Strong Well-Posedness of a Model for an Ionic Exchange Process

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To My Parents

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Introduction

It is a well known fact that chemically reacting systems can be described by means of systems of reaction diffusion equations on the microscopic scale. These kinds of equations have been studied in great detail by many authors during the last three decades. Although in chemical engineering the focus is overriding on the macroscopic scale, leading mostly to ordinary differential equations, in many problems one has to take into account effects of diffusion, convection and dispersion, or physical effects caused by electrical charges (e.g. electromigration). The mass balance equations then become reaction-diffusion-convection equations which are coupled with equations arising from considered physical processes. This leads to systems of partial differential equations in three dimensions which can be very complicated. Therefore, models have been developed on the macroscopic scale which allow for the essential information about the physical processes taking place. However, many chemical processes involve two or more phases, which typically means, the reacting species flow into the *Continuously-flow Stirred Tank Reactor* (CSTR) and at least one of these species must be transferred to another phase through an interface. In such situations it is of importance to take into consideration mass transport in order to arrive at reliable models. However, this brings about the coupling between the macroscopic reactor scale and the microscopic processes.

In the last decades, efforts have been made to account for electrical forces between particles. This approach seems to be reasonable for the reacting species not being electrically neutral, and particularly if electrical interactions can not be neglected in the chemical process. This applies in case that electrical forces are of the same magnitude as the other driving forces, e.g. diffusion or convection. However this involves a new unknown quantity, namely, the so-called electrical potential which is caused by the charged particles. Including this item leads to a strong coupling of the equations for the charged species. One possibility for incorporating these effects into the model is the assumption of *electroneutrality*, which demands that the total charge has to be zero everywhere at any time. This means that, for concentrations c_i of reacting species and corresponding charges $z_i \in \mathbb{Z}$ the following algebraic constraint must hold

$$\sum_i z_i c_i(t,x) \;=\; 0, \quad t \in J, \quad x \in \Omega\,. \tag{1}$$

Thus, the reaction diffusion equations are augmented with an algebraic equation. The effect of electromigration was first taken into account by Henry and Louro [14]. To all appearances there are only a very few papers about electrochemical systems in the mathematical literature, e.g. see [2], [6], [15], [23], [38] and [4]. Therefore, it is this physical feature which is to play a decisive role in our treatise.

In this thesis we are concerned with a mathematical model resulting from a regenerative ionic exchanger, see [21] or [5] for more chemical background. The model will describe in detail the regeneration of the weak acidic cation exchanger-resins *Amberlite IRC*-86® (called pellets) charged with Cu^{2+}-ions via hydrochloric acid HCl in a well stirred tank (CSTR). In fact, the pellets are suspended in a liquid bulk phase, where the acid is fed into the reactor continuously via a carrying liquid and dissociates into H^+ and Cl^-. The exchange of cations H^+ and Cu^{2+} is connected with a subsequent reaction of neutralisation between the moving protons H^+ arising from the acid and the attached ions $\overline{COO^-}$. The chemical reaction equation reads as follows:

$$R(COO^-)_2Cu^{2+} \; + \; 2H^+ \; \longrightarrow \; 2RCOOH \; + \; Cu^{2+}\,. \tag{2}$$

The model is illustrated schematically by the following figure

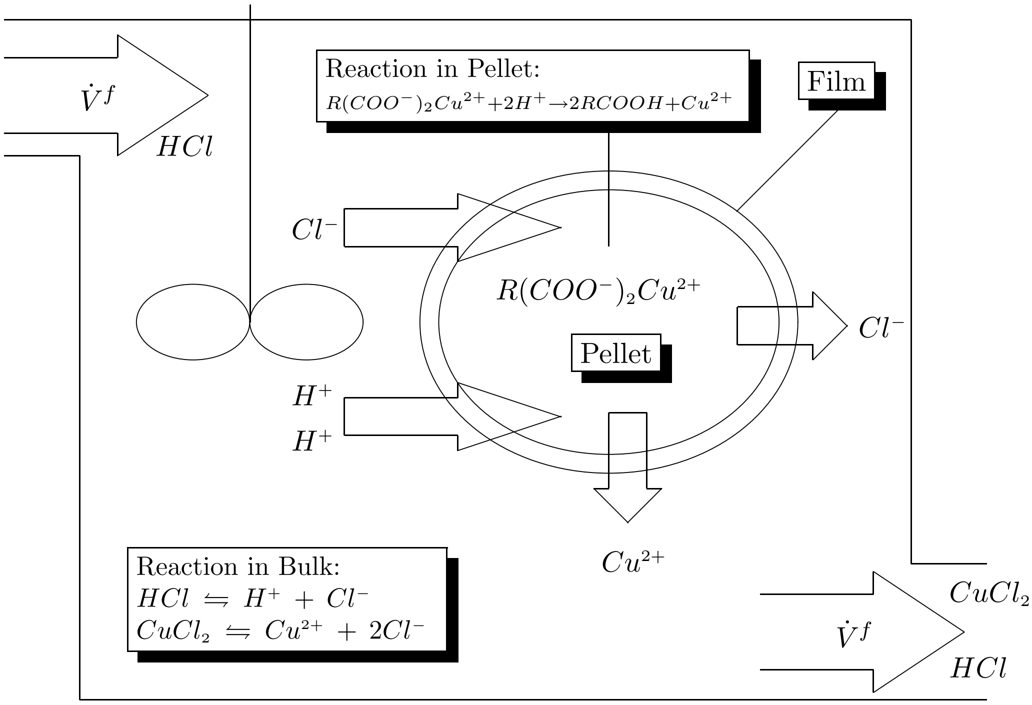

Figure 1: Processes in the CSTR

As visualised above, the underlying chemical system consists of three phases: the almost perfectly mixed bulk phase, the porous pellet and the film. The balance of each phase has to take into consideration coupling of mass transport for all species and chemical reactions. The resulting equations yield systems of heterogeneous reaction diffusion equations in each phase which are connected to boundary conditions. In the end, a system of parabolic equations for concentrations in film and pellet is obtained, and ordinary differential equations reproduce the situation in the bulk phase. As mentioned above, the effect of electromigration caused by considering charged species is to be involved, which in turn requires the electroneutrality condition (1).

Now, we shall describe the equations modelling the above situation. Let Ω be a bounded domain in $\mathbb{R}^3$ which decomposes according to $\Omega = \overline{\Omega}_P \cup \Omega_F$ and their boundaries $\Gamma_P := \partial\Omega_P$ and $\partial\Omega_F = \Gamma_P \cup \Gamma$, $\Gamma := \partial\Omega$, are C^2-smooth with dist $(\Gamma_P, \Gamma) > 0$. The domain Ω_P represents a typical pellet and Ω_F its surrounding liquid film. For the unknown functions $u^k : [0,T] \times \Omega_k \to$

$\mathbb{R}^N$, $k = P, F$, $u^b : [0,T] \to \mathbb{R}^N$ and $\phi^k : [0,T] \times \Omega_k \to \mathbb{R}$, $k = P, F$, we are concerned with the problem

$$
\begin{aligned}
&\partial_t u^P - \nabla \cdot (D^P \nabla u^P) - \nabla \cdot (M^P u^P \otimes \nabla \phi^P) = R^P(t,x,u^P), \quad (t,x) \in J \times \Omega_P,\\
&\partial_t u^F - \nabla \cdot (D^F \nabla u^F) - \nabla \cdot (M^F u^F \otimes \nabla \phi^F) = R^F(t,x,u^F), \quad (t,x) \in J \times \Omega_F,\\
&D^P \partial_\nu u^P + M^P u^P \partial_\nu \phi^P = D^F \partial_\nu u^F + M^F u^F \partial_\nu \phi^F, \quad (t,x) \in J \times \Gamma_P,\\
&\big[\ln(\gamma_i^P u_i^P)\big]_{1 \le i \le N} + \lambda_0 \phi^P z = \big[\ln(\gamma_i^F u_i^F)\big]_{1 \le i \le N} + \lambda_0 \phi^F z, \quad (t,x) \in J \times \Gamma_P,\\
&u^F = u^b, \quad (t,x) \in J \times \Gamma,\\
&\frac{d}{dt} u^b = \frac{1}{\tau}\left(u^f - u^b\right) + R^b(u^b) - a^b \int\limits_\Gamma D^F \partial_\nu u^F + M^F u^F \partial_\nu \phi^F \, d\sigma, \quad t \in J,\\
&u^P(0,x) = u_0^P(x), \quad x \in \overline{\Omega}_P, \quad u^F(0,x) = u_0^F(x), \quad x \in \overline{\Omega}_F, \quad u^b(0) = u_0^b,
\end{aligned} \tag{3}
$$

and

$$
z^T \cdot u^k(t,x) = 0, \quad (t,x) \in J \times \overline{\Omega}_k, \quad k = P, F\,, \quad z^T \cdot u^b(t) = 0, \quad t \in J\,. \tag{4}
$$

Let us explain the relevant quantities and constants. The functions u^k and ϕ^k, $k = P, F, b$ denote the concentration vectors and electrical potentials, respectively, where the superscripts indicate the corresponding phase. The diffusion coefficients d_i^k summarised to the matrix D^k are known functions of (t,x), and the quantity $m_i^k(t,x) := \lambda_0 \cdot d_i^k(t,x) \cdot z_i$ is the so-called electrochemical mobility. We set $M^k = \operatorname{diag}[m_i^k]_{1 \le i \le N}$. Finally, the constant $\lambda_0 := F/RT$ is positive, where F denotes the Faraday constant, T the absolute temperature and R the gas constant.

The charge of species i denoted by z_i is the same in each phase. The first boundary condition is caused by the continuity of fluxes on Γ_P, whereas the second boundary condition is due to continuity of chemical potentials. At the outer surface of the film Γ continuity of concentrations is imposed. The next equation describes the evolution of the bulk concentrations. The feeds u_i^f are time-dependent non-negative functions and the constant a^b comprises among other things the total number of pellets in the bulk volume. Finally, the functions R_i^k, $k = P, F, b$ and $i = 1, \ldots, N$ designate the production rate densities of species i due to the chemical reactions in phase k. The purpose consists in finding functions $u = (u^P, u^F, u^b)$ and $\phi = (\phi^P, \phi^F)$ satisfying the above problem which possess the regularity

$$
u \in Z^T := Z_P^T \times Z_F^T \times Z_b^T, \quad \phi \in \mathcal{Z}^T := \{(\phi^P, \phi^F) \in \mathcal{Z}_P^T \times \mathcal{Z}_F^T : \ \gamma_{|\Gamma_P}(\phi^P - \phi^F) \in Y_{1,\Gamma_P}^T\},
$$

with

$$
Z_k^T := \mathrm{H}_p^1(J; \mathrm{L}_p(\Omega_k; \mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\Omega_k; \mathbb{R}^N)), \quad \mathcal{Z}_k^T := \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^1(\Omega_k)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\Omega_k)),
$$

and Y_{1,Γ_P}^T denotes a certain trace space.

The first question which is raised here is: "What are the determining equations for the electrical potentials ϕ^P, ϕ^F?". It is well known that assumption (1) implies an equation for the electrical potentials, consequently we obtain a closed model. Taking the inner product of (3) with z in $\mathbb{R}^N$ and accounting for the electroneutrality condition (4) yields the elliptic boundary value problem

$$
\begin{aligned}
&\nabla\cdot(z^T\!\cdot M^P u^P\nabla\phi^P)+\nabla\cdot(z^T\!\cdot D^P(t,x)\nabla u^P)=0, \quad (t,x)\in J\times\Omega_P,\\
&\nabla\cdot(z^T\!\cdot M^F u^F\nabla\phi^F)+\nabla\cdot(z^T\!\cdot D^F(t,x)\nabla u^F)=0, \quad (t,x)\in J\times\Omega_F,\\
&\phi^P-\phi^F=\frac{1}{\lambda_0|z|^2}\left[\sum_i z_i\ln\left(\gamma_i^F(t,x)u_i^F\right)-\sum_i z_i\ln\left(\gamma_i^P(t,x)u_i^P\right)\right], \quad (t,x)\in J\times\Gamma_P, \qquad (5)\\
&z^T\!\cdot M^P u^P\partial_\nu\phi^P-z^T\!\cdot M^F u^F\partial_\nu\phi^F=z^T\!\cdot D^F(t,x)\partial_\nu u^F-z^T\!\cdot D^P(t,x)\partial_\nu u^P, \quad (t,x)\in J\times\Gamma_P,\\
&\phi^F=0, \quad (t,x)\in J\times\Gamma\,.
\end{aligned}
$$

It turns out that adding these elliptic equations to problem (3) is an equivalent formulation of (3) with electroneutrality (4). An important issue of this boundary value problem is the regularity of (ϕ^P,ϕ^F) in regard to the additional dependence on variable t. We will see that the electrical potentials possess *half a time derivative* although all terms appearing in the elliptic equations belong to $\mathrm{L}_p(J;\mathrm{L}_p(\Omega_k))$.

Now, we want to dwell on the difficulties we have to overcome. We immediately perceive that the above problem leads to a strongly coupled quasilinear parabolic-elliptic system with nonlinear boundary condition of Dirichlet type, nonlinear transmission condition, dynamical boundary conditions and nonlinear reaction rates. The most interesting difficulty of our problem becomes manifest in the nonlinear transmission condition

$$D^P\partial_\nu u^P+M^P u^P\partial_\nu\phi^P=D^F\partial_\nu u^F+M^F u^F\partial_\nu\phi^F, \quad (t,x)\in J\times\Gamma_P\,. \qquad (6)$$

Almost all quantities are involved in this boundary equation (except for the vector of concentration u^b), all coefficients of unknown functions are different and only terms of highest order occur. Hence, this circumstance naturally leads to a strong coupling between the concentrations and electrical potentials of each phase. Rigorous investigations of multiphase processes including electroneutrality condition (1) and nonlinear boundary conditions, e.g. transmission condition (6), are apparently missing. We would like to mention that a one-dimensional problem (and its modelling), $\Omega\subset\mathbb{R}$ bounded, was treated by Bothe and Prüss [4].

Now, we want to point out where the potential difficulties are hidden. In principle, there are two approaches to solve a parabolic-elliptic system. Either we take the concentration vectors (u^P,u^F,u^b) for granted, solve the elliptic problem and gain a solution formula in terms of the electrical potentials which has to be inserted in the parabolic equations or we consider the reverse. However, this method has an essential disadvantage which is caused by the multiphase situation. In fact, solving the elliptic problem supplies a nonlocal solution operator Φ which acts on (u^P,u^F) linearly and additionally depends on these functions nonlinearly, i.e. we have

$$(\phi^P,\phi^F)=\Phi(u^P,u^F)(u^P,u^F)\,.$$

This representation does not yet provide an insight into the linear part of the nonlinear tranmission condition as in contrast to partial differential equations in domains Ω_P and Ω_F. Here all nonlinear terms of highest order can be treated by using certain projections which correspond with replacing electroneutrality condition by the elliptic equations for potentials. For $k=P,F$ we can define the projections

$$\Pi^k(t,x,u^k):=I-\frac{M^k(t,x)u^k(t,x)\otimes z}{z^T\!\cdot M^k(t,x)u^k(t,x)}\,.$$

Applying these projections to equations in domains Ω_P, Ω_F and utilising electroneutrality condition $z^T \cdot u^k = 0$ entails

$$\partial_t u^k - \Pi^k(t,x,u^k) D^k(t,x) \Delta u^k = \Pi^k(t,x,u^k) R^k(u^k) + \Pi^k(t,x,u^k) \nabla D^k(t,x) \nabla u^k \\ + \Pi^k(t,x,u^k) \nabla [M^k(t,x) u^k(t,x)] \nabla \phi^k, \quad (t,x) \in J \times \Omega_k \,.$$

This shows that only terms of lower order in respect of the nonlocal operator $\Phi(u^P, u^F) = (\phi^P, \phi^F)$ remain, and the quasilinear structure appears. To treat the nonlinear transmission condition we can not employ this approach since both concentrations and both potentials appear in this equation. Moreover, the solution operator Φ of the boundary value problem does not meet with success either. Since the operator Φ is not given analytically, we are not able to compute the expression $\partial_\nu \Phi(u^P, u^F)(u^P, u^F)$ in view of extracting the highest order terms, i.e. $\partial_\nu u^k$. This circumstance is revealed by transforming the problem into the half space via localisation, changing of coordinates and perturbation. In this situation the transmission can be written as

$$(\Pi^F D^F) \partial_y u^F + (D_n + 1)^{1/2} (\Pi \tilde{U}_F^{-1}) u^F - (\Pi^P D^P) \partial_y u^P = g,$$

where the operator $(D_n + 1)^{1/2}$ denotes the square root of the shifted Laplacian in $\mathbb{R}^n$ and $\Pi \tilde{U}_F^{-1}$ is a certain projection. We perceive that the pseudo-differential operator $(D_n + 1)^{1/2}$ is responsible for getting into difficulties and, of course, justifies our approach by means of considering the localised problem. Another difficulty contained in the above equation is caused by the non-commuting coefficient matrices. To be able to solve this two phase problem it depends on figuring out the equation

$$\Big[\sum_{k=1,2} ((\Pi_k D_k)^{-1} \partial_t + D_n + 1)^{1/2} (\Pi_k D_k)^{1/2} \tilde{U}_k^{-1} + (D_n + 1)^{1/2} \Pi \Big] \tilde{U}_F^{-1} c_F = \overline{g},$$

which is linked to the above transmission condition. The purpose consists in determining the unknown function c_F. The difficulty we encounter here are the matrices $\Pi^P D^P$, $\Pi^F D^F$, $\tilde{U}_P^{-1}$ and $\tilde{U}_F^{-1}$ which do not commute. However, the symbol of the operator satisfies a certain lower estimate which entails its invertibility.

Now, we present a summary of the contents of this thesis and put across the essential ideas. In *Chapter 1* we derive the model by considering the principle of conservation of mass, prescribing suggestive boundary conditions and by accounting for mass transport between bulk and pellets. Here we perceive that the equation for concentration of the exchanger-resin which makes up the pellet and the equations for the hydrochloric acid HCl and the salt $CuCl_2$ decouple from the remaining system. After introducing the assumptions for given functions we set about seeking the corresponding linear problem. This proceeding is caused by solving the nonlinear problem via the contraction mapping principle and the fact that the solution operator resulting from the linear problem places us in a position to formulate the original problem (3) as a fixed point equation. With the aid of the contraction mapping principle and for sufficiently small time-intervalls a unique fixed point is then obtained. The term of solution space and other important function spaces related to our problem are introduced here.

The purpose of *Chapter 2* is to compile tools needed for solving the linear problem. A large part of this chapter is devoted to sectorial operators admitting bounded imaginary powers or a bounded $\mathcal{H}^\infty$- calculus. Furthermore, we will focus on $\mathcal{R}$-boundedness of operator families, Fourier multipliers and maximal L_p-regularity. The Mikhlin multiplier theorem in the operator-valued version proven recently by Weiss [36] will play an important role for proving optimal regularity. Another important tool which matters for treating the linear and the nonlinear problem are embedding theorems. Furthermore, certain function spaces are shown to form multiplication algebras. Subsequently, we deal with two general model problems which naturally arise by using techniques of localisation in order to treat the linear problem.

Chapter 3 is devoted to the linear problem and essentially comprises the proof of maximal regularity. We start with considering a problem in the full space induced by localisation in the interior of domain Ω_k, and studying a half space problem as a result of the boundary Γ. These model problems consist of parabolic problems coupled with elliptic equations arising from the electrical potentials which have to be determined as well. Finally, the boundary Γ_P brings about a so-called two phase problem being the gist of this thesis. The particular features of this model problem are transmission condition (6) and the jump condition caused by continuity of the chemical potentials. For solving this intricate problem it boils down to study a boundary equation which is composed of a sum of operators having bounded imaginary powers. These operators are quadratic matrices of dimension $N \times N$ that are not commuting. Owing to this circumstance the Dore-Venni Theorem is not applicable, however, the Mikhlin multiplier theorem in the operator-valued version applies. Maximal L_p regularity of each model problem supplies a solution operator which will be used for representing solutions of local problems with variable coefficients.

Thereafter, we make available techniques of localisation needed for proving maximal regularity of the linear problem in a bounded domain. The process of localisation reduces this task to the model problems treated before. In the end, with the aid of local solutions we are able to construct solution of the original problem.

In *Chapter 4* we tackle the nonlinear problem by means of the contraction mapping principle. As above noted the results of Chapter 3 enter here to attain a fixed point equation equivalent to the original problem. Theorem 4.1 proves existence and uniqueness of a general three-phase not including the equations for concentrations of HCl, $CuCl_2$ and the exchanger-resin. Moreover, we show that a solution (potentials and positive concentrations) has a maximal interval of existence, and defines a local semiflow. To achieve a selfmapping we have to choose a small time interval. To obtain positivity for the concentrations the maximum principle is utilised. By means of continuation we obtain a maximal interval of existence.

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Chapter 1

The Model

In this chapter we first will introduce the problem and explain the physical-chemical background. Then, we start with the derivation of a model considering all relevant effects of physics and chemistry. The resulting conservation equations contain terms to model diffusion, reaction and migration. In literature reaction diffusion systems are often discussed, whereas the latter effect is disregarded. Thereafter, we introduce all assumptions concerning coefficients and nonlinearities and introduce our concept of solutions. Since modelling entails nonlinear equations and we embark on the strategy to solve this problem by linearisation, we are looking for the corresponding linearised equations. This will be done in Section 1.4.

1.1 Regeneration of Ionic Exchangers

For 60 years ionic exchange processes have been applied to regenerative technologies and are typically implemented to purify water, e.g. for softening, decarbonation, decolourisation as well as desalination. A second application playing an important role is the purification of industrial water as well as sewage. Toxic ionics of heavy metal for instance, non-ferrous metal, cyanide or metal complexes, which were disposed from rinsing water or effluent of galvanic industries, can be rendered harmless and partially regained with the aid of ionic exchangers. This process is also suitable for the decontamination of radioactive sewage arising in nuclear power plants, nuclear facilities and factories for recycling of nuclear fuel. Although the technical realisation of ionic exchange processes is not difficult, there is an increased requirement of researching chemical and physical processes, cp. [19]. This justifies the interest in mathematical modelling and numerical simulations of ionic exchangers [17].

For our considerations we devote ourselves to studying the organic exchanger resins. These exchanger particles consist of an irregular, three-dimensional matrix of chained hydrocarbon molecules which give the resin a hydrophobic character. This water-insoluble matrix can be commuted into an electrolytic ionic exchanger by integrating hydrophilic groups into the matrix. These groups, which possess a certain charge, get tied into the three-dimensional matrix and lead to a positive or negative charge of the entire matrix. Consequently, mobile ionics with opposite charge can be fixed to these groups. It turns out that this network polymer was transformed into a reactive polymer due to integrating charged groups. It is exactly this property that is of interest to various industries as mentioned above.

We now come to phenomena responsible for the ionic exchange process. An ionic exchanger basically consists of two phases, namely, the electrolyte and the porous polymer matrix which is pervaded by the electrolyte. Hence, the liquid phase continues inside the insoluble polymer matrix. For considering diffusive forces, the irregular structure and variation of pore diameter of such polymers have to be taken into account, resulting in the micropore diffusion and macrospore diffusion. According to the hydrodynamic size of an ion and the pore diameter we have to consider interactions between pore wall and ions. In addition, interionic forces as repulsion and attraction have an effect on the mobility of ionics, e.g. charge carriers induce an electrical field affecting the other ionics. Hence, the transport of various charged species in electrolytes involves electromotive forces, which can become large if the system deviates from the electroneutrality condition. In other words, these forces quickly bring about the state of electroneutrality respectively inhibit a variation of electroneutrality. The effect of electrical fields on diffusion processes of charged particles, which is called “electromigration”, can be described by means of the “Nernst-Planck Equation” [22]. Models of various ion exchange processes and membrane processes employing the Nernst-Planck Equation can be found in [8], [18].

Finally, let us point out that the “Stefan-Maxwell-Equations”, taking into consideration pressure diffusion and interionic interactions, specify these transport processes in electrolyte and non-electrolyte systems. These approaches were applied to several ionic exchange processes and membrane processes, cf. [11], [37].

1.2 Modelling of an Ionic Exchanger

Now, we consider a concrete ionic exchanger arising in chemistry. The underlying situation is that a certain irreversible chemical reaction, $\mathrm{B} + 2\mathrm{H}^+ = 2\mathrm{P} + \mathrm{Cu}^{2+}$, is to take place inside a pellet of high porosity, carrying ions of type $\mathrm{B} := [\mathrm{R}(\mathrm{COO}^-)_2\mathrm{Cu}^{2+}]$ which are immobile. These particles of small size are suspended in a liquid bulk phase. Typically it is assumed that a stagnant boundary layer is present around the pellets, separating these particles from the region of turbulent liquid, such that the overall system consists of three different phases: the porous pellet, the film and the bulk volume. The film alludes to the fact that due to viscosity there is a transport resistance close to the surface of pellets. The bulk phase reflects a well stirred tank filled with an acid HCl and a salt CuCl_2. These chemicals are fed into the reactor continuously via a carrying liquid and dissociate into H^+, C^- and Cu^{2+}. The reaction equations read as follows

$$\begin{array}{rccccccc}
\text{Pellet:} & \mathrm{B} & + & 2\mathrm{H}^+ & \longrightarrow & 2\mathrm{P} & + & \mathrm{Cu}^{2+} \\
\text{Bulk:} & & & \mathrm{HCl} & \rightleftarrows & \mathrm{H}^+ & + & \mathrm{Cl}^- \\
 & & & \mathrm{CuCl}_2 & \rightleftarrows & \mathrm{Cu}^{2+} & + & 2\mathrm{Cl}^-
\end{array}$$

In the following we will use the abbreviations $\mathrm{H} := [\mathrm{H}^+]$, $\mathrm{C} := [\mathrm{Cl}^-]$, $\mathrm{HC} := [\mathrm{HCl}]$ and $\mathrm{AC} := [\mathrm{CuCl}_2]$. In order that the reaction in the pellet takes place, H has to diffuse through the stagnant film to the surface of the pellet-core and into its interior. Here the copper ion, fixed to the polymer matrix, is replaced by two protons so we get two products, a copper ion and two electric neutral molecules $P := [\mathrm{RCOOH}]$. The latter one does not effect subsequent reactions and is therefore not needed for further balancing. The reaction proceeds as long as B is present. After the reaction the mobile copper ion can diffuse into the interior of the pellets or to the surface and then into the bulk phase. For a realistic model of such processes, we have to take into consideration interfacial mass transport, diffusion and reactions inside the pellet and in the bulk. The principle of conservation of mass for all species leads to the conservations laws

$$\partial_t u_i^P(t,x) \;+\; \nabla \cdot J_i^P(t,x) = R_i^P, \quad (t,x) \in J \times \Omega_P\,, \tag{1.1}$$
$$\partial_t u_B^P(t,x) \qquad\qquad\qquad\quad = R_B^P, \quad (t,x) \in J \times \Omega_P\,, \tag{1.2}$$
$$\partial_t u_i^F(t,x) \;+\; \nabla \cdot J_i^F(t,x) = R_i^F, \quad (t,x) \in J \times \Omega_F\,, \tag{1.3}$$

for $i = \mathrm{H}, \mathrm{A}, \mathrm{C}$. The nonnegative concentration of a species i is denoted by u_i^k and the capital letters P and F indicate the phase. Here we assume that all pellets including the stagnant boundary layer are of the same shape given by a certain bounded set $\Omega \subset \mathbb{R}^3$ with C^2-boundary $\Gamma := \partial\Omega$. Furthermore, we set Ω_P for the pellets and Ω_F for the film. This implies that the boundary of Ω_F splits in two parts, namely the boundary of Ω_P denoted by Γ_P and the boundary of Ω. These boundaries are supposed to satisfy the condition $\operatorname{dist}(\Gamma_P, \Gamma) > 0$. The following picture makes the underlying situation clear.

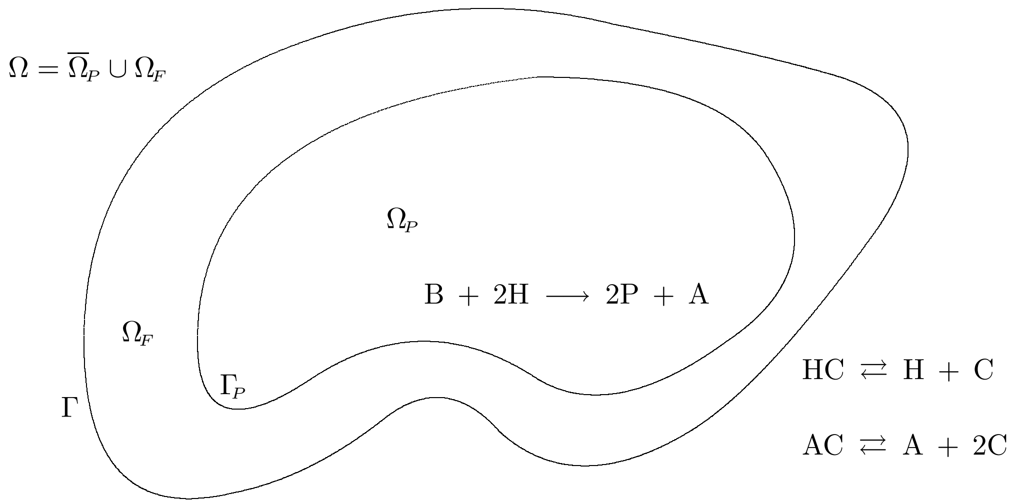

Figure 1.1: Pellet

The function R_i^k designates the production rate density of species i due to the chemical reactions in phase k. The flux vector of a species i is given by

$$J_i^k(t,x) := -d_i^k(t,x)\nabla u_i^k - m_i^k(t,x) u_i^k \nabla \phi^k\,.$$

It consists of a diffusion term according to Fick's law and a migration term, as we want to take into account electrical forces between the charged ions as well. Here, the function ϕ^k denotes the electrical potential in phase k, which is generated by the charged ions. All diffusion coefficients are known functions of (t,x) and the quantity $m_i^k(t,x) := \lambda_0 \cdot d_i^k(t,x) \cdot z_i$ is the so-called electrochemical mobility. Here z_i denotes the charge of species i and it is clear, that this value is the same in each phase. Finally, the constant $\lambda_0 := F/RT$ is positive, where F denotes the Faraday constant, T the temperature and R the gas constant.

Now we shall discuss the boundary conditions. We impose continuity of fluxes on Γ_P and continuity of chemical potentials, respectively. The latter boundary condition leads to a jump of the concentrations at the surface of pellets. At the outer surface of the film Γ we demand that the concentration of a specific species i equals to the corresponding bulk concentration, in other words, continuity of concentrations. Putting together all boundary conditions yields

$$J_i^P(t,x)\cdot\nu = J_i^F(t,x)\cdot\nu \quad \text{on } \Gamma_P\,, \tag{1.4}$$
$$\mu_i^P(t,x) = \mu_i^F(t,x) \quad \text{on } \Gamma_P\,, \tag{1.5}$$
$$u_i^F(t,x) = u_i^b(t) \quad \text{on } \Gamma\,, \tag{1.6}$$

where ν denotes the outer normal. In (1.6) the function $u_i^b(t)$ denotes the concentration of species i in bulk volume V^b. Since we consider a perfectly mixed tank, these concentrations do not depend on space variable x. The chemical potentials in the second boundary condition are typically modelled by

$$\mu_i^k(t,x) := \mu_i^0 + R\cdot T\ln(\gamma_i^k(t,x)u_i^k(t,x)) + z_i\cdot F\cdot\phi^k(t,x)\,,$$

where the standard potentials are independent of concentrations, i.e. $\mu_i^0 := \mu_i^{P,0} = \mu_i^{F,0}$, and the chemical activities γ_i^P, γ_i^F are assumed to be positive. Evolution of the bulk concentrations is described by the system of ordinary differential equations

$$\frac{d}{dt}u_i^b(t) = \frac{1}{\tau}\Big(u_i^f(t) - u_i^b(t)\Big) + a^b\int_\Gamma J_i^F(t,x)\cdot\nu\,d\sigma + R_i^b, \quad t\in J,\ i=\mathrm{H,A,C} \tag{1.7}$$

$$\frac{d}{dt}u_i^b(t) = \frac{1}{\tau}\Big(u^f(t) - u^b(t)\Big) + R_i^b, \quad t\in J, \quad i=\mathrm{HC,AC} \tag{1.8}$$

where the constant τ denotes residence time, i.e. the ratio $V_b/\dot V_b^f$ between the liquid volume V_b and the liquid flow rate $\dot V_b^f$ The feeds u_i^f are time-dependent nonnegative functions and the factor a^b is the ratio N_P/V_b between the total number of pellets N_P and the liquid volume. The integral term reflects mass transport into the pellets and R_i^b is the production rate density of a species i. Of course we have to complete the system by initial data for all species i in each phase $k = P, F, b$. If we set for $k = P, F, b$

$$u^k := \Big(u_1^k, u_2^k, u_3^k\Big) := \Big(u_\mathrm{H}^k, u_\mathrm{A}^k, u_\mathrm{C}^k\Big)^T \in \mathbb{R}^N,\ N=3, \tag{1.9}$$

then the unknown functions are the concentrations vectors u^P, u^F, u^b, the concentration u_B^P of polymer matrix B, the concentration u_HC^b of the acid HC, the concentration u_AC^b of the salt AC, and finally the electrical potential $\phi = (\phi^P, \phi^F)$. However, these quantities will not be determined by the evolution system (1.1)-(1.2), (1.7), (1.8), by the boundary conditions and by the initial data

$$u_\mathrm{B}^P(0,x) = u_{\mathrm{B},0}^P(x)\,, \quad x\in\overline\Omega_P\,, \quad u_\mathrm{HC}^b(0) = u_{\mathrm{HC},0}^b\,, \quad u_\mathrm{AC}^b(0) = u_\mathrm{AC}^b\,, \tag{1.10}$$
$$u^k(0,x) = u_0^k(x)\,, \quad x\in\overline\Omega_k\,, \quad k=P,F, \quad u^b(0) = u_0^b\,, \tag{1.11}$$

since the underlying problem is under-determined. The reason for this defect is induced by the unknown electric potentials ϕ^P and ϕ^F. By disregarding magnetic fields these quantities are exactly determined by the Poisson equation $\Delta\phi^k = F/\varepsilon\sum_i z_i u_i^k$ in domain Ω_k and certain boundary conditions, where F denotes again the Faraday constant and $\varepsilon := \varepsilon_0\cdot\varepsilon_r$ the permittivity (or dielectric constant) depending on the material. Owing to the largeness of factor F/ε, slight charge seperation would give rise to a strong electric field. On the other hand, charge gradients are counterbalanced very fast leading to the assumption of electroneutrality as a first approximation. In fact, the total charge of the solute, given by the sum $\sum_i z_i u_i^k$, has be zero everywhere at any time. So the system should be complemented by the equations

$$\sum_{i=1}^{N} z_i u_i^k(t,x) = 0, \quad (t,x) \in J \times \overline{\Omega}_k, \quad k = P, F\,, \tag{1.12}$$

$$\sum_{i=1}^{N} z_i u_i^b(t) = 0, \quad t \in J\,. \tag{1.13}$$

We shall end up with specifying charges z_i, the production rates R_i^k arising in the area Ω_k and R_i^b in the bulk volume V_b.

$$(z_1, z_2, z_3)^T := (z_{\mathrm{H}}, z_{\mathrm{A}}, z_{\mathrm{C}})^T = (+1, +2, -1)^T, \quad z_{\mathrm{B}} = z_{\mathrm{HC}} = z_{\mathrm{AC}} = 0\,, \tag{1.14}$$

$$R^P(u^P) := (R_1^P, R_2^P, R_3^P)^T := (R_{\mathrm{H}}^P, R_{\mathrm{A}}^P, R_{\mathrm{C}}^P)^T = r_{\mathrm{B}}^P \cdot (2, -1, 0)^T, \quad r_{\mathrm{B}}^P = -k_{\mathrm{B}} u_{\mathrm{B}}^P u_1^P\,, \tag{1.15}$$

$$R^F(u^F) := (0,0,0)^T, \tag{1.16}$$

$$R^b(u^b) := (R_1^b, R_2^b, R_3^b)^T := (R_{\mathrm{H}}^b, R_{\mathrm{A}}^b, R_{\mathrm{C}}^b)^T = (-r_{\mathrm{HC}}^b, -r_{\mathrm{AC}}^b, -r_{\mathrm{HC}}^b - 2r_{\mathrm{AC}}^b)^T, \tag{1.17}$$

$$r_{\mathrm{HC}}^b = k_{\mathrm{H}}(u_1^b u_3^b - K_{\mathrm{H}} u_{\mathrm{HC}}^b)\,, \quad r_{\mathrm{AC}}^b = k_{\mathrm{A}}(u_2^b u_3^b - K_{\mathrm{A}} u_{\mathrm{AC}}^b)\,. \tag{1.18}$$

Incidentally, it is easy to see that the reaction functions R^P and R^b satisfy the electroneutrality condition as well. Besides, if we look at evolution equations for the concentrations u_{B}^P, u_{HC}^b and u_{AC}^b, then we pinpoint that these ordinary differential equations can be solved explicitly. The solution formulae read as follows

$$u_{\mathrm{B}}^P(t,x) = \exp\left(-\int_0^t k_{\mathrm{B}} u_2(s,x)\,ds\right) u_{\mathrm{B},0}^P(0,x)\,, \tag{1.19}$$

$$u_{\mathrm{HC}}^b(t) = e^{-\left(\frac{1}{\tau}+k_{\mathrm{H}}K_{\mathrm{H}}\right)\cdot t} u_{\mathrm{HC},0}^b + \int_0^t \left[e^{-\left(\frac{1}{\tau}+k_{\mathrm{H}}K_{\mathrm{H}}\right)\cdot(t-s)} u_{\mathrm{HC}}^f(s) + k_{\mathrm{H}} u_1(s) u_3(s)\right] ds\,, \tag{1.20}$$

$$u_{\mathrm{AC}}^b(t) = e^{-\left(\frac{1}{\tau}+k_{\mathrm{A}}K_{\mathrm{A}}\right)\cdot t} u_{\mathrm{AC},0}^b + \int_0^t \left[e^{-\left(\frac{1}{\tau}+k_{\mathrm{A}}K_{\mathrm{A}}\right)\cdot(t-s)} u_{\mathrm{AC}}^f(s) + k_{\mathrm{A}} u_2(s) u_3(s)\right] ds\,. \tag{1.21}$$

That is the reason for omitting these functions in the arguments of nonlinear reaction rates $R^P(u^P)$ and $R^b(u^b)$. Therefore it remains to determine the electrical potential $\phi = (\phi^P, \phi^F)$ and the unknown concentration vectors u^P, u^F and u^b satisfying differential equations (1.1), (1.3), (1.7) with general nonlinear reaction rates, boundary conditions (1.4) - (1.6), initial data (1.10) and the electroneutrality condition. As a result, we have to solve a three phase problem, more precisely we are looking for three concentration vectors defined in several domains and coupled by means of boundary conditions.

Now, we want to formulate a result of existence and uniqueness for the concentrations and electrical potentials in case of constant coefficients d_i^k and γ_i^k, $k = P, F$, $i = 1, 2, 3$. We will show that the concentration vector $(u^P, u^F, u^b, u^P_{\mathrm{B}}, u^b_{\mathrm{HC}}, u^b_{\mathrm{AC}})$ belongs to the class of maximal regularity

$$
\begin{aligned}
Z(J_0) &:= Z_P(J_0) \times Z_F(J_0) \times \mathrm{H}^1_p(J_0; \mathbb{R}^N) \times \mathrm{C}^{3/2}(J_0; \mathrm{C}(\overline{\Omega}_P)) \times (\mathrm{H}^1_p(J_0))^2 \\
Z_k(J_0) &:= \mathrm{H}^1_p(J_0; \mathrm{L}_p(\Omega_k; \mathbb{R}^N)) \cap \mathrm{L}_p(J_0; \mathrm{H}^2_p(\Omega_k; \mathbb{R}^N)).
\end{aligned}
$$

Furthermore, the natural phase space for this problem is the space

$$
V := \mathrm{B}^{2-2/p}_{pp}(\Omega_P; E_+) \times \mathrm{B}^{2-2/p}_{pp}(\Omega_F; E_+) \times E_+ \times \mathrm{C}(\overline{\Omega}_P; \mathbb{R}_+) \times \mathbb{R}_+ \times \mathbb{R}_+,
$$

where E_+ denotes the positive cone of the hyperplane $E = \{\eta \in \mathbb{R}^N : z^T \cdot \eta = 0\}$. Hence, by uniqueness of the solution the map

$$
(u^P_0, u^F_0, u^b_0, u^P_{\mathrm{B},0}, u^b_{\mathrm{HC},0}, u^b_{\mathrm{AC},0}) \longrightarrow (u^P(t), u^F(t), u^b(t), u^P_{\mathrm{B}}(t), u^b_{\mathrm{HC}}(t), u^b_{\mathrm{AC}}(t)) \tag{1.22}
$$

defines a local semiflow on $V(E_+)$. A version of the main result reads as follows.

Theorem 1.1 *Let Ω_P, Ω_F be bounded domains in $\mathbb{R}^n$ with C^2- boundary, $\Gamma_P := \partial\Omega_P$, $\partial\Omega_F = \Gamma_P \cup \Gamma$ and $\operatorname{dist}(\Gamma_P, \Gamma) > 0$. Assume that $n + 2 < p < \infty$ and $u^f \in \mathrm{L}_p(J; E_+)$. Suppose that the initial data $(u^P_0, u^F_0, u^b_0, u^P_{\mathrm{B},0}, u^b_{\mathrm{HC},0}, u^b_{\mathrm{AC},0})$ belong to V and the following compatibility conditions are satisfied*

1. $\ln(\gamma_i^P(0)u^P_{i,0}) + \lambda_0 z_i \phi^P_0 = \ln(\gamma_i^F(0)u^F_{i,0}) + \lambda_0 z_i \phi^F_0$ *in* $\mathrm{B}^{2-3/p}_{pp}(\Gamma_P)$ *for* $i \in \{1, \dots, N\}$*;*
2. $D^P(0)\partial_\nu u^P_0 + M^P(0)u^P_0\partial_\nu\phi^P_0 = D^F(0)\partial_\nu u^F_0 + M^F(0)u^F_0\partial_\nu\phi^F_0$ *in* $\mathrm{B}^{1-3/p}_{pp}(\Gamma_P; \mathbb{R}^N)$*, where* (ϕ^P_0, ϕ^F_0) *is given as solution of (4.4).*
3. $u^F_0(x) = u^b_0$ *in* $\mathrm{B}^{2-3/p}_{pp}(\Gamma; E_+)$.

Then there exists $t_{max} > 0$ such that for any $T_0 < t_{max}$ the problem (1.1)-(1.8), (1.10)-(1.13) admits a unique solution $(u^P, u^F, u^b, u^P_{\mathrm{B}}, u^b_{\mathrm{HC}}, u^b_{\mathrm{AC}})$ on $J_0 := [0, T_0]$ in the maximal regularity class $Z(J_0)$.

Moreover, the concentrations $(u^P, u^F, u^b, u^P_{\mathrm{B}}, u^b_{\mathrm{HC}}, u^b_{\mathrm{AC}})$ are positive and the map (1.22) defines a local semiflow in the natural phase space V.

The first open problem concerns global existence ($t_{max} = \infty$). Another interesting problem regards the asymptotic behaviour of concentrations. Furthermore, there are no results on existence and uniqueness of stationary states and their stability for the evolution problem.

1.3 The Mathematical Formulation

Throughout this thesis let J be a compact time interval of $\mathbb{R}_+$ containing 0 and $\Omega \subset \mathbb{R}^{n+1}$ be an open bounded domain with C^2-boundary denoted by Γ. Furthermore let $\Omega = \overline{\Omega}_P \cup \Omega_F$ be made up of two bounded domains Ω_P and Ω_F, such that $\partial\Omega_F = \Gamma \cup \Gamma_P$, $\Gamma_P := \partial\Omega_P$, where we assume a positive distance between the boundaries, i.e. $\operatorname{dist}(\Gamma_P, \Gamma) > 0$, see Figure (1.1) for illustrating the underlying situation.

Now we come to the assumptions on the coefficients arising in the partial differential operators and boundary conditions. Concerning the diffusion coefficients d_i^k we assume that

$$d_i^k \in \mathrm{C}^{1/2}(J;\mathrm{C}^1(\overline{\Omega}_k)), \quad i=1,\dots,N \text{ and } k=P,F\,, \tag{1.23}$$
$$d_i^k(t,x) > 0, \quad (t,x)\in J\times\overline{\Omega}_k, \quad i=1,\dots,N \text{ and } k=P,F\,. \tag{1.24}$$

A consequence of these assumptions is that for $i=1,\dots,N$ and $k=P,F$ we have

$$m_i^k := \lambda_0 z_i d_i^k \in \mathrm{C}^{1/2}(J;\mathrm{C}^1(\overline{\Omega}_k)), \quad z_i m_i^k(t,x) \geq m_0 > 0, \quad (t,x)\in J\times\overline{\Omega}_k\,. \tag{1.25}$$

The latter assertion follows by means of (1.23), (1.24), positivity of λ_0, and compactness of $\overline{\Omega}_k$. We now define the diagonal matrices D^k and M^k which inherit the regularity of d_i^k.

$$\begin{aligned} D^k &:= \mathrm{diag}[d_i^k]_{1\leq i\leq N} \in \mathrm{C}^{1/2}(J;\mathrm{C}^1(\overline{\Omega}_k;\mathcal{L}is(\mathbb{R}^N)))\,,\\ M^k &:= \mathrm{diag}[m_i^k]_{1\leq i\leq N} \in \mathrm{C}^{1/2}(J;\mathrm{C}^1(\overline{\Omega}_k;\mathcal{L}(\mathbb{R}^N)))\,. \end{aligned} \tag{1.26}$$

Due to positivity of d_i^k we deduce that $\sigma(D^k(t,x)) = \{d_i^k(t,x):\ 1\leq i\leq N\}\subset\mathbb{R}_+$ and thus in particular $D^k\in\mathcal{L}is(\mathbb{R}^N)$ for all $(t,x)\in J\times\overline{\Omega}_k$. Moreover, we suppose that γ_i^k is positive for each $i=1,...,N$ and $k=P,F$ and lies in a certain trace space.

$$\gamma_i^k \in \mathrm{Y}_{1,\Gamma_P}(\mathbb{R}_+) := \mathrm{B}_{pp}^{1-1/2p}(J;\mathrm{L}_p(\Gamma_P;\mathbb{R}_+)) \cap \mathrm{L}_p(J;\mathrm{B}_{pp}^{2-1/p}(\Gamma_P;\mathbb{R}_+)) \tag{1.27}$$

In the next chapter, we will explain the choice of this trace space. It remains to impose some regularity and positivity assumptions of the nonlinearities R^k, $k=P,F$ and R^b.

(R1) $R^k: J\times\overline{\Omega}_k\times\mathbb{R}^N\to\mathbb{R}^N$ is assumed to be a Caratheodory function, i.e.
$R^k(\cdot,\cdot,u)$ is measurable $\forall u\in\mathbb{R}^N$,
$R^k(t,x,\cdot)$ is continuous for a.a. $(t,x)\in J\times\Omega_k$;

(R2) For each $L_k>0$ there is a function $l^k\in\mathrm{L}_p(J;\mathrm{L}_p(\Omega_k))$ such that
$|R^k(t,x,u)-R^k(t,x,\overline{u})|\leq l^k(t,x)|u-\overline{u}|$, for all $(t,x)\in J\times\Omega_k$,
u, $\overline{u}\in\mathbb{R}^N$, $|u|$, $|\overline{u}|\leq L_k$;

(R3) Let $u_i\geq 0$ for all i and $u_j=0$, then $R_j^k\geq 0$;
R^k leaves $E=\{\eta\in\mathbb{R}^N:\ z^T\cdot\eta=0\}$ invariant, i.e. $z^T\cdot R^k(t,x,u^k)=0$ a.a. $(t,x)\in J\times\Omega_k$, and for all $u^k\in E$.

The conditions for nonlinearity R^b are of similar type. We suppose that

(R4) $R^b: J\times\mathbb{R}^N\to\mathbb{R}^N$ is a Caratheodory function, i.e.
$R^b(\cdot,u)$ is measurable $\forall u\in\mathbb{R}^N$ and $R^b(t,\cdot)$ is continuous for a.a. $t\in J$;

(R5) For each $L_b>0$ there is a function $l^b\in\mathrm{L}_p(J)$ such that
$|R^b(t,u)-R^b(t,\overline{u})|\leq l^b(t)\|u-\overline{u}\|$, for all $t\in J$, u,$\overline{u}\in\mathbb{R}^N$, $|u|$, $|\overline{u}|\leq L_b$.

(R6) Let $u_i\geq 0$ for all i and $u_k=0$, then $R_k^b\geq 0$;
R^b leaves E invariant, i.e. $z^T\cdot R^b(t,u^b)=0$ a.a. $t\in J$, and for all $u^b\in E$.

In order to avoid writing partial differential operators at full length and be able to use a vector-valued notation, we shall introduce abbreviations. For this purpose, let $(t,x)\in J\times\Omega_k$, $u:\overline{\Omega}_k\to\mathbb{R}^N$ and $\phi:\overline{\Omega}_k\to\mathbb{R}$. We then set for $k=P,F$

$$\begin{aligned} \mathcal{A}_{k1}(D)u = \mathcal{A}_{k1}(t,x,D)u &:= -\nabla\cdot(D^k(t,x)\nabla u),\\ \mathcal{A}_{k2}(D)\phi = \mathcal{A}_{k2}(t,x,u^k_,D)\phi &:= -\nabla\cdot(M^k(t,x)u^k(t,x)\otimes\nabla\phi). \end{aligned} \tag{1.28}$$

In the latter definition we have used the symbol $\otimes$ in order to denote the dyadic product of two vectors. Recall that for $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^n$, the dyadic product of a and b is defined by $(a \otimes b)_{i,j} = a_i b_j$. Supposing $a \in C^1(\Omega; \mathbb{R}^N)$ and $b,\, c \in C^1(\Omega; \mathbb{R}^n)$ then we can verify the following rules

1. $(a \otimes b)^T = b \otimes a$, $(a \otimes b)c = a\,(b^T \cdot c)$,
2. $\nabla \cdot (a(x) \otimes b(x)) = a(x)(\nabla \cdot b(x)) + \nabla a(x) \cdot b(x)$,

where b^T denotes the transpose of vector b and $b^T \cdot c$ designates the inner product of the vectors b and c.

Similarly, we define linear boundary operators by

$$\begin{aligned} \mathcal{B}_{k1}(D)u = \mathcal{B}_{k1}(t, x, D)u &:= D^k(t, x)\partial_\nu u\,, \\ \mathcal{B}_{k2}(D)\phi = \mathcal{B}_{k2}(t, x, u^k, D)\phi &:= M^k(t, x)u^k(t, x)\partial_\nu \phi\,, \end{aligned} \tag{1.29}$$

which will be understood in the sense of traces concerning the spatial variable x. Here ν denotes the outer normal on boundary Γ_P or Γ. Using these notations, the system (1.1), (1.3)-(1.7), (1.11) - (1.13) for concentration vectors u^P, u^F, u^b and electrical potentials ϕ^P, ϕ^F takes the form

$$\begin{aligned}
&\partial_t u^P + \mathcal{A}_{P1}(t, x, D)u^P + \mathcal{A}_{P2}(t, x, u^P, D)\phi^P = R^P(t, x, u^P), \quad (t, x) \in J \times \Omega_P, \\
&\partial_t u^F + \mathcal{A}_{F1}(t, x, D)u^F + \mathcal{A}_{F2}(t, x, u^F, D)\phi^F = 0, \quad (t, x) \in J \times \Omega_F, \\
&\mathcal{B}_{P1}(t, x, D)u^P + \mathcal{B}_{P2}(t, x, u^P, D)\phi^P = \mathcal{B}_{F1}(t, x, D)u^F + \mathcal{B}_{F2}(t, x, u^F, D)\phi^F, \quad (t, x) \in J \times \Gamma_P, \\
&\left[\ln(\gamma_i^P u_i^P)\right]_{1 \le i \le N} + \lambda_0 \phi^P z = \left[\ln(\gamma_i^F u_i^F)\right]_{1 \le i \le N} + \lambda_0 \phi^F z, \quad (t, x) \in J \times \Gamma_P, \\
&u^F = u^b, \quad (t, x) \in J \times \Gamma, \\
&\frac{d}{dt}u^b = \frac{1}{\tau}\left(u^f - u^b\right) + R^b(u^b) - a^b \int_\Gamma [\mathcal{B}_{F1}(t, x, D)u^F + \mathcal{B}_{F2}(t, x, u^F, D)\phi^F]d\sigma, \quad t \in J, \\
&u^P(0, x) = u_0^P(x), \quad x \in \overline{\Omega}_P, \quad u^F(0, x) = u_0^F(x), \quad x \in \overline{\Omega}_F, \quad u^b(0) = u_0^b, \\
&z^T \cdot u^k(t, x) = 0, \quad (t, x) \in J \times \overline{\Omega}_k, \quad k = P, F\,, \quad z^T \cdot u^b(t) = 0, \quad t \in J\,.
\end{aligned} \tag{1.30}$$

1.3.1 Solution Spaces

We now address the issue of choosing solution spaces for concentration vectors u^k, $k = P, F, b$ and potentials ϕ^P, ϕ^F. Beforehand, let us introduce some abbreviations for spaces being relevant to inhomogeneities and initial data of the evolution problem. Define an $(N-1)$ dimensional linear subspace of $\mathbb{R}^N$, the space of electroneutrality, by

$$E := \{\eta \in \mathbb{R}^N :\ z^T \cdot \eta = \sum_{i=1}^N z_i \eta_i = 0\}.$$

Since we are interested in non-negative solutions, we have to introduce the positive cone of E denoted by $E_+ := \{\eta \in E :\ \eta_i \ge 0\,,\ i = 1, ..., N\}$. Let $J = [0, T]$ be a compact time interval.

If we set $u := (u^P, u^F, u^b)$ and $\phi := (\phi^P, \phi^F)$, then (u, ϕ) is called a strong solution of (1.30), provided that u belongs to $Z_P^T \times Z_F^T \times Z_b^T$, i.e.

$$\begin{aligned} u^P \in Z_P^T &:= \mathrm{H}_p^1(J; \mathrm{L}_p(\Omega_P; \mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\Omega_P; \mathbb{R}^N)), \\ u^F \in Z_F^T &:= \mathrm{H}_p^1(J; \mathrm{L}_p(\Omega_F; \mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\Omega_F; \mathbb{R}^N)), \\ u^b \in Z_b^T &:= \mathrm{H}_p^1(J; \mathbb{R}^N), \end{aligned} \tag{1.31}$$

ϕ lies in $\mathcal{Z}^T := \left\{ (\phi^P, \phi^F) \in \mathcal{Z}_P^T \times \mathcal{Z}_F^T : \gamma_{|\Gamma_P} \phi^P - \gamma_{|\Gamma_P} \phi^F \in Y_{1,\Gamma_P}^T \right\}$, where we have set

$$\begin{aligned} \mathcal{Z}_P^T &:= \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^1(\Omega_P)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\Omega_P)), \\ \mathcal{Z}_F^T &:= \mathrm{H}_p^{1/2}(J; \mathrm{H}_{p,\Gamma}^1(\Omega_F)) \cap \mathrm{L}_p(J; \mathrm{H}_{p,\Gamma}^2(\Omega_F)), \\ Y_{1,\Gamma_P}^T &:= \mathrm{B}_{pp}^{1-1/2p}(J; \mathrm{L}_p(\Gamma_P)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{2-1/p}(\Gamma_P)), \end{aligned} \tag{1.32}$$

and (1.30) holds a.e. The function spaces B_{pp}^s appearing in Y_{1,Γ_P}^T are called Besov spaces and coincide with the Slobodeckij spaces W_p^s for $s \notin \mathbb{N}$. These kind of anisotropic spaces arise as natural regularity classes for inhomogeneities in boundary conditions of parabolic problems, see chapter two. If the electroneutrality condition is integrated into solution spaces Z_k^T, then we write $Z_k^T(E)$. Furthermore, let u^k be a function in Z_k^T, then in consequence of *the mixed derivative theorem* which is due to Sobolevskii [31], see Section (2.2), we deduce that the gradient of u^k belongs to

$$Z_{k,\nabla}^T := \mathrm{H}_p^{1/2}(J; \mathrm{L}_p(\Omega_k; \mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{H}_p^1(\Omega_k; \mathbb{R}^N)). \tag{1.33}$$

By the definition of $\mathcal{Z}_k^T$ this result follows for gradients of ϕ^P and ϕ^F as well, whereas these functions take values in $\mathbb{R}$. If $J = \mathbb{R}_+$ resp. the length of J is not decisive, then we will omit the index T. The notation $\gamma_{|\Gamma}$ arising in the definition of $\mathcal{Z}^T$ has the meaning of a trace operator concerning boundary Γ and by means of $\gamma_{|t}$ we denote the time trace operator, i.e. $\gamma_{|t} u(t) := u(t)_{|t=0} = u(0)$. In the definition of $\mathcal{Z}_F^T$ we have used the abbreviations

$$\mathrm{H}_{p,\Gamma}^1(\Omega_F) := \left\{ \phi \in \mathrm{H}_p^1(\Omega_F) : \gamma_{|\Gamma} \phi = 0 \right\}, \quad \mathrm{H}_{p,\Gamma}^2(\Omega_F) := \mathrm{H}_{p,\Gamma}^1(\Omega_F) \cap \mathrm{H}_p^2(\Omega_F). \tag{1.34}$$

In the end, if $\mathcal{F}$ is any of the above function spaces then we set ${}_0\mathcal{F} := \{v \in \mathcal{F} : \gamma_{|t} v = v(0) = 0\}$ whenever traces exist.

