l(B, C) \leq 0.$$

It follows that

$$\lambda \left(g^u(A, B) + g^u(B, C) \right) + (1 - \lambda) \left(g^l(A, B) + g^l(B, C) \right) \leq 0.$$

Due to the triangle inequality of g^u and g^l (see (i) above), we immediately obtain

$$\lambda g^u(A, C) + (1 - \lambda) g^l(A, C) \leq 0,$$

which corresponds to $A \preceq_D^\lambda C$. That means that $\preceq_D^\lambda$ is transitive. Hence, $\preceq_D^\lambda$ is a preorder.

(iv) Let $A \preceq_D^\lambda B$. From Definition 2.4.2, one has

$$\lambda g^u(A, B) + (1 - \lambda) g^l(A, B) \leq 0.$$

This, together with (ii) shows that

$$\lambda g^u(\alpha A, \alpha B) + (1 - \lambda) g^l(\alpha A, \alpha B) = \alpha \left(\lambda g^u(A, B) + (1 - \lambda) g^l(A, B) \right) \leq 0,$$

for all $\alpha \geq 0$, as required. □

We provide an example below to illustrate the new relation $\preceq_D^\lambda$ and discuss the role of the parameter λ.

Example 2.4.7 ([16, Example 2.10])**.** *Let $A := [\underline{a}, \overline{a}]$ and $B := [\underline{b}, \overline{b}]$ be compact sets in $\mathbb{R}$. We choose $D = \mathbb{R}_+$ and $k = 1$. We have*

$$\begin{aligned} g^u(A,B) &= \sup_{a\in A}\inf_{b\in B} z^{D,k}(a-b) = \sup_{a\in A}\inf_{b\in B}\inf\{t\in\mathbb{R} \mid a-b\leq t\} \\ &= \sup_{a\in A}\inf_{b\in B}\,(a-b) = \sup_{a\in A}\, a - \sup_{b\in B} b = \overline{a}-\overline{b}, \\ g^l(A,B) &= \underline{a}-\underline{b}, \quad g^u(B,A) = \overline{b}-\overline{a}, \quad g^l(B,A) = \underline{b}-\underline{a}. \end{aligned}$$

Let, for example, $\underline{a} = 5$, $\overline{a} = 10$, $\underline{b} = 0$, $\overline{b} = 11$. Then $B \not\preceq^u_D A$, but $B \preceq^l_D A$. Also, $A \preceq^u_D B$, but $A \not\preceq^l_D B$. However, we can see that the "amount" of B that is bigger than the supremum of A is very small compared to how the lower bound of B is smaller than the lower bound of A. In that sense, when a decision-maker has no clear understanding of how to choose a set, the new set relation $\preceq^\lambda_D$ can be helpful. We have $g^u(A,B) = -1$, $g^l(A,B) = 5$. So, in order for $A \preceq^\lambda_D B$ to hold, $\lambda \in \left[\frac{5}{6}, 1\right]$. Similarly, as $g^u(B,A) = 1$, $g^l(B,A) = -5$, $\lambda \in \left[0, \frac{5}{6}\right]$ for $B \preceq^\lambda_D A$ to hold true.

2.4.2 Formulation of Set Optimization Problems Using the New Set Relation

For the remainder of Chapter 2.4, we assume that $S \subseteq \mathbb{R}^n$. We use the following definition of semicontinuity of a set-valued map w.r.t. a preorder $\preceq$ (see [56]).

Definition 2.4.8 (Semicontinuity)**.** *Let $S \subseteq \mathbb{R}^n$. The set-valued mapping $F\colon S \rightrightarrows \mathbb{R}^m$ is called* ***semicontinuous*** *at $\bar{x} \in S$ w.r.t. the preorder $\preceq$ if $F(\bar{x}) \in \mathcal{V}$, where $\mathcal{V} := \{T \in \mathcal{A} \mid T \not\preceq V\}$ for some $V \in \mathcal{P}(\mathbb{R}^m)$, implies that there exists a neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ such that $F(x) \in \mathcal{V}$ for all $x \in U$. In other words, F is semicontinuous at $\bar{x}$ if*

$$F(\bar{x}) \not\preceq V \text{ for some } V \in \mathcal{P}(\mathbb{R}^m) \quad\Longrightarrow\quad \exists\, U(\bar{x}):\; F(x) \not\preceq V \;\forall\;\; x \in U.$$

F is called semicontinuous w.r.t. $\preceq$ if F is semicontinuous w.r.t. $\preceq$ at every $\bar{x} \in S$.

We provide the following example for a set-valued mapping that is semicontinuous w.r.t. the the weighted set relation $\preceq^\lambda_D$ introduced in Section 2.4.2.

Example 2.4.9 ([16, Example 3.3])**.** *Let $S = \mathbb{R}$ and consider the set-valued mapping $F : S \rightrightarrows \mathbb{R}^2$ given by*

$$F(x) := [(1-x, x), (1,1)]\,,$$

where $[(a,b),(c,d)]$ is the line segment between (a,b) and (c,d), and the preorder $\preceq^\lambda_D$, $D = \mathbb{R}^2_+$ and $k \in D \setminus \{0\}$. We can see that F is semicontinuous w.r.t. the weighted set relation $\preceq^\lambda_D$ for any $\lambda \in [0,1]$: For $\lambda = 1$, we have, for $V = \{(0,0)\}$, $F(x) \not\preceq^\lambda_D V$ for all $x \in \mathbb{R}$. Therefore, for $\lambda = 1$, F is semicontinuous w.r.t. $\preceq^\lambda_D$. Similarly, for $\lambda = 0$ and by choosing $V = \{(0,0)\}$, we get that $F(x) \not\preceq^\lambda_D V$ for all $x \in \mathbb{R}$. That means that for $\lambda = 0$, F is semicontinuous w.r.t. $\preceq^\lambda_D$. Since $g^u(F(x),V) > 0$ and $g^l(F(x),V) > 0$ for all $x \in \mathbb{R}$, we also obtain that $\lambda g^u(F(x),V) + (1-\lambda)g^l(F(x),V) > 0$ for arbitrary $\lambda \in [0,1]$. Therefore, we can conclude that F is semicontinuous w.r.t. $\preceq^\lambda_D$ for any $\lambda \in [0,1]$.

Definition 2.4.10 (Upper / Lower Semicontinuity, see [6]). *Let $S \subset \mathbb{R}^n$. A set-valued mapping $F : S \rightrightarrows \mathbb{R}^m$ is said to be*

(i) upper semicontinuous at $\bar{x} \in S$ if, for any neighborhood V of $F(\bar{x})$, there exists a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$F(u) \subseteq V, \ \ \forall u \in U(\bar{x}).$$

(ii) lower semicontinuous at $\bar{x} \in S$ if, for any $x \in F(\bar{x})$ and any neighborhood V of x, there exists a neighborhood $U(\bar{x})$ of $\bar{x}$ such that

$$F(u) \cap V \neq \emptyset, \ \ \forall u \in U(\bar{x}).$$

We say that F is upper semicontinuous and lower semicontinuous on S if it is upper semicontinuous and lower semicontinuous at each point $\bar{x} \in S$, respectively. We say that F is continuous on S if it is both upper semicontinuous and lower semicontinuous on S.

For the sake of brevity, we give the following assumptions.

Assumption 2.4.11. *Let D be a closed proper convex cone of $\mathbb{R}^m$ with nonempty interior, $k \in \operatorname{int} D$, and the mapping $F \colon S \rightrightarrows \mathbb{R}^m$ be nonempty and compact-valued (i.e., for each $x \in S$, $F(x) \in \mathcal{P}(\mathbb{R}^m)$ is a nonempty compact set).*

Assumption 2.4.12. *Let $x \in S$. If $F(x) \not\preceq_D^\lambda V$ for some $V \in \mathcal{P}(\mathbb{R}^m)$, there exists a closed and bounded set $\bar{V} \in \mathcal{P}(\mathbb{R}^m)$ such that $F(x) \not\preceq_D^\lambda \bar{V}$.*

Now we have the following existence result for problem (1.10) w.r.t. the new set relation $\preceq_D^\lambda$ introduced in Definition 2.4.2.

Corollary 2.4.13 ([16, Corollary 3.7]). *Let Assumption 2.4.11 be satisfied. Suppose that S is compact and that F is semicontinuous w.r.t. the preorder $\preceq_D^\lambda$ on S. Then the problem (1.10) has a minimal solution w.r.t. the preorder $\preceq_D^\lambda$.*

Proof. The result follows immediately by [56, Theorem 5.1], since $\preceq_D^\lambda$ is a preorder due to Proposition 2.4.6. □

Proposition 2.4.14 ([16, Proposition 3.8]). *Let Assumptions 2.4.11 and 2.4.12 be satisfied and F be continuous on S. Then F is semicontinuous w.r.t. the preorder $\preceq_D^\lambda$ on S.*

Proof. Let $\bar{x} \in S$. If $F(\bar{x}) \not\preceq_D^\lambda V$ for some closed and bounded set $V \in \mathcal{P}(\mathbb{R}^m)$, then

$$\lambda g^u(F(\bar{x}), V) + (1-\lambda) g^l(F(\bar{x}), V) > 0.$$

From [6, Propositions 19 and 21], $\lambda g^u(F(x), V) + (1-\lambda) g^l(F(x), V)$ is continuous w.r.t. x on S. So, there exists a neighborhood U of $\bar{x}$ with $U_S := U \cap S$ such that

$$\lambda g^u(F(x), V) + (1-\lambda) g^l(F(x), V) > 0, \ \forall\ x \in U_S.$$

Definition 2.4.2 implies that $F(x) \not\preceq_D^\lambda V$ for all $x \in U_S$. □

By the definition of minimal solutions of the problem (1.10) w.r.t. the preorder $\preceq_D^\lambda$, we know that if $\bar{x} \in S$ is a minimal solution of the problem (1.10) w.r.t. the preorder $\preceq_D^\lambda$ and $F(\tilde{x}) \preceq_D^\lambda F(\bar{x})$ for some $\tilde{x} \in S$, then $\tilde{x}$ is a minimal solution of the problem (1.10) w.r.t. the preorder $\preceq_D^\lambda$. Denote

$$[F(\bar{x})]^{-1} := \left\{ x \in S : \ F(x) \preceq_D^\lambda F(\bar{x}), \ F(\bar{x}) \preceq_D^\lambda F(x) \right\}.$$

Let $\lambda \in [0,1]$. We now define a function $g_\lambda : S \times S \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g_\lambda(x,y) := \lambda g^u(F(x),F(y)) + (1-\lambda) g^l(F(x),F(y)). \tag{2.13}$$

Below, we propose a sufficient and necessary condition for minimal solutions of the problem (1.10) w.r.t. the preorder $\preceq_D^\lambda$.

Theorem 2.4.15 ([16, Theorem 3.9]). *Let Assumption 2.4.11 be satisfied and $\bar{x} \in S$. Then $\bar{x}$ is a minimal solution of the problem* (1.10) *w.r.t. the preorder $\preceq_D^\lambda$ if and only if the following system* (*in the unknown x*)*:*

$$g_\lambda(x,\bar{x}) \le 0, \ x \in S \setminus [F(\bar{x})]^{-1}, \tag{2.14}$$

is impossible.

Proof. Let $\bar{x} \in S$ be a minimal solution of the problem (1.10) w.r.t. the preorder $\preceq_D^\lambda$. For $x \in S \setminus [F(\bar{x})]^{-1}$, we have $F(x) \not\preceq_D^\lambda F(\bar{x})$. It follows from Definition 2.4.2 that

$$g_\lambda(x,\bar{x}) > 0.$$

This implies that the system (2.14) in the unknown x is impossible.

Conversely, assume that the system (2.14) in the unknown x is impossible. Then we have for any $x \in \mathbb{R}^n$, $g_\lambda(x,\bar{x}) > 0$ or, $x \notin S \setminus [F(\bar{x})]^{-1}$. Particularly, one has for $x \in S$,

$$\lambda g^u(F(x),F(\bar{x})) + (1-\lambda) g^l(F(x),F(\bar{x})) > 0 \ \text{ or, } \ x \in [F(\bar{x})]^{-1}.$$

This means that we have for $x \in S$, $F(x) \not\preceq_D^\lambda F(\bar{x})$ or, $x \in [F(\bar{x})]^{-1}$. Consequently, $\bar{x} \in S$ is a minimal solution of the problem (1.10) w.r.t. the preorder $\preceq_D^\lambda$. □

Proposition 2.4.16 ([16, Proposition 3.10]). *Let Assumption 2.4.11 be satisfied and F be continuous on S. Then $g_\lambda(\cdot,x)$ is continuous on S for all $x \in S$, and for each $\bar{x} \in S$, $g_\lambda(\bar{x},\cdot)$ is continuous on S.*

Proof. The result follows immediately by [6, Propositions 19 and 21]. □

Proposition 2.4.17 ([16, Proposition 3.11]). *Let Assumption 2.4.11 be satisfied, $\bar{x} \in S$ and $S \setminus [F(\bar{x})]^{-1} \neq \emptyset$. Suppose that S is convex and that F is continuous on S. If $\bar{x}$ is a minimal solution of the problem* (1.10) *w.r.t. the preorder $\preceq_D^\lambda$, then there exists $\tilde{x} \in [F(\bar{x})]^{-1}$ such that $g_\lambda(\tilde{x},\bar{x}) = 0$.*

Proof. From the proof of Theorem 2.4.15, we know that

$$g_\lambda(x,\bar{x}) > 0, \ x \in S \setminus [F(\bar{x})]^{-1}.$$

Note that $g_\lambda(\bar{x},\bar{x}) \le 0$. It thus follows from Proposition 2.4.16 that there exists $\tilde{x} \in [x,\bar{x}] \cap [F(\bar{x})]^{-1}$, where $x \in S \setminus [F(\bar{x})]^{-1}$ and $[x,\bar{x}]$ is the line segment between x and $\bar{x}$, such that $g_\lambda(\tilde{x},\bar{x}) = 0$. □

2.4.3 Computing Approximations of Minimal Elements of Set Optimization Problems Using a New Set Relation

As already pointed out, the set relation $\preceq_D^\lambda$ is reflexive and transitive under the requirements in Assumption 2.4.1, such that a descent method is an appropriate tool to generate approximations of minimal solutions of problem (1.10) w.r.t. $\preceq_D^\lambda$, where $S = \mathbb{R}^n$. The idea is that for a given starting point x^0, neighboring points x are tested whether they fulfill $F(x) \preceq_D^\lambda F(x^0)$. The advantage of the new set relation is that it provides us with a possibility to decide whether two sets fulfill the set relation or not in a numerical manner and even give a quantification by means of the extremal points of the functional values $g_\lambda(x, x^0) =: \text{extremal}_{term}$. The point x with the smallest value extremal_{term} is then chosen. This procedure continues until a maximal number of iterations is reached. Moreover, the parameter λ is chosen from $[0, 1]$, such that the output generates approximations of minimal solutions of problem (1.10) w.r.t. $\preceq_D^\lambda$ in terms of λ. This gives the practitioner the chance to choose a solution based on his preferences.

Algorithm 2.4.18. *(A descent method for finding an approximation of a minimal solution of the set-valued problem* (1.10)*)*

Input: $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $S = \mathbb{R}^n$, *ordering cone* D *with nonempty interior, preorder* $\preceq_D^\lambda$,
starting point $x^0 \in \mathbb{R}^n$, $k \in \operatorname{int} D$, *maximal number* i_{max} *of iterations, number of search*
directions n_s, *maximal number* j_{max} *of iterations for the determination of the step size,*
initial step size h_0 *and minimum step size* $h_{\min}$, $\{\lambda_1, \ldots, \lambda_N\} \subset [0, 1]$
for $p = 1 : 1 : N$ ***do***
  % initialization for the descent method
  $i := 0$, $h := h_0$
  choose n_s *points* $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^{n_s}$ *on the unit sphere around* $0_{\mathbb{R}^n}$
  % iteration loop
  while $i \leq i_{max}$ ***do***
    check $F(x^i + h\tilde{x}^j) \preceq_D^{\lambda_p} F(x^i)$ *for every* $j \in \{1, \ldots, n_s\}$ *by evaluating the extremal*
    term (e. g. $\lambda_p g^u(A, B) + (1 - \lambda_p) g^l(A, B)$ *for* $A = F(x^i + h\tilde{x}^j)$ *and* $B = F(x^i)$*).*
    Choose the index $n_0 := j$ *with the smallest function value* extremal_{term}*.*
    if $\text{extremal}_{term} \leq 0$ ***then***
      $x^{i+1} := x^i + h\tilde{x}^{n_0}$ *% new iteration point*
      $j := 1$
      while $F(x^i + (j+1)h\tilde{x}^{n_0}) \preceq_D^{\lambda_p} F(x^i + jh\tilde{x}^{n_0})$ *and* $j \leq j_{max}$ ***do***
        $j := j + 1$
        $x^{i+1} := x^{i+1} + h\tilde{x}^{n_0}$ *% new iteration point*
      end while
    else
      $h := h/2$

23:       ***if*** $h \leq h_{\min}$ ***then***
24:         ***STOP***. *Output:* $x(\lambda_p) := x^i$
25:       ***end if***
26:     ***end if***
27:     $i := i + 1$
28:   ***end while***
29: ***end for***
30: *Output: A set of approximations x of minimal solutions of the set-valued problem (1.10) w.r.t. $\preceq_D^\lambda$ depending on λ.*

For one given starting point x^0, Algorithm 2.4.18 approximates for each considered $\lambda \in [0, 1]$ one minimal solution of problem (1.10). Algorithm 2.4.18 is a descent method. For one given starting point x^0 and an approximation x^i, we get that $F(x^i) \preceq_D^\lambda F(x^0)$ due to the transitivity of the set relation $\preceq_D^\lambda$.

We have the following easy corollary.

Corollary 2.4.19 ([16, Corollary 4.1])**.** *Algorithm 2.4.18 is well-defined for arbitrary input values.*

In the following we use the definition of the functional $g_\lambda : S \times S \to \mathbb{R} \cup \{\pm\infty\}$ given in (2.13) by

$$g_\lambda(x, y) = \lambda g^u(F(x), F(y)) + (1 - \lambda) g^l(F(x), F(y)).$$

We now set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In order to show a convergence result for Algorithm 2.4.18, we need the following modifications in the algorithm:

(A) Assume that the pattern contains at least one direction of descent whenever a set $F(x^i)$ ($i \in \mathbb{N}_0$) can be improved.

(B) Let some $\beta \in (0, 1)$ and an arbitrary null sequence $(\epsilon^i)_{i \in \mathbb{N}_0}$ with $\epsilon^i < 0$ for all $i \in \mathbb{N}_0$ be given. While $g_\lambda(x^{i+1}, x^i) \leq \epsilon^i$, set $h := \beta^q h$ for $q := 0, 1, 2, \ldots$ after line 27 of Algorithm 2.4.18.

Specification (A) means that if x^i ($i \in \mathbb{N}_0$) is not the final iteration point, then there exists a descent direction and a point $\overline{x} \in \mathbb{R}^n$ such that $g_\lambda(\overline{x}, x^i) < 0$. Due to Proposition 2.4.16, the functional g_λ is continuous under Assumption 2.4.11. It follows that there exists some ball $B(\overline{x}, \delta)$ around $\overline{x}$ with radius δ such that for all $x \in B(\overline{x}, \delta)$, $g_\lambda(x, x^i) < 0$. Therefore, refining the grid (that means allowing for different step sizes and more search directions) will eventually lead to a descent direction, and specification (A) can easily be fulfilled.

Specification (B) characterizes a certain kind of step length control.

Theorem 2.4.20 ([16, Theorem 4.2])**.** *Let $\lambda \in [0, 1]$ be given. Furthermore, let Assumption 2.4.11 be satisfied, let Algorithm 2.4.18 with the additional specifications (A) and (B) generate an iteration sequence $(x^i)_{i \in \mathbb{N}_0}$ and let the level set*

$$L_{x^0} := \left\{ x \in \mathbb{R}^n \mid F(x) \preceq_D^\lambda F(x^0) \right\}$$

be compact, where x^0 denotes the initial iteration point in Algorithm 2.4.18. Then

$$\limsup_{i\to+\infty}\ g_\lambda(x^{i+1}, x^i) = 0.$$

Proof. Since Algorithm 2.4.18 is a descent method by construction and the set relation $\preceq^\lambda_D$ is transitive by our assumptions, we have $x^i \in L_{x^0}$ for all $i \in \mathbb{N}_0$. Since the function g_λ is continuous due to Proposition 2.4.16, and because of the compactness of the level set L_{x^0}, the function values $g_\lambda(x^{i+1}, x^i)$ with $i \in \mathbb{N}_0$ are bounded. Consequently, the limit superior exists. We assume now that $\limsup_{i\to+\infty}\ g_\lambda(x^{i+1}, x^i) \neq 0$. Then there exists a subsequence $(x^{i_r})_{r\in\mathbb{N}}$ with $\lim_{r\to+\infty}\ g_\lambda(x^{i_r+1}, x^{i_r}) =: \alpha \neq 0$. By the specification (A), we have $F(x^{i+1}) \preceq^\lambda_D F(x^i)$ for all $i \in \mathbb{N}_0$, and we immediately obtain by definition of the preorder

$$\forall\ i \in \mathbb{N}_0 :\quad g_\lambda(x^{i+1}, x^i) \leq 0.$$

This means that $\alpha < 0$. Then there exists some $N_1 \in \mathbb{N}$ with

$$\forall\ r \geq N_1 :\quad g_\lambda(x^{i_r+1}, x^{i_r}) \leq \frac{\alpha}{2} < 0.$$

As $(\epsilon^i)_{i\in\mathbb{N}_0}$ is a null sequence, there is some $N_2 \in \mathbb{N}$ with the property

$$\forall\ r \geq N_2 :\quad \frac{\alpha}{2} \leq \epsilon^{i_r} < 0.$$

This results in

$$\forall\ r \geq \max\{N_1, N_2\} :\quad g_\lambda(x^{i_r+1}, x^{i_r}) \leq \frac{\alpha}{2} \leq \epsilon^{i_r}.$$

This is a contradiction to specification (B). □

2.5 Representation of Set Relations in Real Linear Spaces

Until now, we have performed our analysis on the real linear *topological* space Y. Quite recently (see [43] and the references therein), the nonlinear scalarizing functional $z^{D,k}$ has been extended to the case where no topology on the space Y is assumed. It is our goal in this section to characterize the set relations from Section 2.2 in a real linear space. The findings presented in this section rely mainly on [45].

2.5.1 Preliminaries

Throughout Section 2.5, let Y be a real linear space. For a nonempty set $F \subset Y$, we denote by

$$\operatorname{cor} F := \{y \in Y \mid \forall v \in Y\ \exists \lambda > 0 \text{ s.t. } y + [0, \lambda]v \subset F\}$$

the algebraic interior of F and by

$$\operatorname{vcl} F := \{y \in Y \mid \exists v \in Y\ \forall \lambda > 0\ \exists \lambda' \in [0, \lambda] \text{ s.t. } y + \lambda' v \in F\}$$

its vector closure. For any given $k \in Y$, let

$$\operatorname{vcl}_k F := \{y \in Y \mid \forall \lambda > 0 \ \exists \lambda' \in [0, \lambda] \text{ s.t. } y + \lambda' k \in F\}$$

and

$$\operatorname{ovcl}_k^{+\infty} F := \{y \in Y \mid \forall \lambda > 0 \ \exists \lambda' \in [\lambda, +\infty] \text{ s.t. } y + \lambda' k \in F\}.$$

We say that F is k-vectorially closed if $\operatorname{vcl}_k F = F$, F is vectorially closed if $\operatorname{vcl} F = F$ and F is algebraically solid if $\operatorname{cor} F \neq \emptyset$. Obviously, it holds $F \subset \operatorname{vcl}_k F$ for all $k \in Y$.

Throughout Chapter 2.5, we consider the following set relations for $A, B \in Y$, $A, B \neq \emptyset$, $A, B \neq Y$:

1. generalized upper set less relation: $A \preceq^u_{D,rls} B :\Longleftrightarrow A \subseteq B - D$;
2. generalized lower set less relation: $A \preceq^l_{D,rls} B :\Longleftrightarrow A + D \supseteq B$;
3. generalized set less relation: $A \preceq^s_{D,rls} B :\Longleftrightarrow A \preceq^u_{D,rls} B$ and $A \preceq^l_{D,rls} B$,

where the notion $\preceq^{\cdot}_{\cdot,rls}$ indicates that we are working in a real linear space.

Let us recall the functional $z^{D,k}$ under more general assumptions: Let $\emptyset \neq D \subset Y$ and $k \in Y \setminus \{0\}$. Then, we define $z^{D,k} \colon Y \to \overline{\mathbb{R}}$ from Gerstewitz [32] (see Section 2.1)

$$z^{D,k}(y) := \begin{cases} +\infty & \text{if } y \notin \mathbb{R}k - D, \\ \inf\{t \in \mathbb{R} \mid y \in tk - D\} & \text{otherwise}. \end{cases}$$

Below we provide some properties of the functional $z^{D,k}$ without any topology posed on Y.

Proposition 2.5.1 ([44]). *Let D and E be nonempty subsets of Y, and let $k \in Y \setminus \{0\}$. Then the following properties hold.*

(a) $\forall\, y \in Y: \ z^{D,k}(y) \leq 0 \Longleftrightarrow y \in (-\infty, 0]k - \operatorname{vcl}_k D$.

(b) $\forall\, y \in Y: \ z^{D,k}(y) < 0 \Longleftrightarrow y \in (-\infty, 0)k - \operatorname{vcl}_k D$.

(c) $z^{D,k}$ *is E-monotone if and only if* $E + D \subset [0, +\infty)k + \operatorname{vcl}_k D$.

(d) $\forall\, y \in Y, \ \forall\, r \in \mathbb{R} : z^{D,k}(y + rk) = z^{D,k}(y) + r$.

As the above set relations rely on set inclusions where the set D is attached pointwise to the considered sets $A, B \in \mathcal{P}(Y)$, we consider the following corollary that relates $A+D$ and $A - D$ respectively by means of the functional $z^{D,k}$ in a real linear space Y which is not a priori equipped with a given topology.

Corollary 2.5.2 ([45, Corollary 2.3]). *Let $D \subset Y$ a convex cone, $A \in \mathcal{P}(Y)$ and $k \in Y \setminus \{0\}$. Then it holds*

$$\sup_{a \in A} z^{D,k}(a) = \sup_{y \in A - D} z^{D,k}(y) \ \text{ and } \ \inf_{a \in A} z^{D,k}(a) = \inf_{y \in A + D} z^{D,k}(y).$$

Proof. We will only prove the first assertion, as the second one can be proven in a similar manner. Let $c \in D$ and $a \in A$ be given. Because D is a convex cone, $D + D \subseteq D \subseteq \operatorname{vcl}_k D = \{0\} + \operatorname{vcl}_k D \subseteq [0, +\infty)k + \operatorname{vcl}_k D$ holds true. Then one can use the D-monotonicity of the functional $z^{D,k}$ (see Proposition 2.5.1 (c) with $E = D$) to show

$$z^{D,k}(a) \geq z^{D,k}(a - c),$$

as $a - c \in a - D$ implies $z^{D,k}(a - c) \leq z^{D,k}(a)$. Then we directly obtain

$$\sup_{a \in A} z^{D,k}(a) \geq \sup_{y \in A - D} z^{D,k}(y).$$

The converse, i.e., $\sup_{a\in A} z^{D,k}(a) \leq \sup_{y\in A-D} z^{D,k}(y)$, follows directly from the definition of the supremum and $0 \in D$, or in particular $A \subseteq A - D$. $\square$

2.5.2 Representation of Set Relations in a Real Linear Space

The following theorem shows a first connection between the upper set less relation and the nonlinear scalarizing functional $z^{D,k}$, where the space Y is not a priori equipped with a topology.

Theorem 2.5.3 ([45, Theorem 3.2]). *Let $D \subset Y$ be a convex cone, $A, B \in \mathcal{P}(Y)$ and $k \in Y \setminus \{0\}$. Then*

$$A \preceq^u_{D,rls} B \implies \sup_{a\in A} z^{D,k}(a) \leq \sup_{b \in B} z^{D,k}(b).$$

Proof. Because D is a convex cone, it holds $D + D \subseteq D \subseteq \operatorname{vcl}_k D = \{0\} + \operatorname{vcl}_k D \subseteq [0, +\infty)k + \operatorname{vcl}_k D$. Due to Proposition 2.5.1 (c), we obtain the D-monotonicity property of $z^{D,k}$. Now let $A \subseteq B - D$. Then for all $a \in A$, there exists $b \in B$ such that $a \in b - D$. This immediately yields $\sup_{a\in A} z^{D,k}(a) \leq \sup_{b\in B} z^{D,k}(b)$. $\square$

The converse implication in Theorem 2.5.3 is not generally fulfilled, even if the underlying sets are convex, see [66, Example 3.2]. However, we have the following result.

Theorem 2.5.4 ([45, Theorem 3.3]). *Let $D \subset Y$. For two sets $A, B \in \mathcal{P}(Y)$ and $k \in Y \setminus \{0\}$, it holds*

$$A \preceq^u_{D,rls} B \quad \implies \quad \sup_{a\in A} \inf_{b\in B} z^{D,k}(a - b) \leq 0\,.$$

Assume on the other hand that there exists a $k_0 \in Y \setminus \{0\}$ such that $\inf_{b\in B} z^{D,k_0}(a - b)$ is attained for all $a \in A$, D is k_0-vectorially closed and $[0, +\infty)k_0 + D \subseteq D$. Then

$$\sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a - b) \leq 0 \quad \implies \quad A \preceq^u_{D,rls} B\,.$$

Proof. Let $A \subseteq B - D$. This corresponds to

$$\forall\, a \in A \;\; \exists\, b \in B : \;\; a - b \in -D \subseteq -\operatorname{vcl}_k D\,.$$

Due to Proposition 2.5.1 (a), we obtain

$$\forall\, a \in A \;\; \exists\, b \in B : \;\; z^{D,k}(a-b) \le 0$$

and this directly implies the assertion, i.e., $\sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b) \le 0$. Conversely, let $\sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a-b) \le 0$. This means that for all $a \in A$, we have $\inf_{b\in B} z^{D,k_0}(a-b) \le 0$. Because for all $a \in A$, $\inf_{b\in B} z^{D,k_0}(a-b)$ is attained, we obtain

$$\forall a \in A\; \exists \bar{b} \in B : \;\; z^{D,k_0}(a-\bar{b}) = \inf_{b\in B} z^{D,k_0}(a-b) \le 0.$$

This implies

$$\forall a \in A\; \exists \bar{b} \in B : \;\; a - \bar{b} \in (-\infty, 0]k_0 - \operatorname{vcl}_{k_0} D.$$

Therefore, $A \subseteq B + (-\infty, 0]k_0 - \operatorname{vcl}_{k_0} D \subseteq B - D$. □

Remark 2.5.5 ([45, Remark 3.4]). *(1) Note that for any $A, B \in \mathcal{P}(Y)$, the set relation $A \preceq^u_{D,rls} B$ by Theorem 2.5.4 also implies $\sup_{k\in Y\setminus\{0\}} \sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b) \le 0$.*
(2) Let $A, B \in \mathcal{P}(Y)$ and $D \subset Y$. If there exists an element $k_0 \in D \setminus \{0\}$ such that $\inf_{b\in B} z^{D,k_0}(a-b)$ is attained for all $a \in A$, D is k_0-vectorially closed and $[0,+\infty)k_0 + D \subseteq D$, then it follows from Theorem 2.5.4 that

$$\begin{aligned} A \preceq^u_{D,rls} B &\iff \sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a-b) \le 0 \\ &\iff \sup_{k\in Y\setminus\{0\}} \sup_{a\in A} \inf_{b\in B} z^{D,k}(a-b) \le 0. \end{aligned}$$

The following example shows that the attainment property in Theorem 2.5.4 cannot be omitted.

Example 2.5.6 ([45, Example 3.5]). *Let $Y = \mathbb{R}^2$, $A := \{(0,0)^T\}$, $B := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1, y_2 \in (-1, 0)\}$, $D = \mathbb{R}^2_+$ and $k_0 = (1,1)^T$. We have $\operatorname{vcl}_{k_0} D = D$ and $[0,+\infty)k_0 + D \subseteq D$. It holds that $\sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a-b) \le 0$, however, $A \not\preceq^u_{D,rls} B$. This is because $\inf_{b\in B} z^{D,k_0}(a-b)$ is not attained for $a = (0,0)^T$.*

In the second part of Theorem 2.5.4, we need the assumption that there exists a $k_0 \in Y \setminus \{0\}$ such that $\inf_{b\in B} z^{D,k_0}(a-b)$ is attained for all $a \in A$. As already mentioned in Remark 2.3.8, sufficient conditions for such an attainment property, i.e., assertions concerning the existence of solutions of the corresponding optimization problems (extremal principles) are given in the literature. Since the functional z^{D,k_0} is studied here in the context of real linear spaces that are not endowed with a particular topology, we cannot rely on continuity assumptions. Therefore, we propose the following theorem without any attainment property.

