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of Riemann and von Mangoldt

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# On the formulas for $\pi(x)$ and $\psi(x)$ of Riemann and von Mangoldt

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## Abstract

Using the Mellin transform and the complex exponential integral we derive various representation formulas for the factors of the entire functions in Hadamards product theorem. The application of these results on Riemann's zeta function leads to a new derivation of Riemann's prime number formula for  $\pi(x)$ . We will thereby present a correct version of this formula, which is given in a wrong way in the literature. Using the nontrivial zeros of the  $\zeta$  function we also obtain explicit formulas for regularizations of von Mangoldt's function  $\psi(x)$ . These regularizations are based on cardinal B-splines and Gaussian integration kernels, which are related by the Central Limit Theorem. Our results will then be generalized to a windowed Mellin or Fourier transform with a Gaussian window function.

## 1 Introduction

The purpose of this article is threefold.

First we give a detailed introduction to the theory of the complex exponential integral and derive some related Mellin and Fourier transforms which can be used for the study of the entire functions in Hadamards product theorem. Parts of these results are widely spread in literature or can only be found in mathematical tables of higher functions, see for example the handbook of Abramowitz and Stegun [1]. But the author has also applied these results in [8] to derive new factorization theorems for the Riemann  $\zeta$ -function and to obtain new representations for fundamental functions in number theory.

One of the various representations of the exponential integral is

$$\text{Ei}(z) := \gamma + \log z + \text{Ei}_0(z) \quad (1.1)$$

with Euler's constant  $\gamma = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \log n \right)$  and the entire function

$$\text{Ei}_0(z) := \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!} = \int_0^z \frac{e^t - 1}{t} dt = \int_0^1 \frac{e^{uz} - 1}{u} du. \quad (1.2)$$

In contrast to the entire function  $\text{Ei}_0$ , the functions  $\text{Ei}$  and  $\log$  are only defined on the cut plane  $\mathbb{C}_- := \{z \in \mathbb{C} \mid z \notin (-\infty, 0]\} = \mathbb{C} \setminus (-\infty, 0]$ .

The second purpose is the application of the theory developed here on the Riemann  $\zeta$  function and the primes. The exponential integral rather than the logarithmic integral plays a key role for the correct formulation of Riemann's famous formulas obtained in [11] for the number  $\pi(x)$  of primes less than or equal  $x > 1$ . It was C.F. Gauss who has observed from extensive calculations of prime tables that the logarithmic integral  $\text{Li}(x) := \text{Ei}(\log x)$  is a "very good" approximation for  $\pi(x)$ . And indeed, it was shown in 1896 by Hadamard and independently by de la Vallée-Poussin that for appropriate constant  $c > 0$

$$\text{Li}(x) = \pi(x) + O(xe^{-c\sqrt{\log x}}) \quad \text{for } x \rightarrow \infty. \quad (1.3)$$

In the case that Riemann's hypothesis is true,  $\text{Li}(x) - \pi(x)$  has the much better error bound  $O(x^{1/2+\varepsilon})$  for all  $\varepsilon > 0$ , see Conrey's overview article [5].

From the study of the complex  $\zeta$  function and their zeros Riemann obtained 1859 in [11] the following explicit formula for  $\pi_*(x) := \sum_{k=1}^{\infty} \frac{1}{k} \pi(\sqrt[k]{x})$ , which is valid at each point  $x > 1$  of continuity

$$\pi_*(x) = \text{Li}(x) - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \text{Ei}(\rho \log x) + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log t} - \log 2. \quad (1.4)$$

Due to an incorrect use of the analytic continuation principle this formula is formulated in a wrong way in the literature, where the expressions  $\text{Ei}(\rho \log x)$  are replaced by  $\text{Li}(x^\rho)$ .

From the formula for  $\pi_*(x)$ , which is denoted by  $f(x)$  in Riemann's article [11], one can also obtain explicit formulas for  $\pi(x)$  by using the Möbius inversion formula

$$\pi(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \pi_*(\sqrt[k]{x}). \quad (1.5)$$

There are two major ingredients for the derivation of explicit prime number formulas involving the zeros of the zeta function. The first one is the relation of the  $\zeta$  function to number theory via Euler's product decomposition valid for  $\Re s > 1$ ,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \exp\left(s \int_1^{\infty} \frac{\pi_*(x)}{x^{s+1}} dx\right). \quad (1.6)$$

The second one is the following product decomposition of the  $\zeta$  function, which was proved and generalized by Hadamard,

$$\zeta(s) = \frac{1}{s-1} \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)} \cdot \frac{1}{2} \lim_{T \rightarrow \infty} \prod_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \left(1 - \frac{s}{\rho}\right). \quad (1.7)$$

According to Riemann's approach in [11], the desired explicit representation formulas for the primes result from a comparison of both product decompositions by using appropriate Mellin- or Fourier inversion formulas.

Riemann's formulas for  $\pi_*(x)$  or  $\pi(x)$  are often neglected nowadays, because they turned out to be equivalent to the simpler formula of von Mangoldt (for its derivation see the textbooks of Edwards [6] and Ingham [7]), also valid at each point  $x > 1$  of continuity, namely

$$\psi(x) := \sum_{p^k \leq x} \log p = x - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{x^\rho}{\rho} + \frac{1}{2} \log \frac{x^2}{x^2-1} - \log(2\pi), \quad (1.8)$$

where the sum is performed with respect to all prime powers  $p^k \geq 2$  with an exponent  $k \geq 1$ . Von Mangoldt's derivation of Riemann's formula for  $\pi_*(x)$  can be found in the textbook of Edwards [6], its starting point is the following modification of (1.8) which is valid for  $r > 0$ ,  $r \neq 1$  and for almost all  $x > 1$ ,

$$\int_0^x x^{-r} d\psi(x) = \frac{x^{1-r}}{1-r} - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{x^{\rho-r}}{\rho-r} - \sum_{n=1}^{\infty} \frac{x^{-2n-r}}{-2n-r} - \frac{\zeta'(r)}{\zeta(r)}. \quad (1.9)$$

However, this formula is also difficult to prove. We will present an interesting alternative derivation of (1.4) by reducing it to von Mangoldt's original formula (1.8). For this purpose we prove a representation which is generally useful for the study of Hadamard's product decomposition of certain entire functions with exponential growth like  $e^s(s-1)\zeta(s)$ ,

$$\left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho}\right) = \exp\left(-s^2 \int_1^\infty \frac{\varphi_\rho(x) \cdot \log x - \frac{x^\rho}{\rho}}{x^{s+1}} dx\right) \quad (1.10)$$

with the following abbreviation for  $x > 1$

$$\varphi_\rho(x) := \gamma + \log(-\rho) + \log(\log x) + \text{Ei}_0(\rho \log x). \quad (1.11)$$

Formula (1.10) is valid for  $\Re \rho < \Re s$  and  $\rho \in \mathbb{C} \setminus [0, \infty)$ , where the integral exists in the Lebesgue sense.

Moreover, beside (1.7) and (1.10) we use the following representation, which is valid for  $\Re s > 1$

$$e^s (s-1) \zeta(s) = \exp \left( s^2 \int_1^\infty \frac{(\pi_*(x) - \text{Li}(x)) \cdot \log x - (\psi(x) - x)}{x^{s+1}} dx \right). \quad (1.12)$$

Finally we will also investigate *explicit formulas for regularizations* of von Mangoldt's function  $\psi(x)$  with cardinal B-splines and Gaussian kernels in terms of the nontrivial zeros of the zeta function which are certain counterparts of the von Mangoldt formula (1.8). Cardinal B-splines and Gaussian kernels are important tools in the wavelet-Fourier theory, see the textbook of Chui [4]. More precisely, we consider regularizations of the Lebesgue integrable function  $\eta : (0, \infty) \rightarrow \mathbb{R}$  with

$$\eta(x) := \psi(x) - \chi_{(1, \infty)}(x) \cdot \left( x + \frac{1}{2} \log \frac{x^2}{x^2 - 1} - \log(2\pi) \right), \quad (1.13)$$

where  $\chi_{(1, \infty)}$  is the characteristic function of the interval  $(1, \infty)$ . We show that the following Gaussian mollifier of the function  $\eta$

$$(G_\delta \eta)(x) := \frac{1}{\sqrt{2\pi\delta}} \int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{2\delta}\right) \eta(xe^u) du \quad (1.14)$$

results in a very natural way from the cardinal B-spline mollifiers and the Central Limit Theorem. Then we derive the following explicit formula for the Gaussian mollifier

$$(G_\delta \eta)(x) = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho} \exp\left(\frac{\delta}{2} \rho^2\right) \Phi\left(\frac{\rho\delta + \log x}{\sqrt{\delta}}\right), \quad (1.15)$$

where  $\Phi$  is the complex Gaussian error function

$$\Phi(z) := \int_0^\infty \frac{e^{-\frac{(z-u)^2}{2}}}{\sqrt{2\pi}} du, \quad z \in \mathbb{C}. \quad (1.16)$$

We will derive a very efficient asymptotic relation for (1.15). The representation (1.15) will also be generalized to some kind of windowed Mellin or Fourier transform with the Gaussian window function in (1.14).

## 2 Representation formulas for the zeta function and their logarithmic derivative via the Euler product formula

In the following two sections we collect some well known basic facts about the Riemann zeta function and Hadamard's factorization which are needed in Section 5. For a proof we refer to the textbook of Edwards [6], further references are given below. The Riemann zeta function is given for  $s \in \mathbb{C}$  with  $\Re s > 1$  by the absolutely convergent series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (2.1)$$

### Theorem 2.1

There holds the following Euler product formula for  $\Re s > 1$

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}. \quad (2.2)$$

### Theorem 2.2

For any integer number  $n \geq 1$  we define the von Mangoldt function

$$\Lambda(n) := \begin{cases} \log p & , \text{ for } n = p^m, \ m \geq 1, \ p \text{ prime} \\ 0 & , \text{ otherwise,} \end{cases}$$

and for  $x \geq 1$  the functions

$$\psi(x) := \sum_{n \leq x} \Lambda(n), \quad \pi(x) := \sum_{p \leq x, \ p \text{ prime}} 1, \quad \pi_*(x) := \sum_{1 < n \leq x} \frac{\Lambda(n)}{\log n} = \sum_{n=1}^{\infty} \frac{\pi(\sqrt[n]{x})}{n},$$

where  $\pi(x)$  is the number of primes  $\leq x$ . Then we obtain for  $\Re s > 1$

$$\zeta(s) = \exp\left(\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}\right) = \exp\left(s \int_1^{\infty} \frac{\pi_*(x)}{x^{s+1}} dx\right) \quad (2.3)$$

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \Lambda(n) n^{-s} = s \int_1^{\infty} \frac{\psi(x)}{x^{s+1}} dx. \quad (2.4)$$

For the study of the Riemann  $\zeta$  function some properties of the complex  $\Gamma$  function are needed, which are proven in an elegant way in the textbook of Andrews, Askey and Roy [2]:

**Theorem 2.3** *For all complex numbers  $s \neq 0, -1, -2, \dots$ , the  $\Gamma$  function is defined by*

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{n! n^s}{s(s+1)(s+2)\dots(s+n)}.$$

The  $\Gamma$  function has the following properties:

(a)  $\Gamma(n+1) = n!$  and  $\Gamma(s+1) = s\Gamma(s)$  for  $n \in \mathbb{N}_0$ ,  $s \neq 0, -1, -2, \dots$ .