Now let us make some remarks about choise of these spaces. For this purpose we consider the partial differential equation in Ω_P

$$\partial_t u^P - \nabla \cdot (D^P \nabla u^P) - \nabla \cdot (M^P u^P \otimes \nabla \phi^P) = R^P. \quad (t, x) \in J \times \Omega_P.$$

If we assume that R^P belongs to $\mathrm{L}_p(J; \mathrm{L}_p(\Omega_P; \mathbb{R}^N))$ and demand this regularity from each term on the left-hand side, then we need at least two spatial derivatives and one temporal derivative of u^P in L_p. For the electrical potential it suffices to ask for two spatial derivatives belonging to L_p. This consideration completely explains that we are looking for the concentration vectors u^P and u^F in Z_P^T and Z_F^T, respectively. However, it does not illustrate the first space of $\mathcal{Z}_k^T$. To see this regularity we apply $z^T \cdot$ to the above equation and take into account the electroneutrality condition resulting in

$$-\nabla \cdot (z^T \cdot D^P \nabla u^P) - \nabla \cdot (z^T \cdot M^P u^P \nabla \phi^P) = 0, \quad (t, x) \in J \times \Omega_P.$$

Multiplying with $v \in \overset{\circ}{\mathrm{H}}{}^1_{p'}(\Omega_P)$ and partial integration yields

$$\int\limits_{\Omega_P} z^T \cdot M^P u^P \nabla\phi^P \nabla v \, dx = -\int\limits_{\Omega_P} z^T \cdot D^P \nabla u^P \nabla v \, dx \le \|v\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\Omega_P)} \|z^T \cdot D^P \nabla u^P\|_{\mathrm{L}_p(\Omega_P)} .$$

By the above remarks and continuity assumptions for D^P we have $z^T \cdot D^P \nabla u^P \in \mathrm{H}^{1/2}_p(J; \mathrm{L}_p(\Omega_P))$ which implies $z^T \cdot M^P u^P \nabla\phi^P \in \mathrm{H}^{1/2}_p(J; \mathrm{L}_p(\Omega_P))$ due to duality. Lastly, the vector u^b satisfies an ordinary differential equation and therefore it only seems natural to seek solutions in $\mathrm{H}^1_p(J; \mathbb{R}^N)$.

1.4 The Linearisation

Since, metaphorically speaking, one has the perception that nonlinear problems can be approximated by a linear one, it appears natural to study a corresponding linear problem. Hence, this section is devoted to looking for an appropriate linearisation of (1.30). First, we shall attend the jump condition on Γ_P which, by the logarithmic terms of u^P and u^F, are nonlinear. In order to linearise this boundary condition, we expand the logarithm function as follows: Let $\tilde{u}^k$ be a positive function in $Z_k(E)$ ($\tilde{u}^k \in Z_k(E_+)$, $\tilde{u}^k_i > 0 \ \forall i$) with $\gamma_{|t} \tilde{u}^k_i := \tilde{u}^k_i(0,x) = u^k_{0,i} > 0$, $i \in \{1, \dots, N\}$, then for every given $\varepsilon > 0$ there exists a sufficiently small $T > 0$, such that

$$C \|u^k - \tilde{u}^k\|_{{}_0\mathcal{U}^T_k} \le \|u^k - \tilde{u}^k\|_{{}_0 Z^T_k} < \varepsilon \, . \tag{1.35}$$

This is the result of embedding $Z^T_k \hookrightarrow \mathcal{U}^T_k := \mathrm{C}^{1/2}(J; \mathrm{C}(\overline{\Omega}_k)) \cap \mathrm{C}(J; \mathrm{C}^1(\overline{\Omega}_k))$ proved in Section 2.4 and of $u^k(0,x) - \tilde{u}^k(0,x) = 0$. For expanding the logarithm, we shall use this auxiliary functions in the following way

$$\begin{aligned} \ln\left(u^k_i\right) &= \ln\left(\tilde{u}^k_i\right) + \ln\left(\frac{u^k_i}{\tilde{u}^k_i}\right) = \ln\left(\tilde{u}^k_i\right) + \frac{u^k_i - \tilde{u}^k_i}{\tilde{u}^k_i} + r\left(\frac{u^k_i - \tilde{u}^k_i}{\tilde{u}^k_i}\right) \\ &= \frac{u^k_i}{\tilde{u}^k_i} - 1 + \ln\left(\tilde{u}^k_i\right) + r\left(\frac{u^k_i - \tilde{u}^k_i}{\tilde{u}^k_i}\right), \end{aligned}$$

where the function $r(\cdot)$ belongs to $\mathrm{C}^\infty(\mathbb{R}_+)$ with $r(0) = 0$ and $r'(0) = 0$. It is to be noted that r is only well-defined for $u^k_i > 0$. Thus we obtain

$$\ln\left(u^P_i\right) - \ln\left(u^F_i\right) = \frac{u^P_i}{\tilde{u}^P_i} - \frac{u^F_i}{\tilde{u}^F_i} + \ln\left(\frac{\tilde{u}^P_i}{\tilde{u}^F_i}\right) + r\left(\frac{u^P_i - \tilde{u}^P_i}{\tilde{u}^P_i}\right) - r\left(\frac{u^F_i - \tilde{u}^F_i}{\tilde{u}^F_i}\right).$$

We collect the functions $r((u^k_i - \tilde{u}^k_i)/\tilde{u}^k_i)$ for $i \in \{1, \dots, N\}$ to the vector function

$$R_{\tilde{u}^k}(u^k) := \left[r\left(\frac{u^k_i - \tilde{u}^k_i}{\tilde{u}^k_i}\right)\right]_{1 \le i \le N}, \quad k = P, F \, . \tag{1.36}$$

All properties of $r(\cdot)$ transfer to the vector-valued function R. In fact, we have

$$\begin{aligned} R_{\tilde{u}^k}(\tilde{u}^k) = 0, &\quad \gamma_{|t} R_{\tilde{u}^k}(u^k) = R_{u^k_0}(u^k_0) = 0, \\ R'_{\tilde{u}^k}(\tilde{u}^k) = 0, &\quad \gamma_{|t} R'_{\tilde{u}^k}(u^k) = R'_{u^k_0}(u^k_0) = 0. \end{aligned}$$

If we put

$$\gamma(t,x) := \left[\ln\left(\frac{\gamma_i^F(t,x)\tilde{u}_i^F(t,x)}{\gamma_i^P(t,x)\tilde{u}_i^P(t,x)}\right)\right]_{1\le i\le N},\ H(t,x,u^P,u^F) := \gamma(t,x) - R_{\tilde{u}^P}(u^P) + R_{\tilde{u}^F}(u^F) \tag{1.37}$$

and

$$\tilde{U}_k^{-1}(t,x) := \mathrm{diag}[(\tilde{u}_i^k(t,x))^{-1}]_{1\le i\le N}\,, \qquad \text{for } k = P, F\,,$$

then the jump condition (1.5) takes the form

$$\tilde{U}_P^{-1}(t,x)u^P - \tilde{U}_F^{-1}(t,x)u^F + \lambda_0(\phi^P - \phi^F)z = H(t,x,u^P,u^F)\,. \tag{1.38}$$

Since the matrices $\tilde{U}_P^{-1}$ and $\tilde{U}_F^{-1}$ are diagonal, and each component is positive, we conclude $\tilde{U}_k^{-1}(t,x) \in \mathcal{L}is(\mathbb{R}^N)$ for a.a. $(t,x) \in J \times \overline{\Omega}_k$.

The quasilinear structure we find in the differential equations and transmission condition stems from operators acting on potentials ϕ^P and ϕ^F due to the dependence on unknown functions u^P, u^F. Therefore we approximate u^k by means of $\tilde{u}^k$ and define

$$\begin{aligned}\mathcal{A}_{k2}(D)\phi = \mathcal{A}_{k2}(t,x,D)\phi &:= \mathcal{A}_{k2}(t,x,\tilde{u}^k,D)\phi,\\ \mathcal{B}_{k2}(D)\phi = \mathcal{B}_{k2}(t,x,D)\phi &:= \mathcal{B}_{k2}(t,x,\tilde{u}^k,D)\phi.\end{aligned}$$

Now, we can formulate the linear problem associated with nonlinear evolution problem (1.30). Let the inhomogeneities f^P, f^F, g, h^P, h^F, f^b be given. Using the above linearisation the linear problem reads as follows

$$\begin{aligned}
&\partial_t w^P + \mathcal{A}_{P1}(D)w^P + \mathcal{A}_{P2}(D)\psi^P = f^P(t,x), \quad (t,x) \in J\times\Omega_P,\\
&\partial_t w^F + \mathcal{A}_{F1}(D)w^F + \mathcal{A}_{F2}(D)\psi^F = f^F(t,x), \quad (t,x) \in J\times\Omega_F,\\
&\mathcal{B}_{P1}(D)w^P + \mathcal{B}_{P2}(D)\psi^P = \mathcal{B}_{F1}(D)w^F + \mathcal{B}_{F2}(D)\psi^F + g(t,x), \quad (t,x)\in J\times\Gamma_P,\\
&\tilde{U}_P^{-1}w^P - \tilde{U}_F^{-1}w^F + \lambda_0 z(\psi^P - \psi^F) = h^P(t,x), \quad (t,x)\in J\times\Gamma_P,\\
&w^F = h^F(t,x), \quad (t,x)\in J\times\Gamma,\\
&w^P(0,x) = u_0^P(x), \quad x\in\overline{\Omega}_P, \quad w^F(0,x) = u_0^F(x), \quad x\in\overline{\Omega}_F,
\end{aligned} \tag{1.39}$$

$$\begin{aligned}
&\frac{d}{dt}w^b + \frac{1}{\tau}w^b = \frac{1}{\tau}u^f(t) + f^b(t) - a^b\int_\Gamma \mathcal{B}_{F1}(D)w^F + \mathcal{B}_{F2}(D)\psi^F d\sigma, \quad t\in J,\\
&w^b(0) = u_0^b,
\end{aligned} \tag{1.40}$$

and

$$z^T\!\cdot w^P = 0, \quad (t,x)\in J\times\overline{\Omega}_P, \quad z^T\!\cdot w^F = 0, \quad (t,x)\in J\times\overline{\Omega}_F, \quad z^T\!\cdot w^b = 0, \quad t\in J. \tag{1.41}$$

At this point one can already realise that the linear problem has an essential advantage, as compared to the nonlinear case, apart from the linear structure of course. On closer inspection we perceive that system (1.39) for (w^P, w^F), (ψ^P, ψ^F) and evolution equation (1.40) for w^b are decoupled. More precisely, after solving (1.39) the known functions w^F, ψ^F can be put in the boundary integral over Γ and thus all terms of the right-hand side of (1.40) are given.

Another question arising here concerns the determination of the electrical potentials. By applying $z^T\cdot$ to the evolution equation (1.39), using electroneutrality condition (1.41) and the commuting property of z^T with all differential operators, we obtain the following equations

$$\begin{aligned}
&\nabla\cdot(a^P\nabla\psi^P)+\nabla\cdot(z^T\!\cdot D^P\nabla w^P)=-z^T\!\cdot f^P,\quad (t,x)\in J\times\Omega_P,\\
&\nabla\cdot(a^F\nabla\psi^F)+\nabla\cdot(z^T\!\cdot D^F\nabla w^F)=-z^T\!\cdot f^F,\quad (t,x)\in J\times\Omega_F,\\
&a^P\partial_\nu\psi^P+z^T\!\cdot D^P\partial_\nu w^P=a^F\partial_\nu\psi^F+z^T\!\cdot D^F\partial_\nu w^F+z^T\!\cdot g,\quad (t,x)\in J\times\Gamma_P,\\
&\psi^P-\psi^F=\frac{1}{\lambda_0|z|^2}\Big[z^T\!\cdot\,\tilde U_F^{-1}w^F-z^T\!\cdot\,\tilde U_P^{-1}w^P\Big]+\frac{1}{\lambda_0|z|^2}z^T\!\cdot h^P,\quad (t,x)\in J\times\Gamma_P,\\
&\psi^F=0,\quad (t,x)\in J\times\Gamma\,.
\end{aligned}\tag{1.42}$$

Here we have used the notations

$$a^k(t,x):=z^T\!\cdot M^k(t,x)\tilde u^k(t,x)=\sum_{i=1}^N\lambda_0 z_i^2 d_i^k\tilde u_i^k\,,\quad (t,x,\tilde u^k)\in J\times\overline{\Omega}_k\times Z_k(E_+)\,,\tag{1.43}$$

In view of positivity of d_i^k we conclude that coefficients a^P and a^F are positive if and only if $\tilde u_i^k>0$ for all $(t,x)\in J\times\overline{\Omega}_k$ and $1\le i\le N$. This explains the assumption $\tilde u^k\in Z_k(E_+)$. The boundary condition on the outer surface of Ω_F can not be obtained by the above procedure, but it is contained implicitly. Since we consider a perfectly mixed tank, the electrical potential in the bulk phase has to be constant and by normalising we can assume it as zero. This leads to the Dirichlet boundary condition on Γ.

The following lemma connects the electroneutrality condition for the concentrations w^P, w^F, w^b to the corresponding boundary value problem (1.42). Obviously, the concentration vector w^b does not appear in the elliptic equations, however the right hand side of the ode-equation (1.40) is involved implicitly. In fact, integrating the above equations over Ω_k, using the divergence theorem and boundary conditions we obtain the identity

$$\int_{\Omega_P} z^T\!\cdot f^P dx+\int_{\Omega_F} z^T\!\cdot f^F dx+\int_{\Gamma_P} z^T\!\cdot g\,d\sigma=-\int_{\Gamma}[a^F\partial_\nu\psi^F+z^T\!\cdot D^F\partial_\nu w^F]d\sigma\,,\quad t\in J\,.\tag{1.44}$$

The integral over Γ also occurs in the ode-problem (1.40). Applying the electroneutrality condition to these equations leads to

$$z^T\!\cdot f^b(t)-a^b\int_\Gamma[a^F\partial_\nu\psi^F+z^T\!\cdot D^F\partial_\nu w^F]d\sigma=0,\quad t\in J,$$

and combining (1.44) with the above indentity yields

$$\int_{\Omega_P} z^T\!\cdot f^P(t,x)\,dx+\int_{\Omega_F} z^T\!\cdot f^F(t,x)\,dx+\int_{\Gamma_P} z^T\!\cdot g(t,x)\,d\sigma+\frac{1}{a^b}z^T\!\cdot f^b(t)=0,\quad t\in J\,.$$

Note that this equation is trivially satisfied in the nonlinear case. The next lemma shows that the elliptic problem (1.42) and the above equation for f^P, f^F, g, f^b are equivalent to the electroneutrality condition (1.41).

Lemma 1.1 *Let J be a compact time interval and u^f belongs to E. Then (w, ψ) is a strong solution of (1.39)-(1.41) on J if and only if (w, ψ) is a strong solution of (1.39), (1.40) and (1.42), with $z^T \cdot u_0^k = 0$ for $k = P, F, b$ and the inhomogenieties f^P, f^F, g, f^b satisfy the compatibility condition*

$$\int_{\Omega_P} z^T \cdot f^P(t,x)\, dx + \int_{\Omega_F} z^T \cdot f^F(t,x)\, dx + \int_{\Gamma_P} z^T \cdot g(t,x)\, d\sigma + \frac{1}{a^b} z^T \cdot f^b(t) = 0, \quad t \in J\,. \tag{1.45}$$

Proof. Let (w, ψ) be a strong solution of (1.39)-(1.40) on J. As above shown the elliptic problem (1.42 follows by applying $z^T\cdot$ to (1.39) and the compatibility condition (1.45) by integration this boundary value problem over Ω.

For proving the converse part we assume that (w, ψ) is a strong solution of (1.39), (1.40) on J with $z^T \cdot w^k(0) = 0$ for $k = P, F, b$ and let the compatibility condition (1.45) be satisfied. By applying $z^T \cdot$ to (1.39), taking into account the elliptic equations of (1.42), we find that

$$\frac{d}{dt} z^T \cdot (w^k(t,x)) = 0\,, \quad (t,x) \in J \times \Omega_k\,, \quad k = P, F\,,$$

and

$$\frac{d}{dt} z^T \cdot w^b(t) + \frac{1}{\tau} z^T \cdot w^b = z^T \cdot f^b(t) - a^b \int_{\Gamma} \left[a^F \partial_\nu \psi^F + z^T \cdot D^F \partial_\nu w^F \right] d\sigma\,, \quad t \in J\,.$$

By integrating the first equation from 0 to $t \in J$ and using $z^T \cdot w^k(0,x) = 0$, it follows that $z^T \cdot w^k(t,x) = 0$ for $(t,x) \in J \times \overline{\Omega}_k$. Furthermore the elliptic problem (1.42) implies the relation (1.44) and in combination with the compatibility condition (1.45) we obtain the ode-problem

$$\frac{d}{dt} z^T \cdot w^b(t) + \frac{1}{\tau} z^T \cdot w^b = 0\,, \quad t \in J\,, \quad z^T \cdot w^b(0) = 0\,,$$

which is solved uniquely by $z^T \cdot w^b(t) = 0$ and this means $w^b \in E$.

□

Chapter 2

Preliminaries

In this second chapter we introduce some basic and powerful results needed to tackle the linear problem (1.39). The chapter consists of three parts. In sections (2.2) and (2.3) below we present the basic notion of sectorial operators, such as operators with bounded imaginary powers, operators which admit a bounded $\mathcal{H}^\infty$-calculus and operators with $\mathcal{RH}^\infty$-calculus. Certain properties of these classes are made available. An important reference will be the paper Denk, Hieber and Prüss [7].

Thereafter we study certain trace spaces and the solution spaces Z_k, $\mathcal{Z}_k$. We will show that for p sufficiently large, each space forms a multiplication algebra. We need this property for example to establish boundedness of multiplication operators $\tilde{U}_P^{-1}$, $\tilde{U}_F^{-1}$ in space $Y_{1,\Gamma_P}(\mathbb{R}^N)$. Moreover, we will see that it is satisfactory to demand $\gamma_i^k \in Y_{1,\Gamma_P}$.

In Section 2.5 we will treat some model problems which are a natural outcome of solving the linear problem (1.39). For proving maximal regularity of these model problems we will employ the results supplied in Sections 2.2 and 2.3. Let us begin with some remarks about notations and conventions.

2.1 Notations and Conventions

Let us start by explaining some of the notations, which will be used throughout this thesis, and recollect some basic definitions and function spaces.

By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of natural numbers, real and complex numbers, respectively. In addition we use the notations $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re\lambda > 0\}$, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$ and $\mathbb{R}^{n+1}_- = \mathbb{R}^n \times \mathbb{R}_-$. X, Y, Z, ... will usually be Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, ... In a unique context, we will omit the indices or reduce them to index of the space, e.g. $\|\cdot\|_{\mathrm{L}_p(\Omega)}$ for $\|\cdot\|_p$ or $\|\cdot\|_{\mathrm{C}(\Omega)} := \|\cdot\|_\infty$. Given two Banach spaces X and Y, $\mathcal{B}(X, Y)$ will designate the Banach space of all linear and bounded operators from X to Y, $\mathcal{B}(X) = \mathcal{B}(X, X)$. $\mathcal{L}is(X, Y)$ denotes the space of continuous isomorphisms from X to Y, we write $\mathcal{L}is(X) = \mathcal{L}is(X, X)$ for short. Furthermore $\mathcal{B}(X, Y)$, if not explicity mentioned, will always be equipped with the natural norm-topology and $\|A\|_{\mathcal{B}(X,Y)}$ designates the norm of an operator $A \in \mathcal{B}(X, Y)$. The domain, range and kernel of an operator A in X is denoted by $D(A)$, $R(A)$ and $N(A)$, respectively. If A is closed, we will denote by D_A the domain of A equipped with the graph norm, $\|x\|_{D(A)} := \|x\| + \|Ax\|$.

Given $G \subset \mathbb{R}^n$, G open or closed, we let $\mathrm{C}(G; X)$ and $\mathrm{BUC}(G; X)$ denote the space of all continuous resp. bounded uniformly continuous functions $f : G \to X$. Also, $\mathrm{C}^s(\overline{G}; X)$

denotes the space of all Hölder continuous functions of order $s \in (0,1)$, we write $\mathrm{C}^{1-}(\overline{G};X)$ for the space of locally Lipschitz continuous functions. Further, if $m \in \mathbb{N}$, $\mathrm{C}^m(\overline{G};X)$ designates the space of all functions $f : G \to X$ which admit continuous partial derivatives and $\partial^\alpha f$ in G has continuous extension to $\overline{G}$, for each $|\alpha| \le m$. For $f \in \mathrm{C}(\overline{G};X)$ the *support* of f is defined by $\operatorname{supp} f = \overline{\{x \in G : \ f(x) \neq 0\}}$. As usual $\mathrm{C}_0^\infty(G;X)$ means the space of *test function* on G with values in X.

If (Ω, Σ, μ) is a-measurable space then $\mathrm{L}_p(\Omega;X) := \mathrm{L}_p(\Omega,\Sigma,\mu;X)$, denotes the space of all Bochner-measurable functions $f : \Omega \to X$ such that $\|f(\cdot)\|_X^p$ is integrable. $\mathrm{L}_p(\Omega;X)$ is normed by $\|f\|^p_{\mathrm{L}_p(\Omega;X)} = \int_\Omega \|f(x)\|_X^p d\mu(x)$, $1 \le p < \infty$.

Let X be a Banach space and Ω is Lebesgue measurable subset of $\mathbb{R}^n$, $s > 0$ and $1 < p < \infty$, by $\mathrm{H}_p^s(\Omega;X)$ and $\mathrm{B}_{pp}^s(\Omega;X)$ we signify the vector-valued Bessel potential space resp. Sobolev-Slobodeckij space of X-valued functions on Ω; see Amann [1], Schmeisser [30], Štrkalj [32], and Zimmermann [40]. In case $X = \mathbb{C}$ we refer to Runst and Sickel [29], and Triebel [33]-[35]. It turns out that most results which are known from the scalar case can be transferred to the vector-valued case, for the H-scale at least if X is a UMD space. In particular, embeddings and real interpolation work as in the case $X = \mathbb{C}$.

Finally, by C, M and c we denote various constants which may differ from line to line, but which are always independent of the free variables.

2.2 The classes $\mathcal{S}(X)$, $\mathcal{BIP}(X)$ and $\mathcal{H}^\infty(X)$

Sectorial operators form an important basic class of unbounded operators appearing in partial differential equations. Therefore we begin with the definition of these operators.

Definition 2.1 *Let X be a complex Banach space, and A a closed linear operator in X. A is called sectorial if the following two conditions are satisfied*

- **(S1)** $\overline{D(A)} = X$, $N(A) = \{0\}$, $\overline{R(A)} = X$, $(-\infty, 0) \subset \rho(A)$;
- **(S2)** $|t(t+A)^{-1}| \le M$ *for all* $t > 0$, *and some* $M < \infty$.

The class of sectorial operators in X will be denoted by $\mathcal{S}(X)$. If only **(S2)** *holds, then A is said to be pseudo-sectorial.*

Assume that A is a sectorial operator. Then, by using the Neumann series, we can verify $\Sigma_\theta \subset \rho(-A)$, for some $\theta > 0$, and $\sup\{|\lambda(\lambda+A)^{-1}| : |\arg\lambda\,| < \theta\} < \infty$. Here $\Sigma_\theta \subset \mathbb{C}$ denotes the standard sector, more precisely

$$\Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : \ |\arg\lambda| < \theta\}.$$

Therefore it makes sense to define the *spectral angle* ϕ_A of $A \in \mathcal{S}(A)$ by

$$\phi_A := \inf\{\phi : \ \Sigma_{\pi-\phi} \subset \rho(-A), \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}| < \infty\}.$$

Obviously, it holds that $\phi_A \in [0,\pi)$ and $\phi_A \ge \sup\{|\arg\lambda| : \ \lambda \in \sigma(A)\}$.

Now we come to the $\mathcal{H}^\infty$-calculus. Let $\phi \in (0,\pi]$ and define the algebra of holomorphic functions on Σ_ϕ denoted by $\mathcal{H}(\Sigma_\phi) = \{f : \Sigma_\phi \to \mathbb{C}\,\text{holomorphic}\ \}$. The space $\mathcal{H}^\infty(\Sigma_\phi) = \{f :$

$\Sigma_\phi \to \mathbb{C}$ holomorphic and bounded$\}$ equipped with norm $|f|_\infty^\phi = \sup\{|f(\lambda)| : |\arg(\lambda)| < \phi\}$ is a Banach algebra. Furthermore we set $\mathcal{H}_0(\Sigma_\phi) = \bigcup_{\alpha,\beta<0} \mathcal{H}_{\alpha,\beta}(\Sigma_\phi)$, where

$$\mathcal{H}_{\alpha,\beta}(\Sigma_\phi) = \{f \in \mathcal{H}(\Sigma_\phi) : \, |f|_{\alpha,\beta}^\infty := \sup_{|\lambda|\le 1} |\lambda^\alpha f(\lambda)| + \sup_{|\lambda|\ge 1} |\lambda^{-\beta} f(\lambda)| < \infty\}.$$

We assume that A is a sectorial operator with $\phi \in (\phi_A, \pi)$. Then we choose any $\psi \in (\phi_A, \phi)$ and denote by $\Gamma = (\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$ the integration path surrounding $\sigma(A)$. The Dunford integral

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - A)^{-1} d\lambda, \qquad \text{for all } f \in \mathcal{H}_0(\Sigma_\phi),$$

converges in $\mathcal{B}(X)$ and defines via $\Phi_A(f) = f(A)$ a functional calculus $\Phi_A : \mathcal{H}_0(\Sigma_\phi) \to \mathcal{B}(X)$ which is a bounded algebra homomorphism.

Definition 2.2 *The operator A is said to admit an $\mathcal{H}^\infty$-calculus if there are $\phi > \phi_A$ and a constant $K_\phi < \infty$ such that*

$$|f(A)| \le K_\phi |f|_\infty^\phi \, , \quad \textit{for all } f \in \mathcal{H}_0(\Sigma_\phi) \, . \tag{2.1}$$

The class of sectorial operators A which admit $\mathcal{H}^\infty$ will be denoted by $\mathcal{H}^\infty(X)$. The $\mathcal{H}^\infty$-angle of A is defined by

$$\phi_A^\infty = \inf\{\phi > \phi_A : \, (2.1) \textit{ is valid } \} \, .$$

If this is the case, then the functional calculus for A on $\mathcal{H}_0(\Sigma_\phi)$ extends uniquely to $\mathcal{H}^\infty(\Sigma_\phi)$.

We now consider operators of $\mathcal{S}(X)$ having boundary imaginary powers. This class of operators $\mathcal{BIP}(X)$ has been introduced by Prüss and Sohr [28]. Since we can define complex powers for any operator A in $\mathcal{S}(X)$, it makes sense to study such operators.

Definition 2.3 *Suppose $A \in \mathcal{S}(X)$. Then, A is said to admit bounded imaginary powers if $A^{is} \in \mathcal{B}(X)$ for each $s \in \mathbb{R}$, and there is a constant $C > 0$ such that $|A^{is}| \le C$ for $|s| \le 1$. The class of such operators will be denoted by $\mathcal{BIP}(X)$.*

Due to the fact that the functions $f_s(z) = z^{is}$ belong to $\mathcal{H}^\infty(\Sigma_\phi)$, for any $s \in \mathbb{R}$ and $\phi \in (0,\pi)$, we evidently have the inclusions $\mathcal{H}^\infty(X) \subset \mathcal{BIP}(X) \subset \mathcal{S}(X)$, and the inequalities $\underline{\phi_A^\infty} \ge \theta_A \ge \phi_A$. Here, θ_A denotes the growth bound of group $\{A^{is} : \, s \in \mathbb{R}\}$, i.e. $\theta_A := \overline{\lim}_{|s|\to\infty} |s|^{-1} \log |A^{is}|$; it will be called the *power angle* of A.

A first application of the class $\mathcal{BIP}(X)$ establishes a relationship for the fractional power spaces

$$X_\alpha = X_{A^\alpha} = (D(A^\alpha), |\cdot|_\alpha), \quad |x|_\alpha = |A^\alpha x| + |x|, \quad 0 < \alpha < 1,$$

where $A \in \mathcal{S}(X)$. If A belongs to $\mathcal{BIP}(X)$, a characterisation of X_α in terms of complex interpolation spaces can be derived.

Theorem 2.1 *Assume $A \in \mathcal{BIP}(X)$. Then*

$$X_\alpha \cong [X, X_A]_\alpha, \quad \alpha \in (0,1),$$

the complex interpolation space between X and $X_A \hookrightarrow X$ of order α.

For a proof we refer to Triebel [34, pp. 103-104] or Yagi [39]. Below we want to study real interpolation spaces $(X, X_\alpha)_{\beta,p}$, $0<\alpha,\beta<1$, $1\le p\le\infty$, defined by the K-method. At first, we recall that the real interpolation space $(X, D_A)_{\beta,p}$ is isomorphic to the space $D_A(\beta,p)$ for $A\in\mathcal{S}(X)$, $\beta\in(0,1)$ and $1\le p\le\infty$. The latter space is defined by means of

$$\begin{aligned} D_A(\beta,p) &:= \{x\in X:\ [x]_{\beta,p} := (\int_0^\infty |t^\beta A(t+A)^{-1}x|_X^p \tfrac{dt}{t})^{1/p}<\infty\}, \quad 1\le p<\infty \\ D_A(\beta,\infty) &:= \{x\in X:\ [x]_{\beta,\infty} := \sup_{t>0} |t^\beta A(t+A)^{-1}x|_X\}. \end{aligned} \tag{2.2}$$

Assume that A belongs to $\mathcal{BIP}(X)$. Employing Theorem (2.1) and the reiteration theorem, see Triebel [34], we conclude that

$$(X, X_\alpha)_{\beta,p} = (X, [X, D_A]_\alpha)_{\beta,p} = (X, D_A)_{\alpha\beta,p}, \quad 0<\alpha,\beta<1, \quad 1\le p\le\infty. \tag{2.3}$$

Another important application for operators belonging to the class $\mathcal{BIP}(X)$ concerns sums of closed operators, which leads us to the concept of maximal regularity. Let X be a Banach space, A, B closed linear operators in X, and let $A+B$ be defined by

$$(A+B)x = Ax+Bx, \quad x\in D(A+B)=D(A)\cap D(B).$$

If $0\in\rho(A+B)$, which implies that $A+B$ is closed, then the equation $y=Ax+Bx$ admits a unique solution $x\in D(A+B)$ for all $y\in X$, i.e. the solution has *maximal regularity.* The closed graph theorem shows the *a priori* estimate

$$|Ax|+|Bx|\le C|Ax+Bx|, \quad \text{for all } x\in D(A+B).$$

The Dore-Venni theorem gives conditions for maximal regularity. Before we state a version of this result we have to remind the meaning of commuting resolvents and Banach spaces of class $\mathcal{HT}$. Two closed linear operators A, B in X are said to *commute*, if there exist $\lambda\in\rho(A)$, $\mu\in\rho(B)$ such that

$$(\lambda-A)^{-1}(\mu-B)^{-1} = (\mu-B)^{-1}(\lambda-A)^{-1}.$$

A Banach space X is said to be of class $\mathcal{HT}$, if the Hilbert transform is bounded on $\mathrm{L}_p(\mathbb{R};X)$ for some (and then all) $p\in(1,\infty)$. The Hilbert transform H of function $f\in\mathcal{S}(\mathbb{R};X)$ is defined by

$$(Hf)(t) = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|s|\ge\epsilon}\frac{f(t-s)}{s}ds, \quad t\in\mathbb{R},$$

where the limit is to be understood in the L_p-sense. These spaces are often called UMD Banach spaces, where UMD stands for unconditional martingale differences. It is a well known theorem that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces. Let $1<p<\infty$ and $(\Omega,\Sigma,d\mu)$ a measure space, then $\mathrm{L}_p(\Omega,d\mu;X)$ is a Banach space of class $\mathcal{HT}$ if $X\in\mathcal{HT}$.

We are now in position to state a variant of the Dore-Venni Theorem, cf. [9], [24], [25].

Theorem 2.2 *Suppose X belongs to the class $\mathcal{HT}$, and assume $A,B\in\mathcal{BIP}(X)$ commute and satisfy the strong parabolicity condition $\theta_A+\theta_B<\pi$, and let $t>0$. Then* *(i)* $A + tB$ *is closed and sectorial;*

(ii) $A + tB \in \mathcal{BIP}(X)$ *with* $\theta_{A+tB} \le \max\{\theta_A, \theta_B\}$

(iii) there is a constant $C > 0$*, independent of* $t > 0$*, such that*

$$|Ax| \; + \; t|Bx| \; \le \; C|Ax + tBx| \,, \quad x \in D(A) \cap D(B). \tag{2.4}$$

In particular, if A *or* B *is invertible, then* $A + tB$ *is invertible as well.*

If we weaken the assumption on B and in return strengthen the assumption on A, then the result can be maintained. More precisely, let X be an arbitrary Banach space and assume that $A \in \mathcal{H}^\infty(X)$ and $B \in \mathcal{RS}(X)$, see [7], with $\phi_A^\infty + \phi_B^{\mathcal{R}} < \pi$. Then the sum $A+B$ is closed as well. This result has been proved by Kalton and Weis [20].

Some consequences of Theorem 2.2 concerning complex interpolation are contained in the following corollary, see Prüss [25]. The proof can be found in the forthcoming monograph of Hieber and Prüss [16].

Corollary 2.1 *Suppose* X *belongs to the class* $\mathcal{HT}$*, and assume that* A*,* $B \in \mathcal{BIP}(X)$ *are commuting in the resolvent sense. Further suppose the strong parabolicity condition* $\theta_A + \theta_B < \pi$*. Let* A *or* B *be invertible and* $\alpha \in (0,1)$*. Then*

(i) $A^\alpha(A+B)^{-\alpha}$ *and* $B^\alpha(A+B)^{-\alpha}$ *are bounded in* X*;*

(ii) $D((A+B)^\alpha) = [X, D(A+B)]_\alpha = [X, D(A)]_\alpha \cap [X, D(B)]_\alpha = D(A^\alpha) \cap D(B^\alpha)$.

The next result has been proved by Grisvard [12], in an even more general context. For a proof we also refer to [16].

Proposition 2.1 *Suppose that* A*,* B *are sectorial operators in a Banach space* X*, commuting in the resolvent sense. Then*

$$(X, D(A) \cap D(B))_{\alpha,p} = (X, D(A))_{\alpha,p} \cap (X, D(B))_{\alpha,p} \,,$$

for all $\alpha \in (0,1)$*,* $p \in [1, \infty]$*.*

The next result is known as the *mixed derivative theorem* and is due to Sobolevskii [31]

Proposition 2.2 *Suppose* A *and* B *are sectorial linear operators in a Banach space* X *with spectral angles* $\phi_A + \phi_B < \pi$*, which commute and are coercively positive, i.e.* $A + tB$ *with natural domain* $D(A + tB) = D(A) \cap D(B)$ *is closed for each* $t > 0$ *and there is a constant* $M > 0$ *such that*

$$\|Ax\|_X + t\|Bx\|_X \le M\|Ax + tBx\|_X, \quad \text{for all } x \in D(A) \cap D(B),\ t > 0.$$

Then there is a constant $C > 0$ *such that*

$$\|A^\alpha B^{1-\alpha} x\|_X \le C\|Ax + Bx\|_X, \quad \text{for all } x \in D(A) \cap D(B),\ \alpha \in [0,1].$$

2.3 Operator-Valued Fourier Multipliers and $\mathcal{R}$-Bounded Functional Calculus

In this section we will introduce the notion of $\mathcal{R}$-boundedness and we will state some elementary properties. In order to extend the classical Mikhlin Theorem for the scalar case to an operator-valued Fourier multiplier theorem, we need the concept of $\mathcal{R}$-boundedness.

We commence with the definition of $\mathcal{R}$-boundedness.

Definition 2.4 *Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called $\mathcal{R}$-bounded, if there exists a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality*

$$|\sum_{j=1}^{N} \varepsilon_j T_j x_j|_{\mathrm{L}_p(\Omega;Y)} \le C|\sum_{j=1}^{N} \varepsilon_j x_j|_{\mathrm{L}_p(\Omega;X)}$$

is valid. The smallest constant C is called $\mathcal{R}$-bound of $\mathcal{T}$, which we denote by $\mathcal{R}(\mathcal{T})$.

One can show that this definition is independent of $p \in [1, \infty)$, which follows from *Kahane's inequality.*

The next result shows that $\mathcal{R}$-bounds behave like norms.

Proposition 2.3 *(a) Let X, Y be Banach spaces, and $\mathcal{T}$, $\mathcal{S} \subset \mathcal{B}(X, Y)$ be $\mathcal{R}$-bounded. Then*

$$\mathcal{T} + \mathcal{S} = \{T + S : T \ \in \mathcal{T}, S \in \mathcal{S}\}$$

is $\mathcal{R}$-bounded as well, and $\mathcal{R}(\mathcal{T} + \mathcal{S}) \le \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S})$.
(b) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be $\mathcal{R}$-bounded. Then

$$\mathcal{S}\mathcal{T} = \{ST : T \in \mathcal{T}, S \in \mathcal{S}\}$$

is $\mathcal{R}$-bounded, and $\mathcal{R}(\mathcal{S}\mathcal{T}) \le \mathcal{R}(\mathcal{S})\mathcal{R}(\mathcal{T})$.

Now, we shall approach the operator-valued Fourier multiplier theorem. Let X be a Banach space and $1 < p < \infty$. We denote by $\mathcal{D}(\mathbb{R}; X)$ the space of X-valued C^∞ functions with compact support and we let $\mathcal{D}'(\mathbb{R}; X) := \mathcal{B}(\mathcal{D}(\mathbb{R}), X)$ designate the space of X-valued distributions. The X-valued Schwartz spaces $\mathcal{S}(\mathbb{R}; X)$ and $\mathcal{S}'(\mathbb{R}; X)$ are defined similarly. Let Y be another Banach space. Then, given $M \in \mathrm{L}_{1,loc}(\mathbb{R}; \mathcal{B}(X, Y))$, we may define an operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X) \to \mathcal{S}'(\mathbb{R}; X)$ by means of

$$T_M\phi \ := \ \mathcal{F}^{-1}M\mathcal{F}\phi\,, \quad \text{for all } \mathcal{F}\phi \in \mathcal{D}(\mathbb{R}; X)\,, \tag{2.5}$$

where $\mathcal{F}$ denotes the Fourier transform. Note that $\mathcal{F}^{-1}\mathcal{D}(\mathbb{R}; X)$ is dense in $\mathrm{L}_p(\mathbb{R}; X)$, consequently the operator T_M is well-defined and linear on a dense subspace of $\mathrm{L}_p(\mathbb{R}; X)$.

Now the question arises on what terms the operator T_M is bounded in L_p, i.e. $T_M \in \mathcal{B}(\mathrm{L}_p(\mathbb{R}; X), \mathrm{L}_p(\mathbb{R}; Y))$. The following theorem contains the operator-valued version of the famous Mikhlin Fourier multiplier theorem in one variable, which is due to Weis [36]. A shorter proof of this theorem can be found in [7].

Theorem 2.3 *Suppose that X, Y are spaces of class $\mathcal{HT}$ and let $1 < p < \infty$. Let $M \in \mathrm{C}^1(\mathbb{R}\backslash\{0\}; \mathcal{B}(X,Y))$ be such that the following conditions are satisfied*

(i) $\mathcal{R}(\{M(\rho) : \rho \in \mathbb{R}\backslash\{0\}\}) =: \kappa_0 < \infty$;

(ii) $\mathcal{R}(\{\rho M'(\rho) : \rho \in \mathbb{R}\backslash\{0\}\}) =: \kappa_1 < \infty$.

Then the operator T defined by (2.5) is bounded from $\mathrm{L}_p(\mathbb{R}; X)$ into $\mathrm{L}_p(\mathbb{R}; Y)$ with norm $|T|_{\mathcal{B}(\mathrm{L}_p(\mathbb{R};X),\mathrm{L}_p(\mathbb{R};Y))} \leq C(\kappa_0 + \kappa_1)$, where $C > 0$ depends only on p, X, Y.

Remark 2.1 This result can be extended to the n-dimensional case, i.e. Mikhlin's theorem in n variables. Here we refer to [7] as well.

Now we want to address the issue of verifying $\mathcal{R}$-boundedness conditions as stated in the above theorem. In applications we often encounter symbols of the form $M(\rho, A)$. That means, $M(\rho, A) \in \mathcal{B}(X,Y)$ is induced by an unbounded operator A. It turns out that conditions as stated in Theorem 2.3 are easy to verify for operators which admit an $\mathcal{R}$-bounded functional calculus and functions $M_\rho(\cdot) := M(\rho, \cdot) \in \mathcal{H}^\infty(\Sigma_\theta)$ being uniformly bounded concerning ρ.

We now want to connect $\mathcal{R}$-boundedness to the $\mathcal{H}^\infty$-calculus.

Definition 2.5 *Let X be a Banach space and suppose that $A \in \mathcal{H}^\infty(X)$. The operator A is said to admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$ -calculus if the set*

$$\left\{h(A) : h \in \mathcal{H}^\infty(\Sigma_\theta)\ ,\ |h|_\infty^\theta \leq 1\right\}$$

is $\mathcal{R}$-bounded for some $\theta > 0$. We denote the class of such operators by $\mathcal{RH}^\infty(X)$ and define the $\mathcal{RH}^\infty$-angle $\phi_A^{R_\infty}$ of A as the infimum of such angles θ.

The importance of this class of operators is justified by the following proposition.

Proposition 2.4 *Let X be a Banach space, $A \in \mathcal{RH}^\infty(X)$ and suppose that $\{h_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded, for some $\theta > \phi_A^{R_\infty}$, where Λ is an arbitrary index set. Then $\{h_\lambda(A) : \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded.*

This result will be useful for proving $\mathcal{R}$-boundedness conditions just like in the Mikhlin theorem.

2.4 Multiplication Algebras

In this section we shall tackle the mapping properties of partial differential operators $\mathcal{A}_{ki}$ and boundary operators $\mathcal{B}_{ki}$, $\tilde{U}_k^{-1}$. In order to treat the latter operators we need the definition of Besov spaces on manifolds. In fact, we will explain the meaning of $\mathrm{L}_p(\partial\Omega)$ and $\mathrm{B}^s_{pp}(\partial\Omega)$, where $\Omega \subset \mathbb{R}^{n+1}$ is bounded with C^k-boundary $\Gamma := \partial\Omega$. At first, $\mathrm{L}_p(\Gamma)$ has the usual meaning where the measure on $\partial\Omega$ is the usual surface measure induced by the Lebesgue measure in $\mathbb{R}^n$. To define Besov spaces on boundaries some preparations are needed. Let $(\varphi_j)_{j=0}^M$ be a resolution of unity with respect to Ω with following properties:

1. $\overline{\Omega} \subset \bigcup_{j=0}^M U_j$, $\varphi_0 \in \mathrm{C}_0^\infty(U_0)$, $U_0 \subset \Omega$;
2. $\varphi_j \in \mathrm{C}_0^\infty(U_j)$, $U_j \cap \partial\Omega \neq$ for $j = 1, \ldots, M$.

Further, we denote by $h_j(x)$ the C^k-diffeomorphism defined in $\overline{U}_j$ such that $y = h_j(x)$ is a one-to-one mapping from U_j onto a bounded domain in $\mathbb{R}^{n+1}$, where the set $\partial\Omega \cap U_j$ is mapped onto a bounded part of the hyperplane $\mathbb{R}^n = \{y \in \mathbb{R}^{n+1} : y_{n+1} = 0\}$. Then one sets

$$\mathrm{B}^s_{pq}(\partial\Omega) := \{f \in \mathrm{L}_p(\partial\Omega) : f_j(y) := [G^j \varphi_j f](y) = (\varphi_j f)(h_j^{-1}(y)) \in \mathrm{B}^s_{pq}(\mathbb{R}^n),\ j = 1, \ldots, M\}\,,$$

$$\|f\|_{\mathrm{B}^s_{pq}(\partial\Omega)} := \sum_{j=1}^{M} \|f_j(\cdot)\|_{\mathrm{B}^s_{pq}(\mathbb{R}^n)}\,.$$

$\mathrm{B}^s_{pq}(\partial\Omega)$ is a Banach space. In the sense of equivalent norms, $\mathrm{B}^s_{pq}(\partial\Omega)$ is independent of the choice of covering $\{U_j\}_{j=1}^M$ and the choice of partition. For general treatises on Besov spaces we refer to the books by Triebel [33], [34], [35]. Since we only consider Besov spaces of type $\mathrm{B}^s_{pp}(\Omega)$, which coincide with the Slobodeckij spaces $\mathrm{W}^s_p(\Omega)$ for $s \notin \mathbb{N}$, we can use as norm

$$\|f\|^p_{\mathrm{B}^s_{pp}(\Omega)} = \|f\|^p_{\mathrm{L}_p(\Omega)} + \sum_{|\alpha|=[s]} \int_\Omega\int_\Omega \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{(n+1)+\{s\}p}}\, dx\, dy\,.$$

Note that here we have put $s = [s] + \{s\}$, $[s]$ integer and $0 \le \{s\} < 1$. With aid of this result we are able to show the following lemma.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with C^k-boundary. Suppose that u and v belong to $\mathrm{B}^s_{pp}(\partial\Omega)$, with $s = [s] + \{s\}$, $[s]$ integer part and $0 < \{s\} < 1$. Let $k \ge [s]$ and $\{s\} - (n+1)/p > 0$. Then $\mathrm{B}^s_{pp}(\partial\Omega)$ forms a multiplication algebra and*

$$\|u \cdot v\|_{\mathrm{B}^s_{pp}(\partial\Omega)} \le C\left[\|u\|_{\mathrm{C}^{[s]}(\partial\Omega)}\|v\|_{\mathrm{B}^s_{pp}(\partial\Omega)} + \|u\|_{\mathrm{B}^s_{pp}(\partial\Omega)}\|v\|_{\mathrm{C}^{[s]}(\partial\Omega)}\right]\,. \tag{2.6}$$

Remark 2.2 Of course, the condition $\{s\} - (n+1)/p > 0$ is more stringent than required. Actually, in order to prove that $\mathrm{B}^s_{pp}(\partial\Omega)$ forms a multiplication algebra, $s - (n+1)/p > 0$ is needed. However, (2.6) need no longer be valid. Since we have in mind to show that Y_1 and Y_2, see (2.8), form multiplication algebras and estimation (2.6) plays a decisive role, we restrict to this case.

Proof. Let u,v in $\mathrm{B}^s_{pp}(\Omega)$ be given and $\{s\} - (n+1)/p > 0$. The latter condition implies the continuous embeddings $\mathrm{B}^s_{pp}(\Omega) \hookrightarrow \mathrm{C}^{[s]}(\Omega)$ and $\mathrm{B}^{\{s\}}_{pp}(\Omega) \hookrightarrow \mathrm{C}(\Omega)$. Now we use the covering of Ω and the partition of unity as described above. Then, the norm of the product $u \cdot v$ takes the form

$$\|uv\|^p_{\mathrm{B}^s_{pp}(\partial\Omega)} = \sum_{j=1}^{M}\left\{\sum_{|\alpha|=[s]} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^\alpha(\varphi_j uv)(h_j^{-1}(y)) - D^\alpha(\varphi_j uv)(h_j^{-1}(z))|^p}{|y-z|^{n+\{s\}p}}\, dy\, dz \right.$$
$$\left. + \|(\varphi_j uv)(h_j^{-1}(\cdot))\|^p_{\mathrm{L}_p(\mathbb{R}^n)}\right\}\,.$$

This means that the highest derivatives of $(\varphi_j uv)(h_j^{-1}(y))$ have to be in $\mathrm{B}^{\{s\}}_{pp}(\partial\Omega)$. Taking into account the mapping property of h_j and the continuous embedding $\mathrm{B}^{\{s\}}_{pp}(\Omega) \hookrightarrow \mathrm{C}(\Omega)$, then the L_p-norm can be estimated as follows

$$\|(\varphi_j uv)(h_j^{-1}(\cdot))\|^p_{\mathrm{L}_p(\mathbb{R}^n)} \le \max_{x \in \partial\Omega\cap U_j} |u(x)|^p \|(\varphi_j v)(h_j^{-1}(\cdot))\|^p_{\mathrm{L}_p(\mathbb{R}^n)}$$
$$\le \|u\|^p_{\mathrm{C}(\partial\Omega)}\|(\varphi_j v)(h_j^{-1}(\cdot))\|^p_{\mathrm{L}_p(\mathbb{R}^n)}\,.$$

To treat the double integral observing that

$$D^\alpha[(\varphi_j uv)(h_j^{-1}(y))] = \sum_{1\le|\gamma|\le|\alpha|} q_{\alpha\gamma}(y)[D^\gamma(\varphi_j uv)](h_j^{-1}(y))\,.$$

Here the coefficients $q_{\alpha\gamma}(y)$ are homogeneous polynomials of degree $|\gamma|$ in derivatives of h_j^{-1} of order between 1 and $|\alpha|-|\gamma|+1$, consequently they are bounded and continuous. Furthermore, Leibniz's product formula also supplies lower order terms in u and v, and top order derivatives for both functions

$$D^\gamma(\varphi_j uv) = [D^\gamma,\varphi_j](uv) + \varphi_j D^\gamma(uv) = \text{“lower order”} + \varphi_j v D^\gamma u + \varphi_j u D^\gamma v\,, \quad |\gamma| = |\alpha|\,.$$

On the whole, we obtain

$$\begin{aligned}
&D^\alpha[(\varphi_j uv)(h_j^{-1}(y))] - D^\alpha[(\varphi_j uv)(h_j^{-1}(z))] = \text{“differences of lower order terms”}+\\
&\sum_{|\gamma|=|\alpha|}\Big\{q_{\alpha\gamma}(y)[\varphi_j v](h_j^{-1}(y))\,[D^\gamma u](h_j^{-1}(y)) - q_{\alpha\gamma}(z)[\varphi_j v](h_j^{-1}(z))\,[D^\gamma u](h_j^{-1}(z))\\
&\quad + q_{\alpha\gamma}(y)[\varphi_j u](h_j^{-1}(y))\,[D^\gamma v](h_j^{-1}(y)) - q_{\alpha\gamma}(z)[\varphi_j u](h_j^{-1}(z))\,[D^\gamma v](h_j^{-1}(z))\Big\}\,. \qquad (2.7)
\end{aligned}$$

All terms of lower order are at least once continuously differentiable, since they belong to $\mathrm{B}_{pp}^{s-|\gamma|}(\mathbb{R}^n)$ for $|\gamma| \le |\alpha| - 1 = [s]-1$. Hence these products being made up of lower order terms can be estimated by

$$\begin{aligned}
&C\sum_{|\gamma|\le|\alpha|-1}\Big\{\|u\|_{\mathrm{C}^{|\gamma|}(\partial\Omega)}\|v\|_{\mathrm{B}_{pp}^{|\gamma|+\{s\}}(\partial\Omega)} + \|v\|_{\mathrm{C}^{|\gamma|}(\partial\Omega)}\|u\|_{\mathrm{B}_{pp}^{|\gamma|+\{s\}}(\partial\Omega)}\Big\}\\
&\qquad\qquad \le C_1\Big\{\|u\|_{\mathrm{C}^{[s]}(\partial\Omega)}\|v\|_{\mathrm{B}_{pp}^{s}(\partial\Omega)} + \|v\|_{\mathrm{C}^{[s]}(\partial\Omega)}\|u\|_{\mathrm{B}_{pp}^{s}(\partial\Omega)}\Big\}\,.
\end{aligned}$$

Therefore, it suffices to look at the differences of top order terms. W.l.o.g. we only consider the first difference of (2.7). By using the triangle inequality, we obtain

$$\begin{aligned}
&\left|\sum_{|\gamma|=[s]} q_{\alpha\gamma}(y)[\varphi_j v](h_j^{-1}(y))\,[D^\gamma u](h_j^{-1}(y)) - q_{\alpha\gamma}(z)[\varphi_j v](h_j^{-1}(z))\,[D^\gamma u](h_j^{-1}(z))\right| \le\\
&\qquad\sum_{|\gamma|=[s]}\Big|q_{\alpha\gamma}(y)[\varphi_j v](h_j^{-1}(y))\Big|\,\Big|[D^\gamma u](h_j^{-1}(y)) - [D^\gamma u](h_j^{-1}(z))\Big| +\\
&\qquad\qquad\Big|[D^\gamma u](h_j^{-1}(z))\Big|\,\Big|q_{\alpha\gamma}(y)[\varphi_j v](h_j^{-1}(y)) - q_{\alpha\gamma}(z)[\varphi_j v](h_j^{-1}(z))\Big|\,.
\end{aligned}$$

Each function in front of differences is at least continuous and thus they can be estimated in L_∞. Furthermore, the function $[D^\gamma u](h_j^{-1}(\cdot))$ appearing in first difference belongs exactly to $\mathrm{B}_{pp}^{\{s\}}(\partial\Omega)$ for all $j = 1,\ldots,M$, whereas the function in the second difference belongs to

$\mathrm{B}^s_{pp}(\partial\Omega)$. Keeping all this in mind, we can estimate as follows

$$\sum_{j=1}^{M}\sum_{|\alpha|=[s]}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|D^\alpha(\varphi_j uv)(h_j^{-1}(y))-D^\alpha(\varphi_j uv)(h_j^{-1}(z))|^p}{|y-z|^{n+\{s\}p}}\,dy\,dz$$

$$\begin{aligned}&\le C_1\left(\|u\|_{\mathrm{C}^{[s]}(\partial\Omega)}\|v\|_{\mathrm{B}^s_{pp}(\partial\Omega)}+\|v\|_{\mathrm{C}^{[s]}(\partial\Omega)}\|u\|_{\mathrm{B}^s_{pp}(\partial\Omega)}\right)\\ &\quad+C_2\Big(\|v\|_{\mathrm{C}(\partial\Omega)}\|u\|_{\mathrm{B}^s_{pp}(\partial\Omega)}+\|u\|_{\mathrm{C}^{[s]}(\partial\Omega)}\|v\|_{\mathrm{B}^{\{s\}}_{pp}(\partial\Omega)}\\ &\qquad+\|u\|_{\mathrm{C}(\partial\Omega)}\|v\|_{\mathrm{B}^s_{pp}(\partial\Omega)}+\|v\|_{\mathrm{C}^{[s]}(\partial\Omega)}\|u\|_{\mathrm{B}^{\{s\}}_{pp}(\partial\Omega)}\Big)\,,\end{aligned}$$

which shows inequality (2.6).

□

Now, we shall investigate a similar result for Bessel potential space which is needed for proving that Z_∇ forms an algebra.

Lemma 2.2 *Let Ω be a bounded domain in $\mathbb{R}^{n+1}$ and $J=[0,T]$. Let further $0<s<1$, $1<p<\infty$ and u, v be arbitrary functions in $\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))\cap\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))$. Then there exists a constant $C>0$ not depending on T such that*

$$\|uv\|_{\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))}\le C\left(\|u\|_{\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))}\|v\|_{\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))}+\|v\|_{\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))}\|u\|_{\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))}\right).$$

Proof. Let u, v be any functions in $\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))\cap\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))$. The subsequent expressions define a norm in $\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))$, cf. Triebel [35], as well as Runst and Sickel [29]. We set

$$[v]_{\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))}:=\Big(\int_\Omega\int_0^T\Big(\int_0^1\sigma^{-2s}\Big(\frac{1}{V(t,\sigma)}\int_{V(t,\sigma)}|v(t+h,x)-v(t,x)|\,dh\Big)^2\frac{d\sigma}{\sigma}\Big)^{p/2}dt\,dx\Big)^{1/p},$$

where $V(t,\sigma)=\{h\in\mathbb{R}:|h|<\sigma\text{ and }t+h\in J\}$, and

$$\|v\|_{\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))}:=[v]_{\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))}+\|v\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega))}.$$

With $|v|_\infty:=\|v\|_{\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))}$ we have

$$\begin{aligned}&|v(t+h,x)u(t+h,x)-v(t,x)v(t,x)|\le|(v(t+h,x)-v(t,x))u(t,x)|+\\ &\quad|(u(t+h,x)-u(t,x))v(t,x)|\le|u|_\infty|v(t+h,x)-v(t,x)|+|v|_\infty|u(t+h,x)-u(t,x)|,\end{aligned}$$

for t, $t+h\in J$ and a.a. $x\in\Omega$. By using this estimate we obtain

$$\begin{aligned}\|uv\|^p_{\mathrm{H}^s_p(J;\mathrm{L}_p(\Omega))}\le&\int_\Omega\int_0^T\Big(\int_0^1\sigma^{-2s}\Big(\frac{1}{V(t,\sigma)}\int_{V(t,\sigma)}|u|_\infty|v(t+h,x)-v(t,x)|\,dh\\ &+\frac{1}{V(t,\sigma)}\int_{V(t,\sigma)}|v|_\infty|u(t+h,x)-u(t,x)|\,dh\Big)^2\frac{d\sigma}{\sigma}\Big)^{p/2}dt\,dx\\ &+|u|_\infty\|v\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega))}.\end{aligned}$$

Employing Minkowski's inequality leads to the desired result.