Theorem 2.5.7 ([45, Theorem 3.6]). *Let $D \subset Y$, $A, B \in \mathcal{P}(Y)$ and $k_0 \in Y \setminus \{0\}$ such that $(-\infty, 0)k_0 - \operatorname{vcl}_{k_0} D \subseteq -D$ and $\operatorname{vcl}_{-k_0}(B - D) \subseteq B - D$. Then*

$$\sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a-b) \leq 0 \quad \Longrightarrow \quad A \preceq^u_{D,rls} B\,.$$

Proof. Let $\sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a-b) \leq 0$. This means that for all $a \in A$, the inequality $\inf_{b\in B} z^{D,k_0}(a-b) \leq 0$ holds true. Therefore, for all $\epsilon > 0$ and for all $a \in A$ there exists $b \in B$ such that $z^{D,k}(a-b) < \epsilon$. By means of Proposition 2.5.1 (d), we obtain

$$\forall\, \epsilon > 0,\ \forall\, a \in A,\ \exists\, b \in B:\ z^{D,k_0}(a - b - \epsilon k_0) < 0.$$

This implies by Proposition 2.5.1 (b) that

$$\forall\, \epsilon > 0,\ \forall\, a \in A,\ \exists\, b \in B:\ a - b - \epsilon k_0 \in (-\infty, 0)k_0 - \operatorname{vcl}_{k_0} D.$$

This results in

$$\forall\, \epsilon > 0:\ A \subseteq B + \epsilon k_0 + (-\infty, 0)k_0 - \operatorname{vcl}_{k_0} D \subseteq B + \epsilon k_0 - D \subseteq \operatorname{vcl}_{-k_0}(B - D) \subseteq B - D.$$

□

We illustrate by the example below that the assumption $\operatorname{vcl}_{-k_0}(B - D) \subseteq B - D$ in Theorem 2.5.7 cannot be dropped.

Example 2.5.8 ([45, Example 3.7]). *We return to Example 2.5.6. We have $\operatorname{vcl}_{k_0} D = D$, and because $k_0 \in \operatorname{cor} D$, $(-\infty, 0)k_0 - \operatorname{vcl}_{k_0} D \subseteq -D$ holds true. Moreover, the inequality $\sup_{a\in A} \inf_{b\in B} z^{D,k_0}(a-b) \leq 0$ is fulfilled. Because $A \not\preceq^u_{D,rls} B$, due to Theorem 2.5.7, $\operatorname{vcl}_{-k_0}(B-D) \subseteq B-D$ cannot be satisfied. This is immediate, as $\operatorname{vcl}_{-k_0}(B-D) = -\mathbb{R}^2_+$.*

Because $A \preceq^u_{D,rls} B$ is equivalent to $-B \preceq^l_{D,rls} -A$, we obtain the following corollaries from Theorems 2.5.3, 2.5.4 and 2.5.7.

Corollary 2.5.9 ([45, Corollary 3.9]). *Let $D \subset Y$ be a convex cone, $A, B \in \mathcal{P}(Y)$ and $k \in Y \setminus \{0\}$. Then*

$$A \preceq^l_{D,rls} B \implies \inf_{a\in A} z^{D,k}(a) \leq \inf_{b\in B} z^{D,k}(b).$$

Corollary 2.5.10 ([45, Corollary 3.10]). *Let $D \subset Y$. For two sets $A, B \in \mathcal{P}(Y)$ and $k \in Y \setminus \{0\}$, it holds*

$$A \preceq^l_{D,rls} B \quad \Longrightarrow \quad \sup_{b\in B} \inf_{a\in A} z^{D,k}(a-b) \leq 0\,.$$

Assume on the other hand that there exists a $k_0 \in Y \setminus \{0\}$ such that $\inf_{a\in A} z^{D,k_0}(a-b)$ is attained for all $b \in B$, D is k_0-vectorially closed and $[0, +\infty)k_0 + D \subseteq D$. Then

$$\sup_{b\in B} \inf_{a\in A} z^{D,k_0}(a-b) \leq 0 \quad \Longrightarrow \quad A \preceq^l_{D,rls} B\,.$$

Corollary 2.5.11 ([45, Corollary 3.11]). *Let $D \subset Y$, $A, B \in \mathcal{P}(Y)$ and $k_0 \in Y \setminus \{0\}$ such that $(-\infty, 0)k_0 - \operatorname{vcl}_{k_0} D \subseteq -D$ and $\operatorname{vcl}_{-k_0}(-A - D) \subseteq -A - D$. Then*

$$\sup_{b \in B} \inf_{a \in A} z^{D,k_0}(a - b) \leq 0 \quad \Longrightarrow \quad A \preceq^l_{D,rls} B\,.$$

For the *set less relation*, we immediately obtain the following results.

Corollary 2.5.12 ([45, Corollary 3.13]). *Let $D \subset Y$ be a convex cone, $A, B \in \mathcal{P}(Y)$ and $k \in Y \setminus \{0\}$. Then*

$$A \preceq^s_{D,rls} B \implies \sup_{a \in A} z^{D,k}(a) \leq \sup_{b \in B} z^{D,k}(b) \text{ and } \inf_{a \in A} z^{D,k}(a) \leq \inf_{b \in B} z^{D,k}(b).$$

Corollary 2.5.13 ([45, Corollary 3.14]). *Let $D \subset Y$. For two sets $A, B \in \mathcal{P}(Y)$ and $k \in Y \setminus \{0\}$, it holds*

$$A \preceq^s_{D,rls} B \quad \Longrightarrow \quad \sup_{a \in A} \inf_{b \in B} z^{D,k}(a - b) \leq 0 \text{ and } \sup_{b \in B} \inf_{a \in A} z^{D,k}(a - b) \leq 0\,.$$

Assume on the other hand that there exists a $k_0 \in Y \setminus \{0\}$ such that $\inf_{b \in B} z^{D,k_0}(a - b)$ is attained for all $a \in A$, and there exists $k_1 \in Y \setminus \{0\}$ such that $\inf_{a \in A} z^{D,k_1}(a - b)$ is attained for all $b \in B$, D is both k_0- and k_1-vectorially closed, $[0, +\infty)k_0 + D \subseteq D$ and $[0, +\infty)k_1 + D \subseteq D$. Then

$$\sup_{a \in A} \inf_{b \in B} z^{D,k_0}(a - b) \leq 0 \text{ and } \sup_{b \in B} \inf_{a \in A} z^{D,k_1}(a - b) \leq 0 \quad \Longrightarrow \quad A \preceq^s_{D,rls} B\,.$$

Corollary 2.5.14 ([45, Corollary 3.15]). *Let $D \subset Y$, $A, B \in \mathcal{P}(Y)$ and $k_0, k_1 \in Y \setminus \{0\}$ such that $(-\infty, 0)k_0 - \operatorname{vcl}_{k_0} D \subseteq -D$, $(-\infty, 0)k_1 - \operatorname{vcl}_{k_1} D \subseteq -D$, $\operatorname{vcl}_{-k_0}(B - D) \subseteq B - D$ and $\operatorname{vcl}_{-k_1}(-A - D) \subseteq -A - D$. Then*

$$\sup_{a \in A} \inf_{b \in B} z^{D,k_0}(a - b) \leq 0 \text{ and } \sup_{b \in B} \inf_{a \in A} z^{D,k_1}(a - b) \leq 0 \quad \Longrightarrow \quad A \preceq^s_{D,rls} B\,.$$

Now we intend to study set optimization problems in terms of (1.10) with Y being a real linear space. By the definition of minimal solutions of the problem (1.10) w.r.t. the relation $\preceq$, we know that if $\overline{x} \in S$ is a minimal solution of the problem (1.10) w.r.t. $\preceq$ and $F(\tilde{x}) \preceq F(\overline{x})$ for some $\tilde{x} \in S$, then $\tilde{x}$ is a minimal solution of the problem (1.10) w.r.t. $\preceq$ as well. Therefore, let us denote

$$[F(\overline{x})]^{-1}_{\preceq} := \{x \in S : \ F(x) \preceq F(\overline{x}),\ F(\overline{x}) \preceq F(x)\}\,.$$

We now consider problem (1.10) with $\preceq = \preceq^u_{D,rls}$, where $D \subset Y$ is a nonempty set (not necessarily convex). We define a function $g^u : S \times S \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g^u(x, \overline{x}) := \sup_{y \in F(x)} \inf_{\overline{y} \in F(\overline{x})} z^{D,k}(y - \overline{y}).$$

Assumption 2.5.15. *For $D \subset Y$, $k \in Y \setminus \{0\}$, and $\overline{x} \in S$ we assume that*

1. *D is k-vectorially closed, $[0,+\infty)k + D \subseteq D$, and for all $x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^u_{D,rls}}$ and $y \in F(x)$, $\inf_{\overline{y} \in F(\overline{x})} z^{D,k}(y - \overline{y})$ is attained; or*

2. *$(-\infty, 0)k - \mathrm{vcl}_k D \subseteq -D$ and $\mathrm{vcl}_{-k}(F(\overline{x}) - D) = F(\overline{x}) - D$.*

The following proposition will be useful in the theorem below.

Proposition 2.5.16. *$\overline{x} \in S$ is a minimal solution of the problem* (1.10) *w.r.t. $\preceq$ if and only if for any $x \in S \setminus [F(\overline{x})]^{-1}_{\preceq}$, we have $F(x) \npreceq F(\overline{x})$.*

Proof. First note that $x \in S \setminus [F(\overline{x})]^{-1}_{\preceq}$ means that $x \in S$ such that $F(x) \npreceq F(\overline{x})$ or $F(\overline{x}) \npreceq F(x)$. Let $\overline{x} \in S$ be a minimal solution of the problem (1.10) w.r.t. $\preceq$. Then we have to consider two cases:
Case 1: For $x \in S$ and $F(x) \npreceq F(\overline{x})$, there is nothing left to show.
Case 2: For $x \in S$ and $F(\overline{x}) \npreceq F(x)$, we obtain $F(x) \npreceq F(\overline{x})$ due to $\overline{x}$'s minimality, as desired.

Conversely, assume that for all $x \in S \setminus [F(\overline{x})]^{-1}_{\preceq}$, $F(x) \npreceq F(\overline{x})$ holds true. Suppose, by contradiction, that $\overline{x}$ is not a minimal solution of the problem (1.10) w.r.t. $\preceq$. This implies the existence of some $x \in S$ with the properties $F(x) \preceq F(\overline{x})$ and $F(\overline{x}) \npreceq F(x)$, in contradiction to the assumption. $\square$

We next present a sufficient and necessary condition for minimal solutions of the problem (1.10) w.r.t. the relation $\preceq^u_{D,rls}$.

Theorem 2.5.17 ([45, Theorem 4.3])**.** *Let Assumption 2.5.15 be satisfied. Then $\overline{x}$ is a minimal solution of the problem* (1.10) *w.r.t. $\preceq^u_{D,rls}$ if and only if the following system* (*in the unknown x*)

$$g^u(x, \overline{x}) \leq 0, \ x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^u_{D,rls}},$$

is impossible.

Proof. First note that, due to Proposition 2.5.16, $\overline{x} \in S$ is a minimal solution of the problem (1.10) w.r.t. $\preceq^u_{D,rls}$ if and only if for $x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^u_{D,rls}}$, we have $F(x) \npreceq^u_{D,rls} F(\overline{x})$. Furthermore, we have

$$\begin{aligned}
& g^u(x, \overline{x}) \leq 0,\, x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^u_{D,rls}} \text{ is impossible} \\
\iff \quad & \nexists x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^u_{D,rls}} \ : \ \sup_{y \in F(x)} \inf_{\overline{y} \in F(\overline{x})} z^{D,k}(y - \overline{y}) \leq 0 \\
\iff \quad & \forall x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^u_{D,rls}} \ : \ \sup_{y \in F(x)} \inf_{\overline{y} \in F(\overline{x})} z^{D,k}(y - \overline{y}) > 0 \\
\iff \quad & \forall x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^u_{D,rls}} \ : \ F(x) \npreceq^u_{D,rls} F(\overline{x}).
\end{aligned}$$

$\square$

Furthermore, let us consider problem (1.10) with $\preceq = \preceq^l_{D,rls}$. We define the function $g^l : S \times S \to \mathbb{R} \cup \{\pm\infty\}$ by

$$g^l(x, \overline{x}) := \sup_{\overline{y} \in F(\overline{x})} \inf_{y \in F(x)} z^{D,k}(y - \overline{y}).$$

Assumption 2.5.18. *For $D \subset Y$, $k \in Y \setminus \{0\}$, and $\overline{x} \in S$ we assume that*

1. *D is k-vectorially closed, $[0, +\infty)k + D \subseteq D$, and for all $x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^l_{D,rls}}$ and $\overline{y} \in F(\overline{x})$, $\inf_{y \in F(x)} z^{D,k}(y - \overline{y})$ is attained; or*
2. *$(-\infty, 0)k - \mathrm{vcl}_k D \subseteq -D$ and for all $x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^l_{D,rls}}$, $\mathrm{vcl}_{-k}(-F(x) - D) = -F(x) - D$.*

In the following, we present a sufficient and necessary condition for minimal solutions of the problem (1.10) w.r.t. $\preceq^l_{D,rls}$.

Corollary 2.5.19 ([45, Corollary 4.5])**.** *Let Assumption 2.5.18 be satisfied. Then $\overline{x}$ is a minimal solution of the problem* (1.10) *w.r.t. $\preceq^l_{D,rls}$ if and only if the following system* (*in the unknown x*)

$$g^l(x, \overline{x}) \leq 0, \; x \in S \setminus [F(\overline{x})]^{-1}_{\preceq^l_{D,rls}},$$

is impossible.

Finally, we have the following result for minimal solutions of the problem (1.10) w.r.t. $\preceq^s_{D,rls}$.

Corollary 2.5.20 ([45, Corollary 4.6])**.** *Let Assumptions 2.5.15 and 2.5.18 be satisfied for the same $k \in Y \setminus \{0\}$. Then $\overline{x}$ is a minimal solution of the problem* (1.10) *w.r.t. $\preceq^s_{D,rls}$ if and only if the following system* (*in the unknown x*)*:*

$$g^u(x, \overline{x}) \leq 0 \text{ and } g^l(x, \overline{x}) \leq 0, \; x \in S \setminus \left([F(\overline{x})]^{-1}_{\preceq^u_{D,rls}} \cup [F(\overline{x})]^{-1}_{\preceq^l_{D,rls}} \right),$$

is impossible.

Chapter 3

Variable Domination Structures in Set Optimization

3.1 Introduction

It is well known that for certain applications, the solution concept given in Definition 1.2.11 together with one of the set relations from Chapter 2 is not sufficient. As one possible resolution, variable domination structures have been introduced when defining a solution concept. This chapter, which is based on [65], presents a concept for dealing with variable ordering structures in set optimization by equipping the upper set less relation $\preceq_D^u$ with a variable cone D. Note that in this thesis, the notions variable domination structure and variable ordering structure are used simultaneously.

Going back to Yu [112], variable domination structures generalize the concept of ordering structures in vector optimization and have since been intensely studied in the field of vector optimization. Motivated by applications in medical image registration [24, 25], variable domination structures in vector optimization gained recognition as they allow to introduce a specification of the decision-maker's preferences into the model. Due to these important applications, variable domination structures have gained increasing interest, compare Durea, Strugariu, Tammer [20], or Eichfelder, Bao, Soleimani, Tammer [7] for an analysis of Ekeland's variational principle with variable ordering structures. Note that Chen et al. [15] consider a vector approach to set optimization with a variable ordering structure. In addition, Bouza and Tammer [14] have introduced a nonlinear scalarizing functional to characterize and compute minimal points of a set with respect to a variable domination structure.

Variable domination structures play a crucial role, for example, in medical image registration, which has been used widely in medical treatment, for instance in radiotherapy (treatment verification, treatment planning, treatment guidance), orthopaedic surgery, and surgical microscope. The problem of image registration is finding a transformation matching two given sets of data (images). The similarity of the transformed data set to the target set can then be measured by several distance measures. As a multitude of measures exist that evaluate distinct characteristics, such as the sum of square differences, mutual information or cross-correlation, it is necessary to decide which distance measure to use. It is well known that different measures can lead to different best transformations. According to [25], some measures fail on special data sets, i.e. they lead to mathematically correct, but useless results. Thus it is important to combine several measures. Possible approaches are a weighted sum of different measures. But difficulties appear, such as badly scaled or nonconvex functions. Instead, Wacker [103] proposed to collect all available distance measures in a vector-valued function and minimizes this function. This leads to a vector-valued optimization problem. This connection with variable domination structures in vector programming has first been analyzed in Wacker [103] and further developed by Eichfelder [24] (see also [25, Section 10.3]). Recently, variable domination structures have been introduced to set optimization problems in [64, 65, 70]. This is particularly useful if uncertainties appear in the objective function, i.e., in the function that comprises the distance measures, for example due to inaccuracies of the data, or movements of the patient during the procedure. Then it is possible to convert the uncertain vector optimization problem into a set optimization problem and compute, for example, *robust* solutions. This is one of the main motivations why recently it has also been of great interest to consider set-valued optimization problems equipped with a variable domination structure by following a set approach. Recently, Köbis [64, 65] and Eichfelder and Pilecka [27] have introduced several set relations for the case that the order is given by a cone-valued map. A very general scalarization scheme for solving set optimization problems w.r.t. variable domination structures has been proposed in [70] (see also [71]).

In this chapter, we modify the upper set less relation given by Kuroiwa [77, 76] (see Definition 2.2.1, where D is a convex cone) in order to compare sets. We equip the upper set less relation with a variable domination structure in Section 3.2, formulate an optimality concept in Section 3.3 and we discuss optimal elements of sections of feasible elements in Section 3.4. Furthermore, Section 3.5 is concerned with providing scalarization results. We conclude this chapter with an application to image registration in medical engineering. Note that some of the results presented in this chapter can be formulated for other set relations, too; compare [64] for an overview.

3.2 Variable Upper Set Less Relation

Throughout this chapter, let Y be a real linear space.

Let us now recall the definition of variable ordering by Eichfelder [25] (see also Yu [112]) in vector optimization.

Definition 3.2.1 (Variable Domination Structures, [25]). *Let $C:\ Y \rightrightarrows Y$ be a set-valued map such that for every $y \in Y$, $C(y)$ is a convex cone. Then we define for $y_1, y_2 \in Y$*

$$y_1 \leq_1 y_2 :\Longleftrightarrow y_1 \in y_2 - C(y_1), \tag{3.1}$$

$$y_1 \leq_2 y_2 :\Longleftrightarrow y_1 \in y_2 - C(y_2). \tag{3.2}$$

Now our aim is to combine the above definition with the upper set less relation $\preceq^u_D$ and hence introduce an upper set less relation for variable ordering cones as follows:

Let $C: Y \rightrightarrows Y$ be a set-valued map such that for every $y \in Y$, $C(y)$ is a convex cone. Then we define two variants of the **upper set less order relation with variable ordering** for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq^u_{C,1} B :\Longleftrightarrow \forall\, a \in A,\ \exists\, b \in B:\ a \in b - C(a),$$
$$A \preceq^u_{C,2} B :\Longleftrightarrow \forall\, a \in A,\ \exists\, b \in B:\ a \in b - C(b).$$

These definitions were recently introduced and further investigated by Eichfelder, Pilecka in [27, 28]. From a practical perspective, however, it may not seem appropriate to obtain $A \preceq^u_{C,1} B$, but $A \not\preceq^u_{C,2} B$. Therefore, one may wish to exclude these cases. To this end, we define:

$$\begin{aligned} A \preceq^u_{C,1,2} B &:\Longleftrightarrow \forall\, a \in A,\ \exists\, b \in B:\ a \in b - C(a) \text{ and } a \in b - C(b) \\ &\Longleftrightarrow \forall\, a \in A,\ \exists\, b \in B:\ a \in b - \underbrace{C(a) \cap C(b)}_{=:D(a,b)}, \end{aligned}$$

with $D : Y \times Y \rightrightarrows Y$. Note that for every $y_1, y_2 \in Y$, $D(y_1, y_2)$ is a convex cone as it represents the intersection of convex cones $C(y_1)$ and $C(y_2)$. This motivates the following more general definition:

Definition 3.2.2 ([65, Definition 2.3])**.** *Let $A, B \in \mathcal{P}(Y)$, and let $D : Y \times Y \rightrightarrows Y$. Then we define the **variable upper set less relation** by*

$$A \preceq^u_{D,v} B :\Longleftrightarrow \forall\, a \in A,\ \exists\, b \in B:\ a \in b - D(a, b). \tag{3.3}$$

Note that the v in the notion $\preceq^u_{D,v}$ indicates that we are dealing with a variable domination structure.

Remark 3.2.3 (Compare also [65, Remark 2.4])**.** *If D in Definition 3.2.2 is given by a constant map, then the above definition coincides with Definition 2.2.1. Therefore, Definition 3.2.2 extends the definition of the generalized upper set less relation.*

Our approach can be justified by the following example.

Example 3.2.4 (Modeling of Uncertainties, [65, Example 2.5])**.** *As discussed in the literature (see, for instance, [53, p. 384] and [22, 50, 51]), set optimization is an important application of uncertain vector optimization. As an example, we consider the case where the data of a vector $\widetilde{a} \in \mathbb{R}^2$ is perturbed and only an approximation $\widetilde{A} \subset \mathbb{R}^2$ is known (see Figure 3.1). Similarly, the data of a vector $\widetilde{b}$ is disturbed and only an estimated set $\widetilde{B}$ can be generated. In order to compare the set $\widetilde{A}$ to the set $\widetilde{B}$, the upper set less relation $\preceq^u_D$ with $D = \mathbb{R}^2_+$ shall be used. The relation $\preceq^u_D$ compares sets based on their upper bounds. Assume that, moreover, we would also like to compare $\widetilde{A}$ and $\widetilde{B}$ with respect to their lower bounds, such that $\widetilde{B} \subseteq \widetilde{A} + D$. This is the lower set less relation (see Kuroiwa [77] and Definition 2.2.9 for the generalized case), which corresponds to $-\widetilde{B} \preceq^u_D -\widetilde{A}$. This relation ensures that the lower bounds of $\widetilde{B}$ are not "worse" than those of $\widetilde{A}$.*

We can see in Figure 3.1, (a), that $\widetilde{A} \preceq^u_D \widetilde{B}$ and $-\widetilde{B} \preceq^u_D -\widetilde{A}$. Since the data are uncertain, it seems likely that there exist undesired elements located far from where most uncertain data is found. If there exists such an element $\bar{a}$ belonging to the set $A := \widetilde{A} \cup \{\bar{a}\}$, then $A \preceq^u_D \widetilde{B}$ may not hold anymore (see Figure 3.1).

Similarly, we assume that there exists an element $\bar{b} \notin \widetilde{B}$ which is located far away from $\widetilde{B}$, such that $-B \not\preceq^u_D -A$, where $B := \widetilde{B} \cup \{\bar{b}\}$. In order to still include $\bar{a}$ and $\bar{b}$ in the analysis but to obtain the result that the set A is, for the "most" part, preferred to B, a planner can introduce a variable ordering structure in the following way: Let $\underline{b} \in \widetilde{B}$, $\underline{a} \in \widetilde{A}$ and $D : Y \times Y \rightrightarrows Y$ with

$$D(y_1, y_2) := \begin{cases} D_1 & \text{if } y_1 = \bar{a},\ y_2 = \underline{b}, \\ D_2 & \text{if } y_1 = -\bar{b},\ y_2 = -\underline{a}, \\ \mathbb{R}^2_+ & \text{else,} \end{cases}$$

where D_1 is a cone with the property $\bar{a} \in \{\underline{b}\} - D_1$ ($D_1 = D(\bar{a}, \underline{b})$) and D_2 is a cone which fulfills $\bar{b} \in \{\underline{a}\} + D_2$ ($D_2 = D(-\bar{b}, -\underline{a})$, see Figure 3.1, (b)). Then we have $A \preceq^u_{D,v} B$ and $-B \preceq^u_{D,v} -A$. This ensures that all estimated elements are taken into account, as nondesired elements can be handled by using variable ordering cones that depend on two variables.

Remark 3.2.5 ([65, Remark 2.6])**.** *If we replace A by $\{y_1\}$ and B by $\{y_2\}$ in Definition 3.2.2 and if $D_1 : Y \rightrightarrows Y$ is given as a set-valued map that only depends on the first variable such that $D_1(y_1) := D(y_1, y_2)$ is a convex cone for all $y_1, y_2 \in Y$, then we have the following equivalence:*

$$\{y_1\} \preceq^u_{D_1} \{y_2\} \iff y_1 \in \{y_2\} - D_1(y_1) \iff y_1 \leq_1 y_2,$$

where $\leq_1$ is given by equivalence (3.1) *for $C = D_1$. If, on the other hand, $D_2 : Y \rightrightarrows Y$ is given as a set-valued map that only depends on the second variable such that $D_2(y_2) := D(y_1, y_2)$ is a convex cone for all $y_1, y_2 \in Y$, then we have the following equivalence:*

$$\{y_1\} \preceq^u_{D_2} \{y_2\} \iff y_1 \in \{y_2\} - D_2(y_2) \iff y_1 \leq_2 y_2,$$

where $\leq_2$ is given by (3.2) *for $C = D_2$.*

The following properties of the relation $\preceq^u_{D,v}$ defined in equivalence (3.3) are to be mentioned. Note that for the special case considered in Remark 3.2.5 above, some of these properties are proved in [25, Lemma 1.10], and the proof that we provide below is inspired by [25, Lemma 1.10].

Proposition 3.2.6 ([65, Proposition 2.7])**.** *Let $D : Y \times Y \rightrightarrows Y$ be a set-valued map such that for every $y_1, y_2 \in Y$, $D(y_1, y_2)$ is a cone. Then the following assertions hold true.*

1. *The relation $\preceq^u_{D,v}$ defined in* (3.3) *is reflexive.*
2. *The relation $\preceq^u_{D,v}$ defined in* (3.3) *is transitive if for all $y_1, y_2, y_3 \in Y$ and for every $d_1 \in D(y_1, y_3)$, $d_2 \in D(y_3, y_2)$, it holds*

$$D(y_1, y_1 + d_1) + D(y_2 - d_2, y_2) \subseteq D(y_1, y_2). \tag{3.4}$$

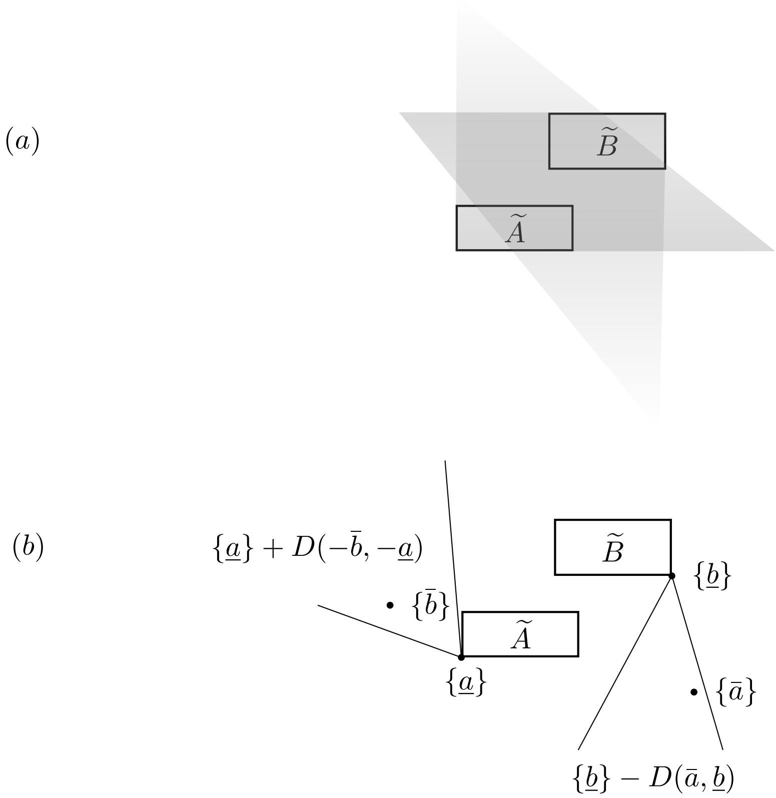

Figure 3.1: Visualization of the problem formulated in Example 3.2.4.

3. *Suppose that for every $y_1, y_2 \in Y$, we have $D(y_1, y_2) = D(y_2, y_1)$. Furthermore, for $A, B \in \mathcal{P}(Y)$, suppose that*

$$D[A \times B] := \bigcup_{a \in A,\ b \in B} D(a, b).$$

is a convex cone. Then

$$A \preceq^u_{D,v} B \text{ and } B \preceq^u_{D,v} A \Longrightarrow A - D[A \times B] = B - D[A \times B].$$

4. *(Compatibility with nonnegative scalar multiplication). Let $A, B \in \mathcal{P}(Y)$. Then*

$$A \preceq^u_{D,v} B,\ \lambda > 0 \implies \forall\ a \in A,\ \exists\ b \in B :\ \lambda a \in \{\lambda b\} - D(\lambda a, \lambda b)$$

holds if

$$\forall\ a \in A,\ \forall\ b \in B,\ \forall\ \lambda > 0 :\ D(a, b) \subseteq D(\lambda a, \lambda b). \tag{3.5}$$

5. *(Compatibility with addition). Let $A, B, C, E \in \mathcal{P}(Y)$. Then*

$$A \preceq^u_{D,v} B,\ C \preceq^u_{D,v} E \implies A + C \preceq^u_{D,v} B + E$$

holds if

$$\forall\, a \in A,\ \forall\, b \in B,\ \forall\, c \in C,\ \forall\, e \in E:\ D(a,b) + D(c,e) \subseteq D(a+c, b+e). \quad (3.6)$$

Proof. 1. Holds true because for every $y \in Y$, $D(y,y)$ is assumed to be a cone, and thus $0 \in D(y,y)$.

2. $A \preceq^u_{D,v} B$ is equivalent to: For all $a \in A$ there exists $b \in B:\ a \in \{b\} - D(a,b)$. Similarly, $B \preceq^u_{D,v} C$ means that for all $b \in B$ there exists $c \in C:\ b \in \{c\} - D(b,c)$. (3.4) yields $D(a, a+d_1) + D(c-d_2, c) \subseteq D(a,c)$ for $d_1 := b - a \in D(a,b)$ and $d_2 := c - b \in D(b,c)$, which altogether yields $A \preceq^u_{D,v} C$.

3. $A \preceq^u_{D,v} B$ implies that for all $a \in A$, $a \in B - D[A \times B]$. This leads to $A \subseteq B - D[A \times B]$. Similarly, $B \preceq^u_{D,v} A$ implies $B \subseteq A - D[A \times B]$. Thus $A - D[A \times B] \subseteq B - D[A \times B]$ and $B - D[A \times B] \subseteq A - D[A \times B]$, as $D[A \times B]$ was assumed to be a convex cone.

4. Let $a \in A$ and $b \in B$. $b - a \in D(a,b)$ and $\lambda > 0$ is equivalent to $\lambda(b-a) \in D(a,b)$, as $D(a,b)$ is assumed to be a cone, and the inclusion (3.5) yields the assertion.

5. $A \preceq^u_{D,v} B$ is defined by

$$\forall\, a \in A,\ \exists\, b \in B:\ a \in \{b\} - D(a,b)$$

and $C \preceq^u_{D,v} E$ corresponds to

$$\forall\, c \in C,\ \exists\, e \in E:\ c \in \{e\} - D(c,e).$$

This yields

$$\forall\, a \in A,\ \forall\, c \in C,\ \exists\, b \in B,\ \exists\, e \in E:\ a + c \in \{b\} + \{e\} - D(a,b) - D(c,e).$$

Then the inclusion (3.6) implies the assertion.

□

Remark 3.2.7 ([65, Remark 2.8]). *Let $D : Y \times Y \rightrightarrows Y$ be a set-valued map such that for every $y_1, y_2 \in Y$, $D(y_1,y_2)$ is a cone. Then the subadditivity property on the whole space Y*

$$\forall\, y_1, y_2, y_3, y_4 \in Y:\ D(y_1,y_2) + D(y_3,y_4) \subseteq D(y_1+y_3, y_2+y_4) \quad (3.7)$$

implies that D is a constant map (compare [25, Lemma 3.23]): Set $y_1 = -y_3$ and $y_2 = -y_4$ in (3.7). Then $D(y_1,y_2) + D(-y_1,-y_2) \subseteq D(0,0)$ for every $y_1, y_2 \in Y$. Since $D(-y_1,-y_2)$ is a cone, it holds $0 \in D(-y_1,-y_2)$, and this yields $D(y_1,y_2) \subseteq D(0,0)$ for every $y_1, y_2 \in Y$. On the other hand, (3.7) *implies that $D(y_1,y_2) + D(0,0) \subseteq D(y_1,y_2)$, and hence $D(0,0) \subseteq D(y_1,y_2)$, and we obtain $D(y_1,y_2) = D(0,0)$ for all $y_1, y_2 \in Y$.*

Remark 3.2.8 ([65, Remark 2.9]). *Let $A, B \in \mathcal{P}(Y)$ be given, and let $D_1, D_2 : Y \times Y \rightrightarrows Y$. Suppose that for all $a \in A$ and $b \in B$, $D_1(a,b) \subseteq D_2(a,b)$ is satisfied. Then the implication $A \preceq^u_{D_1,v} B \Longrightarrow A \preceq^u_{D_2,v} B$ holds.*

3.3 Optimality Notions

From now on, we assume that $D : Y \times Y \rightrightarrows Y$ is a set-valued map such that for all $y_1, y_2 \in Y$, $D(y_1, y_2)$ is a convex cone. Recall that the algebraic interior of a nonempty set $\Omega \subset Y$ is the set

$$\begin{aligned}\operatorname{cor}\Omega := \{\bar{y} \in \Omega \mid & \text{ for every } y \in Y \text{ there exists } \bar{\lambda} > 0 \text{ with } \bar{y} + \lambda y \in \Omega \\ & \text{ for all } \lambda \in [0, \bar{\lambda}]\}.\end{aligned}$$

Whenever we deal with $\operatorname{cor} D(y_1, y_2)$, we suppose that this set is nonempty. We introduce a set-valued map $F : X \rightrightarrows Y$ that we wish to minimize on a nonempty set $S \subseteq X$, where X is a linear space. We denote the minimization problem

$$\min_{x \in S} \; F(x) \tag{P_{var}}$$

and define optimal solutions of (P_{var}) in the following way.