(b) With Euler's constant  $\gamma := \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \log n \right) = 0.5772156\dots$  we have

$$\frac{1}{s\Gamma(s)} = \frac{1}{\Gamma(s+1)} = e^{\gamma s} \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right) e^{-s/k} = \prod_{k=1}^{\infty} \frac{1 + \frac{s}{k}}{\left(1 + \frac{1}{k}\right)^s}.$$

(c) If  $s \neq 0, -1, -2, \dots$ , then  $-\frac{\Gamma'(s)}{\Gamma(s)} = \gamma + \frac{1}{s} + \sum_{k=1}^{\infty} \left( \frac{1}{s+k} - \frac{1}{k} \right)$ , and especially  $\Gamma'(1) = -\gamma$ .

(d) There holds Euler's reflection formula

$$\frac{1}{\Gamma(1+s)\Gamma(1-s)} = \frac{\sin(\pi s)}{\pi s}.$$

(e) For  $\Re s > 0$

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt.$$

(f) There holds Legendre's duplication formula

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s}{2}\right).$$



The following theorem shows that the Riemann  $\zeta$  function is everywhere defined in the complex plane, except at  $s = 1$ , where  $\zeta(s)$  has a simple pole.

**Theorem 2.4** Analytic continuation of  $\zeta(s)$ , functional equation

(a) For  $\Re s > -1$  the  $\zeta$  function is analytically continued by the expression

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \lim_{N \rightarrow \infty} \int_1^N \frac{x - [x] - \frac{1}{2}}{x^{s+1}} dx.$$

Thus the  $\zeta$  function has a simple pole at  $s = 1$  with residue 1.

(b) For  $-1 < \Re s < 0$  the  $\zeta$  function is given by

$$\zeta(s) = s \lim_{N \rightarrow \infty} \int_0^N \frac{[x] - x + \frac{1}{2}}{x^{s+1}} dx.$$

(c) For  $\Re s < 0$  the continuation of the  $\zeta$  function is given by

$$\zeta(s) = 2(2\pi)^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s).$$

Since the  $\Gamma$ -function has no zeros and only simple poles at  $s = 0, -1, -2, \dots$ , we conclude that the zeta function has the so called trivial zeros at  $s = -2, -4, -6, \dots$

$$\zeta(-2n) = 0 \quad \forall n \in \mathbb{N}.$$

(d) If we define the analytic function  $\xi$  on the entire complex plane by

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

then the functional equation for the Riemann zeta function can be written for all  $s \in \mathbb{C}$  in the more symmetric form  $\xi(s) = \xi(1-s)$ .

For a proof of this theorem see also Titchmarsh [13]. In this textbook there are also several alternative proofs for the functional equation (c) of the Riemann zeta function.

The following theorem is due to Hadamard and shows that  $\zeta(s)$  has no zero on the axis  $\Re s = 1$ .

**Theorem 2.5** Zeros of  $\zeta(s)$  and  $\xi(s)$  and special values at  $s = 0$

- (a) Beside the trivial zeros at  $s = -2, -4, -6, \dots$ ,  $\zeta(s)$  has only zeros in the critical strip  $0 < \Re s < 1$ . There holds  $\zeta(0) = -\frac{1}{2}$ . Here  $\zeta(s)$  is defined for  $s \in \mathbb{C} \setminus \{1\}$ .
- (b) The function  $\xi(s)$  defined in Theorem 2.3 has only zeros in the critical strip  $0 < \Re s < 1$ . It holds  $\xi(0) = \frac{1}{2}$ , and  $\xi(s)$  is defined for all  $s \in \mathbb{C}$ .

### 3 A special version of Hadamard's product representation

In the previous section we have given number theoretic representations of the zeta function and their logarithmic derivative based on Euler's product representation over the prime numbers.

In this section we will recall analytic representations using Hadamard's product theorem and the zeros of the zeta function. Roughly speaking, the comparison of both representations is needed for the prime number formulas of von Mangoldt and Riemann, but also for the new representation formulas obtained in Section 5. We formulate only a special version of Hadamard's product representation for certain entire functions, which is sufficient for our applications.

**Theorem 3.1** Hadamard's product representation, special version

Let be  $f : \mathbb{C} \rightarrow \mathbb{C}$  be analytic in the entire complex plane with  $f(0) \neq 0$  and zeros  $\rho_1, \rho_2, \rho_3, \dots$ , such that each zero of order  $m$  is included  $m$  times in this sequence.

There may be a constant  $0 < \alpha < 2$  such that the growing rate of  $f$  is bounded by

$$(*) \quad |f(s)| \leq \exp(|s|^\alpha) \quad \text{for } |s| \gg 1.$$

Then there are constants  $a, b \in \mathbb{C}$  with  $f(s) = e^{as+b} \prod_{\rho_k} \left(1 - \frac{s}{\rho_k}\right) e^{s/\rho_k}$

such that  $\sum_{\rho_k} \frac{1}{|\rho_k|^2} < \infty$ .

The following three examples will be used for our application on the prime number formulas.

**Example 1:** With the analytic continuation  $\lim_{\substack{s \rightarrow 0 \\ s \neq 0}} \frac{\sin(\pi s)}{\pi s} = 1$  there holds the following product representation for all  $s \in \mathbb{C}$

$$f(s) := \frac{\sin(\pi s)}{\pi s} = \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{s}{k}\right) e^{s/k} = \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{k^2}\right).$$

Here inequality (\*) is valid for any  $\alpha > 1$ .

**Example 2:** We define  $1/\infty := 0$  and  $1/0 := \infty$  to formulate for all  $s \in \mathbb{C}$  the product representation of the  $\Gamma$  function in Theorem 2.3(a) as

$$f(s) := \frac{1}{s\Gamma(s)} = \frac{1}{\Gamma(s+1)} = e^{\gamma s} \prod_{k=1}^{\infty} \left(1 + \frac{s}{k}\right) e^{-s/k}.$$

In this case (\*) is satisfied for all  $\alpha > 1$ .

**Example 3:** For all  $s \in \mathbb{C}$  we have

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \lim_{T \rightarrow \infty} \prod_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \left(1 - \frac{s}{\rho}\right),$$

in short notation written as  $\xi(s) = \frac{1}{2} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$ , where (\*) is again satisfied for all  $\alpha > 1$ . Recall that the value  $\xi(0) = \frac{1}{2}$  was already evaluated in Theorem 2.5(b).

For a special derivation of the product representation of  $\xi(s)$  see Edwards [6]. It was already known to Riemann [11].

## 4 The exponential integral and representations for $1/(s-1)$ and $(1 - \frac{s}{\rho})$

In Hadamard's product formula Theorem 3.1 for the entire function  $f$  under consideration there occur the factors  $(1 - \frac{s}{\rho})$  with the roots  $\rho$  of  $f$ . Recall the application of this theorem on the  $\xi$ -function in Example 3 of the last section and the formula (2.3) for  $\zeta(s)$  in Theorem 2.2, namely

$$\zeta(s) = \exp\left(s \int_1^{\infty} \frac{\pi_*(x)}{x^{s+1}} dx\right), \quad \Re s > 1. \quad (4.1)$$

In order to obtain Riemann's explicit formula for  $\pi_*(x)$  and  $\pi(x)$ , one compares the product formula for  $\xi(s)$  with the formula (4.1) for  $\zeta(s)$ . Therefore it is quite natural to look first for representations of the form

$$1 - \frac{s}{\rho} = \exp\left(-s \int_1^{\infty} \frac{\varphi_{\rho}(x)}{x^{s+1}} dx\right), \quad \Re s > 1, \quad (4.2)$$

with appropriate complex, but constant values  $\rho$ . This will be done next, i.e. for some given values of  $\rho$  a function  $\varphi_{\rho} : (1, \infty) \rightarrow \mathbb{C}$  will be calculated such that (4.2) is satisfied. By the way, we will also obtain a useful representation of  $1/(s-1)$  in order to handle the pole of the zeta function.

We first define the exponential integral

$$\text{Ei}(z) := \gamma + \log z + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}. \quad (4.3)$$

We only use the logarithmic main branch  $\log : \mathbb{C}_- \rightarrow \mathbb{C}$  on the cut plane

$$\mathbb{C}_- := \{z \in \mathbb{C} \mid z \notin (-\infty, 0]\} = \mathbb{C} \setminus (-\infty, 0]. \quad (4.4)$$

Thus we conclude that  $\text{Ei}(z)$  is also defined for  $z \in \mathbb{C}_-$ .

We also need the regular part of the exponential integral, the entire function

$$\text{Ei}_0(z) := \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!} = \int_0^z \frac{e^t - 1}{t} dt = \int_0^1 \frac{e^{uz} - 1}{u} du. \quad (4.5)$$

These representations of  $\text{Ei}_0(z)$  are defined for all  $z \in \mathbb{C}$ .

In order to calculate the exponential integral we generalize  $\text{Ei}_0$  and define for  $n \in \mathbb{N}_0$  the entire functions  $\text{Ei}_n : \mathbb{C} \rightarrow \mathbb{C}$  by

$$\text{Ei}_n(z) := \int_0^1 (1-u)^n \cdot \frac{e^{uz} - 1}{u} du \quad (4.6)$$

as well as the entire functions  $Q_n : \mathbb{C} \rightarrow \mathbb{C}$  with

$$Q_n(z) := \int_0^1 (1-u)^n \cdot e^{uz} du = e^z \int_0^1 u^n \cdot e^{-uz} du. \quad (4.7)$$

We obtain by induction for all  $n \in \mathbb{N}_0$  and  $z \in \mathbb{C} \setminus \{0\}$  the important recurrence equations

$$Q_0(z) = \frac{e^z - 1}{z}, \quad Q_{n+1}(z) = \frac{n+1}{z} Q_n(z) - \frac{1}{z}. \quad (4.8)$$

This implies

$$Q_n(z) = \frac{n!}{z^{n+1}} \left( e^z - \sum_{k=0}^n \frac{z^k}{k!} \right). \quad (4.9)$$

The next theorem states that the functions  $Q_n$  and  $Ei_n$  enable an efficient calculation of  $Ei_0(z)$  for any  $z \in \mathbb{C}$  by using a combination of the recurrence equations (4.8) with Taylor expansions of  $Ei_n$ .