□

Before we establish that Z and Z_∇ are Banach algebras for p sufficiently large, we want to characterise traces of functions belonging to $Z = \mathrm{H}^1_p(J;\mathrm{L}_p(\Omega)) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\Omega))$. Furthermore, we are also interested in continuous embeddings. We define

$$\begin{aligned} \mathcal{U}^{\beta,T}_\Omega &:= \mathrm{C}^\beta(J;\mathrm{C}(\overline{\Omega})) \cap \mathrm{C}(J;\mathrm{C}^{2\beta}(\overline{\Omega})), \quad 0<\beta<1, \quad \mathcal{U}^T := \mathcal{U}^{1/2,T}_\Omega, \\ \mathrm{Y}^T_{1,\Gamma} &:= \mathrm{B}^{1-1/2p}_{pp}(J;\mathrm{L}_p(\Gamma)) \cap \mathrm{L}_p(J;\mathrm{B}^{2-1/p}_{pp}(\Gamma)), \\ \mathrm{Y}^T_{2,\Gamma} &:= \mathrm{B}^{1/2-1/2p}_{pp}(J;\mathrm{L}_p(\Gamma)) \cap \mathrm{L}_p(J;\mathrm{B}^{1-1/p}_{pp}(\Gamma)). \end{aligned} \tag{2.8}$$

If $J = \mathbb{R}_+$ resp. the length of $J=[0,T]$ is not decisive, then we will omit the index T. We will use the shortened notation $\mathcal{U}_k$, $k=P,F$, in case of the bounded domains Ω_P, Ω_F. If $\mathcal{F}$ is any of the above function spaces, then $f \in \mathcal{F}(X)$ means that f belongs to the corresponding space with values in X.

Proposition 2.5 *Let J be $\mathbb{R}_+$ or a compact time interval $[0,T]$ and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain with C^2-boundary Γ. Suppose that u belongs to Z and g lies in $Y_{1,\Gamma}$. Let $0 \le \beta < 1$ and $\frac{(n+1)+2}{2(1-\beta)} < p < \infty$. Then it holds*

1. *$\gamma_{|t} u \in \mathrm{B}^{2-2/p}_{pp}(\Omega)$, $\gamma_{|\Gamma} u \in Y_{1,\Gamma}$, ($\gamma_{|\Gamma} \in \mathcal{B}(Z, Y_{1,\Gamma})$) and $\partial_\nu u \in Y_{2,\Gamma}$, ($\gamma_{|\Gamma} \in \mathcal{B}(Z_\nabla, Y_{2,\Gamma})$);*
2. *$Z \hookrightarrow \mathcal{U}^\beta_\Omega$, $\|u\|_{\mathcal{U}^\beta_\Omega} \le C_1 \|u\|_Z$;*
3. *$Y_{1,\Gamma} \hookrightarrow \mathcal{U}^\beta_\Gamma$, $\|g\|_{\mathcal{U}^\beta_\Gamma} \le C_3 \|g\|_{Y_{1,\Gamma}}$.*

If $\gamma_{|t} u = 0$ on $\overline{\Omega}$ and $\gamma_{|t} g = 0$ on Γ, then constants C_1 and C_2 are independent of the length of time interval J.

Proof: Step 1. Concerning the first assertion we refer to [24], where various model problems are treated and traces of functions $u \in Z$ are determined.
Step 2. Let $u \in Z$ be given and $0 \le 2\beta < 2 - \frac{n+3}{p}$, which is equivalent to assumption of p. Then, by the *mixed derivative theorem*, we obtain $Z \hookrightarrow \mathrm{H}^\theta_p(J;\mathrm{H}^{2(1-\theta)}_p(\Omega))$ for $0<\theta<1$. To guarantee the embedding

$$\mathrm{H}^\theta_p(J;\mathrm{H}^{2(1-\theta)}_p(\Omega)) \hookrightarrow \mathrm{C}^\beta(J;\mathrm{C}(\overline{\Omega}))$$

we have to impose $\theta - 1/p > \beta > 0$ and $2(1-\theta) - (n+1)/p > 0$. These conditions are equivalent to $2 - \frac{n+1}{p} > 2\theta > 2\beta + \frac{2}{p}$. Choosing $2\beta < 2 - \frac{n+3}{p}$, we then find $\theta \in (0,1)$ satisfying the above condition. The second embedding

$$\mathrm{H}^\theta_p(J;\mathrm{H}^{2(1-\theta)}_p(\Omega)) \hookrightarrow \mathrm{C}(J;\mathrm{C}^{2\beta}(\overline{\Omega}))$$

can be achieved by choosing $\theta - \frac{1}{p} > 0$ and $2(1-\theta) - \frac{n+1}{p} > 2\beta$. These two inequalities imply $2(1-\beta) - \frac{n+1}{p} > 2\theta > \frac{2}{p}$. This condition for θ is satisfied whenever we ensure $2 - \frac{n+3}{p} > 2\beta$. All embeddings have been continuous so that the inequality $\|u\|_{\mathcal{U}^\beta_\Omega} \le C\|u\|_Z$ holds. Statement 3. follows in the same way. That constant C is independent of T if

$u \in {}_0Z^T := \{u \in Z^T : u(0) = 0\}$ can be seen from the following. Define operator E_+ by means of

$$E_+u := \begin{cases} u(t,x) & : \ t \in [0,T) \\ u(2T-t,x) & : \ t \in [T,2T) \\ 0 & : \ t \in [2T,\infty) \end{cases} .$$

After observing that E_+ is a bounded extension operator with norm $\|E_+\|_{\mathcal{B}({}_0Z^T,{}_0Z(\mathbb{R}_+))} \le 2$ one estimates as follows

$$\|u\|_{{}_0\mathcal{U}_\Omega^{\beta,T}} \le \|E_+u\|_{{}_0\mathcal{U}_\Omega^{\beta}} \le C\|E_+u\|_{{}_0Z} \le 2C\|u\|_{{}_0Z^T} = C_1\|u\|_{{}_0Z^T}.$$

The constant C comes from the embedding for $J = \mathbb{R}_+$ and hence is independent of T.

The last statement can also be proved by means of *the mixed derivative theorem*. To detect the independence of T of constant C_2 in case $g \in {}_0Y_1^T$ we consider the problem

$$\begin{aligned} \partial_t u(t,x) - \Delta u(t,x) = 0\,, \quad (t,x) \in \mathbb{R}_+ \times \Omega\,, \\ u(t,x) = E_+g(t,x)\,, \quad (t,x) \in \mathbb{R}_+ \times \Gamma\,, \\ u(0,x) = 0\,, \quad x \in \overline{\Omega}\,. \end{aligned}$$

Due to [24, Theorem 5] there exists a unique solution $u = L(0, E_+g, 0) \in {}_0Z$, where L denotes the solution operator. Then, we may estimate as follows

$$\begin{aligned} \|g\|_{{}_0\mathcal{U}_\Gamma^{\beta,T}} \le & \|E_+g\|_{{}_0\mathcal{U}_\Gamma^{\beta}} = \|\gamma_{|\Gamma}u\|_{{}_0\mathcal{U}_\Gamma^{\beta}} \le \|\gamma_{|\Gamma}\|_{\mathcal{B}(\mathcal{U}_\Omega^\beta,\mathcal{U}_\Gamma^\beta)}\|u\|_{{}_0\mathcal{U}_\Omega^{\beta}} \le C\|L(0,E_+g,0)\|_{{}_0Z} \\ \le & C\|L\|_{\mathcal{B}({}_0Y_{1,\Gamma},{}_0Z)}\|E_+g\|_{{}_0Y_{1,\Gamma}} \le 2C\|L\|_{\mathcal{B}({}_0Y_{1,\Gamma},{}_0Z)}\|g\|_{{}_0Y_{1,\Gamma}^T} = C_2\|g\|_{{}_0Y_{1,\Gamma}^T}\,. \end{aligned}$$

This shows that the constant C_2 is independent of T.

□

Proposition 2.6 *Let J be a compact time interval or $\mathbb{R}_+$ and $G \subset \mathbb{R}^{n+1}$ be a bounded domain with C^2-boundary Γ. Let $(n+1)+2 < p < \infty$. Then Z, Z_∇, $Y_{1,\Gamma}$ and $Y_{2,\Gamma}$ form multiplication algebras and the following estimations are valid.*

1. $\|u \cdot v\|_Z \le C_1(\|u\|_Z\|v\|_{\mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}))} + \|v\|_Z\|u\|_{\mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}))})$, $\forall\ u,\ v \in Z$;
2. $\|u \cdot v\|_{Z_\nabla} \le C_1(\|u\|_{Z_\nabla}\|v\|_{\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))} + \|v\|_{Z_\nabla}\|u\|_{\mathrm{C}(J;\mathrm{C}(\overline{\Omega}))})$, $\forall\ u,\ v \in Z_\nabla$;
3. $\|g \cdot h\|_{Y_{1,\Gamma}} \le C_2(\|g\|_{Y_{1,\Gamma}}\|h\|_{\mathrm{C}(J;\mathrm{C}^1(\Gamma))} + \|h\|_{Y_{1,\Gamma}}\|g\|_{\mathrm{C}(J;\mathrm{C}^1(\Gamma))})$, $\forall\ g,\ h \in Y_{1,\Gamma}$;
4. $\|g \cdot h\|_{Y_{2,\Gamma}} \le C_3(\|g\|_{Y_{2,\Gamma}}\|h\|_{\mathrm{C}(J;\mathrm{C}(\Gamma))} + \|h\|_{Y_{2,\Gamma}}\|g\|_{\mathrm{C}(J;\mathrm{C}(\Gamma))})$, $\forall\ g,\ h \in Y_{2,\Gamma}$;

Proof. By assumption $(n+1)+2 < p < \infty$ we may apply Proposition (2.5) with $\beta = 1/2$, which implies the embedding $Z \hookrightarrow \mathcal{U}$. Let $u,\ v \in Z$ be given, we then have

$$\begin{aligned} \|uv\|_Z := & \|\partial_t(uv)\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} + \|\nabla^2(uv)\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} + \|uv\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} \\ \le & \|u\|_{\mathrm{C}(J\times G)}\|\partial_t v\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} + \|v\|_{\mathrm{C}(J\times G)}\|\partial_t v\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} \\ & + \|u\|_{\mathrm{C}(J\times G)}\|\nabla^2 v\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} + \|v\|_{\mathrm{C}(J\times G)}\|\nabla^2 u\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} \\ & + 2\||\nabla u|\|_{\mathrm{C}(J\times G)}\||\nabla v|\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} + \|u\|_{\mathrm{C}(J\times G)}\|v\|_{\mathrm{L}_p(J;\mathrm{L}_p(G))} \\ \le & C(\|u\|_Z\|v\|_{\mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}))} + \|v\|_Z\|u\|_{\mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}))}) \le C\|u\|_Z\|v\|_Z\,. \end{aligned}$$

To prove the second statement we remind of the embedding $Z_\nabla \hookrightarrow \mathrm{C}(J \times \overline{\Omega})$ which is needed to employ Lemma 2.2. Using these facts we can proceed in a similar way as above.

We consider the third statement. Since the space $Y_{1,\Gamma}$ is an intersection of two Banach spaces, see definition (2.8), we separately have to attend to each intersection space of $Y_{1,\Gamma}$. For this purpose we employ Lemma 2.1 with $n = 0$, $\Omega = [0, T]$ to $\mathrm{B}_{pp}^{1-1/2p}(J)$ and obtain for a.a. $x \in G$ an estimation of the form

$$\begin{aligned}\|g(\cdot,x)\cdot h(\cdot,x)\|_{\mathrm{B}_{pp}^{1-1/2p}(J)} \le C \Big[&\|g(\cdot,x)\|_{\mathrm{C}(J)}\|h(\cdot,x)\|_{\mathrm{B}_{pp}^{1-1/2p}(J)}\\ &+\|h(\cdot,x)\|_{\mathrm{C}(J)}\|g(\cdot,x)\|_{\mathrm{B}_{pp}^{1-1/2p}(J)}\Big]\,.\end{aligned}$$

Due to Proposition 2.5 we even know that g and h are continuous and hence we may deduce

$$\begin{aligned}\|g\cdot h\|_{\mathrm{B}_{pp}^{1-1/2p}(J;\mathrm{L}_p(\Gamma))} \le &C\Big[\|g\|_{\mathrm{C}(J\times\Gamma)}\|h\|_{\mathrm{B}_{pp}^{1-1/2p}(J;\mathrm{L}_p(\Gamma))} + \|h\|_{\mathrm{C}(J\times\Gamma)}\|g\|_{\mathrm{B}_{pp}^{1-1/2p}(J;\mathrm{L}_p(\Gamma))}\Big]\\ \le &C\|g\|_{Y_{1,\Gamma}}\|h\|_{Y_{1,\Gamma}}\,.\end{aligned}$$

For the second space of Y_1 we can proceed analogously. Choosing $\Omega = G \subset \mathbb{R}^{n+1}$ and applying once again Lemma 2.1 and Proposition 2.5 leads to

$$\begin{aligned}\|g\cdot h\|_{\mathrm{L}_p(J;\mathrm{B}_{pp}^{2-1/p}(\Gamma))} \le &C\Big[\|g\|_{\mathrm{C}(J;\mathrm{C}^1(\Gamma))}\|h\|_{\mathrm{L}_p(J;\mathrm{B}_{pp}^{2-1/p}(\Gamma))} + \|h\|_{\mathrm{C}(J;\mathrm{C}^1(\Gamma))}\|g\|_{\mathrm{L}_p(J;\mathrm{B}_{pp}^{2-1/p}(\Gamma))}\Big]\\ \le &C\|g\|_{Y_{1,\Gamma}}\|h\|_{Y_{1,\Gamma}}\,.\end{aligned}$$

Combining both inequalities implies the desired result. Statement 3. follows in the same way. □

Remark 2.3 Now we are in the position to discuss the term $\gamma_{|\Gamma_P} \ln(\gamma_i^k(t,x)u_i^k(t,x))$ appearing in boundary condition of Dirichlet type. Since we are looking for solutions in regularity class Z and these functions have traces in Y_{1,Γ_P} in virtue of Proposition 2.5, the boundary condition of Dirichlet type has to be considered in Y_{1,Γ_P}. Therefore we have to ensure that each term lies in this space. Proposition 2.6 and $\ln(r) \in C^\infty(0,\infty)$ imply that $\gamma_{|\Gamma_P} \ln(\gamma_i^k(t,x)\cdot u_i^k(t,x)) \in Y_{1,\Gamma_P}$ if $u_i^k \in Z$, $u_i^k > 0$ and $\gamma_i^k \in Y_{1,\Gamma_P}$, $\gamma_i^k > 0$.

We now are going to study the mapping properties of the operators $\mathcal{A}_{ki}$ and boundary operators $\mathcal{B}_{ki}$ and $\tilde{U}_k^{-1}$.

Lemma 2.3 *Let J be $\mathbb{R}_+$ or a compact time interval and $\Omega_P, \Omega_F \subset \mathbb{R}^{n+1}$ be bounded domains with C^2-boundary, $\Gamma_P := \partial\Omega_P$, $\partial\Omega_F = \Gamma_P \cup \Gamma$ and* $\operatorname{dist}(\Gamma_P, \Gamma) > 0$. *Suppose that the smoothness assumptions (1.23) and (1.25) for matrices D^k and M^k are satisfied and $2+(n+1) < p < \infty$. Assuming that $v^k \in Z_k$ and $\tilde{u}^k \in Z_k(E_+)$ then the following statements are valid for $k = P, F$.*

1. $\mathcal{A}_{k1}(\cdot,\cdot,D) \in \mathcal{B}(Z_k, \mathcal{X}_k)$;
2. $[v^k \to \mathcal{A}_{k2}(\cdot,\cdot,v^k,D)] \in \mathrm{C}^{1-}(\mathcal{U}_k(\mathbb{R}^N); \mathcal{B}(\mathcal{Z}_k, \mathcal{X}_k))$;
3. $\mathcal{B}_{k1}(\cdot,\cdot,D) \in \mathcal{B}(Z_k, Y_{2,\Gamma_P}(\mathbb{R}^N))$;
4. $[v^k \to \mathcal{B}_{k2}(\cdot,\cdot,v^k,D)] \in \mathrm{C}^{1-}(\mathcal{U}_k(\mathbb{R}^N); \mathcal{B}(\mathcal{Z}_k, Y_{2,\Gamma_P}(\mathbb{R}^N)))$;
5. $\gamma_{|\Gamma_P}\tilde{U}_k^{-1} \in \mathcal{B}(Z_k, Y_{1,\Gamma_P}(\mathbb{R}^N))$.

Proof. Since all statements are really plausible we only establish the second and fourth assertion to see Lipschitz-continuity of operators $\mathcal{A}_{k2}$ and $\mathcal{B}_{k2}$, and the last assertion.

Let $\phi \in \mathcal{Z}_k$ and $v^k, \overline{v}^k \in Z_k$ be given. Due to Proposition 2.5 we can ensure the embedding $Z_k \hookrightarrow \mathcal{U}_k(\mathbb{R}^N)$ for $p > (n+1)+2$ and consequently $v^k, \overline{v}^k$ lie in $\mathcal{U}_k(\mathbb{R}^N)$. Taking into account the regularity assumption for D^k and that $M^k := \lambda_0 \operatorname{diag}[z_i d_i^k]_{1\le i\le N}$, we may estimate as follows

$$\begin{aligned}
\|[\mathcal{A}_{k2}(v^k; D) - \mathcal{A}_{k2}(\overline{v}^k; D)]\phi\|_{\mathcal{X}_k} &\le \|M^k(v^k - \overline{v}^k)\Delta\phi\|_{X_k} + \|\nabla[M^k(v^k - \overline{v}^k)]\nabla\phi\|_{X_k} \\
&\quad + \|\nabla\cdot(z^T \cdot M^k(v^k - \overline{v}^k))\nabla\phi)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))} \\
&\le \|M^k(v^k - \overline{v}^k)\|_{\mathrm{C}(J\times\overline{\Omega}_k;\mathbb{R}^N)}\|\Delta\phi\|_{X_k} \\
&\quad + \|\nabla[M^k(v^k - \overline{v}^k)]\|_{\mathrm{C}(J\times\overline{\Omega}_k;\mathbb{R}^{N\times n+1})}\|\nabla\phi\|_{X_k} \\
&\quad + \|z^T \cdot M^k(v^k - \overline{v}^k)\nabla\phi)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} \\
&\le C\Big(\|v^k - \overline{v}^k\|_{\mathcal{U}_k(\mathbb{R}^N)}\|\Delta\phi\|_{X_k} + \|v^k - \overline{v}^k\|_{\mathrm{C}(J\times\overline{\Omega}_k;\mathbb{R}^N)}\|\nabla\phi\|_{X_k} \\
&\quad + \|z^T \cdot M^k(v^k - \overline{v}^k)\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))}\|\nabla\psi\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k;\mathbb{R}^{n+1}))}\Big) \\
&\le C\|v^k - \overline{v}^k\|_{\mathcal{U}_k(\mathbb{R}^N)}\|\phi\|_{\mathcal{Z}_k}.
\end{aligned}$$

Before proving the fourth claim, note that $\nabla\phi$ belongs to $\mathcal{Z}_{k,\nabla}$, consequently $\partial_\nu\phi$ lies in Y_{2,Γ_P} and the following estimation holds.

$$\|\partial_\nu\phi\|_{Y_{2,\Gamma_P}} \le C\|\nabla\phi\|_{\mathcal{Z}_{k,\nabla}} \le C\|\phi\|_{\mathcal{Z}_k}$$

Furthermore, the space $\mathcal{U}_{\Gamma_P} = \mathrm{C}^{1/2}(J;\mathrm{C}(\Gamma_P)) \cap \mathrm{C}(J;\mathrm{C}^1(\Gamma_P))$ possesses the multiplicator property concerning Y_{2,Γ_P}, i.e. there is a constant C such that

$$\|mu\|_{Y_{2,\Gamma_P}} \le C\|m\|_{\mathcal{U}_{\Gamma_P}}\|g\|_{Y_{2,\Gamma_P}}, \quad \forall m \in \mathcal{U}_{\Gamma_P}, \quad \forall g \in Y_{2,\Gamma_P}\,.$$

Thus we have

$$\begin{aligned}
\|[\mathcal{B}_{k2}(v^k; D) - \mathcal{B}_{k2}(\overline{v}^k; D)]\phi\|_{Y_{2,\Gamma_P}(\mathbb{R}^N)} &= \|M^k(v^k - \overline{v}^k)\partial_\nu\phi\|_{Y_{2,\Gamma_P}(\mathbb{R}^N)} \\
&\le \|M^k\|_{\mathcal{U}_{\Gamma_P}(\mathcal{B}(\mathbb{R}^N))}\|v^k - \overline{v}^k\|_{\mathcal{U}_{\Gamma_P}(\mathbb{R}^N)}\|\partial_\nu\phi\|_{Y_{2,\Gamma_P}} \\
&\le C\|v^k - \overline{v}^k\|_{\mathcal{U}_k(\mathbb{R}^N)}\|\phi\|_{\mathcal{Z}_k}\,,
\end{aligned}$$

which shows the assertion.

For proving the last claim we can first see that $\tilde{u}_i^k > 0 \in Z_k$ implies $(\tilde{u}_i^k)^{-1} \in Z_k$ in case of $p > 2 + (n+1)$. Moreover, owing to Proposition 2.5 we have $\gamma_{|\Gamma_P} \in \mathcal{B}(Z_k, Y_{1,\Gamma_P})$ and Proposition 2.6 provides the multiplication algebra of Z_k. This makes possible to estimate as follows

$$\begin{aligned}
\|\gamma_{|\Gamma_P}\tilde{U}_k^{-1}v^k\|_{Y_{1,\Gamma_P}(\mathbb{R}^N)} &\le \|\gamma_{|\Gamma_P}\|_{\mathcal{B}(Z_k,Y_{1,\Gamma_P})}\|[v_i^k/\tilde{u}_i^k]_{1\le i\le N}\|_{Z_k} \\
&\le C\sum_{i=1}^N\Big(\|(\tilde{u}_i^k)^{-1}\|_{Z_k}\|v_i^k\|_{\mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}_k))} + \|v_i^k\|_{Z_k}\|(\tilde{u}_i^k)^{-1}\|_{\mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}_k))}\Big) \\
&\le C\|v^k\|_{Z_k}.
\end{aligned}$$

$\square$

2.5 Model Problems

In this section we want to study some model problems which arise out of solving linear problem (1.39 via localisation, changing coordinates and perturbation. At first, we are concerning with

$$\begin{aligned} \dot{u}(t) + A(t)u(t) &= f(t)\,, \quad t \in J, \\ u(0) &= u_0\,. \end{aligned} \tag{2.9}$$

This model problem was considered in Prüss and Schnaubelt [26].

Theorem 2.4 *Suppose Y is a Banach space of class $\mathcal{HT}$, $1 < p < \infty$ and $J = [0, T]$. Suppose that $A(\cdot)$ is continuous in J and $D(A(t)) = D(A(0)) =: D(A)$ for all $t \in J$. Let $A(t) \in \mathcal{RS}(Y)$ be invertible with $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} < \pi/2$ for all $t \in J$. Then (2.9) has precisely one solution in $Z_J := \mathrm{H}_p^1(J; Y) \cap \mathrm{L}_p(J; D_A)$ if and only if the following two conditions are satisfied.*

1. *$f \in X := \mathrm{L}_p(J; Y)$;*
2. *$u_0 \in D_A(1 - 1/p, p)$.*

□

We now consider two abstract second order problems which play an essential role for treating of parabolic problems with inhomogeneous boundary data. The next theorem concerns the problem

$$\begin{aligned} -u''(y) + F^2 u(y) &= f(y), \quad y > 0, \\ u(0) &= \phi, \end{aligned} \tag{2.10}$$

in $\mathrm{L}_p(\mathbb{R}_+; X)$.

Theorem 2.5 *Suppose X is a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$. Let $F \in \mathcal{BIP}(X)$ be invertible with power angle $\theta_F < \pi/2$, and let D_F^j denote the domain $\mathcal{D}(F^j)$ of F^j equipped with its graph norm, $j = 1, 2$.*

Then (2.10) has unique solution u in $Z := \mathrm{H}_p^2(\mathbb{R}_+; X) \cap \mathrm{L}_p(\mathbb{R}_+; D_F^2\ )$ if and only if the following two conditions are satisfied

1. *$f \in \mathrm{L}_p(J; X)$;*
2. *$\phi \in D_F(2 - 1/p, p)$.*

If this is the case we have in addition $u \in \mathrm{H}_p^1(\mathbb{R}_+; D_F)$.

This result is due to Prüss, cf. [26, Theorem 3]. Having in mind that $D_F(2 - 1/p, p) = \{g \in D(F) : Fg \in D_F(1 - 1/p, p)\}$. There is a corresponding result for the abstract second order problem with abstract Robin condition

$$\begin{aligned} -u''(y) + F^2 u(y) &= f(y), \quad y > 0, \\ -u'(0) + Du(0) &= \psi, \end{aligned} \tag{2.11}$$

in $\mathrm{L}_p(\mathbb{R}_+; X)$.

Theorem 2.6 *Suppose X is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$. Let $F \in \mathcal{BIP}\,(X)$ be invertible with power angle $\theta_F < \pi/2$, and let D_F^j denote the domain $\mathcal{D}(F^j)$ of F^j equipped with its graph norm, $j = 1,2$. Suppose that D is pseudo-sectorial in X, belongs to $\mathcal{BIP}\left(\overline{\mathcal{R}(D)}\right)$, commutes with F, and is such that $\theta_F + \theta_D < \pi$.*

Then (2.11) has unique solution u in $Z := \mathrm{H}_p^2(\mathbb{R}_+;X) \cap \mathrm{L}_p(\mathbb{R}_+;D_F^2\ )$ with $u(0) \in \mathcal{D}(D)$ and $Du(0) \in D_F(1-1/p,p)$ if and only if the following two conditions are satisfied

1. *$f \in \mathrm{L}_p(J;X)$;*
2. *$\psi \in D_F(1-1/p,p)$.*

If this is the case we have in addition $u \in \mathrm{H}_p^1(\mathbb{R}_+;D_F)$.

This result is also due to Prüss, see [26, Theorem 4].

Our next result concerns the parabolic problem

$$\begin{aligned} \partial_t u - L\partial_y^2 u + Au &= f(t,y), \quad t \in J, \quad y > 0, \\ u_{|y=0} &= g(t), \quad t \in J, \\ u_{|t=0} &= u_0(y)\,, \quad y > 0, \end{aligned} \tag{2.12}$$

in $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+;Y))$. Here again Y is a Banach space of class $\mathcal{HT}$, $J = [0,T]$ a compact time interval, A denotes a sectorial operator in Y and L is invertible in Y. We are interested in solutions u which belong to the maximal regularity space Z, more precisely

$$u \in Z := \mathrm{H}_p^1(J;\mathrm{L}_p(\mathbb{R}_+;X)) \cap \mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_+;Y)) \cap \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+;D_A)).$$

The above problem is a variation of a parabolic problem which has been considered by Prüss in [24]. Therefore we are in need of only a few modifications to show the following result.

Theorem 2.7 *Suppose Y is a Banach space of class $\mathcal{HT}$, $p \in (1,\infty)$, $p \neq 3/2$ and $J = [0,T]$. Let $A \in \mathcal{BIP}\,(Y)$ be invertible with power angle $\theta_A < \pi/2$. Assume further $L \in \mathcal{B}(Y)$ be invertible with $\sigma\,(L) \subset \Sigma_{\theta_L}$, $\theta_L < \pi/2$, and commute with A. Then problem (2.12) has exactly one solution*

$$u \in \mathrm{H}_p^1(J;\mathrm{L}_p(\mathbb{R}_+;Y)) \cap \mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_+;Y)) \cap \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+;D_A))$$

if and only if the data f, g, u_0 satisfy the following conditions.

1. *$f \in \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+;Y))$;*
2. *$g \in \mathrm{B}_{pp}^{1-1/2p}(J;Y) \cap \mathrm{L}_p(J;D_A(1-1/2p,p))$;*
3. *$u_0 \in \mathrm{B}_{pp}^{2-2/p}(\mathbb{R}_+;Y) \cap \mathrm{L}_p(\mathbb{R}_+;D_A(1-1/p,p))$;*
4. *$g(0) = u_0(0) \in D_A(1-3/2p,p)$ in case $p > 3/2$.*

Proof. The proof of this Theorem runs as in [24, Theorem 5]. Therefore, the necessary part can be also adopted from there.

At first we solve the corresponding Dirichlet problem, i.e. $u_{|y=0} = 0$ for $t \in J$ and $u_0(0) = 0$ due to compatibility. For this purpose we define $G = -L\partial_y^2$ with domain $D(G) = \mathrm{H}_p^2(\mathbb{R}_+;Y) \cap {}_0\mathrm{H}_p^1(\mathbb{R}_+;Y)$; then G belongs to $\mathcal{BIP}\,(\mathrm{L}_p(\mathbb{R}_+;Y))$ with power angle $\theta_G < \pi/2$, since $L \in \mathcal{B}(Y)$ and $\sigma\,(L) \subset \Sigma_{\theta_L}$ with $\theta_L < \pi/2$. Let B denote the natural extension of A

to $\mathrm{L}_p(\mathbb{R}_+;Y)$, with domain $D(B)=\mathrm{L}_p(\mathbb{R}_+;D_A)$. Then B is also in $\mathcal{BIP}\,(\mathrm{L}_p(\mathbb{R}_+;Y))$ with power angle $\theta_B<\pi/2$. Since both operators commute, Theorem 2.2 yields that $\mathcal{A}=G+B$ with domain $D(\mathcal{A})=D(G)\cap D(B)$ belongs to $\mathcal{BIP}\,(\mathrm{L}_p(\mathbb{R}_+;Y))$ with power angle $\theta_{\mathcal{A}}<\pi/2$. Therefore, by Theorem 2.4, $u_1(t)=e^{-\mathcal{A}t}u_0+e^{-\mathcal{A}t}*f$ is the unique solution of the Dirichlet problem which lies in

$$\mathrm{H}^1_p(J;\mathrm{L}_p(\mathbb{R}_+;Y))\cap\mathrm{L}_p(J;\mathrm{H}^2_p(\mathbb{R}_+;Y))\cap\mathrm{L}_p(J;{}_0\mathrm{H}^1_p(\mathbb{R}_+;Y))\cap\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+;D_A)).$$

Now, we tackle the case of vanishing initial data, $f=0$ and $g(0)=0$.

$$\begin{aligned}\partial_t u-L\partial_y^2u+Au&=0,\quad t\in J,\quad y>0,\\ u_{|_{y=0}}&=g(t),\quad t\in J\\ u_{|_{t=0}}&=0,\quad y>0\,.\end{aligned}$$

Define $\mathcal{A}$ in $\mathrm{L}_p(J;D_A)$ by pointwise extension, and set $\mathcal{D}=\partial_t$ with domain $D(\mathcal{D})={}_0\mathrm{H}^1_p(J;Y)$. Both operators are sectorial, they commute and belong to $\mathcal{BIP}\,(\mathrm{L}_p(J;Y))$ with $\theta_{\mathcal{D}}+\theta_{\mathcal{A}}<\pi$. Hence, by Theorem 2.2, $L^{-1}(\mathcal{D}+\mathcal{A})$ with domain $D(\mathcal{D})\cap D(\mathcal{A})$ is invertible and belongs to $\mathcal{BIP}\,(\mathrm{L}_p(J;Y))$ with power angle small than $\pi/2+\theta_L<\pi$. The above problem can be written as

$$\begin{aligned}-\partial_y^2u+F^2u&=0,\quad y>0,\\ u_{|_{y=0}}&=g,\end{aligned}$$

with $F=L^{-1/2}(\mathcal{D}+\mathcal{A})^{1/2}$. Now we are in the position to apply Theorem 2.5 to the result, that $u_2=e^{-Fy}g$ is the unique solution of the above problem which lies in

$$\mathrm{H}^2_p(\mathbb{R}_+;\mathrm{L}_p(J;Y))\cap\mathrm{L}_p(\mathbb{R}_+;{}_0\mathrm{H}^1_p(J;Y))\cap\mathrm{L}_p(\mathbb{R}_+;\mathrm{L}_p(J;D_A)).$$

In the end, if the compatibility condition $g(0)=u_0(0)$ is valid, the unique solution of (2.12) can be written as follows

$$\begin{aligned}u(t,y)&=e^{-\mathcal{A}t}\left[u_0(y)-e^{-L^{-1/2}A^{1/2}y/\sqrt{2}}u_0(0)\right]+(e^{-\mathcal{A}t}*f)(t,y)\\&\quad+e^{-Fy}\left[g(t)-e^{-A\frac{t}{2}}u_0(0)\right]+e^{-L^{-1/2}A^{1/2}\frac{y}{\sqrt{2}}}e^{-A\frac{t}{2}}u_0(0)\\&=T(t)\left[u_0(y)-\Xi(y)u_0(0)\right]+(T*f)(t,y)\\&\quad+\Upsilon(y)\left[g(t)-S(t)u_0(0)\right]+\Xi(y)S(t)u_0(0)\,.\end{aligned}\tag{2.13}$$

Here, we have used the notations $T(t):=e^{-\mathcal{A}t}$, $S(t):=e^{-A\frac{t}{2}}$, $\Xi(y):={}^{-L^{-1/2}A^{1/2}y/\sqrt{2}}$ and $\Upsilon(y):=e^{-Fy}$. Finally, it is left to check that each term belongs to Z. In view of regularisation of semigroups $T(t)$ and $\Upsilon(y)$, we have to verify that $S(\cdot)u_0(0)\in D_F(2-1/p,p)$ and $\Xi(\cdot)u_0(0)\in D_A(1-1/p,p)$. W.l.o.g. we show

$$S(t)u_0(0)\in D_F(2-1/p,p)=D_{F^2}(1-1/2p,p)=D_{\mathcal{D}}(1-1/2p)\cap D_A(1-1/2p,p).$$

Due to assumption $u_0(0)\in D_A(1-3/2p,p)$ and $1-3/2p<1-1/p$ we may employ [13, Theorem 3] which implies $e^{-A\frac{t}{2}}u_0(0)\in\mathrm{L}_p(J;D_A(1-1/2p,p))$. That means, the trajectory $v(t):=e^{-A\cdot\frac{t}{2}}u_0(0)$ gains $1/p$ spatial regularity. Moreover, this fact gives rise to

$v \in \mathrm{B}_{pp}^{1-1/2p}(J;X) = D_{\mathcal{D}}(1-1/2p,p)$ as well. Considering the norm of $D_{\mathcal{D}}(1-1/2p,p)$, see (2.2), and setting $\theta = 1-1/2p$, we then obtain

$$v \in D_{\mathcal{D}}(1-1/2p) \Leftrightarrow \int_0^\infty \|\tau^{1-\theta}\mathcal{D}\mathcal{T}(\tau)v\|^p_{\mathrm{L}_p(\mathbb{R}_+;X)}\frac{d\tau}{\tau} = \int_0^\infty\int_0^\infty |\tau^{1-\theta}\mathcal{D}\mathcal{T}(\tau)v(t)|^p_X\frac{d\tau}{\tau}\,dt < \infty$$

Here $\mathcal{T}(t)$ denotes the left translation semigroup. After some evaluations and using Fubini, we find the relation

$$\int_0^\infty \|\tau^{1-\theta}\mathcal{D}S(\tau)v(t)\|^p_{\mathrm{L}_p(\mathbb{R}_+;X)}\frac{d\tau}{\tau} = \int_0^\infty\int_0^\infty |\tau^{1-\theta}AT(\tau)v(t)|^p_X\frac{d\tau}{\tau}dt.$$

However the existence of integral above is equivalent to $v \in \mathrm{L}_p(J;D_A(1-1/2p,p))$.

□

Chapter 3

Maximal L_p-Regularity for the Linear Problem

In this chapter we solve the linear problem (1.39)-(1.41), that means we prove existence and uniqueness of functions $(w^P, w^F, w^b) \in Z_P \times Z_F \times Z_b$ and $(\psi^P, \psi^F) \in \mathcal{Z}$ satisfying the evolution equations (1.39) and (1.40). We will first consider a problem on the full space $\mathbb{R}^{n+1}$. After this a half space and a two phase problem in $\mathbb{R}^{n+1}_+$ will be studied. Finally, in section (3.4) we solve the problem for the domain Ω via localisation, perturbation, and changing coordinates in order to reduce the problem to related problems on the full and half space. Afterwards, we sum up all local solutions and have to establish that the so constructed function is the unique solution of the original problem.

3.1 A Full Space Problem

In this section we study the full space problem

$$\begin{aligned} &\partial_t w - D[\Delta - 1]w - M\tilde{u}[\Delta - 1]\psi = f, \quad (t,x) \in J \times \mathbb{R}^{n+1}, \\ &w(0,x) = u_0(x), \quad z^T\cdot w = 0, \quad (t,x) \in J \times \mathbb{R}^{n+1} \end{aligned} \tag{3.1}$$

in $\mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}^{n+1}))$, with $J = [0,T]$. All coefficients are constant in time and space, where the denotation indicates to the meaning of the coefficients appearing in the linear problem (1.39) - (1.41), e.g. we have $a := z^T\cdot M\tilde{u} > 0$. We look for unique solutions (w, ψ) in the maximal regularity space $Z^T \times \mathcal{Z}^T$ defined by

$$\begin{aligned} Z^T &:= \mathrm{H}^1_p(J; \mathrm{L}_p(\mathbb{R}^{n+1}; \mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\mathbb{R}^{n+1}; \mathbb{R}^N)), \\ \mathcal{Z}^T &:= \mathrm{H}^{1/2}_p(J; \mathrm{H}^1_p(\mathbb{R}^{n+1})) \cap \mathrm{L}_p(J; \mathrm{H}^2_p(\mathbb{R}^{n+1})). \end{aligned}$$

To obtain this regularity we have to impose on the inhomogeneity the condition

$$f \in \mathcal{X}^T := \{f \in X^T := \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}^{n+1}; \mathbb{R}^N)) : \; z^T\cdot f \in \mathrm{H}^{1/2}_p(J; \mathrm{H}^{-1}_p(\mathbb{R}^{n+1}))\}$$

equipped with the norm

$$\|f\|_{\mathcal{X}^T} := \|f\|_{X^T} + \|z^T\cdot f\|_{\mathrm{H}^{1/2}_p(J; \mathrm{H}^{-1}_p(\mathbb{R}^{n+1}))}.$$

The goal of this section is to prove the following result.

Theorem 3.1 *Let $J = [0,T]$, $1 < p < \infty$ and assume that d_i, $\tilde{u}_i$ for $i = 1,\dots,N$ are positive. Then problem 3.1 has exactly one solution (w,ψ) in the space $Z^T \times \mathcal{Z}^T$ if and only if the data $f(t,x)$ and $u_0(x)$ satisfy the following conditions*

1. $f \in \mathcal{X}^T$;
2. $u_0 \in V(E) := \mathrm{B}_{pp}^{2-2/p}(\mathbb{R}^{n+1}; E)$.

Moreover, there exists an isomorphism $\mathcal{S}$ between space of data and $Z^T \times \mathcal{Z}^T$.

Proof. We start with the necessity part. Suppose that $(w,\psi) \in Z^T \times \mathcal{Z}^T$ solves (3.1). Then it follows $f = \partial_t w + \mathcal{A}_1(D)w + \mathcal{A}_2(D)\psi \in \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}^{n+1}; \mathbb{R}^N))$, where we have set

$$\mathcal{A}_1(D) := -D[\Delta - 1], \quad \mathcal{A}_2(D) := -M\tilde{u}[\Delta - 1]\,. \tag{3.2}$$

To verify $z^T\!\cdot f \in \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^{-1}(\mathbb{R}^{n+1}))$ we use duality for the highest order terms and the embedding $\mathrm{L}_p(\mathbb{R}^{n+1}) \hookrightarrow \mathrm{H}_p^{-1}(\mathbb{R}^{n+1})$ in case of lower order.

$$\begin{aligned}\|z^T\!\cdot f\|_{\mathrm{H}_p^{-1}(\mathbb{R}^{n+1})} &\le \|z^T\!\cdot D[\Delta - 1]w\|_{\mathrm{H}_p^{-1}(\mathbb{R}^{n+1})} + \|z^T\!\cdot M\tilde{u}[\Delta - 1]\psi\|_{\mathrm{H}_p^{-1}(\mathbb{R}^{n+1})} \\ &\le C\left(\|\nabla w\|_{\mathrm{L}_p(\mathbb{R}^{n+1})} + \|\nabla \psi\|_{\mathrm{L}_p(\mathbb{R}^{n+1})} + \|w\|_{\mathrm{L}_p(\mathbb{R}^{n+1})} + \|\psi\|_{\mathrm{L}_p(\mathbb{R}^{n+1})}\right)\end{aligned}$$

By regularity assumption of w and ψ we deduce that $\nabla\psi$, ∇w belong to the space $Z_\nabla^T := \mathrm{H}_p^{1/2}(J; \mathrm{L}_p(\mathbb{R}^{n+1})) \cap \mathrm{L}_p(J; \mathrm{H}_p^1(\mathbb{R}^{n+1}))$ which implies that each term lies in $\mathrm{H}_p^{1/2}(J)$. The last condition is a consequence of Theorem 2.4 and the fact that $z^T\!\cdot w(t,x) = 0$ for all $t \in [0,T]$ and $x \in \mathbb{R}^{n+1}$.

Sufficiency. Observe that (3.1) is equivalent to the subsequent problem, whereas the electroneutrality condition is replaced by the corresponding elliptic equation. Thus, we study the evolution problem

$$\begin{aligned}\partial_t w - D[\Delta - 1]w - M\tilde{u}[\Delta - 1]\psi = f, &\quad (t,x) \in J \times \mathbb{R}^{n+1}, \\ w(0,x) = u_0(x), &\quad x \in \mathbb{R}^{n+1}, \\ -z^T\!\cdot D[\Delta - 1]w - a[\Delta - 1]\psi = z^T\!\cdot f, &\quad (t,x) \in J \times \mathbb{R}^{n+1}.\end{aligned}$$

Next we introduce an important projection, which map $\mathbb{R}^N$ into E. This projection is needed to solve the above evolution problem. It turns out that we can utilise this projection to eliminate the expression $M\tilde{u}[\Delta - 1]\psi$ in the parabolic equation. We define

$$b := \frac{M\tilde{u}}{a}, \quad a := z^T\!\cdot M\tilde{u}, \quad Q := b \otimes z, \quad \Pi := I - Q\,. \tag{3.3}$$

Lemma 3.1 *Suppose that d_i, $\tilde{u}_i$ for $i = 1,\dots,N$ are positive. Then ΠD is a mapping from $\mathbb{R}^N$ into E with $\sigma\left((\Pi D)_{|E}\right) \subset (0,\infty)$.*

Proof. 1. *$im(\Pi D) \subset E$.* Let $\xi \in \mathbb{R}^N$ be given, then we have

$$z^T\!\cdot \Pi D\xi = z^T\!\cdot(D\xi - QD\xi) = z^T\!\cdot D\xi - z^T\!\cdot b z^T\!\cdot D\xi = 0,$$

since $z^T\!\cdot b = 1$.

2. In case D is diagonal we are able to transform ΠD into a symmetric matrix. To achieve this, we define multiply ΠD by $\tilde{U}^{1/2} = \mathrm{diag}[\sqrt{\tilde{u}_i}]_{1\le i\le N}$ from right and by $\tilde{U}^{-1/2}$ from left. Then we obtain

$$\begin{aligned}\tilde{U}^{-1/2}\,(\Pi D)\,\tilde{U}^{1/2} &= \tilde{U}^{-1/2}D\tilde{U}^{1/2} - \frac{\tilde{U}^{-1/2}M\tilde{u}\otimes\tilde{U}^{1/2}Dz}{z^T\cdot M\tilde{u}} = D - \frac{\tilde{U}^{-1/2}D\lambda_0\tilde{U}z\otimes\tilde{U}^{1/2}Dz}{z^T\cdot\lambda_0 D\tilde{U}z}\\ &= D - \frac{\tilde{U}^{1/2}Dz\otimes\tilde{U}^{1/2}Dz}{\left|D^{1/2}\tilde{U}^{1/2}z\right|^2} =: D - b_0\otimes b_0\,.\end{aligned}$$

3. There are only positive eigenvalues. Suppose λ is an eigenvalue and $\eta\neq 0$ the corresponding eigenvector. Multiplying the eigenvalue-equation by η yields

$$\lambda|\eta|^2 = \left|D^{1/2}\eta\right|^2 - |\langle b_0,\eta\rangle|^2 = \left|D^{1/2}\eta\right|^2\left[1 - \left|\langle D^{-\frac{1}{2}}b_0, \frac{D^{1/2}\eta}{|D^{1/2}\eta|}\rangle\right|^2\right]\,.$$

The vectors $D^{-\frac{1}{2}}b_0$ and $\frac{D^{1/2}\eta}{|D^{1/2}\eta|}$ have norm 1 and so we see $\lambda\geq 0$. Furthermore, if we assume that $\lambda = 0$ is an eigenvalue, then $\eta = \alpha D^{-1}b_0$ for some $\alpha\in\mathbb{R}$. Since η should be in $\tilde{U}^{-1/2}E$, this implies

$$0 = z^T\cdot\tilde{U}^{-1/2}\eta = \alpha\, z^T\cdot\tilde{U}^{-1/2}D^{-1}b_0 = \alpha\, z^T\cdot\tilde{U}^{-1/2}D^{-1}D\tilde{U}^{1/2}z = \alpha\,|z|^2\,.$$

This relation can only be satisfied for $\alpha = 0$, however this means that $\eta = 0$.

$\square$

Now the projection $\Pi = I - \frac{M\tilde{u}\otimes z}{a}$ comes into operation. By applying Π to the above equation we can eliminate the term $M\tilde{u}[\Delta - 1]\psi$ due to the fact $M\tilde{u}\in\ker(\Pi)$, and obtain

$$\begin{aligned}\partial_t w - \Pi D[\Delta-1]w &= \Pi f, \quad (t,x)\in J\times\mathbb{R}^{n+1},\\ w(0) &= u_0, \quad x\in\mathbb{R}^{n+1}.\end{aligned}\tag{3.4}$$

This formulation is equivalent to the original one, whereas now the electroneutrality condition is incorporated into the solution space. The equivalence results from the identity

$$\begin{aligned}\mathcal{A}_1(D)w + \mathcal{A}_2(D)\psi - f &= \Pi\mathcal{A}_1(D)w - f + Q\mathcal{A}_1(D)w + Q\mathcal{A}_2(D)\psi\\ &= \Pi\mathcal{A}_1(D)w - f + b[z^T\cdot\mathcal{A}_1(D)w + z^T\cdot\mathcal{A}_2(D)\psi]\\ &= \Pi\mathcal{A}_1(D)w - \Pi f.\end{aligned}$$

At first, we solve the evolution problem (3.4) for w. The given data satisfy all conditions of Theorem 2.4 so that it remains to check the assumption that $A := \Pi\mathcal{A}_1(D) = \Pi D[D_{n+1}+1]$ belongs to $\mathcal{BIP}\left(\mathrm{L}_p(\mathbb{R}^{n+1})\right)$. Here we denote by D_{n+1} the negative Laplacian in $\mathbb{R}^{n+1}$. Lemma 3.1 supplies that the constant coefficient matrix $\Pi D_{|E}$ has only positive eigenvalues, which implies $\Pi D\in\mathcal{BIP}\left(\mathrm{L}_p(\mathbb{R}^{n+1};E)\right)$ with power angle $\theta_{\Pi D} = 0$. Since $D_{n+1}+1$ belongs to $\mathcal{BIP}\left(\mathrm{L}_p(\mathbb{R}^{n+1})\right)$ with power angle $\theta_{D_{n+1}+1} = 0$ we may conclude that A lies in $\mathcal{BIP}\left(\mathrm{L}_p(\mathbb{R}^{n+1};E)\right)$ as well, with power angle $\theta = \theta_{D_{n+1}+1} + \theta_{\Pi D} = 0$. Hence, we obtain a unique solution $w\in Z^T(E) := \{v\in Z^T : z^T\cdot v = 0\}$ of (3.4) given by

$$w(t,x) = (T*\Pi f)(t,x) + T(t)u_0(x),$$

with $T(t) = e^{-\Pi D(D_{n+1}+1)t}$. We stress that this function solves (3.1) since all transformations were equivalent.

Now we want to solve the elliptic problem for ψ. It is clear that

$$-z^T \cdot \mathcal{A}_1(D)w + z^T \cdot f \in \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}^{n+1})) \cap \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^{-1}(\mathbb{R}^{n+1})),$$

where w is given by the above formula. This can be seen by the assumption $f \in \mathcal{X}^T$, which implies $z^T \cdot f \in \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}^{n+1})) \cap \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^{-1}(\mathbb{R}^{n+1}))$, and in the proof of necessity we have established that $z^T \cdot \mathcal{A}_1(D)w$ possesses this regularity as well.

The elliptic equation for ψ reads as follows

$$[1 - \Delta]\psi = a^{-1}\left(-z^T \cdot \mathcal{A}_1(D)w(t,x) + z^T \cdot f(t,x)\right),$$

where $a > 0$ results from the positivity of $\tilde{u}$. The operator $I_s := (1-\Delta)^{s/2}$ with symbol $(1+|\xi|^2)^{s/2}$ possesses the lift property, i.e. I_s is a continuous one-to-one mapping from $H_p^s(\mathbb{R}^{n+1})$ onto $H_p^{s-2}(\mathbb{R}^{n+1})$, for $s \in \mathbb{R}$ and $p \in (1,\infty)$, see [33, Theorem 2.3.4.]. Consequently, the solution of (3.4) belongs to $\mathcal{Z}^T$ and is given by

$$\psi(t,x) = (1-\Delta)^{-1}\left\{a^{-1}\left(-z^T \cdot \mathcal{A}_1(D)w(t,x) + z^T \cdot f(t,x)\right)\right\}.$$

The solution formulae of w and ψ supply the solution operator $\mathcal{S}$. Necessity and sufficiency of the inhomogeneities entail that $\mathcal{S}$ is an isomorphism between space of data and $Z^T \times \mathcal{Z}^T$, i.e.

$$\mathcal{S} \in \mathcal{L}is(\mathcal{X}^T \times V(E), Z^T \times \mathcal{Z}^T).$$

□

3.2 A Half Space Problem

This paragraph is devoted to a parabolic-elliptic problem of second order in half space, where the coefficients are again constant and the differential operator consist only of their main parts. We will deal with

$$\begin{aligned}
&\partial_t w + \mathcal{A}_1(D)w + A_2(D)\psi = f, \quad (t,y) \in J \times \mathbb{R}_+^{n+1},\\
&w(t,y',0) = h(t,y'), \quad \psi(t,y',0) = 0, \quad (t,y') \in J \times \mathbb{R}^n,\\
&w(0,y) = u_0(y), \quad y \in \mathbb{R}_+^{n+1},\\
&z^T \cdot w(t,y) = 0, \quad (t,y) \in J \times \mathbb{R}_+^{n+1}.
\end{aligned} \tag{3.5}$$

Here, we have used the same notations for partial differential operators $\mathcal{A}_1(D)$, $\mathcal{A}_2(D)$ as in Section 3.1. The maximal regularity result of this problem reads as follows.

Theorem 3.2 *Let $J = [0,T]$, $1 < p < \infty$, $p \neq 3/2$, $J = [0,T]$ and assume that d_i, $\tilde{u}_i$ are positive for $i = 1,\dots,N$. Then problem (3.5) has exactly one solution*

$$\begin{aligned}
w &\in Z_+^T := \mathrm{H}_p^1(J; \mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\mathbb{R}_+^{n+1};\mathbb{R}^N)),\\
\psi &\in \mathcal{Z}_{+,0}^T := \mathrm{H}_p^{1/2}(J; \mathring{\mathrm{H}}_p^1(\mathbb{R}_+^{n+1})) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\mathbb{R}_+^{n+1}) \cap \mathring{\mathrm{H}}_p^1(\mathbb{R}_+^{n+1}))
\end{aligned}$$

if and only if the data f, h and u_0 satisfy the following conditions

1. $f \in \mathcal{X}_+^T := \{f \in \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}_+^{n+1}; \mathbb{R}^N)) : \ z^T \cdot f \in \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^{-1}(\mathbb{R}_+^{n+1}))\}$*;*

2. $h \in \mathrm{Y}_1^T(E) := \mathrm{B}_{pp}^{1-1/2p}(J; \mathrm{L}_p(\mathbb{R}^n; E)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{2-1/p}(\mathbb{R}^n; E))$*;*

3. $u_0 \in V_+(E) := \mathrm{B}_{pp}^{2-2/p}(\mathbb{R}_+^{n+1}; E)$*;*

4. $u_{0|_{y_{n+1}=0}} = h_{|t=0} \in \mathrm{B}_{pp}^{2-3/p}(\mathbb{R}^n; E)$ *in case* $p > 3/2$*.*

Furthermore, there exists a solution operator $\mathcal{S}_+$ *of problem (3.5) with*

$$\mathcal{S}_+ \in \mathcal{L}is(\mathcal{X}_+^T \times \{(h, u_0) \in Y_1^T(E) \times V_+(E) :\ h_{|t=0} = u_{0|_{y_{n+1}=0}}\}, Z_+^T \times \mathcal{Z}_{+,0}^T)\,.$$

Proof. We start again with the necessity part. For proving $f \in \mathcal{X}^T$ and $u_0 \in V_+(E)$ we refer to the proof of Theorem 3.1. So it remains to check $h \in Y_1^T(E)$ and the compatibility condition. According to the trace Theorem 2.5 we see the regularity of h and the compatibility condition $h_{|t=0} = u_{0|_{y_{n+1}=0}} \in \mathrm{B}_{pp}^{2-3/p}(\mathbb{R}^n; E)$ whenever $p > 3/2$. Due to the linearity of the boundary condition we conclude $h \in E$.

The sufficiency part. Let the data f, h, u_0 be given. Then, as proceeded in the proof of Theorem 3.1, we replace the electroneutrality condition by the corresponding boundary value problem and apply the projection Π to the parabolic equation. The latter problem has now to be considered in E.

$$\begin{aligned} \partial_t w - \Pi D \partial_{y_{n+1}}^2 w + \Pi D[D_n + 1]w = \Pi f, \quad (t, y) \in J \times \mathbb{R}_+^{n+1}, \\ w(t, y', 0) = h(t, y'), \quad (t, y') \in J \times \mathbb{R}^n, \\ w(0, y) = u_0(y), \quad y \in \mathbb{R}_+^{n+1}. \end{aligned} \tag{3.6}$$

Here D_n denotes the negative Laplacian on $\mathbb{R}^n$. Lemma 3.1 provides that ΠD is invertible with $\sigma(\Pi D) \subset \mathbb{R}_+$ and thus Theorem 2.7 is applicable. We obtain a unique solution of the above problem in $Z_+^T(E)$ with the representation

$$\begin{aligned} w(t, y) = T(t)\left[u_0 - \Xi(y_{n+1}) u_{0|_{y_{n+1}=0}}\right] + [T * \Pi f](t, y) + S(t)\Xi(y_{n+1}) u_{0|_{y_{n+1}=0}} \\ + \Upsilon(y_{n+1})\left[h - S(t) u_{0|_{y_{n+1}=0}}\right], \end{aligned} \tag{3.7}$$

where the definitions of semigroups $T(t)$, $S(t)$ and $\Xi(y)$ can be founded in the proof of Theorem 2.7.