Definition 3.3.1 (Optimality, [65, Definition 3.1]). *$x^0 \in S$ is called an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{D,v}$ if*

$$\nexists\, x \in S \setminus \{x^0\} : \; F(x) \preceq^u_{[D\setminus\{0\}/D/\operatorname{cor} D, v]} F(x^0),$$

which is equivalent to

$$\nexists\, x \in S \setminus \{x^0\} : \; \forall\, y \in F(x), \; \exists\, y^0 \in F(x^0) : \; y \in \{y^0\} - E(y, y^0), \tag{3.8}$$

where $E(y, y^0) := D(y, y^0) \setminus \{0\}$ ($E(y, y^0) := D(y, y^0)$, $E(y, y^0) := \operatorname{cor} D(y, y^0)$, respectively). If D is given by a constant set-valued map C, then we say that $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_C$ if condition (3.8) is fulfilled with $E = C\setminus\{0\}$ ($E = C$, $E = \operatorname{cor} C$, respectively).

Remark 3.3.2 ([65, Remark 3.2]). *Note that the above definition of optimality is an extension of the concepts of **nondominated** and **minimal** solutions given in Yu [112] and Chen et al. [15], respectively, see also Eichfelder [25]. If the objective to be minimized is not a set-valued map, but a vector function $f : X \to Y$, and the ordering cone is given by a set-valued map $D_1 : Y \rightrightarrows Y$, then a solution $f(x^0) = y^0$ is called **nondominated** (**weakly nondominated**) if there does not exist $x \in X$ with $f(x) = y$ such that*

$$y^0 \in \{y\} + D_1(y) \setminus \{0\} \quad (y^0 \in \{y\} + \operatorname{cor} D_1(y), \; \textit{respectively}),$$

*coinciding with the above definition of optimality (weak optimality) for the special case $D := D_1$ and $F := f$. Furthermore, a solution $f(x^0) = y^0$ is called **minimal** (**weakly minimal**) if there does not exist any $x \in X$ with $f(x) = y$ such that*

$$y^0 \in \{y\} + D_1(y^0) \setminus \{0\} \quad (y^0 \in \{y\} + \operatorname{cor} D_1(y^0), \; \textit{respectively}).$$

Remark 3.3.3 (See also [65, Remark 3.3]). *Notice that Definition 3.3.1 differs from the usual definition of optimal solutions in set optimization. Typically, a solution $x^0 \in S$ is called* minimal *w.r.t. $\preceq$ if the following implication is fulfilled (see Definition 1.2.11):*

$$x \in S: \; F(x) \preceq F(x^0) \Longrightarrow F(x^0) \preceq F(x).$$

Here, in this chapter, we work with a stronger notion than minimality, which has also been used in [22, 50, 51] for fixed ordering cones.

Coming back to Definition 3.3.1, we notice the following connection between optimal solutions:

Lemma 3.3.4 ([65, Lemma 3.4]). *If x^0 is a strictly optimal solution of (P_{var}) w.r.t. $\preceq^u_{D,v}$, then x^0 is an optimal solution w.r.t. $\preceq^u_{D,v}$. If x^0 is an optimal solution of (P_{var}) w.r.t. $\preceq^u_{D,v}$, then x^0 is a weakly optimal solution of (P_{var}) w.r.t. $\preceq^u_{D,v}$.*

Proof. Follows from $\operatorname{cor} D(y_1, y_2) \subseteq D(y_1, y_2) \setminus \{0\} \subseteq D(y_1, y_2)$ for every $y_1, y_2 \in Y$. □

In order to get an insight into the issue of set optimization problems equipped with a variable ordering structure, we provide an example below, for a visualization see Figure 3.2.

Example 3.3.5 ([65, Example 3.5]). *In this example we are looking for optimal solutions of a set-valued optimization problem with respect to the variable upper set less relation $\preceq^u_{D,v}$ introduced in Definition 3.2.2. The problem reads*

$$\min_{x \in S} F(x), \tag{P_1}$$

with $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, 1]$ and $F : S \rightrightarrows Y$ is given by

$$F(x) := \begin{cases} [(1,1),(2,2)] & \text{if } x = 0, \\ [(0,0),(3,3)] & \text{if } x \in (0,1), \\ [(1,-1),(1.5,-1.5)] & \text{if } x = 1, \end{cases}$$

where $[(a,b),(c,d)] := \{(y^1, y^2) \in \mathbb{R}^2 \mid a \leq y^1 \leq c, \; b \leq y^2 \leq d\}$ denotes a rectangle. Furthermore, the variable ordering is given by

$$D(y_1, y_2) = \begin{cases} \{\alpha(1,1) + \beta(0.5,1) \mid \alpha, \beta \geq 0\} & \text{if } y_2 \in [(1,1),(2,2)], \; y_1 \in \mathbb{R}^2, \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

Then the strictly optimal solutions of (P_1) w.r.t. $\preceq^u_{D,v}$ in the sense of Definition 3.3.1 are $S^0 := \{0, 1\}$, because

$$\nexists\, x \in S \setminus \{0\} : \; \forall\, y \in F(x), \; \exists\, y^0 \in F(0) : \; y \in \{y^0\} - D(y, y^0) \text{ and}$$
$$\nexists\, x \in S \setminus \{1\} : \; \forall\, y \in F(x), \; \exists\, y^0 \in F(1) : \; y \in \{y^0\} - D(y, y^0).$$

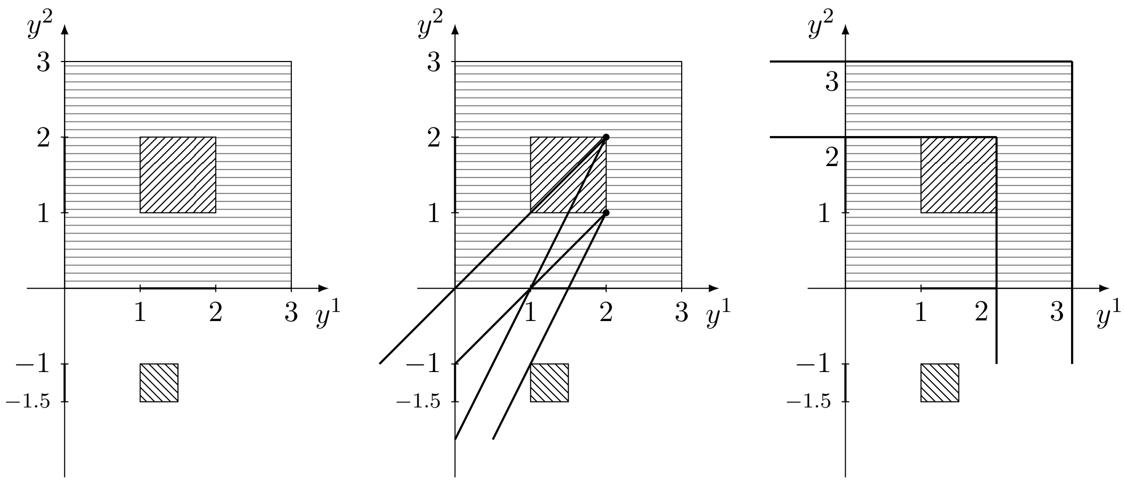

Figure 3.2: Image set of F, as described in Example 3.3.5. The left plot shows the three sets $F(0)$, $F(1)$ and $F(x)$ for $x \in (0,1)$. The middle picture illustrates the sets $\{(2,2)\} - D\big((0,0),(2,2)\big)$ and $\{(2,1)\} - D\big((0,0),(2,1)\big)$. The right plot depicts the sets $\{(2,2)\} - \mathbb{R}^2_+$ and $\{(3,3)\} - \mathbb{R}^2_+$.

If we replace D by the constant ordering cone $\mathbb{R}^2_+$, then $x^0 = 1$ is the only strictly optimal solution w.r.t. $\preceq^u_{\mathbb{R}^2_+}$, since

$$\nexists\; x \in S \setminus \{1\} : \;\forall\; y \in F(x), \;\exists\; y^0 \in F(1) : \; y \in \{y^0\} - \mathbb{R}^2_+.$$

$x^0 = 0$ is not strictly optimal w.r.t. $\preceq^u_{\mathbb{R}^2_+}$, as

$$\exists\; x = 1 : \;\forall\; y \in F(1), \;\exists\; y^0 \in F(0) : \; y \in \{y^0\} - \mathbb{R}^2_+.$$

The following remark shows the connection between optimal solutions w.r.t. different ordering cones:

Remark 3.3.6 ([65, Remark 3.6])**.** *Let $D_1, D_2 : Y \times Y \rightrightarrows Y$ be set-valued maps such that for all $y_1, y_2 \in Y$, $D_1(y_1,y_2)$ and $D_2(y_1,y_2)$ are convex cones with nonempty algebraic interior. Let $A, B \in \mathcal{P}(Y)$ be given. From Remark 3.2.8, we obtain that if it holds*

$$\forall\; a \in A, \;\forall\; b \in B, \; D_1(a,b) \subseteq D_2(a,b),$$

then

$$A \preceq^u_{D_1,v} B \Longrightarrow A \preceq^u_{D_2,v} B.$$

Based on this relation, the following property holds: Assume that

$$\forall\; y_1, y_2 \in Y : \; D_1(y_1,y_2) \subseteq D_2(y_1,y_2)$$

is satisfied. If $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{D_2,v}$, then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D_1,v}$.

The assertions in Remark 3.3.6 can be generalized as follows:

Proposition 3.3.7 ([65, Proposition 3.7]). *Let $x^0 \in S$ and two set-valued maps $D_1, D_2 : Y \times Y \rightrightarrows Y$ satisfy*

$$\forall\, y_1 \in \bigcup_{x \in S \setminus \{x^0\}} F(x),\ \forall\, y_2 \in F(x^0):\ D_1(y_1, y_2) \subseteq D_2(y_1, y_2). \tag{3.9}$$

If $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{D_2,v}$, then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D_1,v}$.

Proof. We proceed by contraposition. Let $x^0 \in S$ not be an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D_1,v}$. Then there exists $x \in S \setminus \{x^0\}$ such that

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y \in \{y^0\} - E(y, y^0), \tag{3.10}$$

where $E(y, y^0) := D_1(y, y^0) \setminus \{0\}$ ($E(y, y^0) := D_1(y, y^0)$, $E(y, y^0) := \operatorname{cor} D_1(y, y^0)$, respectively). (3.10) together with the inclusion (3.9) implies

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y \in \{y^0\} - G(y, y^0),$$

where $G(y, y^0) := D_2(y, y^0) \setminus \{0\}$ ($G(y, y^0) := D_2(y, y^0)$, $G(y, y^0) := \operatorname{cor} D_2(y, y^0)$, respectively). That means $x^0 \in S$ is not an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D_2,v}$, as was to be proved. $\square$

For the next results we introduce the abbreviations

$$A := F[S] := \bigcup_{x \in S} F(x) \tag{3.11}$$

$$D[A \times A] := \bigcup_{y_1, y_2 \in A} D(y_1, y_2), \tag{3.12}$$

$$\overline{D} := \bigcap_{y_1, y_2 \in A} D(y_1, y_2). \tag{3.13}$$

We assume that $D[A \times A]$ is a convex cone with nonempty algebraic interior, and we suppose furthermore that $\operatorname{cor} \overline{D} \neq \emptyset$.

Remark 3.3.8 ([65, Remark 3.8]). *The convexity assumption of $D[A \times A]$ will be important in Theorem 3.4.2, yet this assumption seems to be very strong. A sufficient condition that ensures the convexity of $D[A \times A]$ is the following: Let D_1, D_2 be convex cones in $\mathbb{R}^2$ such that there exists some k, $k \neq 0$, with $k \in D_1 \cap D_2$. Furthermore, for all $d_1 \in D_1$ and $d_2 \in D_2$ with $d_1, d_2 \notin D_1 \cap D_2$, we suppose that d_1 and d_2 are linearly independent. Then $D_1 \cup D_2$ is a convex cone. Figure 3.3 illustrates this result.*

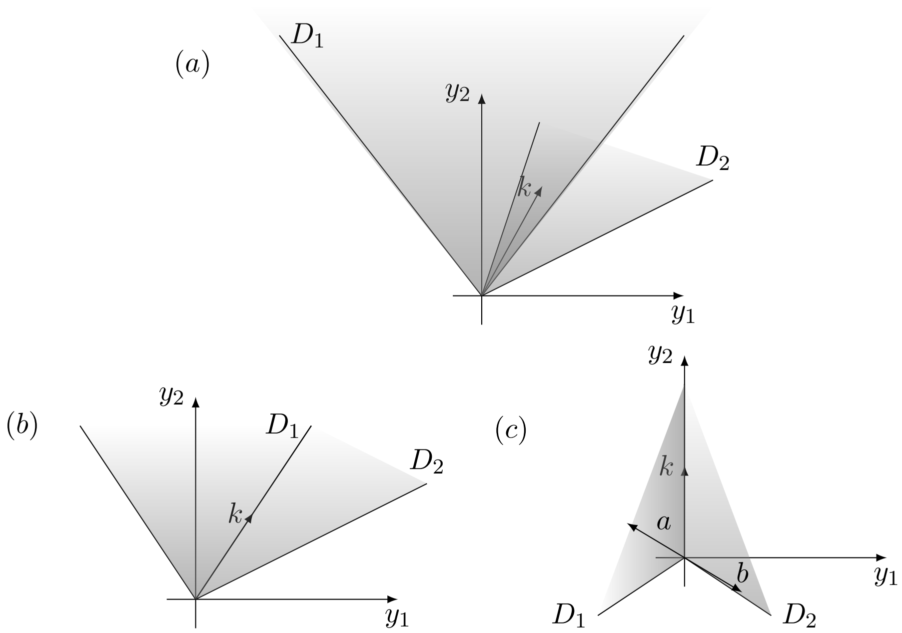

Figure 3.3: Visualization of two convex cones D_1 and D_2. (a) and (b) show that $D_1 \cup D_2$ is a convex cone, because $k \neq 0$ lies in both D_1 and D_2. In (c), $D_1 \cup D_2$ is not convex, because the vectors a and b ($a \neq b$) are not linearly independent.

Example 3.3.9 ([65, Example 3.9]). *We illustrate the variable upper set less relation with the above sets $D[A \times A]$ and $\overline{D}$ in a small example. Let $S := \{x_1, x_2\} \subseteq \mathbb{R}^n$, $F : S \rightrightarrows \mathbb{R}^2$, $F(x_1) := \{(0,0)\}$ and $F(x_2) := \{(1,1)\}$. Then, with the above notation, $A = \{(0,0),(1,1)\}$. Let furthermore the variable ordering be given by $D : Y \times Y \rightrightarrows \mathbb{R}^2$ with*

$$D(y_1, y_2) = \begin{cases} D_1 & \text{if } y_1 = (0,0),\ y_2 = (1,1), \\ D_2 & \text{otherwise,} \end{cases}$$

where $D_1 := \{d \in \mathbb{R}^2 | d = \lambda_1(0.5,1) + \lambda_2(1,0.5),\ \lambda_1, \lambda_2 \geq 0\}$ and $D_2 := \{d \in \mathbb{R}^2 | d = \lambda_1(0.5,0.5) + \lambda_2(2,0.5),\ \lambda_1, \lambda_2 \geq 0\}$. Then $D[A \times A] = D_1 \cup D_2 = \{d \in \mathbb{R}^2 | d = \lambda_1(0.5,1) + \lambda_2(2,0.5),\ \lambda_1, \lambda_2 \geq 0\}$ and $\overline{D} = \{d \in \mathbb{R}^2 | d = \lambda_1(0.5,0.5) + \lambda_2(1,0.5),\ \lambda_1, \lambda_2 \geq 0\}$. Then we have $F(x_2) \not\preceq^u_{D,v} F(x_1)$, therefore x_1 is strictly optimal w.r.t. the variable upper set less order relation. It also holds $F(x_1) \preceq^u_{D,v} F(x_2)$. The relations $F(x_1) \preceq^u_{D[A\times A]} F(x_2)$ and $F(x_1) \preceq^u_{\overline{D}} F(x_2)$ correspond to $F(x_1) \subseteq F(x_2) - D[A \times A]$ and $F(x_1) \subseteq F(x_2) - \overline{D}$, respectively. However, we have $F(x_2) \not\preceq^u_{D[A\times A]} F(x_1)$ as well as $F(x_2) \not\preceq^u_{\overline{D}} F(x_1)$. Therefore, x_1 is strictly optimal w.r.t. $\preceq^u_{D[A\times A]}$ and $\preceq^u_{\overline{D}}$.

x_2 is not strictly optimal w.r.t. $\preceq^u_{D[A\times A]}$ and $\preceq^u_{\overline{D}}$, as $F(x_1) \preceq^u_{D[A\times A]} F(x_2)$ as well as $F(x_1) \preceq^u_{\overline{D}} F(x_2)$. Finally, it holds $F(x_1) \not\preceq^u_{\operatorname{cor} \overline{D}} F(x_2)$. Thus, x_2 is weakly optimal w.r.t. $\preceq^u_{\overline{D}}$.

The following theorem can be derived by using Proposition 3.3.7.

Theorem 3.3.10 ([65, Theorem 3.10]). *(a) If $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{D[A\times A]}$, then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D,v}$.*

(b) Let $\widetilde{D} \supset D[A \times A]$ be a convex cone in Y with nonempty algebraic interior. If $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{\widetilde{D}}$, then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D[A\times A]}$.

(c) If $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{D,v}$, then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{\overline{D}}$.

(d) Let $\overline{\overline{D}} \subset \overline{D}$ be a convex cone with nonempty algebraic interior. If $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{\overline{D}}$, then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{\overline{\overline{D}}}$.

(e) Let $x^0 \in S$ be an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D(y^0,y^0)}$ for every $y^0 \in F(x^0)$. Furthermore, suppose that for all $y \in \bigcup_{x\in S} F(x)$ and for every $y^0 \in F(x^0)$, the inclusion

$$D(y, y^0) \subseteq D(y^0, y^0)$$

holds true. Then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{D,v}$.

(f) Define $\overline{D}(x) := \bigcap_{y\in F(x)} D(y, y)$ for some $x \in S$. Let $x^0 \in S$ be an optimal (a strictly optimal / a weakly optimal) solution of (P_{var}) w.r.t. $\preceq^u_{D,v}$ and suppose that

$$\forall\ y \in \bigcup_{x\in S} F(x),\ \forall\ y^0 \in F(x^0):\ D(y^0, y^0) \subseteq D(y, y^0).$$

Then $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{\overline{D}(x^0)}$.

Proof. The assertions are proved using Proposition 3.3.7 with $y_1 \in \cup_{x\in S\setminus\{x^0\}} F(x)$ and $y_2 \in F(x^0)$. In (a), the desired result can be obtained by choosing $D_1(y_1, y_2) = D(y_1, y_2)$ and $D_2(y_1, y_2) = D[A \times A]$ in Proposition 3.3.7. (b) can be derived with $D_1(y_1, y_2) = D[A \times A]$ and $D_2(y_1, y_2) = \widetilde{D}$. Moreover, (c) is proved by selecting $D_1(y_1, y_2) = \overline{D}$ and $D_2(y_1, y_2) = D(y_1, y_2)$. For the result in (d), we choose $D_1(y_1, y_2) = \overline{\overline{D}}$ and $D_2(y_1, y_2) = \overline{D}$. (e) can be obtained by $D_1(y_1, y_2) = D(y_1, y_2)$ and $D_2(y_1, y_2) = D(y_2, y_2)$. Finally, (f) is derived by choosing $D_1(y_1, y_2) = \overline{D}(x^0)$ (where $\overline{D}(x^0) \subseteq D(y^0, y^0)$ for all $y^0 \in F(x^0)$) and $D_2(y_1, y_2) = D(y_1, y_2)$. $\square$

3.4 Optimal Elements of Sections

In the following we use the optimality notion from Definition 3.3.1 to define efficient solution sets.

Definition 3.4.1 ([65, Definition 4.1])**.** *A solution set $F(x^0)$ is **efficient (strictly efficient, weakly efficient**, respectively) w.r.t. $\preceq^u_{D,v}$ if x^0 is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_{D,v}$. If D is given by a constant set-valued map C, then we say that the solution set $F(x^0)$ is efficient (strictly efficient, weakly efficient, respectively) w.r.t. $\preceq^u_C$ if $x^0 \in S$ is an optimal (a strictly optimal / a weakly optimal) solution of the set-valued problem (P_{var}) w.r.t. $\preceq^u_C$.*

In vector optimization, often so-called sections are considered. A section in vector optimization is a set $(\overline{y}-C)\cap A$, where $\overline{y} \in Y$, $A \subset Y$ is the image set of feasible elements and $C \subset Y$ is a convex cone. Then an efficient element (in the sense of vector optimization) in this section is also an efficient element (in the sense of vector optimization) of the whole set A (see [53, Lemma 6.2]). Eichfelder [25] formulated a corresponding result for vector optimization problems equipped with a variable ordering structure. We extend this analysis to set optimization with the variable upper set less relation. Theorem 3.4.2 shows that the corresponding result goes in line with the standard theory from vector optimization. To this end, we introduce for an element $B \in \mathcal{P}(Y)$ the set

$$A_B := (B - D[A \times A]) \cap A,$$

which is denoted as a section w.r.t. B. Note that A and $D[A \times A]$ are defined by (3.11) and (3.12), respectively. For a visualization of a section A_B in set optimization, see Figure 3.4.

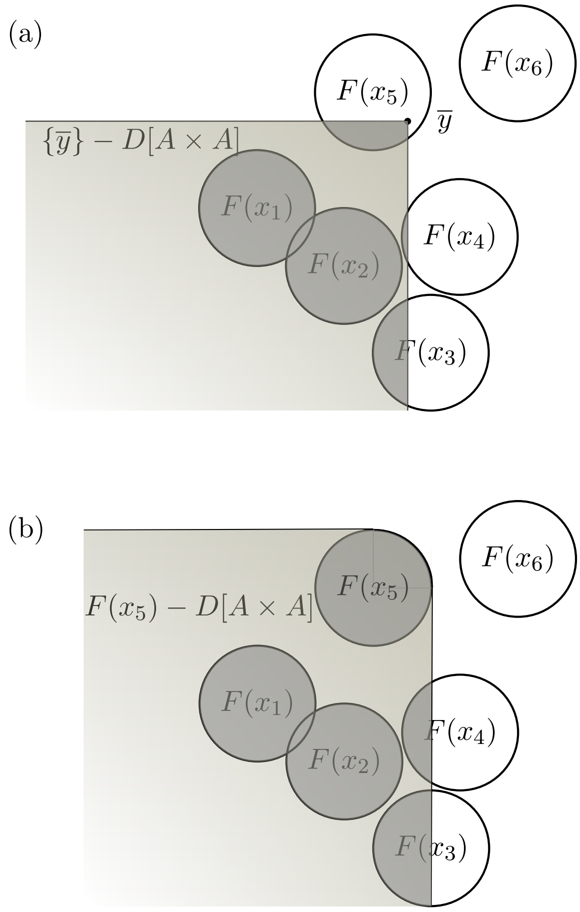

Figure 3.4: Visualization of a section $(B - D[A \times A]) \cap A$ with $B = \{\overline{y}\}$ in (a) and $B = F(x_5)$ in (b), where $D[A \times A] = \mathbb{R}^2_+$.

Theorem 3.4.2 ([65, Theorem 4.2])**.** *If $F(x^0) \subset A_B$ is efficient (strictly efficient / weakly efficient) in A_B for an element $B \in \mathcal{P}(Y)$ w.r.t. $\preceq^u_{D[A\times A]}$, then $F(x^0)$ is efficient (strictly efficient / weakly efficient) in A w.r.t. $\preceq^u_{D,v}$.*

Proof. Let $F(x^0)$ be efficient (strictly efficient / weakly efficient) in A_B w.r.t. $\preceq^u_{D[A\times A]}$, i.e., there does not exist $x \in S \setminus \{x^0\}$ with $F(x) \subset A_B$ such that for $G := D[A \times A] \setminus \{0\}$ ($G := D[A \times A]$, $G := \operatorname{cor} D[A \times A]$, respectively)

$$F(x) \subseteq F(x^0) - G. \tag{3.14}$$

Suppose that $F(x^0)$ is not efficient (strictly efficient / weakly efficient) in A w.r.t. $\preceq^u_{D,v}$. Then there exists $x \in S \setminus \{x^0\}$ such that for $E(y, y^0) := D(y, y^0) \setminus \{0\}$ ($E(y, y^0) :=$

$D(y,y^0)$, $E(y,y^0) := \operatorname{cor} D(y,y^0)$, respectively)

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y \in \{y^0\} - E(y,y^0),$$

implying

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y \in \{y^0\} - E(y,y^0) \subseteq F(x^0) - G \subseteq A_B - G,$$

because $F(x^0) \subseteq A_B$. Since $D[A \times A]$ is a convex cone, we conclude with $F(x) \subseteq A_B$ and $F(x) \subseteq F(x^0) - G$, a contradiction to the inclusion (3.14). □

The converse statement of the preceding theorem is possible by means of an appropriate selection of the ordering cone.

Theorem 3.4.3 ([65, Theorem 4.3])**.** *If $F(x^0) \subset A_B$ for some $B \in \mathcal{P}(Y)$ and $F(x^0)$ is efficient (strictly efficient / weakly efficient) in A w.r.t. $\preceq^u_{D,v}$, then $F(x^0)$ is efficient (strictly efficient / weakly efficient) in A_B w.r.t. $\preceq^u_{\overline{D}}$, where $\overline{D}$ is given by* (3.13)*.*

Proof. Suppose that $F(x^0)$ is not efficient (strictly efficient / weakly efficient) in A_B w.r.t. $\preceq^u_{\overline{D}}$. Then there exists $x \in S \setminus \{x^0\}$ with $F(x) \subseteq A_B$ such that for $G := \overline{D} \setminus \{0\}$ ($G := \overline{D}$, $G := \operatorname{cor} \overline{D}$, respectively)

$$F(x) \subseteq F(x^0) - G,$$

implying

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y \in \{y^0\} - G \subseteq \{y^0\} - E(y,y^0),$$

where $E(y,y^0) := D(y,y^0) \setminus \{0\}$ ($E(y,y^0) := D(y,y^0)$, $E(y,y^0) := \operatorname{cor} D(y,y^0)$, respectively), in contradiction to $F(x^0)$'s efficiency (strict efficiency / weak efficiency) in A w.r.t. $\preceq^u_{D,v}$. □

3.5 Scalarization

Let Y again be a real linear space, where Y' denotes the algebraic dual space. The algebraic dual cone to a cone $D(y_1,y_2)$, $y_1,y_2 \in A$ and A defined by (3.11), is denoted by

$$D'(y_1,y_2) := \{y' \in Y' \mid \forall\, d \in D(y_1,y_2):\ y'(d) \geq 0\}.$$

Note that $D'(y_1,y_2)$ might reduce to the trivial cone if $D(y_1,y_2)$ is not pointed.

The algebraic quasi-interior of $D'(y_1,y_2)$, $y_1,y_2 \in A$, is defined as

$$D^{\#}_{Y'}(y_1,y_2) := \{y' \in Y' \mid \forall\, d \in D(y_1,y_2) \setminus \{0\}:\ y'(d) > 0\}.$$

Now we set

$$\overline{D}' := \bigcap_{y_1,y_2 \in A} D'(y_1,y_2),$$
$$\overline{D}^{\#}_{Y'} := \bigcap_{y_1,y_2 \in A} D^{\#}_{Y'}(y_1,y_2).$$

Theorem 3.5.1 ([65, Theorem 5.1])**.** *Consider for $y' \in \overline{D}'$ the scalar minimization problem*

$$\min_{x \in S} \sup_{y \in F(x)} y'(y). \tag{$P_{y'}^u$}$$

(a) If $x^0 \in S$ is an optimal solution of $(P_{y'}^u)$ with $y' \in \overline{D}_{Y'}^{\#}$ and $\max_{y \in F(x)} y'(y)$ exists for all $x \in S$, then x^0 is an optimal solution of (P_{var}) w.r.t. $\preceq_{D,v}^u$.

(b) If $x^0 \in S$ is a unique optimal solution of $(P_{y'}^u)$ with $y' \in \overline{D}' \setminus \{0\}$, then x^0 is a strictly optimal solution of (P_{var}) w.r.t. $\preceq_{D,v}^u$.

(c) If $x^0 \in S$ is an optimal solution of $(P_{y'}^u)$ with $y' \in \overline{D}' \setminus \{0\}$ and $\max_{y \in F(x)} y'(y)$ exists for all $x \in S$, then x^0 is a weakly optimal solution of (P_{var}) w.r.t. $\preceq_{D,v}^u$.

Proof. Suppose, to the contrary, that $x^0 \in S$ is not an optimal (a strictly optimal, a weakly optimal, respectively) solution of (P_{var}) w.r.t. $\preceq_{D,v}^u$. This is equivalent to the existence of some $x \in S \setminus \{x^0\}$ such that

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y \in \{y^0\} - E(y, y^0),$$

which is equivalent to

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y^0 - y \in E(y, y^0),$$

where $E(y, y^0) := D(y, y^0) \setminus \{0\}$ ($E(y, y^0) := D(y, y^0)$, $E(y, y^0) := \operatorname{cor} D(y, y^0)$, respectively). Because $y' \in \overline{D}_{Y'}^{\#}$ ($y' \in \overline{D}' \setminus \{0\}$, $y' \in \overline{D}' \setminus \{0\}$, respectively), we have $y' \in D_{Y'}^{\#}(y, y^0)$ ($y' \in D'(y, y^0) \setminus \{0\}$, $y' \in D'(y, y^0) \setminus \{0\}$, respectively). This implies

$$\forall\, y \in F(x),\ \exists\, y^0 \in F(x^0):\ y'(y)\ [< / \leq / <]\ y'(y^0),$$

which yields

$$\sup_{y \in F(x)} y'(y)\ [< / \leq / <] \sup_{y^0 \in F(x^0)} y'(y^0),$$

in contradiction to the assumption that x^0 is an optimal (the unique, an optimal, respectively) solution of $(P_{y'}^u)$. $\square$

At this point it is interesting to investigate whether it is possible to provide assumptions that ensure the inverse statement in Theorem 3.5.1 (b) to hold true. Below we follow an approach by Jahn [54, Lemma 2.1] which we adapt to our variable ordering setting. To this end, let Y be a real locally convex linear topological space, where Y^* denotes the dual space. The dual cone to the cone $D(y_1, y_2)$, $y_1, y_2 \in A$, is denoted by $D^*(y_1, y_2) := \{y^* \in Y^* \mid \forall\, d \in D(y_1, y_2):\ y^*(d) \geq 0\}$. We set $\overline{D}^* := \bigcap_{y_1, y_2 \in A} D^*(y_1, y_2)$ and assume that $\overline{D}^* \neq \{0\}$.

