OTTO-VON-GUERICKE-UNIVERSITÄT MAGDEBURG

PREPRINT DER FAKULTÄT FÜR MATHEMATIK

REDUCED SET THEORY

Matthias Kunik

Datum der Einreichung: 22.06.2023

REDUCED SET THEORY

MATTHIAS KUNIK

Abstract. We present a new fragment of axiomatic set theory for pure sets and for the iteration of power sets within given transitive sets. It turns out that this formal system admits an interesting hierarchy of models with true membership relation and with only finite or countably infinite ordinals. Still a considerable part of mathematics can be formalized within this system.

Keywords: Formal mathematical systems, axiomatic set theory.

Mathematics Subject Classification: 03F03, 03E30

1. Introduction

In this article we present a generalization of Zermelo-Fraenkel set theory (ZFC), starting with a fragment of axiomatic set theory which we will call RST, for reduced set theory. We are only dealing with sets whose members are sets again. First we will list the principles how we are dealing with sets in RST without using a formal language. These principles will be given a precise form as axioms in Section 4, where we use the formal systems from [1, Sections 3,4]. For this purpose we need some crucial set constructions given in Section 2.

A set U is called *transitive* iff $Y \subseteq U$ for all $Y \in U$. Every set can be extended to a transitive set by Theorem 2.2. By $\mathcal{P}[Y] = \{V : V \subseteq Y\}$ we denote the power set of Y. We say a set U is *subset-friendly* iff

1. $\emptyset \in U$.
2. U is transitive .
3. For all $Y \in U$ we have $\mathcal{P}[Y] \in U$.
4. For all $Y, Z \in U$ we have a transitive set $V \in U$ with $\{Y, Z\} \subseteq V$.

Date: June 22, 2023.

Now we are listing six principles according to which we are dealing with sets in RST. For sets A, B, U, V, Y these are given by

P1. *Principle of extensionality.* If A and B have the same elements, then $A = B$.

P2. *Subset principle.* If $\mathcal{F}$ is a property which may depend on previously given sets, then we can form the subset of A given by $U = \{Y : \text{there holds } Y \in A \text{ and } Y \text{ has property } \mathcal{F}\}$.
Especially the empty set $\emptyset$ can be obtained from this principle.

P3. *Principle of regularity.* If U is not the empty set, then we have $Y \in U$ with $U \cap Y = \emptyset$.

P4. *Principle for pairing of sets.* If A and B are given, then we can find a set U with $\{A, B\} \subseteq U$. We can combine this with (P2) to form $U = \{A, B\}$.

P5. *Principle for subset-friendly sets.* If A is given, then we have a subset-friendly set U with $A \in U$.

P6. *Principle of choice.* If U has only nonempty and pairwise disjoint elements then we can find a set Y with the following property: For every member $A \in U$ there exists exactly one set V with $Y \cap A = \{V\}$.

(P1)-(P6) will be given an exact form in Section 4 with corresponding RST-axioms (A1)-(A6). In Section 2 we will give a motivation for these principles. The novel feature of (P5) is that it contains the set A as parameter. Hence we can use it step by step. We will first provide a subset-friendly set U with $A = \emptyset \in U$. Then we can apply (P5) to $A = U$ again, and so on. The correctness of (P5) is guaranteed by Theorem 2.5.

To specify the property $\mathcal{F}$ in (P2) exactly, we use the formal language of the predicate calculus. The language of set theory in Section 4 consists of a set $X = \{\, \mathbf{x_1}\,, \mathbf{x_2}\,, \mathbf{x_3}\,, \dots \}$ of variables, the equality predicate $\sim$ and a binary predicate $\in$ for membership relation. Using variables $x, y \in X$ we start with atomic formulas $\sim x, y$ and $\in x, y$. Let formulas F, G be constructed previously. Then we can form step by step the connectives

$$\neg F\,, \quad \to FG\,, \quad \& FG\,, \quad \vee FG\,, \quad \leftrightarrow FG$$

and the formulas

$$\forall x\, F\,,\ \exists x\, F\,.$$

The Hilbert-style predicate calculus we use is only slightly different from that of Shoenfield in his textbook [3]. In Section 3 we provide results from [3] concerning conservative extensions of formal mathematical systems. This is applied in Section 4 to enrich the formal language of RST with new symbols like $\emptyset$, $\subseteq$, $\cup$, $\mathcal{P}$, without changing the provability of the original formulas in RST. We present this result in Theorem 4.10. Extensions of RST which we will call subset-friendly theories are naturally included in our approach, see Definition 4.2. In these theories the subset axioms still remain valid with new symbols in the formulas. Theorem 4.10 also says that all axioms for sets in ZFC given in [3, Chapter 9] are already provable in RST, apart from the replacement axioms. In Section 4 we will use a semi formal approach to ensure these results, i.e. we will only employ well known results from elementary proof theory for the formal system RST and its extensions. The key idea behind the axioms (A5) for subset-friendly sets is that we can use them iteratively. In this way we have a sufficiently large set as background available. Within this set we can now perform the operations listed in Remark 4.11. Then we apply the subset axioms directly instead of the replacement axioms.

In Section 5 we study a hierarchy of models for RST. The universe of each model is a subset-friendly set $\mathcal{U}_n$ given in (5.1), and the membership relation in each model is the true membership relation between the individuals in the universe $\mathcal{U}_n$. This is shown in Theorem 5.1. These are only the simplest models. All of them have only finite or countably infinite ordinals, see Theorem 5.4. We will see that a considerable part of mathematics can be formalized within RST. On the other hand, the models for RST without uncountable ordinals from Theorem 5.1 violate Zermelo's well-ordering theorem. Zermelo's well-ordering theorem states that for every set A there is a bijective mapping from an ordinal to A.

At this place it is instructive to compare our approach with the study of a so called Zermelo universe given in Moschovakis [2]. A Zermelo universe is a special model of a seminal axiomatic set theory originally given by Zermelo without using replacement axioms. In our context with pure sets let ω be the set of all finite ordinals and let U be a transitive set. Then U is called a Zermelo universe if $\omega \in U$ and if U is closed under the operations of pairing, union and power sets. This looks similar to our definition of subset-friendly sets U, apart from the fact that for our application of (P5) we can already start with $A = \emptyset$. Based on the formal theory ZFC for pure sets which makes use of the replacement axioms (5.3), the corresponding set constructions given in (2.2), Theorem 2.2, Theorem 2.5 are indeed the same as in [2, 11.10] and [2, 11.15], respectively. For these set constructions we use the replacement axioms only implicitly, without mentioning them. For example, if we want to build up the set $\bigcup_{n=0}^{\infty} A_n$ with given sets A_n, then we will first use the replacement axioms to form the set

$$\{A_0, A_1, A_2, \ldots\}\,,$$

and after that we form the union of this set. However, in RST we do not make use of the replacement axioms. Hence we really need the new Definition 2.1(b) of subset-friendly sets for the axioms of RST in its given form. Our definition guarantees that any given subset-friendly set has the desired closure properties and sufficiently many transitive sets available as elements. Another difference in our approach is this: In our context the subset-friendly sets obtained from subsequent applications of (P5) are only members of a larger universe for RST, and *not* universes of RST-models. We use the set operation $\mathcal{SP}[\,\cdot\,]$ from (2.2) and illustrate this for the construction of the simplest model set $\mathcal{U}_0$ of RST in (5.1), where the subset-friendly sets

$$V_{0,0} \in V_{0,1} \in V_{0,2} \ldots$$

are only special individuals of its universe

$$\mathcal{U}_0 = \bigcup_{k=0}^{\infty} V_{0,k}\,.$$

Then $A = \emptyset$, $U = \mathcal{SP}[\emptyset] = V_{0,0}$ and $A = V_{0,k-1}$, $U = \mathcal{SP}[V_{0,k-1}] = V_{0,k}$ for $k \in \mathbb{N}$ satisfy (P5), respectively.

2. Some notations and constructions with pure sets

Throughout the whole paper we will only consider *pure sets* without urelements. In addition we require for every set $A = A_0$ that it does not allow an infinite descending sequence of sets

$$\ldots \in A_3 \in A_2 \in A_1 \in A_0 \,. \tag{2.1}$$

This is guaranteed by the principle of regularity which states that for every set $U \neq \emptyset$ there exists a set Y with $Y \in U$ and $U \cap Y = \emptyset$. Assume that we have an infinite sequence $A_0, A_1, A_2, \ldots$ satisfying (2.1) and form the set $U = \{A_0, A_1, A_2, \ldots\}$. If we apply the principle of regularity to U, then we can choose $Y = A_k$ with $k \in \mathbb{N}_0$ and $U \cap Y = \emptyset$ and obtain the contradiction $A_{k+1} \in A_k = Y$, $A_{k+1} \in U \cap Y$.
The union of a set A is

$$\cup[A] = \{C : \text{there exists } B \in A \text{ with } C \in B\}\,,$$

and its power set is $\mathcal{P}[A] = \{B : B \subseteq A\}$. More generally we have

$$\cup^0[A] = A \;\text{ and }\; \cup^n\,[A] = \cup[\cup^{n-1}[A]] \;\text{ for all } n \in \mathbb{N}$$

as well as

$$\mathcal{P}^0[A] = A \;\text{ and }\; \mathcal{P}^n[A] = \mathcal{P}[\mathcal{P}^{n-1}[A]] \;\text{ for all } n \in \mathbb{N}\,.$$

For a given set A we also define

$$\mathcal{TC}[A] = \bigcup_{n=0}^{\infty} \cup^n[A] \quad\text{and}\quad \mathcal{SP}[A] = \bigcup_{n=0}^{\infty} \mathcal{P}^n[A]\,. \tag{2.2}$$

If $A \neq \emptyset$, then we have

$$\cap[A] = \{C : C \in B \;\text{ for all }\; B \in A\}\,.$$

Definition 2.1. The following two definitions are crucial for the interpretation of new set axioms which will be introduced in Section 4.

(a) A set U is called *transitive* iff $Y \subseteq U$ for all $Y \in U$.
(b) A set U is called *subset-friendly* iff
   1. $\emptyset \in U$.
   2. U is transitive.
   3. For all $Y \in U$ we have $\mathcal{P}[Y] \in U$.

4. For all $Y, Z \in U$ we have a transitive set $V \in U$ with $\{Y, Z\} \subseteq V$.

Theorem 2.2. *The following statements are equivalent for any set* T.

(a) T *is transitive*,
(b) $T \subseteq \mathcal{P}[T]$,
(c) $\cup[T] \subseteq T$,
(d) $T = \mathcal{TC}[T]$.