Now, we consider the elliptic problem for ψ which can be reduced to the abstract second order problem

$$\begin{aligned} -\partial_{y_{n+1}}^2 \psi + A^2 \psi = \tilde{f}, \quad y_{n+1} \in \mathbb{R}_+, \\ \psi(0) = 0, \end{aligned}$$

with

$$\tilde{f}(t, y) := a^{-1}\left[z^T \cdot f(t, y) - z^T \cdot \mathcal{A}_1(D) w\right],$$

and $A^2 = D_n + 1$ with domain $D(A^2) = \mathrm{H}_p^2(\mathbb{R}^n)$. Let $R : \mathrm{L}_p(\mathbb{R}_+; X) \to \mathrm{L}_p(\mathbb{R}; X)$ denote the operator of even extension, i.e.

$$(Rf)(y_{n+1}) := \left\{ \begin{array}{rl} f(-y_{n+1}) : & y_{n+1} < 0 \\ f(y_{n+1}) : & y_{n+1} \geq 0 \end{array} \right. ,$$

and $P_+ : \mathrm{L}_p(\mathbb{R}; X) \to \mathrm{L}_p(\mathbb{R}_+; X)$ the restriction to $\mathbb{R}_+$. These operators are bounded. Then, solutions of the elliptic problem can be written as follows

$$
\begin{aligned}
\psi(t,y) = P_+(D_{n+1}+1)^{-1}R\tilde{f} &= P_+\frac{1}{2}A^{-1}\int_{-\infty}^{\infty} e^{-A|y_{n+1}-s|}(R\tilde{f})(t,s)\,ds \\
&= \frac{1}{2}A^{-1}\int_{0}^{\infty}\left[e^{-A|y_{n+1}-s|} - e^{-A(y_{n+1}+s)}\right]\tilde{f}(t,s)\,ds.
\end{aligned}
\tag{3.8}
$$

D_{n+1} denotes the negative Laplacian in $\mathbb{R}^{n+1}$. Now, we can bring forward the same arguments used in the previous section to establish the regularity. In fact, $(D_{n+1}+1)^{-1}$ possesses the lift property, i.e. $(D_{n+1}+1)^{-1}$ is a continuous one-to-one mapping from $H^s_p(\mathbb{R}^{n+1})$ onto $H^{s+2}_p(\mathbb{R}^{n+1})$, for $s \in \mathbb{R}$ and $p \in (1,\infty)$. Consequently, $P_+(D_{n+1}+1)^{-1}R\tilde{f}$ belongs to $\mathcal{Z}^T$ due to the regularity

$$
R\tilde{f} \in \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}^{n+1})) \cap \mathrm{H}^{1/2}_p(J; \mathrm{H}^{-1}_p(\mathbb{R}^{n+1})).
$$

Finally, the solution formulae (3.8) and (3.7) provide a solution operator $\mathcal{S}$. Necessity and sufficiency of the data entail that $\mathcal{S}$ is an continuous one-to-one mapping from

$$
\mathcal{X}^T_+ \times \{(h,u_0) \in Y^T_1(E) \times V_+(E) :\ h_{|t=0} = u_{0|_{y_{n+1}=0}}\}
$$

to $Z^T_+ \times \mathcal{Z}^T_{+,0}$.

□

3.3 A Two Phase Problem

In this section we study a two phase problem in $\mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-$ which arises from the localisation at the boundary Γ_P. Consequently the functions w^k and ψ^k, $k = P, F$ are involved and coupled by the boundary conditions. In the following all coefficients are constant and bear again the meaning as in the linear problem (1.39).

$$
\begin{aligned}
&\partial_t w^P - D^P[\Delta_y - 1]w^P - M^P\tilde{u}^P[\Delta_y - 1]\psi^P = f^P, \quad (t,y) \in J \times \mathbb{R}^{n+1}_+,\\
&\partial_t w^F - D^F[\Delta_y - 1]w^F - M^F\tilde{u}^F[\Delta_y - 1]\psi^F = f^F, \quad (t,y) \in J \times \mathbb{R}^{n+1}_-\\
&-D^P\partial_{y_{n+1}}w^P - M^P\tilde{u}^P\partial_{y_{n+1}}\psi^P = -D^F\partial_{y_{n+1}}w^F - M^F\tilde{u}^F\partial_{y_{n+1}}\psi^F + g, \quad (t,y) \in J \times \mathbb{R}^n \times \{0\},\\
&\tilde{U}_P^{-1}w^P - \tilde{U}_F^{-1}w^F + \lambda_0 z(\psi^P - \psi^F) = h, \quad (t,y) \in J \times \mathbb{R}^n \times \{0\},\\
&w^P(0,y) = u^P_0(y), \quad y \in \mathbb{R}^{n+1}_+, \quad w^F(0,y) = u^F_0(y), \quad y \in \mathbb{R}^{n+1}_-,\\
&z^T\cdot w^P(t,y) = 0, \quad (t,y) \in J \times \mathbb{R}^{n+1}_+, \quad z^T\cdot w^F(t,y) = 0, \quad (t,y) \in J \times \mathbb{R}^{n+1}_-.
\end{aligned}
\tag{3.9}
$$

Take into account that the compatibility condition involves the electrical potential $\psi^k_0 := \psi^k(0)$, $k = P, F$, which have to be determined as well. For this problem the maximal regularity result in $\mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}_+; Y)) \times \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}_-; Y))$, with $Y = \mathrm{L}_p(\mathbb{R}^n; \mathbb{R}^N)$, reads as follows.

Theorem 3.3 *Let $J=[0,T]$ and $1<p<\infty$, $p\neq 3/2$, 3 and assume that d_i, $\tilde{u}_i$ are positive for $i=1,\dots,N$. Then problem (3.9) has precisely one solution*

$$\begin{aligned} w^P \in Z_+^T &:= \mathrm{H}_p^1(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N)) \cap \mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1};\mathbb{R}^N)) \\ w^F \in Z_-^T &:= \mathrm{H}_p^1(J;\mathrm{L}_p(\mathbb{R}_-^{n+1};\mathbb{R}^N)) \cap \mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_-^{n+1};\mathbb{R}^N)), \\ (\psi^P,\psi^F) \in \mathcal{Z}_{+,-}^T &:= \{(\phi^P,\phi^F) \in \mathcal{Z}_+^T \times \mathcal{Z}_-^T :\ \gamma_{|\mathbb{R}^n}(\phi^P-\phi^F) \in Y_1^T\}, \end{aligned}$$

with the spaces $Y_1^T := \mathrm{B}_{pp}^{1-1/2p}(J;\mathrm{L}_p(\mathbb{R}^n)) \cap \mathrm{L}_p(J;\mathrm{B}_{pp}^{2-1/p}(\mathbb{R}^n))\}$ and $\mathcal{Z}_\pm^T := \mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\mathbb{R}_\pm^{n+1}))\cap \mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_\pm^{n+1}))$, if and only if the data f^P, f^F, g, h, u_0^P, u_0^F satisfy the following conditions

1. *$f^P \in \mathcal{X}_+^T := \{f \in \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N)) :\ z^T \cdot f \in \mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\mathbb{R}_+^{n+1}))\}$;*
2. *$f^F \in \mathcal{X}_-^T := \{f \in \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_-^{n+1};\mathbb{R}^N)) :\ z^T \cdot f \in \mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\mathbb{R}_-^{n+1}))\}$;*
3. *$g \in \mathcal{Y}_2^T := \{g \in Y_2^T(\mathbb{R}^N) := \mathrm{B}_{pp}^{1/2-1/2p}(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^N)) \cap \mathrm{L}_p(J;\mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^n;\mathbb{R}^N)) :$ $z^T \cdot g \in \mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\mathbb{R}^n))\}$;*
4. *$h \in Y_1^T(\mathbb{R}^N) := \mathrm{B}_{pp}^{1-1/2p}(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^N)) \cap \mathrm{L}_p(J;\mathrm{B}_{pp}^{2-1/p}(\mathbb{R}^n;\mathbb{R}^N))$;*
5. *$u_0^P \in V_+(E) := \mathrm{B}_{pp}^{2-2/p}(\mathbb{R}_+^{n+1};E)$, $u_0^F \in V_-(E) := \mathrm{B}_{pp}^{2-2/p}(\mathbb{R}_-^{n+1};E)$;*
6. *$\tilde{U}_P^{-1}u_0^P - \tilde{U}_F^{-1}u_0^F + \lambda_0 z(\psi_0^P - \psi_0^F) = h_{|t=0}$ in $\mathrm{B}_{pp}^{2-3/p}(\mathbb{R}^n;\mathbb{R}^N)$ if $p>3/2$;*
7. *$-D^P\partial_{y_{n+1}}u_0^P - M^P\tilde{u}^P\partial_{y_{n+1}}\psi_0^P = -D^F\partial_{y_{n+1}}u_0^F - M^F\tilde{u}^F\partial_{y_{n+1}}\psi_0^F + g_{|t=0}$, in $\mathrm{B}_{pp}^{1-3/p}(\mathbb{R}^n;\mathbb{R}^N)$ if $p>3$, where (ψ_0^P,ψ_0^F) solves the elliptic problem (3.10) after taking trace $t=0$, see proof.*

Furthermore, there exists a solution operator $\mathcal{S}_{+,-}$ of problem (3.9) with

$$\mathcal{S}_{+,-} \in \mathcal{L}is\Big(\mathcal{X}_+^T \times \mathcal{X}_-^T \times \{\, (g,h,u_0^P,u_0^F) \in \mathcal{Y}_2^T \times Y_1^T(\mathbb{R}^N) \times V_+(E) \times V_-(E) \colon (g,h,u_0^P,u_0^F)$$
$$\text{enjoy the compatibility conditions 6. \& 7.}\},\ Z_+^T \times Z_-^T \times \mathcal{Z}_{+,-}^T\Big).$$

Proof. *(a)* We begin with the necessity part. Suppose (w^P,w^F) and (ψ^P,ψ^F) solve (3.9) and belong to $Z_+^T \times Z_-^T \times \mathcal{Z}_{+,-}^T$. The regularity of data f^k, $k=P,F$ can be established as in the proof of Theorem 3.1. The regularities of g, h, u_0 follow by Theorem 2.5, where one has to keep in mind that $\psi^P(t,0)-\psi^F(t,0) \in Y_1^T$ and $\nabla\psi^P \in Z_{+,\nabla}^T$, $\nabla\psi^F \in Z_{-,\nabla}^T$ imply $M^k\tilde{u}^k\partial_{y_{n+1}}\psi^k(t,0) \in Y_2^T(\mathbb{R}^N)$, $k=P,F$. We have still to check the condition $z^T \cdot g \in \mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\mathbb{R}^n))$. Here, we have to study the weak formulation of the boundary value problem obtained by applying $z^T\cdot$ to (3.9). W.l.o.g. we can set $f^k=0$. After multiplying with a test function $v \in \mathring{\mathrm{H}}_{p'}^1(\mathbb{R}^{n+1})$, integrating by parts, making use of transmission condition and accounting for the support of v, we obtain the identity

$$\int_{\mathbb{R}_+^{n+1}} \Big\{ v(y)[a^P\psi^P(t,y) + z^T \cdot D^P w^P(t,y)] + \nabla v(y) \cdot \nabla[a^P\psi^P(t,y) + z^T \cdot D^P w^P(t,y)] \Big\} dy$$
$$+ \int_{\mathbb{R}_-^{n+1}} \Big\{ v(y)[a^F\psi^F(t,y) + z^T \cdot D^F w^F(t,y)] + \nabla v(y) \cdot \nabla[a^F\psi^F(t,y) + z^T \cdot D^F w^F(t,y)] \Big\} dy =$$
$$\int_{\mathbb{R}^n} v(y',0)\, z^T \cdot g(t,y')\, dy'$$

for all $v \in \overset{\circ}{\mathrm{H}}{}^1_{p'}(\mathbb{R}^{n+1})$ and $t \in J$. With a view to establishing the time regularity $\mathrm{H}^{1/2}_p(J)$ of $z^T\!\cdot g$ we shall consider differences of the above equation respect to the time variable. Moreover, the left hand side of these differences will be estimated by using Hölder's inequality, and we obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \tilde{v}(y')[z^T\!\cdot g(t+h,y') - z^T\!\cdot g(t,y')]\,dy' &\le C\|v\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\mathbb{R}^{n+1})} \\ &\cdot \Big\{ \|(\psi^P,\psi^F)(t+h) - (\psi^P,\psi^F)(t)\|_{\mathrm{H}^1_p(\mathbb{R}^{n+1}_+)\times \mathrm{H}^1_p(\mathbb{R}^{n+1}_-)} \\ &\quad + \|(w^P,w^F)(t+h) - (w^P,w^F)(t)\|_{\mathrm{H}^1_p(\mathbb{R}^{n+1}_+;\mathbb{R}^N)\times \mathrm{H}^1_p(\mathbb{R}^{n+1}_-;\mathbb{R}^N)} \Big\},\end{aligned}$$

with $v_{|\mathbb{R}^n} = \tilde{v}$ and $t+h,\ t \in J$. By taking the infimum and using

$$\inf\{\|v\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\mathbb{R}^{n+1})} : v_{|\mathbb{R}^n} = \tilde{v}\} \le C\|\tilde{v}\|_{\mathrm{B}^{1-1/p'}_{p'p'}(\mathbb{R}^n)}$$

we get

$$\begin{aligned}\int\limits_{\mathbb{R}^n} \tilde{v}(y')[z^T\!\cdot g(t+h,y') - z^T\!\cdot g(t,y')]\,dy' &\le C\|\tilde{v}\|_{\mathrm{B}^{1-1/p'}_{p'p'}(\mathbb{R}^n)} \\ &\cdot \Big\{ \|(\psi^P,\psi^F)(t+h) - (\psi^P,\psi^F)(t)\|_{\mathrm{H}^1_p(\mathbb{R}^{n+1}_+)\times \mathrm{H}^1_p(\mathbb{R}^{n+1}_-)} \\ &\quad + \|(w^P,w^F)(t+h) - (w^P,w^F)(t)\|_{\mathrm{H}^1_p(\mathbb{R}^{n+1}_+;\mathbb{R}^N)\times \mathrm{H}^1_p(\mathbb{R}^{n+1}_-;\mathbb{R}^N)} \Big\},\end{aligned}$$

for all $\tilde{v} \in \mathrm{B}^{1-1/p'}_{p'p'}(\mathbb{R}^n)$, which means $z^T\!\cdot g(t+h) - z^T\!\cdot g(t) \in \mathrm{B}^{-1/p}_{pp}(\mathbb{R}^n)$ due to duality. In the end, after employing the norm of $\mathrm{H}^{1/2}_p(J)$ being defined by means of differences we gain the estimate

$$\begin{aligned}\Big\| \sup_{\substack{\tilde{v}\in \mathrm{B}^{1-1/p'}_{p'p'}(\mathbb{R}^n) \\ \|\tilde{v}\|\le 1}} \int\limits_{\mathbb{R}^n} \tilde{v}(y') z^T\!\cdot g(\cdot,y')dy' \Big\|_{\mathrm{H}^{1/2}_p(J)} &\equiv \|z^T\!\cdot g\|_{\mathrm{H}^{1/2}_p(J;\mathrm{B}^{-1/p}_{pp}(\mathbb{R}^n))} \\ &\le C\Big\{ \|(\psi^P,\psi^F)\|_{\mathrm{H}^{1/2}_p(J;\mathrm{H}^1_p(\mathbb{R}^{n+1}_+)\times \mathrm{H}^1_p(\mathbb{R}^{n+1}_-))} \\ &\quad + \|(w^P,w^F)\|_{\mathrm{H}^{1/2}_p(J;\mathrm{H}^1_p(\mathbb{R}^{n+1}_+;\mathbb{R}^N)\times \mathrm{H}^1_p(\mathbb{R}^{n+1}_-;\mathbb{R}^N))} \Big\}.\end{aligned}$$

Last but not least, the compatibility conditions follow from the regularity assumptions and the embeddings

$$Y^T_1(\mathbb{R}^N) \hookrightarrow \mathrm{C}(J;\mathrm{B}^{2-3/p}_{pp}(\mathbb{R}^n;\mathbb{R}^N)), \quad p > 3/2, \quad Y^T_2(\mathbb{R}^N) \hookrightarrow \mathrm{C}(J;\mathrm{B}^{1-3/p}_{pp}(\mathbb{R}^n;\mathbb{R}^N)), \quad p > 3.$$

(b) We come now to the sufficiency part. The first task consists in solving the elliptic problem for (ψ^P,ψ^F). Furthermore, we are interested in finding a solution formula which

provides an insight into the transmission condition, i.e. makes it possible to compute the normal derivatives of the potentials. Assume that w^k are already known.

By applying $z^T\cdot$ to all equations of (3.9) and using the electroneutrality we obtain a two phase boundary value problem, which reads as follows

$$\begin{aligned}
&-a^P(\partial^2_{y_{n+1}} - (D_n+1))\psi^P - z^T\cdot D^P(\partial^2_{y_{n+1}} - (D_n+1))w^P = z^T\cdot f^P\,, \quad (t,y) \in J\times\mathbb{R}^{n+1}_+\\
&-a^F(\partial^2_{y_{n+1}} - (D_n+1))\psi^F - z^T\cdot D^F(\partial^2_{y_{n+1}} - (D_n+1))w^F = z^T\cdot f^F\,, \quad (t,y) \in J\times\mathbb{R}^{n+1}_-\\
&z^T\cdot D^P\partial_{y_{n+1}}w^P + a^P\partial_{y_{n+1}}\psi^P = z^T\cdot D^F\partial_{y_{n+1}}w^F + a^F\partial_{y_{n+1}}\psi^F - z^T\cdot g\,, \quad (t,y)\in J\times\mathbb{R}^n\times\{0\}\\
&\psi^P - \psi^F = \frac{1}{\lambda_0|z|^2}\left[z^T\cdot\tilde{U}_F^{-1}w^F - z^T\cdot\tilde{U}_P^{-1}w^P\right] + \frac{1}{\lambda_0|z|^2}z^T\cdot h\,, \quad (t,y)\in J\times\mathbb{R}^n\times\{0\}\,.
\end{aligned} \tag{3.10}$$

Here we used again the notation $a^k := z^T\cdot M^k\tilde{u}^k$ for $k=P,F$. The operator D_n denotes the negative Laplacian in $\mathbb{R}^n$. For solving the ordinary differential equations concerning the variable y_{n+1}, we introduce the auxiliary function $\rho^k := a^k\psi^k + z^T\cdot D^k w^k$. Hence, the differential equations and transmission condition takes the form

$$\begin{aligned}
-\partial^2_{y_{n+1}}\rho^P + (D_n+1)\rho^P &= z^T\cdot f^P\,, \quad (t,y_{n+1})\in J\times\mathbb{R}_+\\
-\partial^2_{y_{n+1}}\rho^F + (D_n+1)\rho^F &= z^T\cdot f^F\,, \quad (t,y_{n+1})\in J\times\mathbb{R}_-\\
\partial_{y_{n+1}}\rho^P &= \partial_{y_{n+1}}\rho^F - z^T\cdot g\,, \quad (t,y_{n+1})\in J\times\{0\}.
\end{aligned}$$

Solutions of this system are given by

$$\begin{aligned}
a^P\psi^P(t,y) + z^T\cdot D^P w^P(t,y) &= \frac{1}{2}(D_n+1)^{-1/2}\int_0^\infty [T(|y_{n+1}-s|) + T(y_{n+1}+s)]\, z^T\cdot f^P(t,y',s)ds\\
&\quad + T(y_{n+1})\chi^P(t,y'), \quad (t,y)\in J\times\mathbb{R}^{n+1}_+\\
a^F\psi^F(t,y) + z^T\cdot D^F w^F(t,y) &= \frac{1}{2}(D_n+1)^{-1/2}\int_{-\infty}^0 [T(|y_{n+1}-s|) + T(-y_{n+1}-s)]\, z^T\cdot f^F(t,y',s)ds\\
&\quad + T(-y_{n+1})\chi^F(t,y'), \quad (t,y)\in J\times\mathbb{R}^{n+1}_-\,,
\end{aligned} \tag{3.11}$$

where the auxiliary functions ρ^P, ρ^F were replaced. Further on, $T(y_{n+1})$, $y_{n+1}\geq 0$ denotes again the bounded analytic C_0-semigroup generated by $-(D_n+1)^{1/2}$. The unknown functions χ^P and χ^F are determined by transmission condition and boundary condition of Dirichlet type. Due to the transmission condition $\partial_{y_{n+1}}\rho^P = \partial_{y_{n+1}}\rho^F - z^T\cdot g$ we obtain the first relation

$$\chi^P(t,y') + \chi^F(t,y') = (D_n+1)^{-1/2}z^T\cdot g(t,y').$$

The second equation for χ^P, χ^F is caused by the boundary condition of Dirichlet type. At first, observe that

$$\begin{aligned}
\psi^P(t,y',0) - \psi^F(t,y',0) &= -\frac{z^T\cdot D^P w^P(t,y',0)}{a^P} + \frac{z^T\cdot D^F w^F(t,y',0)}{a^F} + \frac{\chi^P(t,y')}{a^P} - \frac{\chi^F(t,y')}{a^F}\\
&+ \frac{1}{a^P}(D_n+1)^{-1/2}\int_0^\infty T(s)z^T\cdot f^P(t,y',s)ds - \frac{1}{a^F}(D_n+1)^{-1/2}\int_{-\infty}^0 T(-s)z^T\cdot f^F(t,y',s)ds,
\end{aligned}$$

and substituting the boundary condition in the left hand side gives the second equation for χ^P, χ^F. Solving this linear system of equations yields

$$\begin{aligned}
\chi^P(t,y') &= \frac{a^P}{a^P+a^F}(D_n+1)^{-1/2} z^T\!\cdot g(t,y') + \frac{a^P a^F}{a^P+a^F}\,\chi(t,y')\,, \\
\chi^F(t,y') &= \frac{a^F}{a^P+a^F}(D_n+1)^{-1/2} z^T\!\cdot g(t,y') - \frac{a^P a^F}{a^P+a^F}\,\chi(t,y')\,,
\end{aligned}$$

with

$$\begin{aligned}
\chi(t,y') := &\left[\frac{z^T\!\cdot D^P w^P(t,y',0)}{a^P} - \frac{z^T\!\cdot D^F w^F(t,y',0)}{a^F}\right] \\
&+ \frac{1}{\lambda_0|z|^2}\left[z^T\!\cdot \tilde{U}_F^{-1} w^F(t,y',0) - z^T\!\cdot \tilde{U}_P^{-1} w^P(t,y',0)\right] + \frac{1}{\lambda_0|z|^2} z^T\!\cdot h(t,y') \\
&+ (D_n+1)^{-1/2}\left[\frac{1}{a^F}\int_{-\infty}^{0} T(-s) z^T\!\cdot f^F(t,y',s)\,ds - \frac{1}{a^P}\int_0^{\infty} T(s) z^T\!\cdot f^P(t,y',s)\,ds\right].
\end{aligned}$$

Note that by formula (3.11) we can compute (ψ_0^P,ψ_0^F) after taking trace $t=0$. Furthermore, we perceive that (ψ_0^P,ψ_0^F) is completely determined by the initial data $u_0^k \in \mathrm{B}_{pp}^{2-3/p}(\mathbb{R}^{n+1})$ and inhomogeneities $z^T\!\cdot f^k(0) \in \mathrm{H}_p^{-1}(\mathbb{R}_\pm^{n+1})$, $z^T\!\cdot g(0) \in \mathrm{B}_{pp}^{-1/p}(\mathbb{R}^n) \cap \mathrm{B}_{pp}^{1-3/p}(\mathbb{R}^n)$, and $z^T\!\cdot h(0) \in \mathrm{B}_{pp}^{2-3/p}(\mathbb{R}^n)$ whenever traces exist.

Now, we want to verify the regularity stated in the theorem. This will be carried out by means of the solution formula of ψ^P. Let $R : \mathrm{L}_p(\mathbb{R}_+;X) \to \mathrm{L}_p(\mathbb{R};X)$ denote the operator of antisymmetric extension at 0, i.e.

$$(Rf)(y) := \begin{cases} f(y): & y \geq 0 \\ -f(-y): & y < 0 \end{cases},$$

and $P_+ : \mathrm{L}_p(\mathbb{R};X) \to \mathrm{L}_p(\mathbb{R}_+;X)$ the restriction to $\mathbb{R}_+$. These operators are bounded. Furthermore, D_{n+1} denotes again the negative Laplacian in $\mathbb{R}^{n+1}$ and set $A := (D_n+1)^{1/2}$. Let $\varrho(t,y)$ denote the first part of the solution formula, i.e. we put

$$\varrho(t,y) := \frac{1}{2}A^{-1}\int_0^{\infty}\left[T(|y_{n+1}-s|) + T(y_{n+1}+s)\right] z^T\!\cdot f^P(t,y',s)ds.$$

Then $\varrho(t,y)$ is equal to

$$P_+(D_{n+1}+1)^{-1}(R\,z^T\!\cdot f^P)(t,y) = P_+\frac{1}{2}A^{-1}\int_{-\infty}^{\infty} e^{-A|y_{n+1}-s|}(R\,z^T\!\cdot f^P)(t,y',s)\,ds,$$

and solves the problem

$$\begin{aligned}
-\partial_{y_{n+1}}^2\varrho + A^2\varrho &= z^T\!\cdot f^P, \quad y_{n+1} > 0, \\
-\partial_{y_{n+1}}\varrho(0) &= 0.
\end{aligned}$$

Here, we see again that $\varrho \in \mathcal{Z}_+^T$ due to the lift property of $(D_{n+1}+1)^{-1}$ and $Rz^T\!\cdot f^P \in \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1})) \cap \mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\mathbb{R}^{n+1}))$.

The next term we want to discuss is $-z^T \cdot D^P w^P$. At first glance, we perceive that this function enjoy the regularity in view of $w^P \in Z_+^T$. Finally, we want to study the function $T(y_{n+1})\chi^P$. Since the semigroup $T(y_{n+1})$ is a continuous mapping from $\mathrm{B}_{pp}^{s-1/p}(\mathbb{R}^n)$ to $\mathrm{H}_p^s(\mathbb{R}_+^{n+1})$ for $s > 1/p$ and $1 < p < \infty$, it remains to check that χ^k belongs to $Y_3^T := \mathrm{H}_p^{1/2}(J; \mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{2-1/p}(\mathbb{R}^n))$ in order to conclude $T(y_{n+1})\chi^k \in \mathcal{Z}_\pm$. The function χ^k comprises the terms $A^{-1}z^T \cdot g$ and χ, except for certain constants. Due to the regularity assumptions of g we know that $z^T \cdot g \in \mathrm{H}_p^{1/2}(J; \mathrm{B}_{pp}^{-1/p}(\mathbb{R}^n)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^n))$ which entails that $A^{-1}z^T \cdot g \in Y_3^T$. To verify that χ lies in Y_3^T we first note that $z^T \cdot D^k w^k(t, y', 0)$ and $z^T \cdot h$ belong to Y_1^T. Hence, there are continuous extension of these function which belong to Z. Using the embedding $Z \hookrightarrow \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^1(\mathbb{R}_+^{n+1})) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\mathbb{R}_+^{n+1}))$ and continuity of the trace operator we see that these functions lie in Y_3^T. As a result of the above considerations we deduce that the integrals appearing in the definition of χ lie in Y_3^T. Applying the semigroup $T(y_{n+1})$ yields the assertion.

Hence, we have established that (ψ^P, ψ^F) belongs to $\mathcal{Z}_{+,-}^T := \{(\phi^P, \phi^F) \in \mathcal{Z}_+^T \times \mathcal{Z}_-^T : \gamma_{|\mathbb{R}^n}(\phi^P - \phi^F) \in Y_1^T\}$, where the claim $\gamma_{|\mathbb{R}^n}(\psi^P - \psi^F) \in Y_1^T$ follows by the regularity of data of boundary condition.

(c) Now, we determine $M^k \tilde{u}^k \partial_{y_{n+1}} \psi^k(t, y', 0) \equiv a^k b^k \partial_{y_{n+1}} \psi^k(t, y', 0)$. By using the solution formula (3.11) we compute

$$\begin{aligned} a^P b^P \partial_{y_{n+1}} \psi^P - a^F b^F \partial_{y_{n+1}} \psi^F = & - Q^P D^P \partial_{y_{n+1}} w^P + Q^F D^F \partial_{y_{n+1}} w^F \\ & - (D_n + 1)^{1/2} \left(b^P \chi^P + b^F \chi^F \right), \end{aligned}$$

with

$$\begin{aligned} -(D_n + 1)^{1/2} \left(b^P \chi^P + b^F \chi^F \right) = & - \frac{1}{a^P + a^F} \left(a^P Q^P + a^F Q^F \right) g \\ & - \frac{a^P a^F}{a^P + a^F} (b^P - b^F)(D_n + 1)^{1/2} \chi . \end{aligned}$$

Using these identities the transmission condition takes the form

$$\begin{aligned} (\Pi^F D^F)\partial_{y_{n+1}} w^F - (\Pi^P D^P)\partial_{y_{n+1}} w^P &= g - \frac{1}{a^P + a^F} \left(a^P Q^P + a^F Q^F \right) g \\ & \quad - \frac{a^P a^F}{a^P + a^F} (b^P - b^F)(D_n + 1)^{1/2} \chi \\ &= \Pi_1 g - \frac{a^P a^F}{a^P + a^F} (b^P - b^F)(D_n + 1)^{1/2} \chi, \end{aligned}$$

with

$$\Pi_1 := I - \frac{1}{a^P + a^F} \left(a^P Q^P + a^F Q^F \right) = \tfrac{1}{a^P + a^F} \left(a^P \Pi^P + a^F \Pi^F \right).$$

We investigate the last term containing χ. It turns out that this function produces expressions containing w^k which have to be worked into the left-hand side of the transmission condition. Before simplifying the function χ we want to derive some useful equations. Firstly, we consider the jump boundary condition. After applying $\Pi_0 := I - \frac{z \otimes z}{|z|^2}$ to this equation we obtain

$$\tilde{U}_P^{-1} w^P = \tilde{U}_F^{-1} w^F - \frac{z}{|z|^2} \left(z^T \cdot \tilde{U}_F^{-1} w^F - z^T \cdot \tilde{U}_P^{-1} w^P \right) + \Pi_0 h.$$

By taking the inner product of the above equation with $\lambda_0^{-1}(b^P)^T$ we arrive at

$$\begin{aligned}\lambda_0^{-1}(b^P)^T\cdot\tilde{U}_P^{-1}w^P &= \lambda_0^{-1}(b^P)^T\cdot\tilde{U}_F^{-1}w^F - \frac{1}{\lambda_0|z|^2}\left(z^T\cdot\tilde{U}_F^{-1}w^F - z^T\cdot\tilde{U}_P^{-1}w^P\right)\\ &\quad + \lambda_0^{-1}(b^P)^T\cdot\Pi_0 h,\end{aligned} \tag{3.12}$$

where we have used $(b^P)^T\cdot z = 1$. Evaluation of the left hand side gives

$$\lambda_0^{-1}(b^P)^T\cdot\tilde{U}_P^{-1}w^P = \frac{\lambda_0^{-1}\sum_i b_i^P(\tilde{u}_i^P)^{-1}w_i^P}{a^P} = \frac{\sum_i z_i d_i^P\tilde{u}_i^P(\tilde{u}_i^P)^{-1}w_i^P}{a^P} = \frac{z^T\cdot D^P w^P}{a^P}. \tag{3.13}$$

Using this identity and

$$\lambda_0^{-1}(b^P)^T\cdot\Pi_0 h = \lambda_0^{-1}(b^P)^T\cdot h - \frac{(b^P)^T\cdot z}{\lambda_0|z|^2}z^T\cdot h = \lambda_0^{-1}(b^P)^T\cdot h - \frac{z^T\cdot h}{\lambda_0|z|^2},$$

we derive from (3.12) the equation

$$\begin{aligned}\frac{z^T\cdot D^P w^P}{a^P} + \frac{1}{\lambda_0|z|^2}\left(z^T\cdot\tilde{U}_F^{-1}w^F - z^T\cdot\tilde{U}_P^{-1}w^P\right) + \frac{z^T\cdot h}{\lambda_0|z|^2} &= \lambda_0^{-1}(b^P)^T\cdot\tilde{U}_F^{-1}w^F +\\ &\quad \lambda_0^{-1}(b^P)^T\cdot h.\end{aligned} \tag{3.14}$$

The left hand side appears in the function χ, so that substituting the above equation in χ results in

$$\begin{aligned}\chi(t,y') &= \frac{1}{\lambda_0}\left[b^P - b^F\right]^T\cdot\tilde{U}_F^{-1}w^F(t,y',0) + (D_n+1)^{-1/2}\frac{1}{a^F}\int_{-\infty}^{0} T(-s)z^T\cdot f^F(t,y',s)\,ds\\ &\quad - (D_n+1)^{-1/2}\frac{1}{a^P}\int_0^\infty T(s)z^T\cdot f^P(t,y',s)\,ds + \frac{1}{\lambda_0}\left(b^P\right)^T\cdot h(t,y'),\end{aligned}$$

where we have used once more the identity $z^T\cdot D^F w^F/a^F = \lambda_0^{-1}(b^F)^T\cdot\tilde{U}_F^{-1}w^F$, cp. (3.13). Now, we shall introduce some new matrices in order to attain a more convenient form of the transmission condition. We define

$$\begin{aligned}&\Pi := \tfrac{1}{\lambda_0}\tfrac{a^P a^F}{a^P+a^F}\left(b^P - b^F\right)\otimes\left(b^P - b^F\right), \quad \Pi_2 := \tfrac{1}{\lambda_0|z|^2}\tfrac{a^P a^F}{a^P+a^F}\left(b^P - b^F\right)\otimes b^P,\\ &\Pi_3 := \tfrac{a^P a^F}{a^P+a^F}(b^P - b^F)\otimes z.\end{aligned}$$

All these matrices are projections from $\mathbb{R}^N$ to the space of electroneutrality E, where $E\subset\ker(\Pi_j)$ for $j = 1,2,3$ and Π is symmetric. Together with the above definitions and the representation of χ we obtain

$$\begin{aligned}-(D_n+1)^{1/2}\frac{a^P a^F}{a^P+a^F}\left(b^P - b^F\right)\chi(t,y') &= -(D_n+1)^{1/2}\Pi\tilde{U}_F^{-1}w^F(t,y',0) - (D_n+1)^{1/2}\Pi_2 h(t,y')\\ &\quad - \Pi_3\left[\frac{1}{a^F}\int_{-\infty}^{0} T(-s)f^F(t,y',s)\,ds - \frac{1}{a^P}\int_0^\infty T(s)f^P(t,y',s)\,ds\right].\end{aligned}$$

Hence, the transmission condition takes the form

$$(\Pi^F D^F)\partial_{y_{n+1}} w^F + (D_n+1)^{1/2}(\Pi\tilde{U}_F^{-1})w^F - (\Pi^P D^P)\partial_{y_{n+1}} w^P = \overline{g}\,. \tag{3.15}$$

Of course, this equation has to be considered in E, since the boundary value problem (3.10) is involved here. Furthermore, the new inhomogeneity $\overline{g}$ defined by means of

$$\overline{g} = \Pi_1 g - (D_n+1)^{1/2}\Pi_2\, h - \Pi_3\left[\frac{1}{a^F}\int_{-\infty}^{0} T(-s)f^F(s)\,ds - \frac{1}{a^P}\int_{0}^{\infty} T(s)f^P(s)\,ds\right] \tag{3.16}$$

contains the data f^P, f^F, g and h. Now, we want to verify that $\overline{g}$ belongs to $Y_2^T(E) := \{g \in Y_2^T(\mathbb{R}^N) : z^T\!\cdot g = 0\}$ as well. We immediately detect that each term of $\overline{g}$ lies in E due to the projections Π_i, $i = 1,2,3$. Concerning regularity we point out that the first two terms belong to $Y_2^T(E)$ in view of the postulated regularity of g and h. To reveal the regularity of both integrals, we will revert to the results obtained by proving the regularity of (ψ^P, ψ^F). There we have shown

$$T(y_{n+1})A^{-1}\int_0^\infty T(s)z^T\!\cdot f^P\,ds \in \mathcal{Z}_+^T,$$

which implies the assertion by taking into account that $\Pi_3\eta = \frac{a^P a^F}{a^P + a^F}(b^P - b^F)z^T\!\cdot\eta$, for $\eta \in \mathbb{R}^N$. Finally, since all calculations above were equivalent we may replace the transmission condition by (3.15).

(d) Now, we turn our attention to problem (3.9). The purpose is to find an equivalent problem in E such that all terms containing the potentials ψ^P, ψ^F are eliminated. The first step has been performed resulting in a new boundary condition. In order to eliminate top order terms of ψ^k arising in partial differential equations, we again employ the projections Π^P and Π^F. To remove potentials in boundary condition of Dirichlet type, we use the projection Π_0. After taking all actions we achieve a model problem for (w^P, w^F).

$$\begin{aligned}
&\partial_t w^P - \Pi^P D^P \partial^2_{y_{n+1}} w^P + \Pi^P D^P (D_n+1) w^P = \Pi^P f^P, \quad (t,y) \in J \times \mathbb{R}^{n+1}_+,\\
&\partial_t w^F - \Pi^F D^F \partial^2_{y_{n+1}} w^F + \Pi^F D^F (D_n+1) w^F = \Pi^F f^F, \quad (t,y) \in J \times \mathbb{R}^{n+1}_-,\\
&\Pi^F D^F \partial_{y_{n+1}} w^F + \Pi\tilde{U}_F^{-1}(D_n+1)^{1/2} w^F - \Pi^P D^P \partial_{y_{n+1}} w^P = \overline{g}, \quad (t,y) \in J \times \mathbb{R}^n \times \{0\}\\
&\Pi_0 \tilde{U}_P^{-1} u^P - \Pi_0 \tilde{U}_F^{-1} u^F = \Pi_0 h, \quad (t,y) \in J \times \mathbb{R}^n \times \{0\},\\
&w^P(0,y) = u_0^P(y), \quad y \in \mathbb{R}^{n+1}_+,\\
&w^F(0,y) = u_0^F(y), \quad y \in \mathbb{R}^{n+1}_-.
\end{aligned} \tag{3.17}$$

The goal is to show maximal regularity of this problem, i.e. we seek solutions (w^P, w^F) in the maximal regularity class $Z_+^T(E) \times Z_-^T(E)$, where the electroneutrality condition is incorporated into the space. We have seen in the proof of Lemma 3.1 that $\Pi_k D_k$ is related to a symmetric matrix, which can be obtained by multiplying $\Pi_k D_k$ from left with $\tilde{U}_k^{-1/2}$ and from right with $\tilde{U}_k^{1/2}$. This fact will be decisive, now subsequently.

First we consider (3.17) with $\Pi^P f^P = \Pi^F f^F = 0$, vanishing initial data and inhomogeneities $\Pi_0 h$, $\overline{g}$ satisfying $\Pi_0 h(0,y') = \overline{g}(0,y') = 0$. This is the most important step of the proof.

$$\begin{aligned}
&\partial_t w^P - \Pi^P D^P \partial^2_{y_{n+1}} w^P + \Pi^P D^P (D_n+1) w^P = 0, \quad t>0, \quad y \in \mathbb{R}^{n+1}_+,\\
&\partial_t w^F - \Pi^F D^F \partial^2_{y_{n+1}} w^F + \Pi^F D^F (D_n+1) w^F = 0, \quad t>0, \quad y \in \mathbb{R}^{n+1}_-,\\
&\Pi^F D^F \partial_{y_{n+1}} w^F + (D_n+1)^{1/2} \Pi \tilde{U}_F^{-1} w^F - \Pi^P D^P \partial_{y_{n+1}} w^F = \overline{g}, \quad t>0, \quad y \in \mathbb{R}^n \times \{0\},\\
&\Pi_0 \tilde{U}_P^{-1} w^P - \Pi_0 \tilde{U}_F^{-1} w^F = \Pi_0 h, \quad t>0, \quad y \in \mathbb{R}^n \times \{0\}\\
&w^P(0,y) = 0\,, \quad y \in \mathbb{R}^{n+1}_+\,, \quad w^F(0,y) = 0\,, \quad y \in \mathbb{R}^{n+1}_-\,.
\end{aligned} \tag{3.18}$$

By this we define $B_k := \Pi^k D^k (D_n+1)$ in $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n;E))$ by pointwise extension. Then B_k is invertible, sectorial, and belongs to $\mathcal{BIP}\,(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n;E)))$ with power angle $\theta_{B_k} = \theta_{\Pi^k D^k} = 0$, see Lemma 3.1. Let $G := \partial_t$ with domain ${}_0\mathrm{H}^1_p(J;\mathrm{L}_p(\mathbb{R}^n;E))$; then G is also sectorial, belongs to $\mathcal{BIP}\,(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n;E)))$ with power angle $\theta_G \le \pi/2$. By Theorem 2.2, $G + B_k$ with domain $D(G) \cap D(B_k)$ is invertible, sectorial, and belongs to $\mathcal{BIP}\,(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n;E)))$, with power angle $\theta < \pi/2$. Define $F_k := (\Pi_k D_k)^{-1/2}\sqrt{G+B_k} = \sqrt{(\Pi^k D^k)^{-1} G + (D_n+1)}$ with domain $D(F_k) = {}_0\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^n;E)) \cap \mathrm{L}_p(J;\mathrm{H}^1_p(\mathbb{R}^n;E))$, by Corollary 2.1, then solutions of (3.18) take the form

$$w^P(t,y) = e^{-F_P y_{n+1}} c^P, \quad w^F(t,y) = e^{+F_F y_{n+1}} c^F.$$

Using both boundary conditions to determine c^P and c^F, we get a linear system of equations

$$\begin{aligned}
F_P \Pi^P D^P c^P + F_F \Pi^F D^F c^F + (D_n+1)^{1/2} \Pi \tilde{U}_F^{-1} c^F &= \overline{g}\,,\\
\Pi_0 \tilde{U}_P^{-1} c^P - \Pi_0 \tilde{U}_F^{-1} c^F &= \Pi_0 h\,.
\end{aligned}$$

From the second equation we want to derive a new equation. By using $\Pi_0 = I - \frac{z\otimes z}{|z|^2}$ and the fact that $\tilde{U}_P z \in \mathbb{R}^N$ lies in $\ker(\Pi^P D^P)$, thus $\Pi^P D^P \tilde{U}_P z = 0$, we obtain an equivalent formulation of the second equation.

$$\Pi^P D^P c^P = \Pi^P D^P \tilde{U}_P \tilde{U}_F^{-1} c^F + \Pi^P D^P \tilde{U}_P \Pi_0 h.$$

Substituting $\Pi^P D^P c^P$ in the first equation yields

$$\left[\sum_{k=P,F} F_k \Pi^k D^k \tilde{U}_k + (D_n+1)^{1/2}\Pi\right] \tilde{U}_F^{-1} c^F \;=\; \overline{g}(t) - F_P \Pi^P D^P \tilde{U}_P^{-1} \Pi_0 h(t). \tag{3.19}$$

Looking at the right hand side we realise that this equation has to be considered in $Y_2(E)$. In fact, the matrix $\Pi^P D^P \tilde{U}_1^{-1}$ leaves E invariant and does not change the regularity, whereas F_P maps $D_{F_P}(2-1/p,p) = Y_1$ to $D_{F_P}(1-1/p,p) = Y_2$. Consequently, we have $\overline{g} - F_P \Pi^P D^P \tilde{U}_P^{-1} \Pi_0 h \in Y_2$ due to the regularity assumptions of the data $\overline{g}$ and $\Pi_0 h$.

Now, we consider the operator L defined by

$$L(G,D_n) := \sum_{k=P,F} F_k(G,D_n) \Pi^k D^k \tilde{U}_k + (D_n+1)^{1/2}\Pi,$$

which coincides exactly with the operator we find in the brackets of equation (3.19). We will see that L satisfies a lower estimate. For this purpose we look at the symbol $l(\lambda,\xi)$ of L. Taking the n-dimensional Fourier transform in y' and the Laplace transform in t we then obtain

$$l(\lambda,\xi) = \sum_{k=P,F} \Big(\lambda + (\Pi^k D^k)(|\xi|^2+1)\Big)^{1/2} (\Pi^k D^k)^{1/2} \tilde{U}_k + (|\xi|^2+1)^{1/2}\Pi\,. \tag{3.20}$$

We perceive that $l(\lambda,\xi)$ belongs to $\mathcal{B}(\tilde{U}_F^{-1}E, E)$ due to the projections Π and Π^k. To achieve a formulation in which domain and range of $l(\lambda,\xi)$ are equal, we multiply (3.19) with $\tilde{U}_F^{-1/2}$ from left and consider $\tilde{U}_F^{-1/2} l(\lambda,\xi)\tilde{U}_F^{-1/2}$ which now belongs to $\mathcal{B}(\tilde{E})$, with $\tilde{E} = \tilde{U}_F^{-1/2}E$. Thus, let $\eta \in \tilde{E}$ be given with $\|\eta\| = 1$. Having in mind the fact that Π is symmetric and nonnegative, we may proceed as follows

$$\begin{aligned}
\|\tilde{U}_F^{-1/2} l(\lambda,\xi)\tilde{U}_F^{-1/2}\eta\|_{\tilde{E}} &\geq Re\left\langle \eta, \tilde{U}_F^{-1/2} l(\lambda,\xi)\tilde{U}_F^{-1/2}\eta\right\rangle_{\tilde{E}} \\
&= \sum_{k=P,F} Re\left\langle \eta, \tilde{U}_F^{-1/2}\Big(\lambda + (|\xi|^2+1)(\Pi^k D^k)\Big)^{1/2}(\Pi^k D^k)^{1/2}\tilde{U}_k\tilde{U}_F^{-1/2}\eta\right\rangle_{\tilde{E}} \\
&\quad + Re\left\langle \eta, \tilde{U}_F^{-1/2}(|\xi|+1)^{1/2}\Pi\tilde{U}_F^{-1/2}\eta\right\rangle_{\tilde{E}} \\
&\geq \sum_{k=P,F} Re\left\langle \eta, \tilde{U}_F^{-1/2}\Big(\lambda + (|\xi|^2+1)(\Pi^k D^k)\Big)^{1/2}(\Pi^k D^k)^{1/2}\tilde{U}_k\tilde{U}_F^{-1/2}\eta\right\rangle_{\tilde{E}}.
\end{aligned}$$

In the next step we want to show that both summands are bounded below in $\tilde{E}$. For this purpose we rewrite as

$$\begin{aligned}
\Big(\lambda + (|\xi|^2+1)(\Pi^k D^k)\Big)^{1/2} (\Pi_k D_k)^{1/2}\tilde{U}_k = \tilde{U}_k^{1/2}\Big(&\tilde{U}_k^{-1/2}\left(\lambda + (|\xi|^2+1)(\Pi_k D_k)\right)^{1/2}\tilde{U}_k^{1/2} \\
&\tilde{U}_k^{-1/2}(\Pi^k D^k)^{1/2}\tilde{U}_k^{1/2}\Big)\,\tilde{U}_k^{1/2}.
\end{aligned}$$

After defining $S_k := \tilde{U}_k^{-1/2}(\Pi^k D^k)\tilde{U}_k^{1/2}$ and keeping in mind that S_k is selfadjoint, positive definite on $\tilde{E}$ by Lemma 3.1, we obtain by using the spectral mapping theorem for normal operators

$$\begin{aligned}
\sum_{k=P,F} Re\,&\left\langle \eta, \tilde{U}_F^{-1/2}\tilde{U}_k^{1/2}(\lambda + (|\xi|^2+1)S_k)^{1/2} S_k^{1/2}\tilde{U}_k^{1/2}\tilde{U}_F^{-1/2}\eta\right\rangle_{\tilde{E}} \\
&\geq \sum_{k=P,F} \min_{s_k\in\sigma(S_k)} Re\,(\lambda + (|\xi|^2+1)s_k)^{1/2}(s_k)^{1/2}\|\tilde{U}_F^{-1/2}\tilde{U}_k^{1/2}\eta\|_{\tilde{E}}^2 \\
&\geq c\sum_{k=P,F} \min_{s_k\in\sigma(S_k)} Re\,(\lambda + (|\xi|^2+1)s_k)^{1/2}(s_k)^{1/2} > 0.
\end{aligned}$$

All in all we then get

$$
\begin{aligned}
\|\tilde{U}_F^{-1/2} l(\lambda,\xi)\tilde{U}_F^{-1/2}\|_{\mathcal{B}(\tilde{E})} &\geq c \sum_{k=P,F} \min_{s_k\in\sigma(S_k)} Re\,(\lambda + (|\xi|^2+1)s_k)^{1/2}(s_k)^{1/2} \\
&\geq c \sum_{k=P,F} \min_{s_k\in\sigma(S_k)} |s_k|^{1/2} \cos\left(\tfrac{\pi}{4} + \tfrac{\theta_{S_k}}{2}\right) |\lambda + (|\xi|^2+1)s_k|^{1/2} \\
&\geq c \sum_{k=P,F} \min_{s_k\in\sigma(S_k)} |s_k|^{1/2} \cos\left(\tfrac{\pi}{4} + \tfrac{\theta_{S_k}}{2}\right) c(\theta_\lambda, \theta_{s_k})(|\lambda| + (|\xi|^2+1)|s_k|)^{1/2} \\
&\geq C_l(|\lambda| + |\xi|^2+1)^{1/2},
\end{aligned} \tag{3.21}
$$

for $\theta_\lambda + \theta_{s_k} < \pi$. Note that S_k has only positive eigenvalues which implies $\theta_{s_k} = 0$. By considering the new symbol $(\lambda + |\xi|^2 + 1)^{-1/2}\tilde{U}_F^{-1/2} l(\lambda,\xi)\tilde{U}_F^{-1/2}$ we arrive at

$$
\|(\lambda + |\xi|^2+1)^{-1/2}\tilde{U}_F^{-1/2} l(\lambda,\xi)\tilde{U}_F^{-1/2}\|_{\mathcal{B}(\tilde{E})} \geq C_l \frac{(|\lambda| + |\xi|^2+1)^{1/2}}{|\lambda + |\xi|^2 + 1|^{1/2}} \geq C_l,
$$

for $Re\lambda \geq 0$. This inequality implies that the set $\{(\lambda + |\cdot|^2+1)^{1/2}[\tilde{U}_F^{-1/2} l(\lambda,\cdot)\tilde{U}_F^{-1/2}]^{-1}\}_{\lambda\in\overline{\mathbb{C}}_+} \subset \mathcal{H}^\infty(\Sigma_\theta)$ is uniformly bounded, for some $\theta > \phi_{D_n}^{\mathcal{R}_\infty} = 0$, i.e. we have

$$
\|(\lambda + |\xi|^2+1)^{1/2}[\tilde{U}_F^{-1/2} l(\lambda,\xi)\tilde{U}_F^{-1/2}]^{-1}\|_{\mathcal{B}(\tilde{E})} \leq C_l^{-1}, \quad (\lambda,\xi) \in \overline{\mathbb{C}}_+ \times \Sigma_\theta\,. \tag{3.22}
$$

We are now going to show that the operator $(G + D_n + 1)^{1/2}[\tilde{U}_F^{-1/2} L(G,D_n)\tilde{U}_F^{-1/2}]^{-1}$ is bounded. Employing Proposition 2.4 for $(D_n+1) \in \mathcal{RH}^\infty(\mathrm{L}_p(\mathbb{R}^n;E))$ provides $\mathcal{R}$-boundedness of $\{(i\rho + D_n + 1)^{1/2}[\tilde{U}_F^{-1/2} l(i\rho,D_n)\tilde{U}_F^{-1/2}]^{-1} : \rho \in \mathbb{R}_+\}$. We put $X_0 := \mathrm{L}_p(\mathbb{R}_+;\mathrm{L}_p(\mathbb{R}^n;\tilde{E}))$ and define the operator-valued symbol

$$
M(\rho) := (i\rho + D_n + 1)^{1/2} l_F^{-1}(i\rho, D_n), \quad \rho \in \mathbb{R}\,,
$$

with $l_F^{-1}(i\rho,D_n) := [\tilde{U}_F^{-1/2} l(i\rho,D_n)\tilde{U}_F^{-1/2}]^{-1}$. To establish that

$$
(G + D_n + 1)^{1/2}[\tilde{U}_F^{-1/2} L(G,D_n)\tilde{U}_F^{-1/2}]^{-1} \in \mathcal{B}(X_0)
$$

we will employ the operator-valued version of the Mikhlin Fourier multiplier Theorem 2.3 to the symbol $M(\rho)$. Hence, we have to show that the sets $\mathcal{M} := \{M(\rho) : \rho \in \mathbb{R}\backslash\{0\}\}$ and $\mathcal{M}' := \{\rho M'(\rho) : \rho \in \mathbb{R}\backslash\{0\}\}$ are $\mathcal{R}$-bounded. By the remarks above we have seen that the first of these sets is $\mathcal{R}$-bounded. To check the second condition we have to compute $\rho M'(\rho)$. By decomposing and factorising, with a view to using $\mathcal{R}$-boundedness of $\mathcal{M}$, we get

$$
\begin{aligned}
\rho M'(\rho) &= \frac{1}{2} i\rho \left[\frac{d}{d(i\rho)}(i\rho + D_n + 1)^{1/2}\right] l_F^{-1}(i\rho,D_n) - i\rho(i\rho + D_n+1)^{1/2}\frac{d}{di\rho} l_F^{-1}(i\rho,D_n) \\
&= \frac{1}{2}(i\rho)(i\rho + D_n + 1)^{-1}(i\rho + D_n + 1)^{1/2} l_F^{-1}(i\rho,D_n) \\
&\quad - i\rho(i\rho + D_n + 1)^{1/2} l_F^{-1}(i\rho,D_n)\frac{d\,l_F(i\rho,D_n)}{d(i\rho)} l_F^{-1}(i\rho,D_n) \\
&= \frac{1}{2}(i\rho)(i\rho + D_n + 1)^{-1} M(\rho) - M(\rho)(i\rho)^{1/2}\frac{d\,l_F(i\rho,D_n)}{d(i\rho)}(i\rho)^{1/2} l_F^{-1}(i\rho,D_n).
\end{aligned}
$$

To conclude that $\mathcal{M}'$ is $\mathcal{R}$-bounded, we still have to prove this property for $(i\rho)^{1/2} l_F^{-1}(i\rho, D_n)$ and $(i\rho)^{1/2} \frac{d}{d(i\rho)} l_F(i\rho, D_n)$ in order to be able to employ Lemma 2.3. The $\mathcal{R}$-boundedness of the vector-valued symbol $(i\rho)^{1/2} l_F^{-1}(i\rho, D_n)$ follows from the lower estimate of $l_F(\lambda, \xi)$.

$$\|(\lambda)^{-1/2} l_F(\lambda, \xi)\|_{\mathcal{B}(\tilde{E})} \geq C_l \frac{(|\lambda| + |\xi|^2 + 1)^{1/2}}{|\lambda|^{1/2}} \geq C_l, , \quad (\lambda, \xi) \in \overline{\mathbb{C}}_+ \times \Sigma_\theta$$

To treat $(i\rho)^{1/2} \frac{d}{d(i\rho)} l_F(i\rho, D_n)$ we again look at the symbol.

$$\frac{d}{d\lambda} l_F(\lambda, \xi) = \sum_{k=P,F} \tilde{U}_F^{-1/2} \left(\frac{d}{d\lambda} (\lambda + (|\xi|^2 + 1)(\Pi^k D^k))^{1/2} \right) (\Pi^k D^k)^{1/2} \tilde{U}_k \tilde{U}_F^{-1/2}$$

First we set $H(\lambda) := (\lambda + (|\xi|^2 + 1)(\Pi^k D^k))^{1/2}$. By using the identity

$$-\frac{d}{d\lambda} H(\lambda) = H(\lambda) \left[\frac{d}{d\lambda} H^{-1}(\lambda) \right] H(\lambda)$$

it remains to compute $\frac{d}{d\lambda} H^{-1}(\lambda)$. Since differentiation is a local property we consider a neighbourhood of $\lambda \in \overline{\Sigma}_{\pi/2}$. Consequently, the spectrum of $\sigma(\lambda + (|\xi|^2 + 1)(\Pi^k D^k))$ is a compact subset of Σ_ϕ, $0 < \phi < \pi/2$, and there exists a simple closed path Γ in Σ_ϕ surrounding $\sigma(\lambda + (|\xi|^2 + 1)(\Pi^k D^k))$ counterclockwise. Then, by the Functional calculus we have

$$\begin{aligned} \frac{d}{d\lambda} H^{-1}(\lambda) &= \frac{d}{d\lambda} \frac{1}{2\pi i} \int_\Gamma \mu^{-1/2} (\mu + \lambda + (|\xi|^2 + 1)(\Pi^k D^k))^{-1} d\mu \\ &= \frac{d}{d\lambda} \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - \lambda)^{-1/2} (\mu + (|\xi|^2 + 1)(\Pi^k D^k))^{-1} d\mu, \end{aligned}$$

where Γ_λ denotes the transformed path caused by changing variables. Using Cauchy's theorem we deform the integration path Γ_λ into Γ_0 in Σ_ϕ not depending on λ. Differentiating under the integral sign yields

$$\begin{aligned} \frac{d}{d\lambda} H^{-1}(\lambda) &= \frac{1}{2} \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - \lambda)^{-3/2} (\mu + (|\xi|^2 + 1)(\Pi^k D^k))^{-1} d\mu \\ &= \frac{1}{2} (\lambda + (|\xi|^2 + 1)(\Pi^k D^k))^{-3/2}, \end{aligned}$$

which is justified by boundedness of the integrand. This implies

$$-\frac{d}{d\lambda} H(\lambda) = H(\lambda) \left[\frac{d}{d\lambda} H^{-1}(\lambda) \right] H(\lambda) = \frac{1}{2} (\lambda + (|\xi|^2 + 1)(\Pi^k D^k))^{-1/2},$$

and thus we have shown

$$\begin{aligned} \frac{d}{d\lambda} l_F(\lambda, \xi) &= \frac{1}{2} \sum_{k=P,F} \tilde{U}_F^{-1/2} \left(\lambda + (|\xi|^2 + 1)(\Pi^k D^k) \right)^{-1/2} (\Pi^k D^k)^{1/2} \tilde{U}_k \tilde{U}_F^{-1/2} \\ &= \frac{1}{2} \sum_{k=P,F} \tilde{U}_F^{-1/2} (\Pi^k D^k)^{1/2} \tilde{U}_k \left[(\lambda + (|\xi|^2 + 1)(\Pi^k D^k))^{1/2} (\Pi^k D^k)^{1/2} \tilde{U}_k \right]^{-1} \\ &\qquad \cdot (\Pi^k D^k)^{1/2} \tilde{U}_k \tilde{U}_F^{-1/2} \\ &= \frac{1}{2} \sum_{k=P,F} \tilde{U}_F^{-1/2} \tilde{U}_k^{1/2} S_k^{1/2} \left[(\lambda + (|\xi|^2 + 1) S_k)^{1/2} S_k^{1/2} \right]^{-1} S_k^{1/2} \tilde{U}_k^{1/2} \tilde{U}_F^{-1/2}. \end{aligned}$$

Observe that each summand belongs to $\mathcal{B}(\tilde{E})$ and satisfies an estimate of the form (3.21), i.e. the operator $\frac{d}{d\lambda} l_F(\lambda, D_n)$ behaves as $l_F^{-1}(i\rho, D_n)$. Consequently the set

$$\left\{ (i\rho)^{1/2} \frac{d}{d(i\rho)} l_F(i\rho, D_n) : \ \rho \in \mathbb{R}\backslash\{0\} \right\}$$

is $\mathcal{R}$-bounded as well. After employing Lemma (2.3) to $\mathcal{M}'$ we arrive at

$$\begin{aligned} \mathcal{R}(\mathcal{M}') \leq \frac{1}{2}\mathcal{R}(\mathcal{D})\mathcal{R}(\mathcal{M}) + \mathcal{R}(\mathcal{M})\mathcal{R}&\left(\{(i\rho)^{1/2}\frac{d}{d(i\rho)} l_F(i\rho, D_n) : \ \rho \in \mathbb{R}\backslash\{0\}\}\right) \\ \cdot \mathcal{R}&\left(\{(i\rho)^{1/2} l_F^{-1}(i\rho, D_n) : \ \rho \in \mathbb{R}\backslash\{0\}\}\right), \end{aligned}$$

with $\mathcal{D} := \{i\rho(i\rho + D_n + 1)^{-1} : \ \rho \in \mathbb{R}\backslash\{0\}\}$. On the whole we have proved

$$(G + D_n + 1)^{1/2}[\tilde{U}_F^{-1/2} L(G, D_n)\tilde{U}_F^{-1/2}]^{-1} \in \mathcal{B}(X_0). \tag{3.23}$$

If we define the operator $\mathcal{F} = \sqrt{G + D_n + 1}$ with natural domain $D(\mathcal{F}) = D(G^{1/2}) \cap D(D_n^{1/2})$ then the operators $\mathcal{F}$ and $L_F^{-1} := [\tilde{U}_F^{-1/2} L(G, D_n)\tilde{U}_F^{-1/2}]^{-1}$ commute and (3.23) implies $L_F^{-1} \in \mathcal{B}(D(\mathcal{F}; \tilde{E}), D(\mathcal{F}^2; \tilde{E}))$. Here, $D(\mathcal{F}; \tilde{E})$ denotes the space of all $\tilde{E}$-valued functions which belong to $D(\mathcal{F})$. After using real interpolation we get

$$\mathcal{L} := \tilde{U}_F^{1/2} L_F^{-1} \tilde{U}_F^{-1/2} \in \mathcal{B}(D_{\mathcal{F}}(1 - 1/p, p), D_{\mathcal{F}}(2 - 1/p, p)) = \mathcal{B}(Y_1(E), Y_2(E)). \tag{3.24}$$

Turning to equation 3.19 and using the above results yields

$$c^F = \mathcal{L}\left(\overline{g}(t) - F_P \Pi^P D^P \tilde{U}_P^{-1} \Pi_0 h(t)\right),$$

this means that the unknown functions c^P and c^F are determined uniquely.

