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L_p–L_q-Theory for a Quasilinear Non-isothermal Westervelt Equation

Mathias Wilke[1]

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Abstract
We investigate a quasilinear system consisting of the Westervelt equation from nonlinear acoustics and Pennes bioheat equation, subject to Dirichlet or Neumann boundary conditions. The concept of maximal regularity of type L_p–L_q is applied to prove local and global well-posedness. Moreover, we show by a parameter trick that the solutions regularize instantaneously. Finally, we compute the equilibria of the system and investigate the long-time behaviour of solutions starting close to equilibria.

Keywords Non-isothermal Westervelt equation · Optimal regularity · Quasilinear parabolic system · Exponential stability

1 Introduction

Thermo-acoustic lensing describes the effect of how the speed of acoustic waves and the pressure of a region are influenced by the temperature of the underlying tissue. A meanwhile well-accepted model which takes care of this effect consists of the Westervelt equation [27]

$$u_{tt} - c^2(\theta)\Delta u - b(\theta)\Delta u_t = k(\theta)(u^2)_{tt}, \tag{1.1}$$

describing the propagation of sound in fluidic media, coupled with the so-called bioheat equation proposed by Pennes [20]

$$\rho_a C_a \theta_t - \kappa_a \Delta\theta + \rho_b C_b W(\theta - \theta_a) = Q(u_t). \tag{1.2}$$

✉ Mathias Wilke
mathias.wilke@mathematik.uni-halle.de

1 Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg, Halle (Saale), Germany

In (1.1), the function $u = u(t, x)$ denotes the acoustic pressure fluctuation from an ambient value at time t and position x. Furthermore, $c(\theta) > 0$ denotes the speed of sound, $b(\theta) > 0$ the diffusivity of sound and $k(\theta) > 0$ the parameter of nonlinearity.

The physical meaning of the parameters in (1.2) are as follows: $\rho_a > 0$ and $\kappa_a > 0$ denote the ambient density and thermal conductivity, respectively. $C_a > 0$ is the ambient heat capacity and $\theta_a > 0$ stands for the constant ambient temperature, $\rho_b > 0$ is the density of blood, $C_b > 0$ is the heat capacity of blood and W denotes the perfusion rate (cooling by blood flow).

The nonlinear function Q models the acoustic energy being absorbed by the surrounding tissue and Q is typically of quadratic type, see Remark 1.2.

Considering (1.1)–(1.2) in a bounded framework, we have to equip these equations with suitable boundary conditions. In this article, we propose either Dirichlet or Neumann boundary conditions on u and θ. Altogether, we end up with the following system

$$\begin{aligned} u_{tt} - c^2(\theta)\Delta u - b(\theta)\Delta u_t &= k(\theta)(u^2)_{tt}, && \text{in } (0,T)\times\Omega,\\ \rho_a C_a\theta_t - \kappa_a\Delta\theta + \rho_b C_b W(\theta-\theta_a) &= Q(u_t), && \text{in } (0,T)\times\Omega,\\ \mathcal{B}_j u &= g_j, && \text{in } (0,T)\times\partial\Omega,\\ \mathcal{B}_\ell\theta &= h_\ell, && \text{in } (0,T)\times\partial\Omega,\\ (u(0), u_t(0)) &= (u_0, u_1), && \text{in } \Omega,\\ \theta(0) &= \theta_0, && \text{in } \Omega, \end{aligned} \tag{1.3}$$

where $(j, \ell) \in \{0, 1\} \times \{0, 1\}$,

- $\mathcal{B}_0 v = v|_{\partial\Omega}$ (Dirichlet boundary conditions),
- $\mathcal{B}_1 v = \partial_\nu v$ (Neumann boundary conditions),

and u_0, u_1, θ_0 denote the initial conditions for u, u_t, θ at $t = 0$.

We observe that as long as $b(\theta) > 0$, the term $b(\theta)\Delta u_t$ renders (1.1) into a strongly damped wave equation which is of parabolic type. Since

$$(u^2)_{tt} = 2u_{tt}u + 2(u_t)^2,$$

we see that parabolicity is preserved as long as $|u|$ is sufficiently close to zero. It follows that (1.3) represents a quasilinear parabolic system for the variables (u, u_t, θ). Therefore, it is reasonable to apply L_p–L_q-theory in order to solve (1.3).

The Westervelt equation (with constant temperature) has been subject to a variety of articles over the last decades, see e.g. [4, 9–12, 14, 15, 25], which is just a selection.

To the best knowledge of the author, there is only the article [17] which provides analytical results for (1.3) in case of homogeneous Dirichlet boundary conditions for both u and θ and provided that the diffusivity of sound b *does not* depend on θ. The analysis in [17] is based on L_2-theory and some (higher-order) energy estimates. To this end, the authors have to equip the initial data with more regularity than is actually needed.

Within the present article, we are interested in the existence and uniqueness of strong solutions to (1.3) having maximal regularity of type L_p–L_q. In particular, we present optimal conditions on the initial data (u_0, u_1, θ_0) and the boundary data (g_j, h_ℓ), thereby improving the assumptions on (u_0, u_1, θ_0) in [17] (for details, see below). Additionally, we investigate the temporal regularity of the solutions to (1.3) as well as their long-time behaviour.

Our article is structured as follows. In Sect. 2 we consider a suitable linearization of (1.3) and we prove optimal regularity results of type L_p-L_q for the resulting parabolic problems. Section 3 is devoted to the proof of the following main-result concerning well-posedness of (1.3) under optimal conditions on the data $(u_0, u_1, \theta_0, g_j, h_\ell)$.

Theorem 1.1 *Let $d \in \mathbb{N}$, $T \in (0, \infty)$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega \in C^2$ and suppose that $c, b, k \in C^1(\mathbb{R})$ with $b(\tau) \geq b_0 > 0$ for all $\tau \in \mathbb{R}$. Assume furthermore that $p, q, r, s \in (1, \infty)$ such that*

$$\frac{d}{q} < 2, \quad \frac{2}{r} + \frac{d}{s} < 2$$

and

$$Q \in C^1\left(W_p^1((0,T); L_q(\Omega)) \cap L_p((0,T); W_q^2(\Omega)); L_r((0,T); L_s(\Omega))\right),$$

with $Q(0) = 0$. Let $1 - j/2 - 1/2q \neq 1/p$ and $1 - \ell/2 - 1/2s \neq 1/r$.

Then there exists $\delta = \delta(T) > 0$ such that for all

$$\begin{aligned}
&u_0 \in W_q^2(\Omega), \quad u_1 \in B_{qp}^{2-2/p}(\Omega), \quad \theta_0 \in B_{sr}^{2-2/r}(\Omega),\\
&g_j \in F_{pq}^{2-j/2-1/2q}((0,T); L_q(\partial\Omega)) \cap W_p^1((0,T); W_q^{2-j-1/q}(\partial\Omega)) =: Y_j(0,T),\\
&h_\ell \in F_{rs}^{1-\ell/2-1/2s}((0,T); L_s(\partial\Omega)) \cap L_r((0,T); W_s^{2-\ell-1/s}(\partial\Omega)),
\end{aligned}$$

with

- $\mathcal{B}_j u_0 = g_j(0)$,
- $\mathcal{B}_j u_1 = \partial_t g_j(0)$ *if* $1 - j/2 - 1/2q > 1/p$,
- $\mathcal{B}_\ell \theta_0 = h_\ell(0)$ *if* $1 - \ell/2 - 1/2s > 1/r$,

and

$$\|u_0\|_{W_q^2(\Omega)} + \|u_1\|_{B_{qp}^{2-2/p}(\Omega)} + \|g_j\|_{Y_j(0,T)} \leq \delta,$$

there exists a unique solution

$$\begin{aligned}
u &\in W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega))\\
\theta &\in W_r^1((0,T); L_s(\Omega)) \cap L_r((0,T); W_s^2(\Omega))
\end{aligned}$$

of (1.3). Moreover, the solution (u, θ) is C^1 with respect to the data $(g_j, u_0, u_1, h_\ell, \theta_0)$.

Remark 1.2 The nonlinear function Q can for instance be modeled by

$$Q(u_t) = C \cdot (u_t)^2$$

or

$$Q(u_t) = \frac{C}{T} \int_0^T (u_t)^2 dt$$

for some constant $C > 0$, see e.g. [6, 7, 19]. In these cases it can be readily checked that $Q(0) = 0$ and

$$Q \in C^1 \left(W_p^1((0, T); L_q(\Omega)) \cap L_p((0, T); W_q^2(\Omega)); L_r((0, T); L_s(\Omega)) \right).$$

provided that

$$\frac{2}{p} + \frac{d}{q} < 2 + \frac{1}{r} + \frac{d}{2s}.$$

For the proof of Theorem 1.1 we employ the implicit function theorem and the results on optimal regularity of the linearization from Sect. 2. In order to compare our results in Theorem 1.1 with [17, Theorem 4.1], we consider the very special case $d \in \{1, 2, 3\}$, $p = q = s = 2$ and $g_j = h_\ell = 0$ in Theorem 1.1.