#### Theorem 4.1

(a) For any  $z \in \mathbb{C}$  and any  $n \in \mathbb{N}_0$  we have

$$Ei_0(z) = \sum_{m=0}^{n-1} \left( Q_m(z) - \frac{1}{m+1} \right) + Ei_n(z).$$

(b) For any  $z \in \mathbb{C}$  and any  $n \in \mathbb{N}_0$  there holds the Taylor expansion

$$Ei_n(z) = n! \sum_{k=1}^{\infty} \frac{z^k}{k \cdot (n+k)!} = \sum_{k=1}^{\infty} \frac{1}{k} \prod_{\nu=1}^k \frac{z}{n+\nu}.$$

Remark: The recurrence equations (4.8) can be used in combination with the Taylor expansion of  $Ei_n$  by choosing  $n := \lfloor |z| \rfloor$  for given  $z \in \mathbb{C}$  in Theorem 4.1 to calculate  $Ei_0(z)$ .

Proof: (a) From (4.7) we immediately obtain for any  $z \in \mathbb{C}$  and any  $m \in \mathbb{N}_0$  that

$$\int_0^1 (1-u)^m \cdot (e^{uz} - 1) du = Q_m(z) - \frac{1}{m+1}. \quad (4.10)$$

For  $0 < u < 1$  we evaluate the finite geometric series

$$\sum_{m=0}^{n-1} (1-u)^m = \frac{1 - (1-u)^n}{u} \quad (4.11)$$

and conclude

$$\sum_{m=0}^{n-1} \int_0^1 (1-u)^m (e^{uz} - 1) du = \int_0^1 \frac{1 - (1-u)^n}{u} (e^{uz} - 1) du \quad (4.12)$$

in order to establish Theorem 4.1(a) with (4.10).

(b) From  $\text{Ei}_n$  in (4.6) we take the  $k$ -th derivative for  $k \geq 1$ ,

$$\text{Ei}_n^{(k)}(z) = \int_0^1 u^{k-1} (1-u)^n \cdot e^{uz} du, \quad (4.13)$$

and obtain at  $z = 0$  that  $\text{Ei}_n^{(k)}(0) = \int_0^1 u^{k-1} (1-u)^n du = \frac{n! \cdot k!}{k \cdot (n+k)!}$ .

Regarding also  $\text{Ei}_n(0) = 0$  we also obtain part (b) of Theorem 4.1.  $\blacksquare$

Now we are also able to calculate the exponential integral  $\text{Ei}(z)$  with arbitrary accuracy, even for large  $|z|$ . However, this method has to be complemented by asymptotic formulas for the exponential integral, which will also be derived in this section.

**Lemma 4.2** *Let  $s$  be any complex number with  $\Re s > 0$ .*

(a) *For all  $n \in \mathbb{N}_0$  there holds*

$$\frac{1}{s^{n+1}} = \int_1^\infty \frac{(\log x)^n}{n!} \frac{dx}{x^{s+1}}.$$

(b) *Assume that  $\rho \in \mathbb{C}$  satisfies  $\Re \rho < \Re s$ . Then  $\text{Ei}_0(\rho \log x)/x^{s+1}$  is Lebesgue integrable on the interval  $(1, \infty)$ , and there holds the relation*

$$\frac{1}{1 - \frac{\rho}{s}} = \exp \left( s \int_1^\infty \frac{\text{Ei}_0(\rho \log x)}{x^{s+1}} dx \right).$$

(c) *The expression  $(\gamma + \log(\log x))/x^{s+1}$  is Lebesgue integrable on the interval  $(1, \infty)$ , and there holds the relation*

$$\frac{1}{s} = \exp \left( s \int_1^\infty \frac{\gamma + \log(\log x)}{x^{s+1}} dx \right).$$

Proof: (a) The integral on the right hand side is obviously well defined. The statement is true for  $n = 0$ . Assume the statement is already established for some  $n \in \mathbb{N}_0$ . Then we obtain by partial integration with zero boundary terms

$$\frac{1}{s^{n+1}} = \int_1^\infty \frac{(\log x)^n}{n!} \frac{dx}{x^{s+1}} = \int_1^\infty \frac{d}{dx} \left[ \frac{(\log x)^{n+1}}{(n+1)!} \right] \frac{dx}{x^s} = s \int_1^\infty \frac{(\log x)^{n+1}}{(n+1)!} \frac{dx}{x^{s+1}}.$$

This is the proof of part (a).

(b) For  $|z| \leq 1$  we have  $|\text{Ei}_0(z)| \leq \text{Ei}_0(1)$  from the Taylor expansion (4.5), and for  $|z| > 1$  we obtain

$$\begin{aligned} |\text{Ei}_0(z)| &\leq \left| \int_0^{\frac{z}{|z|}} \frac{e^t - 1}{t} dt \right| + \left| \int_{\frac{z}{|z|}}^z \frac{e^t - 1}{t} dt \right| \\ &\leq \text{Ei}_0(1) + |z| \max(e + 1, e^{\Re z} + 1). \end{aligned}$$

We put  $z := \rho \log x$  in the last inequality in order to conclude from  $\Re s > 0$  and  $\Re \rho < \Re s$  that  $\frac{1}{1 - \frac{\rho}{s}}$  and  $\int_1^\infty \frac{\text{Ei}_0(\rho \log x)}{x^{s+1}} dx$  are well defined.

Next we show that the formula in Lemma 4.2(b) is true in the special case  $\Re s > |\rho|$ . Then the formula is also true for  $\Re \rho < \Re s$  by analytic continuation.

In order to evaluate the integral in Lemma 4.2(b) we perform partial integration, regard  $\frac{d}{dx} \text{Ei}_0(\rho \log x) = (x^\rho - 1)/(x \log x)$ , and obtain with vanishing boundary terms

$$s \int_1^\infty \frac{\text{Ei}_0(\rho \log x)}{x^{s+1}} dx = \int_1^\infty \frac{x^\rho - 1}{\log x} \cdot \frac{dx}{x^{s+1}}. \quad (4.14)$$

Here  $x^\rho$  is an abbreviation for  $\exp(\rho \log x)$ . Using the Taylor expansion of the exponential function we obtain immediately

$$\frac{x^\rho - 1}{\log x} = \sum_{n=1}^\infty \frac{\rho^n (\log x)^{n-1}}{n \cdot (n-1)!}. \quad (4.15)$$

For our restriction  $\Re s > |\rho|$  we have  $|\rho/s| < 1$  and conclude from part (a)

$$\sum_{n=1}^\infty \int_1^\infty \frac{\rho^n (\log x)^{n-1}}{n \cdot (n-1)!} \cdot \frac{dx}{x^{s+1}} = \sum_{n=1}^\infty \frac{1}{n} \left( \frac{\rho}{s} \right)^n = \log \frac{1}{1 - \frac{\rho}{s}}. \quad (4.16)$$

Next we show that summation and integration in (4.16) can be interchanged. Then we have shown part (b) of the Lemma due to (4.14), (4.15), (4.16).

We define for  $N \in \mathbb{N}$  and  $x > 1$  the  $L_1$  functions  $f_N$  by

$$f_N(x) := \sum_{n=1}^N \frac{\rho^n (\log x)^{n-1}}{n \cdot (n-1)!} \cdot \frac{1}{x^{s+1}}.$$

Then we have already shown the pointwise convergence to

$$f(x) := \lim_{N \rightarrow \infty} f_N(x) = \frac{x^\rho - 1}{\log x} \cdot \frac{1}{x^{s+1}} \quad \forall x > 1.$$

Moreover, we have for all  $N \in \mathbb{N}$  and  $x > 1$

$$|f_N(x)| \leq \sum_{n=1}^{\infty} \frac{|\rho|^n (\log x)^{n-1}}{n \cdot (n-1)!} \cdot \frac{1}{|x^{s+1}|} = F(x) := \frac{x^{|\rho|} - 1}{\log x} \cdot \frac{1}{x^{\Re s+1}}$$

with  $F \in L_1(1, \infty)$  due to our restriction  $\Re s > |\rho|$ . We finally conclude by the Lebesgue dominated convergence theorem that

$$\lim_{N \rightarrow \infty} \int_1^{\infty} f_N(x) dx = \int_1^{\infty} f(x) dx.$$

This means the interchange of summation and integration in (4.16), and we have shown part (b).

In order to prove part (c) we employ the substitution  $u = \log x$  in the integral and obtain

$$\int_1^{\infty} \frac{\gamma + \log(\log x)}{x^{s+1}} dx = \int_0^{\infty} (\gamma + \log u) e^{-su} du. \quad (4.17)$$

It follows first from  $\Re s > 0$  that these integrals are well defined.

However, in contrast to the integral in part (b), we have a singular behaviour of  $\log(\log x)$  at  $x = 1$  and of  $\log u$  at  $u = 0$ , but the integrals in (4.17) still converge in the absolute sense.

Next we define the auxiliary function

$$g(s) := s \int_0^{\infty} e^{-su} \log u du \quad (4.18)$$



and form its derivative by applying partial integration and regarding  $\frac{d}{du}(u \log u - u) = \log u$ ,

$$\begin{aligned}
g'(s) &= \int_0^{\infty} e^{-su} \log u \, du - s \int_0^{\infty} e^{-su} u \log u \, du \\
&= \int_0^{\infty} e^{-su} \log u \, du + \int_0^{\infty} (u \log u - u) \cdot (-s)e^{-su} \, du \\
&\quad + \int_0^{\infty} u \cdot (-s)e^{-su} \, du = -\frac{1}{s}.
\end{aligned}$$

We conclude that there is a constant  $C \in \mathbb{C}$  such that

$$g(s) = s \int_0^{\infty} e^{-su} \log u \, du = -C + \log \frac{1}{s}, \quad \Re s > 0. \quad (4.19)$$

In order to determine  $C$  we put  $s := 1$  and obtain first that

$$C = - \int_0^{\infty} e^{-u} \log u \, du. \quad (4.20)$$

Due to Theorem 2.3(e) and (c) we have

$$\Gamma'(1) = \int_0^{\infty} e^{-u} \log u \, du = -\gamma,$$

this means  $C = \gamma$ , and from (4.19) and (4.20) we finally conclude the part (c) of Lemma 4.2. ■

**Theorem 4.3** *Let  $s$  be any complex number with  $\Re s > 0$ .*

(a) *Then  $\text{Ei}(\log x)/x^{s+1}$  is Lebesgue integrable on the interval  $(1, \infty)$ , and there holds the relation*

$$\frac{1}{s-1} = \exp \left( s \int_1^{\infty} \frac{\text{Ei}(\log x)}{x^{s+1}} dx \right).$$

(b) *Assume that  $\rho \in \mathbb{C} \setminus [0, \infty)$  and  $\Re \rho < \Re s$ . Then*

$$1 - \frac{s}{\rho} = \exp \left( -s \int_1^{\infty} \frac{\gamma + \log(-\rho) + \log(\log x) + \text{Ei}_0(\rho \log x)}{x^{s+1}} dx \right).$$

Proof: Part (a) follows from Lemma 4.2(b) with  $\rho := 1$  and from Lemma 4.2(c) by multiplication.