For every set A *the so called transitive closure* $\mathcal{TC}[A]$ *of* A *is a transitive set with* $A \subseteq \mathcal{TC}[A]$, *the smallest transitive set* T *with* $A \subseteq T$, *i.e.* $\mathcal{TC}[A] = \cap[\{T : A \subseteq T \text{ and } T \text{ is a transitive set}\}]$.

Proof. T is transitive $\iff$ ($B \in T \implies B \subseteq T$ for all sets B) $\iff$ ($B \in T \implies B \in \mathcal{P}[T]$ for all sets B) $\iff$ $T \subseteq \mathcal{P}[T]$.
Hence (a) and (b) are equivalent. We have for all sets A and T:
$A \in \cup[\mathcal{P}[T]] \iff$ (there exists $B \in \mathcal{P}[T]$ with $A \in B$) $\iff$ (there exists $B \subseteq T$ with $A \in B$) $\iff A \in T$.
We obtain $\cup[\mathcal{P}[T]] = T$ for all sets T. Especially for transitive T we can use (b) and conclude that $\cup[T] \subseteq T$. Now let $\cup[T] \subseteq T$ and assume that T is not transitive. Then there exists A with $A \in T$ which violates $A \subseteq T$. Hence there exists $B \in A$ with $B \notin T$. Here we obtain the contradiction $B \in \cup[T] \subseteq T$. We see that (a)-(c) are equivalent. The condition $A \subseteq \mathcal{TC}[A]$ is clear from the definition of $\mathcal{TC}[A]$. In order to prove that $\mathcal{TC}[A]$ is transitive we assume that $\cup[\mathcal{TC}[A]]$ is not a subset of $\mathcal{TC}[A]$. Then we have $C \in \cup[\mathcal{TC}[A]]$ with $C \notin \mathcal{TC}[A]$, and there exists $B \in \mathcal{TC}[A]$ with $C \in B$ and $B \in \cup^n[A]$ for a certain $n \in \mathbb{N}_0$. We obtain the contradiction $C \in \cup[\cup^n[A]] = \cup^{n+1}[A] \subseteq \mathcal{TC}[A]$ and conclude that $\mathcal{TC}[A]$ is transitive. Finally, if $A \subseteq T$ for a transitive set T, then $\mathcal{TC}[A] \subseteq \mathcal{TC}[T]$ and $\mathcal{TC}[T] = T$ from (c). □

Theorem 2.3. *For all sets* A, B *we have*

(a) $A \in B \implies A \neq B$,
(b) $A \notin \mathcal{TC}[A]$.

Proof. (a) Assume $A \in B$ and $A = B$. Then $A \in A$ contradicts the principle of regularity, if applied to $U = \{A\}$. (b) We see by induction

over $n \in \mathbb{N}_0$ that $B \in \cup^n[A]$ iff for all $k \in \{0, \ldots, n\}$ there are sets A_k with $A_k \in \cup^k[\mathcal{P}[A]]$ and $B \in A_n \in \ldots \in A_0 = A$.
Assume that $A \in \mathcal{TC}[A]$. Then $A \in \cup^n[A]$ for some $n \in \mathbb{N}_0$, and from $A \in A_n \in \ldots \in A_0 = A$ with sets $A_k \in \cup^k[\mathcal{P}[A]]$ for $k \in \{0, \ldots, n\}$ we obtain a forbidden periodic sequence of the form given in (2.1). □

Definition 2.4. For any set A we define

(a) its successor $A^+ = \{A\} \cup A$,

(b) the transitive set $\mathcal{S}_+[A] = \mathcal{TC}[\{A\}] = \{A\} \cup \mathcal{TC}[A]$. Let T be any further set and put $\mathcal{S}_-[T] = \cup[T \setminus \cup[T]]$. Then we see from Theorem 2.3 that $\mathcal{S}_-[\mathcal{S}_+[A]] = A$. Note that $A^+ = \mathcal{S}_+[A]$ if A is transitive, see Theorem 2.2.

(c) With a further set B let $\langle A, B\rangle = \{\{A\}, \{A, B\}\}$ be the ordered pair of A and B.

Theorem 2.5. *Let A be a set and $T = \mathcal{TC}[A]$. Then $\mathcal{SP}[T]$ is a subset-friendly set with $A \in \mathcal{SP}[T]$. Now $\mathcal{SP}[T]$ is the smallest subset-friendly set U with $A \in U$, i.e.*

$$\mathcal{SP}[T] = \cap\,[\,\{U\ :\ A \in U \text{ and } U \text{ is a subset-friendly set}\}\,]\,.$$

Proof. We make use of (2.2) and Theorem 2.2. We see $A \subseteq T = \mathcal{TC}[A]$ and hence $A \in \mathcal{P}[T]$, $A \in \mathcal{SP}[T]$. Next we will check that $\mathcal{SP}[T]$ satisfies the four properties for a subset-friendly set.

1. If $U \neq \emptyset$ is a transitive set then we have a set $B \in U$ with $B \cap U = \emptyset$ from the principle of regularity. We have $B = \emptyset$ since $C \in B$ implies $C \in U$ from the transitivity of U and the contradiction $C \in B \cap U$. Therefore the conditions $U \neq \emptyset$ and $\emptyset \in U$ are equivalent for each transitive set U, especially for U in Definition 2.1(b).
2. We see from Theorem 2.2 by complete induction that $\mathcal{P}^n[T]$ is a transitive set for all $n \in \mathbb{N}_0$ with $\mathcal{P}^n[T] \subseteq \mathcal{P}^{n+1}[T]$. We conclude that $\mathcal{SP}[T]$ is a nonempty transitive set with $\emptyset \in \mathcal{SP}[T]$.
3. Let $Y \in \mathcal{SP}[T]$. Then $Y \in \mathcal{P}^n[T]$ for some $n \in \mathbb{N}_0$ and hence $Y \subseteq \mathcal{P}^n[T]$ as well as $\mathcal{P}[Y] \subseteq \mathcal{P}^{n+1}[T]$ from the transitivity of $\mathcal{P}^n[T]$. We obtain that $\mathcal{P}[Y] \in \mathcal{P}^{n+2}[T] \subseteq \mathcal{SP}[T]$.

4. For $Y, Z \in \mathcal{SP}[T]$ we have $Y, Z \in \mathcal{P}^j[T]$ and $\{Y, Z\} \subseteq \mathcal{P}^j[T]$ for sufficiently large $j \in \mathbb{N}_0$ with the transitive set $\mathcal{P}^j[T] \in \mathcal{SP}[T]$.

We see that $\mathcal{SP}[T]$ is a subset-friendly set with $A \in \mathcal{SP}[T]$. Now let U be a subset-friendly set with $A \in U$. We recall the function $\mathcal{S}_+$ in Definition 2.4. Let $V \in U$ be transitive with $\{A\} \subseteq V$, i.e. $A \in V$, see condition 4 in Definition 2.1(b) with $Y = Z = A$. Then we must have $\mathcal{S}_+[A] = \{A\} \cup T \subseteq V$ from $\{A\} \subseteq V$, $T \subseteq V$, $\mathcal{P}^n[T] \subseteq \mathcal{P}^n[V] \subseteq U$ for all $n \in N_0$, using that U is a subset-friendly set. We conclude that $\mathcal{SP}[T]$ is the smallest subset-friendly set U with $A \in U$. $\square$

Remark 2.6. In this section we have summarized basic properties of subset-friendly sets which serve as a guideline for Sections 4, 5.

(a) Theorem 2.5 guarantees that for every set A there is a subset-friendly set U with $A \in U$. This will be stated as a new set axiom in Section 4.

(b) If U is a subset-friendly set and $V \in U$, then $\mathcal{P}[V] \in U$ and hence $\mathcal{P}[V] \subseteq U$ from the transitivity of U. If moreover $A \subseteq V$, then $A \in \mathcal{P}[V]$ and $A \in U$ from $\mathcal{P}[V] \subseteq U$. We see that $\{Y, Z\} \in U$ for the set $\{Y, Z\}$ in Condition 4 of Definition 2.1(b). The transitivity of V in Condition 4 guarantees in addition that $Y \cup Z \subseteq V$ and hence $Y \cup Z \in U$.

3. Formal mathematical systems

We use notations and results from [1, Sections 3,4] and from Shoenfield's textbook [3]. In [1, Section 3] a recursive system S closely related to Smullyan's elementary formal systems in [4] is embedded into a formal mathematical system M. In [1, (3.13)] we use five rules of inference, namely rules (a)-(e). Rule (e) enables formal induction with respect to the recursively enumerable relations generated by the underlying recursive system S. For our application to axiomatic set theory in Section 4 we do not need the general syntax described in [1, (3.1)-(3.15)] and impose three restrictions on our formal systems.

The first restriction. We put $S = S_\emptyset = [[\,];[\,];[\,]]$ in order to avoid the use of rule (e). Then we can shortly write $M = [A; P; B]$ instead of $M = [S_\emptyset; A; P; B]$ for our formal systems. Here A is the set of constants and function symbols, P the set of predicate symbols and B the given set of axioms.

The second restriction. To each predicate symbol $p \in P$ we assign a fixed arity $n \in \mathbb{N}_0$ which will be given in the informal description of the formal system.

The third restriction. In [1, (3.15)] formal mathematical systems $[M;\mathcal{L}]$ with restrictions in the argument lists of the formulas are introduced. The set of restricted argument lists $\mathcal{L}$ contains the variables and is closed with respect to substitutions. To each constant or function symbol $a \in A$ we assign a fixed arity $n \in \mathbb{N}_0$. For $n = 0$ we say that a is a constant symbol, and for $n \geq 1$ we say that a is an n-ary function symbol. Then $\mathcal{L}$ consists only on terms which are generated by the following rules.

1. We have $x \in \mathcal{L}$ for all variables $x \in X = \{\, \mathbf{x_1}\,,\, \mathbf{x_2}\,,\, \mathbf{x_3}\,,\, \ldots \}$.
2. We have $a \in \mathcal{L}$ for all constant symbols $a \in A$.
3. Let $n > 0$ and let a be an n-ary function symbol in A.
   Then $a(\lambda_1 \ldots \lambda_n) \in \mathcal{L}$ for all terms $\lambda_1, \ldots, \lambda_n \in \mathcal{L}$.

Let $\Pi(M;\mathcal{L})$ be the set of formulas provable in $[M;\mathcal{L}]$ by using only the rules of inference (a),(b),(c),(d). Under these restrictions we make use of the following definition and of three subsequent theorems.

Definition 3.1. Given are two formal mathematical systems $[M;\mathcal{L}]$ and $[M';\mathcal{L}']$ with $M = [A;P;B]$ and $M' = [A';P';B']$.

