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To My Parents

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Chapter 1

Introduction

The present thesis is devoted to the study of the L_p-theory of a class of quasilinear parabolic problems with nonlinear boundary conditions. The main objective here is to prove existence and uniqueness of local (in time) strong solutions of these problems. To achieve this we establish optimal regularity estimates of type L_p for an associated linear problem which allow us to reformulate the original problem as a fixed point equation in the desired regularity class, and we show that under appropriate assumptions the contraction mapping principle is applicable, provided the time-interval is sufficiently small.

We describe now the class of equations to be studied. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2-smooth boundary Γ which decomposes according to $\Gamma = \Gamma_D \cup \Gamma_N$ with $\mathrm{dist}(\Gamma_D, \Gamma_N) > 0$. For the unknown scalar function $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, we consider the subsequent problem:

$$\left\{\begin{array}{rl} \partial_t u + dk * (\mathcal{A}(u) : \nabla^2 u) = F(u) + dk * G(u), & t \geq 0,\, x \in \Omega \\ \mathcal{B}_D(u) = 0, & t \geq 0,\, x \in \Gamma_D \\ \mathcal{B}_N(u) = 0, & t \geq 0,\, x \in \Gamma_N \\ u|_{t=0} = u_0, & x \in \Omega. \end{array}\right. \tag{1.1}$$

Here, $(dk*w)(t,x) = \int_0^t dk(\tau)w(t-\tau,x)$, $t \geq 0$, $x \in \Omega$, $\partial_t u$ means the partial derivative of u w.r.t. t, $\nabla u = \nabla_x u$ is the gradient of u w.r.t. the spatial variables, $\nabla^2 u$ denotes its Hessian matrix, that is $(\nabla^2 u)_{ij} = \partial_{x_i}\partial_{x_j} u$, $i, j \in \{1, \ldots, n\}$, and $B : C = \sum_{i=1,\, j=1}^{n} B_{ij} C_{ij}$ stands for the double scalar product of two matrices B, $C \in \mathbb{R}^{n\times n}$. Furthermore, we have the substitution operators

$$\begin{aligned} \mathcal{A}(u)(t,x) &= -a(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\ x \in \Omega, \\ F(u)(t,x) &= f(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\ x \in \Omega, \\ G(u)(t,x) &= g(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\ x \in \Omega, \\ \mathcal{B}_D(u)(t,x) &= b^D(t,x,u(t,x)),\ t \geq 0,\ x \in \Gamma_D, \\ \mathcal{B}_N(u)(t,x) &= b^N(t,x,u(t,x),\nabla u(t,x)),\ t \geq 0,\ x \in \Gamma_N, \end{aligned}$$

where a is $\mathbb{R}^{n\times n}$-valued, and f, g, b^D, b^N are all scalar functions. The scalar-valued kernel k is of bounded variation on each compact interval $[0,T]$ with $k(0) = 0$, and belongs to a certain kernel class with parameter $\alpha \in [0,1)$ which contains, roughly speaking, all 'regular' kernels that behave like t^α for $t\,(>0)$ near zero. Note that this formulation includes the special case $k(t) = 1$, $t > 0$, in which (1.1) amounts to the quasilinear initial-boundary value problem

$$\left\{\begin{array}{rl} \partial_t u + \mathcal{A}(u) : \nabla^2 u = H(u), & t \geq 0,\, x \in \Omega \\ \mathcal{B}_D(u) = 0, & t \geq 0,\, x \in \Gamma_D \\ \mathcal{B}_N(u) = 0, & t \geq 0,\, x \in \Gamma_N \\ u|_{t=0} = u_0, & x \in \Omega, \end{array}\right. \tag{1.2}$$

where $H(u) = F(u) + G(u)$. Observe further that the case $k(t) = t$, $t \geq 0$, is not admissible; in our setting, this kernel would lead to a hyperbolic problem.

Although there is a wide literature on problems of the form (1.1), not much seems to be known towards an L_p-theory in the integrodifferential case with *nonlinear boundary conditions*, even in the linear situation with inhomogeneous Dirichlet and/or Neumann boundary conditions. Before presenting the main result concerning (1.1) and commenting on available results in the literature we give some motivation for the study of these problems.

Equations of the form (1.1) appear in a variety of applied problems. They typically arise in mathematical physics by some constitutive laws pertaining to materials with memory when combined with the usual conservation laws such as balance of energy or balance of momentum. To illustrate this point, we give an example from the theory of heat conduction with memory. For details concerning the underlying physical principles, we refer to Nohel [59]. See also Clément and Nohel [23], Clément and Prüss [25], Lunardi [54], Nunziato [60], and Prüss [63] for work on this subject.

Example: (Nonlinear heat flow in a material with memory)
Consider the heat conduction in a 3-dimensional rigid body which is represented by a bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class C^1. Let $\varepsilon(t, x)$ denote the density of internal energy at time $t \in \mathbb{R}$ and position $x \in \Omega$, $q(t, x)$ the heat flux vector field, $u(t, x)$ the temperature, and $h(t, x)$ the external heat supply. The law of balance of energy then reads as

$$\partial_t \varepsilon(t, x) + \operatorname{div} q(t, x) = h(t, x), \quad t \in \mathbb{R},\, x \in \Omega. \tag{1.3}$$

Equation (1.3) has to be supplemented by boundary conditions; these are basically either prescribed temperature or prescribed heat flux through the boundary, that is to say

$$u(t, x) = u_b(t, x), \quad t \in \mathbb{R},\, x \in \Gamma_b, \tag{1.4}$$

$$-q(t, x) \cdot n(x) = q_f(t, x), \quad t \in \mathbb{R},\, x \in \Gamma_f, \tag{1.5}$$

where Γ_b and Γ_f are assumed to be disjoint closed subsets of $\partial\Omega$ with $\Gamma_b \cup \Gamma_f = \partial\Omega$, and $n(x)$ denotes the outer normal of Ω at $x \in \partial\Omega$. In order to complete the system we have to add constitutive equations for the internal energy and the heat flux reflecting the properties of the material the body is made of. In what is to follow we shall consider an isotropic and homogeneous material with memory. Following [23], [39], [60] and many other authors, we will use the laws

$$\varepsilon(t, x) = \int_0^\infty dm(\tau) u(t - \tau, x), \quad t \in \mathbb{R},\, x \in \Omega, \tag{1.6}$$

$$q(t, x) = -\int_0^\infty dc(\tau) \sigma(\nabla u(t - \tau, x)), \quad t \in \mathbb{R},\, x \in \Omega, \tag{1.7}$$

where m, $c \in BV_{loc}(\mathbb{R}_+)$, and $\sigma \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ are given functions. Note that the heat flux here depends *nonlinearly* on the history of the gradient of u. It is physically reasonable to assume that m, c are bounded functions of the form $m(t) = m_0 + (1 * m_1)(t)$, $t > 0$, $m(0) = 0$, and $c(t) = c_0 + (1 * c_1)(t)$, $t > 0$, $c(0) = 0$, respectively, with $m_0 > 0$, $c_0 \geq 0$, and m_1, $c_1 \in L_1(\mathbb{R}_+)$. Here and in the sequel, $f_1 * f_2$ denotes the convolution of two functions defined by $(f_1 * f_2)(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$, $t \geq 0$.

Without loss of generality we may assume that the material is at zero temperature up to time $t = 0$, and is then exposed to a sudden change of temperature $u(0,x) = u_0(x)$, $x \in \Omega$; otherwise one has to add a known forcing term in both (1.8) and (1.10) below that incorporates the history of the temperature up to time $t = 0$. Then (1.3)-(1.7) yield

$$\begin{aligned} \partial_t(dm * u) - dc * (\operatorname{div} \sigma(\nabla u)) &= h,\ t > 0,\ x \in \Omega, && (1.8)\\ u &= u_b,\ t > 0,\ x \in \Gamma_b, && (1.9)\\ dc * \sigma(\nabla u) &= q_f,\ t > 0,\ x \in \Gamma_f, && (1.10)\\ u|_{t=0} &= u_0,\ x \in \Omega. && (1.11) \end{aligned}$$

We show now that (1.8)-(1.11) can be transformed to a problem of the form (1.1), see also [23]. Note first that without restriction of generality we may assume $m_0 = 1$. By integrating (1.8) with respect to time we obtain

$$u + m_1 * u - c * (\operatorname{div} \sigma(\nabla u)) = 1 * h + u_0, \quad t \geq 0,\ x \in \Omega. \tag{1.12}$$

Define the resolvent kernel $r \in L_{1,loc}(\mathbb{R}_+)$ associated with m_1 as the unique solution of the convolution equation

$$r + m_1 * r = m_1, \quad t \geq 0.$$

Application of the operator $(I - r*)$ to (1.12) then results in

$$u - (c - r * c) * (\operatorname{div} \sigma(\nabla u)) = 1 * (h - r * h - r u_0) + u_0. \tag{1.13}$$

Using (formally) the chain rule yields

$$\operatorname{div} \sigma(\nabla u(t,x)) = D\sigma(\nabla u(t,x)) : \nabla^2 u(t,x), \quad t \geq 0,\ x \in \Omega,$$

$D\sigma$ denoting the Jacobian of σ. Hence, with $k = c - r * c$, $f = h - r * h - r u_0$, and $a = D\sigma$, it follows by differentiation of (1.13) that

$$\partial_t u - dk * (a(\nabla u) : \nabla^2 u) = f, \quad t \geq 0,\ x \in \Omega,$$

which is a special form of the integrodifferential equation in (1.1). Lastly, if c belongs to a certain class of 'nice' kernels, one can invert the convolution with the measure dc and thus rewrite (1.10) as a nonlinear boundary condition of non-memory type as in (1.1). □

Another important application is the theory of viscoelasticity; here problems of the form (1.1) naturally occur when balance of momentum is combined with nonlinear stress-strain relations of memory type. General treatises on this field are, for example, Antman [3], Christensen [12], and Renardy, Hrusa, and Nohel [71], but we also refer the reader to Chow [11], Engler [33], and Prüss [63]. A short account of the basic equations in the linear vector-valued case is given in Chapter 5.

Having motivated the investigation of (1.1) by examples from mathematical physics, we describe next the main result to (1.1), which is stated in Theorem 6.1.2. For $T > 0$ and $1 < p < \infty$, set $J = [0,T]$ and define the space Z^T by

$$Z^T = H_p^{1+\alpha}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)).$$

Here $H_p^s(J; L_p(\Omega))$ $(s > 0)$ means the vector-valued Bessel potential space of functions on J taking values in the Lebesgue space $L_p(\Omega)$. We assume $n + 2/(1+\alpha) < p < \infty$, a condition which ensures that the embedding $Z^T \hookrightarrow C(J; C^1(\overline{\Omega}))$ is valid. Theorem 6.1.2 now asserts that under suitable assumptions on the nonlinearities and the initial data, problem (1.1) admits a unique local in time strong solution in the following sense: there exists $T > 0$ such that there is one and only one function $u \in Z^T$ that satisfies (1.1), the integrodifferential equation almost everywhere on $J \times \Omega$, the initial and boundary conditions being fulfilled pointwise on the entire sets considered.

As to literature, there has been a substantial amount of work on nonlinear Volterra and integrodifferential equations. We can only mention some of the main results here. Using maximal C^α-regularity for linear parabolic differential equations, in 1985 Lunardi and Sinestrari [56] were able to prove local existence and uniqueness in spaces of Hölder continuity for a large class of fully nonlinear integrodifferential equations with a homogeneous linear boundary condition. However, to make their approach work, they assume (in our terminology) that the kernel k has a jump at $t = 0$, a property which is not required in this thesis. Concerning C^α-theory for Volterra and integrodifferential equations, we further refer the reader to Da Prato, Iannelli, Sinestrari [28], Lunardi [53], Lunardi and Sinestrari [55], Prüss [63]; for the case of fractional differential equations see also Clément, Gripenberg, Londen [17], [18], [19], and the survey article Clément, Londen [21]. The standard reference for parabolic partial differential equations in this context is Lunardi [52].

In the L_p-setting, quasilinear integrodifferential equations were first studied by Prüss [68]. He also employs the method of maximal regularity, now in spaces of integrable functions, to obtain existence and uniqueness of strong solutions of the scalar problem

$$\left\{\begin{array}{l} \partial_t u(t,x) = \int_0^t dk(\tau)\{\operatorname{div} g(x, \nabla u(t-\tau, x)) + f(t-\tau, x)\}, \quad t \in J,\, x \in \Omega \\ \quad u(t,x) = 0, \quad t \in J,\, x \in \partial\Omega \\ \quad u(0,x) = u_0(x), \quad x \in \Omega \end{array}\right. \tag{1.14}$$

in the class $H_p^1(J; L_q(\Omega)) \cap L_p(J; H_q^2(\Omega))$ provided that either T or the data u_0, f are sufficiently small. In the latter case he further shows existence and uniqueness for the corresponding problem on the line. The kernel $k \in BV_{loc}(\mathbb{R}_+)$ involved is assumed to be 1-regular in the sense of [68, p. 405] and to fulfill an angle condition of the form

$$|\arg \widehat{dk}(\lambda)| \le \theta < \frac{\pi}{2},\ \operatorname{Re}\lambda > 0, \tag{1.15}$$

where the hat indicates Laplace transform. So, e.g., the important case $k(t) = t^\alpha$, $t \ge 0$, with $\alpha \in (0,1)$ is covered. The author's approach to maximal regularity basically relies on the inversion of the convolution operator in L_p-spaces (see Section 2.8), on the Dore-Venni theorem about the sum of two operators with bounded imaginary powers (see Section 2.3), and on results of Prüss and Sohr [70] about bounded imaginary powers of second order elliptic operators. We point out that these tools will also play an important role in the present work.

For $\Omega = (0,1)$, g not depending on x, and with $k = 1 * k_1$, that is $dk * w = k_1 * w$, global existence of strong solutions of (1.14) $(J = \mathbb{R}_+)$ with $u \in L_{2,loc}(\mathbb{R}_+; H_2^2([0,1]))$ was established by Gripenberg under different assumptions on g and the kernel k_1; in [37] he considers kernels k_1 satisfying (1.15), while in [38] k_1 is assumed to be nonnegative, nonincreasing, convex, and more singular at 0 than $t^{-1/2}$. Engler [34] extended the results of the latter work by treating also higher space dimensions and by allowing for a larger class of nonlinear functions g.

We give now an overview of the contents of the thesis and present the principal ideas in greater detail. The text is divided into three main parts, devoted respectively to preliminaries (Chapter 2), linear theory (Chapters 3, 4, 5), and nonlinear problems (Chapter 6).

Chapter 2 collects the basic tools needed for the investigation of the linear equations to be studied. After fixing some notations, in Section 2.2 we review important classes of sectorial operators, among others, operators which admit a bounded $\mathcal{H}^\infty$-calculus, operators with bounded imaginary powers, and $\mathcal{R}$-sectorial operators. We further discuss some properties of the fractional powers of such operators in connection with real and complex interpolation, and prove that the power A^α, $\alpha \in \mathbb{R}$, of an $\mathcal{R}$-sectorial operator A with $\mathcal{R}$-angle ϕ_A^R is $\mathcal{R}$-sectorial, too, as long as the inequality $|\alpha|\phi_A^R < \pi$ holds (Proposition 2.2.1); the latter result seems to be missing in the literature. In Section 2.3, which is devoted to sums of closed linear operators, we state a variant of the Dore-Venni theorem. Section 2.4 is concerned with the joint $\mathcal{H}^\infty$-calculus for pairs of sectorial operators. In particular, we look at the calculus for the pair $(\partial_t, -\Delta_x)$ in the space $L_p(\mathbb{R}_+ \times \mathbb{R}^n)$, which proves extremely useful in establishing optimal regularity results in Chapter 5. Section 2.5 deals with operator-valued Fourier multipliers. The central result here is the Mikhlin multiplier theorem in the operator-valued version, which was proven recently by Weis [80]. In Section 2.6 we introduce the class of $\mathcal{K}$-kernels consisting of all 1-regular, sectorial kernels k whose Laplace transform $\hat{k}(\lambda)$ behaves like $\lambda^{-\alpha}$ as $\lambda \to 0, \infty$ for some $\alpha \geq 0$; an example is given by $k(t) = t^{\alpha-1}e^{-\beta t}$, $t \geq 0$, with $\alpha > 0$ and $\beta \geq 0$. Kernels of that type have already been studied by Prüss [63] in the context of Volterra operators in L_p, which is the subject of Section 2.8. Before, in Section 2.7 we give a short account of the abstract Volterra equation

$$u(t) + \int_0^t a(t-s)Au(s)\,ds = f(t), \quad t \geq 0, \tag{1.16}$$

where $a \in L_{1,loc}(\mathbb{R}_+)$ is a scalar kernel, and A is a closed linear operator in a Banach space X. We explain the notion of parabolicity of (1.16), give the definition of resolvents, and recall the variation of constants formula. Section 2.8 is devoted to convolution operators in L_p associated to a $\mathcal{K}$-kernel. After stating two fundamental theorems from Prüss [63] on the inversion of such operators in $L_p(\mathbb{R}; X)$ with $1 < p < \infty$ and $X \in \mathcal{HT}$, we consider restrictions of them to $L_p(J; X)$, where $J = [0, T]$ or $\mathbb{R}_+$. The main facts about these operators are summarized in Corollary 2.8.1. It asserts that for every $\mathcal{K}$-kernel k with angle $\theta_k < \pi$ there is a unique sectorial operator $\mathcal{B}$ in $L_p(J; X)$ inverting the convolution $(k*)$, and that this operator - assuming in addition $k \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$ - is invertible and satisfies $\mathcal{B}^{-1}w = k * w$ for all $w \in L_p(J; X)$; it further says that $\mathcal{B} \in \mathcal{BIP}(L_p(J; X))$ and that its domain $\mathcal{D}(\mathcal{B})$ equals the space ${}_0H_p^\alpha(J; X)$, where $\alpha \geq 0$ refers to the order of k in the sense describe above. So, we have precise information about the mapping properties of the convolution operators under study and see that their inverse operators are accessible to the Dore-Venni theorem. In Section 2.8 we further recognize the fractional derivative $(d/dt)^\alpha$ of order $\alpha \in (0, 1)$ to be the inverse convolution operator associated with the standard kernel $t^{\alpha-1}/\Gamma(\alpha)$. Besides, we introduce equivalent norms for the spaces $H_p^\alpha(J; X)$ and consider operators of the form $(I - k*)$, which appear in connection with transformations of Volterra equations, cf. the above example on heat conduction.

The main purpose of *Chapter 3* is to establish maximal regularity results of type L_p for equation (1.16) as well as for a class of abstract linear Volterra equations on an infinite strip $J \times \mathbb{R}_+$ with inhomogeneous boundary condition of Dirichlet resp. (abstract)

Robin type. Unique existence of solutions of these problems in certain spaces of optimal regularity is characterized in terms of regularity and compatibility conditions on the given data. The main result concerning (1.16), Theorem 3.1.4, is proven in Section 3.1. To describe it for the case $J = [0, T]$, let $1 < p < \infty$, $\kappa \in [0, 1/p)$, X be a Banach space of class $\mathcal{HT}$, A an $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle ϕ_A^R, and a a $\mathcal{K}$-kernel (with angle θ_a) of order $\alpha \in (0, 2)$ such that $\alpha + \kappa \notin \{1/p, 1 + 1/p\}$. Let further D_A denote the domain of A equipped with the graph norm of A. Assume the parabolicity condition $\theta_a + \phi_A^R < \pi$. Then (1.16) has a unique solution u in the space $H_p^{\alpha+\kappa}(J; X) \cap H_p^\kappa(J; D_A)$ if and only if the function f satisfies the subsequent conditions:

$$\begin{aligned} (i) &\quad f \in H_p^{\alpha+\kappa}(J; X); \\ (ii) &\quad f(0) \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{p\alpha}, p), \text{ if } \alpha + \kappa > 1/p; \\ (iii) &\quad \dot{f}(0) \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{\alpha} - \tfrac{1}{p\alpha}, p), \text{ if } \alpha + \kappa > 1 + 1/p. \end{aligned}$$

Here, $D_A(\gamma, p)$ stands for the real interpolation space $(X, D_A)_{\gamma,p}$. In the special case $a \equiv 1$ (i.e. $\alpha = 1$) and $\kappa = 0$, by putting $g = \dot{f}$ and $u_0 = f(0)$, we recover the main theorem on maximal L_p-regularity for the abstract evolution equation

$$\dot{u} + Au = g, \ t \in J, \quad u(0) = u_0, \tag{1.17}$$

stating that in the above setting, unique solvability of (1.17) in the space $H_p^1(J; X) \cap L_p(J; D_A)$ is equivalent to the conditions $g \in L_p(J; X)$ and $u_0 \in D_A(1 - 1/p, p)$. We remark that the motivation for considering also the case $\kappa > 0$ comes from the problem studied in Chapter 5 which involves *two* independent kernels.

The proof of Theorem 3.1.4 essentially relies on techniques developed in Prüss [64] using the representation of the resolvent S for (1.16) via Laplace transform, as well as on the Mikhlin theorem in the operator-valued version. With the aid of the latter result and an approximation argument, we succeed in showing $L_p(\mathbb{R}; X)$-boundedness of the operator corresponding to the symbol $M(\rho) = A((\hat{a}(i\rho))^{-1} + A)^{-1}$, $\rho \in \mathbb{R} \setminus \{0\}$; this operator is closely related to the variation of parameters formula.

After proving a rather general embedding theorem in Section 3.2, we continue the study of (1.16), now focusing on the case $\kappa \in (1/p, 1 + 1/p)$, and establish a result corresponding to Theorem 3.1.4. This is done in Section 3.3. In Section 3.4 we collect some known results on maximal L_p-regularity of abstract problems on the halfline. Among others, we consider two abstract second order equations that play a crucial role in the treatment of problems on a strip which are respectively of the form

$$\left\{ \begin{array}{r} u - a * \partial_y^2 u + a * Au = f, \ t \in J, \ y > 0, \\ u(t, 0) = \phi(t), \ t \in J, \end{array} \right. \quad \left\{ \begin{array}{r} u - a * \partial_y^2 u + a * Au = f, \ t \in J, \ y > 0, \\ -\partial_y u(t, 0) + Du(t, 0) = \phi(t), \ t \in J, \end{array} \right. \tag{1.18}$$

where a is a $\mathcal{K}$-kernel of order $\alpha \in (0, 2)$, and A and D are sectorial resp. pseudo-sectorial operators in a Banach space X with $D_{A^{1/2}} \hookrightarrow D_D$. The investigation of these problems is pursued in Section 3.5. We prove results characterizing unique solvability of (1.18) in the regularity class $H_p^\alpha(J; L_p(\mathbb{R}_+; X)) \cap L_p(J; H_p^2(\mathbb{R}_+; X)) \cap L_p(J; L_p(\mathbb{R}_+; D_A))$ in terms of regularity and compatibility conditions on the data. Besides the results concerning (1.16) and that from Section 3.4 we make here repeatedly use of the inversion of the convolution, the Dore-Venni theorem, as well as properties of real interpolation.

Chapter 4 is devoted to the study of linear scalar problems of second order in the space $L_p(J \times \Omega)$, $J = [0, T]$ and Ω a domain in $\mathbb{R}^n$, with general inhomogeneous boundary conditions of order ≤ 1. Sections 4.1 and 4.2 deal with the full resp. half space case. The theory from Chapter 3 is applied to find necessary and sufficient conditions on the data that characterize maximal L_p-regularity of the solutions. The strategy is to look first at problems with constant coefficients and differential operators consisting only of their principle parts, and then to use pointwise multiplication properties of the function spaces involved together with perturbation arguments to extend the results to the general case. In Section 4.3 we study the case of an arbitrary domain $\Omega \subset \mathbb{R}^n$ with compact C^2-smooth boundary Γ decomposing into two disjoint closed parts Γ_D and Γ_N on which inhomogeneous boundary conditions of zeroth resp. first order have to be satisfied. The basic idea here is to employ the localization method to reduce the problem to related problems on $\mathbb{R}^n$ and $\mathbb{R}^n_+$. Proceeding this way we obtain a characterization of unique solvability of the problem

$$\left\{\begin{aligned} v + k * \mathcal{A}(\cdot, x, D_x)v &= f, \quad t \in J,\ x \in \Omega, \\ v &= g, \quad t \in J,\ x \in \Gamma_D, \\ \mathcal{B}(t, x, D_x)v &= h, \quad t \in J,\ x \in \Gamma_N, \end{aligned}\right. \tag{1.19}$$

with

$$\mathcal{A}(t, x, D_x) = -a(t, x) : \nabla_x^2 + a_1(t, x) \cdot \nabla_x + a_0(t, x),\ t \in J,\ x \in \Omega,$$

$$\mathcal{B}(t, x, D_x) = b(t, x) \cdot \nabla_x + b_0(t, x),\ t \in J,\ x \in \Gamma_N,$$

in the regularity class $H^\alpha_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega))$. Here as before, k is a $\mathcal{K}$-kernel of order $\alpha \in (0, 2)$. As to the regularity of the top order coefficients, we only assume $a \in C(J \times \overline{\Omega}, \mathbb{R}^{n\times n})$ and that a has a limit as $|x| \to \infty$ uniformly w.r.t. $t \in J$.

In *Chapter 5* we are concerned with a linear parabolic problem of second order which appears in the theory of viscoelasticity. In comparison to the problems investigated in Chapter 4, it has two new challenging features: first it is a *vector*-valued problem, and second it contains *two independent kernels*. Once more we characterize unique existence of the solution in a certain class of optimal regularity in terms of regularity and compatibility conditions on the given data. Section 5.1 gives a short account of the basic equations of linear viscoelasticity. In Section 5.2 we state the problem and discuss the assumptions on the kernels, which are stronger than in the previous chapters, since the method of proof relies heavily on $\mathcal{H}^\infty$-calculus. Section 5.3 is devoted to the thorough investigation of a half space case of the problem under study, which reads

$$\left\{\begin{aligned} \partial_t v - da * (\Delta_x v + \partial_y^2 v) - (db + \tfrac{1}{3}da) * (\nabla_x \nabla_x \cdot v + \partial_y \nabla_x w) &= f_v \quad && (J \times \mathbb{R}^{n+1}_+) \\ \partial_t w - da * \Delta_x w - (db + \tfrac{4}{3}da) * \partial_y^2 w - (db + \tfrac{1}{3}da) * \partial_y \nabla_x \cdot v &= f_w \quad && (J \times \mathbb{R}^{n+1}_+) \\ -da * \gamma \partial_y v - da * \gamma \nabla_x w &= g_v \quad && (J \times \mathbb{R}^n) \\ -(db - \tfrac{2}{3}da) * \gamma \nabla_x \cdot v - (db + \tfrac{4}{3}da) * \gamma \partial_y w &= g_w \quad && (J \times \mathbb{R}^n) \\ v|_{t=0} &= v_0 \quad && (\mathbb{R}^{n+1}_+) \\ w|_{t=0} &= w_0 \quad && (\mathbb{R}^{n+1}_+), \end{aligned}\right.$$

where the unknown functions v and w are $\mathbb{R}^n$- resp. scalar-valued, and γ denotes the trace operator at $y = 0$. To solve this problem, we introduce an appropriate auxiliary function by which the system can be decoupled. Using the results from Chapter 3, the problem is further reduced to an equation on the boundary, which can be solved by means of the joint $\mathcal{H}^\infty$-calculus for the pair $(\partial_t, -\Delta_x)$ in the space $L_p(\mathbb{R}_+ \times \mathbb{R}^n)$. The essential difficulty is the estimate for the principal symbol of the problem. To be precise, we have to show that there exist $c > 0$ and $\eta \in (0, \pi/2)$ such that the inequality

$$\left|\frac{1}{\hat{a}(z)\tau^2} + 2\right| \leq c \left| \frac{1}{\hat{a}(z)\tau^2} + \frac{\frac{4\hat{b}(z)+\frac{4}{3}\hat{a}(z)}{\hat{b}(z)+\frac{4}{3}\hat{a}(z)}\sqrt{\frac{1}{\hat{a}(z)\tau^2}+1}}{\sqrt{\frac{1}{\hat{a}(z)\tau^2}+1}+\sqrt{\frac{1}{(\hat{b}(z)+\frac{4}{3}\hat{a}(z))\tau^2}+1}} \right|, \ (z,\tau) \in \Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta$$

holds true; here $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$. This crucial estimate is obtained by a careful function theoretic analysis.

Finally, in *Chapter 6* we study the nonlinear problem (1.1) described at the beginning and prove the last main result of the present thesis, Theorem 6.1.2, by means of the contraction mapping principle employing the optimal regularity results obtained for the linear problem (1.19). Section 6.1 deals with the fixed point construction and the basic estimates. It also contains the list of all assumptions needed for our treatment of (1.1). The proof of the harder estimates concerning in particular the nonlinearities on the boundary is deferred to Section 6.2.

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Chapter 2

Preliminaries

2.1 Some notation, function spaces, Laplace transform

In this section we fix some of the notations used throughout this thesis, recall some basic definitions and give references concerning function spaces and the Laplace transform.

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of natural numbers, integers, real and complex numbers, respectively. Let further $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > 0\}$. X, Y, Z will usually be Banach spaces; $|\cdot|_X$ designates the norm of the Banach space X. The symbol $\mathcal{B}(X, Y)$ means the space of all bounded linear operators from X to Y, we write $\mathcal{B}(X) = \mathcal{B}(X, X)$ for short. If A is a linear operator in X, $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ stand for domain, range, and null space of A, respectively, while $\rho(A)$, $\sigma(A)$ designate resolvent set and spectrum of A. For a closed operator A we denote by D_A the domain of A equipped with the graph norm.

In what follows let X be a Banach space. For $\Omega \subset \mathbb{R}^n$ open or closed, $C(\Omega; X)$ and $BUC(\Omega; X)$ stand for the continuous resp. bounded uniformly continuous functions $f : \Omega \to X$. Further, if $\Omega \subset \mathbb{R}^n$ is open and $k \in \mathbb{N}$, $C^k(\overline{\Omega}; X)$ $(BUC^k(\overline{\Omega}; X))$ designates the space of all functions $f : \overline{\Omega} \to X$ for which the partial derivative $\partial^\alpha f$ exists on Ω and can be continuously extended to a function belonging to $C(\overline{\Omega}; X)$ $(BUC(\overline{\Omega}; X))$, for each $0 \le |\alpha| \le k$.

If Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ and $1 \le p < \infty$, then $L_p(\Omega; X)$ denotes the space of all (equivalence classes of) Bochner-measurable functions $f : \Omega \to X$ with $|f|_p := (\int_\Omega |f(y)|_X^p \, dy)^{1/p} < \infty$. $L_p(\Omega; X)$ is a Banach space when normed by $|\cdot|_p$.

For an interval $J \subset \mathbb{R}$, $s > 0$ and $1 < p < \infty$, by $H_p^s(J; X)$ and $B_{pp}^s(J; X)$ we mean the vector-valued Bessel potential space resp. Sobolev-Slobodeckij space of X-valued functions on J, see Amann [6], Schmeisser [73], Štrkalj [76], and Zimmermann [83]. Concerning the scalar case, we refer further to Runst and Sickel [72], Triebel [78], [79]. In Section 2.8 we give a definition of the spaces $H_p^s(J; X)$ in the situation where X belongs to the class $\mathcal{HT}$ (cf. Section 2.3); this will always be the case when we are to consider vector-valued Bessel potential spaces. We will frequently use the property that the Sobolev-Slobodeckij spaces appear as real interpolation spaces between the spaces L_p and H_p^s; more precisely $(L_p(J; X), H_p^s(J; X))_{\theta, p} = B_{pp}^{\theta s}(J; X)$ for all $1 < p < \infty$, $s > 0$, and $\theta \in (0, 1)$. For general treatises on interpolation theory we refer to Bergh and Löfström [9], and Triebel [78].

If $\mathcal{F}$ is any of the above function spaces, then $f \in \mathcal{F}_{loc}$ means that f belongs to the corresponding space when restricted to compact subsets of its domain. In the scalar case $X = \mathbb{R}$ or $X = \mathbb{C}$ we usually omit the image space in the function space notation.

Following these conventions, $BV_{loc}(\mathbb{R}_+)$ designates the space of all scalar functions that are locally of bounded variation on $\mathbb{R}_+$.

If not indicated otherwise, by $f * g$ we mean the *convolution* defined by $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$, $t \geq 0$, of two functions f, g supported on the halfline.

For $u \in L_{1,loc}(\mathbb{R}_+; X)$ of exponential growth, i.e. $\int_0^\infty e^{-\omega t}|u(t)|\,dt < \infty$ with some $\omega \in \mathbb{R}$, the *Laplace transform* of u is defined by

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\,dt, \quad \operatorname{Re}\lambda \geq \omega.$$

For a comprehensive account of the vector-valued Laplace transform we refer to Hille and Phillips [44], and Arendt, Batty, Hieber, Neubrander [7]; see also Prüss [63]. For the classical Laplace transform, one of the standard references is Doetsch [30].

We conclude this section by stating a result on the inversion of the vector-valued Laplace transform. It is due to Prüss, see [64, Corollary 1].

Proposition 2.1.1 *Let X be a Banach space. Suppose $g : \mathbb{C}_+ \to X$ is holomorphic and satisfies*

$$|g(\lambda)| + |\lambda g'(\lambda)| \leq c|\lambda|^{-\beta}, \quad \operatorname{Re}\lambda > 0, \tag{2.1}$$

for some $\beta > 0$. Then, with $n := [\beta]$, there is an n-times continuously differentiable function $u : (0,\infty) \to X$ such that $\hat{u}(\lambda) = g(\lambda)$ for all $\lambda \in \mathbb{C}_+$. Moreover,

$$|u^{(k)}(t)| \leq M\, t^{\beta-k-1}, \;\; t > 0,\, 0 \leq k \leq n, \tag{2.2}$$

where $M > 0$ is a constant depending only on c and β.

2.2 Sectorial operators

This section contains the definitions and certain known properties of sectorial operators, operators which admit a bounded $\mathcal{H}^\infty$-calculus, operators with bounded imaginary powers, $\mathcal{R}$-sectorial operators, and operators with $\mathcal{R}$-bounded functional calculus. A general reference for the material presented here is the extensive work by Denk, Hieber and Prüss [29].

We begin with the definition of sectorial operators. Let X be a complex Banach space, and A be a closed linear operator in X. Then A is called *pseudo-sectorial* if $(-\infty, 0)$ is contained in the resolvent set of A and the resolvent estimate

$$|t(t+A)^{-1}|_{\mathcal{B}(X)} \leq M, \quad t > 0,$$

holds, for some constant $M > 0$. If in addition $\mathcal{N}(A) = \{0\}$, $\overline{\mathcal{D}(A)} = X$, and $\overline{\mathcal{R}(A)} = X$, then A is called *sectorial.* The class of sectorial operators in X is denoted by $\mathcal{S}(X)$. We recall that in case X is reflexive and A is pseudo-sectorial, the space X decomposes according to $X = \mathcal{N}(A) \oplus \overline{\mathcal{R}(A)}$. Thus in such a situation A is sectorial on $\overline{\mathcal{R}(A)}$. Putting

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \theta\}$$

it follows by means of the Neumann series that if $A \in \mathcal{S}(X)$, then $\rho(-A) \supset \Sigma_\theta$, for some $\theta > 0$ and $\sup\{|\lambda(\lambda+A)^{-1}| : |\arg\lambda| < \theta\} < \infty$. Therefore one may define the *spectral angle* ϕ_A of $A \in \mathcal{S}(X)$ by

$$\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}| < \infty\}.$$

Clearly, $\phi_A \in [0, \pi)$ and $\phi_A \geq \sup\{|\arg \lambda| : \lambda \in \sigma(A)\}$.

We turn now to the $\mathcal{H}^\infty$-calculus. For $\phi \in (0, \pi]$, we define the space of holomorphic functions on Σ_ϕ by $\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic}\}$, and the space

$$\mathcal{H}^\infty(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C} \text{ holomorphic and bounded}\},$$

which when equipped with the norm $|f|_\infty^\phi = \sup\{|f(\lambda)| : |\arg \lambda| < \phi\}$ becomes a Banach algebra. We further let $\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha, \beta < 0} \mathcal{H}_{\alpha, \beta}(\Sigma_\phi)$, where $\mathcal{H}_{\alpha, \beta}(\Sigma_\phi) := \{f \in \mathcal{H}(\Sigma_\phi) : |f|_{\alpha, \beta}^\phi < \infty\}$, and $|f|_{\alpha, \beta}^\phi := \sup_{|\lambda| \leq 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda| \geq 1} |\lambda^{-\beta} f(\lambda)|$. Now suppose that $A \in \mathcal{S}(X)$ and $\phi \in (\phi_A, \pi)$. We select any $\psi \in (\phi_A, \phi)$ and denote by Γ_ψ the oriented contour defined by $\Gamma_\psi(t) = -te^{i\psi}$, $-\infty < t \leq 0$, and $\Gamma_\psi(t) = te^{-i\psi}$, $0 \leq t < \infty$. Then the Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\psi} f(\lambda)(\lambda - A)^{-1} \, d\lambda, \quad f \in \mathcal{H}_0(\Sigma_\phi),$$

converges in $\mathcal{B}(X)$ and does not depend on the choice of ψ. Further, it defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$ which is an algebra homomorphism. The following definition is in accordance with McIntosh [57].

Definition 2.2.1 *A sectorial operator A in X admits a* bounded $\mathcal{H}^\infty$-calculus *if there are $\phi > \phi_A$ and a constant $K_\phi < \infty$ such that*

$$|f(A)| \leq K_\phi |f|_\infty^\phi, \quad \text{for all } f \in \mathcal{H}_0(\Sigma_\phi). \tag{2.3}$$

The class of sectorial operators which admit an $\mathcal{H}^\infty$-calculus will be denoted by $\mathcal{H}^\infty(X)$. The $\mathcal{H}^\infty$-angle *ϕ_A^∞ of $A \in \mathcal{H}^\infty(X)$ is defined by*

$$\phi_A^\infty = \inf\{\phi > \phi_A : (2.3) \text{ is valid}\}.$$

If $A \in \mathcal{H}^\infty(X)$, then the functional calculus for A on $\mathcal{H}_0(\Sigma_\phi)$ extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$.

We consider next operators with bounded imaginary powers. This subclass of $\mathcal{S}(X)$ has been introduced in Prüss and Sohr [69]. To justify the subsequent definition, we first note that for any $A \in \mathcal{S}(X)$ one can define complex powers A^z, where $z \in \mathbb{C}$ is arbitrary; cf. Komatsu [48], Prüss [63, Section 8.1] or Denk, Hieber, Prüss [29, Section 2.2].

Definition 2.2.2 *A sectorial operator A in X is said to admit* bounded imaginary powers *if $A^{is} \in \mathcal{B}(X)$ for each $s \in \mathbb{R}$ and there is a constant $C > 0$ such that $|A^{is}| \leq C$ for $|s| \leq 1$. The class of such operators will be denoted by $\mathcal{BIP}(X)$.*

Since A^{is} has the group property (see e.g. Prüss [63, Proposition 8.1]), it is evident that A admits bounded imaginary powers if and only if $\{A^{is} : s \in \mathbb{R}\}$ forms a strongly continuous group of bounded linear operators in X. The growth bound θ_A of this group, that is

$$\theta_A = \limsup_{|s| \to \infty} \frac{1}{|s|} \log |A^{is}|,$$

will be called the *power angle* of A. Owing to the fact that the functions f_s defined by $f_s(z) = z^{is}$ belong to $\mathcal{H}^\infty(\Sigma_\phi)$, for any $s \in \mathbb{R}$ and $\phi \in (0, \pi)$, we clearly have the inclusions

$$\mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{S}(X),$$

and the inequalities

$$\phi_A^\infty \geq \theta_A \geq \phi_A \geq \sup\{|\arg\lambda| : \lambda \in \sigma(A)\}.$$

Operators with bounded imaginary powers are of overriding importance in the context of sums of commuting linear operators. This finds expression in the Dore-Venni theorem, which is one of the fundamental results in this connection. We will state a version of it in the next section.

Another important application of the class $\mathcal{BIP}(X)$ concerns the fractional power spaces

$$X_\alpha = X_{A^\alpha} = (\mathcal{D}(A^\alpha), |\cdot|_\alpha),\ |x|_\alpha = |x| + |A^\alpha x|, \quad 0 < \alpha < 1,$$

where $A \in \mathcal{S}(X)$. If A belongs to $\mathcal{BIP}(X)$, one can derive a characterization of X_α in terms of complex interpolation spaces.

Theorem 2.2.1 *Let $A \in \mathcal{BIP}(X)$. Then*

$$X_\alpha = [X, D_A]_\alpha, \quad \alpha \in (0,1),$$

the complex interpolation space between X and $D_A \hookrightarrow X$ of order α.

For a proof we refer to Triebel [78, pp. 103-104], or Yagi [82].

At this point let us state a very useful property of the real interpolation spaces $(X, X_\alpha)_{\beta,p}$, $0 < \alpha,\ \beta < 1$, $1 \leq p \leq \infty$, between X and the fractional power spaces X_α associated with an operator $A \in \mathcal{S}(X)$, defined by, e.g. the K-method. Recall that for $A \in \mathcal{S}(X)$, $1 \leq p \leq \infty$, and $\gamma \in (0,1)$, the real interpolation space $(X, D_A)_{\gamma,p}$ coincides with the space $D_A(\gamma, p)$ which is defined by means of

$$D_A(\gamma, p) := \{x \in X : [x]_{D_A(\gamma,p)} < \infty\},$$

where

$$[x]_{D_A(\gamma,p)} = \begin{cases} (\int_0^\infty (t^\gamma |A(t+A)^{-1}x|_X)^p \frac{d}{dt})^{\frac{1}{p}} & : \ 1 \leq p < \infty \\ \sup_{t>0} t^\gamma |A(t+A)^{-1}x|_X & : \ p = \infty, \end{cases} \tag{2.4}$$

see e.g. [16, Prop. 3].

Suppose now that $A \in \mathcal{BIP}(X)$. By Theorem 2.2.1 and the reiteration theorem (see e.g. Amann [5, Section 2.8]), we deduce that

$$(X, X_\alpha)_{\beta,p} = (X, [X, D_A]_\alpha)_{\beta,p} = (X, D_A)_{\alpha\beta,p},\ 0 < \alpha,\ \beta < 1,\ 1 \leq p \leq \infty. \tag{2.5}$$

Since $A \in \mathcal{S}(X)$ implies $A^\alpha \in \mathcal{S}(X)$ for all $\alpha \in (0,1)$, we conclude from (2.5) that $D_{A^\alpha}(\beta, p) = D_A(\alpha\beta, p)$. One can show that this relation is even valid for all $A \in \mathcal{S}(X)$, cf. Komatsu [49, Thm. 3.2].

Theorem 2.2.2 *Let $A \in \mathcal{S}(X)$. Then*

$$D_{A^\alpha}(\beta, p) = D_A(\alpha\beta, p), \quad \alpha, \beta \in (0,1),\ 1 \leq p \leq \infty.$$

We come now to $\mathcal{R}$-sectorial operators. First we have to recall the definition of $\mathcal{R}$-*bounded* families of bounded linear operators.

Definition 2.2.3 *Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called* $\mathcal{R}$-bounded, *if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality*

$$|\sum_{j=1}^{N} \varepsilon_j T_j x_j|_{L_p(\Omega;Y)} \leq C|\sum_{j=1}^{N} \varepsilon_j x_j|_{L_p(\Omega;X)} \tag{2.6}$$

is valid. The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$.

The notion of $\mathcal{R}$-sectorial operators is obtained by replacing *bounded* with *$\mathcal{R}$-bounded* in the definition of sectorial operators.

Definition 2.2.4 *Let X be a complex Banach space, and assume A is a sectorial operator in X. Then A is called* $\mathcal{R}$-sectorial *if*

$$\mathcal{R}_A(0) := \mathcal{R}\{t(t + A)^{-1} : t > 0\} < \infty.$$

The $\mathcal{R}$-angle *ϕ_A^R of A is defined by means of*

$$\phi_A^R := \inf\{\theta \in (0, \pi) : \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda + A)^{-1} : |\arg \lambda| \leq \theta\}.$$

The class of $\mathcal{R}$-sectorial operators in X will be denoted by $\mathcal{RS}(X)$.

The $\mathcal{R}$-angle of an $\mathcal{R}$-sectorial operator A is well-defined and it is not smaller than the spectral angle of A, cp. Denk, Hieber and Prüss [29, Definition 4.1].

The following fundamental result, which has been proven in Clément and Prüss [24], says that the class of $\mathcal{R}$-sectorial operators contains the class of operators with bounded imaginary powers, provided that the underlying Banach space X belongs to the class $\mathcal{HT}$, see Section 2.3 for the definition of the latter.

Theorem 2.2.3 *Let X be a Banach space of class $\mathcal{HT}$ and suppose that $A \in \mathcal{BIP}(X)$ with power angle θ_A. Then A is $\mathcal{R}$-sectorial and $\phi_A^R \leq \theta_A$.*

For sectorial operators A one knows that the powers A^α with $\alpha \in \mathbb{R}$ and $|\alpha| < \pi/\phi_A$ are sectorial as well and $\phi_{A^\alpha} \leq |\alpha|\phi_A$, see e.g. [29, Thm. 2.3]. It turns out that there is a corresponding result for the class $\mathcal{RS}(X)$.

Proposition 2.2.1 *Let X be a complex Banach space. Suppose $A \in \mathcal{RS}(X)$ and $\alpha \in \mathbb{R}$ is such that $|\alpha| < \pi/\phi_A^R$. Then A^α is also $\mathcal{R}$-sectorial and $\phi_{A^\alpha}^R \leq |\alpha|\phi_A^R$.*

Proof. In view of $A^{-\alpha} = (A^{-1})^\alpha$, it suffices to consider positive α. In fact, for $A \in \mathcal{RS}(X)$ and $\phi > \phi_A^R$, the relation

$$\lambda(\lambda + A^{-1})^{-1} = \lambda A(1 + \lambda A)^{-1} = A(\lambda^{-1} + A)^{-1},\ \lambda \in \Sigma_{\pi-\phi},$$

shows that $A^{-1} \in \mathcal{RS}(X)$ and $\phi_{A^{-1}}^R = \phi_A^R$. So let $\alpha \in (0, \pi/\phi_A^R)$ be fixed. Since $\mathcal{RS}(X) \subset \mathcal{S}(X)$, it follows that $A^\alpha \in \mathcal{S}(X)$ with spectral angle $\phi_{A^\alpha} \leq \alpha\phi_A$.

Let now $\phi_\alpha < \pi - \alpha\phi_A^R$ and $\mu \in \Sigma_{\phi_\alpha}$. Then the function $g_\mu(\lambda) = \mu/(\mu + \lambda^\alpha)$ belongs to $\mathcal{H}^\infty(\Sigma_\phi)$ as long as $\phi_\alpha + \alpha\phi < \pi$. By means of the *extended* functional calculus (cf.

[29, Section 2.1]) we have $g_\mu(A) = \mu(\mu + A^\alpha)^{-1}$; the problem is to show that the family $\{g_\mu(A) : \mu \in \Sigma_{\phi_\alpha}\} \subset \mathcal{B}(X)$ is $\mathcal{R}$-bounded.

To this purpose we consider first appropriate approximations of A. For $\varepsilon > 0$ set $A_\varepsilon = (\varepsilon + A)(1 + \varepsilon A)^{-1}$. Then A_ε is bounded, sectorial and invertible, for each $\varepsilon > 0$, and $\phi_{A_\varepsilon} \leq \phi_A$, see [29, Prop. 1.4]. Furthermore, A_ε is also $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi^R_{A_\varepsilon} \leq \phi^R_A$, and the $\mathcal{R}$-bounds $\mathcal{R}_{A_\varepsilon}(\phi)$ are uniformly with respect to $\varepsilon > 0$, for each fixed $\phi < \pi - \alpha\phi^R_A$. To see this verify that the subsequent relation is valid.

$$\lambda(\lambda + A_\varepsilon)^{-1} = \frac{\lambda}{\lambda + \varepsilon}\,\varphi_\varepsilon(\lambda)(\varphi_\varepsilon(\lambda) + A)^{-1} + \frac{\varepsilon\lambda}{1 + \varepsilon\lambda} A(\varphi_\varepsilon(\lambda) + A)^{-1},\ \lambda \in \Sigma_\phi,$$

where $\varphi_\varepsilon(\lambda) = (\varepsilon + \lambda)/(1 + \varepsilon\lambda)$. Here it is essential that the functions φ_ε leave all sectors Σ_ϕ invariant. The claim follows then by the rule $\mathcal{R}(\mathcal{T}_1 + \mathcal{T}_2) \leq \mathcal{R}(\mathcal{T}_1) + \mathcal{R}(\mathcal{T}_2)$ and Kahane's contraction principle, cf. [29, Section 3.1].

We next show that $\mathcal{R}_{A_\varepsilon^\alpha}(\phi_\alpha) \leq C < \infty$ uniformly w.r.t. $\varepsilon > 0$. To this end we employ the following representation formula for the operators $g_\mu(A_\varepsilon)$, cf. the proof of Theorem 2.3 in [29].

$$\begin{aligned} g_\mu(A_\varepsilon) =& \frac{\mu}{2\pi i}\int_0^\infty \Big(\frac{e^{i\theta\alpha} - e^{i(\theta-2\pi)\alpha}}{(\mu + r^\alpha e^{i\alpha(\theta-2\pi)})(\mu + r^\alpha e^{i\alpha\theta})}\Big) r^\alpha e^{i\theta}(re^{i\theta} - A_\varepsilon)^{-1}dr \\ &+ \mu\sum_{j=1}^n \lambda_j^{1-\alpha}(\lambda_j - A_\varepsilon)^{-1}/\alpha. \end{aligned} \tag{2.7}$$

This formula is obtained by contracting the contour Γ_ψ from the Dunford integral for $g_\mu(A_\varepsilon)$ to a suitable halfray $\Gamma_\alpha = [0, \infty)e^{i\theta}$, with $\phi^R_A < \psi < \theta \leq \pi$, and using Cauchy's theorem as well as residue calculus. The numbers $\lambda_j = \lambda_j(\mu)$, $j = 1, \ldots, n$ denote the zeros of $\mu + \lambda^\alpha$; note that there are only finitely many of them, and $n = 0$ means that there are none. The angle $\theta = \theta(\mu)$ is chosen such that for some $\delta > 0$, we have with $\varphi = \arg\mu$ the inequalities $|\varphi - \alpha\theta(\mu) - (2k+1)\pi| \geq \delta$ and $|\varphi + 2\alpha\pi - \alpha\theta(\mu) - (2k+1)\pi| \geq \delta$ for all $\mu \in \Sigma_{\phi_\alpha}$ and $k \in \mathbb{Z}$. From (2.7) we get with $\mu + \lambda_j^\alpha = 0$ and the change of variables $r = (|\mu|s)^{1/\alpha}$

$$\begin{aligned} g_\mu(A_\varepsilon) = &\int_0^\infty \underbrace{\frac{e^{i\varphi}(e^{i(2\pi-\theta)\alpha} - e^{-i\theta\alpha})(1+s)^2}{2\pi\alpha i\,(e^{i(\varphi-\alpha\theta)} + s)(e^{i(\varphi+2\alpha\pi-\alpha\theta)} + s)}}_{|\cdot|\leq C} (|\mu|s)^{\frac{1}{\alpha}} e^{i\theta}((|\mu|s)^{\frac{1}{\alpha}}e^{i\theta} - A_\varepsilon)^{-1}\frac{ds}{(1+s)^2} \\ &- \sum_{j=1}^n \lambda_j(\lambda_j - A_\varepsilon)^{-1}/\alpha. \end{aligned}$$

Hence $g_\mu(A_\varepsilon) \in C_0\overline{\mathrm{aco}}\,(\{\lambda(\lambda + A_\varepsilon)^{-1} : \lambda \in \Sigma_{\pi-\psi}\})$, where $\overline{\mathrm{aco}}\,(\mathcal{T})$ means the closure in the strong operator topology of the absolute convex hull of the family $\mathcal{T}$. Proposition 3.8 in [29] and the above observation concerning the $\mathcal{R}$-bounds $\mathcal{R}_{A_\varepsilon}(\phi)$ then yield

$$\mathcal{R}_{A_\varepsilon^\alpha}(\phi_\alpha) \leq 2C_0\mathcal{R}_{A_\varepsilon}(\psi) \leq C < \infty,$$

uniformly w.r.t. $\varepsilon > 0$.

The assertion $\mathcal{R}_{A^\alpha}(\phi_\alpha) < \infty$ can now be established by the following approximation argument, which relies on $g_\mu(A_\varepsilon)x \to g_\mu(A)x$ as $\varepsilon \to 0+$ on $\mathcal{D}(A) \cap R(A)$, which is a dense subset of X, see [29, Thm. 2.1]. Set $\mathcal{T} = \{g_\mu(A) : \mu \in \Sigma_{\phi_\alpha}\}$ and $\mathcal{T}_\varepsilon = \{g_\mu(A_\varepsilon) :$

$\mu \in \Sigma_{\phi_\alpha}\}$. Let $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and suppose that ε_j are independent, symmetric, $\{-1,1\}$-valued random variables on a probability space $(\Omega, \mathcal{M}, \mu)$, $j = 1, \dots, n$. Let further $T_{\varepsilon,j} \in \mathcal{T}_\varepsilon$ be the approximation of T_j, that is, for $T_j = g_{\mu_j}(A)$ we put $T_{\varepsilon,j} = g_{\mu_j}(A_\varepsilon)$. Also, we choose for each x_j a sequence $\{x_{j,k}\}_{k=1}^\infty \subset \mathcal{D}(A) \cap R(A)$ such that $x_{j,k} \to x_j$ as $k \to \infty$. By uniform $\mathcal{R}$-boundedness of $\mathcal{T}_\varepsilon$, we may then estimate

$$\begin{aligned} |\sum_{j=1}^N \varepsilon_j T_j x_j|_{L_p(\Omega;X)} &\le |\sum_{j=1}^N \varepsilon_j T_{\varepsilon,j}\, x_j|_{L_p(\Omega;X)} + |\sum_{j=1}^N \varepsilon_j (T_j - T_{\varepsilon,j}) x_{j,k}|_{L_p(\Omega;X)} \\ &\quad + |\sum_{j=1}^N \varepsilon_j (T_j - T_{\varepsilon,j})(x_j - x_{j,k})|_{L_p(\Omega;X)} \\ &\le C\,|\sum_{j=1}^N \varepsilon_j x_j|_{L_p(\Omega;X)} + \sum_{j=1}^N |(T_j - T_{\varepsilon,j}) x_{j,k}|_X + 2C \sum_{j=1}^N |x_j - x_{j,k}|_X, \end{aligned} \tag{2.8}$$

where C does not depend on ε, j, k. Now let $\varepsilon \to 0+$ in (2.8) with k being fixed. This makes the second summand disappear. The third one vanishes if we then send $k \to \infty$. It remains the desired inequality expressing $\mathcal{R}$-boundedness of $\{g_\mu(A) : \mu \in \Sigma_{\phi_\alpha}\}$ for each $\phi_\alpha < \pi - \alpha\phi_A^R$. □

Connecting the concept of $\mathcal{R}$-boundedness to the $\mathcal{H}^\infty$-calculus, leads to the notion of operators with $\mathcal{R}$-bounded functional calculus.

Definition 2.2.5 *Let X be a complex Banach space and suppose that $A \in \mathcal{H}^\infty(X)$. The operator A is said to admit an* $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus *if the set*

$$\{f(A) : f \in \mathcal{H}^\infty(\Sigma_\theta), |f|_\infty^\theta \le 1\}$$

is $\mathcal{R}$-bounded for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$ and define the $\mathcal{RH}^\infty$-angle $\phi_A^{R\infty}$ of A as the infimum of such angles θ.

One important application of such operators concerns the joint functional calculus of sectorial operators, see Section 2.4.

2.3 Sums of closed linear operators

Let X be a Banach space, A, B closed linear operators in X, and consider the problem

$$Ax + Bx = y. \tag{2.9}$$

Given $y \in X$ one seeks a unique strict solution x of (2.9) in the sense that $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$, that is x possesses the regularity induced by A as well as that coming from B. In this situation we say that the solution has *maximal regularity*. Furthermore it is desirable to have an *a priori* estimate of the form

$$|Ax| + |Bx| \le C|Ax + Bx| \quad \text{for all } x \in \mathcal{D}(A) \cap \mathcal{D}(B), \tag{2.10}$$

where C does not depend on x.

Let us define the sum operator $A+B$ by

$$(A+B)x = Ax + Bx, \quad x \in \mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B).$$

If $0 \in \rho(A+B)$, which in particular means that $A+B$ is closed, equation (2.9) is solvable in the strict sense for all $y \in X$, and the closed graph theorem shows (2.10) with some $C > 0$. For the latter it suffices to know only that $A+B$ is injective and closed. If $A+B$ is merely closable but not closed, and $0 \in \rho(\overline{A+B})$, then (2.9) only admits generalized solutions in the sense that there exist sequences $(x_n) \subset \mathcal{D}(A) \cap \mathcal{D}(B)$, $x_n \to x$, and $y_n \to y$ satisfying

$$Ax_n + Bx_n = y_n, \quad n \in \mathbb{N}.$$

In general, nothing can be said on $A+B$. It may even happen that it is not closable. In order to prove positive results in this direction, further assumptions on A and B have to be imposed.

In 1975 Da Prato and Grisvard ([26]) were able to show that if A and B are commuting sectorial operators satisfying the *parabolicity condition* $\phi_A + \phi_B < \pi$ then $A+B$ is closable, and the closure $L := \overline{A+B}$ is a sectorial operator with $\phi_L \leq \max\{\phi_A, \phi_B\}$, see also [16], [63, Section 8]. Recall that two closed linear operators are said to commute (in the resolvent sense) if there are $\lambda \in \rho(A)$, $\mu \in \rho(B)$ such that

$$(\lambda - A)^{-1}(\mu - A)^{-1} = (\mu - A)^{-1}(\lambda - A)^{-1}.$$

By strengthening the assumptions on A, B and X Dore and Venni [31], [32] succeeded in proving closedness of $A+B$. Prüss and Sohr [69] improved their result by removing some extra assumptions. Before we repeat a version of the Dore-Venni theorem we have to recall what it means for a Banach space X to belong to the class $\mathcal{HT}$.

A Banach space X is said to be of class $\mathcal{HT}$, if the Hilbert transform is bounded on $L_p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform Hf of a function $f \in \mathcal{S}(\mathbb{R}; X)$, the Schwartz space of rapidly decreasing X-valued functions, is defined by

$$(Hf)(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon \leq |s| \leq 1/\epsilon} f(t-s) \frac{ds}{s}, \quad t \in \mathbb{R},$$

where the limit is to be understood in the L_p-sense. There is a well known theorem which says that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of *UMD* spaces, where *UMD* stands for *unconditional martingale difference property*. It is further known that $\mathcal{HT}$-spaces are reflexive. Every Hilbert space belongs to the class $\mathcal{HT}$, and if (Ω, Σ, μ) is a measure space, $1 < p < \infty$, then $L_p(\Omega, \Sigma, \mu; X)$ is an $\mathcal{HT}$-space. For all these results see the survey article by Burkholder [10].

We state now a variant of the Dore-Venni theorem, cf. [31], [65], [69].