Part (b) also follows from Lemma 4.2(b),(c) by multiplication

$$\begin{aligned} 1 - \frac{s}{\rho} &= \frac{s}{(-\rho)} \cdot \left(1 - \frac{\rho}{s}\right) = \exp \left( -s \int_1^{\infty} \frac{\gamma + \log(-\rho) + \log(\log x)}{x^{s+1}} dx \right) \\ &\quad \cdot \exp \left( -s \int_1^{\infty} \frac{\text{Ei}_0(\rho \log x)}{x^{s+1}} dx \right). \end{aligned}$$

■

#### **Theorem 4.4**

*Let  $s, \alpha$  be any complex numbers with  $\Re s > 0, \Re \alpha > 0$ . Then*

$$(a) \exp \left( s \int_1^{\infty} \frac{x \int_x^{\infty} \frac{t^{-(\alpha+1)} dt}{\log t}}{x^{s+1}} dx \right) = 1 + \frac{s}{\alpha}.$$

(b) *For any real  $x > 1$  there holds*

$$- \int_x^{\infty} \frac{t^{-(\alpha+1)} dt}{\log t} = \gamma + \log \alpha + \log(\log x) + \text{Ei}_0(-\alpha \log x).$$

Proof: (a) The left hand side in (b) has only a logarithmic singularity at  $x = 1$ , and thus the left hand side in (a) defines for any fixed  $s$  with  $\Re s > 0$  an analytic function in the half plane  $\Re \alpha > 0$ . We conclude that

$$\begin{aligned} \frac{\partial}{\partial \alpha} \left\{ s \int_1^{\infty} \frac{x \int_1^{\infty} \frac{t^{-(\alpha+1)} dt}{\log t}}{x^{s+1}} dx \right\} &= -s \int_1^{\infty} \frac{x \int_1^{\infty} t^{-(\alpha+1)} dt}{x^{s+1}} dx \\ &= -s \int_1^{\infty} \frac{x^{-\alpha}}{x^{s+1}} dx = \frac{1}{s + \alpha} - \frac{1}{\alpha}, \end{aligned}$$

and therefore

$$s \int_1^{\infty} \frac{x \int_1^{\infty} \frac{t^{-(\alpha+1)} dt}{\log t}}{x^{s+1}} dx = f_1(s) + \log \left( 1 + \frac{s}{\alpha} \right)$$

with an integration constant  $f_1(s)$  depending only on  $s$ . But  $f_1(s)$  must be zero for all  $s$ , because both sides of the last equality must be zero in the limit  $\alpha \rightarrow \infty$  along the positive real axis.

(b) Differentiation with respect to  $x > 1$  on both sides in (b) gives the same result  $\frac{x^{-(\alpha+1)}}{\log x}$ . This means that both sides in (b) can only differ in an integration constant  $f_2(\alpha)$ . We put  $\rho = -\alpha$  in Theorem 4.3(b) in order to conclude with (a) that  $f_2(\alpha) = 0$  for all real  $\alpha > 0$ . The general result for  $\Re \alpha > 0$  follows by analytic continuation.  $\blacksquare$

In the next theorem we derive some integral representations for the exponential integral. A resulting asymptotic expansion for  $\text{Ei}(z)$  is very efficient for  $|\Im z| \gg |\Re z|$ .

**Theorem 4.5** Integral representations and asymptotic expansion for  $\text{Ei}(z)$

(a) Let  $\alpha$  be any complex number with  $\Re \alpha > 0$  and  $\Im \alpha \neq 0$ . Then we obtain for all  $y > 0$

$$\text{Ei}(-\alpha y) = -i\pi \text{sign}(\Im \alpha) - \int_y^{\infty} \frac{e^{-\alpha u}}{u} du.$$

(b) Let be  $\sigma, t \in \mathbb{R}$  and  $t \neq 0$ . Then we put  $z := \sigma + it$  and obtain

$$\operatorname{Ei}(z) = i\pi \operatorname{sign}(\Im z) + \int_0^\infty \frac{e^{z-u}}{z-u} du.$$

Moreover, for all  $n \in \mathbb{N}_0$  we have the asymptotic expansion

$$\int_0^\infty \frac{e^{z-u}}{z-u} du = \frac{e^z}{z} \sum_{k=0}^n \frac{k!}{z^k} + (n+1)! \int_0^\infty \frac{e^{z-u}}{(z-u)^{n+2}} du,$$

and for all  $k \in \mathbb{N}_0$  the estimate

$$\left| \int_0^\infty \frac{e^{z-u}}{(z-u)^k} du \right| \leq \frac{e^\sigma}{|t|^k}.$$

(c) For  $\Im z > 0$  we have

$$\operatorname{Ei}(z) = i\pi \operatorname{sign}(\Im z) - \lim_{T \rightarrow \infty} \int_{\Im z}^T \frac{e^{\sigma+i\vartheta}}{\sigma+i\vartheta} i d\vartheta.$$

Proof: (a) The left and right hand side of the equation in (a) define analytic functions in the quarter plane  $\Re \alpha > 0$  and  $\Im \alpha > 0$ . By performing the derivative with respect to  $\alpha$  on both sides we obtain that

$$\frac{\partial}{\partial \alpha} \operatorname{Ei}(-\alpha y) = \frac{\partial}{\partial \alpha} \left( \mp i\pi - \int_y^\infty \frac{e^{-\alpha u}}{u} du \right) = \frac{e^{-\alpha y}}{\alpha}.$$

It remains to determine the constants  $\mp i\pi$  on the right hand side. We apply the substitution  $u = \log t$  on the integral in Theorem 4.4(b) and obtain for the number  $x > 1$  given there that

$$- \int_{\log x}^\infty \frac{e^{-\alpha u}}{u} du = \gamma + \log \alpha + \log(\log x) + \operatorname{Ei}_0(-\alpha \log x).$$

In this equation we put  $y := \log x > 0$  and conclude for  $\Im \alpha > 0$

$$- \int_y^\infty \frac{e^{-\alpha u}}{u} du = \log \alpha + \log y - \log(-\alpha y) + \operatorname{Ei}(-\alpha y) = i\pi + \operatorname{Ei}(-\alpha y).$$

We obtain Theorem 4.5(a) by regarding that  $\overline{\text{Ei}(-\alpha y)} = \text{Ei}(-\bar{\alpha}y)$ .

Note that the condition  $\Im\alpha > 0$  guarantees that the argument  $-\alpha y$  does not pass the cut of the logarithm function in the exponential integral. This condition is not necessary in Theorem 4.4(b), which makes use of the entire function  $\text{Ei}_0$ .

It is not difficult to check that the representations for  $\text{Ei}(z)$  in (b) and (c) define analytic functions in the half planes  $t = \Im z > 0$  and  $t = \Im z < 0$ . Differentiation with respect to  $z = \sigma + it$  reduces the proof of the representation formulas to the determination of the integration constants, which just follow from part (a) in the limit  $\sigma \rightarrow -\infty$ . From the representation for  $\text{Ei}(z)$  in (b) we also obtain the asymptotic expansion by partial integration. The estimate for the asymptotic expansion of  $\text{Ei}(z)$  in (b) results from

$$\left| \int_0^\infty \frac{e^{z-u}}{(z-u)^k} du \right| \leq \int_0^\infty \frac{|e^{(\sigma-u)+it}|}{|(\sigma-u)+it|^k} du \leq \int_0^\infty \frac{e^{\sigma-u}}{|t|^k} du = \frac{e^\sigma}{|t|^k}.$$

■

The functions  $\text{Ei}$  and  $\text{Ei}_0$  are closely related to the cosine and sine integral functions, for which we obtain the following

#### Corollary 4.6

The cosine and sine integral functions  $\text{Ci} : \mathbb{C}_- \rightarrow \mathbb{C}$  and  $\text{Si} : \mathbb{C} \rightarrow \mathbb{C}$  are given by

$$\begin{aligned} \text{Ci}(z) &:= \gamma + \log z + \int_0^z \frac{\cos(t) - 1}{t} dt, \\ \text{Si}(z) &:= \int_0^z \frac{\sin t}{t} dt = \frac{\text{Ei}_0(iz) - \text{Ei}_0(-iz)}{2i}. \end{aligned}$$

They satisfy the asymptotic relations

$$\lim_{x \rightarrow \infty} \text{Ci}(x) = 0, \quad \lim_{x \rightarrow \infty} \text{Si}(x) = \frac{\pi}{2}.$$

■

### Theorem 4.7

Let  $s, \rho$  be any complex numbers with  $\Re s > 0$  and  $\Re \rho < \Re s$ . By  $\frac{d \log}{ds} f(s) := \frac{f'(s)}{f(s)}$  we denote the logarithmic derivative of an analytic expression  $f(s)$ . Then

$$(a) \quad s \int_1^{\infty} \frac{(x^\rho - 1)/\rho}{x^{s+1}} dx = \frac{d \log}{ds} \left(1 - \frac{s}{\rho}\right) = \frac{s}{\rho(s - \rho)} - \frac{1}{\rho} = \frac{1}{s - \rho},$$

$$(b) \quad s \int_1^{\infty} \frac{\gamma + \log(x - 1)}{x^{s+1}} dx = -\frac{d \log}{ds} \Gamma(s) = -\frac{\Gamma'(s)}{\Gamma(s)},$$

$$(c) \quad s \int_1^{\infty} \frac{\frac{1}{2} \log \frac{x^2}{x^2 - 1}}{x^{s+1}} dx = \sum_{k=1}^{\infty} \frac{s}{2k(s + 2k)} = \frac{\gamma}{2} + \frac{d \log}{ds} \Gamma\left(\frac{s}{2} + 1\right).$$

Proof: (a) is trivial, (b) follows with Lemma 4.2(a) and Theorem 2.3(c) from  $\log(x - 1) = \log x - \log \frac{1}{1 - 1/x} = \log x - \sum_{k=1}^{\infty} \frac{x^{-k}}{k}$ ,  $x > 1$ , and the proof of (c) is analogous to (b). ■

## 5 Regularizations for von Mangoldt's $\psi(x)$

In the sequel we assume that

$$\sigma_0, \sigma_1 \in \mathbb{R} \cup \{-\infty, +\infty\} \quad \text{and} \quad \sigma_0 < \sigma_1. \quad (5.1)$$

Depending on  $\sigma_0, \sigma_1$  we define the strip  $\mathcal{S}(\sigma_0, \sigma_1)$  in the complex plane by

$$\mathcal{S}(\sigma_0, \sigma_1) := \{s \in \mathbb{C} \mid \sigma_0 < \Re s < \sigma_1\}. \quad (5.2)$$

In the preceding section there often occurs the Mellin transform of a given function  $f : (0, \infty) \rightarrow \mathbb{C}$ ,

$$(\mathcal{M}f)(s) := \int_0^{\infty} \frac{f(x)}{x^{s+1}} dx, \quad (5.3)$$

defined for all  $s$  in an appropriate strip  $\mathcal{S}(\sigma_0, \sigma_1)$ , provided that  $f(x)/x^{s+1}$  is Lebesgue integrable on  $(0, \infty)$  for all fixed  $s \in \mathcal{S}(\sigma_0, \sigma_1)$ . There are slightly

different but equivalent definitions of a Mellin transform, however the definition above is better suited for our study of the  $\zeta$  function and the primes.