(e) Now we turn to complete problem, i.e. we consider non-vanishing initial data and inhomogeneities enjoying the compatibility conditions. If we set

$$\begin{aligned} &T^k(t) := e^{-\Pi^k D^k(-\partial^2_{y_{n+1}} + D_n + 1)t}, \quad S^k(t) := e^{-\Pi^k D^k (D_n+1)\frac{t}{2}}, \quad t \geq 0, \qquad \text{for } k = P, F\,, \\ &\Xi^P(y_{n+1}) := e^{-(\Pi^P D^P)^{-1/2}(D_n+1)^{1/2}\frac{y_{n+1}}{\sqrt{2}}}, \quad \Upsilon^P(y_{n+1}) := e^{-F_P y_{n+1}}, \quad y_{n+1} \geq 0\,, \\ &\Xi^F(y_{n+1}) := e^{+(\Pi^F D^F)^{-1/2}(D_n+1)^{1/2}\frac{y_{n+1}}{\sqrt{2}}}, \quad \Upsilon^F(y_{n+1}) := e^{+F_F y_{n+1}}, \quad y_{n+1} \leq 0\,, \\ &T(t) := \operatorname{diag}[T^P(t), T^F(t)], \quad S(t) := \operatorname{diag}[S^P(t), S^F(t)], \quad t \geq 0\,, \\ &\Xi(y_{n+1}) := \operatorname{diag}[\Xi^P(y_{n+1}), \Xi^F(y_{n+1})], \quad \Upsilon(y_{n+1}) := \operatorname{diag}[\Upsilon^P(y_{n+1}), \Upsilon^F(y_{n+1})]\,, \end{aligned}$$

then the solution can be written in the form

$$\begin{aligned} (w^P(t,y), w^F(t,y)) = {}& T(t)(\alpha^P(y), \alpha^F(y)) + [T * (\Pi^P f^P, \Pi^F f^F)](t,y) \\ &+ S(t)\Xi(y_{n+1})(u_0^P(y',0), u_0^F(y'0)) \\ &+ \Upsilon(y_{n+1})(\beta^P(t,y'), \beta^F(t,y')) + \Upsilon(y_{n+1})(\gamma^P(t,y'), \gamma^F(t,y')). \end{aligned} \tag{3.25}$$

If we restrict the above function to $J = [0, T]$ and take into account that the convolution operator and $\mathcal{L}$ are causal, we perceive that the so-constructed function solves the original problem (3.17). For a better understanding, we now shall explain all terms contained in the formula (3.25). The first and second function of the above decomposition solve the parabolic problem

$$
\begin{aligned}
&\partial_t w^P - \Pi^P D^P \partial^2_{y_{n+1}} w^P + \Pi^P D^P (D_n+1) w^P = \Pi^P f^P, \quad t>0, \quad y \in \mathbb{R}^{n+1}_+,\\
&\partial_t w^F - \Pi^F D^F \partial^2_{y_{n+1}} w^F + \Pi^F D^F (D_n+1) w^F = \Pi^P f^P, \quad t>0, \quad y \in \mathbb{R}^{n+1}_-,\\
&w^P(t,y',0) = w^F(t,y',0) = 0, \quad t>0, \quad y' \in \mathbb{R}^n,\\
&w^P(0,y) = \alpha^P(y), \quad y \in \mathbb{R}^{n+1}_+, \quad w^F(0,y) = \alpha^F(y), \quad y \in \mathbb{R}^{n+1}_-,
\end{aligned}
$$

with the initial data

$$
\alpha^k(y) = u_0^k(y) - \Xi^k(y) u_0^k(y',0), \quad k = P, F. \tag{3.26}
$$

The third function is a special solution which satisfies $\partial_t v^k - \Pi^k D^k \partial^2_{y_{n+1}} v^k + \Pi^k D^k (D_n+1) v^k = 0$ and certain boundary conditions which can be computed explicitly. The second last function solves (3.18) with the inhomogeneity $\Pi_0 \tilde{h}$ in the boundary condition of Dirichlet type and 0 in the transmission condition, where we have set

$$
\Pi_0 \tilde{h}(t,y') := \Pi_0 \left[h(t,y') - \tilde{U}_P^{-1} S^P(t) u_0^P(y',0) + \tilde{U}_F^{-1} S^F(t) u_0^F(y',0) \right].
$$

At first glance, we realise that $\Pi_0 \tilde{h}(0,y') = 0$ in view of compatibility condition 6. The functions $\beta^P(t,y')$ and $\beta^F(t,y')$ are given by

$$
\begin{aligned}
(\Pi^P D^P)\beta^P(t,y') &= (\Pi^P D^P)\tilde{U}_P \tilde{U}_F^{-1} \beta^F(t,y') + (\Pi^P D^P)\tilde{U}_P \Pi_0 \tilde{h}(t,y'),\\
\beta^F(t,y') &= -\mathcal{L} F_P (\Pi^P D^P) \tilde{U}_P^{-1} \Pi_0 \tilde{h}(t,y').
\end{aligned}
$$

Note that $\Pi^P D^P$ is invertible in E. Last but not least $\Upsilon(y_{n+1})(\gamma^P(t,y'), \gamma^F(t,y'))$ solves (3.18) with the inhomogeneity

$$
\begin{aligned}
g_0(t,y') := \overline{g}(t,y') - (D_n+1)^{1/2} \Bigg[&\Pi \tilde{U}_F^{-1} S^F(t) u_0^F(y',0) + \frac{1}{\sqrt{2}} (\Pi^P D^P) S^P(t) u_0^P(y',0) +\\
&\frac{1}{\sqrt{2}} (\Pi^F D^F) S^F(t) u_0^F(y',0) \Bigg] + \partial_{y_{n+1}} [T^P * (\Pi^P D^P) f^P + T^P \alpha^P](t,y',0)\\
&- \partial_{y_{n+1}} [T^F * (\Pi^F D^F) f^F + T^F \alpha^F](t,y',0)
\end{aligned}
$$

in the transmission condition and 0 in the jump condition. It is easy to check that $g_0(0,y') = 0$ is caused by the compatibility condition 7. The functions $\gamma^P(t,y')$ and $\gamma^F(t,y')$ are given by

$$
(\Pi^P D^P)\gamma^P(t,y') = (\Pi^P D^P)\tilde{U}_P \tilde{U}_F^{-1} \gamma^F(t,y'), \quad \gamma^F(t,y') = \mathcal{L} g_0(t,y').
$$

Finally, formula (3.25) is written for the case $p > 3$, in which the compatibility conditions are involved. In case $3/2 < p < 3$ simply set $g_0(t,y') = \overline{g}(t,y')$ and for $p < 3/2$ set $u_0^k(y',0) = 0$. Hence, it remains to verify the regularity of each function. According to Theorem 2.4 the first and second term of formula (3.25) belong to $Z_P^T(E) \times Z_F^T(E)$. The last both functions possess this regularity as well in view of the assumptions for data, the mapping property $\mathcal{L} \in \mathcal{B}(Y_2^T(E), Y_1^T(E))$ and Theorem 2.7. We now come to the third function $(v^P(t,y),\, v^F(t,y)) :=$

$(S^P(t)\Xi^P(y_{n+1})u_0^P(y',0), S^F(t)\Xi^F(y_{n+1})u_0^F(y',0))$. Observe that each component satisfies the parabolic problem

$$
\begin{aligned}
\partial_t v^k - \Pi^k D^k \partial^2_{y_{n+1}} v^k + \Pi^k D^k (D_n+1) v^k = 0, \quad t>0,\ y\in\mathbb{R}^{n+1}_+, \text{if } k=P,\ y\in\mathbb{R}^{n+1}_-, \text{if } k=F,\\
v^k(t,y',0) = S^k(t)u_0^k(y',0), \quad t>0,\ y'\in\mathbb{R}^n\\
v^k(0,y) = \Xi^k(y_{n+1})u_0^k(y',0), \quad y'\in\mathbb{R}^n,\ y_{n+1}\geq 0 \text{ if } k=P,\\
y_{n+1}\leq 0 \text{ if } k=F.
\end{aligned}
$$

We know that $u_0^k(y)\in D_{D_n}(1-1/p,p)$ due to condition four and consequently after taking trace in y_{n+1} we have $u_0^k(y',0)\in D_{D_n}(1-3/2p,p)$. As in the proof of the previous theorem we are able to show $S^k(t)u_0^k(y',0)\in D_F(2-1/p,p)$ and $\Xi^k(y_{n+1})u_0^k(y',0)\in D_{B_k}(1-1/p,p)$, which provides $v^k\in Z_k^T(E)$.

Finally, the solution formulae (3.25) and (3.11) provide a solution operator $\mathcal{S}$. Necessity and sufficiency of the data entail that $\mathcal{S}$ is an continuous one-to-one mapping from

$$
\mathcal{X}_+^T\times\mathcal{X}_-^T\times\{\,(g,h,u_0^P,u_0^F)\in\mathcal{Y}_2^T\times Y_1^T\times V_+\times V_- : (g,h,u_0^P,u_0^F) \text{ enjoy the compatibility conditions 5. and 6. stated in Theorem 3.9}\}
$$

to $Z_+^T\times Z_-^T\times\mathcal{Z}_{+,-}^T$. Thus the proof is complete.

□

3.4 The linear problem in domain

Before we are going to approach the linear problem on the domain, we make available the method of localisation.

3.4.1 Localisation Techniques for Bounded Domains

In the following, let $\Omega\subset\mathbb{R}^{n+1}$ be an open connected domain with compact C^2-boundary $\partial\Omega$. Now we want to comment on variable transformations. Let $x_j\in\partial\Omega$ and consider local coordinates corresponding to x_j (which) are defined as coordinates obtained by rotation and shifting, which moves x_j to the origin such that the exterior normal at x_j has the direction of the negative x_{n+1}-axis. By definition of a C^2-boundary an open neighbourhood $U_j=U_n^j\times U_1^j\subset R^{n+1}$ exists containing x_j with $U_n^j\subset\mathbb{R}^n$ and $U_1^j\subset\mathbb{R}^1$ open and a function $h_j\in C^2(\overline{U}_n^j;\mathbb{R})$ satisfying

$$
\begin{aligned}
\Omega\cap U_j &= \{x=(x',x_{n+1})\in U_j : x_{n+1}>h_j(x')\}\,,\\
\partial\Omega\cap U_j &= \{x\in U_j : x_{n+1}=h_j(x')\}\,.
\end{aligned}
\tag{3.27}
$$

Setting

$$
g_j(x) := \begin{pmatrix} x' \\ x_{n+1}-h_j(x') \end{pmatrix} \;:\; U_j\cap\Omega\to\mathbb{R}^{n+1}_+ \tag{3.28}
$$

we obtain an injection $g_j\in C^2(\overline{U}_j;\mathbb{R}^{n+1})$ where relations (3.27) can be written as $\Omega\cap U_j=\{x\in U_j : g_{n+1}(x)>0\}$ and $\partial\Omega\cap U_j=\{x\in U_j : g_{n+1}(x)=0\}$. By compactness of $\partial\Omega$, all derivatives of g_j and g_j^{-1}, defined on $\tilde{U}_j:=g_j(U_j)$, up to order 2 are bounded by a constant independent of x_j. Now we need an extension of g_j. For this we extend $h_j\in C^2(\overline{U}_n^j;\mathbb{R})$

to a function $\tilde{h}_j \in C^2(\mathbb{R}^n;\mathbb{R})$ with compact support and set, further on using coordinates corresponding to x_j,

$$\Omega_{x_j} := \left\{x \in \mathbb{R}^{n+1} : x_{n+1} > \tilde{h}_j(x')\right\} \subset \mathbb{R}^{n+1}_+ \ .$$

Defining $\tilde{g}_j$ again by (3.28) with $\tilde{h}_j$ instead of h_j and we obtain a C^2-diffeomorphism $\tilde{g}_j : \Omega_{x_j} \to \mathbb{R}^{n+1}_+$ with $\tilde{g}_{j|_{U_j}} = g_j$. It is easily seen that $D\tilde{g}_j(x_j) = I_{n+1}$, due to the special choice of local coordinates corresponding x_j which implies $\nabla_{x'}\tilde{h}_j(x'_j) = 0$.

For a function $u : \Omega \cap U_j \to E$ consider the push-forward operator defined on $\tilde{U}_j \cap \mathbb{R}^{n+1}_+$ by $v(y) := (\mathcal{G}_j u)(y) = u(g_j^{-1}(y))$. If $u \in \mathrm{H}^2_p(\Omega \cap U_j; E)$ then the classical formula for the derivatives of v holds,

$$(D^\alpha v)(y) = \sum_{1\le|\gamma|\le|\alpha|} q_{\alpha\gamma}(y)(D^\gamma u)(g_j^{-1}(y)) \text{ for almost all } y \in \overline{U_j} \cap \mathbb{R}^{n+1}_+ \ .$$

Here $q_{\alpha\gamma}$ are homogeneous polynomials of degree $|\gamma|$ in derivatives of g_j^{-1} of order between 1 and $|\alpha|-|\gamma|+1$. Owing to boundedness of derivatives for g_j and g_j^{-1} we have that $\mathcal{G}_k$ induces isomorphisms between $\mathrm{H}^k_p(\Omega \cap U_j; E)$ and $\mathrm{H}^k_p(\mathbb{R}^{n+1}_+ \cap \tilde{U}_j; E)$ for $k = 0, 1, 2$ and $p \in [1, \infty]$. The same holds for the linear transformation given by $\tilde{g}_k$ which induces isomorphisms $\tilde{\mathcal{G}}_j : \mathrm{H}^k_p(\Omega_{x_j}; E) \to \mathrm{H}^k_p(\mathbb{R}^{n+1}_+; E)$.

Now let us consider an open covering of the form

$$\partial\Omega \subset \bigcup_{x_j \in \partial\Omega} g_j^{-1}\left(B_{r_j}(y_j)\right)$$

with $y_j = g_j(x_j)$. By compactness we can choose a finite sub-covering

$$\partial\Omega \subset \bigcup_{j=1}^{M'} U_j$$

where we have set $U_j := g_j^{-1}\left(B_{r_j}(y_j)\right)$ for $j = 1, \ldots, M'$. We cover the compact set $\Omega \backslash \bigcup_{j=1}^{M'} U_j$ by finitely many $U_j = B_{r_j}(x_j)$, $j = M'+1, \ldots, M$ with $x_j \in \Omega$. We get a finite covering for our domain $\overline{\Omega}$ of the form

$$\overline{\Omega} \subset \bigcup_{j=1}^{M'} g_j^{-1}\left(B_{r_j}(y_j)\right) \cup \bigcup_{j=M'+1}^{M} B_{r_j}(x_j) \ .$$

Now we consider a general partial differential operator $\mathcal{A}(x, D) := \sum_{|\alpha|\le 2} a_\alpha(x) D^\alpha$ acting on Ω which is transformed into operator $\mathcal{A}^{g_j}(y, D) = \mathcal{G}_j \mathcal{A}(x, D)\mathcal{G}_j^{-1}$ for $j = 1, \ldots, M'$. In the same way we can define transformed boundary differential operators $\mathcal{B}_l^{g_j}(y, D) = \mathcal{G}_j \mathcal{B}_l(x, D)\mathcal{G}_j^{-1}$ for $j = 1, \ldots, M'$, where $\mathcal{B}_l(x, D) = \sum_{|\beta|\le m_l} b_{\beta l}(x) D^\beta$, $l = 1, \ldots, m$. Obviously $\mathcal{A}^{g_j}$ and $\mathcal{B}_l^{g_j}$ are partial differential operators of order 2 and m_j, respectively and act on functions defined on $\tilde{U}_j \cap \mathbb{R}^{n+1}_+$. By splitting these operators in main part and terms of lower derivatives we may write

$$\mathcal{A}^{g_j} = \mathcal{A}^j + \mathcal{A}^{low} = \sum_{|\alpha|=2} \tilde{a}_\alpha(y) D^\alpha + \sum_{|\alpha|<2} \tilde{a}_\alpha(y) D^\alpha$$

and

$$\mathcal{B}_l^{g_j} = \mathcal{B}_l^j + \mathcal{B}_l^{low} = \sum_{|\beta|=m_l} \tilde{b}_{\beta l}(y) D^\beta + \sum_{|\beta|<m_l} \tilde{b}_{\beta l}(y) D^\beta .$$

We can extend the main parts $\mathcal{A}^j$ and $\mathcal{B}_l^j$ to the half space $\mathbb{R}_+^{n+1}$ by extension of the coefficients. In fact, for the coefficients of $\mathcal{A}^j$ we can use e.g. the reflection method, i.e., we define

$$a_\alpha^j(y) := \begin{cases} \tilde{a}_\alpha(y) \,, & y \in \overline{B_{r_j}(y_j)} \cap \overline{\mathbb{R}_+^{n+1}} \\ \tilde{a}_\alpha(y_j + r_j^2 \frac{y-y_j}{|y-y_j|^2}) \,, & y \in \overline{\mathbb{R}_+^{n+1}} \backslash \overline{B_{r_j}(y_j)} \end{cases}$$

For the coefficients of the boundary operators $\mathcal{B}_l^j$ we fix $\chi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\chi \equiv 1$ for $|x| \leq 1$ and $\chi(x) \equiv 0$ for $|x| \geq 2$ and set for $j = 1, \ldots, M'$, $l = 1, \ldots, m$ and all $|\beta| = m_l$

$$b_{\beta l}^j(y) := \tilde{b}_{\beta l} \left(y_j + \chi\left(\frac{y-y_j}{r_j} \right) \cdot (y - y_j) \right) \,, \; y \in \overline{\mathbb{R}_+^{n+1}}.$$

For $j = M'+1, \ldots, M$ boundary conditions do not appear, and we only need an extension to the whole space $\mathbb{R}^{n+1}$ for the main part of $\mathcal{A}$. Hence we define coefficients a_α^j of local operators $\mathcal{A}^j$ again by reflection, in fact

$$a_\alpha^j(x) = \begin{cases} a_\alpha(x) & x \in \overline{B_{r_j}(x_j)} \,, \\ a_\alpha\left(x_j + r_j \frac{x-x_j}{|x-x_j|^2} \right) \,, & x \in \mathbb{R}^{n+1} \backslash \overline{B_{r_j}(x_j)} \end{cases} . \tag{3.29}$$

By the smoothness properties of the functions $a_\alpha(x)$, $|\alpha| = 2$, there exists $r_j(\varepsilon) > 0$ such that

$$\begin{aligned} &\sum_{|\alpha|=2} |a_\alpha^j(y) - a_\alpha^j(y_j)| < \varepsilon \,, \; y \in \overline{\mathbb{R}_+^{n+1}} \,, \; j = 1, \ldots, M' \\ &\sum_{|\alpha|=2} |a_\alpha^j(x) - a_\alpha^j(x_j)| < \varepsilon \,, \; x \in \mathbb{R}^{n+1} \,, \; j = M'+1, \ldots, M \\ &\sum_{|\beta|=m_j} |b_{\beta l}^j(y) - b_{\beta j}^j(y_j)| < \varepsilon \,, \; y \in \overline{\mathbb{R}_+^{n+1}} \,, \; j = 1, \ldots, M' \end{aligned} \tag{3.30}$$

for any prescribed $\varepsilon > 0$.

3.4.2 Existence and Uniqueness

The purpose of this section is to establish maximal regularity for the linear problem in domain $\Omega = \overline{\Omega}_P \cup \Omega_F$. By using operators $\mathcal{A}_{ki}(D)$ and $\mathcal{B}_{ki}(D)$ defined in (1.28), (1.29) the linear problem reads as follows

$$\begin{aligned} &\partial_t w^P + \mathcal{A}_{P1}(D) w^P + \mathcal{A}_{P2}(D)\psi^P = f^P(t,x), \quad (t,x) \in J \times \Omega_P, \\ &\partial_t w^F + \mathcal{A}_{F1}(D) w^F + \mathcal{A}_{F2}(D)\psi^F = f^F(t,x), \quad (t,x) \in J \times \Omega_F, \\ &\mathcal{B}_{P1}(D) w^P + \mathcal{B}_{P2}(D)\psi^P = \mathcal{B}_{F1}(D) w^F + \mathcal{B}_{F2}(D)\psi^F + g(t,x), \quad (t,x) \in J \times \Gamma_P, \\ &\tilde{U}_P^{-1}(t,x) w^P - \tilde{U}_F^{-1}(t,x) w^F + \lambda_0 z(\psi^P - \psi^F) = h^P(t,x), \quad (t,x) \in J \times \Gamma_P, \\ &w^F = h^F(t,x), \quad \psi^F = 0, \quad (t,x) \in J \times \Gamma, \\ &w^P(0,x) = u_0^P(x), \quad x \in \overline{\Omega}_P, \quad w^F(0,x) = u_0^F(x), \quad x \in \overline{\Omega}_F, \end{aligned} \tag{3.31}$$

$$
\begin{aligned}
&\frac{d}{dt}w^b + \frac{1}{\tau}w^b = \frac{1}{\tau}u^f + f^b(t) - a^b \int\limits_{\Gamma} \mathcal{B}_{F1}(D)w^F + \mathcal{B}_{F2}(D)\psi^F d\sigma, \quad t \in J \\
&w^b(0) = u_0^b,
\end{aligned}
\tag{3.32}
$$

and

$$
z^T \cdot w^P = 0, \quad (t,x) \in J \times \overline{\Omega}_P, \quad z^T \cdot w^F = 0, \quad (t,x) \in J \times \overline{\Omega}_F, \quad z^T \cdot w^b = 0, \quad t \in J. \tag{3.33}
$$

Before we turn to existence and uniqueness of the whole problem we want to study the ode-equation (3.32), since the concentrations (w^P, w^F) and the electrical potentials (ψ^P, ψ^F) are not determined by w^b. Therefore, we are able to compute w^b with aid of these functions. Assume that (w^P, w^F) and (ψ^P, ψ^F) are known then the function w^b is given uniquely by the following solution formula

$$
\begin{aligned}
w^b(t) = e^{-\frac{1}{\tau}t}u_0^b - a^b \int\limits_0^t e^{-\frac{1}{\tau}(t-s)} \int\limits_{\Gamma} \mathcal{B}_{F1}(s,x,D)w^F(s,x) + \mathcal{B}_{F2}(s,x,D)\psi^F(s,x)\, d\sigma\, ds \\
+ \int\limits_0^t e^{-\frac{1}{\tau}(t-s)} \left[\frac{1}{\tau}u^f(s) + f^b(s)\right] ds.
\end{aligned}
\tag{3.34}
$$

Lemma 3.2 *Let $(n+1)+2 < p < \infty$ and $u^f \in \mathrm{L}_p(J;E)$. Suppose that (w^F, ψ^F) belongs to $Z_F \times \mathcal{Z}_F$. Then the problem (3.32) with the condition $z^T \cdot w^b(t) = 0$ has precisely one solution $w^b \in Z_b = \mathrm{H}_p^1(J;\mathbb{R}^N)$ if and only if the data f^b, u_0^b satisfy the following conditions*

1. *$f^b \in \mathrm{L}_p(J;\mathbb{R}^N)$ and $f^b(t) - a^b \int\limits_{\Gamma} \mathcal{B}_{F2}(D)\psi^F + \mathcal{B}_{F1}(D)w^F\, d\sigma \in E$;*
2. *$u_0^b \in E$.*

Proof. Suppose that $w^b \in Z_b$ is given. Let us consider the boundary integral over Γ appearing in the right hand side of (3.32). The integrand belongs to $Y_{2,\Gamma} \hookrightarrow \mathrm{L}_p(J;\mathrm{C}(\Gamma))$ for $p > n$, and thus the integral exists a.e. Furthermore, the temporal regularity does not change, resulting in

$$
a^b \int\limits_{\Gamma} \mathcal{B}_{F1}(t,x,D)w^F + \mathcal{B}_{F2}(t,x,D)\psi^F d\sigma \in \mathrm{L}_p(J;\mathbb{R}^N)\,.
$$

Consequently, we see $f^b \in \mathrm{L}_p(J;\mathbb{R}^N)$. Applying $z^T\cdot$ to (3.32) we get

$$
z^T \cdot f^b - a^b \int_{\Gamma} z^T \cdot \mathcal{B}_{F1}(t,x,D)w^F + z^T \cdot \mathcal{B}_{F2}(t,x,D)\psi^F d\sigma = 0.
$$

Due to the regularity of w^b and electroneutrality for all $t \in [0,T]$, we may take trace and obtain $w^b(0) \in E$. Conversely, let $u_0^b \in E$ and $f^b \in \mathrm{L}_p(J;\mathbb{R}^N)$ be given. Then, the unique solution of (3.32) is given by formula (3.34), and here one can verify that w^b belongs to Z_b with $z^T \cdot w^b(t) = 0$.

□

Now, we shall prove the main result which reads as follows.

Theorem 3.4 *Let Ω_P, Ω_F be open bounded domains in $\mathbb{R}^{n+1}$ with C^2- boundary, $\Gamma_P := \partial\Omega_P$, $\partial\Omega_F = \Gamma_P \cup \Gamma$ and* $\operatorname{dist}(\Gamma_P, \Gamma) > 0$*. Let $J = [0,T]$ and $2 + (n+1) < p < \infty$. Suppose that the assumptions (1.23)-(1.25) are satisfied. Then problem (1.39-(1.41) has exactly one solution*

$$
\begin{aligned}
&(w^P, w^F, w^b) \in Z_P^T \times Z_F^T \times Z_b^T\,,\\
&(\psi^P, \psi^F) \in \mathcal{Z}^T := \{(\phi^P, \phi^F) \in \mathcal{Z}_P^T \times \mathcal{Z}_F^T :\ \gamma_{|\Gamma_P}(\phi^P - \phi^F) \in Y_{1,\Gamma_P}^T\}
\end{aligned}
$$

if and only if the data f^P, f^F, g, h^P, h^F, f^b, u_0^P, u_0^F, u_0^b satisfy the following conditions

1. $f^P \in \mathcal{X}_P^T := \{f \in \mathrm{L}_p(J; \mathrm{L}_p(\Omega_P; \mathbb{R}^N)) :\ z^T\cdot f \in \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^{-1}(\Omega_P))\}$*;*
2. $f^F \in \mathcal{X}_F^T := \{f \in \mathrm{L}_p(J; \mathrm{L}_p(\Omega_F; \mathbb{R}^N)) :\ z^T\cdot f \in \mathrm{H}_p^{1/2}(J; \mathrm{H}_p^{-1}(\Omega_F))\}$*;*
3. $f^b \in X_b^T := \mathrm{L}_p(J; \mathbb{R}^N)$*;*
4. $g \in \mathcal{Y}_2^T := \{g \in Y_{2,\Gamma_P}^T(\mathbb{R}^N) := \mathrm{B}_{pp}^{1/2-1/2p}(J; \mathrm{L}_p(\Gamma_P; \mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{1-1/p}(\Gamma_P; \mathbb{R}^N)) :$
   $z^T\cdot g \in \mathrm{H}_p^{1/2}(J; \mathrm{B}_{pp}^{-1/p}(\Gamma_P))\}$*;*
5. $h^P \in Y_{1,\Gamma_P}^T(\mathbb{R}^N) := \mathrm{B}_{pp}^{1-1/2p}(J; \mathrm{L}_p(\Gamma_P; \mathbb{R}^N)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{2-1/p}(\Gamma_P; \mathbb{R}^N))$*;*
6. $h^F \in Y_{1,\Gamma}^T(E) := \mathrm{B}_{pp}^{1-1/2p}(J; \mathrm{L}_p(\Gamma; E)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{2-1/p}(\Gamma; E))$*;*
7. $u_0^P \in V_P(E_+) := \mathrm{B}_{pp}^{2-2/p}(\Omega_P; E_+)$*,* $u_0^F \in V_F(E_+) := \mathrm{B}_{pp}^{2-2/p}(\Omega_F; E_+)$*,* $u_0^b \in E_+$*, and* $u_0^k > 0$*,* $k = P, F, b$*;*
8. $\lambda_0 z(\psi_0^P - \psi_0^F) = h^P(0)$ *in* $\mathrm{B}_{pp}^{2-3/p}(\Gamma_P; \mathbb{R}^N)$ *and* $u_0^F = h^F(0,x)$ *in* $\mathrm{B}_{pp}^{2-3/p}(\Gamma; E_+)$*;*
9. $\mathcal{B}_{P1}(0,D)u_0^P + \mathcal{B}_{P2}(0,u_0^P,D)\psi_0^P - \mathcal{B}_{F1}(0,D)u_0^F - \mathcal{B}_{F2}(0,u_0^F,D)\psi_0^F = g(0)$
   in $\mathrm{B}_{pp}^{1-3/p}(\Gamma_P; \mathbb{R}^N)$*, where (ψ_0^P, ψ_0^F) is the unique solution of the elliptic problem (3.35) (see Remark 3.1);*
10. f^P*,* f^F*,* g *and* f^b *fulfil the compatibility condition*

$$
\int_{\Omega_P} z^T\cdot f^P(t,x)\,dx + \int_{\Omega_F} z^T\cdot f^F(t,x)\,dx + \int_{\Gamma_P} z^T\cdot g(t,x)\,d\sigma + \frac{1}{a^b} z^T\cdot f^b(t) = 0, \quad t \in J\,.
$$

Moreover, there exists an isomorphism between the space of data including the compatibility conditions and the regularity class $Z_P^T \times Z_F^T \times Z_b^T \times \mathcal{Z}^T$.

Remark 3.1 1. We want to discuss the compatibility conditions on the boundary Γ_P. Taking trace $t = 0$ in the boundary conditions on Γ_P involves the new functions $(\psi_0^P, \psi_0^F) := (\psi^P(0), \psi^F(0))$. In fact, ψ^k does not make for a known function as in contrast to w^k. The resource is to consider the elliptic problem after taking trace in $t = 0$, where the function $(\psi^P(0), \psi^F(0))$ is accounted the weak solution of this problem. Note that this procedure is justified by the time regularity of the data. The problem reads as follows

$$
\begin{aligned}
&\nabla\cdot(a_0^P\nabla\psi^P(0)) + \nabla\cdot(z^T\cdot D^P(0)\nabla u_0^P) = -z^T\cdot f^P(0), \quad x \in \Omega_P,\\
&\nabla\cdot(a_0^F\nabla\psi^F(0)) + \nabla\cdot(z^T\cdot D^F(0)\nabla u_0^F) = -z^T\cdot f^F(0), \quad x \in \Omega_F,\\
&a_0^P\partial_\nu\psi^P(0) + z^T\cdot D^P(0)\partial_\nu u_0^P = a_0^F\partial_\nu\psi^F(0) + z^T\cdot D^F(0)\partial_\nu u_0^F + z^T\cdot g(0), \quad x \in \Gamma_P,\\
&\psi^P(0) - \psi^F(0) = \frac{1}{\lambda_0|z|^2} z^T\cdot h^P(0), \quad x \in \Gamma_P,\\
&\psi^F(0) = 0, \quad x \in J\times\Gamma,
\end{aligned}
\tag{3.35}
$$

with $a_0^k = z^T\cdot M^k(0,x)u_0^k(x) \in \mathrm{C}^1(\overline{\Omega}_k)$ due to the embedding $\mathrm{B}_{pp}^{2-2/p}(\Omega_k) \hookrightarrow \mathrm{C}^1(\overline{\Omega}_k)$ for $p > (n+1)+2$.

Furthermore, for solving this problem we have to ensure that a_0^k is positive which is accomplished by the condition $u_0^k(x) > 0$. This elliptic problem is solved implicitly during the proof, cp. to the proof of Theorem 3.3.

2. Note that the compatibility condition 10. is only needed to ensure that $z^T\cdot w^b = 0$ for $t \in J$, cf. Lemma 3.2 and 1.1.

Proof. Step 1 - the necessary part. Let $(w,\psi) := (w^P, w^F, \psi^P, \psi^F)$ be a strong solution of (3.31)-(3.33) with the regularity stated above. Then it follows $f^k = \partial_t w^k + \mathcal{A}_{k1}w^k + \mathcal{A}_{k2}\psi^k \in \mathrm{L}_p(J;\mathrm{L}_p(\Omega_k;\mathbb{R}^N))$ for $k = P, F$. To show that f^b belongs to $\mathrm{L}_p(J;\mathbb{R}^N)$ we refer to the proof of Lemma 3.2. Furthermore, we have to verify that $z^T\cdot f^k$ belongs to $\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))$. In view of the divergence form we obtain by using duality

$$\begin{aligned}\|z^T\cdot f^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))} \leq& \|z^T\cdot D^k\nabla w^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k;\mathbb{R}^{n+1}))} + \|a^k\nabla\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k;\mathbb{R}^{n+1}))}\\ \leq& C\left(\|w^k\|_{Z_k^T} + \|\psi^k\|_{\mathcal{Z}_k^T}\right).\end{aligned}$$

Thereby, have in mind the regularity assumption (1.23) for D^k and $\tilde{u}^k \in Z_k(E_+)$, which imply $a^k \in \mathrm{C}^{1/2}(J;\mathrm{C}(\Omega_k))$.

Now, we prove the fourth condition. By Proposition 2.3 we know that $\mathcal{B}_{k1}(D)w^k \in Y_{2,\Gamma_P}^T$ and $\mathcal{B}_{k2}(D)\psi^k \in Y_{2,\Gamma_P}^T$ which entails $g \in Y_{2,\Gamma_P}$. To prove $z^T\cdot g \in \mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\Gamma_P))$ we have to study the weak formulation of the elliptic boundary value problem obtained by employing $z^T\cdot$ to (3.31). The space $\mathrm{B}_{pp}^{-1/p}(\Gamma_P)$ is interpreted as the dual space of $\mathrm{B}_{p'p'}^{1-1/p'}(\Gamma_P)$ where the measure on Γ_P is the usual surface measure induced by the Lebesgue measure in $\mathbb{R}^n$, see Section 2.1. Further on, we set $f^k = 0$. Before studying the weak formulation, let us make some preparations. Given $p \in (1,\infty)$, we denote p' the dual exponent, that is $p' := p/(p-1)$ and by $\langle,\rangle_\Omega$ the duality pairing

$$\langle\cdot,\cdot\rangle_\Omega : \mathrm{L}_{p'}(\Omega;\mathbb{R}^l) \times \mathrm{L}_p(\Omega;\mathbb{R}^l) \to \mathbb{R}\,, \quad (v',v) \to \int_\Omega v'\cdot v\,dx\,,$$

where it will always be clear from the context which $l \in \mathbb{N}\backslash\{0\}$ has to be chosen. As usual, we get the weak formulation by multiplying the differential equation in Ω_k with a test function $\tilde{v} \in \mathring{\mathrm{H}}_{p'}^1(\Omega)$, i.e. $\tilde{v} \in \mathrm{H}_{p'}^1(\Omega)$ with compact support in Ω. Integrating by parts, and making use of $\tilde{v} = 0$ on $\Gamma := \partial\Omega$ we obtain

$$\sum_{k=P,F}\int_{\Omega_k} \nabla\tilde{v}(x)\cdot[a^k(t,x)\nabla\psi^k(t,x)dx + z^T\cdot D^k(t,x)\nabla w^k(t,x)]\,dx = \int_{\Gamma_P}\tilde{v}(x)z^T\cdot g(t,x)\,d\sigma\,.$$

From the above identity we derive

$$\begin{aligned}\int_{\Gamma_P} v(x)[z^T\cdot g(t+h,x) - z^T\cdot g(t,x)]\,d\sigma \leq& \|\nabla\tilde{v}\|_{\mathrm{L}_p(\Omega;\mathbb{R}^{n+1})}\\ &\cdot\sum_{k=P,F}\Big\{\|a^k(t+h)\nabla\psi^k(t+h) - a^k(t)\nabla\psi^k(t)\|_{\mathrm{L}_p(\Omega_k;\mathbb{R}^{n+1})} +\\ &\|z^T\cdot D^k(t+h)\nabla w^k(t+h) - z^T\cdot D^k(t)\nabla w^k(t)\|_{\mathrm{L}_p(\Omega_k;\mathbb{R}^{n+1})}\Big\},\end{aligned}$$

with $\tilde{v}_{|_{\Gamma_P}} = v$, for all $\tilde{v} \in \overset{\circ}{\mathrm{H}}{}^1_{p'}(\Omega)$ and $t+h,\, t \in J$. By taking the infimum and using

$$\inf\left\{\|\tilde{v}\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\Omega)} : \tilde{v} = v \text{ on } \Gamma_P\right\} \le C\|v\|_{\mathrm{B}^{1-1/p'}_{p'p'}(\Gamma_P)}$$

we get

$$\int_{\Gamma_P} v(x)\ [z^T\cdot g(t+h,x) - z^T\cdot g(t,x)]\, d\sigma \le C\|v\|_{\mathrm{B}^{1-1/p'}_{p'p'}(\Gamma_P)}$$
$$\cdot \sum_{k=P,F}\Big\{\|a^k(t+h)\nabla\psi^k(t+h) - a^k(t)\nabla\psi^k(t)\|_{\mathrm{L}_p(\Omega_k;\mathbb{R}^{n+1})} +$$
$$\|z^T\cdot D^k(t+h)\nabla w^k(t+h) - z^T\cdot D^k(t)\nabla w^k(t)\|_{\mathrm{L}_p(\Omega_k;\mathbb{R}^{n+1})}\Big\},$$

for all $v \in \mathrm{B}^{1-1/p'}_{p'p'}(\Gamma_P)$. This inequality implies $z^T\cdot g(t+h) - z^T\cdot g(t) \in \mathrm{B}^{-1/p}_{pp}(\Gamma_P)$ owing to duality. Finally, applying the norm of $\mathrm{H}^{1/2}_p(J)$, which incorporates the above differences, and taking into account the higher regularity of the coefficients a^k and D^k we obtain

$$\|z^T\cdot g\|_{\mathrm{H}^{1/2}_p(J;\mathrm{B}^{-1/p}_{pp}(\Gamma_P))} \le C\sum_{k=P,F}\Big\{\|a^k\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))}\|\psi^k\|_{\mathrm{H}^{1/2}_p(J;\mathrm{H}^1_p(\Omega_k))}$$
$$+ \|D^k\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k;\mathcal{B}(\mathbb{R}^N,)))}\|w^k\|_{\mathrm{H}^{1/2}_p(J;\mathrm{H}^1_p(\Omega_k;\mathbb{R}^N))}\Big\}.$$

The conditions 5. - 7. are consequences of Theorem 2.5, $\nabla\psi^k \in Z^T_{k,\nabla}$, $\psi^P - \psi^F \in Y^T_{1,\Gamma_P}$ and the embedding $\mathrm{H}^1_p(J) \hookrightarrow \mathrm{C}(J)$. The compatibility conditions 8. and 9. follow from the embeddings

$$Y^T_{1,\Gamma_P}(\mathbb{R}^N) \hookrightarrow \mathrm{C}(J;\mathrm{B}^{2-3/p}_{pp}(\Gamma_P;\mathbb{R}^N)), \quad Y^T_{2,\Gamma_P}(\mathbb{R}^N) \hookrightarrow \mathrm{C}(J;\mathrm{B}^{1-3/p}_{pp}(\mathbb{R}^N)).$$

Finally, the condition 10. results from the boundary value problem, see proof of the Lemma 1.1.

Step 2 - the sufficiency part. Let the data f^k, u^k_0, for $k = P, F, b$ and g, h^P, h^F be given. Assume that we have already determined the functions (w^P, w^F) and (ψ^P, ψ^F). Then we may employ Lemma 3.2 to the ode-equation resulting in $w^b \in Z^T_b$. To get $w^b \in E$ we have still to check that $f^b(t) - a^b\int_\Gamma \mathcal{B}_{F1}(D)w^F + \mathcal{B}_{F2}(D)\psi^F d\sigma$ lies in E. This can be seen by using compatibility condition 10. and identity (1.44) derived in the proof of Lemma 1.1.

Now, we will solve the linear problem (3.31) for a small time interval, that means, we choose an appropriate T such that all arguments work, which use this fact. This can be always reached by decomposing J into finitely many intervals $[ih,(i+1)h]$, $i = 0,\dots,I$, with h being sufficiently small. Then, we solve (3.31) in each of these intervals, as it is carried out for the time interval $[0,h]$. Every solution $w_{[ih,(i+1)h]}(t,x)$ belongs to the space of maximal regularity, since the function $w_{[ih,(i+1)h]}(ih,x)$ lies again in the space of initial data $V_P(E) \times V_F(E)$. Consequently, w belongs to $Z_P([0,T]) \times Z_F([0,T])$. The electrical potentials $\psi_{[ih,(i+1)h]}(t,x)$, $i = 0,\dots,I$ belong to $\mathcal{Z}$ on this interval and $\psi_{[ih,(i+1)h]}(ih,x)$ lies in $\mathrm{B}^{2-2/p}_{pp}(\Omega_P) \times \mathrm{B}^{2-2/p}_{pp}(\Omega_F)$. This can be seen by the embedding

$$Z_{k,\nabla} = \mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Omega_k)) \cap \mathrm{L}_p(J;\mathrm{H}^1_p(\Omega_k)) \hookrightarrow \mathrm{C}(J;D_{D_{n+1}}(1/2-1/p,p)) = \mathrm{C}(J;\mathrm{B}^{1-2/p}_{pp}(\Omega_k))\,,$$

where D_{n+1} denotes the negative Laplacian in $\mathrm{L}_p(\Omega_k)$. Since $\nabla\psi^k$ belongs to $Z_{k,\nabla}$ and the differential operator ∇ is a bounded mapping from $\mathrm{B}_{pp}^{2-2/p}(\Omega_k)$ into $\mathrm{B}_{pp}^{1-2/p}(\Omega_k)$, we may deduce $\psi_0^k \in V_k := \mathrm{B}_{pp}^{2-2/p}(\Omega_k)$. Consequently, ψ belongs to $\mathcal{Z}([0,T])$ due to connecting conditions.

(a) Localisation. Now, we will localise the problem as in Section 3.4.1. We choose a partition of unity $\varphi_j \in \mathrm{C}_0^\infty(\mathbb{R}^{n+1})$, $j = 1, \dots, M_4$, with $0 \le \varphi_j \le 1$ and $\operatorname{supp}\varphi_j \subset U_j$, such that the domain is covered in the following way

$$\overline{\Omega} \subset \bigcup_{j=1}^{M_4} U_j\,, \quad \Gamma_P \subset \bigcup_{j=M_1+1}^{M_2} U_j\,, \quad \Gamma \subset \bigcup_{j=M_3+1}^{M_4} U_j$$

$$\Omega_P \setminus \left(\bigcup_{j=M_1+1}^{M_2} U_j \right) \subset \bigcup_{j=1}^{M_1} U_j\,, \quad \Omega_F \setminus \left(\bigcup_{j=M_1+1}^{M_2} U_j \cup \bigcup_{j=M_3+1}^{M_4} U_j \right) \subset \bigcup_{j=M_2+1}^{M_3} U_j\,.$$

U_j are chosen as described in Section 3.4.1. Then (w, ψ) is a solution of (3.31) if and only if for $j = 1, \dots, M_4$ they satisfy parabolic equations in domain Ω_k

$$\partial_t(\varphi_j w^P) + \varphi_j \mathcal{A}_{P1}(D) w^P + \varphi_j \mathcal{A}_{P2}(D)\psi^P = \varphi_j f^P, \quad (t,x) \in J \times \Omega_P \cap U_j, \tag{3.36}$$

$$\partial_t(\varphi_j w^F) + \varphi_j \mathcal{A}_{F1}(D) w^F + \varphi_j \mathcal{A}_{F2}(D)\psi^F = \varphi_j f^F, \quad (t,x) \in J \times \Omega_F \cap U_j, \tag{3.37}$$

transmission condition on Γ_P

$$\varphi_j \mathcal{B}_{P1}(D) w^P + \varphi_j \mathcal{B}_{P2}(D)\psi^P = \varphi_j \mathcal{B}_{F1}(D) w^F + \varphi_j \mathcal{B}_{F2}(D)\psi^F + \varphi_j g, \quad (t,x) \in J \times \Gamma_P \cap U_j, \tag{3.38}$$

jump condition on Γ_P and boundary condition of Dirichlet type on Γ

$$\tilde{U}_P^{-1}(\varphi_j w^P) - \tilde{U}_F^{-1}(\varphi_j w^F) + \lambda z\big[\varphi_j \psi^P - \varphi_j \psi^F\big] = \varphi_j h^P, \quad (t,x) \in J \times \Gamma_P \cap U_j, \tag{3.39}$$

$$\varphi_j w^F = \varphi_j h^F, \quad \varphi_j \psi^F = 0, \quad (t,x) \in J \times \Gamma \cap U_j, \tag{3.40}$$

and (w^P, w^F) satisfy the initial data

$$\varphi_j w^P(0,x) = \varphi_j u_0^P(x), \quad x \in \overline{\Omega}_P \cap U_j, \quad \varphi_j w^F(0,x) = \varphi_j u_0^F(x), \quad x \in \overline{\Omega}_F \cap U_j, \tag{3.41}$$

and enjoys the electroneutrality condition

$$z^T \cdot \varphi_j w^P(0,x) = 0, \quad x \in \overline{\Omega}_P \cap U_j, \quad z^T \cdot \varphi_j w^F(0,x) = 0, \quad x \in \overline{\Omega}_F \cap U_j. \tag{3.42}$$

For the case $j = 1, \dots, M_1$ and $j = M_2 + 1, \dots, M_3$ boundary conditions do not appear, and we rewrite (3.36), (3.37) by commuting φ_j with differential operators in the form

$$\begin{aligned} \partial_t(\varphi_j w^k) + \mathcal{A}_{k1}(D)(\varphi_j w^k) + \mathcal{A}_{k2}(D)(\varphi_j \psi^k) &= \varphi_j f^k + C_j^k(w^k, \psi^k), \quad (t,x) \in J \times \Omega_k \cap U_j, \\ \varphi_j w^k(0,x) &= \varphi_j u_0^k(x), \quad x \in \overline{\Omega}_k \cap U_j, \\ z^T \cdot \varphi_j w^k(0,x) &= 0, \quad x \in \overline{\Omega}_k \cap U_j. \end{aligned} \tag{3.43}$$

Here one has to set $k = P$ for the case $j = 1, \dots, M_1$ and $k = F$ for $j = M_2 + 1, \dots, M_3$. The differential operators $\mathcal{A}_{k1}$ and $\mathcal{A}_{k2}$ are not yet split in a main part and lower terms

Furthermore, with a view to getting the invertibility of $-\Delta$ in the full space, we have shifted this operator by 1. In fact, we set

$$\begin{aligned}\mathcal{A}_{k1}(t,x,D)w &= -\nabla\cdot(D^k\nabla w)+D^k w\\ \mathcal{A}_{k2}(t,x,D)\psi &= -\nabla\cdot(M^k\tilde{u}^k\otimes\nabla\psi)+M^k\tilde{u}^k\psi\end{aligned}$$

All terms of lower order are combined as

$$\begin{aligned}C_j^k(t,x,w^k,\psi^k) &:= C_{j,1}^k(t,x,w^k)+C_{j,2}^k(t,x,\psi^k)\\ &:= [\mathcal{A}_{k1}(t,x,D),\varphi_j]w^k+\varphi_j D^k w^k\\ &\quad+[\mathcal{A}_{k2}(t,x,D),\varphi_j]\psi^k+\varphi_j M^k\tilde{u}^k\psi^k\\ &=\nabla\cdot(D^k w^k\nabla\varphi_j)+D^k\nabla w^k\cdot\nabla\varphi_j+\varphi_j D^k w^k\\ &\quad+\nabla\cdot(M^k\tilde{u}^k\psi^k\nabla\varphi_j)+M^k\tilde{u}^k\nabla\psi^k\cdot\nabla\varphi_j+\varphi_j M^k\tilde{u}^k\psi^k.\end{aligned}\tag{3.44}$$

(b) Full space problems. We turn to the localised evolution problem (3.43). By extension of partial differential operators to the whole space $\mathbb{R}^{n+1}$ as performed in (3.29), we obtain local operators $\mathcal{A}_{k1}^j(t,x,D)$, $\mathcal{A}_{k2}^j(t,x,D)$. After putting $w_j^k:=\varphi_j w^k$, $\psi_j^k:=\varphi_j\psi^k$ and $f_j^k=\varphi_j f^k+C_j^k(w^k,\psi^k)$ we can write

$$\begin{aligned}&\partial_t w_j^k+\mathcal{A}_{k1}^j(t,x,D)w_j^k+\mathcal{A}_{k2}^j(t,x,D)\psi_j^k=f_j^k,\quad (t,x)\in J\times\mathbb{R}^{n+1},\\ &w_j^k(0,x)=\varphi_j u_0^k(x),\quad z^T\!\cdot w_j^k=0\,,\quad (t,x)\in J\times\mathbb{R}^{n+1}\,.\end{aligned}\tag{3.45}$$

By using arguments of perturbation, this problem is solved by the full space problem considered in Section 3.1. In fact, employing Theorem 3.1 to the perturbed problem of (3.45) leads to

$$(w_j^k,\psi_j^k)=\mathcal{S}(f_j^k,\varphi_j u_0^k)+K_j^k(w_j^k,\psi_j^k),$$

with

$$\begin{aligned}K_j^k(w_j^k,\psi_j^k):=\mathcal{S}\Big(&\nabla\cdot\big([D_j^k(t,x)-D_j^k(0,x_j)]\,\nabla w_j^k\big)+[D_j^k(t,x)-D_j^k(0,x_j)]\,w_j^k\\ &+\nabla\cdot\big([(M_j^k\tilde{u}_j^k)(t,x)-(M_j^k\tilde{u}_j^k)(0,x_j)]\,\nabla\psi_j^k\big)\\ &+[(M_j^k\tilde{u}_j^k)(t,x)-(M_j^k\tilde{u}_j^k)(0,x_j)]\,\psi_j^k,\,0\Big)\end{aligned}$$

and $\mathcal{S}\in\mathcal{L}is(\mathcal{X}^T\times V(E_+),Z^T\times\mathcal{Z}^T)$. It is easily seen that $I-K_j^k\in\mathcal{B}({}_0Z^T\times{}_0\mathcal{Z}^T)$. The task consists in estimating K_j^k to aim at $\|K_j^k\|\le k<1$. By using maximal regularity of $\mathcal{S}$ it depends on estimating terms of perturbation in $\mathcal{X}^T$. The smallness of these terms can be obtained by continuity assumptions of the coefficients combined with the techniques of localisation. At first, we consider the perturbation in ψ^k. In the following, for the sake of simplicity we suppress the indices k and j, where it is possible. Having in mind the embedding

$\mathcal{Z}^T \hookrightarrow \mathrm{C}(J; \mathrm{C}^1(\mathbb{R}^{n+1}))$ we obtain

$$\begin{aligned}
\|\nabla\cdot([(M\tilde{u})(0,x_j)-(M\tilde{u})(\cdot,\cdot)]\nabla\psi)\|_{{}_0\mathcal{X}^T} &:= \|\nabla\cdot([(M\tilde{u})(0,x_j)-(M\tilde{u})(\cdot,\cdot)]\nabla\psi)\|_{X^T}+ \\
&\quad \|\nabla\cdot([a(0,x_j)-a(\cdot,\cdot)]\nabla\psi)\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{H}^{-1}_p(\mathbb{R}^{n+1}))} \\
&\le C\big(\|(M\tilde{u})(0,x_j)-(M\tilde{u})(\cdot,\cdot)\|_{\mathrm{C}(J;\mathrm{C}(\mathbb{R}^{n+1};\mathbb{R}^N))}\|\Delta\psi\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1}))} \\
&\quad + T^{1/2}\|\nabla(M\tilde{u})\|_{\mathrm{C}(J;\mathrm{C}(\mathbb{R}^{n+1};\mathbb{R}^{N\times n+1}))}\|\nabla\psi\|_{\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))}\big) \\
&\quad + \|\nabla\cdot([a(0,x_j)-a(\cdot,\cdot)]\nabla\psi)\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{H}^{-1}_p(\mathbb{R}^{n+1}))}
\end{aligned}$$

By using duality the last norm can be estimated by

$$\begin{aligned}
\|\nabla\cdot([a(0,x_j)-a(\cdot,\cdot)]\nabla\psi)\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{H}^{-1}_p(\mathbb{R}^{n+1}))} &\le \|a(0,y_j)-a(\cdot,\cdot)\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\mathbb{R}^{n+1}))} \\
&\quad \cdot\|\nabla\psi\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1};\mathbb{R}^{n+1}))} \\
&\le \|a(0,y_j)-a(\cdot,\cdot)\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\mathbb{R}^{n+1}))}\|\psi\|_{{}_0\mathcal{Z}^T}.
\end{aligned}$$