Corollary 1.3 *Let $T \in (0, \infty)$, $d \in \{1, 2, 3\}$, $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega \in C^2$ and suppose that $c, b, k \in C^1(\mathbb{R})$ with $b(\tau) \geq b_0 > 0$ for all $\tau \in \mathbb{R}$. Assume furthermore that $r \in (1, \infty)$ such that*

$$\frac{2}{r} + \frac{d}{2} < 2$$

and

$$Q \in C^1 \left(W_2^1((0, T); L_2(\Omega)) \cap L_2((0, T); W_2^2(\Omega)); L_r((0, T); L_2(\Omega)) \right),$$

with $Q(0) = 0$. Let $3/4 - \ell/2 \neq 1/r$.

Then there exists $\delta = \delta(T) > 0$ such that for all

$$u_0 \in W_2^2(\Omega), \quad u_1 \in W_2^1(\Omega), \quad \theta_0 \in B_{2r}^{2-2/r}(\Omega),$$

with

- $\mathcal{B}_j u_0 = 0$,
- $\mathcal{B}_j u_1 = 0$ *if* $3/4 - j/2 > 1/2$,
- $\mathcal{B}_\ell \theta_0 = 0$ *if* $3/4 - \ell/2 > 1/r$,

and

$$\|u_0\|_{W_2^2(\Omega)} + \|u_1\|_{W_2^1(\Omega)} \leq \delta,$$

there exists a unique solution

$$\begin{aligned} u &\in W_2^2((0,T); L_2(\Omega)) \cap W_2^1((0,T); W_2^2(\Omega)) \\ \theta &\in W_r^1((0,T); L_2(\Omega)) \cap L_r((0,T); W_2^2(\Omega)) \end{aligned}$$

of (1.3) *with* $g_j = h_\ell = 0$.

Let us compare the well-posedness result [17, Theorem 4.1] concerning (1.3) with homogeneous Dirichlet boundary conditions with our result. In [17], the authors assume that

$$u_0 \in W_2^3(\Omega), \quad u_1, \theta_0 \in W_2^2(\Omega),$$

(plus compatibility conditions on $\partial\Omega$). Since

$$W_2^2(\Omega) = B_{22}^2(\Omega) \hookrightarrow B_{2r}^2(\Omega) \hookrightarrow B_{2r}^{2-2/r}(\Omega)$$

for any $r \geq 2$, we were able to reduce the regularity of the initial data (u_0, u_1, θ_0). Moreover, a crucial assumption in [17] is that the mapping $[\tau \mapsto b(\tau)]$ is constant and furthermore, only homogeneous Dirichlet boundary conditions for u and θ are considered in [17]. In summary, Theorem 1.1 generalizes [17, Theorem 4.1] considerably.

In Sect. 4 we study the regularity of the solution with respect to the temporal variable t. We use a parameter trick which goes back to Angenent [3], combined with the implicit function theorem to prove that the solution enjoys higher regularity with respect to t as soon as $t > 0$, see Theorem 4.1. This result reflects the parabolic regularization effect.

Finally, in Sect. 5, we compute the equilibria of the system (1.3) if $g_j = 0$ and $h_\ell = (1-\ell)\theta_a$ and investigate the long-time behaviour of solutions starting close to equilibria. For the case of Dirichlet boundary conditions for u, we prove in Theorem 5.1 that the corresponding equilibria are exponentially stable. Since our assumptions on the initial data (u_0, u_1, θ_0) as well as on the nonlinearities are less restrictive compared to [18], Theorem 5.1 may be understood of a generalization of [18, Theorems 2.2 and 2.3].

The definitions and basic properties of the functions spaces being used in the analysis of (1.3) are provided in the Appendix A.

2 Maximal Regularity of a Linearization

Let us consider the two linear problems

$$\begin{aligned} \rho_a C_a \theta_t - \kappa_a \Delta\theta + \rho_b C_b W\theta &= f_1, && \text{in } (0,T)\times\Omega, \\ \mathcal{B}_\ell \theta &= h_\ell, && \text{in } (0,T)\times\partial\Omega, \\ \theta(0) &= \theta_0, && \text{in } \Omega, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} u_{tt} - a_1(t,x)\Delta u_t - a_2(t,x)\Delta u &= f_2, && \text{in } (0,T)\times\Omega, \\ \mathcal{B}_j u &= g_j, && \text{in } (0,T)\times\partial\Omega, \\ (u(0), u_t(0)) &= (u_0, u_1), && \text{in } \Omega. \end{aligned} \tag{2.2}$$

Here $\rho_a, C_a, \rho_b, C_b, \kappa_a, W$ are positive parameters, $a_1, a_2, f, g, u_0, u_1, \theta_0$ are given functions and $(j, \ell) \in \{0,1\}\times\{0,1\}$, where

- $\mathcal{B}_0 v = v|_{\partial\Omega}$ (Dirichlet boundary conditions) or
- $\mathcal{B}_1 v = \partial_\nu v$ (Neumann boundary conditions).

For the linear problems (2.1) and (2.2) we have the following results.

Lemma 2.1 *Let $r, s \in (1,\infty)$, $\Omega \subset \mathbb{R}^d$ be a bounded C^2-domain and let $T \in (0,\infty)$. Suppose that $1 - \ell/2 - 1/2s \neq 1/r$.*

Then there exists a unique solution

$$\theta \in W_r^1((0,T); L_s(\Omega)) \cap L_r((0,T); W_s^2(\Omega))$$

of (2.1) if and only if

(1) $f_1 \in L_r((0,T); L_s(\Omega))$;
(2) $h_\ell \in F_{rs}^{1-\ell/2-1/2s}((0,T); L_s(\partial\Omega)) \cap L_r((0,T); W_s^{2-\ell-1/s}(\partial\Omega))$;
(3) $\theta_0 \in B_{sr}^{2-2/r}(\Omega)$
(4) $\mathcal{B}_\ell\theta_0 = h_\ell(0)$ *if* $1 - \ell/2 - 1/2s > 1/r$.

Proof The proof follows from [5, Theorem 2.3]. □

Lemma 2.2 *Let $p, q \in (1,\infty)$, $\Omega \subset \mathbb{R}^d$ be a bounded C^2-domain and let $T \in (0,\infty)$. Suppose furthermore that $a_1, a_2 \in C([0,T]\times\overline{\Omega})$ and $a_1(t,x) \geq \alpha > 0$ for all $(t,x) \in [0,T]\times\overline{\Omega}$. Assume that $1 - j/2 - 1/2q \neq 1/p$.*

Then there exists a unique solution

$$u \in W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega))$$

of (2.2) if and only if

(1) $f_2 \in L_p((0,T); L_q(\Omega))$;
(2) $g_j \in F_{pq}^{2-j/2-1/2q}((0,T); L_q(\partial\Omega)) \cap W_p^1((0,T); W_q^{2-j-1/q}(\partial\Omega))$;

(3) $u_0 \in W^2_q(\Omega)$, $u_1 \in B^{2-2/p}_{qp}(\Omega)$
(4) $\mathcal{B}_j u_0 = g_j(0)$ *for all* $p, q \in (1, \infty)$ *and*
(5) $\mathcal{B}_j u_1 = \partial_t g_j(0)$ *if* $1 - j/2 - 1/2q > 1/p$.

Proof We start with the necessity part. If

$$u \in W^2_p((0,T); L_q(\Omega)) \cap W^1_p((0,T); W^2_q(\Omega))$$

is a solution of (2.2), then clearly $f \in L_p((0,T; L_q(\Omega))$ by the assumptions on a_j and by the first equation in (2.2). Furthermore,

$$\begin{aligned} &W^2_p((0,T); L_q(\Omega)) \cap W^1_p((0,T); W^2_q(\Omega)) \\ &\quad \hookrightarrow W^1_p((0,T); W^2_q(\Omega)) \hookrightarrow C([0,T]; W^2_q(\Omega)) \end{aligned}$$

see e.g. [2, Theorem VII.2.6.6 (ii)], hence $u_0 = u(0) \in W^2_p(\Omega)$. Since

$$\partial_t u \in W^1_p((0,T); L_q(\Omega)) \cap L_p((0,T); W^2_q(\Omega)),$$

it follows that $u_1 = \partial_t u(0) \in B^{2-2/p}_{qp}(\Omega)$, see e.g. [22, Theorem 3.4.8].

Concerning the boundary data g_j, note that $\mathcal{B}_j u \in W^1_p((0,T); W^{2-j-1/q}_q(\partial\Omega))$ and

$$\mathcal{B}_j \partial_t u \in F^{1-j/2-1/2q}_{pq}((0,T); L_q(\partial\Omega)) \cap L_p((0,T); W^{2-j-1/q}_q(\partial\Omega)),$$

see e.g. [2, Chapter VIII], [5, Section 6] or [22, Section 6.2].

