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Differentiability Properties for Boundary Control of Fluid-Structure Interactions of Linear Elasticity with Navier-Stokes Equations with Mixed-Boundary Conditions in a Channel

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Abstract
In this paper we consider a fluid-structure interaction problem given by the steady Navier Stokes equations coupled with linear elasticity taken from (Lasiecka et al. in Nonlinear Anal 44:54–85, 2018). An elastic body surrounded by a liquid in a rectangular domain is deformed by the flow which can be controlled by the Dirichlet boundary condition at the inlet. On the walls along the channel homogeneous Dirichlet boundary conditions and on the outflow boundary do-nothing conditions are prescribed. We recall existence results for the nonlinear system from that reference and analyze the control to state mapping generalizing the results of (Wollner and Wick in J Math Fluid Mech 21:34, 2019) to the setting of the nonlinear Navier-Stokes equation for the fluid and the situation of mixed boundary conditions in a domain with corners.

Keywords Fluid-structure interaction · Boundary control · Differentiability properties · Navier Stokes equation · Mixed boundary conditions · Domain with corners

Mathematics Subject Classification 74F10

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1 Introduction

The paper deals with fluid-structure interaction (FSI) problems given by a fluid flow around an elastic body in a rectangular channel with fixed walls in two space dimensions. The elastic body deforms under the flow and is modelled by linear elasticity, for the fluid we consider the steady Navier-Stokes equation with Dirichlet condition at the inlet, no-slip condition on the wall, and do-nothing condition on the outlet. The configuration is taken from Lasiecka, Szulc, and Zochoswki [27] who analyze existence of solutions to this FSI problem and existence of an optimal inflow profile, considered as a boundary control, which minimizes the drag at the interface of the elastic body and the fluid. Let g denote the Dirichlet inflow boundary values and (u, w, p) be the solution of the FSI problem after transforming the variables for the fluid to a reference domain, that means u solves the elasticity equation, (w, p) is the solution of the Navier-Stokes equation and both equations are coupled via the traction force at the interface and via coefficients in the Navier-Stokes equation: We show that the control to state map of the FSI problem

$$B_r(\mathcal{G}_{3/2}) \to X^p, \quad g \mapsto (u, w, p) \tag{1.1}$$

with ball $B_r(\mathcal{G}_{3/2})$ around zero with radius $r > 0$ in the space $\mathcal{G}_{3/2}$ defined in (2.28) and X^p, $p > 2$, defined in (3.17) is continuously Fréchet differentiable for sufficiently small r. The exact statement is formulated in Theorem 4.

The differentiability is a crucial property to derive first-order optimality conditions which are usually the starting point for characterizing optimal controls and numerical schemes to solve such type of optimal control problems. While the formal derivation of these optimality conditions for similar settings has been considered, see below, we leave the rigorous derivation of optimality conditions for this specific case for future work. Difficulties in the analysis to derive Fréchet differentiability arise from the fact that (i) we consider the nonlinear Navier-Stokes equation, (ii) the problem is formulated in a polygonal domain, (iii) we have mixed Dirichlet-Neumann boundary conditions, and (iv) the analysis is considered in a higher regularity setting. Differentiability of FSI problems with respect to data has been considered for the Stokes equation with Dirichlet boundary conditions in smooth domains coupled with linear elasticity in Wick and Wollner [34]. There the differentiability is obtained by the implicit function theorem which we apply also here following their ideas. Therefore, the linearized Navier-Stokes operator needs to be an isomorphism in suitable spaces; hence, main parts of the paper deal with the derivation of regularity results for the linearized equation. We proceed in three steps following the procedure in [27]: In (i) we derive a lower regularity result for the velocity pressure pair in $W^{1,2} \times L^2$ based on Lax-Milgram arguments. In (ii) we derive a higher regularity result in $W^{2,2} \times W^{1,2}$ based on Benes and Kucera [8, Appendix] who prove $W^{2,2} \times W^{1,2}$ regularity for the solution of the Stokes equation in rectangular domains with mixed boundary conditions. They use a construction which explicitly relies on the angle at the corner and apply results from Agmon, Douglis, and Nirenberg [1] for ellitpic systems. In (iii) we derive higher p-integrability, namely $W^{2,p} \times W^{1,p}$ on compact subsets using commutator analysis.

For the analysis of linear elasticity we rely on classical theory.

We remark that in contrast to [27] in our setting the traction force at the interface involves not only the pressure but also the normal derivative of the velocity as considered in Grandmont [19, Equation (8)].

We Give an Overview About Related Literature. *On FSI problems*: Galdi and Kyed [17] analyze existence of steady FSI problems in smooth domains. Wick and Wollner [34] derived as mentioned the differentiability of steady FSI problems with respect to the problem data in smooth domains. For an introduction to evolutionary FSI problems we refer to Kaltenbacher et al [26]; moreover, see, e.g., Gunzburger et al. [12, 13], Grandmont and Maday [18], and Ignatova, Kukavica, Lasiecka, and Tuffaha [25].

On Optimal Control and FSI: In [27] boundary control of a FSI problem with stationary Navier-Stokes equation is considered. The authors show existence of a unqiue solution of the underlying equation under a smallness condition as well as of an optimal control. This paper extends Grandmot [19] in the sense that the problem is considered in a domain with corners and with mixed boundary conditions. In the later reference an elastic body surrounds the fluid and an additional volume constraint is imposed while in the former paper the elastic body is surrounded by the fluid, furthermore, a radial unbounded cost is considered. Rigorously derived first order optimality conditions have been, to the best knowledge of the authors, not been stated yet for the problem under consideration. Numerics including formally derived optimality conditions are considered, e.g., in Richter and Wick [32] where optimal control and parameter estimation for stationary FSI problems are considered.

For a control problem for a dynamic version of the considered model within regular domains we refer to Bociu et al. [5]. For further references on control of evolutionary FSI problems see, e.g. Feiler, Meidner, and Vexler [16] who consider linear FSI systems with coupled Stokes and wave equation and derive optimality conditions as well as Moubachir and Zolesio [30] who derive for an optimal control problem for nonlinear time-dependent FSI problem necessary optimality conditions formally. Existence of optimal controls for the problem of minimizing flow turbulence in the case of a nonlinear fluid-structure interaction model is considered in Bociu et al. [6].

As mentionend above a challenge of the considered problem is due to regularity properties of the Stokes equation in a rectangular domain with mixed-boundary conditions. For results on general Lipschitz domain we refer to Brown et al. [9].

Finally, we remark that differentiablity properties of shape optimization problems for fluid-structure interation has been considered in Haubner, Ulbrich, and Ulbrich [24].

Notation: Throughout the paper we use the usual notation for Lebesgue and Sobolev spaces. For spaces of type $W^{s,p}(\Omega)^2$ ($W^{s,p}(\Omega)^{2\times 2}$ resp.) we often omit the dimension. We use the usual definition for smooth functions with compact support $C_c^\infty(\Omega)$. We define the symbolic expression

$$(w \cdot \nabla)w := (w_i \partial_i w_1, w_i \partial_i w_2)^\top \tag{1.2}$$

for $w \in W^{1,2}(\Omega_{\mathrm{f}})^2$ using Einstein summation convention, and we write $\operatorname{div} w := \partial_1 w_1 + \partial_2 w_2$. We denote $\nabla \cdot \sigma := \left(\sum_{j=1}^2 \frac{\partial \sigma_{ij}}{\partial x_j}\right)_{1 \le i \le 2}$ for $\sigma \in W^{1,2}(\Omega_{\mathrm{f}})^{2\times 2}$. For matrices B_1 and B_2 in $\mathbb{R}^{2\times 2}$ we denote the Frobenius product by $A \cdot B := \sum_{i,j=1}^{2} A_{ij} B_{ij}$. Sometimes we write 0 for the zero map. The dependence of a function f on another function g is indicated by $f[g]$ while the dependence on the spatial variable x by $f(x) = f[g](x)$. We use the following notation for the Jacobian of the flow map Φ as a function of u

$$\nabla\Phi := \nabla\Phi[u] := D\Phi[u] := \begin{pmatrix} \partial_1\Phi_1 & \partial_1\Phi_2 \\ \partial_2\Phi_1 & \partial_2\Phi_2 \end{pmatrix}[u] \tag{1.3}$$

and for the cofactor matrix and determinant of the Jacobian

$$\begin{aligned} K[u] &:= \det(D\Phi[u]) D\Phi[u]^{-\top} =: \operatorname{cof}(D\Phi[u]), \\ J[u] &:= \det(D\Phi[u]). \end{aligned} \tag{1.4}$$

Moreover, we set

$$A[u] := J[u]^{-1} K[u]^\top K[u]. \tag{1.5}$$

Further, we use the notation

$$\partial_{A[u],n} w := (A[u]\nabla w) n_\mathrm{f} \tag{1.6}$$

with outer normal n_f to Ω_f. With

$$c[u](v, w, z) := ((v \cdot K[u]\nabla) w, z)_{L^2(\Omega)} \tag{1.7}$$

we simplify the notation for the case u equal zero to $c(\cdot,\cdot,\cdot) := c[0](\cdot,\cdot,\cdot)$. We set for matrix $K \in \mathbb{R}^{n,n}$ the expression

$$\operatorname{div}_{K^\top} w := \operatorname{div}(K^\top w). \tag{1.8}$$

For functions f and e and operators D we write for the commutator $[f, D]e := fDe + D(fe)$. The space of linear bounded mappings from Banach space X_1 to Banach space X_2 we denote by $L(X_1, X_2)$.

The ball of radius $r > 0$ around zero in a Banach space W we denote by $B_r(W)$. Finally, $c > 0$ denotes a generic constant and $c_\varepsilon > 0$ a constant depending on $\varepsilon > 0$. The Euclidean norm in $\mathbb{R}^d$ is denoted by $\|\cdot\|$.

Structure of the paper: In Sect. 2 we introduce the physical setting as well as the flow map and transformation rules between the physical and reference domain, in Sect. 3 we introduce the Navier-Stokes system, the elasticity system, and the fluid-structure interaction system and prove existence of solutions, in Sect. 4 we state the main result of the paper, in Sect. 5 we show existence and a priori estimates for the linearized system in higher Sobolev norms, and in Sect. 6 we show the differentiability of the control to state mapping for the FSI system. In the appendix we recall the transformation of the Navier-Stokes equation and its linearization to the reference domain.

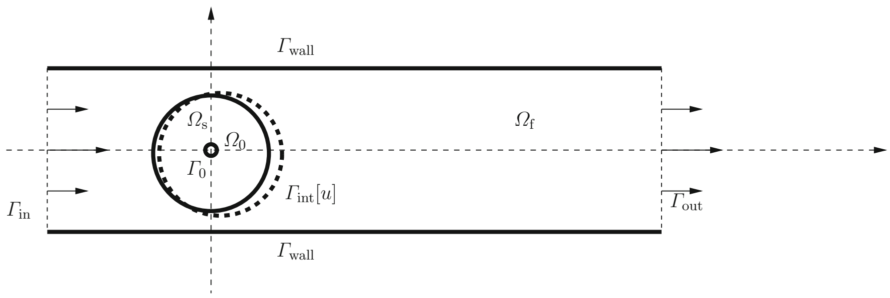

Fig. 1 Domain

Table 1 Variables in physical and reference domain

	Domains		Variables
Original domain	Ω_{s}	Ω_{f}	
Physical domain	$\Omega_{\mathrm{s}}[u]$	$\Omega_{\mathrm{f}}[u]$	$(\tilde{w}, \tilde{p})$
Reference domain	Ω_{s}	Ω_{f}	(w, p)

2 The Domain

We recall the problem setting from Lasiecka et al. [27]. Let $D \subset \mathbb{R}^2$ be a bounded domain with piecewise regular boundary ∂D and straight corners as shown in Fig. 1. Further, let

$$D = \Omega_{\mathrm{s}} \cup \Omega_{\mathrm{f}} \cup \Omega_0 \tag{2.1}$$

with Ω_{s} and Ω_{f} be subsets of D with Ω_{s} being a domain with a hole Ω_0 and boundary $\partial\Omega_{\mathrm{s}} := \Gamma_{\mathrm{int}} \cup \Gamma_0$. The exterior boundary of Ω_{f} is denoted by $\Gamma_{\mathrm{ext}} := \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{out}}$.

In Ω_{s} we consider a *problem of linear elasticity* for an elastic body with u denoting the displacement field. In the exterior subdomain Ω_{f} we consider a *Navier-Stokes problem* for the motion of a fluid with velocity field denoted by $\tilde{w}$ and pressure $\tilde{p}$.

We consider a parallel fluid flow in the channel D containing the elastic body in Ω_{s} which deforms due to the influence of surface forces by the fluid. The original boundary $\Gamma_{\mathrm{int}} = \Gamma_{\mathrm{int}}[0]$ of Ω_{s} transforms itself into $\Gamma_{\mathrm{int}}[u]$ with elastic displacement u on Γ_{int}, more precisely

$$\Gamma_{\mathrm{int}} \to \Gamma_{\mathrm{int}}[u], \quad x \mapsto x + u(x). \tag{2.2}$$

This leads to a new domain $\Omega_{\mathrm{f}}[u]$ with boundaries Γ_{in}, Γ_{out}, Γ_{wall}, and $\Gamma_{\mathrm{int}}[u]$. Variables in the physical domain are denoted with a tilde, cf. Table 1. The outer normal to Ω_{f} is denoted by n_{f} and the one to $\Omega_{\mathrm{f}}[u]$ by $n_{\mathrm{f}}[u]$. The outer normal to Ω_{s} is denoted by n_{s}.

2.1 The Flow Map and Some Transformation Rules

In this section we introduce the flow map and study the transformation between the physical and reference domain. At first, we recall some standard operators. The trace operator (cf. [15, Thm. B.54])

$$\gamma\colon W^{2,p}(\Omega_{\mathrm{s}}) \to W^{2-1/p,p}(\Gamma_{\mathrm{int}}), \qquad 2 \le p < \infty, \tag{2.3}$$

is surjective and satisfies for $u \in W^{2,p}(\Omega_{\mathrm{s}})$

$$\|\gamma u\|_{W^{2-1/p,p}(\Gamma_{\mathrm{int}})} \le c\,\|u\|_{W^{2,p}(\Omega_{\mathrm{s}})}\,. \tag{2.4}$$

The corresponding trace operator for any open subset $\omega \subset \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{wall}}$ we denote by γ_ω.

Proposition 1 (Dirichlet harmonic extension) *For $2 \le p < \infty$ the harmonic extension*

$$\mathcal{D}\colon W^{2-1/p,p}(\Gamma_{int}) \to W^{2,p}(\Omega_f), \quad \eta_i \mapsto \mathcal{D}\eta_i := \phi_i[\eta_i], \quad i = 1, 2 \tag{2.5}$$

defined by

$$\Delta\phi_i = 0 \text{ in } \Omega_f, \quad \phi_i = \eta_i \text{ on } \Gamma_{int}, \quad \phi_i = 0 \text{ on } \partial\Omega_f \setminus \Gamma_{int} \tag{2.6}$$

is well-posed and satisfies the estimate

$$\|\phi_i\|_{W^{s,p}(\Omega_f[\eta])} \le C\left\|\gamma_{\Gamma_{int}}\eta_i\right\|_{W^{s-1/p,p}(\Gamma_{int})}, \quad \text{for } i = 1, 2. \tag{2.7}$$

Proof See Amrouche and Moussaoui [3] for an overview about results in domains with smooth and nonsmooth boundaries relying in particular on Necas [31] and Grivsvard [20–22]. We use the fact that Γ_{int} is smooth and that Γ_{ext} has straight angles. □

In the following we set $\phi[\eta] := (\phi_1[\eta_1], \phi_2[\eta_2])^\top$ for ϕ_i defined in (2.5).
In the rest of the paper we assume

$$2 < p < \infty. \tag{2.8}$$

Definition 1 (*Flow map*) For $u \in W^{2,p}(\Omega_{\mathrm{s}})$, and ϕ defined in (2.6) the flow map is given by

$$\Phi\colon W^{2,p}(\Omega_{\mathrm{s}}) \to W^{2,p}(\Omega_{\mathrm{f}}), \quad \Phi[u] := \mathrm{id} + \phi(\gamma_{\Gamma_{\mathrm{int}}}u). \tag{2.9}$$

Here, $\Phi[u](x)$ lifts the boundary trace $u|_{\Gamma_{\mathrm{int}}} = \gamma_{\Gamma_{\mathrm{int}}}u \in W^{1-1/p,p}(\Gamma_{\mathrm{int}})$ from the interface Γ_{int} into $\Omega_{\mathrm{f}}[u] = \Phi[u](\Omega_{\mathrm{f}})$, in particular we have $\Omega_{\mathrm{f}} = \Omega_{\mathrm{f}}[0] = \Phi[u]^{-1}(\Omega_{\mathrm{f}}[u])$.