On the same lines the other terms of K^k_j can be treated, so that we obtain an estimation of the form

$$\|K^k_j(w^k_j,\psi^k_j)\|_{{}_0Z^T\times{}_0\mathcal{Z}^T} \le C(\varepsilon,T)\|(w^k_j,\psi^k_j)\|_{{}_0Z^T\times{}_0\mathcal{Z}^T}.$$

Choosing ε, T sufficiently small we achieve $C(\varepsilon,T)<1$ and consequently $\|K^k_j\|\le k<1$. Applying the Neumann series leads to the invertibility of $I-K^k_j$ in ${}_0Z^T\times{}_0\mathcal{Z}^T$. The operator $\mathcal{S}^k_j := (I-K^k_j)^{-1}\mathcal{S}$ defines again an isomorphism between space of data and $Z^T\times\mathcal{Z}^T$, i.e. we have

$$\mathcal{S}^k_j \in \mathcal{L}is(\mathcal{X}^T\times V(E_+), Z^T\times\mathcal{Z}^T),$$

where $(w^k_j,\psi^k_j)=\mathcal{S}^k_j(\varphi_j u^k_0,\, f^k_j)$ solves the problem (3.45). In order to enable a more convenient notation for writing the entire solution (w,ψ) we set

$$\begin{aligned}
\mathcal{S}_j &:= (\mathcal{S}^P_j,0), \quad j=1,\dots,M_1\,, \\
\mathcal{S}_j &:= (0,\mathcal{S}^F_j), \quad j=M_2+1,\dots,M_3\,.
\end{aligned}$$

(c) Half space problems. Next, we turn to $j=M_3+1,\dots,M_4$. Here we have to solve a parabolic problem with boundary condition of Dirichlet type. In fact, the equations (3.37), (3.40), (3.41) and (3.42) are left from system (3.31). After commuting φ_j with differential operators, using local coordinates as defined in Subsection 3.4.1 and perturbation, we obtain

$$\begin{aligned}
&\partial_t w^F_j - D^F_j(0,y_j)[\Delta-1]w^F_j - (M^F_j\tilde{u}^F_j)(0,y_j)\psi^F_j = \tilde{f}^F_j, \quad (t,y)\in J\times\mathbb{R}^{n+1}_+, \\
&w^F_j(t,y',0)=h^F_j(t,y'), \quad \psi^F_j(t,y',0)=0, \quad (t,y')\in J\times\mathbb{R}^n, \\
&w^F_j(0,y)=u^F_{0,j}(y), \quad y\in\mathbb{R}^{n+1}_+, \\
&z^T\cdot w^F_j(t,y)=0, \quad (t,y)\in J\times\mathbb{R}^{n+1}_+\,.
\end{aligned} \tag{3.46}$$

Here, we used the notations $w^F_j := G^j\varphi_j w^F$, $u^F_{0,j} := G^j\varphi_j u^F_0$, $h^F_j := G^j\varphi_j h^F$ and

$$\tilde{f}^F_j := f^F_j + \mathcal{A}^{j,\varepsilon}_F(D)(w^F_j,\psi^F_j) := G^j\varphi_j f^F + G^j C^F_j + \mathcal{A}^{j,\varepsilon}_F(D)(w^F_j,\psi^F_j).$$

The operator of perturbation C_j^F containing the commutators $[\mathcal{A}_{Fi}(D), \varphi_j]$, $i = 1, 2$, is defined by (3.44). The partial differential operator $\mathcal{A}_F^{j,\varepsilon}(D)$ comprises all terms of perturbation caused by changing coordinates and passing into constant coefficients. In order to specify this perturbation we are going to compute the transformation of an operator having a divergence form. It is easily verified that

$$\begin{aligned}-G^j\nabla_x \cdot \big(a\nabla_x u\big) = &- \nabla_y \cdot \big(a\nabla_y u\big) - \nabla_{y'} \cdot \big(a\,\partial_{y_{n+1}} u\,\nabla_{y'} h(y')\big)\\ &- \partial_{y_{n+1}}\big(a\nabla_{y'} u \cdot \nabla_{y'} h(y') + a\,\partial_{y_{n+1}} u\,|\nabla_{y'} h(y')|^2\big)\end{aligned}$$

where a denotes any coefficient. After putting

$$\begin{aligned}A^{j,sm}(a, D)u := &- \partial_{y_{n+1}}\big(a\nabla_{y'} u \cdot \nabla_{y'} h(y') + a\,\partial_{y_{n+1}} u\,|\nabla_{y'} h(y')|^2\big)\\ &- \nabla_{y'} \cdot \big(a\,\partial_{y_{n+1}} u\,\nabla_{y'} h(y')\big)\end{aligned} \tag{3.47}$$

the operator $\mathcal{A}_F^{j,\varepsilon}(D)$ can be written as follows

$$\begin{aligned}\mathcal{A}_F^{j,\varepsilon}(D)(w,\psi) := &- \nabla \cdot ([D_j^F(0,y_j) - D_j^F(t,y)]\nabla w) + [D_j^F(0,y_j) - D_j^F(t,y)]w\\ &- A^{j,sm}(D_j^F, D)w - \nabla \cdot ([(M_j^F\tilde{u}_j^F)(0,y_j) - (M_j^F\tilde{u}_j^F)(t,y)] \otimes \nabla\psi)\\ &+ [(M_j^F\tilde{u}_j^F)(0,y_j) - (M_j^F\tilde{u}_j^F)(t,y)]\psi - A^{j,sm}(M_j^F\tilde{u}_j^F, D)\psi.\end{aligned} \tag{3.48}$$

At first glance, we perceive that the divergence structure is maintained which is going to play an important role for studying these terms in $\mathcal{X}_+^T$. Employing Theorem 3.2 to this evolution problem leads to the equivalent formulation

$$(w_j^F, \psi_j^F) = \mathcal{S}_+(f_j^F, h_j^F, u_{0,j}^F) + \mathcal{S}_+(\mathcal{A}_F^{j,\varepsilon}(D)(w_j^F, \psi_j^F), 0, 0).$$

Note that the data $(h_j^F, u_{0,j}^F)$ and $(0,0)$ satisfy the compatibility condition. As in case of the full space we define the operator K_j^F by means of

$$K_j^F(w_j^F, \psi_j^F) := \mathcal{S}_+\left((\mathcal{A}_F^{j,\varepsilon}(D)(w_j^F, \psi_j^F), 0, 0)\right),$$

with $\mathcal{S}_+ \in \mathcal{L}is(\mathcal{X}_+^T \times \{(h, u_0) \in Y_1^T(E) \times V_+(E):\ h_{|t=0} = u_{0|_{y_{n+1}=0}}\}, Z_+^T \times \mathcal{Z}_{+,0}^T)$, and have to verify the smallness of $\|K_j^F\|$. All second order terms of $\mathcal{A}_F^{j,\varepsilon}(D)w$ having differences become small by applying the same arguments used in the full space problem. The perturbation operator $A^{j,sm}(D)$ contains the function h_j picturing the manifold Γ. By the construction of transformation G^j, we have arranged that $h_j \in \mathrm{C}^2(\mathbb{R}^n)$, $\operatorname{supp} h_j \subset \operatorname{supp}\varphi_j$ and $\nabla h_j(y_j') = 0$. The latter property gives rise to the smallness of $A^{j,sm}(D)$ in $\mathcal{X}^T$, where we have to exploit the divergence structure for estimating in $\mathrm{H}_p^{1/2}(J; \mathrm{H}_p^{-1}(\mathbb{R}_+^{n+1}))$. At first, we consider the space $X_+^T := \mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}_+^{n+1}; \mathbb{R}^N))$.

$$\begin{aligned}\|A^{j,sm}(M^F\tilde{u}^F, D)\psi^F\|_{X_+^T} \le\ &\|\nabla_{y'} h\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}\|\partial_{y_{n+1}}(M^F\tilde{u}^F \otimes \nabla_{y'}\psi^F)\|_{X_+^T}\\ &+ \|\nabla_{y'} h\|^2_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}\|\partial_{y_{n+1}}(M^F\tilde{u}^F\partial_{y_{n+1}}\psi^F)\|_{X_+^T}\\ &+ \|\nabla_{y'} h\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}\|\nabla_{y'}(M^F\tilde{u}^F\partial_{y_{n+1}}\psi^F)\|_{X_+^T}\\ &+ \|\Delta_{y'} h\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}\|M^F\tilde{u}^F\partial_{y_{n+1}}\psi^F\|_{X_+^T}\\ \le\ &c(\|\nabla_{y'} h\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)})\|\psi^F\|_{\mathcal{Z}_F^T} + C\|\Delta_{y'} h\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}T^{1/2}\|\psi^F\|_{\mathcal{Z}_F^T}\\ =\ &c(\varepsilon, T)\|\psi^F\|_{\mathcal{Z}_F^T}\end{aligned}$$

In order to estimate $z^T\!\cdot A^{j,sm}(M^F\tilde{u}^F\!, D)\psi^F$ in $\mathrm{H}^{1/2}_p(J;\mathrm{H}^{-1}_p(\mathbb{R}^{n+1}_+))$ we make use of duality arguments due to the divergence structure of this term. By doing so we obtain

$$\begin{aligned}\|z^T\!\cdot A^{j,sm}(M^F\tilde{u}^F\!, D)\psi^F\|_{\mathrm{H}^{1/2}_p(J;\mathrm{H}^{-1}_p(\mathbb{R}^{n+1}_+))} &\leq \|\nabla_{y'}h\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}\|a^F\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\mathbb{R}^{n+1}_+))} \\ &\cdot \big(\|\partial_{y_{n+1}}\psi^F\|_{\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1}_+))} + \|\nabla_{y'}\psi^F\|_{\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1}_+;\mathbb{R}^n))}\big) \\ + \|\nabla_{y'}h\|^2_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}\|a^F\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\mathbb{R}^{n+1}_+))}&\|\partial_{y_{n+1}}\psi^F\|_{\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1}_+))} \leq c(\varepsilon)\|\psi^F\|_{\mathcal{Z}^T_F}.\end{aligned}$$

Hence, we obtain an estimation of the form

$$\|A^{j,sm}(D^F_j, D)w^F_j + A^{j,sm}(M^F_j\tilde{u}^F_j, D)\psi^F_j\|_{{}_0\mathcal{X}^T_+} \leq c(\varepsilon, T)\|(w^F_j, \psi^F_j)\|_{{}_0Z^T_+\times{}_0\mathcal{Z}^T_+},$$

and after summarising all estimations we attain

$$\|K^F_j(w^F_j, \psi^F_j)\|_{{}_0Z^T_+\times{}_0\mathcal{Z}^T_+} \leq C(\varepsilon, T)\|(w^F_j, \psi^F_j)\|_{{}_0Z^T_+\times{}_0\mathcal{Z}^T_+},$$

with $C(\varepsilon, T) < 1$ for sufficiently small ε and T. Applying the Neumann series yields

$$(w^F_j, \psi^F_j) = (I - K^F_j)^{-1}\mathcal{S}_+(f^F_j, h^F_j, u^F_{0,j}).$$

We set $\mathcal{S}^F_j := (I - K^F_j)^{-1}\mathcal{S}_+$ and this operator defines an isomorphism between space of data and $Z^T_+ \times \mathcal{Z}^T_{+,0}$, i.e. we have

$$\mathcal{S}^F_j \in \mathcal{L}is(\mathcal{X}^T_+ \times \{(h, u_0) \in Y^T_1(E)\times V_+(E_+)\colon\, h_{|t=0} = u_{0|_{y_{n+1}=0}}\}, Z^T_+ \times \mathcal{Z}^T_{+,0}),$$

for $j = M_3 + 1, \ldots, M_4$; putting again $\mathcal{S}_j := (0, \mathcal{S}^F_j)$.

(d) Two phase problems. We are now concerned with the last case $j = M_1 + 1, \ldots, M_2$. For this we have to study a system for $(w^P\!, w^F)$ and $(\psi^P\!, \psi^F)$ coupled by boundary conditions on Γ_P. After proceeding as before, i.e. commuting again φ_j with differential operators and matrices, applying transformations G^j and perturbation, we obtain

$$\begin{aligned}&\partial_t w^P_j - D^P_j(0, y_j)[\Delta - 1]w^P_j - (M^P_j\tilde{u}^P_j)(0, y_j)[\Delta - 1]\psi^P_j = \tilde{f}^P_j, \quad (t, y) \in J \times \mathbb{R}^{n+1}_+ \\ &\partial_t w^F_j - D^F_j(0, y_j)[\Delta - 1]w^F_j - (M^F_j\tilde{u}^F_j)(0, y_j)[\Delta - 1]\psi^F_j = \tilde{f}^F_j, \quad (t, y) \in J \times \mathbb{R}^{n+1}_- \\ &- D^P_j(0, y_j)\partial_{y_{n+1}}w^P_j - (M^P_j\tilde{u}^P_j)(0, y_j)\partial_{y_{n+1}}\psi^P_j = -D^F_j(0, y_j)\partial_{y_{n+1}}w^F_j \\ &\qquad\qquad - (M^F_j\tilde{u}^F_j)(0, y_j)\partial_{y_{n+1}}\psi^F_j + \tilde{g}_j, \quad (t, y) \in J \times \mathbb{R}^n \times \{0\} \\ &\tilde{U}^{-1}_{P,j}(0, y_j)w^P_j - \tilde{U}^{-1}_{F,j}(0, y_j)w^F_j + \lambda_0 z(\psi^P_j - \psi^F_j) = \tilde{h}^P_j, \quad (t, y) \in J \times \mathbb{R}^n \times \{0\} \\ &w^P_j(0) = u^P_{0,j}, \quad y \in \mathbb{R}^{n+1}_+, \quad w^F_j(0) = u^F_{0,j}, \quad y \in \mathbb{R}^{n+1}_- \\ &z^T\!\cdot w^P_j = 0, \quad (t, y) \in J \times \mathbb{R}^{n+1}_+, \quad z^T\!\cdot w^F_j = 0, \quad (t, y) \in J \times \mathbb{R}^{n+1}_-\end{aligned} \tag{3.49}$$

The support of φ_j was transformed onto $\mathbb{R}^{n+1}$, where the boundary Γ_P turns into the hyperplane $\{(y', 0) : y' \in \mathbb{R}^n\}$. The functions $w^P_j = G^j\varphi_j w^P$ and $\psi^P_j = G^j\varphi_j\psi^P$ live in the upper half space $\mathbb{R}^{n+1}_+$, whereas $w^F_j = G^j\varphi_j w^F$ and $\psi^F_j = G^j\varphi_j\psi^F$ live in the lower half space $\mathbb{R}^{n+1}_-$. Furthermore, we have set $\tilde{U}^{-1}_{k,j}(t, y) := G^j\tilde{U}^{-1}_k(t, x)$ and $h^P_j := G^j\varphi_j h^P$. The right hand sides are defined by

$$
\begin{aligned}
\tilde{f}_j^k &:= f_j^k + \mathcal{A}_k^{j,\varepsilon}(D)(w_j^k, \psi_j^k) := G^j \varphi_j f^k + G^j C_j^k(w^k, \psi^k) + \mathcal{A}_k^{j,\varepsilon}(D)(w_j^k, \psi_j^k). \\
\tilde{g}_j &:= g_j + \mathcal{B}^{j,\varepsilon}(D)(w_j^P, \psi_j^P, w_j^F, \psi_j^F) := G^j \varphi_j g + G^j C_j(w^P, \psi^P, w^F, \psi^F) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad - \mathcal{B}^{j,\varepsilon}(D)(w_j^P, \psi_j^P, w_j^F, \psi_j^F) \\
\tilde{h}_j^P &:= h_j^P + \left(\tilde{U}_{P,j}^{-1}(0, y_j) - \tilde{U}_{P,j}^{-1}(t, y', 0)\right) w_j^P - \left(\tilde{U}_{F,j}^{-1}(0, y_j) - \tilde{U}_{F,j}^{-1}(t, y', 0)\right) w_j^F .
\end{aligned}
\tag{3.50}
$$

By C_j we have summarised all terms arising from interchanging φ_j with boundary operators $\mathcal{B}_{k1}(D)$, $k = P, F$ and $i = 1, 2$.

$$
\begin{aligned}
C_j(t, x, w^P, \psi^P, w^F, \psi^F) &:= C_{j,1}(t, x, w^P, w^F) + C_{j,2}(t, x, \psi^P, \psi^F) \\
&:= [\mathcal{B}_{P1}(t, x, D), \varphi_j]w^P + [\mathcal{B}_{P2}(t, x, D), \varphi_j]\psi^P \\
&\quad - [\mathcal{B}_{F1}(t, x, D), \varphi_j]w^F - [\mathcal{B}_{F2}(t, x, D), \varphi_j]\psi^F .
\end{aligned}
$$

The operators $\mathcal{A}_k^{j,\varepsilon}$, $k = P, F$ are defined in (3.48). So, it remains to explain the operator of perturbation $\mathcal{B}^{j,\varepsilon}(D)$ appearing in the transmission condition. The normal derivative $G^j \nabla u(x) \cdot \nu(x)$ transforms to $-\partial_{y_{n+1}}(G^j u) + B^{j,sm}(D)(G^j u)$, with

$$
B^{j,sm}(D)u := \nabla_{y'} u \cdot \nabla_{y'} h_j(y') - |\nabla_{y'} h_j(y')|^2 \partial_{y_{n+1}} u.
$$

Hence, $\mathcal{B}^{j,\varepsilon}(D)$ can be written in the form

$$
\begin{aligned}
\mathcal{B}^{j,\epsilon}(D)(w^P, \psi^P, w^F, \psi^F) &:= \sum_{k=P,F} \delta_k \Big\{ [D_j^k(0, y_j) - D_j^k(t, y', 0)]\partial_{n+1} w^k + [(M_j^k \tilde{u}_j^k)(0, y_j) - \\
&(M_j^k \tilde{u}_j^k)(t, y', 0)]\partial_{n+1}\psi^k - D_j^k(t, y', 0) B^{j,sm}(D) w^k - (M_j^k \tilde{u}_j^k)(t, y', 0) B^{j,sm}(D)\psi^k \Big\},
\end{aligned}
\tag{3.51}
$$

with $\delta_P = 1$ and $\delta_F = -1$.

Now, we turn our attention to problem (3.49). So according to Theorem 3.2 this evolution problem can be solved in the maximal regularity class $Z_+^T \times Z_+^T \times \mathcal{Z}_{+,-}^T$. Using the solution operator $\mathcal{S}_{+,-}$ we obtain the equivalent formulation

$$
(w_j^P, \psi_j^P, w_j^F, \psi_j^F) = \mathcal{S}_{+,-}\left(f_j^P, f_j^F, g_j, h_j^P, u_{0,j}^P, u_{0,j}^F\right) + K_j(w_j^P, \psi_j^P, w_j^F, \psi_j^F),
$$

with

$$
\begin{aligned}
K_j(w_j^P, \psi_j^P, w_j^F, \psi_j^F) := \mathcal{S}_{+,-}\Big(&\mathcal{A}_P^{j,\varepsilon}(w_j^P, \psi_j^P), \mathcal{A}_F^{j,\varepsilon}(w_j^F, \psi_j^F), \mathcal{B}^{j,\varepsilon}(w_j^P, \psi_j^P, w_j^F, \psi_j^F), \\
&[\tilde{U}_{P,j}^{-1}(0, y_j) - \tilde{U}_{P,j}^{-1}(t, y', 0)]w_j^P - [\tilde{U}_{F,j}^{-1}(0, y_j) - \tilde{U}_{F,j}^{-1}(t, y', 0)]w_j^F, 0, 0\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{S}_{+,-} \in \mathcal{L}is(\mathcal{X}_+^T \times \mathcal{X}_-^T \times \{\, (g, h, u_0^P, u_0^F) \in \mathcal{Y}_2^T \times Y_1^T(\mathbb{R}^N) \times V_+(E) \times V_-(E) \colon (g, h, u_0^P, u_0^F) \\
\text{enjoy the compatibility conditions 6. \& 7.}\}, Z_+^T \times Z_-^T \times \mathcal{Z}_{+,-}^T) .
\end{aligned}
$$

Note that the data enjoy the compatibility conditions of Theorem 3.3 as well as the perturbed data due to considering null initial data. The perturbation will be treated via a Neumann series. The smallness of operators $\mathcal{A}_k^{j,\varepsilon}(D)$ were already discussed when considering the full and half space problem. We now investigate the perturbation in the boundary condition of Dirichlet type. The postulated regularity of $\tilde{u}_j^k$ entails that $\tilde{U}_{k,j}^{-1}$ is a linear bounded operator in $Y_1^T(\mathbb{R}^N)$ as well as in Z_+^T, Z_-^T for $p > 2 + (n+1)$, see Proposition 2.3. Further on, using the continuity of the trace operator leads to

$$\sum_{k=P,F} \|[\tilde{U}_{k,j}^{-1}(0, y_j) - \tilde{U}_{k,j}^{-1}(\cdot,\cdot)]w_j^k\|_{{}_0Y_2^T(\mathbb{R}^N)} \le C(\|[\tilde{U}_{P,j}^{-1}(0, y_j) - \tilde{U}_{P,j}^{-1}(\cdot,\cdot)]w_j^P\|_{{}_0Z_+^T} + \|[\tilde{U}_{F,j}^{-1}(0, y_j) - \tilde{U}_{F,j}^{-1}(\cdot,\cdot)]w_j^F\|_{{}_0Z_-^T}).$$

In the following we consider the case $k = P$ and do without any indications. Proceeding as in the proof of Lemma 2.6 yields

$$\begin{aligned}
\|[\tilde{U}^{-1}(0, y_j) - \tilde{U}^{-1}]w\|_{{}_0Z_+^T} \le\ & C(\|\tilde{U}^{-1}(0, y_j) - \tilde{U}^{-1}\|_{\mathrm{C}(J;\mathrm{C}(\mathbb{R}_+^{n+1};\mathcal{B}(\mathbb{R}^N)))}\|\partial_t w\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N))} \\
& + \|w\|_{{}_0\mathrm{C}(J;\mathrm{C}(\mathbb{R}_+^{n+1};\mathbb{R}^N))}\|\partial_t\tilde{U}^{-1}\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathcal{B}(\mathbb{R}^N)))} \\
& + \|w\|_{\mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1};\mathbb{R}^N))}\|\tilde{U}^{-1}(0, y_j) - \tilde{U}^{-1}\|_{\mathrm{C}(J;\mathrm{C}(\mathbb{R}_+^{n+1};\mathcal{B}(\mathbb{R}^N)))} \\
& + \|w\|_{{}_0\mathrm{C}(J;\mathrm{C}(\mathbb{R}_+^{n+1};\mathbb{R}^N))}\|\tilde{U}^{-1}\|_{\mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1};\mathcal{B}(\mathbb{R}^N)))} \\
& + \|\tilde{U}^{-1}\|_{\mathrm{C}(J;\mathrm{C}^1(\mathbb{R}_+^{n+1};\mathcal{B}(\mathbb{R}^N)))}\|w\|_{\mathrm{L}_p(J;\mathrm{H}_p^1(\mathbb{R}_+^{n+1};\mathbb{R}^N))}) \\
\le\ & C(\|\tilde{U}^{-1}(0, y_j) - \tilde{U}^{-1}\|_{\mathrm{C}(J;\mathrm{C}(\mathbb{R}^{n+1};\mathcal{B}(\mathbb{R}^N)))} + T^{1/2})\|w\|_{{}_0Z_+^T}.
\end{aligned}$$

Lastly, it remains to examine the perturbation of boundary operators in the space $\mathcal{Y}_2^T$. At first, we have to ascertain the smallness of $\mathcal{B}^{j,\varepsilon}(D)(w, \psi)$ in $Y_2^T(\mathbb{R}^N)$. In view of continuity of the trace operator, i.e. $\gamma_{|\mathbb{R}^n} \in \mathcal{B}(Z_\nabla^T, Y_2^T)$, it suffices to study these terms in Z_∇^T. Terms containing differences are treated in a similar way as above, where the estimations in $\mathrm{H}_p^{1/2}(J; \mathrm{L}_p(\mathbb{R}_\pm^{n+1}))$ are coped with Lemma 2.6. Furthermore, by making use of the fact that $|\nabla_{y'} h_j(y')|$ tends to zero in view of $\nabla_{y'} h_j(y'_j) = 0$ we can control the operator $B^{j,sm}(D)$ in $Y_2^T(\mathbb{R}^N)$. Exploiting these facts we accomplish the desired result.

Now, we deal with the estimation of $z^T \cdot \mathcal{B}^{j,\varepsilon}(D)(w, \psi)$ in $\mathrm{H}_p^{1/2}(J; \mathrm{B}_{pp}^{-1/p}(\mathbb{R}))$, whereas in contrast to the above approaches we have not direct access. The purpose consists in deriving a relation between the perturbations $\mathcal{A}_k^{j,\varepsilon}(D)$ and $\mathcal{B}^{j,\varepsilon}(D)$. The idea bases on the fact that the perturbation $\mathcal{A}_k^{j,\varepsilon}(D)$ has divergence structure and in a way is compatible with the perturbation of transmission condition. At first, we examine the expression

$$\int_{\mathbb{R}_+^{n+1}} z^T \cdot \mathcal{A}_P^{j,\varepsilon}(D)(w_j^P, \psi_j^P)\tilde{v}\, dy + \int_{\mathbb{R}_-^{n+1}} z^T \cdot \mathcal{A}_F^{j,\varepsilon}(D)(w_j^F, \psi_j^F)\tilde{v}\, dy,$$

where $\tilde{v}$ denotes a test function belonging to the space $\overset{\circ}{\mathrm{H}}{}_{p'}^1(\mathbb{R}^{n+1})$. In the following we will suppress subscript j for the sake of simplicity. By means of the case $k = P$ we shall compute the above integral, see (3.48) for the definition of $\mathcal{A}_P^{j,\varepsilon}(D)$. Due to the divergence structure of this perturbation we can integrate by parts which entails

$$\int\limits_{\mathbb{R}^{n+1}_+} z^T\cdot \mathcal{A}_P^{j,\varepsilon}(D)(w^P,\psi^P)\tilde v\,dy = \int\limits_{\mathbb{R}^{n+1}_+}\left[z^T\cdot(D^P(0,y_j)-D^P(t,y))w^P+(a^P(0,y_j)-a^P(t,y))\psi^P\right]\tilde v\,dy$$

$$+\int\limits_{\mathbb{R}^n}\Big(z^T\cdot(D^P(0,y_j)-D^P(t,y',0))\partial_{y_{n+1}}w^P+(a^P(0,y_j)-a^P(t,y',0))\partial_{y_{n+1}}\psi^P\Big)\,\tilde v\,dy'$$

$$+\int\limits_{\mathbb{R}^{n+1}_+}\left(z^T\cdot(D^P(0,y_j)-D^P(t,y))\nabla w^P+(a^P(0,y_j)-a^P(t,y))\nabla\psi^P\right)\cdot\nabla\tilde v\,dy$$

$$+\int\limits_{\mathbb{R}^n}\Big(z^T\cdot D^P(t,y',0)\nabla_{y'}w^P\cdot\nabla_{y'}h(y')+z^T\cdot D^P(t,y',0)\partial_{y_{n+1}}w^P|\nabla_{y'}h(y')|^2\Big)\,\tilde v dy'$$

$$+\int\limits_{\mathbb{R}^{n+1}_+}\Big(z^T\cdot D^P(t,y)\nabla_{y'}w^P\cdot\nabla_{y'}h(y')+z^T\cdot D^P(t,y)\partial_{y_{n+1}}w^P|\nabla_{y'}h(y')|^2\Big)\,\partial_{y_{n+1}}\tilde v dy'$$

$$+\int\limits_{\mathbb{R}^n}\Big(a^P(t,y',0)\nabla_{y'}\psi^P\cdot\nabla_{y'}h(y')+a^P(t,y',0)\partial_{y_{n+1}}\psi^P|\nabla_{y'}h(y')|^2\Big)\,\tilde v dy'$$

$$+\int\limits_{\mathbb{R}^{n+1}_+}\Big(a^P(t,y)\nabla_{y'}\psi^P\cdot\nabla_{y'}h(y')+a^P(t,y)\partial_{y_{n+1}}\psi^P|\nabla_{y'}h(y')|^2\Big)\,\partial_{y_{n+1}}\tilde v dy'$$

$$+\int\limits_{\mathbb{R}^{n+1}_+}\Big(z^T\cdot D^P(t,y)\partial_{y_{n+1}}w^P\,\nabla_{y'}h(y')\Big)\cdot\nabla_{y'}\tilde v\,dy+\int\limits_{\mathbb{R}^{n+1}_+}\Big(a^P(t,y)\partial_{y_{n+1}}\psi^P\,\nabla_{y'}h(y')\Big)\cdot\nabla_{y'}\tilde v\,dy.$$

On closer inspection it turns out that the perturbation $\mathcal{B}^{j,\varepsilon}(D)(w_j^P,\psi_j^P,w_j^F,\psi_j^F)$ exactly coincides with functions of the above boundary integrals, of course by adding the case $k=F$. This fact results in the identity

$$\int\limits_{\mathbb{R}^{n+1}_+} z^T\cdot\mathcal{A}_P^{j,\varepsilon}(D)(w_j^P,\psi_j^P)\tilde v\,dy+\int\limits_{\mathbb{R}^{n+1}_-} z^T\cdot\mathcal{A}_F^{j,\varepsilon}(D)(w_j^F,\psi_j^F)\tilde v\,dy=\int\limits_{\mathbb{R}^n} z^T\cdot\mathcal{B}^{j,\varepsilon}(D)(w_j^P,\psi_j^P,w_j^F,\psi_j^F)\tilde v\,dy'$$
$$+I_\varepsilon(w_j^P,\psi_j^P,w_j^F,\psi_j^F,\tilde v),$$

where I_ε comprises the above integrals over half space $\mathbb{R}^{n+1}_+$ and in case $k=F$ over $\mathbb{R}^{n+1}_-$. The latter perturbation can be estimated by using Hölder's inequality as follows

$$I_\varepsilon(w_j^P,\psi_j^P,w_j^F,\psi_j^F,\tilde v)\le\|\tilde v\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\mathbb{R}^{n+1})}\sum_{k=P,F}\Bigg\{\Big[\|z^T\cdot(D_j^k(0,y_j)-D_j^k(t,\cdot))\|_{\mathrm{C}(\mathbb{R}^{n+1}_\pm;\mathbb{R}^N)}$$
$$+\|z^T\cdot D_j^k(t)\|_{\mathrm{C}(\mathbb{R}^{n+1}_\pm;\mathbb{R}^N)}(\|\nabla_{y'}h_j\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}+\|\nabla_{y'}h_j\|^2_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)})\Big]\,\|w_j^k(t)\|_{\mathrm{H}^1_p(\mathbb{R}^{n+1}_\pm;\mathbb{R}^N)}$$
$$+\Big[\|a_j^k(0,y_j)-a_j^k(t,\cdot)\|_{\mathrm{C}(\mathbb{R}^{n+1}_\pm)}+\|a_j^k(t)\|_{\mathrm{C}(\mathbb{R}^{n+1}_\pm)}(\|\nabla_{y'}h_j\|_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)}+\|\nabla_{y'}h_j\|^2_{\mathrm{C}(\mathbb{R}^n;\mathbb{R}^n)})\Big]\cdot$$
$$\|\psi_j^k\|_{\mathrm{H}^1_p(\mathbb{R}^{n+1}_\pm)}\Bigg\},$$

and thus

$$\|I_\varepsilon(w_j^P,\psi_j^P,w_j^F,\psi_j^F,\tilde v)\|_{\mathrm{H}_p^{1/2}(J)} \le c(\varepsilon)\|\tilde v\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\mathbb{R}^{n+1})}\|(w_j^P,w_j^F,\psi_j^P,\psi_j^F)\|_{Z_P^T\times Z_F^T\times \mathcal{Z}^T}.$$

By using duality concerning the perturbations $\mathcal{A}_k^{j,\varepsilon}(D)(w_j^k,\psi_j^k)$ we get

$$\begin{aligned}\int_{\mathbb{R}^n} z^T\cdot\mathcal{B}^{j,\varepsilon}(w_j^P,\psi_j^P,w_j^F,\psi_j^F)v\,dy' \le \|\tilde v\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\mathbb{R}^{n+1})}\Big\{&\|z^T\cdot\mathcal{A}_P^{j,\varepsilon}(D)(w_j^P,\psi_j^P)\|_{\mathrm{H}_p^{-1}(\mathbb{R}_+^{n+1})}\\
+\|z^T\cdot\mathcal{A}_F^{j,\varepsilon}(D)(w_j^F,\psi_j^F)\|_{\mathrm{H}_p^{-1}(\mathbb{R}_-^{n+1})}+c(\varepsilon)\big[&\|(w_j^P,w_j^F)\|_{\mathrm{H}_p^1(\mathbb{R}_+^{n+1};\mathbb{R}^N)\times\mathrm{H}_p^1(\mathbb{R}_-^{n+1};\mathbb{R}^N)}+\\
&\|(\psi_j^P,\psi_j^F)\|_{\mathrm{H}_p^1(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^1(\mathbb{R}_-^{n+1})}\big]\Big\},\end{aligned}$$

where we set $\tilde v_{|\mathbb{R}^n}=v$. As in the necessary part of this proof the above calculations can be carried out for the times $t+h$, $t\in J$, and we can again consider the difference of both expressions. After taking the infimum over $\|\tilde v\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\mathbb{R}^{n+1})}$ and using duality we obtain

$$\begin{aligned}\|z^T\cdot\mathcal{B}^{j,\varepsilon}(w_j^P,\psi_j^P,w_j^F,\psi_j^F)(t+h)-z^T\cdot\mathcal{B}^{j,\varepsilon}(w_j^P,\psi_j^P,w_j^F,\psi_j^F)(t)\|_{\mathrm{B}_{pp}^{-1/p}(\mathbb{R}^n)}\le\\
\|z^T\cdot\mathcal{A}_P^{j,\varepsilon}(D)(w_j^P,\psi_j^P)(t+h)-z^T\cdot\mathcal{A}_P^{j,\varepsilon}(D)(w_j^P,\psi_j^P)(t)\|_{\mathrm{H}_p^{-1}(\mathbb{R}_+^{n+1})}\\
+\|z^T\cdot\mathcal{A}_F^{j,\varepsilon}(D)(w_j^F,\psi_j^F)(t+h)-z^T\cdot\mathcal{A}_F^{j,\varepsilon}(D)(w_j^F,\psi_j^F)(t)\|_{\mathrm{H}_p^{-1}(\mathbb{R}_-^{n+1})}\\
+c(\varepsilon)\big\{\|(w_j^P,w_j^F)(t+h)-(w_j^P,w_j^F)(t)\|_{\mathrm{H}_p^1(\mathbb{R}_+^{n+1};\mathbb{R}^N)\times\mathrm{H}_p^1(\mathbb{R}_-^{n+1};\mathbb{R}^N)}+\\
\|(\psi_j^P,\psi_j^F)(t+h)-(\psi_j^P,\psi_j^F)(t)\|_{\mathrm{H}_p^1(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^1(\mathbb{R}_-^{n+1})}\big\}.\end{aligned}$$

We now employ that equivalent norm of $\mathrm{H}_p^{1/2}(J)$ which is characterised by differences. Thus we attain $\mathcal{B}^{j,\varepsilon}(w_j^P,\psi_j^P,w_j^F,\psi_j^F)\in{}_0\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\mathbb{R}^n))$, and this operator can be compared with perturbations coming from the half spaces $\mathbb{R}_+^{n+1}$ and $\mathbb{R}_-^{n+1}$. In fact, we have shown the following estimate

$$\begin{aligned}\|z^T\cdot\mathcal{B}^{j,\varepsilon}(w_j^P,\psi_j^P,w_j^F,\psi_j^F)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\mathbb{R}^n))}\le\|z^T\cdot\mathcal{A}_P^{j,\varepsilon}(D)(w_j^P,\psi_j^P)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\mathbb{R}_+^{n+1}))}+\\
\|z^T\cdot\mathcal{A}_F^{j,\varepsilon}(D)(w_j^F,\psi_j^F)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\mathbb{R}_-^{n+1}))}+c(\varepsilon)\big\{\|(w_j^P,w_j^F)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\mathbb{R}_+^{n+1};\mathbb{R}^N)\times\mathrm{H}_p^1(\mathbb{R}_-^{n+1};\mathbb{R}^N))}\\
+\|(\psi_j^P,\psi_j^F)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^1(\mathbb{R}_-^{n+1}))}\big\}\\
\le C(T,\varepsilon)\|(w_j^P,w_j^F,\psi_j^P,\psi_j^F)\ \|_{{}_0Z_+^T\times{}_0Z_-^T\times{}_0\mathcal{Z}_{+,-}^T}.\end{aligned}$$

All things considered, we have achieved an estimation of the form $\|K_j\|\le k_0<1$. After employing the Neumann series, we obtain a unique solution $(w_j^P,\psi_j^P,w_j^F,\psi_j^F)$ of (3.49) given by

$$(w_j^P,\psi_j^P,w_j^F,\psi_j^F)=\mathcal{S}_j\left(f_j^P,f_j^F,g_j,h_j^P,u_{0,j}^P,u_{0,j}^F\right),$$

with $\mathcal{S}_j:=[I-K_j]^{-1}\mathcal{S}_{+,-}$ and

$$\begin{aligned}\mathcal{S}_j\in\mathcal{L}is(\mathcal{X}_+^T\times\mathcal{X}_-^T\times\{\,(g,h,u_0^P,u_0^F)\in\mathcal{Y}_2^T\times Y_1^T(\mathbb{R}^N)\times V_+(E_+)\times V_-(E_+):\ (g,h,u_0^P,u_0^F)\\
\text{enjoy the compatibility conditions }\},Z_+^T\times Z_-^T\times\mathcal{Z}_{+,-}^T).\end{aligned}$$

(e) Problem in domain. Now we address the issue of constructing a function with the aid of local solutions. Subsequently, we have to check that the so designed function solves (3.31) and is unique. For this purpose we choose another partition of unity $\tilde{\varphi}_j \in C_0^\infty(\mathbb{R}^n)$ such that $\tilde{\varphi}_j \equiv 1$ on $\operatorname{supp}\varphi_j$ and $\operatorname{supp}\tilde{\varphi}_j \subset U_j$. Summing over j we receive a solution formula of the linear problem.

$$(w,\psi) = \sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j G^j \varphi_j \rho + \sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j G^j \mathcal{C}_j(w,\psi) := \mathcal{S}_0 \varrho + \mathcal{S}_1(w,\psi) \tag{3.52}$$

The inhomogeneities and initial data data were summarised to $\varrho := (f^P, f^F, g, h^P, h^F, u_0^P, u_0^F)$, whereas $\mathcal{C}_j$ comprises all terms of lower order, i.e. we set

$$\mathcal{C}_j(w,\psi) := \begin{cases} (C_j^P(w^P,\psi^P), 0) : & j = 1, \dots, M_1 \\ (C_j^P(w^P,\psi^P), C_j^F(w^F,\psi^F), C_j(w^P,\psi^P,w^F,\psi^F), 0, 0, 0) : & j = M_1+1, \dots, M_2 \\ (C_j^F(w^F,\psi^F), 0) : & j = M_2+1, \dots, M_3 \\ (C_j^F(w^F,\psi^F), 0, 0) : & j = M_3+1, \dots, M_4 \end{cases} \tag{3.53}$$

Furthermore, we want to remind of the mapping properties of solution operators $\mathcal{S}_j$.

$$\begin{aligned} &\mathcal{S}_j \in \mathcal{L}is(\mathcal{X}^T \times V(E_+), Z^T \times \mathcal{Z}^T), \quad j = 1, \dots, M_1 \text{ and } j = M_2+1, \dots, M_3, \\ &\mathcal{S}_j \in \mathcal{L}is\big(\mathcal{X}_+^T \times \mathcal{X}_-^T \times \{(g,h,u_0^P,u_0^F) \in \mathcal{Y}_2^T \times Y_1^T(\mathbb{R}^N) \times V_+(E_+) \times V_-(E_+) :\ (g,h,u_0^P,u_0^F) \\ &\qquad\qquad \text{enjoy the compatibility conditions }\}, Z_+^T \times Z_-^T \times \mathcal{Z}_{+,-}^T\big), \quad j = M_1+1, \dots, M_2, \\ &\mathcal{S}_j \in \mathcal{L}is(\mathcal{X}_+^T \times \{(h,u_0) \in Y_1^T(E) \times V_+(E_+) :\ h_{|t=0} = u_{0|_{y_{n+1}=0}}\}, Z_+^T \times \mathcal{Z}_{+,0}^T), \\ &\qquad j = M_3+1, \dots, M_4 \,. \end{aligned}$$

The two-phase problem (3.3) can be written abstractly as

$$\mathcal{L}(w,\psi) := \varrho\,. \tag{3.54}$$

We have to establish that solution formula (3.52) leads to the inverse operator of $\mathcal{L}$. Firstly, we prove the existence of a left inverse, i.e. the only solution of $(w,\psi) = \mathcal{S}_0\varrho + \mathcal{S}_1(w,\psi)$ has to vanish for zero data. The goal consists in establishing that $I - \mathcal{S}_1$ is invertible in ${}_0Z_P^T \times {}_0Z_F^T \times {}_0\mathcal{Z}^T$. We are going to see that the operator of perturbation $\mathcal{S}_1$ can not treated as before, namely by applying the Neumann series. It turns out that not all lower order terms become small for T tending to zero. This circumstance is caused by the elliptic problem for ψ which is involved due to the electroneutrality condition. Therefore we have to study this problem separately. In spite of this fact we want to estimate $\mathcal{S}_1$ in order to which terms do not become small. By using the mapping properties of solution operators $\mathcal{S}_j$ we have

$$\begin{aligned} \|\mathcal{S}_1(w,\psi)\|_{{}_0Z_P^T \times {}_0Z_F^T \times {}_0\mathcal{Z}^T} \le C \Bigg(&\sum_{j=1}^{M_1} \|C_j^P\|_{{}_0\mathcal{X}_P^T} + \sum_{j=M_1+1}^{M_2} \sum_{k=P,F} \|C_j^k\|_{{}_0\mathcal{X}_k^T} + \|C_j\|_{{}_0\mathcal{Y}_2^T} \\ &+ \sum_{j=M_2+1}^{M_4} \|C_j^F\|_{{}_0\mathcal{X}_F^T} \Bigg)\,. \end{aligned}$$

We start considering $C_j(w,\psi)$ in ${}_0Y^T_{2,\Gamma_P}$. Lower order terms with respect to the transmission condition are the functions w^k and ψ^k with certain coefficients. In fact, we have set

$$\begin{aligned}C_j(w,\psi) &:= C_{j,1}(w^P,w^F) + C_{j,2}(\psi^P,\psi^F)\\ &:= D^P w^P \partial_\nu \varphi_j + M^P \tilde{u}^P \psi^P \partial_\nu \varphi_j - D^F w^F \partial_\nu \varphi_j - M^F \tilde{u}^F \psi^F \partial_\nu \varphi_j.\end{aligned}$$

Due to the continuity of the trace operator $\gamma_{|_{\Gamma_P}}$ we are able to carry out the estimations in Z^T_∇ resulting in

$$\|D^k w^k \partial_\nu \varphi_j\|_{{}_0Y^T_{2,\Gamma_P}(\mathbb{R}^N)} \le C\|w^k\|_{{}_0Z^T_{k,\nabla}} \le CT^{1/2}\|w^k\|_{{}_0Z^T_k}.$$

To treat lower order terms of ψ^k, we have to take into account $\gamma_{|_{\Gamma_P}}\psi^k \in {}_0\mathrm{H}^{1/2}_p(J;\mathrm{B}^{1-1/p}_{pp}(\Gamma_P))$ and the continuous embeddings

$$\mathrm{H}^{1/2}_p(J) \hookrightarrow \mathrm{B}^{1/2}_{pp}(J) \hookrightarrow \mathrm{B}^{1/2-1/2p}_{pp}(J), \quad p \ge 2.$$

We thus get

$$\begin{aligned}\|M^k\tilde{u}^k\psi^k\partial_\nu\varphi_j\|_{{}_0Y^T_{2,\Gamma_P}(\mathbb{R}^N)} &\le C\|M^k\tilde{u}^k\|_{\mathcal{U}^T_k(\mathbb{R}^N)}\Big(\|\psi^k\|_{{}_0\mathrm{B}^{1/2-1/2p}_{pp}(J;\mathrm{L}_p(\Gamma_P))} + \|\psi^k\|_{\mathrm{L}_p(J;\mathrm{B}^{1-1/p}_{pp}(\Gamma_P))}\Big)\\ &\le C(T^{1/2p}\|\psi^k\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Gamma_P))} + T^{1/2}\|\psi^k\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{B}^{1-1/p}_{pp}(\Gamma_P))})\\ &\le C(T^{1/2p}+T^{1/2})\|\psi^k\|_{{}_0\mathcal{Z}^T_k}.\end{aligned}$$

Now, we want to tackle the estimation of $z^T\cdot C_j(w,\psi)$ in $\mathrm{H}^{1/2}_p(J;\mathrm{B}^{-1/p}_{pp}(\Gamma_P))$. Observe that the estimate

$$\|z^T\cdot D^k w^k\partial_\nu\varphi_j\|_{\mathrm{H}^{1/2}_p(J;\mathrm{B}^{-1/p}_{pp}(\Gamma_P))} \le C\|z^T\cdot D^k w^k\|_{\mathrm{H}^{1/2}_p(J;\mathrm{B}^{2-2\theta-1/p}_{pp}(\Gamma_P))}$$

holds for $1 > 2-2\theta-1/p > 0$ and $\theta > 1/2$. Then, we continue with

$$\begin{aligned}\|z^T\cdot D^k w^k\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{B}^{2-2\theta-1/p}_{pp}(\Gamma_P))} &\le C\|D^k\|_{\mathrm{C}^{1/2}(J;\mathrm{C}^1(\Omega_k;\mathcal{B}(\mathbb{R}^N)))}\|w^k\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{H}^{2-2\theta}_p(\Omega_k))}\\ &\le CT^{\theta-1/2}\|w^k\|_{{}_0\mathrm{H}^\theta_p(J;\mathrm{H}^{2-2\theta}_p(\Omega_k))} \le CT^{\theta-1/2}\|w^k\|_{{}_0Z^T_k},\end{aligned}$$

where $1-1/2p > \theta > 1/2$. Carrying out this estimation for ψ^k does not lead to a factor involving a power of T. In fact, we obtain

$$\begin{aligned}\|z^T\cdot M^k\tilde{u}^k\psi^k\partial_\nu\varphi_j\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{B}^{-1/p}_{pp}(\Gamma_P))} &\le C\|a^k\psi^k\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Gamma_P))} \le C\|\psi^k\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Gamma_P))}\\ &\le C\|\psi^k\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{H}^s_p(\Omega_k))},\end{aligned}$$

for $1/p < s \le 1$. Since ψ has not additional time regularity we can not follow up the above estimation. Now, we want to discuss $C^k_j(w^k,\psi^k)$ in $\mathcal{X}^T_k$ to aim at gaining a constant tending to zero for $T \to 0$. Observe that

$$\begin{aligned}C^k_j &= \nabla\cdot(D^k w^k\nabla\varphi_j) + D^k\nabla w^k\cdot\nabla\varphi_j + \varphi_j D^k w^k\\ &\quad + \nabla\cdot(M^k\tilde{u}^k\psi^k\nabla\varphi_j) + M^k\tilde{u}^k\nabla\psi^k\cdot\nabla\varphi_j + \varphi_j M^k\tilde{u}^k\psi^k.\end{aligned}$$

In the following we use the embedding $Z_k^T, \mathcal{Z}_k^T \hookrightarrow \mathrm{C}(J; \mathrm{C}^1(\overline{\Omega}_k))$, cf. Lemma 2.5, regularity of the coefficients, and the additional temporal regularity which provides the constant $T^{1/2}$.

$$
\begin{aligned}
\|C_j^k\|_{X_k^T} &\leq C\left(\|w^k\|_{X_k^T} + \|\nabla w^k\|_{X_k^T} + \|\psi^k\|_{X_k^T} + \|\nabla\psi^k\|_{X_k^T}\right) \\
&\leq CT^{1/2}\left(\|\nabla w^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} + \|\psi^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} + \|\nabla\psi^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))}\right) \\
&\quad + CT\|w^k\|_{{}_0\mathrm{H}_p^{1}(J;\mathrm{L}_p(\Omega_k))} \\
&\leq CT^{1/2}\left(\|w^k\|_{{}_0Z_k^T} + \|\psi^k\|_{{}_0\mathcal{Z}_k^T}\right)
\end{aligned}
$$

For estimating $z^T\!\cdot C_j^k$ in $\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))$ with intent to provide a small constant we will use the duality pairing. In order to keep the effort small, we pick the "worst terms" concerning regularity. We shall consider $(z^T\!\cdot D^k\nabla w^k + a^k\nabla\psi^k)\nabla\varphi_j$ and $\nabla\cdot[(z^T\!\cdot D^k w^k + a^k\psi^k)\nabla\varphi_j]$. Taking into account the smoothness assumption of d_i^k, $a^k = z^T M^k\tilde{u}^k$, using Lemma 2.2 and $\tilde{u}^k \in Z_k(E_+) \hookrightarrow \mathcal{U}_k^T(E_+)$ which imply $a^k \in \mathcal{U}_k^T$, we obtain

$$
\begin{aligned}
&\|\nabla\cdot[(z^T\!\cdot D^k w^k + a^k\psi^k)\nabla\varphi_j]\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))} \leq C\|z^T\!\cdot D^k w^k + a^k\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} \\
&\quad\leq C\left(\|D^k\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))}\|w^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} + \|a^k\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))}\|\psi^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))}\right) \\
&\qquad\qquad\leq CT^{1/2}\|w^k\|_{{}_0\mathrm{H}_p^{1}(J;\mathrm{L}_p(\Omega_k))} + C\|\psi^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))}.
\end{aligned}
$$

Now, we are going to treat $(z^T\!\cdot D^k\nabla w^k + a^k\nabla\psi^k)\nabla\varphi_j$, where we use the above arguments.

$$
\begin{aligned}
&\|z^T D^k\nabla w^k\nabla\varphi_j + a^k\nabla\psi^k\nabla\varphi_j\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))} \leq C\|D^k\|_{\mathrm{C}^{1/2}(J;\mathrm{C}^1(\Omega_k))}\|w^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} \\
&\qquad + C\|\|a^k\|_{\mathrm{H}_p^1(\Omega_k)}\|\psi^k\|_{\mathrm{L}_p(\Omega_k)}\|_{{}_0\mathrm{H}_p^{1/2}(J)} \leq CT^{1/2}\|w^k\|_{{}_0\mathrm{H}_p^1(J;\mathrm{L}_p(\Omega_k))} \\
&\quad + \|a^k\|_{\mathrm{C}(J;\mathrm{H}_p^1(\Omega_k))}\|\psi^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} + \|a^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\Omega_k))}\|\psi^k\|_{{}_0\mathrm{C}(J;\mathrm{L}_p(\Omega_k))} \\
&\qquad\qquad \leq C\big(T^{1/2}\|w^k\|_{{}_0Z_k^T} + T^{1/2-1/p}\|\psi^k\|_{{}_0\mathcal{Z}_k^T} + \|\psi^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))}\big).
\end{aligned}
$$

The above inequalities lead to

$$
\|C_j^k\|_{\mathcal{X}_k^T} \leq c(T)\|(w^k,\psi^k)\|_{{}_0Z_k^T\times{}_0\mathcal{Z}_k^T} + C\|\psi^k\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))},
$$

with $c(T)\to 0$ for $T\to 0$. Finally, summarising all estimations yields

$$
\|\mathcal{S}_1(w,\psi)\|_{{}_0Z_P^T\times{}_0Z_F^T\times{}_0\mathcal{Z}^T} \leq c(T)\|(w,\psi)\|_{{}_0Z_P^T\times{}_0Z_F^T\times{}_0\mathcal{Z}^T} + C\|\psi\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^s(\Omega_P)\times\mathrm{H}_p^s(\Omega_F))},
$$

which shows that not all terms become small for T tending to zero. This fact is a natural consequence of the elliptic problem caused by the electroneutrality condition. More precisely, up to now we have not checked on solvability of the boundary value problem in domain Ω for the electrical potentials (ψ^P,ψ^F). Therefore we have to study this problem separately.

(f) An elliptic problem. We shall consider the following two phase problem

$$
\begin{aligned}
&\lambda\psi^P - \nabla\cdot(a^P\nabla\psi^P) = g^P, \quad (t,x)\in J\times\Omega_P, \\
&\lambda\psi^F - \nabla\cdot(a^F\nabla\psi^F) = g^F, \quad (t,x)\in J\times\Omega_F, \\
&a^P\partial_\nu\psi^P - a^F\partial_\nu\psi^F = h_N, \quad (t,x)\in J\times\Gamma_P, \\
&\psi^P - \psi^F = h_D, \quad (t,x)\in J\times\Gamma_P, \\
&\psi^F = h_D^F, \quad (t,x)\in J\times\Gamma,
\end{aligned}
\tag{3.55}
$$

where $\lambda > 0$, and the variable $t \in J$ can be seen as a parameter. We are looking for solutions in space $\mathcal{Z}_P \times \mathcal{Z}_F$ which is to be equipped with the norm

$$\|(\psi^P, \psi^F)\|_\lambda := \sum_{k=P,F} \|\psi^k\|_{\mathcal{Z}_k} + \lambda \|\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k)) \cap \mathrm{L}_p(J;\mathrm{L}_p(\Omega_k))}.$$

The inhomogeneities are summarised to σ belonging to $\mathrm{H}_p^{1/2}(J;\mathcal{X}_\omega) \cap \mathrm{L}_p(J;\mathcal{X})$, where we set

$$\begin{aligned}
\mathcal{X}_\omega &:= \mathrm{H}_p^{-1}(\Omega_P) \times \mathrm{H}_p^{-1}(\Omega_F) \times \mathrm{B}_{pp}^{-1/p}(\Gamma_P) \times \mathrm{B}_{pp}^{1-1/p}(\Gamma_P) \times \mathrm{B}_{pp}^{1-1/p}(\Gamma),\\
\mathcal{X} &:= \mathrm{L}_p(\Omega_P) \times \mathrm{L}_p(\Omega_F) \times \mathrm{B}_{pp}^{1-1/p}(\Gamma_P) \times \mathrm{B}_{pp}^{2-1/p}(\Gamma_P) \times \mathrm{B}_{pp}^{2-1/p}(\Gamma).
\end{aligned}$$

We shall associate this two phase problem with the abstract equation

$$\lambda \mathcal{J}(\psi^P, \psi^F) + \mathcal{E}(t,D)(\psi^P, \psi^F) = \sigma,$$

with

$$\mathcal{J} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T.$$

The existence and uniqueness result reads as follows.