Theorem 2.3.1 *Suppose X is a Banach space of class $\mathcal{HT}$, and assume $A, B \in \mathcal{BIP}(X)$ commute in the resolvent sense and satisfy the* strong parabolicity condition $\theta_A + \theta_B < \pi$. *Let further $\mu > 0$. Then*

- *(i)* *$A + \mu B$ is closed and sectorial;*
- *(ii)* *$A + \mu B \in \mathcal{BIP}(X)$ with $\theta_{A+\mu B} \leq \max\{\theta_A, \theta_B\}$;*
- *(iii)* *there exists a constant $C > 0$, independent of $\mu > 0$, such that*

$$|Ax| + \mu|Bx| \leq C|Ax + \mu Bx|, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B). \tag{2.11}$$

In particular, if A or B is invertible, then $A + \mu B$ is invertible as well.

We remark that a theorem of the Dore-Venni type for noncommuting operators has been established by Monniaux and Prüss [58]. Another remarkable result has been obtained by Kalton and Weiss [47]. They could show, without restriction on the underlying Banach space X, that $A+B$ is closed, provided that $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$ commute with $\phi_A^\infty + \phi_B^R < \pi$.

Some consequences of Theorem 2.3.1 concerning complex interpolation are contained in the following corollary, see Prüss [65, Cor. 1]. For a proof we refer to the forthcoming monograph Hieber and Prüss [43].

Corollary 2.3.1 *Suppose X belongs to the class $\mathcal{HT}$, and assume that $A, B \in \mathcal{BIP}(X)$ are commuting in the resolvent sense. Further suppose the* strong parabolicity condition $\theta_A + \theta_B < \pi$. *Let A or B be invertible and $\alpha \in (0,1)$. Then*

(i) $A^\alpha(A+B)^{-\alpha}$ *and* $B^\alpha(A+B)^{-\alpha}$ *are bounded in X;*

(ii) $\mathcal{D}((A+B)^\alpha) = [X, \mathcal{D}(A+B)]_\alpha = [X, \mathcal{D}(A)]_\alpha \cap [X, \mathcal{D}(B)]_\alpha = \mathcal{D}(A^\alpha) \cap \mathcal{D}(B^\alpha)$.

We conclude this section with two results which are also very useful in connection with the method of sums. The first of these has been established by Grisvard [40], even in a more general situation.

Proposition 2.3.1 *Suppose that A, B are sectorial operators in a Banach space X, commuting in the resolvent sense. Then*

$$(X, \mathcal{D}(A) \cap \mathcal{D}(B))_{\alpha,p} = (X, \mathcal{D}(A))_{\alpha,p} \cap (X, \mathcal{D}(B))_{\alpha,p},$$

for all $\alpha \in (0,1)$, $p \in [1, \infty]$.

The following result is known as *the mixed derivative theorem* and is due to Sobolevskii [75].

Proposition 2.3.2 *Suppose A, B are sectorial operators in a Banach space X, commuting in the resolvent sense. Assume that their spectral angles satisfy the parabolicity condition $\phi_A + \phi_B < \pi$. Further suppose that the pair (A, B) is* coercively positive, *i.e. $A + \mu B$ with natural domain $\mathcal{D}(A+\mu B) = \mathcal{D}(A) \cap \mathcal{D}(B)$ is closed for each $\mu > 0$ and there is a constant $M > 0$ such that*

$$|Ax|_X + \mu|Bx|_X \le M|Ax + \mu Bx|_X, \quad \textit{for all}\, x \in \mathcal{D}(A) \cap \mathcal{D}(B),\ \mu > 0.$$

Then there exists a constant $C > 0$ such that

$$|A^\alpha B^{1-\alpha} x|_X \le C|Ax + Bx|_X, \quad \textit{for all}\, x \in \mathcal{D}(A) \cap \mathcal{D}(B),\ \alpha \in [0,1].$$

In particular, if A or B is invertible, then $A^\alpha B^{1-\alpha}(A+B)^{-1}$ is bounded in X, for each $\alpha \in [0,1]$.

2.4 Joint functional calculus

This section is devoted to the joint $\mathcal{H}^\infty$-calculus for a pair of sectorial operators A, B on X with commuting resolvents. It was first introduced by Albrecht [1] and is a natural two-variable analogue of McIntosh's $\mathcal{H}^\infty$-calculus, which we have already discussed in Section 2.2. For proofs and many more details we refer to [1], [2], [51], and [47].

Given ϕ, $\phi' \in (0,\pi]$ we denote by $\mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$ the Banach algebra of all bounded holomorphic scalar-valued functions on $\Sigma_\phi \times \Sigma_{\phi'}$ equipped with the norm $|f|_\infty^{\phi,\phi'} := \sup\{|f(\lambda,\lambda')| : |\arg\lambda| < \phi, |\arg\lambda'| < \phi'\}$, and we put $\mathcal{H}_0(\Sigma_\phi \times \Sigma_{\phi'}) = \{f \in \mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'}) : \exists (f_1, f_2) \in \mathcal{H}_0(\Sigma_\phi) \times \mathcal{H}_0(\Sigma_{\phi'}),\ f_1$ and f_2 non-vanishing, and $f(f_1 f_2)^{-1} \in \mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})\}$. Let A, $B \in \mathcal{S}(X)$ with spectral angles ϕ_A and ϕ_B, respectively, commute in the resolvent sense. For $\phi \in (\phi_A, \pi)$, $\phi' \in (\phi_B, \pi)$, and $f \in \mathcal{H}_0(\Sigma_\phi \times \Sigma_{\phi'})$ one defines

$$f(A,B) = -\frac{1}{4\pi^2}\int_{\Gamma_\psi \times \Gamma_{\psi'}} f(\lambda,\lambda')(\lambda - A)^{-1}(\lambda - B)^{-1} d\lambda\, d\lambda', \tag{2.12}$$

where $(\psi, \psi') \in (\phi_A, \phi) \times (\phi_B, \phi')$. This integral converges in $\mathcal{B}(X)$ and does not depend on the choice of ψ and ψ'. Via $\Phi_{A,B}(f) = f(A,B)$, it defines a joint functional calculus $\Phi_{A,B} : \mathcal{H}_0(\Sigma_\phi \times \Sigma_{\phi'}) \to \mathcal{B}(X)$. In analogy to Definition 2.2.1, we say that (A,B) admits a *bounded joint* $\mathcal{H}^\infty$*-calculus* (symbolized by $(A,B) \in \mathcal{H}^\infty(X)$) if there exist $\phi \in (\phi_A, \pi)$, $\phi' \in (\phi_B, \pi)$, and a constant $K_{\phi,\phi'} < \infty$ such that

$$|f(A,B)| \le K_{\phi,\phi'} |f|_\infty^{\phi,\phi'} \quad \text{for all } f \in \mathcal{H}_0(\Sigma_\phi \times \Sigma_{\phi'}). \tag{2.13}$$

If $(A,B) \in \mathcal{H}^\infty(X)$, then the functional calculus for (A,B) on $\mathcal{H}_0(\Sigma_\phi \times \Sigma_{\phi'})$ extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi \times \Sigma_{\phi'})$.

An interesting question is the following: what are the Banach spaces X for which (A,B) admits a bounded joint $\mathcal{H}^\infty$-calculus as soon as A and B, each, admit a bounded $\mathcal{H}^\infty$-calculus? In [1] Albrecht was able to prove that this is the case if $X = L_p(\Omega, \Sigma, \mu)$, $1 < p < \infty$, where (Ω, Σ, μ) is a σ-finite measure space, see also [2, Section 5]. Lancien *et al.* [51] extended this result to a class of Banach spaces which enjoy a certain geometric property. They also give an example for a Banach space not possessing the *joint calculus property*. We would further like to mention a result by Kalton and Weis [47] which asserts, without additional assumption on X, that if $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RH}^\infty(X)$ with commuting resolvents, then (A,B) admits a bounded joint $\mathcal{H}^\infty$-calculus.

We consider now an important example.

Example 2.4.1 Let $1 < p < \infty$, $X = L_p(\mathbb{R}_+ \times \mathbb{R}^n)$, and denote the independent variables by $t\,(\in \mathbb{R}_+)$ resp. $x\,(\in \mathbb{R}^n)$. Take $B = \partial_t$ with domain $\mathcal{D}(B) = {}_0H_p^1(\mathbb{R}_+; L_p(\mathbb{R}^n))$, and define A as the natural extension of $-\Delta_x$ in $L_p(\mathbb{R}^n)$ with $\mathcal{D}(-\Delta_x) = H_p^2(\mathbb{R}^n)$ to X, i.e. $\mathcal{D}(A) = L_p(\mathbb{R}_+; H_p^2(\mathbb{R}^n))$ and $Af = -\Delta_x f$, $f \in \mathcal{D}(A)$. Then A and B commute in the resolvent sense, and A, $B \in \mathcal{H}^\infty(X)$ with $\mathcal{H}^\infty$-angles $\phi_A^\infty = 0$ resp. $\phi_B^\infty = \pi/2$. Since X has the joint calculus property, we have $(A,B) \in \mathcal{H}^\infty(X)$. More precisely, for each $\eta \in (0, \pi/2)$, there exists $C_\eta > 0$ such that for all $f \in \mathcal{H}^\infty(\Sigma_\eta \times \Sigma_{\frac{\pi}{2}+\eta})$, $f(A,B) \in \mathcal{B}(X)$ and $|f(A,B)|_{\mathcal{B}(X)} \le C_\eta |f|_\infty^{\eta,\pi/2+\eta}$.

Regarding functions in X as elements in $L_p(\mathbb{R}_+; L_p(\mathbb{R}^n))$, the resolvent of B admits the kernel representation

$$(\lambda - B)^{-1} w(t) = -\int_0^t e^{\lambda(t-s)} w(s)\, ds, \quad t \in \mathbb{R}_+,$$

for all $w \in X$. Thus, for $f \in \mathcal{H}_0(\Sigma_\eta \times \Sigma_{\frac{\pi}{2}+\eta})$, the operators $f(A,B)$ admit a kernel representation as well, namely

$$f(A,B)w(t) = \int_0^t \left(\frac{-1}{2\pi i} \int_{\Gamma_{\psi'}} e^{\lambda'(t-s)} f(A,\lambda') d\lambda' \right) w(s)\, ds, \quad t \in \mathbb{R}_+. \tag{2.14}$$

Here $f(A,\cdot) \in \mathcal{H}_0(\Sigma_{\frac{\pi}{2}+\eta}; \mathcal{B}(X))$ results from the functional calculus of A:

$$f(A,\lambda') = \frac{1}{2\pi i}\int_{\Gamma_\psi} f(\lambda,\lambda')(\lambda - A)^{-1} d\lambda, \quad \lambda' \in \Sigma_{\frac{\pi}{2}+\eta}.$$

Since the resolvent of A is a pointwise operator with respect to $t \in \mathbb{R}_+$, we deduce from (2.14) that $f(A,B)$ is *causal* for all $f \in \mathcal{H}_0(\Sigma_\eta \times \Sigma_{\frac{\pi}{2}+\eta})$. For an arbitrary function $f \in \mathcal{H}^\infty(\Sigma_\eta \times \Sigma_{\frac{\pi}{2}+\eta})$, this property of $f(A,B)$ can be seen by approximating f with a sequence $(f_n) \subset \mathcal{H}_0(\Sigma_\eta \times \Sigma_{\frac{\pi}{2}+\eta})$.

2.5 Operator-valued Fourier multipliers

Let $1 < p < \infty$ and X be a Banach space. We denote by $\mathcal{D}(\mathbb{R}; X)$ the space of X-valued C^∞-functions with compact support on $\mathbb{R}$. Let $\mathcal{D}'(\mathbb{R}; X) := \mathcal{B}(\mathcal{D}(\mathbb{R}), X)$ be the space of X-valued distributions on the real line. Further we denote by $\mathcal{S}(\mathbb{R}; X)$ the Schwartz space of smooth rapidly decreasing X-valued functions on $\mathbb{R}$, see e.g. Amann [5, p. 129]. By $\mathcal{S}'(\mathbb{R}; X) := \mathcal{B}(\mathcal{S}(\mathbb{R}), X)$ we mean the space of X-valued temperate distributions. Let Y be another Banach space. Given $M \in L_{1,loc}(\mathbb{R}; \mathcal{B}(X,Y))$, one may define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X) \to \mathcal{S}'(\mathbb{R}; Y)$ by means of

$$T_M\phi := \mathcal{F}^{-1}M\mathcal{F}\phi, \quad \text{for all } \mathcal{F}\phi \in \mathcal{D}(\mathbb{R}; X), \tag{2.15}$$

$\mathcal{F}$ denoting the Fourier transform. Note that $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X)$ is dense in $L_p(\mathbb{R}; X)$. Thus T_M is well-defined and linear on a dense subset of $L_p(\mathbb{R}; X)$.

One can now ask what conditions have to be imposed on M so that T_M is bounded in L_p-norm, i.e. $T_M \in \mathcal{B}(L_p(\mathbb{R}; X), L_p(\mathbb{R}; Y))$. The following theorem, which is due to Weis [80], contains the operator-valued version of the famous Mikhlin Fourier multiplier theorem in one variable.

Theorem 2.5.1 *Suppose X and Y are Banach spaces of class $\mathcal{HT}$ and let $1 < p < \infty$. Let $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X,Y))$ be such that the following conditions are satisfied.*

$$\begin{array}{ll} (i) & \mathcal{R}(\{M(\rho) : \rho \in \mathbb{R} \setminus \{0\}\}) =: \kappa_0 < \infty; \\ (ii) & \mathcal{R}(\{\rho M'(\rho) : \rho \in \mathbb{R} \setminus \{0\}\}) =: \kappa_1 < \infty. \end{array}$$

Then the operator T_M defined by (2.15) is bounded from $L_p(\mathbb{R}; X)$ into $L_p(\mathbb{R}; Y)$ with norm $|T_M|_{\mathcal{B}(L_p(\mathbb{R};X),L_p(\mathbb{R};Y))} \le C(\kappa_0 + \kappa_1)$, where $C > 0$ depends only on p, X, Y.

A rather short and elegant proof of this theorem is given in [29].

2.6 Kernels

In this section we collect some of the basic definitions and properties concerning scalar kernels which we need for the treatment of parabolic Volterra equations. Definition 2.6.1 and 2.6.2 as well as Lemma 2.6.1 were taken from the monograph Prüss [63, Sections 3 and 8].

Let $a \in L_{1,loc}(\mathbb{R}_+)$. We say that a is of *subexponential growth* if for all $\varepsilon > 0$, $\int_0^\infty e^{-\varepsilon t}|a(t)|\,dt < \infty$. If this is the case, then it is readily seen that the Laplace transform $\hat{a}(\lambda)$ exists for $\operatorname{Re}\lambda > 0$.

Definition 2.6.1 ([63, Def. 3.2]) *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth and suppose $\hat{a}(\lambda) \neq 0$ for all $\operatorname{Re}\lambda > 0$. a is called* sectorial with angle $\theta > 0$ *(or merely* θ-sectorial*) if*

$$|\arg \hat{a}(\lambda)| \leq \theta \qquad \text{for all } \operatorname{Re}\lambda > 0. \tag{2.16}$$

Here, $\arg \hat{a}(\lambda)$ is defined as the imaginary part of a fixed branch of $\log \hat{a}(\lambda)$, and θ in (2.16) is allowed to be greater than π. In case a is sectorial, we always choose that branch of $\log \hat{a}(\lambda)$ which yields the smallest angle θ; in particular, if $\hat{a}(\lambda)$ is real for real λ we choose the principal branch.

The next definition introduces an appropriate notion of regularity of kernels.

Definition 2.6.2 ([63, Def. 3.3]) *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. a is called* k-regular *if there is a constant $c > 0$ such that*

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \leq c\,|\hat{a}(\lambda)| \qquad \text{for all } \operatorname{Re}\lambda > 0,\ 0 \leq n \leq k. \tag{2.17}$$

It is not difficult to see that convolutions of k-regular kernels are again k-regular. Furthermore, k-regularity is preserved by integration and differentiation, while sums and differences of k-regular kernels need not be k-regular. However, if a and b are k-regular and

$$|\arg \hat{a}(\lambda) - \arg \hat{b}(\lambda)| \leq \theta < \pi, \quad \operatorname{Re}\lambda > 0, \tag{2.18}$$

then $a + b$ is k-regular as well (see Lemma 2.6.2(ii)).

Some important properties of 1-regular kernels are contained in the following lemma.

Lemma 2.6.1 ([63, Lemma 8.1]) *Suppose $a \in L_{1,loc}(\mathbb{R}_+)$ is of subexponential growth and 1-regular. Then*

(i) $\hat{a}(i\rho) := \lim_{\lambda\to i\rho} \hat{a}(\lambda)$ *exists for each* $\rho \neq 0$;

(ii) $\hat{a}(\lambda) \neq 0$ *for each* $\operatorname{Re}\lambda \geq 0$;

(iii) $\hat{a}(i\cdot) \in W^{\infty}_{1,loc}(\mathbb{R} \setminus \{0\})$;

(iv) $|\rho \hat{a}'(i\rho)| \leq c|\hat{a}(i\rho)|$ *for a.a.* $\rho \in \mathbb{R}$;

(v) there is a constant $c > 0$ such that

$$c|\hat{a}(|\lambda|)| \leq |\hat{a}(\lambda)| \leq c^{-1}|\hat{a}(|\lambda|)|, \quad \operatorname{Re}\lambda \geq 0,\ \lambda \neq 0;$$

(vi) $\lim_{r\to\infty} \hat{a}(re^{i\phi}) = 0$ *uniformly for* $|\phi| \leq \frac{\pi}{2}$.

With regard to Volterra operators in L_p (see Section 2.8), we now introduce the subsequent class of kernels.

Definition 2.6.3 *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth, and assume $r \in \mathbb{N}$, $\theta_a > 0$, and $\alpha \geq 0$. Then a is said to belong to the class $\mathcal{K}^r(\alpha, \theta_a)$ if*

($\mathcal{K}1$) a is r-regular;

($\mathcal{K}2$) a is θ_a-sectorial;

($\mathcal{K}3$) $\limsup_{\mu\to\infty} |\hat{a}(\mu)|\,\mu^{\alpha} < \infty$, $\liminf_{\mu\to\infty} |\hat{a}(\mu)|\,\mu^{\alpha} > 0$, $\liminf_{\mu\to 0} |\hat{a}(\mu)| > 0$.

Further, $\mathcal{K}^\infty(\alpha,\theta_a) := \{a \in L_{1,loc}(\mathbb{R}_+) : a \in \mathcal{K}^r(\alpha,\theta_a)$ *for all* $r \in \mathbb{N}\}$*. The kernel* a *is called a* $\mathcal{K}$-kernel *if there exist* $r \in \mathbb{N}$, $\theta_a > 0$, *and* $\alpha \geq 0$, *such that* $a \in \mathcal{K}^r(\alpha,\theta_a)$.

An example for a $\mathcal{K}$-kernel is the so-called *standard kernel*:

Example 2.6.1 (standard kernel) *Let* $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, *with* $\alpha > 0$. *Then* $a \in L_{1,loc}(\mathbb{R}_+)$, *a is of subexponential growth, and its Laplace transform is given by* $\hat{a}(\lambda) = \lambda^{-\alpha}$, $\operatorname{Re}\lambda > 0$. *Hence,* $a \in \mathcal{K}^\infty(\alpha, \alpha\frac{\pi}{2})$.

From [58, p. 4793] it follows that $\mathcal{K}^r(\alpha,\theta) \neq \emptyset$ entails the inequality $\theta \geq \alpha\frac{\pi}{2}$. In view of Example 2.6.1 we thus get the following equivalence:

Remarks 2.6.1 *Let* $\alpha > 0$. *Then* $\mathcal{K}^r(\alpha,\theta) \neq \emptyset$ *if and only if* $\theta \geq \alpha\frac{\pi}{2}$.

The subsequent lemma collects some important algebraic properties of $\mathcal{K}$-kernels.

Lemma 2.6.2 *Suppose* $a \in \mathcal{K}^r(\alpha,\theta_a)$, $b \in \mathcal{K}^s(\beta,\theta_b)$, *and* $\omega > 0$. *Let* a_ω *be defined by* $a_\omega(t) = a(t)e^{-\omega t}$, $t \geq 0$. *Then the following statements hold true.*

(i) $a_\omega \in \mathcal{K}^r(\alpha,\theta_a) \cap L_1(\mathbb{R}_+)$, *and there exist positive constants* C_1, C_2 *such that*

$$\frac{C_1}{|\lambda+\omega|^\alpha} \leq |\hat{a}_\omega(\lambda)| \leq \frac{C_2}{|\lambda+\omega|^\alpha}, \quad \operatorname{Re}\lambda > 0. \tag{2.19}$$

(ii) If a *and* b *satisfy (2.18), then* $a + b \in \mathcal{K}^{\min\{r,s\}}(\min\{\alpha,\beta\}, \max\{\theta_a,\theta_b\})$.

(iii) $a * b \in \mathcal{K}^{\min\{r,s\}}(\alpha+\beta, \theta_a+\theta_b)$.

(iv) If $\alpha > \beta$ *and* $\liminf_{\mu\to 0} |\hat{a}(\mu)/\hat{b}(\mu)| > 0$, *then there exists a unique kernel* $c \in \mathcal{K}^{\min\{r,s\}}(\alpha-\beta, \theta_a+\theta_b)$ *such that* $a = b * c$. *If in addition* $\operatorname{Im}\hat{a}(\lambda)\cdot\operatorname{Im}\hat{b}(\lambda) \geq 0$ *for all* $\operatorname{Re}\lambda > 0$, *then* $c \in \mathcal{K}^{\min\{r,s\}}(\alpha-\beta, \max\{\theta_a,\theta_b\})$.

(v) If $\alpha > 0$ *and* $\theta_a < \pi$, *then there is a unique kernel* $\xi \in \mathcal{K}^r(\alpha,\theta_a)$ *such that* $\xi + \omega\xi * a = a$.

(vi) If $a \in L_1(\mathbb{R}_+)$ *and* $\epsilon := \omega|a|_{L_1(\mathbb{R}_+)} < 1$, *then there is a unique kernel* $\xi \in \mathcal{K}^r(\alpha, \theta_a + \arcsin(\epsilon)) \cap L_1(\mathbb{R}_+)$ *such that* $\xi - \omega\xi * a = a$.

Proof. Suppose $a \in \mathcal{K}^r(\alpha,\theta_a)$ and $\omega > 0$. Then it is evident that a_ω lies in $L_1(\mathbb{R}_+)$ and is of subexponential growth. Further, $\hat{a}_\omega(\lambda) = \hat{a}(\lambda+\omega)$, $\operatorname{Re}\lambda > 0$. Thus, for all $k \in \mathbb{N}$ with $1 \leq k \leq r$ and $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} |\lambda^k \hat{a}_\omega^{(k)}(\lambda)| &= |\lambda^k \hat{a}^{(k)}(\lambda+\omega)| = |(\lambda+\omega)^k \hat{a}^{(k)}(\lambda+\omega)| \left|\frac{\lambda}{\lambda+\omega}\right|^k \\ &\leq C|\hat{a}(\lambda+\omega)| = C|\hat{a}_\omega(\lambda)|, \end{aligned}$$

where $C > 0$ is the constant of r-regularity of a, i.e. a_ω is r-regular. We easily see θ_a-sectoriality of a_ω, too. Moreover, Lemma 2.6.1(v) implies $|\hat{a}_\omega(\lambda)| \leq c|\hat{a}_\omega(|\lambda|)|$ for all $\lambda \in \mathbb{C}_+$, with c not depending on λ. So, owing to continuity and the asymptotic behaviour of $\hat{a}$ on $\mathbb{R}_+$ described in ($\mathcal{K}3$), there exist two positive constants C_1 and C_2 such that $|\hat{a}_\omega(\lambda)(\lambda+\omega)^\alpha| = |\hat{a}(\lambda+\omega)(\lambda+\omega)^\alpha| \in [C_1, C_2]$ for all $\operatorname{Re}\lambda > 0$. This shows (2.19). As for property ($\mathcal{K}3$), by Lemma 2.6.1(ii), we have $\liminf_{\mu\to 0} |\hat{a}_\omega(\mu)| = |\hat{a}(\omega)| > 0$. The other two properties in $(\mathcal{K}3)$ follow immediately from (2.19) and $\lim_{\mu\to\infty}|\mu^\alpha/(\mu+\omega)^\alpha| = 1$. Hence, assertion (i) of Lemma 2.6.2 is proved.

To show (ii), suppose $a \in \mathcal{K}^r(\alpha,\theta_a)$ and $b \in \mathcal{K}^s(\beta,\theta_b)$. Trivially, $a+b \in L_{1,loc}(\mathbb{R}_+)$, and $a+b$ is of subexponential growth. We then note that by (2.18) there exists a constant $c > 0$ such that $|\hat{a}(\lambda)| + |\hat{b}(\lambda)| \le c|\hat{a}(\lambda)+\hat{b}(\lambda)|$, $\mathrm{Re}\,\lambda > 0$. Therefore, for all $k \in \mathbb{N}$ with $1 \le k \le \min\{r,s\}$ and $\lambda \in \mathbb{C}_+$,

$$|\lambda^k(\hat{a}(\lambda)+\hat{b}(\lambda))| \le |\lambda^k\hat{a}(\lambda)| + |\lambda^k\hat{b}(\lambda)| \le C(|\hat{a}(\lambda)| + |\hat{b}(\lambda)|) \le C\,c|\hat{a}(\lambda)+\hat{b}(\lambda)|,$$

where C only depends on the constants of $\min\{r,s\}$-regularity of a and b. This proves $\min\{r,s\}$-regularity of $a+b$. From θ_a-sectoriality of a, θ_b-sectoriality of b, and (2.18) it follows that $|\arg(\hat{a}(\lambda)+\hat{b}(\lambda))| \le \max\{\theta_a,\theta_b\}$ for all $\mathrm{Re}\,\lambda > 0$. Thanks to (2.18) we further have

$$\liminf_{\mu\to 0}|\hat{a}(\mu)+\hat{b}(\mu)| \ge c^{-1}\liminf_{\mu\to 0}(|\hat{a}(\mu)| + |\hat{b}(\mu)|) > 0,$$

which shows the third condition in $(\mathcal{K}3)$ for $a+b$. W.l.o.g. we may then assume $\alpha \le \beta$ and obtain the estimates

$$\liminf_{\mu\to\infty}|\hat{a}(\mu)+\hat{b}(\mu)|\mu^\alpha \ge c^{-1}\liminf_{\mu\to\infty}(|\hat{a}(\mu)| + |\hat{b}(\mu)|)\mu^\alpha \ge c^{-1}\liminf_{\mu\to\infty}|\hat{a}(\mu)|\mu^\alpha > 0,$$

$$\limsup_{\mu\to\infty}|\hat{a}(\mu)+\hat{b}(\mu)|\mu^\alpha \le \limsup_{\mu\to\infty}|\hat{a}(\mu)|\mu^\alpha + \limsup_{\mu\to\infty}|\hat{b}(\mu)|\mu^\beta\mu^{\alpha-\beta} < \infty.$$

So assertion (ii) is also established.

We now come to (iii). Suppose $a \in \mathcal{K}^r(\alpha,\theta_a)$, $b \in \mathcal{K}^s(\beta,\theta_b)$. By Young's inequality, $a*b \in L_{1,loc}(\mathbb{R}_+)$ and given $\varepsilon > 0$, we have

$$|(a*b)e^{-\varepsilon\cdot}|_{L_1(\mathbb{R}_+)} = |(a_\varepsilon * b_\varepsilon)|_{L_1(\mathbb{R}_+)} \le |a_\varepsilon|_{L_1(\mathbb{R}_+)}\,|b_\varepsilon|_{L_1(\mathbb{R}_+)} < \infty,$$

i.e. $a*b$ is of subexponential growth. Further, $(a*b)\hat{} = \hat{a}\hat{b}$ due to the convolution theorem. So $a*b$ is $(\theta_a+\theta_b)$-sectorial. Assuming $k \in \mathbb{N}$, $1 \le k \le \min\{r,s\}$ Leibniz' formula in combination with r-regularity of a and b yields

$$|\lambda^k(\hat{a}(\lambda)\hat{b}(\lambda))^{(k)}| \le \sum_{i=0}^{k}\binom{k}{i}|\lambda^i\hat{a}^{(i)}(\lambda)|\,|\lambda^{(k-i)}\hat{b}^{(k-i)}(\lambda)| \le C < \infty, \;\; \mathrm{Re}\,\lambda > 0. \tag{2.20}$$

Thus $a*b$ is $\min\{r,s\}$-regular. Finally,

$$\limsup_{\mu\to\infty}|\hat{a}(\mu)\hat{b}(\mu)|\mu^{\alpha+\beta} \le (\limsup_{\mu\to\infty}|\hat{a}(\mu)|\mu^\alpha)(\limsup_{\mu\to\infty}|\hat{b}(\mu)|\mu^\beta) < \infty.$$

The other two conditions in $(\mathcal{K}3)$ are shown similarly. Hence, $a*b$ satisfies $(\mathcal{K}3)$ with exponent $\alpha+\beta$, and so (iii) is proved.

Next we show (iv). Suppose $a \in \mathcal{K}^r(\alpha,\theta_a)$, $b \in \mathcal{K}^s(\beta,\theta_b)$, and $\alpha > \beta$. For any fixed $\omega > 0$ we know from (i) that $a_\omega \in \mathcal{K}^r(\alpha,\theta_a)$ and $b_\omega \in \mathcal{K}^s(\beta,\theta_b)$, in particular $\hat{b}_\omega(\lambda) \ne 0$, $\mathrm{Re}\lambda > 0$, due to Lemma 2.6.1(ii). So we can define $g_\omega(\lambda) := \hat{a}_\omega(\lambda)/\hat{b}_\omega(\lambda)$, $\mathrm{Re}\lambda > 0$. The function g_ω is holomorphic in $\mathbb{C}_+$, and by 1-regularity of a_ω and b_ω we get

$$\begin{aligned} |\lambda g_\omega'(\lambda)| &= \left|\frac{\lambda\hat{a}_\omega'(\lambda)\hat{b}_\omega(\lambda) - \lambda\hat{a}_\omega(\lambda)\hat{b}_\omega'(\lambda)}{\hat{b}_\omega(\lambda)^2}\right| \le \left(\left|\frac{\lambda\hat{a}_\omega'(\lambda)}{\hat{a}_\omega(\lambda)}\right| + \left|\frac{\lambda\hat{b}_\omega'(\lambda)}{\hat{b}_\omega(\lambda)}\right|\right)|g_\omega(\lambda)| \\ &\le C_1|g_\omega(\lambda)|, \qquad \mathrm{Re}\,\lambda > 0. \end{aligned} \tag{2.21}$$

Using (2.19) for a_ω and b_ω yields $|g_\omega(\lambda)| \leq C_2|\lambda+\omega|^{\beta-\alpha} \leq C_2|\lambda|^{\beta-\alpha}$, $\mathrm{Re}\lambda > 0$, where $C_2 > 0$ is independent of λ. In view of (2.21) we thus obtain

$$|\lambda g_\omega'(\lambda)| + |g_\omega(\lambda)| \leq \frac{C}{|\lambda|^{\alpha-\beta}}, \qquad \mathrm{Re}\,\lambda > 0, \tag{2.22}$$

with $C > 0$ not depending on λ. Proposition 2.1.1 then allows us to define $u_\omega \in C(0,\infty) \cap L_{1,loc}(\mathbb{R}_+)$ by means of $\hat{u}_\omega(\lambda) = g_\omega(\lambda)$, $\mathrm{Re}\,\lambda > 0$. Estimate (2.2) implies that u_ω is of subexponential growth. So we have $\hat{u}(\lambda)\hat{b}_\omega(\lambda) = \hat{a}_\omega(\lambda)$, $\lambda \in \mathbb{C}_+$, i.e. $(u_\omega * b_\omega)(t) = a_\omega(t)$, $t > 0$. With $u(t) := u_\omega(t)e^{\omega t}$, $t > 0$, we thus arrive at $u * b = a$. Observe that the construction of u is independent of the chosen $\omega > 0$. Further, u is of subexponential growth, for u_ω possesses this property for each $\omega > 0$. Since $\hat{u}(\lambda) = \hat{a}(\lambda)/\hat{b}(\lambda)$, $\mathrm{Re}\lambda > 0$, it is clear that u is $(\theta_a + \theta_b)$-sectorial, even $\max\{\theta_a, \theta_b\}$-sectorial provided that $\mathrm{Im}\,\hat{a}(\lambda)\cdot\mathrm{Im}\,\hat{b}(\lambda) \geq 0$ for all $\lambda \in \mathbb{C}_+$.

We now show $\min\{r,s\}$-regularity of u. To this purpose we put $m = \min\{r,s\}$ and $\hat{h}(\lambda) = 1/\hat{b}(\lambda)$, $\lambda \in \mathbb{C}_+$. Then 1-regularity of b yields an estimate

$$|\lambda^k \hat{h}^{(k)}(\lambda)| \leq C|\hat{h}(\lambda)|, \qquad \lambda \in \mathbb{C}_+, \tag{2.23}$$

for $k = 1$. Here the constant C does not depend on λ. Let us now assume that (2.23) holds true for all $k \in \mathbb{N}$ with $1 \leq k \leq n$, where $n \in \mathbb{N}$ and $n < m$. If we then differentiate $(n+1)$ times both sides of the equation $\hat{b}\hat{h} = 1$ and use Leibniz' formula it becomes apparent that

$$\hat{h}^{(n+1)}(\lambda) = -\frac{1}{\hat{b}(\lambda)} \sum_{i=1}^{n+1} \binom{n+1}{i} \hat{b}^{(i)}(\lambda)\hat{h}^{(n+1-i)}(\lambda), \quad \mathrm{Re}\,\lambda > 0.$$

Thus, by hypothesis and m-regularity of b,

$$|\lambda^{n+1}\hat{h}^{(n+1)}(\lambda)| \leq \sum_{i=1}^{n+1} \binom{n+1}{i} \left|\frac{\lambda^i \hat{b}^{(i)}(\lambda)}{\hat{b}(\lambda)}\right| \left|\lambda^{n+1-i}\hat{h}^{(n+1-i)}(\lambda)\right| \leq C_1|\hat{h}(\lambda)|$$

for all $\lambda \in \mathbb{C}_+$, with C_1 not depending on λ. So, induction over $n \leq m$ establishes (2.23) for all $k \leq m$. Using this fact and m-regularity of a we can argue as in the proof of (iii) (cp. (2.20)) to see that u is m-regular.

The function u also fulfills property $(\mathcal{K}3)$ with exponent $\alpha - \beta$. In fact, we have $\liminf_{\mu\to 0} |\hat{u}(\mu)| > 0$ by assumption. Moreover, owing to $a_1 \in \mathcal{K}^r(\alpha, \theta_a)$, $b_1 \in \mathcal{K}^s(\beta, \theta_b)$, and (2.19), we have

$$\limsup_{\mu\to\infty} \left|\frac{\hat{a}(\mu)}{\hat{b}(\mu)}\right| \mu^{\alpha-\beta} = \limsup_{\mu\to\infty} \left|\frac{\hat{a}(\mu+1)(\mu+1)^\alpha}{\hat{b}(\mu+1)(\mu+1)^\beta}\right| < \infty.$$

Likewise we see $\liminf_{\mu\to\infty} |\hat{u}(\mu)|\mu^{\alpha-\beta} > 0$. Hence, the kernel $c := u$ possesses all properties claimed in (iv). Uniqueness follows from the unique inverse of the Laplace transform. The proof of (iv) is complete.

Our next aim is to show (v). Suppose $a \in \mathcal{K}^r(\alpha, \theta_a)$, $\omega, \alpha > 0$, and $\theta_a < \pi$. Further fix an arbitrary $\eta > 0$. Due to the last assumption there exists a constant $c > 0$ not depending on λ such that

$$|1 + \omega\hat{a}(\lambda)| \geq c, \quad \mathrm{Re}\,\lambda > 0. \tag{2.24}$$

Letting $g_\eta(\lambda) = \hat{a}_\eta(\lambda)/(1 + \omega \hat{a}_\eta(\lambda))$, $\mathrm{Re}\lambda > 0$, we thus obtain $|g_\eta(\lambda)| \leq C_0|\lambda|^{-\alpha}$ for all $\lambda \in \mathbb{C}_+$, where $C_0 > 0$ is independent of λ. Here, we also made use of (2.19) for the kernel a_η. Observe that

$$\lambda g_\eta'(\lambda) = g_\eta(\lambda)\, \frac{\lambda \hat{a}_\eta'(\lambda)}{\hat{a}_\eta(\lambda)} \cdot \frac{1}{1 + \omega \hat{a}_\eta(\lambda)}, \quad \mathrm{Re}\,\lambda > 0.$$

Thanks to (2.24) and 1-regularity of a_η, we therefore get an estimate

$$|\lambda g_\eta'(\lambda)| + |g_\eta(\lambda)| \leq \frac{C}{|\lambda|^\alpha}, \qquad \mathrm{Re}\,\lambda > 0, \tag{2.25}$$

which, together with $\alpha > 0$ and holomorphy of g_η in $\mathbb{C}_+$, implies existence of $u_\eta \in C(0, \infty) \cap L_{1,\,loc}(\mathbb{R}_+)$ satisfying $\hat{u}_\eta(\lambda) = g_\eta(\lambda)$, $\mathrm{Re}\,\lambda > 0$, by Proposition 2.1.1. (2.2) entails that u_η is of subexponential growth. By inversion of the Laplace transform we then obtain $u_\eta + \omega a_\eta * u_\eta = a_\eta$, hence $u + \omega a * u = a$, where $u(t) := u_\eta(t)e^{\eta t}$, $t > 0$. As in the proof of (iv), we see that the construction of u is independent of $\eta > 0$, and that u is of subexponential growth. From $\hat{u} = \hat{a}/(1 + \omega \hat{a})$ and $\omega > 0$ one deduces that u is θ_a-sectorial. Furthermore, the function $h(\lambda) := 1 + \omega \hat{a}(\lambda)$ defined on $\mathbb{C}_+$ satisfies (2.23) with a suitable constant $C > 0$ for $k = 1, \ldots, r$. In fact,

$$|\lambda^k \hat{h}^{(k)}(\lambda)| = \omega^k |\lambda^k \hat{a}^{(k)}(\lambda)| \leq C|\hat{a}(\lambda)| \leq C|\hat{h}(\lambda)| \left| \frac{1}{\omega + 1/\hat{a}(\lambda)} \right| \leq C_1|\hat{h}(\lambda)|$$

for all $\lambda \in \mathbb{C}_+$ and $k = 1, \ldots, r$, in virtue of $\theta_a < \pi$ and r-regularity of a. By the considerations in the proof of (iv), that property of h is passed on to the function $\bar{h}(\lambda) := 1/h(\lambda)$, $\lambda \in \mathbb{C}_+$, which is well-defined in view of (2.24). Then $\hat{u} = \hat{a}\bar{h}$, and as above, with the aid of Leibniz' formula, we see that u is r-regular. Concerning $(\mathcal{K}3)$, the assumption $\zeta := \liminf_{\mu \to 0} |\hat{a}(\mu)| > 0$ implies $\liminf_{\mu \to 0} |\hat{u}(\mu)| > 0$, since

$$\liminf_{\mu \to 0} |\hat{u}(\mu)| \geq \liminf_{\mu \to 0} (\omega + |1/\hat{a}(\mu)|)^{-1} = (\omega + 1/\zeta)^{-1}.$$

Besides, by (2.24), $\limsup_{\mu \to \infty} |\hat{a}(\mu)|\mu^\alpha < \infty$, and (2.19) applied to a_1, we have

$$\limsup_{\mu \to \infty} |\hat{u}(\mu)|\mu^\alpha \leq c^{-1} \limsup_{\mu \to \infty} |\hat{u}(\mu)|\mu^\alpha < \infty,$$

as well as

$$\liminf_{\mu \to \infty} |\hat{u}(\mu)|\mu^\alpha \geq \liminf_{\mu \to \infty} \frac{(\mu + 1)^\alpha}{\omega + |1/\hat{a}(\mu + 1)|} \geq \liminf_{\mu \to \infty} \frac{(\mu + 1)^\alpha}{\omega + C(\mu + 1)^\alpha} > 0.$$

Hence, the kernel $\xi := u$ satisfies $\xi + \omega a * \xi = a$ and belongs to $\mathcal{K}^r(\alpha, \theta_a)$. Uniqueness follows again from the unique inverse of the Laplace transform. Thus (v) is shown.

It remains to prove (vi). Suppose $a \in \mathcal{K}^r(\alpha, \theta_a) \cap L_1(\mathbb{R}_+)$, $\omega > 0$, and $\epsilon := \omega|a|_{L_1(\mathbb{R}_+)} < 1$. We proceed as in the previous part. Fix an arbitrary $\eta > 0$ and define $g_\eta(\lambda) := \hat{a}_\eta(\lambda)/(1 - \omega \hat{a}_\eta(\lambda))$, $\mathrm{Re}\lambda > 0$. This time the assumption $\epsilon < 1$ ensures that the denominator in the definition of g_η is bounded away from zero. Using this and 1-regularity of a_η yields (2.25) in a similar fashion as above. With the aid of Proposition 2.1.1, existence of $u \in L_{1,\,loc}(\mathbb{R}_+)$ with $u - \omega a * u = a$ can then be seen. By the same line of arguments as in the previous part one further shows that u is r-regular. Validity of the conditions in $(\mathcal{K}3)$ with exponent α can be proved for u similarly as above. By elementary trigonometry, $|\arg{(1 - \omega \hat{a}(\lambda))}| \leq \arcsin(\epsilon)$ for all $\lambda \in \mathbb{C}_+$, i.e. u is $(\theta_a + \arcsin(\epsilon))$-sectorial. Last but not least, $u \in L_1(\mathbb{R}_+)$ follows from $\epsilon < 1$ and $a \in L_1(\mathbb{R}_+)$ by the Paley-Wiener theorem. Hence, $\xi := u$ possesses all properties claimed in (vi), uniqueness being evident as above. □

2.7 Evolutionary integral equations

Let X be a Banach space, A a closed linear, but in general unbounded operator in X with dense domain $\mathcal{D}(A)$, and $a \in L_{1,loc}(\mathbb{R}_+)$ a scalar kernel of subexponential growth which is not identically zero. We consider the Volterra equation

$$u(t) + \int_0^t a(t-s)Au(s)\,ds = f(t), \quad t \geq 0, \tag{2.26}$$

where $f : \mathbb{R}_+ \to X$ is a given function, strongly measurable and locally integrable, at least. Observe that in case $a(t) \equiv 1$ and f differentiable, (2.26) is equivalent to the Cauchy problem

$$\dot{u}(t) + Au(t) = \dot{f}(t),\ t \geq 0, \quad u(0) = f(0).$$

Following Prüss [66] (see also [63, Def. 3.1]), we call (2.26) *parabolic* if $\hat{a}(\lambda) \neq 0$ for $\operatorname{Re}\lambda > 0$, $-1/\hat{a}(\lambda) \in \rho(A)$, and there is a constant $M > 0$ such that

$$|(I + \hat{a}(\lambda)A)^{-1}| \leq M \quad \text{for } \operatorname{Re}\lambda > 0.$$

If A belongs to the class $\mathcal{S}(X)$ with spectral angle ϕ_A, and a is θ_a-sectorial, then (2.26) is parabolic provided that $\theta_A + \phi_A < \pi$, cf. [63, Prop. 3.1].

An important property of parabolic Volterra equations consists in that they admit bounded *resolvents* whenever the kernel a is 1-regular, see [63, Thm 3.1]. By a resolvent for (2.26) we mean a family $\{S(t)\}_{t\geq 0}$ of bounded linear operators in X which satisfy the following conditions:

(S1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;

(S2) $S(t)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $AS(t)x = S(t)Ax$ for all $x \in \mathcal{D}(A)$, $t \geq 0$;

(S3) $S(t)x + A(a * Sx)(t) = x$, for all $x \in X$, $t \geq 0$.

(S3) is called *resolvent equation*, cf. [63, Def. 1.3, Prop. 1.1]. One can show that (2.26) admits at most one resolvent, and if it exists, then (2.26) has a unique mild solution u represented by the variation of parameters formula

$$u(t) = \frac{d}{dt}\int_0^t S(t-s)f(s)\,ds, \quad t \geq 0, \tag{2.27}$$

at least for such f for which (2.27) is meaningful, see [63, Section 1.1 and 1.2].

2.8 Volterra operators in L_p

This paragraph looks at convolution operators in L_p which are associated to a $\mathcal{K}$-kernel. After stating two fundamental theorems from the monograph Prüss [63] on the inversion of the convolution in $L_p(\mathbb{R};X)$, we will consider restrictions of it to $L_p(J;X)$ and use them to introduce equivalent norms for the vector-valued Bessel-potential spaces $H_p^\alpha(J;X)$. We will also study operators of the form $(I - a*)$ in these spaces. Such operators occur in connection with transformations of Volterra equations.

For $J = [0,T]$ and $J = \mathbb{R}_+$, we identify in the sequel $L_p(J;X)$ with the subspace $\{f \in L_p(\mathbb{R};X) : \operatorname{supp} f \subseteq \mathbb{R}_+\}$ of $L_p(\mathbb{R};X)$.

Theorem 2.8.1 (Prüss [63, Thm. 8.6]) *Suppose X belongs to the class $\mathcal{HT}$, $p \in (1,\infty)$, and let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth. Assume that a is 1-regular and θ-sectorial, where $\theta < \pi$. Then there is a unique operator $B \in \mathcal{S}(L_p(\mathbb{R};X))$ such that*

$$(Bf)\tilde{}(\rho) = \frac{1}{\hat{a}(i\rho)}\tilde{f}(\rho), \quad \rho \in \mathbb{R},\ \tilde{f} \in C_0^\infty(\mathbb{R}\setminus\{0\};X). \tag{2.28}$$

Moreover, B has the following properties:

(i) B commutes with the group of translations;

(ii) $(\mu + B)^{-1}L_p(\mathbb{R}_+;X) \subset L_p(\mathbb{R}_+;X)$ for each $\mu > 0$, i.e. B is causal;

(iii) $B \in \mathcal{BIP}(L_p(\mathbb{R};X))$, and $\theta_B = \phi_B = \theta_a$, where

$$\theta_a = \sup\{|\arg \hat{a}(\lambda)| \ :\ \operatorname{Re}\lambda > 0\}; \tag{2.29}$$

(iv) $\sigma(B) = \overline{\{1/\hat{a}(i\rho) \ :\ \rho \in \mathbb{R}\setminus\{0\}\}}$.

The next theorem provides information about the domain of the operator B.

Proposition 2.8.1 (Prüss [63, Cor. 8.1]) *Suppose X belongs to the class $\mathcal{HT}$, $p \in (1,\infty)$. Assume $a \in \mathcal{K}^1(\alpha,\theta)$ with $\theta < \pi$, and let B be defined by (2.28). Then $\mathcal{D}(B) = H_p^\alpha(\mathbb{R};X)$.*

Here $H_p^\alpha(\mathbb{R};X) := \mathcal{D}(B_0^{\alpha/2})$, $\alpha \in \mathbb{R}_+$, where $B_0 = -(d^2/dt^2) \in \mathcal{BIP}(L_p(\mathbb{R};X))$, cf. [63, p. 226].

Suppose the assumptions of Proposition 2.8.1 hold. Let $J = [0,T]$ or $J = \mathbb{R}_+$. We put $H_p^\alpha(J;X) = \{f|_J : f \in H_p^\alpha(\mathbb{R};X)\}$ and endow this space with the norm $|f|_{H_p^\alpha(J;X)} = \inf\{|g|_{H_p^\alpha(\mathbb{R};X)} : g|_J = f\}$. We further introduce the subspace ${}_0H_p^\alpha(J;X)$ by means of ${}_0H_p^\alpha(J;X) = \{f|_J : f \in H_p^\alpha(\mathbb{R};X) \text{ and } \operatorname{supp} f \subseteq \mathbb{R}_+\}$. Define then the operator $\mathcal{B} \in \mathcal{S}(L_p(J;X))$ as the restriction of the operator B constructed in Theorem 2.8.1 to $L_p(J;X)$. This makes sense in virtue of causality. In fact, we have

$$\begin{aligned}
\mathcal{D}(\mathcal{B}) &= \mathcal{D}(B|_{L_p(J;X)}) = \{f \in L_p(J;X) \cap \mathcal{D}(B) :\ Bf \in L_p(J;X)\} \\
&= \{f \in L_p(J;X) \cap H_p^\alpha(\mathbb{R};X) :\ Bf \in L_p(J;X)\} \\
&= \{f|_J :\ f \in H_p^\alpha(\mathbb{R};X),\ \operatorname{supp} f \subseteq \mathbb{R}_+,\ Bf \in L_p(\mathbb{R};X), \text{ and } \operatorname{supp} Bf \subseteq \mathbb{R}_+\} \\
&= \{f|_J :\ f \in H_p^\alpha(\mathbb{R};X) \text{ and } \operatorname{supp} f \subseteq \mathbb{R}_+\} \\
&= {}_0H_p^\alpha(J;X),
\end{aligned}$$

the equals sign before the last following from the causality of B. Assuming in addition $a \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$, by Young's inequality, the operator $\mathcal{B}$ is invertible and $\mathcal{B}^{-1}w = a * w$ for all $w \in L_p(J;X)$. From Theorem 2.8.1 we further see that $\mathcal{B} \in \mathcal{BIP}(L_p(J;X))$ and $\theta_{\mathcal{B}} \le \theta_B = \theta_a$. We summarize these observations in the subsequent corollary.

Corollary 2.8.1 *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, and $J = [0,T]$ be a compact interval or $J = \mathbb{R}_+$. Suppose $a \in \mathcal{K}^1(\alpha,\theta)$ with $\theta < \pi$, and assume in addition $a \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$. Then the restriction $\mathcal{B} := B|_{L_p(J;X)}$ of the operator B constructed in Theorem 2.8.1 to $L_p(J;X)$ is well-defined. The operator $\mathcal{B}$ belongs to the class $\mathcal{BIP}(L_p(J;X))$ with power angle $\theta_{\mathcal{B}} \le \theta_B = \theta_a$ and is invertible satisfying $\mathcal{B}^{-1}w = a * w$ for all $w \in L_p(J;X)$. Moreover $\mathcal{D}(\mathcal{B}) = {}_0H_p^\alpha(J;X)$.*

Example 2.8.1 For $J = [0,T]$ and $a_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$, $\alpha \in (0,1)$, the operator $\mathcal{B}$ in Corollary 2.8.1 takes the form

$$\mathcal{B}u(t) = \frac{d}{dt}\int_0^t a_{1-\alpha}(t-s)u(s)\,ds, \quad t > 0,\ u \in {}_0H_p^\alpha(J;X),$$

thus coincides with $(d/dt)^\alpha$, the *derivation operator of (fractional) order* α. The function $\mathcal{B}u$ is called the *fractional derivative* of u of order α.

We point out that Corollary 2.8.1 will prove extremely useful in establishing maximal L_p-regularity results for parabolic Volterra equations. On the one hand it describes precisely the mapping properties of the convolution operators associated to a $\mathcal{K}$-kernel, on the other hand it enables us to apply the Dore-Venni theorem to operator sums in $L_p(J;X)$ which involve an inverse convolution operator $\mathcal{B}$.

We next show how Volterra operators can be used to introduce equivalent norms for the vector-valued Bessel-potential spaces $H_p^\alpha(J;X)$. Suppose we are in the situation of Corollary 2.8.1, where we restrict ourselves to the case $J = [0,T]$. Assume further that $\alpha \in (0,2) \setminus \{1/p, 1+1/p\}$ and let $\mu > 0$. For $f \in H_p^\alpha(J;X)$, we put with some abuse of language

$$|f|^{(a,\mu)}_{H_p^\alpha(J;X)} = \begin{cases} |\mathcal{B}f|_{L_p(J;X)} & : \alpha \in (0,\frac{1}{p}) \\ |\mathcal{B}(f-f(0))|_{L_p(J;X)} + \mu|f(0)|_X & : \alpha \in (\frac{1}{p}, 1+\frac{1}{p}) \\ |\mathcal{B}(f-f(0)-t\dot{f}(0))|_{L_p(J;X)} + \mu|f(0)|_X + \mu|\dot{f}(0)|_X & : \alpha \in (1+\frac{1}{p}, 2). \end{cases} \tag{2.30}$$

This is a well-defined expression in view of Sobolev's embedding theorem. Observe that $|\cdot|^{(a,\mu)}_{H_p^\alpha(J;X)}$ enjoys the properties of a norm for the space $H_p^\alpha(J;X)$. To verify that it is equivalent to the usual norm $|\cdot|_{H_p^\alpha(J;X)}$ we can employ Corollary 2.8.1 and Sobolev embeddings. Indeed, if $\alpha > 1+1/p$ and $f \in H_p^\alpha(J;X)$, we may estimate

$$\begin{aligned} |f|_{H_p^\alpha(J;X)} &\le |f-f(0)-t\dot{f}(0)|_{H_p^\alpha(J;X)} + |f(0)|_{H_p^\alpha(J;X)} + |t\dot{f}(0)|_{H_p^\alpha(J;X)} \\ &= |a * \mathcal{B}(f-f(0)-t\dot{f}(0))|_{H_p^\alpha(J;X)} + |\{t \mapsto 1\}|_{H_p^\alpha(J)}|f(0)|_X \\ &\quad + |\{t \mapsto t\}|_{H_p^\alpha(J)}|\dot{f}(0)|_X \\ &\le c_1\,(|\mathcal{B}(f-f(0)-t\dot{f}(0))|_{L_p(J;X)} + \mu|f(0)|_X + \mu|\dot{f}(0)|_X) \\ &= c_1\,|f|^{(a,\mu)}_{H_p^\alpha(J;X)}, \end{aligned}$$

with c_1 not depending on f. Conversely, by

$$|f(0)|_X + |\dot{f}(0)|_X \le |f|_{C^1(J;X)} \le C_{Sob}|f|_{H_p^\alpha(J;X)} \tag{2.31}$$

and

$$\begin{aligned} |\mathcal{B}(f-f(0)-t\dot{f}(0))|_{L_p(J;X)} &\le c|f-f(0)-t\dot{f}(0)|_{H_p^\alpha(J;X)} \\ &\le c(|f|_{H_p^\alpha(J;X)} + |f(0)|_{H_p^\alpha(J;X)} + |t\dot{f}(0)|_{H_p^\alpha(J;X)}) \\ &\le c(|f|_{H_p^\alpha(J;X)} + \tilde{c}(|f(0)|_X + |\dot{f}(0)|_X)), \end{aligned}$$

we also get an inequality of the form

$$|f|^{(a,\mu)}_{H_p^\alpha(J;X)} \le c_2\,|f|_{H_p^\alpha(J;X)},$$

where c_2 does not depend on f.

To see the equivalence of the norms for $\alpha < 1 + 1/p$, replace the non-existing traces in the above estimates with zero and use $H_p^\alpha(J;X) \hookrightarrow C(J;X)$ instead of (2.31), if $\alpha \in (1/p, 1+1/p)$. We thus have proved

Corollary 2.8.2 *Let the assumptions of Corollary 2.8.1 hold. Let $J = [0,T]$, $\mu > 0$, and assume that $\alpha \in (0,2) \setminus \{1/p, 1+1/p\}$. Then (2.30) defines an equivalent norm for $H_p^\alpha(J;X)$.*

We continue to consider the setting of Corollary 2.8.1, where $J = [0,T]$ and $\alpha \in (0,2) \setminus \{1/p, 1+1/p\}$.

Let us additionally assume that $a \in L_1(\mathbb{R}_+)$ with $\nu := |a|_{L_1(\mathbb{R}_+)} < 1$, and define the operator $\mathcal{T}_a$ in $L_p(J;X)$ by

$$\mathcal{T}_a f = f - a * f, \quad f \in L_p(J;X). \tag{2.32}$$

By Young's inequality,

$$|a * f|_{L_p(J;X)} \leq |a|_{L_1(J)} |f|_{L_p(J;X)} \leq \nu |f|_{L_p(J;X)}, \; f \in L_p(J;X),$$

i.e. $\mathcal{T}_a \in \mathcal{B}(L_p(J;X))$ and $|\mathcal{T}_a|_{\mathcal{B}(L_p(J;X))} \leq 1 + \nu$. Since $\nu < 1$, we also see that $\mathcal{T}_a$ is invertible with $|\mathcal{T}_a^{-1}|_{\mathcal{B}(L_p(J;X))} \leq 1/(1-\nu)$.

Suppose now that $f \in Y := H_p^\alpha(J;X)$. Then trivially $f \in L_p(J;X)$, and thus Corollary 2.8.1 yields $\mathcal{T}_a f \in Y$. Assuming $\mu \geq \nu^{-1} T^{1/p} \max\{1, T/(1+p)^{1/p}\}$ in (2.30), we obtain in the case $\alpha > 1 + 1/p$,

$$\begin{aligned}
|a * f|_Y^{(a,\mu)} &= |\mathcal{B}(a * f)|_{L_p(J;X)} = |f|_{L_p(J;X)} \\
&\leq |f - f(0) - t\dot{f}(0)|_{L_p(J;X)} + |f(0)|_{L_p(J;X)} + |t\dot{f}(0)|_{L_p(J;X)} \\
&\leq |a * \mathcal{B}(f - f(0) - t\dot{f}(0))|_{L_p(J;X)} + T^{\frac{1}{p}} |f(0)|_X + (\frac{T^{p+1}}{p+1})^{\frac{1}{p}} |\dot{f}(0)|_X \\
&\leq |a|_{L_1(J)} |\mathcal{B}(f - f(0) - t\dot{f}(0))|_{L_p(J;X)} + \mu\nu(|f(0)|_X + |\dot{f}(0)|_X) \\
&\leq \nu \, |f|_Y^{(a,\mu)}.
\end{aligned}$$

In the same way, we see that $|a * f|_Y^{(a,\mu)} \leq \nu |f|_Y^{(a,\mu)}$ is valid for $\alpha < 1 + 1/p$. This shows $\mathcal{T}_a \in \mathcal{B}(Y)$ and $|\mathcal{T}_a|_{\mathcal{B}(Y)} \leq 1 + \nu$, where the operator norm is induced by $|\cdot|_Y^{(a,\mu)}$, which is a norm for Y, due to Corollary 2.8.2. Moreover, in view of $\nu < 1$, it follows that $\mathcal{T}_a$ is invertible in Y and $|\mathcal{T}_a^{-1}|_{\mathcal{B}(Y)} \leq 1/(1-\nu)$. This is also true in the subspace ${}_0Y := {}_0H_p^\alpha(J;X)$. Here (2.30) reduces to $|f|_{{}_0Y}^{(a,\mu)} = |\mathcal{B}f|_{L_p(J;X)}$, $f \in {}_0Y$.

We record these properties of $\mathcal{T}_a$ in

Corollary 2.8.3 *Let the assumptions of Corollary 2.8.1 hold. Let $J = [0,T]$, and assume $\alpha \in (0,2) \setminus \{1/p, 1+1/p\}$. Suppose further $a \in L_1(\mathbb{R}_+)$ and $\nu := |a|_{L_1(\mathbb{R}_+)} < 1$. Then the operator $\mathcal{T}_a$ defined by (2.32) is an isomorphism of the spaces $Y = L_p(J;X)$, $H_p^\alpha(J;X)$, and ${}_0H_p^\alpha(J;X)$, satisfying*

$$|\mathcal{T}_a|_{\mathcal{B}(Y)} \leq 1 + \nu, \quad |\mathcal{T}_a^{-1}|_{\mathcal{B}(Y)} \leq \frac{1}{1-\nu},$$

where in case $Y = H_p^\alpha(J;X)$ or ${}_0H_p^\alpha(J;X)$, the operator norm is induced by (2.30) with $\mu \geq \nu^{-1} T^{1/p} \max\{1, T/(1+p)^{1/p}\}$.