Hypothesis 1 *Throughout the paper we assume that*

$$\det\Phi[u] > 0. \tag{2.10}$$

This can be guaranteed by considering only small u.

Remark 1 We restrict the presentation to the case $n = 2$. Several results in this paper also hold for the case $n \in \{2, 3\}$ with

$$n < p \begin{cases} < \infty, & \text{if } n = 2, \\ \leq 6, & \text{if } n = 3; \end{cases} \tag{2.11}$$

however Hypothesis 1 requires pointwise positivity of the gradient of $\Phi[u]$ which requires $W^{1,\infty}(\Omega_{\mathrm{f}})$ regularity. This is given in the case $n = 2$ with the continuous embedding $W^{2,p}(\Omega_{\mathrm{f}}) \subset C^{0,1}(\overline{\Omega_{\mathrm{f}}})$, in dimension $n = 3$ we only have $W^{2,p}(\Omega_{\mathrm{f}}) \subset C^{0,\frac{1}{2}+\varepsilon}(\overline{\Omega_{\mathrm{f}}})$ with small $\varepsilon > 0$.

We define

$$U^p := W^{2,p}(\Omega_{\mathrm{s}}). \tag{2.12}$$

From Grandmont [19] we recall the following properties stated there for a three dimensional spatial setting.

Lemma 1 (i) *The mapping* $K\colon W^{2,p}(\Omega_s) \to W^{1,p}(\Omega_f)$

$$K[u] := \mathrm{cof}(\nabla \Phi[u]) \tag{2.13}$$

is of class C^∞ *with cofactor defined in* (1.4).

(ii) *The mapping* $G\colon W^{2,p}(\Omega_s) \to W^{1,p}(\Omega_f)$

$$F[u] := \nabla \Phi[u] \tag{2.14}$$

is of class C^∞. *There exists a* $r_1 > 0$ *such that for all* $u \in B_{r_1}(U^p)$ *we have*

$$F[u] = \nabla(\mathrm{id} + \phi(\gamma_{\Gamma_{int}} u) = \mathrm{id} + \nabla(\phi(\gamma_{\Gamma_{int}} u)) \tag{2.15}$$

is an invertible matrix in $W^{1,p}(\Omega_f)$. *Moreover, we have*

(ii.a) $\Phi[u] = \mathrm{id} + \phi(\gamma_{\Gamma_{int}} u)$ *is injective on* $\overline{\Omega_f}$,
(ii.b) $\Phi[u]\colon \Omega_f \to \Phi[u](\Omega_f)$ *is a* C^1*-diffeomorphism.*

(iii) *The mapping* $A\colon B_{r_1}(U^p) \to W^{1,p}(\Omega_f)$, *with*

$$A[u] := (\nabla(\phi[u]))^{-1} \,\mathrm{cof}(\nabla(\phi[u])) \tag{2.16}$$

is of class C^∞.

Moreover, A satisfies a condition of uniform ellipticity *over* $B_{r_1}(U^p)$, *i.e. there exists a constant* $\beta > 0$ *such that*

$$A(u)(x) \geq \beta \mathrm{id}, \quad \textit{for all } u \in B_{r_1}(U^p), \quad \textit{and all } x \in \Omega_f. \tag{2.17}$$

Proof (i) The mapping $K[u]$ belongs to $W^{1,p}(\Omega_\mathrm{f})$ since $W^{1,p}(\Omega_\mathrm{f})$ is an algebra (see Lemma 17). As a composition of C^∞ mappings it is smooth. (ii) For the first statement we apply the same arguments as in (i). For the second, we use that

$$\Phi[u] = \mathrm{id} + \phi(\gamma_{\Gamma_\mathrm{int}} u) \in W^{2,p}(\Omega_\mathrm{f}), \quad \text{for all } u \in W^{2,p}(\Omega_\mathrm{s}). \tag{2.18}$$

Choosing r_1 such that

$$\|u\|_{W^{2,p}(\Omega_s)} \leq r_1 \quad \text{implies} \quad \left\|\nabla(\phi(\gamma_{\Gamma_\mathrm{int}} u))\right\|_{W^{1,p}(\Omega_\mathrm{f})} < \frac{1}{c}, \tag{2.19}$$

where c is the constant in Lemma 17, then $\mathrm{id} + \nabla(\phi(\Gamma_\mathrm{int}(b)))$ is an invertible matrix in $W^{1,p}(\Omega_s)$ and we get the result.

For the proof of (ii.a) and (ii.b) we refer to Grandmont [19, Lem. 2].

(iii) We recall the ideas from [19, Lem. 3]. Let $b \in B_p$. That $A[u] \in W^{1,p}(\Omega_\mathrm{f})$ follows from point (ii). As for the regularity of A, it is sufficient to show that the mapping:

$$W^{1,p}(\Omega_\mathrm{f}) \to W^{1,p}(\Omega_\mathrm{f}), \quad T \mapsto T^{-1} \tag{2.20}$$

is infinitely differentiable at any invertible matrix of $W^{1,p}(\Omega_\mathrm{f})$. This can be proven by standard arguments, see [10, Chap. I]. The condition of uniform ellipticity of A over $B_{r_1}(U^p)$ derives from continuity and compactness arguments ($W^{1,p}(\Omega_\mathrm{f})$ is compactly embedded in $C(\bar{\Omega}_2)$).

For the estimate for the derivative we use the boundedness of A on the bounded set $B_{r_1}(U^p)$. □

2.2 Transformation of Integrals

We recall some properties on the transformation of integrals and derivatives under a reference map.

For function $\tilde{\pi}$ on the physical domain $\Omega_\mathrm{f}[u]$ we define the transformed function on the reference domain $\Omega_\mathrm{f} = \Phi[u]^{-1}(\Omega_\mathrm{f}[u])$ (for given u) by

$$\pi(x) := \tilde{\pi}(y), \quad y = \Phi[u](x) \tag{2.21}$$

which is well-defined by Lemma 1 (ii). Moreover, we denote the determinant of the gradient of the flow map by

$$J[\cdot] := \det(D\Phi[\cdot]). \tag{2.22}$$

As a direct consequence we have $A[u] = J[u]^{-1} K[u]^\top K[u]$.

Lemma 2 *Let $u \in B_{r_1}(U^p)$ and Φ be defined by Proposition 2.9. Then, the following relations hold:*

(i) Volume elements transform as

$$\int_{\Omega_f[u]} 1 \mathrm{d}y = \int_{\Omega_f} J[u](x) \mathrm{d}x, \tag{2.23}$$

(ii) Boundary elements transform with $J_\Gamma[u] := \left\| K[u] n_f \right\|$ as

$$\int_{\Gamma_{out}[u]} 1 \mathrm{d}s_y = \int_{\Gamma_{out}} J_\Gamma[u] \mathrm{d}s_x. \tag{2.24}$$

(iii) The gradient transforms as

$$\nabla_y \tilde{f}(y) = D\Phi[u]^{-\top} \nabla_x f(x) \quad \textit{iff} \quad \nabla_y = \frac{1}{J[u]} K[u] \nabla_x. \tag{2.25}$$

(iv) For the outer normal $n_f[u]$ to $\Omega_f[u]$ and n_f to Ω_f we have

$$n_f[u] = \frac{D\Phi[u]^{-\top} n_f}{\left\| D\Phi[u]^{-\top} n_f \right\|} = \frac{K[u] n_f}{\left\| K[u] n_f \right\|}, \tag{2.26}$$

$$\int_{\Gamma_{int}[u]} \left(\nu \nabla \tilde{w} n_f[u] - \tilde{p} n_f[u] \right) \mathrm{d}s_y = \int_{\Gamma_{int}} \left(\nu (A[u] \nabla w) n_f - p K[u] n_f \right) \mathrm{d}s_x. \tag{2.27}$$

Proof We refer to [19, Equation (8)] and [27, Appendix A.1]. □

2.3 Transformation of the Navier-Stokes Equation

We consider the Navier-Stokes system in $\mathbb{R}^2$ with viscosity $\nu > 0$. We define

$$\mathcal{G}_\mu := \left\{ g \in W^{\mu,2}(\Gamma_{\mathrm{in}}) \ : \ g|_{\partial \Gamma_{\mathrm{in}}} = 0 \right\}, \quad \text{for } \mu \in \left\{ \frac{1}{2}, \frac{3}{2} \right\}. \tag{2.28}$$

Let $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)^\top$ the fluid velocity and $\tilde{p}$ the pressure in the physical domain $\Omega_{\mathrm{f}}[u] = \Phi[u](\Omega_{\mathrm{f}})$ satisfying

$$\begin{cases} -\nu \Delta_x \tilde{w}_1 + \tilde{w}^\top \nabla \tilde{w}_1 + (\nabla \tilde{p})_1 = 0 & \text{in } \Omega_{\mathrm{f}}[u], \\ -\nu \Delta_x \tilde{w}_2 + \tilde{w}^\top \nabla \tilde{w}_2 + (\nabla \tilde{p})_2 = 0 & \text{in } \Omega_{\mathrm{f}}[u], \\ \operatorname{div} \tilde{w} = 0 & \text{in } \Omega_{\mathrm{f}}[u], \\ \tilde{w} = g & \text{on } \Gamma_{\mathrm{in}}, \\ \tilde{w} = 0 & \text{on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}[u], \\ -\nu \nabla \tilde{w} n_{\mathrm{f}}[u] + \tilde{p} n_{\mathrm{f}}[u] = 0 & \text{on } \Gamma_{\mathrm{out}} \end{cases} \tag{2.29}$$

for given data $g \in \mathcal{G}_{1/2}$. Let $\Gamma_{\mathrm{bd}} := \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{out}}$. By (2.6) we have $\Phi = \mathrm{id}_x$ on Γ_{bd} such that for trial functions $\tilde{\psi}_1$ and $\tilde{\psi}_2$ vanishing on Γ_{bd} also the transformed ψ_1 and ψ_2 vanish on Γ_{bd}. The transformed strong form of the Navier-Stokes system in

Ω_{f} is given by (cf. [27, Appendix A.1]), see also Appendix A,

$$\begin{cases} -\nu \operatorname{div}(A[u]\nabla w) + (w \cdot K[u]\nabla)w + K[u]\nabla p = 0 & \text{in } \Omega_{\mathrm{f}}, \\ \operatorname{div}(K[u]^\top w) = 0 & \text{in } \Omega_{\mathrm{f}}, \\ w = g & \text{on } \Gamma_{\mathrm{in}}, \\ w = 0 & \text{on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}, \\ -\nu(A[u]\nabla w)n_{\mathrm{f}} + pK[u]n_{\mathrm{f}} = 0 & \text{on } \Gamma_{\mathrm{out}}. \end{cases} \tag{2.30}$$

3 Existence of Solutions for the Considered Systems

In this section we consider the nonlinear Navier-Stokes system, the linear elasticity system, as well as the fluid-structure interaction model.

3.1 The Navier-Stokes System

Let

$$\Omega_{\mathrm{f}}^C := \left\{\Omega \;:\; \exists \Omega^c \subset D \text{ compact with } \Omega = \Omega^c \cap \Omega_{\mathrm{f}}\right\}. \tag{3.1}$$

For $m = 0, 1, 2$ we introduce

$$W_c^{m,p}(\Omega_{\mathrm{f}}) := \{v \in W^{m,2}(\Omega_{\mathrm{f}}) \;:\; v \in W^{m,p}(\Omega^c) \text{ for all } \Omega^c \in \Omega_{\mathrm{f}}^C\}, \tag{3.2}$$

and further the spaces,

$$W^p := W_c^{2,p}(\Omega_{\mathrm{f}}) \times W_c^{1,p}(\Omega_{\mathrm{f}}), \quad W := W^{2,2}(\Omega_{\mathrm{f}}) \times W^{1,2}(\Omega_{\mathrm{f}}). \tag{3.3}$$

For given $\Omega^c \in \Omega_{\mathrm{f}}^C$ we write

$$\begin{aligned} &W_{w,\Omega^c} := W^{2,p}(\Omega^c) \cap W^{2,2}(\Omega_{\mathrm{f}}), \quad W_{p,\Omega^c} := W^{1,p}(\Omega^c) \cap W^{1,2}(\Omega_{\mathrm{f}}); \\ &W_{\Omega^c}^p := W_{w,\Omega^c} \times W_{p,\Omega^c}; \end{aligned} \tag{3.4}$$

note the different meaning of p here as upper and lower index.

Theorem 1 *One can choose $r > 0$, $r_1 > 0$, and $r_2 > 0$ such that for all $g \in B_r(\mathcal{G}_{3/2})$ and $u \in B_{r_1}(U^p)$ there exists a unique solution (w, p) in $B_{r_2}(W^p)$ of (2.30). Moreover, for any $\Omega^c \in \Omega_f^C$ the solution $(w, p) \in B_{r_2}(W_{\Omega^c}^p)$ depends continuously on g.*

Proof We follow closely ideas from [27]. We consider the fixed point equation

$$\mathcal{M}_g \colon B_{r_1}(W_{\Omega^c}^p) \to B_{r_1}(W_{\Omega^c}^p), \quad (w, p) = \mathcal{M}_g(\bar{w}, \bar{p}), \tag{3.5}$$

where $\mathcal{M}_g$ maps for given $g \in B_r(\mathcal{G}_{3/2})$ the point $(\bar{w}, \bar{p})$ to the solution (w, p) of

$$\begin{cases} -\nu \operatorname{div}(\nabla w) + \nabla p = -\nu \operatorname{div}((-A[u] + \mathrm{id})\nabla \bar{w}) & \\ \qquad - (\bar{w} \cdot (K[u] - \mathrm{id})\nabla)\bar{w} - (K[u] - \mathrm{id})\nabla \bar{p} & \text{in } \Omega_{\mathrm{f}}, \\ \operatorname{div} w = -((K[u] - \mathrm{id})\nabla)^\top \bar{w} & \text{in } \Omega_{\mathrm{f}}, \\ w = g & \text{on } \Gamma_{\mathrm{in}}, \\ w = 0 & \text{on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}, \\ -\nu \partial_n w + p n_{\mathrm{f}} = \nu(\partial_{A[u],n} - \partial_n)\bar{w} - \bar{p}(K[u] - \mathrm{id})n_{\mathrm{f}} & \text{on } \Gamma_{\mathrm{out}}. \end{cases} \tag{3.6}$$

Existence follows by Banach's fixed point theorem, see [27, (68),(85)], using smallness of the data g.