Proposition 3.1 *Let Ω_P, Ω_F be open bounded domains in $\mathbb{R}^{n+1}$ with C^2- boundary, $\Gamma_P := \partial\Omega_P$, $\partial\Omega_F = \Gamma_P \cup \Gamma$ and* $\operatorname{dist}(\Gamma_P, \Gamma) > 0$. *Let $J = [0,T]$ and $2 + (n+1) < p < \infty$. Suppose that the assumptions (1.23)-(1.25) are satisfied and $\tilde{u}$ belongs to $Z_P^T \times Z_F^T$. Then the boundary value problem (3.55), for $\lambda = 0$, has exactly one solution $(\psi^P, \psi^F) \in \mathcal{Z}_P^T \times \mathcal{Z}_F^T$ if and only if the data $\sigma := (g^P, g^F, h_N, h_D, h_D^F)$ belongs to $\mathrm{H}_p^{1/2}(J;\mathcal{X}_\omega) \cap \mathrm{L}_p(J;\mathcal{X})$. Moreover, if σ even lies in $\mathrm{H}_p^{1/2}(J;\mathcal{X})$ then the unique solution (ψ^P, ψ^F) belongs to*

$$\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\Omega_P)) \times \mathrm{H}_p^{1/2}(J;\mathrm{H}_{p,\Gamma}^2(\Omega_F)).$$

Proof of the Proposition. Since the necessary part of the proof is obviously, we directly address the sufficiency part. The problem (3.55) can be localised as at the beginning of this section, so that we obtain full and half space problems. Proceeding as in the proofs of Theorems (3.1)-(3.3) yields solution formulae for the local functions (ψ_j^P, ψ_j^F). Note that all coefficients contained in the operator $\mathcal{E}(t,D)$ possess enough time regularity for considering the parameter t in spaces $\mathrm{H}_p^\theta(J)$, $\theta \in [0,1/2]$. More precisely, it follows that

$$\begin{aligned}
\mathcal{E}(\cdot, D) &\in \mathrm{H}_p^{1/2}(J;\mathcal{B}(\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F), \mathcal{X})),\\
\mathcal{E}(\cdot, D) &\in \mathrm{C}^{1/2}(J;\mathcal{B}(\mathrm{H}_p^1(\Omega_P) \times \mathrm{H}_{p,\Gamma}^1(\Omega_F), \mathcal{X}_\omega)).
\end{aligned} \tag{3.56}$$

Consequences of these properties are

$$\begin{aligned}
&\mathcal{E} \in \mathcal{B}(\mathrm{H}_p^\theta(J;\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F)), \mathrm{H}_p^\theta(J;\mathcal{X})), \quad \theta \in [0,1/2]\\
&\mathcal{E} \in \mathcal{B}(\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\Omega_P) \times \mathrm{H}_{p,\Gamma}^1(\Omega_F)), \mathrm{H}_p^{1/2}(J;\mathcal{X}_\omega)),
\end{aligned} \tag{3.57}$$

where the first assertion results from Lemma 2.2. In fact, we have by interpolation

$$\begin{aligned}
\|\mathcal{E}\psi\|_{\mathrm{H}_p^\theta(J;\mathcal{X})} &\le \| \|\mathcal{E}(\cdot)\|_{\mathcal{B}(\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F), \mathcal{X})} \|\psi(\cdot)\|_{\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F)} \|_{\mathrm{H}_p^\theta(J)}\\
&\le C(\|\mathcal{E}\|_{\mathrm{H}_p^{1/2}(J;\mathcal{B}(\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F), \mathcal{X}))} \|\psi\|_{\mathrm{C}(J;\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F))}\\
&\quad + \|\mathcal{E}\|_{\mathrm{C}(J;\mathcal{B}(\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F), \mathcal{X}))} \|\psi\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F))}).
\end{aligned}$$

We shall prove the first assertion of (3.56). We remind of the regularity assumption of the coefficient a^k, i.e. we have $a^k \in Z_k^T$ and $a^k \in \mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\Omega_k))$. Let $\psi \in \mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F))$ be given. By using Lemma 2.2 we may estimate as follows

$$\begin{aligned}
\|\mathcal{E}\psi\|_{\mathrm{H}_p^{1/2}(J;\mathcal{X})} \leq & \sum_{k=P,F} (\|z^T\cdot\mathcal{A}_{k2}(D)\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} + \|z^T\cdot\mathcal{B}_{k2}(D)\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{1-1/p}(\Gamma_P))}) \\
& + \|\psi^P - \psi^F\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{2-1/p}(\Gamma_P))} \\
\leq & \sum_{k=P,F} \|a^k\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))}\| \;\Delta\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} \\
& + \|\nabla a^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))}\|\nabla\psi^k\|_{\mathrm{C}(J\times\overline{\Omega}_k)} + \|\nabla a^k\|_{\mathrm{C}(J\times\overline{\Omega}_k)}\|\nabla\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_k))} \\
& \|a^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{1-1/p}(\Gamma_P))}\|\partial_\nu\psi^k\|_{\mathrm{C}(J\times\Gamma_P)} + \|a^k\|_{\mathrm{C}(J\times\Gamma_P)}\|\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{1-1/p}(\Gamma_P))} \\
& + \|\psi^P\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{2-1/p}(\Gamma_P))} + \|\psi^F\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{2-1/p}(\Gamma_P))} \\
\leq & \, C\|\psi\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\Omega_P)\times\mathrm{H}_p^2(\Omega_F))}.
\end{aligned}$$

We turn to the local problems. The perturbed problems can be treated as it was carried out for the primary problem. By doing so we attain the presentation

$$\psi = \sum_j \tilde{\varphi}_j (G^j)^{-1}\mathcal{S}_j^e G^j \varphi_j \sigma + \sum_j \tilde{\varphi}_j (G^j)^{-1}\mathcal{S}_j^e G^j \mathcal{C}_j^e(\psi) =: L\sigma + L^\varepsilon\psi,$$

where $\mathcal{S}_j^e$ denotes the local solution operators (the local resolvents). The lower order terms $\mathcal{C}_j^e(\psi)$ are given by

$$\mathcal{C}_j^e(\psi) := \left\{ \begin{array}{ll} [z^T\cdot\mathcal{A}_{P2}(t,D),\varphi_j]\psi^P : & j = 1,\dots,M_1 \\ \big([z^T\cdot\mathcal{A}_{P2}(t,D),\varphi_j]\psi^P, [z^T\cdot\mathcal{A}_{F2}(t,D),\varphi_j]\psi^F, & \\ \ [z^T\cdot\mathcal{B}_{P2}(t,D),\varphi_j]\psi^P - [z^T\cdot\mathcal{B}_{F2}(t,D),\varphi_j]\psi^F, 0, 0\big) : & j = M_1+1,\dots,M_2 \\ [z^T\cdot\mathcal{A}_{F2}(t,D),\varphi_j]\psi^F : & j = M_2+1,\dots,M_3 \\ ([z^T\cdot\mathcal{A}_{F2}(t,D),\varphi_j]\psi^F, 0) : & j = M_3+1,\dots,M_4 \end{array} \right. ,$$

which coincide with the perturbations $z^T\cdot C_{j,2}^k(\psi^k)$ and $z^T\cdot C_{j,2}(\psi^P,\psi^F)$ (!). Now, we have to establish that the above solution formula leads to the inverse operator of $\lambda\mathcal{J} + \mathcal{E}$. Firstly, we prove the existence of a left inverse. The goal consists in establishing that $I - L^\varepsilon$ is invertible in $\mathcal{Z}_P \times \mathcal{Z}_F$ by means of Neumann's series. By using the mapping properties of the resolvents $\mathcal{S}_j^e$ we obtain

$$\begin{aligned}
\|L^\varepsilon\psi\|_\lambda &= \|L^\varepsilon\psi\|_{\mathcal{Z}_P\times\mathcal{Z}_F} + \lambda\|L^\varepsilon\psi\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_P)\times\mathrm{H}_p^{-1}(\Omega_F))\cap\mathrm{L}_p(J;\mathrm{L}_p(\Omega_P)\times\mathrm{L}_p(\Omega_F))} \\
&\leq C\sum_{j=1}^{M_4} \|\mathcal{C}_j^e(\psi)\|_{\mathrm{H}_p^{1/2}(J;\mathcal{X}_\omega)\cap\mathrm{L}_p(J;\mathcal{X})},
\end{aligned}$$

and continue with

$$\begin{aligned}\sum_{j=1}^{M_4}\|\mathcal{C}_j^e(\psi)\|_{\mathrm{H}_p^{1/2}(J;\mathcal{X}_\omega)\cap\mathrm{L}_p(J;\mathcal{X})} \le C\{&\sum_{j=1}^{M_1}\|[z^T\cdot\mathcal{A}_{P2}(D),\varphi_j]\psi^P\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_P))\cap\mathrm{L}_p(J;\mathrm{L}_p(\Omega_P))}\\
&+\sum_{j=M_1+1}^{M_2}\big[\sum_{k=P,F}\|[z^T\cdot\mathcal{A}_{k2}(D),\varphi_j]\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))\cap\mathrm{L}_p(J;\mathrm{L}_p(\Omega_k))}\\
&\quad+\|[z^T\cdot\mathcal{B}_{k2}(D),\varphi_j]\psi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\Gamma_P))\cap\mathrm{L}_p(J;\mathrm{B}_{pp}^{1-1/p}(\Gamma_P))}\big]\\
&\quad+\sum_{j=M_2+1}^{M_4}\|[z^T\cdot\mathcal{A}_{F2}(D),\varphi_j]\psi^F\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_F))}\}\\
\le C_1\|(\psi^P,\psi^F)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^s(\Omega_P)\times\mathrm{H}_p^s(\Omega_F))}&+C_2\|(\psi^P,\psi^F)\|_{\mathrm{L}_p(J;\mathrm{H}_p^1(\Omega_P)\times\mathrm{H}_p^1(\Omega_F))},\end{aligned}$$

with $s-1/p>0$, cf. the estimation of $C_j^k(w,\psi)$ and $C_j(w,\psi)$. Furthermore, there is $\theta\in(0,1)$ so that $\mathrm{H}_p^s(\Omega_k)=[\mathrm{H}_p^{-1}(\Omega_k),\mathrm{H}_p^1(\Omega_k)]_\theta$, $\mathrm{H}_p^1(\Omega_k)=[\mathrm{L}_p(\Omega_k),\mathrm{H}_p^2(\Omega_k)]_{1/2}$, and the interpolation inequalities

$$\|\psi^k\|_{\mathrm{H}_p^s(\Omega_k)}\le c_\theta\|\psi^k\|_{\mathrm{H}_p^{-1}(\Omega_k)}^{1-\theta}\|\psi^k\|_{\mathrm{H}_p^1(\Omega_k)}^{\theta},\quad \|\psi^k\|_{\mathrm{H}_p^1(\Omega_k)}\le c\|\psi^k\|_{\mathrm{L}_p(\Omega_k)}^{1/2}\|\psi^k\|_{\mathrm{H}_p^2(\Omega_k)}^{1/2}$$

hold. Hence, by means of Young's inequality we obtain

$$\begin{aligned}\|\psi^k\|_{\mathrm{H}_p^s(\Omega_k)}&\le C_\varepsilon\|\psi^k\|_{\mathrm{H}_p^{-1}(\Omega_k)}+\varepsilon\|\psi^k\|_{\mathrm{H}_p^1(\Omega_k)},\\
\|\psi^k\|_{\mathrm{H}_p^1(\Omega_k)}&\le C_\varepsilon\|\psi^k\|_{\mathrm{L}_p(\Omega_k)}+\varepsilon\|\psi^k\|_{\mathrm{H}_p^2(\Omega_k)},\end{aligned}$$

and consequently

$$\begin{aligned}\|L^\varepsilon\psi\|_{\mathcal{Z}_P\times\mathcal{Z}_F}\le C&\Big[\varepsilon\|(\psi^P,\psi^F)\|_{\mathcal{Z}_P\times\mathcal{Z}_F}+C_\varepsilon(\|(\psi^P,\psi^F)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_P)\times\mathrm{H}_p^{-1}(\Omega_F))}\\
&+\|(\psi^P,\psi^F)\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega_P))\times\mathrm{L}_p(J;\mathrm{L}_p(\Omega_F))})\Big]\\
\le C&\max\{\varepsilon,\frac{C_\varepsilon}{\lambda}\}\|\psi\|_{\mathcal{Z}_\lambda}.\end{aligned}$$

Choosing ε sufficiently small and $\lambda>0$ large enough we may invert $I-L^\varepsilon$ due to Neumann's series and thus ψ has to vanish for data $\sigma=0$.

To show that $\lambda\mathcal{J}+\mathcal{E}(t,D)$ is surjective, i.e. there also exists a right inverse, we have to establish that the function $\psi=(I-L^\varepsilon)^{-1}L\sigma$ satisfying $\psi=L\sigma+L^\varepsilon\psi$ solves the problem (3.55). Applying $\lambda\mathcal{J}+\mathcal{E}(D)$ to solution formula and commuting $\tilde\varphi_j$ with $\lambda\mathcal{J}+\mathcal{E}(t,D)$ yields

$$\begin{aligned}(\lambda\mathcal{J}+\mathcal{E}(t,D))\psi=&\sum_j\tilde\varphi_j(G^j)^{-1}(\lambda\mathcal{J}+\mathcal{E}_j(t,D))\mathcal{S}_j^e\{G^j\varphi_j\sigma+G^j\mathcal{C}_j^e(\psi)\}\\
&+\sum_j[\mathcal{E}(t,D),\tilde\varphi_j](G^j)^{-1}\mathcal{S}_j^e\{G^j\varphi_j\sigma+G^j\mathcal{C}_j^e(\psi)\},\end{aligned}$$

where $\mathcal{E}_j(t,D)$ denotes the localised operator of $\mathcal{E}(t,D)$. Since $\psi_j=\mathcal{S}_j^e\{G^j\varphi_j\sigma+G^j\mathcal{C}_j^e(\psi)\}$ solves the problem $(\lambda\mathcal{J}+\mathcal{E}_j(t,D))\psi_j=G^j\varphi_j\sigma+G^j\mathcal{C}_j^e(\psi)$ we obtain

$$(\lambda\mathcal{J}+\mathcal{E}(t,D))\psi=\sigma+\sum_j[\mathcal{E}(t,D),\tilde\varphi_j](G^j)^{-1}\mathcal{S}_j^e\{G^j\varphi_j\sigma+G^j\mathcal{C}_j^e(\psi)\}+\sum_j\tilde\varphi_j\mathcal{C}_j^e(\psi).$$

Furthermore, the operator of perturbation $\mathcal{C}_j^e$ inherits the compact support of φ_j which entails

$$\sum_j \tilde{\varphi}_j \mathcal{C}_j^e \psi = \sum_j \mathcal{C}_j^e \psi = \sum_j [\mathcal{E}(t,D), \varphi_j]\psi = 0.$$

By using the left inverse resp. the representation $\psi = (I - L^\varepsilon)^{-1} L\sigma$, we get the identity

$$(\lambda\mathcal{J} + \mathcal{E}(t,D))(I - L^\varepsilon)^{-1}L\sigma = \sigma + \sum_j [\mathcal{E}(t,D), \tilde{\varphi}_j](G^j)^{-1}\mathcal{S}_j^e\{G^j\varphi_j + G^j\mathcal{C}_j^e(I - L^\varepsilon)^{-1}L\}\sigma.$$

Now, we set $K(t)\sigma := \sum_j [\mathcal{E}(t,D), \tilde{\varphi}_j](G^j)^{-1}\mathcal{S}_j^e\left\{G^j\varphi_j + \mathcal{C}_j^e(I - L^\varepsilon)^{-1}L\right\}\sigma$ and define $\overline{\sigma} := \sigma + K(t)\sigma$. The operator K only contains lower order terms so that we achieve $\|K\| \le k < 1$ by choosing λ large enough. Employing the Neumann's series we are able to rewrite the above problem as follows

$$(\lambda\mathcal{J} + \mathcal{E}(t,D))(I - L^\varepsilon)^{-1}L(I + K(t))^{-1}\overline{\sigma} = \overline{\sigma}, \tag{3.58}$$

which shows that (3.58) gives rise to a right inverse $(I - L^\varepsilon)^{-1}L(I + K(t))^{-1}$ of $\lambda\mathcal{J} + \mathcal{E}(t,D)$, i.e. we have surjectivity. Since right and left inverse have to be equal, it must hold that $K(t) = 0$. We have proved that there is $\lambda > 0$ so that problem (3.55) has a unique solution, i.e. $\lambda \in \rho_{\mathcal{J}}(\mathcal{E}(t,D))$ for all $t \in J$.

Now, we are going to show that $0 \in \rho_{\mathcal{J}}(\mathcal{E}(t,D))$. It suffices to establish the injectivity in L_2 in view of the embedding $L_p \subset L_2$, for $p > 2$ and bounded domains. For this, let $(\psi^P, \psi^F) \in \mathcal{D}(\mathcal{E}(t,D))$ be given with $\mathcal{E}(t,D)(\psi^P, \psi^F) = 0$. We multiply the first differential equation by ψ^P and the second one by ψ^F. Integrating and summing up of both equations yields

$$\int_{\Omega_P} \nabla\cdot(a^P\nabla\psi^P)\psi^P\, dx + \int_{\Omega_F} \nabla\cdot(a^F\nabla\psi^F)\psi^F\, dx = 0\,.$$

By using the *Gaussian divergence theorem* we then obtain

$$\sum_{k=P,F} \int_{\Omega_k} a^k|\nabla\psi^k|^2\, dx = \int_{\Gamma_P} a^P\partial_\nu\psi^P\psi^P - a^F\partial_\nu\psi^F\cdot\psi^F\, d\sigma + \int_{\Gamma} a^F\partial_\nu\psi^F\cdot\psi^F\, d\sigma\,.$$

Taking into account boundary conditions of $\mathcal{E}(t,D)$ we see

$$\int_{\Omega_P} a^P|\nabla\psi^P|^2\, dx + \int_{\Omega_F} a^F|\nabla\psi^F|^2\, dx = 0\,.$$

Thus, it follows that $\psi^P = c_P$ and $\psi_2^F = c_F$. If we once again use the boundary condition $\psi^P - \psi^F = 0$ for $x \in \Gamma_P$ and $\psi^F = 0$ for $x \in \Gamma$, we deduce $c_P = c_F = 0$. Hence the operator $\mathcal{E}(t,D)$ is injective. The idea of establishing surjectivity bases on the following facts.

Claim 1. Let X, Y and Z be three Banach spaces. Let A be a bounded operator from Z to X. Supposing that $B \in \mathcal{B}(Z,Y)$ is a retraction, i.e. there exists an operator (corretraction) $B^c \in \mathcal{B}(Y,Z)$ so that $BB^c = id$, it follows that the operator $\mathcal{A} := (A,B)$ is an isomorphism from Z onto $X \times Y$ if and only if $A_B := A$ with $D(A_B) = \{u \in Z : Bu = 0\}$ is an isomorphism from $D(A_B)$ onto X.

Proof of the claim. The equivalence of both problems arises from the existence of a corretraction B^c which implies surjectivity of B. Thus the problem $(A, B)w = (f, g)$ is equivalent to $Av = f - AB^c g =: \tilde{f}, Bv = 0$, and $w = v + B^c g$.

Claim 2. Let $A : X_A \subset X \to X$ be linear and injective, where X_A denotes the domain $D(A)$ equipped with the graph norm of A. Assume that $\rho(A) \neq \varnothing$ and X_A is compactly embedded into X. Then the operator A is surjective, i.e. $R(A) = X$.

Proof of the claim. Let $\lambda \in \rho(A)$. Then we can write A in the following way

$$A = \lambda + A - \lambda = (\lambda + A)\left[I - \lambda(\lambda + A)^{-1}\right] =: (\lambda + A)(I + K) .$$

The operator K defined by $K := -\lambda(\lambda + A)^{-1}$ belongs to $\mathcal{K}(X_A)$, the set of compact linear operators of X_A into X_A, due to compact embedding $X_A \hookrightarrow\hookrightarrow X$. The Fredholm alternative states that $R(I + K)$ is closed and $ind(K) = 0$, i.e. $dimN(I + K) = codim(I + K)$. If we can show $dimN(I + K) = 0$, then in virtue of $R(I + K) = \overline{R(I + K)}$ we obtain that $I + K$ is surjective. Let $x \in X_A$, $x \neq 0$ and satisfy $x + Kx = 0$. By definition of K this equation is equivalent to $Ax = 0$, but injectivity of A implies $x = 0$. Since λ belongs to the resolvent set, we know that $\lambda + A$ is surjective, in fact $(\lambda + A)X_A = X$. The remarks above also showed that $(I + K)X_A = X_A$, consequently we may conclude

$$AX_A = (\lambda + A)(I + K)X_A = (\lambda + A)X_A = X ,$$

which means that A is surjective.

$\square$

By studying the problem (3.55) we have seen that $\rho_{\mathcal{J}}(\mathcal{E}(t, D)) \neq \varnothing$ which in particular implies that the boundary operators are retractions. Hence, by Claim 1 it suffices to consider the elliptic problem with homogeneous boundary conditions. Furthermore, the solution spaces $\mathrm{H}^1_p(\Omega_P) \times \mathrm{H}^1_{p,\Gamma}(\Omega_F)$ and $\mathrm{H}^2_p(\Omega_P) \times \mathrm{H}^2_{p,\Gamma}(\Omega_F)$ are compactly embedded into $\mathrm{H}^{-1}_p(\Omega_P) \times \mathrm{H}^{-1}_p(\Omega_F)$ and $\mathrm{L}_p(\Omega_P) \times \mathrm{L}_p(\Omega_F)$, respectively. According to the second claim we may conclude that $0 \in \rho(\mathcal{E}(t, D))$ for all $t \in J$ which is equivalent to $0 \in \rho_{\mathcal{J}}(\mathcal{E}(t, D))$ due to the first claim. Combining these results with (3.57) yields

$$\mathcal{E} \in \mathcal{L}is(\mathrm{H}^\theta_p(J; \mathrm{H}^2_p(\Omega_P) \times \mathrm{H}^2_{p,\Gamma}(\Omega_F)), \mathrm{H}^\theta_p(J; \mathcal{X})), \quad \theta \in [0, 1/2],$$
$$\mathcal{E} \in \mathcal{L}is(\mathrm{H}^{1/2}_p(J; \mathrm{H}^1_p(\Omega_P) \times \mathrm{H}^1_{p,\Gamma}(\Omega_F)), \mathrm{H}^{1/2}_p(J; \mathcal{X}_\omega)),$$

which shows $(\psi^P, \psi^F) \in \mathcal{Z}_P \times \mathcal{Z}_F$ by choosing $\theta = 0$, whereas the choice $\theta = 1/2$ implies the second statement of the proposition.

$\square$

Let us return to the solution formula (3.52). We shall split the lower order terms $G^j\mathcal{C}_j(w,\psi)$ in three parts.

$$G^j\mathcal{C}_j(w, \psi) = G^j\mathcal{C}_{j,1}(w) + [G^j\mathcal{C}_{j,2}(\psi) - \mathcal{T}_j\tilde{\psi}_j] + \mathcal{T}_j\tilde{\psi}_j$$

The first part comprises only terms of $w = (w^P, w^F)$, which become small due to the more temporal regularity, cp. the estimation of $\mathcal{C}_j(w, \psi)$. The second part is composed of all lower order terms of $\psi = (\psi^P, \psi^F)$ denoted by $\mathcal{C}_{j,2}(\psi)$ and a suitable function $\tilde{\psi}_j$, so that this expression belongs to the space of electroneutrality E. This has the advantage that we only need to estimate this term in space $X^T_P \times X^T_F \times Y^T_2$. Note that the lower order terms of ψ

become small in this spaces. Now, we comment on the function $\tilde{\psi}_j$ and the operator $\mathcal{T}_j$. We put

$$\begin{aligned}
\mathcal{T}_j\tilde{\psi}_j^P &:= \mathcal{A}^j_{P2}(t,D)\tilde{\psi}_j^P : && j=1,\dots,M_1\\
\mathcal{T}_j(\tilde{\psi}_j^P,\tilde{\psi}_j^F) &:= \big(\mathcal{A}^j_{P2}(t,D)\tilde{\psi}_j^P,\ \mathcal{A}^j_{F2}(t,D)\tilde{\psi}_j^F,\\
&\qquad \mathcal{B}^j_{P2}(t,D)\tilde{\psi}_j^P-\mathcal{B}^j_{F2}(t,D)\tilde{\psi}_j^F,\ \lambda_0 z\gamma_{|\mathbb{R}^n}(\tilde{\psi}_j^P-\tilde{\psi}_j^F)\big) : && j=M_1+1,\dots,M_2\\
\mathcal{T}_j\tilde{\psi}_j^F &:= \mathcal{A}^j_{F2}(t,D)\tilde{\psi}_j^F : && j=M_2+1,\dots,M_3\\
\mathcal{T}_j\tilde{\psi}_j^F &:= \big(\mathcal{A}^j_{F2}(t,D)\tilde{\psi}_j^F,\ \gamma_{|\mathbb{R}^n}\tilde{\psi}_j^F\big) : && j=M_3+1,\dots,M_4.
\end{aligned}$$

To ensure that $G^j\mathcal{C}_{j,2}(\psi)-\mathcal{T}_j\tilde{\psi}_j$ lies in E the function $\tilde{\psi}_j$ has to solve the local elliptic problems $z^T\cdot\mathcal{T}_j\tilde{\psi}_j = G^j z^T\cdot\mathcal{C}_{j,2}(\psi)$, i.e.

$$\begin{aligned}
z^T\cdot\mathcal{A}^j_{P2}(t,D)\tilde{\psi}_j^P &= G^j z^T\cdot C^P_{j,2}(\psi^P), && (t,y)\in J\times\mathbb{R}^{n+1} : && j=1,\dots,M_1\\
z^T\cdot\mathcal{A}^j_{F2}(t,D)\tilde{\psi}_j^F &= G^j z^T\cdot C^F_{j,2}(\psi^F), && (t,y)\in J\times\mathbb{R}^{n+1} : && j=M_2+1,\dots,M_3\\
z^T\cdot\mathcal{A}^j_{F2}(t,D)\tilde{\psi}_j^F &= G^j z^T\cdot C^F_{j,2}(\psi^F), && (t,y)\in J\times\mathbb{R}^{n+1}_+, &&\\
\tilde{\psi}_j^F &= 0, && (t,y)\in J\times\mathbb{R}^n\times\{0\} : && j=M_3+1,\dots,M_4
\end{aligned}$$

and for $j=M_1+1,\dots,M_2$

$$\begin{aligned}
z^T\cdot\mathcal{A}^j_{P2}(t,D)\tilde{\psi}_j^P &= G^j z^T\cdot C^P_{j,2}(\psi^P), \quad (t,y)\in J\times\mathbb{R}^{n+1}_+,\\
z^T\cdot\mathcal{A}^j_{F2}(t,D)\tilde{\psi}_j^F &= G^j z^T\cdot C^F_{j,2}(\psi^F), \quad (t,y)\in J\times\mathbb{R}^{n+1}_-\\
z^T\cdot\mathcal{B}^j_{P2}(t,D)\tilde{\psi}_j^P - z^T\cdot\mathcal{B}^j_{F2}(t,D)\tilde{\psi}_j^F &= G^j z^T\cdot C_{j,2}(\psi^P,\psi^F), \quad (t,y)\in J\times\mathbb{R}^n\times\{0\},\\
\tilde{\psi}_j^P-\tilde{\psi}_j^F &= 0, \quad (t,y)\in J\times\mathbb{R}^n\times\{0\}.
\end{aligned}$$

Observe that the inhomogeneities $G^jC^k_{j,2}(\psi^k)$ belong to $\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1}))$ respectively in the half spaces $\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1}_\pm))$, and $G^jC_{j,2}(\psi^P,\psi^F)\in\mathrm{H}^{1/2}_p(J;\mathrm{B}^{1-1/p}_{pp}(\mathbb{R}^n))$. According to the second statement of Proposition 3.1, the unique solutions $\tilde{\psi}_j$ of these local problems belong to

$$\begin{aligned}
\mathrm{H}^{1/2}_p(J;\mathrm{H}^2_p(\mathbb{R}^{n+1})), &\quad \text{for } j=1,\dots,M_1,M_2+1,\dots,M_3,\\
\mathrm{H}^{1/2}_p(J;\mathrm{H}^2_p(\mathbb{R}^{n+1}_+)\times\mathrm{H}^2_p(\mathbb{R}^{n+1}_-)), &\quad \text{for } j=M_1+1,\dots,M_2,\\
\mathrm{H}^{1/2}_p(J;\mathrm{H}^2_p(\mathbb{R}^{n+1}_+)\cap\overset{\circ}{\mathrm{H}}{}^1_p(\mathbb{R}^{n+1}_+)), &\quad \text{for } j=M_3+1,\dots,M_4.
\end{aligned}$$

Now, we are in the position to treat the solution formula (3.52), i.e. to show injectivity. Considering vanishing data, $\rho=0$, and using the above decomposition of $\mathcal{C}_j(w,\psi)$ leads to the representation

$$(w,\psi)=\sum_{j=1}^{M_4}\tilde{\varphi}_j(G^j)^{-1}\mathcal{S}_jG^j\mathcal{C}_{j,1}(w)+\sum_{j=1}^{M_4}\tilde{\varphi}_j(G^j)^{-1}\mathcal{S}_j[G^j\mathcal{C}_{j,2}(\psi)-\mathcal{T}_j\tilde{\psi}_j]+\sum_{j=1}^{M_4}\tilde{\varphi}_j(G^j)^{-1}\mathcal{S}_j\mathcal{T}_j\tilde{\psi}_j.$$

At first, we want to delve into the last sum. The functions $(w_j,\psi_j)=\mathcal{S}_j\mathcal{T}_j\tilde{\psi}_j$ solve local full, half and two phase problems arising from the localisation, where the inhomogeneities are given by $\mathcal{T}_j\tilde{\psi}_j$. If we decompose the local electrical potentials as follows $\psi_j = \overline{\psi}_j + \tilde{\psi}_j$ then we see that $(w_j, \psi_j) = \mathcal{S}_j\mathcal{T}_j\tilde{\psi}_j$ is equivalent to

$$(w_j, \overline{\psi}_j) = \mathcal{S}_j 0 \equiv 0 \quad \text{and} \quad \tilde{\psi}_j = S_j^e G^j z^T \cdot \mathcal{C}_{j,2}(\psi).$$

Here $S_j^e \equiv (z^T \cdot \mathcal{T}_j)^{-1}$, $j = 1, \ldots, M_4$, denote the local solution operators (resolvents) of the local elliptic problems. Consequently, we obtain

$$\sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j \mathcal{T}_j \tilde{\psi}_j = \left(0, \sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j^e G^j z^T \cdot \mathcal{C}_{j,2}(\psi)\right) =: (0, K(t)\psi),$$

and thus

$$(w, \psi - K(t)\psi) = \sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j G^j \mathcal{C}_{j,1}(w) + \sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j [G^j \mathcal{C}_{j,2}(\psi) - \mathcal{T}_j \tilde{\psi}_j].$$

It is obvious that $K(t)$ is a compact operator for all $t \in J$ in view of the lower order terms $\mathcal{C}_{j,2}(\psi)$. More precisely, we have

$$K(\cdot) \in \mathrm{C}(J; \mathcal{K}(\mathrm{H}_p^2(\Omega_P) \times \mathrm{H}_{p,\Gamma}^2(\Omega_F))) \cap \mathrm{C}^{1/2}(J; \mathcal{K}(\mathrm{H}_p^1(\Omega_P) \times \mathrm{H}_{p,\Gamma}^1(\Omega_F))),$$

where $\mathcal{K}(X)$ denotes the set of all compact operators from X into X. The time regularity of $K(t)$ arises from the regularity of the coefficients a^k. In Proposition 3.1 we have seen that these coefficients have enough time regularity in order to study the elliptic problem in $\mathrm{H}_p^\theta(J)$, $\theta \in [0, 1/2]$. This fact implies the above property of $K(t)$.

Thus, the goal consists in using the Fredholm alternative in order to invert $I - K(t)$ and by the above time regularity to conclude that $(I - K(\cdot))^{-1}$ is bounded in $\mathcal{Z}_P \times \mathcal{Z}_F$. Therefore, we have to establish that $I - K(t)$ is injective, i.e. $\psi - K(t)\psi = 0$ implies $\psi = 0$. However, this property is also an trivial consequence of $0 \in \rho_{\mathcal{J}}(\mathcal{E}(t, D))$. Observe that $\mathcal{E}(t, D)\psi = 0$ is equivalent to

$$\mathcal{E}_j(t, D) G^j \varphi_j \psi = G^j \mathcal{C}_j^e(\psi) \equiv G^j z^T \cdot \mathcal{C}_{j,2}(\psi).$$

After employing the resolvent $\mathcal{S}_j^e$ and summing up we obtain

$$\psi = \sum_j \varphi_j (G^j)^{-1} \mathcal{S}_j^e G^j z^T \cdot \mathcal{C}_{j,2}(\psi) \equiv K(t)\psi.$$

The equivalence of the latter action arises from $0 \in \rho_{\mathcal{J}}(\mathcal{E}(t, D))$. Since $\mathcal{E}(t, D)\psi = 0$ is only satisfied for $\psi = 0$ we deduce that $I - K(t)$ is injective for all $t \in J$. We have achieved

$$\begin{aligned}(w, \psi) = (I, (I - K(t))^{-1}) &\left\{ \sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j G^j \mathcal{C}_{j,1}(w) + \sum_{j=1}^{M_4} \tilde{\varphi}_j (G^j)^{-1} \mathcal{S}_j [G^j \mathcal{C}_{j,2}(\psi) - \mathcal{T}_j \tilde{\psi}_j] \right\} \\ =: \mathcal{S}_\varepsilon(w, \psi).\end{aligned}$$

Now, we are able to estimate the right hand side so that all terms become small for T tending to zero.

$$\begin{aligned}\|\mathcal{S}_\varepsilon(w,\psi)\|_{{}_0Z_P^T\times{}_0Z_F^T\times{}_0\mathcal{Z}^T} \le C\|(I-K)^{-1}\|_{\mathcal{B}(\mathcal{Z}_P\times\mathcal{Z}_F)}\Big\{&\sum_{j=1}^{M_4}\big[\|\mathcal{C}_{j,1}(w)\|_{\mathcal{X}_P^T\times\mathcal{X}_F^T\times\mathcal{Y}_2^T}\\
&+\|\mathcal{C}_{j,2}(\psi)\ \|_{X_P^T\times X_F^T\times Y_{2,\Gamma_P}^T}\big]+\sum_{j=1}^{M_1}\|\mathcal{T}_j\tilde{\psi}_j^P\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1};\mathbb{R}^N))}\\
&+\sum_{j=M_1+1}^{M_2}\|\mathcal{T}_j\tilde{\psi}_j\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N))\times\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_-^{n+1};\mathbb{R}^N))\times Y_2^T(\mathbb{R}^n)}\\
&+\sum_{j=M_2+1}^{M_3}\|\mathcal{T}_j\tilde{\psi}_j^F\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n+1};\mathbb{R}^N))}+\sum_{j=M_3+1}^{M_4}\|\mathcal{T}_j\tilde{\psi}_j^F\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N))}\Big\}\end{aligned}$$

Here, we have already used that the term $\mathcal{C}_{j,2}(\psi)-\mathcal{T}_j\tilde{\psi}_j$ lies in E, and thus we only need to carry out the estimation in $X_P^T\times X_F^T\times Y_{2,\Gamma_P}^T$. By estimating $\mathcal{S}_1(w,\psi)$ we have seen that the lower order terms of w gives rise to a constant $c(T)$ getting small for $T\to 0$ as well as the lower order terms of ψ in spaces X_P^T, X_F^T and Y_{2,Γ_P}^T, i.e. we have

$$\|\mathcal{C}_{j,1}(w)\|_{\mathcal{X}_P^T\times\mathcal{X}_F^T\times\mathcal{Y}_2^T}+\|\mathcal{C}_{j,2}(\psi)\ \|_{X_P^T\times X_F^T\times Y_{2,\Gamma_P}^T}\le c(T)\|(w,\psi)\|_{Z_P^T\times Z_F^T\times\mathcal{Z}^T}.$$

Therefore, it remains to show the smallness of the norms $\|\mathcal{T}_j\tilde{\psi}_j\|$. For discussing these terms we restrict to the second sum in order to take the effort low. In virtue of Proposition 3.55, we know that the local functions $\tilde{\psi}_j=\mathcal{S}_j^eG^jz^T\!\cdot\mathcal{C}_{j,2}(\psi)$ belong to the regularity class $\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^2(\mathbb{R}_-^{n+1}))$ due to the fact that $G^jz^T\!\cdot\,\mathcal{C}_{j,2}(\psi)$ possess the regularity $\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}_+^{n+1})\times\mathrm{L}_p(\mathbb{R}_-^{n+1})\times\mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^n))$. Using this extra time regularity gives the estimate

$$\begin{aligned}\|\mathcal{T}_j\tilde{\psi}_j\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N))\times\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_-^{n+1};\mathbb{R}^N))\times Y_2^T(\mathbb{R}^n)}&\le C\big(\|(\tilde{\psi}_j^P,\tilde{\psi}_j^F)\|_{\mathrm{L}_p(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^2(\mathbb{R}_-^{n+1}))}\\
&\quad+\sum_{k=P,F}\|\partial_{y_{n+1}}\tilde{\psi}_j^k\|_{\mathrm{B}_{pp}^{1/2-1/2p}(J;\mathrm{L}_p(\mathbb{R}^n))\cap\mathrm{L}_p(J;\mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^n))}\big)\\
&\le C\big(T^{1/2}\|(\tilde{\psi}_j^P,\tilde{\psi}_j^F)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^2(\mathbb{R}_-^{n+1}))}\\
&\quad+(T^{1/2p}+T^{1/2})\sum_{k=P,F}\|\partial_{y_{n+1}}\tilde{\psi}_j^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^n))}\big)\\
&\le C\max\{T^{1/2},T^{1/2p}\}\|(\tilde{\psi}_j^P,\tilde{\psi}_j^F)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^2(\mathbb{R}_-^{n+1}))}.\end{aligned}$$

Continuity of the solution operator $\mathcal{S}_j^e$ entails

$$\begin{aligned}\|(\tilde{\psi}_j,\tilde{\psi}_j)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^2(\mathbb{R}_-^{n+1}))}&=\|S_j^eG^jz^T\!\cdot\mathcal{C}_{j,2}(\psi)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^2(\mathbb{R}_+^{n+1})\times\mathrm{H}_p^2(\mathbb{R}_-^{n+1}))}\\
&\le C\|z^T\!\cdot\mathcal{C}_{j,2}(\psi)\|_{\mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega_P)\times\mathrm{L}_p(\Omega_F)\times\mathrm{B}_{pp}^{1-1/p}(\Gamma_P))}\le C\|\psi\|_{\mathcal{Z}_P\times\mathcal{Z}_F},\end{aligned}$$

and in the end

$$\|\mathcal{T}_j\tilde{\psi}_j\|_{\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_+^{n+1};\mathbb{R}^N))\times\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}_-^{n+1};\mathbb{R}^N))\times Y_2^T(\mathbb{R}^n)}\le c(T)\|\psi\|_{\mathcal{Z}_P\times\mathcal{Z}_F}.$$

Finally, all estimations imply that $\mathcal{S}_\varepsilon$ becomes small for $T \to 0$ such that the Neumann's series results in $(I+\mathcal{S}_\varepsilon)^{-1} \in \mathcal{B}({}_0Z_P^T \times {}_0Z_F^T \times {}_0\mathcal{Z}^T)$. This shows that (w,ψ) is equal zero if $\varrho = 0$, i.e. there is a left inverse of $\mathcal{L}$ denoted by $\mathcal{S}_L$.

To show that $\mathcal{S}_L$ is surjective, i.e. there also exists a right inverse of $\mathcal{L}$, we have to establish that the function $(w,\psi) = \mathcal{S}_L\varrho$ satisfying $(w,\psi) = \mathcal{S}_0\rho + \mathcal{S}_\varepsilon(w,\psi)$ solves (3.31). To make use of local problems we have to create the operator which implicates the left hand sides of the local problems. Therefore, we introduce the operator $\mathcal{L}^s(w,\psi)$ with $\mathcal{A}_{k1}(D)$, $\mathcal{A}_{k2}(D)$ replaced by

$$\mathcal{A}_{k1}^s(D) := \mathcal{A}_{k1}(D) + D^k, \quad \mathcal{A}_{k2}^s(D) := \mathcal{A}_{k2}(D) + M^k\tilde{u}^k.$$

By this means we have shifted the operators $\mathcal{A}_{k1}(D)$ and $\mathcal{A}_{k2}(D)$, cp. with the local problems. We further set $\mathcal{L}^{low} := \mathcal{L}^s - \mathcal{L}$.

We turn to the surjectivity respectively (w,ψ) given by (3.52) solves the problem $\mathcal{L}(w,\psi) = \varrho$. The main idea consists in commuting $\tilde{\varphi}_j$ with $\mathcal{L}$ in order to exploit that $(w_j,\psi_j) = \mathcal{S}_jG^j\varphi_j\rho$ solves a local problem. After applying $\mathcal{L}^s$ to $\mathcal{S}_0\rho$ we get

$$\sum_{j=1}^{M_4} \mathcal{L}^s\tilde{\varphi}_j(G^j)^{-1}(w_j,\psi_j) = \sum_{j=1}^{M_4} \tilde{\varphi}_j(G^j)^{-1}\mathcal{L}^j(w_j,\psi_j) + [\mathcal{L}^s,\tilde{\varphi}_j](G^j)^{-1}(w_j,\psi_j)\,.$$

The local transformed operator $\mathcal{L}^j$ coincides exactly with the left hand side of each local problem, so that we may deduce $\mathcal{L}^j(w_j,\psi_j) = G^j\varphi_j\rho + G^j\mathcal{C}_j(w,\psi)$. However, this implies

$$\sum_{j=1}^{M_4} \tilde{\varphi}_j(G^j)^{-1}\mathcal{L}^j(w_j,\psi_j) = \rho + \sum_{j=1}^{M_4} \tilde{\varphi}_j\mathcal{C}_j(w,\psi)$$

due to $\tilde{\varphi}_j \equiv 1$ on $\operatorname{supp}\varphi_j$. Furthermore, the operator of perturbation $\mathcal{C}_j$, see (3.53), equals $[\mathcal{L}^s,\varphi_j] - \varphi_j\mathcal{L}^{low}$. This can be traced back to the fact that all equations without any differential operators do not generate terms of lower order.

By combining these facts with $\sum_j \varphi_j = 1$ and $\sum_j [\mathcal{L}^s,\varphi_j] = 0$ leads to the relation

$$\sum_j \tilde{\varphi}_j\mathcal{C}_j(w,\psi) = \sum_j [\mathcal{L}^s,\varphi_j](w,\psi) - \sum_j \varphi_j\mathcal{L}^{low}(w,\psi) = -\mathcal{L}^{low}(w,\psi).$$

Finally, the above evaluations imply the identity

$$\mathcal{L}^s(w,\psi) = \rho - \mathcal{L}^{low}(w,\psi) + \sum_{j=1}^{M_4} [\mathcal{L}^s,\tilde{\varphi}_j](G^j)^{-1}(w_j,\psi_j)$$

and consequently

$$\mathcal{L}(w,\psi) = \rho + \sum_{j=1}^{M_4} [\mathcal{L}^s,\tilde{\varphi}_j](G^j)^{-1}(w_j,\psi_j).$$

By using $(w,\psi) = \mathcal{S}_L\varrho$ and keeping in mind that $(w_j,\psi_j) = \mathcal{S}_jG^j\varphi_j\varrho + \mathcal{S}_j\mathcal{C}_j\mathcal{S}_L\varrho$, we have shown

$$\mathcal{L}\mathcal{S}_L\varrho = \varrho + \sum_{j=1}^{M_4} [\mathcal{L}^s,\tilde{\varphi}_j](G^j)^{-1}\mathcal{S}_j\left\{G^j\varphi_j + \mathcal{C}_j\mathcal{S}_L\right\}\varrho.$$

Now, we set $\mathcal{K}\varrho := \sum_{j=1}^{M_4}[\mathcal{L}^s, \tilde{\varphi}_j](G^j)^{-1}\mathcal{S}_j \left\{G^j\varphi_j + \mathcal{C}_j\mathcal{S}_L\right\} \varrho$ and define $\overline{\rho} := \rho + \mathcal{K}\rho$. The operator $\mathcal{K}$ only contains lower order terms so that we can achieve $\|\mathcal{K}\| \leq k < 1$ by choosing T small. Since this procedure was frequently carried out we want to spare ourself of repeating it. Hence, the above problem can be rewritten as follows

$$\mathcal{L}\mathcal{S}_L(I + \mathcal{K})^{-1}\overline{\varrho} = \overline{\varrho}, \tag{3.59}$$

which shows that (3.59) gives rise to a right inverse $\mathcal{S}_L(I+\mathcal{K})^{-1}$ of $\mathcal{L}$, i.e. we have surjectivity. Since right and left inverse have to be equal, it must hold that $\mathcal{S}_L = \mathcal{S}_L(I + \mathcal{K})^{-1}$, but this means $\mathcal{K} = 0$. The inverse operator of $\mathcal{L}$ may be written as the Neumann series

$$\mathcal{L}^{-1} = \sum_{l=0}^{\infty} \mathcal{S}_\varepsilon^l \mathcal{S}_0\,, \tag{3.60}$$

which completes the proof.

□

Chapter 4

The Nonlinear Problem

In this chapter we intend to solve the nonlinear problem (1.30). To achieve this, we apply the maximal L_p regularity result of the linear problems (1.39), (1.40) via the contraction mapping principle. The latter technique prompts us to derive a fixed point equation which is associated with the original problem. At this point the linearisation (1.39), (1.40) will enter. Strictly speaking, we make use of the bijectivity of the solution operator defined by the linear problem (1.39), (1.40). After rewriting the nonlinear system into this fix point equation, it boils down to carry out the estimations which are required for establishing contraction and self-mapping.

4.1 Reformulation

The reformulation of the nonlinear problem (1.30) is carried out in two steps. On the one hand we use the linearisation developed in Section 1.4 in an appropriate manner, and on the other hand we invert the operator arising from this procedure. The latter action is justified by the main result of Chapter 3, Theorem 3.4, which provides invertibility. In other words, the main idea consists in creating the left hand-side of linear problem (1.39), (1.40). Of course, this approach produces new terms on the right hand side.

Let $J_0 = [0, T_0]$ be a compact time interval and set $(u, \phi) := (u^P, u^F, u^b, \phi^P, \phi^F)$. Then the rewritten problem for (u, ϕ) reads as follows

$$
\begin{aligned}
&\partial_t u^P + \mathcal{A}_{P1}(D)u^P + \mathcal{A}_{P2}(D)\phi^P = F^P(t, x, u^P, \phi^P), \quad (t, x) \in J_0 \times \Omega_P \\
&\partial_t u^F + \mathcal{A}_{F1}(D)u^F + \mathcal{A}_{F2}(D)\phi^F = F^P(t, x, u^F, \phi^F), \quad (t, x) \in J_0 \times \Omega_F \\
&\mathcal{B}_{P1}(D)u^P + \mathcal{B}_{P2}(D)\phi^P = \mathcal{B}_{F1}(D)u^F + \mathcal{B}_{F2}(D)\phi^F + G(t, x, u^P, u^F, \phi), \quad (t, x) \in J_0 \times \Gamma_P \\
&\tilde{U}_P^{-1}(t, x)u^P - \tilde{U}_F^{-1}(t, x)u^F + \lambda_0 z(\phi^P - \phi^F) = H(t, x, u^P, u^F), \quad (t, x) \in J_0 \times \Gamma_P \\
&u^F = u^b, \quad (t, x) \in J_0 \times \Gamma \\
&u^P(0, x) = u_0^P(x), \quad x \in \overline{\Omega}_P, \quad u^F(0, x) = u_0^F(x), \quad x \in \overline{\Omega}_F, \\
&z^T \cdot u^P(t, x) = 0, \quad (t, x) \in J_0 \times \overline{\Omega}_P, \quad z^T \cdot u^F(t, x) = 0, \quad (t, x) \in J_0 \times \overline{\Omega}_F,
\end{aligned}
\tag{4.1}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}u^b(t) + \frac{1}{\tau}u^b(t) + a^b \int_\Gamma \mathcal{B}_{F1}(D)u^F + \mathcal{B}_{F2}(D)\phi^F d\sigma = F^b(t, u^b, u^F, \phi^F), \quad t \in J_0 \\
&u^b(0) = u_0^b, \quad z^T \cdot u^b(t) = 0, \quad t \in J_0.
\end{aligned}
\tag{4.2}
$$

The right-hand side terms F^P, F^F, G, H, F^b are defined by

$$
\begin{aligned}
&F^k(t,x,u^k,\phi^k) := [\mathcal{A}_{k2}(t,x,\tilde{u}^k,D) - \mathcal{A}_{k2}(t,x,u^k,D)]\phi^k + R^k(t,x,u^k), \quad k = P, F\\
&G(t,x,u^P,u^F,\phi) := [\mathcal{B}_{P2}(t,x,\tilde{u}^P,D) - \mathcal{B}_{P2}(t,x,u^P,D)]\phi^P\\
&\qquad\qquad\qquad\qquad - [\mathcal{B}_{F2}(t,x,\tilde{u}^F,D) - \mathcal{B}_{F2}(t,x,u^F,D)]\phi^F,\\
&H(t,x,u^P,u^F) := \gamma(t,x,\tilde{u}^P,\tilde{u}^F) - R_{\tilde{u}^P}(t,u^P) + R_{\tilde{u}^F}(t,u^F),\\
&F^b(t,u^b,u^F,\phi^F) := \frac{1}{\tau}u^f(t) + R^b(t,u^b(t)) + a^b\int_\Gamma [\mathcal{B}_{F2}(t,x,u^F,D) - \mathcal{B}_{F2}(t,x,\tilde{u}^F,D)]\phi^F\,d\sigma.
\end{aligned}
\tag{4.3}
$$

Call to mind that $\tilde{u}^k$ plays the part of an approximation of u^k, i.e. $\tilde{u}^k$ belongs to $Z_k(E_+)$ with $\gamma_{|_t}\tilde{u}^k := u_0^k > 0$, see Section 1.4. We shall remind of the definitions of $\gamma(t,x)$ and $R_{\tilde{u}^k}(t,u^k)$. We have set

$$
\gamma(t,x,\tilde{u}^P,\tilde{u}^F) := \left[\ln\left(\frac{\gamma_i^F(t,x)\tilde{u}_i^F}{\gamma_i^P(t,x)\tilde{u}_i^P}\right)\right]_{1\le i\le N}, \quad R_{\tilde{u}^k}(u^k) := \left[r\left(\frac{u_i^k - \tilde{u}_i^k}{\tilde{u}_i^k}\right)\right]_{1\le i\le N}, \quad k = P, F,
$$

where r comes from expanding the logarithm, see (1.36), (1.37). Our next task will consist in finding an abstract formulation of the equations above with the aim of putting it into a fixed point problem. For this, we put together the initial data u_0^k for $k = P, F, b$ to the vector $u_0 := (u_0^P, u_0^F, u_0^b)$ belonging to

$$
V(E_+) := \mathrm{B}_{pp}^{2-2/p}(\Omega_P; E_+) \times \mathrm{B}_{pp}^{2-2/p}(\Omega_F; E_+) \times E_+
$$

and set

$$
\begin{aligned}
&Z^{T_0} := Z_P^{T_0} \times Z_F^{T_0} \times Z_b^{T_0},\\
&\mathcal{Z}^{T_0} := \left\{(\phi^P,\phi^F) \in \mathcal{Z}_P^{T_0} \times \mathcal{Z}_F^{T_0} :\ \gamma_{|_{\Gamma_P}}\phi^P - \gamma_{|_{\Gamma_P}}\phi^F \in Y_{1,\Gamma_P}^{T_0}\right\},\\
&\mathcal{M}^{T_0} := \mathcal{X}_P^{T_0} \times \mathcal{X}_F^{T_0} \times X_b^{T_0} \times \{(g,h^P,h^F,u_0) \in \mathcal{Y}_2^{T_0} \times Y_{1,\Gamma_P}^{T_0}(\mathbb{R}^N) \times Y_{1,\Gamma}^{T_0}(E_+) \times V(E_+) :\\
&\qquad\qquad\qquad\qquad (g,h^P,h^F,u_0) \text{ enjoy the compatibility conditions }\},\\
&\mathcal{U}^{T_0} := \mathcal{U}_P^{T_0}(E) \times \mathcal{U}_F^{T_0}(E) \times \mathrm{C}(J_0;E), \quad \mathcal{U}_k^{T_0}(E) := \mathrm{C}^{1/2}(J_0;\mathrm{C}(\overline{\Omega}_k;E)) \cap \mathrm{C}(J_0;\mathrm{C}^1(\overline{\Omega}_k;E)).
\end{aligned}
$$

Now, we comment on the compatibility conditions stated in the space $\mathcal{M}^{T_0}$. As in the linear problem compatibility conditions are obtained by taking trace $t = 0$ in the boundary conditions. In doing so, the condition $u_0^F(x) = u_0^b$ has to be satisfied on boundary Γ. The compatibility conditions on boundary Γ_P take the form

$$
\begin{aligned}
&d_i^P(0,x)\partial_\nu u_{0,i}^P(x) + m_i^P(0,x)u_{0,i}^P(x)\partial_\nu\phi^P(0,x) = d_i^F(0,x)\partial_\nu u_{0,i}^F(x) + m_i^F(0,x)u_{0,i}^F(x)\partial_\nu\phi^F(0,x),\\
&\lambda_0 z_i(\phi^P(0,x) - \phi^F(0,x)) = \ln\left(\frac{\gamma_i^F(0,x)u_{0,i}^F(x)}{\gamma_i^P(0,x)u_{0,i}^P(x)}\right), \quad x \in \Gamma_P, \quad i = 1,\dots,N.
\end{aligned}
$$

where the latter equation is equivalent to

$$
u_{0,i}^P(x) = u_{0,i}^F(x)\tfrac{\gamma_i^F(0,x)}{\gamma_i^P(0,x)}e^{\lambda_0 z_i(\phi^F(0,x) - \phi^P(0,x))}, \quad x \in \Gamma_P, \quad i = 1,\dots,N.
$$

In compatibility conditions on boundary Γ_P the electrical potentials $\phi^P(0,x)$, $\phi^F(0,x)$ are involved making up one degree of freedom. More precisely, each condition comprises N equations due to N species, whereas the electrical potentials satisfy a scalar-valued elliptic problem including these quantities. The potentials at $t=0$ can be interpreted as the weak solution of the following elliptic problem

$$
\begin{aligned}
z^T\cdot\mathcal{A}_{P2}(0,u_0^P,D)\phi_0^P + z^T\cdot\mathcal{A}_{P1}(0,D)u_0^P &= 0, \quad x\in\Omega_P,\\
z^T\cdot\mathcal{A}_{F2}(0,u_0^F,D)\phi_0^F + z^T\cdot\mathcal{A}_{F1}(0,D)u_0^F &= 0, \quad x\in\Omega_F,\\
z^T\cdot\mathcal{B}_{P2}(0,u_0^P,D)\phi_0^P + z^T\cdot\mathcal{B}_{P1}(0,D)u_0^P &= z^T\cdot\mathcal{B}_{F2}(0,u_0^F,D)\phi_0^F + z^T\cdot\mathcal{B}_{F1}(0,D)u_0^F, \quad x\in\Gamma_P,\\
\lambda_0|z|^2(\phi_0^P-\phi_0^F) &= z^T\cdot\gamma(0,x,u_0^P,u_0^F), \quad x\in\Gamma_P,\\
\phi_0^F &= 0, \quad x\in\Gamma,
\end{aligned}
\tag{4.4}
$$

which can be obtained from (1.30) after applying $z^T\cdot$ and taking traces in $t=0$.