Theorem 3.5.2 ([65, Theorem 5.2]). *For every $x \in S$, let the set $F(x) - \overline{D}$, with $\overline{D}$ given by (3.13), be closed and convex. If x^0 is a strictly optimal solution of (P_{var}) w.r.t. $\preceq^u_{D,v}$, then there does not exist $x \in S \setminus \{x^0\}$ such that for all $y^* \in \overline{D}^* \setminus \{0\}$*

$$\sup_{\overline{y} \in F(x)} y^*(\overline{y}) \leq \sup_{y^0 \in F(x^0)} y^*(y^0).$$

Proof. $x^0 \in S$ is a strictly optimal solution of (P_{var}) w.r.t. $\preceq^u_{D,v}$ if

$$\nexists\, x \in S \setminus \{x^0\} : \ \forall\, y \in F(x),\ \exists\, y^0 \in F(x^0) : \ y \in \{y^0\} - D(y, y^0),$$

in other words,

$$\forall x \in S \setminus \{x^0\} : \ \exists\, y \in F(x),\ \forall\, y^0 \in F(x^0) : \ y \notin \{y^0\} - D(y, y^0).$$

This implies

$$\forall x \in S \setminus \{x^0\} : \ \exists\, y \in F(x),\ \forall\, y^0 \in F(x^0) : \ y \notin \{y^0\} - \overline{D},$$

which leads to

$$\forall x \in S \setminus \{x^0\} : \ F(x) \not\subseteq F(x^0) - \overline{D}.$$

Since $F(x^0) - \overline{D}$ is closed and convex, we use a separation theorem such that we get

$$\begin{aligned} \forall\, x \in S \setminus \{x^0\} : \ \exists\, y \in F(x),\ \exists\, y^* \in Y^* \setminus \{0\},\ \exists\, \alpha \in \mathbb{R}\ \ \forall\, \overline{y} \in F(x^0) - \overline{D} : \\ y^*(y) > \alpha \geq y^*(\overline{y}), \end{aligned} \tag{3.15}$$

and this yields

$$\begin{aligned} \forall\, x \in S \setminus \{x^0\} : \ \exists\, y^* \in Y^* \setminus \{0\},\ \alpha \in \mathbb{R} : \\ \sup_{\overline{y} \in F(x)} y^*(\overline{y}) > \alpha \geq \sup_{\overline{y} \in F(x^0) - \overline{D}} y^*(\overline{y}). \end{aligned} \tag{3.16}$$

Furthermore,

$$\sup_{\overline{y} \in F(x^0) - \overline{D}} y^*(\overline{y}) = \sup_{y^0 \in F(x^0)} y^*(y^0) + \sup_{d \in -\overline{D}} y^*(d) = \sup_{y^0 \in F(x^0)} y^*(y^0). \tag{3.17}$$

To show that $y^* \in \overline{D}^*$, suppose that $y^* \notin \overline{D}^*$, which means that there is some $d \in \overline{D}$ such that $y^*(d) < 0$. With (3.15), we obtain for any $y^0 \in F(x^0)$ and some $\lambda \geq 0$

$$\alpha \geq y^*(y^0 - \lambda d) = y^*(y^0) - \lambda y^*(d) \overset{\lambda \to +\infty}{\to} +\infty,$$

a contradiction. (3.16) and (3.17) imply

$$\forall\, x \in S \setminus \{x^0\}\ \exists\, y^* \in \overline{D}^* \setminus \{0\} : \ \sup_{\overline{y} \in F(x)} y^*(\overline{y}) > \sup_{y^0 \in F(x^0)} y^*(y^0).$$

□

3.6 Application to Image Registration

Here we consider an application of set optimization problems with a variable order relation in medical image registration. The importance of variable ordering structures in set optimization in the medical field, specifically for intensity-modulated radiation therapy, has also recently been discussed by Eichfelder and Pilecka [27, 28]. Given two sets of data, A and B, the problem consists of finding a transformation t of all possible transformations T which gives a sufficient characterization of the sets. This problem was first modeled in [103] and was the motivation of introducing a variable ordering structure in vector optimization [23, 24, 25]. The objective in [103] is to minimize a vector-valued function $f: T \times A \times B \to \mathbb{R}^m$, which comprises m *distance functions.* The problem can be formulated as

$$\min_{t \in T} f(t, A, B).$$

Eichfelder [23] noted that using the natural ordering cone $\mathbb{R}^m_+$ for defining optimality for this problem is not adequate here, as solutions with one minimal distance measure, but several other relatively high function values could be chosen to be optimal. While this corresponds to the usual definition of efficiency, it is not satisfying in this application. Therefore, a variable ordering structure is introduced in order to define optimality.

The original approach discussed in [23] attaches to every value $y := f(t, A, B)$ a nonnegative weight vector $w(y) \in \mathbb{R}^m_+$, which depends on the element y in the objective space $\mathbb{R}^m$. This weight vector is incorporated in the variable ordering $D : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ in the following way:

$$D^w := D(w(y)) := \{d \in \mathbb{R}^m \mid \sum_{i=1}^{m} \operatorname{sgn}(d^i) w^i(y) \geq 0\},$$

where

$$\operatorname{sgn}(d^i) := \begin{cases} 1 & (d^i > 0) \\ 0 & (d^i = 0) \\ -1 & (d^i < 0). \end{cases}$$

D^w is called the cone of preferred directions. Notice that $w^i(y) \in \mathbb{R}_+$ for every $i = 1, \ldots, m$. The cone D^w for all possible weights $w \in \mathbb{R}^2_+$ is calculated in Eichfelder [23].

In contrast to the original approach, we suppose that the objective function is given by a set-valued map $F: T \times A \times B \rightrightarrows \mathbb{R}^m$, where the individual distance measures are supposed to be set-valued, i.e., $F_i: T \times A \times B \rightrightarrows \mathbb{R}$, $i = 1, \ldots, m$. This could be due to incomplete information of the acquired data, uncertain data in the sensors or movements of the patient during the tomography, and it means that we wish to consider all possible values that the distance map may attain. This also fits into the set optimization framework, as uncertain multi-objective optimization actually is an application of set optimization, as was noted in [53, Example 14.3], compare also [51].

Regarding the calculations for an optimal radiation treatment in intensity-modulated radiation therapy, Eichfelder and Pilecka [27, 28] discuss the significance of variable ordering structures for set optimization problems. They explain that for safety purposes one might prefer to do necessary calculations based on several data sets. Moreover, for calculating the dose stress various approaches exist that need to be taken into consideration simultaneously.

Coming back to the image registration topic, we assign every two elements $y_1, y_2 \in \mathbb{R}^m$ in the objective space a weight vector $w(y_1, y_2) \in \mathbb{R}^m_+$, which depends on both y_1 and y_2. The cone of preferred directions for each given pair $y_1, y_2 \in \mathbb{R}^m$ is introduced as a set-valued map $D_{set} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ defined by

$$D^w_{set} := D_{set}(w(y_1, y_2)) := \{d \in \mathbb{R}^m \mid \sum_{i=1}^{m} \operatorname{sgn}(d^i) w^i(y_1, y_2) \geq 0\}.$$

Then the problem of finding an adequate transformation function in image registration can now be formulated as

$$\min_{t \in T} F(t, A, B). \tag{SP}$$

Now that the problem has been formulated, it is necessary to define an optimality concept for solving (SP). Since the problem is set-valued, there are several possibilities.

On the one hand, a vector approach can be considered. That means that the set $\mathcal{Y} := \cup_{t \in T} F(t, A, B)$ is computed and optimal elements (in the sense of vector optimization) can be determined. By taking into account the variable ordering cone D^w, we arrive at two optimality notions: A transformation $t^0 \in T$ is called *nondominated* if $t \not\leq_1 t^0$ for all $t \in T \setminus \{t^0\}$ (see (3.1) in Definition 3.2.1). An element $t^0 \in T$ is called *minimal* if $t \not\leq_2 t^0$ for all $t \in T \setminus \{t^0\}$ (see (3.2)). However, by using such a vector approach, solution sets $F(t^0, A, B)$ are chosen based on their "best" element $y^0 \in F(t^0, A, B)$. This approach does not take the whole set $F(t^0, A, B)$ into account. Therefore, a set approach is considered to be more appropriate.

By using such a set approach, it is necessary to determine how to compare the sets $F(t, A, B)$, $t \in T$. There exist several set relations in the literature, for example the upper set less relation (see Definition 2.2.1), the lower set less relation (see Definition 2.2.9), or the certainly less relation (see Definition 2.2.19). However, none of the known set relations incorporate a variable ordering cone. If a fixed ordering cone is used instead, the information gathered in the variable ordering cone D^w_{set} would be lost. Therefore, we suggest to use the variable upper set less relation $\preceq^u_D$ introduced in Definition 3.2.2.

Then a feasible transformation $t^0 \in T$ of the set optimization problem (SP) is optimal (strictly optimal / weakly optimal) w.r.t. $\preceq^u_{D^w_{set}}$ if

$$\nexists\, t \in T \setminus \{t^0\} : \ \forall\, y \in F(t, A, B)\ \exists\, y^0 \in F(t^0, A, B) : \ y \in y^0 - E(y, y^0),$$

where $E(y, y^0) = D^w_{set}(y, y^0) \setminus \{0\}$ ($E(y, y^0) = D^w_{set}(y, y^0)$, $E(y, y^0) = \operatorname{cor} D^w_{set}(y, y^0)$, respectively).

From a practical perspective, using the *variable upper set less relation* leads to a robust approach, since the sets of *worst cases* in every set $F(t, A, B)$ are compared with one another. For the single objective case, this corresponds to the *robust optimization* approach studied by Ben-Tal, El Ghaoui and Nemirovski [9], and was extended to uncertain multi-objective optimization in Ehrgott, Ide, Schöbel [22], see also [50, 51, 63]. Of course, it is also possible to perform the analysis proposed in this chapter for other set relations. The usefulness for certain applications then needs to be discussed.

Remark 3.6.1. *Let us mention that the application of variable domination structures in the field of uncertain programming has also been investigated in [4, Section 5], where a characterization of optimistic solutions of uncertain vector optimization problems is derived.*

Chapter 4

Approximate Solutions of Set-Valued Optimization Problems Using Set-Criteria

4.1 Motivation

It is well known that minimal elements of vector and set optimization problems do not always exist, and hence some approximate minimality notions have been widely investigated in the literature (see [3, 19, 41, 59, 81] and the references therein). This chapter is devoted to introducing and studying approximate minimal elements of set optimization problems. We consider a family of nonempty sets and define three notions of approximate minimality by following a set approach using the lower-type set relation, which is a binary relation among sets (see [76, 77, 79] and Definition 2.2.9). Our definitions extend those given in the literature; we also show that some of the approximate minimality notions known from the literature are obtained as special cases.

This chapter is based on the results derived in [42] and is organized as follows: Section 4.2 recalls some definitions and a preliminary result on approximate minimal elements of vector and set optimization problems. We introduce three new notions of approximate minimal element for a family of sets. In Section 4.3, we propose solution methods for obtaining approximate minimal elements of a family of sets. In this section, the family of sets may be given by an infinite number of sets. We divide our analysis into two parts: In Subsection 4.3.1, we propose linear scalarization results to obtain approximate minimal elements, where convexity of the considered sets plays an important role. Omitting any convexity assumptions, we present nonlinear scalarization results in Subsection 4.3.2. In case that the number of given sets is finite, we derive efficient solution procedures for obtaining approximate minimal elements in Section 4.4.

4.2 Preliminaries on Approximate Minimality

Throughout this chapter, let Y be a real topological vector space. Here, we study approximate minimal elements w.r.t. the generalized lower set relation $\preceq^l_D$ (compare Definition 2.2.9).

It is well known that usually, the existence of minimal elements (see Definition 1.2.11) can only be guaranteed under additional assumptions (for an existence result of minimal elements, see, for example, [56]). Because the set $\mathcal{A}_{min}$ may be empty, it is necessary to introduce a weaker notion of minimality. For this reason, we introduce three new notions of approximate minimality.

Definition 4.2.1 ([42, Definition 2.4]). *Let $\mathcal{A}$ be a family of elements of $\mathcal{P}(Y)$, $D, H \in \mathcal{P}(Y)$, and $D, H \neq Y$.*

*(a) $\overline{A} \in \mathcal{A}$ is called an H^1–**approximate minimal element** of $\mathcal{A}$ w.r.t. $\preceq^l_D$ if*

$$A \preceq^l_D \overline{A},\ A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq^l_D A + H\,.$$

The set of all H^1–approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ will be denoted by $\mathcal{A}_{H^1}$.

*(b) $\overline{A} \in \mathcal{A}$ is called an H^2–**approximate minimal element** of $\mathcal{A}$ w.r.t. $\preceq^l_D$ if*

$$A + H \preceq^l_D \overline{A},\ A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq^l_D A + H\,.$$

We call the set of all H^2–approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ $\mathcal{A}_{H^2}$.

*(c) $\overline{A} \in \mathcal{A}$ is called an H^3–**approximate minimal element** of $\mathcal{A}$ w.r.t. $\preceq^l_D$ if $A + H \not\preceq^l_D \overline{A}$, for all $A \in \mathcal{A}$. We call the set of all H^3–approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ $\mathcal{A}_{H^3}$.*

Definition 4.2.1 (a) is a natural formulation for approximate minimality. It states that $\overline{A}$ is H^1-approximate minimal if the statement "an element A dominates $\overline{A}$" ($A \preceq^l_D \overline{A}$) implies that "$\overline{A}$ dominates a perturbation of A" ($\overline{A} \preceq^l_D A + H$). Definition 4.2.1 (b) is derived from the standard notion of approximate efficiency for vector-valued maps (see part 3 of Remark 4.2.2) where the minimality notion in the vector case is replaced by the minimality notion for families of elements of $\mathcal{P}(Y)$ from Definition 1.2.11. Definition 4.2.1 (c) is an approximate version of the well-known nondomination concept of vector optimization. Note that, if $H = \{0\}$, Definition 4.2.1 (a) and (b) recover Definition 1.2.11 given in Section 1.2.2.

Remark 4.2.2 ([42, Remark 2.5]). *1. Definition 4.2.1 (b) is equivalent to*

$$\nexists\ A \in \mathcal{A}:\ A + H \preceq^l_D \overline{A} \text{ and } \overline{A} \not\preceq^l_D A + H.$$

Then it is obvious that $\mathcal{A}_{H^3} \subseteq \mathcal{A}_{H^2}$. The inclusion $\mathcal{A}_{H^2} \subseteq \mathcal{A}_{H^3}$ does not generally hold; see Examples 4.2.5 and 4.2.6. On the other hand, if $H = \{0\}$, then the definitions of H^1– and H^2–approximate minimal element coincide with Definition 1.2.11.

2. *Note that* $\mathcal{A}_{H^2} \subseteq \mathcal{A}_{H^1} \subseteq \mathcal{A}_{min}$ *if* $0 \in H$. *Moreover, if* $0 \in H + D$, *then the set of* H^3*–approximate minimal elements is empty, since for all* $\overline{A} \in \mathcal{A}$, $\overline{A} + H \preceq^l_D \overline{A}$.

3. *In case the family* $\mathcal{A}$ *and* H *consist of single-valued sets, Definition 4.2.1 is closely related to well-known notions of approximate efficiency. For example, if* $H = \{\varepsilon\}$, *then set* $\mathcal{A}_{H^1}$ *coincides with a notion of approximate solution of vector optimization problems due to White (see [106]).*

   Furthermore, if $0 \in D$, D *is pointed (i.e.,* $D \cap (-D) = \{0\}$*) and* $H = \{\varepsilon\}$, *the concept of* H^2*-approximate minimality introduced in Definition 4.2.1 (b) coincides with the concept introduced by Kutateladze in [80], the most popular notion of approximate efficient solutions in vector optimization. Indeed, from Definition 4.2.1 (b) for this special case, we get that* $\bar{y} \in \mathcal{A}_{H^2}$, *if*

   $$\{y\} + \{\varepsilon\} \preceq^l_D \{\bar{y}\},\ y \in \mathcal{A} \quad \Longrightarrow \quad \{\bar{y}\} \preceq^l_D \{y\} + \{\varepsilon\},$$

   i.e., taking into account the definition of $\preceq^l_D$

   $$\{\bar{y}\} \subseteq \{y\} + \{\varepsilon\} + D,\ y \in \mathcal{A} \quad \Longrightarrow \quad \{y\} + \{\varepsilon\} \subseteq \{\bar{y}\} + D.$$

   Since D *is pointed, it holds that* $\bar{y} \in \mathcal{A}_{H^2}$, *if*

   $$y \in \bar{y} - \varepsilon - D,\ y \in \mathcal{A} \quad \Longrightarrow \quad y = \bar{y} - \varepsilon.$$

4. *Let* $K \subset Y$ *be a convex cone with nonempty topological interior,* $\delta \in \mathbb{R}$, $\delta > 0$, $D = \operatorname{int} K$, $e \in \operatorname{int} K$, $\varepsilon := \delta \cdot e$ *and* $H := \{\varepsilon\}$. *In [19], the authors call* $\overline{A} \in \mathcal{A}$ ε*-l-weak efficient if there is no* $A \in \mathcal{A}$ *such that* $A + H \preceq^l_D \overline{A}$. *Thus, it is clear that this definition is a particular case of Definition 4.2.1 (c).*

5. *Suppose that* Y *is normed. Then, Definition 4.2.1 (b) contains the notion of* ε*-lower minimality presented in [3] if the set* H *is single-valued: Due to [3], given* $\varepsilon > 0$, *a set* $\overline{A} \in \mathcal{A}$ *is said to be an* ε*-lower minimal set of* $\mathcal{A}$ *if there exists* $\alpha_\varepsilon \in Y$ *with* $||\alpha_\varepsilon|| < \varepsilon$ *such that*

   $$A \preceq^l_D \overline{A} - \alpha_\varepsilon,\ A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} - \alpha_\varepsilon \preceq^l_D A\,.$$

   Then, if we denote the unit open ball of Y *by* $\mathcal{B}$, *it follows that* $\overline{A} \in \mathcal{A}$ *is an* ε*-lower minimal set of* $\mathcal{A}$ *if and only if there exists* $y \in \varepsilon\mathcal{B}$ *such that it is a* $\{y\}^2$*-approximate minimal element of* $\mathcal{A}$.

Remark 4.2.3 ([42, Remark 2.6])**.** *It is easy to observe that if* $H \subseteq \widetilde{H}$, *then* $\mathcal{A}_{\widetilde{H}^1} \subseteq \mathcal{A}_{H^1}$ *holds true.*

Lemma 4.2.4 ([42, Lemma 2.7])**.** *Assume that* D *is a convex cone and* $H \subseteq D$. *Then the inclusion* $\mathcal{A}_{min} \subseteq \mathcal{A}_{H^1} \subseteq \mathcal{A}_{H^2}$ *holds.*

Proof. Choose $\overline{A} \in \mathcal{A}_{min}$ and assume that there exists some $A \in \mathcal{A}$ such that $A \preceq^l_D \overline{A}$. By the minimality of $\overline{A}$, we get $\overline{A} \preceq^l_D A$. This means $A \subseteq \overline{A} + D$, which implies $A + H \subseteq \overline{A} + H + D \subseteq \overline{A} + D$. This corresponds to $\overline{A} \preceq^l_D A + H$.

Now let $\overline{A} \in \mathcal{A}$ be an H^1–approximate minimal element of $\mathcal{A}$ w.r.t. $\preceq^l_D$ and assume that there exists some $A \in \mathcal{A}$ with $A + H \preceq^l_D \overline{A}$. This means that $\overline{A} \subseteq A + H + D$. As $H \subseteq D$ and D is a convex cone, we obtain $\overline{A} \subseteq A + D$, corresponding to $A \preceq^l_D \overline{A}$. By the H^1–approximate minimality of $\overline{A}$, we obtain $\overline{A} \preceq^l_D A + H$. □

The inverse inclusion, namely $\mathcal{A}_{H^2} \subseteq \mathcal{A}_{H^1}$, does not generally hold, as the following example shows (see also the numerical tests in Example 4.4.11 on page 97).

Example 4.2.5 ([42, Example 2.8])**.** *Consider $Y = \mathbb{R}^2$, the family $\mathcal{A} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1, y_2 \geq 1\}$ consisting of infinitely many single-valued sets, $D = \mathbb{R}^2_+$, $\varepsilon = 1$, $e = (0.5, 1)$ and $H = \{\varepsilon e\}$. The set of all H^1–approximate minimal elements of $\mathcal{A}$ is $\mathcal{A}_{H^1} = \mathcal{A} \cap \{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \leq 2,\ y_1 \leq 1.5\}$. We can see that $\mathcal{A}_{H^2} = \mathcal{A}_{H^1} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid 1.5 \leq y_1, 1 \leq y_2 < 2\} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid 2 \leq y_2,\ 1 \leq y_1 < 1.5\}$. Moreover, it holds that $\mathcal{A}_{min} = \{(1, 1)\}$. Notice that $\mathcal{A}_{H^3} = \mathcal{A}_{H^2} \backslash \{(1.5, 2)\}$, because, for a family of single-valued sets, the set relation $\preceq^l_D$ reduces to the usual partial order (i.e., $\preceq^l_D$ is antisymmetric) in vector optimization. See Figure 4.1 for an illustration.*

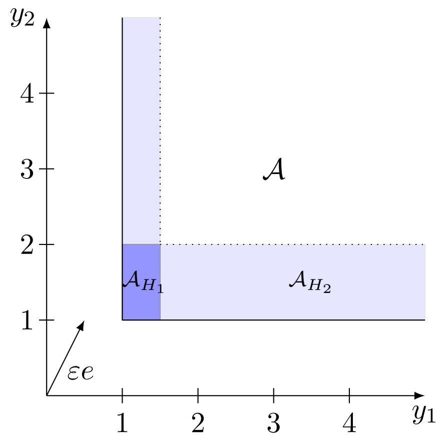

Figure 4.1: $\mathcal{A}_{H^2} \subseteq \mathcal{A}_{H^1}$ does not hold (see Example 4.2.5).

The next example also shows that $\mathcal{A}_{H^2} \subseteq \mathcal{A}_{H^3}$ does not hold true in general.

Example 4.2.6 ([42, Example 2.9])**.** *Let $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $H = \{(0, 2)\}$, $\overline{A} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1, y_2 \in [0, 2]\}$, $A = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 \leq y_1 \leq 1,\ -2 \leq y_2 \leq -1\}$, $\mathcal{A} = \{\overline{A}, A\}$. Then $\overline{A}$ is an H^2–approximate minimal element of $\mathcal{A}$, but $\overline{A} \notin \mathcal{A}_{H^3}$. For an illustration, see Figure 4.2.*

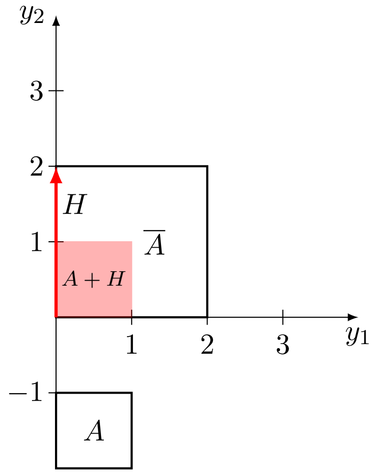

Figure 4.2: $\overline{A} \in \mathcal{A}_{H^2}$, but $\overline{A} \notin \mathcal{A}_{H^3}$ (see Example 4.2.6).

4.3 Scalarization Results

In this section, we present some characterizations for approximate minimal elements of a family of sets w.r.t. $\preceq^l_D$ by means of linear and nonlinear scalarization methods.

4.3.1 Linear Scalarization

Throughout this subsection, $D \in \mathcal{P}(Y)$ is taken such that $D \neq Y$; for some results, however, we need $D \in \mathcal{P}(Y)$ to be a convex cone and this particular assumption will be stated in the respective assertion.

The following result is closely related to a vectorization approach by Jahn [54, Lemma 2.1].

Theorem 4.3.1 ([42, Theorem 3.1]). *If for all $A \in \mathcal{A}\backslash\{\overline{A}\}$ there exists some $\ell \in D^*\backslash\{0\}$ such that*

$$\inf_{\overline{a}\in\overline{A}} \ell(\overline{a}) < \inf_{a\in A} \ell(a) \tag{4.1}$$

holds, then $\overline{A} \in \mathcal{A}_{min}$.

It is straightforward that under the assumptions stated in Lemma 4.2.4, that is, if D is a convex cone and $H \subseteq D$, then $\overline{A} \in \mathcal{A}_{H^1} \cap \mathcal{A}_{H^2}$ also holds whenever condition (4.1) is fulfilled due to the inclusion $\mathcal{A}_{min} \subseteq \mathcal{A}_{H^1} \subseteq \mathcal{A}_{H^2}$. Similar to Theorem 4.3.1, we will provide a scalarization approach for obtaining H^3-approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ in Theorem 4.3.5. First, we are interested in obtaining H^1-approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ via linear scalarization. After the following result, we will discuss the single-valued case and note that the assumptions in Theorem 4.3.2 are quite strong.

Theorem 4.3.2 ([42, Theorem 3.2]). *Let Y be a real locally convex vector space. Assume that $D \subset Y$ is a convex cone. If for all $A \in \mathcal{A}$ and for all $\ell \in D^*$*

$$\inf_{\overline{a} \in \overline{A}} \ell(\overline{a}) \leq \inf_{a \in A+H} \ell(a) \tag{4.2}$$

holds and $\overline{A} + D$ is closed and convex, then $\overline{A} \in \mathcal{A}_{H^1}$.

Proof. Suppose to the contrary that there exists some $A \in \mathcal{A}$ such that $A \preceq^l_D \overline{A}$, but $\overline{A} \not\preceq^l_D A + H$. This means that $A + H \not\subseteq \overline{A} + D$. Thus, there exists some $a_0 \in A + H$ such that $a_0 \notin \overline{A} + D$. Since $\overline{A} + D$ is closed and convex, we are able to apply a classical separation argument (for example, [53, Theorem 3.18]) such that there exists $\alpha \in \mathbb{R}$, $\ell \in Y^* \setminus \{0\}$ such that

$$\forall\, \overline{a} \in \overline{A} + D:\ \ell(a_0) < \alpha \leq \ell(\overline{a}). \tag{4.3}$$

Because D is a convex cone, we get $\ell \in D^* \setminus \{0\}$. The inequality (4.3) implies

$$\inf_{a \in A+H} \ell(a) < \alpha \leq \inf_{\overline{a} \in \overline{A}} \ell(\overline{a}),$$

contradicting the assumption (4.2). □

Note that under the assumptions stated in Theorem 4.3.2 and Lemma 4.2.4 (that is, if $H \subseteq D$), $\overline{A} \in \mathcal{A}_{H^2}$ holds as well if condition (4.2) is true.

Theorem 4.3.2 leads to the following result for $H = \{0\}$ and if the sets in $\mathcal{A}$ are single-valued.

Corollary 4.3.3 ([42, Corollary 3.3]). *Assume that D is a closed convex cone in the real locally convex vector space Y and $H = \{0\}$. Furthermore, let all elements in $\mathcal{A}$ be single-valued. If for all $a \in \mathcal{A}$ and for all $\ell \in D^*$*

$$\ell(\overline{a}) \leq \ell(a) \tag{4.4}$$

holds, then $\overline{a} \in \mathcal{A}_{min}$.

Let us mention that for Example 4.2.5, all H^1-approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ can be found by means of Theorem 4.3.2, as can be easily seen by visualization. This observation is a motivation for the following consequence of Theorem 4.3.2:

Corollary 4.3.4 ([42, Corollary 3.4]). *Let Y be a real locally convex vector space. Assume that $D \subset Y$ is a closed convex cone, $\varepsilon \in Y$ and $H = \{\varepsilon\}$. Furthermore, let all elements in $\mathcal{A}$ be single-valued. If for all $a \in \mathcal{A}$ and for all $\ell \in D^*$*

$$\ell(\overline{a}) \leq \ell(a + \varepsilon)$$

holds, then $\overline{a} \in \mathcal{A}_{H^1}$.

It is clear that the assumptions in Corollaries 4.3.3 and 4.3.4 are very strong; in fact, under the assumptions of Corollary 4.3.3 and if $\bar{a} \in \mathcal{A}$ satisfies (4.4), it follows by the bipolar theorem (see, for instance, [29, Proposition 2.1 (1)]) that $\bar{a} \preceq^l_D a$, for all $a \in \mathcal{A}$, i.e., $\bar{a}$ is a strongly minimal element of $\mathcal{A}$ (see [53, Definition 4.8(a)]). Analogously, if the assumptions of Corollary 4.3.4 are satisfied, we deduce that $\bar{a} \preceq^l_D a + \varepsilon$, for all $a \in \mathcal{A}$, so in this case $\bar{a}$ is an ε-strongly minimal element of $\mathcal{A}$ (see [29]). Therefore, a recourse to actually obtain H^1-approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ is presented in Section 4.3.2 using a nonlinear scalarization functional.

Theorem 4.3.5 presents a sufficient condition for H^3-approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$.

Theorem 4.3.5 ([42, Theorem 3.5]). *If for all $A \in \mathcal{A}\backslash\{\overline{A}\}$ there exists some $\ell \in D^*\backslash\{0\}$ such that*

$$\inf_{\overline{a}\in\overline{A}} \ell(\overline{a}) < \inf_{a\in A+H} \ell(a)$$

holds, then $\overline{A} \in \mathcal{A}_{H^3}$.

Proof. Suppose by contradiction that $\overline{A} \notin \mathcal{A}_{H^3}$. Then there exists some $A \in \mathcal{A}$ such that $A + H \preceq^l_D \overline{A}$. From this relation, we immediately obtain for all $\ell \in D^* \setminus \{0\}$ the inequality $\inf_{a\in A+H} \ell(a) \leq \inf_{\overline{a}\in\overline{A}} \ell(\overline{a})$, contradicting the assumption. □

The following result gives a necessary condition for the H^2-approximate minimal elements.

Theorem 4.3.6 ([42, Theorem 3.6]). *Let Y be a real locally convex vector space and let $D \subset Y$ be a convex cone. Assume that for all $A \in \mathcal{A}$, $A + H + D$ is a closed convex set and let $\overline{A} \in \mathcal{A}_{H^2}$. Then for all $A \in \mathcal{A}$ there exists some $\ell \in D^* \setminus \{0\}$ with the property*

$$\inf_{\overline{a}\in\overline{A}} \ell(\overline{a}) \leq \inf_{a\in A+H} \ell(a).$$

Proof. Suppose to the contrary that there exists some $A \in \mathcal{A}$ such that

$$\forall\ \ell \in D^* \setminus \{0\} : \inf_{\overline{a}\in\overline{A}} \ell(\overline{a}) > \inf_{a\in A+H} \ell(a). \tag{4.5}$$

Then we obtain by [54, Lemma 2.1] $A + H \preceq^l_D \overline{A}$. By the H^2-minimality of $\overline{A}$, we get $\overline{A} \preceq^l_D A + H$. This directly yields the equality $\overline{A} + D = A + H + D$, in contradiction with (4.5). □

The proof of the following theorem is skipped, as it is similar to that of Theorem 4.3.6 (or can be directly obtained from Lemma 4.2.4).

Theorem 4.3.7 ([42, Theorem 3.7]). *Let Y be a real locally convex vector space and let $D \subset Y$ be a convex cone. Assume that $H \subset D$ and that for all $A \in \mathcal{A}$, $A + H + D$ is a closed convex set and let $\overline{A} \in \mathcal{A}_{H^1}$. Then for all $A \in \mathcal{A}$ there exists some $\ell \in D^* \setminus \{0\}$ with the property*

$$\inf_{\overline{a}\in\overline{A}} \ell(\overline{a}) \leq \inf_{a\in A+H} \ell(a).$$

Under less restrictive assumptions, we have the following assertion.

Theorem 4.3.8 ([42, Theorem 3.8]). *Let Y be a real locally convex vector space and let $D \subset Y$ be a convex cone. Assume that $0 \in H$ and that for all $A \in \mathcal{A}$, $A + D$ is a closed convex set and let $\overline{A} \in \mathcal{A}_{H^1}$. Then for all $A \in \mathcal{A}$ there exists some $\ell \in D^* \setminus \{0\}$ with the property*

$$\inf_{\overline{a} \in \overline{A}} \ell(\overline{a}) \leq \inf_{a \in A} \ell(a).$$

Remark 4.3.9 ([42, Remark 3.9]). *The convexity assumptions of $A + H + D$ and $A + D$ for all $A \in \mathcal{A}$ in Theorems 4.3.7 and 4.3.8, respectively, are indeed necessary. Consider, for instance, the space $Y = \mathbb{R}^2$, the sets $D = \mathbb{R}^2_+$, $H = \{0\}$, $A = \{(1,1), (2, 0.5)\}$, $\overline{A} = \{(1.5, 0.8)\}$ and $\mathcal{A} = \{A, \overline{A}\}$. Then we have $A \not\preceq^l_D \overline{A}$, thus, $\overline{A} \in \mathcal{A}_{H^1}$, but $\inf_{a \in A} \ell(a) < \inf_{\overline{a} \in \overline{A}} \ell(\overline{a})$ for all $\ell \in D^* \setminus \{0\}$.*

4.3.2 Nonlinear Scalarization

In this section, we present characterizations of approximate minimal elements of $\mathcal{A}$ w.r.t. $\preceq^l_D$ without any convexity assumptions. Throughout this section, we assume that $D \subset Y$ is a convex cone with nonempty interior, $k \in \operatorname{int} D$, and $\overline{A} \in \mathcal{P}(Y)$ is D-proper. Let $\Delta_{D,k,\overline{A}} : \mathcal{P}(Y) \to 2^{\mathbb{R}}$ be defined by

$$\forall\ A \in \mathcal{P}(Y) :\ \Delta_{D,k,\overline{A}}(A) := \{t \in \mathbb{R} \mid A \preceq^l_D tk + \overline{A}\}.$$

We recall the following nonlinear scalarizing functional $z_{D,k,\overline{A}} : \mathcal{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, given by

$$z_{D,k,\overline{A}}(A) := \begin{cases} +\infty & \text{if } \Delta_{D,k,\overline{A}}(A) = \emptyset, \\ \inf\{t \in \mathbb{R} \mid A \preceq^l_D tk + \overline{A}\} & \text{otherwise.} \end{cases}$$

Some useful properties of the functional $z_{D,k,\overline{A}}$ are collected below.

Proposition 4.3.10 ([43, 46]). *1. Monotonicity property:*

$$A_1, A_2 \in \mathcal{P}(Y),\ A_1 \preceq^l_D A_2 \Rightarrow z_{D,k,\overline{A}}(A_1) \leq z_{D,k,\overline{A}}(A_2).$$

2. Let $A \in \mathcal{P}(Y)$, $r \in \mathbb{R}$. Then $z_{D,k,\overline{A}}(A) \leq r \iff rk + \overline{A} \subseteq \operatorname{cl}(A + D)$. In particular we have $z_{D,k,\overline{A}}(\overline{A}) \leq 0$.

The following result is easily obtained.