The continuous dependence on the data follows by the contraction property of $\mathcal{M}_g$ and the continuous dependence of the iterates on g. □

Hypothesis 2 *For given $r_2 > 0$ let $r > 0$ and $r_1 > 0$ be sufficiently small such that for all $g \in B_r(\mathcal{G}_{3/2})$ and $u \in B_{r_1}(U^p)$ the Navier-Stokes equation (2.30) has a unique solution (w, p) in $B_{r_2}(W^p)$.*

3.2 The Elasticity System and the Traction Force

We introduce the Piola Kirchhoff stress tensor

$$\sigma[u] := \lambda \operatorname{tr}(\varepsilon[u])\mathrm{id} + 2\mu\varepsilon[u], \quad \varepsilon[u] := \frac{1}{2}\left(\nabla u + \nabla u^\top\right), \tag{3.7}$$

with Lamé parameters λ and μ. We set

$$W^{2,p}_{\Gamma_0}(\Omega_{\mathrm{s}}) := \{\zeta \in W^{2,p}(\Omega_{\mathrm{s}}) \ : \ \zeta|_{\Gamma_0} = 0\} \tag{3.8}$$

and define the Neumann harmonic extension

$$\mathcal{N}\colon W^{1-1/p,p}(\Gamma_{\mathrm{int}}) \to W^{2,p}_{\Gamma_0}(\Omega_{\mathrm{s}}), \quad v \mapsto u =: \mathcal{N}v, \tag{3.9}$$

with u be the solution of

$$\begin{cases} -\operatorname{div}\sigma[u] = 0 & \text{in } \Omega_{\mathrm{s}}, \\ u = 0 & \text{on } \Gamma_0, \\ \sigma[u]n_{\mathrm{s}} = v & \text{on } \Gamma_{\mathrm{int}} \end{cases} \tag{3.10}$$

with outer normal n_{s} to Ω_{s}, and vector n_{s} is the unit outward normal along Γ_{int} pointing from Ω_{s} to Ω_{f}. We call u the displacement field and will also consider the system with inhomogeneous right hand side

$$\begin{cases} -\operatorname{div}\sigma[u] = f_1 & \text{in } \Omega_{\mathrm{s}}, \\ u = 0 & \text{on } \Gamma_0, \\ \sigma[u]n_{\mathrm{s}} = v & \text{on } \Gamma_{\mathrm{int}} \end{cases} \tag{3.11}$$

denoting the solution operator again by $\mathcal{N}$ as a function of f_1 and v.

Theorem 2 *(i) For $f_1 \in L^q(\Omega_s)$, $q \geq 2$, and $v \in W^{1-1/q,q}(\Gamma_{int})$ system (3.11) has a unique solution $u \in W^{2,q}(\Omega_s)$, i.e. we have*

$$\|\mathcal{N}[f_1, v]\|_{W^{2,q}(\Omega_s)} \leq c \left(\|v\|_{W^{1-1/q,q}(\Gamma_{int})} + \|f_1\|_{L^q(\Omega_s)}\right). \tag{3.12}$$

(ii) Moreover,

$$\mathcal{N} \colon L^q(\Omega_s) \times W^{1-1/q,q}(\Gamma_{int}) \to W^{2,q}(\Omega), \quad (f_1, v) \mapsto u \tag{3.13}$$

is continuously differentiable.

Proof (i) We refer to Ciarlet [11, Thm. 6.3-6 and p. 298], note that Γ_{int} has positive distance to $\Gamma_{\text{wall}} \cup \Gamma_{\text{out}} \cup \Gamma_{\text{in}}$.

(ii) Follows from the linearity of the mapping. □

The inhomogeneous system (3.11) is considered in the proof of Theorem 4.

We introduce $\kappa \in C_c^\infty(D)$ and $\Omega_\kappa := \operatorname{supp}(\kappa) \cap \Omega_{\text{f}}$ to localize $v \in W^{m,p}(\Omega_{\text{f}})$ away from the external boundary ∂D by considering $\kappa v \in W^{m,p}(\Omega_\kappa)$.

Next, we define the traction force on the interface Γ_{int}.

Definition 2 (*Traction map*) We define the traction force by

$$\begin{aligned} &t \colon W^{2,p}(\Omega_{\text{s}}) \times W_c^{2,p}(\Omega_{\text{f}}) \times W_c^{1,p}(\Omega_{\text{f}}) \to W^{1-1/p,p}(\Gamma_{\text{int}}), \\ &(u, w, p) \mapsto t[u, w, p] := \nu(A[u]\nabla w)n_{\text{f}} - pK[u]n_{\text{f}} \text{ on } \Gamma_{\text{int}} \end{aligned} \tag{3.14}$$

with $K[u]$ given by (1.4).

In particular we have for $(u, p) \in W^{2,p}(\Omega_{\text{s}}) \times W^{1,p}(\Omega_\kappa) \cap W^{1,2}(\Omega_{\text{f}})$, that

$$\begin{aligned} \|t[u, \kappa w, \kappa p]\|_{W^{1-1/p,p}(\Gamma_{\text{int}})} &\leq c\, \|A[u]\|_{L^\infty(\Omega_{\text{f}})}\, \|\kappa w\|_{W^{1,p}(\Omega_{\text{f}})} \\ &\quad + c\, \|A[u]\|_{W^{1,p}(\Omega_{\text{f}})}\, \|\kappa w\|_{L^\infty(\Omega_{\text{f}})} \\ &\leq c\, \|\kappa p\|_{W^{1,p}(\Omega_{\text{f}})}\, \|K[u]\|_{L^\infty(\Omega_{\text{f}})} \\ &\quad + c\, \|\kappa p\|_{L^\infty(\Omega_{\text{f}})}\, \|K[u]\|_{W^{1,p}(\Omega_{\text{f}})} \\ &\leq c\, \|A[u]\|_{W^{1,p}(\Omega_{\text{f}})}\, \|\kappa w\|_{W^{1,p}(\Omega_{\text{f}})} \\ &\quad + c\, \|\kappa p\|_{W^{1,p}(\Omega_{\text{f}})}\, \|K[u]\|_{W^{1,p}(\Omega_{\text{f}})}. \end{aligned} \tag{3.15}$$

Remark 2 Here, in contrast to Lasiecka et al. [27] we define the traction force not only with the term involving the pressure, i.e. $pK[u]n_{\text{s}}$. Hence, the results cited from this reference have to be adapted to the modified definition. That means, [27, Equation (89)] has to be modified but the argument works also with the definition of the traction force used here.

3.3 The Fluid-Structure Interation System

For $\nu > 0$, $g \in B_r(\mathcal{G}_{3/2})$ we can state the fluid-structure interaction model given as

$$\begin{cases} (2.30) \text{ together with } u = \mathcal{N} t[u, p] \text{ in } \Omega_s \\ \text{with } \mathcal{N} \text{ and } t \text{ defined in (3.9) and (3.14).} \end{cases} \tag{3.16}$$

For $\Omega_f^c \in \Omega_f^C$ we introduce for $2 < p < \infty$ the spaces

$$X_1^p := U^p \times W_{\Omega^c}^p, \quad X_2^p := U^p \times W, \quad X^p := X_1^p \cap X_2^p. \tag{3.17}$$

Theorem 3 *For any $\tilde{r} > 0$ there exist an $r > 0$ such that for $g \in B_r(\mathcal{G}_{3/2})$ problem (3.16) has a unique solution $(u, w, p) \in B_{\tilde{r}}(X^p)$ which depends continuously on the data.*

Proof We refer to [27, Thm. 3.2]. The proof uses a fixed-point argument based on estimates which we already cited in the proof of Theorem 1. □

4 Main Results

We state the main result of this paper on the continuous differentiability of the data-to-solution-map for the fluid-structure interation problem.

Theorem 4 (Continuous differentiability of the control-to-state mapping) *Let $\nu > 0$, $p > 2$, $g \in B_r(\mathcal{G}_{3/2})$ with $r > 0$ sufficiently small. Then, the mapping*

$$\Pi\colon B_r(\mathcal{G}_{3/2}) \to X^p, \quad g \mapsto (u[g], w[g], p[g]) \tag{4.1}$$

which maps the inflow Dirichlet condition g to the solution $(u[g], w[g], p[g])$ of the fluid-structure interaction problem (3.16) is continuously differentiable.

The proof will be presented in the following sections. In Sect. 5 we analyze the linearized equations which will be considered in Sect. 6 where we apply the implicit function theorem to prove Theorem 4.

5 The Linearized Equations

In this section we analyze the linearized Navier-Stokes equation in the domain Ω_f and derive regularity results for its solution using techniques from [27] which are applied there for the Navier-Stokes equation.

We introduce the space

$$H := \{w \in W^{1,2}(\Omega_f)^2 \ : w = 0 \text{ on } \Gamma_{in} \cup \Gamma_{int} \cup \Gamma_{wall}\} \tag{5.1}$$

and recall the property $W^{1,p}(\Omega_\mathrm{f}) \subset L^\infty(\Omega_\mathrm{f})$. Moreover, for $u \in U^p$ and $(v, w, y) \in \Pi_{i=1}^3 W^{1,2}(\Omega_\mathrm{f})^2$ we define

$$c[u](v, w, y) := (v \cdot (K[u]\nabla)w, y)_{L^2(\Omega_\mathrm{f})}. \tag{5.2}$$

Lemma 3 *Let $u \in U^p$ and $(v, w, y) \in \Pi_{i=1}^3 W^{1,2}(\Omega_f)^2$, then (5.2) can be estimated as*

$$|c[u](v, w, y)| \le c(u) \, \|w\|_{W^{1,2}(\Omega_f)^{2\times 2}} \, \|v\|_{L^4(\Omega_f)^2} \, \|y\|_{L^4(\Omega_f)^2} \, . \tag{5.3}$$

Proof We have $\partial_j v_i \in L^2(\Omega_\mathrm{f})$ and by Sobolev's embedding that w_j and z_i belong to $L^4(\Omega_\mathrm{f})$ and hence,

$$\int_{\Omega_\mathrm{f}} \left|v_j \partial_j w_i z_i\right| \mathrm{d}x \le \int_{\Omega_\mathrm{f}} \left|\partial_j w_i\right|^2 \mathrm{d}x \int_{\Omega_\mathrm{f}} \left|v_j\right|^4 \mathrm{d}x \int_{\Omega_\mathrm{f}} |z_i|^4 \,\mathrm{d}x \tag{5.4}$$

and we conclude. □

In the following we write $c(v, w, y)$ for $c[0](v, w, y)$ with 0 denoting the zero map.

5.1 Linearized State Equation: Coefficients Equal to One

Let

$$f \in L^2(\Omega_\mathrm{f}), \quad f_2 \in L^2(\Omega_\mathrm{f}), \quad f_3 \in W^{-1/2,2}(\Gamma_\mathrm{out}), \quad \delta g \in \mathcal{G}_{1/2}. \tag{5.5}$$

Let $(\hat{w}, \hat{p}) \in W^p$ solution of the Navier-Stokes equation (2.30) be given. We consider the linearized Navier-Stokes system around this point with inhomogeneous right hand side given by

$$\begin{cases} -\nu\Delta z_w + (\hat{w}\cdot\nabla)z_w + (z_w\cdot\nabla)\hat{w} + \nabla z_p = f & \text{in } \Omega_\mathrm{f}, \\ -\operatorname{div} z_w = f_2 & \text{in } \Omega_\mathrm{f}, \\ z_w = \delta g & \text{on } \Gamma_\mathrm{in}, \\ z_w = 0 & \text{on } \Gamma_\mathrm{wall} \cup \Gamma_\mathrm{int}, \\ -\nu\partial_n z_w + z_p n_\mathrm{f} = f_3 & \text{on } \Gamma_\mathrm{out}. \end{cases} \tag{5.6}$$

Let $L\colon W^{1,2}(\Omega_\mathrm{f}) \to \mathbb{R}$ with

$$L(v) := \int_{\Omega_\mathrm{f}} (f + f_2)\cdot v \mathrm{d}x + \int_{\Gamma_\mathrm{out}} f_3 \cdot v \mathrm{d}y, \quad v \in W^{1,2}(\Omega_\mathrm{f}), \tag{5.7}$$

and for $\hat{w} \in H$ we define $b_{\hat{w}}\colon H \times H \to \mathbb{R}$, by

$$b_{\hat{w}}(w, v) := \nu \int_{\Omega_\mathrm{f}} \nabla w \cdot \nabla v \mathrm{d}x + c(\hat{w}, w, v) + c(w, \hat{w}, v). \tag{5.8}$$

To address the linearized terms a smallness condition on the velocity $\hat{w}$ is made, see also de los Reyes and Yousept [14].

Lemma 4 *For $r_2 > 0$ sufficiently small we have*

$$b_{\hat{w}}(v, v) \geq c \, \|v\|^2_{W^{1,2}(\Omega_f)} \text{ for all } v \in W^{1,2}(\Omega_f); \tag{5.9}$$

moreover, the bilinear form $b_{\hat{w}}(\cdot, \cdot)$ is continuous.

Proof By Lemma 3 there exists an $\varepsilon = \varepsilon(r_2) > 0$ such that

$$\begin{aligned} &\nu \int_{\Omega_\mathrm{f}} \nabla v \cdot \nabla v \mathrm{d}x + c(v, \hat{w}, v) + c(\hat{w}, v, v) \\ &\geq \nu \, \|\nabla v\|^2_{L^2(\Omega_\mathrm{f})} - \varepsilon \, \|v\|^2_{L^4(\Omega_\mathrm{f})} - \varepsilon \, \|\nabla v\|_{L^2(\Omega_\mathrm{f})} \, \|v\|_{L^4(\Omega_\mathrm{f})} \\ &\geq \frac{\nu}{2} \, \|v\|_{W^{1,2}(\Omega_\mathrm{f})} \,. \end{aligned} \tag{5.10}$$

The continuity follows again from Lemma 3 and Sobolev's embedding. □

The weak formulation for (5.6) is given as follows: Find $z_w \in H$ solution of

$$\begin{cases} b_{\hat{w}}(z_w, v) - \displaystyle\int_{\Omega_\mathrm{f}} z_p \nabla v \mathrm{d}x = L(v) \quad \text{for all } v \in H, \\ \operatorname{div} z_w = f_2 \text{ in } \Omega_\mathrm{f}, \qquad z_w = \delta g \text{ on } \Gamma_\mathrm{in}. \end{cases} \tag{5.11}$$

Theorem 5 *For $\left\| \hat{w} \right\|_{W^{1,2}(\Omega_f)}$ sufficiently small system (5.6) (resp. (5.11)) has a unique solution $(z_w, z_p) \in W^{1,2}(\Omega_f) \times L^2(\Omega_f)$ with*

$$\begin{aligned} \|z_w\|_{W^{1,2}(\Omega_f)} + \left\|z_p\right\|_{L^2(\Omega_f)} &\leq c \, \|f\|_{W^{-1,2}(\Omega_f)} + c \, \|f_2\|_{L^2(\Omega_f)} \\ &\quad + c \, \|f_3\|_{W^{-1/2,2}(\Gamma_{out})} + c \, \|\delta g\|_{W^{1/2,2}(\Gamma_{in})} \,. \end{aligned} \tag{5.12}$$

Note, that this lower regularity existence and the estimate follows by classical Lax-Milgram arguments, see [27, Step 1] and also [29, Theorem 11.1.2], together with Lemma 4.

Hypothesis 3 *Let $r_2 > 0$ be sufficiently small such that for $\hat{w} \in B_{r_2}(W^{2,p}_c(\Omega_f))$ equation (5.6) has a unique solution $(z_w, z_p) \in W^{1,2}(\Omega_f) \times L^2(\Omega_f)$.*

Note, that here we consider a higher norm for $\hat{w}$ than necessary in comparison to Theorem 5. This is due to the fact that later we will also estimate higher norms of (z_w, z_p).

5.2 The Linearized State Equation

Let $(\hat{w}, \hat{p})$ be given solution of the Navier-Stokes equation (2.30). We consider the in this point linearized equation with inhomogeneous right hand sides satisfying (5.5)

$$\begin{cases} -\nu \operatorname{div}(A[u]\nabla z_w) + (\hat{w} \cdot K[u]\nabla) z_w & \\ \qquad +(z_w \cdot K[u]\nabla)\hat{w} + K[u]\nabla z_p = f & \text{in } \Omega_{\mathrm{f}}, \\ \operatorname{div}(K[u]^\top z_w) = f_2 & \text{in } \Omega_{\mathrm{f}}, \\ z_w = \delta g & \text{on } \Gamma_{\mathrm{in}}, \\ z_w = 0 & \text{on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}, \\ -\nu \partial_{A[u],n} z_w + z_p K[u] n_{\mathrm{f}} = f_3 & \text{on } \Gamma_{\mathrm{out}}. \end{cases} \tag{5.13}$$

To analyze this equation we follow [27] and take ideas from Grandmont [19] into account. We recall a technical result which follows by a Taylor argument.