Mellin transforms include the Fourier-Laplace transforms, which can be seen by applying the integral substitution  $x = e^u$  in (5.3), namely

$$(\mathcal{M}f)(s) = \int_{-\infty}^{\infty} f(e^u) e^{-su} du. \quad (5.4)$$

Note that  $f(x) = 0$  for  $0 < x < 1$  leads to the lower integration limit 0 instead of  $-\infty$  in (5.4), such that (5.4) contains the usual Laplace transformation for  $\Im s = 0$  and  $\Re s > 0$  as a special case.

The relation between the Mellin transform and the Fourier-Laplace transform is important, because we can use the Fourier-theory in order to obtain the following uniqueness result and inversion formula:

**Theorem 5.1** *The function  $f : (0, \infty) \rightarrow \mathbb{C}$  may be given in such a way that  $|f(x)|/x^{\sigma+1}$  is Lebesgue integrable on  $(0, \infty)$  for any fixed  $\sigma \in (\sigma_0, \sigma_1)$ . Then there hold*

(a) *If  $(\mathcal{M}f)(\sigma + it) = 0$  for all  $t \in \mathbb{R}$  and fixed  $\sigma \in (\sigma_0, \sigma_1)$ , then  $f(x) \equiv 0$  for almost all  $x > 0$ .*

(b) *Mellin inversion formula*

*Assume that the Mellin transform  $(\mathcal{M}f)(\sigma + it)$  is Lebesgue integrable on  $\mathbb{R}$  with respect to  $t \in \mathbb{R}$  for fixed  $\sigma \in (\sigma_0, \sigma_1)$ . Then the function  $g : (0, \infty) \rightarrow \mathbb{C}$  with*

$$g(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{M}f)(\sigma + it) x^{\sigma+it} dt = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{(\mathcal{M}f)(s) x^s}{2\pi i} ds$$

*is well defined and continuous, and there holds  $g(x) = f(x)$  for almost all  $x > 0$ .*

Remark: The uniqueness result in part (a) of this theorem is a special case of part (b). Here we have just used the Fourier inversion formula for Lebesgue integrable Fourier transforms. ■

Recall the definition of von Mangoldt's function  $\psi$  in Theorem 2.2. Due to Theorem 2.3(a) and Theorem 2.4(d) we can write

$$\zeta(s) = \frac{\xi(s)}{s-1} \cdot \frac{\pi^{s/2}}{\Gamma(\frac{s}{2} + 1)}. \quad (5.5)$$

Due to Hadamard's product representations for  $\xi(s)$  and  $\Gamma(\frac{s}{2} + 1)$  the Mellin transform of  $\psi$  is given for  $\Re s > 1$  by the absolutely convergent series

$$-\frac{1}{s} \frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(s-\rho)} + \sum_{k=1}^{\infty} \frac{1}{2k(s+2k)} - \frac{\zeta'(0)}{s\zeta(0)}. \quad (5.6)$$

Having Theorem 4.7 in mind, it is natural to state von Mangoldt's formula at each point  $x > 1$  of continuity, namely

$$\psi(x) = x - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{x^\rho}{\rho} + \frac{1}{2} \log \frac{x^2}{x^2-1} - \frac{\zeta'(0)}{\zeta(0)}. \quad (5.7)$$

A technical difficult proof of (5.7) was presented by von Mangoldt, see also the textbook of Edwards [6]. The numerical constant in (5.7) turns out to be

$$\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi). \quad (5.8)$$

The following theorem is used for the derivation of von Mangoldt's formula (5.7), but we will also use the resulting Lemma 5.3 for Riemann's prime number formula in the next section. It provides some information about the vertical distribution of the zeros of the zeta function in the critical strip. The following result was used by Riemann in [11] and gives the asymptotic density of the nontrivial roots. It was proved by von Mangoldt [10] and then simplified by Backlund [3].

**Theorem 5.2** *For  $T \geq 2$  let  $N(T)$  be the number of zeros  $\rho = \sigma + it$  of the  $\zeta$ -function in the critical strip  $0 < \sigma < 1$  with  $0 < t \leq T$ , regarding the multiplicity of the roots. Then*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \quad (5.9)$$

■



Theorem 5.2 can be applied very easily for the proof of the following Lemma by using the Stieltjes calculus.

**Lemma 5.3** *There hold the following asymptotic relations for  $T \rightarrow \infty$*

$$(a) \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{1}{|\rho|} = \frac{1}{2\pi} (\log T)^2 + O(\log T),$$

$$(b) \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| > T}} \frac{1}{|\Im \rho|^2} = \frac{1}{\pi} \frac{\log T}{T} + O(1/T).$$

Proof: We use Stieltjes integral representations as follows in order to apply Theorem 5.2

$$\sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{1}{|\Im \rho|} = 2 \int_0^T \frac{dN(t)}{t} = 2 \int_0^T \frac{N(t)}{t^2} dt + 2 \frac{N(T)}{T} = \frac{(\log T)^2}{2\pi} + O(\log T),$$

where the boundary term of partial integration is zero because  $\zeta(s)$  has no zero on the line segment  $0 < s < 1$ . The first part of the Lemma is now a consequence of

$$\frac{1}{|\Im \rho|} - \frac{1}{|\Im \rho|^2} < \frac{1}{|\Im \rho| + 1} < \frac{1}{|\rho|} < \frac{1}{|\Im \rho| - 1} < \frac{1}{|\Im \rho|} + \frac{1}{|\Im \rho|^2},$$

since the imaginary part of  $\rho$  is larger than 1, while the real part is between 0 and 1. For the second part we obtain

$$\sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| > T}} \frac{1}{|\Im \rho|^2} = 2 \int_T^\infty \frac{dN(t)}{t^2} = 4 \int_T^\infty \frac{N(t)}{t^3} dt - 2 \frac{N(T)}{T^2} = \frac{1}{\pi} \frac{\log T}{T} + O(1/T).$$

■

Without going into details we will present the key idea in order to obtain a short proof of the von Mangoldt formula (5.7). On one hand it is based on the Fourier- or Mellin-inversion formula for integrable and piecewise smooth

functions, a variant of Theorem 5.1 in order to handle the jump discontinuities of  $\psi(x)$ . On the other hand there holds

$$\lim_{T \rightarrow \infty} \int_{\sigma-iT}^{\sigma+iT} \left[ \frac{\xi'(s)}{\xi(s)} - \frac{\xi'(0)}{\xi(0)} \right] \frac{x^s}{s} ds = \lim_{T \rightarrow \infty} \int_{\sigma-iT}^{\sigma+iT} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^s}{\rho(s-\rho)} ds. \quad (5.10)$$

The main difficulty is to prove that termwise integration in (5.10) is possible. But this can be justified by regarding Theorem 5.2 and the identity

$$\int_{\sigma-iT}^{\sigma+iT} \frac{x^s}{\rho(s-\rho)} ds = \int_{\sigma-iT}^{\sigma+iT} \frac{\varphi_s(x)}{(s-\rho)^2} ds + \left. \frac{s \varphi_s(x)}{\rho(s-\rho)} \right|_{s=\sigma-iT}^{s=\sigma+iT}, \quad (5.11)$$

where  $s = \sigma + it$  for some  $\sigma > 1$ ,  $t \in \mathbb{R}$  and for  $x > 1$

$$\varphi_s(x) := \text{Ei}(s \log x) - i\pi \text{sign}(\Im s) = \int_0^1 \frac{x^s du}{s \log x + \log u} \sim \frac{x^s}{s \log x} \text{ for } |t| \rightarrow \infty.$$

We will now derive new von Mangoldt like prime number formulas for appropriate regularizations of  $\psi(x)$ .

Assume that  $f(x)/x^{s+1}$  is Lebesgue integrable on  $(0, \infty)$  with respect to  $x$  on some strip  $\sigma_0 < \Re s < \sigma_1$  and that  $f(x) = 0$  for  $0 < x < \delta$  and some constant  $\delta > 0$ . We define for a positive parameter  $\varepsilon$  and all  $x > 0$  the regularization of  $f$  by the integral mean value

$$(R_\varepsilon f)(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x \exp(\varepsilon u)) du. \quad (5.12)$$

Then the Mellin transform of  $(R_\varepsilon f)$  is

$$\int_0^\infty \frac{(R_\varepsilon f)(x)}{x^{s+1}} dx = \frac{\sinh(\frac{s\varepsilon}{2})}{(\frac{s\varepsilon}{2})} \int_0^\infty \frac{f(y)}{y^{s+1}} dy. \quad (5.13)$$

Note that  $f(x) = 0$  for  $0 < x < \delta$  leads to a vanishing boundary term in the application of the partial integration above.

Next we define the Lebesgue integrable function  $\eta : (0, \infty) \rightarrow \mathbb{R}$  by

$$\eta(x) := \psi(x) - \chi_{(1, \infty)}(x) \cdot \left( x + \frac{1}{2} \log \frac{x^2}{x^2 - 1} - \log(2\pi) \right), \quad (5.14)$$

where  $\chi_{(1,\infty)}$  is the characteristic function of the interval  $(1, \infty)$ . Due to Theorem 4.7 and (5.6) the Mellin transform of  $\eta$  is for  $\Re s > 1$

$$\int_0^\infty \frac{\eta(y)}{y^{s+1}} dy = \int_1^\infty \frac{\eta(y)}{y^{s+1}} dy = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(s-\rho)}. \quad (5.15)$$

We define for  $\varepsilon > 0$  and all  $n \in \mathbb{N}_0$  the regularizations  $R_\varepsilon^{(n)} : (0, \infty) \rightarrow \mathbb{R}$  of the function  $\eta$  by the recurrence relations

$$R_\varepsilon^{(0)}\eta := \eta, \quad R_\varepsilon^{(n+1)}\eta := R_\varepsilon(R_\varepsilon^{(n)}\eta). \quad (5.16)$$

From (5.13) and (5.15) we obtain for all  $\varepsilon > 0$  and all  $n \in \mathbb{N}_0$

$$\int_0^\infty \frac{(R_\varepsilon^{(n)}\eta)(x)}{x^{s+1}} dx = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(s-\rho)} \left( \frac{\sinh(\frac{s\varepsilon}{2})}{(\frac{s\varepsilon}{2})} \right)^n. \quad (5.17)$$

In the sequel we assume that  $n \geq 1$ . On the right hand side of (5.17) the Mellin inversion integral formula can be applied termwise due to  $n \geq 1$  to recover  $(R_\varepsilon^{(n)}\eta)(x)$  according to

$$(R_\varepsilon^{(n)}\eta)(x) = -\frac{1}{2\pi i} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{\rho(s-\rho)} \left( \frac{\sinh(\frac{s\varepsilon}{2})}{(\frac{s\varepsilon}{2})} \right)^n ds. \quad (5.18)$$

The summation of the integrals is performed with respect to the nontrivial zeros of the  $\zeta$ -function, where the ordering of the terms is arbitrary thanks to the absolute convergence. Here  $\sigma$  is any fixed real number larger than 1.