(a) We say that $[M';\mathcal{L}']$ is an *extension* of $[M;\mathcal{L}]$ if

$$A \subseteq A'\,, \quad P \subseteq P'\,, \quad \mathcal{L} \subseteq \mathcal{L}' \ \text{ and } \ \Pi(M;\mathcal{L}) \subseteq \Pi(M';\mathcal{L}')\,.$$

(b) Let $[M';\mathcal{L}']$ be an extension of $[M;\mathcal{L}]$. If we have in addition

$$F \in \Pi(M';\mathcal{L}') \implies F \in \Pi(M;\mathcal{L})$$

for all formulas F in $[M;\mathcal{L}]$, then $[M';\mathcal{L}']$ is called a *conservative extension* of $[M;\mathcal{L}]$.

The proofs of the following three theorems are analogous to that of Shoenfield's theorems on functional extensions and extensions by definitions in [3, Section 4.5] and [3, Section 4.6].

Theorem 3.2. *Let* $[M;\mathcal{L}]$ *be a formal mathematical system. We write* $M=[A;P;B]$. *We choose a new* n*-ary predicate symbol* $p\notin P$ *with* $n\in\mathbb{N}_0$ *and form* $P'=P\cup\{p\}$. *Let* $x_1,\dots,x_n$ *be distinct variables, and let* G *be a formula in* $[M;\mathcal{L}]$ *in which no variable other than* $x_1,\dots,x_n$ *is free. We put* $B'=B\cup\Big\{\leftrightarrow p\,x_1,\dots,x_n\,G\Big\}$ *and* $M'=[A;P';B']$ *(for* $n=0$ *we have* $p\,x_1,\dots,x_n=p$*). Then* $[M';\mathcal{L}]$ *is a conservative extension of* $[M;\mathcal{L}]$.

Theorem 3.3. *Let* $[M;\mathcal{L}]$ *with* $M=[A;P;B]$ *be a formal mathematical system. Choose a new constant symbol* $c\notin A$, *form* $\tilde{A}=A\cup\{c\}$ *and* $\tilde{\mathcal{L}}=\{\lambda\frac{c}{z}\,:\,\lambda\in\mathcal{L}\ \text{and}\ z\in X\}$. *Let* $u,v\in X$ *be distinct variables and* G *a formula in* $[M;\mathcal{L}]$ *with free*$(G)\subseteq\{u\}$. *Assume that* v *is not occurring bound in* G.

(a) *We put* $B'=B\cup\{G\frac{c}{u}\}$, $M'=[\tilde{A};P;B']$. *If the formula* $\exists uG$ *is provable in* $[M;\mathcal{L}]$, *then* $[M';\tilde{\mathcal{L}}]$ *is a conservative extension of* $[M;\mathcal{L}]$.

(b) *Put* $B''=B\cup\Big\{\leftrightarrow\sim u,c\ G\Big\}$ *and* $M''=[\tilde{A};P;B'']$. *If* $\exists uG$ *and* $\to G\to G\frac{v}{u}\sim u,v$ *are both provable in* $[M;\mathcal{L}]$, *then* $[M'';\tilde{\mathcal{L}}]$ *is a conservative extension of* $[M;\mathcal{L}]$.

Theorem 3.4. *Let* $[M;\mathcal{L}]$ *be a formal mathematical system. We write* $M=[A;P;B]$. *We choose a new* n*-ary function symbol* $f\notin A$ *with* $n\in\mathbb{N}$ *and form* $\tilde{A}=A\cup\{f\}$. *Let* $\tilde{\mathcal{L}}$ *be the smallest set of terms satisfying*

- $\mathcal{L}\subseteq\tilde{\mathcal{L}}$,
- $f(y_1\dots y_n)\in\tilde{\mathcal{L}}$ *for all* $y_1,\dots,y_n\in X$,
- $\lambda\frac{\mu}{z}\in\tilde{\mathcal{L}}$ *for all* $z\in X$ *and for all* $\lambda,\mu\in\tilde{\mathcal{L}}$.

Let $u,v,x_1,\dots,x_n$ *be distinct variables, and let* G *be a formula in* $[M;\mathcal{L}]$ *in which no variable other than* $u,x_1,\dots,x_n$ *is free. Assume that* $v,x_1,\dots,x_n$ *are not occurring bound in* G.

(a) *Put* $B'=B\cup\Big\{G\,\dfrac{f(x_1\dots x_n)}{u}\Big\}$ *and* $M'=[\tilde{A};P;B']$. *If the formula* $\exists uG$ *is provable in* $[M;\mathcal{L}]$, *then* $[M';\tilde{\mathcal{L}}]$ *is a conservative extension of* $[M;\mathcal{L}]$.

(b) *Put* $B'' = B \cup \Big\{ \leftrightarrow\sim u, f(x_1 \ldots x_n)\, G \Big\}$ *and* $M'' = [\tilde{A}; P; B'']$. *If* $\exists u G$ *and* $\to\ G \to\ G^{\,v}_{u} \sim u, v$ *are both provable in* $[M;\mathcal{L}]$, *then* $[M''; \tilde{\mathcal{L}}]$ *is a conservative extension of* $[M;\mathcal{L}]$.

4. The system RST of reduced set theory

Now we put $M^{(0)} = [\,[\ ]; [\in]; B^{(0)}]$ and $\mathcal{L}^{(0)} = X$ with the set of all variables given in [1, (1.1)(c)]. The formal set axioms in $B^{(0)}$ will be given below. In addition to [1, (3.3)(a)] we will only allow prime formulas $\sim r, s$ and $\in r, s$ with variables $r, s \in X$. The 2-ary symbol $\in$ will be used in the formal system RST as well as for the membership relation in our informal english text, which will not lead to confusion. First we define the formal mathematical system RST $= [M^{(0)}; \mathcal{L}^{(0)}]$ with the following axioms for $B^{(0)}$, where $t, u, v, w, x, y, z \in X$ are distinct variables which may vary and which may be chosen arbitrarily.

A1. Axioms of extensionality.

$$\to\ \forall y\ \leftrightarrow \in y, u\ \in y, v\ \ \sim u, v$$

A2. Subset axioms.

$$\exists u\, \forall y\ \leftrightarrow \in y, u\, \&\in y, x\, F$$

with RST-formulas F and $u, x \notin \mathrm{var}(F)$.

A3. Axioms of regularity.

$$\to\ \exists y\ \in y, u\ \exists y\ \&\in y, u\ \neg\, \exists z\, \&\ \in z, u\ \in z, y$$

A4. Axioms for pairing of sets. $\quad \exists u\ \&\in x, u \in y, u$

A5. Axioms for subset-friendly sets.

$$\exists u\ \ \&\,\&\,\&\ \in x, u$$
$$\forall y\ \to \in y, u\ \forall z\ \to \in z, y\ \in z, u$$
$$\forall y\ \to \in y, u\ \exists z\ \&\ \in z, u$$
$$\forall v\ \ \leftrightarrow\ \in v, z\ \forall w\ \to \in w, v \in w, y$$
$$\forall y\ \ \to\ \in y, u\ \forall z\ \ \to\ \in z, u$$
$$\exists v\ \&\ \& \in v, u\ \& \in y, v \in z, v$$
$$\forall w \to \in w, v\ \forall t \to \in t, w \in t, v$$

A6. Axioms of choice.

$$\to\ \forall x\ \to \in x, u\ \exists w \in w, x$$

$$\to\ \forall x \forall y\ \to \in x, u$$

$$\to \in y, u$$

$$\to\ \exists w\, \&\ \in w, x\ \in w, y\ \sim x, y$$

$$\exists y\, \forall x\ \to \in x, u$$

$$\exists v\, \&\& \in v, x \in v, y$$

$$\forall w\ \to \& \in w, x \in w, y\ \sim v, w$$

Remark 4.1. In RST we can apply rule (c) for the collision-free substitution of free variables as well as the replacement of bound variables. Therefore it would be sufficient to choose a single set of distinct variables $t, u, v, w, x, y, z \in X$ for the formulation of the axioms. However, the more general choice of axioms is better suited for our purposes.
We use a Hilbert-style calculus for the formal mathematical systems [M; $\mathcal{L}$]. Let F be a formula in [M; $\mathcal{L}$] and $x \in X$. Then $F \in \Pi(\mathrm{M}; \mathcal{L})$ iff $\forall x\, F \in \Pi(\mathrm{M}; \mathcal{L})$ from [1, (3.11)(a),(3.13)(b)(d)]. Hence we can use open formulas with free variables in our axioms. Now $\Pi(RST)$ denotes the set of formulas provable in RST.

Definition 4.2. Let $[M; \mathcal{L}]$ be an extension of RST (including RST itself). We say that $[M; \mathcal{L}]$ is a subset-friendly theory, or sf-theory for short, if in addition the formulas $\exists u\, \forall y\ \leftrightarrow \in\ y, u\, \&\ \in\ y, x\, F$ are provable in $[M; \mathcal{L}]$ for all collections of distinct variables x, y, u and for all formulas F in $[M; \mathcal{L}]$ with $u, x \notin \mathrm{var}(F)$.

Lemma 4.3. *Let $[M; \mathcal{L}]$ be a sf-theory.*

(a) *With distinct variables $y, u, v \in X$ the formula*

$$\leftrightarrow\ \forall y\ \leftrightarrow \in y, u\ \in y, v\ \sim u, v$$

is provable in $[M; \mathcal{L}]$.

(b) *With distinct variables $x, y, u \in X$ the formula*

$$\exists u\, \forall y\ \leftrightarrow \in y, u\, \& \in y, x\, F$$

is provable in $[M; \mathcal{L}]$ for all $[M; \mathcal{L}]$-formulas F with $u \notin$ free(F) .