From (A.1), (A.2) and [2, Theorem VII.5.2.3 (iv)] we obtain the embedding

$$W^1_p((0,T); W^{2-j-1/q}_q(\partial\Omega)) \hookrightarrow F^{1-j/2-1/2q}_{pq}((0,T); L_q(\partial\Omega)).$$

This readily implies

$$g_j, \partial_t g_j \in F^{1-j/2-1/2q}_{pq}((0,T); L_q(\partial\Omega)) \cap L_p((0,T); W^{2-j-1/q}_q(\partial\Omega)),$$

hence

$$g_j \in F^{2-j/2-1/2q}_{pq}((0,T); L_q(\partial\Omega)) \cap W^1_p((0,T); W^{2-j-1/q}_q(\partial\Omega)),$$

by [2, Theorem VII.5.5.1].

Since $\mathcal{B}_j u = g_j \in W^1_p((0,T); W^{2-j-1/q}_q(\partial\Omega))$ and

$$W^1_p((0,T); W^{2-j-1/q}_q(\partial\Omega)) \hookrightarrow C([0,T]; W^{2-j-1/q}_q(\partial\Omega)),$$

([2, Theorem VII.2.6.6 (ii)]) we necessarily have $\mathcal{B}_j u_0 = g_j(0)$ for all $p, q \in (1, \infty)$. Furthermore,

$$\mathcal{B}_j \partial_t u = \partial_t g_j \in F_{pq}^{1-j/2-1/2q}((0,T); L_q(\partial\Omega)) \cap L_p((0,T); W_q^{2-j-1/q}(\partial\Omega))$$

and

$$F_{pq}^{1-j/2-1/2q}((0,T); L_q(\partial\Omega)) \hookrightarrow C([0,T]; L_q(\partial\Omega))$$

by [16, Proposition 7.4], provided $1 - j/2 - 1/2q > 1/p$, which readily implies $\mathcal{B}_j u_1 = \partial_t g_j(0)$.

We now prove that the conditions in Lemma 2.2 are also sufficient. To this end, we first consider the problem

$$\begin{aligned} v_t - a_1(t,x)\Delta v &= f_2, && \text{in } (0,T)\times\Omega,\\ \mathcal{B}_j v &= \partial_t g_j, && \text{in } (0,T)\times\partial\Omega,\\ v(0) &= u_1, && \text{in } \Omega. \end{aligned} \tag{2.3}$$

By [5, Theorem 2.3] there exists a unique solution

$$v \in W_p^1((0,T); L_q(\Omega)) \cap L_p((0,T); W_q^2(\Omega))$$

of (2.3). Define

$$u(t,x) = u_0(x) + \int_0^t v(s,x)ds, \quad t \in [0,T].$$

Then

$$u \in W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega)),$$

$u(0,x) = u_0(x)$, $\mathcal{B}_j u(t,x) = g_j(t,x)$ (by the compatibility condition on u_0) and $\partial_t^k u(t,x) = \partial_t^{k-1} v(t,x)$ for $k \in \{1,2\}$. Consequently, the function u is the unique solution of the problem

$$\begin{aligned} u_{tt} - a_1(t,x)\Delta u_t &= f_2, && \text{in } (0,T)\times\Omega,\\ \mathcal{B}_j u &= g_j, && \text{in } (0,T)\times\partial\Omega,\\ (u(0), u_t(0)) &= (u_0, u_1), && \text{in } \Omega. \end{aligned} \tag{2.4}$$

Uniqueness can be seen as follows. If u_1 and u_2 are two solutions of (2.4), then $u_1 - u_2$ solves (2.4) with $(f_2, g_j, u_0, u_1) = 0$ and therefore, $\partial_t(u_1 - u_2)$ solves (2.3) with $(f_2, g_j, u_1) = 0$, wherefore $\partial_t(u_1 - u_2) = 0$. Since $(u_1 - u_2)(0) = 0$, it follows that $u_1 - u_2 = 0$, hence $u_1 = u_2$.

Next, we consider the problem

$$\begin{aligned} w_{tt} - a_1(t,x)\Delta w_t - a_2(t,x)\Delta w &= \tilde{f}_2, && \text{in } (0,T)\times\Omega, \\ \mathcal{B}_j w &= 0 && \text{in } (0,T)\times\partial\Omega, \\ (w(0), w_t(0)) &= (0,0), && \text{in } \Omega, \end{aligned} \tag{2.5}$$

for given $\tilde{f}_2 \in L_p((0,T); L_q(\Omega))$. Note that for a sufficiently smooth solution, it holds that $\mathcal{B}_j w_t = 0$ in $(0,T)\times\partial\Omega$. We reformulate (2.5) as a first order system. To this end, let $z = (z_1, z_2) = (w, w_t)$ and $F = (0, \tilde{f}_2)$. Then

$$z_t = \begin{pmatrix} 0 & I \\ 0 & a_1(t,x)\Delta \end{pmatrix} z + \begin{pmatrix} 0 & 0 \\ a_2(t,x)\Delta & 0 \end{pmatrix} z + F, \tag{2.6}$$

with the initial condition $z(0) = 0$ in Ω and the boundary condition $\mathcal{B}_j z = 0$ in $(0,T)\times\partial\Omega$. Let

$$D(\Delta_j) = \{w \in W_q^2(\Omega) \mid \mathcal{B}_j w = 0 \text{ on } \partial\Omega\}$$

and define $X_0 = D(\Delta_j) \times L_q(\Omega)$ as well as $X_1 = D(\Delta_j) \times D(\Delta_j)$. Furthermore, let

$$A_1(t) = \begin{pmatrix} 0 & I \\ 0 & a_1(t,\cdot)\Delta \end{pmatrix} \quad \text{and} \quad A_2(t) = \begin{pmatrix} 0 & 0 \\ a_2(t,\cdot)\Delta & 0 \end{pmatrix}.$$

Then, we have $A_1 \in C([0,T]; \mathcal{L}(X_1, X_0))$ and $A_2 \in C([0,T]; \mathcal{L}(X_0, X_0))$. Moreover, $A_1(t)$ has the property of L_p-maximal regularity in X_0 for any $t \in [0,T]$.

By [21, Theorem 3.1] there exists a unique solution

$$z \in W_p^1((0,T); X_0) \cap L_p((0,T); X_1)$$

of Eq. (2.6) subject to the initial condition $z(0) = 0$. This in turn yields the existence and uniqueness of a solution

$$w \in W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega)),$$

of (2.5). Finally, we solve (2.4) to obtain a solution

$$\tilde{u} \in W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega)).$$

Then, we solve (2.5) with $\tilde{f}_2 := a_2\Delta\tilde{u} \in L_p((0,T); L_q(\Omega))$ to obtain a solution

$$\tilde{w} \in W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega)).$$

It is readily checked that the sum

$$u := \tilde{u} + \tilde{w} \in W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega))$$

is the unique solution of (2.2). □

Finally, let us consider the following coupled linear problem

$$\begin{aligned} u_{tt} - a_1(t,x)\Delta u_t - a_2(t,x)\Delta u &= f_2, && \text{in } (0,T)\times\Omega,\\ \rho_a C_a\theta_t - \kappa_a\Delta\theta + \rho_b C_b W\theta + Bu_t &= f_1, && \text{in } (0,T)\times\Omega,\\ \mathcal{B}_j u &= g_j, && \text{in } (0,T)\times\partial\Omega,\\ \mathcal{B}_\ell\theta &= h_\ell, && \text{in } (0,T)\times\partial\Omega,\\ (u(0),u_t(0)) &= (u_0,u_1), && \text{in } \Omega,\\ \theta(0) &= \theta_0, && \text{in } \Omega. \end{aligned} \tag{2.7}$$

Lemma 2.3 *Let $\Omega\subset\mathbb{R}^d$ be a bounded C^2-domain, $T\in(0,\infty)$ and let $p,q,r,s\in(1,\infty)$ such that*

$$B : W_p^1((0,T);L_q(\Omega))\cap L_p((0,T);W_q^2(\Omega)) \to L_r((0,T);L_s(\Omega))$$

is linear and bounded. Suppose furthermore that $a_1,a_2\in C([0,T]\times\overline{\Omega})$ and $a_1(t,x)\geq\alpha>0$ for all $(t,x)\in[0,T]\times\overline{\Omega}$. Assume that $1-j/2-1/2q\neq 1/p$ and $1-\ell/2-1/2s\neq 1/r$.

Then there exists a unique solution

$$u\in W_p^2((0,T);L_q(\Omega))\cap W_p^1((0,T);W_q^2(\Omega)),$$
$$\theta\in W_r^1((0,T);L_s(\Omega))\cap L_r((0,T);W_s^2(\Omega))$$

of (2.7) if and only if

(1) $f_1\in L_r((0,T);L_s(\Omega))$;
(2) $f_2\in L_p((0,T);L_q(\Omega))$;
(3) $g_j\in F_{pq}^{2-j/2-1/2q}((0,T);L_q(\partial\Omega))\cap W_p^1((0,T);W_q^{2-j-1/q}(\partial\Omega))$;
(4) $h_\ell\in F_{rs}^{1-\ell/2-1/2s}((0,T);L_s(\partial\Omega))\cap L_r((0,T);W_s^{2-\ell-1/s}(\partial\Omega))$;
(5) $u_0\in W_q^2(\Omega)$, $u_1\in B_{qp}^{2-2/p}(\Omega)$;
(6) $\theta_0\in B_{sr}^{2-2/r}(\Omega)$;
(7) $\mathcal{B}_j u_0 = g_j(0)$ *for all* $p,q\in(1,\infty)$;
(8) $\mathcal{B}_j u_1 = \partial_t g_j(0)$ *if* $1-j/2-1/2q>1/p$;
(9) $\mathcal{B}_\ell\theta_0 = h_\ell(0)$ *if* $1-\ell/2-1/2s>1/r$.