We conclude this section by illustrating the usefulness of the operators $\mathcal{T}$ in connection with transformations of Volterra equations.

Let X be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, and $J = [0, T]$ be a compact interval. We consider in X the Volterra equation

$$u + a * Au = f, \quad t \in J, \tag{2.33}$$

where A is a closed linear operator in X with domain $\mathcal{D}(A)$, and the kernel a is assumed to belong to the class $a \in \mathcal{K}^r(\alpha, \theta_a)$ for some $r \in \mathbb{N}$, $\alpha \geq 0$, and $0 < \theta_a < \theta$. Given $\lambda > 0$, our aim is to transform (2.33) in such a way that the operator A is shifted to $\lambda + A$.

To this end, we choose an $\omega \geq 0$ such that a_ω defined by $a_\omega(t) = a(t)e^{-\omega t}$, $t \geq 0$, is an $L_1(\mathbb{R}_+)$-function and $\nu := \lambda |a_\omega|_{L_1(\mathbb{R}_+)} < 1$, as well as $\theta_\nu := \theta_a + \arcsin(\nu) < \theta$. According to Lemma 2.6.2, we have $a_\omega \in \mathcal{K}^r(\alpha, \theta_a)$, and there is a unique kernel $b \in \mathcal{K}^r(\alpha, \theta_\nu) \cap L_1(\mathbb{R}_+)$ such that $b - \lambda b * a_\omega = a_\omega$. Further, the kernel λa_ω fulfills the assumptions of Corollary 2.8.3, hence the operator $\mathcal{T} := \mathcal{T}_{\lambda a_\omega} := (I - \lambda a_\omega *)$ is well-defined and is an isomorphism of $L_p(J; X)$, of ${}_{(0)}H^\alpha_p(J; X)$, and also of $L_p(J; \mathcal{D}(A))$. Observe that $a_\omega * g = b * \mathcal{T} g$ for all $g \in L_p(J; X)$. Multiply now (2.33) by $e^{-\omega t}$, put $u_\omega(t) = u(t)e^{-\omega t}$ as well as $f_\omega(t) = f(t)e^{-\omega t}$ and add a zero-term to obtain

$$u_\omega - \lambda a_\omega * u_\omega + a_\omega * (\lambda + A)u_\omega = f_\omega, \quad t \in J. \tag{2.34}$$

If we set $v = \mathcal{T} u_\omega$, then $b * v = a_\omega * u_\omega$ and so (2.34) transforms to

$$v + b * (\lambda + A)v = f_\omega, \quad t \in J. \tag{2.35}$$

This equation is equivalent to (2.33). The kernel b enjoys the same properties as a, and instead of A we have now the shifted operator $\lambda + A$.

Chapter 3

Maximal Regularity for Abstract Equations

3.1 Abstract parabolic Volterra equations

In this section we study the abstract Volterra equation

$$u + a * Au = f, \qquad t \geq 0, \tag{3.1}$$

on a Banach space X. Here A is an $\mathcal{R}$-sectorial operator in X, the kernel a belongs to the class $\mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0,2)$, and we assume the parabolicity condition $\theta_a + \phi_A^R < \pi$. Our aim is to find conditions on the given function f which are necessary and sufficient for the existence of a unique solution u of (3.1) in the space

$$H_p^{\alpha+\kappa}(J;X) \cap H_p^{\kappa}(J;D_A),$$

where J is $\mathbb{R}_+$ or a compact time-interval $[0,T]$, D_A denotes the domain of A equipped with the graph norm of A, and κ is a real parameter belonging to the interval $[0,1/p)$.

We begin with the special case of vanishing traces at $t=0$.

Theorem 3.1.1 *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, and A an $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle ϕ_A^R. Further let J be $\mathbb{R}_+$ or a compact time-interval $[0,T]$. Suppose that a belongs to $\mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (0,2)$ and that in addition $a \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$. Further let $\kappa \in [0,1/p)$, $\alpha+\kappa \notin \{1/p, 1+1/p\}$, and suppose the* parabolicity condition $\theta_a + \phi_A^R < \pi$.

Then (3.1) has a unique solution in $Z := {}_0H_p^{\alpha+\kappa}(J;X) \cap H_p^{\kappa}(J;D_A)$ if and only if $f \in {}_0H_p^{\alpha+\kappa}(J;X)$.

Proof. Suppose that $u \in Z$ is a solution of (3.1). This clearly implies $Au \in H_p^{\kappa}(J;X) = {}_0H_p^{\kappa}(J;X)$. From Corollary 2.8.1 we then deduce that $a * Au \in {}_0H_p^{\alpha+\kappa}(J;X)$. This, together with $u \in {}_0H_p^{\alpha+\kappa}(J;X)$, entails $f \in {}_0H_p^{\alpha+\kappa}(J;X)$. Hence, the necessity part is established.

To prove the converse, we first consider the case $\kappa = 0$. Suppose $f \in {}_0H_p^{\alpha}(J;X)$ is given. From Section 2.7 we know that equation (3.1) is parabolic and admits a resolvent $S(\cdot)$, with the aid of which the mild solution u of (3.1) can be represented by the variation of parameters formula (2.27). According to Corollary 2.8.1 it makes sense to define $\mathcal{B} \in \mathcal{S}(L_p(J;X))$ as the inverse convolution operator associated with the kernel a. The operator $\mathcal{B}$ is invertible and we have $\mathcal{B}^{-1}w = a * w$ for all $w \in L_p(J;X)$. Furthermore, $\mathcal{D}(\mathcal{B}) = {}_0H_p^{\alpha}(J;X)$ so that we may set $g = \mathcal{B}f$. Then $g \in L_p(J;X)$ and

$$u(t) = \frac{d}{dt}(S * (a * g))(t), \quad t \in J.$$

Let $E_J : L_p(J;X) \to L_p(\mathbb{R};X)$ denote the operator of extension by 0, i.e.

$$(E_J h)(t) = h(t), \quad t \in J, \quad (E_J)(t) = 0, \quad t \notin J,$$

let $P_J : L_p(\mathbb{R};X) \to L_p(J;X)$ be the restriction to J, and define the operator-valued kernel K by means of $K(t) = (S * a)(t)\chi_{[0,\,\infty)}(t)$, $t \in \mathbb{R}$. Then the solution u can be written in terms of a convolution operator on $L_p(\mathbb{R};X)$:

$$u = P_J \frac{d}{dt}(K * E_J g). \tag{3.2}$$

In order to show that $Au \in L_p(J;X)$, we study the symbol of the operator $(A\frac{d}{dt}K*)$, which reads

$$M(\rho) = A\left(\frac{1}{\hat{a}(i\rho)} + A\right)^{-1}, \quad \rho \in \mathbb{R},\ \rho \neq 0. \tag{3.3}$$

By Lemma 2.6.1, for each $\rho \neq 0$, $\hat{a}(i\rho) := \lim_{\lambda \to i\rho} \hat{a}(\lambda)$ exists and does not vanish. Besides, $\hat{a}(i\cdot) \in W^1_{\infty,\,loc}(\mathbb{R} \setminus \{0\})$, and the sectoriality of a implies $|\arg(\hat{a}(i\rho))| \leq \theta_a$ for all $\rho \neq 0$. The idea is to apply the Mikhlin multiplier theorem in the operator-valued version, Theorem 2.5.1, to the symbol M. But it is not clear that $M \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(L_p(J;X)))$, so we introduce the sequence of symbols

$$M_n(\rho) := A\left(\frac{1}{\hat{a}((i+\frac{1}{n})\rho)} + A\right)^{-1}, \quad \rho \in \mathbb{R},\ n \in \mathbb{N}.$$

Since A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi_A^R < \pi - \theta_a$, we deduce that $\mathcal{R}(\{M_n(\rho) : \rho \in \mathbb{R} \setminus \{0\}\}) \leq \kappa < \infty$ for all $n \in \mathbb{N}$ with κ not depending on n. From

$$\rho M_n'(\rho) = \frac{(i+\frac{1}{n})\rho \hat{a}'((i+\frac{1}{n})\rho)}{\hat{a}((i+\frac{1}{n})\rho)^2}\left(\frac{1}{\hat{a}((i+\frac{1}{n})\rho)} + A\right)^{-1} M_n(\rho), \quad \rho \in \mathbb{R},$$

using 1-regularity of a, $\mathcal{R}$-sectoriality of A, and Kahane's contraction principle (see [29, Lemma 3.5]), we obtain

$$\mathcal{R}(\{\rho M_n'(\rho) : \rho \in \mathbb{R} \setminus \{0\}\}) \leq C\kappa(1+\kappa),$$

for all n with C not depending on n. By Theorem 2.5.1, it follows that the operators T_n defined by

$$T_n \phi = \mathcal{F}^{-1}(M_n \mathcal{F}\phi), \quad \text{for all } \mathcal{F}\phi \in \mathcal{D}(\mathbb{R};X),$$

are uniformly L_p-bounded, i.e.

$$|\,T_n|_{\mathcal{B}(L_p(\mathbb{R};X))} \leq \kappa_0, \quad n \in \mathbb{N}.$$

Furthermore we have, for all $\rho \neq 0$, $\lim_{n\to\infty} M_n(\rho) = M(\rho)$ and $|M_n(\rho) - M(\rho)| \leq 2\kappa$. Thus, by Lebesgue's dominated convergence theorem, we conclude $M_n \to M$ in

$L_{1,\,loc}(\mathbb{R};\mathcal{B}(X))$ as $n \to \infty$. It follows now by an approximation result, cp. Clément and Prüss [24, p. 6] or [29, Proposition 3.18], that $(A\frac{d}{dt}K*)$, which corresponds to the symbol M, is a bounded linear operator on $L_p(\mathbb{R};X)$ with $|A\frac{d}{dt}K*|_{\mathcal{B}(L_p(\mathbb{R};X))} \le \kappa_0$. Since E_J and P_J are bounded as well we see that $Au \in L_p(J;X)$. As in the necessity part, this implies $a*Au \in {}_0H^\alpha_p(J;X)$, i.e. $u = f - a*Au \in {}_0H^\alpha_p(J;X)$. Hence $u \in Z$.

We now consider the case $\kappa \in (0, 1/p)$. Suppose that $f \in {}_0H^{\alpha+\kappa}_p(J;X)$. Putting $b(t) = e^{-t}t^{\kappa-1}$, $t > 0$, yields $b \in \mathcal{K}^1(\kappa, \kappa\pi/2) \cap L_1(\mathbb{R}_+)$. Let $\mathcal{B}_\kappa$ be the operator constructed in Corollary 2.8.1 associated with the kernel b. Then $\mathcal{B}_\kappa f \in {}_0H^\alpha_p(J;X)$. Sufficiency being already established for $\kappa = 0$, we may define $v \in {}_0H^\alpha_p(J;X) \cap L_p(J;D_A)$ as the solution of

$$v + a*Av = \mathcal{B}_\kappa f, \qquad t \ge 0.$$

Since $\mathcal{B}_\kappa$ commutes with both A and the Volterra operator corresponding to the kernel a, we see that $u := b*v$ solves (3.1) and lies in Z. $\square$

Remarks 3.1.1 (i) Although not explicitly stated in Theorem 3.1.1, we have an estimate of the form

$$C^{-1}|f|_{{}_0H^{\alpha+\kappa}_p(J;X)} \le |u|_Z \le C|f|_{{}_0H^{\alpha+\kappa}_p(J;X)}, \quad f \in {}_0H^{\alpha+\kappa}_p(J;X),$$

where C is a positive constant not depending on f. This follows immediately from the above proof. Note that the subsequent theorems on linear problems have to be understood in the same sense: whenever necessary resp. sufficient conditions are stated in terms of regularity classes, this, by convention, means that the corresponding *a priori* estimates hold true.

(ii) Observe that the statement of Theorem 3.1.1 remains true if $\kappa \ge 1/p$ and Z is defined by ${}_0H^{\alpha+\kappa}_p(J;X) \cap {}_0H^\kappa_p(J;D_A)$.

(iii) In the case of a compact interval J, one can weaken the assumption on A. In view of the transformation property of (3.1) discussed at the end of Section 2.8, it suffices to know that $\mu + A \in \mathcal{RS}(X)$ with $\theta_a + \phi^R_{\mu+A} < \pi$ for some $\mu \ge 0$.

(iv) If $\kappa = 0$, equation (3.1) is equivalent to $\mathcal{B}u + \mathcal{A}u = \mathcal{B}f$ in the space $Y := L_p(J;X)$, where $\mathcal{A}$ stands for the natural extension of A to Y. If one additionally assumes that $A \in \mathcal{BIP}(X)$ and $\theta_A + \theta_a < \pi$, the assertion of Theorem 3.1.1 can also be proved by means of the Dore-Venni theorem, Theorem 2.3.1. This approach has been used in Prüss [63, Thm. 8.7].

(v) In the case $J = \mathbb{R}_+$ there is a variant of Theorem 3.1.1 which does not need the assumption $a \in L_1(\mathbb{R}_+)$. Instead one assumes that A is invertible and that f in (3.1) is of the form $f = a*g$. In this situation, existence of a unique solution of (3.1) in Z is equivalent to the condition $g \in {}_0H^\kappa_p(\mathbb{R}_+;X)$. In fact, if $g \in L_p(\mathbb{R}_+;X)$, then, as seen in the above proof, we have $Au \in L_p(\mathbb{R}_+;X)$, which by invertibility of A entails $u \in L_p(\mathbb{R}_+;X)$. From $\mathcal{B}u + \mathcal{A}u = g$, we then deduce that $\mathcal{B}u \in L_p(\mathbb{R}_+;X)$. So $u \in {}_0H^\alpha_p(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+;D_A)$. The converse direction is trivial, and the case $\kappa > 0$ can be reduced to the case $\kappa = 0$ as above.

(vi) The idea to use Theorem 2.5.1 to show that the linear operator corresponding to the symbol (3.3) is bounded in $L_p(\mathbb{R};X)$ goes back to Clément and Prüss [24]. However, they do not give a detailed proof including the approximation argument, by the aid of which one can surmount the technical difficulty consisting in the fact that Theorem 2.5.1 cannot be applied directly to (3.3).

We now turn our attention to situations where the function f or its derivative $\dot{f}$, if it exists, has a non-vanishing trace at $t = 0$. If $f \in H^{\alpha+\kappa}_p(J;X)$ and $\alpha + \kappa > 1/p$, then

$x := f(0) \in X$ exists and we are led to ask under what conditions on $x \in X$ the solution of the problem

$$u(t) + (a * Au)(t) = x, \quad t \geq 0, \tag{3.4}$$

lies in the space $Z = H_p^{\alpha+\kappa}(J;X) \cap H_p^{\kappa}(J;D_A)$. In case $\alpha + \kappa > 1 + 1/p$ we even have to take into account the trace $y := \dot{f}(0)$. Thus we also have to examine the problem

$$u(t) + (a * Au)(t) = ty, \quad t \geq 0, \tag{3.5}$$

with given $y \in X$.

Observe that the solution u of (3.4) is given by $u(t) = S(t)x$, $t \geq 0$, while that of (3.5) equals $(1 * S)(\cdot)y$. This follows immediately from the variation of parameters formula (2.27).

The next theorem gives conditions on the traces which ensure that the solutions of (3.4) and (3.5), respectively, are contained in Z. Take notice of the fact that here the operator A is only assumed to be sectorial.

Theorem 3.1.2 *Let X be a Banach space of class $\mathcal{HT}$, $J = [0,T]$ a compact time-interval, $p \in (1,\infty)$, and A a sectorial operator in X with spectral angle ϕ_A. Suppose that $\kappa \in [0, 1/p)$, $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (1/p - \kappa, 2)$, $\alpha + \kappa \neq 1 + 1/p$. Further suppose that $\theta_a + \phi_A < \pi$. Then*

$$x \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{p\alpha}, p) \Rightarrow S(\cdot)x \in Z = H_p^{\alpha+\kappa}(J;X) \cap H_p^{\kappa}(J;D_A), \tag{3.6}$$

and if $\alpha + \kappa > 1 + 1/p$,

$$y \in D_A(1 + \tfrac{\kappa}{\alpha} - \tfrac{1}{\alpha} - \tfrac{1}{p\alpha}, p) \Rightarrow (1 * S)(\cdot)y \in Z. \tag{3.7}$$

Proof. We first show (3.6). In case $0 < \kappa < 1/p$ we let $\mathcal{B}_\kappa \in \mathcal{S}(L_p(J;X))$ be the inverse convolution operator associated with the kernel $b(t) = t^{\kappa-1}/\Gamma(\kappa)$, $t > 0$. If $\kappa = 0$, we set $\mathcal{B}_\kappa = I$. Suppose $x \in \mathcal{D}(A)$. Letting $u(\cdot) = S(\cdot)x$ our goal is to show that $|\mathcal{B}_\kappa Au|_{L_p(J;X)}$ is bounded above by the $D_A(1 + \kappa/\alpha - 1/p\alpha, p)$-norm of x. Since $\mathcal{D}(A)$ is densely embedded into $D_A(1 + \kappa/\alpha - 1/p\alpha, p)$, the assertion then follows by an approximation argument.

To establish the desired estimate we use the representation of the resolvent S via Laplace transform. Recall that $\hat{S}$ is given by

$$\hat{S}(\lambda) = \frac{1}{\lambda}\left(1 + \hat{a}(\lambda)A\right)^{-1}, \quad \mathrm{Re}\lambda > 0.$$

Let Γ denote a contour $\gamma + i(-\infty, \infty)$ with some $\gamma > 0$. We have

$$\begin{aligned} \widehat{t\mathcal{B}_\kappa S}(\lambda) &= -\frac{d}{d\lambda}\widehat{\mathcal{B}_\kappa S}(\lambda) \\ &= -\frac{d}{d\lambda}\left(\lambda^\kappa \cdot \frac{1}{\lambda}\left(1 + \hat{a}(\lambda)A\right)^{-1}\right) \\ &= \frac{1-\kappa}{\lambda^{2-\kappa}}\left(1 + \hat{a}(\lambda)A\right)^{-1} + \frac{1}{\lambda^{1-\kappa}}\hat{a}'(\lambda)A\left(1 + \hat{a}(\lambda)A\right)^{-2}. \end{aligned}$$

With

$$\phi(\lambda) = 1 - \kappa + \frac{\lambda\hat{a}'(\lambda)}{\hat{a}(\lambda)} \cdot \hat{a}(\lambda)A\left(1 + \hat{a}(\lambda)A\right)^{-1}, \quad \mathrm{Re}\lambda > 0, \tag{3.8}$$

and

$$G_\kappa(\lambda) = \lambda^\kappa A(1+\hat{a}(\lambda)A)^{-1}, \quad \mathrm{Re}\lambda > 0, \tag{3.9}$$

we then obtain

$$(t\mathcal{B}_\kappa AS)\hat{\ }(\lambda) = \frac{\phi(\lambda)}{\lambda^2}\, G_\kappa(\lambda), \quad \mathrm{Re}\lambda > 0.$$

Inversion of the Laplace transform now yields for $t > 0$

$$\begin{aligned}
t\mathcal{B}_\kappa AS(t)x &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(t\mathcal{B}_\kappa AS)\hat{\ }(\lambda)x\, d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\gamma+i\rho)t}\,\phi(\gamma+i\rho)\, G_\kappa(\gamma+i\rho)x\, \frac{d\rho}{(\gamma+i\rho)^2} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\gamma t+i\sigma}\,\phi(\gamma+i\frac{\sigma}{t})\, G_\kappa(\gamma+i\frac{\sigma}{t})x\, \frac{t\, d\sigma}{(\gamma t+i\sigma)^2},
\end{aligned}$$

where we used the change of variables $\sigma = t\rho$. By 1-regularity of a and parabolicity of (3.4) we get a bound $|\phi(\lambda)| \leq C$, for all $\mathrm{Re}\,\lambda > 0$. Using this estimate and choosing $\gamma = \frac{1}{t}$ we obtain

$$|\mathcal{B}_\kappa AS(t)x|_X \leq C\int_{-\infty}^{\infty} |G_\kappa((1+i\rho)/t)x|_X \frac{d\sigma}{1+\sigma^2}, \;\; t > 0.$$

Taking the L_p-norm on the interval $J = [0,T]$ and applying the continuous version of Minkowski's inequality yields

$$|\mathcal{B}_\kappa AS(\cdot)x|_{L_p(J;X)} \leq C\int_{-\infty}^{\infty}(\int_0^T |G_\kappa((1+i\rho)/t)x|_X^p\, dt)^{1/p}\frac{d\sigma}{1+\sigma^2}.$$

Now we employ the change of variables $s = \sqrt{1+\sigma^2}/t$ for the inner integral and enlarge its interval of integration to get

$$\begin{aligned}
|\mathcal{B}_\kappa AS(\cdot)x|_{L_p(J;X)} &\leq C\int_{-\infty}^{\infty}(\int_{\frac{1}{T}}^{\infty}(s^{-\frac{2}{p}}|G_\kappa(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})x|_X)^p\, ds)^{1/p}\frac{d\sigma}{(1+\sigma^2)^{1-\frac{1}{2p}}} \\
&\leq C\sup_{\sigma\in\mathbb{R}}\{(\int_{\frac{1}{T}}^{\infty}(s^{-\frac{1}{p}}|G_\kappa(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})x|_X)^p\,\frac{ds}{s})^{1/p}\}.
\end{aligned}$$

This shows that $\mathcal{B}_\kappa AS(\cdot)x \in L_p(J;X)$ whenever

$$\eta := \sup_{\sigma\in\mathbb{R}}\{(\int_{\frac{1}{T}}^{\infty}(s^{\kappa-\frac{1}{p}}|\hat{a}(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})|^{-1}|A(1/\hat{a}(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p}\} < \infty. \tag{3.10}$$

Now we have by the resolvent equation

$$\begin{aligned}
A\left(\frac{1}{\hat{a}(\lambda)}+A\right)^{-1} &= A(|\lambda|^\alpha+A)^{-1} + \\
&\quad + \left(|\lambda|^\alpha - \frac{1}{\hat{a}(\lambda)}\right)\left(\frac{1}{\hat{a}(\lambda)}+A\right)^{-1} A(|\lambda|^\alpha+A)^{-1},
\end{aligned}$$

for $\mathrm{Re}\,\lambda > 0$, thus using the parabolicity of (3.4)

$$\begin{aligned}
|A(1/\hat{a}(\lambda)+A)^{-1}x| &\leq |A(|\lambda|^\alpha+A)^{-1}x| + \\
&\quad +|\,|\lambda|^\alpha\hat{a}(\lambda)-1|\,|(1+\hat{a}(\lambda)A)^{-1}|\,|A(|\lambda|^\alpha+A)^{-1}x| \\
&\leq (C_1 + C_2|\hat{a}(\lambda)|\,|\lambda|^\alpha)\,|A(|\lambda|^\alpha+A)^{-1}x|,
\end{aligned}$$

with two positive constants C_1, C_2 not depending on λ. Therefore, we can estimate η as follows.

$$\begin{aligned}\eta &\leq C_1 \sup_{\sigma\in\mathbb{R}}\{(\int_{\frac{1}{T}}^{\infty}(s^{\kappa-\frac{1}{p}}|\hat{a}(\tfrac{(1+i\sigma)s}{\sqrt{1+\sigma^2}})|^{-1}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p}\}+\\ &\quad +C_2(\int_{\frac{1}{T}}^{\infty}(s^{\alpha+\kappa-\frac{1}{p}}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p}\\ &=: \eta_1+\eta_2.\end{aligned}$$

As for η_2, by employing the change of variables $r = s^\alpha$ we get

$$\eta_2 \leq C_2(\int_{(\frac{1}{T})^\alpha}^{\infty}(r^{1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha}}|A(r+A)^{-1}x|_X)^p\,\frac{dr}{r\alpha})^{1/p} \leq \tilde{C}_2|x|_{D_A(1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha},p)}.$$

Concerning η_1, there exists $C_3 > 0$ independent of λ such that $|\hat{a}(|\lambda|)\,| \leq C_3|\hat{a}(\lambda)|$, for all $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$, see Lemma 2.6.1. Thus,

$$\eta_1 \leq C_1C_3(\int_{\frac{1}{T}}^{\infty}(s^{\kappa-\frac{1}{p}}|\hat{a}(s)|^{-1}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p}.$$

Exploiting the assumption $\liminf_{\mu\to\infty}|\,\hat{a}(\mu)|\,\mu^\alpha > 0$ for a bound $|\hat{a}(s)|\,s^\alpha \geq C_4 > 0$, $1/T \leq s < \infty$, we deduce that

$$\eta_1 \leq \frac{C_1C_3}{C_4}(\int_{\frac{1}{T}}^{\infty}(s^{\alpha+\kappa-\frac{1}{p}}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s})^{1/p},$$

i.e. we have the same expression as above for η_2, hence the desired estimate follows. So if $x \in D_A(1+\kappa/\alpha-1/p\alpha, p)$, then $\mathcal{B}_\kappa Au \in L_p(J;X)$, which is equivalent to $Au \in H_p^\kappa(J;X) = {}_0H_p^\kappa(J;X)$, by Corollary 2.8.1. Applying this once more it follows then that $a * Au \in {}_0H_p^{\alpha+\kappa}(J;X)$, thus $u = x - a * Au \in H_p^{\alpha+\kappa}(J;X)$. Hence $u \in Z$.

We now prove (3.7). Suppose $y \in \mathcal{D}(A)$, and put $u(\cdot) = (1 * S)(\cdot)y$. To show that the $L_p(J;X)$-norm of $\mathcal{B}_\kappa Au$ can be estimated above by the $D_A(1+\kappa/\alpha-1/\alpha-1/p\alpha, p)$-norm of y, we use once more the representation of u via Laplace transform. With

$$\begin{aligned}(t\mathcal{B}_\kappa(1*S))\hat{\ }(\lambda) &= -\frac{d}{d\lambda}(\mathcal{B}_\kappa(1*S))\hat{\ }(\lambda) \;=\; -\frac{d}{d\lambda}\left(\frac{\lambda^\kappa}{\lambda^2}\,(1+\hat{a}(\lambda)A)^{-1}\right)\\ &= \frac{2-\kappa}{\lambda^{3-\kappa}}\,(1+\hat{a}(\lambda)A)^{-1} + \frac{1}{\lambda^{2-\kappa}}\,\hat{a}'(\lambda)A\,(1+\hat{a}(\lambda)A)^{-2}\end{aligned}$$

and ϕ as well as G_κ from above (see (3.8),(3.9)), we have this time

$$(t\mathcal{B}_\kappa A(1*S))\hat{\ }(\lambda) = \frac{\phi(\lambda)+1}{\lambda^3}\,G_\kappa(\lambda), \quad \operatorname{Re}\lambda > 0.$$

By repeating all steps from the first part of the proof, we get

$$\begin{aligned}|\mathcal{B}_\kappa Au|_{L_p(J;X)} &\leq C\int_{\frac{1}{T}}^{\infty}(s^{\alpha+\kappa-1-\frac{1}{p}}|A(s^\alpha+A)^{-1}x|_X)^p\,\frac{ds}{s}\\ &= C\int_{(\frac{1}{T})^\alpha}^{\infty}(r^{1+\frac{\kappa}{\alpha}-\frac{1}{\alpha}-\frac{1}{p\alpha}}|A(r+A)^{-1}x|_X)^p\,\frac{dr}{r\alpha} \leq \tilde{C}|y|_{D_A(1+\kappa/\alpha-1/\alpha-1/p\alpha,p)}.\end{aligned}$$

So by approximation, we see that $y \in D_A(1+\kappa/\alpha-1/\alpha-1/p\alpha, p)$ implies $\mathcal{B}_\kappa Au \in L_p(J;X)$. As in the first part, we obtain $u \in Z$. □

Remarks 3.1.2 (i) It is not difficult to see that the second assertion of Theorem 3.1.2 remains true if we allow κ to lie in $[1/p, 1+1/p)$.

(ii) Let $\kappa = 0$, A be invertible, and assume that the kernel a admits in addition the estimate $C \le s^{-\alpha}|\hat{a}(s)|$, $s > 0$, for some constant $C > 0$. Then by taking $J = \mathbb{R}_+$ in the above lines and by employing this additional estimate, one obtains the implications

$$\begin{aligned} x \in D_A(1-\tfrac{1}{p\alpha}, p) &\Rightarrow S(\cdot)x \in L_p(\mathbb{R}_+; D_A), \text{ if } \alpha > \tfrac{1}{p}; \\ y \in D_A(1-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha}, p) &\Rightarrow (1*S)(\cdot)y \in L_p(\mathbb{R}_+; D_A), \text{ if } \alpha > 1+\tfrac{1}{p}, \end{aligned}$$

see also Prüss [64, Theorem 6].

(iii) The proof of Theorem 3.1.2 is inspired by the estimates derived in the proof of Theorem 6 in Prüss [64].

We next want to show that these conditions on the traces are also necessary. In the following theorem, again, the operator A is only sectorial. For technical reasons, we have to assume maximal regularity of the solutions of (3.4) and (3.5), respectively, on the whole halfline. Later on we shall extend this result to problems on compact time-intervals where the operator A is $\mathcal{R}$-sectorial.

Theorem 3.1.3 *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, and A a sectorial operator in X with spectral angle ϕ_A. Suppose that $\kappa \in [0, 1/p)$, $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (1/p-\kappa, 2)$, $\alpha+\kappa \neq 1+1/p$. Further let $\omega \ge 0$ and assume that $\theta_a + \phi_A < \pi$. Then*

$$e^{-\omega\cdot} AS(\cdot)x \in H^\kappa_p(\mathbb{R}_+; X) \Rightarrow x \in D_A(1+\tfrac{\kappa}{\alpha}-\tfrac{1}{p\alpha}, p), \tag{3.11}$$

and if $\alpha+\kappa > 1+1/p$,

$$e^{-\omega\cdot} A(1*S)(\cdot)y \in H^\kappa_p(\mathbb{R}_+; X) \Rightarrow y \in D_A(1+\tfrac{\kappa}{\alpha}-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha}, p). \tag{3.12}$$

Proof. The main idea of the proof is to use Proposition 1 in [64], which says that for every function g in $L_p(\mathbb{R}_+; X)$, the Laplace transform $\hat{g}(\lambda)$ exists for $\operatorname{Re}\lambda > 0$, and with $p^{-1}+q^{-1} = 1$ the estimate

$$\int_0^\infty |\hat{g}(\lambda)|^p \lambda^{p-2}\, d\lambda \le |g|^p_{L_p(\mathbb{R}_+;X)} \Gamma(1/q)^p \tag{3.13}$$

holds true.

We first show implication (3.11). Let $\mathcal{B}_\kappa$ be defined as in the proof of Theorem 3.1.2. Suppose that $g(t) := \mathcal{B}_\kappa(e^{-\omega t}AS(t)x)$, $t \ge 0$, is contained in $L_p(\mathbb{R}_+; X)$. Then g is Laplace transformable, according to the proposition mentioned above. From the resolvent equation for $S(\cdot)$, $S(t)x + (a*AS)(t)x = x$, $t \ge 0$, it follows by the convolution theorem that

$$\hat{g}(\lambda) = \frac{\lambda^\kappa}{\lambda+\omega} A(1+\hat{a}_\omega(\lambda)A)^{-1}x, \quad \operatorname{Re}\lambda > 0.$$

Here, $a_\omega(t) := a(t)e^{-\omega t}$, $t \ge 0$. Therefore, using (3.13) we obtain

$$\eta := \int_0^\infty \Big(\frac{\lambda^{1+\kappa-\frac{1}{p}}}{\lambda+\omega} |A(1+\hat{a}_\omega(\lambda)A)^{-1}x|_X\Big)^p \frac{d\lambda}{\lambda} \le C|g|^p_{L_p(\mathbb{R}_+;X)}. \tag{3.14}$$

From Lemma 2.6.2 we know that $a_\omega \in \mathcal{K}^1(\alpha, \theta_a)$. So, in similar manner as in the proof of Theorem 3.1.2, we can derive the following estimate from the resolvent equation for A and parabolicity of (3.4):

$$|A(|\lambda|^\alpha + A)^{-1}x| \le (C_1 + C_2(|\hat{a}_\omega(\lambda)|\,|\lambda|^\alpha)^{-1})\,|A(1/\hat{a}_\omega(\lambda)+A)^{-1}x|,\ \operatorname{Re}\lambda > 0,$$

with two positive constants C_1 and C_2. Employing this estimate as well as the inequality $\lambda/(\lambda+\omega) \geq C_0 > 0$, $\lambda \geq 1$, we deduce that

$$\begin{aligned}\eta &\geq C_0^p \int_1^\infty (\frac{\lambda^{\kappa-\frac{1}{p}}}{|\hat{a}_\omega(\lambda)|}|A(1/\hat{a}_\omega(\lambda)+A)^{-1}x|)^p \frac{d\lambda}{\lambda} \\ &\geq C_0^p \int_1^\infty (\frac{\lambda^{\alpha+\kappa-\frac{1}{p}}}{C_1|\hat{a}_\omega(\lambda)|\lambda^\alpha + C_2}|A(\lambda^\alpha+A)^{-1}x|)^p \frac{d\lambda}{\lambda}.\end{aligned}$$

We now exploit the assumption $\limsup_{\mu\to\infty} |\hat{a}(\mu)|\,\mu^\alpha < \infty$ to get an upper bound $|\hat{a}_\omega(\lambda)|\,\lambda^\alpha \leq C_3 < \infty$, $1 \leq \lambda < \infty$, and thus arrive at

$$\begin{aligned}C|g|^p_{L_p(\mathbb{R}_+;X)} &\geq \int_1^\infty (\lambda^{\alpha+\kappa-\frac{1}{p}}|A(\lambda^\alpha+A)^{-1}x|)^p \frac{d\lambda}{\lambda} \\ &= \int_1^\infty (r^{1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha}}|A(r+A)^{-1}x|_X)^p \,\frac{dr}{r\alpha},\end{aligned}$$

i.e. $x \in D_A(1+\kappa/\alpha - 1/p\alpha, p)$.

The proof of (3.12) is similar. Suppose $g(t) := \mathcal{B}_\kappa(e^{-\omega t}A(1*S)(t)x)$, $t \geq 0$, lies in $L_p(\mathbb{R}_+;X)$. Then g is Laplace transformable, on account of the proposition mentioned at the beginning of the proof. Integrating the resolvent equation for $S(\cdot)$ yields with $S_1 := 1 * S$ the relation $S_1(t)x + (a * AS_1)(t)x = tx$, $t \geq 0$. Thus, by the convolution theorem, we obtain that

$$\hat{g}(\lambda) = \frac{\lambda^\kappa}{(\lambda+\omega)^2}A(1+\hat{a}_\omega(\lambda)A)^{-1}x, \quad \operatorname{Re}\lambda > 0.$$

Using (3.13) then yields

$$\int_1^\infty (\frac{\lambda^{1+\kappa-\frac{1}{p}}}{(\lambda+\omega)^2}|A(1+\hat{a}_\omega(\lambda)A)^{-1}x|_X)^p \frac{d\lambda}{\lambda} \leq C|g|^p_{L_p(\mathbb{R}_+;X)},$$

in consequence of which we arrive at

$$\int_1^\infty (\lambda^{\kappa-1-\frac{1}{p}}|A(1+\hat{a}_\omega(\lambda)A)^{-1}x|_X)^p \frac{d\lambda}{\lambda} \leq \tilde{C}|g|^p_{L_p(\mathbb{R}_+;X)},$$

thanks to the inequality $\lambda/(\lambda+\omega) \geq C_0 > 0$, $\lambda \geq 1$. By the same line of conclusions as in the first part of the proof we then obtain

$$\int_1^\infty (r^{1+\frac{\kappa}{\alpha}-\frac{1}{\alpha}-\frac{1}{p\alpha}}|A(r+A)^{-1}x|_X)^p \,\frac{dr}{r\alpha} \leq \tilde{C}|g|^p_{L_p(\mathbb{R}_+;X)}.$$

Hence $x \in D_A(1+\kappa/\alpha - 1/\alpha - 1/p\alpha, p)$. □

We have now got all important ingredients of the main theorem concerning (3.1) which reads as follows.

Theorem 3.1.4 *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, J a compact time-interval $[0,T]$ or $\mathbb{R}_+$, and A an $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle ϕ_A^R. Suppose that a belongs to $\mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (0,2)$ and that in addition $a \in L_1(\mathbb{R}_+)$ in case $J = \mathbb{R}_+$. Further let $\kappa \in [0,1/p)$ and $\alpha+\kappa \notin \{1/p, 1+1/p\}$. Assume the* parabolicity condition

$\theta_a + \phi_A^R < \pi$. *Then (3.1) has a unique solution in* $Z := H_p^{\alpha+\kappa}(J;X) \cap H_p^{\kappa}(J;D_A)$ *if and only if the function* f *satisfies the subsequent conditions.*

$$\begin{array}{ll} (i) & f \in H_p^{\alpha+\kappa}(J;X); \\ (ii) & f(0) \in D_A(1+\frac{\kappa}{\alpha}-\frac{1}{p\alpha},p),\ \textit{if}\ \alpha+\kappa > 1/p; \\ (iii) & \dot{f}(0) \in D_A(1+\frac{\kappa}{\alpha}-\frac{1}{\alpha}-\frac{1}{p\alpha},p),\ \textit{if}\ \alpha+\kappa > 1+1/p. \end{array}$$

Proof. We consider three cases with respect to the sum $\alpha + \kappa$.

Case 1: $\alpha + \kappa < 1/p$. Because here $H_p^{\alpha+\kappa}(J;X) = {}_0H_p^{\alpha+\kappa}(J;X)$, we are in the situation of Theorem 3.1.1.

Case 2: $1/p < \alpha + \kappa < 1 + 1/p$. We begin with the necessity part. Suppose that $u \in Z$ is a solution of (3.1). Then $Au \in H_p^{\kappa}(J;X) = {}_0H_p^{\kappa}(J;X)$, which entails $a * Au \in {}_0H_p^{\alpha+\kappa}(J;X)$, thanks to Corollary 2.8.1. Thus $f = u + a * Au \in H_p^{\alpha+\kappa}(J;X)$, i.e. condition (i) is proved.

To show the second condition we extend u in the case $J = [0,T]$ to a function v on $\mathbb{R}_+$ such that

$$v \in Z_{\mathbb{R}_+} := H_p^{\alpha+\kappa}(\mathbb{R}_+;X) \cap H_p^{\kappa}(\mathbb{R}_+;D_A).$$

If $J = \mathbb{R}_+$, we simply set $v := u$. From $a \in \mathcal{K}^1(\alpha,\theta_A)$, it follows by Lemma 2.6.2 that the kernel a_1 defined by $a_1(t) = a(t)e^{-t}$, $t \geq 0$, belongs to $\mathcal{K}^1(\alpha,\theta_A)$, too. Further, $a_1 \in L_1(\mathbb{R}_+)$. Thus we deduce that $a_1 * Av \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$. Therefore, $g := v + a_1 * Av \in H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$ as well as $g - g(0)e^{-\cdot} \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$. Define now v_1 by means of the equation $v_1 + a_1 * Av_1 = g - g(0)e^{-t}$, $t \in \mathbb{R}_+$. This makes sense owing to Theorem 3.1.1, which also yields $v_1 \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X) \cap H_p^{\kappa}(\mathbb{R}_+;D_A)$. Then $v_2 := v - v_1 \in Z_{\mathbb{R}_+}$ is the solution of the equation $w + a_1 * Aw = g(0)e^{-t}$, $t \in \mathbb{R}_+$. Denoting the resolvent of (3.1) by $S(\cdot)$ it is not difficult to see that $v_2(t) = e^{-t}S(t)g(0)$, $t \geq 0$. Consequently, by Theorem 3.1.3, we get $g(0) = v(0) = u(0) = f(0) \in D_A(1 + \kappa/\alpha - 1/p\alpha, p)$. This establishes condition (ii).

To prove the converse, suppose that the conditions (i) and (ii) are satisfied. Uniqueness is a direct consequence of Theorem 3.1.1. Concerning existence, we construct a solution of (3.1) in the following way. If $J = [0,T]$, we define $u_1, u_2 \in Z$ as the solutions of the problems $w_1 + a * Aw_1 = f - f(0)$, $t \in J$, and $w_2 + a * Aw_2 = f(0)$, $t \in J$, respectively. These solutions exist and lie in Z, by virtue of Theorem 3.1.1 and Theorem 3.1.2. Thus $u := u_1 + u_2$ has the desired regularity and solves (3.1). In case $J = \mathbb{R}_+$ let v_1 be the solution of $w_1 + a * Aw_1 = f(0)$, $t \in [0,1]$. Condition (ii) implies $v_1 \in H_p^{\alpha+\kappa}([0,1];X) \cap H_p^{\kappa}([0,1];D_A)$, by Theorem 3.1.2. We extend v_1 to a function $u_1 \in Z$. Then $g := u_1 + a * Au_1 - f \in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$, and the solution u_2 of the problem $w_2 + a * Aw_2 = g$, $t \in \mathbb{R}_+$, lies in Z, thanks to Theorem 3.1.1. Hence, $u := u_1 + u_2 \in Z$ is a solution of (3.1).

Case 3: $\alpha + \kappa > 1 + 1/p$. We first prove the necessity part. Suppose that $u \in Z$ is a solution of (3.1). Then condition (i) can be derived as in the second case. Our next objective is to show (iii). The idea behind the following argument is a reduction to a situation of Case 2.

For this purpose we extend u to a function $v \in H_p^{\alpha+\kappa}(\mathbb{R};X) \cap H_p^{\kappa}(\mathbb{R};D_A)$. Define $\mathcal{A}$ as the natural extension of A to $Y := L_p(\mathbb{R};X)$ and let $\mathcal{G} := (I - D_t^2)^{\alpha/2}$ with domain $\mathcal{D}(\mathcal{G}) = H_p^{\alpha}(\mathbb{R};X)$. Then the operators $\mathcal{A}, \mathcal{G}$ are sectorial in Y with spectral angles $\phi_{\mathcal{A}} \leq \phi_A^R$ and $\phi_{\mathcal{G}} = 0$. Thus, $\phi_{\mathcal{A}} + \phi_{\mathcal{G}} < \pi$. Furthermore the resolvents of $\mathcal{A}$ and $\mathcal{G}$ commute, and the pair $(\mathcal{G}, \mathcal{A})$ is coercively positive. This allows us to apply the mixed derivative theorem to this pair of operators acting on Y to obtain

$$|\mathcal{A}^{1-\frac{1-\kappa}{\alpha}}\mathcal{G}^{\frac{1-\kappa}{\alpha}}y|_Y \le C|\mathcal{A}y+\mathcal{G}y|_Y, \ \text{ for all } y \in \mathcal{D}(\mathcal{A})\cap\mathcal{D}(\mathcal{G}), \tag{3.15}$$

where $C>0$ is a constant not depending on y. By definition, $\mathcal{D}(\mathcal{G}^{\frac{1-\kappa}{\alpha}}) = H_p^{1-\kappa}(\mathbb{R};X)$. This together with (3.15) implies

$$H_p^{\alpha}(\mathbb{R};X)\cap L_p(\mathbb{R};D_A) \hookrightarrow H_p^{1-\kappa}(\mathbb{R};D_{A^{1-\frac{1-\kappa}{\alpha}}}).$$

Thus, by the boundedness of $D_t\mathcal{G}^{-\frac{1}{\alpha}}$ in $L_p(\mathbb{R};D_{A^{1-\frac{1-\kappa}{\alpha}}})$,

$$\dot{v} = D_t\mathcal{G}^{-\frac{1}{\alpha}}\mathcal{G}^{\frac{1-\kappa}{\alpha}}\mathcal{G}^{\frac{\kappa}{\alpha}}v \in H_p^{\alpha+\kappa-1}(\mathbb{R};X)\cap L_p(\mathbb{R};D_{A^{1-\frac{1-\kappa}{\alpha}}}).$$

To determine the regularity of $\dot{v}(0)$ we now use the necessity part of the second case. Let $b(t)=t^{\alpha+\kappa-2}$, $t\ge 0$. Then $b\in\mathcal{K}^1(\alpha+\kappa-1,(\alpha+\kappa-1)\pi/2)$. Since A is $\mathcal{R}$-sectorial, the operator $\mathcal{C}:=A^{1-(1-\kappa)/\alpha}$ is $\mathcal{R}$-sectorial as well, and

$$\phi_{\mathcal{C}}^R \le (1-\tfrac{1-\kappa}{\alpha})\phi_A^R \le (\tfrac{\alpha+\kappa-1}{\alpha})(\pi-\alpha\tfrac{\pi}{2}) = (\tfrac{\alpha+\kappa-1}{\alpha})\pi - (\tfrac{\alpha+\kappa-1}{2})\pi,$$

by Proposition 2.2.1 and the parabolicity condition, combined with Remark 2.6.1. So we see that $\theta_b+\phi_{\mathcal{C}}^R<\pi$. By the necessity part of Case 2, applied to the equation

$$\dot{v}(t) + (b*\mathcal{C}\dot{v})(t) = g(t), \quad t\in[0,1],$$

with some $g\in H_p^{\alpha+\kappa-1}([0,1];X)$, we get $\dot{f}(0)=\dot{v}(0)\in D_{\mathcal{C}}(1-1/p(\alpha+\kappa-1),p)$. It follows now, by Theorem 2.2.2, that $\dot{f}(0)$ lies in

$$\begin{aligned} D_{A^{1-\frac{1-\kappa}{\alpha}}}\left(1-\tfrac{1}{p(\alpha+\kappa-1)},p\right) &= D_A\left((1-\tfrac{1-\kappa}{\alpha})(1-\tfrac{1}{p(\alpha+\kappa-1)}),p\right) \\ &= D_A\left(1+\tfrac{\kappa}{\alpha}-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha},p\right). \end{aligned}$$

Hence condition (iii) is satisfied.

It remains to show (ii). Put $J_1:=[0,1]$ in case $J=\mathbb{R}_+$ and $J_1:=J$, otherwise. Define w_1 by means of

$$w_1(t)+(a*Aw_1)(t) = t\dot{f}(0), \quad t\in J_1.$$

Then, due to Theorem 3.1.2, it follows from condition (iii) that $u|_{J_1}-w_1\in H_p^{\alpha+\kappa}(J_1;X)\cap H_p^{\kappa}(J_1;D_A)$. We extend $u|_{J_1}-w_1$ to a function $v\in Z_{\mathbb{R}_+}$ and put $h(t):=v(t)+(a_1*Av)(t)$, $t\in\mathbb{R}_+$, where a_1 is defined as in Case 2. As above, we see that $h\in H_p^{\alpha+\kappa}(\mathbb{R}_+;X)$. By construction, we have $\dot{v}(0)=\dot{h}(0)=0$, i.e. $h-\psi(\cdot)h(0)\in {}_0H_p^{\alpha}(\mathbb{R}_+;X)$, where $\psi(t)=(1+t)e^{-t}$, $t\ge 0$. In fact, $\psi\in H_p^{\alpha+\kappa}(\mathbb{R}_+)$, $\psi(0)=1$, $\dot{\psi}(t)=-te^{-t}$, $t\ge0$, in particular $\dot{\psi}(0)=0$. Define now the function v_1 by means of $v_1+a_1*Av_1=h-\psi h(0)$, $t\in\mathbb{R}_+$. This is possible in view of Theorem 3.1.1, which also gives $v_1\in {}_0H_p^{\alpha+\kappa}(\mathbb{R}_+;X)\cap H_p^{\kappa}(\mathbb{R}_+;D_A)$. Consequently, $v_2:=v-v_1\in Z_{\mathbb{R}_+}$, and v_2 solves the equation $w+a_1*Aw=\psi h(0)$, $t\in\mathbb{R}_+$. If $S(\cdot)$ denotes the resolvent for (3.1), then one verifies that

$$v_2(t) = e^{-t}S(t)h(0) + e^{-t}(1*S)(t)h(0), \quad t\ge 0.$$

Letting $k(t) = e^{-t}$ and $\xi(t) = e^{-t}S(t)h(0)$, $t \geq 0$, we thus have

$$v_2(t) = \xi(t) + (k * \xi)(t), \quad t \geq 0. \tag{3.16}$$

Define now $r \in L_{1,\,loc}(\mathbb{R}_+)$ by means of

$$r(t) + (r * k)(t) = k(t), \qquad t \geq 0.$$

Then (3.16) implies

$$\xi(t) = v_2(t) - (r * v_2)(t), \qquad t \geq 0,$$

in particular

$$A\xi(t) = Av_2(t) - (r * Av_2)(t), \qquad t \geq 0.$$

Since $k \in L_1(\mathbb{R}_+)$ and

$$1 + \hat{k}(\lambda) = 1 + \tfrac{1}{\lambda+1} \neq 0, \quad \operatorname{Re}\lambda \geq 0,$$

it follows by the halfline Paley-Wiener theorem (see [39, Chapter 2, Theorem 4.1, p. 45]) that $r \in L_1(\mathbb{R}_+)$. So, letting $b(t) = t^{\kappa-1}e^{-t}$, $t \geq 0$, and $\mathcal{B}_\kappa = (b*)^{-1} \in \mathcal{S}(L_p(\mathbb{R}_+;X))$, we see with the aid of Young's inequality and Corollary 2.8.1 that

$$r * Av_2 = r * b * \mathcal{B}_\kappa Av_2 = b * r * \mathcal{B}_\kappa Av_2 \in H^\kappa_p(\mathbb{R}_+;X).$$

Therefore $A\xi \in H^\kappa_p(\mathbb{R}_+;X)$, and so, by Theorem 3.1.3, we get $h(0) = v(0) = u(0) = f(0) \in D_A(1 + \kappa/\alpha - 1/p\alpha, p)$. This proves condition (ii).

To prove the converse direction, suppose that the conditions (i),(ii) and (iii) are fulfilled. Uniqueness is an immediate consequence of Theorem 3.1.1. As for existence, we build a solution $u \in Z$ of (3.1) as follows. Put $x = f(0)$ and $y = \dot{f}(0)$. If $J = [0,T]$, let u_1 be the solution of

$$w(t) + (a * Aw)(t) = f(t) - x - ty, \quad t \in J.$$

This solution exists and lies in ${}_0H^{\alpha+\kappa}_p(J;X) \cap H^\kappa_p(J;D_A)$, thanks to Theorem 3.1.1. By Theorem 3.1.2, we also have $S(\cdot)x$, $(1 * S)(\cdot)y \in Z$, so that $u := u_1 + Sx + 1 * Sy \in Z$. It is easy to see that u solves (3.1). In case of $J = \mathbb{R}_+$, we extend v_1, $v_2 \in H^{\alpha+\kappa}_p([0,1];X) \cap L_p([0,1];D_A)$ defined by $v_1(t) = S(t)x$, $v_2(t) = (1 * S)(t)$, $t \in [0,1]$, to functions u_1, $u_2 \in Z$ and set $f_1 = f - (u_1 + a * Au_1) - (u_2 + a * Au_2)$. By construction, $f_1 \in {}_0H^{\alpha+\kappa}_p(\mathbb{R}_+;X)$. So, due to Theorem 3.1.1, we can define $u_3 \in Z$ as the solution of $v + a * Av = f_1$, $t \in \mathbb{R}_+$. Clearly, $u := u_1 + u_2 + u_3 \in Z$ solves (3.1). Hence the proof is complete. $\square$

Remarks 3.1.3 (i) In Section 3.3 we will prove a corresponding result for the case of a compact time-interval J where $\kappa \in (1/p, 1 + 1/p)$.

(ii) In the case of a compact interval J, one can weaken the assumptions on both a and A. Due to the transformation property of (3.1) discussed at the end of Section 2.8, it suffices to assume that $\mu + A \in \mathcal{RS}(X)$ with $\theta_a + \phi^R_{\mu+A} < \pi$ for some $\mu \geq 0$. As for the kernel, the theorem is also true, if a is of the form $a = b + dk * b$, where b is like a in the statement of Theorem 3.1.4 and $k \in BV_{loc}(\mathbb{R}_+)$ with $k(0) = k(0+) = 0$. This follows from a straightforward perturbation argument, cp. [63, Section 8.5].

3.2 A general trace theorem

Let X be a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $J = [0,T]$ or $\mathbb{R}_+$, and A be an $\mathcal{R}$-sectorial operator in X with *arbitrary* $\mathcal{R}$-angle $\phi_A^R < \pi$. Further suppose $\gamma \in [0,1/p)$, $s > 1/p - \gamma$, and

$$u \in Z := H_p^{s+\gamma}(J;X) \cap H_p^{\gamma}(J;D_A). \tag{3.17}$$

Our first aim is to prove that $u(0) \in D_A(1+\gamma/s - 1/ps, p)$. To do so, note that, by the mixed derivative theorem, we have the embedding

$$H_p^{s+\gamma}(J;X) \cap H_p^{\gamma}(J;D_A) \hookrightarrow H_p^{(1-\theta)s+\gamma}(J;D_{A^\theta}), \quad \theta \in (0,1). \tag{3.18}$$

We now distinguish two cases.

Case 1: $1/p - \gamma < s < 2(1/p-\gamma)$. By hypothesis and (3.18), we see that

$$u \in H_p^{s+\gamma}(J;X) \cap H_p^{(1-\theta)s+\gamma}(J;D_{A^\theta}), \quad \theta \in (0,1).$$

If we put $\kappa = (1-\theta)s+\gamma$, $B = A^\theta$, and $a(t) = e^{-t}t^{\theta s - 1}$, $t > 0$, then we are in the situation of Theorem 3.1.4, provided that $\kappa < 1/p$ and $\theta s\pi/2 + \phi_B^R < \pi$. But these two conditions are fulfilled with $\theta = 1/2$. In fact, in case of $\theta = 1/2$, we estimate $\kappa = s/2 + \gamma < (1/p - \gamma) + \gamma = 1/p$, as well as

$$\theta s \tfrac{\pi}{2} + \phi_B^R \le \tfrac{s\pi}{4} + \tfrac{1}{2}\phi_A^R < \tfrac{\pi}{2} + \tfrac{\pi}{2} = \pi,$$

where the inequality $\phi_B^R \le \phi_A^R/2$ follows from Proposition 2.2.1. Further,

$$1 + \tfrac{\kappa}{\theta s} - \tfrac{1}{\theta s p} = 2 + \tfrac{2\gamma}{s} - \tfrac{2}{ps},$$

and so, by Theorem 3.1.4 and Theorem 2.2.2, we obtain

$$u(0) \in D_B(2 + \tfrac{2\gamma}{s} - \tfrac{2}{ps}, p) = D_{A^{\frac{1}{2}}}(2 + \tfrac{2\gamma}{s} - \tfrac{2}{ps}, p) = D_A(1 + \tfrac{\gamma}{s} - \tfrac{1}{ps}, p).$$

Case 2: $s \ge 2(1/p - \gamma)$. Here, we look at the regularity with respect to the operator A. By hypothesis and (3.18),

$$u \in H_p^{(1-\theta)s+\gamma}(J;D_{A^\theta}) \cap H_p^{\gamma}(J;D_A), \quad \theta \in (0,1).$$

Therefore, if we choose the base space $X_\theta = \mathcal{D}(A^\theta)$, then we get

$$u \in H_p^{(1-\theta)s+\gamma}(J;X_\theta) \cap H_p^{\gamma}(J;D_{A^{1-\theta}}), \quad \theta \in (0,1).$$

Again, we want to apply Theorem 3.1.4. So we have to ensure that $1/p < (1-\theta)s+\gamma$ and $(1-\theta)s\pi/2 + (1-\theta)\phi_A^R < \pi$. But these conditions are satisfied for

$$\theta = 1 - \tfrac{\eta}{s}\left(\tfrac{1}{p} - \gamma\right)$$

with an arbitrary $\eta \in (1, 2/(1+1/p-\gamma)]$. We namely have in this case

$$(1-\theta)s + \gamma = \eta\left(\tfrac{1}{p} - \gamma\right) + \gamma > \tfrac{1}{p}$$

as well as

$$(1-\theta)s\,\tfrac{\pi}{2} + (1-\theta)\phi_A^R < \eta\left(\tfrac{1}{p}-\gamma\right)\cdot\tfrac{\pi}{2} + \tfrac{\eta}{s}\left(\tfrac{1}{p}-\gamma\right)\pi \le \eta\left(\tfrac{1}{p}-\gamma\right)\cdot\tfrac{\pi}{2} + \tfrac{\eta}{2}\pi \le \pi,$$

where we used the assumption $s \geq 2(1/p - \gamma)$ for the second summand. Furthermore,

$$1 + \tfrac{\gamma}{(1-\theta)s} - \tfrac{1}{(1-\theta)ps} = 1 - \tfrac{1}{\eta}.$$

Thus, Theorem 3.1.4 yields

$$u(0) \in (X_\theta, D_A)_{1-\frac{1}{\eta},p} = (\mathcal{D}(A^\theta), \mathcal{D}(A))_{1-\frac{1}{\eta},p},$$

which entails, by the reiteration theorem (cf. Amann [5, Section 2.8]),

$$u(0) \in (X, \mathcal{D}(A))_{\theta\frac{1}{\eta}+(1-\frac{1}{\eta}),p} = D_A(1 - \tfrac{1-\theta}{\eta}, p) = D_A(1 + \tfrac{\gamma}{s} - \tfrac{1}{ps}, p).$$

Hence, $u(0) \in D_A(1 + \gamma/s - 1/ps, p)$ is established for all $s > 1/p$.

Suppose now that $s + \gamma > n + 1/p$ with $n \in \mathbb{N}$. If $u \in Z$, then $u^k(0)$ exists for all $0 \leq k \leq n$. Taking $\theta = 1 - (k - \gamma)/s$ in (3.18) shows that

$$u^{(k)} \in H_p^{s+\gamma-k}(J; X) \cap L_p(J; D_{A^{1-\frac{k-\gamma}{s}}}).$$

So, with $B = A^{1-\frac{k-\gamma}{s}}$, the above mapping property of the trace operator implies that

$$u^{(k)}(0) \in D_B(1 - \tfrac{1}{(s+\gamma-k)p}, p) = D_{A^{1-\frac{k-\gamma}{s}}}(1 - \tfrac{1}{(s+\gamma-k)p}, p) = D_A(1 + \tfrac{\gamma}{s} - \tfrac{k}{s} - \tfrac{1}{ps}, p), \tag{3.19}$$

using once more Theorem 2.2.2.

If we replace in (3.17) and (3.19) the operator A by A^s, assuming $A \in \mathcal{RS}(X)$ and $\phi_A^R < \pi/s$, we see that the composition of D_t^k and the trace operator tr

$$tr \circ D_t^k : \ H_p^{s+\gamma}(J; X) \cap H_p^\gamma(J; D_{A^s}) \to D_A(s + \gamma - k - \tfrac{1}{p}, p) \tag{3.20}$$

is bounded. Thus, by real interpolation, we obtain boundedness of

$$tr \circ D_t^k : \ B_{pp}^{s+\gamma}(J; X) \cap H_p^\gamma(J; D_A(s, p)) \to D_A(s + \gamma - k - \tfrac{1}{p}, p). \tag{3.21}$$

Strong continuity of the translation group then yields (3.22) and (3.23) in the following

Theorem 3.2.1 *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, $\gamma \in [0, 1/p)$, and $s + \gamma > n + 1/p$ with $n \in \mathbb{N}_0$. Let further $J = [0, T]$ or $\mathbb{R}_+$, and A be an $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle $\phi_A^R < \pi/s$. Then for all $0 \leq k \leq n$,*

$$H_p^{s+\gamma}(J; X) \cap H_p^\gamma(J; D_{A^s}) \hookrightarrow BUC^k(J; D_A(s + \gamma - k - \tfrac{1}{p}, p)) \tag{3.22}$$

and

$$B_{pp}^{s+\gamma}(J; X) \cap H_p^\gamma(J; D_A(s, p)) \hookrightarrow BUC^k(J; D_A(s + \gamma - k - \tfrac{1}{p}, p)). \tag{3.23}$$

The proof of the previous result is inspired by [43]. Theorem 3.2.1 is an extension of [65, Proposition 3], where $\kappa = 0$ and A is assumed to have bounded imaginary powers.