Proof. (a) Since $[M;\mathcal{L}]$ is an extension of RST, the extensionality axioms of RST are provable in $[M;\mathcal{L}]$. Now the statement simply follows from the axioms of equality in [1, (3.10)].
(b) Let F be a formula in $[M;\mathcal{L}]$ with $u \notin \mathrm{free}(F)$. Choose a variable $x' \notin \mathrm{var}(F)$ different from x, y, u and put $F' = F\frac{x'}{x}$. We replace all the variables occurring bound in F' by new variables, different from x, y, u and different from all variables in F'. There results a new formula F'', and from [1, (3.17)(b) Theorem] we have $\leftrightarrow F'F'' \in \Pi(M;\mathcal{L})$. Due to [1, (3.16) Lemma, part (a)] we can apply rule (c) to the last formula in order to replace the variable x' by x. Then we have $\leftrightarrow\ F'\dfrac{x}{x'}F''\dfrac{x}{x'}\ = \leftrightarrow\ FF''\dfrac{x}{x'} \in \Pi(M;\mathcal{L})$. Now $u, x \notin \mathrm{var}(F'')$, and $\exists u\,\forall y\ \leftrightarrow \in y, u\,\&\in y, x\,F''$ is provable in $[M;\mathcal{L}]$ due to Definition 4.2. To this formula we can again apply rule (c) in order to replace the variable x' by x. Then

$$\exists u\,\forall y\ \leftrightarrow \in y, u\,\&\in y, x\,F''\frac{x}{x'}$$

and $\leftrightarrow\ FF''\dfrac{x}{x'}$ are both provable in $[M;\mathcal{L}]$. We apply the equivalence theorem [1, (3.17)(a)] and obtain that $\exists u\,\forall y\ \leftrightarrow \in y, u\,\& \in y, x\,F$ is provable in $[M;\mathcal{L}]$. □

Lemma 4.4. *Let F_1, F_2 be formulas in a formal mathematical system $[M;\mathcal{L}]$ and let u, y be distinct variables. If $\rightarrow F_1\,F_2 \in \Pi(M;\mathcal{L})$ then*

(a) $\rightarrow \forall yF_1\ \forall yF_2\ \in \Pi(M;\mathcal{L})$,
(b) $\rightarrow \exists yF_1\ \exists yF_2\ \in \Pi(M;\mathcal{L})$,
(c) $\rightarrow \exists u\forall yF_1\ \exists u\forall yF_2\ \in \Pi(M;\mathcal{L})$.

Proof. The following formulas are provable in $[M;\mathcal{L}]$.

1) $\forall y \rightarrow F_1\,F_2$ from $\rightarrow F_1\,F_2 \in \Pi(M;\mathcal{L})$ and rule (d).
2) $\rightarrow\ \forall y \rightarrow F_1\,F_2\ \ \rightarrow \forall yF_1\ \forall yF_2$ from [1, Theorem (3.18)(11)].
3) $\rightarrow \forall yF_1\ \forall yF_2$ from rule (b) with 1) and 2), which shows (a).
4) $\rightarrow\ \forall y \rightarrow F_1\,F_2\ \ \rightarrow \exists yF_1\ \exists yF_2$ from [1, Theorem (3.18)(12)].
5) $\rightarrow \exists yF_1\ \exists yF_2$ from rule (b) with 1) and 4), which shows (b).
6) $\rightarrow \exists u\forall yF_1\ \exists u\forall yF_2$, if we apply part (b) of the lemma on 3) .

□

Lemma 4.5. *Let $[M;\mathcal{L}]$ be a sf-theory, u, v, y be different variables and F be a formula in $[M;\mathcal{L}]$. Then the following formula is provable in $[M;\mathcal{L}]$:*

$$\begin{aligned}\to\ &\forall y \leftrightarrow \in y,u\ F\\ \to\ &\forall y \leftrightarrow \in y,v\ F\\ &\sim u,v\,.\end{aligned}$$

Proof.

1) Due to the propositional calculus

$$\to\ \&\ \leftrightarrow \in y,u\ F \leftrightarrow \in y,v\ F\ \leftrightarrow \in y,u\ \in y,v$$

is provable in $[M;\mathcal{L}]$.

2) From 1) and Lemma 4.4(a) we obtain that

$$\to\ \forall y\ \&\ \leftrightarrow \in y,u\ F \leftrightarrow \in y,v\ F\ \forall y \leftrightarrow \in y,u\ \in y,v$$

is provable in $[M;\mathcal{L}]$.

3) Using Lemma 4.3(a) and the equivalence theorem [1, (3.17)(a)] we obtain from 2) that

$$\to\ \forall y\ \&\ \leftrightarrow \in y,u\ F \leftrightarrow \in y,v\ F\ \sim u,v$$

is provable in $[M;\mathcal{L}]$.

4) Due to 3) and [1, (3.17)(a),(3.18)(15)] the formula

$$\begin{aligned}\to\ \&\ &\forall y \leftrightarrow \in y,u\ F\\ &\forall y \leftrightarrow \in y,v\ F\\ &\sim u,v\end{aligned}$$

is provable in $[M;\mathcal{L}]$.

Now the statement of the lemma is a consequence of 4), using the propositional calculus. □

Lemma 4.6. *Let $[M;\mathcal{L}]$ be a sf-theory, u, y be different variables and F be a formula in $[M;\mathcal{L}]$ with $u \notin free(F)$. Then the following formula is provable in $[M;\mathcal{L}]$:*

$$\begin{aligned}\leftrightarrow\ &\exists u\,\forall y\ \to\ F\ \in y,u\\ &\exists u\,\forall y\ \leftrightarrow \in y,u\ F\,.\end{aligned}$$

Proof. Let $v \notin \mathrm{var}(F)$, v different from u, y be a new variable. Then the following formulas are provable in $[M; \mathcal{L}]$:

1) The axiom of the propositional calculus

$$\begin{aligned} \to \; & \leftrightarrow \in y,u \;\&\; \in y,vF \\ \to \; & \to F \in y,v \\ & \leftrightarrow \in y,u\; F\,. \end{aligned}$$

2) From 1) and Lemma 4.4(a)

$$\begin{aligned} \to \forall y \; & \leftrightarrow \in y,u \;\&\; \in y,vF \\ \forall y \; & \to \to F \in y,v \\ & \leftrightarrow \in y,u\; F\,. \end{aligned}$$

3) From [1, (3.18)(11)]

$$\begin{aligned} & \to \forall y \to \to F \in y,v \\ & \quad \leftrightarrow \in y,u\; F \\ & \to \; \forall y \to F \in y,v \\ & \quad \forall y \leftrightarrow \in y,u\; F\,. \end{aligned}$$

4) From 2), 3) and the propositional calculus

$$\begin{aligned} & \to \forall y \leftrightarrow \in y,u \;\&\; \in y,v\,F \\ & \to \; \forall y \to F \in y,v \\ & \quad \forall y \leftrightarrow \in y,u\; F\,. \end{aligned}$$

5) From 4) and Lemma 4.4(b)

$$\begin{aligned} & \to \; \exists u \forall y \leftrightarrow \in y,u \;\&\; \in y,v\,F \\ & \quad \exists u \to \; \forall y \to F \in y,v \\ & \qquad \forall y \leftrightarrow \in y,u\; F\,. \end{aligned}$$

6) From Lemma 4.3(b) due to $u \notin \mathrm{free}(F)$

$$\exists u \forall y \leftrightarrow \in y,u \;\&\; \in y,v\,F\,.$$

7) From 5), 6) and rule (b)

$$\exists u \rightarrow \ \forall y \ \rightarrow \ F \ \in y, v$$
$$\forall y \ \leftrightarrow \in y, u \ F \, .$$

8) From [1, (3.18)(21)] and 7) with $J = \rightarrow$, $u \notin \text{free}(F)$

$$\rightarrow \ \forall y \ \rightarrow \ F \ \in y, v$$
$$\exists u \ \forall y \ \leftrightarrow \in y, u \ F \, .$$

9) From 8) and rule (c) with $v \mapsto u$

$$\rightarrow \ \forall y \ \rightarrow \ F \ \in y, u$$
$$\exists u \ \forall y \ \leftrightarrow \in y, u \ F \, ,$$

using that $v \notin \text{var}(F)$.

10) From 9) and rule (d)

$$\forall u \rightarrow \ \forall y \ \rightarrow \ F \ \in y, u$$
$$\exists u \ \forall y \ \leftrightarrow \in y, u \ F \, .$$

11) From 10) and [1, (3.18)(18)]

$$\rightarrow \exists u \forall y \ \rightarrow \ F \ \in y, u$$
$$\exists u \ \forall y \ \leftrightarrow \in y, u \ F \, .$$

12) With $F_1 = \leftrightarrow \in y, u \ F$ and $F_2 = \rightarrow \ F \ \in y, u$ from Lemma 4.4(c)

$$\rightarrow \exists u \ \forall y \ \leftrightarrow \in y, u \ F$$
$$\exists u \forall y \ \rightarrow \ F \ \in y, u \, .$$

From 11) and 12) we obtain the desired result. □

Lemma 4.7. *Let* $[M;\mathcal{L}]$ *with* $M = [A;P;B]$ *be a sf-theory. Let* $[M_c;\mathcal{L}_c]$ *with* $M_c = [A \cup \{c\};P;B]$ *result from* $M = [A;P;B]$ *by adding a new constant symbol* $c \notin A$. *Then* $[M_c;\mathcal{L}_c]$ *is a conservative and subset-friendly extension of* $[M;\mathcal{L}]$.

Proof. It follows from [1, (4.9) Corollary] that $[M_c;\mathcal{L}_c]$ is a conservative extension of $[M;\mathcal{L}]$. Let x, y, u be distinct variables and F be a formula in $[M_c; \mathcal{L}_c]$ with $u, x \notin \text{var}(F)$. Let $v \notin \text{var}(F) \cup \{x, y, u\}$ be a variable and let F' result from F if we replace everywhere in F the constant c by v. Then F' is a formula in $[M; \mathcal{L}]$ with $u, x \notin \text{var}(F')$, and

$$H = \exists u \, \forall y \ \leftrightarrow \in y, u \,\&\in y, x \, F'$$

is provable in the sf-theory $[M; \mathcal{L}]$. We see that

$$H \frac{c}{v} = \exists u \, \forall y \ \leftrightarrow \in y, u \,\&\in y, x \, F$$

is provable in $[M_c; \mathcal{L}_c]$ from the substitution rule (c). □

Theorem 4.8. *Let $[M; \mathcal{L}]$ be a sf-theory and let x, y, z, v, w be distinct variables. Then the formula*

$$\rightarrow \ \exists v \in v, x \ \exists w \, \forall z \ \leftrightarrow \in z, w \ \forall y \ \rightarrow \in y, x \in z, y$$

is provable in $[M; \mathcal{L}]$.