Proof Necessity of the conditions follows as in the proofs of Lemmas 2.1 and 2.2.

To prove sufficiency, one first solves $(2.7)_{1,3,5}$ for u by Lemma 2.2. Then, by the assumption on B, it follows that $Bu_t\in L_r((0,T);L_s(\Omega))$ is a given function. Therefore, we may solve $(2.7)_{2,4,6}$ by Lemma 2.1 to obtain θ. □

3 Proof of Theorem 1.1

We will prove Theorem 1.1 by means of the implicit function theorem. To this end, for fixed but arbitrary $T > 0$, let us first introduce the function spaces

$$\begin{aligned}
\mathbb{E}_0^u &:= L_p((0,T); L_q(\Omega)), \quad \mathbb{E}_0^\theta := L_r((0,T); L_s(\Omega)),\\
\mathbb{E}_1^u &:= W_p^2((0,T); L_q(\Omega)) \cap W_p^1((0,T); W_q^2(\Omega)),\\
\mathbb{E}_1^\theta &:= W_r^1((0,T); L_s(\Omega)) \cap L_r((0,T); W_s^2(\Omega)),\\
Y_j^u &:= F_{pq}^{2-j/2-1/2q}((0,T); L_q(\partial\Omega)) \cap W_p^1((0,T); W_q^{2-j-1/q}(\partial\Omega)),\\
Y_\ell^\theta &:= F_{rs}^{1-\ell/2-1/2s}((0,T); L_s(\partial\Omega)) \cap L_r((0,T); W_s^{2-\ell-1/s}(\partial\Omega)),\\
X_\gamma^u &:= W_q^2(\Omega) \times B_{qp}^{2-2/p}(\Omega), \quad X_\gamma^\theta := B_{sr}^{2-2/r}(\Omega),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{Y}_j^u &:= \{(\tilde{g}_j, (\tilde{u}_0, \tilde{u}_1)) \in Y_j^u \times X_\gamma^u : \mathcal{B}_j \tilde{u}_1 = \partial_t \tilde{g}_j(0) \text{ if } 1 - j/2 - 1/2q > 1/p,\ \mathcal{B}_j \tilde{u}_0 = \tilde{g}_j(0)\},\\
\mathbb{Y}_\ell^\theta &:= \{(\tilde{h}_\ell, \tilde{\theta}_0) \in Y_\ell^\theta \times X_\gamma^\theta : \mathcal{B}_\ell \tilde{\theta}_0 = \tilde{h}_\ell(0) \text{ if } 1 - \ell/2 - 1/2s > 1/r\}.
\end{aligned}$$

Next, we define a function

$$\Phi : \mathbb{E}_1^u \times \mathbb{E}_1^\theta \times \mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta \to \mathbb{E}_0^u \times \mathbb{E}_0^\theta \times \mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta,$$

by

$$\Phi(u, \theta, g_j, u_0, u_1, h_\ell, \theta_0) = \begin{pmatrix} u_{tt} - c^2(\theta)\Delta u - b(\theta)\Delta u_t - k(\theta)(u^2)_{tt} \\ \rho_a C_a \theta_t - \kappa_a \Delta\theta + \rho_b C_b W(\theta - \theta_a) - Q(u_t) \\ \mathcal{B}_j u - g_j \\ u(0) - u_0 \\ u_t(0) - u_1 \\ \mathcal{B}_\ell \theta - h_\ell \\ \theta(0) - \theta_0 \end{pmatrix}.$$

Note that

$$(u^2)_{tt} = 2u_{tt} \cdot u + 2(u_t)^2$$

for each $u \in \mathbb{E}_1^u$. Since (by assumption) $d/q < 2$, it holds that

$$\mathbb{E}_1^u \hookrightarrow W_p^1((0,T); W_q^2(\Omega)) \hookrightarrow C([0,T]; W_q^2(\Omega)) \hookrightarrow C([0,T]; C(\overline{\Omega})),$$

hence

$$\|u_{tt} \cdot u\|_{\mathbb{E}_0^u} \leq C \cdot \|u\|_{\mathbb{E}_1^u}^2,$$

for some constant $C > 0$. Let

$$\dot{\mathbb{E}}_1^u := W_p^1((0,T); L_q(\Omega)) \cap L_p((0,T); W_q^2(\Omega)).$$

Then,

$$\dot{\mathbb{E}}_1^u \hookrightarrow L_{2p}((0,T); L_{2q}(\Omega))$$

provided $1/p + d/2q < 2$, which is satisfied, since $d/q < 2$ and $p > 1$. Therefore

$$\|(u_t)^2\|_{\mathbb{E}_0^u} \leq C\|u_t\|^2_{\dot{\mathbb{E}}_1^u} \leq C\|u\|^2_{\mathbb{E}_1^u},$$

for some constant $C > 0$. Finally, note that

$$\mathbb{E}_1^\theta \hookrightarrow C([0,T]; C(\overline{\Omega}))$$

since (by assumption) $2/r + d/s < 2$. It follows that

$$\|k(\theta)(u^2)_{tt}\|_{\mathbb{E}_0^u} \leq \|k(\theta)\|_{L_\infty((0,T);L_\infty(\Omega))}\|(u^2)_{tt}\|_{\mathbb{E}_0^u} \leq C\|k(\theta)\|_{L_\infty((0,T);L_\infty(\Omega))}\|u\|^2_{\mathbb{E}_1^u},$$

as well as

$$\|b(\theta)\Delta u_t\|_{\mathbb{E}_0^u} \leq \|b(\theta)\|_{L_\infty((0,T);L_\infty(\Omega))}\|\Delta u_t\|_{\mathbb{E}_0^u} \leq C\|b(\theta)\|_{L_\infty((0,T);L_\infty(\Omega))}\|u\|_{\mathbb{E}_1^u}$$

for some constant $C > 0$, since $b, k \in C(\mathbb{R})$. Similarly, we obtain

$$\|c^2(\theta)\Delta u\|_{\mathbb{E}_0^u} \leq C\|c^2(\theta)\|_{L_\infty((0,T);L_\infty(\Omega))}\|u\|_{\mathbb{E}_1^u}.$$

In summary, the mapping Φ is well-defined and

$$\Phi \in C^1(\mathbb{E}_1^u \times \mathbb{E}_1^\theta \times \mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta; \mathbb{E}_0^u \times \mathbb{E}_0^\theta \times \mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta),$$

by the assumptions on b, c, k and Q.

Let $(h_\ell^*, \theta_0^*) \in \mathbb{Y}_\ell^\theta$ be given and denote by $\theta^* \in \mathbb{E}_1^\theta$ the unique solution of

$$\begin{aligned} \rho_a C_a \theta_t^* - \kappa_a \Delta\theta^* + \rho_b C_b W(\theta^* - \theta_a) &= 0, && \text{in } (0,T)\times\Omega, \\ \mathcal{B}_\ell \theta^* &= h_\ell^*, && \text{in } (0,T)\times\partial\Omega, \\ \theta^*(0) &= \theta_0^*, && \text{in } \Omega, \end{aligned} \tag{3.1}$$

which exists thanks to Lemma 2.1. Then, obviously, $\Phi(0, \theta^*, 0, 0, 0, h_\ell^*, \theta_0^*) = 0$ and

$$D_{(u,\theta)}\Phi(0, \theta^*, 0, 0, 0, h_\ell^*, \theta_0^*)(\hat{u}, \hat{\theta}) = \begin{pmatrix} \hat{u}_{tt} - c^2(\theta^*)\Delta\hat{u} - b(\theta^*)\Delta\hat{u}_t \\ \rho_a C_a \hat{\theta}_t - \kappa_a \Delta\hat{\theta} + \rho_b C_b W \hat{\theta} - Q'(0)\hat{u}_t \\ \mathcal{B}_j \hat{u} \\ \hat{u}(0) \\ \hat{u}_t(0) \\ \mathcal{B}_\ell \hat{\theta} \\ \hat{\theta}(0) \end{pmatrix},$$

where $D_{(u,\theta)}\Phi$ denotes the total derivative of Φ with respect to (u, θ). By Lemma 2.3, the linear operator

$$D_{(u,\theta)}\Phi(0, \theta^*, 0, 0, 0, h_\ell^*, \theta_0^*) : \mathbb{E}_1^u \times \mathbb{E}_1^\theta \to \mathbb{E}_0^u \times \mathbb{E}_0^\theta \times \mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta$$

is invertible. Hence, the implicit function theorem yields some $\delta > 0$ and the existence of a C^1-function

$$\psi : \mathbb{B}_{\mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta}((0, 0, 0, h_\ell^*, \theta_0^*), \delta) \to \mathbb{E}_1^u \times \mathbb{E}_1^\theta$$

such that $(0, \theta^*) = \psi(0, 0, 0, h_\ell^*, \theta_0^*)$ and

$$\Phi(\psi(g_j, u_0, u_1, h_\ell, \theta_0), (g_j, u_0, u_1, h_\ell, \theta_0)) = 0$$

for all

$$(g_j, u_0, u_1, h_\ell, \theta_0) \in \mathbb{B}_{\mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta}((0, 0, 0, h_\ell^*, \theta_0^*), \delta).$$

This completes the proof of Theorem 1.1.