3.3 More time regularity for Volterra equations

This section deals with the question under what conditions the solution of equation (3.1) lies in the space

$$H_p^{\alpha+\kappa}(J;X)\cap H_p^{\kappa}(J;D_A),$$

where, in contrast to Section 3.1, we assume $\kappa\in(1/p,1+1/p)$ and $\alpha+\kappa<2+1/p$.

Theorem 3.3.1 *Let X be a Banach space of class $\mathcal{HT}$, $p\in(1,\infty)$, J a compact time-interval $[0,T]$, and A an $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle ϕ_A^R. Suppose that a belongs to $\mathcal{K}^1(\alpha,\theta_a)$ with $\alpha\in(0,2)$. Further let $\kappa\in(1/p,1+1/p)$, $\alpha+\kappa<2+1/p$ and $\alpha+\kappa\neq 1+1/p$. Assume the* parabolicity condition *$\theta_a+\phi_A^R<\pi$. Then (3.1) has a unique solution in $Z:=H_p^{\alpha+\kappa}(J;X)\cap H_p^{\kappa}(J;D_A)$ if and only if*

(i) $f(0)\in\mathcal{D}(A)$;

(ii) $f=h+(1*a)Af(0)$, *with* $h\in H_p^{\alpha+\kappa}(J;X)$, *and* $\dot h(0)\in D_A(1+\frac{\kappa}{\alpha}-\frac{1}{\alpha}-\frac{1}{p\alpha},p)$ *in case* $\alpha+\kappa>1+1/p$.

Proof. We first show the necessity part. Suppose that $u\in Z$ solves (3.1). Then, $\kappa>1/p$ entails $\alpha+\kappa>1/p$. Thus, the trace $x:=u(0)\in X$ exists. Furthermore we see that $Au\in C(J;X)$. So, by the closedness of A, $x\in\mathcal{D}(A)$ and we infer from (3.1) that

$$u(t)+(a*A(u-x))(t)=f(t)-(1*a)(t)Ax,\quad t\in J.$$

From $A(u-x)\in {}_0H_p^{\kappa}(J;X)$, it follows that $a*A(u-x)\in {}_0H_p^{\alpha+\kappa}(J;X)$, in view of Corollary 2.8.1. In addition, $1*a$ is absolutely continuous on J and vanishes at $t=0$. Hence, $f(0)=x$ as well as $f=h+(1*a)Af(0)$, where h is defined by $h=u+a*A(u-x)\in H_p^{\alpha+\kappa}(J;X)$. This proves (i) and the first part of (ii).

To verify the second condition in (ii), suppose that $\alpha+\kappa>1+1/p$. Owing to $a*A(u-x)\in {}_0H_p^{\alpha+\kappa}(J;X)$, we have $\dot h(0)=\dot u(0)$. We now consider two cases. In case of $\kappa\in(1/p,1)$, we can argue as in the proof of Theorem 3.1.4, Case 3, to see that $\dot u(0)\in D_A(1+\kappa/\alpha-1/\alpha-1/p\alpha,p)$. If $\kappa\in[1,1+1/p)$, then

$$\dot u\in H_p^{\alpha+\kappa-1}(J;X)\cap H_p^{\kappa-1}(J;D_A),$$

and we obtain the desired regularity of $\dot u(0)$ with the aid of Theorem 3.1.4, applied to the equation

$$\dot u(t)+(a*A\dot u)(t)=g(t),\quad t\in J,$$

with some $g\in H_p^{\alpha+\kappa-1}(J;X)$. Hence, the necessity part is complete.

We now want to prove the converse. Uniqueness follows immediately from Theorem 3.1.1. Concerning sufficiency, we suppose validity of (i) and (ii) and distinguish two cases.

If $\alpha+\kappa<1+1/p$, we set $u=x+u_1$, where u_1 is defined as solution of $v+a*Av=h-x$ on J. Condition (i) ensures that the constant function $w(t)=x$, $t\in J$ lies in Z. From the second condition, we deduce $h(0)=f(0)=x$ and $h-x\in {}_0H_p^{\alpha+\kappa}(J;X)$. So, due to Remark 3.1.1(ii), we obtain $u_1\in {}_0H_p^{\alpha+\kappa}(J;X)\cap {}_0H_p^{\kappa}(J;D_A)$. Thus $u\in Z$. Further,

$$u+a*Au=(x+(1*a)Ax)+(h-x)=f,$$

by condition (ii). Hence, $u\in Z$ is indeed the solution of (3.1).

We now assume $\alpha + \kappa > 1 + 1/p$. Put this time $u = x + 1 * S\dot{h}(0) + u_1$, where u_1 solves (3.1) with right-hand side $g(t) := h(t) - x - t\dot{h}(0)$ on J. Because of (ii), we have $g \in {}_0H_p^{\alpha+\kappa}(J; X)$. So it follows by Remark 3.1.1(ii) that $u_1 \in {}_0H_p^{\alpha+\kappa}(J; X) \cap {}_0H_p^{\kappa}(J; D_A)$. Further, the property $\dot{h}(0) \in D_A(1 + \kappa/\alpha - 1/\alpha - 1/p\alpha, p)$ implies $1 * S\dot{h}(0) \in Z$, thanks to Remark 3.1.2(i). Finally, as in the first case, condition (i) ensures that the constant function $w(t) = x$, $t \in J$ lies in Z. So we conclude that $u \in Z$. From

$$u(t) + (a * Au)(t) = (x + (1 * a)(t)Ax) + t\dot{h}(0) + (h(t) - x - t\dot{h}(0)) = f(t), \ t \in J,$$

we see that u solves (3.1). □

It should be mentioned that, in the situation of Theorem 3.3.1, we have in general $1 * a \notin H_p^{\alpha+\kappa}(J)$. As illustration, we consider the following example.

Example 3.3.1 Let $0 \leq \kappa \neq 1/p$, $\alpha \in (0, 1 - \kappa)$, and take $a(t) = t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$. Further put $b = 1 * a$. Since $b(0) = 0$, we see that $b \in H_p^{\alpha+\kappa}(J)$ if and only if $b \in {}_0H_p^{\alpha+\kappa}(J)$. So, letting $k(t) = t^{-(\alpha+\kappa)}/\Gamma(1 - (\alpha + \kappa))$, $t > 0$, we have

$$b \in {}_0H_p^{\alpha+\kappa}(J) \Leftrightarrow \frac{d}{dt}(k * b) \in L_p(J) \Leftrightarrow k * a \in L_p(J).$$

But $(k * a)(t) = t^{-\kappa}/\Gamma(1 - \kappa)$, $t > 0$, so that $k * a \in L_p(J)$ if and only if $\kappa < 1/p$. Hence, $1 * a \notin H_p^{\alpha+\kappa}(J)$ whenever $\kappa > 1/p$.

3.4 Abstract equations of first and second order on the halfline

In this paragraph we collect some known results on maximal L_p-regularity of abstract problems on the *halfline*.

The first theorem, which is due to Weis [81], concerns the abstract Cauchy problem

$$\dot{u} + Au = f, \qquad t > 0, \qquad u(0) = u_0, \tag{3.24}$$

in a Banach space X.

Theorem 3.4.1 *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, and A be an invertible and $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle $\phi_A^R < \pi/2$.*

Then (3.24) has a unique solution in $Z := H_p^1(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_A)$ if and only if

$$(i) \quad f \in L_p(\mathbb{R}_+; X); \qquad (ii) \quad u_0 \in D_A(1 - 1/p, p).$$

Proof. The assertion follows immediately from Remark 3.1.1(iv), Remark 3.1.2(ii) and Theorem 3.1.4 with $a \equiv 1$. □

In the remainder of this section, we consider two abstract second order problems, which play an essential role in the treatment of abstract parabolic problems with inhomogeneous boundary data. By the aid of the two subsequent key results, in Section 3.5, we will succeed in finding the natural regularity classes for the data on the boundary.

The following theorem concerns the problem with Dirichlet condition

$$\left\{ \begin{array}{rl} -u''(y) + F^2u(y) = f(y), & y > 0, \\ u(0) = \phi, & \end{array} \right. \tag{3.25}$$

in $L_p(\mathbb{R}_+; X)$.

Theorem 3.4.2 *Suppose X is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$. Let $F \in \mathcal{BIP}(X)$ be invertible with power angle $\theta_F < \pi/2$, and let D_F^j denote the domain $\mathcal{D}(F^j)$ of F^j equipped with its graph norm, $j = 1, 2$.*

Then (3.25) has a unique solution u in $Z := H_p^2(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_F^2)$ if and only if the following two conditions are satisfied.

$$(i) \quad f \in L_p(\mathbb{R}_+; X); \qquad (ii) \quad \phi \in D_F(2 - \tfrac{1}{p}, p).$$

If this is the case we have in addition $u \in H_p^1(\mathbb{R}_+; D_F^1)$.

This result has been obtained by Prüss, cf. [65, Theorem 3]. Recall that $D_F(2-1/p, p) = \{g \in \mathcal{D}(F) : Fg \in D_F(1-1/p, p)\}$.

There is a corresponding result for the abstract second order problem with abstract Robin condition

$$\begin{cases} -u''(y) + F^2 u(y) = f(y), \quad y > 0, \\ -u'(0) + Du(0) = \psi, \end{cases} \tag{3.26}$$

in $L_p(\mathbb{R}_+; X)$. For $D = 0$, the Robin condition becomes the Neumann condition.

Theorem 3.4.3 *Suppose X is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$. Let $F \in \mathcal{BIP}(X)$ be invertible with power angle $\theta_F < \pi/2$, and let D_F^j denote the domain $\mathcal{D}(F^j)$ of F^j equipped with its graph norm, $j = 1, 2$. Suppose that D is pseudo-sectorial in X, belongs to $\mathcal{BIP}(\overline{\mathcal{R}(D)})$, commutes with F, and is such that $\theta_F + \theta_D < \pi$.*

Then (3.26) has a unique solution u in $Z := H_p^2(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_F^2)$ with $u(0) \in \mathcal{D}(D)$ and $Du(0) \in D_F(1-1/p, p)$ if and only if the following two conditions are satisfied.

$$(i) \quad f \in L_p(\mathbb{R}_+; X); \qquad (ii) \quad \psi \in D_F(1 - \tfrac{1}{p}, p).$$

If this is the case we have in addition $u \in H_p^1(\mathbb{R}_+; D_F^1)$.

This result is also due to Prüss, see [65, Theorem 4].

3.5 Parabolic Volterra equations on an infinite strip

We now study the vector-valued problem

$$\begin{cases} u - a * \partial_y^2 u + a * Au = f, \quad t \in J,\ y > 0, \\ u(t,0) = \phi(t), \quad t \in J, \end{cases} \tag{3.27}$$

in $L_p(J; L_p(\mathbb{R}_+; X))$. Here X is a Banach space which belongs to the class $\mathcal{HT}$, $J = [0,T]$ is a compact time-interval, A is a sectorial operator in X and the kernel a belongs to the class $\mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0,2)$. The data f and ϕ are given. Our aim is to characterize unique existence of a solution u in the maximal regularity class of type L_p, i.e.

$$u \in Z := H_p^\alpha(J; L_p(\mathbb{R}_+; X)) \cap L_p(J; H_p^2(\mathbb{R}_+; X)) \cap L_p(J; L_p(\mathbb{R}_+; D_A))$$

in terms of regularity classes for the data. Recall that D_A denotes the space $\mathcal{D}(A)$ equipped with the graph norm of A. The main result reads as follows.

Theorem 3.5.1 *Let $p \in (1,\infty)$ and X be a Banach space of class $\mathcal{HT}$. Suppose that $a \in \mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (0,2) \setminus \left\{\frac{1}{p},\ \frac{2}{2p-1},\ 1+\frac{1}{p},\ 1+\frac{3}{2p-1}\right\}$ and $A \in \mathcal{BIP}(X)$ with power angle θ_A. Assume further $\theta_a + \theta_A < \pi$. Then the problem (3.27) has a unique solution in Z if and only if the data f and ϕ satisfy the following conditions.*

$$
\begin{array}{ll}
(i) & f \in H_p^\alpha(J; L_p(\mathbb{R}_+; X));\\
(ii) & \phi \in B_{pp}^{\alpha(1-\frac{1}{2p})}(J;X) \cap L_p(J; D_A(1-\frac{1}{2p},p));\\
(iii) & f|_{t=0} \in B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+; D_A(1-\frac{1}{p\alpha},p)),\ \text{if}\ \alpha > \frac{1}{p};\\
(iv) & f|_{t=0,\,y=0} = \phi|_{t=0},\ \text{if}\ \alpha > \frac{2}{2p-1};\\
(v) & \partial_t f|_{t=0} \in B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\mathbb{R}_+;X) \cap L_p(\mathbb{R}_+; D_A(1-\frac{1}{\alpha}-\frac{1}{p\alpha},p)),\ \text{if}\ \alpha > 1+\frac{1}{p};\\
(vi) & \partial_t f|_{t=0,\,y=0} = \dot{\phi}|_{t=0},\ \text{if}\ \alpha > 1+\frac{3}{2p-1}.
\end{array}
$$

If this is the case, then additionally

$$
\begin{array}{ll}
(vii) & f|_{t=0,\,y=0},\ \phi|_{t=0} \in D_A(1-\frac{1}{2p}-\frac{1}{p\alpha},p),\ \text{if}\ \alpha > \frac{2}{2p-1};\\
(viii) & \partial_t f|_{t=0,\,y=0},\ \dot{\phi}|_{t=0} \in D_A(1-\frac{1}{2p}-\frac{1}{\alpha}-\frac{1}{p\alpha},p),\ \text{if}\ \alpha > 1+\frac{3}{2p-1}.
\end{array}
$$

Proof. We begin with the necessity part. Suppose that $u \in Z$ is a solution of (3.27). Then clearly $f = u - a * \partial_y^2 u + a * Au \in H_p^\alpha(J; L_p(\mathbb{R}_+;X))$, i.e. the first condition is satisfied.

Next we extend u w.r.t. y to all of $\mathbb{R}$ by $u(t,y) = 3u(t,-y) - 2u(t,-2y), y < 0$, resulting in a function (again denoted by u) that belongs to

$$H_p^\alpha(J; L_p(\mathbb{R};X)) \cap L_p(J; H_p^2(\mathbb{R};X)) \cap L_p(J; L_p(\mathbb{R}; D_A)).$$

Define $\mathcal{A}$ as the natural extension of A to $Y := L_p(\mathbb{R};X)$ and let $G = -\partial_y^2$ with domain $\mathcal{D}(G) = H_p^2(\mathbb{R};X)$. Then G is sectorial and belongs to the class $\mathcal{BIP}(Y)$ with power angle $\theta_G = 0$. Since both operators commute in the resolvent sense, Theorem 2.3.1 yields that

$$\Lambda := \mathcal{A} + G \tag{3.28}$$

with domain $\mathcal{D}(\Lambda) = \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(G)$ is sectorial and belongs to $\mathcal{BIP}(Y)$ with power angle $\theta_\Lambda \le \theta_A$, in particular $\Lambda \in \mathcal{RS}(Y)$ with $\mathcal{R}$-angle $\phi_\Lambda^R \le \theta_A$, by Theorem 2.2.3. It now follows from Theorem 3.1.4 that $u|_{t=0} \in D_\Lambda(1-1/p\alpha, p)$, if $\alpha > 1/p$. Proposition 2.3.1 gives

$$
\begin{aligned}
D_\Lambda(1-\tfrac{1}{p\alpha},p) &= D_G(1-\tfrac{1}{p\alpha},p) \cap D_{\mathcal{A}}(1-\tfrac{1}{p\alpha},p)\\
&= B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R};X) \cap L_p(\mathbb{R}; D_A(1-\tfrac{1}{p\alpha},p)),
\end{aligned} \tag{3.29}
$$

which entails the third condition, by restriction to $y \in \mathbb{R}_+$. If $\alpha > 1+1/p$, then Theorem 3.1.4 further yields $\partial_t u|_{t=0} \in D_\Lambda(1-1/\alpha-1/p\alpha, p)$. By employing Proposition 2.3.1 once more, we get

$$D_\Lambda(1-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha},p) = B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\mathbb{R};X) \cap L_p(\mathbb{R}; D_A(1-\tfrac{1}{\alpha}-\tfrac{1}{p\alpha},p)). \tag{3.30}$$

Thus, after restriction to $y \in \mathbb{R}_+$ we arrive at condition (v).

Our next objective is to show necessity of (ii). For this purpose we choose an extension of the solution $u \in Z$ w.r.t. t to all of $\mathbb{R}$ (again denoted by u) which lies in the regularity class

$$Z_1 := H_p^\alpha(\mathbb{R}; L_p(\mathbb{R}_+; X)) \cap L_p(\mathbb{R}; H_p^2(\mathbb{R}_+; X)) \cap L_p(\mathbb{R}; L_p(\mathbb{R}_+; D_A)). \tag{3.31}$$

Set $A_1 = I + A$ with domain $\mathcal{D}(A_1) = \mathcal{D}(A)$ and let $\mathcal{A}_1$ be the natural extension of A_1 to $Y := L_p(\mathbb{R}; X)$. Then the operator A_1 is invertible, and by Theorem 2.3.1 it belongs to the class $\mathcal{BIP}(X)$ with power angle $\theta_{A_1} \leq \theta_A$; thus $\mathcal{A}_1$ is invertible as well and contained in $\mathcal{BIP}(Y)$ with power angle $\theta_{\mathcal{A}_1} \leq \theta_A$. Further let $B \in \mathcal{BIP}(Y)$ be the inverse Volterra operator from Theorem 2.8.1. Then the resolvents of $\mathcal{A}_1$ and B commute, and we have $\theta_B + \theta_{\mathcal{A}_1} \leq \theta_a + \theta_A < \pi$. This allows us to apply Theorem 2.3.1 to the pair $(B, \mathcal{A}_1)$ in the space Y yielding that $B + \mathcal{A}_1$ with domain $\mathcal{D}(B) \cap \mathcal{D}(\mathcal{A}_1) = H_p^\alpha(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A)$ is invertible and contained in $\mathcal{BIP}(Y)$ with power angle $\theta_{B+\mathcal{A}_1} \leq \max\{\theta_a, \theta_A\}$. The function $u \in Z_1$ now satisfies a problem of the form

$$\begin{aligned} Bu - \partial_y^2 u + A_1 u &= g, \quad y > 0,\ t \in \mathbb{R}, \\ u(t,0) &= \varphi(t), \quad t \in \mathbb{R}, \end{aligned}$$

with $g \in L_p(\mathbb{R}; Y)$ and some φ, which is an extension of ϕ to all of $\mathbb{R}$. To determine the regularity of φ, we apply Theorem 3.4.2 to the invertible operator

$$F := \sqrt{B + \mathcal{A}_1}, \tag{3.32}$$

which belongs again to $\mathcal{BIP}(Y)$ and has power angle $\theta_F \leq \max\{\theta_a, \theta_A\}/2 < \pi/2$. This results in $\varphi \in D_F(2 - 1/p, p)$. Due to Theorem 2.2.2 and Proposition 2.3.1, we have

$$D_F(\gamma, p) = D_{(B+\mathcal{A}_1)^{1/2}}(\gamma, p) = D_{B+\mathcal{A}_1}(\tfrac{\gamma}{2}, p) = D_B(\tfrac{\gamma}{2}, p) \cap D_{\mathcal{A}_1}(\tfrac{\gamma}{2}, p), \quad \gamma \in (0,1).$$

Therefore

$$D_F(1 - \tfrac{1}{p}, p) = B_{pp}^{\alpha(\frac{1}{2} - \frac{1}{2p})}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)), \tag{3.33}$$

which implies

$$D_F(2 - \tfrac{1}{p}, p) = B_{pp}^{\alpha(1 - \frac{1}{2p})}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(1 - \tfrac{1}{2p}, p)). \tag{3.34}$$

Here we employ the embeddings

$$B_{pp}^{\alpha(1-\frac{1}{2p})}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(1 - \tfrac{1}{2p}, p)) \hookrightarrow B_{pp}^{\alpha(\frac{1}{2} - \frac{1}{2p})}(\mathbb{R}; D_{A^{\frac{1}{2}}}), \tag{3.35}$$

$$B_{pp}^{\alpha(1-\frac{1}{2p})}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(1 - \tfrac{1}{2p}, p)) \hookrightarrow H_p^{\frac{\alpha}{2}}(\mathbb{R}; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)), \tag{3.36}$$

which follow from the mixed derivative theorem and real interpolation. Hence, we have shown that

$$\varphi \in B_{pp}^{\alpha(1-\frac{1}{2p})}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(1 - \tfrac{1}{2p}, p)).$$

By restriction to J, we see that the second condition in Theorem 3.5.1 is necessary.

To derive condition (iv), we notice that $u \in Z$ satisfies

$$u \in H_p^{\alpha s}(J; H_p^{2(1-s)}(\mathbb{R}_+; X)), \tag{3.37}$$

for each $s \in [0,1]$. This follows from the mixed derivative theorem. The space in (3.37) embeds into $BUC(J \times \mathbb{R}_+; X)$ if $1/p < s\alpha$ and $1/p < 2 - 2s$, i.e. if $1/p\alpha < s < 1 - 1/2p$. This shows that the compatibility condition (iv) is necessary in case $\alpha > 2/(2p-1)$.

We proceed with the determination of the regularity of $\phi|_{t=0}$ in case $\alpha > 2/(2p-1)$. Observe that this inequality is equivalent to $1/p < \alpha(1 - 1/2p)$. In light of Theorem 3.2.1, we have the embedding

$$B^{s\gamma}_{pp}(J; X) \cap L_p(J; D_A(\gamma, p)) \hookrightarrow C(J; D_A(\gamma - \tfrac{1}{p\,s}, p)), \tag{3.38}$$

if $\gamma, \gamma - 1/ps \in (0,1)$ and $s\gamma \in (1/p, 2)$. Thus, by taking $s = \alpha$ and $\gamma = 1 - 1/2p$, we see that $\phi|_{t=0} \in D_A(1 - 1/2p - 1/p\alpha, p)$. Hence property (vii) is established.

It remains to show (vi) and (viii). Let $\alpha > 1 + 3/(2p-1)$, which in particular means that $\alpha > 1 + 1/p$. Exploiting (3.37) with $s = 1/\alpha$ yields

$$\partial_t u \in H^{\alpha-1}_p(J; L_p(\mathbb{R}_+; X)) \cap L_p(\mathbb{R}; H^{2(1-\frac{1}{\alpha})}_p(\mathbb{R}_+; X)),$$

which in turn entails

$$\partial_t u \in H^{(\alpha-1)s}_p(J; H^{2(1-\frac{1}{\alpha})(1-s)}_p(\mathbb{R}_+; X)),$$

for each $s \in [0,1]$, by the mixed derivative theorem. This space is embedded into $BUC(J \times \mathbb{R}_+; X)$ whenever $(\alpha-1)s > 1/p$ and $2(1-1/\alpha)(1-s) > 1/p$, i.e. if $1/p(\alpha-1) < s < 1 - \alpha/2p(\alpha-1)$. An easy calculation shows that existence of such an s is equivalent to $\alpha > 1 + 3/(2p-1)$, which we just assumed. Therefore the trace $\partial_t u(0,0)$ exists, and we see that the compatibility condition (vi) is necessary.

Last but not least, we restrict $\partial_t f|_{t=0}$ to some finite interval $J_1 = [0, y_0]$, which results in a function that belongs to

$$B^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}_{pp}(J_1; X) \cap L_p(J_1; D_A(1 - \tfrac{1}{\alpha} - \tfrac{1}{p\alpha}, p)),$$

owing to property (v). Then we apply again (3.38), this time with $\gamma = 1 - 1/\alpha - 1/p\alpha$ and $s = 2$. This is possible on account of $\alpha > 1 + 3/(2p-1)$. We immediately see that $\partial_t f|_{t=0,\,y=0}$ enjoys the regularity claimed in (viii). Hence the necessity part of Theorem 3.5.1 as well as the additional properties (vii) and (viii) are established.

We turn now to the sufficiency part of Theorem 3.5.1. Let the data f and ϕ be given such that the conditions (i)-(vi) are satisfied. We will build the solution $u \in Z$ of (3.27) as a sum of two functions in Z.

At first we will construct a function $u_1 \in Z$ such that

$$u_1 - a * \partial_y^2 u_1 + a * A u_1 = f, \quad t \in J,\ y > 0, \tag{3.39}$$

i.e. u_1 solves the first equation of (3.27). For this purpose we extend the function f w.r.t. y to all of $\mathbb{R}$ in such a way that (i),(iii),(v) are fulfilled with $\mathbb{R}_+$ replaced by $\mathbb{R}$. Let $Y := L_p(\mathbb{R}; X)$ and $\Lambda \in \mathcal{BIP}(Y)$ be the operator defined in (3.28). Then we have $f|_{t=0} \in D_\Lambda(1 - 1/p\alpha, p)$, if $\alpha > 1/p$, and $\partial_t f|_{t=0} \in D_\Lambda(1 - 1/\alpha - 1/p\alpha, p)$ in case $\alpha > 1 + 1/p$, owing to (3.29) and (3.30). Thus, by Theorem 3.1.4, the Volterra equation

$$v + a * \Lambda v = f, \quad t \in J, \tag{3.40}$$

admits a unique solution v_1 in the regularity class

$$H^\alpha_p(J; L_p(\mathbb{R}; X)) \cap L_p(J; H^2_p(\mathbb{R}; X)) \cap L_p(J; L_p(\mathbb{R}; D_A)).$$

Let u_1 be the restriction of v_1 to $y \in \mathbb{R}_+$. Then clearly $u_1 \in Z$, and u_1 satisfies (3.39).

Next we put $\phi_1 := u_1|_{y=0}$. Due to the necessity part of Theorem 3.5.1 (conditions (ii),(iv),(vi)), we see that

$$\phi_1 \in B_{pp}^{\alpha(1-\frac{1}{2p})}(J;X) \cap L_p(J; D_A(1-\tfrac{1}{2p},p)),$$

and $f|_{t=0,\,y=0} = \phi_1|_{t=0}$, if $\alpha > \frac{2}{2p-1}$, as well as $\partial_t f|_{t=0,\,y=0} = \dot{\phi}_1|_{t=0}$, in case $\alpha > 1 + 3/(2p-1)$. Therefore we deduce that

$$\phi - \phi_1 \in {}_0B_{pp}^{\alpha(1-\frac{1}{2p})}(J;X) \cap L_p(J; D_A(1-\tfrac{1}{2p},p)).$$

Consider now the problem

$$\left\{ \begin{array}{rl} v - a * \partial_y^2 v + a * Av = 0, & t \in J,\ y > 0, \\ v(t,0) = \phi - \phi_1, & t \in J. \end{array} \right. \tag{3.41}$$

Define this time $\mathcal{A}$ as the natural extension of A to $Y_1 := L_p(J;X)$. Let further $\mathcal{B} \in \mathcal{BIP}(L_p(J;X))$ be the inverse Volterra operator from Corollary 2.8.1 associated with the kernel a. Then $\mathcal{B}$ is invertible, $\mathcal{A}$ and $\mathcal{B}$ belong to $\mathcal{BIP}(Y_1)$, their resolvents commute, and $\theta_{\mathcal{A}} + \theta_{\mathcal{B}} \le \theta_A + \theta_a < \pi$. Therefore, by Theorem 2.3.1, the operator $\mathcal{B} + \mathcal{A}$ with domain $\mathcal{D}(\mathcal{B}) \cap \mathcal{D}(\mathcal{A}) = {}_0H_p^\alpha(J;X) \cap L_p(J;D_A)$ is invertible and contained in $\mathcal{BIP}(Y_1)$ with power angle $\theta_{\mathcal{B}+\mathcal{A}} \le \max\{\theta_a, \theta_A\}$. Moreover, the operator $F_1 := \sqrt{\mathcal{B}+\mathcal{A}}$ is also invertible and belongs to $\mathcal{BIP}(Y_1)$ with power angle $\theta_{F_1} \le \max\{\theta_a,\theta_A\}/2 < \pi/2$. We can now rewrite (3.41) as

$$-v''(y) + F_1^2 v(y) = 0,\ y > 0, \quad v(0) = \phi - \phi_1. \tag{3.42}$$

In the same way as above for the operator F, one can show that

$$D_{F_1}(1-\tfrac{1}{p},p) = {}_0B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(\mathbb{R};X) \cap L_p(\mathbb{R}; D_A(\tfrac{1}{2}-\tfrac{1}{2p},p)) \tag{3.43}$$

and

$$D_{F_1}(2-\tfrac{1}{p},p) = {}_0B_{pp}^{\alpha(1-\frac{1}{2p})}(J;X) \cap L_p(J; D_A(1-\tfrac{1}{2p},p)). \tag{3.44}$$

Compare this with (3.33) and (3.34). Thus we have $\phi - \phi_1 \in D_{F_1}(2-1/p,p)$, which implies existence of a solution u_2 of (3.41) in the space

$${}_0H_p^\alpha(J;L_p(\mathbb{R}_+;X)) \cap L_p(J;H_p^2(\mathbb{R}_+;X)) \cap L_p(J;L_p(\mathbb{R}_+;D_A)),$$

by virtue of Theorem 3.4.2.

Finally let $u := u_1 + u_2$. Then $u \in Z$, and by construction u is a solution of problem (3.27).

Uniqueness follows from Theorem 3.4.2. To see this, rewrite (3.27) with zeros on the right-hand side as (3.42) with initial value zero. This completes the proof of Theorem 3.5.1. □

There is a corresponding result for the problem

$$\left\{ \begin{array}{rl} u - a * \partial_y^2 u + a * Au = f, & t \in J,\ y > 0, \\ -\partial_y u(t,0) + Du(t,0) = \phi(t), & t \in J. \end{array} \right. \tag{3.45}$$

Here the operator D is pseudo-sectorial in X, and we assume that $D_{A^{1/2}} \hookrightarrow D_D$. As above we seek a solution in

$$Z = H_p^\alpha(J; L_p(\mathbb{R}_+; X)) \cap L_p(J; H_p^2(\mathbb{R}_+; X)) \cap L_p(J; L_p(\mathbb{R}_+; D_A)).$$

Before we state the theorem concerning (3.45) we note that any function v in the space Z automatically satisfies $v|_{y=0} \in L_p(J; D_D)$ and

$$Dv(\cdot, 0) \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; X) \cap L_p(J; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)). \tag{3.46}$$

In fact, if $v \in Z$, then it follows, by Theorem 3.5.1, that

$$v|_{y=0} \in L_p(J; D_A(1 - \tfrac{1}{2p}, p)),$$

which entails the first claim, for we have the embeddings

$$D_A(1 - \tfrac{1}{2p}, p) \hookrightarrow D_{A^{\frac{1}{2}}} \hookrightarrow D_D.$$

As for (3.46), we see with the aid of the mixed derivative theorem (Proposition 2.3.2) that for $w := A^{1/2}v$,

$$w \in H_p^{\frac{\alpha}{2}}(J; L_p(\mathbb{R}_+; X)) \cap L_p(J; H_p^1(\mathbb{R}_+; X)) \cap L_p(J; L_p(\mathbb{R}_+; D_{A^{\frac{1}{2}}})). \tag{3.47}$$

We extend w w.r.t. t to all of $\mathbb{R}$ such that (3.47) holds true, with J replaced by $\mathbb{R}$. As above, set $Y = L_p(\mathbb{R}; X)$ and define the operator F by (3.32). Then we know that F is invertible and lies in the class $\mathcal{BIP}(Y)$ with power angle $\theta_F < \pi/2$. Further, $w \in H_p^1(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D_F)$. Thus, Theorem 3.4.1 yields $w|_{y=0} \in D_F(1 - 1/p, p)$, which by restriction to $t \in J$ implies

$$A^{\frac{1}{2}}v(\cdot, 0) \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; X) \cap L_p(J; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)),$$

in view of (3.33). Assertion (3.46) now follows on account of $D_{A^{1/2}} \hookrightarrow D_D$.

Theorem 3.5.2 *Let $p \in (1, \infty)$, X be a Banach space of class $\mathcal{HT}$, $a \in \mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0, 2) \setminus \left\{\frac{1}{p}, \frac{2}{p-1}, 1 + \frac{1}{p}\right\}$, and $A \in \mathcal{BIP}(X)$ with power angle θ_A. Let further D be a pseudo-sectorial operator in X such that $D \in \mathcal{BIP}(\overline{\mathcal{R}(D)})$ with power angle θ_D. Suppose that A and D commute in the resolvent sense, and that $D_{A^{1/2}} \hookrightarrow D_D$. Assume further the angle conditions $\theta_a + \theta_A < \pi$ and $\theta_D < \pi - \max\{\theta_a, \theta_A\}/2$. Then the problem (3.45) has a unique solution in Z if and only if the data f and ϕ are subject to the following conditions.*

(*i*) $f \in H_p^\alpha(J; L_p(\mathbb{R}_+; X))$;

(*ii*) $\phi \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; X) \cap L_p(J; D_A(\frac{1}{2} - \frac{1}{2p}, p))$;

(*iii*) $f|_{t=0} \in B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_A(1 - \frac{1}{p\alpha}, p)),$ *if* $\alpha > \frac{1}{p}$;

(*iv*) $f|_{t=0,\, y=0} \in \mathcal{D}(D)$ *and* $-\partial_y f|_{t=0,\, y=0} + Df|_{t=0,\, y=0} = \phi|_{t=0},$ *if* $\alpha > \frac{2}{p-1}$;

(*v*) $\partial_t f|_{t=0} \in B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\mathbb{R}_+; X) \cap L_p(\mathbb{R}_+; D_A(1 - \frac{1}{\alpha} - \frac{1}{p\alpha}, p)),$ *if* $\alpha > 1 + \frac{1}{p}$.

If this is the case, then additionally

(*vi*) $f|_{t=0,\, y=0} \in D_A(1 - \frac{1}{2p} - \frac{1}{p\alpha}, p),$ *if* $\alpha > \frac{2}{2p-1}$;

(*vii*) $Df|_{t=0,\, y=0},\ \phi|_{t=0} \in D_A(\frac{1}{2} - \frac{1}{2p} - \frac{1}{p\alpha}, p),$ *if* $\alpha > \frac{2}{p-1}$.

Proof. We begin with the necessity part. Suppose we have a function $u \in Z$ which is a solution of (3.45). Then u is also a solution of the problem

$$\begin{cases} u - a * \partial_y^2 u + a * Au = f, \quad t \in J,\ y > 0, \\ \qquad\qquad u(t,0) = \varphi(t), \quad t \in J, \end{cases}$$

where $\varphi := u|_{y=0}$. Therefore, conditions (i),(iii) and (v) are necessary, by the first part of the proof of Theorem 3.5.1, where the case $\alpha = 2/(2p-1)$ is admissible, too. Moreover, we see that $f|_{t=0,\, y=0} \in D_A(1 - 1/2p - 1/p\alpha, p)$, if $\alpha > 2/(2p-1)$, i.e. property (vi) is fulfilled.

To show condition (ii), we proceed similarly as in the proof of Theorem 3.5.1. We extend u w.r.t. t to all of $\mathbb{R}$ such that $u \in Z_1$ (see (3.31) for the definition), $u|_{y=0} \in L_p(\mathbb{R}; D_D)$, and

$$Du(\cdot,0) \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(\mathbb{R}; X) \cap L_p(\mathbb{R}; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)).$$

Let $Y := L_p(\mathbb{R}; X)$ and define $\mathcal{D}$ as the natural extension of D to this space. Let F be as in (3.32). Then the space $D_F(1-1/p, p)$ is given by (3.33). Thus we see that $u \in H_p^2(\mathbb{R}_+, Y) \cap L_p(\mathbb{R}_+; D_{F^2})$, $u|_{y=0} \in \mathcal{D}(\mathcal{D})$, and $\mathcal{D}u|_{y=0} \in D_F(1-1/p, p)$. Now consider u as the solution of the problem

$$-u''(y) + F^2 u(y) = g, \quad y > 0, \quad -u'(0) = \bar{\phi} - \mathcal{D}u(0),$$

with some $g \in L_p(\mathbb{R}_+; Y)$ and an extension $\bar{\phi}$ of ϕ on the real line. Since $\theta_F \leq \max\{\theta_a, \theta_A\}/2 < \pi$, it then follows from Theorem 3.4.3 that $\bar{\phi} - \mathcal{D}u(0) \in D_F(1-1/p, p)$, i.e. $\bar{\phi} \in D_F(1-1/p, p)$, which after restriction to $t \in J$ yields condition (ii).

Finally, we have to prove (iv) and (vii). By the mixed derivative theorem, it follows from $u \in Z$ that

$$\partial_y u \in H_p^{\frac{\alpha}{2}}(J; L_p(\mathbb{R}_+; X)) \cap L_p(J; H_p^1(\mathbb{R}_+; X)),$$

and further

$$\partial_y u \in H_p^{\frac{\alpha s}{2}}(J; H_p^{1-s}(\mathbb{R}_+; X)),$$

for all $s \in [0,1]$. This space embeds into $BUC(J \times \mathbb{R}_+; X)$ if $\alpha s/2 > 1/p$ and $1 - s > 1/p$, i.e. in case $2/p\alpha < s < 1 - 1/p$. Such an s exists if and only if $\alpha > 2/(p-1)$ as a short computation shows. Thus in this situation the trace $(\partial_y u)(0,0) \in X$ exists. Further,

$$Du|_{y=0},\ \phi \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; X) \cap L_p(J; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p))$$

and

$$B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; X) \cap L_p(J; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)) \hookrightarrow C(J; D_A(\tfrac{1}{2} - \tfrac{1}{2p} - \tfrac{1}{p\alpha}, p)),$$

thanks to Theorem 3.2.1 and the remark before Theorem 3.5.2. Therefore, by the closedness of D, $f|_{t=0,\, y=0} \in \mathcal{D}(D)$ and

$$-\partial_y f|_{t=0,\, y=0} + Df|_{t=0,\, y=0} = \phi|_{t=0},$$

where each term in this equation lies in the space $D_A(1/2 - 1/2p - 1/p\alpha, p)$. Hence, conditions (iv) and (vii) are proved. This completes the necessity part of the proof.

We come now to sufficiency. Suppose we are given the data f and ϕ which fulfill the conditions (i)-(v). We proceed similarly as in the proof of Theorem 3.5.1.

Firstly, let $u_1 \in Z$ be a function which satisfies (3.39). Such a function was constructed in the proof of Theorem 3.5.1. Put $\phi_1 := -\partial_y u_1|_{y=0} + Du_1|_{y=0}$ and consider the problem

$$\begin{cases} v - a * \partial_y^2 v + a * Av = 0, \quad t \in J,\ y > 0, \\ -\partial_y v(t,0) + Dv(t,0) = \phi(t) - \phi_1(t), \quad t \in J. \end{cases} \tag{3.48}$$

Owing to the necessity part of Theorem 3.5.2 and the above remark concerning (3.46), ϕ_1 is well-defined and lies in the same regularity class as the data ϕ. In addition, we see that $(\phi-\phi_1)|_{t=0} = 0$, if $\alpha > 2/(p-1)$, by construction of ϕ_1 and thanks to condition (iv). That is, the compatibility condition is satisfied. To solve (3.48), we let $Y_1 = L_p(J;X)$, $\mathcal{D}$ be the natural extension of D to this space, and define the operator F_1 as in the paragraph following problem (3.41). Then (3.48) can be rewritten as

$$-v''(y) + F_1^2 v(y) = 0,\ y > 0, \quad -v'(0) + \mathcal{D}v(0) = \phi - \phi_1. \tag{3.49}$$

Observe that $\mathcal{D}$ is pseudo-sectorial in Y_1 and $\mathcal{D} \in \mathcal{BIP}(\overline{\mathcal{R}(\mathcal{D})})$ with power angle $\theta_{\mathcal{D}} \leq \theta_D$. In view of (3.43), $\phi - \phi_1 \in D_{F_1}(1-1/p,p)$. Further, we have $\theta_{\mathcal{D}} + \theta_{F_1} \leq \theta_D + \max\{\theta_a, \theta_A\}/2 < \pi$. By Theorem 3.4.3, problem (3.48) thus admits a unique solution u_2 in the space

$$ {}_0H_p^\alpha(J; L_p(\mathbb{R}_+;X)) \cap L_p(J; H_p^2(\mathbb{R}_+;X)) \cap L_p(J; L_p(\mathbb{R}_+; D_A))$$

with

$$Du_2(\cdot,0) \in D_{F_1}(1-1/p,p) = {}_0B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J;X) \cap L_p(J; D_A(\tfrac{1}{2} - \tfrac{1}{2p}, p)).$$

Finally put $u := u_1 + u_2$. Then u clearly possesses the desired regularity and solves (3.45).

Uniqueness follows by Theorem 3.4.3. In fact, rewrite (3.45) with zero data as (3.49) with zeros on the right-hand side. Then it is apparent that the zero-function is the only solution of (3.45) in Z. $\square$

Remarks 3.5.1 (i) Theorem 3.5.1 and Theorem 3.5.2 are also true, if the kernel a is of the form $a = b + dk * b$, where b is like a above and $k \in BV_{loc}(\mathbb{R}_+)$ with $k(0) = k(0+) = 0$.

(ii) The proofs of the two foregoing theorems are inspired by Prüss [65]. In the case $a \equiv 1$ Theorem 3.5.1 is equivalent with a version of [65, Thm. 5] for compact J, while Theorem 3.5.2, roughly speaking, can be regarded as an extension of [65, Thm. 6], at least in the case $D_{A^{1/2}} \hookrightarrow D_D$.

Chapter 4

Linear Problems of Second Order

This chapter is devoted to the study of linear problems of second order on $L_p(J \times \Omega)$, Ω a domain in $\mathbb{R}^n$, with general inhomogeneous boundary conditions of order ≤ 1. We shall apply abstract results proven in Chapter 3 to characterize maximal L_p-regularity of the solutions in terms of regularity and compatibility conditions for the data. We will first consider problems on the full space $\mathbb{R}^n$. This will be followed by the investigation of the half space case. Finally, we study the case of an arbitrary domain. Here we use the localization method to reduce the problem to related problems on $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

4.1 Full space problems

In this section we study the Volterra equation

$$v + k * \mathcal{A}(\cdot, x, D_x)v = f, \quad t \in J, \ x \in \mathbb{R}^n, \tag{4.1}$$

in $L_p(J; L_p(\mathbb{R}^n))$. Here $J = [0, T]$, the kernel k belongs to the class $\mathcal{K}^1(\alpha, \theta)$ with $\alpha \in (0, 2)$, $\theta < \pi$, and $\mathcal{A}(t, x, D_x)$ is a differential operator of second order with variable coefficients:

$$\mathcal{A}(t, x, D_x) = -a(t, x) : \nabla_x^2 + b_1(t, x) \cdot \nabla_x + b_0(t, x), \ t \in J, \ x \in \mathbb{R}^n. \tag{4.2}$$

By $\nabla_x^2 v$ we mean the Hessian matrix of v w.r.t. x, that is $(\nabla_x^2 v(t, x))_{ij} = \partial_{x_i} \partial_{x_j} v(t, x)$, $i, j = 1, \ldots, n$. The double scalar product $a : b$ of two matrices $a, b \in \mathbb{C}^{n \times n}$ is defined by $a : b = \sum_{i,j=1}^n a_{ij} b_{ij}$. We further denote the principal part of the differential operator (4.2) by $\mathcal{A}_\#(t, x, D_x)$, that is $\mathcal{A}_\#(t, x, D_x) = -a(t, x) : \nabla_x^2$, and we write $\mathcal{A}(t, x, D_x) = \mathcal{A}_\#(t, x, D_x) + \mathcal{A}_R(t, x, D_x)$.

For an unbounded domain $\Omega \subset \mathbb{R}^n$ and a Banach space X, we set

$$C_{ul}(J \times \overline{\Omega}; X) = \{g \in C(J \times \overline{\Omega}; X) : \lim_{|x| \to \infty} g(t, x) \text{ exists uniformly for all } t \in J\}.$$

The symbol $\mathrm{Sym}\{n\}$ stands for the space of n-dimensional real symmetric matrices.

The goal of this section is to prove the following result.

Theorem 4.1.1 *Let $1 < p < \infty$ and $n \in \mathbb{N}$. Suppose the differential operator $\mathcal{A}(t, x, D_x)$ is given by (4.2). Assume the following properties.*

(H1) $k \in \mathcal{K}^1(\alpha, \theta)$, where $\alpha \in (0, 2) \setminus \{1/p, 1 + 1/p\}$, $\theta < \pi$;

(H2) $a \in C_{ul}(J \times \mathbb{R}^n, \mathrm{Sym}\{n\})$, $b_1 \in L_\infty(J \times \mathbb{R}^n, \mathbb{R}^n)$, $b_0 \in L_\infty(J \times \mathbb{R}^n)$;

(H3) $\exists a_0 > 0 : a(t,x)\xi \cdot \xi \geq a_0|\xi|^2,\ t \in J,\ x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n$.

Then the Volterra equation (4.1) admits a unique solution in the space

$$Z := H_p^\alpha(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^2(\mathbb{R}^n))$$

if and only if the function f is subject to the subsequent conditions.

$$\begin{array}{ll} (i) & f \in H_p^\alpha(J; L_p(\mathbb{R}^n)); \\ (ii) & f|_{t=0} \in \gamma_0 Z := B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R}^n),\ \textit{if}\ \alpha > 1/p; \\ (iii) & \partial_t f|_{t=0} \in \gamma_1 Z := B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\mathbb{R}^n),\ \textit{if}\ \alpha > 1 + 1/p. \end{array}$$

Proof. We start with the necessity part. Suppose that $v \in Z$ solves (4.1). By assumption (H2), we immediately verify that $\mathcal{A}(t,x,D_x)v \in L_p(J; L_p(\mathbb{R}^n))$. Thus, in view of (H1), $f = v - k * \mathcal{A}(\cdot,x,D_x)v \in H_p^\alpha(J; L_p(\mathbb{R}^n))$, i.e. condition (i) is satisfied. To see (ii) and (iii), apply Theorem 3.2.1 to the space $L_p(\mathbb{R}^n)$ and the operator $A = -\Delta$ with domain $H_p^2(\mathbb{R}^n)$, and use $(L_p(\mathbb{R}^n), H_p^2(\mathbb{R}^n))_{s,p} = B_{pp}^{2s}(\mathbb{R}^n)$, $s \in (0,1)$, together with $\partial_t^j v|_{t=0} = \partial_t^j f|_{t=0}$ in case $\alpha > j + 1/p$, $j = 0,1$.

The sufficiency part is more involved. Suppose f satisfies (i)-(iii). In order to prove existence of a unique solution of (4.1) in Z, we use localization and perturbation to reduce (4.1) to related equations with constant coefficients.

Given $\eta > 0$, assumption (H2) allows us to select a large ball $B_{r_0}(0) \subset \mathbb{R}^n$ such that

$$|a(t,x) - a(t,\infty)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \leq \frac{\eta}{2}, \quad \text{for all } t \in J,\ x \in \mathbb{R}^n, |x| \geq r_0.$$

Putting $U_0 = \mathbb{R}^n \setminus \overline{B_{r_0}(0)}$ we can further cover $\overline{B_{r_0}(0)}$ by finitely many balls $U_j = B_{r_j}(x_j)$, $j = 1,\ldots,N$, and choose a partition $0 =: T_0 < T_1 < \ldots < T_{M-1} < T_M := T$ such that for all $i = 0,\ldots,M-1$, $j = 1,\ldots,N$,

$$|a(t,x) - a(T_i,x_j)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \leq \eta, \quad t \in [T_i,T_{i+1}],\ x \in \overline{B_{r_j}(x_j)},$$

and

$$|a(t,\infty) - a(T_i,\infty)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \leq \frac{\eta}{2}, \quad t \in [T_i,T_{i+1}].$$

Define coefficients of spatially local operators $\mathcal{A}_\#^j(t,x,D_x) = -a^j(t,x) : \nabla_x^2$ e.g. by reflection, that is

$$a^0(t,x) := \left\{ \begin{array}{ll} a(t,x) & : t \in J,\ x \notin B_{r_0}(0) \\ a(t, r_0^2 \frac{x}{|x|^2}) & : t \in J,\ x \in B_{r_0}(0) \end{array} \right.$$

and

$$a^j(t,x) := \left\{ \begin{array}{ll} a(t,x) & : t \in J,\ x \in B_{r_j}(x_j) \\ a(t, x_j + r_j^2 \frac{x-x_j}{|x-x_j|^2}) & : t \in J,\ x \notin B_{r_j}(x_j) \end{array} \right.$$

for each $j = 1,\ldots,N$. With $x_0 = \infty$, we then have

$$|a^j(t,x) - a(T_i,x_j)|_{\mathcal{B}(\mathbb{R}^{n\times n})} \leq \eta, \quad t \in [T_i,T_{i+1}],\ x \in \mathbb{R}^n,$$

for all $i = 0,\ldots,M-1$, $j = 0,\ldots,N$.

The strategy for solving (4.1) is now as follows. First we determine the solution of (4.1) on the interval $[0, T_1]$. Let us denote it by v^1. Now suppose we already know the solution v^i of (4.1) on the interval $[0, T_i]$, where $1 \leq i \leq M-1$. We then seek the solution v^{i+1} of (4.1) on the larger interval $[0, T_{i+1}]$ which equals v^i on $[0, T_i]$. The last step is repeated as long as $i < M$. Proceeding in this way we finally obtain the solution of (4.1) on the entire interval $[0, T]$. This is the basic idea concerning the localization in *time*. Besides, we will also localize (4.1) with respect to the *space* variable. This will be done in each single time step by means of a partition of unity $\{\varphi_j\}_{j=0}^N \subset C^\infty(\mathbb{R}^n)$ which enjoys the properties $\sum_{j=0}^N \varphi_j \equiv 1$, $0 \leq \varphi_j(x) \leq 1$ and $\operatorname{supp} \varphi_j \subset U_j$. We will also make use of a fixed family $\{\psi_j\}_{j=0}^N \subset C^\infty(\mathbb{R}^n)$ that satisfies $\psi_j \equiv 1$ on an open set V_j containing $\operatorname{supp} \varphi_j$, and $\operatorname{supp} \psi_j \subset U_j$.

Suppose we are in the $(i+1)$th time step of the above procedure. Set ${}_{(0)}Z_{i+1} = {}_{(0)}H_p^\alpha([0, T_{i+1}]; L_p(\mathbb{R}^n)) \cap L_p([0, T_{i+1}]; H_p^2(\mathbb{R}^n))$. If $i = 0$ (initial time step), we have to find v^1 in the space

$$Z_1(v^0) := \{w \in Z_1 : \partial_t^m w|_{t=0} = \partial_t^m f|_{t=0}, \text{ if } \alpha > m + 1/p, \, m = 0, 1\}.$$

If $i > 0$, we assume that $v^i =: \mathcal{V}_i^{-1} f$ lies in Z_i. Here $\mathcal{V}_i$ refers to the operator $I + k * \mathcal{A}(\cdot, x, D_x)$ on $[0, T_i]$. Using the notation

$$Z_{i+1}(\tilde{w}) := \{w \in Z_{i+1} : w|_{[0,T_i]} = \tilde{w}\} \quad \text{for } \tilde{w} \in Z_i, \tag{4.3}$$

our aim is then to determine v^{i+1} in the space $Z_{i+1}(v^i)$. To achieve this, observe that with $d(w_1, w_2) = |w_1 - w_2|_{Z_{i+1}}$, $(Z_{i+1}(v^i), d)$ is a complete metric space. Our plan is to transform (4.1) to an appropriate fixed point equation in $Z_{i+1}(v^i)$ and to apply the contraction principle.

We first derive the local equations associated with $\{\varphi_j\}_{j=0}^N$. Note that (4.1) is equivalent to

$$v + k * \mathcal{A}_\#(\cdot, x, D_x)v = f - k * \mathcal{A}_R(\cdot, x, D_x)v,$$

which when multiplied by φ_j becomes

$$\begin{aligned} \varphi_j v + k * \mathcal{A}_\#^j(\cdot, x, D_x)\varphi_j v &= \varphi_j f - k * \varphi_j \mathcal{A}_R(\cdot, x, D_x)v \\ &\quad + k * [\mathcal{A}_\#(\cdot, x, D_x), \varphi_j]v. \end{aligned} \tag{4.4}$$

We freeze the coefficients of the local operator $\mathcal{A}_\#^j(t, x, D_x)$ at the point (T_i, x_j) to get the homogeneous differential operator with constant coefficients $\mathcal{A}^{ij}(D_x) := \mathcal{A}_\#^j(T_i, x_j, D_x)$. Then (4.4) can be written as

$$\begin{aligned} \varphi_j v + k * \mathcal{A}^{ij}(D_x)\varphi_j v &= \varphi_j f - k * \varphi_j \mathcal{A}_R(\cdot, x, D_x)v + k * [\mathcal{A}_\#(\cdot, x, D_x), \varphi_j]v \\ &\quad + k * (\mathcal{A}_\#^j(T_i, x_j, D_x) - \mathcal{A}_\#^j(\cdot, x, D_x))\varphi_j v. \end{aligned} \tag{4.5}$$

Let A^{ij} be the L_p-realization of the differential operator $\mathcal{A}^{ij}(D_x)$. For $l \in \{1, \ldots, M\}$, we put $X_l = L_p([0, T_l]; L_p(\mathbb{R}^n))$ and define the space Ξ_l as the set of all functions $g \in H_p^\alpha([0, T_l]; L_p(\mathbb{R}^n))$ satisfying $g|_{t=0} \in \gamma_0 Z$ in case $\alpha > 1/p$, and $\partial_t g|_{t=0} \in \gamma_1 Z$ in case $\alpha > 1 + 1/p$. Ξ_l endowed with the norm

$$|g|_{\Xi_l} = |g|^{(k,1)}_{H_p^\alpha([0,T_l]; L_p(\mathbb{R}^n))} + \chi_{(\frac{1}{p}, 2)}(\alpha)|g|_{t=0}|_{\gamma_0 Z} + \chi_{(1+\frac{1}{p}, 2)}(\alpha)|\partial_t g|_{t=0}|_{\gamma_1 Z}$$

is a Banach space, cp. Corollary 2.8.2. Recall that for $g \in {}_0\Xi_l := {}_0H^\alpha_p([0,T_l];L_p(\mathbb{R}^n))$, one has $|g|^{(k,1)}_{H^\alpha_p([0,T_l];L_p(\mathbb{R}^n))} = |\mathcal{B}_k g|_{X_l}$, $\mathcal{B}_k$ denoting the inverse convolution operator in X_l associated with the kernel k. For the spaces Z_l, $l = 1,\dots,M$, we choose the norm

$$|w|_{Z_l} = |w|^{(k,1)}_{H^\alpha_p([0,T_l];L_p(\mathbb{R}^n))} + |\nabla^2_x w|_{X_l^{n^2}}.$$

Claim 1: Let $i \in \{0,\dots,M-1\}$, $j \in \{0,\dots,N\}$, and $l \in \{1,\dots,M\}$. Then

$$w + k * A^{ij} w = g, \quad t \in [0,T_l],\, x \in \mathbb{R}^n, \tag{4.6}$$

possesses a unique solution $w =: \mathcal{L}^{ij}_l g$ in the space Z_l if and only if $g \in \Xi_l$. Further, there exists a constant $C > 0$ not depending on i, j, l such that

$$|\mathcal{L}^{ij}_l g|_{Z_l} \le C|g|_{\Xi_l}, \quad \forall g \in {}_0\Xi_l. \tag{4.7}$$

Claim 1 is an immediate consequence of Theorem 3.1.4. Note that after a rotation and a stretch of the spatial coordinates, the elliptic operator A^{ij} becomes the negative Laplacian. The constant C in (4.7) can be selected to be independent of i and j, due to the uniform ellipticity assumption (H3); the independence on l is clear, because in (4.7), functions g in the subspace ${}_0\Xi_l$ are considered, only.

By applying the solution operator $\mathcal{L}^{ij}_{i+1}$ to (4.5) we get

$$(I - S^{ij})\varphi_j v = \mathcal{L}^{ij}_{i+1}(\varphi_j f) + \mathcal{L}^{ij}_{i+1}\, k * \mathcal{C}_j(\cdot,x,D_x)v =: h^{ij}(f,v), \tag{4.8}$$

where

$$S^{ij} w = \mathcal{L}^{ij}_{i+1}\, k * (\mathcal{A}^j_\#(T_i,x_j,D_x) - \mathcal{A}^j_\#(\cdot,x,D_x))w, \tag{4.9}$$

and

$$\mathcal{C}_j(t,x,D_x) = [\mathcal{A}_\#(t,x,D_x),\varphi_j] - \varphi_j \mathcal{A}_R(t,x,D_x). \tag{4.10}$$

One immediately verifies that $S^{ij} \in \mathcal{B}(Z_{i+1})$. Furthermore, S^{ij} enjoys the subsequent properties, which will be shown at the end of this proof.

Claim 2: There exists $\eta_0 > 0$ such that whenever $\eta \le \eta_0$, $i \in \{0,\dots,M-1\}$, and $j \in \{0,\dots,N\}$,

(a) $|S^{ij} w|_{Z_{i+1}} \le \frac{1}{2}|w|_{Z_{i+1}}$ for all $w \in Z_{i+1}(0)$ ($Z_1(0) := {}_0Z_1$);

(b) if $w \in Z_{i+1}(0)$ and $w_0 = (I - S^{ij})w$, then $|w|_{Z_{i+1}} \le 2|w_0|_{Z_{i+1}}$;

(c) for each $g \in \Xi_{i+1}$ and $v \in Z_{i+1}(\mathcal{V}_i^{-1}g)$, the equation $(I - S^{ij})w = h^{ij}(g,v)$ admits a unique solution $w =: (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} g)} h^{ij}(g,v)$ in $Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} g)$. Here $Z_1(\mathcal{V}_0^{-1} g) := \{v \in Z_1 : \partial_t^m v|_{t=0} = \partial_t^m g|_{t=0},\ \text{if}\ \alpha > m + 1/p,\ m = 0,1\}$.

Let $\eta \le \eta_0$. By employing the operators $(I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j v^i)}$ desribed in Claim 2 we infer from (4.8) that

$$\varphi_j v = (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j v^i)} h^{ij}(f,v).$$

Since $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$, we may multiply this equation by ψ_j resulting in

$$\varphi_j v = \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j v^i)} h^{ij}(f, v).$$

Summing over j then yields

$$v = \sum_{j=0}^{N} \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} f)} \mathcal{L}^{ij}_{i+1} (\varphi_j f + k * \mathcal{C}_j(\cdot, x, D_x) v) =: \mathcal{G}(v), \tag{4.11}$$

which is a fixed point equation for $v \in Z_{i+1}(v^i)$.

Due to Claim 2(c), $\mathcal{G}$ is a self-mapping of $Z_{i+1}(v^i)$. Thus the contraction principle is applicable to (4.11), if we can verify that $\mathcal{G}$ is a strict contraction. It turns out that this can be achieved by choosing a yet finer partition $0 = T_0 < T_1 < \ldots < T_{M-1} < T_M = T$, more precisely, by making $\delta := \max_i |T_{i+1} - T_i|$ sufficiently small. The following observation is crucial in this connection.