Proof. We add a new constant symbol c to $[M; \mathcal{L}]$ and form $[M_c; \mathcal{L}_c]$ as in Lemma 4.7. Let $[M'; \mathcal{L}']$ result from $[M_c; \mathcal{L}_c]$ by adding the new basis axiom $\exists v \in v, c$ to $[M_c; \mathcal{L}_c]$. We see from Lemma 4.7 that $[M'; \mathcal{L}']$ is a subset-friendly extension of $[M; \mathcal{L}]$. Let $[M''; \mathcal{L}'']$ result from $[M'; \mathcal{L}']$ by adding a new constant symbol d and the new basis axiom $\in d, c$ to $[M'; \mathcal{L}']$. Due to Lemma 4.7 and Theorem 3.3(a) we obtain that $[M''; \mathcal{L}'']$ is a conservative and subset-friendly extension of $[M'; \mathcal{L}']$. Due to Definition 4.2 the following formula is provable in $[M''; \mathcal{L}'']$:

$$\exists w \, \forall z \ \leftrightarrow \in z, w \ \& \in z, d \ \forall y \ \rightarrow \in y, c \in z, y \, .$$

Since $\in d, c$ is provable in $[M''; \mathcal{L}'']$, we conclude that

$$\leftrightarrow \ \& \in z, d \ \forall y \ \rightarrow \in y, c \in z, y \ \ \forall y \ \rightarrow \in y, c \in z, y$$

is provable in $[M''; \mathcal{L}'']$. We see that

$$\exists w \, \forall z \ \leftrightarrow \in z, w \ \forall y \ \rightarrow \in y, c \in z, y$$

is provable in $[M''; \mathcal{L}'']$ and hence in $[M'; \mathcal{L}']$. Due to the deduction theorem [1, (4.3)] the formula

$$\rightarrow \ \exists v \in v, c \ \exists w \, \forall z \ \leftrightarrow \in z, w \ \forall y \ \rightarrow \in y, c \in z, y$$

is provable in $[M_c; \mathcal{L}_c]$. Now [1, (4.9) Corollary] allows the generalization of the constant c in the last formula, which concludes the proof of the theorem. □

Now we extend RST=$[M^{(0)};\mathcal{L}^{(0)}]$ by the following steps.

E1. $[M^{(1)};\mathcal{L}^{(1)}]$ results from $[M^{(0)};\mathcal{L}^{(0)}]$ if we add the constant symbol $\emptyset$ and the following axioms to $[M^{(0)};\mathcal{L}^{(0)}]$:

$$\leftrightarrow\,\sim u,\emptyset\;\;\forall y\,\neg\in y,u\,.$$

Here $u,y\in X$ range over all pairs of distinct variables.

E2. $[M^{(2)};\mathcal{L}^{(2)}]$ results from $[M^{(1)};\mathcal{L}^{(1)}]$ with $\mathcal{L}^{(2)}=\mathcal{L}^{(1)}$ if we add the 2-ary predicate symbol $\subseteq$ and the following axioms to $[M^{(1)};\mathcal{L}^{(1)}]$:

$$\leftrightarrow\,\subseteq u,v\;\;\forall y\;\rightarrow\in y,u\in y,v\,.$$

Here $u,v,y\in X$ range over all triples of distinct variables.

E3. $[M^{(3)};\mathcal{L}^{(3)}]$ results from $[M^{(2)};\mathcal{L}^{(2)}]$ if we add the 1-ary function symbol $\cup$ and the following axioms to $[M^{(2)};\mathcal{L}^{(2)}]$:

$$\leftrightarrow\,\sim u,\cup(x)\;\;\forall z\;\leftrightarrow\in z,u\;\exists y\,\&\in z,y\in y,x$$

for all quadruples $u,x,y,z\in X$ of distinct variables.

E4. $[M^{(4)};\mathcal{L}^{(4)}]$ results from $[M^{(3)};\mathcal{L}^{(3)}]$ if we add the 1-ary function symbol σ and the following axioms to $[M^{(3)};\mathcal{L}^{(3)}]$:

$$\leftrightarrow\,\sim u,\sigma(x)\;\;\forall y\;\leftrightarrow\in y,u\;\;\sim y,x$$

for all triples $u,x,y\in X$ of distinct variables.
Now $\sigma(x)$ denotes the set $\{x\}$.

E5. $[M^{(5)};\mathcal{L}^{(5)}]$ results from $[M^{(4)};\mathcal{L}^{(4)}]$ if we add the 2-ary function symbol σ_2 and the following axioms to $[M^{(4)};\mathcal{L}^{(4)}]$:

$$\leftrightarrow\,\sim u,\sigma_2(x\,y)\;\;\forall z\;\leftrightarrow\in z,u\,\vee\,\sim z,x\sim z,y\,.$$

Here $u,x,y,z\in X$ range over all quadruples of distinct variables. Now $\sigma_2(xy)$ denotes the set $\{x,y\}$. *We introduce the following abbreviation, which is not part of the formal language: We define the successor* $x^+=\cup(\sigma_2(x\,\sigma(x)))$ *of* x.

E6. $[M^{(6)};\mathcal{L}^{(6)}]$ results from $[M^{(5)};\mathcal{L}^{(5)}]$ if we add the 1-ary function symbol $\mathcal{P}$ and the following axioms to $[M^{(5)};\mathcal{L}^{(5)}]$:

$$\leftrightarrow\,\sim z,\mathcal{P}(x)\;\;\forall v\;\leftrightarrow\in v,z\;\forall w\;\rightarrow\in w,v\in w,x\,.$$

Here $v,w,x,z\in X$ range over all quadruples of distinct variables.

We finally obtain the new system $\mathrm{RST}_{ext} = [\, M^{(6)}; \mathcal{L}^{(6)} \,]$ with $M^{(6)} = [\, A^{(6)}; P^{(6)}; B^{(6)} \,]$. The symbols are $A^{(6)} = [\, \emptyset; \cup; \sigma; \sigma_2; \mathcal{P} \,]$ and $P^{(6)} = [\, \in; \subseteq \,]$.

The axioms of $B^{(6)}$ are given in A1-A6 and E1-E6, and $\mathcal{L}^{(6)}$ is the set of terms constructed from the constant $\emptyset$, the 1-ary function symbols $\cup$, σ, $\mathcal{P}$ and the 2-ary function symbol σ_2.

Remark 4.9. Using the formal system RST_{ext} we can rewrite the axioms A5 for subset-friendly sets in the slightly simpler form

$$\begin{aligned} &\exists u \;\; \& \& \& \in x, u \\ &\quad \forall y \to \in y, u \;\subseteq y, u \\ &\quad \forall y \to \in y, u \;\in \mathcal{P}(y), u \\ &\quad \forall y \;\to \in y, u \;\forall z \;\to \in z, u \\ &\qquad \exists v \;\& \;\& \in v, u \;\subseteq \sigma_2(yz), v \\ &\qquad\quad \forall w \to \in w, v \;\subseteq w, v \,. \end{aligned}$$

The purpose of the following theorem is twofold. First it shows that every formula provable in RST_{ext} can be replaced by an equivalent formula in RST which is already provable in RST. Secondly it says that all axioms for sets in ZFC given in [3, Chapter 9] are already provable in RST, apart from the replacement axioms.

Theorem 4.10. *RST_{ext} is a sf-theory and a conservative extension of RST. The following formulas are provable in RST and more generally in every sf-theory $[M; \mathcal{L}]$ for all collections of distinct variables $x, y, z, u, v, w \in X$.*

(a) *Existence of the empty set.*

$$\exists u \, \forall y \, \leftrightarrow \in y, u \, \& \in y, x \neg \in y, x \quad \textit{and} \quad \exists u \, \forall y \neg \in y, u$$

(b) *Existence of unions.* $\exists u \, \forall z \leftrightarrow \in z, u \; \exists y \, \& \in z, y \in y, x$

(c) *Existence of pair sets.*

$$\exists u \, \forall z \leftrightarrow \in z, u \, \vee \, \sim z, x \sim z, y$$

(d) *Existence of power sets.*

$$\exists z \, \forall v \;\; \leftrightarrow \, \in v, z \, \forall w \; \to \in w, v \in w, x$$

(e) *Existence of an inductive set.*

$$\exists x \,\&\, \exists y \;\&\in y,x \,\forall z \,\neg \in z,y$$
$$\forall y \rightarrow \in y,x \;\exists z \,\&\; \in z,x$$
$$\forall v \;\leftrightarrow \in v,z \vee \in v,y \sim v,y$$

Proof. To obtain the first part of the theorem we show for $j = 1, \ldots, 6$ that $[M^{(j)}; \mathcal{L}^{(j)}]$ is a conservative extension of $[M^{(j-1)}; \mathcal{L}^{(j-1)}]$. For this purpose we make use of Theorems 3.2, 3.3(b) and 3.4(b), which allows us to replace formulas with new symbols step by step with equivalent formulas from the previous systems by using the equivalence theorem from [1, (3.17)(a)], the axioms of equality and the substitution rule (c). We use [1, (3.17)(b)] for the replacement of bound variables to obtain the general formulation of the axioms with different collections of variables. Then we see that for $j = 1, \ldots, 6$ each conservative extension $[M^{(j)}; \mathcal{L}^{(j)}]$ is a sf-theory. We will see that the existence conditions in Theorem 3.3(b) and Theorem 3.4(b) for the extensions $[M^{(j)}; \mathcal{L}^{(j)}]$ with a new constant or function symbol are directly provable in RST and hence in $[M^{(j-1)}; \mathcal{L}^{(j-1)}]$. In each case the corresponding uniqueness conditions will automatically result from Lemma 4.5. Then the existence conditions (a)-(d) of the theorem are obtained as a by-product. The proof of part (e) requires a little bit more effort.

E1. Since RST=$[M^{(0)}; \mathcal{L}^{(0)}]$ is a sf-theory, we obtain from Lemma 4.3(b) that $\exists u \,\forall y \;\leftrightarrow \in y,u \,\&\, \in y,x \,\neg \in y,x$ is provable in $[M^{(0)}; \mathcal{L}^{(0)}]$. Using that $\leftrightarrow \leftrightarrow \in y,u \,\&\, \in y,x \,\neg \in y,x \,\neg \in y,u$ is an axiom of the propositional calculus, we obtain from the equivalence theorem [1, (3.17)(a)] that the existence condition $\exists u \,\forall y \,\neg \in y,u$ is provable in $[M^{(0)}; \mathcal{L}^{(0)}]$. But the latter formula is equivalent to the first existence condition. We see that the two formulas in part (a) are provable in RST and RST_{ext}.

E2. This extension has the desired properties due to Theorem 3.2.