Remark 3.1 (1) It is possible to generalize (1.3) to the case where the nonlinearities c, b or k in (1.3) depend not only on θ but also on $\nabla\theta$. In this case, the condition

$$\frac{2}{r} + \frac{d}{s} < 2$$

in Theorem 1.1 has to be replaced by the stronger condition

$$\frac{2}{r} + \frac{d}{s} < 1,$$

since in this case $B_{sr}^{2-2/r}(\Omega) \hookrightarrow C^1(\overline{\Omega})$. Then all assertions of Theorem 1.1 remain valid provided $c, b, k \in C^1(\mathbb{R} \times \mathbb{R}^d)$.

(2) The nonlinearity $(u^2)_{tt}$ in (1.3) can be replaced by the more general formulation $(f(u)u_t)_t$, where $f \in C^2(\mathbb{R})$ with $f(0) = 0$. This kind of nonlinearity has been derived in [13]. If $f(s) = 2s$, we are in the situation of (1.3).

4 Higher Regularity

We intend to prove that the solution (u, θ) in Theorem 1.1 enjoys more time regularity as soon as $t > 0$.

Let $(u_*, \theta_*) \in \mathbb{E}_1^u \times \mathbb{E}_1^\theta$ be the unique solution to (1.3) with $g_j = h_\ell = 0$ on the interval $[0, T]$ which exists thanks to Theorem 1.1. For fixed $\varepsilon \in (0, 1)$ and $t \in [0, T/(1+\varepsilon)]$, $\lambda \in (1-\varepsilon, 1+\varepsilon)$, we define $u_\lambda(t) := u_*(\lambda t)$ and $\theta_\lambda(t) := \theta_*(\lambda t)$. Then $(u_\lambda, \theta_\lambda)$ is a solution of

$$
\begin{aligned}
\partial_t^2 u_\lambda - \lambda^2 c^2(\theta_\lambda)\Delta u_\lambda - \lambda b(\theta_\lambda)\Delta\partial_t u_\lambda &= k(\theta_\lambda)(u_\lambda^2)_{tt}, && \text{in } (0, T_\varepsilon) \times \Omega,\\
\rho_a C_a \partial_t \theta_\lambda - \lambda\kappa_a \Delta\theta_\lambda + \lambda\rho_b C_b W(\theta_\lambda - \theta_a) &= \lambda Q(\lambda^{-1}\partial_t u_\lambda), && \text{in } (0, T_\varepsilon) \times \Omega,\\
\mathcal{B}_j u_\lambda &= 0, && \text{in } (0, T_\varepsilon) \times \partial\Omega,\\
\mathcal{B}_\ell \theta_\lambda &= 0, && \text{in } (0, T_\varepsilon) \times \partial\Omega,\\
(u_\lambda(0), \partial_t u_\lambda(0)) &= (u_0, \lambda u_1), && \text{in } \Omega,\\
\theta_\lambda(0) &= \theta_0, && \text{in } \Omega,
\end{aligned}
\tag{4.1}
$$

where $T_\varepsilon := T/(1+\varepsilon)$, $(u_0, u_1) \in X_\gamma^u$, $\theta_0 \in X_\gamma^\theta$ with

$$
\mathcal{B}_j u_1 = 0 \text{ if } 1 - j/2 - 1/2q > 1/p,\ \mathcal{B}_j u_0 = 0
$$

and $\mathcal{B}_\ell \theta_0 = 0$ if $1 - \ell/2 - 1/2s > 1/r$. For those fixed initial data, we define a function

$$
\Phi : (1-\varepsilon, 1+\varepsilon) \times \mathbb{E}_1^u \times \mathbb{E}_1^\theta \to \mathbb{E}_0^u \times \mathbb{E}_0^\theta \times \mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta
$$

by

$$
\Phi(\lambda, u, \theta) = \begin{pmatrix}
u_{tt} - \lambda^2 c^2(\theta)\Delta u - \lambda b(\theta)\Delta u_t - k(\theta)(u^2)_{tt}\\
\rho_a C_a \theta_t - \lambda\kappa_a \Delta\theta + \lambda\rho_b C_b W(\theta - \theta_a) - \lambda Q(\lambda^{-1} u_t)\\
\mathcal{B}_j u\\
u(0) - u_0\\
u_t(0) - \lambda u_1\\
\mathcal{B}_\ell \theta\\
\theta(0) - \theta_0
\end{pmatrix}.
$$

Under the conditions of Theorem 1.1, the mapping Φ is C^1. Furthermore, we observe $\Phi(1, u_*, \theta_*) = 0$ and

$$D_{(u,\theta)}\Phi(1, u_*, \theta_*)(\hat{u}, \hat{\theta}) = \begin{pmatrix} \hat{u}_{tt} - c^2(\theta_*)\Delta\hat{u} - b(\theta_*)\Delta\hat{u}_t - A_1(u_*, \theta_*)\hat{\theta} - A_2(u_*, \theta_*)\hat{u} \\ \rho_a C_a \hat{\theta}_t - \kappa_a \Delta\hat{\theta} + \rho_b C_b W \hat{\theta} - Q'((u_*)_t)\hat{u}_t \\ \mathcal{B}_j \hat{u} \\ \hat{u}(0) \\ \hat{u}_t(0) \\ \mathcal{B}_\ell \hat{\theta} \\ \hat{\theta}(0) \end{pmatrix},$$

where

$$A_1(u_*, \theta_*)\hat{\theta} := [2c'(\theta_*)c(\theta_*)\Delta u_* + b'(\theta_*)\Delta(u_*)_t + k'(\theta_*)((u_*)^2)_{tt}]\hat{\theta}$$

and $A_2(u_*, \theta_*)\hat{u} = 2k(\theta_*)(u_*\hat{u})_{tt}$.

A Neumann series argument implies that

$$D_{(u,\theta)}\Phi(1, u_*, \theta_*) : \mathbb{E}_1^u \times \mathbb{E}_1^\theta \to \mathbb{E}_0^u \times \mathbb{E}_0^\theta \times \mathbb{Y}_j^u \times \mathbb{Y}_\ell^\theta$$

is invertible provided that the norm $\|u_*\|_{\mathbb{E}_1^u}$ is sufficiently small, which follows readily by decreasing $\|(u_0, u_1)\|_{X_\gamma^u}$, if necessary. Note that then also $\|\theta_* - \theta^*\|_{\mathbb{E}_1^\theta}$ is small, where θ^* solves (3.1) with $h_\ell^* = 0$ and $\theta_0^* = \theta_0$.

Therefore, by the implicit function theorem, there exists $r \in (0, \varepsilon)$ and a unique mapping $\phi \in C^1((1-r, 1+r); \mathbb{E}_1^u \times \mathbb{E}_1^\theta)$ such that $\Phi(\lambda, \phi(\lambda)) = 0$ for all $\lambda \in (1-r, 1+r)$ and $\phi(1) = (u_*, \theta_*)$. By uniqueness, it holds that $(u_\lambda, \theta_\lambda) = \phi(\lambda)$, hence

$$[\lambda \mapsto (u_\lambda, \theta_\lambda)] \in C^1((1-r, 1+r); \mathbb{E}_1^u \times \mathbb{E}_1^\theta).$$

Since $\partial_\lambda(u_\lambda(t), \theta_\lambda(t))|_{\lambda=1} = t\partial_t(u_*, \theta_*)$, we obtain

$$[t \mapsto t\partial_t(u_*(t), \theta_*(t))] \in \mathbb{E}_1^u \times \mathbb{E}_1^\theta.$$

In particular, this yields

$$u_* \in W_p^3((\tau, T); L_q(\Omega)) \cap W_p^2((\tau, T); W_q^2(\Omega)),$$
$$\theta_* \in W_r^2((\tau, T); L_s(\Omega)) \cap W_r^1((\tau, T); W_s^2(\Omega)),$$

for each $\tau \in (0, T)$, as $\varepsilon \in (0, 1)$ was arbitrary.