Lemma 5 *For r_u and r_w positive and $\bar{u} \in B_{r_u}(U^p)$ and $\bar{w} \in B_{r_w}(X^p)$ and some $s \geq 1$ the following estimates hold:*

$$(i) \quad \|A[\bar{u}] - \mathrm{id}\|_{L^\infty(\Omega_f)} \leq c r_u^s, \qquad (ii) \quad \|A(\bar{u}) - \mathrm{id}\|_{W^{1,p}(\Omega_f)} \leq c r_u^s,$$
$$(iii) \quad \|K(\bar{u})\|_{L^\infty(\Omega_f)} \leq c(1 + r_w^s), \qquad (iv) \quad \|K(\bar{u}) - \mathrm{id}\|_{L^\infty(\Omega_f)} \leq c r_u^s,$$

as well as

$$(v) \quad \|\operatorname{div}((A(\bar{u}) - \mathrm{id})\nabla)\bar{w}\|_{L^q(\Omega_f)} \leq c r_u^s \|\bar{w}\|_{W^{2,q}(\Omega_f)}$$

for some $s \geq 1$ and $q \geq 2$.

Proof For the proof we refer to [27, Lem. 4.1]. □

Remark 3 We mention that in the cited reference the power s arise purely from the higher order terms.

We follow ideas in [27, Prop. 4.2, Lem. 4.3, Lem 4.4, and Lem. 4.5] developed there for the Navier-Stokes equation to analyze the linearized equation in (5.13). We start with a preliminary consideration which is later used in (5.29).

Lemma 6 *For $v \in W^{1,p}(\Omega_f)$ and $s \in L^2(\Gamma_{out})$ we have*

$$\|vs\|_{W^{-1/2,2}(\Gamma_{out})} \leq \|v\|_{L^\infty(\Omega_f)} \|s\|_{W^{-1/2,2}(\Gamma_{out})}. \tag{5.14}$$

Proof Since $W^{1,p} \subset C(\overline{\Omega})$ continuous the product of the trace of v on Γ_{out} with s is in $L^2(\Gamma_{\mathrm{out}}) \subset W^{-1/2,2}(\Gamma_{\mathrm{out}})$ and we have

$$\begin{aligned} \|vs\|_{W^{-1/2,2}(\Gamma_{\mathrm{out}})} &= \sup_{\|\eta\|_{W^{1/2,2}(\Gamma_{\mathrm{out}})}=1} (|v||s|, |\eta|)_{L^2(\Gamma_{\mathrm{out}})} \\ &\leq \|v\|_{L^\infty(\Gamma_{\mathrm{out}})} \|s\|_{W^{-1/2,2}(\Gamma_{\mathrm{out}})}. \end{aligned} \tag{5.15}$$

Again using that v is continuous up to the boundary we conclude. □

For given $u \in B_{r_1}(U^p)$ we define a map

$$T = T_u : W^p \to W^p, \quad (\bar{z}_w, \bar{z}_p) \mapsto (z_w, z_p) \tag{5.16}$$

by rewriting (5.13) as

$$\begin{cases} -\nu \operatorname{div}(\nabla z_w) + (\hat{w} \cdot \nabla) z_w + (z_w \cdot \nabla)\hat{w} + \nabla z_p & \\ \qquad = -\nu \operatorname{div}((-A[u] + \mathrm{id})\bar{z}_w) & \\ \qquad\quad - (\hat{w} \cdot (K[u] - \mathrm{id})\nabla)\bar{z}_w & \\ \qquad\quad - (\bar{z}_w \cdot (K[u] - \mathrm{id})\nabla)\hat{w} & \\ \qquad\quad - (K[u] - \mathrm{id})\nabla \bar{z}_p + f & \text{in } \Omega_\mathrm{f}, \\ \operatorname{div} z_w = -((K[u] - \mathrm{id})\nabla)^\top \bar{z}_w + f_2 & \text{in } \Omega_\mathrm{f}, \\ z_w = \delta g & \text{on } \Gamma_\mathrm{in}, \\ z_w = 0 & \text{on } \Gamma_\mathrm{wall} \cup \Gamma_\mathrm{int}, \\ -\nu \partial_n z_w + z_p n_x = \nu \partial_{A[u]-\mathrm{id},n} \bar{z}_w - \bar{z}_p (K[u] - \mathrm{id}) n_\mathrm{f} + f_3 & \text{on } \Gamma_\mathrm{out}; \end{cases} \tag{5.17}$$

this will allow to define a sequence $((z_{w,n}, z_{p,n}))_{n \in \mathbb{N}}$ with $(z_{w,0}, z_{p,0})$ equal to some $(\bar{z}_w, \bar{z}_p) \in W^p$ which we further analyze in Sect. 5.5 to obtain existence of a solution for (5.13).

Lemma 7 *Let r_u and r_w positive. For $u \in B_{r_u}(U^p)$ and $v \in B_{r_w}(W^{1,p}(\Omega_f))$ we have*

$$\|(v \cdot K(u)\nabla) z_w\|_{L^p(\Omega_f)} \le c(1 + r_u^s) r_w \|z_w\|_{W^{1,p}(\Omega_f)} . \tag{5.18}$$

Proof We have

$$\begin{aligned} \|(v \cdot K(u)\nabla z_w)\|_{L^p(\Omega_\mathrm{f})} &\le \|K(u)\|_{L^\infty(\Omega_\mathrm{f})} \|v\|_{L^\infty(\Omega_\mathrm{f})} \|\nabla z_w\|_{L^p(\Omega_\mathrm{f})} \\ &\le (1 + r_u^s) r_w \|z_w\|_{W^{1,p}(\Omega_\mathrm{f})} \end{aligned} \tag{5.19}$$

and conclude with Lemma 5. □

5.3 Lower Regularity

We have the following a priori $W^{1,2} \times L^2$-estimate without having to take into account the special situation of mixed boundary conditions.

Lemma 8 *Let Hypothesis 3 be satisfied. For the solution (z_w, z_p) of (5.17) we have the estimate*

$$\begin{aligned} \|z_w\|_{W^{1,2}(\Omega_f)} + \left\|z_p\right\|_{L^2(\Omega_f)} \le{}& c\,\|f\|_{L^2(\Omega_f)} + c\,\|f_2\|_{L^2(\Omega_f)} + c\,\|g\|_{W^{1/2,2}(\Gamma_{in})} \\ &+ c\,\|f_3\|_{W^{-1/2,2}(\Omega_f} + c r_1^s \|\bar{z}_w\|_{W^{2,2}(\Omega_f)} \\ &+ c r_1^s \left\|\bar{z}_p\right\|_{W^{1,2}(\Omega_f)} \end{aligned} \tag{5.20}$$

with constant c depending on r_2 and $s \geq 1$.

Proof By Theorem 5 we have existence of a unique solution and the following lower regularity result for the solution (z_w, z_p) given by

$$\begin{aligned}\|z_w\|_{W^{1,2}(\Omega_f)} + \left\|z_p\right\|_{L^2(\Omega_f)} \leq c\Big(&\|f\|_{L^2(\Omega_f)} + \|f_2\|_{W^{1,2}(\Omega_f)} + \|g\|_{W^{1/2,2}(\Gamma_{in})} \\ &+ \|f_3\|_{W^{-1/2,2}(\Omega_f)} + \left\|F(u,\bar{z}_w,\bar{z}_p)\right\|_{L^2(\Omega_f)} + \|F_2(\bar{z}_w,u)\|_{L^2(\Omega_f)} \\ &+ \left\|\partial_{A[u]-\mathrm{id},n}\bar{z}_w\right\|_{W^{-1/2,2}(\Gamma_{out})} + \left\|\bar{z}_p(K[u]-\mathrm{id})\right\|_{W^{-1/2,2}(\Gamma_{out})}\Big),\end{aligned} \tag{5.21}$$

where

$$\begin{aligned}F(u,\bar{z}_w,\bar{z}_p) &:= -\nu\,\mathrm{div}((-A[u]+\mathrm{id})\nabla z_w) \\ &\quad - (\bar{z}_w\cdot(K[u]-\mathrm{id})\nabla)\hat{w} - (\hat{w}\cdot(K[u]-\mathrm{id})\nabla)\bar{z}_w - (K[u]-\mathrm{id})\nabla\bar{z}_p, \\ F_2(u,\bar{z}_w) &:= \mathrm{div}_{(\mathrm{id}-K[u]^\top)}\,\bar{z}_w = ((\mathrm{id}-K[u])\nabla)^\top\cdot\bar{z}_w.\end{aligned} \tag{5.22}$$

We estimate each term separately. Differently to [27] we have to estimate the linearized convection term

$$\begin{aligned}\left\|(\bar{z}_w\cdot(-K[u]-\mathrm{id})\nabla)\hat{w}\right\|_{L^2(\Omega_f)} &\leq c\,\|-K[u]-\mathrm{id}\|_{L^\infty(\Omega_f)}\,\|\bar{z}_w\|_{L^\infty(\Omega_f)}\,\left\|\nabla\hat{w}\right\|_{L^2(\Omega_f)} \\ &\leq cr_1^s r_2\,\|\bar{z}_w\|_{L^\infty(\Omega_f)}\end{aligned} \tag{5.23}$$

and accordingly,

$$\left\|(\hat{w}\cdot(-K[u]-\mathrm{id})\nabla)\bar{z}_w\right\|_{L^2(\Omega_f)} \leq cr_1^s\,\|\bar{z}_w\|_{W^{1,2}(\Omega_f)}\,\left\|\hat{w}\right\|_{L^\infty(\Omega_f)}. \tag{5.24}$$

The other terms are treated in the same way, for simplicity we recall here the main steps. For some $s \geq 1$ using Lemma 5 4. we have for the diffusion term

$$\|\nu\,\mathrm{div}((A[u]-\mathrm{id})\nabla z_w)\|_{L^2(\Omega_f)} \leq cr_1^s\,\|z_w\|_{W^{2,2}(\Omega_f)}. \tag{5.25}$$

Again by [27, Lem. 4.1] we obtain for the term involving the pressure

$$\left\|(K[u]-\mathrm{id})\nabla\bar{z}_p\right\|_{L^2(\Omega_f)} \leq \|K[u]-\mathrm{id}\|_{L^\infty(\Omega_f)}\,\left\|\nabla\bar{z}_p\right\|_{L^2(\Omega_f)} \leq cr_1^s\,\left\|\bar{z}_p\right\|_{W^{1,2}(\Omega_f)}. \tag{5.26}$$

By $\mathrm{div}_{\mathrm{id}-K[u]^\top}\,w = (\mathrm{id}-K[u]^\top)\cdot\nabla w$, cf. Appendix C, we have

$$\left\|\mathrm{div}_{\mathrm{id}-K[u]^\top}\,\bar{z}_w\right\|_{L^2(\Omega_f)} \leq \left\|\mathrm{id}-K[u]^\top\right\|_{L^\infty(\Omega_f)}\,\|\bar{z}_w\|_{W^{1,2}(\Omega_f)} \leq cr_1^s\,\|\bar{z}_w\|_{W^{1,2}(\Omega_f)} \tag{5.27}$$

and for the boundary terms

$$\begin{aligned}\left\|\partial_{A-\mathrm{id},n}\bar{z}_w\right\|_{W^{1/2,2}(\Gamma_{\mathrm{out}})} &\leq \|A[u]-\mathrm{id}\|_{L^\infty(\Omega_{\mathrm{f}})}\,\|\partial_n\bar{z}_w\|_{W^{1/2,2}(\Gamma_{\mathrm{out}})}\\ &+\|A[u]-\mathrm{id}\|_{W^{1,p}(\Omega_{\mathrm{f}})}\,\|\bar{z}_w\|_{W^{2,2}(\Omega_{\mathrm{f}})} \leq cr_1^s\,\|\bar{z}_w\|_{W^{2,2}(\Omega_{\mathrm{f}})}\end{aligned} \tag{5.28}$$

using for the latter estimate the Neumann trace estimate; note, that we estimate the trace in a higher norm than necessary here. Moreover, with estimate (5.14)

$$\begin{aligned}\left\|\bar{z}_p(K[u]-\mathrm{id})\right\|_{W^{-1/2,2}(\Gamma_{\mathrm{out}})} &\leq \|K[u]-\mathrm{id}\|_{L^\infty(\Omega_{\mathrm{f}})}\,\left\|\bar{z}_p\right\|_{W^{-1/2,2}(\Gamma_{\mathrm{out}})}\\ &\leq cr_1^s\,\left\|\bar{z}_p\right\|_{W^{-1/2,2}(\Gamma_{\mathrm{out}})}.\end{aligned} \tag{5.29}$$

Consequently, with (5.25)–(5.28) we obtain the result. □

5.4 Higher Regularity

For $F\in L^2(\Omega_{\mathrm{f}})$, $F_2\in W^{1,2}(\Omega_{\mathrm{f}})$, $\delta g\in\mathcal{G}_{3,2}$, and $F_3\in W^{1/2,2}(\Gamma_{\mathrm{out}})$ we consider

$$\begin{cases} -\Delta v+\nabla q = F, & \text{in } \Omega_{\mathrm{f}},\\ \operatorname{div} v = F_2, & \text{in } \Omega_{\mathrm{f}},\\ v = g, & \text{in } \Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{wall}},\\ -\partial_n v+qn_{\mathrm{f}} = F_3, & \text{in } \Gamma_{\mathrm{out}}.\end{cases} \tag{5.30}$$

Let $\kappa\in C_c^\infty(D)$ localize v away from the external boundary $\partial\Omega_{\mathrm{f}}$ and set $\Omega_\kappa := \operatorname{supp}(\kappa)$. Here, we rely on estimates provided in Lasiecka et al. [27, equation (44)] given by

$$\begin{aligned}\|(1-\kappa)v\|_{W^{2,2}(\Omega_{\mathrm{f}})}+\|(1-\kappa)q\|_{W^{1,2}(\Omega_{\mathrm{f}})} &\leq c\,\|(1-\kappa)F\|_{L^2(\Omega_{\mathrm{f}})}\\ +\|(1-\kappa)F_2\|_{W^{1,2}(\Omega_{\mathrm{f}})}+\|g\|_{W^{3/2,2}(\Gamma_{\mathrm{in}})}&+\|F_3\|_{W^{1/2,2}(\Gamma_{\mathrm{out}})}.\end{aligned} \tag{5.31}$$

Remark 4 The authors in [27] refer here to the notion of ellipticity for systems introduced in Agmon, Douglis, and Nirenberg [1], see also Maz'ya and Rossmann [29, Sec. 1.1.3], and Bouchev and Gunzburger [7, Appendix D]. Following Beneš and Kučera [8, Appendix] the regularity is established at first locally for boundary points on the Dirichlet boundary part, the Neumann boundary part, and then for the two corners where the different types of boundary conditions meet (the less standard result), see [27, Appendix A.3]. With cut-off functions the solutions are localized and the estimates are derived using [7, Thm. D.1]. Using the compactness of D global regularity is achieved.