Theorem 4.3.11 ([42, Theorem 3.11]). *If $z_{D,k,\overline{A}}(A) > 0$ for all $A \in \mathcal{A} \setminus \{\overline{A}\}$, then $\overline{A} \in \mathcal{A}_{min}$.*

Proof. Suppose the opposite. Then there exists some $A \in \mathcal{A} \setminus \{\overline{A}\}$ such that $A \preceq^l_D \overline{A}$ and $\overline{A} \not\preceq^l_D A$. By the monotonicity property of $z_{D,k,\overline{A}}$, we obtain $z_{D,k,\overline{A}}(A) \leq z_{D,k,\overline{A}}(\overline{A}) \leq 0$. But this is a contradiction to the assumption, and the result follows. □

Under the assumption stated in Theorem 4.3.11, $\overline{A}$ also belongs to $\mathcal{A}_{H^1} \cap \mathcal{A}_{H^2}$ for any $H \subseteq D$. Notice that if the family of sets $\mathcal{A}$ is only given by single-valued sets, Theorem 4.3.11 reduces to a separation theorem for not necessarily convex sets (see [37, Theorem 2.3.6]). A sufficient condition for H^3-approximate minimal elements is given in Theorem 4.3.12. The proof is similar to the one of Theorem 4.3.11 and is therefore omitted.

Theorem 4.3.12 ([42, Theorem 3.12])**.** *If $z_{D,k,\overline{A}}(A + H) > 0$ for all $A \in \mathcal{A} \setminus \{\overline{A}\}$, then $\overline{A} \in \mathcal{A}_{H^3}$.*

The following theorem describes a necessary condition for H^1-approximate minimal elements.

Theorem 4.3.13 ([42, Theorem 3.13])**.** *For all $A \in \mathcal{A}$, assume that $A + D$ is closed. If $\overline{A} \in \mathcal{A}_{H^1}$, then $z_{D,k,\overline{A}}(A) > 0$ for all $A \in \mathcal{A} \setminus \{\overline{A}\}$ with $\overline{A} \npreceq^l_D A + H$.*

Proof. Suppose by contradiction that there exists some $A \in \mathcal{A} \setminus \{\overline{A}\}$ with $\overline{A} \npreceq^l_D A + H$ such that $z_{D,k,\overline{A}}(A) \leq 0$. This is equivalent to $A \preceq^l_D \overline{A}$. Using the H^1-approximate minimality of $\overline{A}$, we immediately obtain that $\overline{A} \preceq^l_D A + H$, a contradiction. So, the result is proved. □

The following result gives a necessary condition for H^2-approximate minimal elements.

Theorem 4.3.14 ([42, Theorem 3.14])**.** *Assume that for all $A \in \mathcal{A}$, $A + H + D$ is closed. If $\overline{A} \in \mathcal{A}_{H^2}$, then $z_{D,k,\overline{A}}(A + H) > 0$ for all $A \in \mathcal{A} \setminus \{\overline{A}\}$ with $\overline{A} \npreceq^l_D A + H$.*

Proof. Suppose that there exists some $A \in \mathcal{A} \setminus \{\overline{A}\}$ with $\overline{A} \npreceq^l_D A + H$ such that $z_{D,k,\overline{A}}(A + H) \leq 0$. This is equivalent to $A + H \preceq^l_D \overline{A}$, which implies by the H^2-approximate minimality of $\overline{A}$ that $\overline{A} \preceq^l_D A + H$, in contradiction to the assumption. Then, the proof is complete. □

Remark 4.3.15 ([42, Remark 3.15])**.** *The results obtained in this section can be helpful to derive numerical methods for computing approximate minimal elements of a family of (possibly infinitely many) sets. In case that the family of sets is given by a finite number of sets, we develop efficient numerical methods in Section 4.4.*

4.4 Finding H^1- and H^2-Approximate Minimal Elements of a Family of Finitely Many Elements

In this section, we present algorithms that filter out elements of a family of finitely many sets which cannot be H^1- (H^2-, respectively) approximate minimal and that find all H^1- and H^2-approximate minimal elements. A first approach as a filter method for sorting out non-minimal elements of a family of finitely many sets has been proposed in Section 2.3.2. In this section, the idea of such a method is generalized to set optimization problems, where we assume that a family of finitely many sets $\mathcal{A}$ is given and we are looking for H^1- (H^2-, respectively) approximate minimal elements of $\mathcal{A}$.

The following algorithm is a reduction method that excludes sets that cannot be H^1- (H^2-, respectively) approximate minimal. Moreover, it is a self learning method that improves with each step. The reason for this is that if the if-statement of Algorithm 4.4.1 is not fulfilled for some A_j ($j \in \{2,\dots,m\}$), then A_j is not added to the set $\mathcal{T}$ and therefore, it can be excluded from all further investigation. In addition, due to the if-statement of Algorithm 4.4.1, each element is compared only with elements that have been under consideration before, that is, which belong to the set $\mathcal{T}$. That implies that it is not necessary to compare all elements with each other pairwise according to the definition of H^1- (H^2-, respectively) approximate minimality, which can strongly reduce the computation time of determining approximate minimal elements.

Algorithm 4.4.1 ([42, Algorithm 4.1])**.** *(Method for sorting out elements of a family of finitely many sets which are not H^1- (H^2-, respectively) approximate minimal elements)*

Input: $\mathcal{A} := \{A_1, \dots, A_m\}$, *set relation* $\preceq^l_D$, $H \subset Y$
% initialization
$\mathcal{T} := \{A_1\}$,
% iteration loop
for $j = 2:1:m$ ***do***
  if $\left(A \preceq^l_D A_j,\ A \in \mathcal{T} \implies A_j \preceq^l_D A + H\right)$
   $\Big(\left(A + H \preceq^l_D A_j,\ A \in \mathcal{T} \implies A_j \preceq^l_D A + H\right)$, *respectively*$\Big)$, ***then***
   $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
  end if
end for
Output: $\mathcal{T}$

Below we note that all H^1- (H^2-, respectively) approximate minimal elements of the family of sets $\mathcal{A}$ are contained in the output set $\mathcal{T}$ generated by Algorithm 4.4.1. We refrain from giving a proof, as the results can be proven in a similar way as Theorem 2.3.6.

Theorem 4.4.2 ([42, Theorem 4.2])**.**

1. *Algorithm 4.4.1 is well-defined.*
2. *Algorithm 4.4.1 generates a nonempty set $\mathcal{T} \subseteq \mathcal{A}$.*
3. *Every H^1- (H^2-, respectively) approximate minimal element of $\mathcal{A}$ also belongs to the set $\mathcal{T}$ generated by Algorithm 4.4.1.*

It was shown in Chapter 2.3.2 that all minimal elements of a family of finitely many sets are found if the if-loop in Algorithm 2.3.5 is run backwards on the set $\mathcal{T}$ and if the set relation is antisymmetric and the set of minimal elements is externally stable. Since $\preceq^l_D$ is not antisymmetric, we cannot rely on our previously derived procedure, and hence we propose the following algorithm that does not rely on antisymmetry or external stability. The basic idea of the proposed method can be found in Eichfelder [25, Algorithm 1] in the context of finding minimal elements in vector optimization, where the domination structure is equipped with a variable ordering cone. The following algorithm consists of the first if-loop of Algorithm 4.4.1, the backwards iteration of the if-loop which finds a set $\mathcal{U} \subseteq \mathcal{T}$, and a third for-loop, which is added to compare the elements that were obtained with all remaining elements in $\mathcal{A} \setminus \mathcal{U}$.

We further need the following assumption.

Assumption 4.4.3. *Let D be a convex cone and $H \subseteq D$.*

With the above assumption, the implications

$$\begin{aligned} \forall\, A_i \in \mathcal{U} \setminus \{A_j\} :\ A_i \preceq^l_D A_j \quad &\Longrightarrow \quad A_j \preceq^l_D A_i + H \\ (\forall\, A_i \in \mathcal{U} \setminus \{A_j\} :\ A_i + H \preceq^l_D A_j \quad &\Longrightarrow \quad A_j \preceq^l_D A_i + H, \text{ respectively}), \end{aligned}$$

and

$$\begin{aligned} \forall\, A_i \in \mathcal{U} :\ A_i \preceq^l_D A_j \quad &\Longrightarrow \quad A_j \preceq^l_D A_i + H \\ (\forall\, A_i \in \mathcal{U} :\ A_i + H \preceq^l_D A_j \quad &\Longrightarrow \quad A_j \preceq^l_D A_i + H, \text{ respectively}), \end{aligned}$$

are equivalent, which will be useful in Lemma 4.4.5 and Theorems 4.4.6 and 4.4.10.

Algorithm 4.4.4 ([42, Algorithm 4.4])**.** *(Method for finding all H^1- (H^2-, respectively) approximate minimal elements of a family of finitely many sets under Assumption 4.4.3)*

Input: $\mathcal{A} := \{A_1, \ldots, A_m\}$, *set relation* $\preceq^l_D$, $H \subset Y$
% initialization
$\mathcal{T} := \{A_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ ***do***
  if $\left(A \preceq^l_D A_j,\ A \in \mathcal{T} \implies A_j \preceq^l_D A + H\right)$
   $\Big(\left(A + H \preceq^l_D A_j,\ A \in \mathcal{T} \implies A_j \preceq^l_D A + H\right)$, *respectively*$\Big)$, ***then***
   $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
  end if
end for
$\{A_1, \ldots, A_p\} := \mathcal{T}$
$\mathcal{U} := \{A_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ ***do***
  if $\left(A \preceq^l_D A_j,\ A \in \mathcal{U} \implies A_j \preceq^l_D A + H\right)$
   $\Big(\left(A + H \preceq^l_D A_j,\ A \in \mathcal{U} \implies A_j \preceq^l_D A + H\right)$, *respectively*$\Big)$, ***then***
   $\mathcal{U} := \mathcal{U} \cup \{A_j\}$
  end if
end for
$\{A_1, \ldots, A_q\} := \mathcal{U}$

$\mathcal{V} := \emptyset$
% final comparison
for $j = 1 : 1 : q$ ***do***
  if $\big(A \preceq^l_D A_j,\ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq^l_D A + H\big)$
  $\Big(\big(A + H \preceq^l_D A_j,\ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq^l_D A + H\big)$, *respectively*$\Big)$, ***then***
    $\mathcal{V} := \mathcal{V} \cup \{A_j\}$
  end if
end for
Output: $\mathcal{V}$

The following lemma will be useful for showing that every element of the set $\mathcal{V}$ is an H^1- (H^2-, respectively) approximate minimal element of $\mathcal{A}$.

Lemma 4.4.5 ([42, Lemma 4.5]). *Let Assumption 4.4.3 be fulfilled. Every element of $\mathcal{U}$ generated by Algorithm 4.4.4 is also an H^1- (H^2-, respectively) approximate minimal element of $\mathcal{U}$.*

Proof. Let $A_j \in \mathcal{U} = \{A_1, \ldots, A_q\}$. By the forward iteration, we obtain

$$\begin{aligned} &\forall\ i < j\ (i \geq 1) : A_i \preceq^l_D A_j \Longrightarrow A_j \preceq^l_D A_i + H \\ &(\forall\ i < j\ (i \geq 1) : A_i + H \preceq^l_D A_j \Longrightarrow A_j \preceq^l_D A_i + H,\ \text{respectively}). \end{aligned}$$

The backward iteration yields

$$\begin{aligned} &\forall\ i > j\ (i \leq q) : A_i \preceq^l_D A_j \Longrightarrow A_j \preceq^l_D A_i + H \\ &(\forall\ i > j\ (i \leq q) : A_i + H \preceq^l_D A_j \Longrightarrow A_j \preceq^l_D A_i + H,\ \text{respectively}). \end{aligned}$$

This means that

$$\begin{aligned} &\forall\ i \neq j\ (1 \leq i \leq q) : A_i \preceq^l_D A_j \Longrightarrow A_j \preceq^l_D A_i + H \\ &(\forall\ i \neq j\ (1 \leq i \leq q) : A_i + H \preceq^l_D A_j \Longrightarrow A_j \preceq^l_D A_i + H,\ \text{respectively}). \end{aligned} \tag{4.6}$$

(4.6) implies that

$$\begin{aligned} &\forall\ A_i \in \mathcal{U} \setminus \{A_j\} : A_i \preceq^l_D A_j \quad \Longrightarrow \quad A_j \preceq^l_D A_i + H \\ &(\forall\ A_i \in \mathcal{U} \setminus \{A_j\} : A_i + H \preceq^l_D A_j \quad \Longrightarrow \quad A_j \preceq^l_D A_i + H,\ \text{respectively}). \end{aligned}$$

Then A_j is an H^1- (H^2-, respectively) approximate minimal element of $\mathcal{U}$. □

Now we are ready to show that all elements in the output $\mathcal{V}$ of Algorithm 4.4.4 are H^1- (H^2-, respectively) approximate minimal elements of $\mathcal{A}$.

Theorem 4.4.6 ([42, Theorem 4.6]). *Let Assumption 4.4.3 be fulfilled. Algorithm 4.4.4 generates exactly all H^1- (H^2-, respectively) approximate minimal elements of $\mathcal{A}$.*

Proof. Let A_j be an arbitrary element in $\mathcal{V}$. Then $A_j \in \mathcal{U}$, as $\mathcal{V} \subseteq \mathcal{U}$, and

$$\begin{aligned} &A \preceq^l_D A_j,\ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq^l_D A + H \\ &(A + H \preceq^l_D A_j,\ A \in \mathcal{A} \setminus \mathcal{U} \implies A_j \preceq^l_D A + H, \text{ respectively}). \end{aligned} \tag{4.7}$$

Suppose that A_j is not H^1- (H^2-, respectively) approximate minimal in $\mathcal{A}$. Then there exists some $A \in \mathcal{A}$ such that $A \preceq^l_D A_j$ ($A + H \preceq^l_D A_j$, respectively) and $A_j \not\preceq^l_D A + H$. If $A \notin \mathcal{U}$, then this is a contradiction to (4.7). If $A \in \mathcal{U}$, then due to the H^1- (H^2- , respectively) approximate minimality of A_j in $\mathcal{U}$ (see Lemma 4.4.5), we obtain from $A \preceq^l_D A_j$ ($A + H \preceq^l_D A_j$, respectively) that $A_j \preceq^l_D A + H$, a contradiction.

Conversely, let A_j be H^1- (H^2-, respectively) approximate minimal in $\mathcal{A}$. Then we get

$$\begin{aligned} &A \preceq^l_D A_j,\ A \in \mathcal{A} \implies A_j \preceq^l_D A + H \\ &(A + H \preceq^l_D A_j,\ A \in \mathcal{A} \implies A_j \preceq^l_D A + H, \text{ respectively}). \end{aligned}$$

Now suppose that $A_j \notin \mathcal{V}$. Thus, there exists some $A \in \mathcal{A} \setminus \mathcal{U}$ with $A \preceq^l_D A_j$ ($A + H \preceq^l_D A_j$, respectively) and $A_j \not\preceq^l_D A + H$. As A_j is H^1- (H^2-, respectively) approximate minimal in $\mathcal{A}$, we get $A_j \preceq^l_D A + H$, a contradiction. □

The next algorithm produces from a family of finitely many sets those that are improved for some other sets in the family in the approximate minimality sense. It is motivated by a similar one introduced in [68] that computes a so-called *antisymmetric subfamily* of $\mathcal{A}$. An *antisymmetric subfamily* of $\mathcal{A}$ is a family of sets $\mathcal{A}^* \subseteq \mathcal{A}$ upon which the set relation $\preceq^l_D$ fulfills the antisymmetry condition (although $\preceq^l_D$ is not antisymmetric itself). In the context of H^1- (H^2-, respectively) approximate minimality, we introduce a similar notion called *proper subfamily* $\mathcal{A}^*$ of $\mathcal{A}$. Later on, we will compute all H^1- (H^2-, respectively) approximate minimal elements of $\mathcal{A}^*$ without making use of the third for-loop in Algorithm 4.4.4.

Algorithm 4.4.7 ([42, Algorithm 4.7])**.** *(Method for finding a proper subfamily $\mathcal{A}^*$ of $\mathcal{A}$)*

Input: $\mathcal{A} := \{A_1, \dots, A_m\}$, *set relation* $\preceq^l_D$, $H \subseteq Y$
% initialization
$\mathcal{A}^* := \emptyset$
% iteration loop
for $i = 1 : 1 : m$ *do*
    if $\nexists\ A \in \mathcal{A} \setminus \{A_i\}$ *such that* $A_i \preceq^l_D A$ *and* $A \preceq^l_D A_i + H$
      ($A_i + H \preceq^l_D A$ *and* $A \preceq^l_D A_i + H$*, respectively)*
        then $\mathcal{A}^* = \mathcal{A}^* \cup \{A_i\}$
    end if
end for
Output: $\mathcal{A}^*$

Let $\mathcal{A}^*$ denote a proper subfamily of $\mathcal{A}$. By $\mathcal{A}^*_{H^1}$ ($\mathcal{A}^*_{H^2}$, respectively), we denote the set of H^1- (H^2-, respectively) approximate minimal elements of $\mathcal{A}^*$.

Now it is our goal to apply the for-loop in Algorithm 4.4.1 backwards in order to obtain a method which determines all H^1- (H^2-, respectively) approximate minimal elements of a family of sets under an external stability assumption.

Definition 4.4.8 ([42, Definition 4.8]). *If for all elements $A \in \mathcal{A} \setminus \mathcal{A}_{H^1}$ ($A \in \mathcal{A} \setminus \mathcal{A}_{H^2}$, respectively) there exists some $\overline{A} \in \mathcal{A}_{H^1}$ ($\overline{A} \in \mathcal{A}_{H^2}$, respectively) with $\overline{A} \preceq^l_D A$ ($\overline{A}+H \preceq^l_D A$, respectively), then $\mathcal{A}$ is called **externally stable** by $\mathcal{A}_{H^1}$ ($\mathcal{A}_{H^2}$, respectively).*

Algorithm 4.4.9 ([42, Algorithm 4.9]). *(Method for finding H^1- (H^2-, respectively) approximate minimal elements of a proper subfamily $\mathcal{A}^*$ of finitely many sets, where $\mathcal{A}^*$ is externally stable by $\mathcal{A}^*_{H^1}$ ($\mathcal{A}^*_{H^2}$, respectively) under Assumption 4.4.3)*

Input: $\mathcal{A}^* := \{A_1, \dots, A_m\}$, *set relation* $\preceq^l_D$, $H \subset Y$
% initialization
$\mathcal{T} := \{A_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ ***do***
    if $\left(A \preceq^l_D A_j,\ A \in \mathcal{T} \implies A_j \preceq^l_D A + H\right)$
    $\Big(\left(A + H \preceq^l_D A_j,\ A \in \mathcal{T} \implies A_j \preceq^l_D A + H\right)$, *respectively*$\Big)$, ***then***
        $\mathcal{T} := \mathcal{T} \cup \{A_j\}$
    end if
end for
$\{A_1, \dots, A_p\} := \mathcal{T}$
$\mathcal{U} := \{A_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ ***do***
    if $\left(A \preceq^l_D A_j,\ A \in \mathcal{U} \implies A_j \preceq^l_D A + H\right)$
    $\Big(\left(A + H \preceq^l_D A_j,\ A \in \mathcal{U} \implies A_j \preceq^l_D A + H\right)$, *respectively*$\Big)$, ***then***
        $\mathcal{U} := \mathcal{U} \cup \{A_j\}$
    end if
end for
Output: $\mathcal{U}$

Theorem 4.4.10 ([42, Theorem 4.10]). *Let Assumption 4.4.3 be fulfilled. Let the set $\mathcal{A}^*$ be a proper subfamily of $\mathcal{A}$ externally stable by the set of H^1- (H^2-, respectively) approximate minimal elements $\mathcal{A}^*_{H^1}$ ($\mathcal{A}^*_{H^2}$, respectively). Then the output $\mathcal{U}$ of Algorithm 4.4.9 consists of exactly all H^1- (H^2-, respectively) approximate minimal elements of the family of sets $\mathcal{A}^*$.*

Proof. Let $\mathcal{U} := \{A_1, \dots, A_q\}$. By part 3 of Theorem 4.4.2, it is clear that all H^1- (H^2-, respectively) approximate minimal elements of $\mathcal{A}^*$ are contained in $\mathcal{T}$ as well as in $\mathcal{U}$.

Conversely, we prove that every element of $\mathcal{U}$ is also an H^1- (H^2-, respectively) approximate minimal element of the set $\mathcal{A}^*$. Let $A_j \in \mathcal{U}$ be arbitrarily chosen. Then, by Lemma 4.4.5, A_j is an H^1- (H^2-, respectively) approximate minimal element of $\mathcal{U}$. Now suppose that A_j is not an H^1- (H^2-, respectively) approximate minimal element in $\mathcal{A}^*$, then $A_j \notin \mathcal{A}^*_{H^1}$ ($A_j \notin \mathcal{A}^*_{H^2}$, respectively). Thus, as $\mathcal{A}^*$ was assumed to be externally stable by $\mathcal{A}^*_{H^1}$ ($\mathcal{A}^*_{H^2}$, respectively), there exists an H^1- (H^2-, respectively) approximate minimal element A (especially, $A \neq A_j$) with the property $A \preceq^l_D A_j$ ($A + H \preceq^l_D A_j$, respectively). Since A is an H^1- (H^2-, respectively) approximate minimal element in $\mathcal{A}^*$, part 3 of Theorem 4.4.2 implies that $A \in \mathcal{U}$. Because A_j is an H^1- (H^2-, respectively) approximate minimal element of $\mathcal{U}$, we get $A_j \preceq^l_D A + H$. But this is a contradiction since $\mathcal{A}^*$ is proper and the proof finishes. □

Example 4.4.11 ([42, Example 4.11]). *We exemplarily demonstrate the usefulness of our proposed algorithms by a numerical example. In Figure 4.3, we have randomly generated 150 single-valued sets in the box* $[0, 50] \times [0, 50] \subset \mathbb{R}^2$. *Let* $D = \mathbb{R}^2_+$ *and* $H = \{(5, 5)^T\}$. *Concerning* H^1*-approximate minimal elements, Algorithm 4.4.1 first generates 20 elements in the set* $\mathcal{T}$*, which are the darkly filled circles in the left image, while the backward iteration in Algorithm 4.4.4 finds 12 elements in the set* $\mathcal{U}$*, denoted with a cross. The third for-loop, i.e., the final comparison, finds that* $\mathcal{U} = \mathcal{V}$.

In the right picture in Figure 4.3, we deal with H^2*-approximate minimal elements by Algorithms 4.4.1 and 4.4.4. We can see all 39 elements in* $\mathcal{T}$ *and 35 elements of the set* $\mathcal{U}$*, which are denoted by a cross. The final for-loop finds that* $\mathcal{U} = \mathcal{V}$*. The minimal elements of the family of single-valued sets are depicted in the lower illustration in Figure 4.3 for comparison. They have been obtained using the standard Jahn-Graef-Younes method (see [53]), where all 14 elements of the set* $\mathcal{T}$ *are plotted in black circles and all 8 minimal elements are noted by a cross.*

Example 4.4.12 ([42, Example 4.12]). *In this example, we compute* H^1*- (*H^2*-, respectively) approximate minimal elements of a family of finitely many starshaped (not necessarily convex) sets, where* $D = \mathbb{R}^2_+$*. Using a radial function, we construct piecewise starshaped sets* $F(x)$*, where* $x \in \mathbb{R}^2$*. First we clarify the notion of piecewise starshapedness. A nonempty compact set* $A \subset \mathbb{R}^m$ *is called* piecewise starshaped*, if there are finitely many compact subsets* $A_1, \ldots, A_r$ *of* A *(with* $r \in \mathbb{N}$*) so that* $A = \bigcup_{i=1}^{r} A_i$ *and every subset* A_i *(with* $i \in \{1, \ldots, r\}$*) is starshaped with respect to some* $\hat{y}^i \in A_i$*, i.e.*

$$\lambda y + (1 - \lambda)\hat{y}^i \in A_i \text{ for all } \lambda \in [0, 1] \text{ and all } y \in A_i.$$

Given a set-valued map $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ *with piecewise starshaped images, it is our goal to select* H^1*- (*H^2*-, respectively) approximate minimal of this family of finitely many sets. To this end, we choose an element* $\hat{x} = (\hat{x}_1, \hat{x}_2)^T \in \mathbb{R}^2$ *and compute its reference point* $\hat{y}(\hat{x})$ *in the objective space. In this example, such an element is given by*

$$\begin{pmatrix} \hat{y}_1(\hat{x}) \\ \hat{y}_2(\hat{x}) \end{pmatrix} = \begin{pmatrix} 2 \cdot \hat{x}_1^2 \\ 2 \cdot \hat{x}_2^2 \end{pmatrix}.$$

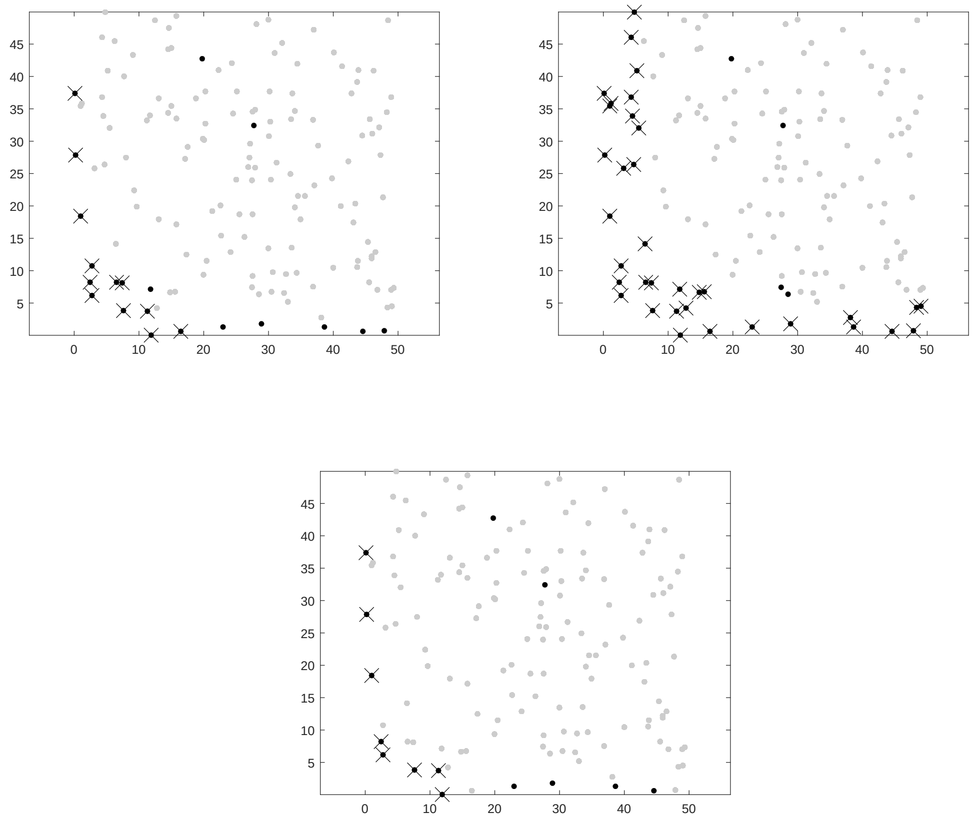

Figure 4.3: Left: H^1-approximate minimal elements. Right: H^2-approximate minimal elements. Below: Minimal elements (see Example 4.4.11).

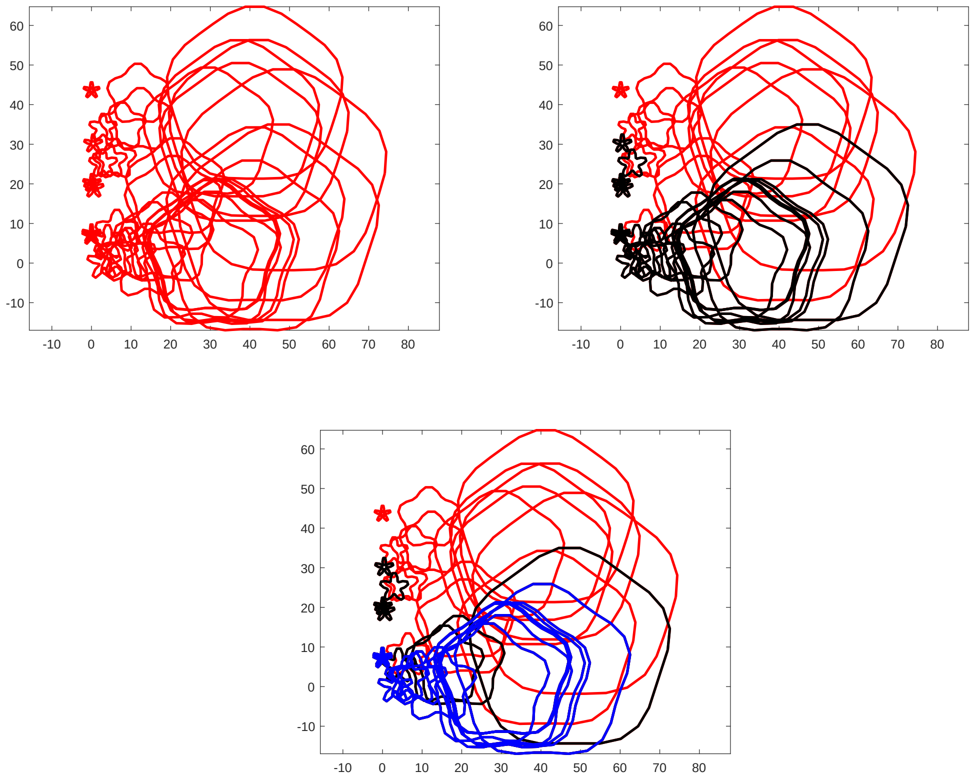

Figure 4.4: H^1-approximate minimal elements of a family of star-shaped sets (see Example 4.4.12).

In a next step, the following radial function is selected:

$$R(x,t) := \frac{9}{x_2+5} \cdot \sin(5 \cdot t) + 1.2 + x_1^2 \quad \text{for } t \in [0, 2\pi].$$

Now we are able to construct star-shaped sets $F(x)$ w.r.t. a reference point $\hat{y}$. In order to discretize the elements on the boundary of the sets $F(x)$, we compute for $x \in \mathbb{R}^2$

$$\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} \hat{y}_1(\hat{x}) + R(x,t) \cdot \cos(t) \\ \hat{y}_2(\hat{x}) + R(x,t) \cdot \sin(t) \end{pmatrix}.$$

At first, 40 reference points are selected randomly and the corresponding sets are constructed (see Figure 4.4). In the left picture, all sets are visible. We first choose $H = \{(5,5)^T\}$. Concerning H^1-approximate minimal elements, 23 elements belong to the set $\mathcal{T}$ generate by Algorithm 4.4.4 (highlighted in black), and all 14 elements which are H^1-approximate minimal are contained in the set $\mathcal{U} = \mathcal{V}$ (depicted in blue color).

There exist 29 H^2-approximate minimal elements, and in this example, we have $\mathcal{T} = \mathcal{U} = \mathcal{V}$ (see Figure 4.5).

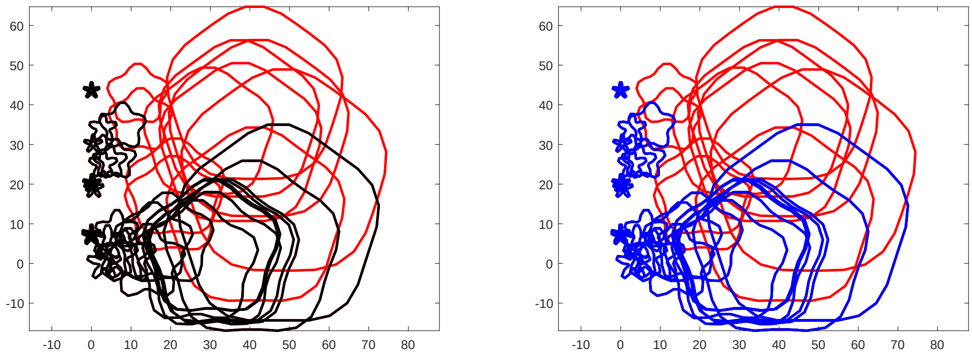

Figure 4.5: H^2-approximate minimal elements of a family of star-shaped sets (see Example 4.4.12).

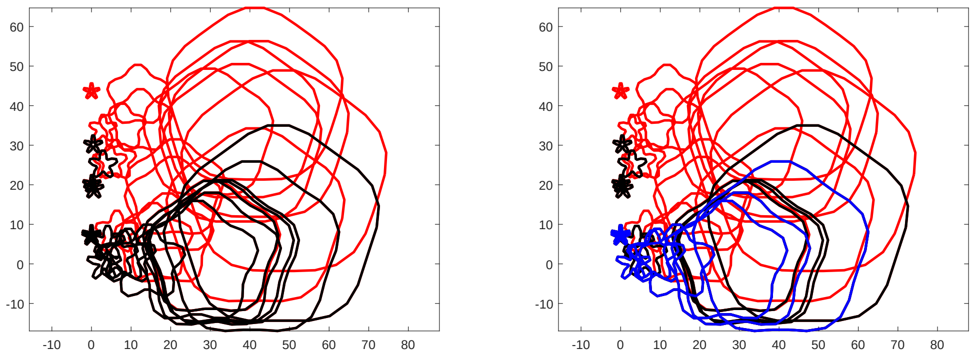

Figure 4.6: Minimal elements of a family of star-shaped sets (see Example 4.4.12), which were obtained using Algorithm 4.4.4.

Now we choose $H = \{(0,0)^T\}$ (see Figure 4.6). Out of all 40 sets, 19 sets are selected within the set $\mathcal{T}$ (in black color). Finally, 9 sets belong to $\mathcal{U} = \mathcal{V}$ (depicted in blue color).