Suppose $u \in Z_{i+1}(0)$. By causality, it is clear that $\mathcal{B}_k u|_{[0,T_i]} = 0$. So we can write

$$u = k * \mathcal{B}_k u = (k \chi_{[0, T_{i+1} - T_i]}) * \mathcal{B}_k u.$$

Thus, by Young's inequality,

$$|u|_{X_{i+1}} \le |k|_{L_1(0, T_{i+1} - T_i)} |\mathcal{B}_k u|_{X_{i+1}} \le |k|_{L_1(0,\delta)} |u|_{Z_{i+1}} \quad \forall u \in Z_{i+1}(0). \tag{4.12}$$

Another observation concerns the operators $\mathcal{C}_j(t, x, D_x)$. Since those are at most of first order and have bounded coefficients, for each $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|\mathcal{C}_j(t, x, D_x) w|_{X_{i+1}} \le \varepsilon |\nabla_x^2 w|_{X^{n^2}_{i+1}} + C_\varepsilon |w|_{X_{i+1}} \tag{4.13}$$

for all $i = 0, \ldots, M-1$, $j = 0, \ldots, N$, and $w \in L_p([0, T_{i+1}]; H^2_p(\mathbb{R}^n))$.

We are now ready to prove the contractivity of $\mathcal{G}$. Let $v, \bar{v} \in Z_{i+1}(v^i)$. In view of (4.7), Claim 2(b), (4.12), and (4.13), we may estimate

$$\begin{aligned}
|\mathcal{G}(v) - \mathcal{G}(\bar{v})|_{Z_{i+1}} &= |\sum_{j=0}^{N} \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(0)} \mathcal{L}^{ij}_{i+1} k * \mathcal{C}_j(\cdot, x, D_x)(v - \bar{v})|_{Z_{i+1}} \\
&\le C_0 \sum_{j=0}^{N} |\mathcal{L}^{ij}_{i+1} \, k * \mathcal{C}_j(\cdot, x, D_x)(v - \bar{v})|_{Z_{i+1}} \\
&\le C_0 C_1 \sum_{j=0}^{N} |\mathcal{C}_j(\cdot, x, D_x)(v - \bar{v})|_{X_{i+1}} \\
&\le C_0 C_1 \sum_{j=0}^{N} (\varepsilon |\nabla_x^2 (v - \bar{v})|_{X^{n^2}_{i+1}} + C_\varepsilon |(v - \bar{v})|_{X_{i+1}}) \\
&\le C_0 C_1 (N+1) \Big(\varepsilon + C_\varepsilon |k|_{L_1(0,\delta)} \Big) |v - \bar{v}|_{Z_{i+1}} =: \kappa(\varepsilon, \delta) |v - \bar{v}|_{Z_{i+1}},
\end{aligned} \tag{4.14}$$

with C_0, C_1 and N being independent of δ. This shows existence of a *left* inverse $\mathcal{Q}_{i+1}$ for the operator

$$\mathcal{V}_{i+1} = I + k * \mathcal{A}(\cdot, x, D_x) : Z_{i+1}(v^i) \to \Xi_{i+1}(\mathcal{V}_i v^i), \tag{4.15}$$

provided that $\kappa < 1$, that is, if the numbers ε and δ are selected sufficiently small. The symbol $\Xi_{i+1}(\mathcal{V}_i v^i)$ in (4.15) has to be understood like the corresponding one for Z defined in (4.3).

We still have to show that $v^{i+1} = \mathcal{Q}_{i+1} f$ indeed solves (4.1) on $[0, T_{i+1}]$, i.e. that $\mathcal{V}_{i+1}$ is a surjection. To this purpose we define the linear operator $\mathcal{K}_{i+1} : \Xi_{i+1} \to {}_0\Xi_{i+1}$ by

$$\mathcal{K}_{i+1} g = k * \sum_{j=0}^{N} [\mathcal{A}_\#(\cdot, x, D_x), \psi_j](I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} g)} h^{ij}(g, \mathcal{Q}_{i+1} g).$$

The commutators $[\mathcal{A}_\#(t, x, D_x), \psi_j]$ are differential operators of order ≤ 1. Thus for δ sufficiently small, we see that the mapping $g \mapsto f - \mathcal{K}_{i+1} g$ is a strict contraction in the space $\{g \in \Xi_{i+1} : \partial_t^m g|_{t=0} = \partial_t^m f|_{t=0},\ \text{if } \alpha > m + 1/p, m = 0, 1\}$. That means for such δ, there exists $g \in \Xi_{i+1}$ satisfying $g + \mathcal{K}_{i+1} g = f$. We apply now $\mathcal{V}_{\#, i+1} := I + k * \mathcal{A}_\#(\cdot, x, D_x)$ to $v = Q_{i+1} g$ in (4.11) with f replaced by g. This gives

$$\begin{aligned} \mathcal{V}_{\#, i+1} \mathcal{Q}_{i+1} g &= \sum_{j=0}^{N} \mathcal{V}_{\#, i+1} \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}_i^{-1} g)} h^{ij}(g, \mathcal{Q}_{i+1} g) \\ &= \sum_{j=0}^{N} \psi_j (\varphi_j g + k * \mathcal{C}_j(\cdot, x, D_x) \mathcal{Q}_{i+1} g) + \mathcal{K}_{i+1} g. \end{aligned}$$

From $\sum_j \varphi_j = 1$ we infer that $\sum_j [\mathcal{A}_\#(t, x, D_x), \varphi_j] = 0$. Using this, together with the fact that $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$, we see that

$$\sum_{j=0}^{N} \psi_j (\varphi_j g + k * \mathcal{C}_j(\cdot, x, D_x) \mathcal{Q}_{i+1} g) = g - k * \mathcal{A}_R(\cdot, x, D_x) \mathcal{Q}_{i+1} g.$$

Therefore

$$\mathcal{V}_{i+1} \mathcal{Q}_{i+1} g = g + \mathcal{K}_{i+1} g = f. \tag{4.16}$$

Hence $\mathcal{V}_{i+1}$ is surjective, provided that δ is sufficiently small.

Concluding, we have proven that (4.1) admits a unique solution $v \in Z$.

It remains to prove Claim 2. Let $w \in Z_{i+1}(0)$. Thanks to Claim 1 we may estimate

$$\begin{aligned} |S^{ij} w|_{Z_{i+1}} &= |\mathcal{L}^{ij}_{i+1} k * (\mathcal{A}^j_\#(T_i, x_j, D_x) - \mathcal{A}^j_\#(\cdot, x, D_x)) w|_{Z_{i+1}} \\ &\leq C |(\mathcal{A}^j_\#(t, x, D_x) - \mathcal{A}^j_\#(T_i, x_j, D_x)) w|_{X_{i+1}} \\ &\leq C\, |(a^j(\cdot, \cdot) - a(T_i, x_j)) : \nabla_x^2 w|_{L_p([T_i, T_{i+1}]; L_p(\mathbb{R}^n))} \\ &\leq C\eta\, |\nabla_x^2 w|_{X^{n^2}_{i+1}} \\ &\leq C\eta\, |w|_{Z_{i+1}} \leq \tfrac{1}{2} |w|_{Z_{i+1}}, \end{aligned} \tag{4.17}$$

provided that $\eta \leq \eta_0 := 1/2C$. This shows (a). Suppose now that $w \in Z_{i+1}(0)$ and $w_0 = (I - S^{ij}) w$. Then in view of (a)

$$|w|_{Z_{i+1}} \leq |S^{ij} w|_{Z_{i+1}} + |w_0|_{Z_{i+1}} \leq \tfrac{1}{2} |w|_{Z_{i+1}} + |w_0|_{Z_{i+1}}.$$

Hence (b) holds.

Turning to (c), let $g \in \Xi_{i+1}$ and $v \in Z_{i+1}(\mathcal{V}_i^{-1}g)$. Evidently $h^{ij}(g,v) \in Z_{i+1}$. If $i > 0$, we further have

$$\begin{aligned} h^{ij}(g,v)|_{[0,T_i]} &= (\mathcal{L}_{i+1}^{ij}(\varphi_j g + k * \mathcal{C}_j(\cdot, x, D_x)v))|_{[0,T_i]} \\ &= \mathcal{L}_i^{ij}(\varphi_j g|_{[0,T_i]} + k * \mathcal{C}_j(\cdot, x, D_x)\mathcal{V}_i^{-1}g) \qquad \text{(here } * \text{ is meant on } [0,T_i]) \\ &= \varphi_j \mathcal{V}_i^{-1}g - \mathcal{L}_i^{ij} k * (\mathcal{A}_{\#}^j(\cdot, x, D_x) - \mathcal{A}_{\#}^j(T_i, x_j, D_x))\varphi_j \mathcal{V}_i^{-1}g. \end{aligned}$$

Thus, $w \in Z_{i+1}(\varphi_j \mathcal{V}_i^{-1}g)$ implies $F(w) := S^{ij}w + h^{ij}(g,v) \in Z_{i+1}(\varphi_j \mathcal{V}_i^{-1}g)$. In fact,

$$\begin{aligned} (F(w))|_{[0,T_i]} &= \mathcal{L}_i^{ij} k * (\mathcal{A}_{\#}^j(\cdot, x, D_x) - \mathcal{A}_{\#}^j(T_i, x_j, D_x))\varphi_j \mathcal{V}_i^{-1}g + h^{ij}(g,v)|_{[0,T_i]} \\ &= \varphi_j \mathcal{V}_i^{-1}g. \end{aligned}$$

So F is a self-mapping of $Z_{i+1}(\varphi_j \mathcal{V}_i^{-1}g)$. This is true for $i = 0$, too. Compare the initial values of all terms occurring above to see this. In view of (4.17), F is also a strict contraction. Hence the assertion follows by the contraction principle. □

4.2 Half space problems

This section is devoted to parabolic problems of second order in a half space subject to general boundary conditions. In the first subsection we study the case in which the coefficients are constant and the differential operators consist only of their principal parts. Then we shall prove pointwise multiplication properties for the function spaces arising as the natural regularity classes on the boundary. These results allow us to treat also the case of variable coefficients by means of perturbation arguments. This will be done in the last part of this paragraph.

4.2.1 Constant coefficients

Let $J = [0,T]$ and $\mathbb{R}_+^{n+1} = \{x := (x', y) \in \mathbb{R}^{n+1} : x' \in \mathbb{R}^n,\ y > 0\}$. We separately consider the problems

$$\left\{ \begin{aligned} u - k * a : \nabla_x^2 u = f, &\quad t \in J,\ x \in \mathbb{R}_+^{n+1}, \\ u = g, &\quad t \in J,\ x' \in \mathbb{R}^n,\ y = 0, \end{aligned} \right. \tag{4.18}$$

$$\left\{ \begin{aligned} u - k * a : \nabla_x^2 u = f, &\quad t \in J,\ x \in \mathbb{R}_+^{n+1}, \\ -\partial_y u + b \cdot \nabla_{x'} u = h, &\quad t \in J,\ x' \in \mathbb{R}^n,\ y = 0, \end{aligned} \right. \tag{4.19}$$

where k is as in Section 4.1, a is an $(n+1)$-dimensional real matrix, and $b \in \mathbb{R}^n$. We look for unique solutions u in the maximal regularity space

$$Z := H_p^\alpha(J; L_p(\mathbb{R}_+^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}_+^{n+1})).$$

As to (4.18), we have the following result.

Theorem 4.2.1 *Let $1 < p < \infty$, $n \in \mathbb{N}$, and $a \in \mathrm{Sym}\{n+1\}$. Let further $k \in \mathcal{K}^1(\alpha, \theta)$, where $\theta < \pi$ and $\alpha \in (0,2) \setminus \left\{\frac{1}{p},\ \frac{2}{2p-1},\ 1+\frac{1}{p},\ 1+\frac{3}{2p-1}\right\}$. Assume that a is positive definite.*

Then (4.18) has a unique solution in the space Z if and only if the data f and g are subject to the subsequent conditions.

$$\begin{array}{ll} (i) & f \in H_p^{\alpha}(J; L_p(\mathbb{R}_+^{n+1})); \\ (ii) & g \in B_{pp}^{\alpha(1-\frac{1}{2p})}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{2-\frac{1}{p}}(\mathbb{R}^n)); \\ (iii) & f|_{t=0} \in B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R}_+^{n+1}), \ \text{if}\ \alpha > \frac{1}{p}; \\ (iv) & f|_{t=0,\, y=0} = g|_{t=0}, \ \text{if}\ \alpha > \frac{2}{2p-1}; \\ (v) & \partial_t f|_{t=0} \in B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\mathbb{R}_+^{n+1}), \ \text{if}\ \alpha > 1+\frac{1}{p}; \\ (vi) & \partial_t f|_{t=0,\, y=0} = \partial_t g|_{t=0}, \ \text{if}\ \alpha > 1+\frac{3}{2p-1}. \end{array}$$

Proof. By means of a variable transformation of the form $\bar{x} = Q^T \Lambda Q x$, where Q is a rotation matrix and Λ is diagonal with $\Lambda_{ii} = 1/\sqrt{\lambda_i}$ ($\lambda_1, \ldots, \lambda_{n+1}$ denoting the positive eigenvalues of the matrix a), problem (4.18) can be reduced to a problem of the same structure but with $a = I_{n+1}$. The assertion follows then from Theorem 3.5.1 applied to $X = L_p(\mathbb{R}^n)$ and the operator $A = -\Delta_{x'}$, which belongs to $\mathcal{BIP}(X)$ and has power angle 0, and from the fact that all function spaces occurring in that theorem are preserved under the above variable transformation. □

The corresponding result for (4.19) reads

Theorem 4.2.2 *Let $1 < p < \infty$, $n \in \mathbb{N}$, $a \in \mathrm{Sym}\{n+1\}$, and $b \in \mathbb{R}^n$. Suppose $k \in \mathcal{K}^1(\alpha, \theta)$, where $\theta < \pi$ and $\alpha \in (0,2) \setminus \left\{\frac{1}{p},\ \frac{2}{p-1},\ 1+\frac{1}{p}\right\}$. Assume that a is positive definite.*

Then (4.19) possesses a unique solution in the space Z if and only if the functions f and h satisfy the following conditions.

$$\begin{array}{ll} (i) & f \in H_p^{\alpha}(J; L_p(\mathbb{R}_+^{n+1})); \\ (ii) & h \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)); \\ (iii) & f|_{t=0} \in B_{pp}^{2-\frac{2}{p\alpha}}(\mathbb{R}_+^{n+1}), \ \text{if}\ \alpha > \frac{1}{p}; \\ (iv) & -\partial_y f|_{t=0,\, y=0} + b \cdot \nabla_{x'} f|_{t=0, y=0} = h|_{t=0}, \ \text{if}\ \alpha > \frac{2}{p-1}; \\ (v) & \partial_t f|_{t=0} \in B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\mathbb{R}_+^{n+1}), \ \text{if}\ \alpha > 1+\frac{1}{p}. \end{array}$$

Proof. Use the variable transformation described in the proof of Theorem 4.2.1 and normalize the coefficient in front of the normal derivative to reduce (4.19) to a problem of the same structure with $a = I_{n+1}$. The assertion is then a consequence of Theorem 3.5.1 applied to $X = L_p(\mathbb{R}^n)$, $A = -\Delta_{x'}$, and $D = b \cdot \nabla_{x'}$ ($b \in \mathbb{R}^n$). Note that $D \in \mathcal{BIP}(\overline{\mathcal{R}(D)})$ with power angle $\theta_D \le \pi/2$ (cp. Prüss [65, Section 3]) and thus $\theta_D < \pi - \theta/2 = \pi - \max\{\theta, \theta_A\}/2$ showing the second angle condition in Theorem 3.5.1. □

4.2.2 Pointwise multiplication

Let $1 < p < \infty$, $s_1, s_2 \in (0,1)$, $0 < T_2$, $0 \le T_1 \le T_2$, and Ω be an arbitrary domain in $\mathbb{R}^n$. We are interested in pointwise multipliers for the intersection space

$$Y^{T_2} := Y_1^{T_2} \cap Y_2^{T_2} := B_{pp}^{s_1}([0;T_2]; L_p(\Omega)) \cap L_p([0,T_2]; B_{pp}^{s_2}(\Omega))$$

endowed with the norm $|\cdot|_{Y^{T_2}}$ defined by

$$|\cdot|_{Y^T} = |\cdot|_{L_p([0,T]\times\Omega)} + [\cdot]_{Y_1^T} + [\cdot]_{Y_2^T}, \qquad T > 0 \tag{4.20}$$

where

$$[f]_{Y_1^T} = (\int_0^T \int_0^T \int_\Omega \frac{|f(t,x)-f(\tau,x)|^p}{|t-\tau|^{1+s_1 p}}\, dx\, d\tau\, dt)^{\frac{1}{p}}, \tag{4.21}$$

$$[f]_{Y_2^T} = (\int_0^T \int_\Omega \int_\Omega \frac{|f(t,x)-f(t,y)|^p}{|x-y|^{n+s_2 p}}\, dx\, dy\, dt)^{\frac{1}{p}}. \tag{4.22}$$

We consider first products with bounded factors. Suppose that $m, f \in Y^{T_2} \cap L_\infty([0,T_2]\times\Omega) =: Y^{T_2} \cap L_\infty$. In what is to follow we estimate $|mf|_{Y^{T_2}}$, using among other things terms referring to norms of functions on $[0,T_1]$ and $[T_1,T_2]$, respectively. Note that in case $T_1 = 0$ respectively $T_1 = T_2$ these terms have to be regarded as zero.

It is readily seen that

$$\begin{aligned} |mf|_{L_p([0,T_2]\times\Omega)} &\leq |mf|_{L_p([0,T_1]\times\Omega)} + |mf|_{L_p([T_1,T_2]\times\Omega)} \\ &\leq |m|_{L_\infty([0,T_1]\times\Omega)}|f|_{L_p([0,T_1]\times\Omega)} + |m|_{L_\infty([T_1,T_2]\times\Omega)}|f|_{L_p([0,T_2]\times\Omega)}. \end{aligned}$$

By employing the trivial identity

$$m(t,x)f(t,x) - m(t,y)f(t,y) = m(t,x)(f(t,x)-f(t,y)) + (m(t,x)-m(t,y))f(t,y)$$

and Minkowski's inequality, we further get

$$\begin{aligned} [mf]_{Y_2^{T_2}} \leq\ & |m|_{L_\infty([0,T_1]\times\Omega)}[f]_{Y_2^{T_1}} + [m]_{Y_2^{T_1}}|f|_{L_\infty([0,T_1]\times\Omega)} \\ & + |m|_{L_\infty([T_1,T_2]\times\Omega)}[f]_{Y_2^{T_2}} + [m]_{L_p([T_1,T_2];B_{pp}^{s_2}(\Omega))}|f|_{L_\infty([0,T_2]\times\Omega)}. \end{aligned}$$

Proceeding similarly as above, we also obtain

$$\begin{aligned} [mf]_{Y_1^{T_2}} &= (\int_0^{T_2}\int_0^{T_2}\int_\Omega \frac{|m(t,x)f(t,x)-m(\tau,x)f(\tau,x)|^p}{|t-\tau|^{1+s_1 p}}\, dx\, d\tau\, dt)^{\frac{1}{p}} \\ &\leq (\int_0^{T_1}\int_0^{T_1}\int_\Omega \cdots)^{\frac{1}{p}} + 2(\int_{T_1}^{T_2}\int_0^{T_2}\int_\Omega \cdots)^{\frac{1}{p}} \\ &\leq |m|_{L_\infty([0,T_1]\times\Omega)}[f]_{Y_1^{T_1}} + [m]_{Y_1^{T_1}}|f|_{L_\infty([0,T_1]\times\Omega)} \\ &\quad +2(|m|_{L_\infty([T_1,T_2]\times\Omega)}[f]_{Y_1^{T_2}} + [m]_{Y_1^{T_1,T_2}}|f|_{L_\infty([0,T_2]\times\Omega)}), \end{aligned} \tag{4.23}$$

where we have set

$$[m]_{Y_1^{T_1,T_2}} = (\int_{T_1}^{T_2}\int_0^{T_2}\int_\Omega \frac{|m(t,x)-m(\tau,x)|^p}{|t-\tau|^{1+s_1 p}}\, dx\, d\tau\, dt)^{\frac{1}{p}}. \tag{4.24}$$

So with $|\cdot|_{Y^{T_i}\cap L_\infty} = |\cdot|_{Y^{T_i}} + |\cdot|_{L_\infty([0,T_i]\times\Omega)}$, $i = 1,2$, and

$$|m|_{Y^{T_1,T_2}\cap L_\infty} = |m|_{L_\infty([T_1,T_2]\times\Omega)} + [m]_{Y_1^{T_1,T_2}} + [m]_{L_p([T_1,T_2];B_{pp}^{s_2}(\Omega))}, \tag{4.25}$$

we have thus proved

Lemma 4.2.1 *Let $1 < p < \infty$, $s_1, s_2 \in (0,1)$, $0 < T_2$, $0 \le T_1 \le T_2$, and Ω be an arbitrary domain in $\mathbb{R}^n$. Then*

$$|mf|_{Y^{T_2}} \le |m|_{Y^{T_1}\cap L_\infty}|f|_{Y^{T_1}\cap L_\infty} + 2|m|_{Y^{T_1,\,T_2}\cap L_\infty}|f|_{Y^{T_2}\cap L_\infty} \tag{4.26}$$

for all $m,\, f \in Y^{T_2} \cap L_\infty([0,T_2]\times\Omega)$.

We turn now to products where one factor might be unbounded. Such a constellation arises for example when $s_1 < 1/p$. We confine ourselves to the case $\Omega = \mathbb{R}^n$. By means of extension and restriction, the subsequent multiplication property can be transferred to domains with sufficiently smooth boundary.

Suppose $f \in Y^{T_2}$ and $m \in C^{r_1}([0,T_2];C(\mathbb{R}^n)) \cap C([0,T_2];C^{r_2}(\mathbb{R}^n)) =: M^{T_2}$ for some $s_i < r_i < 1$, $i = 1,2$. Letting J be a subinterval of $[0,T_2]$, we then estimate

$$\begin{aligned}[][mf]_{L_p(J;B^{s_2}_{pp}(\mathbb{R}^n))} &= \Big(\int_J\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|m(t,x)f(t,x)-m(t,y)f(t,y)|^p}{|x-y|^{n+s_2p}}\,dx\,dy\,dt\Big)^{\frac1p}\\ &\le |m|_{L_\infty(J\times\Omega)}[f]_{L_p(J;B^{s_2}_{pp}(\mathbb{R}^n))} + I(m,f),\end{aligned}$$

where

$$\begin{aligned} I(m,f) &\le \Big(\int_J\int_{\mathbb{R}^n}\int_{B_1(y)}\frac{|m(t,x)-m(t,y)|^p|f(t,y)|^p}{|x-y|^{n+s_2p}}\,dx\,dy\,dt\Big)^{\frac1p} + \Big(\int_J\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B_1(y)}\cdots\Big)^{\frac1p}\\ &=: I_1(m,f) + I_2(m,f).\end{aligned}$$

We put $[m]_{C(J;\,C^{r_2}(\mathbb{R}^n))} = \sup_{t\in J,\,x,\,y\in\mathbb{R}^n}|m(t,x)-m(t,y)|\,|x-y|^{-r_2}$. By hypothesis $1 - s_2/r_2 > 0$. If $\eta \in [0, 1-s_2/r_2)$, then $(1-\eta)r_2 - s_2 > 0$, and we obtain

$$\begin{aligned} I_1(m,f) &\le (2|m|_{L_\infty(J\times\mathbb{R}^n)})^\eta [m]^{1-\eta}_{C(J;\,C^{r_2}(\mathbb{R}^n))}\Big(\int_J\int_{\mathbb{R}^n}\int_{B_1(y)}\frac{|f(t,y)|^p\,dx\,dy\,dt}{|x-y|^{n-((1-\eta)r_2-s_2)p}}\Big)^{\frac1p}\\ &\le C(p,r_2,s_2,n)|m|^\eta_{L_\infty(J\times\mathbb{R}^n)}[m]^{1-\eta}_{C(J;\,C^{r_2}(\mathbb{R}^n))}|f|_{L_p(J\times\mathbb{R}^n)}.\end{aligned}$$

Further,

$$\begin{aligned} I_2(m,f) &\le 2|m|_{L_\infty(J\times\mathbb{R}^n)}\Big(\int_J\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B_1(y)}\frac{|f(t,y)|^p\,dx\,dy\,dt}{|x-y|^{n+s_2p}}\Big)^{\frac1p}\\ &\le C(p,s_2,n)|m|_{L_\infty(J\times\mathbb{R}^n)}|f|_{L_p(J\times\mathbb{R}^n)}.\end{aligned}$$

Hence

$$\begin{aligned}[][mf]_{Y_2^{T_2}} &\le [mf]_{L_p([0,T_1];B^{s_2}_{pp}(\mathbb{R}^n))} + [mf]_{L_p([T_1,T_2];B^{s_2}_{pp}(\mathbb{R}^n))}\\ &\le C(|m|_{L_\infty([0,T_1]\times\mathbb{R}^n)} + [m]_{C([0,T_1];\,C^{r_2}(\mathbb{R}^n))})(|f|_{L_p([0,T_1]\times\mathbb{R}^n)} + [f]_{Y_2^{T_1}})\\ &\quad + |m|_{L_\infty([T_1,T_2]\times\mathbb{R}^n)}\Big([f]_{Y_2^{T_2}} + C(1+[m]_{C([T_1,T_2];\,C^{r_2}(\mathbb{R}^n))})|f|_{L_p([0,T_2]\times\mathbb{R}^n)}\Big).\end{aligned}$$

Let now $[m]_{C^{r_1}(J;C(\mathbb{R}^n))} = \sup_{t,\,\tau\in J,\,x\in\mathbb{R}^n}|m(t,x)-m(\tau,x)|\,|t-\tau|^{-r_1}$. To estimate $[mf]_{Y_1^{T_2}}$, we take (4.23) with $\Omega = \mathbb{R}^n$ as starting point, apply Fubini's theorem, and use the same estimation techniques as for $[mf]_{Y_2^{T_2}}$ to the result

$$\begin{aligned}[][mf]_{Y_1^{T_2}} &\le |m|_{L_\infty([0,T_1]\times\mathbb{R}^n)}[f]_{Y_1^{T_1}} + C_1[m]_{C^{r_1}([0,T_1];\,C(\mathbb{R}^n))}|f|_{L_p([0,T_1]\times\mathbb{R}^n)}\\ &\quad + 2|m|_{L_\infty([T_1,T_2]\times\mathbb{R}^n)}([f]_{Y_1^{T_2}} + C_2[m]_{C^{r_1}([T_1,T_2];\,C(\mathbb{R}^n))}|f|_{L_p([0,T_2]\times\mathbb{R}^n)}).\end{aligned}$$

We remark that the constant C_2 stems from finding an upper bound for the integral $\int_{T_1}^{T_2} |t-\tau|^{-\beta}\,dt$, $\tau \in [0,T_2]$, where $\beta < 1$ is a fixed number. Thus if T_1 and T_2 vary within the set $[0,T]$, C_2 can be chosen to be independent of those numbers.

Set

$$|m|_{M^{T_1}} = |m|_{L_\infty([0,T_1]\times\mathbb{R}^n)} + [m]_{C^{r_1}([0,T_1];\,C(\mathbb{R}^n))} + [m]_{C([0,T_1];\,C^{r_2}(\mathbb{R}^n))}$$

and

$$[m]_{M^{T_1,T_2}} = [m]_{C^{r_1}([T_1,T_2];\,C(\mathbb{R}^n))} + [m]_{C([T_1,T_2];\,C^{r_2}(\mathbb{R}^n))}.$$

Then our observations can be summarized as follows.

Lemma 4.2.2 *Suppose that $1 < p < \infty$, $0 < T$, $0 \le T_1 \le T_2 \le T$, and $0 < s_i < r_i < 1$ for $i = 1,2$. Let $Y^{T_2} = B^{s_1}_{pp}([0;T_2];L_p(\mathbb{R}^n)) \cap L_p([0,T_2];B^{s_2}_{pp}(\mathbb{R}^n))$, and $M^{T_2} = C^{r_1}([0,T_2];C(\mathbb{R}^n)) \cap C([0,T_2];C^{r_2}(\mathbb{R}^n))$. Then there exists a constant $C > 0$ not depending on T_1 and T_2, such that*

$$|mf|_{Y^{T_2}} \le C\Big(|m|_{M^{T_1}}|f|_{Y^{T_1}} + |m|_{L_\infty([T_1,T_2]\times\mathbb{R}^n)}(1+[m]_{M^{T_1,T_2}})|f|_{Y^{T_2}}\Big) \tag{4.27}$$

for all $m \in M^{T_2}$ and $f \in Y^{T_2}$.

4.2.3 Variable coefficients

The goal of this subparagraph is to extend the results proven in Subsection 4.2.1 to variable coefficients. Besides we no longer stick to differential operators consisting only of the principal part but consider general operators of second (Volterra equation) respectively first order (boundary condition).

To start with, recall the notation $\mathbb{R}^{n+1}_+ = \{(x',y) \in \mathbb{R}^{n+1} : x' \in \mathbb{R}^n,\ y > 0\}$. Define the operators $\mathcal{A}(t,x,D_x)$ and $\mathcal{B}(t,x',D_x)$ by

$$\mathcal{A}(t,x,D_x) = -a(t,x) : \nabla_x^2 + a_1(t,x)\cdot\nabla_x + a_0(t,x),\ t \in J,\ x \in \mathbb{R}^{n+1}_+,$$

and

$$\mathcal{B}(t,x',D_x) = -\partial_y + b(t,x')\cdot\nabla_{x'} + b_0(t,x'),\ t \in J,\ x' \in \mathbb{R}^n, \tag{4.28}$$

respectively. We are concerned with the two separate problems

$$\begin{cases} v + k * \mathcal{A}(\cdot,x,D_x)v = f, & t \in J,\ x \in \mathbb{R}^{n+1}_+, \\ v = g, & t \in J,\ x' \in \mathbb{R}^n,\ y = 0 \end{cases}, \tag{4.29}$$

$$\begin{cases} v + k * \mathcal{A}(\cdot,x,D_x)v = f, & t \in J,\ x \in \mathbb{R}^{n+1}_+, \\ \mathcal{B}(t,x',D_x)v = h, & t \in J,\ x' \in \mathbb{R}^n,\ y = 0 \end{cases}, \tag{4.30}$$

and seek, as in Subsection 4.2.1, unique solutions in the regularity class

$$Z = H^\alpha_p(J;L_p(\mathbb{R}^{n+1}_+)) \cap L_p(J;H^2_p(\mathbb{R}^{n+1}_+)).$$

Concerning (4.29), we have the following result.

Theorem 4.2.3 *Let $1 < p < \infty$, $J = [0,T]$, $n \in \mathbb{N}$, and $k \in \mathcal{K}^1(\alpha,\theta)$, where $\theta < \pi$, and $\alpha \in (0,2) \setminus \left\{\frac{1}{p},\ \frac{2}{2p-1},\ 1+\frac{1}{p},\ 1+\frac{3}{2p-1}\right\}$. Suppose $a \in C_{ul}(J \times \overline{\mathbb{R}^{n+1}_+}, \mathrm{Sym}\{n+1\})$, $a_1 \in L_\infty(J\times\mathbb{R}^{n+1}_+,\mathbb{R}^{n+1})$, $a_0 \in L_\infty(J\times\mathbb{R}^{n+1}_+)$, and assume further that there exists $c_0 > 0$ such that $a(t,x)\xi\cdot\xi \ge c_0|\xi|^2$, $t \in J$, $x \in \mathbb{R}^{n+1}$, $\xi \in \mathbb{R}^{n+1}$.*

Then (4.29) has a unique solution in the space Z if and only if the data f and g are subject to the conditions (i)-(vi) stated in Theorem 4.2.1.

In order to formulate the corresponding result for (4.30), we put $s_1 = \alpha(\frac{1}{2} - \frac{1}{2p})$, $s_2 = 1 - \frac{1}{p}$, as well as $Y = Y_1 \cap Y_2$, where $Y_1 = B^{s_1}_{pp}(J; L_p(\mathbb{R}^n))$ and $Y_2 = L_p(J; B^{s_2}_{pp}(\mathbb{R}^n))$.

Theorem 4.2.4 *Let $1 < p < \infty$, $J = [0,T]$, $n \in \mathbb{N}$, and $k \in \mathcal{K}^1(\alpha,\theta)$, where $\theta < \pi$, and $\alpha \in (0,2) \setminus \left\{\frac{1}{p},\ \frac{2}{p-1},\ 1 + \frac{1}{p}\right\}$. Let further a, a_1, and a_0 be as in Theorem 4.2.3. Assume $b \in C_{ul}(J \times \mathbb{R}^n)$, as well as $(b, b_0) \in Y^{n+1}$, if $p > n + 1 + 2/\alpha$, and $(b, b_0) \in (C^{r_1}(J; C(\mathbb{R}^n)) \cap C(J; C^{r_2}(\mathbb{R}^n)))^{n+1}$ with some $r_i > s_i$, $i = 1, 2$, otherwise.*

Then (4.30) has a unique solution in the space Z if and only if the data f and h satisfy the conditions (i)-(iii) and (v) stated in Theorem 4.2.2, as well as

$$(iv) \quad \mathcal{B}(0, x', D_x) f|_{t=0,\, y=0} = h|_{t=0}, \ \text{if}\ \alpha > \tfrac{2}{p-1}.$$

We only prove Theorem 4.2.3. Theorem 4.2.4 can be established by means of the same techniques; the proof is even much simpler, since one does not have to consider perturbations on the boundary.

Proof of Thm. 4.2.4. We begin with the "only if" part. Suppose $v \in Z$ is a solution of (4.30). By the regularity assumptions on $\mathcal{A}(t, x, D_x)$, it is evident that $\mathcal{A}(t, x, D_x)v \in L_p(J; L_p(\mathbb{R}^{n+1}_+))$, which in turn implies $k * \mathcal{A}(t, x, D_x)v \in {}_0H^\alpha_p(J; L_p(\mathbb{R}^{n+1}_+))$, according to Corollary 2.8.1. Thus $f \in H^\alpha_p(J; L_p(\mathbb{R}^{n+1}_+))$. Since $\partial^j_t v|_{t=0} = \partial^j_t f|_{t=0}$ in case $\alpha > j + 1/p$, $j = 0, 1$, Theorem 4.2.1 shows (iii) and (v). Another consequence of that theorem is that $v|_{y=0} \in Y$. Also, $\nabla_x v|_{y=0} \in Y^{n+1}$, by Theorem 4.2.2. Since $p > n + 1 + 2/\alpha$ entails $Y \hookrightarrow C(J \times \mathbb{R}^n)$, the coefficients b and b_0 exhibit just the regularity needed for the application of Lemma 4.2.1 and Lemma 4.2.2, respectively. Hence condition (ii) is necessary. Last but not least, the compatibility condition (iv) follows from the regularity assumptions on b, b_0, and the embedding $Y \hookrightarrow C(J; L_p(\mathbb{R}^n))$, which is valid whenever $\alpha > 2/(p-1)$, cp. the proof of Theorem 3.5.2 and that of Theorem 4.2.2.

We come now to the sufficiency part. Although problem (4.30) is far more complicated than (4.1), the strategy we are going to follow is basically the same as in the proof of Theorem 4.1.1. Unless stated otherwise and apart from trivial modifications, the notation used here is adopted from that proof. Note that we decompose the boundary differential operator as $\mathcal{B}(t, x', D_x) = \mathcal{B}_\#(t, x', D_x) + b_0(t, x')$.

Given $\eta > 0$, the assumptions on a and b permit us to choose a large ball $B_{r_0}(0) \subset \mathbb{R}^{n+1}$ such that

$$|a(t,x) - a(t,\infty)|_{\mathcal{B}(\mathbb{R}^{(n+1)^2})} \le \frac{\eta}{2}, \quad \text{for all } t \in J,\ x \in \overline{\mathbb{R}^{n+1}_+}, |x| \ge r_0,$$

as well as

$$|b(t,x') - b(t,\infty)|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R})} \le \frac{\eta}{2}, \quad \text{for all } t \in J,\ x' \in \mathbb{R}^n, |x'| \ge r_0.$$

Put $U_0 = \{x \in \mathbb{R}^{n+1} : |x| > r_0\}$. We can cover $\overline{B_{r_0}(0)} \cap \overline{\mathbb{R}^{n+1}_+}$ by finitely many balls $U_j = B_{r_j}(x_j)$, $j = 1, \ldots, N_b + N$, and select a partition $0 =: T_0 < T_1 < \ldots < T_{M-1} < T_M := T$ such that for all $i = 0, \ldots, M-1$, the following conditions are fulfilled:

$$\exists x'_j \in \mathbb{R}^n : B_{r_j}(x_j) = B_{r_j}((x'_j, 0)),\ 1 \le j \le N_b;$$

$$B_{r_j}(x_j) \cap \{(x', y) \in \mathbb{R}^{n+1} : y = 0\} = \emptyset,\ N_b + 1 \le j \le N_b + N;$$

$$|a(t,x) - a(T_i, x_j)|_{\mathcal{B}(\mathbb{R}^{(n+1)^2})} \le \eta,\ t \in [T_i, T_{i+1}],\ x \in \overline{\mathbb{R}^{n+1}_+} \cap \overline{B_{r_j}(x_j)},\ 1 \le j \le N_b + N;$$

$$|b(t,x') - b(T_i, x'_j)|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R})} \le \eta,\ t \in [T_i, T_{i+1}],\ (x', 0) \in \overline{B_{r_j}(x_j)},\ 1 \le j \le N_b;$$

$$|a(t,\infty) - a(T_i,\infty)|_{\mathcal{B}(\mathbb{R}^{(n+1)^2})}, |b(t,\infty) - b(T_i,\infty)|_{\mathcal{B}(\mathbb{R}^n,\mathbb{R})} \le \frac{\eta}{2},\ t \in [T_i, T_{i+1}].$$

One can then construct functions $a^j \in C(J \times \overline{\mathbb{R}^{n+1}_+})$, $0 \le j \le N_b$, as well as $a^j \in C(J \times \mathbb{R}^{n+1})$, $N_b + 1 \le j \le N_b + N$, which are subject to $a^0(t,x) = a(t,x)$, $t \in J$, $x \in \overline{U_0} \cap \overline{\mathbb{R}^{n+1}_+}$; $a^j(t,x) = a(t,x)$, $t \in J$, $x \in \overline{U_j} \cap \overline{\mathbb{R}^{n+1}_+}$, $j > 0$; and

$$|a^j(t,x) - a(T_i, x_j)|_{\mathcal{B}(\mathbb{R}^{(n+1)^2})} \le \eta, \quad t \in [T_i, T_{i+1}],\ x \in \mathbb{R}^{n+1}_+ \text{ resp. } \mathbb{R}^{n+1},$$

for all $i = 0, \ldots, M-1$ and $j = 0, \ldots, N_b + N$, where $x_0 = \infty$. Similarly, one finds functions b^j, $0 \le j \le N_b$, defined on $J \times \mathbb{R}^n$ enjoying the same regularity properties as b, and which are such that b^j equals b on $J \times \{x' \in \mathbb{R}^n : (x', 0) \in U_j\}$ as well as

$$|b^j(t,x') - b(T_i, x'_j)|_{\mathcal{B}(\mathbb{R}^n, \mathbb{R})} \le \eta, \quad t \in [T_i, T_{i+1}],\ x' \in \mathbb{R}^n,$$

for all $i = 0, \ldots, M-1$ and $j = 0, \ldots, N_b$. With these functions we define spatially local operators by $\mathcal{A}^j_{\#}(t,x,D_x) = -a^j(t,x) : \nabla^2_x$, $0 \le j \le N_b + N$, and $\mathcal{B}^j_{\#}(t,x',D_x) = -\partial_y + b^j(t,x') \cdot \nabla_{x'}$, $0 \le j \le N_b$. Further, let $\mathcal{A}^{ij}(D_x) = \mathcal{A}^j_{\#}(T_i, x_j, D_x)$ and $\mathcal{B}^{ij}(D_x) = \mathcal{B}^j_{\#}(T_i, x'_j, D_x)$.

We next choose a partition of unity $\{\varphi_j\}_{j=0}^{N_b+N} \subset C^\infty(\mathbb{R}^{n+1})$ which has the properties $\sum_{j=0}^{N_b+N} \varphi_j \equiv 1$ on $\overline{\mathbb{R}^{n+1}_+}$, $0 \le \varphi_j(x) \le 1$ and $\operatorname{supp} \varphi_j \subset U_j$. Fix also a family $\{\psi_j\}_{j=0}^{N_b+N} \subset C^\infty(\mathbb{R}^{n+1})$ that satisfies $\psi_j \equiv 1$ on an open set V_j containing $\operatorname{supp} \varphi_j$, and $\operatorname{supp} \psi_j \subset U_j$. To derive the local equations associated with $\{\varphi_j\}_{j=0}^{N_b+N}$ we multiply both equations in (4.30) by φ_j. For the Volterra equation, this again results in (4.5). Concerning the boundary, for $j = N_b + 1, \ldots, N_b + N$ a condition does not appear, whereas for $j = 0, \ldots, N_b$ we obtain

$$\begin{aligned} \mathcal{B}^{ij}(D_x)\varphi_j v \quad &= \quad \varphi_j h - \varphi_j b_0(\cdot, x')v + (\mathcal{B}_{\#}(t,x',D_x)\varphi_j)v \\ &\quad + (\mathcal{B}^j_{\#}(T_i, x'_j, D_x) - \mathcal{B}^j_{\#}(t,x',D_x))\varphi_j v. \end{aligned} \tag{4.31}$$

Rephrasing, for $j = N_b + 1, \ldots, N_b + N$ we encounter the *full* space problems which were already considered in the proof of Theorem 4.1.1 and which, with η being sufficiently small, gave rise to the equations

$$\varphi_j v = \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j v^i)} h^{ij}(f, v). \tag{4.32}$$

In case $0 \le j \le N_b$ we are led to *half* space problems of the form

$$\left\{ \begin{array}{rl} w + k * \mathcal{A}^{ij}(D_x) w = g, & t \in [0, T_l],\ x \in \mathbb{R}^{n+1}_+, \\ -\partial_y w + b(T_i, x'_j) \cdot \nabla_{x'} w = \phi, & t \in [0, T_l],\ x' \in \mathbb{R}^n,\ y = 0, \end{array} \right. \tag{4.33}$$

where $1 \le i+1,\ l \le M$. By Theorem 4.2.2, (4.33) has a unique solution $w =: \mathcal{L}^{ij}_{H,l}(g, \phi)^T$ in the space Z_l if and only if g and ϕ are subject to the conditions (i)-(v) enunciated therein with $J = [0, T_l]$. Let us write for the latter $(g, \phi) \in \Xi_{H,l}$. Moreover, an *a priori* estimate for $|w|_{Z_l}$ in terms of the norms of the data g and ϕ holds, and the constant in this estimate is independent of i, j, l, if the data belong to the spaces with vanishing traces at $t = 0$. Note that for Z_l, we use as in the proof of Theorem 4.1.1 the norm

$$|w|_{Z_l} = |w|^{(k,1)}_{H^\alpha_p([0,T_l]; L_p(\mathbb{R}^{n+1}_+))} + |\nabla^2_x w|_{X_l^{(n+1)^2}}.$$

Here $X_l = L_p([0, T_l] \times \mathbb{R}^{n+1}_+)$. The corresponding norm for

$$Y_l := B^{\alpha(\frac{1}{2} - \frac{1}{2p})}_{pp}([0, T_l]; L_p(\mathbb{R}^n)) \cap L_p([0, T_l]; B^{1-\frac{1}{p}}_{pp}(\mathbb{R}^n)),$$

the natural regularity space of ϕ in (4.33), is equivalent to

$$|\cdot|_{Y_l} = |\cdot|_{L_p([0,T_l]\times\mathbb{R}^n)} + [\cdot]_{Y_1^{T_l}} + [\cdot]_{Y_2^{T_l}},$$

see (4.21) and (4.22) for the definition of $[\cdot]_{Y_1^{T_l}}$, $[\cdot]_{Y_2^{T_l}}$.

Taking now for (g,ϕ) in (4.33) the right-hand sides of (4.5) and (4.31), application of the solution operator $\mathcal{L}^{ij}_{H,\,i+1}$ yields

$$(I - S^{ij}_H)\varphi_j v = \mathcal{L}^{ij}_{H,\,i+1}\begin{pmatrix} \varphi_j f + k * \mathcal{C}_j(\cdot,x,D_x)v \\ \varphi_j h + \gamma_y \mathcal{C}_{H,\,j}(\cdot,x')v \end{pmatrix} =: h^{ij}_H(f,h,v), \tag{4.34}$$

where

$$S^{ij}_H w = \mathcal{L}^{ij}_{H,\,i+1}\begin{pmatrix} k * (\mathcal{A}^j_\#(T_i,x_j,D_x) - \mathcal{A}^j_\#(\cdot,x,D_x))w \\ \gamma_y(\mathcal{B}^j_\#(T_i,x'_j,D_x) - \mathcal{B}^j_\#(\cdot,x',D_x))w \end{pmatrix},$$

and

$$\mathcal{C}_{H,\,j}(t,x) = \mathcal{B}_\#(t,x',D_x)\varphi_j - \varphi_j b_0, \tag{4.35}$$

γ_y denoting the trace operator at $y=0$. One can then show an analogue to Claim 2 (cf. the proof of Theorem 4.1.1) asserting in particular existence of a small $\eta_0 > 0$ which is such that whenever $\delta = \max_i |T_{i+1} - T_i|$, $\eta \le \eta_0$, the equation $(I - S^{ij}_H)w = h^{ij}_H(f,h,v)$ admits a unique solution

$$w =: (I - S^{ij}_H)|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(f,h))} h^{ij}_H(f,h,v)$$

in $Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(f,\,h))$ for all $(f,h) \in \Xi_{H,\,i+1}$, $v \in Z_{i+1}(\mathcal{V}^{-1}_{H,\,i}(f,\,h))$, $i = 0,\dots,M-1$, and $j = 0,\dots,N_b$. Here $\mathcal{V}^{-1}_{H,\,i}(f,\,h)$, $i \ge 1$, refers to the solution of (4.30) on the time-interval $[0,T_i]$, which is already known in the $(i+1)$th time step. Further, $Z_1(\varphi_j \mathcal{V}^{-1}_{H,\,0}(f,\,h)) := \{\varphi_j w : w \in Z_1,\ \partial_t^m w|_{t=0} = \partial_t^m f|_{t=0},\ \text{if } \alpha > m + 1/p,\ m = 0,1\}$.

The proof of these properties is similar to that of Claim 2 above, which is why we only consider the part that involves estimates on the boundary.

Let $w \in Z_{i+1}(0)$. If $p > n+1+2/\alpha$, that is $Y \hookrightarrow BUC([0,T]\times\mathbb{R}^n)$, Lemma 4.2.1 yields

$$\begin{aligned} |\gamma_y(\mathcal{B}^j_\#(T_i,x'_j,D_x) - \mathcal{B}^j_\#(t,x',D_x))w|_{Y_{i+1}} &= |(b(T_i,x'_j) - b^j(\cdot,\cdot))\cdot\nabla_{x'}\gamma_y w|_{Y_{i+1}} \\ &\le C|b(T_i,x'_j) - b^j(\cdot,\cdot)|_{(Y^{T_i,\,T_{i+1}}\cap L_\infty)^n}|\,|\nabla_{x'}\gamma_y w|_{(Y_{i+1}\cap L_\infty)^n} \\ &\le C_1|b(T_i,x'_j) - b^j(\cdot,\cdot)|_{(Y^{T_i,\,T_{i+1}}\cap L_\infty)^n}|w|_{Z_{i+1}} =: \kappa_1|w|_{Z_{i+1}}, \end{aligned}$$

where the constant C_1 does not depend on i, j. Similarly, if $p \le n+1+2/\alpha$, Lemma 4.2.2 shows that

$$\begin{aligned} &|\gamma_y(\mathcal{B}^j_\#(T_i,x'_j,D_x) - \mathcal{B}^j_\#(t,x',D_x))w|_{Y_{i+1}} \le \\ &\le C|(b(T_i,x'_j) - b^j(\cdot,\cdot))|_{L_\infty([T_i,T_{i+1}]\times\mathbb{R}^n)^n}(1 + [b(T_i,x'_j) - b^j(\cdot,\cdot)]_{(M^{T_i,\,T_{i+1}})^n})|w|_{Z_{i+1}} \\ &\le C_2\eta|w|_{Z_{i+1}} =: \kappa_2|w|_{Z_{i+1}}, \end{aligned}$$

again with a constant not depending on i, j. In view of (4.24) and (4.25), it is clear that κ_1, κ_2 tend to zero if η and δ do so. This, together with Theorem 4.2.2 and an estimate analogous to (4.17), shows that $S_H^{ij} : Z_{i+1}(\varphi_j \mathcal{V}_i^{-1}(f,\,h)) \to Z_{i+1}$ is contractive if both η and δ are sufficiently small.

Assuming this, we thus obtain, aside from (4.32),

$$\varphi_j v = (I - S_H^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(f,h))} h_H^{ij}(f,h,v),\ j = 0,\dots,N_b.$$

Multiplying these equations by ψ_j and summing over all j then results in

$$\begin{aligned} v = \mathcal{G}_H(v) :=& \sum_{j=0}^{N_b} \psi_j (I - S_H^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(f,h))} h_H^{ij}(f,h,v) \\ &+ \sum_{j=N_b+1}^{N_b+N} \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(f,h))} h^{ij}(f,v), \end{aligned} \tag{4.36}$$

a fixed point equation for $v \in Z_{i+1}(v^i)$. Since $\mathcal{G}_H$ leaves this space invariant, the contraction principle is applicable, provided that $\mathcal{G}_H$ is a strict contraction.

To verify that this can be arranged by selecting δ sufficiently small, we let $v,\,\bar{v} \in Z_{i+1}(v^i)$ and estimate with the aid of Theorem 4.1.1 and 4.2.2

$$\begin{aligned} |\mathcal{G}_H(v) - \mathcal{G}_H(\bar{v})|_{Z_{i+1}} =& \Big| \sum_{j=0}^{N_b} \psi_j (I - S_H^{ij})|^{-1}_{Z_{i+1}(0)} \mathcal{L}^{ij}_{H,\,i+1} \begin{pmatrix} k * \mathcal{C}_j(\cdot,x,D_x)(v-\bar{v}) \\ \gamma_y \mathcal{C}_{H,\,j}(\cdot,x)(v-\bar{v}) \end{pmatrix} \\ &+ \sum_{j=N_b}^{N_b+N} \psi_j (I - S^{ij})|^{-1}_{Z_{i+1}(0)} \mathcal{L}^{ij}_{i+1} k * \mathcal{C}_j(\cdot,x,D_x)(v-\bar{v}) \Big|_{Z_{i+1}} \\ \le& C(\sum_{j=0}^{N_b+N} |\mathcal{C}_j(\cdot,x,D_x)(v-\bar{v})|_{X_{i+1}} + \sum_{j=0}^{N_b} |\gamma_y \mathcal{C}_{H,\,j}(t,x)(v-\bar{v})|_{Y_{i+1}}), \end{aligned}$$

with $C > 0$ not depending on δ. By extension to $J \times \mathbb{R}^{n+1}$, estimate (4.14), and restriction to $J \times \mathbb{R}^{n+1}_+$, we obtain for the first sum

$$\sum_{j=0}^{N_b+N} |\mathcal{C}_j(\cdot,x,D_x)(v-\bar{v})|_{X_{i+1}} \le C_1(\varepsilon + C_\varepsilon |k|_{L_1(0,\delta)})|v-\bar{v}|_{Z_{i+1}}, \tag{4.37}$$

where $\varepsilon > 0$ can be chosen arbitrary small and $C_1,\,C_\varepsilon > 0$ do not depend on δ.

Turning to the second sum, we introduce the space

$${}_0Z^{1/2}_{i+1} = {}_0H_p^{\frac{\alpha}{2}}(J; L_p(\mathbb{R}^{n+1}_+)) \cap L_p(J; H^1_p(\mathbb{R}^{n+1}_+)),$$

which is normed by

$$|w|_{{}_0Z^{1/2}_{i+1}} = |\mathcal{B}_{k_{\alpha/2}} w|_{X_{i+1}} + |\nabla_x w|_{X^{n+1}_{i+1}},$$

where $k_{\alpha/2}(t) = t^{\alpha/2-1}$, $t > 0$, and $\mathcal{B}_{k_{\alpha/2}} = (k_{\alpha/2}*)^{-1}$ in X_{i+1}. Suppose that $u \in Z_{i+1}(0)$. By causality, $\mathcal{B}^2_{k_{\alpha/2}} u|_{[0,T_i]} = 0$, and so we have

$$\begin{aligned} |u|_{{}_0Z^{1/2}_{i+1}} &= |(k_{\alpha/2}\chi_{[0,T_{i+1}-T_i]}) * \mathcal{B}^2_{k_{\alpha/2}} u|_{X_{i+1}} + |\nabla_x u|_{X^{n+1}_{i+1}} \\ &\le |k_{\alpha/2}|_{L_1(0,T_{i+1}-T_i)} |\mathcal{B}^2_{k_{\alpha/2}} u|_{X_{i+1}} + \varepsilon |\nabla_x^2 u| + C_\varepsilon |u|_{X_{i+1}} \\ &\le C_0 |k_{\alpha/2}|_{L_1(0,\delta)} |\mathcal{B}_k u|_{X_{i+1}} + \varepsilon |\nabla_x^2 u| + C_\varepsilon |k|_{L_1(0,\delta)} |\mathcal{B}_k u|_{X_{i+1}} \\ &\le (C_0 |k_{\alpha/2}|_{L_1(0,\delta)} + \varepsilon + C_\varepsilon |k|_{L_1(0,\delta)}) |u|_{Z_{i+1}} \quad \forall u \in Z_{i+1}(0), \end{aligned} \tag{4.38}$$

where C_0 is independent of δ ($u \in Z_{i+1}(0)!$) and $\varepsilon > 0$ can be chosen arbitrary small, cf. (4.12) for the last term. Theorem 4.2.1, the results in Section 4.2.2 on pointwise multiplication, and (4.38) allow us now to estimate

$$\sum_{j=0}^{N_b} |\gamma_y \mathcal{C}_{H,j}(t,x)(v-\bar{v})|_{Y_{i+1}} \le C_2(N_b+1)|\gamma_y(v-\bar{v})|_{Y_{i+1}} \le C_2(N_b+1)C_3|v-\bar{v}|_{{}_0Z^{1/2}_{i+1}}$$
$$\le C_4(C_0|k_{\alpha/2}|_{L_1(0,\delta)} + \varepsilon + C_\varepsilon |k|_{L_1(0,\delta)})|v-\bar{v}|_{Z_{i+1}}, \quad (4.39)$$

the constants C_0, C_4 being independent of δ. In view of (4.37) and (4.39), it is now apparent that $\mathcal{G}_H$ is a strict contraction for δ sufficiently small. Consequently, for such δ, (4.36) possesses a unique solution $v =: \mathcal{Q}_{H,\,i+1}(f,h)$ in the space $Z_{i+1}(\mathcal{V}_i^{-1}(f,\,h))$. In other words, $\mathcal{Q}_{H,\,i+1}$ is a left inverse for the operator

$$\mathcal{V}_{H,\,i+1} := \big(I + k * \mathcal{A}(\cdot, x, D_x), \gamma_y \mathcal{B}(t, x', D_x)\big) :$$

$$Z_{i+1}(v^i) \to \Xi_{i+1}(\mathcal{V}_i v^i) \times Y_{i+1}(\gamma_y \mathcal{B}(t,x',D_x)v^i).$$

Here the symbols $\Xi_{i+1}(\psi)$ and $Y_{i+1}(\psi)$ have to be understood like the corresponding one for Z defined in (4.3).

To show that $\mathcal{V}_{H,\,i+1}$ is a surjection, we proceed as in the proof of Theorem 4.1.1. Define the linear operator $\mathcal{K}_{H,\,i+1}$ by means of

$$\mathcal{K}_{H,\,i+1}(g,g_b) = (\mathcal{K}^{(1)}_{H,\,i+1}(g,g_b), \mathcal{K}^{(2)}_{H,\,i+1}(g,g_b))$$
$$= \Big(k * \sum_{j=0}^{N_b} [\mathcal{A}_\#(\cdot,x,D_x),\psi_j](I - S_H^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(g,g_b))} h_H^{ij}(g,g_b,\mathcal{Q}_{H,\,i+1}(g,g_b))$$
$$+ k * \sum_{j=N_b+1}^{N_b+N} [\mathcal{A}_\#(\cdot,x,D_x),\psi_j](I - S_\lambda^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(g,g_b))} h^{ij}(g,g_b,\mathcal{Q}_{H,\,i+1}(g,g_b)),$$
$$\gamma_y \sum_{j=0}^{N_b} (\mathcal{B}_\#(t,x',D_x)\psi_j)(I - S_H^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(g,g_b))} h_H^{ij}(g,g_b,\mathcal{Q}_{H,\,i+1}(g,g_b))\Big).$$

Observe that $\mathcal{K}_{H,\,i+1}$ maps pairs $(g,g_b) \in \Xi_{i+1} \times Y_{i+1}$ satisfying the compatibility condition (iv) into ${}_0\Xi_{i+1} \times {}_0Y_{i+1}$. Indeed, if $\alpha > 2/(p-1)$, the function

$$w := (I - S_H^{ij})|^{-1}_{Z_{i+1}(\varphi_j \mathcal{V}^{-1}_{H,\,i}(g,g_b))} h_H^{ij}(g,g_b,\mathcal{Q}_{H,\,i+1}(g,g_b))$$

has temporal trace $w|_{t=0} = \varphi_j g|_{t=0}$. Since $\psi_j \equiv 1$ on an open set V_j containing $\operatorname{supp}\varphi_j$, it follows that $\mathcal{B}_\#(t,x',D_x)\psi_j \equiv 0$ on $\operatorname{supp}\varphi_j$. Hence $(\mathcal{K}^{(2)}_{H,\,i+1}(g,g_b))|_{t=0} = 0$.