We define

$$\mathcal{S}^{p'} := W_c^{0,p'}(\Omega_{\mathrm{f}})\cap L^2(\Omega_{\mathrm{f}})\times W_c^{1,p'}(\Omega_{\mathrm{f}})\times\mathcal{G}_{3/2}\times W^{1/2,2}(\Gamma_{\mathrm{out}}),\quad p'\geq 2, \tag{5.32}$$

and assume

$$(f, f_2, g, f_3) \in \mathcal{S}^{p'}. \tag{5.33}$$

We introduce

$$\begin{aligned} z_{w,a} &:= \kappa z_w, \quad z_{w,b} := (1-\kappa) z_w, \\ z_{p,a} &:= \kappa z_p, \quad z_{p,b} := (1-\kappa) z_p \end{aligned} \tag{5.34}$$

implying $z_w = z_{w,a} + z_{w,b}$ and $z_p = z_{p,a} + z_{p,b}$ and write the solution (z_w, z_p) of (5.17) as the sum of $(z_{w,a}, z_{p,a})$ and $(z_{w,b}, z_{p,b})$ being solutions of the following two systems localized in the interior and close to the boundary:

$$\begin{cases} -\nu \operatorname{div}(\nabla z_{w,a}) + \nabla z_{p,a} = -\nu \operatorname{div}((-A[\bar{u}] + \mathrm{id})\nabla \bar{z}_{w,a}) + \nu[\operatorname{div}((-A[u] + \mathrm{id})\nabla), \kappa]\bar{z}_w \\ \qquad -\kappa(\bar{z}_w \cdot K[\bar{u}]\nabla)\hat{w} - \kappa(\hat{w} \cdot K[\bar{u}]\nabla)\bar{z}_w \\ \qquad - \kappa(K[\bar{u}] - \mathrm{id})\nabla \bar{z}_p - [\kappa, \nu\nabla^2] z_w + [\kappa, \nabla] z_p + \kappa f, \\ \operatorname{div} z_{w,a} = \operatorname{div}_{\mathrm{id}-K[\bar{u}]} \bar{z}_{w,a} + [\kappa, \operatorname{div}] z_w, \\ \qquad + [\operatorname{div}_{\mathrm{id}-K^\top[\bar{u}]}, \kappa] z_w + \kappa f_2, \\ z_{w,a} = 0 \text{ on } \partial\Omega_{\mathrm{f}}, \end{cases} \tag{5.35}$$

(note that $\partial\Omega_{\mathrm{f}} = \Gamma_{\mathrm{int}} \cup \Gamma_{\mathrm{out}}$) and

$$\begin{cases} -\nu \operatorname{div}(\nabla z_{w,b}) + \nabla z_{p,b} = -(1-\kappa)\Big(\nu \operatorname{div}((-A[\bar{u}] + \mathrm{id})\nabla \bar{z}_w) \\ \qquad -(\bar{z}_w \cdot K[\bar{u}]\nabla)\hat{w} - (\hat{w} \cdot K[\bar{u}]\nabla)\bar{z}_w \\ \qquad - (K[\bar{u}] - \mathrm{id})\nabla \bar{z}_p\Big) + [1-\kappa, \nu\nabla^2] z_w \\ \qquad - [1-\kappa, \nabla] z_p + f, \\ \operatorname{div} z_{w,b} = (1-\kappa)(\operatorname{div}_{\mathrm{id}-K^\top[\bar{u}]} \bar{z}_w) + [1-\kappa, \operatorname{div}] z_w + f_2, \\ z_{w,b} = (1-\kappa)\delta g = g \text{ on } \Gamma_{\mathrm{in}}, \\ z_{w,b} = 0 \text{ on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}, \\ -\nu\partial_n z_{w,b} + z_{p,b} = -\partial_{A[\bar{u}]-\mathrm{id},n} \bar{z}_w + \bar{z}_p(K[\bar{u}] - \mathrm{id}) + f_3 \text{ on } \Gamma_{\mathrm{out}}. \end{cases} \tag{5.36}$$

Lemma 9 *Let Hypothesis 3 be satisfied. For every $\varepsilon > 0$ we have for $p' = 2$, and $s \geq 1$ that*

$$\begin{aligned} \|z_{w,b}\|_{W^{2,2}(\Omega_f)} + \|z_{p,b}\|_{W^{1,2}(\Omega_f)} &\leq c\|(f, f_2, g, f_3)\|_{\mathcal{S}^2} + c r_1^s \|\bar{z}_{w,b}\|_{W^{2,2}(\Omega_f)} \\ &+ c r_1 \|\bar{z}_p\|_{W^{1,2}(\Omega_f)} + \varepsilon \|z_w\|_{W^{2,2}(\Omega_f)} + c_\varepsilon \|z_w\|_{L^2(\Omega_f)} \\ &+ c \|z_p\|_{L^2(\Omega_f)} + c r_2 \|\bar{z}_w\|_{W^{2,2}(\Omega_f)}. \end{aligned} \tag{5.37}$$

Proof In the following we omit the first term in the estimate on the right hand side, since its derivation follows easily. By (5.31) we have for the solution of equation (5.36)

$$\|(1-\kappa)z_w\|_{W^{2,2}(\Omega_\mathrm{f})} + \left\|(1-\kappa)z_p\right\|_{W^{1,2}(\Omega_\mathrm{f})} \le c\left\|F^b\right\|_{L^2(\Omega_\mathrm{f})} + \left\|F_2^b\right\|_{W^{1,2}(\Omega_\mathrm{f})} + \left\|F_3^1\right\|_{W^{1/2,2}(\Gamma_\mathrm{out})} + \left\|F_3^2\right\|_{W^{1/2,2}(\Gamma_\mathrm{out})}, \tag{5.38}$$

where

$$\begin{aligned} F^b :=&\ (1-\kappa)\Big(\nu\,\mathrm{div}((-A[\bar u]+\mathrm{id})\nabla\bar z_w)+\hat w((-K[\bar u])\nabla)\bar z_w \\ &+\bar z_w((-K[\bar u])\nabla)\hat w - (K[\bar u]-\mathrm{id})\nabla\bar z_p\Big) \\ &+[1-\kappa,\nu\nabla^2]z_w - [1-\kappa,\nabla]z_p =: \sum_{i=1}^{6} I_i, \end{aligned} \tag{5.39}$$

$$\begin{aligned} F_2^b &:= (1-\kappa)(\mathrm{div}_{\mathrm{id}-K^\top[\bar u]}\,\bar z_w) + [1-\kappa,\mathrm{div}]w =: I_7 + I_8,\\ F_3^1 &:= \bar z_p(K[\bar u]-\mathrm{id})\cdot n_x,\\ F_3^2 &:= \partial_{A[\bar u]-\mathrm{id},n}\bar w. \end{aligned}$$

Note, that in the following we consider (besides for boundary terms) general L^q, $q\ge 2$, and not only L^2 estimates to include also estimates needed for the subsequential lemma in which instead of $(z_w, z_p)\in W$ the pair $(z_{w,a}, z_{p,a})\in W^p$ will be considered implying that below higher regularity has to be assumed for terms involving $\|z_w\|_{W^{2,p}(\Omega_\mathrm{f})}$.

We have with $2<q<\infty$ for the linearized convection term using Sobolev embedding $W^{2,2}(\Omega_\mathrm{f})\subset W^{1,q}(\Omega_\mathrm{f})$

$$\begin{aligned} \|I_2+I_3\|_{L^q(\Omega_\mathrm{f})} &\le \|K[u]\|_{W^{1,q}(\Omega_\mathrm{f})}\left\|\hat w\right\|_{L^\infty(\Omega_\mathrm{f})}\|\bar z_w\|_{W^{2,2}(\Omega_\mathrm{f})} \\ &\quad+\|K[u]\|_{W^{1,q}(\Omega_\mathrm{f})}\|\bar z_w\|_{L^\infty(\Omega_\mathrm{f})}\left\|\hat w\right\|_{W^{2,2}(\Omega_\mathrm{f})} \\ &\le cr_2(\|\bar z_w\|_{W^{2,2}(\Omega_2)}+\|\bar z_w\|_{L^\infty(\Omega_2)}). \end{aligned} \tag{5.40}$$

Moreover, following [27], with Hölder's inequality with suitable $q_1\ge 1$ and $q_2\ge 1$ satisfying $1/q = 1/q_1 + 1/q_2$ and Lemma 7

$$\begin{aligned} \|I_1\|_{L^q(\Omega_\mathrm{f})} &\le c\,\|A[\bar u]-\mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})}\|\bar z_w\|_{W^{2,q}(\Omega_\mathrm{f})} \\ &\quad+c\,\|A[\bar u]-\mathrm{id}\|_{W^{1,q_1}(\Omega_\mathrm{f})}\|\bar z_w\|_{W^{1,q_2}(\Omega_\mathrm{f})} \\ &\le cr_1^s\,\|\bar z_w\|_{W^{2,q}(\Omega_\mathrm{f})}; \end{aligned} \tag{5.41}$$

for the later estimate we used that for $q_2=(qq_1)/(q_1-q)$ the inclusion $W^{2,q}(\Omega_\mathrm{f})\subset W^{1,q_1}(\Omega_\mathrm{f})$ is continuous. Using that the appearing commutator loses one order of differentiability we get

$$\|I_5 + I_6\|_{L^q(\Omega_\mathrm{f})} \leq c(\|z_w\|_{W^{1,q}(\Omega_\mathrm{f})} + \|z_p\|_{L^q(\Omega_\mathrm{f})}) \tag{5.42}$$

which can be further estimated in the case $q = 2$ by (5.20). Further, we have for $q = 2$ that

$$\|I_4\|_{L^2(\Omega_\mathrm{f})} \leq \|K[\bar{u}] - \mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})} \|\nabla z_p\|_{L^2(\Omega_\mathrm{f})} \leq cr_1^s \|\nabla z_p\|_{L^2(\Omega_\mathrm{f})}. \tag{5.43}$$

We have with Hölder's inequality for $\tilde{q} \geq 2$ that

$$\begin{aligned} \left\|F_3^1\right\|_{W^{1/2,2}(\Gamma_\mathrm{out})} &\leq c \|z_p(K[u] - \mathrm{id})n_\mathrm{f}\|_{W^{1/2,2}(\Gamma_\mathrm{out})} \\ &\leq c \|z_p\|_{W^{1,2}(\Omega_\mathrm{f})} \|K[u] - \mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})} \\ &\quad + \|z_p\|_{L^{\frac{2\tilde{q}}{\tilde{q}-2}}(\Omega_\mathrm{f})} \|K[u] - \mathrm{id}\|_{W^{1,\tilde{q}}(\Omega_\mathrm{f})}. \end{aligned} \tag{5.44}$$

The composition for $q \geq 2$

$$\nabla \circ \mathcal{D} \circ \gamma_{\Gamma_\mathrm{int}} \colon W^{2,q}(\Omega_\mathrm{s}) \to W^{1,q}(\Omega_\mathrm{f}), \quad u \mapsto \nabla \Phi[u] \tag{5.45}$$

defines a continuous inclusion. Using the representation of $\nabla\Phi[u]$ and $A[u]$, see [27, (129) and (138)], and that $K[u] = \nabla\Phi[u]A[u]$, we have, cf. [27, Proof of Lem. 4.1],

$$\|K[u] - \mathrm{id}\|_{W^{1,q}(\Omega_\mathrm{f})} \leq c \|u\|^s_{W^{2,q}(\Omega_\mathrm{f})} \tag{5.46}$$

and we can conclude

$$\begin{aligned} \left\|F_3^1\right\|_{W^{1/2,2}(\Gamma_\mathrm{out})} &\leq \|z_p\|_{W^{1,2}(\Omega_\mathrm{f})} \left(\|K[u] - \mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})} + \|K[u] - \mathrm{id}\|_{W^{1,p}(\Omega_\mathrm{f})}\right) \\ &\leq cr_1^s \|z_p\|_{W^{1,2}(\Omega_\mathrm{f})}. \end{aligned} \tag{5.47}$$

Next, we have as in (5.28) the estimate

$$\left\|F_3^2\right\|_{W^{1/2,2}(\Gamma_\mathrm{out})} \leq cr_1^s \|\bar{z}_{w,b}\|_{W^{2,2}(\Omega_\mathrm{f})}. \tag{5.48}$$

For $q \geq 2$ we have using Appendix C and (5.46) that

$$\begin{aligned} \|I_7\|_{W^{1,q}(\Omega_\mathrm{f})} &\leq c \|\bar{u}\|_{W^{2,q}(\Omega_\mathrm{s})} \|\bar{z}_{w,b}\|_{W^{2,q}(\Omega_\mathrm{f})} \\ &\quad + \left\|\mathrm{id} - K^\top(\bar{u})\right\|_{L^\infty(\Omega)} \|\bar{z}_{w,b}\|_{W^{2,q}(\Omega_\mathrm{f})} \\ &\leq cr_1 \|\bar{z}_{w,b}\|_{W^{2,q}(\Omega_\mathrm{f})}. \end{aligned} \tag{5.49}$$

Moreover, we have

$$\|I_8\|_{W^{1,2}(\Omega_\mathrm{f})} \leq \|[1 - \kappa, \mathrm{div}]w\|_{W^{1,2}(\Omega_\mathrm{f})} \leq c \|w\|_{W^{1,2}(\Omega_\mathrm{f})} \tag{5.50}$$

using that $[(1-\kappa), \operatorname{div}]w = \nabla(1-\kappa)\cdot w$. The norm on the right hand side can be further estimated using again (5.20).

Now, setting $q = 2$ we conclude. □

5.4.1 Interior Estimates

For references on L^p–estimates for the Stokes equation we refer to Amrouche and Rejaiba [4], Hieber and Saal [23], Solonnikov [33]. We recall an interior estimate for the Stokes equation, note that in this case there arises no difficulty from mixed boundary conditions. We set

$$\begin{aligned} F^a &:= -\nu \operatorname{div}((-A[\bar u]+\mathrm{id})\nabla \bar z_{w,a}) - \nu[\operatorname{div}((-A[\bar u]+\mathrm{id})\nabla), \kappa]\bar z_w \\ &\quad - \kappa\Big((\bar z_w\cdot K[\bar u]\nabla)\hat w + (\hat w\cdot K[\bar u]\nabla)\bar z_w + (K[\bar u]-\mathrm{id})\nabla \bar z_p\Big) \\ &\quad + [\kappa, \nu\nabla_x^2]z_w + [\kappa,\nabla]z_p + \kappa f, \\ F_2^a &:= \operatorname{div}_{\mathrm{id}-K[\bar u]^\top}\bar z_{w,a} + [\kappa,\operatorname{div}]z_w + [\operatorname{div}_{\mathrm{id}-K[\bar u]},\kappa]z_w + \kappa f_2. \end{aligned} \tag{5.51}$$

Lemma 10 *Choosing $p = p' > 2$ we have*

$$\|\kappa z_w\|_{W^{2,p}(\Omega_f)} + \left\|\kappa z_p\right\|_{W^{1,p}(\Omega_f)} \le c\,\|F_a\|_{L^p(\Omega_f)} + \|D_a\|_{W^{1,p}(\Omega_f)}\,. \tag{5.52}$$

Proof For a proof see [28, Thm 11.3.4]; we use the fact that $\kappa w \in W_0^{1,2}(\Omega_\kappa)$. □

Lemma 11 *Let Hypothesis 3 be satisfied. Then, we have for solution (z_w, z_p) of (5.35) for $\varepsilon > 0$*

$$\begin{aligned} &\left\|z_{w,a}\right\|_{W^{2,p}(\Omega_f)} + \left\|z_{p,a}\right\|_{W^{1,p}(\Omega_f)} \le c\,\|(\kappa f,\kappa f_2)\|_{L^p(\Omega_\kappa)\times W^{1,p}(\Omega_\kappa)} \\ &\quad + c\,\|(f, f_2, g, f_3)\|_{\mathcal{S}^2} + cr_1\left\|\bar z_{w,a}\right\|_{W^{2,p}(\Omega_f)} + cr_2\,\|\bar z_w\|_{W^{2,2}(\Omega_f)} \\ &\quad + cr_1^s\left(\left\|\bar z_{p,a}\right\|_{W^{1,p}(\Omega_f)} + \left\|\bar z_p\right\|_{W^{1,2}(\Omega_f)}\right) + \varepsilon\left(\|z_w\|_{W^{2,2}(\Omega_f)} + \left\|z_p\right\|_{W^{1,2}(\Omega_f)}\right) \\ &\quad + c_\varepsilon\left(\|z_w\|_{W^{1,2}(\Omega_f)} + \left\|z_p\right\|_{L^2(\Omega_f)}\right). \end{aligned} \tag{5.53}$$

Proof (i) We start with (5.52). Recalling ideas from [27], to estimate $\left\|\kappa(K[u]-\mathrm{id})\nabla z_p\right\|_{L^p(\Omega_\mathrm{f})}$ we cannot use an estimate as (5.43) in a higher L^p–norm, since we have no $W^{1,p}(\Omega_\mathrm{f})$ regularity of the pressure up to the boundary. Hence, we use the property of the communtator that

$$\kappa(K[u]-\mathrm{id})\nabla z_p = (K[u]-\mathrm{id})\nabla z_{p,a} + (K[u]-\mathrm{id})[\nabla,\kappa]z_p \tag{5.54}$$

and that the commutator looses one derivative

$$[\nabla,\kappa]\,z_p = \nabla(\kappa z_p) - \kappa\nabla z_p = (\nabla\kappa)z_p + \kappa\nabla z_p - \kappa\nabla z_p = z_p\nabla\kappa \tag{5.55}$$

implying that

$$
\begin{aligned}
\left\|\kappa(K[u]-\mathrm{id})\nabla z_p\right\|_{L^p(\Omega_\mathrm{f})} &\le c\,\|K(u)-\mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})}\left\|z_{p,a}\right\|_{W^{1,p}(\Omega_\mathrm{f})}\\
&\quad + c(\kappa)\,\|K[u]-\mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})}\left\|z_p\right\|_{L^p(\Omega_\mathrm{f})}\\
&\le c\,\|K[u]-\mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})}\left\|z_{p,a}\right\|_{W^{1,p}(\Omega_\mathrm{f})}\\
&\quad + c\,\|K[u]-\mathrm{id}\|_{L^\infty(\Omega_\mathrm{f})}\left\|z_p\right\|_{W^{1,2}(\Omega_\mathrm{f})},
\end{aligned}
\tag{5.56}
$$

using the continuous embedding $W^{1,2}(\Omega_\mathrm{f}) \subset L^p(\Omega_\mathrm{f})$. This term can then be estimated as in (5.43).