Chapter 5

Application: Unified Approaches to Uncertain Programming

In this chapter, we will present unified concepts to uncertain programming problems based on three approaches, namely, the vector-valued approach, set-valued approach and by using the nonlinear functional $z^{D,k}$ defined in (2.2). Allowing uncertain parameters in optimization problems is extremely important, as most real-world problems are contaminated by uncertainty and the computed solutions can highly depend on it. In particular, using the set-valued approach shows that it is possible to handle a number of concepts from uncertain programming using the theory derived in this thesis. We will describe each approach separately in Section 5.1. The results presented here can, in more detail, be found in Klamroth et al. [61].

Many optimization problems involve uncertainties that are, for example, due to unknown future developments, measurement and/or manufacturing errors, or incomplete information in model development. Uncertainties can be induced by future demands that have to be predicted in order to adapt a production process or the design of a network. In risk theory, assets are naturally affected by uncertainties due to market changes, changing preferences of customers and unforeseeable events. If such uncertainties are not taken into account when solving practical optimization problems, this may result in solutions that perform very poorly under some scenarios, or that are even infeasible in some cases.

Two prominent approaches for handling uncertain optimization problems are *robust optimization* and *stochastic programming*, respectively. In robust optimization it is typically assumed that the uncertain parameters belong to a set that is known prior to solving the optimization problem. The focus lies on looking at the worst case or the worst case regret of a solution, hence no probability distribution for the uncertain data is needed. The goal typically is to ensure that the solution is feasible and performs reasonably well in every possible future scenario, regardless of how likely this scenario may be. Robust optimization problems were introduced by Soyster [100] in 1973 and have been extensively studied in the literature. We refer to Kouvelis and Yu [74] and to Ben-Tal, El Ghaoui and Nemirovski [9] for extensive collections of results. For a survey on recent developments and new robustness concepts, see Goerigk and Schöbel [38].

On the other hand, stochastic programming assumes that the uncertain parameter is probabilistic with known probability distribution. Instead of focusing on the worst case scenario, objective functions are usually based on the expected performance of a solution, or on criteria induced by stochastic dominance. Stochastic programming models often involve two-stage and multi-stage processes that reflect the situation that the knowledge on the realization of the uncertain paramters increases over time. We refer to Birge and Louveaux [13] for a general introduction to stochastic programming.

Several related concepts can be mentioned. Examples are *online optimization*, where decisions have to be made ad hoc in real time and without knowing all problem parameters (see, for example, Grötschel, Krumke and Rambau [39]), and a posteriori approaches like *parametric optimization* (see, for example, Klatte and Kummer [62]).

As a consequence of the fundamental difference in modeling assumptions (probability distribution not known or known, respectively), robust optimization and stochastic programming have mostly been treated separately in the literature. However, an analysis from the perspective of multiobjective optimization reveals that, assuming that the scenario set is finite and defining one objective function for each scenario, both concepts lead to solutions that are nondominated with respect to the same multiobjective counterpart problem. Taking this perspective, Klamroth et al. [60] develop a unifying framework covering both robust and stochastic optimization. Moreover, it is shown that the nonlinear scalarizing functionals introduced in Gerstewitz [32], applied to the multiobjective counterpart problems, induce many of the classical concepts from robust optimization and stochastic programming.

For *finite uncertainty sets*, the interrelation between uncertain scalar optimization problems and associated deterministic multiobjective counterparts has already been discussed in earlier works, see, for example, Hites et al. [48] for a critical evaluation. There often is a particular focus on specific robustness concepts and on transferring ideas from one modeling paradigm to another. As an example, Kouvelis and Sayin [73, 98] transfer methods from uncertain scalar optimization to deterministic multiobjective optimization to derive efficient solution algorithms for the latter. Conversely, multiobjective counterpart problems have been used to derive approximation algorithms for several classes of robust optimization problems, see, for example, Aissi, Bazgan and Vanderpooten [1, 2]. Similarly, multiobjective counterpart problems can be used to motivate new robustness concepts, see, for example, Ogryczak [86, 87, 88], Ogryczak and Śliwiński [89], and Perny et al. [91].

For *infinite uncertainty sets*, a deterministic multiobjective counterpart problem with a finite number of objectives is in general not sufficient to fully represent the problem. It is the goal of this chapter to discuss possible generalizations of the ideas derived in [60] to the infinite dimensional case. We show that concepts from vector optimization can be used in place of multiobjective models, and we analyze the relation to concepts from set-based optimization. Nonlinear scalarizing functionals are again a versatile tool to relate deterministic vector optimization counterparts with many classical concepts from robust optimization and stochastic programming, as has been demonstrated in [61].

Parts of our analysis on deterministic vector optimization counterparts are closely related to recent work of Rockafellar and Royset [93, 94, 95, 96] who investigate decision making under uncertainty in a unified framework involving risk measures. This relation will be discussed in more detail in Remark 5.1.5 below. Deterministic vector optimization counterparts have also been analyzed in Engau [30], who generalizes the concept of proper efficiency to the case of a countably infinite number of objective functions. In this way, weakly efficient solutions can be avoided, not only in the context of uncertain optimization problems. A review on recent developments in robust optimization, also including a discussion on the relation between different modeling paradigms, can be found in Gabrel, Murat and Thiele [31].

We consider optimization problems $(Q(\xi))$ which depend on uncertain parameters $\xi \in \mathcal{U} \subseteq \mathbb{R}^L$. For fixed parameters $\xi \in \mathcal{U}$ (called a *scenario*) the problem to be solved is given as

$$\begin{aligned} & f(x,\xi) \to \inf \\ & \text{s.t. } F_i(x,\xi) \leq 0, \; i = 1,\dots,m, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{$Q(\xi)$}$$

where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \;\; F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}, \; i = 1,\dots,m.$

Let us denote the set of feasible solutions of $(Q(\xi))$ by

$$\mathcal{X}(\xi) = \{x \in \mathbb{R}^n : F_i(x,\xi) \leq 0, \; i = 1,\dots,m\}.$$

We assume that for every fixed scenario $\xi \in \mathcal{U}$ the optimization problem $(Q(\xi))$ has an optimal solution; in particular, $\mathcal{X}(\xi) \neq \emptyset$.

ξ models the parameters which are uncertain. Such uncertainties occur in many real-world optimization problems and can e.g. be caused by measuring errors, modeling assumptions or simply because a future parameter is not known prior to solving an optimization problem. Throughout this work, we assume that the parameters ξ are unknown, but stem from an uncertainty set $\mathcal{U}$ that is nonempty and compact, and that is in general not finite. This is a common assumption in the context of robust optimization. Examples include interval based uncertainties (e.g. [12]), polyhedral uncertainties (e.g. [99]), or ellipsoidal uncertainty sets (e.g. [9]).

An uncertain optimization problem $P(\mathcal{U})$ is defined as a family of parametrized optimization problems

$$(Q(\xi), \xi \in \mathcal{U}). \tag{5.1}$$

We denote by $\hat{\xi} \in \mathcal{U}$ the *nominal value*, i.e., the value of ξ that we believe is true today. $(Q(\hat{\xi}))$ is called the *nominal problem.*

Concepts for handling uncertain optimization problems often rely on the formulation of a *deterministic counterpart* problem in order to identify a most preferred solution under varying modeling assumptions. In the following we illustrate this approach by giving a small example from robust optimization and from stochastic programming, respectively.

Example 5.0.1 (Uncertain linear optimization problem, [61, Example 1])**.** *Consider* $f(x,\xi) := c(\xi)^T x$, *where* $x \in \mathbb{R}^n$ *is the decision variable and* $c(\xi) = c^0 + \sum_{i=1}^{L} \xi_i c^i \in \mathbb{R}^n$

for given $c^i \in \mathbb{R}^n$, $i = 0, \dots, L$ and $\xi \in \mathbb{R}^L$. $\xi \in \mathcal{U}$ is the uncertain parameter and $\mathcal{U} := \{\xi \in \mathbb{R}^L | -1 \leq \xi_i \leq 1,\ i = 1, \dots, L\}$. We would like to solve

$$(c^0 + \sum_{i=1}^{L} \xi_i c^i)^T x \to \inf_{x \in \mathbb{R}^n},$$

where $\xi \in \mathcal{U}$ is unknown, leaving us with an uncertain optimization problem $(Q(\xi),\ \xi \in \mathcal{U})$.

Strictly robust solution. *A prominent robustness concept is to minimize the worst case objective function value, i.e., we aim at solving*

$$\sup_{\xi \in \mathcal{U}} (c^0 + \sum_{i=1}^{L} \xi_i c^i)^T x \to \inf_{x \in \mathbb{R}^n}.$$

Minimizing the expectation. *If a probability distribution over $\mathcal{U}$ is known we can also minimize the expected objective value, i.e., solve*

$$\mathbb{E}\big((c^0 + \sum_{i=1}^{L} \xi_i c^i)^T x\big) \to \inf_{x \in \mathbb{R}^n}.$$

In this chapter, we propose a unifying framework for uncertain optimization problems. We choose *strict robustness*, *optimistic robustness*, *regret robustness*, *reliability* and *adjustable robustness* as examples, and refer to [61] for a detailed analysis on a larger number of concepts for uncertainty. In [61], it is also shown that our framework motivates the definition of new concepts, and some that might be useful in practice are presented there. As an example, we introduce here the concept *certain robustness* as a novel approach to uncertain programming using set optimization techniques.

5.1 Three Unifying Concepts for Uncertain Optimization

In this section we present the basic ideas which allow for a unified treatment of concepts from robust optimization and stochastic programming.

5.1.1 Vector Optimization as Unifying Concept

The first concept generalizes the idea of multiobjective counterpart problems, formulated for finite uncertainty sets in Klamroth et al. [60], to the case of infinite uncertainty sets. The underlying idea of this approach is not new, see Rockafellar and Royset [93, 94, 95, 96] or Engau [30], but it has never been studied as broad as we will do in Section 5.2 and in [61].

If $\mathcal{U} = \{\xi^1, \dots, \xi^q\}$ is finite, each scenario can be interpreted as an objective function. For a solution $x \in \mathbb{R}^n$ we then obtain a vector $F_x \in \mathbb{R}^q$ which contains $f(x, \xi^i)$ in its ith coordinate. In order to compare two solutions x and y, order relations for the vectors F_x and F_y are used. In this way, many concepts of robust optimization and of stochastic programming can be characterized using multiobjective counterpart problems, see [60]. If $\mathcal{U}$ is not a finite set, we obtain not vectors but functions, i.e., $F_x : \mathcal{U} \to \mathbb{R}$ where $F_x(\xi) = f(x, \xi)$ contains the objective function value of x in scenario ξ, $\xi \in \mathcal{U}$. In order to compare two solutions x and y, we hence need order relations in the real linear functional space $\mathbb{R}^{\mathcal{U}}$ of all mappings $F : \mathcal{U} \to \mathbb{R}$.

In order to describe the vector optimization approach formally, let $(Q(\xi), \xi \in \mathcal{U})$ be the given uncertain optimization problem. Let $Y = \mathbb{R}^{\mathcal{U}}$ be the space of all functions $F : \mathcal{U} \to \mathbb{R}$. For a fixed solution $x \in \mathbb{R}^n$ we define

$$F_x \in Y : \quad F_x(\xi) := f(x, \xi).$$

In order to compare elements of Y, we consider different order relations on the space Y, which are denoted by $\leq$. In the context of vector optimization, (partial) order relations can, for example, be defined based on cones: Let $C \subseteq Y$ be a proper ($C \neq Y$ and $C \neq \{0\}$), closed, convex, and pointed cone. Such a cone C induces an order relation $\leq := \leq_C$ by

$$y_1 \leq_C y_2 \;:\Longleftrightarrow\; y_1 \in y_2 - C \quad (\Longleftrightarrow y_2 \in y_1 + C)$$

see, for example, [53], or (1.7). Whenever we are working with the interior of an ordering cone, we assume that $Y = C(\mathcal{U}, \mathbb{R})$, i.e., that the functions $F_x = f(x, \xi)$ are continuous in ξ for all feasible values of x. A particular order relation which is of interest later on is given in the next definition.

Definition 5.1.1 ([61, Definition 1]). *The* natural order relation $\leq_{Y^+}$ *(see* (1.7) *with $C = Y^+$) is given by the cone*

$$Y^+ := \{F \in Y | \forall \xi \in \mathcal{U} : \; F(\xi) \geq 0\}$$

inducing for all $F, G \in Y$ that

$$\begin{aligned} F \leq_{Y^+} G \quad &\Longleftrightarrow \quad G \in F + Y^+ \\ &\Longleftrightarrow \quad F(\xi) \leq G(\xi) \textit{ for all } \xi \in \mathcal{U}. \end{aligned}$$

In the following it will be important to identify (weakly) minimal elements in subsets of Y (compare Definition 1.2.5). To this end, let $\mathcal{F}$ be a nonempty subset of Y.

Definition 5.1.2. *Let $\mathcal{F} \subseteq Y$ and let $\leq$ be an order relation on Y. $F \in \mathcal{F}$ is a* ***minimal*** *element of $\mathcal{F}$ in Y w.r.t. $\leq$ if*

$$\textit{for all } G \in \mathcal{F} : \; G \;\leq\; F \implies F \;\leq\; G.$$

Moreover, if $\leq$ is induced by a proper closed convex cone C in Y with $\operatorname{int}(C) \neq \emptyset$*, then we set $\leq =: \leq_C$ and call $F \in \mathcal{F}$ a* ***weakly minimal*** element *of $\mathcal{F}$ in Y w.r.t. $\leq_C$ if*

$$(F - \operatorname{int}(C)) \cap \mathcal{F} = \emptyset.$$

Note that if $\leq$ is induced by a cone C, we define $\leq_C:=\leq$. Then an element $F \in \mathcal{F}$ is a minimal element of $\mathcal{F}$ in Y w.r.t. $\leq_C$ if and only if $(F-C)\cap\mathcal{F} \subseteq F+C$. Moreover, if C is a proper closed convex cone with $\operatorname{int}(C) \neq \emptyset$, then minimality implies weak minimality.

Given an order relation $\leq$ and a set $\mathcal{F}$, the *vector optimization problem* asks for minimal elements of $\mathcal{F}$ in Y w.r.t. $\leq$. According to [61], many concepts for uncertain optimization can be interpreted as solving such a vector optimization problem, and conversely, every order $\leq$ induces a concept for handling uncertainty. While not all such concepts necessarily have a meaningful interpretation in the context of uncertain optimization, this relationship provides a systematic means of devising and understanding deterministic counterparts of an uncertain optimization problem.

Example 5.1.3 ([61, Example 2]). *In the case of the natural order relation $\leq_{Y^+}$ of Y introduced in Definition 5.1.1, an element $F \in \mathcal{F}$ is a minimal element of $\mathcal{F}$ in Y w.r.t. $\leq_{Y^+}$ if and only if*

$$\nexists G \in \mathcal{F} \setminus \{F\} : \ \forall \xi \in \mathcal{U} : \ (G-F)(\xi) \leq 0.$$

If $Y = C(\mathcal{U}, \mathbb{R})$, then $\operatorname{int}(Y^+) = \{F \in Y | \forall \xi \in \mathcal{U} : \ F(\xi) > 0\}$*, and an element $F \in \mathcal{F}$ is a weakly minimal element of $\mathcal{F}$ in Y w.r.t. $\leq_{Y^+}$ if and only if*

$$\nexists G \in \mathcal{F} : \ \forall \xi \in \mathcal{U} : \ (G-F)(\xi) < 0. \tag{5.2}$$

Remark 5.1.4 ([61, Remark 1]). *Note that the concept of minimality introduced above is also known as* (Edgeworth) Pareto *minimality in the context of vector or multi-objective optimization. In particular, a solution which is* minimal with respect to the natural order relation $\leq_{Y^+}$ *is often called a* Pareto solution.

Remark 5.1.5 ([61, Remark 26]). *As mentioned before, the general idea of the vector optimization approach to uncertain optimization problems is not new. However, to the best of our knowledge it has not been applied at the same level of broadness in the literature. We mention the related works of Rockafellar and Royset [93, 94, 95, 96] who suggest a similar unifying concept for handling uncertainty in a decision making process. By interpreting the uncertain outcome F_x of a solution $x \in \mathbb{R}^n$ as a random variable, they show that many concepts from robust optimization and stochastic programming can be represented by risk measures. More precisely, let Y be a space of random variables. A measure of risk is a functional $R : Y \to \mathbb{R} \cup \{\pm\infty\}$ that assigns to a response or cost random variable y a number $R(y)$ as a quantification of the risk in y. An examination of superquantiles and their broad applications to risk and random variables is given in [93]. In [94], the authors present how risk measures provide an enlarged set of models for handling uncertainty, covering for example worst case optimization (see Section 5.2) and expected value minimization. In [95], the definition of* measures of residual risk *is introduced, which extends the notion of risk measures by considering an additional random variable. This approach is motivated by tradeoffs detected by forecasters and investors,* *and it provides profound connections to regression, surrogate models and distributional robustness. Other risk measures considered in [93, 94, 95, 96] include quantiles, safety margins, superquantiles and utility functions.*

In the recent work of Engau [30], uncertain optimization problems with a countably infinite uncertainty set are considered, and a vector optimization problem is formulated as a deterministic counterpart also in this case. This counterpart problem is then used to apply generalized concepts of proper efficiency in the context of uncertain optimization problems in order to avoid weakly efficient solutions.

5.1.2 Set-based Optimization as Unifying Concept

In this section, we derive an approach to uncertain scalar optimization based on set-valued optimization. In the literature, there already exist some concepts for uncertain vector (or multi-objective) optimization that deal with concepts from set-valued optimization. Ide et al. [51] investigate uncertain vector-valued optimization problems and relationships to set order relations (compare also Ide, Köbis [50] and Crespi et al. [18]). They define robust solutions to families of uncertain vector-valued problems and use techniques from set optimization. Specifically, in [18] the authors use an embedding approach from set optimization as a vectorization, that is, to transfer the set optimization problem into a vector optimization problem. Moreover, well-posedness by means of certain convexity notions on the objective map is studied in [18]. In this chapter, we focus on uncertain *scalar* optimization problems with infinite uncertainty set and their correspondence to vector-valued, set-valued and scalarizing counterparts. The relation to these counterparts is naturally derived by means of the uncertainty set. The counterparts are then deterministic. The set-based unifying concept, which will be described in this section, is a special case of the approach considered in [18] for uncertain scalar optimization problems.

In our second approach we associate a solution x with the set of possible objective values which can occur if x is chosen. These objective values are given by

$$B_x := f(x,\mathcal{U}) := \{f(x,\xi)\,|\,\xi \in \mathcal{U}\} \subseteq \mathbb{R}.$$

In order to compare two solutions x and y in this setting we have to define order relations between their corresponding sets B_x and B_y.

Let $Z := \overline{\mathcal{P}}(\mathbb{R})$ be the set of all subsets of $\mathbb{R}$. For a given $x \in \mathbb{R}^n$ the set $B_x \in Z$ is hence the image of the mapping F_x under $\mathcal{U}$. Note that $B_x \subseteq \mathbb{R}$ is an interval for example in case that $f(x,\cdot)$ is a continuous function on a convex uncertainty set $\mathcal{U}$.

In order to compare elements of Z we consider set order relations, which we denote by $\preceq$. Many examples for set order relations can be found, for example, in [59] and [26].

Example 5.1.6 ([61, Example 3])**.** *An example for a set order relation is the* lower-type set-relation *introduced in [76, 77, 79] and defined as follows (see also Definition 2.2.9): Let $A, B \in Z$ be nonempty sets. Then*

$$\begin{aligned} A \preceq^l_{\mathbb{R}_+} B \quad &:\Longleftrightarrow \quad B \subseteq A + \mathbb{R}_+ \\ &\Longleftrightarrow \quad \inf A \leq \inf B. \end{aligned} \tag{5.3}$$

We are again interested in finding minimal elements of Z. To this end, let $\mathcal{B}$ be a nonempty subset of Z. Given an order relation $\preceq$ and a set $\mathcal{B} \subseteq Z$, *set-valued optimization* asks for minimal elements (in the sense of Definition 1.2.11) of $\mathcal{B}$ in Z w.r.t. $\preceq$. It is reported in [61] that some (but not all) concepts for uncertain optimization can be interpreted as solving such a set-valued optimization problem, and that every set order relation $\preceq$ induces a concept for handling uncertainty.

In the special case $\mathcal{U} = \{\xi_1, \dots, \xi_q\}$ ($q \in \mathbb{N}$) we obtain that

$$B_x = \{f(x,\xi) \,|\, \xi \in \mathcal{U}\} = \{f(x,\xi_1), \dots, f(x,\xi_q)\} \subseteq \mathbb{R}$$

is a set of finitely many elements in $\mathbb{R}$.

5.1.3 The Nonlinear Scalarizing Functional as Unifying Concept

Many concepts of robust and stochastic optimization can also be interpreted using the following nonlinear scalarizing functional for general vector optimization problems. Intuitively, whenever a robust counterpart problem generates (weakly) minimal elements of an associated vector optimization problem in the sense of Section 5.1.1, these solutions can also be generated using appropriate scalarizing functionals. Depending on the properties of the involved parameters, these scalarizing functionals possess a variety of useful properties, especially monotonicity and continuity properties. Consequently, this gives rise to a third unifying framework.

Let Y be a linear space, $k \in Y \setminus \{0\}$ and let $\mathcal{F}$ be a nonempty, proper subset of Y (denoting the set of feasible elements of Y). In the following, we are particularly interested in the special case that $Y = \mathbb{R}^{\mathcal{U}}$ is the linear space of all real-valued functions $F : \mathcal{U} \to \mathbb{R}$, c.f. Section 5.1.1. Now let B be a closed proper subset of Y satisfying

$$B + [0, +\infty) \cdot k \subseteq B \tag{5.4}$$

and introduce the functional $z^{B,k} : Y \to \overline{\mathbb{R}}$ (compare Chapter 2.1),

$$z^{B,k}(y) := \inf\{t \in \mathbb{R} | y \in tk - B\}. \tag{5.5}$$

An important question is whether the nonlinear scalarizing functional $z^{B,k}$ can be used as a tool to characterize solutions of the robust counterpart problems. For a finite uncertainty set, the answer to this question is positive, as discussed in [63]. Using the scalarizing functional we can define the following minimization problem, which will be used later on to represent concepts of robust and stochastic optimization.

Definition 5.1.7 ([61, Definition 4])**.** *Let $\mathcal{F} \subseteq Y$ and let $z^{B,k}$ be defined as in* (5.5)*. An element $F \in \mathcal{F}$ is a **minimal** element of $\mathcal{F}$ in Y w.r.t. $z^{B,k}$ if*

$$\forall G \in \mathcal{F} : \; z^{B,k}(F) \leq z^{B,k}(G),$$

i.e., F solves the scalar optimization problem

$$z^{B,k}(F) \to \inf_{F \in \mathcal{F}}. \tag{$P_{k,B,\mathcal{F}}$}$$

We remark that many scalarization concepts that are suggested in the literature are special cases of the above nonlinear scalarization concept. For example, in the case of (finite-dimensional) multiobjective optimization, this scalarization method comprises weighted-sum, Tschebyscheff- and ϵ-constraint-scalarizations, and many others.

The functional has many interesting properties, some of which we collect below in the case that Y is a linear topological space and B is a proper closed convex cone in Y with nonempty interior and $k \in \operatorname{int} B$ (see Theorem 2.1.2 for the more general case when B is not necessarily a proper closed convex cone with nonempty interior).

Lemma 5.1.8 ([37]). *Let Y be a linear topological space, B be a proper closed convex cone in Y with nonempty interior and $k \in \operatorname{int} B$. Then $z = z^{B,k}$, defined by* (5.5)*, is a finite-valued continuous sublinear and strictly* $(\operatorname{int} B)$*-monotone functional such that*

$$\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y) \leq r \Longleftrightarrow y \in rk - B,$$
$$\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y) < r \Longleftrightarrow y \in rk - \operatorname{int} B.$$

5.2 Strict Robustness

The three general concepts introduced in Section 5.1 allow for a unified treatment of a large variety of models from robust optimization and stochastic programming. In the following, we exemplarily review the classical and most prominent concept, namely *strict robustness*, and interpret this model in terms of vector optimization, set-based optimization and using nonlinear scalarizing functionals. A similar analysis is performed for a wide range of concepts from uncertain programming in [61]. As reported in [61], it turns out that, under relatively mild assumptions, solutions that are optimal for robust optimization or stochastic programming models are typically obtained as (weakly) minimal solutions of an appropriately formulated deterministic vector optimization counterpart. Similarly, nonlinear scalarizing functionals, which yield (weakly) minimal solutions of the respective vector optimization counterparts, can be applied to achieve similar results.

Strict robustness (also called *minmax robustness*) has been introduced by Soyster [100] and extensively researched since then, see Ben-Tal et al. [9] for a collection of results on various uncertainty sets. Strict robustness is a conservative concept in which a robust solution is required to be feasible for every scenario $\xi \in \mathcal{U}$. In the objective function one considers the worst case. Formally, the **strictly robust counterpart** (RC) of the uncertain optimization problem $(Q(\xi), \xi \in \mathcal{U})$ is defined by

$$\begin{aligned} &\rho_{\mathrm{RC}}(x) = \sup_{\xi \in \mathcal{U}} f(x,\xi) \to \inf \\ &\text{s.t. } \forall \xi \in \mathcal{U}:\ F_i(x,\xi) \leq 0,\ i = 1, \dots, m, \\ &\qquad x \in \mathbb{R}^n. \end{aligned} \tag{RC}$$

A feasible solution to (RC) is called *strictly robust* and we denote the set of strictly robust solutions by

$$\mathfrak{A}_{\text{strict}} := \{x \in \mathbb{R}^n |\ \forall\ \xi \in \mathcal{U}:\ F_i(x,\xi) \leq 0,\ i = 1, \dots, m\}.$$

5.2.1 Vector Optimization Approach for Strict Robustness

The strictly robust counterpart (RC) can be formulated as a vector optimization problem in the functional space $Y = \mathbb{R}^{\mathcal{U}}$ as follows. We denote the set of *strictly robust outcome functions* in Y by

$$\mathcal{F}_{\text{strict}} := \{F_x \in Y \mid x \in \mathfrak{A}_{\text{strict}}\}. \tag{5.6}$$

Let two functions $F_x, F_y \in Y$ be given. We consider the following order relation on Y:

$$F_x \ \leq^{\text{sup}} \ F_y :\Longleftrightarrow \sup_{\xi \in \mathcal{U}} F_x(\xi) \leq \sup_{\xi \in \mathcal{U}} F_y(\xi).$$

In the special case of a finite uncertainty set $\mathcal{U} = \{\xi_1, \ldots, \xi_q\}$, $q \in \mathbb{N}$, $\leq^{\text{sup}}$ corresponds to the max-order relation in multiobjective optimization (see, for example, Ehrgott [21]). We will thus refer to $\leq^{\text{sup}}$ as the *sup-order relation* in the following. As in the finite dimensional case, the sup-order relation $\leq^{\text{sup}}$ is not compatible with addition, i.e., for three elements $F_x, F_y, F_z \in Y$, $F_x \leq^{\text{sup}} F_y$ does not necessarily imply $(F_x + F_z) \leq^{\text{sup}} (F_y + F_z)$. Consequently, $\leq^{\text{sup}}$ cannot be represented by an ordering cone. Nevertheless, it has the following properties.

Remark 5.2.1 ([61, Remark 2])**.** *$\leq^{\text{sup}}$ is reflexive and transitive. Furthermore, $\leq^{\text{sup}}$ is a total preorder.*

The order relation $\leq^{\text{sup}}$ allows to represent the strictly robust optimization problem as a vector optimization problem.

Theorem 5.2.2 ([61, Theorem 1])**.** *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (RC) *if and only if F_x is a minimal element of $\mathcal{F}_{\text{strict}}$ with respect to the sup-order relation $\leq^{\text{sup}}$.*

Proof. Let $x \in \mathfrak{A}_{\text{strict}}$. Then

$$\begin{aligned} x \text{ is an optimal solution to (RC)} \quad &\Longleftrightarrow \quad \sup_{\xi \in \mathcal{U}} f(x, \xi) \leq \sup_{\xi \in \mathcal{U}} f(\overline{x}, \xi) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}} \\ &\Longleftrightarrow \quad \sup_{\xi \in \mathcal{U}} F_x(\xi) \leq \sup_{\xi \in \mathcal{U}} F_{\overline{x}}(\xi) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}} \\ &\Longleftrightarrow \quad F_x \leq^{\text{sup}} F_{\overline{x}} \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}, \\ &\Longleftrightarrow \quad F_x \leq^{\text{sup}} G \text{ for all } G \in \mathcal{F}_{\text{strict}}, \end{aligned}$$

and the result follows since $\leq^{\text{sup}}$ is a total preorder. □

This means that optimal solutions of the strictly robust counterpart (RC) correspond to outcome functions whose suprema are minimal.

We now analyze the relation between the sup-order relation $\leq^{\text{sup}}$ and the natural order relation $\leq_{Y^+}$ introduced in Definition 5.1.1.

Remark 5.2.3 ([61, Remark 3])**.** *$F \leq_{Y^+} G \Longrightarrow F \ \leq^{\text{sup}} \ G$ for $F, G \in Y$.*

However, this does in general not imply that every minimal element w.r.t. $\leq^{\sup}$ is also a minimal element w.r.t. $\leq_{Y^+}$, or vice versa, or, in other words, an optimal solution to (RC) need not be a Pareto solution, or vice versa. Under some additional assumptions, Iancu and Trichakis [49] have shown that there exist optimal solutions to (RC) which are Pareto, and call them *PRO* robust solutions.

The only general relation on (Pareto) minimal elements is the following:

Lemma 5.2.4 ([61, Lemma 2]). *Let $Y = C(\mathcal{U}, \mathbb{R})$. Assume that every $F \in \mathcal{F}_{\text{strict}}$ attains its supremum on $\mathcal{U}$. If $F \in \mathcal{F}_{\text{strict}}$ is a minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. $\leq^{\sup}$, then F is a weakly minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. the natural order relation $\leq_{Y^+}$.*

Proof. Let $F \in \mathcal{F}_{\text{strict}}$ be a minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. $\leq^{\sup}$. Since $\leq^{\sup}$ is a total preorder, this means that

$$\sup_{\xi \in \mathcal{U}} F(\xi) \leq \sup_{\xi \in \mathcal{U}} G(\xi) \text{ for all } G \in \mathcal{F}_{\text{strict}}. \tag{5.7}$$

Now suppose that F is not a weakly minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. the natural order relation $\leq_{Y^+}$ of Y. Thus, there exists $G \in \mathcal{F}_{\text{strict}}$ s.t.

$$\forall\, \xi \in \mathcal{U}: \; G(\xi) < F(\xi),$$

see, (5.2). Since G attains its supremum on $\mathcal{U}$, this means that

$$\sup_{\xi \in \mathcal{U}} G(\xi) = G(\bar{\xi}) < F(\bar{\xi}) \leq \sup_{\xi \in \mathcal{U}} F(\xi),$$

with some $\bar{\xi} \in \mathcal{U}$, a contradiction to (5.7). $\square$

Using this relation together with Theorem 5.2.2 we obtain that F_x is weakly Pareto minimal for all optimal solutions x to (RC).

Corollary 5.2.5 ([61, Corollary 1]). *Let $Y = C(\mathcal{U}, \mathbb{R})$ and let the worst case be attained for every solution $x \in \mathfrak{A}_{\text{strict}}$. Then for every optimal solution x to the strictly robust counterpart* (RC)*, F_x is a weakly minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. the natural order relation $\leq_{Y^+}$ in Y.*

The following example illustrates the preceding results. Other concepts of robustness, as presented in [61], can be discussed analogously.

Example 5.2.6 ([61, Example 4]). *In many applications in mathematical finance, the risk is to be minimized. Especially one could use the variance as risk measure such that one has an uncertain quadratic optimization problem of the following type. We consider the uncertain quadratic optimization problem with linear constraints*

$$\begin{aligned} & x^T A(\xi) x \to \inf \\ & \text{s.t. } (D(\xi)x - d(\xi))_i \leq 0, \; i = 1, \ldots, m, \\ & \qquad x \in \mathbb{R}^n \end{aligned} \tag{5.8}$$

where $A(\xi) \in \mathbb{R}^{(n,n)}$ is the covariance matrix, which is assumed to be positive definite, $D(\xi) \in \mathbb{R}^{(m,n)}$, $d(\xi) \in \mathbb{R}^m$, and $\xi \in \mathcal{U}$ for a given uncertainty set $\mathcal{U}$. The strictly robust counterpart of (5.8) *reads*

$$\begin{aligned} &\sup_{\xi\in\mathcal{U}} x^T A(\xi)x \to \inf \\ &\text{s.t. } \forall \xi \in \mathcal{U}: \ (D(\xi)x - d(\xi))_i \leq 0, \ i = 1,\dots,m, \\ &\qquad x \in \mathbb{R}^n. \end{aligned} \tag{5.9}$$

For $x \in \mathbb{R}^n$ let $F_x(\xi) := x^T A(\xi)x$ and $\mathcal{F}_{\text{strict}} := \{F_x \in \mathbb{R}^{\mathcal{U}} \mid \forall \xi \in \mathcal{U}: \ (D(\xi)x - d(\xi))_i \leq 0, \ i = 1,\dots,m\}$. Theorem 5.2.2 says that $x \in \mathbb{R}^n$ is an optimal solution to (5.9) *if and only if F_x is a minimal element of $\mathcal{F}_{\text{strict}}$ with respect to the sup-order relation $\leq^{\sup}$. Moreover, if we assume $Y = C(\mathcal{U},\mathbb{R})$ (that means every function F_x is continuous in ξ for each $x \in \mathbb{R}^n$), then Lemma 5.2.4 states the following. Assume that every $F \in \mathcal{F}_{\text{strict}}$ attains its supremum on $\mathcal{U}$. If $F_x \in \mathcal{F}_{\text{strict}}$ is a minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. $\leq^{\sup}$ (that means that x is an optimal solution to problem* (5.9)*), then F_x is a weakly minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. the natural order relation $\leq_{Y^+}$. That means that for computing weakly minimal solutions of $\mathcal{F}_{\text{strict}}$ w.r.t. $\leq_{Y^+}$, we can make use of the scalar problem* (5.9) *(see Corollary 5.2.5).*

5.2.2 Set-Valued Optimization Approach for Strict Robustness

In this section we interpret the strictly robust counterpart (RC) as a set-valued optimization problem. We denote the set of *strictly robust outcome sets* in the power set $Z = \overline{\mathcal{P}}(\mathbb{R})$ by

$$\mathcal{B}_{\text{strict}} := \{B_x \in Z \mid x \in \mathfrak{A}_{\text{strict}}\}.$$

For $B_x, B_y \in Z$, the *upper-type set-relation* $\preceq^u_{\mathbb{R}_+}$ is defined as

$$\begin{aligned} B_x \ \preceq^u_{\mathbb{R}_+} \ B_y \quad &:\Longleftrightarrow \quad B_x \subseteq B_y - \mathbb{R}_+ \\ &\Longleftrightarrow \quad \sup B_x \leq \sup B_y, \end{aligned}$$

see Kuroiwa [76, 77] and Kuroiwa et al. [79] (compare also Definition 2.2.1).