The commutators $[\mathcal{A}_\#(t,x,D_x),\psi_j]$ are differential operators of first order, while multiplying pointwise by $\mathcal{B}_\#(t,x',D_x)\psi_j$ can be regarded as an operator of order zero. So we see that by choosing δ small enough, the mapping $(g,g_b) \mapsto (f,h) - \mathcal{K}_{H,\,i+1}(g,g_b)$ becomes a strict contraction in the space $\{(g,g_b) \in \Xi_{i+1} \times Y_{i+1} : \partial_t^m g|_{t=0} = \partial_t^m f|_{t=0},$ if $\alpha > m + 1/p$, $m = 0,1$; $g_b|_{t=0} = h|_{t=0}$, if $\alpha > 2/(p-1)\}$; thus for such δ we find a pair $(g,g_b) \in \Xi_{i+1} \times Y_{i+1}$ satisfying

$$(g,g_b) + \mathcal{K}_{H,\,i+1}(g,g_b) = (f,h).$$

Apply now $\mathcal{V}_{\#,\,i+1} := I + k * \mathcal{A}_\#(\cdot, x, D_x)$ to $\mathcal{Q}_{H,\,i+1}(g, g_b)$; by (4.36) this gives

$$\begin{aligned}\mathcal{V}_{\#,\,i+1}\mathcal{Q}_{H,\,i+1}(g,g_b) &= \sum_{j=0}^{N_b+N} \psi_j(\varphi_j g + k * \mathcal{C}_j(\cdot,x,D_x)\mathcal{Q}_{H,\,i+1}(g,g_b)) + \mathcal{K}^{(1)}_{H,\,i+1}(g,g_b)\\ &= g - k * \mathcal{A}_R(\cdot,x,D_x)\mathcal{Q}_{H,\,i+1}(g,g_b)) + \mathcal{K}^{(1)}_{H,\,i+1}(g,g_b),\end{aligned}$$

whence

$$\mathcal{V}_{i+1}\mathcal{Q}_{H,\,i+1}(g,g_b) = g + \mathcal{K}^{(1)}_{H,\,i+1}(g,g_b) = f.$$

Similarly,

$$\begin{aligned}\gamma_y\mathcal{B}_\#(t,x',D_x)\mathcal{Q}_{H,\,i+1}(g,g_b) &= \sum_{j=0}^{N_b} \psi_j(\varphi_j g_b + \gamma_y\mathcal{C}_{H,\,j}(t,x)\mathcal{Q}_{H,\,i+1}(g,g_b)) + \mathcal{K}^{(2)}_{H,\,i+1}(g,g_b)\\ &= g_b - b_0\gamma_y\mathcal{Q}_{H,\,i+1}(g,g_b) + \mathcal{K}^{(2)}_{H,\,i+1}(g,g_b),\end{aligned}$$

which entails

$$\gamma_y\mathcal{B}(t,x',D_x)\mathcal{Q}_{H,\,i+1}(g,g_b) = g_b + \mathcal{K}^{(2)}_{H,\,i+1}(g,g_b) = h.$$

This proves surjectivity of $\mathcal{V}_{H,\,i+1}$, provided δ is sufficiently small.

All in all, we have shown that there exists a unique solution v of (4.30) in the space Z. □

4.3 Problems in domains

In this section let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with compact C^2-boundary Γ which decomposes according to $\Gamma = \Gamma_D \cup \Gamma_N$ and $\mathrm{dist}(\Gamma_D, \Gamma_N) > 0$. Let further $1 < p < \infty$ and $J = [0, T]$. We consider the problem

$$\left\{\begin{aligned} v + k * \mathcal{A}(\cdot,x,D_x)v &= f, \quad t \in J,\ x \in \Omega,\\ v &= g, \quad t \in J,\ x \in \Gamma_D,\\ \mathcal{B}(t,x,D_x)v &= h, \quad t \in J,\ x \in \Gamma_N.\end{aligned}\right. \tag{4.40}$$

Here the differential operators $\mathcal{A}(t,x,D_x)$ and $\mathcal{B}(t,x,D_x)$ are of the form

$$\mathcal{A}(t,x,D_x) = -a(t,x) : \nabla_x^2 + a_1(t,x)\cdot\nabla_x + a_0(t,x),\ t \in J,\ x \in \Omega, \tag{4.41}$$

respectively

$$\mathcal{B}(t,x,D_x) = b(t,x)\cdot\nabla_x + b_0(t,x),\ t \in J,\ x \in \Gamma_N.$$

Put $Y = B^{s_1}_{pp}(J; L_p(\Gamma_N)) \cap L_p(J; B^{s_2}_{pp}(\Gamma_N))$ with $s_1 = \alpha(\frac{1}{2} - \frac{1}{2p})$, $s_2 = 1 - \frac{1}{p}$, as well as $M = \bigcup_{r_i > s_i} C^{r_1}(J; C(\Gamma_N)) \cap C(J; C^{r_2}(\Gamma_N))$. Let further $\nu(x)$ denote the outer unit normal of Ω at $x \in \Gamma$. Then our assumptions read as follows.

(H1) (*kernel*): $k \in \mathcal{K}^1(\alpha,\theta)$ with $\theta < \pi$ and $\alpha \in (0,2)\setminus\left\{\frac{1}{p},\ \frac{2}{2p-1},\ \frac{2}{p-1},\ 1+\frac{1}{p},\ 1+\frac{3}{2p-1}\right\}$;
(H2) (*smoothness of coefficients*): $a \in C_{ul}(J\times\overline{\Omega}, \mathrm{Sym}\{n+1\})$, $a_1 \in L_\infty(J\times\Omega, \mathbb{R}^{n+1})$, $a_0 \in L_\infty(J\times\Omega)$, as well as $(b, b_0) \in Y^{n+2}$ in case $p > n + 1 + 2/\alpha$, and $(b, b_0) \in M^{n+2}$, otherwise;
(H3) (*uniform ellipticity*): $\exists c_0 > 0$ s.t. $a(t,x)\xi\cdot\xi \geq c_0|\xi|^2$, $t \in J$, $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^{n+1}$;
(H4) (*normality*): $b(t,x)\cdot\nu(x) \neq 0$, $t \in J$, $x \in \Gamma_N$.

The aim of this section is to prove the subsequent result.

Theorem 4.3.1 *Let $1 < p < \infty$, $J = [0, T]$, and $\Omega \subset \mathbb{R}^{n+1}$ be a domain with compact C^2-boundary Γ which decomposes according to $\Gamma = \Gamma_D \cup \Gamma_N$ and dist$(\Gamma_D, \Gamma_N) > 0$. Suppose the assumptions (H1)-(H4) are satisfied. Then (4.40) admits a unique solution in the space*

$$Z := H_p^\alpha(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega))$$

if and only if the functions f, g, h are subject to the following conditions.

$$
\begin{array}{rl}
(i) & f \in H_p^\alpha(J; L_p(\Omega));\\
(ii) & g \in B_{pp}^{\alpha(1-\frac{1}{2p})}(J; L_p(\Gamma_D)) \cap L_p(J; B_{pp}^{2-\frac{1}{p}}(\Gamma_D));\\
(iii) & h \in B_{pp}^{\alpha(\frac{1}{2}-\frac{1}{2p})}(J; L_p(\Gamma_N)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\Gamma_N));\\
(iv) & f|_{t=0} \in B_{pp}^{2-\frac{2}{p\alpha}}(\Omega),\ \ \text{if}\ \alpha > \frac{1}{p};\\
(v) & \partial_t f|_{t=0} \in B_{pp}^{2(1-\frac{1}{\alpha}-\frac{1}{p\alpha})}(\Omega),\ \ \text{if}\ \alpha > 1 + \frac{1}{p};\\
(vi) & f|_{t=0} = g|_{t=0}\ \text{on}\ \Gamma_D,\ \ \text{if}\ \alpha > \frac{2}{2p-1};\\
(vii) & \mathcal{B}(0, x, D_x) f|_{t=0} = h|_{t=0}\ \text{on}\ \Gamma_N,\ \ \text{if}\ \alpha > \frac{2}{p-1};\\
(viii) & \partial_t f|_{t=0} = \partial_t g|_{t=0}\ \text{on}\ \Gamma_D,\ \ \text{if}\ \alpha > 1 + \frac{3}{2p-1}.
\end{array}
$$

Before proving Theorem 4.3.1 we recall some general properties of variable transformations. Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain with compact C^2-boundary Γ and $x_0 \in \Gamma$. Without restriction of generality, we may assume that $x_0 = 0$ and that $n(x_0) = (0, \ldots, 0, -1)$; this can always be achieved by a composition of a translation and a rotation in $\mathbb{R}^{n+1}$. It is easy to see that such affine mappings of $\mathbb{R}^{n+1}$ onto itself leave invariant all function spaces under consideration (i.e. L_p, Z, and the regularity classes of the data and of the coefficients), and they also preserve ellipticity (including the ellipticity constant in (H3)) as well as normality. Continuing, by definition of a C^2-boundary, there exist an open neighbourhood $U = U_1 \times U_2 \subset \mathbb{R}^{n+1}$ of x_0 with $U_1 \subset \mathbb{R}^n$ and $U_2 \subset \mathbb{R}$ as well as a function $\zeta \in C^2(\overline{U_1})$ such that

$$
\begin{aligned}
\Gamma \cap U &= \{x = (x', y) \in U : y = \zeta(x')\},\\
\Omega \cap U &= \{x = (x', y) \in U : y > \zeta(x')\}.
\end{aligned}
$$

Define $\vartheta : \bar{U} \to \mathbb{R}^{n+1}$ by

$$\vartheta_k(x) = x'_k \text{ if } k = 1, \ldots, n \quad \text{and} \quad \vartheta_{n+1}(x) = y - \zeta(x'). \tag{4.42}$$

Clearly, $\vartheta \in C^2(\overline{U}, \mathbb{R}^{n+1})$ is one-to-one and satisfies $\Omega \cap U = \{x \in U : \vartheta_{n+1}(x) > 0\}$ as well as $\Gamma \cap U = \{x \in U : \vartheta_{n+1}(x) = 0\}$. By extending ζ to a function $\tilde{\zeta} \in C^2(\mathbb{R}^n)$ with compact support and defining $\tilde{\vartheta}$ by (4.42) with ζ being replaced by $\tilde{\zeta}$, we get a C^2-diffeomorphism $\tilde{\vartheta}$ of $\mathbb{R}^{n+1}$ onto itself extending ϑ and satisfying $\tilde{\vartheta}(x) = x$ for large values of $|x|$. Also, $\tilde{\vartheta}$ is a C^2-diffeomorphic mapping from $\Omega_0 := \{x \in \mathbb{R}^{n+1} : y > \tilde{\zeta}(x')\}$ onto $\mathbb{R}^{n+1}_+$. For the Jacobian $D\tilde{\vartheta}(x)$, one obtains

$$D\tilde{\vartheta}(x) = \begin{pmatrix} I_n & 0 \\ -\nabla_{x'}\tilde{\zeta}(x') & 1 \end{pmatrix}, \quad x \in \mathbb{R}^{n+1},$$

which entails $\det D\tilde{\vartheta}(x) = 1$ for all $x \in \mathbb{R}^{n+1}$. Notice also that $D\tilde{\vartheta}(0) = I_{n+1}$.

Given a function $v \in H^2_p(\mathbb{R}^{n+1}_+)$ we consider the pull-back Θv defined on Ω_0 by $\Theta v(x) = v(\tilde{\vartheta}(x))$. Using now the notation $x = (x_1, \dots, x_{n+1})$, the function $u = \Theta v$ satisfies

$$\partial_{x_i} u(x) = \sum_{k=1}^{n+1} \partial_{\bar{x}_k} v(\tilde{\vartheta}(x)) \partial_{x_i} \tilde{\vartheta}_k(x),$$

$$\partial_{x_i}\partial_{x_j} u(x) = \sum_{k=1}^{n+1} \partial_{\bar{x}_k} v(\tilde{\vartheta}(x)) \partial_{x_i}\partial_{x_j} \tilde{\vartheta}_k(x) + \sum_{k,\,l=1}^{n+1} \partial_{\bar{x}_k}\partial_{\bar{x}_l} v(\tilde{\vartheta}(x)) \partial_{x_i} \tilde{\vartheta}_k(x) \partial_{x_j} \tilde{\vartheta}_l(x)$$

for $x \in \Omega_0$ and $i, j = 1, \dots, n+1$. For a differential expression of the form

$$(\mathcal{E}u)(x) = -c(x) : \nabla_x^2 u(x) + c_1(x) \cdot \nabla_x u(x) + c_0(x) u(x), \quad x \in \Omega_0, \tag{4.43}$$

we thus obtain

$$\begin{aligned}(\mathcal{E}u)(x) = &- (D\tilde{\vartheta}(x) c(x) D\tilde{\vartheta}^T(x)) : \nabla_{\bar{x}}^2 v(\tilde{\vartheta}(x)) \\ &+ (D\tilde{\vartheta}(x) c_1(x) - D^2\tilde{\vartheta}(x) : c(x)) \cdot \nabla_{\bar{x}} v(\tilde{\vartheta}(x)) + c_0(x) v(\tilde{\vartheta}(x)), \quad x \in \Omega_0,\end{aligned}$$

with $(D^2\tilde{\vartheta}(x) : c(x))_k = \sum_{i,\,j=1}^{n+1} c_{ij}(x) \partial_{x_i}\partial_{x_j} \tilde{\vartheta}_k(x)$, $k = 1, \dots, n+1$. So applying the push-forward operator Θ^{-1} to the function $\mathcal{E}u$ on Ω_0 gives

$$\Theta^{-1}\mathcal{E}u = (\Theta^{-1}\mathcal{E}\Theta)\Theta^{-1}u = \mathcal{E}^{\tilde{\vartheta}} v,$$

where $\mathcal{E}^{\tilde{\vartheta}} := \Theta^{-1}\mathcal{E}\Theta$ is the second order differential operator

$$(\mathcal{E}^{\tilde{\vartheta}} w)(\bar{x}) = -c^{\tilde{\vartheta}}(\bar{x}) : \nabla_{\bar{x}}^2 w(\bar{x}) + c_1^{\tilde{\vartheta}}(\bar{x}) \cdot \nabla_{\bar{x}} w(\bar{x}) + c_0^{\tilde{\vartheta}}(\bar{x}) w(\bar{x}), \quad \bar{x} \in \mathbb{R}^{n+1}_+, \tag{4.44}$$

with coefficients

$$\begin{aligned}&c^{\tilde{\vartheta}}(\bar{x}) = (D\tilde{\vartheta}\, c\, D\tilde{\vartheta}^T) \circ \tilde{\vartheta}^{-1}(\bar{x}), \;\; c_1^{\tilde{\vartheta}}(\bar{x}) = (D\tilde{\vartheta} c_1 - D^2\tilde{\vartheta} : c) \circ \tilde{\vartheta}^{-1}(\bar{x}), \\ &c_0^{\tilde{\vartheta}}(\bar{x}) = c_0(\tilde{\vartheta}^{-1}(\bar{x})), \qquad \bar{x} \in \mathbb{R}^{n+1}_+,\end{aligned} \tag{4.45}$$

see also [69, Section 5].

Observe that the preceeding formulas are also valid for functions on $J \times \Omega_0$ resp. $J \times \mathbb{R}^{n+1}_+$ and differential operators (4.43) resp. (4.44) with time-dependent coefficients. In view of (4.45), it follows in particular that for an operator $\mathcal{A}(t, x, D_x)$ of the form (4.41) and satisfying the smoothness and ellipticity conditions in (H2) and (H3) with Ω replaced by Ω_0, the transformed operator $\mathcal{A}^{\tilde{\vartheta}}(t, \bar{x}, D_{\bar{x}})$ defined on $J \times \mathbb{R}^{n+1}_+$ enjoys the same properties, the ellipticity constant c_0 appearing in (H3) remaining unchanged. Further, since $D\tilde{\vartheta} \equiv 1$ and the derivatives of $\tilde{\vartheta}$ and $\tilde{\vartheta}^{-1}$ up to order 2 are bounded, the change of variable formula for the Lebesgue integral shows that Θ induces isomorphisms $\Theta^{(p)} : H^m_p(\mathbb{R}^{n+1}_+) \to H^m_p(\Omega_0)$ for each $p \in (1, \infty)$ and $m = 0, 1, 2$.

Proof of Theorem 4.3.1. We begin this time with the sufficiency part. The overall plan can roughly described as follows. With the aid of localization w.r.t. space and the coordinate transformations discussed above, problem (4.40) is reduced to a finite number of related problems on $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}_+$, respectively. For these problems, solution operators are available thanks to the Theorems 4.1.1, 4.2.3, and 4.2.4. So the local equations can be solved and, by summing over all 'local solutions', we obtain a fixed point equation for v of the form $v = v_0 + R(v) =: \mathcal{G}(v)$, where v_0 is determined by the data, and $R(v)$ contains only terms of lower order, see equation (4.52) below. By means of the contraction principle, this fixed point equation can be solved first on a small interval $[0, T_1]$ (denote the (unique) solution by v^1), then on $[0, T_2]$, where $T_2 > T_1$ and the unknown v equaling v^1 on $[0, T_1]$, and proceeding in this way, finally, after finitely many steps, it can be solved on the entire interval $[0, T]$. Here it is essential that $\max_i |T_{i+1} - T_i|$ is sufficiently small to ensure that, in each step of this procedure, the mapping $\mathcal{G}$ is a strict contraction.

Let us start with the spatial localization. By boundedness of Γ, there exists $r_0 > 0$ such that Γ is entirely contained in the open ball $B_{r_0}(0)$. If Ω is unbounded we set $U_0 = \{x \in \mathbb{R}^{n+1} : |x| > r_0\}$, otherwise we may assume that $\overline{\Omega} \subset B_{r_0}(0)$ and put $U_0 = \emptyset$. The other assumptions on Ω allow us to cover $\overline{B_{r_0}(0)}$ by finitely many open sets U_j, $j = 1, \dots, N$, which are subject to the following conditions.

(L1) $U_j \cap \Gamma = \emptyset$ and $U_j = B_{r_j}(x_j)$ for all $j = 1, \dots, N_1$;

(L2) $U_j \cap \Gamma_D \neq \emptyset$ for $N_1 + 1 \leq j \leq N_2$, $U_j \cap \Gamma_N \neq \emptyset$ for $N_2 + 1 \leq j \leq N$, and for each j in either index set, there exist $x_j \in U_j \cap \Gamma_D$ resp. $U_j \cap \Gamma_N$ and $\tilde{\zeta}_j \in C^2(\mathbb{R}^n)$ with compact support such that - using coordinates corresponding to x_j (i.e. $x_j = 0$ and $n(x_j) = (0, \dots, 0, -1)$) - $\Gamma \cap U_j = \{x = (x', y) \in U_j : y = \tilde{\zeta}_j(x')\}$ as well as $\Omega \cap U_j = \{x = (x', y) \in U_j : y > \tilde{\zeta}_j(x')\}$, and $U_j = \tilde{\vartheta}_j^{-1}(B_{r_j}(x_j))$, where $\tilde{\vartheta}_j$ is related to $\tilde{\zeta}_j$ as described above.

(L3) $U_i \cap U_j = \emptyset$ for all $N_1 + 1 \leq i \leq N_2$ and $N_2 + 1 \leq j \leq N$.

It is then not difficult to construct local operators $\mathcal{A}_j = \mathcal{A}_j(t, x, D_x)$, $j = 0, \dots, N$, and $\mathcal{B}_j = \mathcal{B}_j(t, x, D_x)$, $j = N_2 + 1, \dots, N$, of second resp. first order which enjoy the subsequent properties.

(LO1) $\mathcal{A}_j$ is defined on $J \times \mathbb{R}^{n+1}$ if $0 \leq j \leq N_1$, and on $J \times \Omega_j$ otherwise; here the set Ω_j is given in coordinates corresponding to x_j by means of $\Omega_j = \{x = (x', y) \in \mathbb{R}^{n+1} : y \geq \tilde{\zeta}_j(x')\}$; $\mathcal{B}_j$ is defined on $J \times \Gamma_j$ for all $j = N_2 + 1, \dots, N$, where $\Gamma_j = \{x = (x', y) \in \mathbb{R}^{n+1} : y = \tilde{\zeta}_j(x')\}$;

(LO2) the coefficients of $\mathcal{A}_j$ coincide with the corresponding coefficients of $\mathcal{A}(t, x, D_x)$ on $\overline{\Omega} \cap \overline{U_j}$, for all $j = 0, \dots, N$, and the coefficients of $\mathcal{B}_j$ coincide with those of $\mathcal{B}(t, x, D_x)$ on $\Gamma \cap \overline{U_j}$, for all $j = N_2 + 1, \dots, N$;

(LO3) $\mathcal{A}_j$ satisfies the assumptions of Theorem 4.1.1 for all $j = 0, \dots, N_1$; $\mathcal{A}_j^{\tilde{\vartheta}_j} = \Theta_j^{-1} \mathcal{A}_j \Theta_j$ defined on $J \times \mathbb{R}^{n+1}_+$ fulfills the assumptions of Theorem 4.2.3 for all $j = N_1 + 1, \dots, N$; finally, $\mathcal{B}_j^{\tilde{\vartheta}_j} = \Theta_j^{-1} \mathcal{B}_j \Theta_j$ defined on $J \times \mathbb{R}^n$ satisfies the assumptions of Theorem 4.2.4 for all $j = N_2 + 1, \dots, N$.

Here we use the fact that ellipticity and normality, as well as smoothness of the coefficients of $\mathcal{A}(t, x, D_x)$ resp. $\mathcal{B}(t, x, D_x)$ are preserved in $\overline{\Omega} \cap \overline{U_j}$ resp. $\Gamma \cap \overline{U_j}$ under the coordinate transformations $\bar{x} = \tilde{\vartheta}_j(x)$ for all $j = N_1 + 1, \dots, N$. We refer to [29, Section 8.2], where appropriate extensions of the coefficients are constructed by means of reflection and cut-off techniques.

We choose next a partition of unity $\{\varphi_j\}_{j=0}^N \subset C^\infty(\mathbb{R}^{n+1})$ such that $\sum_{j=0}^N \varphi_j(x) \equiv 1$ on $\overline{\Omega}$, $0 \leq \varphi_j(x) \leq 1$, and $\operatorname{supp} \varphi_j \subset U_j$; we fix also a family $\{\psi_j\}_{j=0}^N \subset C^\infty(\mathbb{R}^{n+1})$ that satisfies $\psi_j \equiv 1$ on an open set V_j containing $\operatorname{supp} \varphi_j$, as well as $\operatorname{supp} \psi_j \subset U_j$. As to localization w.r.t. time, we subdivide the interval $[0, T]$ according to $0 =: T_0 < T_1 < \dots < T_{M-1} < T_M := T$ and put $\delta := \max_i |T_{i+1} - T_i|$. Then, owing to (LO1) and (LO2),

v is a solution of (4.40) on $J_{i+1} := [0, T_{i+1}]$ if and only if

$$\begin{cases} \varphi_j v + k * \mathcal{A}_j(\cdot, x, D_x)\varphi_j v = \varphi_j f + k * [\mathcal{A}_j(\cdot, x, D_x), \varphi_j]v & (J_{i+1} \times \Omega) \quad, 0 \le j \le N \\ \varphi_j v = \varphi_j g & (J_{i+1} \times \Gamma_D),\ N_1+1 \le j \le N_2 \\ \mathcal{B}_j(t, x, D_x)\varphi_j v = \varphi_j h + [\mathcal{B}_j(t, x, D_x), \varphi_j]v & (J_{i+1} \times \Gamma_N),\ N_2+1 \le j \le N. \end{cases} \tag{4.46}$$

In case $j = 0, \dots, N_1$, we have to consider full space problems for the functions $\varphi_j v$. In view of (LO3), we can apply Theorem 4.1.1, which ensures existence of corresponding solution operators $\mathcal{L}^F_{ij}$, thereby obtaining

$$\varphi_j v = \mathcal{L}^F_{ij}(\varphi_j f + k * [\mathcal{A}_j, \varphi_j]v) =: h^F_{ij}(f, v),\ j = 0, \dots, N_1. \tag{4.47}$$

For $j = N_1 + 1, \dots, N_2$, we get problems on crooked half spaces with inhomogeneous Dirichlet boundary condition. Using affine mappings that transform x_j to the origin and $n(x_j)$ to $(0, \dots, 0, -1)$ combined with the variable transformations $\bar{x} = \tilde{\vartheta}_j(x)$ (denote these compositions again by $\tilde{\vartheta}_j$) leads to

$$\begin{cases} \Theta_j^{-1}(\varphi_j v) + k * \mathcal{A}_j^{\tilde{\vartheta}_j}\Theta_j^{-1}(\varphi_j v) = \Theta_j^{-1}(\varphi_j f) + k * \Theta_j^{-1}[\mathcal{A}_j, \varphi_j]v & (J_{i+1} \times \mathbb{R}^{n+1}_+) \\ \Theta_j^{-1}(\varphi_j v) = \Theta_j^{-1}(\varphi_j g) & (J_{i+1} \times \mathbb{R}^n), \end{cases} \tag{4.48}$$

that is, to half space problems for $\Theta_j^{-1}(\varphi_j v)$, for which Theorem 4.2.3 is applicable, in virtue of (LO3). Employing the corresponding solution operators denoted by $\mathcal{L}^D_{ij}$ thus yields

$$\varphi_j v = \Theta_j \mathcal{L}^D_{ij} \begin{pmatrix} \Theta_j^{-1}(\varphi_j f) + k * \Theta_j^{-1}[\mathcal{A}_j, \varphi_j]v \\ \Theta_j^{-1}(\varphi_j g) \end{pmatrix} =: h^D_{ij}(f, g, v),\ j = N_1 + 1, \dots, N_2. \tag{4.49}$$

The situation is similar for $j = N_2 + 1, \dots, N$. In this case we have to consider problems on crooked half spaces with inhomogeneous boundary condition of first order. Using again the variable substitutions $\bar{x} = \tilde{\vartheta}_j(x)$ gives

$$\begin{cases} \Theta_j^{-1}(\varphi_j v) + k * \mathcal{A}_j^{\tilde{\vartheta}_j}\Theta_j^{-1}(\varphi_j v) = \Theta_j^{-1}(\varphi_j f) + k * \Theta_j^{-1}[\mathcal{A}_j, \varphi_j]v & (J_{i+1} \times \mathbb{R}^{n+1}_+) \\ \mathcal{B}_j^{\tilde{\vartheta}_j}\Theta_j^{-1}(\varphi_j v) = \Theta_j^{-1}(\varphi_j h) + [\mathcal{B}_j, \varphi_j]v & (J_{i+1} \times \mathbb{R}^n), \end{cases} \tag{4.50}$$

which are problems on a half space for $\Theta_j^{-1}(\varphi_j v)$. Without loss of generality, we may assume that $b(t, x) \cdot \nu(x) = 1$ for all $t \in J$, $x \in \Gamma_N$; in fact, we can always divide the boundary condition in (4.40) by $(b(t, x) \cdot \nu(x))$ to achieve this without affecting the smoothness of the inhomogeneity and that of the coefficients of the boundary operator, see Section 6.2. As a consequence of this normalization, the operators $\mathcal{B}_j^{\tilde{\vartheta}_j}$ take the form (4.28). By (LO3), we can apply Theorem 4.2.4, which asserts existence of solution operators $\mathcal{L}^N_{ij}$ for the above problems. So we immediately get

$$\varphi_j v = \Theta_j \mathcal{L}^N_{ij} \begin{pmatrix} \Theta_j^{-1}(\varphi_j f) + k * \Theta_j^{-1}[\mathcal{A}_j, \varphi_j]v \\ \Theta_j^{-1}(\varphi_j h) + [\mathcal{B}_j, \varphi_j]v \end{pmatrix} =: h^N_{ij}(f, h, v),\ j = N_2 + 1, \dots, N. \tag{4.51}$$

Multiplying now (4.47), (4.49), and (4.51) by ψ_j and summing over all j yields the formula

$$v = \sum_{j=0}^{N_1} \psi_j\, h^F_{ij}(f, v) + \sum_{j=N_1+1}^{N_2} \psi_j\, h^D_{ij}(f, g, v) + \sum_{j=N_2+1}^{N} \psi_j\, h^N_{ij}(f, h, v) =: \mathcal{G}(v), \tag{4.52}$$

which is necessary for v to be a solution of (4.40) on $[0, T_{i+1}]$.

Let $Z_{i+1} = H_p^\alpha([0, T_{i+1}]; L_p(\Omega)) \cap L_p([0, T_{i+1}]; H_p^2(\Omega))$ for $i = 0, \dots, M-1$, and set as usual $Z_1(v^0) = \{w \in Z_1 : \partial_t^m w|_{t=0} = \partial_t^m f|_{t=0}, \text{ if } \alpha > m + 1/p,\ m = 0, 1\}$, as well as $Z_{i+1}(v^i) := \{w \in Z_{i+1} : w|_{[0,T_i]} = v^i\}$ for $i > 0$, where v^i denotes the unique solution of (4.40) on $[0, T_i]$ determined in the ith time step. In similar fashion as in the proofs of Theorem 4.1.1 and Theorem 4.2.3, one can now show that for each $i = 0, \dots, M-1$, the mapping $\mathcal{G}$ leaves $Z_{i+1}(v^i)$ invariant and is a strict contraction, provided that δ is small enough. In fact, one uses the estimates derived in the above proofs together with the diffeomorphism property of the coordinate transformations $\bar{x} = \tilde{\vartheta}_j(x)$. Observe that we have uniform bounds for $|\Theta_j|$ and $|\Theta_j^{-1}|$ w.r.t. i and j, as only the spatial variables are transformed and by compactness of Γ. So for sufficiently small δ, equation (4.52) admits a unique solution $v =: Q_{i+1}(f, g, h)$ in the space $Z_{i+1}(v^i)$. To see that this function indeed solves (4.40) on $[0, T_{i+1}]$, one can again argue as in the proofs of Theorem 4.1.1 and Theorem 4.2.3. Here one has to consider the linear operators $\mathcal{K}_{i+1}$ defined by

$$\begin{aligned}
\mathcal{K}_{i+1}(f_1, f_2, f_3) &= (\mathcal{K}_{i+1}^{(1)}(f_1, f_2, f_3),\, 0\,, \mathcal{K}_{i+1}^{(3)}(f_1, f_2, f_3)) \\
= \Big(&\sum_{j=0}^{N_1} [\mathcal{A}, \psi_j]\, h_{ij}^F(f_1, Q_{i+1}(f_1, f_2, f_3)) + \sum_{j=N_1+1}^{N_2} [\mathcal{A}, \psi_j]\, h_{ij}^D(f_1, f_2, Q_{i+1}(f_1, f_2, f_3)) \\
+ &\sum_{j=N_2+1}^{N} [\mathcal{A}, \psi_j]\, h_{ij}^N(f_1, f_3, Q_{i+1}(f_1, f_2, f_3)),\, 0\,, \gamma_{\Gamma_N} \sum_{j=N_2+1}^{N} [\mathcal{B}, \psi_j]\, h_{ij}^N(f_1, f_3, Q_{i+1}(f_1, f_2, f_3)) \Big),
\end{aligned}$$

for triples (f_1, f_2, f_3) in the product space of the regularity classes of f, g, h whose temporal traces at $t = 0$ coincide with those of f, g, h (including the first temporal derivative) whenever these traces exist. Since $\mathcal{K}_{i+1}$ contains only terms of lower order, one can prove existence of a triple (f_1, f_2, f_3) satisfying $(f_1, f_2, f_3) + \mathcal{K}_{i+1}(f_1, f_2, f_3) = (f, g, h)$, provided that δ is sufficiently small. By simple computations, one shows then that $Q_{i+1}(f_1, f_2, f_3)$ solves (4.40) on $[0, T_{i+1}]$. Finally, uniqueness implies $v = Q_{i+1}(f, g, h) = Q_{i+1}(f_1, f_2, f_3)$. This establishes the sufficiency part.

We turn now to necessity. Suppose that $v \in Z$ solves (4.40). Clearly, $\varphi_j v \in Z$ for all $j = 0, \dots, N$, which in turn implies $\Theta_j^{-1}(\varphi_j v) \in H_p^\alpha(J; L_p(\mathbb{R}_+^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}_+^{n+1}))$ for all $j = N_1 + 1, \dots, N$. Observe further that all those terms on the right-hand sides of (4.48), (4.50), and the first equation of (4.46) ($i = M - 1$) which involve the function v have the regularity desired for the corresponding inhomogeneity (f, g resp. h) on $J \times \mathbb{R}^{n+1}$ resp. $J \times \mathbb{R}_+^{n+1}$. In view of the Theorems 4.1.1, 4.2.3, 4.2.4, and the diffeomorphism property of the variable transformations $\bar{x} = \tilde{\vartheta}_j(x)$, it thus follows that the functions $\varphi_j f$, $\varphi_j g$, and $\varphi_j h$ enjoy the desired regularity on $J \times \Omega$, $J \times \Gamma_D$, and $J \times \Gamma_N$, respectively, for each j. On account of $\sum_{j=0}^N \varphi_j(x) \equiv 1$ on $\overline{\Omega}$, we eventually obtain the desired regularity for the data f, g, h themselves. The compatibility conditions can be seen in the same way. $\square$

Chapter 5

Linear Viscoelasticity

In this chapter we shall study a linear parabolic problem of second order which arises in the theory of viscoelasticity. In comparison to the problems investigated in the previous chapter, it has two new challenging features: (1) it is a *vector*-valued problem, and (2) it contains *two independent kernels*. As before we shall characterize unique existence of the solution in a certain class of optimal L_p-regularity in terms of regularity and compatibility conditions on the given data.

The chapter is organized as follows. At first we recall the model equations from linear viscoelasticity, here following the presentation given in Prüss [63, Section 5]. In the second part we state the problem and discuss the assumptions on the kernels. The third and main part of this chapter is devoted to the thorough investigation of a half space case of the problem.

For a derivation of the fundamental equations of continuum mechanics and of linear viscoelasticity, we further refer to the books by Christensen [12], Gurtin [41], Pipkin [62], and Gripenberg, Londen, Staffans [39].

5.1 Model equations

Let $\Omega \subset \mathbb{R}^3$ be an open set with boundary $\partial\Omega$ of class C^2. The set Ω shall represent a body, i.e. a solid or fluid material. Acting forces lead to a deformation of the body, displacing every material point $x \in \Omega$ at time t to the point $x + u(t,x)$. The vector field $u : \mathbb{R} \times \Omega \to \mathbb{R}^3$ is called the *displacement field*, or briefly *displacement*. The *velocity* $v(t,x)$ of the material point $x \in \Omega$ at time t is then given by $v(t,x) = \dot{u}(t,x)$, the dot indicating partial derivative w.r.t. t.

The deformation of the body induces a *strain* $\mathcal{E}(t,x)$, which will depend linearly on the gradient $\nabla u(t,x)$, provided that the deformation is small enough. We will put

$$\mathcal{E}(t,x) = \frac{1}{2}(\nabla u(t,x) + (\nabla u(t,x))^T), \quad t \in \mathbb{R},\ x \in \Omega, \tag{5.1}$$

i.e. $\mathcal{E}$ is the symmetric part of the displacement gradient ∇u.

The strain in turn causes *stress* in a way which has to be specified, expressing the properties of the material the body is made of. The stress is described by the symmetric tensor $\mathcal{S}(t,x)$. If ρ denotes the *mass density* and assuming that it is time independent, i.e. $\rho(t,x) = \rho_0(x)$, the balance of momentum law implies

$$\rho_0(x)\ddot{u}(t,x) = \operatorname{div} \mathcal{S}(t,x) + \rho_0(x) f(t,x), \quad t \in \mathbb{R},\ x \in \Omega, \tag{5.2}$$

where f represents external forces acting on the body, like gravity or electromagnetic forces. In components (5.2) reads

$$\rho_0(x)\ddot{u}_i(t,x) = \sum_{j=1}^{3} \partial_{x_j} \mathcal{S}_{ij}(t,x) + \rho_0(x) f_i(t,x), \quad i = 1,2,3.$$

We have to supplement (5.2) by boundary conditions. Possible boundary conditions are either 'prescribed displacement' or 'prescribed normal stress'. Let $\partial\Omega = \Gamma_d \cup \Gamma_s$ with $\overline{\mathring{\Gamma}_d} = \Gamma_d$, $\overline{\mathring{\Gamma}_s} = \Gamma_s$ and $\mathring{\Gamma}_d \cap \mathring{\Gamma}_s = \emptyset$. The boundary conditions then read as follows.

$$u(t,x) = u_d(t,x), \quad t \in \mathbb{R},\, x \in \mathring{\Gamma}_d, \tag{5.3}$$

$$\mathcal{S}(t,x)n(x) = g_s(t,x), \quad t \in \mathbb{R},\, x \in \mathring{\Gamma}_s, \tag{5.4}$$

where $n(x)$ denotes the outer normal of Ω at $x \in \Omega$.

A material is called *incompressible*, if there are no changes of volume in the body Ω during a deformation, i.e. if

$$\det\left(I + \nabla u(t,x)\right) = 1, \quad t \in \mathbb{R},\, x \in \Omega, \tag{5.5}$$

is fulfilled; otherwise the material is called *compressible*. For the linear theory, the nonlinear constraint (5.5) can be simplified to the linear condition

$$\operatorname{div} u(t,x) = 0, \quad t \in \mathbb{R},\, x \in \Omega. \tag{5.6}$$

In the sequel, we shall consider compressible materials.

We still have to describe how the stress $\mathcal{S}(t,x)$ depends on the strain $\mathcal{E}$. This is done by a *constitutive law* or a *stress-strain relation*. Such an equation completes the system inasmuch as it relates the stress $\mathcal{S}(t,x)$ to the unknown u and its derivatives. If the material is purely *elastic*, then the *stress* $\mathcal{S}(t,x)$ will depend (linearly) only on the strain $\mathcal{E}(t,x)$. However, the stress may also depend on the history of the strain and its time derivative; in this case the material is called *viscoelastic*. The general constitutive law for compressible materials is given by

$$\mathcal{S}(t,x) = \int_0^\infty d\mathcal{A}(\tau,x)\, \dot{\mathcal{E}}(t-\tau,x)\, d\tau, \quad t \in \mathbb{R},\, x \in \Omega, \tag{5.7}$$

where $\mathcal{A} : \mathbb{R}_+ \times \Omega \to \mathcal{B}(\mathrm{Sym}\{3\})$ is locally of bounded variation w.r.t. $t \geq 0$. The symbol $\mathrm{Sym}\{n\}$ denotes the space of n-dimensional real symmetric matrices. As a consequence of this, the symmetry relations

$$\mathcal{A}_{ijkl}(t,x) = \mathcal{A}_{jikl}(t,x) = \mathcal{A}_{ijlk}(t,x), \quad t \in \mathbb{R}_+,\, x \in \Omega, \tag{5.8}$$

have to be satisfied for all $i,j,k,l \in \{1,2,3\}$. The function $\mathcal{A}$ is called the *relaxation function* of the material. Its component functions $\mathcal{A}_{ijkl}$, the so-called *stress relaxation moduli*, have to be determined in experiments.

In the following we want to consider the case where the material is *isotropic*, which by definition means that the constitutive law is invariant under the group of rotations. It can be shown that the general isotropic stress relaxation tensor takes the form

$$\mathcal{A}_{ijkl}(t,x) = \frac{1}{3}(3b(t,x) - 2a(t,x))\delta_{ij}\delta_{kl} + a(t,x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{5.9}$$

where δ_{ij} denotes Kronecker's symbol. The function a describes how the material responds to shear, while b determines its behaviour under compression. Therefore, a is called *shear modulus* and b *compression modulus*. The constitutive law (5.7) becomes

$$\mathcal{S}(t,x) = 2\int_0^\infty da(\tau,x)\dot{\mathcal{E}}(t-\tau,x) + \frac{1}{3}\mathcal{I}\int_0^\infty (3db(\tau,x) - 2da(\tau,x))\mathrm{tr}\,\dot{\mathcal{E}}(t-\tau,x).$$

Besides, we want to assume that the material is *homogeneous*, i.e. ρ_0 as well as a and b do not depend on the material points $x \in \Omega$. For simplicity, let us put $\rho_0(x) \equiv 1$, $x \in \Omega$.

To summarize, we obtain the following integro-differential equation for homogeneous and isotropic materials:

$$\ddot{u}(t,x) = \int_0^\infty da(\tau)\Delta\dot{u}(t-\tau,x) + \int_0^\infty (db(\tau) + \frac{1}{3}da(\tau))\nabla\nabla\cdot\dot{u}(t-\tau,x) + f(t,x), \quad (5.10)$$

for all $t \in \mathbb{R}$ and $x \in \Omega$. This equation has to be supplemented by the boundary conditions (5.3),(5.4).

Let us consider a material which is at rest up to time $t = 0$, but is then suddenly moved with the velocity $v_0(x)$, $x \in \Omega$. More precisely, we want to assume that $v(t,x) = \dot{u}(t,x) = 0$, $t < 0$, $x \in \Omega$, and $v(0,x) = v_0(x)$, $x \in \Omega$. Then the problem (5.10),(5.3),(5.4) amounts to

$$\begin{cases} \partial_t v - da * \Delta v - (db + \frac{1}{3}da) * \nabla\nabla\cdot v = f,\ t > 0,\ x \in \Omega \\ v = v_d,\ t > 0,\ x \in \Gamma_d \\ (2da * \dot{\mathcal{E}} + \frac{1}{3}\mathcal{I}(3db - 2da) * \mathrm{tr}\,\dot{\mathcal{E}})\,n = g_s,\ t > 0,\ x \in \Gamma_s \\ v|_{t=0} = v_0,\ x \in \Omega, \end{cases} \quad (5.11)$$

where we use the notation $(dk * \phi)(t) = \int_0^t dk(\tau)\phi(t-\tau)$, $t > 0$.

5.2 Assumptions on the kernels and formulation of the goal

Given f, v_d, g_s, v_0, our goal is to solve (5.11) for v. For this to be possible it is necessary that the convolution terms in (5.11) do not produce terms involving the displacement u, which would be the case, if for example $a(t) = t$, $t \geq 0$. That means the problem must not be hyperbolic. Further we need certain regularity assumptions on a and b so that we can apply the results from Section 3.5. It turns out that the following class of kernels is appropriate for our problem.

Definition 5.2.1 *A function $k : [0,\infty) \to \mathbb{R}$ is said to be of* type (E) *if*

(E1) $k(0) = 0$, and k is of the form $k(t) = k_0 + \int_0^t k_1(\tau)\,d\tau$, $t > 0$, where $k_0 \geq 0$ and $k_1 \in L_{1,loc}(\mathbb{R}_+)$;

(E2) k_1 is completely monotonic, *i.e. $k_1 \in C^\infty(0,\infty)$ and $(-1)^l k_1^{(l)}(t) \geq 0$ for all $t > 0$, $l \in \mathbb{N}_0$;*

(E3) if $k_1 \neq 0$, then $k_1 \in \mathcal{K}^\infty(\alpha, \theta_{k_1})$, for some $\alpha \in [0,1)$ and $\theta_{k_1} < \frac{\pi}{2}$.

Observe that (E1) and (E2) imply that the function k in Definition 5.2.1, restricted to the interval $(0,\infty)$, is a *Bernstein function*, which by definition means that k is $\mathbb{R}_+$-valued, infinitely differentiable on $(0,\infty)$, and that k' is completely monotonic. We further remark that property (E2) already entails r-regularity of k_1 for all $r \in \mathbb{N}$ as well as $\operatorname{Re} \hat{k}_1(\lambda) \geq 0$ for all $\operatorname{Re} \lambda \geq 0$, $\lambda \neq 0$, i.e. k_1 is of *positive type*. For a proof of this fact see Prüss [63, Proposition 3.3]. In comparison to the last inequality, (E3) requires that $|\arg \hat{k}_1(\lambda)| \leq \theta_{k_1} < \pi/2$ for all $\operatorname{Re} \lambda > 0$.

We recall that the Laplace transform of any function which is locally integrable on $\mathbb{R}_+$ and completely monotonic has an analytic extension to the region $\mathbb{C} \backslash \mathbb{R}_-$, see e.g. [39, Thm. 2.6, p. 144]. Thus if k is of type (E), both $\hat{k}$ and $\widehat{dk}$, given by $\hat{k}(\lambda) = (k_0 + \hat{k}_1(\lambda))/\lambda$ resp. $\widehat{dk}(\lambda) = k_0 + \hat{k}_1(\lambda)$, may be assumed to be analytic in $\mathbb{C} \setminus \mathbb{R}$.

In the sequel we will assume that both kernels a and b are of type (E) and that $a \neq 0$. Define the parameters α, θ_{a_1} and β, θ_{b_1} by $a_1 \in \mathcal{K}^\infty(\alpha, \theta_{a_1})$ and $b_1 \in \mathcal{K}^\infty(\beta, \theta_{b_1})$, if $a_1 \neq 0$ respectively $b_1 \neq 0$.

We will study (5.11) in $L_p(J; L_p(\Omega, \mathbb{R}^3))$, where $1 < p < \infty$, and $J = [0, T]$ is a compact time-interval. We are looking for a unique solution v of (5.11) in the regularity class

$$Z := (H_p^{\delta_a}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)))^3,$$

where the exponent δ_a is defined by $\delta_a := 1$, if $a_0 \neq 0$, and $\delta_a := 1 + \alpha$, otherwise. In other words, δ_a gives the regularization order of a in the sense of Corollary 2.8.1. Here, the regularity properties of b are not taken into account, since, in some sense, b only plays a subordinate role in solving problem (5.11) as we shall see below. Nevertheless, if $b \neq 0$, we have to distinguish two principal cases. Letting δ_b be the regularization order of $b \neq 0$, defined in the same way as for a, we have to distinguish the cases $\delta_a \leq \delta_b$ and $\delta_a > \delta_b$. The second case is more difficult, for here the terms involving b have less regularity than those involving a. In order to cope with this defect, supplementary regularity conditions have to be introduced.

It is convenient to define δ_b also in the case $b = 0$. So we put $\delta_b = \infty$ in that case.

The strategy in solving (5.11) is the same as in Chapter 4. (5.11) is studied first in the cases $\Omega = \mathbb{R}^3$ and $\Omega = \mathbb{R}^3_+$, where in the latter situation one has to consider both boundary conditions separately. Having solved those cases, a solution of (5.11) can be constructed by the aid of localization and perturbation arguments. We will restrict our investigation to the half space case with prescribed normal stress. It will become apparent that the techniques used here also apply to the much simpler full space case and the half space case with prescribed velocity.

5.3 A homogeneous and isotropic material in a half space

In this section we consider a homogeneous and isotropic material in a half space with prescribed normal stress. We do not only look at the three-dimensional situation but study the general $(n+1)$-dimensional case, $n \in \mathbb{N}$.

Let $\mathbb{R}^{n+1}_+ = \{(x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n,\ y > 0\}$ and denote the velocity vector by (v, w), where v is $\mathbb{R}^n$-valued and w is a scalar function. From (5.11) we are then led to the problem

$$\begin{cases} \partial_t v - da * (\Delta_x v + \partial_y^2 v) - (db + \frac{1}{3}da) * (\nabla_x \nabla_x \cdot v + \partial_y \nabla_x w) = f_v & (J \times \mathbb{R}_+^{n+1}) \\ \partial_t w - da * \Delta_x w - (db + \frac{4}{3}da) * \partial_y^2 w - (db + \frac{1}{3}da) * \partial_y \nabla_x \cdot v = f_w & (J \times \mathbb{R}_+^{n+1}) \\ -da * \gamma \partial_y v - da * \gamma \nabla_x w = g_v & (J \times \mathbb{R}^n) \\ -(db - \frac{2}{3}da) * \gamma \nabla_x \cdot v - (db + \frac{4}{3}da) * \gamma \partial_y w = g_w & (J \times \mathbb{R}^n) \\ v|_{t=0} = v_0 & (\mathbb{R}_+^{n+1}) \\ w|_{t=0} = w_0 & (\mathbb{R}_+^{n+1}), \end{cases} \tag{5.12}$$

where γ denotes the trace operator at $y = 0$. We seek a unique solution (v, w) of (5.12) in the regularity class

$$Z := (H_p^{\delta_a}(J; L_p(\mathbb{R}_+^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}_+^{n+1})))^{n+1}.$$

5.3.1 The case $\delta_a \leq \delta_b$: necessary conditions

In this and the following subsection, we assume that $\delta_a \leq \delta_b$. On the basis of the results in Section 3.5, we first derive necessary conditions for the existence of a solution (v, w) of (5.12) in the space Z.

Suppose that we are given such a solution. By Corollary 2.8.1, it then follows immediately that

$$(f_v, f_w) \in (H_p^{\delta_a - 1}(J; L_p(\mathbb{R}_+^{n+1})))^{n+1}. \tag{5.13}$$

Taking the temporal trace of (v, w) and $(\partial_t v, \partial_t w)$, respectively, at $t = 0$ gives according to Theorem 3.5.2 (see also the results from Chapter 4)

$$(v_0, w_0) \in (B_{pp}^{2(1 - \frac{1}{p\delta_a})}(\mathbb{R}_+^{n+1}))^{n+1}, \tag{5.14}$$

and

$$(f_v, f_w)|_{t=0} \in (B_{pp}^{2(1 - \frac{1}{\delta_a} - \frac{1}{p\delta_a})}(\mathbb{R}_+^{n+1}))^{n+1} \tag{5.15}$$

in case $\delta_a > 1 + 1/p$. Putting

$$Y = B_{pp}^{\frac{\delta_a}{2}(1 - \frac{1}{p})}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{1 - \frac{1}{p}}(\mathbb{R}^n))$$

Theorem 3.5.2 further yields $(\gamma \partial_y v, \gamma \nabla_x w, \gamma \partial_y w, \gamma \nabla_x \cdot v) \in Y^{2n+2}$. So if we set

$$\phi = -\gamma \partial_y v - \gamma \nabla_x w,$$

then it follows from the first boundary condition in (5.12) that g_v is of the form

$$g_v = da * \phi, \quad \text{with } \phi \in Y^n. \tag{5.16}$$

As, in case $p > 1 + 2/\delta_a$, we have the embedding $B_{pp}^{\frac{\delta_a}{2}(1 - \frac{1}{p})}(J; L_p(\mathbb{R}^n)) \hookrightarrow C(J; L_p(\mathbb{R}^n))$, we see that ϕ satisfies in addition the compatibility condition

$$\phi|_{t=0} = -\gamma \partial_y v_0 - \gamma \nabla_x w_0, \text{ if } p > 1 + \tfrac{2}{\delta_a}. \tag{5.17}$$

Turning to the second boundary condition in (5.12), we put

$$\psi_1 = \tfrac{2}{3} \gamma \nabla_x \cdot v - \tfrac{4}{3} \gamma \partial_y w, \qquad \psi_2 = -\gamma \nabla_x \cdot v - \gamma \partial_y w.$$

Then it follows that g_w is of the form

$$g_w = da * \psi_1 + db * \psi_2, \qquad \text{with } \ \psi_1, \psi_2 \in Y. \tag{5.18}$$

In case that $p > 1 + 2/\delta_a$, we discover the additional compatibility conditions

$$\psi_1|_{t=0} = \tfrac{2}{3}\gamma\nabla_x \cdot v_0 - \tfrac{4}{3}\gamma\partial_y w_0, \quad \psi_2|_{t=0} = -\gamma\nabla_x \cdot v_0 - \gamma\partial_y w_0, \text{ if } p > 1 + \tfrac{2}{\delta_a}. \tag{5.19}$$

All in all we have established necessity of

$$(\mathbf{N1}) \quad (5.13), (5.14), (5.15), (5.16), (5.17), (5.18), (5.19).$$

5.3.2 The case $\delta_a \le \delta_b$: sufficiency of (N1)

We now want to prove the converse. So assume that the data satisfy all conditions in **(N1)**. At first glance, it seems to be a hard task to solve (5.12) for it is a coupled system in the unknown functions v, w and since, in contrast to the previous problems, we have to cope with *two* kernels. To overcome these difficulties, the basic idea is to introduce an appropriate auxiliary function p by the aid of which (5.12) can be decoupled and made amenable to the results from Chapter 3.

To start with, we set $k = b + \frac{4}{3}a$ and introduce the inverse convolution operators $A = (a*)^{-1}$ and $K = (k*)^{-1}$ in $L_p(J;X)$, where X is $L_p(\mathbb{R}^{n+1}_+)$ or $L_p(\mathbb{R}^n)$. This makes sense since a and b are of type (E) and in view of Lemma 2.6.2(ii). Further let $F = (A + D_n)^{\frac{1}{2}}$ and $G = (K + D_n)^{\frac{1}{2}}$ with natural domains, D_n denoting the negative Laplacian on $\mathbb{R}^n$. We proceed now in three steps.

Step 1. Assume for the moment that (v, w) is a solution of (5.12) and define the pressure p by means of

$$p = -\nabla_x \cdot v - \partial_y w. \tag{5.20}$$

Then it follows from (5.12) that

$$\left\{ \begin{array}{rl} \partial_t v - da * (\Delta_x v + \partial_y^2 v) + (db + \frac{1}{3}da) * \nabla_x p = f_v & (J \times \mathbb{R}^{n+1}_+) \\ \partial_t w - da * (\Delta_x w + \partial_y^2 w) + (db + \frac{1}{3}da) * \partial_y p = f_w & (J \times \mathbb{R}^{n+1}_+) \\ -da * \gamma\partial_y v - da * \gamma\nabla_x w = g_v & (J \times \mathbb{R}^n) \\ -\gamma\partial_y w - \gamma\nabla_x \cdot v = \gamma p & (J \times \mathbb{R}^n) \\ v|_{t=0} = v_0 & (\mathbb{R}^{n+1}_+) \\ w|_{t=0} = w_0 & (\mathbb{R}^{n+1}_+) \end{array} \right. \tag{5.21}$$

and

$$-da * \gamma\partial_y w + \tfrac{1}{2}(db - \tfrac{2}{3}da) * \gamma p = \tfrac{1}{2}g_w. \tag{5.22}$$

Applying $-\nabla_x \cdot$ to the first, $-\partial_y$ to the second equation of (5.21) and adding them yields further

$$\left\{ \begin{array}{rl} \partial_t p - (db + \frac{4}{3}da) * (\Delta_x p + \partial_y^2 p) = -\nabla_x \cdot f_v - \partial_y f_w & (J \times \mathbb{R}^{n+1}_+) \\ p|_{t=0} = -\nabla_x \cdot v_0 - \partial_y w_0 & (\mathbb{R}^{n+1}_+), \end{array} \right. \tag{5.23}$$

where here $\nabla_x \cdot$ and ∂_y are meant in the distributional sense. This shows one direction of

$$(5.12), (5.20) \Leftrightarrow (5.21), (5.22), (5.23). \tag{5.24}$$

To see the converse direction, suppose the triple (v, w, p) satisfies (5.21), (5.22), and (5.23). Let $q = -\nabla_x \cdot v - \partial_y w$ and deduce from (5.21), by performing the same calculation as above, that

$$\begin{cases} \partial_t q - da * (\Delta_x q + \partial_y^2 q) - (db + \frac{1}{3}da) * (\Delta_x p + \partial_y^2 p) = -\nabla_x \cdot f_v - \partial_y f_w & (J \times \mathbb{R}_+^{n+1}) \\ \gamma q = \gamma p & (J \times \mathbb{R}^n) \\ q|_{t=0} = -\nabla_x \cdot v_0 - \partial_y w_0 & (\mathbb{R}_+^{n+1}). \end{cases} \tag{5.25}$$

Then we replace the right-hand side of the first and third equation of (5.25) by the corresponding terms in (5.23), to discover

$$\begin{cases} \partial_t (q-p) - da * (\Delta_x + \partial_y^2)(q-p) = 0 & (J \times \mathbb{R}_+^{n+1}) \\ \gamma(q-p) = 0 & (J \times \mathbb{R}^n) \\ (q-p)|_{t=0} = 0 & (\mathbb{R}_+^{n+1}). \end{cases} \tag{5.26}$$

So by uniqueness of the solution of (5.26), we find $q = p$, that is, (5.20) is established. System (5.12) now follows immediately.

Observe that if we once know the boundary value γp, then the pressure p is uniquely determined by (5.23). With p being known, (v, w) can then be obtained via (5.21). So we have to find a formula for γp involving only the given data.

Step 2. To approach our goal, we continue by extending the functions $\psi_f := -\nabla_x \cdot f_v - \partial_y f_w$ and $\psi_0 := -\nabla_x \cdot v_0 - \partial_y w_0$ to all of $\mathbb{R}$ w.r.t. y so that the new functions (again denoted by ψ_f and ψ_0) lie in the corresponding regularity classes on $J \times \mathbb{R}^{n+1}$, that is

- $\psi_f \in H_p^{\delta_a - 1}(J; H_p^{-1}(\mathbb{R}^{n+1}))$;
- $\psi_f|_{t=0} \in B_{pp}^{1-\frac{2}{\delta_a}-\frac{2}{p\delta_a}}(\mathbb{R}^{n+1})$, if $\delta_a > 1 + \frac{1}{p}$;
- $\psi_0 \in B_{pp}^{1-\frac{2}{p\delta_a}}(\mathbb{R}^{n+1})$.

We then consider the problem

$$\begin{cases} \partial_t q - dk * (\Delta_x q + \partial_y^2 q) = \psi_f, \ t \in J, \, x \in \mathbb{R}^n, \, y \in \mathbb{R}, \\ q|_{t=0} = \psi_0, \ x \in \mathbb{R}^n, \, y \in \mathbb{R}, \end{cases} \tag{5.27}$$

on the space $X_{-1} := H_p^{-1}(\mathbb{R}^{n+1})$. By integration we see that (5.27) is equivalent to the Volterra equation

$$q(t) + (k * \Lambda q)(t) = (1 * \psi_f)(t) + \psi_0, \quad t \in J, \tag{5.28}$$

where $\Lambda = D_{n+1}$ with domain $\mathcal{D}(\Lambda) = H_p^1(\mathbb{R}^{n+1})$. One readily verifies that

- $1 * \psi_f + \psi_0 \in H_p^{\delta_a}(J; X_{-1})$;
- $\psi_0 \in (X_{-1}, \mathcal{D}(\Lambda))_{1-1/p\delta_a, p}$;
- $\psi_f(0) \in (X_{-1}, \mathcal{D}(\Lambda))_{1-1/\delta_a - 1/p\delta_a, p}$, if $\delta_a > 1 + \frac{1}{p}$.

So according to Theorem 3.1.4, (5.27) admits a unique solution ϕ_p in the space

$$H_p^{\delta_a}(J; H_p^{-1}(\mathbb{R}^{n+1})) \cap L_p(J; H_p^1(\mathbb{R}^{n+1})).$$

Note that ϕ_p only depends upon the data and the selected extension operator.