(ii) Using estimates from the proof of Lemma 9, estimates for the commutator, and the consideration from (i) we obtain

$$
\begin{aligned}
\left\|F^a\right\|_{L^p(\Omega_\mathrm{f})} &\le c r_1^s \left\|\bar{z}_{w,a}\right\|_{W^{2,p}(\Omega_\mathrm{f})} + c r_2 \|\bar{z}_w\|_{W^{2,2}(\Omega_\mathrm{f})}\\
&\quad + c r_1^s \left(\left\|\bar{z}_{p,a}\right\|_{W^{1,p}(\Omega_\mathrm{f})} + \left\|\bar{z}_p\right\|_{W^{1,2}(\Omega_\mathrm{f})}\right)\\
&\quad + c\,\|z_w\|_{W^{1,p}(\Omega_\mathrm{f})} + c\left\|z_p\right\|_{L^p(\Omega_\mathrm{f})} + c\,\|\kappa f\|_{L^p(\Omega_\kappa)},\\
\left\|F_2^a\right\|_{W^{1,p}(\Omega_\mathrm{f})} &\le c r_1 \left\|\bar{z}_{w,a}\right\|_{W^{2,p}(\Omega_\mathrm{f})} + c\,\|\bar{z}_w\|_{L^p(\Omega_\mathrm{f})} + c\,\|\kappa f_2\|_{W^{1,p}(\Omega_\kappa)}.
\end{aligned}
\tag{5.57}
$$

For the terms $\|z_w\|_{W^{1,p}(\Omega_\mathrm{f})} + c\left\|z_p\right\|_{L^p(\Omega_\mathrm{f})}$ we cannot apply (5.20) directly for $p > 2$. Using Ehrling's lemma we have for $\varepsilon > 0$

$$
\|z_w\|_{W^{1,p}(\Omega_\mathrm{f})} \le \varepsilon\,\|z_w\|_{W^{2,2}(\Omega_\mathrm{f})} + c_\varepsilon\,\|z_w\|_{W^{1,2}(\Omega_\mathrm{f})}
\tag{5.58}
$$

which yields

$$
\begin{aligned}
\|z_w\|_{W^{1,p}(\Omega_\mathrm{f})} + c\left\|z_p\right\|_{L^p(\Omega_\mathrm{f})} &\le \varepsilon(\|z_w\|_{W^{2,2}(\Omega_\mathrm{f})} + \left\|z_p\right\|_{W^{1,2}(\Omega_\mathrm{f})})\\
&\quad + c_\varepsilon(\|z_w\|_{W^{1,2}(\Omega_\mathrm{f})} + \|p\|_{L^2(\Omega_\mathrm{f})})
\end{aligned}
\tag{5.59}
$$

which allows to sublimate the higher order terms and gives, with ε arbitrarily small, the result. □

5.5 Limit Behaviour

The map (5.16) defines an iteration scheme generating a sequence of iterates

$$
(z_{w,n}, z_{p,n}) = (z_{w,a}, z_{p,a}) + (z_{w,b}, z_{p,b}) \in W^p
\tag{5.60}
$$

We will verify that it converges for $n \to \infty$ towards the unique solution $(z_w, z_p) \in W^p$ of (5.17). For a $\eta \in]0, 1[$ we will estimate

$$\left\| z_{w,n+1} - z_{w,n} \right\|_{W_{w,\Omega^c}} + \left\| z_{p,n+1} - z_{p,n} \right\|_{W_{p,\Omega^c}} \leq \eta \Big(\left\| z_{w,n} - z_{w,n-1} \right\|_{W_{w,\Omega^c}} + \left\| z_{p,n} - z_{p,n-1} \right\|_{W_{p,\Omega^c}} \Big) \tag{5.61}$$

for $\Omega_{\mathrm{f}}^c \in \Omega_{\mathrm{f}}^C$. Then, there exists $(\bar{z}_w, \bar{z}_p) \in W^p_{\Omega^c}$ and sequence $((z_{w,n}, z_{p,n}))_{n \in \mathbb{N}} \subset W^p_{\Omega^c}$ such that

$$\begin{aligned} z_{w,n} &\to \bar{z}_w \text{ in } W_{w,\Omega^c} \quad \text{as } n \to +\infty, \\ z_{p,n} &\to \bar{z}_p \text{ in } W_{p,\Omega^c} \quad \text{as } n \to +\infty. \end{aligned} \tag{5.62}$$

with $(\bar{z}_w, \bar{z}_p)$ the unique solution of (5.17). This idea is taken from Grandmont [19].

Next, we show the strategy in detail.

5.5.1 The Linearized State Equation: Contraction Property

Let $Y_1 := (z_w^1, z_p^1)$, $Y_2 = (z_w^2, z_p^2)$, and $\bar{Y}_i := (\bar{z}_w^i, \bar{z}_p^i)$, $i = 1, 2$, with

$$Y_1 = T\bar{Y}_1, \quad Y_2 = T\bar{Y}_2. \tag{5.63}$$

Our aim is to show that

$$\| Y_1 - Y_2 \|_{W^p_{\Omega^c}} = \left\| T(\bar{Y}_1 - \bar{Y}_2) \right\|_{W^p_{\Omega^c}} \leq \eta \left\| \bar{Y}_1 - \bar{Y}_2 \right\|_{W^p_{\Omega^c}} \tag{5.64}$$

where $\eta < 1$ uniform in Ω_{f}^c. From the definition of the map T we write

$$\begin{cases} -\nu \operatorname{div}(\nabla z_w^i) + \nabla z_p^i = -\nu \operatorname{div}((-A[u] + \mathrm{id})\nabla \bar{z}_w^i) \\ \qquad -(\hat{w} \cdot K[u]\nabla)\bar{z}_w^i - (\bar{z}_w^i \cdot K[u]\nabla)\hat{w} \\ \qquad - (K[u] - \mathrm{id})\nabla \bar{z}_p^i + f =: D(\bar{Y}_i) \quad \text{in } \Omega_{\mathrm{f}} \\ \operatorname{div} z_w^i = \operatorname{div}_{\mathrm{id}-K^\top(u)} \bar{z}_w^i + f_2 =: B(\bar{Y}_i) \quad \text{in } \Omega_{\mathrm{f}} \\ z_w^i = g \quad \text{on } \Gamma_{\mathrm{in}} \\ -\nu \partial_n z_w^i + z_p^i n = -\partial_{A[u]-\mathrm{id},n} \bar{z}_w^i + \bar{z}_p^i (K[u] - \mathrm{id})n + f_3 \quad \text{on } \Gamma_{\mathrm{out}} \end{cases} \tag{5.65}$$

for $i = 1, 2$. Denoting $\bar{Y} := \bar{Y}_1 - \bar{Y}_2$ we obtain the equation for

$$Y := Y_1 - Y_2 =: (Z_w, Z_p) \tag{5.66}$$

in terms of $\bar{Y}_i \in B_r(W^p)$:

$$\begin{cases} -\nu \operatorname{div}(\nabla Z_w) + \nabla Z_p = D(\bar{Y}_1) - D(\bar{Y}_2) & \text{in } \Omega_{\mathrm{f}}, \\ \operatorname{div} Z_w = B(\bar{Y}_1) - B(\bar{Y}_2) = \operatorname{div}_{\mathrm{id}-K^\top(u)} \bar{Z}_w & \text{in } \Omega_{\mathrm{f}}, \\ Z_w = 0 & \text{on } \Gamma_{\mathrm{int}} \cup \Gamma_{\mathrm{in}}, \\ -\nu \partial_n Z_w + Z_p n = -\partial_{A(u)-\mathrm{id},n} \bar{Z}_w + \bar{Z}_p (K[u] - \mathrm{id}) n_{\mathrm{f}} & \text{on } \Gamma_{\mathrm{out}}. \end{cases} \tag{5.67}$$

Lemma 12 *Let Hypothesis 3 be satisfied. For the solution of (5.67) we have*

$$\|Z_w\|_{W^{2,2}(\Omega_f)} + \|Z_p\|_{W^{1,2}(\Omega_f)} \le c(r_1^s + r_1 + r_2)(\|\bar{Z}_w\|_{W^{2,2}(\Omega_f)} + \|\bar{Z}_p\|_{W^{1,2}(\Omega_f)}) \tag{5.68}$$

where $\bar{Z}_w := \bar{z}_w^1 - \bar{z}_w^2$ *and* $\bar{Z}_p := \bar{z}_p^1 - \bar{z}_p^2$.

Proof We proceed similarly as in Lemma 9 using also Theorem 8; we estimate

$$\begin{aligned} \|D(Y_1) - D(Y_2)\|_{L^2(\Omega_{\mathrm{f}})} &= \left\|\nu \operatorname{div}((-A[u] + \mathrm{id})\nabla \bar{Z}_w)\right\|_{L^2(\Omega_{\mathrm{f}})} \\ &\quad + \left\|(\hat{w} \cdot (-K[u]\nabla)\bar{Z}_w) - (\bar{Z}_w \cdot (K[u]\nabla)\hat{w})\right\|_{L^2(\Omega_{\mathrm{f}})} \\ &\quad + \left\|(K[u] - \mathrm{id})\nabla \bar{Z}_p\right\|_{L^2(\Omega_{\mathrm{f}})} \\ &\le c(r_1 + r_1^s + r_2)(\left\|\bar{Z}_w\right\|_{W^{2,2}(\Omega_{\mathrm{f}})} + \left\|\bar{Z}_p\right\|_{W^{2,2}(\Omega_{\mathrm{f}})}). \end{aligned} \tag{5.69}$$

For the term $B(Y_i)$ we have by (5.46) that with $1/p_1 + 1/p_2 = 1/2$, $p_1 > 2$,

$$\begin{aligned} &\left\|B(\bar{Y}_1) - B(\bar{Y}_2)\right\|_{W^{1,2}(\Omega_{\mathrm{f}})} = \left\|\operatorname{div}_{\mathrm{id}-K[u]^\top} \bar{Z}_w\right\|_{W^{1,2}(\Omega_{\mathrm{f}})} \\ &\quad \le \|\mathrm{id} - K[u]\|_{W^{1,p_1}(\Omega_{\mathrm{f}})} \left\|\bar{Z}_w\right\|_{W^{1,p_2}(\Omega_{\mathrm{f}})} + \|\mathrm{id} - K[u]\|_{L^\infty(\Omega_{\mathrm{f}})} \left\|\bar{Z}_w\right\|_{W^{2,2}(\Omega_{\mathrm{f}})} \\ &\quad \le \|\mathrm{id} - K[u]\|_{W^{1,p_1}(\Omega_{\mathrm{f}})} \left\|\bar{Z}_w\right\|_{W^{2,2}(\Omega_{\mathrm{f}})} + \|\mathrm{id} - K[u]\|_{L^\infty(\Omega_{\mathrm{f}})} \left\|\bar{Z}_w\right\|_{W^{2,2}(\Omega_{\mathrm{f}})} \\ &\quad \le c r_1^s \left\|\bar{Z}_w\right\|_{W^{2,2}(\Omega_{\mathrm{f}})} \end{aligned} \tag{5.70}$$

and on the boundary Γ_{out}

$$-\nu \partial_n Z_w + Z_p n_{\mathrm{f}} = -\partial_{A[u]-\mathrm{id},n} Z_w + \bar{Z}_p K[u] n_{\mathrm{f}} - \bar{Z}_p n_{\mathrm{f}}. \tag{5.71}$$

From Lemma 9 it follows that

$$\begin{aligned} \|Z_w\|_{W^{2,2}(\Omega_{\mathrm{f}})} + \left\|Z_p\right\|_{W^{1,2}(\Omega_{\mathrm{f}})} &\le c \left\|D(\bar{Y}_1) - D(\bar{Y}_2)\right\|_{L^2(\Omega_{\mathrm{f}})} \\ &+ c \left\|B(\bar{Y}_1) - B(\bar{Y}_2)\right\|_{W^{1,2}(\Omega_{\mathrm{f}})} + c \left\|-\partial_{A[u]-\mathrm{id},n} \bar{Z}_w\right\|_{W^{1/2,2}(\Gamma_{\mathrm{out}})} \\ &+ c \left\|\bar{Z}_p K[\bar{u}] n_{\mathrm{f}}\right\|_{W^{1/2,2}(\Gamma_{\mathrm{out}})} + c \left\|\bar{Z}_p n_{\mathrm{f}}\right\|_{W^{1/2,2}(\Gamma_{\mathrm{out}})}. \end{aligned} \tag{5.72}$$

Using estimates (5.28) and (5.29) for the boundary terms we conclude. □

Lemma 13 *Let Hypothesis 3 be satisfied and additionally, $r_1 > 0$ and $r_2 > 0$ be sufficiently small. Then, the map T defined by (5.16) satisfies for some $0 < \eta < 1$*

$$\|T(Y_1 - Y_2)\|_{W^p_{\Omega^c}} < \eta \left\|\bar{Y}_1 - \bar{Y}_2\right\|_{W^p_{\Omega^c}} \tag{5.73}$$

for $\Omega^c \in \Omega_f^C$.

Proof As a consequence of the previous lemma it remains to prove the contraction property with respect to higer p-integrability on compact subsets.

We recall the function κ. We remark that the commutator has for sufficiently smooth v the property that

$$[\kappa, D_x]v = -D_x(\kappa v) + \kappa D_x v, \quad [\kappa, D_x^2]v = -D_x^2(\kappa v) + \kappa D_x^2 v. \tag{5.74}$$

Hence, we have

$$\begin{aligned} -\nu \operatorname{div}(\nabla Z_{w,a}) + \nabla Z_{p,a} &= \kappa(D(\bar{Y}_1) - D(\bar{Y}_2)) + [\kappa, \nu D_x^2]Z_w \\ &\quad + [\kappa, \nabla]Z_p && \text{in } \Omega_\mathrm{f}, \\ \operatorname{div} Z_{w,a} &= \kappa(B(\bar{Y}_1) - B(\bar{Y}_2)) + [\kappa, \operatorname{div}]Z_w && \text{in } \Omega_\mathrm{f} \\ Z_{w,a} &= 0 && \text{on } \Gamma_\mathrm{int} \cup \Gamma_\mathrm{in} \cup \Gamma_\mathrm{out}. \end{aligned} \tag{5.75}$$

with $Z_{w,a} := z^1_{w,a} - z^2_{w,a}$ and $Z_{p,a} := z^1_{p,a} - z^2_{p,a}$. Since the commutators loose one order of derivative we can derive higher Lebesgue integrability, i.e. for $(w, p) \in W^{2,2}(\Omega_\mathrm{f}) \times W^{1,2}(\Omega_\mathrm{f})$

$$\begin{aligned} \left\|[\kappa, D_x^2]w\right\|_{L^p(\Omega_\mathrm{f})} &\le C\|w\|_{W^{1,p}(\Omega_\mathrm{f})} \le c\|w\|_{W^{2,2}(\Omega_\mathrm{f})}, \\ \|[\kappa, \operatorname{div}]w\|_{W^{1,p}(\Omega_\mathrm{f})} &\le C\|w\|_{W^{1,p}(\Omega_\mathrm{f})} \le c\|w\|_{W^{2,2}(\Omega_\mathrm{f})}, \\ \|[\kappa, \nabla]p\|_{L^p(\Omega_\mathrm{f})} &\le C\|p\|_{W^{1,2}(\Omega_\mathrm{f})}. \end{aligned} \tag{5.76}$$

Similar as in the proof of Lemma 11 we estimate $\left\|\kappa(B(\bar{Y}_1) - B(\bar{Y}_2))\right\|_{W^{1,p}(\Omega_\mathrm{f})}$ and $\left\|\kappa(D(\bar{Y}_1) - D(\bar{Y}_2))\right\|_{W^{1,p}(\Omega_\mathrm{f})}$. Here we use the same trick as in that proof to obtain higher p-integrability, namely we switch around the order of κ and the differential operators in the term with coefficient $A[u]$ as well as in the divergence term and introduce a commutator as correction term.