We now define the nonlinear operator $F_{\tilde u}(u,\phi)$ being composed of the new right-hand side of (4.1) by means of

$$\mathcal{F}_{\tilde u}(u,\phi) := \Big(F^P, F^F, F^b, G, H, u^b\Big)\,.$$

It is an immediate consequence of definitions stated in (4.3) that the nonlinear operator $\mathcal{F}_{\tilde u}(u,\phi)$ is a mapping from $Z^{T_0}\times\mathcal{Z}^{T_0}$ to $\mathcal{X}_P^{T_0}\times\mathcal{X}_F^{T_0}\times X_b^{T_0}\times\mathcal{Y}_2^{T_0}\times Y_{1,\Gamma_P}^{T_0}(\mathbb{R}^N)\times Y_{1,\Gamma}^{T_0}(E_+)$. In fact, if $(u,\phi)\in Z^{T_0}\times\mathcal{Z}^{T_0}$, then this corresponds, as we know from Theorem 3.4, to the regularity classes

$$\mathcal{A}_{k2}\phi^k\in\mathcal{X}_k^{T_0},\quad \mathcal{B}_{k2}(D)\phi^k\in Y_{2,\Gamma_P}^{T_0}(\mathbb{R}^N),\quad \gamma_{|\Gamma_P}u^k\in Y_{1,\Gamma_P}^{T_0}(E),\quad \gamma_{|t}u^k\in \mathrm{B}_{pp}^{2-2/p}(\Omega_k;E).$$

In the following we associate (4.1) and (4.2) with the abstract equation

$$\mathcal{L}(u,\phi) = (\mathcal{F}_{\tilde u}(u,\phi), u_0) \quad \text{in } \mathcal{M}^{T_0}\,. \tag{4.5}$$

The goal consists in inverting the operator $\mathcal{L}$ so that we obtain a fixed point equation. In Section 3.3 maximal regularity has been proved, i.e. $\mathcal{L}$ is a continuous one-to-one mapping from the space of data $\mathcal{M}^{T_0}$ to the class of maximal regularity $Z^{T_0}\times\mathcal{Z}^{T_0}$, i.e

$$\mathcal{L}^{-1}\in\ \mathcal{L}is(\mathcal{M}^{T_0}, Z^{T_0}\times\mathcal{Z}^{T_0}). \tag{4.6}$$

Now, we focus on the operator norm of $\mathcal{L}^{-1}$, in particular on independence from the length of time interval $J_0=[0,T_0]$. This fact is needed since we want to obtain contraction and self-mapping by choosing $T\in(0,T_0]$ sufficiently small. Therefore, we have to guarantee that all constants coming from estimations of $\mathcal{L}^{-1}$ are independent of T. In general one can not prove this assertion, except in case of null initial data. Thus we introduce the spaces

$$
\begin{aligned}
Z_0^T &:= \{w\in Z^{T_0}:\ w(0)=0\}, \quad \mathcal{Z}_0^{T_0} := \big\{\phi\in\mathcal{Z}^{T_0}:\ \phi(0)=0\big\},\\
\mathcal{M}_0^{T_0} &:= \big\{(\rho,u_0)\in\mathcal{M}^{T_0}:\rho(0)=0, u_0=0\big\}, \quad \mathcal{U}_0^{T_0} := \{v\in\mathcal{U}^{T_0}:\ v(0)=0\}.
\end{aligned}
$$

We mean $\rho(0)$ by all components of ρ having trace in t. We consider the linear problem (3.31), (3.33) with initial data $u_0=0$ and $\rho\in\mathcal{X}_P^{T_0}\times\mathcal{X}_F^{T_0}\times X_b^{T_0}\times\mathcal{Y}_2^{T_0}\times Y_{1,\Gamma_P}^{T_0}(\mathbb{R}^N)\times Y_{1,\Gamma}^{T_0}(E_+)$

satisfying compatibility conditions. Observe that the latter assumption implies $(\rho, 0) \in \mathcal{M}_0^{T_0}$. This circumstance enables us to extend the data ρ as follows. If a component $\rho_i(t)$ of ρ only possesses H_p^α-regularity in time with $\alpha < 1/p$ then we set

$$E_{\mathbb{R}_+}\rho_i(t) = \begin{cases} \rho_i(t) & : t \in [0, T_0] \\ 0 & : t \in [T_0, \infty) \end{cases},$$

otherwise we put

$$E_{\mathbb{R}_+}\rho_i(t) = \begin{cases} \rho_i(t) & : t \in [0, T_0] \\ \rho_i(2T_0 - t) & : t \in [T_0, 2T_0] \\ 0 & : t \in [2T_0, \infty) \end{cases}.$$

It is easily seen that $E_{\mathbb{R}_+}$ is an admissible extension for each space appearing in $\mathcal{M}^{T_0}$. That means $E_{\mathbb{R}_+}$ is bounded and the norm does not depend on T, e.g. we have $\|E_{\mathbb{R}_+}\rho\| \leq 2\|\rho\|$. Having this in mind and employing Theorem 3.4 we obtain a unique solution $(w, \psi) \in Z_0 \times \mathcal{Z}_0$ and the following estimation is valid.

$$\begin{aligned} \|(w,\psi)\|_{Z_0^{T_0} \times \mathcal{Z}_0^{T_0}} &\leq \|(w,\psi)\|_{Z_0 \times \mathcal{Z}_0} \leq \|\mathcal{L}^{-1}\|_{\mathcal{B}(\mathcal{M}_0, Z_0 \times \mathcal{Z}_0)} \|(E_{\mathbb{R}_+}\rho, 0)\|_{\mathcal{M}_0} \\ &=: \|\mathcal{L}_{\mathbb{R}_+}^{-1}\| \|(E_{\mathbb{R}_+}\rho, 0)\|_{\mathcal{M}_0} \leq 2\|\mathcal{L}_{\mathbb{R}_+}^{-1}\| \|(\rho, 0)\|_{\mathcal{M}_0^{T_0}} \end{aligned}$$

The solution operator has been subscripted with $\mathbb{R}_+$ in order to refer to the interval being considered here. The above estimation provides $\|\mathcal{L}_{[0,T_0]}^{-1}\| \leq 2\|\mathcal{L}_{\mathbb{R}_+}^{-1}\| =: C_{max}$ which shows the desirable result.

4.2 Existence and Uniqueness

We now come to the result which ensures existence and uniqueness on a maximal interval of existence $[0, t_{max}(u_0))$. This interval is characterised by the condition that $\lim_{t \to t_{max}(u_0)} u(t)$ does not exist in $V(E_+)$, since otherwise we may apply Theorem 3.4 with initial value $u(t_{max}(u_0)) = \lim_{t \to t_{max}(u_0)} u(t)$ to obtain a contradiction to maximality. Moreover, we can show positivity of $u = (u^P, u^F, u^b)$ if the initial data are positive, that means, for all $i \in \{1, \ldots, N\}$ and $k = P, F, b$ we have $u_i^k > 0$ whenever $u_{0,i}^k > 0$.

Theorem 4.1 *Let Ω_P, Ω_F be bounded domains in $\mathbb{R}^{n+1}$ with C^2- boundary, $\Gamma_P := \partial\Omega_P$, $\partial\Omega_F = \Gamma_P \cup \Gamma$ and* $\operatorname{dist}(\Gamma_P, \Gamma) > 0$. *Let $(n+1) + 2 < p < \infty$ and suppose that*

1. *$d_i^k \in \mathrm{C}^{1/2}(J_0; \mathrm{C}^1(\Omega_k))$, $d_i^k(t,x) > 0$ for $(t,x) \in J_0 \times \overline{\Omega}_k$, $i \in \{1, \ldots, N\}$, $k = P, F$;*
2. *$\gamma_i^k \in \mathrm{B}_{pp}^{1-1/2p}(J_0; \mathrm{L}_p(\Gamma_P; \mathbb{R}_+)) \cap \mathrm{L}_p(J_0; \mathrm{B}_{pp}^{2-1/p}(\Gamma_P; \mathbb{R}_+))$, $i \in \{1, \ldots, N\}$, $k = P, F$;*
3. *$u_0 = (u_0^P, u_0^F, u_0^b) \in \mathrm{B}_{pp}^{2-2/p}(\Omega_P; E_+) \times \mathrm{B}_{pp}^{2-2/p}(\Omega_F; E_+) \times E_+$, $u_{0,i}^k > 0$, $k = P, F, b$, $\forall i$;*
4. *$u^f \in \mathrm{L}_p(J_0; E_+)$; R^k, $k = P, F$ and R^b satisfy (R1)-R(3) and (R4)-(R6), respectively;*
5. *compatibility conditions:*
   (a) *$\ln(\gamma_i^P(0,x) u_{i,0}^P(x)) + \lambda_0 z_i \phi_0^P(x) = \ln(\gamma_i^F(0,x) u_{i,0}^F(x)) + \lambda_0 z_i \phi_0^F(x)$ in $\mathrm{B}_{pp}^{2-3/p}(\Gamma_P)$ for $i \in \{1, \ldots, N\}$;*

(b) $D^P(0,x)\partial_\nu u_0^P + M^P(0,x)u_0^P\partial_\nu \phi_0^P = D^F(0,x)\partial_\nu u_0^F + M^F(0,x)u_0^F\partial_\nu\phi_0^F$ *in* $\mathrm{B}_{pp}^{1-3/p}(\Gamma_P;\mathbb{R}^N)$*, where* (ϕ_0^P,ϕ_0^F) *is given as solution of (4.4).*

(c) $u_0^F(x) = u_0^b$.

Then there exists $t_{max} > 0$ *such that for any* $T_0 < t_{max}$ *the nonlinear problem (4.1), (4.2) admits a unique solution* $(u^P, u^F, u^b, \phi^P, \phi^F)$ *on* $J_0 = [0, T_0]$ *in the maximal regularity class* $Z(J_0)\times\mathcal{Z}(J_0)$*. In particular, we have*

$$u^k \in \mathrm{C}^1((0,t_{max});\mathrm{C}(\Omega_k)) \cap \mathrm{C}((0,t_{max});\mathrm{C}^2(\Omega_k)), \quad k = P, F,$$
$$\phi^k \in \mathrm{C}^{1/2}((0,t_{max});\mathrm{C}(\Omega_k)) \cap \mathrm{C}((0,t_{max});\mathrm{C}^2(\Omega_k)), \quad k = P, F\,.$$

Moreover, the solution (u^P, u^F, u^b) *is positive and the map*

$$(u_0^P, u_0^F, u_0^b) \longrightarrow (u^P(t), u^F(t), u^b(t)) \tag{4.7}$$

defines a local semiflow on the natural phase space $V(E_+)$ *in the autonomous case.*

Proof. (a) Unique existence on $[0,T]$ *for* T *sufficiently small.* By the above considerations we have seen that the evolution problem (1.30) can be converted into the equivalent problem (4.5). This equation is solved locally via the contraction mapping theorem. For this purpose we introduce a reference function (w,ψ) defined as the solution of the linear problem

$$\mathcal{L}(w,\psi) = (\mathcal{F}_{\tilde u}(\tilde u, 0), u_0), \quad \text{in } \mathcal{M}^T\,. \tag{4.8}$$

The choice $\tilde u \in Z^T(E_+)$, with $\tilde u(0) = u_0$ and $\phi = 0$ entail that the functions $R_{\tilde u^P}$, $R_{\tilde u^F}$ and all terms containing an electrical potential disappear. In fact, one computes

$$\mathcal{F}_{\tilde u}(\tilde u, 0) = \Big(R^P(t,x,\tilde u^P), R^F(t,x,\tilde u^F), 0, \gamma(t,x,\tilde u^P, \tilde u^F), \tfrac{1}{\tau}u^f(t) + R(t,\tilde u^b)\Big)\,.$$

Note that this right hand side belongs to $\mathcal{M}^T$, in particular, the compatibility conditions are satisfied. So according to Theorem 3.4 we obtain a unique solution (w,ψ) which belongs to the space of maximal regularity. Next we introduce a ball $Z^T\times\mathcal{Z}^T$ with radius δ and center point (w,ψ) as follows

$$\Sigma_{\delta,T} := \left\{(v,\varphi)\in Z^T\times\mathcal{Z}^T : (v(0),\varphi(0)) = (u_0,\phi_0)\,,\ \|(v,\varphi)-(w,\psi)\|_{Z^T\times\mathcal{Z}^T}\le\delta\right\},$$

which is a closed subset of $Z^T\times\mathcal{Z}^T$. We want to show that $\mathcal{L}^{-1}\mathcal{F}_{\tilde u}(\Sigma_{\delta,T})\subset\Sigma_{\delta,T}$ and that $\mathcal{L}^{-1}\mathcal{F}_{\tilde u}$ is a contraction in the norm of $Z^T\times\mathcal{Z}^T$. These two properties can be shown, provided the parameters $T\in(0,T_0]$ and $\delta\in(0,1]$ are chosen properly. Before we will introduce some auxiliary functions depending on parameter T. These will be useful for the upcoming estimates. We set

$$\psi_1(T) := \|(w,\psi)\|_{Z^T\times\mathcal{Z}^T} := \|w\|_{Z^T} + \|\psi\|_{\mathcal{Z}^T}, \quad \psi_2(T) := \|w-\tilde u\|_{\mathcal{U}_0^T} \le C\|w-\tilde u\|_{Z_0^T},$$

$$\psi_3(T) := \max\left\{\sum_{k=P,F}\max_{v\in\Sigma_{\delta,T}}\|R'_{\tilde u^k}(v)\|_{{}_0Y^T_{1,\Gamma_P}(\mathcal{B}(E))},\right.$$
$$\left.\sum_{k=P,F}\max_{s\in[0,1]}\|R'_{\tilde u^k}(w^k+s(\tilde u^k-w^k))\|_{{}_0Y^T_{1,\Gamma_P}(\mathcal{B}(E))}\right\},$$

$$\psi_4^k(T) := \|l^k\|_{\mathrm{L}_p([0,T];\mathrm{L}_p(\Omega_k;\mathbb{R}^N))}, \quad k = P,F\,, \quad \psi_4^b(T) := \|l^b\|_{\mathrm{L}_p(0,T)}.$$

Apparently, $\psi_2(T) \to 0$ as $T \to 0$ due to $w_{|t=0} - \tilde{u}_{|t=0} = 0$. Observe that $\psi_i(T), \psi_4^k(T) \to 0$, $i = 1,3$, $k = P,F,b$ as $T \to 0$ by virtue "integral norms" concerning the time variable $t \in [0,T]$. Now, we come to self-mapping and contraction. Let (u,ϕ), $(\overline{u},\overline{\phi}) \in \Sigma_{\delta,T}$ be given. By using the result of maximal regularity $\mathcal{L}^{-1} \in \mathcal{L}is(\mathcal{M}^T, Z^T \times \mathcal{Z}^T)$ we may estimate as follows

$$\begin{aligned}\|\mathcal{L}^{-1}(\mathcal{F}_{\tilde{u}}(u,\phi),u_0) - (w,\psi)\|_{Z_0^T\times\mathcal{Z}_0^T} &= \|\mathcal{L}^{-1}(\mathcal{F}_{\tilde{u}}(u,\phi) - \mathcal{F}_{\tilde{u}}(\tilde{u},0),0)\|_{Z_0^T\times\mathcal{Z}_0^T} \\ &\le \|\mathcal{L}^{-1}_{\mathbb{R}_+}\| \|(\mathcal{F}_{\tilde{u}}(u,\phi) - \mathcal{F}_{\tilde{u}}(\tilde{u},0),0)\|_{\mathcal{M}_0^T}.\end{aligned}$$

In a similar way we obtain for the contraction

$$\|\mathcal{L}^{-1}(\mathcal{F}_{\tilde{u}}(u,\phi),u_0) - \mathcal{L}^{-1}(\mathcal{F}_{\tilde{u}}(\overline{u},\overline{\phi}),u_0)\|_{Z_0^T\times\mathcal{Z}_0^T} \le \|\mathcal{L}^{-1}_{\mathbb{R}_+}\| \|(\mathcal{F}_{\tilde{u}}(u,\phi) - \mathcal{F}_{\tilde{u}}(\overline{u},\overline{\phi}),0)\|_{\mathcal{M}_0^T}.$$

With a view of both estimations we perceive that it remains to consider differences of functions $(\mathcal{F}_{\tilde{u}},0)$ in $\mathcal{M}_0^T$. In case of self-mapping we find

$$\begin{aligned}\|(\mathcal{F}_{\tilde{u}}(u,\phi) - \mathcal{F}_{\tilde{u}}(\tilde{u},0),0)\|_{\mathcal{M}_0^T} &\le \sum_{k=P,F} \|R^k(\cdot,\cdot,u^k) - R^k(\cdot,\cdot,\tilde{u}^k)\|_{X_k^T} \\ \|R^b(\cdot,u^b) - R^b(\cdot,\tilde{u}^b)\|_{X_b^T} &+ \sum_{k=P,F} \Big\{ \|[\mathcal{A}_{k2}(\cdot,\cdot,\tilde{u}^k,D) - \mathcal{A}_{k2}(\cdot,\cdot,u^k,D)]\phi^k\|_{0\mathcal{X}_k^T} \\ &+ \|[\mathcal{B}_{k2}(\cdot,\cdot,\tilde{u}^k,D) - \mathcal{B}_{k2}(\cdot,\cdot,u^k,D)]\phi^k\|_{0\mathcal{Y}_2^T} + \|R_{\tilde{u}^k}(u^k)\|_{0Y^T_{1,\Gamma_P}(\mathbb{R}^N)} \Big\} \\ &+ a^b \Big\| \int_\Gamma \big[\mathcal{B}_{F2}(\cdot,x,u^F,D) - \mathcal{B}_{F2}(\cdot,x,\tilde{u}^F,D)\big]\phi^F d\sigma \Big\|_{X_b^T}. \end{aligned} \tag{4.9}$$

We are now going to estimate the term $\|R_{\tilde{u}^k}(u^k)\|$. Due to $R'_{\tilde{u}^k}(\tilde{u}^k) = 0$ we have

$$\begin{aligned}\Big|R_{\tilde{u}^k}(u^k)\Big| &\le \Big|R_{\tilde{u}^k}(u^k) - R_{\tilde{u}^k}(w^k)\Big| + \Big|R_{\tilde{u}^k}(w^k) - R_{\tilde{u}^k}(\tilde{u}^k)\Big| \\ &\le \max_{s\in[0,1]} \Big|R'_{\tilde{u}^k}(w^k + s(u^k - w^k))\Big| \Big|u^k - w^k\Big| \\ &\quad + \max_{s\in[0,1]} \Big|R'_{\tilde{u}^k}(w^k + s(\tilde{u}^k - w^k))\Big| \Big|w^k - \tilde{u}^k\Big|,\end{aligned}$$

and this relation implies

$$\sum_{k=P,F} \|R_{\tilde{u}^k}(u^k)\|_{0Y^T_{1,\Gamma_P}(\mathbb{R}^N)} \le C_3\psi_3(T)\big[\delta + \psi_2(T)\big].$$

The last term of (4.9) can be treated as follows

$$\begin{aligned}\Big\|\int_\Gamma \big[\mathcal{B}_{F2}(\cdot,x,\tilde{u}^F,D) - \mathcal{B}_{F2}(\cdot,x,u^F,D)\big]\phi^F d\sigma\Big\|_{X_b^T} &\le |\Gamma| \|M^F(\tilde{u}^F - u^F)\|_{0\mathrm{C}(J;\mathrm{C}(\Gamma;\mathbb{R}^N))} \|\partial_\nu\phi^F\|_{\mathrm{L}_p(J;\mathrm{C}(\Gamma))} \\ &\le CT^{1/2}\|\tilde{u}^F - u^F\|_{0\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_F;\mathbb{R}^N))} \|\phi^F\|_{\mathcal{Z}_F^T} \le CT^{1/2}(\delta + \psi_2(T))(\delta + \psi_1(T)).\end{aligned}$$

Due to the assumptions (R3) and (R7) for nonlinearities R^k, $k = P,F$ and R^b, respectively, we achieve

$$\sum_{k=P,F} \|R^k(\cdot,\cdot,u^k) - R^k(\cdot,\cdot,\tilde{u}^k)\|_{X_k^T} + \|R^b(\cdot,u^b) - R^b(\cdot,\tilde{u}^b)\|_{X_b^T} \le C_1(\delta + \psi_2(T)) \sum_{k=P,F,b} \psi_4^k(T).$$

Taking into account the mapping properties of operators $\mathcal{A}_{k2}$ stated in Lemma 2.3 we bring off

$$\sum_{k=P,F} \|[\mathcal{A}_{k2}(\cdot,\cdot,\tilde{u}^k_, D) - \mathcal{A}_{k2}(\cdot,\cdot,u^k_, D)]\phi^k\|_{0\mathcal{X}_k^T} \leq C\|\tilde{u}^k - u^k\|_{0\mathcal{U}_k^T}\|\phi^k\|_{\mathcal{Z}_k^T}$$
$$\leq C(\delta+\psi_2(T))(\delta+\psi_1(T))$$

In the long run we have to study $\mathcal{B}_{P2}$ in $\mathcal{Y}_2^T$. The estimation in the space ${}_0Y^T_{2,\Gamma_P}(\mathbb{R}^N)$ can be obtained by Lemma 2.3 resulting in

$$\sum_{k=P,F} \|[\mathcal{B}_{k2}(\cdot,\cdot,\tilde{u}^k_, D) - \mathcal{B}_{k2}(\cdot,\cdot,u^k_, D)]\phi^k\|_{0Y^T_{2,\Gamma_P}(\mathbb{R}^N)} \leq C\sum_{k=P,F}\|\tilde{u}^k - u^k\|_{0\mathcal{U}_k^T}\|\partial_\nu\phi^k\|_{Y^T_{2,\Gamma_P}}$$
$$\leq C(\delta+\psi_2(T))(\delta+\psi_1(T))$$

To get over the estimation in $\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\Gamma_P))$ we are going to derive a relation between the data $z^T\!\cdot G(t,x,u^P_, u^F_, \phi)$ and $z^T\!\cdot F^k(t,x,u^k_,\phi^k)$, which is similar to the estimation of $\mathcal{B}^{j,\varepsilon}(D)$ in the proof of Theorem 3.4. At first, we compute the expression

$$\sum_{k=P,F}\int_{\Omega_k} z^T\!\cdot F^k(t,x,u^k,\phi^k)\tilde{v}\,dx = -\sum_{k=P,F}\int_{\Omega_k} \nabla\cdot([a^k(t,x,\tilde{u}^k)-a^k(t,x,u^k)]\nabla\phi^k)\tilde{v}\,dx,$$

with $\tilde{v}\in \overset{\circ}{\mathrm{H}}{}^1_{p'}(\Omega)$. Integrating by parts and using $\tilde{v}=0$ on Γ yields

$$\sum_{k=P,F}\int_{\Omega_k} z^T\!\cdot F^k(t,x,u^k,\phi^k)\tilde{v}\,dx = -\int_{\Gamma_P}\big([a^P(t,x,\tilde{u}^P)-a^P(t,x,u^P)]\partial_\nu\phi^P$$
$$-[a^F(t,x,\tilde{u}^F)-a^F(t,x,u^F)]\partial_\nu\phi^F\big)\tilde{v}\,d\sigma + \sum_{k=P,F}\int_{\Omega_k}\big([a^k(t,x,\tilde{u}^k)-a^k(t,x,u^k)]\nabla\phi^k\big)\nabla\tilde{u}\,dx$$
$$\equiv -\int_{\Gamma_P} z^T\!\cdot G(t,x,u^P_, u^F_,\phi)\tilde{v}\,d\sigma + \sum_{k=P,F}\int_{\Omega_k}\big([a^k(t,x,\tilde{u}^k)-a^k(t,x,u^k)]\nabla\phi^k\big)\nabla\tilde{u}\,dx.$$

We again consider the above identity for the times $t+h$, $t\in J$ and take the difference from each other. By using duality and Hölder's inequality we obtain

$$\int_{\Gamma_P}[z^T\!\cdot G(t+h,x)-z^T\!\cdot G(t,x)]v(x)\,d\sigma \leq \|\tilde{v}\|_{\overset{\circ}{\mathrm{H}}{}^1_{p'}(\Omega)}\Big\{\sum_{k=P,F}\|z^T\!\cdot F^k(t+h)-z^T\!\cdot F^k(t)\|_{\mathrm{H}_p^{-1}(\Omega_k)}+$$
$$\|[a^k(t+h,\tilde{u}^k(t+h))-a^k(t+h,u^k(t+h))]\nabla\phi^k(t+h)-[a^k(t,\tilde{u}^k(t))-a^k(t,u^k(t))]\nabla\phi^k(t)\|_{\mathrm{L}_p(\Omega_k)}\Big\},$$

for all $\tilde{v}\in\mathrm{H}^1_{p'}(\Omega)$, with $\tilde{v}_{|\Gamma_P}=v$. Taking the infimum over $\|\tilde{v}\|_{0\mathrm{H}^1_{p'}(\Omega)}$ and using

$$\inf\{\|\tilde{v}\|_{0\mathrm{H}^1_{p'}(\Omega)} : \tilde{v}_{|\Gamma_P}=v\}\leq C\|v\|_{\mathrm{B}^{1-1/p'}_{p'p'}(\Gamma_P)}$$

gives

$$\|z^T\cdot G(t+h) - z^T\cdot G(t)\|_{\mathrm{B}_{pp}^{-1/p}(\Gamma_P)} \le C\Big\{ \sum_{k=P,F} \|z^T\cdot F^k(t+h) - z^T\cdot F^k(t)\|_{{}_0\mathrm{H}_p^{-1}(\Omega_k)} +$$

$$\|[a^k(t+h,\tilde{u}^k(t+h)) - a^k(t+h,u^k(t+h))]\nabla\phi^k(t+h) - [a^k(t,\tilde{u}^k(t)) - a^k(t,u^k(t))]\nabla\phi^k(t)\|_{\mathrm{L}_p(\Omega_k)}\Big\}.$$

Finally, after applying that norm of $\mathrm{H}_p^{1/2}(J)$ which is characterised by means of differences and taking account that the coefficients a^P, a^F belong to $\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))$ we may estimate as follows

$$\begin{aligned}
\|z^T\cdot G\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{B}_{pp}^{-1/p}(\Gamma_P))} &\le C\Big\{ \|z^T\cdot F^k(u^k,\phi^k)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))} \\
&\qquad + \|a^k(u^k) - a^k(\tilde{u}^k)\|_{{}_0\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))}\|\phi^k\|_{\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^1(\Omega_k))} \\
&\le C\Big\{ \|z^T\cdot F^k(u^k,\phi^k)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))} + \big(\|u^k - w^k\|_{{}_0\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))} \\
&\qquad + \|w^k - \tilde{u}^k\|_{{}_0\mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k))}\big)\cdot\big(\|\phi^k - \psi^k\|_{{}_0\mathcal{Z}_k^T} + \|\psi^k\|_{\mathcal{Z}_k^T}\big)\Big\} \\
&\le C\Big\{ \sum_{k=P,F} \|z^T\cdot F^k(u^k,\phi^k)\|_{{}_0\mathrm{H}_p^{1/2}(J;\mathrm{H}_p^{-1}(\Omega_k))} \\
&\qquad + (\delta + \psi_2(T))(\delta + \psi_1(T))\Big\}.
\end{aligned}$$

Thus the desired result is achieved in view of having estimates of F^k in $\mathcal{X}_k^T$. By taking account all estimates above we accomplish

$$\|(\mathcal{F}_{\tilde{u}}(u,\phi) - \mathcal{F}_{\tilde{u}}(\tilde{u},0),0)\|_{\mathcal{M}_0^T} \le M_1(\delta + \psi_2(T))[\delta + \psi_1(T) + \psi_3(T) + \sum_{k=P,F,b} \psi_4^k(T)].$$

Note that all constants appearing in the above estimations are independent of T. In case of contraction we proceed in the same line. By using the triangle inequality we get

$$\begin{aligned}
&\|(\mathcal{F}_{\tilde{u}}(u,\phi) - \mathcal{F}_{\tilde{u}}(\overline{u},\overline{\phi}),0)\|_{\mathcal{M}_0^T} \le \sum_{k=P,F} \|R^k(\cdot,\cdot,u^k) - R^k(\cdot,\cdot,\overline{u}^k)\|_{X_k^T} + \\
&\quad \|R^b(\cdot,u^b) - R^b(\cdot,\tilde{u}^b)\|_{X_b^T} + \sum_{k=P,F}\Big\{\|[\mathcal{A}_{k2}(\cdot,\cdot,\tilde{u}^k,D) - \mathcal{A}_{k2}(\cdot,\cdot,u^k,D)](\phi^k - \overline{\phi}^k)\|_{{}_0\mathcal{X}_k^T} \\
&\qquad + \|[\mathcal{A}_{k2}(\cdot,\cdot,u^k,D)\phi^k - \mathcal{A}_{k2}(\cdot,\cdot,\overline{u}^k,D)]\overline{\phi}^k\|_{{}_0\mathcal{X}_k^T} \\
&\qquad + \|[\mathcal{B}_{k2}(\cdot,\cdot,\tilde{u}^k,D) - \mathcal{B}_{k2}(\cdot,\cdot,u^k,D)](\phi^k - \overline{\phi}^k)\|_{{}_0\mathcal{Y}_2^T} \\
&\qquad + \|[\mathcal{B}_{k2}(\cdot,\cdot,u^k,D) - \mathcal{B}_{k2}(\cdot,\cdot,\overline{u}^k,D)]\overline{\phi}^P\|_{{}_0\mathcal{Y}_2^T} + \|R_{\tilde{u}^k}(u^k) - R_{\tilde{u}^k}(\overline{u}^k)\|_{{}_0Y_{1,\Gamma_P}^T(\mathbb{R}^N)}\Big\} \\
&\quad + \|u^b - \overline{u}^b\|_{{}_0Y_{1,\Gamma}^T(E)} + a^b\Big\|\int_\Gamma \big[\mathcal{B}_{F2}(\cdot,x,\tilde{u}^F,D) - \mathcal{B}_{F2}(\cdot,x,u^F,D)\big](\phi^F - \overline{\phi}^F)d\sigma\Big\|_{X_b^T} \\
&\qquad + a^b\Big\|\int_\Gamma \big[\mathcal{B}_{F2}(\cdot,x,u^F,D) - \mathcal{B}_{F2}(\cdot,x,\overline{u}^F,D)\big]\overline{\phi}^F d\sigma\Big\|_{X_b^T}.
\end{aligned}$$

We continue with

$$\begin{aligned}\|u^b-\overline{u}^b\|_{{}_0Y^T_{1,\Gamma}(E)} \leq & C\|u^b-\overline{u}^b\|_{{}_0\mathrm{B}^{1-1/2p}_{pp}(J;E)} \leq CT^\alpha\|u^b-\overline{u}^b\|_{{}_0\mathrm{B}^{1-1/2p+\alpha}_{pp}(J;E)}\\ \leq & CT^\alpha\|u^b-\overline{u}^b\|_{Z^T_b}\,,\end{aligned}$$

where we have to demand $0<\alpha<1/2p$ to ensure the embedding $\mathrm{H}^1_p(J)\hookrightarrow \mathrm{B}^{1-1/2p+\alpha}_{pp}(J)$. comparing the norms of self-mapping with the above terms we perceive that this difference is the sole new expression which is added. By treating the other norms as in the case of self-mapping we thus get

$$\|(F_{\tilde u}(u,\phi)-F_{\tilde u}(\overline{u},\overline{\phi}),0)\|_{\mathcal{M}^T_0} \leq M_2[\sum_{i=1}^3\psi_i(T)+\sum_{k=P,F,b}\psi^k_4(T)+\delta+T^\alpha]\|(u,\phi)-(\overline{u},\overline{\phi})\|_{{}_0Z^T\times {}_0\mathcal{Z}^T}.$$

If we choose $(\delta,T)\in(0,1]\times(0,T_0]$ sufficiently small, then we succeed in estimating (4.9) by δ, i.e. $\mathcal{L}^{-1}\mathcal{F}_{\tilde u}$ is a self-mapping. Moreover, by a possibly smaller choice of δ and T we attain $\sum_{i=1}^3\psi_i(T)+\sum_{k=P,F,b}\psi^k_4(T)+\delta+T^\alpha \leq 1/(2M_2)$ which implies contraction. Hence, the contraction mapping principle yields a unique fixed point of equation (4.5) in $\Sigma_{\delta,T}$ which is the unique strong solution on $J=[0,T]$ in the regularity space $Z^T\times\mathcal{Z}^T$.

(b) Continuation, positivity and regularity. In order to carry out the continuation of the solution (u,ϕ), we have to ensure that $u(T)$ belongs to $V(E_+)$. Note that the regularity follows directly from the trace theorem. On the one hand, the positivity of $u(T)$ is required for applying Theorem 3.4 to the linear problem. In fact, the function u^k, which is incorporated in the coefficient $a^k=z^T\!\cdot M^ku^k$ of differential operator $z^T\!\cdot\mathcal{A}_{k2}(D)$, must not vanish in order to guarantee that the elliptic problem for the potentials is regular. On the other hand, u^k is inserted in the logarithm appearing in boundary condition of Dirichlet type on Γ_P. Nevertheless, we can perform the continuation as long as u is positive.

If the right hand sides of parabolic equations have more regularity, e.g. $R^k\in C^1$, then we can establish more regularity of (u,ϕ) in the interior of domain resp. in the open time interval, see e.g. Escher, Prüss and Simonett [10] or Prüss [27]. Employing these methods entails classical solutions in $(0,T)\times\Omega_k$. This places us in a position to apply the maximum principle. In fact, assume that there exists an index $i\in\{1,\dots,N\}$ and a point $(t_0,x_0)\in(0,T)\times\Omega_k$ so that $u^k_i(t_0,x_0)=0$, $u^k_i(t,x)>0$ for all $x\in\Omega_k$ and $t<t_0$, and $u^k_i(t_0,x)\geq 0$ for $x\in\Omega_k$. This assumption implies $\nabla u^k_i(t_0,x_0)=0$ and $\Delta u^k_i(t_0,x_0)\geq 0$. Keeping in mind the assumptions and conclusions we derive the inequality

$$\begin{aligned}\partial_t u^k_i(t_0,x_0)=&d^k_i(t_0,x_0)\Delta u^k_i(t_0,x_0)+\nabla d^k_i(t_0,x_0)\nabla u^k_i(t_0,x_0)+\\ &m^k_i(t_0,x_0)u^k_i(t_0,x_0)\Delta\phi^k(t_0,x_0)+\nabla(m^k_i(t_0,x_0)u^k_i(t_0,x_0))\nabla\phi^k\\ &+R^k_i(t_0,x_0,u^k)\geq R^k_i(t_0,x_0,u^k)\geq 0,\end{aligned}$$

where the latter inequality is a result from assumption (R3). We achieve

$$u^k_i(t_0,x_0)\geq u^k_i(0,x_0)=u^k_{i,0}(x_0)>0,$$

which contradicts to the assumption. Hence, we have shown that u^P and u^F are positive in Ω_k. The next purpose is targeted on positivity on boundary Γ_P. Thus, suppose that there exists an index $i\in\{1,\dots,N\}$ and a point $(t_0,x_0)\in(0,T]\times\Gamma_P$ so that w.l.o.g. $u^P_i(t_0,x_0)=0$,

$u_i^P(t,x) > 0$ for all $x \in \overline{\Omega}_P$ and $t < t_0$, and $u_i^P(t_0,x) \geq 0$ for $x \in \Gamma_P$. Due to the boundary condition of Dirichlet type we can deduce that u_i^F vanishes in (t_0,x_0) as well. This can be seen by the following identity

$$u_i^P(t,x) = \frac{\gamma_i^F(t,x)}{\gamma_i^P(t,x)} u_i^F(t,x) e^{\lambda_0 z_i(\phi^F(t,x)-\phi^P(t,x))}, \quad (t,x) \in [0,T] \times \Gamma_P,$$

which is equivalent to the original boundary condition. Consequently, the transmission condition takes the form in the point (t_0,x_0)

$$d_i^P(t_0,x_0)\partial_\nu u_i^P(t_0,x_0) = d_i^F(t_0,x_0)\partial_\nu u_i^F(t_0,x_0).$$

Note that the embedding $Z_k^T \hookrightarrow \mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}_k;\mathbb{R}^N))$ admits traces in t and x in the above equation. Now, we want to apply Hopf's Lemma to produce a contradiction. In view of the fact that $u_i^P(t_0,x) > u_i^P(t_0,x_0) = 0$ for all $x \in \Omega_P$ we conclude by Hopf's Lemma $\partial_\nu u_i^P(t_0,x_0) < 0$ and thus $d_i^P(t_0,x_0)\partial_\nu u_i^P(t_0,x_0) < 0$. On the other hand we know that $u_i^F(t_0,x) > u_i^F(t_0,x_0) = 0$ for all $x \in \Omega_F$. Keeping in mind that the outer normal on Γ_P concerning the domain Ω_F points to the opposite direction, we then deduce $d_i^F(t_0,x_0)\partial_\nu u_i^F(t_0,x_0) > 0$. Combining these inequalities with the above boundary condition leads to a contradiction in view of the assumption.

Now, we discuss the other boundary segment of Ω_F. As above we suppose that there exists an index $i \in \{1,\dots,N\}$ and a point $(t_0,x_0) \in (0,T] \times \Gamma$ so that $u_i^F(t_0,x_0) = 0$. On the other hand, the boundary condition $u_i^F(t,x) = u_i^b(t)$ on Γ gives rise to

$$u_i^F(t_0,x) = u_i^b(t_0) = u_i^F(t,x_0) = 0, \quad \forall x \in \Gamma,$$

i.e. $u_i^F(t_0,x)$ vanishes on the whole boundary Γ. To derive a contradiction we consider the bulk equation of u_i^b evaluated in t_0. By using the above results, positivity assumption (R6) for R^b and $\partial_\nu u_i^F \leq 0$ we obtain

$$\begin{aligned} \frac{d}{dt}u_i^b(t_0) &= \frac{1}{\tau}(u_i^f(t_0) - u_i^b(t_0)) - a^b \int_\Gamma d_i^F(t_0,x)\partial_\nu u_i^F(t_0,x) + m_i^F(t_0,x)u_i^F(t_0,x)\partial_\nu \phi^F(t_0,x)\, d\sigma \\ &\quad + R_i^b(t_0,u(t_0)) \\ &= \frac{1}{\tau}u_i^f(t_0) - a^b \int_\Gamma d_i^F(t_0,x)\partial_\nu u_i^F(t_0,x)\, d\sigma + R_i^b(t_0,u(t_0)) \geq \frac{1}{\tau}u_i^f(t_0) \geq 0, \end{aligned}$$

which shows positivity of $u_i^b(t_0)$ due to positive initial data, and thus provides a contradiction to $u_i^F(t_0,x_0) = u_i^b(t_0) = 0$.

Now, we turn to the process of continuation. In fact, the nonlinear problem was solved in $[0,T]$ and due to the above considerations the solution $u(t)$ keeps positive, i.e. in particular $u(T) > 0$. Consequently, we can carry on solving the nonlinear problem with the new initial data $u(T) > 0$. This process results in a maximal interval of existence $[0,t_{max}(u_0))$ which is characterised by the condition that $\lim_{t\to T_{max}} u(t)$ does not exist in $V(E_+)$. In turn, this condition is equivalent to $\|u\|_{Z^{t_{max}}} = \infty$. Due to the embedding $Z^T \hookrightarrow \mathrm{C}(J;V(E_+))$ the map $u_0 \to u(t)$ defines a local semiflow on the natural phase space $V(E_+)$ in the autonomous case. Thus the proof is complete.

□

Now we are in the position to treat the example problem as introduced in Chapter 1.

Corollary 4.1 *Let Ω_P, Ω_F be bounded domains in $\mathbb{R}^{n+1}$ with C^2- boundary, $\Gamma_P := \partial\Omega_P$, $\partial\Omega_F = \Gamma_P \cup \Gamma$ and* $\operatorname{dist}(\Gamma_P, \Gamma) > 0$. *Let $(n+1)+2 < p < \infty$ and suppose that*

1. *$d_i^k \in \mathrm{C}^{1/2}(J; \mathrm{C}^1(\Omega_k))$, $d_i^k(t,x) > 0$ for $(t,x) \in J \times \overline{\Omega}_k$, $i \in \{1,2,3\}$, $k = P, F$;*
2. *$\gamma_i^k \in \mathrm{B}_{pp}^{1-1/2p}(J; \mathrm{L}_p(\Gamma_P; \mathbb{R}_+)) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{2-1/p}(\Gamma_P; \mathbb{R}_+))$, $i \in \{1,2,3\}$, $k = P, F$;*
3. *$u_0 = (u_0^P, u_0^F, u_0^b) \in V(E_+)$, $u_{\mathrm{B},0}^P \in \mathrm{C}(\overline{\Omega}_P)$, $u_{i,0}^k > 0$ for $k = P, F, b$, $i = 1,2,3,B$, $u_{\mathrm{AC},0}^P > 0$, $u_{\mathrm{HC},0}^b > 0$, $u^f \in \mathrm{L}_p(J; E_+)$;*
4. *the compatibility conditions of Theorem 4.1 are valid.*

Then there exists $t_{max} > 0$ such that for any $T_0 < t_{max}$ the problem (1.1)-(1.8), (1.10)-(1.13) admits a unique solution $(u^P, u^F, u^b, \phi^P, \phi^F)$, $(u_{\mathrm{B}}^P, u_{\mathrm{HC}}^b, u_{\mathrm{AC}}^b)$ on $J = [0, T_0]$ in the regularity class

$$Z^{T_0}(E_+) \times \mathcal{Z}^{T_0} \times \mathrm{C}^{3/2}([0,T_0]; \mathrm{C}(\overline{\Omega}_P; \mathbb{R}_+)) \times \mathrm{H}_p^1([0,T_0]; \mathbb{R}_+) \times \mathrm{H}_p^1([0,T_0]; \mathbb{R}_+)\,.$$

In particular, we have

$$\begin{aligned} &u^k \in \mathrm{C}^1((0,t_{max}); \mathrm{C}(\Omega_k)) \cap \mathrm{C}((0,t_{max}); \mathrm{C}^2(\Omega_k)), \quad k = P, F, \\ &\phi^k \in \mathrm{C}^{1/2}((0,t_{max}); \mathrm{C}(\Omega_k)) \cap \mathrm{C}((0,t_{max}); \mathrm{C}^2(\Omega_k)), \quad k = P, F\,. \end{aligned} \tag{4.10}$$

Moreover, the vector of concentrations $(u^P, u^F, u^b, u_{\mathrm{B}}^P, u_{\mathrm{HC}}^b, u_{\mathrm{AC}}^b)$ is positive and the map

$$(u_0^P, u_0^F, u_0^b, u_{\mathrm{B},0}^P, u_{\mathrm{HC},0}^b, u_{\mathrm{AC},0}^b) \longrightarrow (u^P(t), u^F(t), u^b(t), u_{\mathrm{B}}^P(t), u_{\mathrm{HC}}^b(t), u_{\mathrm{AC}}^b(t)) \tag{4.11}$$

defines a local semiflow on the natural phase space $V(E_+) \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ in the autonomous case.

Proof. We have seen in Section 1.2 that the ordinary differential equations for concentrations u_{B}^P, u_{HC}^b, and u_{AC}^b can be solved via variation of constants formula. We also perceived that only the functions u_1^P and $u^b = (u_1^b, u_2^b, u_3^b)$ come in. Consequently, this presentation can be used to eliminate u_{B}^P, u_{HC}^b, u_{AC}^b in reaction rates R^P, R^b in order to achieve a problem treated in Theorem 4.1. Therfore, we have to check if R^P and R^b satisfy the assumptions (R1)-(R6). First, we recall the definition of R^P and the solution formula for u_{B}^P.

$$\begin{aligned} &R^P(t,x,u^P) = r_{\mathrm{B}}^P(t,x,u^P)(2,-1,0)^T, \\ &r_{\mathrm{B}}^P(t,x,u^P) = -k_{\mathrm{B}} u_{\mathrm{B}}^P(t,x) u_1^P(t,x), \quad u_{\mathrm{B}}^P(t,x) = e^{-k_{\mathrm{B}} \int_0^t u_1^P(s,x)ds} u_{\mathrm{B},0}^P(x). \end{aligned} \tag{4.12}$$

It is easy to verify that each R^P satisfies the assumptions (R1) and (R3). To see (R2) we estimate as follows

$$\begin{aligned} &\|R^P(u^P) - R^P(\overline{u}^P)\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega_P;\mathbb{R}^N))} \le C\|u_{\mathrm{B},0}^P\|_{\mathrm{C}(\overline{\Omega}_P)} \|u_1^P - \overline{u}_1^P\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega_P))} + \\ &\quad C\|u^P\|_{\mathrm{C}(J\times\overline{\Omega}_P)} \|u_{\mathrm{B}}^P - \overline{u}_{\mathrm{B}}^P\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega_P))} \le CT\|u_{\mathrm{B},0}^P\|_{\mathrm{C}(\overline{\Omega}_P)} \|u^P - \overline{u}^P\|_{Z_P^T} + \\ &\quad C\|\overline{u}^P\|_{\mathrm{C}(J\times\overline{\Omega}_P)} T^{1+1/p} \|u^P - \overline{u}^P\|_{Z_P^T} \le C(T\|u_{\mathrm{B},0}^P\|_{\mathrm{C}(\overline{\Omega}_P)} + L_P T^{1+1/p}) \|u^P - \overline{u}^P\|_{Z_P^T} \end{aligned}$$

In doing so, we have exploited the presentation of u_{B}^P, positivity of u_1, and the condition $|u_1^P|_\infty \le L_P$. Thus, from the above estimate we can read off that $\|l^P\|_{\mathrm{L}_p(J)} = C(T\|u_{\mathrm{B},0}^P\|_\infty + L_P T^{1+1/p})$ which becomes small due to the parameter T.

Regularity and positivity of u^P_{B} can immediately be seen by solution formula, $u^P_{\mathrm{B},0} \in \mathrm{C}(\overline{\Omega}_P;\mathbb{R}_+)$, and the fact that $u^P_1 \in Z^T_P \hookrightarrow \mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_P)) \cap \mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}_P))$.

We now come to the bulk reaction rate R^b given by

$$R^b(t,u^b) = (-r^b_{\mathrm{HC}}(t,u^b), -r^b_{\mathrm{AC}}(t,u^b), -r^b_{\mathrm{HC}}(t,u^b) - 2r^b_{\mathrm{AC}}(t,u^b))^T\,,$$
$$r^b_{\mathrm{HC}}(t,u^b) = k_{\mathrm{H}}(u^b_1u^b_3 - K_{\mathrm{H}}u^b_{\mathrm{HC}})\,,\quad r^b_{\mathrm{AC}}(t,u^b) = k_{\mathrm{A}}(u^b_2u^b_3 - K_{\mathrm{A}}u^b_{\mathrm{AC}})\,,$$

and solution formulae for concentrations of species HC and AC

$$u^b_{\mathrm{HC}}(t) = e^{-\left(\frac{1}{\tau}+k_{\mathrm{H}}K_{\mathrm{H}}\right)\cdot t}u^b_{\mathrm{HC},0} + \int\limits_0^t e^{-\left(\frac{1}{\tau}+k_{\mathrm{H}}K_{\mathrm{H}}\right)\cdot(t-s)}u^f_{\mathrm{HC}}(s) + k_{\mathrm{H}}u^b_1(s)u^b_3(s)\,ds\,,$$
$$u^b_{\mathrm{AC}}(t) = e^{-\left(\frac{1}{\tau}+k_{\mathrm{A}}K_{\mathrm{A}}\right)\cdot t}u^b_{\mathrm{AC},0} + \int\limits_0^t e^{-\left(\frac{1}{\tau}+k_{\mathrm{A}}K_{\mathrm{A}}\right)\cdot(t-s)}u^f_{\mathrm{AC}}(s) + k_{\mathrm{A}}u^b_2(s)u^b_3(s)\,ds\,.$$

We only prove assumption (R5) since the other conditions are trivial. Further on, it suffices to show this condition for r^b_{HC} and r^b_{AC} since each component of R^b is composed of a linear combination of these functions. Let u^b, $\overline{u}^b \in \mathrm{H}^1_p(J;E_+)$ be given with $\|u^b\|_\infty, \|\overline{u}^b\|_\infty \le L_b$. By using the above solution formulae we then obtain

$$\begin{aligned}
\|r^b_{\mathrm{HC}}(u^b) - r^b_{\mathrm{HC}}(\overline{u}^b)\|_{\mathrm{L}_p(J)} &\le C(\|u^b_1u^b_3 - \overline{u}^b_1\overline{u}^b_3\|_{\mathrm{L}_p(J)} + \|u^b_{\mathrm{HC}} - \overline{u}^b_{\mathrm{HC}}\|_{\mathrm{L}_p(J)})\\
&\le CL_b\|u^b - \overline{u}^b\|_{\mathrm{L}_p(J;E)} + \Big\|\int\limits_0^t e^{-\left(\frac{1}{\tau}+k_{\mathrm{H}}K_{\mathrm{H}}\right)\cdot(t-s)}k_{\mathrm{H}}[u_1(s)u_3(s) - \overline{u}_1(s)\overline{u}_3(s)]\,ds\Big\|_{\mathrm{L}_p(J)}\\
&\le CL_b(T + T^{1+1/p})\|u^b - \overline{u}^b\|_{Z^T_b}\,,
\end{aligned}$$

and in the same way we get

$$\|r^b_{\mathrm{AC}}(u^b) - r^b_{\mathrm{AC}}(\overline{u}^b)\|_{\mathrm{L}_p(J)} \le CL_b(T + T^{1+1/p})\|u^b - \overline{u}^b\|_{Z^T_b}\,.$$

We see from the solution formulae that u^b_{HC} and u^b_{AC} are positive, and that the regularity is determined by u^f_i since this function possesses the least regularity. In fact, u^f_i belongs to $\mathrm{L}_p(J)$ and thus the convolution of $e^{-(\frac{1}{\tau}+k_iK_i)}$ with u^f_i lies in $\mathrm{H}^1_p(J)$.

In the end, in order to make use of the regularity theory yielding classical solutions in $(0,t_{max}) \times \Omega_k$, we have to ensure that the right hand sides belong to $\mathrm{C}^1(J \times \Omega_k \times \mathbb{R}^N)$ as well. Since in domain Ω_F chemical reactions do not take place we have $R^F \equiv 0$ which is real analytic. In case $k = P$ the right hand side $R^P_i(t,x,u^P)$ is given by 4.12 and it is easy to see that R^P belongs to the class C^1. So according to Theorem 4.1 we obtain the results stated in the above corollary and the proof is complete.

$\square$

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Matthias Kotschote: *Strong Well-Posedness of a Model for an Ionic Exchange Process (Zusammenfassung)*

Gegenstand dieser Arbeit ist ein mathematisches Modell zur Beschreibung eines Ionenaustauschers. Bei der Modellierung dieses Problems werden, neben den chemischen Reaktionen, die Transportprozesse der ionischen Spezies und der Einfluss elektrischer Felder auf diesen Stofftransport berücksichtigt.

Im folgenden sei Ω ein beschränktes Gebiet im $\mathbb{R}^n$, welches sich zusammensetzt aus einem Kerngebiet Ω_P (Pellet) und einem Streifengebiet Ω_F (Film). Das Gebiet Ω_F umschließt vollständig Ω_P und besitzt dadurch zwei disjunkte Ränder, einen äußeren Rand $\Gamma = \partial\Omega$ and einen inneren Rand $\Gamma_P = \partial\Omega_P$. Diese Ränder sollen zur Klasse C^2 gehören und einen positiven Abstand besitzen. Für die unbekannten Stoffkonzentrationen $u_i^k : J = [0,T] \times \Omega_k \to \mathbb{R}_+$, $i = 1, \ldots, N$, und elektrischen Potenziale $\phi^k : J = [0,T] \times \Omega_k \to \mathbb{R}$ betrachten wir das System von Differentialgleichungen

$$
(1) \quad \begin{cases}
\partial_t u_i^P - \nabla \cdot (d_i^P \nabla u_i^P) - \nabla \cdot (m_i^P u_i^P \nabla \phi^P) = R_i^P, \quad (t,x) \in J \times \Omega_P, \\
\partial_t u_i^F - \nabla \cdot (d_i^F \nabla u_i^F) - \nabla \cdot (m_i^F u_i^F \nabla \phi^F) = R_i^F, \quad (t,x) \in J \times \Omega_F, \\
\frac{d}{dt} u_i^b(t) = -a^b \int\limits_\Gamma [d_i^F \partial_\nu u_i^F + m_i^F u_i^F \partial_\nu \phi^F] \, d\sigma + \frac{1}{\tau} \left(u_i^f(t) - u_i^b(t) \right) + R_i^b(t), \quad t \in J,
\end{cases}
$$

mit den Randbedingungen und den Anfangswerten

$$
(2) \quad \begin{cases}
d_i^P \partial_\nu u_i^P + m_i^P u_i^P \partial_\nu \phi^P = d_i^F \partial_\nu u_i^F + m_i^F u_i^F \partial_\nu \phi^F, \quad (t,x) \in J \times \Gamma_P, \\
\ln(\gamma_i^P u_i^P) + \lambda_0 z_i \phi^P = \ln(\gamma_i^F u_i^F) + \lambda_0 z_i \phi^F, \quad (t,x) \in J \times \Gamma_P, \\
u_i^F(t,x) = u_i^b(t), \quad (t,x) \in J \times \Gamma, \\
u_i^P(0,x) = u_{0,i}^P(x), \quad x \in \overline{\Omega}_P, \\
u_i^F(0,x) = u_{0,i}^F(x), \quad x \in \overline{\Omega}_F, \\
u_i^b(0) = u_{0,i}^b.
\end{cases}
$$

Das obige System wird durch die Elektroneutralitätsbedingung, welche eine algebraische Gleichung ist, vervollständigt:

$$
(3) \quad \left\{ \sum_{i=1}^N z_i u_i^k(t,x) = 0, \quad (t,x) \in J \times \overline{\Omega}_k, \quad k = P, F, \quad \sum_{i=1}^N z_i u_i^b(t) = 0, \quad t \in J \right. .
$$

Hierbei bezeichnet $\partial_t u_i^k$ die partielle Ableitung der Konzentration u_i^k nach t, ∇u_i^k ($\nabla \phi^k$) deN Gradient von u_i^k (ϕ^k) bezüglich der räumlichen Variablen und $\nabla\cdot$ den Divergenzoperator. Ferner werden die Diffusionskoeffizienten $d_i^k(t,x)$ und die chemischen Aktivitäten $\gamma_i^k(t,x)$ als bekannt vorausgestzt. Die sogenannte elektrochemische Mobilität m_i^k ist definiert durch $m_i^k(t,x) := \lambda_0 d_i^k(t,x) z_i$, wobei z_i die elektrische Ladungszahl der Spezie i ist. Die Konstante $\lambda_0 = F/RT$ setzt sich zusammen aus der Faraday Konstante F, der allgemeinen Gaskonstante R und der absoluten Temperatur T. Desweiteren bezeichnet τ die hydrodynamische Verweilzeit und $a^b = N^b/V^b$ die Anzahl der Pellets pro Bulkvolumen. Die nichlineare Funktion R_i^k ist die Produktionsrate der Spezie i in der Phase k.

Das betrachtete Problem kann als ein System von parabolischen Differentialgleichungen angesehen werden, welches aufgrund der Elektroneutralitätsbedingung mit einem elliptischen Randwertproblem gekoppelt ist. Die Konzentrationen verschiedener Phasen werden durch die Randbedingungen miteinander in Beziehung gebracht, wobei die elliptischen Gleichungen eine Kopplung aller Transportgleichungen bewirken. Diese Kopplung ist verantwortlich für die Nichtlinearität der Transmissionsrandbedingung, welche eine besondere Schwierigkeit darstellt und bisher nicht in der Literatur analytisch behandelt wurde.

Aufgrund der allgemeinen Form der Gleichungen (1)-(3) stehen diese stellvertretend für eine ganze Klasse von Problemen, die in der technischen Chemie anzutreffen sind. Insbesondere lässt sich der Ionenaustauscher damit beschreiben.

Unter geeigneten Voraussetzungen an die Systemgrößen d_i^k und γ_i^k, die Nichlinearitäten R_i^k und Anfangswerte $u_{i,0}^k$ wird in der Arbeit nachgewiesen, dass das vorliegende Problem eine eindeutige starke Lösung im L_p-Sinn besitzt. Sei $n+2 < p < \infty$. Dann gibt es ein $t_{max} > 0$, so dass für alle $T < t_{max}$ genau eine Funktion (u^P, u^F, u^b) im Raum

$$Z^T := Z_P^T \times Z_F^T \times \mathrm{H}_p^1([0,T];\mathbb{R}_+^N)$$

und genau ein Potenzial (ϕ^P, ϕ^F) in

$$\mathcal{Z}^T := \{(\varphi^P, \varphi^F) \in \mathcal{Z}_P^T \times \mathcal{Z}_F^T : \gamma_{|\Gamma_P}(\phi^P - \phi^F) \in Y_{1,\Gamma_P}^T\}.$$

existieren, die das System (1)-(3) lösen. Dabei sind $u^k := (u_1^k, \dots, u_N^k)$ für $k = P, F, b$,

$$\begin{aligned} Z_k^T &:= \mathrm{H}_p^1([0,T];\mathrm{L}_p(\Omega_k;\mathbb{R}_+^N)) \cap \mathrm{L}_p([0,T];\mathrm{H}_p^2(\Omega_k;\mathbb{R}_+^N)), \\ \mathcal{Z}_k^T &:= \mathrm{H}_p^{1/2}([0,T];\mathrm{H}_p^1(\Omega_k)) \cap \mathrm{L}_p([0,T];\mathrm{H}_p^2(\Omega_k)), \end{aligned}$$

für $k = P, F$, sowie

$$Y_{1,\Gamma_P}^T := \mathrm{B}_{pp}^{1-1/2p}([0,T];\mathrm{L}_p(\Gamma_P)) \cap \mathrm{L}_p([0,T];\mathrm{B}_{pp}^{2-1/p}(\Gamma_P)).$$

Hierbei bezeichnet $\mathrm{B}_{pp}^s(J;\mathrm{L}_p(\Gamma_P))$ den vektorwertigen Sobolev-Slobodeckijraum von Funktionen auf $J = [0,T]$ mit Werten im Lebesgueraum $\mathrm{L}_p(\Gamma_P)$. Darüberhinaus definiert die Abbildung

$$(u_0^P, u_0^F, u_0^b) \to (u^P(t), u^F(t), u^b(t))$$

einen lokalen Halbfluss im autonomen Fall. Die Idee des Beweises besteht darin, für das nichtlineare Problem (1)-(3) ein verwandtes lineares Problem mit inhomogenen Randdaten herzuleiten und für dieses maximale L_p-Regularität nachzuweisen. Diese Eigenschaft ermöglicht die Umformulierung des nichtlinearen Problemes in eine Fixpunktgleichung im Raum $Z^T \times \mathcal{Z}^T$, die dann mit Hilfe des Kontraktionsprinzips gelöst werden kann. Die Voraussetzung an p stellt dabei sicher, dass die Einbettung $Z_k^T \hookrightarrow \mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega}_k)) \cap \mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}_k))$ gilt. Entscheidend für dieses Vorgehen ist das Finden der Regularitäten der rechten Seiten, die notwendig und hinreichend sind für die Existenz und Eindeutigkeit einer Lösung im Raum der maximalen Regularität. Um die Notwendigkeit der Bedingungen an die Inhomogenitäten einzusehen, werden bekannte Spurensätzen verwendet. Für die Hinlänglichkeit benutzen wir die Methoden der Lokalisierung und Störung, welche das Ausgangsproblem auf Ganz- und Halbraumprobleme mit konstanten Koeffizienten zurückführen. Das Lösen dieser Gleichungen wird mit Hilfe von Sätzen über Operatorsummen (Dore-Venni-Theorie), reeller Interpolation, sowie dem vektorwertigen Multiplikatorensatz von Michlin gewährleistet.

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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