Each single integral in (5.18) can be calculated if we replace for  $T > 0$  the integration path from  $\sigma - i\infty$  to  $\sigma + i\infty$  by the closed rectangular path  $\Gamma_T$  ranging from  $\sigma - iT$  to  $\sigma + iT$ , from  $\sigma + iT$  to  $\sigma - T + iT$ , from  $\sigma - T + iT$  to  $\sigma - T - iT$  and from  $\sigma - T - iT$  to  $\sigma - iT$ . If  $x > 1$  and if  $\varepsilon > 0$  is small enough such that  $\log x > \frac{n\varepsilon}{2}$ , then we obtain by Cauchy's theorem

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{\rho(s-\rho)} \left( \frac{\sinh(\frac{s\varepsilon}{2})}{(\frac{s\varepsilon}{2})} \right)^n ds = \\ & \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\Gamma_T} \frac{x^s}{\rho(s-\rho)} \left( \frac{\sinh(\frac{s\varepsilon}{2})}{(\frac{s\varepsilon}{2})} \right)^n ds = \frac{x^\rho}{\rho} \left( \frac{\sinh(\frac{\rho\varepsilon}{2})}{(\frac{\rho\varepsilon}{2})} \right)^n. \end{aligned} \quad (5.19)$$

For  $\log x > \frac{n\varepsilon}{2}$  there results the formula

$$(R_\varepsilon^{(n)}\eta)(x) = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho} \left( \frac{\sinh(\frac{\rho\varepsilon}{2})}{(\frac{\rho\varepsilon}{2})} \right)^n. \quad (5.20)$$

We conclude at each point  $x > 1$  of continuity for all  $n \geq 1$  that

$$\psi(x) = x - \lim_{\varepsilon \rightarrow 0} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho} \left( \frac{\sinh(\frac{\rho\varepsilon}{2})}{(\frac{\rho\varepsilon}{2})} \right)^n + \frac{1}{2} \log \frac{x^2}{x^2 - 1} - \log(2\pi). \quad (5.21)$$

The regularizations (5.16) can also be rewritten by the  $n$ -fold integral

$$(R_\varepsilon^{(n)}\eta)(x) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{+\frac{1}{2}} \eta(x \exp(\varepsilon(u_1 + \dots + u_n))) du_1 \dots du_n. \quad (5.22)$$

Using the  $n$ -th order cardinal B-splines  $N_n(u)$ ,

$$N_1(u) := \chi_{(0,1)}(u), \quad N_{n+1}(u) := \int_0^1 N_n(u-v) dv, \quad n \geq 1, \quad (5.23)$$

we obtain that

$$(R_\varepsilon^{(n)}\eta)(x) = \int_{-\frac{n}{2}}^{+\frac{n}{2}} N_n(u + \frac{n}{2}) \eta(x \exp(\varepsilon u)) du. \quad (5.24)$$

The  $n$ -th order cardinal B-spline  $N_n(u)$  has support  $[0, n]$ , is normalized, has center  $\frac{n}{2}$  and variance  $n/12$ , i.e.

$$\int_0^n N_n(u) du = 1, \quad \int_0^n u N_n(u) du = \frac{n}{2}, \quad \int_0^n (u - \frac{n}{2})^2 N_n(u) du = \frac{n}{12}, \quad (5.25)$$

and by the Central Limit Theorem we have with uniform convergence in  $u$

$$\lim_{n \rightarrow \infty} \sqrt{\frac{n}{12}} N_n(\sqrt{\frac{n}{12}} u + \frac{n}{2}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right). \quad (5.26)$$

For more details about cardinal B-splines see the textbook of Chui [4].

Thus the regularizations (5.16) are related to the Gaussian regularization

$$(G_\delta \eta)(x) := \frac{1}{\sqrt{2\pi\delta}} \int_{-\infty}^{+\infty} \exp\left(-\frac{u^2}{2\delta}\right) \eta(xe^u) du \quad (5.27)$$

for a constant parameter  $\delta > 0$  with its Mellin transform

$$\int_0^{+\infty} \frac{(G_\delta \eta)(x)}{x^{s+1}} dx = -\exp\left(\frac{\delta}{2} s^2\right) \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(s-\rho)}. \quad (5.28)$$

Here we will not prove (5.28), which holds for  $\Re s > 1$ , because we will obtain a generalization of this result in Theorem 5.6.

To see the relation between  $G_\delta \eta$  and the cardinal B-splines we define for given  $n \geq 1$  and given  $\delta > 0$

$$\varepsilon_n := \sqrt{\frac{12\delta}{n}}. \quad (5.29)$$

Then we obtain with (5.26) and (5.24) that

$$\begin{aligned} (G_\delta \eta)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{v^2}{2}\right) \eta\left(xe^{\sqrt{\delta}v}\right) dv \\ &= \lim_{n \rightarrow \infty} \sqrt{\frac{n}{12}} \int_{-\infty}^{+\infty} N_n\left(\sqrt{\frac{n}{12}}v + \frac{n}{2}\right) \eta(xe^{\sqrt{\delta}v}) dv \\ &= \lim_{n \rightarrow \infty} (R_{\varepsilon_n}^{(n)} \eta)(x). \end{aligned} \quad (5.30)$$

Note that  $(R_{\varepsilon}^{(n)} \eta)(x)$  in (5.16) is for  $x \gg 1$  a mean value of  $\psi(x) - x + \log(2\pi)$ . In the Gaussian limit  $n \rightarrow \infty$  there results

$$\lim_{n \rightarrow \infty} \left( \frac{\sinh\left(\frac{\rho \varepsilon_n}{2}\right)}{\left(\frac{\rho \varepsilon_n}{2}\right)} \right)^n = \exp\left(\frac{\delta}{2} \rho^2\right), \quad (5.31)$$

which can be seen by Taylor expansion of  $\frac{\sinh z}{z}$ . However, we cannot use (5.20) and (5.31) in order to express  $(G_\delta \eta)(x)$  in terms of the zeros of the zeta function due to the restriction  $\log x - \frac{n\varepsilon_n}{2} > 0$ . Fortunately this is not necessary because we can termwise apply the Mellin inversion formula in

Theorem 5.1(b) on the Mellin transform (5.28). For this purpose we need the following

**Lemma 5.4** *For  $x, \delta > 0$ ,  $\rho \in \mathbb{C}$ ,  $\alpha \in \mathbb{R}$  and  $\Re \rho < \alpha$  we have*

$$f_\delta(x, \rho) := \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{x^{s-\rho} \exp\left(\frac{\delta}{2} s^2\right)}{s-\rho} ds = \exp\left(\frac{\delta}{2} \rho^2\right) \Phi\left(\frac{\rho\delta + \log x}{\sqrt{\delta}}\right)$$

with the complex Gaussian error function  $\Phi(z) := \int_0^\infty \frac{e^{-\frac{(z-u)^2}{2}}}{\sqrt{2\pi}} du$ ,  $z \in \mathbb{C}$ .

Proof: First we form the derivative

$$\frac{\partial f_\delta}{\partial x}(x, \rho) = \frac{x^{-\rho-1} \exp\left(-\frac{\log^2 x}{2\delta}\right)}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \exp\left(\frac{\delta}{2} \left(s + \frac{\log x}{\delta}\right)^2\right) ds, \quad (5.32)$$

and conclude that the integral on the right hand side in (5.32) is defined for  $\alpha \in \mathbb{R}$  and independent on  $\alpha$  by Cauchy's Theorem and the rapid decay of the integrand. We put  $\alpha := -\frac{\log x}{\delta}$  and obtain

$$\frac{\partial f_\delta}{\partial x}(x, \rho) = \frac{x^{-\rho-1}}{\sqrt{2\pi\delta}} \exp\left(-\frac{\log^2 x}{2\delta}\right). \quad (5.33)$$

On the other hand we have

$$\lim_{x \rightarrow 0^+} f_\delta(x, \rho) = 0, \quad (5.34)$$

such that

$$f_\delta(x, \rho) = \int_0^x \frac{t^{-\rho-1}}{\sqrt{2\pi\delta}} \exp\left(-\frac{\log^2 t}{2\delta}\right) dt. \quad (5.35)$$

Using the substitution  $u = \log t$  in (5.35) we can easily evaluate the resulting integral in order to obtain Lemma 5.4.  $\blacksquare$

There results the following

**Theorem 5.5**

(a) *For the Gaussian regularization of  $\eta$  we have*

$$(G_\delta \eta)(x) = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho} \exp\left(\frac{\delta}{2} \rho^2\right) \Phi\left(\frac{\rho\delta + \log x}{\sqrt{\delta}}\right).$$

(b) If we define for  $x > 0$  and  $\beta \in \mathbb{C}$  the quantity

$$H_\delta(x, \beta) := \exp\left(\frac{\delta}{2}\beta^2\right) \left(1 - \Phi\left(\frac{\beta\delta + \log x}{\sqrt{\delta}}\right)\right),$$

then we obtain for  $\Re\beta \geq 0$  and  $\beta \neq 0$  the estimate

$$|H_\delta(x, \beta)| \leq \frac{2x^{-\Re\beta}}{|\beta|} \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{\log^2 x}{2\delta}\right).$$