Applying further the estimate of Lemma 12 to the terms (5.76) we obtain finally

$$\|Z_{w,a}\|_{W^{2,p}(\Omega_\mathrm{f})} + \|Z_{p,a}\|_{W^{1,p}(\Omega_\mathrm{f})} \le c(r_1 + r_1^s + r_2)\left\|\bar{Y}\right\|_{W^p_{\Omega_\kappa}}. \tag{5.77}$$

Thus, for $r_1 > 0$ and $r_2 > 0$ sufficiently small we obtain the result. □

Theorem 6 *Let Hypothesis 3 be satisifed and additionally $r_1 > 0$ and $r_2 > 0$ sufficiently small. For data satisfying the regularity assumption in (5.33), $\hat{w} \in$*

$B_{r_2}(W_c^{2,p}(\Omega_f))$, and $u \in B_{r_1}(U^p)$ the linearized equation (5.13) has a unique solution $(z_w, z_p) \in W^p$. Moreover, the solution is bounded by the data, we have

$$\begin{aligned} \left\|(z_w, z_p)\right\|_{W^p_{\Omega^c}} \le c\,\|f\|_{L^p(\Omega^c)\cap L^2(\Omega_f)} + c\,\|f_2\|_{W^{1,p}(\Omega^c)\cap W^{1,2}(\Omega_f)} \\ + c\,\|\delta g\|_{W^{3/2,2}(\Gamma_{int})} + c\,\|f_3\|_{W^{1/2,2}(\Gamma_{int})} \end{aligned} \tag{5.78}$$

for subsets $\Omega^c \in \Omega_f^C$.

Proof The existence follows by the procedure described at the beginning of Sect. 5.5 and the contraction property given in Lemma 13. The estimate follows from the boundedness of the operator T shown in Lemma 9 and 11 and sublimating the with powers of r_i weighted terms by the left hand side. □

Hypothesis 4 *Let $r_1 > 0$ and $r_2 > 0$ be sufficiently small, such that for $\hat{w} \in W_c^{2,p}(\Omega_f)$ and $u \in B_{r_1}(U^p)$ the linearized equation (5.13) has a unique solution in W^p satsfying estimate (5.78).*

6 Differentiability

In this section we show the main result, the differentiability of the mapping which maps the infow profile g to the deformation-velocity-pressure triple (u, w, p) of the fluid-structure interaction system. We follow in parts ideas from [34] where linear elasticity is coupled with the Stokes equation with Dirichlet boundary conditions in a smooth domain. In a first step we consider the differentiability of the data-to-solution map g to (w, p) for the Navier-Stokes system.

We introduce two systems, which will appear to be the linearized systems with respect to inflow data g and with respect to perturbation u, namely

$$\begin{cases} -\nu\,\mathrm{div}(A[u]\nabla\delta w_g) + (\delta w_g \cdot K[u]\nabla)\hat{w} + (\hat{w}\cdot K[u]\nabla)\delta w_g & \\ \qquad +K[u]\nabla\delta p_g = 0 & \text{in } \Omega_{\mathrm{f}}, \\ \mathrm{div}(K[u]^\top \delta w_g) = 0 & \text{in } \Omega_{\mathrm{f}}, \\ \delta w_g = \delta g & \text{on } \Gamma_{\mathrm{in}}, \\ \delta w_g = 0 & \text{on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}, \\ -\nu\partial_{A[u],n}\delta w_g + \delta p_g K[u] n_{\mathrm{f}} = 0 & \text{on } \Gamma_{\mathrm{out}} \end{cases} \tag{6.1}$$

and

$$\begin{cases} -\nu \operatorname{div}(A[\hat{u}]\nabla\delta w_u)+(\delta w_u \cdot K[\hat{u}]\nabla)\hat{w} + (\hat{w}\cdot K[\hat{u}]\nabla)\delta w_u - K[\hat{u}]\nabla\delta p & \\ \quad = -(\hat{w}\cdot K'[\hat{u}]\delta u\nabla)\hat{w} + \gamma\nu \operatorname{div}(A'[\hat{u}]\delta u\nabla\hat{w}) & \\ \quad\quad - K'[\hat{u}]\delta u\nabla\hat{p} & \text{in } \Omega_{\mathrm{f}}, \\ \operatorname{div}(K[\hat{u}]^\top\delta w_u) = \operatorname{div}_{K^\top(\hat{u})\delta u}\hat{w} & \text{in } \Omega_{\mathrm{f}}, \\ w = 0 & \text{on } \Gamma_{\mathrm{in}}, \\ w = 0 & \text{on } \Gamma_{\mathrm{wall}}\cup\Gamma_{\mathrm{int}}, \\ -\nu\partial_{A[\hat{u}]}\delta w_u + \delta p K[\hat{u}]n_{\mathrm{f}} = \nu\partial_{A'[\hat{u}]\delta u,n} w - pK'[\hat{u}]\delta u n_{\mathrm{f}} & \text{on } \Gamma_{\mathrm{out}}. \end{cases} \tag{6.2}$$

For given $(\hat{u}, \hat{g}) \in B_{r_1}(U^p) \times B_r(\mathcal{G}_{3/2})$ we write the Navier-Stokes equation (2.30) as

$$e\colon X^p \times \mathcal{G}_{3/2} \to \mathcal{S}^{p'}, \quad e(u, w, p, g) = 0, \tag{6.3}$$

with

$$e(u, w, p, g) := \begin{pmatrix} -\nu \operatorname{div}(A[u]\nabla w) + (w\cdot K[u]\nabla)w + K[u]\nabla p \\ \operatorname{div}(K[u]^\top w) \\ w|_{\Gamma_{\mathrm{in}}} - g \\ -\nu(A[u]\nabla w)n_{\mathrm{f}} + pK[u]n_{\mathrm{f}} \end{pmatrix}. \tag{6.4}$$

Lemma 14 *The function e defined in (6.3)–(6.4) is continuously differentiable.*

Proof The statement follows by the regularity of the appearing functions and the smoothness of A and K, see Lemma 1. □

To apply the implicit function theorem we show that the derivative of e with respect to (w, p) defines an isomorphism in a solution $(\hat{u}, \hat{w}, \hat{p}, \hat{g})$ of (6.3).

Let $\hat{u} \in W^{2,p}(\Omega_{\mathrm{s}})$ and $(\hat{w}, \hat{p}) \in W^p$ the corresponding solution of the Navier-Stokes equation (2.30). Moreover, let $(F, F_2, g, F_3) \in \mathcal{S}^{p'}$. Recalling Hypothesis 2 and 3, we consider the solution $(z_w, z_p) \in W^p$ of

$$D_{(w,p)}e(\hat{u}, \hat{w}, \hat{p}, \hat{g})(z_w, z_p) = (F, F^2, g, F^3)^\top. \tag{6.5}$$

By Theorem 6 the solution is well-defined and we have

$$\begin{aligned} &\left\|(z_w, z_p)\right\|_{W^{2,p}(\Omega^c)\cap W^{2,2}(\Omega_{\mathrm{f}})\times W^{1,p}(\Omega^c)\cap W^{1,2}(\Omega_{\mathrm{f}})} \le c\,\|F\|_{L^p(\Omega^c)\cap L^2(\Omega_{\mathrm{f}})} \\ &\quad + c\,\|F_2\|_{W^{1,p}(\Omega^c)\cap W^{1,2}(\Omega_{\mathrm{f}})} + c\,\|g\|_{W^{3/2,2}(\Gamma_{\mathrm{int}})} + c\,\|F_3\|_{W^{1/2,2}(\Gamma_{\mathrm{int}})} \end{aligned} \tag{6.6}$$

for $\Omega^c \Subset \Omega_{\mathrm{f}}^C$.

Lemma 15 *Let Hypothesis 2 and 4 hold.*

(ia) The mapping

$$\mathcal{M}\colon B_{r_1}(U^p) \times B_{r_2}(\mathcal{G}_{3/2}) \to W^p_{\Omega^c}, \quad (u, g) \mapsto (w[u, g], p[u, g]) \tag{6.7}$$

is continuously differentiable with $(w[u,g], p[u,g])$ *the solution of* (2.30) *for given* (u,g).

(ib) Let $u \in B_{r_1}(U^p)$ *be fixed. The derivative* $(\delta w_g, \delta p_g)$ *of*

$$B_r(\mathcal{G}_{3/2}) \to W^p_{\Omega^c}, \quad g \mapsto (w[u,g], p[u,g]) \tag{6.8}$$

is given by (6.1).

(ic) Let $g \in B_r(\mathcal{G}_{3/2})$ *be fixed. The derivative* $(\delta w_u, \delta p_u)$ *of*

$$B_{r_1}(U^p) \to W^p_{\Omega^c}, \quad u \mapsto (w[u,g], p[u,g]) \tag{6.9}$$

is given by (6.2).

(ii) The mapping

$$\begin{aligned} \mathcal{F}\colon B_{r_1}(U^p) \times B_r(\mathcal{G}_{3/2}) &\to W^{1-1/p,p}(\Gamma_{int}), \\ (u,g) &\mapsto \nu(A[u]\nabla w[u,g])n_f - p[u,g]K[u]n_f \end{aligned} \tag{6.10}$$

is continuously differentiable.

Proof (ia) To show continuous differentiability of $(w[\cdot], p[\cdot])$, we employ the implicit function theorem. We note that

$$D_{(w,p)}e(u,w,p,g)\colon W^p_{\Omega^c} \to \mathcal{S}^{p'}, \tag{6.11}$$

corresponds to the transformed Stokes operator on the left given by

$$\begin{pmatrix} -\nu\,\mathrm{div}(A[u]\nabla\delta w) + (\delta w \cdot K[u]\nabla)\hat{w} + (\hat{w} \cdot K[u]\nabla)\delta w + K[u]\nabla\delta p \\ \mathrm{div}(K[u]^\top \delta w) \\ \delta w|_{\Gamma_{\mathrm{in}}} \\ -\nu(A[u]\nabla\delta w)n_{\mathrm{f}} + \delta p K[u]n_{\mathrm{f}} \end{pmatrix}. \tag{6.12}$$

We observe that $D_{(w,p)}e(u,w,p,g)\colon W^p_{\Omega^c} \to \mathcal{S}^{p'}$ is an isomorphism by Theorem 6 and estimate given there, cf. (6.6).

(ib) With $D_g e(u,w,p,g)\delta g$ given by

$$\big(0,\, 0,\, \delta g,\, 0\big)^\top \tag{6.13}$$

the derivative $(\delta w_g, \delta p_g)$ with respect to g is given as the solution of

$$D_{(w,p)}e(u,w,p,g)(\delta w_g, \delta p_g) = -D_g e(u,w,p,g)\delta g \tag{6.14}$$

or equivalently by (6.1). A solution exists by Theorem 6 and is bounded by the data, the result follows.

(ic) Analogously, the partial derivative $D_u e(u, w, p, g)\delta u$ is given by

$$\begin{pmatrix} -\nu \operatorname{div}(A'[u]\delta u \nabla w) + (w \cdot K'[u]\delta u \nabla) w + K'[u]\delta u \nabla p \\ \operatorname{div}(K'[u]^\top \delta u w) \\ 0 \\ -\nu (A'[u]\delta u \nabla w) n_{\mathrm{f}} + p K'[u]\delta u n_{\mathrm{f}} \end{pmatrix} \tag{6.15}$$

and (6.2) can be written as

$$D_{(w,p)} e(u, w, p, g)(\delta w_u, \delta p_u) = -D_u e(u, w, p, g) \tag{6.16}$$

or equivalently by (6.2). Since

$$\begin{aligned} -\nu \operatorname{div}(A'[\hat{u}]\delta u \nabla \hat{w}) + (\hat{w} \cdot K'[\hat{u}]\delta u \nabla)\hat{w} + K'[\hat{u}]\delta u \nabla \hat{p} \in W_c^{0,p}(\Omega_{\mathrm{f}}) \cap L^2(\Omega_{\mathrm{f}}), \\ -\nu (A'[u]\delta u \nabla w) n_{\mathrm{f}} + p K'[u]\delta n_{\mathrm{f}} \in W^{1-1/p,2}(\Gamma_{\mathrm{int}}) \end{aligned} \tag{6.17}$$

for $p > 2$, the right hand side in (6.16) has the suitable regularity and we conclude again with Theorem 6.

(ii) Follows directly from (ia). Note, that here we use that in the interior we have higher p-integrability and that Γ_{int} is bounded away from Γ_{ext}. □

Lemma 16 *Let Hypothesis 2 and 4 be satisfied. For $g \in B_r(\mathcal{G}_{3/2})$ and $u \in B_{r_1}(U^p)$ and $\mathcal{F}$ given in (6.10) we have for any $\varepsilon > 0$*

$$\left\| \frac{\mathrm{d}}{\mathrm{d}u} \mathcal{F}(u, g) \right\|_{L_F} \leq \varepsilon, \tag{6.18}$$

with $L_F := L(W^{2,p}(\Omega_s), W^{1-1/p,p}(\Gamma_{int}))$ provided that r and r_1 are sufficiently small.

Proof We write

$$\mathcal{F}(u, g) = t(u, \kappa \mathcal{M}(u, g)). \tag{6.19}$$

By Lemma 15 and applying the chain rule, we get for any direction $\delta u \in W^{2,p}(\Omega_{\mathrm{s}})$ that

$$\frac{\mathrm{d}}{\mathrm{d}u} \mathcal{F}(u, g)\delta u = \frac{\mathrm{d}}{\mathrm{d}u} t(u, \kappa \mathcal{M}(u, g))\delta u. \tag{6.20}$$

By Theorem 1 we can choose for $\delta > 0$ the radii $r > 0$ and $r_1 > 0$ sufficiently small such that $(w, p) \in B_\delta(W^p)$. Using the smoothness of the outer normal on the interface taking into account that Γ_{int} is bounded away from Γ_{ext} and recalling that $p > n$ we have, see (3.15) (omitting the dependencies on u and g) that

$$\begin{aligned}\left\|\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{F}(u,g)\delta u\right\|_{W^{1-1/p,p}(\Gamma_{\mathrm{int}})} &\leq \left\|A[u]\nabla z_{w,a}\right\|_{W^{1,p}(\Omega_{\mathrm{f}})} + \left\|A'[u]\delta u\nabla(\kappa w)\right\|_{W^{1,p}(\Omega_{\mathrm{f}})} \\ &\quad + \left\|z_{p,a}K[u]\right\|_{W^{1,p}(\Omega_{\mathrm{f}})} + \left\|\kappa p K'[u]\delta u\right\|_{W^{1,p}(\Omega_{\mathrm{f}})} \\ &\leq \|A[u]\|_{W^{1,p}(\Omega_{\mathrm{f}})}\left\|z_{w,a}\right\|_{W^{2,p}(\Omega_{\mathrm{f}})} \\ &\quad + \left\|A'[u]\delta u\right\|_{W^{1,p}(\Omega_{\mathrm{f}})}\|\kappa w\|_{W^{2,p}(\Omega_{\mathrm{f}})} \\ &\quad + \left\|z_{p,a}\right\|_{W^{1,p}(\Omega_{\mathrm{f}})}\|K[u]\|_{W^{1,p}(\Omega_{\mathrm{f}})} \\ &\quad + \|\kappa p\|_{W^{1,p}(\Omega_{\mathrm{f}})}\left\|K'[u]\delta u\right\|_{W^{1,p}(\Omega_{\mathrm{f}})}. \end{aligned} \tag{6.21}$$

Note, that in (6.21) we use higher p-integrability of $(\kappa w, \kappa p)$ whose supports are bounded away from the boundary. Now, using the estimate in Theorem 6 applied to (6.2), we have for any $\gamma > 0$ and data sufficiently small that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{F}(u,g)\delta u\right\|_{W^{1-1/p,p}(\Gamma_{\mathrm{int}})} \leq c\gamma\,\|\delta u\|_{W^{2,p}(\Omega_{\mathrm{s}})} \tag{6.22}$$

which shows the assertion. □

Now we can prove Theorem 4.