We now set

$$p_0 := \gamma p - \gamma \phi_p. \tag{5.29}$$

Then clearly, γp can be determined immediately as soon as p_0 is known, and vice versa. Putting $p_1 := \phi_p|_{\mathbb{R}^{n+1}_+} \in Z_{-1} := H^{\delta_a}_p(J; H^{-1}_p(\mathbb{R}^{n+1}_+)) \cap L_p(J; H^1_p(\mathbb{R}^{n+1}_+))$, it further follows from the construction of ϕ_p that (5.23) is equivalent to the identity

$$p = e^{-Gy} p_0 + p_1. \tag{5.30}$$

By the mixed derivative theorem, we have

$$Z_{-1} \hookrightarrow H_p^{\frac{\delta_a}{2}}(J; L_p(\mathbb{R}^{n+1}_+)) \cap L_p(J; H^1_p(\mathbb{R}^{n+1}_+)).$$

Notice also that

$$\begin{aligned} & e^{-Gy} p_0 \in {}_0H_p^{\frac{\delta_a}{2}}(J; L_p(\mathbb{R}^{n+1}_+)) \cap L_p(J; H^1_p(\mathbb{R}^{n+1}_+)) \\ \iff \quad & p_0 \in {}_0Y := {}_0B_{pp}^{\frac{\delta_a}{2}(1-\frac{1}{p})}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)). \end{aligned}$$

Consequently

$$p_0 \in {}_0Y \Rightarrow p \in H_p^{\frac{\delta_a}{2}}(J; L_p(\mathbb{R}^{n+1}_+)) \cap L_p(J; H^1_p(\mathbb{R}^{n+1}_+)).$$

According to Theorem 3.5.2, this regularity of p suffices to obtain $(v, w) \in Z$ when (5.21) is solved for this pair of functions. In fact, let

$$u = \begin{bmatrix} v \\ w \end{bmatrix}, \quad u_0 = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix}, \quad f = \begin{bmatrix} f_v - (db + \frac{1}{3}da) * \nabla_x p \\ f_w - (db + \frac{1}{3}da) * \partial_y p \end{bmatrix},$$

$$h = \begin{bmatrix} A(1 * g_v) \\ \gamma p \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -\nabla_x \\ -\nabla_x \cdot & 0 \end{bmatrix}.$$

Then system (5.21) is equivalent to the following problem for u.

$$\left\{ \begin{aligned} \partial_t u - da * \Delta_x u - da * \partial_y^2 u &= f, \quad t \in J,\ x \in \mathbb{R}^n,\ y > 0 \\ -\gamma \partial_y u + \gamma D u &= h, \quad t \in J,\ x \in \mathbb{R}^n \\ u|_{t=0} &= u_0, \quad x \in \mathbb{R}^n,\ y > 0. \end{aligned} \right. \tag{5.31}$$

Setting

$$f_1 = \begin{bmatrix} f_v - (db + \frac{1}{3}da) * \nabla_x p_1 \\ f_w - (db + \frac{1}{3}da) * \partial_y p_1 \end{bmatrix}, \quad h_1 = \begin{bmatrix} A(1 * g_v) \\ \gamma p_1 \end{bmatrix},$$

as well as

$$f_p = \begin{bmatrix} -A(b + \frac{1}{3}a) * \nabla_x e^{-Gy} p_0 \\ A(b + \frac{1}{3}a) * G e^{-Gy} p_0 \end{bmatrix}, \quad h_p = \begin{bmatrix} 0 \\ p_0 \end{bmatrix},$$

the solution u can be written as

$$u = u_1 + u_p, \tag{5.32}$$

where u_1 is defined by means of

$$\left\{ \begin{aligned} \partial_t u_1 - da * \Delta_x u_1 - da * \partial_y^2 u_1 &= f_1, \quad t \in J,\ x \in \mathbb{R}^n,\ y > 0 \\ -\gamma \partial_y u_1 + \gamma D u_1 &= h_1, \quad t \in J,\ x \in \mathbb{R}^n \\ u_1|_{t=0} &= u_0, \quad x \in \mathbb{R}^n,\ y > 0, \end{aligned} \right. \tag{5.33}$$

and u_p solves

$$\left\{ \begin{array}{rl} Au_p - \Delta_x u_p - \partial_y^2 u_p = f_p, & t \in J,\, x \in \mathbb{R}^n,\, y > 0 \\ -\gamma\partial_y u_p + \gamma D u_p = h_p, & t \in J,\, x \in \mathbb{R}^n. \end{array} \right. \tag{5.34}$$

Observe that u_1 is determined by the data and does not depend on p_0. Note further that the compatibility condition is satisfied in either case. Theorem 3.5.2 yields $u_1 \in Z$.

To summarize we see that step 2 shows the equivalence

$$(5.21), (5.23) \Leftrightarrow (5.30), (5.32), (5.34), \tag{5.35}$$

as well as the implication

$$p_0 \in {}_0Y \Rightarrow (v, w) \in Z. \tag{5.36}$$

Step 3. We will now employ condition (5.22), together with (5.32),(5.34), to derive a formula for p_0.

To begin with, the function u_p can be written in the form

$$u_p(y) = e^{-Fy}(F+D)^{-1}h_p + \frac{1}{2}F^{-1}\int_0^\infty [e^{-F|y-s|} + (F-D)(F+D)^{-1}e^{-F(y+s)}] f_p(s)\, ds$$

(cp. Prüss [65, p. 6]), which implies

$$\gamma u_p = (F+D)^{-1}h_p + (F+D)^{-1}\int_0^\infty e^{-Fs} f_p(s)\, ds.$$

A short computation using the Fourier transform shows that

$$(F+D)^{-1} = \begin{bmatrix} F + ((\nabla_x \nabla_x \cdot) + D_n)F^{-1} & \nabla_x \\ \nabla_x \cdot & F \end{bmatrix} F_1^{-2}, \quad F_1 := (A + 2D_n)^{\frac{1}{2}},$$

so we obtain

$$\begin{aligned} \gamma v_p \;=\;& \nabla_x F_1^{-2} p_0 - F F_1^{-2} \int_0^\infty e^{-Fs} A(b + \tfrac{1}{3}a) * \nabla_x e^{-Gs} p_0\, ds + \\ & + \nabla_x F_1^{-2} \int_0^\infty e^{-Fs} A(b + \tfrac{1}{3}a) * G e^{-Gs} p_0\, ds, \end{aligned}$$

and furthermore

$$\begin{aligned} \gamma \nabla_x \cdot v_p \;=\;& \Delta_x F_1^{-2} p_0 - \Delta_x F F_1^{-2} A(b + \tfrac{1}{3}a) * (F+G)^{-1} p_0 + \\ & + \Delta_x F_1^{-2} A(b + \tfrac{1}{3}a) * G(F+G)^{-1} p_0 \\ =\;& \Delta_x F_1^{-2} p_0 - \Delta_x F_1^{-2} A(b + \tfrac{1}{3}a) * (F-G)(F+G)^{-1} p_0. \end{aligned} \tag{5.37}$$

On the one hand, we now have

$$-\gamma\partial_y w = -\gamma\partial_y w_p - \gamma\partial_y w_1 = \gamma\nabla_x \cdot v_p + p_0 - \gamma\partial_y w_1. \tag{5.38}$$

On the other hand, it follows from (5.22) that

$$\begin{aligned} -\gamma\partial_y w \;=\;& \tfrac{1}{2}A(1 * g_w) - \tfrac{1}{2}A(b - \tfrac{2}{3}a) * \gamma p \\ =\;& \tfrac{1}{2}A(1 * g_w) - \tfrac{1}{2}A(b - \tfrac{2}{3}a) * \gamma p_1 - \tfrac{1}{2}A(b - \tfrac{2}{3}a) * p_0. \end{aligned} \tag{5.39}$$

Combining (5.38) with (5.39), setting

$$q_0 := \tfrac{1}{2}A(1 * g_w) - \tfrac{1}{2}A(b - \tfrac{2}{3}a) * \gamma p_1 + \gamma \partial_y w_1 \tag{5.40}$$

and using (5.37) leads to

$$\begin{aligned} q_0 &= p_0 + \tfrac{1}{2}A(b - \tfrac{2}{3}a) * p_0 + \gamma \nabla_x \cdot v_p \\ &= Aa * p_0 + \tfrac{1}{2}A(b - \tfrac{2}{3}a) * p_0 + \Delta_x F_1^{-2} p_0 - \Delta_x F_1^{-2} A(b + \tfrac{1}{3}a) * (F-G)(F+G)^{-1} p_0 \\ &= \tfrac{1}{2}A(b + \tfrac{4}{3}a) * p_0 - \Delta_x F_1^{-2} \left(-Aa * (F+G) + A(b + \tfrac{1}{3}a) * (F-G)\right)(F+G)^{-1} p_0 \\ &= \tfrac{1}{2}A(b + \tfrac{4}{3}a) * p_0 - \Delta_x F_1^{-2} \left(A(b - \tfrac{2}{3}a) * F - A(b + \tfrac{4}{3}a) * G\right)(F+G)^{-1} p_0. \end{aligned}$$

Further,

$$\begin{aligned} 2Ka * q_0 &= \left(I + 2D_n F_1^{-2}[K(b - \tfrac{2}{3}a) * F - G](F+G)^{-1}\right) p_0 \\ &= \left((A + 2D_n + 2D_n K(b - \tfrac{2}{3}a)*)F + AG\right)(F+G)^{-1} F_1^{-2} p_0 \\ &= \left(A + [2D_n + 2D_n K(b - \tfrac{2}{3}a)*]F(F+G)^{-1}\right) F_1^{-2} p_0 \\ &= \left(A + [2D_n(K(b + \tfrac{4}{3}a)*) + 2D_n K(b - \tfrac{2}{3}a)*]F(F+G)^{-1}\right) F_1^{-2} p_0 \\ &= \left(A + D_n K(4b + \tfrac{4}{3}a) * F(F+G)^{-1}\right) F_1^{-2} p_0 \;=\; L p_0, \end{aligned} \tag{5.41}$$

where the operator L is defined by

$$L = \left(A + D_n K(4b + \tfrac{4}{3}a) * F(F+G)^{-1}\right) F_1^{-2}. \tag{5.42}$$

We claim now that $q_0 \in {}_0Y$. Indeed, in virtue of (5.18),(5.19), there exist $\psi_1, \psi_2 \in Y$ such that

$$\begin{aligned} q_0 &= \tfrac{1}{2}(\psi_1 + Ab * \psi_2) - \tfrac{1}{2}A(b - \tfrac{2}{3}a) * \gamma p_1 + \gamma \partial_y w_1 \\ &= \tfrac{1}{2}(\psi_1 + A(b - \tfrac{2}{3}a) * \psi_2) + \tfrac{1}{3}\psi_2 - \tfrac{1}{2}A(b - \tfrac{2}{3}a) * \gamma p_1 + \gamma \partial_y w_1 \\ &= \tfrac{1}{2}A(b - \tfrac{2}{3}a) * (\psi_2 - \gamma p_1) + (\tfrac{1}{2}\psi_1 + \tfrac{1}{3}\psi_2 + \gamma \partial_y w_1), \end{aligned}$$

and

$$\psi_1|_{t=0} = \tfrac{2}{3}\gamma \nabla_x \cdot v_0 - \tfrac{4}{3}\gamma \partial_y w_0, \qquad \psi_2|_{t=0} = -\gamma \nabla_x \cdot v_0 - \gamma \partial_y w_0 \tag{5.43}$$

in case $p > 1 + 2/\delta_a$. But from (5.43) and the definition of p_1 and w_1, we deduce that

$$\psi_2 - \gamma p_1 \,,\; \tfrac{1}{2}\psi_1 + \tfrac{1}{3}\psi_2 + \gamma \partial_y w_1 \;\in {}_0Y.$$

Hence the claim follows, because $A(b*) \in \mathcal{B}({}_0Y)$.

From $q_0 \in {}_0Y$ and $K(a*) \in \mathcal{B}({}_0Y)$ we conclude further that $K(a * q_0) \in {}_0Y$. That is, to solve (5.41) for p_0, we have to show that L has a bounded inverse on ${}_0Y$. To achieve this, we shall use, aside from extension and restriction, the joint (causal) $\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta)$ - calculus $(0 < \eta < \pi/2)$ of the pair (∂_t, D_n) in $L_p(\mathbb{R}_+ \times \mathbb{R}^n)$, cf. Example 2.4.1.

For this purpose we look at the symbol $l(z, \xi)$ of L (in $L_p(\mathbb{R}_+ \times \mathbb{R}^n)$). Taking the Laplace-transform in t and the Fourier-transform in x we obtain for $l(z, \xi)$:

$$l(z,\xi) = \left(\frac{1}{\hat{a}(z)|\xi|^2} + \frac{\frac{4\hat{b}(z) + \frac{4}{3}\hat{a}(z)}{\hat{b}(z) + \frac{4}{3}\hat{a}(z)} \sqrt{\frac{1}{\hat{a}(z)|\xi|^2} + 1}}{\sqrt{\frac{1}{\hat{a}(z)|\xi|^2} + 1} + \sqrt{\frac{1}{(\hat{b}(z) + \frac{4}{3}\hat{a}(z))|\xi|^2} + 1}} \right) \left(\frac{1}{\hat{a}(z)|\xi|^2} + 2 \right)^{-1}. \tag{5.44}$$

It can be written as

$$l(z,\xi) = \left(\zeta + \frac{4(1-\kappa)\sqrt{\zeta+1}}{\sqrt{\zeta+1}+\sqrt{\kappa\zeta+1}}\right)(\zeta+2)^{-1}, \tag{5.45}$$

where

$$\zeta(z,\xi) := \frac{1}{\hat{a}(z)|\xi|^2}, \quad \kappa(z) := \frac{\hat{a}(z)}{\hat{b}(z)+\frac{4}{3}\hat{a}(z)}.$$

We first study the function k defined by

$$k(z,\tau) = \frac{1}{\hat{a}(z)\tau^2} + \frac{4(1-\kappa(z))\sqrt{\frac{1}{\hat{a}(z)\tau^2}+1}}{\sqrt{\frac{1}{\hat{a}(z)\tau^2}+1}+\sqrt{\frac{1}{(\hat{b}(z)+\frac{4}{3}\hat{a}(z))\tau^2}+1}}, \quad (z,\tau) \in \Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta.$$

Remember that both $\hat{a}$ and $\hat{b}$ are analytic functions in $\mathbb{C} \setminus \mathbb{R}_-$, since a and b are assumed to be of type (E).

Lemma 5.3.1 *There exist $c > 0$, $\eta > 0$ such that*

$$|k(z,\tau)| \geq c\left(\left|\frac{1}{\hat{a}(z)\tau^2}\right| + 1\right), \quad (z,\tau) \in \Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta. \tag{5.46}$$

Proof. Let $\nu = 1/(\hat{a}(z)\tau^2)$. Assume for the moment that z is fixed with $\arg(z) \in [0,\pi/2)$ and $\tau \in (0,\infty)$. Then we have $\arg(1/\widehat{da}(z)) \in [0,\theta_a]$ and $\arg(1/\widehat{db}(z)) \in [0,\theta_b]$. Now we examine two cases.

Case 1: $\arg(1/\widehat{da}(z)) \geq \arg(1/\widehat{db}(z))$. It follows that

$$\arg\left(\frac{1}{\widehat{db}(z)}\right) \leq \arg\left(\frac{1}{\widehat{da}(z)} + \frac{\omega}{\widehat{db}(z)}\right) \leq \arg\left(\frac{1}{\widehat{da}(z)}\right), \quad \forall \omega \geq 0.$$

Thus we have

$$\arg(\kappa) = \arg\left(\frac{\widehat{da}(z)}{\widehat{db}(z)+\frac{4}{3}\widehat{da}(z)}\right) = \arg\left(\frac{\frac{1}{\widehat{db}(z)}}{\frac{1}{\widehat{da}(z)}+\frac{4}{3}\frac{1}{\widehat{db}(z)}}\right) \leq 0$$

as well as $\arg(1-\kappa) \geq 0$. Moreover, it is easy to see that $\arg(1-\kappa) \leq \theta_a$ and $|\kappa| < 1$. From $\arg(1/\widehat{da}(z)) \in [0,\theta_a]$ and $\arg(z) \in [0,\pi/2)$ we infer that $\arg(\nu) \in [0,\pi/2+\theta_a)$. Since $\arg(\kappa) \leq 0$, we have $\arg(\kappa\nu) \leq \arg(\nu)$. This together with $|\kappa\nu| < |\nu|$ and $\arg(\nu) \geq 0$ implies $\arg(\kappa\nu+1) \leq \arg(\nu+1)$. Therefore, $\arg(1+\sqrt{(\kappa\nu+1)/(\nu+1)}) \leq 0$. On the other hand we have $\arg(\kappa\nu) \in [0,\pi/2+\theta_a)$, by definition of κ and the inequality

$$0 \leq \arg\left((\widehat{db}(z)+\tfrac{4}{3}\widehat{da}(z))^{-1}\right) \leq \arg\left(\frac{1}{\widehat{da}(z)}\right).$$

Thus $\arg(1+\sqrt{(\kappa\nu+1)/(\nu+1)}) \in (-(\pi/4+\theta_a/2),0]$. By writing

$$k(z,\tau) = \nu + 4(1-\kappa(z))\left(1+\sqrt{\frac{\kappa\nu+1}{\nu+1}}\right)^{-1}$$

we see that both summands in the formula for $k(z,\tau)$ lie in the sector $\Sigma_{\pi/2+\theta_a} \cap \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$. This means in particular $k(z,\tau) \neq 0$.

Case 2: $\arg(1/\widehat{da}(z)) \leq \arg(1/\widehat{db}(z))$. This time we have $\arg(\kappa) \in [0,\theta_b]$, $\arg(1-\kappa) \leq 0$ and again $|\kappa| < 1$. We write $k(z,\tau)$ as

$$k(z,\tau) = \frac{\hat{b}(z) + \frac{1}{3}\hat{a}(z)}{\hat{a}(z)} \left(\frac{1}{(\hat{b}(z) + \frac{1}{3}\hat{a}(z))\tau^2} + \frac{4\kappa\sqrt{\nu+1}}{\sqrt{\nu+1} + \sqrt{\kappa\nu+1}} \right).$$

Clearly, $1/[(\hat{b}(z) + 1/3\hat{a}(z))\tau^2] \in \Sigma_{\pi/2+\theta_b} \cap \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$. Now we look at the second summand. The inequality $\arg(1-\kappa) \leq 0$ yields

$$\begin{aligned} \arg(1+\kappa\nu) &= \arg((1-\kappa) + \kappa(1+\nu)) \;\leq\; \arg(\kappa(1+\nu)) \\ &= \arg(\kappa) + \arg(1+\nu). \end{aligned} \tag{5.47}$$

By employing (5.47), we get

$$\begin{aligned} \arg\left(\frac{\kappa\sqrt{\nu+1}}{\sqrt{\nu+1} + \sqrt{\kappa\nu+1}} \right) &\geq \arg(\kappa) + \tfrac{1}{2}\arg(1+\nu) - \tfrac{1}{2}\max\{\arg(1+\nu), \arg(1+\kappa\nu)\} \\ &\geq \tfrac{1}{2}\arg(\kappa) \;\geq 0. \end{aligned}$$

On the other hand it is easy to see that

$$\arg\left(\frac{\kappa\sqrt{\nu+1}}{\sqrt{\nu+1} + \sqrt{\kappa\nu+1}} \right) \leq \tfrac{\pi}{4} + \tfrac{3}{2}\,\theta_b.$$

Therefore both summands in parentheses lie in the sector $\Sigma_{\pi/2+\theta_b} \cap \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$.

If $\arg(z) \in (-\pi/2, 0]$ then all signs of the arguments in the above lines change which means that the summands under consideration lie in the corresponding sectors in the lower half plane.

By continuity of the argument function, there exists $\eta > 0$ such that, in each case, the summands under consideration lie in a sector of angle $\theta < \pi$, for all $(z,\tau) \in \Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta$. Consequently, there is $c > 0$ such that

$$|k(z,\tau)| \geq c\left(|\nu| + \left| \frac{4(1-\kappa)\sqrt{\nu+1}}{\sqrt{\nu+1} + \sqrt{\kappa\nu+1}} \right| \right),$$

for all $(z,\tau) \in \Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta$. From the boundedness of the function ψ defined by

$$\psi(\rho) = \frac{1 + \frac{4}{3}\rho}{1 + \frac{1}{3}\rho}, \quad \rho \in \Sigma_{\frac{\pi}{2}+\eta},$$

it follows that $|1-\kappa(z)|$ is bounded away from zero. We also see that the term $\sqrt{(\kappa\nu+1)/(\nu+1)}$ is bounded, for all $(z,\tau) \in \Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta$. Thus we obtain the desired estimate (5.46). □

By (5.46), it follows that the function l_0 defined by

$$l_0(z,\tau) = \frac{\nu+2}{k(z,\tau)}, \quad (z,\tau) \in \Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta \tag{5.48}$$

belongs to $\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta)$. Hence the associated operator is bounded in $L_p(\mathbb{R}_+ \times \mathbb{R}^n)$, by the joint $\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}+\eta} \times \Sigma_\eta)$ - calculus of the pair (∂_t, D_n). By causality, extension and restriction, it is then clear that $L^{-1} \in \mathcal{B}(L_p(J \times \mathbb{R}^n))$. In view of $[F^{-1}, L^{-1}] = 0$ we also have $L^{-1} \in \mathcal{B}(\mathcal{D}(F))$, where $\mathcal{D}(F) = {}_0H_p^{\delta_a/2}(J; L_p(\mathbb{R}^n)) \cap L_p(J; H_p^1(\mathbb{R}^n))$. Since ${}_0Y = (L_p(J; L_p(\mathbb{R}^n)), \mathcal{D}(F))_{1-1/p,p}$, by real interpolation it follows that $L^{-1} \in \mathcal{B}({}_0Y)$.

In sum we have proved

Theorem 5.3.1 *Let $1 < p < \infty$, and suppose that the kernels $a \neq 0$ and b are of type (E). Let δ_a and δ_b denote the regularization order of a and b, respectively, and assume that $\delta_a \le \delta_b$. Suppose further that $\delta_a \notin \{\frac{2}{p-1}, 1 + \frac{1}{p}\}$. Then (5.12) has a unique solution $(v, w) \in Z$ if and only if the conditions* **(N1)** *are satisfied.*

5.3.3 The case $0 < \delta_a - \delta_b < 1/p$

Let $\kappa = \delta_a - \delta_b$ $(= \alpha - \beta)$. Suppose that $(v, w) \in Z$ solves (5.12) and satisfies in addition

$$p = -\nabla_x \cdot v - \partial_y w \in H_p^\kappa(J; H_p^1(\mathbb{R}_+^{n+1})). \tag{5.49}$$

According to Corollary 2.8.1, the latter implies

$$(db * \nabla_x p, db * \partial_y p) \in ({}_0H_p^{\delta_a - 1}(J; L_p(\mathbb{R}_+^{n+1})))^{n+1}.$$

In light of (5.21), we therefore obtain again necessity of (5.13) and (5.15). In the same way as in the case $\delta_a < \delta_b$, we further see that conditions (5.14),(5.16), and (5.17) are necessary. Concerning g_w we deduce from (5.22) that

$$g_w = da * \psi_1 + db * \psi_2, \quad \text{with } \psi_1 \in Y,\ \psi_2 \in Y_\kappa, \tag{5.50}$$

where

$$Y_\kappa = B_{pp}^{\frac{\delta_b}{2}(1-\frac{1}{p})+\kappa}(J; L_p(\mathbb{R}^n)) \cap H_p^\kappa(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)).$$

Observe as well that we have the compatibility conditions

$$\psi_1|_{t=0} = \tfrac{2}{3}\gamma \nabla_x \cdot v_0 - \tfrac{4}{3}\gamma \partial_y w_0, \quad \text{if } p > 1 + \tfrac{2}{\delta_a}, \tag{5.51}$$

$$\psi_2|_{t=0} = -\gamma \nabla_x \cdot v_0 - \gamma \partial_y w_0, \quad \text{if } p > \tfrac{2+\delta_b}{2\kappa+\delta_b}. \tag{5.52}$$

Finally, from (5.49) and (5.23) there emerge the two conditions

$$\nabla_x \cdot v_0 + \partial_y w_0 \in B_{pp}^{1+\frac{2\kappa}{\delta_b}-\frac{2}{p\delta_b}}(\mathbb{R}_+^{n+1}), \tag{5.53}$$

$$(\nabla_x \cdot f_v + \partial_y f_w)_{t=0} \in B_{pp}^{1+\frac{2\kappa}{\delta_b}-\frac{2}{\delta_b}-\frac{2}{p\delta_b}}(\mathbb{R}_+^{n+1}), \quad \text{if } \alpha > \tfrac{1}{p}, \tag{5.54}$$

where $\nabla_x \cdot$ and ∂_y have to be understood in the distributional sense.

In sum we have shown necessity of

$$\textbf{(N2)} \quad (5.13), (5.14), (5.15), (5.16), (5.17), (5.50) - (5.54).$$

Turning to the converse, we suppose that all conditions in (N2) are fulfilled. Let us look first at q_0. In virtue of (5.50),(5.51), and (5.52), we see that

$$q_0 = \tfrac{1}{2}A(b - \tfrac{2}{3}a) * (\psi_2 - \gamma p_1) + (\tfrac{1}{2}\psi_1 + \tfrac{1}{3}\psi_2 + \gamma \partial_y w_1),$$

with $\psi_1 \in Y$, $\psi_2 \in Y_\kappa$ and

$$\psi_2 - \gamma p_1 \in {}_0Y_\kappa\,,\ \tfrac{1}{2}\psi_1 + \tfrac{1}{3}\psi_2 + \gamma\partial_y w_1 \in {}_0Y,$$

where

$${}_0Y_\kappa := {}_0B_{pp}^{\frac{\delta_b}{2}(1-\frac{1}{p})+\kappa}(J; L_p(\mathbb{R}^n)) \cap H_p^\kappa(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)).$$

Thus

$$q_0 \in {}_0B_{pp}^{\frac{\delta_b}{2}(1-\frac{1}{p})}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)),$$

which entails $K(a*q_0) \in {}_0Y_\kappa$. In view of $L^{-1} \in \mathcal{B}({}_0Y_\kappa)$ and $2K(a*q_0) = Lp_0$, it therefore follows that $p_0 \in {}_0Y_\kappa$. Observe now that

$$p_0 \in {}_0Y_\kappa \Leftrightarrow e^{-Gy}p_0 \in {}_0H_p^{\frac{\delta_b}{2}+\kappa}(J; L_p(\mathbb{R}^{n+1}_+)) \cap H_p^\kappa(J; H_p^1(\mathbb{R}^{n+1}_+)).$$

Concerning p_1, we proceed as in the case $\delta_a \le \delta_b$. According to Theorem 3.1.4, it follows from (5.13),(5.53),(5.54) that (5.27) admits a unique solution ϕ_p in the space

$$H_p^{\delta_a}(J; H_p^{-1}(\mathbb{R}^{n+1})) \cap H_p^\kappa(J; H_p^1(\mathbb{R}^{n+1})),$$

which is embedded into

$$H_p^{\frac{\delta_b}{2}+\kappa}(J; L_p(\mathbb{R}^{n+1}_+)) \cap H_p^\kappa(J; H_p^1(\mathbb{R}^{n+1}_+)),$$

by the mixed derivative theorem. Consequently, due to (5.30),

$$p \in H_p^{\frac{\delta_b}{2}+\kappa}(J; L_p(\mathbb{R}^{n+1}_+)) \cap H_p^\kappa(J; H_p^1(\mathbb{R}^{n+1}_+)),$$

as well as $\gamma p \in Y_\kappa$. From

$$\tfrac{\delta_b}{2}(1-\tfrac{1}{p}) + \kappa > \tfrac{\delta_a}{2}(1-\tfrac{1}{p})$$

we then deduce

$$Y_\kappa \hookrightarrow B_{pp}^{\frac{\delta_a}{2}(1-\frac{1}{p})}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)).$$

Hence, p and γp lie in the right regularity classes when (5.21) is solved for (v,w). Using this fact, together with (5.13),(5.14),(5.15),(5.16), and (5.17), Theorem 3.5.2 yields $(v,w) \in Z$.

Theorem 5.3.2 *Let $1 < p < \infty$, and suppose that the kernels $a \ne 0$ and b are of type (E). Let δ_a and δ_b denote the regularization order of a and b, respectively, and assume that $0 < \kappa := \delta_a - \delta_b < 1/p$. Suppose further that $\delta_a \notin \{\frac{2}{p-1}, 1+\frac{1}{p}\}$ as well as $p(2\delta_a - \delta_b) \ne 2 + \delta_b$. Then (5.12) has a unique solution $(v,w) \in Z$ satisfying (5.49) if and only if the data are subject to the conditions* **(N2)**.

5.3.4 The case $\delta_a - \delta_b > 1/p$

Let $\kappa > 1/p$. Suppose that $(v,w) \in Z$ solves (5.12) and satisfies in addition (5.49). Then the latter implies

$$p \in C(J; H_p^1(\mathbb{R}^{n+1}_+)),$$

in particular

$$p|_{t=0} = -\nabla_x \cdot v_0 - \partial_y w_0 \in H_p^1(\mathbb{R}^{n+1}_+), \tag{5.55}$$

which allows us to write the first two equations in (5.21) as

$$\begin{cases} \partial_t v - da * (\Delta_x v + \partial_y^2 v) + db * \nabla_x (p - p|_{t=0}) + \frac{1}{3} da * \nabla_x p = f_v - b(\nabla_x p|_{t=0}), \\ \partial_t w - da * (\Delta_x w + \partial_y^2 w) + db * \partial_y (p - p|_{t=0}) + \frac{1}{3} da * \partial_y p = f_w - b(\partial_y p|_{t=0}). \end{cases} \tag{5.56}$$

Owing to $(v, w) \in Z$ and

$$(\nabla_x (p - p|_{t=0}), \partial_y (p - p|_{t=0})) \in ({}_0H_p^\kappa(J; L_p(\mathbb{R}_+^{n+1})))^{n+1},$$

it is clear that all terms on the left-hand side of (5.56) are functions in the space $H_p^{\delta_a - 1}(J; L_p(\mathbb{R}_+^{n+1}))$. Thus

$$f_v = h_v + b(\nabla_x p|_{t=0}), \qquad f_w = h_w + b(\partial_y p|_{t=0}), \tag{5.57}$$

with

$$(h_v, h_w) \in (H_p^{\delta_a - 1}(J; L_p(\mathbb{R}_+^{n+1})))^{n+1}, \tag{5.58}$$

and furthermore

$$(h_v, h_w)|_{t=0} \in (B_{pp}^{2(1 - \frac{1}{\delta_a} - \frac{1}{p\delta_a})}(\mathbb{R}_+^{n+1}))^{n+1}. \tag{5.59}$$

As in the two cases before one can see that (5.14),(5.16),(5.17) are necessary.

We now consider (5.23). Note that in view of (5.57) and (5.58) we have in the distributional sense

$$\nabla_x \cdot f_v + \partial_y f_w = \nabla_x \cdot h_v + \partial_y h_w + b(\Delta_x + \partial_y^2) p|_{t=0}.$$

So it follows from $(v, w) \in Z$ and (5.49), cp. Theorem 3.3.1, that

$$(\nabla_x \cdot h_v + \partial_y h_w)|_{t=0} \in B_{pp}^{1 + \frac{2\kappa}{\delta_b} - \frac{2}{\delta_b} - \frac{2}{p\delta_b}}(\mathbb{R}_+^{n+1}). \tag{5.60}$$

Turning to g_w, observe first that $(v, w) \in Z$ and (5.49) entail

$$\partial_t p \in H_p^{\frac{\delta_b}{2} + \kappa - 1}(J; L_p(\mathbb{R}_+^{n+1})) \cap L_p(J; H_p^{1 + \frac{2\kappa}{\delta_b} - \frac{2}{\delta_b}}(\mathbb{R}_+^{n+1})), \quad \text{if } \tfrac{\delta_b}{2} + \kappa - 1 > 0,$$

and

$$\gamma p \in B_{pp}^{\frac{\delta_b}{2}(1 - \frac{1}{p}) + \kappa}(J; L_p(\mathbb{R}^n)) \cap H_p^\kappa(J; B_{pp}^{1 - \frac{1}{p}}(\mathbb{R}^n)).$$

Therefore

$$\gamma \partial_t p \in B_{pp}^{\frac{\delta_b}{2}(1 - \frac{1}{p}) + \kappa - 1}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{1 + \frac{2\kappa}{\delta_b} - \frac{2}{\delta_b} - \frac{1}{p}}(\mathbb{R}^n)), \text{ if } \tfrac{\delta_b}{2}(1 - \tfrac{1}{p}) + \kappa > 1,$$

as well as

$$(\gamma \partial_t p)|_{t=0} \in B_{pp}^\eta(\mathbb{R}^n), \quad \text{if } \eta := 1 + \tfrac{2\kappa}{\delta_b} - \tfrac{2}{\delta_b} - \tfrac{2}{p\delta_b} - \tfrac{1}{p} > 0,$$

the latter also being a consequence of (5.60). So we conclude from (5.22) that g_w is of the structure

$$g_w = da * \psi_1 + db * \psi_2, \quad \text{with } \psi_1 \in Y, \ \psi_2 \in Y_\kappa, \tag{5.61}$$

where Y_κ is defined as in the previous case, that is

$$Y_\kappa = B_{pp}^{\frac{\delta_b}{2}(1 - \frac{1}{p}) + \kappa}(J; L_p(\mathbb{R}^n)) \cap H_p^\kappa(J; B_{pp}^{1 - \frac{1}{p}}(\mathbb{R}^n)),$$

and ψ_1, ψ_2 are subject to the compatibility conditions

$$\psi_1|_{t=0} = \tfrac{2}{3}\gamma\nabla_x \cdot v_0 - \tfrac{4}{3}\gamma\partial_y w_0, \tag{5.62}$$

$$\psi_2|_{t=0} = -\gamma\nabla_x \cdot v_0 - \gamma\partial_y w_0, \tag{5.63}$$

and

$$\partial_t\psi_2|_{t=0} = -(\nabla_x \cdot h_v + \partial_y h_w)|_{t=0}, \quad \text{if } \eta > 0. \tag{5.64}$$

All in all we have established necessity of

$$(\mathbf{N3}) \quad (5.14), (5.16), (5.17), (5.55), (5.57) - (5.64).$$

That these conditions are also sufficient for the existence of a unique pair $(v, w) \in Z$ solving (5.12) and satisfying (5.49), is shown in the following.

Suppose that (N3) is fulfilled. We first investigate the regularity of q_0. Using assumptions (5.61)-(5.64) we see that

$$q_0 = \tfrac{1}{2}A(b - \tfrac{2}{3}a) * (\psi_2 - \gamma p_1) + (\tfrac{1}{2}\psi_1 + \tfrac{1}{3}\psi_2 + \gamma\partial_y w_1),$$

with $\psi_1 \in Y$, $\psi_2 \in Y_\kappa$ and

$$\psi_2 - \gamma p_1 \in {}_0Y_\kappa\,,\ \tfrac{1}{2}\psi_1 + \tfrac{1}{3}\psi_2 + \gamma\partial_y w_1 \in {}_0Y,$$

where

$${}_0Y_\kappa := {}_0B_{pp}^{\frac{\delta_b}{2}(1-\frac{1}{p})+\kappa}(J; L_p(\mathbb{R}^n)) \cap {}_0H_p^\kappa(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)).$$

Therefore

$$q_0 \in {}_0B_{pp}^{\frac{\delta_b}{2}(1-\frac{1}{p})}(J; L_p(\mathbb{R}^n)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\mathbb{R}^n)),$$

and so, by the same conclusions as in the previous case, we find that

$$e^{-Gy}p_0 \in {}_0H_p^{\frac{\delta_b}{2}+\kappa}(J; L_p(\mathbb{R}^{n+1}_+)) \cap {}_0H_p^\kappa(J; H_p^1(\mathbb{R}^{n+1}_+)).$$

Next we look at p_1. From (5.14),(5.55),(5.57),(5.58), and (5.60) it follows by Theorem 3.3.1 that (5.27) has a unique solution ϕ_p in the space

$$H_p^{\delta_a}(J; H_p^{-1}(\mathbb{R}^{n+1})) \cap H_p^\kappa(J; H_p^1(\mathbb{R}^{n+1})).$$

So we can argue as in the case $\kappa \in (0, 1/p)$ to see that (5.12) admits a unique solution $(v, w) \in Z$ with (5.49).

Theorem 5.3.3 *Let $1 < p < \infty$, and suppose that the kernels $a \neq 0$ and b are of type (E). Let δ_a and δ_b denote the regularization order of a and b, respectively, and assume that $\kappa = \delta_a - \delta_b > 1/p$. Suppose further that $\delta_a \neq \frac{2}{p-1}$ as well as $p(2\delta_a - \delta_b) \neq 2 + \delta_b + 2p$. Then (5.12) has a unique solution $(v, w) \in Z$ satisfying (5.49) if and only if the data are subject to the conditions* **(N3)**.

Chapter 6

Nonlinear Problems

6.1 Quasilinear problems of second order with nonlinear boundary conditions

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary Γ which decomposes as $\Gamma = \Gamma_D \cup \Gamma_N$ with $\mathrm{dist}(\Gamma_D, \Gamma_N) > 0$. Let further $J_0 = [0, T_0]$ be a compact time-interval and $U_0 \subset \mathbb{R}$, $U_1 \subset \mathbb{R}^n$ be nonempty open convex sets. With the functions $a : J_0 \times \overline{\Omega} \times U_0 \times U_1 \to \mathbb{R}^{n\times n}$, f, $g : J_0 \times \Omega \times U_0 \times U_1 \to \mathbb{R}$, $b^D : J_0 \times \Gamma_D \times U_0 \to \mathbb{R}$, and $b^N : J_0 \times \Gamma_N \times U_0 \times U_1 \to \mathbb{R}$, we put

$$\begin{aligned}
\mathcal{A}(u)(t,x) &= -a(t,x,u(t,x),\nabla u(t,x)),\ t \in J_0,\ x \in \Omega,\\
F(u)(t,x) &= f(t,x,u(t,x),\nabla u(t,x)),\ t \in J_0,\ x \in \Omega,\\
G(u)(t,x) &= g(t,x,u(t,x),\nabla u(t,x)),\ t \in J_0,\ x \in \Omega,\\
\mathcal{B}_D(u)(t,x) &= b^D(t,x,u(t,x)),\ t \in J_0,\ x \in \Gamma_D,\\
\mathcal{B}_N(u)(t,x) &= b^N(t,x,u(t,x),\nabla u(t,x)),\ t \in J_0,\ x \in \Gamma_N,
\end{aligned}$$

where $\nabla = \nabla_x$ refers to the spatial variables, and $u : J_0 \times \overline{\Omega} \to \mathbb{R}$ is a $C(J_0; C^1(\overline{\Omega}))$ function subject to $u(t,x) \in U_0$ and $\nabla u(t,x) \in U_1$, for all $t \in J_0$, $x \in \overline{\Omega}$.

Let further $k \in BV_{loc}(\mathbb{R}_+) \cap \mathcal{K}^1(1+\alpha, \theta)$ with $k(0) = 0$, $\theta < \pi$ and $\alpha \in [0,1)$. Then the problem under consideration reads as

$$\left\{\begin{aligned}
\partial_t u + dk * (\mathcal{A}(u) : \nabla^2 u) &= F(u) + dk * G(u), && t \in J_0,\ x \in \Omega\\
\mathcal{B}_D(u) &= 0, && t \in J_0,\ x \in \Gamma_D\\
\mathcal{B}_N(u) &= 0, && t \in J_0,\ x \in \Gamma_N\\
u|_{t=0} &= u_0, && x \in \Omega.
\end{aligned}\right. \tag{6.1}$$

Our goal is to prove unique existence of a local strong solution, more precisely, we are looking for an interval $J = [0, T]$ with $0 < T \le T_0$ and a unique solution u of (6.1) on J in the space

$$Z^T := H_p^{1+\alpha}(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)).$$

This will be achieved under appropriate assumptions by means of maximal L_p-regularity of a linear problem related to (6.1) and the contraction mapping principle.

To fix notation, we denote the independent variables by $t \in J_0$, $x \in \overline{\Omega}$, $\xi \in U_0$, and $\eta \in U_1$. For what is to follow we need the spaces

$$X^T = L_p(J; L_p(\Omega)), \quad X_1^T = H_p^\alpha(J; L_p(\Omega)), \tag{6.2}$$

$$Z_\nabla^T = H_p^{\frac{1}{2}(1+\alpha)}(J; L_p(\Omega)) \cap L_p(J; H_p^1(\Omega)),$$

$$Y_D^T = B_{pp}^{(1+\alpha)(1-\frac{1}{2p})}(J; L_p(\Gamma_D)) \cap L_p(J; B_{pp}^{2-\frac{1}{p}}(\Gamma_D)), \tag{6.3}$$

$$Y_N^T = B_{pp}^{(1+\alpha)(\frac{1}{2}-\frac{1}{2p})}(J; L_p(\Gamma_N)) \cap L_p(J; B_{pp}^{1-\frac{1}{p}}(\Gamma_N)), \tag{6.4}$$

and

$$Y_0 = B_{pp}^{2-\frac{2}{p(1+\alpha)}}(\Omega), \quad Y_1 = B_{pp}^{2-\frac{2}{1+\alpha}-\frac{2}{p(1+\alpha)}}(\Omega),$$

$\alpha > 1/p$ being assumed in the definition of Y_1. For $\Lambda \in \{Z^T, Z_\nabla^T, Y_D^T, Y_N^T, X_1^T)$, we as usual denote by ${}_0\Lambda$ the subspace of all functions $h \in \Lambda$ with $h|_{t=0} = 0$ and $\partial_t h|_{t=0} = 0$, in case that these traces exist.

If $u \in Z^T$, then this corresponds, as we know from Theorem 4.3.1, to the regularity classes

$$\nabla u \in (Z_\nabla^T)^n, \nabla^2 u \in (X^T)^{n\times n}, \gamma_D u \in Y_D^T, \gamma_N \nabla u \in (Y_N^T)^n, u|_{t=0} \in Y_0, \partial_t u|_{t=0} \in Y_1.$$

Consequently, Y_0 is the natural space for u_0, and if $\alpha > 1/p$ and $u \in Z^T$ is a solution of (6.1), then we have to ensure that $u_1 := \partial_t u|_{t=0}$, which is given by

$$u_1(x) = f(0, x, u_0(x), \nabla u_0(x)),\ x \in \Omega,$$

belongs to Y_1. Of course, we have to assume that

$$u_0(x) \in U_0, \quad \nabla u_0(x) \in U_1, \quad x \in \overline{\Omega}. \tag{6.5}$$

Observe also that

$$Z^T \hookrightarrow C(J; Y_0) \hookrightarrow C(J \times \overline{\Omega})$$

as well as

$$Z_\nabla^T \hookrightarrow C(J; B_{pp}^{1-\frac{2}{p(1+\alpha)}}(\Omega)) \hookrightarrow C(J \times \overline{\Omega}),$$

provided that $1 - 2/p(1+\alpha) > n/p$, which is equivalent to

$$p > \tfrac{2}{1+\alpha} + n \tag{6.6}$$

and which will be assumed throughout this section.

Notice further that we have to take into account the three compatibility conditions

$$\begin{aligned} b^D(0, x, u_0(x)) &= 0,\ x \in \Gamma_D, \\ b^N(0, x, u_0(x), \nabla u_0(x)) &= 0,\ x \in \Gamma_N, \\ b_t^D(0, x, u_0(x)) + b_\xi^D(0, x, u_0(x))u_1(x) &= 0,\ x \in \Gamma_D,\ \text{if } \alpha > \tfrac{3}{2p-1}. \end{aligned}$$

Here and in the subsequent lines we assume that b^D and b^N are as smooth as needed to make the formulas meaningful. Precise regularity statements will be given later on.

We now set out to reformulate (6.1) as a fixed point problem in an appropriate subset of Z^T. To this end, we first put

$$\mathcal{A}_0(x) = -a(0, x, u_0(x), \nabla u_0(x)),\ x \in \overline{\Omega}.$$

Further we fix a function $\phi \in Z^{T_0}$ which satisfies $\phi|_{t=0} = u_0$ and $\partial_t \phi|_{t=0} = u_1$, the latter being demanded in case that $\alpha > 1/p$. In view of (6.5) and (6.6), we see that, for T

sufficiently small, say $T \le T_1 \le T_0$, we have $\phi(t,x) \in U_0$ and $\nabla\phi(t,x) \in U_1$ for all $t \in J$ and $x \in \overline{\Omega}$. So for such T we may define operators $\mathcal{B}_K^\circ(\phi)$ and $\mathcal{R}_K^\phi$, $K = D$, N, by means of

$$\begin{aligned}
\mathcal{B}_D^\circ(\phi)u(t,x) &= b_\xi^D(t,x,\phi(t,x))u(t,x),\ t \in J,\ x \in \Gamma_D, \\
\mathcal{B}_N^\circ(\phi)u(t,x) &= b_\xi^N(t,x,\phi(t,x),\nabla\phi(t,x))u(t,x) \\
&\quad + b_\eta^N(t,x,\phi(t,x),\nabla\phi(t,x)) \cdot \nabla u(t,x),\ t \in J,\ x \in \Gamma_N,
\end{aligned}$$

and

$$\mathcal{R}_K^\phi(u) = \mathcal{B}_K(u) - \mathcal{B}_K(\phi) - \mathcal{B}_K^\circ(\phi)(u-\phi), \quad K = D, N.$$

Obviously, (6.1) restricted to J is equivalent to

$$\left\{\begin{array}{rll}
\partial_t u + dk * \mathcal{A}_0 : \nabla^2 u = & F(u) + dk * G(u) & \\
& +dk * ((\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u) & (J \times \Omega) \\
\mathcal{B}_D^\circ(\phi)u = & -\mathcal{B}_D(\phi) + \mathcal{B}_D^\circ(\phi)\phi - \mathcal{R}_D^\phi(u) & (J \times \Gamma_D) \\
\mathcal{B}_N^\circ(\phi)u = & -\mathcal{B}_N(\phi) + \mathcal{B}_N^\circ(\phi)\phi - \mathcal{R}_N^\phi(u) & (J \times \Gamma_N) \\
u|_{t=0} = & u_0 & (\Omega).
\end{array}\right. \tag{6.7}$$

In other words, $u \in Z^T$ solves a problem of the form

$$\left\{\begin{array}{rll}
\partial_t v + dk * \mathcal{A}_0 : \nabla^2 v = & h & (J \times \Omega) \\
\mathcal{B}_D^\circ(\phi)v = & \psi^D & (J \times \Gamma_D) \\
\mathcal{B}_N^\circ(\phi)v = & \psi^N & (J \times \Gamma_N) \\
v|_{t=0} = & v_0 & (\Omega).
\end{array}\right. \tag{6.8}$$

with certain functions on the right-hand side. Given data h, ψ^D, ψ^N and v_0, (6.8) is a *linear* problem. For our construction, it is essential to understand this problem, in particular, one needs a precise characterization of unique solvability of (6.8) in Z^T in terms of regularity and compatibility conditions for the data.

Let us assume for the moment that we have such a result at our disposal - possibly on a yet smaller time-interval - and that the right-hand sides of the following both problems (6.9) and (6.11), which are of the form (6.8), fulfill the conditions needed to get a unique solution in Z^T in either case. Then it makes sense to define the *reference function* $w \in Z^T$ as solution of the linear problem

$$\left\{\begin{array}{rll}
\partial_t w + dk * \mathcal{A}_0 : \nabla^2 w = & F(\phi) + dk * G(\phi) & (J \times \Omega) \\
\mathcal{B}_D^\circ(\phi)w = & -\mathcal{B}_D(\phi) + \mathcal{B}_D^\circ(\phi)\phi & (J \times \Gamma_D) \\
\mathcal{B}_N^\circ(\phi)w = & -\mathcal{B}_N(\phi) + \mathcal{B}_N^\circ(\phi)\phi & (J \times \Gamma_N) \\
w|_{t=0} = & u_0 & (\Omega).
\end{array}\right. \tag{6.9}$$

Given $\rho > 0$, let

$$\Sigma(\rho, T, \phi) = \{v \in Z^T :\ v|_{t=0} = u_0,\ \partial_t v|_{t=0} = u_1 \text{ (if } \alpha > 1/p),\ |v - w|_{Z^T} \le \rho\},$$

which is a closed subset of Z^T. Since $Z^T \hookrightarrow C(J; C^1(\overline{\Omega}))$, we may further put

$$\mu_w(T) = \max\{|w(t,x) - u_0(x)| + |\nabla w(t,x) - \nabla u_0(x)| : t \in J,\ x \in \overline{\Omega}\}.$$

Apparently, $\mu_w(T) \to 0$ as $T \to 0$, due to $w|_{t=0} = u_0$.

Suppose that $u \in \Sigma(\rho, T, \phi)$. Then for $t \in J$ and $x \in \overline{\Omega}$, we estimate

$$\begin{aligned} |u(t,x) - u_0(x)| + |\nabla u(t,x) - \nabla u_0(x)| &\leq |u - w|_{C(J;\,C^1(\overline{\Omega}))} + \mu_w(T) \\ &\leq M|u - w|_{Z^T} + \mu_w(T) \leq M\rho + \mu_w(T). \end{aligned} \tag{6.10}$$

Here the constant $M > 0$ does not depend on u and $T \in (0, T_1]$, the latter being true because $u - w \in {}_0Z^T$. So (6.10) shows that $u(t,x) \in U_0$ as well as $\nabla u(t,x) \in U_1$ for all $u \in \Sigma(\rho, T, \phi)$, $t \in J$ and $x \in \overline{\Omega}$, provided that T and ρ are sufficiently small, let us say $T \leq T_2 \leq T_1$ and $\rho \leq \rho_1$. Henceforth we will assume that these smallness conditions hold.

The last assumption allows us to define the mapping $\Upsilon : \Sigma(\rho, T, \phi) \to Z^T$ which assigns to every $u \in \Sigma(\rho, T, \phi)$ the solution $v = \Upsilon(u)$ of the linear problem

$$\left\{ \begin{array}{rll} \partial_t v + dk * \mathcal{A}_0 : \nabla^2 v = & F(u) + dk * G(u) & \\ & +dk * ((\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u) & (J \times \Omega) \\ \mathcal{B}_D^\circ(\phi) v = & -\mathcal{B}_D(\phi) + \mathcal{B}_D^\circ(\phi)\phi - \mathcal{R}_D^\phi(u) & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi) v = & -\mathcal{B}_N(\phi) + \mathcal{B}_N^\circ(\phi)\phi - \mathcal{R}_N^\phi(u) & (J \times \Gamma_N) \\ v|_{t=0} = & u_0 & (\Omega). \end{array} \right. \tag{6.11}$$

Observe now that, in view of (6.7), every fixed point u of Υ is a solution of (6.1) and vice versa, at least for some small time interval $J = [0, T]$. So the idea is to apply the contraction principle to Υ.

After these introductory considerations we become now more rigorous. We have not yet shown that w and Υ are really well-defined. Nor is it clear that the contraction principle is applicable.

In order to succeed we need the subsequent assumptions. Here, for short, we put $\zeta = (\xi, \eta)$ and $U = U_0 \times U_1$. For a function $b(x)$ on a boundary segment Γ_K ($K = D$ or N), we denote the surface gradient by b_{x_Γ}.

(H1) (*kernel*): $k \in BV_{loc}(\mathbb{R}_+) \cap \mathcal{K}^1(1+\alpha, \theta)$, $k(0) = 0$, $\theta < \pi$, $\alpha \in [0, 1) \setminus \left\{\frac{1}{p}, \frac{3}{2p-1}\right\}$, $\kappa := (1+\alpha)(1 - \frac{1}{2p}) \neq 1$;

(H2) (*exponents*): $n \geq 2$, $\frac{2}{1+\alpha} + n < p < \infty$;

(H3) (*smoothness of nonlinearities*):

(a) $a \in C(J_0 \times \overline{\Omega} \times U; \mathbb{R}^{n^2})$,

$$|a(t,x,\zeta) - a(t,x,\bar{\zeta})| \leq C|\zeta - \bar{\zeta}|,\ t \in J_0,\ x \in \Omega, \zeta, \bar{\zeta} \in U;$$

(b) $g(\cdot,\cdot,\zeta)$ is measurable on $J_0 \times \Omega$ for all $\zeta \in U$; $\exists C_g \in X^{T_0}$, $C_g \geq 0$, s.t.

$$|g(t,x,\zeta) - g(t,x,\bar{\zeta})| \leq C_g(t,x)|\zeta - \bar{\zeta}|,\ t \in J_0,\ x \in \Omega,\ \zeta, \bar{\zeta} \in U;$$

(c) f is as g in case $\alpha = 0$, otherwise: $f(\cdot,\cdot,\zeta)$ and $f_\zeta(\cdot,\cdot,\zeta)$ are measurable on $J_0 \times \Omega$ for all $\zeta \in U$; $f_\zeta \in L_\infty(J_0 \times \Omega \times U; \mathbb{R}^{n+1})$; $\exists C_f \in L_p(\Omega)$, $C_f \geq 0$ and $\sigma_f > \alpha$ s.t.

$$|f_\zeta(t,x,\zeta) - f_\zeta(\bar{t},x,\bar{\zeta})| \leq C_f(x)|t - \bar{t}|^{\sigma_f} + C|\zeta - \bar{\zeta}|,\ t, \bar{t} \in J_0,\ x \in \Omega,\ \zeta, \bar{\zeta} \in U;$$

(d) $b^D \in C(J_0 \times \Gamma_D \times U_0)$, $\exists C_{b_1} \in L_p(\Gamma_D)$, $C_{b_1} \geq 0$, $\exists C_{b_2} \in L_p(J)$, $C_{b_2} \geq 0$, and $\exists \sigma_2 > 1 - \frac{1}{p}$ such that in case $\kappa < 1$: $\exists \sigma_1 > \kappa$ s.t.

$$|b^D_{x_\Gamma}(t,x,\xi) - b^D_{x_\Gamma}(t,\bar{x},\xi)| \leq C_{b_2}(t)|x-\bar{x}|^{\sigma_2}, \tag{6.12}$$
$$|b^D_{\xi}(t,x,\xi) - b^D_{\xi}(\bar{t},x,\xi)| \leq C_{b_1}(x)|t-\bar{t}|^{\sigma_1},$$
$$|b^D_{x_\Gamma\xi}(t,x,\xi) - b^D_{x_\Gamma\xi}(t,\bar{x},\bar{\xi})| \leq C_{b_2}(t)|x-\bar{x}|^{\sigma_2} + C|\xi-\bar{\xi}|, \tag{6.13}$$
$$|b^D_{\xi\xi}(t,x,\xi) - b^D_{\xi\xi}(t,\bar{x},\bar{\xi})| \leq C(|x-\bar{x}|^{\sigma_2} + |\xi-\bar{\xi}|),$$

and in case $\kappa > 1$: $\exists \sigma_1 > \kappa - 1$ s.t. (6.12), (6.13),

$$|b^D_{t}(t,x,\xi) - b^D_{t}(\bar{t},x,\xi)| \leq C_{b_1}(x)|t-\bar{t}|^{\sigma_1},$$
$$|b^D_{t\xi}(t,x,\xi) - b^D_{t\xi}(\bar{t},x,\bar{\xi})| \leq C_{b_1}(x)|t-\bar{t}|^{\sigma_1} + C|\xi-\bar{\xi}|,$$
$$|b^D_{\xi\xi}(t,x,\xi) - b^D_{\xi\xi}(\bar{t},\bar{x},\bar{\xi})| \leq C(|t-\bar{t}|^{\sigma_1} + |x-\bar{x}|^{\sigma_2} + |\xi-\bar{\xi}|),$$

all these inequalities being true for $t,\bar{t} \in J_0$, $x,\bar{x} \in \Gamma_D$, $\xi,\bar{\xi} \in U_0$; each of the derivatives of b^D occurring above is Carathéodory and essentially bounded on $J_0 \times \Gamma_D \times U_0$;

(e) $b^N \in C(J_0 \times \Gamma_N \times U)$, $b^N_\zeta \in L_\infty(J_0 \times \Gamma_N \times U; \mathbb{R}^{n+1})$ is Carathéodory, $\exists C_{b_1} \in L_p(\Gamma_N)$, $C_{b_1} \geq 0$, $\exists C_{b_2} \in L_p(J)$, $C_{b_2} \geq 0$, $\exists \sigma_1 > (1+\alpha)(\frac{1}{2} - \frac{1}{2p})$, $\exists \sigma_2 > 1 - \frac{1}{p}$, s.t.

$$|b^N(t,x,\zeta) - b^N(\bar{t},\bar{x},\zeta)| \leq C_{b_1}(x)|t-\bar{t}|^{\sigma_1} + C_{b_2}(t)|x-\bar{x}|^{\sigma_2},$$
$$|b^N_\zeta(t,x,\zeta) - b^N_\zeta(\bar{t},\bar{x},\bar{\zeta})| \leq C_{b_1}(x)|t-\bar{t}|^{\sigma_1} + C_{b_2}(t)|x-\bar{x}|^{\sigma_2} + C|\zeta-\bar{\zeta}|,$$

$t,\bar{t} \in J_0$, $x,\bar{x} \in \Gamma_N$, $\zeta,\bar{\zeta} \in U$;

(H4) (*initial data*): $u_0 \in Y_0$; $u_1 \in Y_1$, if $\alpha > \frac{1}{p}$ $(u_1(x) := f(0,x,u_0(x),\nabla u_0(x)),\ x \in \Omega)$; $f(\cdot,\cdot,u_0(\cdot),\nabla u_0(\cdot)), g(\cdot,\cdot,u_0(\cdot),\nabla u_0(\cdot)) \in X^{T_0}$.

(H5) (*compatibility*): $(u_0(x),\nabla u_0(x)) \in U_0 \times U_1$, $x \in \overline{\Omega}$;

$$b^D(0,x,u_0(x)) = 0,\ x \in \Gamma_D;$$
$$b^N(0,x,u_0(x),\nabla u_0(x)) = 0,\ x \in \Gamma_N;$$
$$b^D_t(0,x,u_0(x)) + b^D_\xi(0,x,u_0(x))u_1(x) = 0,\ x \in \Gamma_D,\ \text{if } \alpha > \tfrac{3}{2p-1};$$

(H6) (*ellipticity*): $a(0,x,u_0(x),\nabla u_0(x)) \in \mathrm{Sym}\{n\}$, $x \in \overline{\Omega}$; $\exists c_0 > 0$ s.t.

$$a(0,x,u_0(x),\nabla u_0(x))\varrho \cdot \varrho \geq c_0|\varrho|^2,\ x \in \overline{\Omega},\ \varrho \in \mathbb{R}^n;$$

(H7) (*normality*):

$$b^D_\xi(0,x,u_0(x)) \neq 0,\ x \in \Gamma_D;$$
$$b^N_\eta(0,x,u_0(x),\nabla u_0(x)) \cdot \nu(x) \neq 0,\ x \in \Gamma_N.$$

We have now the following result.

Theorem 6.1.1 *Suppose that the assumptions (H1)-(H7) are satisfied. Let $\phi \in Z^{T_0}$ be as above, and assume that $\rho \leq \rho_1$. Then there exists $T_3 \in (0,T_2]$ such that for each $T \in (0,T_3]$ the following statements hold:*

(i) (6.9) has a unique solution w in Z^T;

(ii) for every $u \in \Sigma(\rho, T, \phi)$, (6.11) has a unique solution $v = \Upsilon(u)$ in Z^T;

(iii) there exist positive constants M and $\mu(T)$ both not depending on ρ, with M being also independent of T and $\mu(T) \to 0$ as $T \to 0$, such that for all $u, v \in \Sigma(\rho, T, \phi) \cup \{\phi|_J\}$ and $K = D, N$, the subsequent inequalities are fulfilled:

$$|(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u|_{X^T} \leq M(\mu(T) + \rho)^2, \tag{6.14}$$

$$|(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u - (\mathcal{A}_0 - \mathcal{A}(v)) : \nabla^2 v|_{X^T} \leq M(\mu(T) + \rho)|u - v|_{Z^T}, \tag{6.15}$$

$$|F(u) - F(v)|_{X_1^T} + |G(u) - G(v)|_{X^T} \leq \mu(T)|u - v|_{Z^T}, \tag{6.16}$$

$$|\mathcal{R}_K^\phi(u) - \mathcal{R}_K^\phi(v)|_{Y_K^T} \leq M(\mu(T) + \rho)|u - v|_{Z^T}. \tag{6.17}$$

Proof. To prove (i) and (ii), we have to consider the linear problem (6.8). Since $\phi|_{t=0} = u_0$, it follows from (H7) and the compactness of Γ that there exist $T_3 \in (0, T_2]$ and $c > 0$ such that $|\mathcal{B}_D^\circ(\phi)(t, x)| \geq c$ as well as $|\mathcal{B}_N^\circ(\phi)(t, x) \cdot \nu(x)| \geq c$ for all $t \in [0, T_3]$ and $x \in \Gamma_D$ resp. $x \in \Gamma_N$. Hereafter, we suppose that $T \in (0, T_3]$. We may then normalize the boundary conditions in (6.8) by dividing by $\mathcal{B}_D^\circ(\phi)(t, x)$ resp. $\mathcal{B}_N^\circ(\phi)(t, x) \cdot \nu(x)$, and integrate the first equation in (6.8) w.r.t. time. This way we can rewrite (6.8) as a problem of the form (4.40). One has now to check that Theorem 4.3.1 is applicable to the reformulations of (6.9) and (6.11).