E3. We obtain from A5 and Lemma 4.4(b) that

$$\exists u \;\&\; \in x,u \;\forall y \;\rightarrow \in y,u \;\forall z \;\rightarrow \in z,y \;\in z,u$$

is provable in RST. We add a 1-ary function symbol λ to RST and apply Theorem 3.4(a) to the last formula. Now the following formulas are provable in a conservative extension RST$'$ of RST:

1. $\& \in x, \lambda(x)\ \forall y \to \in y, \lambda(x)\ \forall z \to \in z, y\ \in z, \lambda(x)$
2. $\in x, \lambda(x)$
3. $\forall y \to \in y, \lambda(x)\ \forall z \to \in z, y\ \in z, \lambda(x)$
4. $\to \in y, \lambda(x)\ \forall z \to \in z, y\ \in z, \lambda(x)$
5. $\to \in x, \lambda(x)\ \forall z \to \in z, x\ \in z, \lambda(x)$
6. $\forall z \to \in z, x\ \in z, \lambda(x)$
7. $\to \in y, x\ \in y, \lambda(x)$
8. $\to \in y, \lambda(x)\ \to \in z, y\ \in z, \lambda(x)$ (4. and [1, (3.16)(c)])
9. $\to\ \& \in z, y \in y, x\ \in z, \lambda(x)$ (7. and 8.)
10. $\forall z \forall y \to\ \& \in z, y \in y, x\ \in z, \lambda(x)$ (9.)
11. $\forall z \to\ \exists y \,\& \in z, y \in y, x\ \in z, \lambda(x)$ (10. and [1, (3.18)(18)])
12. $\exists u \forall z \to\ \exists y \,\& \in z, y \in y, x\ \in z, u$ (11. and [1, (3.19)])

Since RST$'$ is a conservative extension of RST, the last formula is already provable in RST. Now we can apply Lemma 4.6 and obtain that the formula (b) of the theorem for the existence of unions is provable in RST.

E4. The following formulas are provable in RST:

1. $\exists u \,\& \in x, u\ \in y, u$ from A4
2. $\exists u\ \in x, u$ (from 1.)
3. $\to \in x, u\ \to \sim y, x \in y, u$
4. $\forall y \to \in x, u\ \to \sim y, x \in y, u$
5. $\to \in x, u \,\forall y\ \to \sim y, x \in y, u$ (4. and [1, (3.18)(20)])
6. $\to \exists u\ \in x, u\ \exists u \forall y \to \sim y, x \in y, u$ (5. and Lemma 4.4(b))
7. $\exists u \forall y \to \sim y, x \in y, u$ (2. and 6.)
8. $\exists u \forall y \leftrightarrow \in y, u\ \sim y, x$ (7. and Lemma 4.6).

E5. The following formulas are provable in RST:

1. $\exists u \,\& \in x, u\ \in y, u$ from A4
2. $\to\ \& \in x, u\ \in y, u\ \to\ \vee \sim z, x \sim z, y \in z, u$
3. $\to\ \& \in x, u\ \in y, u \,\forall z \to\ \vee \sim z, x \sim z, y \in z, u$
(2. and [1, (3.18)(20)])

4. $\to \exists u \,\&\, \in x, u \, \in y, u \; \exists u \forall z \to \; \vee \sim z, x \sim z, y \in z, u$
(3. and Lemma 4.4(b))
5. $\exists u \forall z \to \; \vee \sim z, x \sim z, y \in z, u$ (1. and 4.)
6. $\exists u \forall z \leftrightarrow \in z, u \vee \sim z, x \sim z, y$ (5. and Lemma 4.6).
The last formula is formula (c) in the theorem.

E6. We obtain from A5 and Lemma 4.4(b) that

$$\exists u \;\&\; \in x, u \;\forall y \;\to \in y, u \;\exists z \,\&\; \in z, u$$
$$\forall v \;\leftrightarrow \in v, z \,\forall w \;\to \in w, v \in w, y$$

is provable in RST. We add a 1-ary function symbol μ to RST and apply Theorem 3.4(a) to the last formula. The following formulas are provable in a conservative extension RST$''$ of RST:

$$\& \;\in x, \mu(x) \;\forall y \;\to \in y, \mu(x) \;\exists z \,\&\; \in z, \mu(x)$$
$$\forall v \;\leftrightarrow \in v, z \,\forall w \;\to \in w, v \in w, y$$

and from $\in x, \mu(x)$ the two formulas

$$\exists z \,\&\; \in z, \mu(x) \;\forall v \;\leftrightarrow \in v, z \,\forall w \;\to \in w, v \in w, x\,,$$

$$\exists z \,\forall v \;\leftrightarrow \in v, z \,\forall w \;\to \in w, v \in w, x\,.$$

The last formula is already provable in RST. It is formula (d) in the theorem. Hence it remains to show that formula (e) in the theorem is provable in RST. For this purpose we use Remark 4.9 and the formal system RST$_{ext}$. Then we will not explicitely mention the use of axioms (E1)-(E6). Let $[\mathcal{M}', \mathcal{L}']$ result from RST$_{ext}$ by adding a new constant symbol c. Let $[\mathcal{M}'', \mathcal{L}'']$ result from $[\mathcal{M}', \mathcal{L}']$ by adding a new constant symbol d. Finally, let $[\mathcal{M}''', \mathcal{L}''']$ result from $[\mathcal{M}'', \mathcal{L}'']$ by adding the new basis axiom $\&\&\&F_1''F_2''F_3''F_4''$, where the formulas $F_1'', F_2'', F_3'', F_4''$ are given by the following abbreviations for a collection $v, w, y, z \in X$

of distinct variables:

$$F_1'' = \quad \in c, d\,,$$
$$F_2'' = \quad \forall y \ \rightarrow \in y, d \ \subseteq y, d\,,$$
$$F_3'' = \quad \forall y \ \rightarrow \in y, d \ \in \mathcal{P}(y), d\,,$$
$$F_4'' = \quad \forall y \ \rightarrow \in y, d \ \forall z \ \rightarrow \in z, d$$
$$\exists v \ \& \ \& \in v, d \ \subseteq \sigma_2(yz), v$$
$$\forall w \rightarrow \in w, v \ \subseteq w, v\,.$$

Then the following formulas are provable in $[\mathcal{M}''', \mathcal{L}''']$:

S$_1$. $\in c, d\,,$
S$_2$. $\rightarrow \in y, d \ \subseteq y, d\,,$
S$_3$. $\rightarrow \in y, d \ \in \mathcal{P}(y), d\,,$
S$_4$.

$$\rightarrow \in y, d \ \rightarrow \in z, d \ \exists v \ \& \ \& \in v, d$$
$$\subseteq \sigma_2(yz), v \ \forall w \rightarrow \in w, v \ \subseteq w, v\,.$$

With given new distinct variables t, y' the following formulas are also provable in $[\mathcal{M}''', \mathcal{L}''']$:

S$_5$. $\rightarrow \subseteq y', y \ \in y', \mathcal{P}(y)\,,$
S$_6$. $\rightarrow \in \mathcal{P}(y), d \ \subseteq \mathcal{P}(y), d$ (from S$_2$) ,
S$_7$. $\rightarrow \subseteq y', y \ \rightarrow \in y, d \in y', d$ (from S$_3$, S$_5$ and S$_6$) ,
S$_8$. $\forall t \ \rightarrow \subseteq \sigma_2(yz), t \ \rightarrow \in t, d \in \sigma_2(yz), d$ (from S$_7$) ,
S$_9$. $\rightarrow \in y, d \ \rightarrow \in z, d \ \in \sigma_2(yz), d$ (from S$_4$ and S$_8$) ,
S$_{10}$.

$$\rightarrow \in y, d \ \rightarrow \in z, d \ \exists v \ \& \ \& \in v, d$$
$$\subseteq \cup(\sigma_2(yz)), v \ \forall w \rightarrow \in w, v \ \subseteq w, v$$

(from S$_4$).
S$_{11}$. $\forall t \ \rightarrow \subseteq \cup(\sigma_2(yz)), t \ \rightarrow \in t, d \in \cup(\sigma_2(yz)), d$ (from S$_7$) ,
S$_{12}$. $\rightarrow \in y, d \ \rightarrow \in z, d \ \in \cup(\sigma_2(yz)), d$ (from S$_{10}$ and S$_{11}$) ,
S$_{13}$. $\forall y \ \rightarrow \in y, d \ \forall z \ \rightarrow \in z, d \ \in \sigma_2(yz), d$ (from S$_9$) ,
S$_{14}$. $\forall y \ \rightarrow \in y, d \ \forall z \ \rightarrow \in z, d \ \in \cup(\sigma_2(yz)), d$ (from S$_{12}$) .
S$_{15}$. $\forall y' \, \forall y \ \rightarrow \subseteq y', y \ \rightarrow \in y, d \in y', d$ (from S$_7$) .

For $j = 1, \dots, 15$ we denote the formula in S_j by G''_j, and we form G'_j from G''_j by replacing everywhere in G''_j the constant d with a new variable u. For $k = 1, \dots, 4$ let F'_k result from F''_k if we replace everywhere in F''_k the constant d by u. We obtain from the deduction theorem [1, (4.3)] that the formula

$$\rightarrow\ \&\&\&F''_1 F''_2 F''_3 F''_4\ \&\&G''_{13} G''_{14} G''_{15}$$

is provable in $[\mathcal{M}'', \mathcal{L}'']$, and from the generalization of the constant symbols d with the variable u that the formula

$$\rightarrow\ \&\&\&F'_1 F'_2 F'_3 F'_4\ \&\&G'_{13} G'_{14} G'_{15}$$

is provable in $[\mathcal{M}', \mathcal{L}']$. We form G_j from G'_j by replacing everywhere in G'_j the constant c with a new variable x. Let F_k result from F'_k if we replace everywhere in F'_k the constant c with x. Here it is only affecting F'_1. From the generalization of the constant c we see that the formulas

$$\rightarrow\ \&\&\&F_1 F_2 F_3 F_4\ \&\&G_{13} G_{14} G_{15}$$

and

$$\forall u\ \rightarrow\ \&\&\&F_1 F_2 F_3 F_4\ \&\&G_{13} G_{14} G_{15}$$

are provable in RST_{ext}. We have

$$\begin{aligned} F_1 = &\ \in x, u\,, \\ F_2 = &\ \ \forall y\ \rightarrow \in y, u\ \subseteq y, u\,, \\ F_3 = &\ \ \forall y\ \rightarrow \in y, u\ \in \mathcal{P}(y), u\,, \\ F_4 = &\ \ \forall y\ \rightarrow \in y, u\ \forall z\ \rightarrow \in z, u \\ &\qquad \exists v\ \&\ \& \in v, u\ \subseteq \sigma_2(yz), v \\ &\qquad\quad \forall w \rightarrow \in w, v\ \subseteq w, v \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} G_{13} &= \forall y\ \rightarrow \in y, u\ \forall z\ \rightarrow \in z, u\ \in \sigma_2(yz), u\,, \\ G_{14} &= \forall y\ \rightarrow \in y, u\ \forall z\ \rightarrow \in z, u\ \in \cup(\sigma_2(yz)), u\,, \\ G_{15} &= \forall y'\, \forall y\ \rightarrow \subseteq y', y\ \rightarrow \in y, u \in y', u\,. \end{aligned} \tag{4.2}$$