Remark 5.2.7 ([61, Remark 4])**.** *$\preceq^u_{\mathbb{R}_+}$ is reflexive and transitive. Furthermore, it is a total preorder.*

We obtain the following relation between $\preceq^u_{\mathbb{R}_+}$ and $\leq^{\sup}$.

Lemma 5.2.8 ([61, Lemma 3])**.** *Let $x, y \in \mathbb{R}^n$ and let F_x, F_y their corresponding outcome functions and B_x, B_y their corresponding outcome sets. Then*

$$B_x \preceq^u_{\mathbb{R}_+} B_y \Longleftrightarrow F_x \leq^{\sup} F_y.$$

Proof.

$$\begin{aligned} B_x \ \preceq^u_{\mathbb{R}_+} \ B_y \ &\Longleftrightarrow \sup\ B_x \ \leq \ \sup\ B_y \\ &\Longleftrightarrow \sup\{F_x(\xi) \mid \xi \in \mathcal{U}\} \leq \sup\{F_y(\xi) \mid \xi \in \mathcal{U}\} \\ &\Longleftrightarrow F_x \leq^{\sup} F_y. \end{aligned}$$

□

The order relation $\preceq^u_{\mathbb{R}_+}$ allows to represent the strictly robust optimization problem as a set-valued optimization problem.

Theorem 5.2.9 ([61, Theorem 2]). *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (RC) *if and only if B_x is a minimal element of $\mathcal{B}_{\text{strict}}$ w.r.t. the order relation $\preceq^u_{\mathbb{R}_+}$.*

Proof. We know from Theorem 5.2.2 that $x \in \mathfrak{A}_{\text{strict}}$ is an optimal solution to (RC) if and only if $F_x \leq^{\sup} F_{\overline{x}}$ for all $\overline{x} \in \mathfrak{A}_{\text{strict}}$. According to Lemma 5.2.8 this is equivalent to $B_x \preceq^u_{\mathbb{R}_+} B_{\overline{x}}$ for all $\overline{x} \in \mathfrak{A}_{\text{strict}}$ and the result follows. □

Example 5.2.10 ([61, Example 5]). *We return to the uncertain quadratic optimization problem* (5.9) *that we discussed in Example 5.2.6. By defining $\mathcal{B}_{\text{strict}} := \{B_x \in Z \mid \forall \xi \in \mathcal{U} : \ (D(\xi)x - d(\xi))_i \leq 0, \ i = 1, \dots, m\}$, we are able to define solutions of* (5.9) *as minimal elements of a set-valued optimization problem. Theorem 5.2.9 says that for every optimal solution $x \in \mathbb{R}^n$ of* (5.9)*, B_x is a minimal element w.r.t. $\preceq^u_{\mathbb{R}_+}$, and vice versa.*

5.2.3 Nonlinear Scalarizing Functional for Strict Robustness

We finally represent the strictly robust counterpart (RC) using the nonlinear scalarizing functional (5.5) introduced in Section 5.1.3. Our basic result again holds for the general case that $Y = \mathbb{R}^{\mathcal{U}}$.

Theorem 5.2.11 ([61, Theorem 3]). *Let $Y = \mathbb{R}^{\mathcal{U}}$, $B := Y^+$, and $k :\equiv 1 \in Y$. Then $x \in \mathbb{R}^n$ is an optimal solution to* (RC) *if and only if F_x solves problem* $(P_{k,B,\mathcal{F}})$ *with $\mathcal{F} = \mathcal{F}_{\text{strict}}$.*

Proof. $B + [0, +\infty) \cdot k \subseteq B$ holds, thus inclusion (5.4) is satisfied and the functional $z^{B,k}$ can be defined. Furthermore, we have

$$\begin{aligned}
z^{B,k}(F_x) &= \inf\{t \in \mathbb{R} | F_x \in tk - B\} \\
&= \inf\{t \in \mathbb{R} | F_x - tk \in -Y^+\} \\
&= \inf\{t \in \mathbb{R} | \forall \xi \in \mathcal{U} : \ F_x(\xi) \leq t\} \\
&= \sup_{\xi \in \mathcal{U}} f(x, \xi).
\end{aligned}$$

Thus, F_x is minimal for $(P_{k,B,\mathcal{F}_{\text{strict}}})$ if and only if $x \in \mathfrak{A}_{\text{strict}}$ minimizes $\sup_{\xi \in \mathcal{U}} f(x, \xi)$, i.e., if and only if x is an optimal solution to (RC). □

Remark 5.2.12 ([61, Remark 5]). *If $Y = C(\mathcal{U}, \mathbb{R})$, we have the following properties. Since $B = Y^+$ is a proper closed convex cone and $k \in \operatorname{int}(Y^+)$, Lemma 5.1.8 implies that the functional $z^{B,k}$ is continuous, finite-valued, Y^+-monotone, strictly* $(\operatorname{int} Y^+)$*-monotone and sublinear, and*

$$\begin{aligned}
&\forall \ F_x \in Y, \ \forall \ t \in \mathbb{R} : \ \ z^{B,k}(F_x) \leq t \iff F_x \in tk - Y^+, \\
&\forall \ F_x \in Y, \ \forall \ t \in \mathbb{R} : \ \ z^{B,k}(F_x) < t \iff F_x \in tk - \operatorname{int}(Y^+).
\end{aligned}$$

Note that in the special case of a discrete uncertainty set $\mathcal{U} = \{\xi_1, \dots, \xi_q\}$, Theorem 5.2.11 simplifies to

$$\min_{F_x \in \mathcal{F}_{\text{strict}}} z^{\overline{B},\overline{k}}(F_x) = \min_{x \in \mathfrak{A}_{\text{strict}}} \max_{\xi \in \mathcal{U}} f(x, \xi)$$

with $\overline{B} := \mathbb{R}^q_+$ and $\overline{k} := (1, \dots, 1)^T$. This is equivalent to a reference point approach of Wierzbicki [107] using the origin as reference point, and in the case that $f(x, \xi) \geq 0$ for all $\xi \in \mathcal{U}$ and $x \in \mathfrak{A}_{\text{strict}}$, to a weighted Tchebycheff scalarization, see Steuer and Choo [101], (with equal weights) applied to the corresponding multiobjective optimization problem

$$\operatorname*{vmin}_{x \in \mathfrak{A}_{\text{strict}}} \; (f(x, \xi_1), \dots, f(x, \xi_q)),$$

where "vmin" is to be understood in the sense of Definition 5.1.2 with an order relation $\leq_{\mathbb{R}^q_+}$ induced by the natural ordering cone $\mathbb{R}^q_+$ in $\mathbb{R}^q$

$$y^1 \;\leq_{\mathbb{R}^q_+}\; y^2 \;:\Longleftrightarrow\; y^2 \in y^1 + \mathbb{R}^q_+$$

for all $y^1, y^2 \in \mathbb{R}^q$.

Example 5.2.13 ([61, Example 6])**.** *We again consider the strictly robust quadratic optimization problem with linear constraints* (5.9)*. We use the same notation as in Example 5.2.6. By Theorem 5.2.11, we know that $x \in \mathbb{R}^n$ is an optimal solution to* (5.9) *if and only if F_x solves the problem $(P_{k,B,\mathcal{F}_{\text{strict}}})$, where B is the natural ordering cone in Y (that is, $B = Y^+$) and k is the constant function $k \equiv 1$.*

Remark 5.2.14 ([61, Remark 6])**.** *If the worst case is attained for every solution $x \in \mathfrak{A}_{\text{strict}}$, Corollary 5.2.5 says that for every optimal solution x of the scalarization problem $(P_{k,B,\mathcal{F}_{\text{strict}}})$, F_x is a weakly minimal element w.r.t. the natural order relation $\leq_{Y^+}$. This is not always satisfactory, and particularly in the context of scalarizing functionals it is common practice to apply methods that guarantee minimal (instead of weakly minimal) elements w.r.t. $\leq_{Y^+}$. This can, for example, be realized by a second stage optimization applied on the set of optimal solutions of $(P_{k,B,\mathcal{F}_{\text{strict}}})$ as suggested in Iancu and Trichakis [49], or by using an appropriate augmentation term for $z^{B,k}$ in the first stage (see, for example, Jahn [53]).*

5.3 Optimistic Robustness

While strict robustness focuses on the worst case and can thus be viewed as a pessimistic model, optimistic robustness aims at minimizing the best realization of the objective value of a feasible solution over all scenarios. We consider the optimization problem

$$\begin{aligned}
&\rho_{\text{oRC}}(x) = \inf_{\xi \in \mathcal{U}} f(x, \xi) \to \inf \\
&\text{s.t. } \forall \xi \in \mathcal{U}: \;\; F_i(x, \xi) \leq 0, \;\; i = 1, \dots, m, \\
&\qquad x \in \mathbb{R}^n,
\end{aligned} \tag{oRC}$$

which we call **optimistically robust counterpart**. (oRC) resembles the optimistic counterpart introduced in Beck and Ben-Tal [8] in the context of duality theory in robust optimization. In contrast to [8], in the present work, a feasible solution to (oRC) is required to be strictly robust, the set of feasible solutions of (oRC) is hence given by $\mathfrak{A}_{\text{strict}}$.

5.3.1 Vector Optimization Approach for Optimistic Robustness

The optimistically robust counterpart (oRC) can be formulated as a vector optimization problem in the functional space $Y = \mathbb{R}^{\mathcal{U}}$ as follows. We again use the set of strictly robust outcome functions $\mathcal{F}_{\text{strict}} = \{F_x \in Y | \; x \in \mathfrak{A}_{\text{strict}}\}$ in Y (see (5.6)). Now let two functions $F_x, F_y \in Y$ be given. We consider the following order relation on Y:

$$F_x \;\leq^{\inf}\; F_y :\Longleftrightarrow \inf_{\xi\in\mathcal{U}} F_x(\xi) \leq \inf_{\xi\in\mathcal{U}} F_y(\xi). \tag{5.10}$$

This order relation will be referred to as the *inf-order relation* in the following. Note that the inf-order relation is closely related to the sup-order relation (5.2.1). More precisely,

$$\begin{aligned} \inf_{\xi\in\mathcal{U}} F(\xi) \leq \inf_{\xi\in\mathcal{U}} G(\xi) \quad &\Longleftrightarrow \quad \sup_{\xi\in\mathcal{U}}(-G(\xi)) \leq \sup_{\xi\in\mathcal{U}}(-F(\xi)), \text{ and thus} \\ F \;\leq^{\inf}\; G \quad &\Longleftrightarrow \quad (-G) \;\leq^{\sup}\; (-F). \end{aligned} \tag{5.11}$$

Hence, the inf-order relation $\leq^{\inf}$ defines a total preorder (see also Remark 5.2.1). Like $\leq^{\sup}$, it cannot be represented by an ordering cone in Y.

Remark 5.3.1 ([61, Remark 7])**.** *$\leq^{\inf}$ is reflexive, transitive, and total. Therefore, $\leq^{\inf}$ is a total preorder.*

Analogous to Theorem 5.2.2, the order relation $\leq^{\inf}$ can be used to characterize the optimistically robust optimization problem (oRC) as a vector optimization problem.

Theorem 5.3.2 ([61, Theorem 4])**.** *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (oRC) *if and only if F_x is a minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. the inf-order relation $\leq^{\inf}$.*

Proof. For $x \in \mathfrak{A}_{\text{strict}}$ we have

$$\begin{aligned} x \text{ is an optimal solution to (oRC)} \quad &\Longleftrightarrow \quad \inf_{\xi\in\mathcal{U}} f(x,\xi) \leq \inf_{\xi\in\mathcal{U}} f(\overline{x},\xi) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}} \\ &\Longleftrightarrow \quad \inf_{\xi\in\mathcal{U}} F_x(\xi) \leq \inf_{\xi\in\mathcal{U}} F_{\overline{x}}(\xi) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}} \\ &\Longleftrightarrow \quad F_x \;\leq^{\inf}\; F_{\overline{x}} \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}, \end{aligned}$$

and the result follows since $\leq^{\inf}$ is a total preorder. □

In order to analyze the relation between the inf-order relation $\leq^{\inf}$ and the natural order relation $\leq_{Y^+}$, first note that the natural order relation $\leq_{Y^+}$ satisfies

$$F \leq_{Y^+} G \quad \Longleftrightarrow \quad (-G) \leq_{Y^+} (-F). \tag{5.12}$$

Together with (5.11) above, Remark 5.2.3, Lemma 5.2.4 and Corollary 5.2.5 can be easily adapted to this case:

Remark 5.3.3 ([61, Remark 8]). *$F \leq_{Y^+} G \Longrightarrow F \leq^{\inf} G$ for $F, G \in Y$.*

Lemma 5.3.4 ([61, Lemma 4]). *Let $Y = C(\mathcal{U}, \mathbb{R})$. Assume that every $F \in \mathcal{F}_{\text{strict}}$ attains its infimum on $\mathcal{U}$. If $F \in \mathcal{F}_{\text{strict}}$ is a minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. $\leq^{\inf}$, then F is a weakly minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. the natural order relation $\leq_{Y^+}$.*

As in Corollary 5.2.5 this means that F_x is weakly Pareto minimal for all optimal solutions x to (oRC).

Corollary 5.3.5 ([61, Corollary 2]). *Let $Y = C(\mathcal{U}, \mathbb{R})$, and let the best-case be attained for every solution $x \in \mathfrak{A}_{\text{strict}}$. Then for every optimal solution x to the optimistic robust counterpart* (oRC)*, F_x is a weakly minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. the natural order relation $\leq_{Y^+}$ in Y.*

5.3.2 Set-Valued Optimization Approach for Optimistic Robustness

Using the *lower-type set-relation*, we can interpret the optimistically robust counterpart (oRC) as a set-valued optimization problem. The set of *optimistically robust outcome sets* in the power set $Z = \overline{\mathcal{P}}(\mathbb{R})$ is $\mathcal{B}_{\text{strict}} := \{B_x \in Z | \ x \in \mathfrak{A}_{\text{strict}}\}$ as before. The optimistically robust counterpart (oRC) minimizes the best case objective value of a solution $x \in \mathfrak{A}_{\text{strict}}$. This corresponds to minimizing the infimum of the outcome sets $B_x \subseteq \mathbb{R}$ with $x \in \mathfrak{A}_{\text{strict}}$ in the set-based interpretation: We hence use the lower-type set-relation $\preceq^l_{\mathbb{R}_+}$ as in (5.3). Like for $\preceq^u_{\mathbb{R}_+}$ we have:

Remark 5.3.6 ([61, Remark 9]). *$\preceq^l_{\mathbb{R}_+}$ is reflexive and transitive. Furthermore, it is a total preorder.*

Lemma 5.3.7 ([61, Lemma 5]). *Let $x, y \in \mathbb{R}^n$ and let F_x, F_y their corresponding outcome functions and B_x, B_y their corresponding outcome sets. Then*

$$B_x \preceq^l_{\mathbb{R}_+} B_y \Longleftrightarrow F_x \leq^{\inf} F_y.$$

Consequently, we can reformulate (oRC) as the following set-valued optimization problem.

Theorem 5.3.8 ([61, Theorem 5]). *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (oRC) *if and only if B_x is a minimal element of $\mathcal{B}_{\text{strict}}$ w.r.t. the order relation $\preceq^l_{\mathbb{R}_+}$.*

5.3.3 Nonlinear Scalarizing Functional for Optimistic Robustness

In the following we show that the optimistically robust counterpart (oRC) can be characterized using the nonlinear scalarizing functional (5.5) from Section 5.1.3. Again, the basic reformulation holds for the case that $Y = \mathbb{R}^{\mathcal{U}}$.

Theorem 5.3.9 ([61, Theorem 6]). *Let $Y = \mathbb{R}^{\mathcal{U}}$, $B_{\inf} := \{F \in Y | \exists \ \xi \in \mathcal{U} : \ F(\xi) \geq 0\}$, and $k \equiv 1 \in Y$. Then $x \in \mathbb{R}^n$ is an optimal solution to* (oRC) *if and only if F_x solves problem $(P_{k, B_{\inf}, \mathcal{F}_{\text{strict}}})$.*

Proof. We first note that $B_{\inf} + [0, +\infty) \cdot k \subseteq B_{\inf}$ holds. Hence the inclusion (2.1) is fulfilled and the functional $z^{B_{\inf},k}$ is well-defined. Now we have the following relation:

$$\begin{aligned} z^{B_{\inf},k}(F_x) &= \inf\{t \in \mathbb{R} | F_x \in tk - B_{\inf}\} \\ &= \inf\{t \in \mathbb{R} | F_x - tk \in -B_{\inf}\} \\ &= \inf\{t \in \mathbb{R} | \exists\ \xi \in \mathcal{U} :\ F_x(\xi) \leq t\} \\ &= \inf_{\xi \in \mathcal{U}} f(x, \xi). \end{aligned}$$

That means that F_x is minimal for $(P_{k,B_{\inf},\mathcal{F}_{\text{strict}}})$ if and only if $x \in \mathfrak{A}_{\text{strict}}$ minimizes $\inf_{\xi \in \mathcal{U}} f(x, \xi)$, i.e., if and only if x is an optimal solution to (oRC). □

Remark 5.3.10 ([61, Remark 10])**.** *If we choose* $Y = C(\mathcal{U}, \mathbb{R})$*, notice that* $B_{\inf}$ *is a proper convex cone and* $k \equiv 1 \in \operatorname{int}(B_{\inf})$ *since* $Y^+ \subseteq B_{\inf}$ *and* $\operatorname{int} Y^+ \neq \emptyset$*. Taking into account* $B_{\inf} = Y \setminus \operatorname{int}(-Y^+)$*, we get the closedness of* $B_{\inf}$*. Therefore, Lemma 5.1.8 implies that the functional* $z^{B_{\inf},k}$ *is continuous, finite-valued,* $B_{\inf}$*-monotone, strictly* $(\operatorname{int} B_{\inf})$*-monotone and sublinear, and it holds*

$$\begin{aligned} &\forall\ F_x \in Y,\ \forall\ t \in \mathbb{R} :\ \ z^{B_{\inf},k}(F_x) \leq t \iff F_x \in tk - B_{\inf}, \\ &\forall\ F_x \in Y,\ \forall\ t \in \mathbb{R} :\ \ z^{B_{\inf},k}(F_x) < t \iff F_x \in tk - \operatorname{int}(B_{\inf}). \end{aligned}$$

For a discrete uncertainty set $\mathcal{U} = \{\xi_1, \ldots, \xi_q\}$ ($q \in \mathbb{N}$), Theorem 5.3.9 reduces to

$$\min_{F_x \in \mathcal{F}_{\text{strict}}} z^{\overline{B}_{opt},\overline{k}}(F_x) = \min_{x \in \mathfrak{A}_{\text{strict}}} \min_{\xi \in \mathcal{U}} f(x, \xi),$$

where $\overline{B}_{opt} := \{y \in \mathbb{R}^q | \exists i \in \{1, \ldots, q\} : y_i \geq 0\} = \mathbb{R}^q \setminus \operatorname{int}(-\mathbb{R}^q_+)$ and $\overline{k} := (1, \ldots, 1)^T$. An optimal solution to (oRC) will thus be minimal for at least one of the individual objective functions. Note that this does not hold for strict robustness in general.

The duality relation between the optimistically robust counterpart (oRC) and the strictly robust counterpart (RC) gives rise to an alternative formulation as a maximization problem using the natural ordering cone Y^+.

Theorem 5.3.11 ([61, Theorem 7])**.** *Let* $Y = \mathbb{R}^{\mathcal{U}}$ *and* $B := Y^+$*,* $k \equiv 1$*. Then* $x \in \mathbb{R}^n$ *is an optimal solution to* (oRC) *if and only if* $F_x \in \mathcal{F}_{\text{strict}}$ *minimizes* $\sup\{t \in \mathbb{R} | F_x \in tk + B\}$*.*

Proof. Analogous to the proof of Theorem 5.2.11 we obtain

$$\begin{aligned} \sup\{t \in \mathbb{R} | F_x \in tk + B\} &= \sup\{t \in \mathbb{R} | F_x - tk \in Y^+\} \\ &= \sup\{t \in \mathbb{R} | \forall \xi \in \mathcal{U} :\ F_x(\xi) \geq t\} \\ &= \inf_{\xi \in \mathcal{U}} f(x, \xi). \end{aligned}$$

Thus, $F_x \in \mathcal{F}_{\text{strict}}$ minimizes $\sup\{t \in \mathbb{R} | F_x \in tk + B\}$ if and only if $x \in \mathfrak{A}_{\text{strict}}$ minimizes $\inf_{\xi \in \mathcal{U}} f(x, \xi)$, i.e., if and only if x is an optimal solution to (oRC). □

Remark 5.3.12 ([61, Remark 11])**.** *As in the case of strict robustness (see Remark 5.2.14), a second stage optimization or an augmented scalarization can be applied to avoid weakly minimal elements w.r.t.* $\leq_{Y^+}$*, see Corollary 5.3.5 above.*

5.4 Regret Robustness

The next robustness concept we consider is known under the names *min max regret robustness* or *deviation robustness.* In this concept one evaluates a solution by comparing it with the best possible solution for the realized scenario in the worst case, i.e., the function to be minimized is $\sup_{\xi\in\mathcal{U}}(f(x,\xi) - f^*(\xi))$, where $f^*(\xi) \in \mathbb{R}$ is the optimal value of problem $(Q(\xi))$ for the fixed parameter $\xi \in \mathcal{U}$. This robustness concept has been used in applications, e.g., in scheduling or location theory, mostly in cases where no uncertainty in the constraints is present (see Kouvelis and Yu, [74]) and has been researched also for spanning trees and in matroids, see Yaman et al. [108, 109].

The **regret robust counterpart** of the uncertain optimization problem $(Q(\xi), \xi \in \mathcal{U})$ can be formulated as

$$\begin{aligned} \rho_{\mathrm{rRC}}(x) &= \sup_{\xi\in\mathcal{U}}(f(x,\xi) - f^*(\xi)) \to \inf \\ \text{s.t. } &\forall\xi \in \mathcal{U}: \; F_i(x,\xi) \le 0, \; i = 1,\dots,m, \\ &x \in \mathbb{R}^n. \end{aligned} \tag{rRC}$$

Note that we require $x \in \mathfrak{A}_{\text{strict}}$, i.e., we only allow strictly robust solutions as feasible solutions for the regret robust counterpart.

Let $f^* \in \mathbb{R}^{\mathcal{U}}$ be the function $f^* : \mathcal{U} \to \mathbb{R}$ given by

$$f^*(\xi) := \inf\{F_x(\xi) | x \in \mathfrak{A}_{\text{strict}}\}. \tag{5.13}$$

Note that the inf can be replaced by min in (5.13) since we assumed that for every fixed scenario ξ an optimal solution exists.

We refer to f^* as the *ideal solution* of problem $(Q(\xi))$. It is in general *not* a feasible solution to (rRC), i.e., there does not exist $x \in \mathfrak{A}_{\text{strict}}$ with $f(x,\xi) = f^*(\xi)$ for all $\xi \in \mathcal{U}$. The regret robust counterpart (rRC) thus minimizes the maximum deviation (over all scenarios) between the objective value of the implemented solution $f(x,\xi)$ and the ideal solution $f^*(\xi)$.

5.4.1 Vector Optimization Approach for Regret Robustness

Similar to the strictly robust counterpart (RC), the regret robust counterpart (rRC) can be directly reformulated as a vector optimization problem in the functional space $Y = \mathbb{R}^{\mathcal{U}}$. We consider $\mathcal{F}_{\text{strict}}$ as in (5.6). For two functions $F_x, F_y \in Y$, we use the order relation $\le^{\text{regret}}$ on Y given by

$$F_x \;\le^{\text{regret}}\; F_y \quad :\Longleftrightarrow \quad \sup_{\xi\in\mathcal{U}}(F_x(\xi) - f^*(\xi)) \le \sup_{\xi\in\mathcal{U}}(F_y(\xi) - f^*(\xi)).$$

We have the following relation to the sup-order relation $\le^{\sup}$ (as introduced in (5.2.1)):

Remark 5.4.1 ([61, Remark 12])**.**

$$F_x \;\le^{\text{regret}}\; F_y \iff (F_x - f^*) \;\le^{\sup}\; (F_y - f^*).$$

In comparison to $\leq^{\text{sup}}$, we can interpret $\leq^{\text{regret}}$ as a *sup-order relation with reference point* f^*, while the sup-order relation $\leq^{\text{sup}}$ has the origin $f^0 :\equiv 0 \in Y$ as reference point. Intuitively, we have shifted the reference point from the origin f^0 (for $\leq^{\text{sup}}$) to the ideal solution f^* (for $\leq^{\text{regret}}$). Analogous to the sup-order relation $\leq^{\text{sup}}$, we have:

Remark 5.4.2 ([61, Remark 13])**.** *The sup-order relation $\leq^{\text{regret}}$ with reference point f^* is reflexive and transitive. Furthermore, $\leq^{\text{regret}}$ is a total preorder.*

Note that $\leq^{\text{regret}}$ is (as $\leq^{\text{sup}}$) not compatible with addition and can thus not be represented by an ordering cone. We first state that $\leq^{\text{regret}}$ can be used to represent regret robustness as a vector-valued optimization problem.

Theorem 5.4.3 ([61, Theorem 8])**.** *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (rRC) *if and only if F_x is a minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. $\leq^{\text{regret}}$.*

Proof. Let $x \in \mathfrak{A}_{\text{strict}}$. Then

$$
\begin{aligned}
&x \text{ is an optimal solution to (rRC)}\\
&\qquad\qquad \iff \sup_{\xi\in\mathcal{U}}(f(x,\xi) - f^*(\xi)) \leq \sup_{\xi\in\mathcal{U}}(f(\overline{x},\xi) - f^*(\xi)) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}\\
&\qquad\qquad \iff (F_x - f^*) \ \leq^{\text{sup}} \ (F_{\overline{x}} - f^*) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}\\
&\qquad\qquad \iff F_x \ \leq^{\text{regret}} \ F_{\overline{x}} \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}\\
&\qquad\qquad \iff F_x \ \leq^{\text{regret}} \ G \text{ for all } G \in \mathcal{F}_{\text{strict}}
\end{aligned}
$$

and the result follows as before since $\leq^{\text{regret}}$ is a total preorder. □

We easily see that:

Remark 5.4.4 ([61, Remark 14])**.** *Let $F, G \in Y$. Then we have: $F \leq_{Y^+} G \implies F \ \leq^{\text{regret}} \ G$.*

Similar to $\leq^{\text{sup}}$ we obtain the following relation between minimal solutions w.r.t. $\leq^{\text{regret}}$ and weakly minimal solutions w.r.t. $\leq_{Y^+}$.

Lemma 5.4.5 ([61, Lemma 6])**.** *Let $Y = C(\mathcal{U}, \mathbb{R})$. Assume that every function $F - f^*$ attains its supremum on $\mathcal{U}$ for all $F \in \mathcal{F}_{\text{strict}}$. If $F \in \mathcal{F}_{\text{strict}}$ is a minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. $\leq^{\text{regret}}$, then F is a weakly minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. the natural order relation $\leq_{Y^+}$.*

Proof. Let $F \in \mathcal{F}_{\text{strict}}$ be a minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. $\leq^{\text{regret}}$. Since $\leq^{\text{regret}}$ is a total preorder, this means that

$$\sup_{\xi\in\mathcal{U}}(F(\xi) - f^*(\xi)) \leq \sup_{\xi\in\mathcal{U}}(G(\xi) - f^*(\xi)) \text{ for all } G \in \mathcal{F}_{\text{strict}}. \tag{5.14}$$

Now suppose that F is not a weakly minimal element of $\mathcal{F}_{\text{strict}}$ in Y w.r.t. the natural order relation $\leq_{Y^+}$ of Y. Thus, there exists $G \in \mathcal{F}_{\text{strict}}$ with

$$\forall\, \xi \in \mathcal{U} : \ G(\xi) < F(\xi), \quad \text{and hence} \quad (G(\xi) - f^*(\xi)) < (F(\xi) - f^*(\xi)).$$

Since $(G - f^*)$ attains its supremum on $\mathcal{U}$, this means that

$$\sup_{\xi\in\mathcal{U}}(G(\xi) - f^*(\xi)) = G(\overline{\xi}) - f^*(\overline{\xi}) < F(\overline{\xi}) - f^*(\overline{\xi}) \leq \sup_{\xi\in\mathcal{U}}(F(\xi) - f^*(\xi)),$$

with some $\overline{\xi} \in \mathcal{U}$, a contradiction to (5.14). □

As in Corollary 5.2.5 we obtain that F_x is weakly Pareto minimal also for all optimal solutions x to (rRC).

Corollary 5.4.6 ([61, Corollary 3])**.** *Let $Y = C(\mathcal{U}, \mathbb{R})$ and let the worst case regret be attained for every solution $x \in \mathfrak{A}_{\text{strict}}$. Then for every optimal solution x to the regret robust counterpart* (rRC)*, F_x is a weakly minimal element of $\mathcal{F}_{\text{strict}}$ w.r.t. the natural order relation $\leq_{Y^+}$ in Y.*

5.4.2 Set-Valued Optimization Approach for Regret Robustness

The set-valued interpretation of the strictly robust counterpart (5.2) presented in Section 5.2.2 does not directly transfer to the case of regret robustness, since we have in general that

$$\sup\{f(x,\xi) - f^*(\xi) | \xi \in \mathcal{U}\} \neq \sup\left(\underbrace{\{f(x,\xi)|\xi \in \mathcal{U}\}}_{=B_x} - \{f^*(\xi)|\xi \in \mathcal{U}\}\right).$$

For $x \in \mathfrak{A}_{\text{strict}}$ we thus define the *regret robust outcome set*

$$B_x^{f-f^*} := \{f(x,\xi) - f^*(\xi)|\xi \in \mathcal{U}\}$$

as the set of possible outcomes of the regret function $f(x,\xi) - f^*(\xi)$, $\xi \in \mathcal{U}$.

$$\mathcal{B}_{\text{regret}} := \{B_x^{f-f^*} \in Z|\ x \in \mathfrak{A}_{\text{strict}}\}$$

is called the set of *regret robust outcome sets.* The regret robust counterpart (rRC) minimizes the worst case regret value of a solution $x \in \mathfrak{A}_{\text{strict}}$, which is equivalent to minimizing the supremum of the regret robust outcome sets $B_x^{f-f^*} \in \mathcal{B}_{\text{regret}}$. This corresponds to using the upper-type set relation $\preceq^u_{\mathbb{R}_+}$ introduced in (5.10) on $\mathcal{B}_{\text{regret}}$.

Theorem 5.4.7 ([61, Theorem 9])**.** *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (rRC) *if and only if $B_x^{f-f^*}$ is a minimal element of $\mathcal{B}_{\text{regret}}$ with respect to the upper-type set order relation $\preceq^u_{\mathbb{R}_+}$.*

Proof. Let $x \in \mathfrak{A}_{\text{strict}}$. We know that $\sup B_x^{f-f^*} = \sup_{\xi\in\mathcal{U}}(f(x,\xi) - f^*(\xi))$. We hence obtain

$$\begin{aligned}
x \text{ is an optimal }&\text{ solution to (rRC)}\\
&\iff \sup_{\xi\in\mathcal{U}}(f(x,\xi) - f^*(\xi)) \leq \sup_{\xi\in\mathcal{U}}(f(\overline{x},\xi) - f^*(\xi)) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}\\
&\iff \sup B_x^{f-f^*} \leq \sup B_{\overline{x}}^{f-f^*} \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}\\
&\iff B_x^{f-f^*} \preceq^u_{\mathbb{R}_+} B_{\overline{x}}^{f-f^*} \text{ for all } \overline{x} \in \mathfrak{A}_{\text{strict}}.\\
&\iff B_x^{f-f^*} \preceq^u_{\mathbb{R}_+} B \text{ for all } B \in \mathcal{B}_{\text{regret}},
\end{aligned}$$

and the result follows since $\preceq^u_{\mathbb{R}_+}$ is a total preorder. $\square$

Note that Theorem 5.4.7 can alternatively be proven using the relation between $\leq^{\text{regret}}$ and $\leq^{\text{sup}}$ and between $\leq^{\text{sup}}$ and $\preceq^u_{\mathbb{R}_+}$, see Lemma 5.2.8.