As to regularity of the data, clearly the initial data u_0 and u_1 belong to the right regularity classes, by assumption (H4). Let us next look at the term which involves the function g. Suppose $u \in \Sigma(\rho, T, \phi) \cup \{\phi|_J\}$. By (H3b) and (H4), we have

$$\begin{aligned} |g(\cdot, \cdot, u, \nabla u)|_{X^T} &\leq |g(\cdot, \cdot, u, \nabla u) - g(\cdot, \cdot, u_0, \nabla u_0)|_{X^T} + |g(\cdot, \cdot, u_0, \nabla u_0)|_{X^T} \\ &\leq |C_g|_{X^T}(|u - u_0|_\infty + |\nabla u - \nabla u_0|_\infty) + |g(\cdot, \cdot, u_0, \nabla u_0)|_{X^T}. \end{aligned}$$

So $G(u) \in X^T$, that is, this term lies in the right regularity class. Furthermore, we have $(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u \in X^T$, in view of (6.14), which will be shown below. Concerning the other terms, we refer to Section 6.2, where we shall prove that the regularity assumptions on f, b^D, and b^N, together with (H4), ensure that these terms enjoy the regularity required for the application of Theorem 4.3.1, i.e. that $F(u) \in X_1^T$ and $\mathcal{B}_K(\phi) - \mathcal{B}_K^\circ(\phi)\phi + \mathcal{R}_K^\phi(u) \in Y_K^T$ for all $u \in \Sigma(\rho, T, \phi) \cup \{\phi|_J\}$, $K = D, N$. We shall also demonstrate that this regularity is also preserved under the above normalization on the boundary.

Observe further that the compatibility conditions are satisfied for (6.9) and (6.11). This follows from (H5), the definition of $\Sigma(\rho, T, \phi)$, and from the fact that $\phi|_{t=0} = u_0$ and $\partial_t \phi|_{t=0} = u_1$ in case $\alpha > 1/p$. Normality has already been discussed above. Hence, (i) and (ii) are established for all $T \in (0, T_3]$.

Turning to (iii), we here only show (6.14), (6.15), and one half of (6.16). The remaining estimates, which take more effort to be proved, are subject of Section 6.2. In the subsequent inequalities, M and $\mu(T)$ are constants, which may differ from line to line, but which are such that both do not depend on ρ, M is independent of T, too, and $\mu(T) \to 0$ as $T \to 0$.

Let $u, v \in \Sigma(\rho, T, \phi) \cup \{\phi|_J\}$. By means of (H3a) we get

$$\begin{aligned} |(\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u|_{X^T} &\leq (|\mathcal{A}_0 - \mathcal{A}(w)|_\infty + |\mathcal{A}(w) - \mathcal{A}(u)|_\infty) \\ &\qquad \times (|\nabla^2 u - \nabla^2 w|_{(X^T)^{n^2}} + |\nabla^2 w|_{(X^T)^{n^2}}) \\ &\leq (\mu(T) + M\rho)(\rho + \mu(T)), \end{aligned}$$

which entails (6.14). Correspondingly,

$$\begin{aligned}
&|(\mathcal{A}_0-\mathcal{A}(u)):\nabla^2 u-(\mathcal{A}_0-\mathcal{A}(v)):\nabla^2 v|_{X^T}\\
&\quad\le |(\mathcal{A}_0-\mathcal{A}(u)):(\nabla^2 u-\nabla^2 v)|_{X^T}+|(\mathcal{A}(u)-\mathcal{A}(v)):(\nabla^2 v-\nabla^2 w)|_{X^T}\\
&\qquad+|(\mathcal{A}(u)-\mathcal{A}(v)):\nabla^2 w|_{X^T}\\
&\quad\le (\mu(T)+M\rho)|u-v|_{Z^T}+M\rho|u-v|_{Z^T}+M\mu(T)|u-v|_{Z^T}\\
&\quad\le M(\mu(T)+\rho)|u-v|_{Z^T},
\end{aligned}$$

showing (6.15). We finally estimate the term $|G(u)-G(v)|_{X^T}$. By virtue of (H3b), we obtain similarly as above

$$|g(\cdot,\cdot,u,\nabla u)-g(\cdot,\cdot,v,\nabla v)|_{X^T}\le |C_g|_{X^T}M|u-v|_{Z^T}\le M\mu(T)|u-v|_{Z^T}.\square$$

Existence and uniqueness of a local strong solution of (6.1) can now be obtained by means of Theorem 6.1.1 and the contraction mapping principle.

Theorem 6.1.2 *Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary Γ which decomposes as $\Gamma=\Gamma_D\cup\Gamma_N$ with dist$(\Gamma_D,\Gamma_N)>0$. Let further $U_0\subset\mathbb{R}$, $U_1\subset\mathbb{R}^n$ be nonempty open convex sets. Suppose that the assumptions (H1)-(H7) are satisfied. Then there exists $T\in(0,T_0]$ such that (6.1) restricted to $J=[0,T]$ admits a unique solution in Z^T.*

Proof. Let $\rho\le\rho_1$ and $T\in(0,T_3]$, so that the reference function $w\in Z^T$ as well as $v=\Upsilon(u)\in Z^T$ are well-defined for each $u\in\Sigma(\rho,T,\phi)$, cf. Theorem 6.1.1. We want to show that, for sufficiently small T and ρ, Υ maps $\Sigma(\rho,T,\phi)$ into itself and is strictly contractive.

To show the first property we have to estimate $v-w$ in the Z^T-norm. By definition of w and $v=\Upsilon(u)$, we see that $v-w$ satisfies

$$\left\{\begin{array}{rll}
\partial_t(v-w)+dk*\mathcal{A}_0:\nabla^2(v-w)= & F(u)-F(\phi)+dk*(G(u)-G(\phi)) & \\
 & +dk*((\mathcal{A}_0-\mathcal{A}(u)):\nabla^2 u) & (J\times\Omega)\\
\mathcal{B}_D^\circ(\phi)(v-w)= & -\mathcal{R}_D^\phi(u) & (J\times\Gamma_D)\\
\mathcal{B}_N^\circ(\phi)(v-w)= & -\mathcal{R}_N^\phi(u) & (J\times\Gamma_N)\\
(v-w)|_{t=0}= & 0 & (\Omega).
\end{array}\right.$$

The maximal regularity estimate for problem (6.8) thus yields

$$\begin{aligned}
|v-w|_{Z^T}\le M_1\Big(&|F(u)-F(\phi)+dk*(G(u)-G(\phi))+dk*((\mathcal{A}_0-\mathcal{A}(u)):\nabla^2 u)|_{X_1^T}\\
&+|\mathcal{R}_D^\phi(u)|_{Y_D^T}+|\mathcal{R}_N^\phi(u)|_{Y_N^T}\Big),
\end{aligned}$$

with a constant $M_1>0$ *not* depending on T ($v-w\in{}_0Z^T$!). Using the estimates from Theorem 6.1.1, combined with $\mathcal{R}_K^\phi(\phi)=0$, $K=D,N$, we obtain an inequality of the form

$$|v-w|_{Z^T}\le M(\rho+\mu(T))^2, \tag{6.18}$$

where $M>0$ is independent of T and ρ, and $\mu(T)>0$ vanishes as $T\to0$. Here we employ the simple inequality $|u-\phi|_{Z^T}\le\rho+|\phi-w|_{Z^T}$, the last term of which behaves like $\mu(T)$. From (6.18) it is clear that Υ is a self-mapping of $\Sigma(\rho,T,\phi)$, if T and ρ are sufficiently small; choose e.g. ρ so small that $4M\rho^2\le\rho$, and diminish T until $\mu(T)\le\rho$.

Strict contractivity of Υ can be established in similar fashion. Let u, $\bar{u} \in \Sigma(\rho, T, \phi)$ and put $v = \Upsilon(u)$ and $\bar{v} = \Upsilon(\bar{u})$. Then $\bar{v} - v$ satisfies

$$\begin{cases} \partial_t(\bar{v}-v) + dk * \mathcal{A}_0 : \nabla^2(\bar{v}-v) = F(\bar{u}) - F(u) + dk * (G(\bar{u}) - G(u)) & (J \times \Omega) \\ \qquad\qquad + dk * (((\mathcal{A}_0 - \mathcal{A}(\bar{u})) : \nabla^2 \bar{u}) - (\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u) & \\ \mathcal{B}_D^\circ(\phi)(\bar{v}-v) = -(\mathcal{R}_D^\phi(\bar{u}) - \mathcal{R}_D^\phi(u)) & (J \times \Gamma_D) \\ \mathcal{B}_N^\circ(\phi)(\bar{v}-v) = -(\mathcal{R}_N^\phi(\bar{u}) - \mathcal{R}_N^\phi(u)) & (J \times \Gamma_N) \\ (\bar{v}-v)|_{t=0} = 0 & (\Omega), \end{cases}$$

whence

$$\begin{aligned} |\bar{v}-v|_{Z^T} \le\; & M_1 \Big(|G(\bar{u}) - G(u) + (((\mathcal{A}_0 - \mathcal{A}(\bar{u})) : \nabla^2 \bar{u}) - (\mathcal{A}_0 - \mathcal{A}(u)) : \nabla^2 u)|_{X^T} \\ & + |F(\bar{u}) - F(u)|_{X_1^T} + |\mathcal{R}_D^\phi(\bar{u}) - \mathcal{R}_D^\phi(u)|_{Y_D^T} + |\mathcal{R}_N^\phi(\bar{u}) - \mathcal{R}_N^\phi(u)|_{Y_N^T} \Big) \\ \le\; & M(\mu(T) + \rho) |\bar{u} - u|_{Z^T}, \end{aligned}$$

by maximal regularity and the estimates from Theorem 6.1.1. Here the constants M and $\mu(T)$ are like those in (6.18). Hence Υ becomes a strict contraction, when ρ and T are selected sufficiently small.

The assertion follows now by the contraction mapping principle and the fact that fixed points of Υ correspond to solutions of (6.1) for small time-intervals $J = [0, T]$. □

Remarks 6.1.1 (i) The statement of Theorem 6.1.2 is also true, if k is of the form $k = k_1 + dl * k_1$, where k_1 is like k in (H1) and $l \in BV_{loc}(\mathbb{R}_+)$ with $l(0) = l(0+) = 0$.

(ii) One can further replace k on the right-hand side of (6.1) by an arbitrary kernel $k_1 \in BV_{loc}(\mathbb{R}_+) \cap \mathcal{K}^1(1 + \alpha_1, \theta_1)$ with $k(0) = 0$, $\alpha_1 \ge \alpha$, $\theta_1 < \pi$, and the theorem still holds true.

6.2 Nemytskij operators for various function spaces

This paragraph can be regarded as an appendix to Section 6.1. Its purpose is to complete the proof of Theorem 6.1.1. We still have to show certain mapping properties and Lipschitz estimates for the substitution operators which involve the nonlinear functions f, b^D and b^N, cf. the beginning of Section 6.1. The Nemytskij operators under consideration act on the function spaces which arose as natural regularity classes for the inhomogeneities in the treatment of the linear problem (6.8), and turned out to be the spaces X_1^T, Y_D^T, and Y_N^T, see (6.2), (6.3), (6.4) for their definitions. These are anisotropic Bessel-potential and Sobolev-Slobodeckij spaces in domains, respectively, compact manifolds in the euclidean space, which means that the subject of this section is not altogether trivial.

We remark that in order to get the desired estimates for Y_D^T and Y_N^T, which are spaces on $J \times \Gamma_D$ resp. $J \times \Gamma_N$, one considers first the corresponding spaces in domains (w.r.t. the spatial variable). The results obtained for the latter can then be transferred to Y_D^T and Y_N^T by means of the standard method of local coordinates. In what is to follow, we shall focus on the first step. As to the second step, we merely point out that the fact that Ω (in Section 6.1) has a (compact) C^2 boundary ensures that the smoothness of b^D resp. b^N w.r.t. the spatial variable x is preserved under the local coordinate transformations studied in Section 4.3, which flatten the boundary.

The reader is further reminded of the embedding $Z^T \hookrightarrow C(J; C^1(\overline{\Omega}))$, which is valid in view of the assumption (H2), and which considerably simplifies the investigation of the substitution operators to be studied in this section. Note that thanks to this embedding, we do not require any growth conditions on the nonlinearities.

It should also be remarked that all nonlinearities appearing in the subsequent estimates are tacitly assumed to be Carathéodory functions so that we do not have to be concerned with measurability questions, cf. Appell and Zabrejko [4, Section 1.4]. Notice that this corresponds to the regularity assumptions in (H4).

We fix now the notation used in this section. Let $J = [0,T]$ $(0 < T \le T_0)$, and Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary. For $p \in (1,\infty)$ and $m \in \mathbb{N}$ we introduce the symbols $X^m := L_p(J \times \Omega, \mathbb{R}^m)$, $X_1^m := H_p^1(J; L_p(\Omega, \mathbb{R}^m))$, and $X_2^m := L_p(J; H_p^1(\Omega, \mathbb{R}^m))$. Our interest lies in the spaces $X_{1,s}^m := H_p^s(J; L_p(\Omega, \mathbb{R}^m))$, $(Y_1^{k,s})^m := B_{pp}^{k+s}(J; L_p(\Omega, \mathbb{R}^m))$, $(Y_2^{k,s})^m := L_p(J; B_{pp}^{k+s}(\Omega, \mathbb{R}^m))$, where $k \in \{0,1\}$ and $s \in (0,1)$. But we will also deal with the space $(C_1^r)^m := C^r(J; C(\overline{\Omega}, \mathbb{R}^m))$, where $r \in [0,1)$. For more brevity, we omit the parameter m in all these notations if $m = 1$, i.e. we write $X = X^1$, $X_i = X_i^1$ and so forth. With $|z| := \sum_{i=1}^m |z_i|$ for $z \in \mathbb{R}^m$, the following seminorms will play a part below:

$$|f|_{\infty,m} = |f|_{L_\infty(J\times\Omega,\mathbb{R}^m)}, \quad [f]_{X_1^m} = |\partial_t f|_{X^m}, \quad [f]_{X_2^m} = \sum_{i=1}^n |\partial_{x_i} f|_{X^m},$$

$$\begin{aligned}
[f]_{(Y_1^{k,s})^m} &= \Big(\int_0^T \int_0^T \int_\Omega \frac{|\partial_t^k f(t,x) - \partial_t^k f(\tau,x)|^p}{|t-\tau|^{1+sp}}\, dx\, d\tau\, dt\Big)^{\frac{1}{p}}, \\
[f]_{(Y_2^{0,s})^m} &= \Big(\int_0^T \int_\Omega \int_\Omega \frac{|f(t,x) - f(t,y)|^p}{|x-y|^{n+sp}}\, dx\, dy\, dt\Big)^{\frac{1}{p}}, \\
[f]_{(Y_2^{1,s})^m} &= \sum_{i=1}^n \Big(\int_0^T \int_\Omega \int_\Omega \frac{|\partial_{x_i} f(t,x) - \partial_{x_i} f(t,y)|^p}{|x-y|^{n+sp}}\, dx\, dy\, dt\Big)^{\frac{1}{p}},
\end{aligned}$$

$$[f]_{(C_1^r)^m} = \sup_{t\neq\tau\in J,\, x\in\overline{\Omega}} \frac{|f(t,x) - f(\tau,x)|}{|t-\tau|^r} \quad (r \in (0,1)),$$

$$[f]_{X_{1,s}^m} = \Big(\int_\Omega \int_0^T \Big(\int_0^1 \sigma^{-2s}\Big(\frac{1}{|V(t,\sigma)|} \int_{V(t,\sigma)} |f(t+h,x) - f(t,x)|\, dh\Big)^2 \frac{d\sigma}{\sigma}\Big)^{\frac{p}{2}} dt\, dx\Big)^{\frac{1}{p}},$$

where $V(t,\sigma) = \{h \in \mathbb{R} : |h| < \sigma \text{ and } t+h \in J\}$. The subsequent expressions are norms in the corresponding spaces:

$$\begin{aligned}
|\cdot|_{(Y_1^{0,s_1}\cap Y_2^{0,s_2})^m} &= |\cdot|_{X^m} + [\cdot]_{(Y_1^{0,s_1})^m} + [\cdot]_{(Y_2^{0,s_2})^m}, \\
|\cdot|_{(Y_1^{0,s_1}\cap Y_2^{1,s_2})^m} &= |\cdot|_{X^m} + [\cdot]_{(Y_1^{0,s_1})^m} + [\cdot]_{X_2^m} + [\cdot]_{(Y_2^{1,s_2})^m}, \\
|\cdot|_{(Y_1^{1,s_1}\cap Y_2^{1,s_2})^m} &= |\cdot|_{X^m} + [\cdot]_{X_1^m} + [\cdot]_{(Y_1^{1,s_1})^m} + [\cdot]_{X_2^m} + [\cdot]_{(Y_2^{1,s_2})^m}, \\
|\cdot|_{X_{1,s}^m} &= |\cdot|_{X^m} + [\cdot]_{X_{1,s}^m},
\end{aligned}$$

cf. Triebel [78], [79], as well as Runst and Sickel [72].

Throughout this section, let further K be an open convex subset of $\mathbb{R}^m$ and $b : J\times\Omega\times K \to \mathbb{R}$, $(t,x,\xi) \mapsto b(t,x,\xi)$. We shall investigate the Nemytskij operator B which assigns to a function f on $J \times \Omega$ with values in K the function $Bf(t,x) = b(t,x,f(t,x))$ which is real-valued and defined on $J \times \Omega$. In what follows w will be a fixed K-valued function defined on $J\times\Omega$, too, which is as smooth as the functions f under consideration, and which serves as a reference function in the following sense. For $F = Y_1^{k_1,s_1} \cap Y_2^{k_2,s_2}$, and given $\rho \in (0, \rho_0]$, let $\Sigma = \Sigma(\rho, w, F)$ be the set of all K-valued f in F^m such that $f - w \in {}_0F^m$ and $|f - w|_{F^m} \leq \rho$. The aim is to show that $Bf \in F$ whenever $f \in \Sigma$ and that

$$|Bf - Bg|_F \leq C(\rho + \mu(T) + |b_\xi(\cdot, \cdot, w)|_{\infty, m})|f - g|_{F^m}, \; f, g \in \Sigma, \tag{6.19}$$

where $C > 0$ is independent of ρ and T, and $0 < \mu(T) \to 0$ as T tends to zero. We further need the property that $Bf \in X_{1,s}$ if $f \in X^m_{1,s_0} \cap (C^0_1)^m$ ($s_0 \in (s, 1)$), and we wish to have a Lipschitz estimate of the form

$$|Bf - Bg|_{X_{1,s}} \leq \mu(T)(|f - g|_{X^m_{1,s_0}} + |f - g|_{\infty, m}), \tag{6.20}$$

for all (K-valued) $f, g \in \Sigma' = \Sigma'(\rho, w) := \{h \in X^m_{1,s_0} \cap (C^0_1)^m : (h - w)|_{t=0} = 0$ and $|h - w|X^m_{1,s_0} + |h - w|_{\infty,m} \leq \rho\}$, where again the constant $\mu(T) > 0$ vanishes as $T \to 0$.

In both cases we shall only establish the Lipschitz estimate. The corresponding mapping property of B follows by means of the same techniques; here the proof is even simpler than for the Lipschitz estimate. When proving Lipschitz estimates we will restrict ourselves to the seminorms of highest order, that is, e.g., if we are to estimate $Bf - Bg$ in the $Y_1^{1,s_1} \cap Y_2^{1,s_2}$-norm, we shall consider only the seminorms $[Bf - Bg]_{Y_1^{1,s_1}}$ and $[Bf - Bg]_{Y_2^{1,s_2}}$. Having proved the desired estimate for these terms, it will then be clear how to obtain it for the seminorm terms of lower order, which are much easier to treat.

We begin now with the spaces

$$(Y_1^{k_1,s_1} \cap Y_2^{k_2,s_2})^m = B^{k_1+s_1}_{pp}(J; L_p(\Omega, \mathbb{R}^m)) \cap L_p(J; B^{k_2+s_2}_{pp}(\Omega, \mathbb{R}^m)),$$

$k_i \in \{0, 1\}, s_i \in (0, 1)$, $i = 1, 2$. Here, the cases $(k_1, k_2) = (0, 0)$, $(0, 1)$, $(1, 1)$ have to be studied. We assume that in each of these cases, p is large enough such that we have the embedding $Y_1^{0,s_1} \cap Y_2^{0,s_2} \hookrightarrow C(J; C(\bar{\Omega}))$, $Y_1^{0,s_1} \cap Y_2^{1,s_2} \hookrightarrow C(J; C^1(\bar{\Omega}))$, and $Y_1^{1,s_1} \cap Y_2^{1,s_2} \hookrightarrow C^r(J; C(\bar{\Omega})) \cap C(J; C^1(\bar{\Omega}))$, respectively, with some number r in $(s_1, 1)$. More precisely, we make the assumption that $p > p_\star(n, k_1, s_1, k_2, s_2)$, where

$$p_\star(n, k_1, s_1, k_2, s_2) := \begin{cases} \frac{1}{s_1} + \frac{n}{s_2} & : \; (k_1, k_2) = (0, 0) \\ \frac{1}{s_1}(1 + \frac{1}{s_2}) + \frac{n}{s_2} & : \; (k_1, k_2) = (0, 1) \\ \max\left\{\frac{1}{s_2}(\frac{1+s_2}{1+s_1} + n), \; 1 + n\,\frac{1+s_1}{1+s_2}\right\} & : \; (k_1, k_2) = (1, 1). \end{cases}$$

Roughly speaking, this condition on the exponent p when translated to the situation of Section 6.1 corresponds to the assumption (H2) therein.

The first lemma is concerned with the case $(k_1, k_2) = (0, 0)$. Here and in the subsequent estimates we denote by $\mu(T)$ a positive constant depending on T such that $\mu(T) \to 0$ as $T \to 0$. Further, M and C denote constants which may differ from line to line, but which do not depend on T and ρ.

Lemma 6.2.1 *Let $w \in (Y_1^{0,s_1} \cap Y_2^{0,s_2})^m$ be a fixed K-valued function and $\rho > 0$. Let further Σ be as described above. Suppose that there exist $C_{HL} > 0$, $r \in (s_1, 1)$, and $C_b \in L_p(\Omega)$ such that*

$$|b_\xi(t, x, \xi) - b_\xi(\tau, x, \eta)| \leq C_{HL}(C_b(x)|t - \tau|^r + |\xi - \eta|),$$

for all $t,\tau \in J$, $\xi,\eta \in K$, and a.a. $x \in \Omega$. Then there exists a constant $C > 0$ not depending on T and ρ such that

$$[b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{Y_1^{0,s_1}} \le C\Big(\rho + \mu(T) + |b_\xi(\cdot,\cdot,w)|_{\infty,m}\Big)|f-g|_{(Y_1^{0,s_1}\cap Y_2^{0,s_2})^m}, \; f,g \in \Sigma.$$

Proof. Let f and g be arbitrary functions in Σ. Put

$$h(t,\tau,x) = b(t,x,f(t,x)) - b(t,x,g(t,x)) - b(\tau,x,f(\tau,x)) + b(\tau,x,g(\tau,x))$$

for $t,\tau \in J$, and a.a. $x \in \Omega$. Then

$$[b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{Y_1^{0,s_1}} = (\int_0^T\int_0^T\int_\Omega \frac{|h(t,\tau,x)|^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt)^{\frac{1}{p}}.$$

Letting $\phi(t,\tau,x,\theta) = b_\xi(t,x,g(\tau,x)+\theta(f(\tau,x)-g(\tau,x)))$, $t,\tau \in J$, $x \in \Omega$, $\theta \in [0,1]$, we write

$$\begin{aligned} h(t,\tau,x) &= \int_0^1 \phi(t,t,x,\theta)\,d\theta\cdot(f(t,x)-g(t,x)) - \int_0^1 \phi(\tau,\tau,x,\theta)\,d\theta\cdot(f(\tau,x)-g(\tau,x)) \\ &= \int_0^1 \phi(t,t,x,\theta)\,d\theta\cdot(f(t,x)-f(\tau,x)-g(t,x)+g(\tau,x)) + \\ &\quad + \int_0^1 (\phi(t,t,x,\theta)-\phi(\tau,\tau,x,\theta))\,d\theta\cdot(f(\tau,x)-g(\tau,x)). \end{aligned}$$

With $\psi(f,g) := \text{ess sup}\{|\phi(t,t,x,\theta)| :\ t \in J,\ x \in \Omega,\ \theta \in [0,1]\}$, we therefore have

$$\begin{aligned} |h(t,\tau,x)| &\le \psi(f,g)|f(t,x)-f(\tau,x)-g(t,x)+g(\tau,x)| + \\ &\quad + \Big(C_{HL}\int_0^1((1-\theta)|g(t,x)-g(\tau,x)| + \theta|f(t,x)-f(\tau,x)|)\,d\theta + \\ &\quad + C_{HL}C_b(x)|t-\tau|^r\Big)\cdot|f(\tau,x)-g(\tau,x)| \\ &\le \psi(f,g)|f(t,x)-f(\tau,x)-g(t,x)+g(\tau,x)| + C_{HL}|f-g|_\infty\cdot \\ &\quad \cdot(|f(t,x)-f(\tau,x)| + |g(t,x)-g(\tau,x)| + C_b(x)|t-\tau|^r) \end{aligned}$$

for all $t,\tau \in J$, and a.a. $x \in \Omega$. On the whole we thus find that

$$\begin{aligned} [b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{Y_1^{0,s_1}} &\le \psi(f,g)[f-g]_{(Y_1^{0,s_1})^m} + C_{HL}|f-g|_{\infty,m}\Big([f]_{(Y_1^{0,s_1})^m} + \\ &\quad + [g]_{(Y_1^{0,s_1})^m} + |C_b|_{L_p(\Omega)}(\int_0^T\int_0^T \frac{d\tau\,dt}{|t-\tau|^{1+(s_1-r)p}})^{\frac{1}{p}}\Big) \\ &= \psi(f,g)[f-g]_{(Y_1^{0,s_1})^m} + C_{HL}|f-g|_{\infty,m}\Big([f]_{(Y_1^{0,s_1})^m} + \\ &\quad + [g]_{(Y_1^{0,s_1})^m} + |C_b|_{L_p(\Omega)}C_1T^{r-s_1+\frac{1}{p}}\Big), \end{aligned}$$

where $C_1 = 2^{1/p}[(r-s_1)p(1+(r-s_1)p)]^{-1/p}$.

By definition of Σ, the Lipschitz estimate for b_ξ, and in view of the embedding $Y_1^{0,s_1} \cap Y_2^{0,s_2} \hookrightarrow C(J;C(\bar\Omega))$, we have the inequalities

$$[f-g]_{\infty,m} \le M|f-g|_{(Y_1^{0,s_1}\cap Y_2^{0,s_2})^m}, \quad [f]_{(Y_1^{0,s_1})^m} + [g]_{(Y_1^{0,s_1})^m} \le 2(\rho + [w]_{(Y_1^{0,s_1})^m}),$$

$$\psi(f,g) \le M\rho + |b_\xi(\cdot,\cdot,w)|_{\infty,m}, \quad f,g \in \Sigma,$$

M being independent of T because $(f-g)|_{t=0} = 0$. Hence the assertion follows with $\mu(T) = [w]_{(Y_1^{0,s_1})^m} + T^{r-s_1+\frac{1}{p}}$. □

By repeating the above considerations with the roles of J and Ω being reversed, one obtains (under the corresponding assumptions, cf. (H4) in Section 6.1) the estimate

$$[b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{Y_2^{0,s_2}} \le C\Big(\rho + \mu(T) + |b_\xi(\cdot,\cdot,w)|_{\infty,m}\Big)|f-g|_{(Y_1^{0,s_1}\cap Y_2^{0,s_2})^m}, \; f,g \in \Sigma.$$

We come now to exponents greater than 1.

Lemma 6.2.2 *Let $w \in (Y_1^{1,s_1} \cap Y_2^{1,s_2})^m$ be a fixed K-valued function and $\rho > 0$. Let further Σ be as described above. Suppose $b_\xi, b_{t\xi} \in L_\infty(J \times \Omega \times K, \mathbb{R}^m)$, $b_{\xi\xi} \in L_\infty(J \times \Omega \times K, \mathbb{R}^{m\times m})$, and assume that there exist $C_{HL} > 0$, $r_1 \in (s_1, 1)$, and $C_b \in L_p(\Omega)$ such that*

$$\begin{aligned} |b_{t\xi}(t,x,\xi) - b_{t\xi}(\tau,x,\eta)| &\le C_{HL}(C_b(x)|t-\tau|^{r_1} + |\xi-\eta|), \\ |b_{\xi\xi}(t,x,\xi) - b_{\xi\xi}(\tau,x,\eta)| &\le C_{HL}(|t-\tau|^{r_1} + |\xi-\eta|), \end{aligned}$$

for all $t,\tau \in J$, $\xi,\eta \in K$, and a.a. $x \in \Omega$. Then there exists a constant $C > 0$ not depending on T and ρ such that

$$[b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{Y_1^{1,s_1}} \le C\Big(\rho + \mu(T) + |b_\xi(\cdot,\cdot,w)|_{\infty,m}\Big)|f-g|_{(Y_1^{1,s_1}\cap Y_2^{1,s_2})^m}, \; f,g \in \Sigma.$$

Proof. For brevity we set $Y = Y_1^{1,s_1} \cap Y_2^{1,s_2}$. Remember that we have the embedding $Y \hookrightarrow C^r(J; C(\bar\Omega))$ for some $r \in (s_1, 1)$. Let now $f, g \in \Sigma$ be arbitrary functions. Put

$$\begin{aligned} h_1(t,\tau,x) &= b_t(t,x,f(t,x)) - b_t(t,x,g(t,x)) - b_t(\tau,x,f(\tau,x)) + b_t(\tau,x,g(\tau,x)), \\ h_2(t,\tau,x) &= b_\xi(t,x,f(t,x)) \cdot f_t(t,x) - b_\xi(t,x,g(t,x)) \cdot g_t(t,x) \\ &\quad - b_\xi(\tau,x,f(\tau,x)) \cdot f_t(\tau,x) + b_\xi(\tau,x,g(\tau,x)) \cdot g_t(\tau,x) \end{aligned}$$

for $t,\tau \in J$, and a.a. $x \in \Omega$. Then

$$\begin{aligned} [b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{Y_1^{1,s_1}} &= \Big(\int_0^T \int_0^T \int_\Omega \frac{(|h_1(t,\tau,x) + h_2(t,\tau,x)|)^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac{1}{p}} \\ &\le \sum_{i=1}^2 \Big(\int_0^T \int_0^T \int_\Omega \frac{|h_i(t,\tau,x)|^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac{1}{p}} =: I_1 + I_2. \end{aligned}$$

Concerning I_1, we may use the estimates from the proof of Lemma 6.2.1, thereby obtaining

$$\begin{aligned} I_1 &\le C\Big((\rho + |b_{t\xi}(\cdot,\cdot,w)|_{\infty,m})[f-g]_{(Y_1^{0,s_1})^m} + (\rho + \mu(T))|f-g|_{\infty,m}\Big) \\ &\le C\Big((\rho_0 + |b_{t\xi}|_{\infty,m})\mu(T)[f-g]_{(C_1^r)^m} + M(\rho + \mu(T))|f-g|_{Y^m}\Big) \\ &\le C(\rho + \mu(T))|f-g|_{Y^m}. \end{aligned}$$

The term I_2 is more sophisticated. We employ the identity

$$aA - bB - cC + dD \;=\; (a-b-c+d)D + (-a+b+c)(A-B-C+D) +$$

$$+(a-c)(A-B)+(a-b)(A-C)$$

to write

$$\begin{aligned}
h_2(t,\tau,x) &= h_{21}(t,\tau,x)\cdot g_t(\tau,x)+\\
&\quad +(-b_\xi(t,x,f(t,x))+b_\xi(t,x,g(t,x))+b_\xi(\tau,x,f(\tau,x)))\cdot h_{22}(t,\tau,x)\\
&\quad +(b_\xi(t,x,f(t,x))-b_\xi(\tau,x,f(\tau,x))\cdot(f_t(t,x)-g_t(t,x))+\\
&\quad +(b_\xi(t,x,f(t,x))-b_\xi(t,x,g(t,x)))\cdot(f_t(t,x)-f_t(\tau,x))\\
&=: I_3(t,\tau,x)+I_4(t,\tau,x)+I_5(t,\tau,x)+I_6(t,\tau,x),
\end{aligned}$$

where

$$h_{21}(t,\tau,x)=b_\xi(t,x,f(t,x))-b_\xi(t,x,g(t,x))-b_\xi(\tau,x,f(\tau,x))+b_\xi(\tau,x,g(\tau,x)),$$

$$h_{22}(t,\tau,x)=f_t(t,x)-g_t(t,x)-f_t(\tau,x)+g_t(\tau,x),\ t,\tau\in J,\ \text{a.a. } x\in\Omega.$$

The summand I_3 can be estimated by mimicking the middle part of the proof of Lemma 6.2.1. Letting

$$h_{23}(t,\tau,x)=f(t,x)-g(t,x)-f(\tau,x)+g(\tau,x)$$

and $r_0=\min\{r,r_1\}$ we get

$$\begin{aligned}
&|h_{21}(t,\tau,x)|\le\\
&\quad\le C(|h_{23}(t,\tau,x)|+|f-g|_{\infty,m}(|f(t,x)-f(\tau,x)|+|g(t,x)-g(\tau,x)|+|t-\tau|^{r_1}))\\
&\quad\le C(|t-\tau|^r[f-g]_{(C_1^r)^m}+|f-g|_{\infty,m}(|t-\tau|^r([f]_{(C_1^r)^m}+[g]_{(C_1^r)^m})+|t-\tau|^{r_1}))\\
&\quad\le C(|t-\tau|^r[f-g]_{Y^m}(1+\rho+[w]_{(C_1^r)^m})+[f-g]_{Y^m}|t-\tau|^{r_1})\\
&\quad\le C|t-\tau|^{r_0}[f-g]_{Y^m}
\end{aligned}$$

for all $t,\tau\in J$, and a.a. $x\in\Omega$. Thus,

$$\begin{aligned}
&\Big(\int_0^T\int_0^T\int_\Omega\frac{|I_3(t,\tau,x)|^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac1p}\le\\
&\quad\le C[f-g]_{Y^m}\Big(\int_0^T\int_\Omega|g_t(\tau,x)|^p\Big(\int_0^T\frac{dt}{|t-\tau|^{1+(s_1-r_0)p}}\Big)\,dx\,d\tau\Big)^{\frac1p}\\
&\quad\le CT^{r_0-s_1}[g]_{X_1^m}|f-g|_{Y^m}\le C\mu(T)|f-g|_{Y^m}.
\end{aligned}$$

Turning to I_4, we immediately see that

$$\Big(\int_0^T\int_0^T\int_\Omega\frac{|I_4(t,\tau,x)|^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac1p}\le M(\rho+|b_\xi(\cdot,\cdot,w)|_{\infty,m})[f-g]_{(Y_1^{1,s})^m}.$$

As for I_5, we estimate

$$\begin{aligned}
|I_5(t,\tau,x)|&\le|b_\xi(t,x,f(t,x))-b_\xi(\tau,x,f(t,x))||f_t(t,x)-g_t(t,x)|+\\
&\quad+|b_\xi(\tau,x,f(t,x))-b_\xi(\tau,x,f(\tau,x))||f_t(t,x)-g_t(t,x)|\\
&\le(|b_{t\xi}|_{\infty,m}|t-\tau|+|b_{\xi\xi}|_{\infty,m^2}[f]_{(C_1^r)^m}|t-\tau|^r)|f_t(t,x)-g_t(t,x)|\\
&\le C|t-\tau|^r|f_t(t,x)-g_t(t,x)|
\end{aligned}$$

for all $t, \tau \in J$, and a.a. $x \in \Omega$. Therefore,

$$(\int_0^T \int_0^T \int_\Omega \frac{|I_5(t,\tau,x)|^p}{|t-\tau|^{1+s_1p}}\, dx\, d\tau\, dt)^{\frac{1}{p}} \leq CT^{r-s_1}[f-g]_{X_1^m}.$$

Finally,

$$|I_6(t,\tau,x)| \leq |b_{\xi\xi}|_{\infty,m^2}|f-g|_{\infty,m}|f_t(t,x)-f_t(\tau,x)| \leq C[f-g]_{Y^m}|f_t(t,x)-f_t(\tau,x)|.$$

for all $t, \tau \in J$, and a.a. $x \in \Omega$. So we deduce

$$\begin{aligned}(\int_0^T \int_0^T \int_\Omega \frac{|I_6(t,\tau,x)|^p}{|t-\tau|^{1+s_1p}}\, dx\, d\tau\, dt)^{\frac{1}{p}} &\leq\ C|f-g|_{Y^m}[f]_{(Y_1^{1,s_1})^m} \\ &\leq C(\rho + [w]_{(Y_1^{1,s_1})^m})|f-g|_{Y^m}\ \leq\ C(\rho+\mu(T))|f-g|_{Y^m}.\end{aligned}$$

The assertion follows now from

$$[b(\cdot,\cdot,f)-b(\cdot,\cdot,g)]_{Y_1^{1,s_1}} \leq I_1 + \sum_{j=3}^{6}(\int_0^T \int_0^T \int_\Omega \frac{|I_j(t,\tau,x)|^p}{|t-\tau|^{1+s_1p}}\, dx\, d\tau\, dt)^{\frac{1}{p}}.\ \square$$

Under the corresponding assumptions, cf. (H4) in Section 6.1, we can repeat the above steps with the roles of J and Ω being reversed to obtain the estimate

$$[b(\cdot,\cdot,f)-b(\cdot,\cdot,g)]_{Y_2^{1,s_2}} \leq C\Big(\rho+\mu(T)+|b_\xi(\cdot,\cdot,w)|_{\infty,m}\Big)|f-g|_{(Y_1^{1,s_1}\cap Y_2^{1,s_2})^m},\ f,g\in\Sigma.$$

It is further not difficult to check that the same line of arguments also yields

$$[b(\cdot,\cdot,f)-b(\cdot,\cdot,g)]_{Y_2^{1,s_2}} \leq C\Big(\rho+\mu(T)+|b_\xi(\cdot,\cdot,w)|_{\infty,m}\Big)|f-g|_{(Y_1^{0,s_1}\cap Y_2^{1,s_2})^m},\ f,g\in\Sigma,$$

in the case $(k_1,k_2)=(0,1)$. Here, the reader should recall that in the proof of Lemma 6.2.2, we employed the embedding $Y_1^{1,s_1}\cap Y_2^{1,s_2} \hookrightarrow C^r(J;C(\bar{\Omega}))$ with some $r\in(s_1,1)$. In the case $(k_1,k_2)=(0,1)$ the situation is more comfortable since, by assumption, we even have the embedding $Y_1^{0,s_1}\cap Y_2^{1,s_2} \hookrightarrow C(J;C^1(\overline{\Omega}))$.

To conclude, we see that the estimate (6.19) holds true for $F = Y_1^{k_1,s_1}\cap Y_2^{k_2,s_2}$. As already mentioned at the beginning of this section, this result can be transferred to the spaces Y_D^T and Y_N^T considered in Section 6.1 by means of the well-known method of local coordinates. If we apply the corresponding result to the function b defined by

$$b(t,x,\xi) = b^D(t,x,\xi)-b^D(t,x,\phi(t,x))-b_\xi(t,x,\phi(t,x))(\xi-\phi(t,x)),\ t\in J,\ x\in\Gamma_D,\ \xi\in U_0,$$

then we get, owing to $b_\xi(t,x,\xi) = b_\xi^D(t,x,\xi)-b_\xi^D(t,x,\phi(t,x))$, an estimate of the form

$$|\mathcal{R}_D^\phi(u)-\mathcal{R}_D^\phi(v)|_{Y_D^T} \leq M(\mu(T)+\rho+|b_\xi^D(\cdot,\cdot,w)-b_\xi^D(\cdot,\cdot,\phi)|_\infty)|u-v|_{Z^T}$$

for all $u,v\in\Sigma(\rho,T,\phi)\cup\{\phi|_J\}$. Since $|b_\xi^D(\cdot,\cdot,w)-b_\xi^D(\cdot,\cdot,\phi)|_\infty \to 0$ as $T\to 0$, the inequality (6.17) with $K=D$ follows. In the same way one can also show the validity of (6.17) with $K=N$.

We turn now to the spaces $X_{1,s}^m = H_p^s(J;L_p(\Omega))$, $0<s<1$.

Lemma 6.2.3 *Let $0 < s < s_0 < 1$, $\rho \in (0, \rho_0]$, and $w \in X^m_{1,\,s_0} \cap C(J; C(\overline{\Omega}))$ be a fixed K-valued function. Let further Σ' be as described above. Suppose that b is as in Lemma 6.2.1 with $r \in (s, 1)$ and $b_\xi \in L_\infty(J \times \Omega \times K, \mathbb{R}^m)$. Then there exists a constant $C > 0$ not depending on T and ρ such that*

$$[b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{X_{1,\,s}} \leq C\mu(T)(|f-g|_{X^m_{1,\,s_0}} + |f-g|_{\infty,\,m}),\ f, g \in \Sigma'.$$

Proof. Let f, g be arbitrary functions in Σ'. Put

$$\omega(t,h,x) = b(t+h,x,f(t+h,x)) - b(t+h,x,g(t+h,x)) - b(t,x,f(t,x)) + b(t,x,g(t,x))$$

for $t, t+h \in J$, and $x \in \Omega$. Then

$$[b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{X_{1,\,s}} = (\int_\Omega \int_0^T (\int_0^1 \sigma^{-2s}(\frac{1}{|V(t,\sigma)|}\int_{V(t,\sigma)} |\omega(t,h,x)|\,dh)^2 \frac{d\sigma}{\sigma})^{\frac{p}{2}} dt\,dx)^{\frac{1}{p}}.$$

Similarly as in the proof of Lemma 6.2.1 we may establish

$$\begin{aligned}|\omega(t,h,x)| \ &\leq\ |b_\xi|_{\infty,\,m}\,|f(t+h,x) - f(t,x) - g(t+h,x) + g(t,x)| + \\ &\quad + C_{HL}|f-g|_{\infty,\,m}(|f(t+h,x) - f(t,x)| + |g(t+h,x) - g(t,x)| + C_b(x)|h|^r)\end{aligned}$$

for all $t, t+h \in J$ and a.a. $x \in \Omega$. With

$$\begin{aligned}\Theta \ &:=\ (\int_0^T (\int_0^1 \sigma^{-2s}(\frac{1}{|V(t,\sigma)|}\int_{V(t,\sigma)} |h|^r\,dh)^2 \frac{d\sigma}{\sigma})^{\frac{p}{2}} dt)^{\frac{1}{p}} \\ &\leq\ (\int_0^T (\int_0^1 \sigma^{-2s}(\frac{1}{|V(t,\sigma)|}\int_{V(t,\sigma)} \sigma^r\,dh)^2 \frac{d\sigma}{\sigma})^{\frac{p}{2}} dt)^{\frac{1}{p}} \ =\ \frac{T^{\frac{1}{p}}}{\sqrt{2(r-s)}},\end{aligned}$$

we thus obtain

$$\begin{aligned}&[b(\cdot,\cdot,f) - b(\cdot,\cdot,g)]_{X_{1,\,s}} \leq \\ &\leq C\Big([f-g]_{X^m_{1,\,s}} + |f-g|_{\infty,\,m}([f]_{X^m_{1,\,s}} + [g]_{X^m_{1,\,s}} + |C_b|_{L_p(\Omega)}\Theta)\Big) \\ &\leq C\Big([f-g]_{X^m_{1,\,s}} + |f-g|_\infty([f-w]_{X^m_{1,\,s}} + [g-w]_{X^m_{1,\,s}} + [w]_{X^m_{1,\,s}} + \mu(T))\Big) \\ &\leq C\mu(T)\Big([f-g]_{X^m_{1,\,s_0}} + |f-g|_{\infty,\,m}([f-w]_{X^m_{1,\,s_0}} + [g-w]_{X^m_{1,\,s_0}} + 1)\Big) \\ &\leq C\mu(T)([f-g]_{X^m_{1,\,s_0}} + |f-g|_{\infty,\,m}).\ \square\end{aligned}$$

Lemma 6.2.3 and the trivial inequality $|Bf - Bg|_X \leq C\mu(T)|f-g|_{\infty,\,m}$ yield the estimate (6.20), which we were aiming at. This completes the proof of the inequality (6.16), since we have $Z^T_\nabla \hookrightarrow H_p^{(1+\alpha)/2}(J; L_p(\Omega)) \cap C(J \times \overline{\Omega})$ and $(1+\alpha)/2 > \alpha$.

We conclude this paragraph with a result on pointwise multiplication which have been used several times in the previous sections. In Section 4.2.2 we have already seen that the space $Y_1^{0,s_1} \cap Y_2^{0,s_2}$ forms a multiplication algebra if the embedding $Y_1^{0,s_1} \cap Y_2^{0,s_2} \hookrightarrow C(J; C(\overline{\Omega}))$ is valid. We will show under the above assumptions on p that this is true also for $Y_1^{k_1,s_1} \cap Y_2^{1,s_2}$ with $k_1 = 0, 1$. As before, we shall only consider the seminorm terms of highest order.

Lemma 6.2.4 *Let $0 < s_1 < r < 1$. Then there exists a constant $C > 0$ not depending on T such that*

$$[fg]_{Y_1^{1,s_1}} \leq C([f]_{Y_1^{1,s_1}}|g|_\infty + |f|_\infty[g]_{Y_1^{1,s_1}} + T^{r-s_1}[f]_{X_1}[g]_{C_1^r} + T^{r-s_1}[f]_{C_1^r}[g]_{X_1})$$

for all $f, g \in Y_1^{1,s_1} \cap C^r(J; C(\overline{\Omega}))$.

Proof. Clearly $Y_1^{1,s_1} \hookrightarrow X_1$. Suppose $f, g \in Y_1^{1,s_1} \cap C^r(J; L_\infty(\Omega))$. We estimate

$$\begin{aligned}
&[fg]_{Y_1^{1,s_1}} = \\
&= \Big(\int_0^T\int_0^T\int_\Omega \frac{|(f_tg)(t,x) + (fg_t)(t,x) - (f_tg)(\tau,x) - (fg_t)(\tau,x)|^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac{1}{p}} \\
&\leq \Big(\int_0^T\int_0^T\int_\Omega \frac{(|f_t(t,x) - f_t(\tau,x)||g(t,x)|)^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac{1}{p}} + \\
&\quad +\Big(\int_0^T\int_0^T\int_\Omega \frac{(|f_t(\tau,x)||g(t,x) - g(\tau,x)|)^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac{1}{p}} + \\
&\quad +\Big(\int_0^T\int_0^T\int_\Omega \frac{(|f(t,x)||g_t(t,x) - g_t(\tau,x)|)^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac{1}{p}} + \\
&\quad +\Big(\int_0^T\int_0^T\int_\Omega \frac{(|f(t,x) - f(\tau,x)||g_t(\tau,x)|)^p}{|t-\tau|^{1+s_1p}}\,dx\,d\tau\,dt\Big)^{\frac{1}{p}} \\
&\leq [f]_{Y_1^{1,s_1}}|g|_\infty + [g]_{C_1^r}\Big(\int_0^T\int_\Omega |f_t(\tau,x)|^p\Big(\int_0^T \frac{dt}{|t-\tau|^{1+(s_1-r)p}}\Big)\,dx\,d\tau\Big)^{\frac{1}{p}} + \\
&\quad +|f|_\infty[g]_{Y_1^{1,s_1}} + [f]_{C_1^r}\Big(\int_0^T\int_\Omega |g_t(\tau,x)|^p\Big(\int_0^T \frac{dt}{|t-\tau|^{1+(s_1-r)p}}\Big)\,dx\,d\tau\Big)^{\frac{1}{p}} \\
&\leq [f]_{Y_1^{1,s_1}}|g|_\infty + C_1T^{r-s_1}[f]_{X_1}[g]_{C_1^r} + |f|_\infty[g]_{Y_1^{1,s_1}} + C_1T^{r-s_1}[f]_{C_1^r}[g]_{X_1},
\end{aligned}$$

where $C_1 = [2/(r-s_1)p]^{1/p}$. $\square$

A corresponding estimate can be obtained for $[fg]_{Y_2^{1,s_2}}$, so together with the inequalities from Section 4.2.2, we see that $Y_1^{k_1,s_1} \cap Y_2^{1,s_2}$ with $k_1 \in \{0,1\}$ is a multiplication algebra provided the above assumptions on p are fulfilled.

We conclude this section by justifying the normalization step which we carried through in Section 6.1 for the coefficients on the boundary. Let f, $g \in Y_N^T$ and $f(t,x) > 0$, $t \in J$, $x \in \Gamma_N$. By compactness of Γ_N, we even have $f(t,x) \geq c$, $t \in J$, $x \in \Gamma_N$, for some positive constant c. Set $K = (c/2, \infty)$ and consider the function $b : K \to \mathbb{R}$ defined by $b(\xi) = 1/\xi$. Clearly f is K-valued, $b \in C^\infty(K)$ and b, b' are bounded. Therefore $1/f \in Y_N^T$. Since Y_N^T is a multiplication algebra, we deduce that $g/f \in Y_N^T$, too. By an analogous argument, this property can also be proved for the space Y_D^T.

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Rico Zacher: ***Quasilinear parabolic problems with nonlinear boundary conditions*** **(Zusammenfassung)**

Die vorliegende Arbeit widmet sich dem Studium der L_p-Theorie für die nachfolgend beschriebene Klasse von quasilinearen parabolischen Problemen mit nichtlinearen Randbedingungen. Sei $\Omega \subset \mathbb{R}^n$ ein beschränktes Gebiet mit C^2-Rand Γ, welcher sich aus zwei disjunkten abgeschlossenen Mengen Γ_D und Γ_N zusammensetzt. Für die unbekannte skalare Funktion $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ betrachten wir das Problem

$$\left\{\begin{array}{rl} \partial_t u + dk * (\mathcal{A}(u) : \nabla^2 u) = F(u) + dk * G(u), & t \geq 0,\, x \in \Omega \\ \mathcal{B}_D(u) = 0, & t \geq 0,\, x \in \Gamma_D \\ \mathcal{B}_N(u) = 0, & t \geq 0,\, x \in \Gamma_N \\ u|_{t=0} = u_0, & x \in \Omega. \end{array}\right. \qquad \text{(NP)}$$

Dabei sind $(dk * w)(t, x) = \int_0^t dk(\tau) w(t - \tau, x)$, $t \geq 0$, $x \in \Omega$, $\partial_t u$ die partielle Ableitung von u nach t, $\nabla u = \nabla_x u$ der Gradient von u bezüglich der räumlichen Variablen und $\nabla^2 u$ die zugehörige Hesse-Matrix, d.h. $(\nabla^2 u)_{ij} = \partial_{x_i} \partial_{x_j} u$, $i, j \in \{1, \ldots, n\}$. Ferner steht $B : C = \sum_{i=1,\, j=1}^n B_{ij} C_{ij}$ für das Doppelskalarprodukt von zwei Matrizen B, $C \in \mathbb{R}^{n \times n}$. Die Substitutionsoperatoren sind gegeben durch

$$\begin{aligned} \mathcal{A}(u)(t, x) &= -a(t, x, u(t, x), \nabla u(t, x)),\ t \geq 0,\, x \in \Omega, \\ F(u)(t, x) &= f(t, x, u(t, x), \nabla u(t, x)),\ t \geq 0,\, x \in \Omega, \\ G(u)(t, x) &= g(t, x, u(t, x), \nabla u(t, x)),\ t \geq 0,\, x \in \Omega, \\ \mathcal{B}_D(u)(t, x) &= b^D(t, x, u(t, x)),\ t \geq 0,\, x \in \Gamma_D, \\ \mathcal{B}_N(u)(t, x) &= b^N(t, x, u(t, x), \nabla u(t, x)),\ t \geq 0,\, x \in \Gamma_N, \end{aligned}$$

wo a eine $\mathbb{R}^{n \times n}$-wertige und f, g, b^D, b^N skalare Funktionen sind. Der skalare Kern $k \in BV_{loc}(\mathbb{R}_+)$ mit $k(0) = 0$ gehört einer gewissen Klasse von Kernen mit Parameter $\alpha \in [0, 1)$ an, welche, grob gesprochen, alle "regulären" Kerne enthält, die sich wie t^α für $t\,(> 0)$ nahe Null verhalten. Der Spezialfall $k(t) = 1$, $t > 0$, wo sich die Integrodifferenzialgleichung zu einer partiellen Differenzialgleichung vereinfacht, ist in dieser Formulierung mit enthalten.

Gleichungen der Form (NP) treten in einer Vielzahl von angewandten Problemen auf. Wichtige Beispiele sind die nichtlineare Viskoelastizität und Wärmeleitung in Materialien mit Gedächtnis. Obwohl es in der Literatur eine Fülle von Resultaten zu Problemen der Form (NP) gibt, scheint nur wenig in Bezug auf eine L_p-Theorie im Falle der Integrodifferenzialgleichung mit *nichtlinearen Randbedingungen* bekannt zu sein.

Unter geeigneten Voraussetzungen an die Nichtlinearitäten und den Anfangswert wird in der Arbeit nachgewiesen, dass das Problem (NP) eine eindeutige lokale starke Lösung in folgendem Sinn besitzt: Sei $n + 2/(1 + \alpha) < p < \infty$. Dann gibt es ein $T > 0$, so dass im Raum $Z^T = H_p^{1+\alpha}([0, T]; L_p(\Omega)) \cap L_p([0, T; H_p^2(\Omega))$ genau eine Funktion u existiert, welche (NP) genügt. Hierbei bezeichnet $H_p^s([0, T]; L_p(\Omega))$ $(s > 0)$ den vektorwertigen Besselpotenzialraum von Funktionen auf $[0, T]$ mit Werten im Lebesgueraum $L_p(\Omega)$. Die obige Voraussetzung an p ist wesentlich; sie stellt sicher, dass die Einbettung $Z^T \hookrightarrow C(J; C^1(\overline{\Omega}))$ gilt.

Die Grundidee des Beweises besteht darin, für ein mit (NP) verwandtes *lineares* Problem (LP) mit *inhomogenen Randdaten* optimale Regularitätsabschätzungen vom L_p-Typ herzuleiten, welche es erlauben, (NP) als Fixpunktgleichung im Raum Z^T zu schreiben. Existenz und Eindeutigkeit eines Fixpunktes werden dann für hinreichend

kleines T mit Hilfe des Kontraktionsprinzips erhalten. Entscheidend ist bei diesem Zugang, Bedingungen an die Inhomogenitäten, insbesondere die Randdaten, zu finden, welche die eindeutige Lösbarkeit von (LP) im Raum der maximalen Regularität charakterisieren. Diese Bedingungen werden mit Hilfe der Lokalisierungsmethode und Störungsargumenten aus Resultaten zu Ganz- und Halbraumproblemen mit konstanten Koeffizienten gewonnen. Letztere folgen aus Sätzen über abstrakte Probleme, deren Analyse einen wesentlichen Bestandteil der vorliegenden Arbeit darstellt.

Zwei Klassen von abstrakten Gleichungen werden dabei untersucht: 1. die abstrakte Volterra-Gleichung

$$u(t) + (a * Au)(t) = f(t), \quad t \geq 0,$$

und 2. Probleme auf einem Streifengebiet $J \times \mathbb{R}_+$ ($J = [0, T]$) von der Form

$$\begin{cases} u - a * \partial_y^2 u + a * Au = f, \ t \in J, \ y > 0, \\ u(t, 0) = \phi(t), \ t \in J, \end{cases} \quad \begin{cases} u - a * \partial_y^2 u + a * Au = f, \ t \in J, \ y > 0, \\ -\partial_y u(t, 0) + Du(t, 0) = \phi(t), \ t \in J, \end{cases}$$

wobei A ein sektorieller und D ein pseudosektorieller Operator in einem Banachraum X sind. Für jede dieser abstrakten Gleichungen werden Bedingungen an die gegebenen Daten hergeleitet, die notwendig und hinreichend für die eindeutige Lösbarkeit des betreffenden Problems in einem bestimmten Raum optimaler Regularität vom L_p-Typ sind. Wesentliche Hilfsmittel sind dabei die Inversion der Faltung, Dore-Venni-Theorie, reelle Interpolation, und der Multiplikatorensatz von Michlin in der operatorwertigen Version. Die Resultate verallgemeinern bekannte Sätze über maximale L_p-Regularität von abstrakten Evolutionsgleichungen.

Die vorliegende Arbeit beschäftigt sich ferner mit dem vektorwertigen Halbraumproblem

$$\begin{cases} \partial_t v - da * (\Delta_x v + \partial_y^2 v) - (db + \frac{1}{3} da) * (\nabla_x \nabla_x \cdot v + \partial_y \nabla_x w) = f_v & (J \times \mathbb{R}_+^{n+1}) \\ \partial_t w - da * \Delta_x w - (db + \frac{4}{3} da) * \partial_y^2 w - (db + \frac{1}{3} da) * \partial_y \nabla_x \cdot v = f_w & (J \times \mathbb{R}_+^{n+1}) \\ -da * \gamma \partial_y v - da * \gamma \nabla_x w = g_v & (J \times \mathbb{R}^n) \\ -(db - \frac{2}{3} da) * \gamma \nabla_x \cdot v - (db + \frac{4}{3} da) * \gamma \partial_y w = g_w & (J \times \mathbb{R}^n) \\ v|_{t=0} = v_0 & (\mathbb{R}_+^{n+1}) \\ w|_{t=0} = w_0 & (\mathbb{R}_+^{n+1}), \end{cases}$$

welches in der Theorie der Viskoelastizität eine Rolle spielt. Die unbekannten Funktionen v und w sind $\mathbb{R}^n$- bzw. $\mathbb{R}$-wertig, γ bezeichnet den Spuroperator bzgl. $y = 0$. Im Gegensatz zu den obigen Problemen tauchen hier *zwei unabhängige Kerne* auf. Einmal mehr charakterisieren wir die eindeutige Lösbarkeit des Problems in einem bestimmten Raum maximaler Regularität vom L_p-Typ in Form von Regularitäts- und Kompatibilitätsbedingungen an die Daten. Dabei verwenden wir die Resultate zu obigen abstrakten Gleichungen und den gemeinsamen $\mathcal{H}^\infty$-Kalkül des Operatorenpaares $(\partial_t, -\Delta_x)$ im Raum $L_p(\mathbb{R}_+ \times \mathbb{R}^n)$. Die wesentliche Schwierigkeit besteht dabei in der Abschätzung für das Hauptsymbol des Problems: Man muss zeigen, dass es Konstanten $c > 0$ und $\eta \in (0, \pi/2)$ gibt, so dass die Ungleichung

$$\left| \frac{1}{\hat{a}(z)\tau^2} + 2 \right| \leq c \left| \frac{1}{\hat{a}(z)\tau^2} + \frac{\frac{4\hat{b}(z) + \frac{4}{3}\hat{a}(z)}{\hat{b}(z) + \frac{4}{3}\hat{a}(z)} \sqrt{\frac{1}{\hat{a}(z)\tau^2} + 1}}{\sqrt{\frac{1}{\hat{a}(z)\tau^2} + 1} + \sqrt{\frac{1}{(\hat{b}(z) + \frac{4}{3}\hat{a}(z))\tau^2} + 1}} \right|, \ (z, \tau) \in \Sigma_{\frac{\pi}{2} + \eta} \times \Sigma_\eta$$

gilt, wobei $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$. Diese entscheidende Abschätzung wird mittels einer sorgfältigen funktionentheoretischen Analyse gezeigt.

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

Halle (Saale), 07. Januar 2003

Rico Zacher

Curriculum Vitae

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Awards/Prizes/Scholarships

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