We see that $\exists\, u\, \&\&\&F_1 F_2 F_3 F_4$ is the formula in Remark 4.9 and that

$$\exists\, u\, \&\&\&F_1 F_2 F_3 F_4 \text{ and } \forall u\ \rightarrow\ \&\&\&F_1 F_2 F_3 F_4\ \&\&G_{13} G_{14} G_{15} \tag{4.3}$$

are both provable in RST_{ext}. Let F and G be formulas in RST_{ext}, and assume that $\exists u\, F$ as well as $\forall u \to FG$ are both provable in RST_{ext}. Then the following formulas are provable in RST_{ext} as well: $\to FG$, $\to F\&FG$ and $\to \exists u\, F\ \exists u\,\&FG$ from Lemma 4.4(b). We obtain that

$$\text{(4.4)} \qquad \begin{cases} \exists u\, F \in \Pi(\mathrm{RST}_{ext}) \quad \text{and} \quad \forall u \to FG \in \Pi(\mathrm{RST}_{ext}) \\ \qquad\qquad \Longrightarrow \exists u\,\&FG \in \Pi(\mathrm{RST}_{ext})\,. \end{cases}$$

We can apply (4.4) to (4.3) to strengthen the existence condition for the formula in Remark 4.9. If we put $x = \emptyset$ in the formulas (4.3), then we obtain the existence of an inductive set from the formal definition of the successor $y^+ = \cup(\sigma_2(y\,\sigma(y))) = \cup(\sigma_2(y\,\sigma_2(yy)))$. □

Remark 4.11. We can also introduce the following abbreviations, which are not part of the formal language: For $n \geq 2$ we put

$$\sigma(x_1 \dots x_n) = \cup(\sigma_2(\sigma(x_1 \dots x_{n-1})\sigma(x_n)))\,.$$

Then $\sigma(x_1 \dots x_n)$ denotes the set $\{x_1, \dots, x_n\}$ and $\cup(\sigma(x_1 \dots x_n))$ the set $x_1 \cup \dots \cup x_n$ for all $n \in \mathbb{N}$, respectively. If we use $\langle x\, y\rangle$ as abbreviation for the ordered pair $\sigma_2(\sigma(x)\sigma_2(xy)) = \sigma(\sigma(x)\sigma(xy))$, and more generally $\langle x_1 \dots x_n\rangle = \langle\langle x_1 \dots x_{n-1}\rangle x_n\rangle$ for $n \geq 3$, then we can easily form cartesian product sets.

Any subset-friendly set U satisfies the following properties:

- If $A \in U$, then $\cup[A] \in U$ and $\mathcal{P}[A] \in U$,
- If $A \subseteq V$ and $V \in U$, then $A \in U$,
- If $A, B \in U$, then $A \cup B \in U$, $A \cap B \in U$ and $A \setminus B \in U$,
- If $A_1, \dots, A_n \in U$, then $\{A_1, \dots, A_n\} \in U$.
- If $A_1, \dots, A_n \in U$, then $A_1 \times \dots \times A_n \in U$.

That these constructions can be done within a subset-friendly set U is also formally provable in RST. This results from Theorem 4.10 and from the provability of the formulas (4.3) in RST_{ext}, see (4.1), (4.2) and (4.4) in the proof of Theorem 4.10. We also see that Theorem 2.3(a) can be formalized immediately in RST. From Theorem 4.8 and Theorem 4.10 we can also prove in RST that there is a smallest inductive set called ω. Thus RST enables formal induction with respect to ω and the introduction of arithmetic operations for $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. We see that a considerable part of mathematics can be formalized in RST.

5. Models for RST

To obtain models for RST we make free use of the intuitive notion of a set as we did it in Section 2. We also accept the principles of regularity and choice in the informal mathematical argumentation. But all the set constructions we use can be formalized in ZFC. We assume that ZFC is consistent, recall (2.2) and define

$$\begin{cases} V_{0,0} = \mathcal{SP}[\emptyset], \\ V_{n,k} = \mathcal{SP}[V_{n,k-1}] \quad \text{for } n \in \mathbb{N}_0 \text{ and } k \in \mathbb{N}, \\ \mathcal{U}_n = \bigcup_{k=0}^{\infty} V_{n,k} \quad \text{for } n \in \mathbb{N}_0, \\ V_{n,0} = \mathcal{SP}[\mathcal{U}_{n-1}] \quad \text{for } n \in \mathbb{N}. \end{cases} \tag{5.1}$$

It follows from Theorem 2.5 by complete induction that $V_{n,k}$ and $\mathcal{U}_n$ are subset-friendly sets for all $n, k \in \mathbb{N}_0$. Note that

$$\mathcal{U}_0 \in \mathcal{U}_1 \in \mathcal{U}_2 \in \dots \quad \text{and hence} \quad \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots$$

from the transitivity of the sets $\mathcal{U}_n$ for $n \in \mathbb{N}_0$ and the regularity principle.

Theorem 5.1. *For any fixed $n \in \mathbb{N}_0$ the set $\mathcal{U}_n$ is the universe of a model for RST with the individuals $A \in \mathcal{U}_n$ and with the true membership relation between these individuals. We call it the $\mathcal{U}_n$-model for short.*

Proof. The logical axioms [1, (3.9),(3.10),(3.11)] are generally valid and rules [1, (3.13)(a)(b)(c)(d)] correspond to correct method of deduction. Hence it is sufficient to check that axioms (A1)-(A6) are valid in the $\mathcal{U}_n$-model. To each member $A \in \mathcal{U}_n$ we choose exactely one name α_A, put $W_n = \{\alpha_A : A \in \mathcal{U}_n\}$ and add all these constant symbols to RST. We denote the resulting formal mathematical system by RST_n. We define $\mathcal{D} : W_n \mapsto \mathcal{U}_n$ by $\mathcal{D}(\alpha_A) = A$ and extend $\mathcal{D}$ in order to assign a truth value $\top$ or $\perp$ to all closed formulas of RST_n as follows.

- For all sets $A, B \in \mathcal{U}_n$ we put $\mathcal{D}(\sim \alpha_A, \alpha_B) = \top$ iff $A = B$ as well as $\mathcal{D}(\in \alpha_A, \alpha_B) = \top$ iff $A \in B$.
- $\mathcal{D}(\neg F) = \top$ iff $\mathcal{D}(F) = \perp$ for all closed formulas F in RST_n.

- $\mathcal{D}(\to FG) = \top$ iff $\mathcal{D}(F) \implies \mathcal{D}(G)$, similarly for $\leftrightarrow$, & and $\vee$. Here F, G are any closed formulas in RST_n.
- $\mathcal{D}(\forall x F) = \top$ iff $\mathcal{D}(F\frac{\alpha_A}{x}) = \top$ for all $A \in \mathcal{U}_n$. Here F is any formula in RST_n with $\mathrm{free}(F) \subseteq \{x\}$.
- $\mathcal{D}(\exists x F) = \top$ iff there exists $A \in \mathcal{U}_n$ with $\mathcal{D}(F\frac{\alpha_A}{x}) = \top$. Here F is any formula in RST_n with $\mathrm{free}(F) \subseteq \{x\}$.

If F is an open formula in RST_n with $\mathrm{free}(F) \subseteq \{x_1, \ldots, x_n\}$ and $n \in \mathbb{N}$, then we say that F is valid in $\mathcal{D}$ iff $\mathcal{D}(F\frac{\alpha_{A_1}}{x_1} \ldots \frac{\alpha_{A_n}}{x_n}) = \top$ for all $A_1, \ldots, A_n \in \mathcal{U}_n$, see also Remark 4.1. Now we prove that $\mathcal{D}$ is the desired $\mathcal{U}_n$-model.

A1. The extensionality axioms are valid in $\mathcal{D}$ since $\mathcal{U}_n$ is a transitive set: Let $U, V \in \mathcal{U}_n$ be any two sets. Then

$$\mathcal{D}(\to\ \forall y\ \leftrightarrow \in y, \alpha_U\ \in y, \alpha_V\ \ \sim \alpha_U, \alpha_V) = \top$$

because we have for all sets Y that $Y \in U$ is equivalent to $Y \in U \,\&\, Y \in \mathcal{U}_n$ and $Y \in V$ is equivalent to $Y \in V \,\&\, Y \in \mathcal{U}_n$.

A2. Let u, x, y be distinct variables and F be a formula in RST_n with $u, x \notin \mathrm{var}(F)$. We replace all the variables in F other than y by arbitrary constants in W_n and obtain a formula G in RST_n with $u, x \notin \mathrm{var}(G)$ and $\mathrm{free}(G) \subseteq \{y\}$. We have to show for all $A \in \mathcal{U}_n$ that $\mathcal{D}(\exists u \forall y\ \leftrightarrow \in y, u \,\&\, \in y, \alpha_A\, G) = \top$. We define the set $U = \{Y \in A : \mathcal{D}(G\frac{\alpha_Y}{y}) = \top\}$. From $U \subseteq A$ and $A \in \mathcal{U}_n$ with the subset-friendly set $\mathcal{U}_n$ we obtain that $U \in \mathcal{U}_n$, see also Remark 2.6(b). Hence we can form the name $\alpha_U \in W_n$ and obtain that $\mathcal{D}(\forall y\ \leftrightarrow \in y, \alpha_U \,\&\, \in y, \alpha_A\, G) = \top$. Therefore the existence condition $\exists u \forall y\ \leftrightarrow \in y, u \,\&\, \in y, \alpha_A\, G$ is true in $\mathcal{D}$ as well.

A3. Here we prescribe any nonempty set $U \in \mathcal{U}_n$. Then we have $\mathcal{D}(\exists y\ \in y, \alpha_U) = \top$ from the transitivity of $\mathcal{U}_n$. From the regularity principle we have a set Y with $Y \in U$ and $U \cap Y = \emptyset$. From $Y \in U \in \mathcal{U}_n$ with the transitive set $\mathcal{U}_n$ we have $\alpha_Y \in W_n$ and obtain that $\mathcal{D}(\& \in \alpha_Y, \alpha_U\ \neg \exists z \,\&\, \in z, \alpha_U\ \in z, \alpha_Y) = \top$. Now we see

$$\mathcal{D}(\to\ \exists y\ \in y, \alpha_U\, \exists y\ \& \in y, \alpha_U\ \neg \exists z \,\&\, \in z, \alpha_U\ \in z, y) = \top\,.$$

A4. We have to show for all $A, Y \in \mathcal{U}_n$ that

$$\mathcal{D}(\exists u \ \& \in \alpha_A, u \in \alpha_Y, u) = \top\,.$$

We have indices $j, k \in \mathbb{N}_0$ with $A \in V_{n,j}$ and $Y \in V_{n,k}$. For $m = \max(j,k)$ we see $A, Y \in V_{n,m} \subseteq \mathcal{U}_n$. From $U = V_{n,m}$ we can form the name $\alpha_U \in W_n$ of $V_{n,m}$ and conclude that

$$\mathcal{D}(\& \in \alpha_A, \alpha_U \in \alpha_Y, \alpha_U) = \top\,.$$

We see that the desired existence condition is true in $\mathcal{D}$ as well.