Moreover, if all nonlinearities c, b, k and Q are C^m-mappings, where $m \in \mathbb{N}$, then also $\phi \in C^m((1-r, 1+r); \mathbb{E}_1^u \times \mathbb{E}_1^\theta)$ by the implicit function theorem. Inductively, this yields

$$[t \mapsto t^m \partial_t^m(u_*(t), \theta_*(t))] \in \mathbb{E}_1^u \times \mathbb{E}_1^\theta$$

and therefore

$$u_* \in W_p^{m+2}((\tau,T);L_q(\Omega))\cap W_p^{m+1}((\tau,T);W_q^2(\Omega)),$$
$$\theta_* \in W_r^{m+1}((\tau,T);L_s(\Omega))\cap W_r^{m}((\tau,T);W_s^2(\Omega)).$$

We have thus proven the following result.

Theorem 4.1 *Let the conditions of Theorem 1.1 be satisfied. Then the unique solution*

$$u \in W_p^2((0,T);L_q(\Omega))\cap W_p^1((0,T);W_q^2(\Omega))$$
$$\theta \in W_r^1((0,T);L_s(\Omega))\cap L_r((0,T);W_s^2(\Omega))$$

of (1.3) *with* $g_j = h_\ell = 0$ *satisfies*

$$u \in W_p^3((\tau,T);L_q(\Omega))\cap W_p^2((\tau,T);W_q^2(\Omega)),$$
$$\theta \in W_r^2((\tau,T);L_s(\Omega))\cap W_r^1((\tau,T);W_s^2(\Omega)),$$

for each $\tau\in(0,T)$.

If, in addition, c, b, k *and* Q *are* C^m*-mappings, it holds that*

$$u \in W_p^{m+2}((\tau,T);L_q(\Omega))\cap W_p^{m+1}((\tau,T);W_q^2(\Omega)),$$
$$\theta \in W_r^{m+1}((\tau,T);L_s(\Omega))\cap W_r^{m}((\tau,T);W_s^2(\Omega)).$$

for each $\tau\in(0,T)$.

Remark 4.2 Under the conditions of Theorem 4.1 one can also prove *joint time–space* regularity by an application of the parameter trick in [22, Section 9.4]. We refrain from giving the details.

5 Equilibria and Long-Time Behaviour

The equilibria (u_*,θ_*) of (1.3) with $g_j = 0$ and $h_\ell = (1-\ell)\theta_a$ are determined by the equations

$$\begin{aligned} -c^2(\theta)\Delta u_* &= 0, && \text{in } \Omega,\\ -\kappa_a\Delta\theta_* + \rho_b C_b W(\theta_*-\theta_a) &= Q(0), && \text{in } \Omega,\\ \mathcal{B}_j u_* &= 0, && \text{on } \partial\Omega,\\ \mathcal{B}_\ell\theta_* &= (1-\ell)\theta_a, && \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

Let us assume that $c^2(\tau)\ge c_0>0$ for all $\tau\in\mathbb{R}$. It follows that $u_*=0$ if $j=0$ or u_* is an arbitrary constant if $j=1$.

Concerning θ, we observe that if $Q(0)=0$, then $\theta_*=\theta_a$ is the unique solution of $(5.1)_{2,4}$. We will show that in case $j=0$, the equilibrium $(u_*,\theta_*)=(0,\theta_a)$ is exponentially stable (in the sense of Lyapunov). In a first step, we define $\tilde{\theta} := \theta - \theta_a$, so that we may consider the problem

$$\begin{aligned} u_{tt} - \tilde{c}^2(\tilde{\theta})\Delta u - \tilde{b}(\tilde{\theta})\Delta u_t &= \tilde{k}(\tilde{\theta})(u^2)_{tt}, && \text{in } (0,T)\times\Omega,\\ \rho_a C_a \tilde{\theta}_t - \kappa_a \Delta\tilde{\theta} + \rho_b C_b W\tilde{\theta} &= Q(u_t), && \text{in } (0,T)\times\Omega,\\ u &= 0, && \text{in } (0,T)\times\partial\Omega,\\ \mathcal{B}_\ell\tilde{\theta} &= 0, && \text{in } (0,T)\times\partial\Omega,\\ (u(0),u_t(0)) &= (u_0,u_1), && \text{in } \Omega,\\ \tilde{\theta}(0) &= \tilde{\theta}_0, && \text{in } \Omega, \end{aligned} \tag{5.2}$$

where $\tilde{\theta}_0 := \theta_0 - \theta_a$ and $\tilde{f}(\tau) := f(\tau + \theta_a)$ for $f \in \{c, b, k\}$. Observe that

$$\theta_0 \in B^{2-2/r}_{sr}(\Omega) \quad\Longleftrightarrow\quad \tilde{\theta}_0 \in B^{2-2/r}_{sr}(\Omega)$$

as θ_a is constant and Ω is bounded.

We define the function spaces

$$\begin{aligned} &\mathbb{E}^u_0(\mathbb{R}_+) := L_p(\mathbb{R}_+; L_q(\Omega)), \quad \mathbb{E}^{\tilde{\theta}}_0(\mathbb{R}_+) := L_r(\mathbb{R}_+; L_s(\Omega)),\\ &\mathbb{E}^u_1(\mathbb{R}_+) := \{u \in W^2_p(\mathbb{R}_+; L_q(\Omega)) \cap W^1_p(\mathbb{R}_+; W^2_q(\Omega)) : u = 0 \text{ on } \partial\Omega\},\\ &\mathbb{E}^{\tilde{\theta}}_1(\mathbb{R}_+) := \{\tilde{\theta} \in W^1_r(\mathbb{R}_+; L_s(\Omega)) \cap L_r(\mathbb{R}_+; W^2_s(\Omega)) : \mathcal{B}_\ell\tilde{\theta} = 0 \text{ on } \partial\Omega\},\\ &\mathbb{X}^u_\gamma := \{(u_0,u_1) \in W^2_q(\Omega) \times B^{2-2/p}_{qp}(\Omega) : u_1|_{\partial\Omega} = 0 \text{ if } 1 - 1/2q > 1/p,\ u_0|_{\partial\Omega} = 0\}, \end{aligned}$$

and

$$\mathbb{X}^{\tilde{\theta}}_\gamma := \{\tilde{\theta}_0 \in B^{2-2/r}_{sr}(\Omega) : \mathcal{B}_\ell\tilde{\theta}_0 = 0 \text{ on } \partial\Omega \text{ if } 1/2 - 1/2s > 1/r\}.$$

For $\mathbb{F} \in \{\mathbb{E}^u_0, \mathbb{E}^u_1, \mathbb{E}^{\tilde{\theta}}_0, \mathbb{E}^{\tilde{\theta}}_1\}$ we define furthermore

$$v \in e^{-\omega}\mathbb{F}(\mathbb{R}_+) \quad :\Longleftrightarrow \quad [t \mapsto e^{\omega t}v(t)] \in \mathbb{F}(\mathbb{R}_+), \quad \omega \geq 0,$$

and a mapping

$$\Phi : e^{-\omega}\mathbb{E}^u_1(\mathbb{R}_+) \times e^{-\omega}\mathbb{E}^{\tilde{\theta}}_1(\mathbb{R}_+) \times \mathbb{X}^u_\gamma \times \mathbb{X}^{\tilde{\theta}}_\gamma \to e^{-\omega}\mathbb{E}^u_0(\mathbb{R}_+) \times e^{-\omega}\mathbb{E}^{\tilde{\theta}}_0(\mathbb{R}_+) \times \mathbb{X}^u_\gamma \times \mathbb{X}^{\tilde{\theta}}_\gamma$$

by

$$\Phi(u, \tilde{\theta}, u_0, u_1, \tilde{\theta}_0) = \begin{pmatrix} u_{tt} - \tilde{c}^2(\tilde{\theta})\Delta u - \tilde{b}(\tilde{\theta})\Delta u_t - \tilde{k}(\tilde{\theta})(u^2)_{tt} \\ \rho_a C_a \tilde{\theta}_t - \kappa_a \Delta\tilde{\theta} + \rho_b C_b W\tilde{\theta} - Q(u_t) \\ u(0) - u_0 \\ u_t(0) - u_1 \\ \tilde{\theta}(0) - \tilde{\theta}_0 \end{pmatrix}.$$

Note that the mapping Φ is well defined and

$$\Phi \in C^1\left(\mathbb{E}_1^u(\mathbb{R}_+) \times \mathbb{E}_1^{\tilde{\theta}}(\mathbb{R}_+) \times \mathbb{X}_\gamma^u \times \mathbb{X}_\gamma^{\tilde{\theta}}; \mathbb{E}_0^u(\mathbb{R}_+) \times \mathbb{E}_0^{\tilde{\theta}}(\mathbb{R}_+) \times \mathbb{X}_\gamma^u \times \mathbb{X}_\gamma^{\tilde{\theta}}\right)$$

provided that

$$Q \in C^1\left(e^{-\omega}\dot{\mathbb{E}}_1^u(\mathbb{R}_+); e^{-\omega}\mathbb{E}_0^{\tilde{\theta}}(\mathbb{R}_+)\right),$$

where

$$\dot{\mathbb{E}}_1^u(\mathbb{R}_+) := W_p^1(\mathbb{R}_+; L_q(\Omega)) \cap L_p(\mathbb{R}_+; W_q^2(\Omega)).$$

Moreover, $\Phi(0, 0, 0, 0, 0) = 0$ and

$$D_{(u,\tilde{\theta})}\Phi(0, 0, 0, 0, 0)(\hat{u}, \hat{\theta}) = \begin{pmatrix} \hat{u}_{tt} - \tilde{c}^2(0)\Delta\hat{u} - \tilde{b}(0)\Delta\hat{u}_t \\ \rho_a C_a \hat{\theta}_t - \kappa_a \Delta\hat{\theta} + \rho_b C_b W \hat{\theta} - Q'(0)\hat{u}_t \\ \hat{u}(0) \\ \hat{u}_t(0) \\ \hat{\theta}(0) \end{pmatrix}.$$

Let us recall that the Dirichlet- as well as the Neumann–Laplacian Δ_m, $m \in \{D, N\}$ has the property of L_r-maximal regularity in $L_s(\Omega)$, see e.g. [22, Section 6]. Since for any $\alpha > 0$, the spectral bound of the operator $(\Delta_m - \alpha I)$ in $L_s(\Omega)$ is strictly negative, it generates an exponentially stable analytic semigroup in $L_s(\Omega)$ with L_r-maximal regularity.