Proof: Part (a) results from Lemma 5.4, the Mellin inversion formula and part (b). In order to show part (b) we conclude for  $\beta \neq 0$

$$\begin{aligned} H_\delta(x, \beta) &= \exp\left(\frac{\delta}{2}\beta^2\right) \left(1 - \int_0^\infty \frac{\exp\left(-\frac{1}{2}\left(\frac{\beta\delta + \log x}{\sqrt{\delta}} - u\right)^2\right)}{\sqrt{2\pi}} du\right) \\ &= x^{-\beta} \exp\left(\frac{\delta}{2}\beta^2 + \beta \log x\right) \int_0^\infty \frac{\exp\left(-\frac{1}{2}\left(\frac{\beta\delta + \log x}{\sqrt{\delta}} + v\right)^2\right)}{\sqrt{2\pi}} dv \\ &= x^{-\beta} \int_0^\infty \frac{\exp\left(-\frac{1}{2}\left(v + \frac{\log x}{\sqrt{\delta}}\right)^2\right)}{\sqrt{2\pi}} \exp\left(-\beta\sqrt{\delta}v\right) dv \\ &= \frac{x^{-\beta}}{\beta} \frac{1}{\sqrt{2\pi\delta}} \int_0^\infty \left(1 - e^{-\beta\sqrt{\delta}v}\right) \left(v + \frac{\log x}{\sqrt{\delta}}\right) \exp\left(-\frac{1}{2}\left(v + \frac{\log x}{\sqrt{\delta}}\right)^2\right) dv. \end{aligned} \tag{5.36}$$

From this equality we finally obtain for  $\Re\beta \geq 0$  due to  $v \geq 0$  that

$$\begin{aligned} |H_\delta(x, \beta)| &\leq \frac{x^{-\Re\beta}}{|\beta|} \frac{1}{\sqrt{2\pi\delta}} \int_0^\infty 2 \left(v + \frac{\log x}{\sqrt{\delta}}\right) \exp\left(-\frac{1}{2}\left(v + \frac{\log x}{\sqrt{\delta}}\right)^2\right) dv \\ &= \frac{2x^{-\Re\beta}}{|\beta|} \frac{1}{\sqrt{2\pi\delta}} \exp\left(-\frac{\log^2 x}{2\delta}\right). \quad \blacksquare \end{aligned}$$

Note that the last inequality is in perfect agreement with the asymptotic prediction (5.31) from the Central Limit Theorem, because it implies

$$\left| - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re\rho < 1}} \frac{x^\rho}{\rho} \exp\left(\frac{\delta}{2}\rho^2\right) - (G_\delta \eta)(x) \right| \leq \frac{c}{\sqrt{2\pi\delta}} \exp\left(-\frac{\log^2 x}{2\delta}\right) \tag{5.37}$$

for a constant  $c > 0$  neither depending on  $x > 0$  nor depending on  $\delta > 0$ . Thus the counterpart of (5.21) reads in the Gaussian limit

$$\psi(x) = x - \lim_{\delta \rightarrow 0} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho} \exp\left(\frac{\delta}{2} \rho^2\right) + \frac{1}{2} \log \frac{x^2}{x^2 - 1} - \log(2\pi). \quad (5.38)$$

Theorem 5.5 can be generalized if we introduce for  $x, \delta > 0$  and  $\lambda \in \mathbb{C}$  the following integral transform of  $\eta$ ,

$$(G_{\delta, \lambda} \eta)(x) := \frac{1}{\sqrt{2\pi\delta}} \int_{-\infty}^{+\infty} e^{-\lambda u} \exp\left(-\frac{u^2}{2\delta}\right) \eta(xe^u) du. \quad (5.39)$$

For  $\lambda = 0$  it reduces to  $G_\delta \eta$ . We obtain

### Theorem 5.6

(a) For  $\Re s > 1$  we have the following Mellin transform of  $G_{\delta, \lambda} \eta$

$$\int_0^\infty \frac{(G_{\delta, \lambda} \eta)(x)}{x^{s+1}} dx = -\exp\left(\frac{\delta}{2} (s - \lambda)^2\right) \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(s - \rho)}.$$

(b) The representation of  $G_{\delta, \lambda} \eta$  in terms of the nontrivial zeros of the  $\zeta$  function reads

$$(G_{\delta, \lambda} \eta)(x) = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho} \exp\left(\frac{\delta}{2} (\rho - \lambda)^2\right) \Phi\left(\frac{(\rho - \lambda)\delta + \log x}{\sqrt{\delta}}\right).$$

Proof: For part (a) we conclude that

$$\int_0^\infty \frac{(G_{\delta, \lambda} \eta)(x)}{x^{s+1}} dx = \int_{-\infty}^{+\infty} \frac{e^{-\lambda u}}{\sqrt{2\pi\delta}} \exp\left(-\frac{u^2}{2\delta}\right) \left(\int_0^\infty \frac{\eta(xe^u)}{x^{s+1}} dx\right) du. \quad (5.40)$$

On the inner integral we apply the substitution  $y = x e^u$  with  $dx/x = dy/y$ ,  $x^{-s} = e^{us} y^{-s}$  and obtain

$$\int_0^\infty \frac{(G_{\delta, \lambda} \eta)(x)}{x^s} \frac{dx}{x} = \int_{-\infty}^{+\infty} \frac{e^{(s-\lambda)u}}{\sqrt{2\pi\delta}} \exp\left(-\frac{u^2}{2\delta}\right) \left(\int_0^\infty \frac{\eta(y)}{y^s} \frac{dy}{y}\right) du. \quad (5.41)$$



From (5.15) we obtain that

$$\begin{aligned}
\int_0^\infty \frac{(G_{\delta,\lambda} \eta)(x)}{x^{s+1}} dx &= - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(s-\rho)} \int_{-\infty}^{+\infty} \frac{e^{(s-\lambda)u}}{\sqrt{2\pi\delta}} \exp\left(-\frac{u^2}{2\delta}\right) du \\
&= - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(s-\rho)} \exp\left(\frac{\delta(s-\lambda)^2}{2}\right). \quad (5.42)
\end{aligned}$$

In order to prove part (b) we will directly apply the Mellin transform on the desired representation, which will also lead to an alternative proof of part (a) in Theorem 5.5. We obtain for each fixed  $\rho$  that

$$\begin{aligned}
& - \frac{e^{\frac{\delta}{2}(\rho-\lambda)^2}}{\rho} \int_0^\infty x^{\rho-s-1} \Phi\left(\frac{\delta(\rho-\lambda) + \log x}{\sqrt{\delta}}\right) dx \\
&= \frac{e^{\frac{\delta}{2}(\rho-\lambda)^2}}{\rho \sqrt{2\pi\delta}} \int_0^\infty \frac{x^{\rho-s}}{\rho-s} \exp\left\{-\frac{1}{2} \left((\rho-\lambda)\sqrt{\delta} + \frac{\log x}{\sqrt{\delta}}\right)^2\right\} \frac{dx}{x} \\
&= \frac{1}{\rho(\rho-s)\sqrt{2\pi\delta}} \int_0^\infty x^{\lambda-s} \exp\left(-\frac{\log^2 x}{2\delta}\right) \frac{dx}{x} \\
&= \frac{1}{\rho(\rho-s)\sqrt{2\pi\delta}} \int_{-\infty}^\infty e^{u(\lambda-s)} e^{-\frac{u^2}{2\delta}} du \\
&= -\frac{1}{\rho(s-\rho)} \exp\left(\frac{\delta(s-\lambda)^2}{2}\right).
\end{aligned}$$

Thus we have shown Theorem 5.6. ■

**Remark:** The series in Theorem 5.6(b) is absolute convergent due to (5.36) and  $|1 - \exp(-\beta\sqrt{\delta}v)| \leq 1 + \exp(-\Re\beta\sqrt{\delta}v)$  with  $\beta := \rho - \lambda$ . The integral transform in (5.39) shows a strong relationship to the windowed Fourier transforms. It permits the study of the local behaviour of  $\eta(x)$  and  $\psi(x)$  in terms of the nontrivial roots of the  $\zeta$  function. The function  $\eta(x)$  and its regularizations describe the fluctuations of the von Mangoldt function  $\psi$ , and these fluctuations are caused by the nontrivial zeros of the zeta function.

These results could be obtained from Euler's and Hadamard's product decomposition, without making use of the von Mangoldt formula. However,

we can employ (5.7) in order to derive a further explicit formula which gives another explanation for the asymptotic behaviour of (5.37).

We start for real  $x > 0$  with the definition of the absolute convergent series

$$g(x) := - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho(1-\rho)}, \quad (5.43)$$

which represents a continuous function for positive  $x$ . We determine  $g$  by performing for  $0 < x < 1$  its termwise derivative and making use of the functional equation  $\xi(s) = \xi(1-s)$  as follows

$$g'(x) = - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{x^{\rho-1}}{1-\rho} = - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{\left(\frac{1}{x}\right)^{1-\rho}}{1-\rho} = \eta\left(\frac{1}{x}\right). \quad (5.44)$$

For  $x = 1$  we obtain

$$g(1) = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{1}{\rho(1-\rho)} = 2(\log(2\pi) - 1) - (\gamma + \log \pi), \quad (5.45)$$

and thus for  $0 < x < 1$

$$g(x) = g(1) + \int_1^x \eta\left(\frac{1}{t}\right) dt. \quad (5.46)$$

We conclude for all  $x > 0$  the convergence of

$$\tilde{\eta}(x) := - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{x^\rho}{\rho} \quad (5.47)$$

by adding for  $0 < x < 1$  termwise the series for  $g(x)$  and von Mangoldt's series for  $-x\eta\left(\frac{1}{x}\right)$ ,

$$g(x) - x\eta\left(\frac{1}{x}\right) = \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \frac{x^\rho \cdot (\rho - 1)}{\rho(1-\rho)} = \tilde{\eta}(x). \quad (5.48)$$

Like  $\eta$ , the function  $\tilde{\eta}$  also describes the fluctuations of von Mangoldt's  $\psi$  function and also has a logarithmic singularity at  $x = 1$ . Note  $\tilde{\eta}(x) = \eta(x)$  for all  $x > 1$ . But the following "reflection formula" for  $\tilde{\eta}$  is valid for all  $x > 0$

$$\tilde{\eta}(x) + x \tilde{\eta}\left(\frac{1}{x}\right) = 2\tilde{\eta}(1) + \int_1^x \tilde{\eta}\left(\frac{1}{t}\right) dt. \quad (5.49)$$

Here we have  $2\tilde{\eta}(1) = g(1)$ . Instead of (5.39) we consider now for  $x, \delta > 0$  and  $\lambda \in \mathbb{C}$  the integral transform

$$(G_{\delta,\lambda} \tilde{\eta})(x) := \frac{1}{\sqrt{2\pi\delta}} \int_{-\infty}^{+\infty} e^{-\lambda u} \exp\left(-\frac{u^2}{2\delta}\right) \tilde{\eta}(xe^u) du. \quad (5.50)$$

Then the following Theorem gives a very simple explicit formula by making essentially use of Theorem 5.6(b):

**Theorem 5.7** *The representation of  $G_{\delta,\lambda} \tilde{\eta}$  in terms of the nontrivial zeros of the  $\zeta$  function reads*

$$(G_{\delta,\lambda} \tilde{\eta})(x) = - \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho}{\rho} \exp\left(\frac{\delta}{2}(\rho - \lambda)^2\right).$$

Proof: This results formally by termwise integration from (5.47) and (5.50). For a rigorous derivation we split the integral (5.50) into two parts, integrating from  $-\log x$  to  $\infty$  and from  $-\infty$  to  $-\log x$ , respectively. The first one is just  $(G_{\delta,\lambda} \eta)(x)$  in Theorem 5.6(b), and the second one can also be evaluated due to Theorem 5.6(b) by using the relation (5.48), the integral substitution  $x e^u = \frac{1}{x} e^{-v}$  and the symmetry relations  $\Phi(z) + \Phi(-z) = 1$ ,  $\xi(s) = \xi(1-s)$ .  
 ■

Both definitions of the fluctuation functions  $\eta$  and  $\tilde{\eta}$  are useful for different purposes. The function  $\eta$  has the advantage that its Mellin transform is well defined, whereas the representation formula for the regularization of  $\tilde{\eta}$  in Theorem 5.7 has a very simple structure.