A5. For all $A \in \mathcal{U}_n$ we have $A \in V_{n,j} \in \mathcal{U}_n$ for some index $j \in \mathbb{N}_0$ with the subset-friendly set $U = V_{n,j}$. This implies

$$\mathcal{D}(\in \alpha_A, \alpha_U) = \top\,.$$

For all sets $Y \in U$ the conditions $Z \in Y$ and $Z \in Y \cap \mathcal{U}_n$ are equivalent, and therefore

$$\mathcal{D}(\forall y \ \rightarrow \in y, \alpha_U \ \forall z \ \rightarrow \in z, y \ \in z, \alpha_U) = \top\,.$$

Similarly we obtain from the properties 3. and 4. in Definition 2.1(b) for the subset-friendly set $U \in \mathcal{U}_n$ that

$$\begin{aligned}
&\mathcal{D}(\forall y \ \rightarrow \in y, \alpha_U \ \exists z \,\& \ \in z, \alpha_U \\
&\quad \forall v \ \leftrightarrow \in v, z \ \forall w \ \rightarrow \in w, v \in w, y) = \top\,, \\
&\mathcal{D}(\forall y \ \rightarrow \in y, \alpha_U \ \forall z \ \rightarrow \in z, \alpha_U \\
&\quad \exists v \ \& \ \& \in v, \alpha_U \ \& \in y, v \in z, v \\
&\qquad \forall w \rightarrow \in w, v \ \forall t \rightarrow \in t, w \in t, v) = \top\,.
\end{aligned}$$

We see that axioms A5 are true in $\mathcal{D}$.

A6. Let $U \in \mathcal{U}_n$ be a set which has only nonempty and pairwise disjoint elements. Using the transitivity of $\mathcal{U}_n$ we obtain from our assumptions

$$\mathcal{D}(\forall x \ \rightarrow \in x, \alpha_U \ \exists w \in w, x) = \top$$

as well as

$$\begin{aligned}
&\mathcal{D}(\forall x \forall y \ \rightarrow \in x, \alpha_U \ \ \rightarrow \in y, \alpha_U \\
&\qquad\qquad \rightarrow \ \exists w \,\& \ \in w, x \ \in w, y \ \sim x, y) = \top\,.
\end{aligned}$$

From the principle of choice we can find a set Y' such that $Y' \cap A$ has exactely one element $V(A) \in A$ for all $A \in U$. Let

$Y = \{V(A) : A \in U\}$. Then $Y \subseteq \cup[U] \in \mathcal{U}_n$ and hence $Y \in \mathcal{U}_n$ with $Y \cap A = \{V(A)\}$ for all $A \in U$. We see that all sets involved other than Y' are members of $\mathcal{U}_n$ and that

$$\begin{aligned}&\mathcal{D}(\exists y\,\forall x\ \to \in x, \alpha_U\\ &\qquad \exists v\,\&\& \in v, x \in v, y\\ &\qquad\quad \forall w\ \to \& \in w, x \in w, y\ \sim v, w) = \top\,.\end{aligned}$$

Therefore axioms A6 are true in $\mathcal{D}$.

□

Due to Shoenfield [3, Chapter 9.3] we say that a set α is an ordinal if α is transitive and if every member of α is transitive. There one can find the following facts about ordinals which we will use now:

- Members of ordinals are again ordinals.
- If α is an ordinal then it is well-ordered by $\in$, i.e. for $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$ we have either $\beta \in \gamma$ or $\gamma \in \beta$, and we do not have an infinite sequence (2.1) with members $A_0, A_1, A_2, \ldots$ of α.
- Using transitivity we obtain for all *ordinals* α, β that the conditions $\beta \subseteq \alpha$ and $[\beta \in \alpha$ or $\beta = \alpha]$ are equivalent.

Lemma 5.2. *Let T be a transitive set and define its subset*

$$\gamma = \{\, \beta \in T \,:\, \beta \textit{ is an ordinal} \}.$$

Then γ is an ordinal, and we have

$$\gamma^+ = \gamma \cup \{\gamma\} = \{\, \alpha \in \mathcal{P}[T] \,:\, \alpha \textit{ is an ordinal} \}.$$

Proof. We define the set

$$\gamma_* = \{\, \alpha \in \mathcal{P}[T] \,:\, \alpha \text{ is an ordinal}\,\} = \{\, \alpha \subseteq T \,:\, \alpha \text{ is an ordinal}\,\}.$$

1. Let $\vartheta \in \cup[\gamma_*]$. Then we have an ordinal $\eta \in \gamma_*$ with $\vartheta \in \eta$ and $\vartheta \subseteq \eta \subseteq T$ from the transitivity of η and from $\eta \in \gamma_*$. Now ϑ is an ordinal with $\vartheta \subseteq T$, i.e. $\vartheta \in \gamma_*$. We have $\cup[\gamma_*] \subseteq \gamma_*$ and see that γ_* is transitive and hence an ordinal.
2. Let $\vartheta \in \cup[\gamma]$. Then we have an ordinal $\eta \in \gamma$ with $\vartheta \in \eta \in T$ and $\eta \subseteq T$ from the transitivity of T. Now ϑ is an ordinal with $\vartheta \in T$, i.e. $\vartheta \in \gamma$. We have $\cup[\gamma] \subseteq \gamma$ and see that γ is transitive and hence an ordinal.

3. γ^+ is also an ordinal, and we have

$$\begin{aligned}\gamma^+ &= \{\alpha \,:\, \alpha \in \gamma^+\} \\ &= \{\alpha \,:\, \alpha \in \gamma \text{ or } \alpha = \gamma\} = \{\alpha \,:\, \alpha \subseteq \gamma \text{ is an ordinal}\}\,.\end{aligned}$$

The latter condition $\alpha \subseteq \gamma$ only holds for ordinals and hence is equivalent with $\alpha \subseteq T$ due to the definition of γ. We obtain $\gamma^+ = \gamma_*$.

□

Lemma 5.3. *Let T be a transitive set. We define*

$$\gamma = \{\, \beta \in T \,:\, \beta \textit{ is an ordinal}\}$$

and

$$\tilde{\gamma} = \{\, \alpha \in \mathcal{SP}[T] \,:\, \alpha \textit{ is an ordinal}\}.$$

If γ is at most countably infinite, then $\tilde{\gamma}$ is a countably infinite set.

Proof. We obtain from Theorem 2.2 that the sets $\mathcal{P}^n[T]$ with $n \in \mathbb{N}_0$ form an increasing chain

$$T = \mathcal{P}^0[T] \subseteq \mathcal{P}^1[T] \subseteq \mathcal{P}^2[T] \ldots$$

of transitive sets with union $\mathcal{SP}[T]$. It follows from Lemma 5.2 and Theorem 2.3(a) that $\mathcal{P}^{n+1}[T] \cap \tilde{\gamma}$ has exactely one ordinal more as a member than $\mathcal{P}^n[T] \cap \tilde{\gamma}$. To conclude the proof we only have to note that

$$\bigcup_{n=0}^{\infty} (\mathcal{P}^n[T] \cap \tilde{\gamma}) = \tilde{\gamma}\,.$$

□

Theorem 5.4. *For $n \in \mathbb{N}_0$ we recall the model set $\mathcal{U}_n$ in* (5.1)*. Then*

$$\{\, \alpha \in \mathcal{U}_n \,:\, \alpha \textit{ is an ordinal}\} \tag{5.2}$$

is a countably infinite ordinal.

Proof. The set of all finite sets is given by $V_{0,0}$ in (5.1). A countably infinite union of countably infinite sets is countable. Starting with $V_{0,0}$, it follows from Lemma (5.3) by complete induction that the set (5.2) is only countably infinite. Since $\mathcal{U}_n$ is transitive, we see from Lemma 5.2 that (5.2) is an ordinal. □

The RST-models given by (5.1) can only serve as an example. Beside the $\mathcal{U}_n$-models there are various other models for RST with only countable ordinals. Note that there are uncountably many countable ordinals. But we can specify further properties of these models by adding step by step appropriate new symbols and axioms to RST. To study extensions of RST is a quite natural approach since there is no such thing as complete axiomatics for set theory anyway. For this reason we have presented the general frame of sf-theories in Section 4, which also includes the formal system ZFC′. Here ZFC′ results from RST by adding the replacement axioms to RST. We see from Theorem 4.10 and Theorem 2.5 that ZFC and ZFC′ are equivalent formal systems. Due to Shoenfield [3, Chapter 9.1] we have the following replacement axioms for ZFC and ZFC′ with given formulas F in RST:

$$\begin{cases} \rightarrow \forall x \, \exists z \, \forall y \; \leftrightarrow \; F \; \in y, z \\ \quad \exists u \, \forall y \; \rightarrow \; \exists x \; \& \in x, v \, F \in y, u \, . \end{cases} \tag{5.3}$$

Here x, y, z, u, v run through all collections of distinct variables with the restriction that u, v, z are not occurring in F. The symbol $\leftrightarrow$ in formula (5.3) can be replaced by $\rightarrow$ according to Lemma 4.6.

Acknowledgement

I am very thankful to Gerald Warnecke for encouragement and support of my work.

References

[1] Kunik, Matthias, "Formal mathematical systems including a structural induction principle", A revised version of the Preprint Nr. 31/2002, Fakultät für Mathematik, Otto-von-Guericke-Universität Magdeburg. Available online, see arXiv:2005.04951 (2020).

[2] Moschovakis, Yiannis N., "Notes on set theory", Springer (1994).

[3] Shoenfield, Joseph R., "Mathematical logic", Association for symbolic logic (1967).

[4] Smullyan, Raymond M., "Theory of formal systems", *Annals of Math. Stud. No.* **47**, Princeton Univ. Press (1961).

Universität Magdeburg, IAN, Gebäude 02, Universitätsplatz 2, D-39106 Magdeburg, Germany

E-mail address: matthias.kunik@ovgu.de