We note furthermore, that $\tilde{c}(0) = c(\theta_a)$ and $\tilde{b}(0) = b(\theta_a)$ are positive constants. Hence, [15, Theorem 2.5] in combination with the exponential stability of the semigroup, generated by $(\Delta_m - \alpha I)$ in $L_s(\Omega)$, implies that there is some $\omega_0 > 0$ such that for all $\omega \in [0, \omega_0)$, the operator

$$\begin{aligned} D_{(u,\tilde{\theta})}\Phi(0, 0, 0, 0, 0) &: e^{-\omega}\mathbb{E}_1^u(\mathbb{R}_+) \times e^{-\omega}\mathbb{E}_1^{\tilde{\theta}}(\mathbb{R}_+) \\ &\to e^{-\omega}\mathbb{E}_0^u(\mathbb{R}_+) \times e^{-\omega}\mathbb{E}_0^{\tilde{\theta}}(\mathbb{R}_+) \times \mathbb{X}_\gamma^u \times \mathbb{X}_\gamma^{\tilde{\theta}} \end{aligned}$$

is invertible. By the implicit function theorem, there exists some $\delta > 0$ and a mapping

$$\psi \in C^1\left(\mathbb{B}_{\mathbb{X}_\gamma^u \times \mathbb{X}_\gamma^{\tilde{\theta}}}((0, 0, 0), \delta); e^{-\omega}\mathbb{E}_1^u(\mathbb{R}_+) \times e^{-\omega}\mathbb{E}_1^{\tilde{\theta}}(\mathbb{R}_+)\right)$$

such that $\psi(0, 0, 0) = (0, 0)$ and

$$\Phi(\psi(u_0, u_1, \tilde{\theta}_0), (u_0, u_1, \tilde{\theta}_0)) = 0$$

for all $(u_0, u_1, \tilde{\theta}_0) \in \mathbb{B}_{\mathbb{X}_\gamma^u \times \mathbb{X}_\gamma^{\tilde{\theta}}}((0, 0, 0), \delta)$. Since $\psi(0, 0, 0) = 0$ and ψ is continuously differentiable, it follows that for each $r \in (0, \delta)$, there exists a constant $C = C(r) > 0$

such that

$$\|\psi(u_0, u_1, \tilde{\theta}_0)\|_{e^{-\omega}\mathbb{E}_1^u(\mathbb{R}_+)\times e^{-\omega}\mathbb{E}_1^{\tilde{\theta}}(\mathbb{R}_+)} \leq C\|(u_0, u_1, \tilde{\theta}_0)\|_{\mathbb{X}_\gamma^u\times\mathbb{X}_\gamma^{\tilde{\theta}}}$$

holds for all $(u_0, u_1, \tilde{\theta}_0) \in \mathbb{B}_{\mathbb{X}_\gamma^u\times\mathbb{X}_\gamma^{\tilde{\theta}}}((0,0,0), r)$.

For the solution $(u, \tilde{\theta}) = \psi(u_0, u_1, \tilde{\theta}_0)$ of (5.2), this implies the estimate

$$\begin{aligned} e^{\omega t}&\left(\|u(t)\|_{W_q^2(\Omega)} + \|u_t(t)\|_{B_{qp}^{2-2/p}(\Omega)} + \|\tilde{\theta}(t)\|_{B_{sr}^{2-2/r}(\Omega)}\right) \\ &\leq C\left(\|u_0\|_{W_q^2(\Omega)} + \|u_1\|_{B_{qp}^{2-2/p}(\Omega)} + \|\tilde{\theta}_0\|_{B_{sr}^{2-2/r}(\Omega)}\right) \end{aligned} \tag{5.3}$$

for all $t \geq 0$. We summarize these considerations in

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with boundary $\partial\Omega \in C^2$ and suppose that $c, b, k \in C^1(\mathbb{R})$ with $b(\tau) \geq b_0 > 0$ and $c^2(\tau) \geq c_0 > 0$ for all $\tau \in \mathbb{R}$. Assume furthermore that $p, q, r, s \in (1, \infty)$ such that*

$$\frac{d}{q} < 2, \quad \frac{2}{r} + \frac{d}{s} < 2$$

and

$$Q \in C^1\left(e^{-\omega}(W_p^1(\mathbb{R}_+; L_q(\Omega)) \cap L_p(\mathbb{R}_+; W_q^2(\Omega))); e^{-\omega}L_r(\mathbb{R}_+; L_s(\Omega))\right),$$

with $Q(0) = 0$. Assume that $1 - 1/2q \neq 1/p$ and $1 - \ell/2 - 1/2s \neq 1/r$.

Then there are $\delta > 0$ and $\omega_0 > 0$ such that for all $\omega \in [0, \omega_0)$,

$$u_0 \in W_q^2(\Omega), \quad u_1 \in B_{qp}^{2-2/p}(\Omega), \quad \theta_0 \in B_{sr}^{2-2/r}(\Omega),$$

with

- $u_0|_{\partial\Omega} = 0$,
- $u_1|_{\partial\Omega} = 0$ *if* $1 - 1/2q > 1/p$,
- $\mathcal{B}_\ell\theta_0 = (1-\ell)\theta_a$ *on* $\partial\Omega$ *if* $1 - \ell/2 - 1/2s > 1/r$

and

$$\|u_0\|_{W_q^2(\Omega)} + \|u_1\|_{B_{qp}^{2-2/p}(\Omega)} + \|\theta_0 - \theta_a\|_{B_{sr}^{2-2/r}(\Omega)} \leq \delta,$$

there exists a unique global solution (u, θ) of (1.3) with

$$\begin{aligned} u &\in e^{-\omega}(W_p^2(\mathbb{R}_+; L_q(\Omega)) \cap W_p^1(\mathbb{R}_+; W_q^2(\Omega))) \\ \theta - \theta_a &\in e^{-\omega}(W_r^1(\mathbb{R}_+; L_s(\Omega)) \cap L_r(\mathbb{R}_+; W_s^2(\Omega))). \end{aligned}$$

Moreover, there exists a constant $C > 0$ such that the estimate

$$\begin{aligned}\|u(t)\|_{W^2_q(\Omega)} + \|u_t(t)\|_{B^{2-2/p}_{qp}(\Omega)} + \|\theta(t)-\theta_a\|_{B^{2-2/r}_{sr}(\Omega)} \le \\ \le Ce^{-\omega t}\left(\|u_0\|_{W^2_q(\Omega)} + \|u_1\|_{B^{2-2/p}_{qp}(\Omega)} + \|\theta_0-\theta_a\|_{B^{2-2/r}_{sr}(\Omega)}\right)\end{aligned}$$

holds for all $t \ge 0$.

Remark 5.2 In [18], the authors proved Theorem 5.1 for the case $p = q = s = 2$, $d \in \{2, 3\}$ under more restrictive assumptions on the initial data (u_0, u_1, θ_0) as well as on the nonlinearities c, k, Q by means of higher order energy methods/estimates. Furthermore, in [18] it is assumed that the function b is constant. Thus, Theorem 5.1 may be understood as a generalization of the results in [18].

Remark 5.3 In case $j = 1$ (Neumann boundary conditions for u), one has to deal with a family of equilibria (u_*, θ_*), where $u_* = \mathsf{r} \in \mathbb{R}$ is constant and $\theta_* = \theta_a$. In this case, one can use the same strategy as in [25] to show that each equilibrium (r, θ_*), with $\mathsf{r} \in \mathbb{R}$ being close to zero, is *normally stable*. We refrain from giving the details and refer the interested reader to [23] and [25].

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Declarations

Conflict of interest The author declares that they have no conflict of interest.

Appendix A

In this section, we collect the definitions and some properties of the function spaces, being used in this paper.