5.4.3 Nonlinear Scalarizing Functional for Regret Robustness

As in the case of strict robustness (see Section 5.2 above), the formulation of the regret robust counterpart (rRC) as a vector optimization problem in the functional space $Y = \mathbb{R}^{\mathcal{U}}$ gives rise to a representation using nonlinear scalarizing functionals.

As compared to Section 5.2.3, the dominating set $B \subseteq Y$ is now shifted by the ideal solution $f^* \in \mathbb{R}^{\mathcal{U}}$ of problem $(Q(\xi))$.

Theorem 5.4.8 ([61, Theorem 10]). *Let $Y = \mathbb{R}^{\mathcal{U}}$, $B_{\text{regret}} := Y^+ - f^*$ and consider again $k \equiv 1$. Then $x \in \mathbb{R}^n$ is an optimal solution to* (rRC) *if and only if F_x solves problem* $(P_{k,B_{\text{regret}},\mathcal{F}_{\text{strict}}})$.

Proof. First note that $B_{\text{regret}} + [0, +\infty) \cdot k \subseteq B_{\text{regret}}$ holds and thus (2.1) is satisfied. Moreover,

$$\begin{aligned} z^{B_{\text{regret}},k}(F_x) &= \inf\{t \in \mathbb{R} | F_x \in tk - B_{\text{regret}}\} \\ &= \inf\{t \in \mathbb{R} | \forall \xi \in \mathcal{U} : \; F_x(\xi) - t \leq f^*(\xi)\} \\ &= \inf\{t \in \mathbb{R} | \forall \xi \in \mathcal{U} : \; f(x, \xi) - f^*(\xi) \leq t\} \\ &= \sup_{\xi \in \mathcal{U}} (f(x, \xi) - f^*(\xi)). \end{aligned}$$

Thus, F_x is minimal for $(P_{k,B_{\text{regret}},\mathcal{F}_{\text{strict}}})$ if and only if x minimizes $\sup_{\xi \in \mathcal{U}}(f(x, \xi) - f^*(\xi))$, i.e., if it is optimal for (rRC). $\square$

Lemma 5.4.9 ([61, Lemma 7]). *Let $Y = C(\mathcal{U}, \mathbb{R})$. Then the functional $z^{B_{\text{regret}},k}$ is continuous, finite-valued, Y^+-monotone, strictly* $(\operatorname{int} Y^+)$*-monotone, convex and*

$$\begin{aligned} &\forall\, F_x \in Y,\ \forall\, t \in \mathbb{R} : \quad z^{B_{\text{regret}},k}(F_x) \leq t \iff F_x \in tk - (Y^+ - f^*), \\ &\forall\, F_x \in Y,\ \forall\, t \in \mathbb{R} : \quad z^{B_{\text{regret}},k}(F_x) < t \iff F_x \in tk - \operatorname{int}(Y^+ - f^*). \end{aligned}$$

Proof. Since $f^* \in Y$ we obtain for all $k \in \operatorname{int} Y^+$

$$z^{Y^+ - f^*,k}(y) = z^{Y^+,k}(y - f^*),$$

i.e., we have $z^{B_{\text{regret}},k}(F_x) = z^{B,k}(F_x - f^*)$ for all $F_x \in Y$. Applying this shift and taking into account that the functional $z^{B,k}$ is continuous, finite-valued, Y^+-monotone, strictly $(\operatorname{int} Y^+)$-monotone and convex (see Theorem 2.1.2), we have these properties for the functional $z^{B_{\text{regret}},k}$ as well. Furthermore, we get

$$\begin{aligned} &\forall\, F_x \in Y,\ \forall\, t \in \mathbb{R} : \quad z^{B_{\text{regret}},k}(F_x) \leq t \iff F_x \in tk - (Y^+ - f^*), \\ &\forall\, F_x \in Y,\ \forall\, t \in \mathbb{R} : \quad z^{B_{\text{regret}},k}(F_x) < t \iff F_x \in tk - \operatorname{int}(Y^+ - f^*). \end{aligned}$$

from the corresponding properties of the functional $z^{B,k}$. $\square$

In the special case of a discrete uncertainty set $\mathcal{U} = \{\xi_1, \dots, \xi_q\}$, $q \in \mathbb{N}$, the scalarization problem $(P_{k,B_{\text{regret}},\mathcal{F}_{\text{strict}}})$ simplifies to

$$\min_{F_x \in \mathcal{F}_{\text{strict}}} z^{\overline{B}_{\text{regret}},\overline{k}}(F_x) = \min_{x \in \mathfrak{A}_{\text{strict}}} \max_{\xi \in \mathcal{U}} (f(x,\xi) - f^*(\xi))$$

with $\overline{B}_{\text{regret}} := \mathbb{R}^q_+ - (f^*(\xi_1), \dots, f^*(\xi_1))^T$ and $\overline{k} := (1, \dots, 1)^T$, see [60]. In light of the associated multiobjective optimization problem $\text{vmin}_{x \in \mathfrak{A}_{\text{strict}}}(f(x,\xi_1), \dots, f(x,\xi_q))$, this is equivalent to a reference point approach with reference point f^*, and if $f(x,\xi) \geq 0$ for all $\xi \in \mathcal{U}$ and $x \in \mathfrak{A}_{\text{strict}}$, to a weighted Tchebycheff scalarization with reference point f^* (and with equal weights).

Remark 5.4.10 ([61, Remark 15])**.** *As before (see Remarks 5.2.14 and 5.3.12), a second stage optimization or an augmented scalarization can be applied also in this case to guarantee minimal (instead of weakly minimal) elements w.r.t.* $\preceq_{Y^+}$.

5.5 Reliability

Requiring a strictly robust solution x is known as a very conservative approach. In case that $\mathfrak{A}_{\text{strict}}$ is empty or only contains solutions with a high objective function value, a less conservative approach is the concept of *reliability.* Here, some buffer is added to the right-hand-side values of the constraints, i.e., the constraints $F_i(x,\xi) \leq 0$ are relaxed to $F_i(x,\xi) \leq \delta_i$ for some given $\delta_i \in \mathbb{R}_+$, $i = 1, \dots, m$. Nevertheless, it is required that the original constraints must be satisfied for the nominal scenario $\hat{\xi}$, i.e., $F_i(x,\hat{\xi}) \leq 0$, $i = 1, \dots, m$. Formally, the **reliable counterpart** of (5.1) proposed by Ben-Tal and Nemirovski in [11], is defined by

$$\begin{aligned} &\rho_{\text{Rely}}(x) = \sup_{\xi \in \mathcal{U}} f(x,\xi) \to \inf \\ &\text{s.t. } F_i(x,\hat{\xi}) \leq 0, \ i = 1, \dots, m, \\ &\qquad \forall \xi \in \mathcal{U}: \ F_i(x,\xi) \leq \delta_i, \ i = 1, \dots, m, \\ &\qquad x \in \mathbb{R}^n. \end{aligned} \tag{Rely}$$

A feasible solution to (Rely) is called *reliable.* If $\delta_i = 0$ for all $i = 1, \dots, m$, the reliable counterpart (Rely) reduces to the strictly robust problem (RC). On the other hand, the reliable counterpart can be interpreted as a strictly robust counterpart with the uncertain constraint functions $F_i'(x,\xi) = F_i(x,\xi) - \delta_i$ and the deterministic constraint $F_i(x,\hat{\xi}) \leq 0$, $i = 1, \dots, m$. This interpretation already suggests that many results can be transferred from strict robustness to reliability. This is in fact the case, and we will hence leave out the proofs in this section and just state the results.

We denote by

$$\mathfrak{A}_{\text{rely}} := \{x \in \mathbb{R}^n | \forall \xi \in \mathcal{U}: \ F_i(x,\xi) \leq \delta_i, \ F_i(x,\hat{\xi}) \leq 0, \ i = 1, \dots, m\}$$

the set of reliable solutions.

5.5.1 Vector Optimization Approach for Reliability

Similar to the strictly robust counterpart (RC), the reliable counterpart (Rely) can be formulated as a vector optimization problem in the functional space $Y = \mathbb{R}^{\mathcal{U}}$. Towards this end, denote the set of *reliable outcome functions* in Y by

$$\mathcal{F}_{\text{rely}} := \{F_x \in Y \mid x \in \mathfrak{A}_{\text{rely}}\}. \tag{5.15}$$

We consider the order relation $\leq^{\sup}$ that has been introduced in Section 5.2.1 and transfer the results obtained there. First, finding a minimal element of $\mathcal{F}_{\text{rely}}$ w.r.t. $\leq^{\sup}$ solves the reliable counterpart:

Theorem 5.5.1 ([61, Theorem 11]). *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (Rely) *if and only if F_x is a minimal element of $\mathcal{F}_{\text{rely}}$ with respect to the sup-order relation $\leq^{\sup}$.*

Also for the concept of reliability we have an analogous relation between optimal reliable solutions and (Pareto) minimal elements w.r.t. the natural ordering $\leq_{Y^+}$.

Lemma 5.5.2 ([61, Lemma 8]). *Let $Y = C(\mathcal{U}, \mathbb{R})$. Assume that every $F \in \mathcal{F}_{\text{rely}}$ attains its supremum on $\mathcal{U}$. If $F \in \mathcal{F}_{\text{rely}}$ is a minimal element of $\mathcal{F}_{\text{rely}}$ w.r.t. $\leq^{\sup}$, then F is a weakly minimal element of $\mathcal{F}_{\text{rely}}$ w.r.t. the natural order relation $\leq_{Y^+}$.*

As before, this means that F_x is weakly Pareto minimal for all optimal solutions x to (Rely).

Corollary 5.5.3 ([61, Corollary 4]). *Let $Y = C(\mathcal{U}, \mathbb{R})$ and let the worst case be attained for every solution $x \in \mathfrak{A}_{\text{rely}}$. Then for every optimal solution x to the reliable counterpart* (Rely)*, F_x is a weakly minimal element of $\mathcal{F}_{\text{rely}}$ w.r.t. the natural order relation $\leq_{Y^+}$ in Y.*

5.5.2 Set-Valued Optimization Approach for Reliability

Analogously, we can interpret the reliable counterpart (Rely) as a set-valued optimization problem. We denote the set of *reliable outcome sets* in the power set $Z = \overline{\mathcal{P}}(\mathbb{R})$ by

$$\mathcal{B}_{\text{rely}} := \{B_x \in Z \mid x \in \mathfrak{A}_{\text{rely}}\}. \tag{5.16}$$

In order to compare two sets of $\mathcal{B}_{\text{rely}}$ we use the *upper-type set-relation* $\preceq^u_{\mathbb{R}_+}$ as introduced in (5.10). The reliable counterpart (Rely) minimizes the worst case objective value of a solution $x \in \mathfrak{A}_{\text{rely}}$ which is equivalent to minimizing the supremum of the outcome sets $B_x \subseteq \mathbb{R}$ with $x \in \mathfrak{A}_{\text{rely}}$ in the set-based interpretation:

Theorem 5.5.4 ([61, Theorem 12]). *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (Rely) *if and only if B_x is a minimal element of $\mathcal{B}_{\text{rely}}$ with respect to the order relation $\preceq^u_{\mathbb{R}_+}$.*

Proof. As for Theorem 5.2.9, this result follows from Theorem 5.5.1 and Lemma 5.2.8. □

5.5.3 Nonlinear Scalarizing Functional for Reliability

The following theorem describes how the reliably robust optimization problem can be represented using the same nonlinear scalarizing functional $z^{B,k}$ as for strict robustness (see Theorem 5.2.11), but with feasible set $\mathcal{F}_{\text{rely}}$ instead of $\mathcal{F}_{\text{strict}}$.

Theorem 5.5.5 ([61, Theorem 13]). *Let $Y = \mathbb{R}^{\mathcal{U}}$, $B = Y^+$, and $k \equiv 1 \in Y$. Then x is an optimal solution to* (Rely) *if and only if F_x solves problem $(P_{k,B,\mathcal{F}_{\text{rely}}})$.*

Since the scalarizations $(P_{k,B,\mathcal{F}_{\text{rely}}})$ and $(P_{k,B,\mathcal{F}_{\text{strict}}})$ (for reliable robustness and for strict robustness, respectively) differ only in their respective feasible sets, they have a similar interpretation in the case that the scenario set is finite: Then $(P_{k,B,\mathcal{F}_{\text{rely}}})$ simplifies to a reference point approach with the origin as reference point, which corresponds to a weighted Tchebycheff scalarization (with equal weights) of the corresponding multiobjective optimization problem $\operatorname{vmin}_{x\in\mathfrak{A}_{\text{rely}}}(f(x,\xi_1),\dots,f(x,\xi_q))$ if $f(x,\xi) \geq 0$ for all $\xi \in \mathcal{U}$ and $x \in \mathfrak{A}_{\text{rely}}$.

Remark 5.5.6 ([61, Remark 16]). *As in the cases of strict, optimistic and regret robustness (c.f. Remarks 5.2.14, 5.3.12 and 5.4.10), the scalarization by the functional $z^{B,k}$ gives rise to a second stage optimization or an appropriate augmentation in order to avoid weakly minimal elements.*

5.6 Adjustable Robustness

Adjustable robustness is a two-stage approach in robust optimization which assumes that there are two types of variables: These are the here-and-now variables x whose values have to be fixed before the scenario realizes and the wait-and-see variables u which may be decided after the scenario becomes known. These variables are then chosen such that the objective function (for the fixed x and the realized scenario ξ) is minimized. Adjustable robustness has first been introduced in Ben-Tal et al. [10] and extensively studied in [9].

The adjustable robust counterpart asks for a solution x which, if adjusted optimally in the second stage, achieves the best overall objective value in the worst case. We hence reformulate $(Q(\xi))$ in terms of the here-and-now variables $x \in \mathbb{R}^n$ and the wait-and-see variables $u \in \mathbb{R}^p$ and obtain an uncertain problem with $n+p$ variables.

$$\begin{aligned} & f(x,u,\xi) \to \inf \\ & \text{s.t. } F_i(x,u,\xi) \leq 0,\ i = 1,\dots,m \\ & x \in \mathbb{R}^n, u \in \mathbb{R}^p. \end{aligned} \tag{$Q(\xi)$}$$

Let us define

$$\mathcal{G}(x,\xi) := \{u \in \mathbb{R}^p : F_i(x,u,\xi) \leq 0,\ i = 1,\dots,m\}$$

as the set of feasible second-stage variables u for given first-stage variables x and a given scenario ξ. Then the *second-stage optimization problem* can be written as

$$\begin{aligned} Q(x,\xi) = \inf\ & f(x,u,\xi) \\ \text{s.t. }\ & u \in \mathcal{G}(x,\xi). \end{aligned} \tag{5.17}$$

We assume that $(Q(\xi))$ has an optimal solution for every fixed scenario $\xi \in \mathcal{U}$, and that $Q(x,\xi)$ also has an optimal solution for all $\xi \in \mathcal{U}$ and $x \in \mathbb{R}^n$. We can hence replace inf by min and $Q(x,\xi)$ is well-defined.

A solution $x \in \mathbb{R}^n$ is called *adjustable robust* if it can be completed to a solution (x,u) which is feasible for $(Q(\xi))$ for every scenario $\xi \in \mathcal{U}$, i.e., if $\mathcal{G}(x,\xi) \neq \emptyset$ for every scenario $\xi \in \mathcal{U}$. The set of adjustable robust solutions is given as

$$\begin{aligned} \mathfrak{A}_{\text{adjust}} &:= \{x \in \mathbb{R}^n \mid \forall\, \xi \in \mathcal{U}\ \exists\, u \in \mathbb{R}^p :\ F_i(x,u,\xi) \leq 0,\ i = 1,\dots,m\} \\ &= \{x \in \mathbb{R}^n \mid \forall\, \xi \in \mathcal{U} : \mathcal{G}(x,\xi) \neq \emptyset\} \end{aligned}$$

and the **adjustable robust counterpart** then reads

$$\rho_{\text{aRC}}(x) = \sup_{\xi \in \mathcal{U}} Q(x,\xi) \to \inf_{x \in \mathfrak{A}_{\text{adjust}}}. \tag{aRC}$$

Note that, similar to the case of reliability (see Section 5.5), the adjustable robust counterpart is closely related to the strictly robust counterpart discussed in Section 5.2.3, only with a modified feasible set.

5.6.1 Vector Optimization Approach for Adjustable Robustness

Let $Y = \mathbb{R}^{\mathcal{U}}$. For $x \in \mathbb{R}^n$ we define the functions $F_x \in Y$ through

$$\begin{aligned} F_x(\xi) &:= Q(x,\xi) \\ &= \inf\{f(x,u,\xi) | u \in \mathcal{G}(x,\xi)\}. \end{aligned}$$

Furthermore, we define

$$\mathcal{F}_{\text{adjust}} := \{F_x \in Y \mid x \in \mathfrak{A}_{\text{adjust}}\}.$$

We obtain

$$\rho_{\text{aRC}}(x) = \sup_{\xi \in \mathcal{U}} F_x(\xi).$$

As ordering relation we again consider the sup order relation

$$F_x \ \leq^{\sup}\ F_y \ :\Longleftrightarrow \left(\sup_{\xi \in \mathcal{U}} F_x(\xi) \ \leq\ \sup_{\xi \in \mathcal{U}} F_y(\xi)\right).$$

Theorem 5.6.1 ([61, Theorem 14])**.** *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (aRC) *if and only if F_x is a minimal element of $\mathcal{F}_{\text{adjust}}$ with respect to the order relation $\leq^{\sup}$.*

Proof. For $x \in \mathfrak{A}_{\text{adjust}}$ we have

$$\begin{aligned} x \text{ is an optimal solution to (aRC)} &\iff \sup_{\xi\in\mathcal{U}} Q(x,\xi) \leq \sup_{\xi\in\mathcal{U}} Q(\overline{x},\xi) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{adjust}} \\ &\iff \sup_{\xi\in\mathcal{U}} F_x(\xi) \leq \sup_{\xi\in\mathcal{U}} F_{\overline{x}}(\xi) \text{ for all } \overline{x} \in \mathfrak{A}_{\text{adjust}} \\ &\iff F_x \leq^{\sup} F_{\overline{x}} \text{ for all } \overline{x} \in \mathfrak{A}_{\text{adjust}} \\ &\iff F_x \leq^{\sup} G \text{ for all } G \in \mathcal{F}_{\text{adjust}} \end{aligned}$$

and the result follows since $\leq^{\sup}$ is a total preorder. □

Also on the new set $\mathcal{F}_{\text{adjust}}$ we can compare elements w.r.t the natural order relation $\leq_{Y^+}$. Since the result of Lemma 5.2.4 only depends on the order relations $\leq_{Y^+}$ and $\leq^{\sup}$ but not on the subset of functions $\mathcal{F} \subseteq Y$ considered, its consequence Corollary 5.2.5 also holds for the case of adjustable robustness. This means for every optimal solution x to (aRC) its outcome function is weakly Pareto minimal, i.e., there is no other solution $\bar{x}$ which is strictly better for every scenario $\xi \in \mathcal{U}$.

Lemma 5.6.2 ([61, Lemma 9]). *Let $Y = C(\mathcal{U},\mathbb{R})$. Assume that every $F \in \mathcal{F}_{\text{adjust}}$ attains its supremum on $\mathcal{U}$. If $F \in \mathcal{F}_{\text{adjust}}$ is a minimal element of $\mathcal{F}_{\text{adjust}}$ w.r.t. $\leq^{\sup}$, then F is a weakly minimal element of $\mathcal{F}_{\text{adjust}}$ w.r.t. the natural order relation $\leq_{Y^+}$.*

Corollary 5.6.3 ([61, Corollary 5]). *Let $Y = C(\mathcal{U},\mathbb{R})$ and let the worst case be attained for every solution $x \in \mathfrak{A}_{\text{adjust}}$. Then for every optimal solution x to the adjustable robust counterpart* (aRC)*, F_x is a weakly minimal element of $\mathcal{F}_{\text{adjust}}$ w.r.t. the natural order relation $\leq_{Y^+}$ in Y.*

5.6.2 Set-Valued Optimization Approach for Adjustable Robustness

A similar analysis can be applied using the set optimization approach. Let $Z = \overline{\mathcal{P}}(\mathbb{R})$, and let $F_x(\xi) = Q(x,\xi)$ as before. We define

$$B_x \ := \ \{Q(x,\xi) | \xi \in \mathcal{U}\} = \{F_x(\xi) | \xi \in \mathcal{U}\}$$

and denote the set of adjustable robust outcome sets by

$$\mathcal{B}_{\text{adjust}} := \{B_x \in Z |\ x \in \mathfrak{A}_{\text{adjust}}\} \subseteq Z.$$

For $B_x, B_y \in Z$ we use the set order relation $\preceq^u_{\mathbb{R}_+}$.

Due to its relation to $\leq^{\sup}$ (see Lemma 5.2.8), we can again use the order relation $\preceq^u_{\mathbb{R}_+}$ for a representation of the adjustable robust counterpart as a set-valued optimization problem.

Theorem 5.6.4 ([61, Theorem 15]). *A solution $x \in \mathbb{R}^n$ is an optimal solution to* (aRC) *if and only if B_x is a minimal element of $\mathcal{B}_{\text{adjust}}$ with respect to the order relation $\preceq^u_{\mathbb{R}_+}$.*

Proof. Let $x \in \mathfrak{A}_{\text{adjust}}$. Then we know from Theorem 5.6.1 that x is an optimal solution to (aRC) if and only if $F_x \leq^{\sup} F_{\overline{x}}$ for all $\overline{x} \in \mathfrak{A}_{\text{adjust}}$. According to Lemma 5.2.8 this is equivalent to $B_x \preceq^u_{\mathbb{R}_+} B_{\overline{x}}$ for all $\overline{x} \in \mathfrak{A}_{\text{adjust}}$. □

5.6.3 Nonlinear Scalarizing Functional for Adjustable Robustness

Theorem 5.6.5 ([61, Theorem 16]). *Let $Y = \mathbb{R}^{\mathcal{U}}$, $B := Y^+$, and $k \equiv 1 \in Y$. Then $x \in \mathbb{R}^n$ is an optimal solution to* (aRC) *if and only if F_x solves problem* $(P_{k,B,\mathcal{F}_{\text{adjust}}})$.

Proof. The proof is completely analogous to the proof of Theorem 5.2.11, with the adapted feasible set $\mathcal{F}_{\text{adjust}}$ instead of $\mathcal{F}_{\text{strict}}$. □

Again, the scalarizations $(P_{k,B,\mathcal{F}_{\text{adjust}}})$ and $(P_{k,B,\mathcal{F}_{\text{strict}}})$ (for adjustable robustness and for strict robustness, respectively) differ only in their feasible sets. Thus, for finite scenario sets problem $(P_{k,B,\mathcal{F}_{\text{adjust}}})$ also corresponds to a reference point approach with the origin as reference point.

Remark 5.6.6 ([61, Remark 17]). *As before, a second stage optimization can be performed, or an appropriate augmentation term can be appended to the scalarizing functional $z^{B,k}$ in order to avoid weakly minimal elements.*

5.7 Certain Robustness as a New Concept Based on Set Relations

In the following, we motivate a new concept of robustness by means of the set-based optimization approach. As is reported in [61], a set-based interpretation given through a set order relation $\preceq$ always implies a corresponding interpretation as a vector optimization problem by defining $F_x \leq F_y :\Leftrightarrow B_x \preceq B_y$. Note that the converse is not true in general, since $\preceq$ is not well-defined through $B_x \preceq B_y :\Leftrightarrow F_x \leq F_y$.

We hence could have motivated the following concept also based on a vector optimization model. However, in the case discussed in this section the set-valued approach is better suited to highlight the particular problem characteristics and their interpretation. Let the sets $B_x, B_y \in Z$ be given. We use an order relation $\preceq^{cert}_{\mathbb{R}_+}$ (see Eichfelder and Jahn [26], compare also Definition 2.2.19) given by

$$\begin{aligned} B_x \ \preceq^{cert}_{\mathbb{R}_+} \ B_y \quad &:\Longleftrightarrow \quad B_x \subseteq B_y - \mathbb{R}_+ \text{ and } B_y \subseteq B_x + \mathbb{R}_+ \\ &\Longleftrightarrow \quad \sup B_x \leq \sup B_y \text{ and } \inf B_x \leq \inf B_y. \end{aligned}$$

This means that a set B_x dominates a set B_y if both the upper bound as well as the lower bound of the set B_x is smaller than the respective upper or lower bounds of the set B_y. We have the following relationship between the order relations $\preceq^u_{\mathbb{R}_+}$, $\preceq^l_{\mathbb{R}_+}$ and $\preceq^{cert}_{\mathbb{R}_+}$:

$$B_x \ \preceq^{cert}_{\mathbb{R}_+} \ B_y \Longrightarrow B_x \ \preceq^{u}_{\mathbb{R}_+} \ B_y \quad \text{as well as} \quad B_x \ \preceq^{l}_{\mathbb{R}_+} \ B_y.$$

Conversely to our approaches in in Section 5.2, 5.3, 5.4, 5.5 and 5.6, we now use $\preceq^{cert}_{\mathbb{R}_+}$ to define *certainly robust solutions.* To this end, let $\mathcal{B}_{\text{strict}} \subseteq Z$ be defined as the set of strictly robust outcome sets (see (5.2.2)).

$$x \text{ is } \textit{certainly robust} :\Longleftrightarrow B_x \text{ is a minimal element of } \mathcal{B}_{\text{strict}} \text{ w.r.t } \preceq^{cert}_{\mathbb{R}_+}.$$

This approach is useful if a decision maker is interested in solutions that are not dominated either by their upper or by their lower bounds. Note that by using the order relation $\preceq^{cert}_{\mathbb{R}_+}$, we obtain a rather weak concept. Nonetheless, the concept of certain minimality filters out solutions which are obviously bad choices.

5.8 Discussion of the Set-Valued Approach to Uncertain Programming

Since the set-based interpretation does not reflect distributional information as needed, for example, when minimizing the expectation or in 2-stage stochastic programming models, its applicability is mainly restricted to concepts from robust optimization. We present a summary in Table 5.1.

Concept	Order relation	$B_x, B_y \in Z: \ B_x \preceq B_y \iff$	$\mathfrak{A}$
Strict robustness (Sec. 5.2.2)	$\preceq^u_{\mathbb{R}_+}$	$\sup B_x \leq \sup B_y$	$\mathfrak{A}_{\text{strict}} = \{x \in \mathbb{R}^n \mid \forall\ \xi \in \mathcal{U}: \ F_i(x,\xi) \leq 0, \ i = 1,\dots,m\}$
Optimistic robustness (Sec. 5.3.2)	$\preceq^l_{\mathbb{R}_+}$	$\inf B_x \leq \inf B_y$	$\mathfrak{A}_{\text{strict}} = \{x \in \mathbb{R}^n \mid \forall\ \xi \in \mathcal{U}: \ F_i(x,\xi) \leq 0, \ i = 1,\dots,m\}$
Regret robustness (Sec. 5.4.2)	$\preceq^u_{\mathbb{R}_+}$	$\sup B_x^{f-f^*} \leq \sup B_y^{f-f^*}$	$\mathfrak{A}_{\text{strict}} = \{x \in \mathbb{R}^n \mid \forall\ \xi \in \mathcal{U}: \ F_i(x,\xi) \leq 0, \ i = 1,\dots,m\}$
Reliability (Sec. 5.5.2)	$\preceq^u_{\mathbb{R}_+}$	$\sup B_x \leq \sup B_y$	$\mathfrak{A}_{\text{rely}} = \{x \in \mathbb{R}^n \mid \forall \xi \in \mathcal{U}: \ F_i(x,\xi) \leq \delta_i, \ F_i(x,\hat{\xi}) \leq 0, \ i = 1,\dots,m\}$
Adjustable robustness (Sec. 5.6.2)	$\preceq^u_{\mathbb{R}_+}$	$\sup B_x \leq \sup B_y$	$\mathfrak{A}_{\text{adjust}} = \{x \in \mathbb{R}^n \mid \forall\ \xi \in \mathcal{U}\ \exists\ u \in \mathbb{R}^p: \ F_i(x,u,\xi) \leq 0, \ i = 1,\dots,m\}$
Certain robustness (Sec. 5.7)	$\preceq^{cert}_{\mathbb{R}_+}$	$\sup B_x \leq \sup B_y$ and $\inf B_x \leq \inf B_y$	$\mathfrak{A}_{\text{strict}} = \{x \in \mathbb{R}^n \mid \forall\ \xi \in \mathcal{U}: \ F_i(x,\xi) \leq 0, \ i = 1,\dots,m\}$

Table 5.1: Summary of interpretations using set-based counterparts

Table 5.1 shows that the role of the order relation $\leq$ in the vector optimization approach is now taken by the set order relation $\preceq$ while the other important characteristic is still the feasible set $\mathfrak{A}$. The relation $\preceq$ is hence important to consider if we want to derive a comparison or classification of different robustness concepts, e.g., a measure for the level of conservatism. This can be illustrated looking at the concepts strict robustness, optimistic robustness, and certain robustness, which use the same sets B_x, the same feasible set $\mathfrak{A}$, but the three different set order relations $\preceq^u_{\mathbb{R}_+}$, $\preceq^l_{\mathbb{R}_+}$, and $\preceq^{cert}_{\mathbb{R}_+}$. Comparing the definitions of these three set order relations, we see that $\preceq^u_{\mathbb{R}_+}$ is only defined by using the supremum (i.e., the worst case over all scenarios), hence it is the most conservative of these three concepts. $\preceq^l_{\mathbb{R}_+}$ does not at all consider the supremum; and in fact, nobody would call this a conservative concept. It is rather a very risky concept, which is well-suited for a risk-affine decision-maker. Finally, $\preceq^{cert}_{\mathbb{R}_+}$ lies somewhere in between, trying to get as much as possible while still looking at the worst case.

Note that we have also seen that the set order relation $\preceq$ is closely connected to the order relation $\leq$: $\preceq^u_{\mathbb{R}_+}$ corresponds to $\leq^{\sup}$ (Lemma 5.2.8), $\preceq^l_{\mathbb{R}_+}$ corresponds to $\leq^{\inf}$ (Lemma 5.3.7). Changing the notation for regret robustness slightly by defining

$$B_x \preceq^{\text{regret}} B_y :\Longleftrightarrow B_x^{f-f^*} \preceq^u_{\mathbb{R}_+} B_y^{f-f*},$$

we analogously receive that

$$B_x \preceq^{\text{regret}} B_y \Longleftrightarrow F_x \leq^{\text{regret}} F_y,$$

i.e., again a correspondence. We remark that *any* set relation $\preceq$ can be used to define a corresponding order relation $\leq$ by setting

$$F_x \leq F_y :\Longleftrightarrow B_x \preceq B_y,$$

but not vice versa. The reason is that the function

$$\vartheta : Y \to Z, \;\; \vartheta(F_x) := \{F_x(\xi) : \xi \in \mathcal{U}\}$$

is well-defined, but not injective, i.e., B_x is uniquely determined by F_x while F_x cannot be determined uniquely from B_x. This shows again that the idea of using vector optimization is more flexible in this context.

Evidently, set order relations $\preceq$ play a significant role in the unifying concept based on set optimization. There are several set order relations known in the literature. A classification of set order relations can be found in Khan et al. [59, Chapter 2.6.2] (see also Chapter 2) and Kuroiwa [76], and an embedding approach of set optimization is presented in Kuroiwa and Nuriya [78]. Of course, it would be very interesting to classify robustness concepts based on a classification of the underlying set order relations $\preceq$.

Chapter 6

Conclusions

This work presents some novel directions in the optimization process of set-valued mappings, with a focus on sets that do not necessarily need to be convex. We present extensions of set relations, where the involved sets as well as the set describing the domination structure are arbitrary, nonempty sets. In particular, characterizations of generalized set relations by means of a nonlinear functional are presented and a new set relation is derived based on these characterizations. This new relation comprises the upper and lower set less relation as special cases and builds a compromise between the two. We moreover show an existence result for this new set relation. Several algorithmic methods are proposed to facilitate computing minimal solutions of set optimization problems. Specifically, we develop a descent method that can be used for solving continuous problems, and Jahn-Graef-Younes-type methods that filter out sets that cannot be minimal and finally return all minimal elements. Moreover, our analysis is extended to the case when the objective space is not a priori equipped with a particular topology. Therefore, our analysis shows that extensions to linear spaces are not only possible, but also useful. In order to include as much of the decision-maker's expertise into the model as is possible and useful, we furthermore propose to include a variable domination structure when defining set relations. Since variable domination structures possess various applications in the medical field, for instance in medical image registration, our concepts prove to be very valuable for decision-making under uncertainty. We analyze optimal elements of sections and give scalarization results for set optimization problems involving a variable domination structure. As it is well known that minimal elements of set optimization problems do not always exist, we investigate several new notions of approximate solutions. These notions are analyzed and treated in term of linear and nonlinear scalarization functionals. When the family of sets is finite, we propose efficient algorithms to compute the approximate minimal solutions. Finally, an application is given in the field of uncertain programming, and it is shown, among others, that a large number of concepts from robust optimization can be treated in a unifying framework using set optimization techniques.

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Selbständigkeitserklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Habilitationsschrift selbständig und ohne fremde Hilfe angefertigt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

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