Remark 5 It is not necessary to assume Hypothesis 2, 3, or 4 explicitly, since by Theorem 3 the existence of a solution of the FSI problem is in a ball of radius $\tilde{r}$ which we can choose arbitrary small if $r > 0$ is chosen accordingly sufficiently small. This guarantees implicitly the existence of a solution to the Navier-Stokes equation making Hypothesis 2 redundant as well as a sufficiently small bound on the velocity of the Navier-Stokes equation and the solution of the elasticity system making Hypothesis 4 and so also Hypothesis 3 redundant.

Proof of Theorem 4 We follow ideas from [34]. Existence of a solution of the fluid-structure interaction problem follows by Theorem 3. We have $(u, w, p) = \Pi(g)$ and

$$u = \mathcal{N}\left[f_1, \mathcal{F}(\phi(\gamma_{\Gamma_{\mathrm{int}}}u), g)\right] \tag{6.23}$$

with $\mathcal{N}$ defined in Theorem 2 and $\mathcal{F}$ given in (6.10). Since (w, p) depends continuously differentiable on (u, g) by Lemma 15, it is sufficient to show differentiability of the mapping $g \mapsto u$ given by the above fix point relation (6.23). We apply the implicit function theorem. We note that

$$D_2\mathcal{N}\left[f_1, \mathcal{F}(\phi(\gamma_{\Gamma_{\mathrm{int}}}u), g)\right]\colon W^{1-1/p,p}(\Omega_{\mathrm{f}}) \to W^{2,p}(\Omega_{\mathrm{f}}) \tag{6.24}$$

corresponds to the solution operator for the elasticity problem (3.10), see Theorem 2 and is hence, bounded. For

$$D_u\mathcal{F}(\phi(\gamma_{\Gamma_{\text{int}}}u), g)(\delta u)\colon W^{2,p}(\Omega_{\text{s}}) \to W^{1-1/p,p}(\Gamma_{\text{int}}) \tag{6.25}$$

we use that by Lemma 16 the norm $\|D_u\mathcal{F}\|_{L_F}$ can be made arbitrarily small choosing r sufficiently small and taking the continuous dependence of the solution of the FSI problem on the data into account, see Theorem 3. Thus, $\text{id} - D_2\mathcal{N} \circ D_u\mathcal{F}$ is invertible. By the implicit function theorem we obtain the continous differentiability of the mapping Π. □

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A: Transformation of the Navier-Stokes Equation

Following [27] we state the strong and weak formulation of the Navier-Stokes equation in the physical and reference domain. We have for the velocity $(\tilde{w}_1, \tilde{w}_2)$ and pressure $\tilde{p}$ in the physical domain $\Omega_{\text{f}}[u]$

$$\begin{aligned}
-\nu\Delta_x\tilde{w}_1 + \tilde{w}^\top\nabla\tilde{w}_1 + (\nabla\tilde{p})_1 &= 0 && \text{in } \Omega_{\text{f}}[u],\\
-\nu\Delta_x\tilde{w}_2 + \tilde{w}^\top\nabla\tilde{w}_2 + (\nabla\tilde{p})_2 &= 0 && \text{in } \Omega_{\text{f}}[u],\\
\operatorname{div}\tilde{w} &= 0 && \text{in } \Omega_{\text{f}}[u],\\
\tilde{w} &= \delta g && \text{on } \Gamma_{\text{in}},\\
\tilde{w} &= 0 && \text{on } \Gamma_{\text{wall}} \cup \Gamma_{\text{int}}[u],\\
-\nu D\tilde{w}n_{\text{f}}[u] + \tilde{p}n_{\text{f}}[u] &= 0 && \text{on } \Gamma_{\text{out}}.
\end{aligned} \tag{A.1}$$

Transforming to a weak form by multiplying with a test function, integration over $\Omega_{\mathrm{f}}[u]$, and apply integration by parts we obtain

$$\begin{aligned}
-\nu \int_{\Gamma_{\mathrm{out}}} \tilde{\psi}_1 \nabla \tilde{w}_1 n_{\mathrm{f}}[u] \mathrm{d}s_y + \nu \int_{\Omega_{\mathrm{f}}[u]} (\nabla \tilde{\psi}_1)^\top (\nabla \tilde{w}_1) \mathrm{d}y \\
+ \int_{\Omega_{\mathrm{f}}[u]} \tilde{\psi}_1 (\tilde{w}^\top \nabla) \tilde{w}_1 \mathrm{d}y + \int_{\Gamma_{\mathrm{out}}} \tilde{\psi}_1 \tilde{p} (n_{\mathrm{f}}[u])_1 \mathrm{d}s_y \\
- \int_{\Omega_{\mathrm{f}}[u]} \tilde{p} (\nabla \tilde{\psi}_1)_1 \mathrm{d}y =: I_1 + I_2 + I_3 + I_4 + I_5 = 0.
\end{aligned} \tag{A.2}$$

We have by (2.24), (2.25), and (2.26) on the do-nothing outflow boundary part

$$\begin{aligned}
I_1 &:= -\nu \int_{\Gamma_{\mathrm{out}}} \tilde{\psi}_1 \nabla \tilde{w}_1 n_{\mathrm{f}}[u] \mathrm{d}s_y \\
&= -\nu \int_{\Gamma_{\mathrm{out}}} \psi_1 (F[u]^{-1} \nabla w_1)^\top \frac{K[u] n_{\mathrm{f}}}{\|K[u] n_{\mathrm{f}}\|} \, \|K[u] n_{\mathrm{f}}\| \, \mathrm{d}s_x \\
&= -\nu \int_{\Gamma_{\mathrm{out}}} \psi_1 (\nabla w_1)^\top \left(\frac{1}{J} K^\top K \right) n_{\mathrm{f}} \mathrm{d}s_x \\
&= -\nu \int_{\Gamma_{\mathrm{out}}} \psi_1 (\nabla w_1)^\top A n_{\mathrm{f}} \mathrm{d}s_x.
\end{aligned} \tag{A.3}$$

For the diffusion term we have using (2.25)

$$\begin{aligned}
I_2 &:= \nu \int_{\Omega_{\mathrm{f}}[u]} (\nabla \tilde{\psi}_1)^\top (\nabla \tilde{w}_1) \mathrm{d}y \\
&= \nu \int_{\Omega_{\mathrm{f}}} \left(\frac{1}{J} K \nabla \psi_1 \right)^\top \left(\frac{1}{J} K \nabla w_1 \right) J \mathrm{d}y \\
&= \nu \int_{\Omega_{\mathrm{f}}} (\nabla \psi_1)^\top A (\nabla w_1) \mathrm{d}x \\
&= \nu \int_{\Gamma_{\mathrm{out}}} \psi_1 (n_{\mathrm{f}}^\top A \nabla w_1) \mathrm{d}s_x - \nu \int_{\Omega_{\mathrm{f}}} \psi_1 \nabla^\top (A \nabla w_1) \mathrm{d}x.
\end{aligned} \tag{A.4}$$

The convection term transforms using (2.25) as follows

$$I_3 := \int_{\Omega_{\mathrm{f}}[u]} \tilde{\psi}_1 (\tilde{w}^\top \nabla) \tilde{w}_1 \mathrm{d}y = \int_{\Omega_{\mathrm{f}}} \psi_1 w^\top \frac{1}{J} K \nabla w_1 J \mathrm{d}x = \int_{\Omega_{\mathrm{f}}} \psi_1 w^\top K \nabla w_1 \mathrm{d}x. \tag{A.5}$$

For the boundary pressure term we have by (2.24) and (2.26)

$$
\begin{aligned}
I_4 &:= \int_{\Gamma_{\text{out}}} \tilde{\psi}_1 \tilde{p} (n_{\text{f}}[u])_1 \mathrm{d}s_y \\
&= \int_{\Gamma_{\text{out}}} \psi_1 \left(p \frac{K n_{\text{f}}}{\|K n_{\text{f}}\|} \right)_1 \|K n_{\text{f}}\| \, \mathrm{d}s_x \\
&= \int_{\Gamma_{\text{out}}} \psi_1 p (K n_{\text{f}})_1 \mathrm{d}s_x .
\end{aligned}
\tag{A.6}
$$

Finally, for the volume pressure term we have

$$
\begin{aligned}
I_5 &:= - \int_{\Omega_{\text{f}}[u]} \tilde{p} (\nabla \tilde{\psi}_1)_1 \mathrm{d}y \\
&= - \int_{\Omega_{\text{f}}} p (K \nabla \psi_1)_1 \mathrm{d}x \\
&= - \int_{\Gamma_{\text{out}}} \psi_1 p (K n_{\text{f}})_1 \mathrm{d}s_x + \int_{\Omega_{\text{f}}} \psi_1 p \, \mathrm{div}_x (K p)_1 \mathrm{d}x ,
\end{aligned}
\tag{A.7}
$$

where

$$
\mathrm{div}_x (K p)_1 := \partial_{x_1} (k_{11} p) + \partial_{x_2} (k_{12} p). \tag{A.8}
$$

Summarizing we obtain the weak formulation

$$
\begin{aligned}
&-\nu \int_{\Gamma_{\text{out}}} \psi_1 (\nabla w_1)^\top A n_{\text{f}} \mathrm{d}s_x + \nu \int_{\Omega_{\text{f}}} (\nabla \psi_1)^\top A (\nabla w_1) \mathrm{d}x \\
&\quad + \int_{\Omega_{\text{f}}} \psi_1 w^\top K \nabla w_1 \mathrm{d}x + \int_{\Gamma_{\text{out}}} \psi_1 p (K n_{\text{f}})_1 \mathrm{d}s_x + \int_{\Omega_{\text{f}}} \psi_1 \, \mathrm{div}_x (K p)_1 \mathrm{d}x = 0
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
&\nu \int_{\Omega_{\text{f}}} (\nabla \psi_1)^\top A (\nabla w_1) \mathrm{d}x + \int_{\Omega_{\text{f}}} \psi_1 w^\top K \nabla w_1 \mathrm{d}x \\
&\quad + \int_{\Omega_{\text{f}}} \psi_1 \, \mathrm{div}_x (K p)_1 \mathrm{d}x = \int_{\Gamma_{\text{out}}} f_3 v \mathrm{d}s + \int_{\Omega_{\text{f}}} f v \mathrm{d}x
\end{aligned}
\tag{A.10}
$$

and equivalently in strong form

$$
\begin{cases}
-\nu \, \mathrm{div}(A[u] \nabla w) + (w \cdot K[u] \nabla) w + K[u] \nabla p = 0 & \text{in } \Omega_{\text{f}}, \\
\mathrm{div}_{K^\top [u]^\top} w = 0 & \text{in } \Omega_{\text{f}}, \\
w = \delta g & \text{on } \Gamma_{\text{in}}, \\
w = 0 & \text{on } \Gamma_{\text{wall}} \cup \Gamma_{\text{int}}, \\
-\nu \partial_{A[u],n} w + p K[u] n_{\text{f}} = 0 & \text{on } \Gamma_{\text{out}}.
\end{cases}
\tag{A.11}
$$

B: Transformation of the Linearized Navier-Stokes Equation

For the velocity $(\tilde{w}_1, \tilde{w}_2)$ and pressure $\tilde{p}$ in the physical domain $\Omega_{\mathrm{f}}[u]$ we have

$$\begin{aligned} -\nu\Delta_x \tilde{z}_{w_1} + \hat{w}^\top \nabla \tilde{z}_{w_1} + \tilde{z}_w^\top \nabla \hat{w}_1 + (\nabla \tilde{z}_p)_1 &= 0 && \text{in } \Omega_{\mathrm{f}}[u], \\ -\nu\Delta_x \tilde{z}_{w_2} + \tilde{w}^\top \nabla \tilde{z}_{w_2} + \tilde{z}_w^\top \nabla \hat{w}_2 + (\nabla \tilde{z}_p)_2 &= 0 && \text{in } \Omega_{\mathrm{f}}[u], \\ \operatorname{div} \tilde{z}_w &= 0 && \text{in } \Omega_{\mathrm{f}}[u], \\ \tilde{z}_w &= \delta g && \text{on } \Gamma_{\mathrm{in}}, \\ \tilde{z}_w &= 0 && \text{on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}[u], \\ -\nu\partial_n \tilde{z}_w + \tilde{z}_p n_{\mathrm{f}}[u] &= 0 && \text{on } \Gamma_{\mathrm{out}}. \end{aligned} \tag{B.1}$$

All linear terms are transformed as for the Navier-Stokes equation. The first term of the linearized convection term transforms using (2.25) as follows

$$\int_{\Omega_{\mathrm{f}}[u]} \tilde{\psi}_1 (\tilde{w}^\top \nabla) z_{\tilde{w}_1} \mathrm{d}y = \int_{\Omega_{\mathrm{f}}} \psi_1 w^\top \frac{1}{J} K \nabla z_{w_1} J \mathrm{d}x = \int_{\Omega_{\mathrm{f}}} \psi_1 w^\top K \nabla z_{w_1} \mathrm{d}x \tag{B.2}$$

and the second one accordingly. That means we have for the transformed equation in strong form

$$\begin{cases} -\nu \operatorname{div}(A[u]\nabla z_w) + (z_w \cdot K[u]\nabla)\hat{w} & \\ \qquad + (\hat{w} \cdot K[u]\nabla) z_w + K[u]\nabla z_p = 0 & \text{in } \Omega_{\mathrm{f}}, \\ \operatorname{div}(K[u]^\top z_w) = 0 & \text{in } \Omega_{\mathrm{f}}, \\ z_w = 0 & \text{on } \Gamma_{\mathrm{in}}, \\ z_w = \delta g & \text{on } \Gamma_{\mathrm{wall}} \cup \Gamma_{\mathrm{int}}, \\ -\nu \partial_{A[u],n_{\mathrm{f}}} z_w + z_p K[u] n_{\mathrm{f}} = 0 & \text{on } \Gamma_{\mathrm{out}}. \end{cases} \tag{B.3}$$

C: Some Properties

Lemma 17 (Algebra property) *For $v \in W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega)$, the product uv belongs to $W^{1,p}(\Omega)$, and we have*

$$\|uv\|_{W^{1,p}(\Omega)} \le c\, \|u\|_{W^{1,p}(\Omega)} \|v\|_{W^{1,p}(\Omega)}\,. \tag{C.1}$$

Proof Immediate. □

With the embedding of Sobolev in Hölder spaces we have for $p > n$

$$W^{2,p}(\Omega_{\mathrm{f}}) \subset C^{0,1}(\bar{\Omega}_2) \tag{C.2}$$

and so [2, p. 338 and p. 325]

$$\|v\|_{W^{1,\infty}(\Omega_{\mathrm{f}})} = \|v\|_{C^{0,1}(\bar{\Omega}_2)} \le c\, \|v\|_{W^{2,p}(\Omega_{\mathrm{f}})} \quad \text{for } v \in W^{2,p}(\Omega_{\mathrm{f}}). \tag{C.3}$$

For $w \in W^{1,2}(\Omega_{\mathrm{f}})^2$ and recalling $K[u]$ we have the following calculus rules:

$$\operatorname{div}(K[u]) = 0 \quad \text{(Piola's identity)},$$
$$\operatorname{div}_{\mathrm{id}-K[u]^\top} w = ((\mathrm{id} - K[u])\nabla)^\top w = \operatorname{div} w - K[u]^\top \cdot \nabla w = (\mathrm{id} - K[u]^\top) \cdot \nabla w. \tag{C.4}$$

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