Definitions

We follow [2, 16, 24]. Let X a Banach space and $\mathcal{S}(\mathbb{R}^d; X)$ the X-valued Schwartz functions. Let $\mathcal{S}'(\mathbb{R}^d; X)$ the X-valued tempered distributions and $\hat{f} := \mathcal{F}f$ the Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d; X)$.

For $m \in \mathbb{N}_0$ and $p \in [1, \infty]$, the **Sobolev space** $W_p^m(\mathbb{R}^d; X)$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^d; X)$ with respect to the norm

$$\| \cdot \|_{W_p^m(\mathbb{R}^d;X)} := \sum_{|\alpha| \le m} \|D^\alpha \cdot \|_{L_p(\mathbb{R}^d;X)}.$$

We note that $W_p^0(\mathbb{R}^d; X) = L_p(\mathbb{R}^d; X)$ is the X-valued **Lebesgue space**.

Choose a sequence $(\varphi_k)_{k\ge 0} \subset \mathcal{S}(\mathbb{R}^d; \mathbb{R})$ with the properties

$$\hat{\varphi}_0 = \hat{\varphi}, \quad \hat{\varphi}_1(\xi) = \hat{\varphi}(\xi/2) - \hat{\varphi}(\xi), \quad \hat{\varphi}_k(\xi) = \hat{\varphi}_1(2^{-k+1}\xi), \quad k \ge 2,$$

and with a generating function $\varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{R})$ satisfying

$$0 \le \hat{\varphi}(\xi) \le 1, \quad \xi \in \mathbb{R}^d, \quad \hat{\varphi}(\xi) = 1 \text{ if } |\xi| \le 1, \quad \hat{\varphi}(\xi) = 0 \text{ if } |\xi| > \frac{3}{2}.$$

For $p, q \in [1, \infty]$, $s \in \mathbb{R}$, the **Besov space** $B_{pq}^s(\mathbb{R}^d; X)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ such that

$$\|f\|_{B_{pq}^s(\mathbb{R}^d;X)} := \left\| \left(2^{ks} (\varphi_k * f)\right)_{k\ge 0} \right\|_{\ell_q(L_p(\mathbb{R}^d;X))} < \infty.$$

For $p \in [1, \infty)$, $q \in [1, \infty]$, $s \in \mathbb{R}$, the **Triebel–Lizorkin space** $F_{pq}^s(\mathbb{R}^d; X)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ such that

$$\|f\|_{F_{pq}^s(\mathbb{R}^d;X)} := \left\| \left(2^{ks} (\varphi_k * f)\right)_{k\ge 0} \right\|_{L_p(\mathbb{R}^d;\ell_q(X))} < \infty.$$

It follows directly from the definitions of B_{pq}^s and F_{pq}^s that

$$B_{pp}^s(\mathbb{R}^d; X) = F_{pp}^s(\mathbb{R}^d; X)$$

for $s \in \mathbb{R}$ and $p \in [1, \infty)$.

For $p \in [1, \infty]$, we define the **Sobolev-Slobodecki spaces** by

$$W_p^s(\mathbb{R}^d; X) := \begin{cases} B_{pp}^s(\mathbb{R}^d; X) & , s \in \mathbb{R}, \ s > 0, \ s \notin \mathbb{N}, \\ W_p^m(\mathbb{R}^d; X) & , s = m \in \mathbb{N}_0. \end{cases}$$

For $s \in \mathbb{R}$ and $p \in (1, \infty)$, the **Bessel potential space** $H_p^s(\mathbb{R}^d; X)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ such that

$$\|f\|_{H_p^s(\mathbb{R}^d;X)} := \|\mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2}\mathcal{F}f]\|_{L_p(\mathbb{R}^d;X)} < \infty.$$

All these function spaces are Banach spaces with respect to the norms defined above, see [2, Chapter VII].

Selected Embeddings

The preceeding definitions imply the elementary embeddings

$$B^{s+\varepsilon}_{pq_0}(\mathbb{R}^d; X) \hookrightarrow B^s_{pq_1}(\mathbb{R}^d; X), \quad p, q_0, q_1 \in [1, \infty],$$

$$F^{s+\varepsilon}_{pq_0}(\mathbb{R}^d; X) \hookrightarrow F^s_{pq_1}(\mathbb{R}^d; X), \quad p \in [1, \infty),\ q_0, q_1 \in [1, \infty], \tag{A.1}$$

and

$$B^s_{pq_0}(\mathbb{R}^d; X) \hookrightarrow B^s_{pq_1}(\mathbb{R}^d; X), \quad p \in [1, \infty],\ 1 \le q_0 \le q_1 \le \infty,$$
$$F^s_{pq_0}(\mathbb{R}^d; X) \hookrightarrow F^s_{pq_1}(\mathbb{R}^d; X), \quad p \in [1, \infty),\ 1 \le q_0 \le q_1 \le \infty,$$

valid for all $s \in \mathbb{R}$ and $\varepsilon > 0$. Furthermore, for all $p \in [1, \infty)$, $q \in [1, \infty]$ and $s \in \mathbb{R}$ it holds that

$$B^s_{p\min\{p,q\}}(\mathbb{R}^d; X) \hookrightarrow F^s_{pq}(\mathbb{R}^d; X) \hookrightarrow B^s_{p\max\{p,q\}}(\mathbb{R}^d; X)$$

see e.g. [16, Proposition 3.11].

For general Banach spaces X, the Sobolev and Bessel potential spaces are related to the B- and F-scale via the following sandwich theorems (see e.g. [2, Chapter VII] or [24, Proposition 2]).

$$A^s_{p1}(\mathbb{R}^d; X) \hookrightarrow H^s_p(\mathbb{R}^d; X) \hookrightarrow A^s_{p\infty}(\mathbb{R}^d; X), \qquad s \in \mathbb{R},\ p \in (1, \infty),$$

$$A^k_{p1}(\mathbb{R}^d; X) \hookrightarrow W^k_p(\mathbb{R}^d; X) \hookrightarrow A^k_{p\infty}(\mathbb{R}^d; X), \qquad k \in \mathbb{N}_0,\ p \in (1, \infty), \tag{A.2}$$

where $A \in \{B, F\}$.

UMD Spaces

It follows from [2, Theorem VII.4.3.2] or [24, Remark 4] that for $k \in \mathbb{N}$, $p \in (1, \infty)$ it holds that

$$W^k_p(\mathbb{R}^d; X) = H^k_p(\mathbb{R}^d; X)$$

if one assumes in addition that X is a *UMD space*, which by definition means that the Hilbert transform is bounded in $L_p(\mathbb{R}; X)$ for some $p \in (1, \infty)$. We list some facts on UMD spaces (cf. [1, Section III.4] or [8, Chapter 4]).

- Every Hilbert space is a UMD space.
- Closed subspaces and the dual of UMD spaces are UMD spaces.

- If X is a UMD space, then $L_p(\mathbb{R}^d; X)$ is a UMD space for $p \in (1, \infty)$.
- If $p, q \in (1, \infty)$, $s \in \mathbb{R}$ and $X = \mathbb{R}$, then the scalar versions H_p^s, B_{pq}^s, F_{pq}^s of the spaces introduced above are UMD spaces.
- Every UMD space is reflexive.

By [24, Remark 5], for $s \in \mathbb{R}$ and $p \in (1, \infty)$, the identity

$$F_{p2}^s(\mathbb{R}^d; X) = H_p^s(\mathbb{R}^d; X)$$

holds if and only if X can be renormed as a Hilbert space. If this is the case, then

$$W_p^k(\mathbb{R}^d; X) = H_p^k(\mathbb{R}^d; X) = F_{p2}^k(\mathbb{R}^d; X)$$

for any $k \in \mathbb{N}_0$, $p \in (1, \infty)$, since every Hilbert space X is of class UMD and in particular it follows that

$$H_2^s(\mathbb{R}^d; X) = F_{22}^s(\mathbb{R}^d; X) = B_{22}^s(\mathbb{R}^d; X) = W_2^s(\mathbb{R}^d; X)$$

for any $s \in \mathbb{R}$ provided X is a Hilbert space.

Restricted Spaces

For open $D \subset \mathbb{R}^d$ and $\mathbb{F} \in \{B_{pq}^s, F_{pq}^s, H_p^s, W_p^m\}$, we define

$$\mathbb{F}(D; X) := \{f \in \mathcal{D}'(D; X) \mid \exists g \in \mathbb{F}(\mathbb{R}^d; X) : g_{|D} = f\}$$

and

$$\|f\|_{\mathbb{F}(D;X)} := \inf\{\|g\|_{\mathbb{F}(\mathbb{R}^d;X)} \mid g_{|D} = f\}.$$

Here, $\mathcal{D}'(D; X)$ is the set of all X-valued distributions on D, see [1, Chapter III].

Finally, if M is an embedded compact hypersurface in $\mathbb{R}^d$ (for example $M = \partial\Omega$ and Ω is a smooth bounded domain in $\mathbb{R}^d$), the spaces $B_{pq}^s(M)$ and $F_{pq}^s(M)$ are defined via local charts, see e.g. [26, Section 3.2.2].

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