## 6 Riemann's formula for $\pi_*(x)$

In terms of Stieltjes differentials there holds the following relation between von Mangoldt's function  $\psi(x)$  and Riemann's function  $\pi_*(x)$ , by Riemann denoted by  $f(x)$ ,

$$d\pi_*(x) \log x = d\psi(x). \quad (6.1)$$

This relation has the interesting consequence

$$\int_1^x \frac{\pi_*(t)}{t} dt = \pi_*(x) \log x - \psi(x), \quad (6.2)$$

which is valid for all  $x > 1$ . Using this identity and partial integration we rewrite the integral

$$\int_1^\infty \frac{\pi_*(x)}{x^{s+1}} dx = \int_1^\infty \frac{\pi_*(x)}{x} \cdot \frac{dx}{x^s} = s \int_1^\infty \left( \int_1^x \frac{\pi_*(t)}{t} dt \right) x^{-(s+1)} dx,$$

and obtain in view of (6.2) and (2.3)

$$\int_1^\infty \frac{\pi_*(x)}{x^{s+1}} dx = s \int_1^\infty \frac{\pi_*(x) \log x - \psi(x)}{x^{s+1}} dx \quad (6.3)$$

as well as

$$\zeta(s) = \exp \left( s^2 \int_1^\infty \frac{\pi_*(x) \log x - \psi(x)}{x^{s+1}} dx \right). \quad (6.4)$$

Assume that  $\rho \in \mathbb{C} \setminus [0, \infty)$  and  $\Re \rho < \Re s$ . Then we define for  $x > 1$

$$\varphi_\rho(x) := \gamma + \log(-\rho) + \log(\log x) + \text{Ei}_0(\rho \log x), \quad (6.5)$$

and obtain from Theorem 4.3 by partial integration analogous as for the derivation of (6.4) the two relations

$$\left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho}\right) = \exp \left( -s^2 \int_1^\infty \frac{\varphi_\rho(x) \log x - \frac{x^\rho}{\rho}}{x^{s+1}} dx \right) \quad (6.6)$$

as well as

$$\frac{\exp(-s)}{s-1} = \exp\left(s^2 \int_1^\infty \frac{\text{Li}(x) \log x - x}{x^{s+1}} dx\right) \quad (6.7)$$

with the logarithmic integral  $\text{Li}(x) := \text{Ei}(\log x)$ . The next equation follows from the product representation of  $1/\Gamma(s/2 + 1)$  and Theorem 4.4(b) if we calculate the sum of the expressions in (6.6) with respect to all trivial roots  $\rho = -2n$  of the  $\zeta$ -function, namely

$$\frac{\exp(-s \frac{\gamma}{2})}{\Gamma(\frac{s}{2} + 1)} = \exp\left(s^2 \int_1^\infty \frac{\left(\log x \cdot \int_x^\infty \frac{dt}{t(t^2-1)\log t} - \frac{1}{2} \log \frac{x^2}{x^2-1}\right)}{x^{s+1}} dx\right). \quad (6.8)$$

Finally we have due to Lemma 4.2(a) the identity

$$\frac{\exp(s \log(2\pi))}{2} = \exp\left(s^2 \int_1^\infty \frac{-(\log 2) \log x + \log(2\pi)}{x^{s+1}} dx\right). \quad (6.9)$$

The last four representations are valid for all complex  $s$  with  $\Re s > 1$ , where all the integrals are well defined in the Lebesgue sense. Especially the term  $\log x$  under the integral in equation (6.8) is very important since it eliminates the pole singularity at  $x = 1$  of the expression

$$\int_x^\infty \frac{dt}{t(t^2-1)\log t}.$$

Now we determine the product of the expressions (6.6) with respect to all nontrivial zeros  $\rho$  of the  $\zeta$ -function and obtain

$$\prod_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \left( \left(1 - \frac{s}{\rho}\right) \exp\left(\frac{s}{\rho}\right) \right) = 2 \xi(s) \pi^{s/2} \exp\left(s \left(1 + \frac{\gamma}{2} - \log(2\pi)\right)\right). \quad (6.10)$$

We note that due Theorem 4.5(b) the following series converges pointwise and absolutely for all  $x > 1$  according to

$$\begin{aligned}
& \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \left( \varphi_\rho(x) \log x - \frac{x^\rho}{\rho} \right) = \\
& \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \left( (\text{Ei}(\rho \log x) - i\pi \text{sign}(\Im \rho)) \log x - \frac{x^\rho}{\rho} \right) = \\
& \log x \int_0^\infty \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1}} \frac{x^\rho e^{-u}}{(\rho(\log x) - u)^2} du. \quad (6.11)
\end{aligned}$$

On the last line we could apply the Lebesgue dominated convergence theorem in view of Lemma 5.3(b).

For  $x > 1$  we denote the expression on the right hand side in (6.11) by  $\Delta(x)$ . We first conclude that  $\Delta(x)/x^{s+1}$  is Lebesgue integrable on each  $x$ -interval  $(x_0, \infty)$  with  $x_0 > 1$  for fixed  $s$  with  $\Re s > 1$ .

Assume for a while we know that  $\Delta(x)/x^{s+1}$  is Lebesgue integrable on the whole  $x$ -interval  $(1, \infty)$ . The product of the four expressions (6.7)-(6.10) gives  $\zeta(s)$  due to equation (5.5). We compare this product with the representation (6.4) for  $\zeta(s)$  to conclude from our assumption and from von Mangoldt's formula for  $\psi(x)$  with (6.11) and Theorem 5.1(a) that Riemann's formula for  $\pi_*(x)$  is also valid, namely

$$\pi_*(x) = \text{Li}(x) - \lim_{T \rightarrow \infty} \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq T}} \text{Ei}(\rho \log x) + \int_x^\infty \frac{dt}{t(t^2 - 1) \log t} - \log 2 \quad (6.12)$$

for all  $x > 1$  such that  $x \neq p^\alpha$  for all prime numbers  $p \geq 2$  and all integer exponents  $\alpha \geq 1$ . We have already shown the pointwise convergence of the right hand side in (6.12) for these values of  $x$ . It only remains to prove for our assumption that  $\Delta(x)/x^{s+1}$  is Lebesgue integrable on a finite interval  $(1, x_0)$  for a constant  $x_0 > 1$  and all  $s \in \mathbb{C}$  with  $\Re s > 1$ .

According to Lemma 5.3 we will choose  $x_0$  with  $1 < x_0 < e$  sufficiently small such that we can find constants  $c_1, c_2 > 0$  with

$$\sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq 1/\log x}} \frac{1}{|\Im \rho|} \leq c_1 \left( \log \left( \frac{1}{\log x} \right) \right)^2 \quad \forall x \in (1, x_0), \quad (6.13)$$

$$\sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| > 1/\log x}} \frac{1}{|\Im \rho|^2} \leq c_2 (\log x) \log \left( \frac{1}{\log x} \right) \quad \forall x \in (1, x_0). \quad (6.14)$$

Then we define the functions  $\Delta_1, \Delta_2 : (1, x_0) \rightarrow \mathbb{R}$  by

$$\Delta_1(x) := \log x \int_0^\infty \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq 1/\log x}} \frac{x^\rho e^{-u} du}{(\rho(\log x) - u)^2}, \quad (6.15)$$

$$\Delta_2(x) := \log x \int_0^\infty \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| > 1/\log x}} \frac{x^\rho e^{-u} du}{(\rho(\log x) - u)^2}. \quad (6.16)$$

Now we use again Theorem 4.5(b) in order to conclude for all  $x \in (1, x_0)$

$$\begin{aligned} |\Delta_1(x)| &\leq \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq 1/\log x}} \left\{ \log x |\varphi_\rho(x)| + \frac{x^{\Re \rho}}{|\rho|} \right\} \\ &\leq \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| \leq 1/\log x}} \left\{ \log x \frac{x^{\Re \rho}}{|(\log x) \Im \rho|} + \frac{x^{\Re \rho}}{|\rho|} \right\} \\ &\leq 2 c_1 x \left( \log \left( \frac{1}{\log x} \right) \right)^2. \end{aligned} \quad (6.17)$$

It follows that  $\Delta_1(x)/x^{s+1}$  is Lebesgue integrable on  $(1, x_0)$ , and in the same way we also conclude that  $\Delta_2(x)/x^{s+1}$  is Lebesgue integrable on  $(1, x_0)$ ,

$$|\Delta_2(x)| \leq \log x \sum_{\substack{\rho: \zeta(\rho)=0 \\ 0 < \Re \rho < 1 \\ |\Im \rho| > 1/\log x}} \frac{x^{\Re \rho}}{|(\log x) \Im \rho|^2} \leq c_2 x \log \left( \frac{1}{\log x} \right). \quad (6.18)$$

Since  $\Delta(x) = \Delta_1(x) + \Delta_2(x)$  for all  $x \in (1, x_0)$ , we have established Riemann's prime number formula (6.12).

Formula (6.12) is formulated in a wrong way in the literature, where the expressions  $\text{Ei}(\rho \log x)$  are replaced by  $\text{Li}(x^\rho)$ . For real  $\beta > 0$  we have indeed

that  $\text{Ei}(\beta \log x) = \text{Li}(x^\beta)$ . The expression  $\text{Ei}(\beta \log x)$  is analytic with respect to  $\beta$  in the half plane  $\Re\beta > 0$ , but **not** so  $\text{Li}(x^\beta)$ ,  $x > 1$  a fixed parameter. This is shown in the following two figures, where the section

$$S := \left\{ \beta = \frac{1}{2} + it \mid -10 \leq t \leq 10 \right\}$$

of the critical line is mapped by  $\text{Ei}(\beta \log x)$  as well as by  $\text{Li}(x^\beta)$  for  $x := 1000$ .

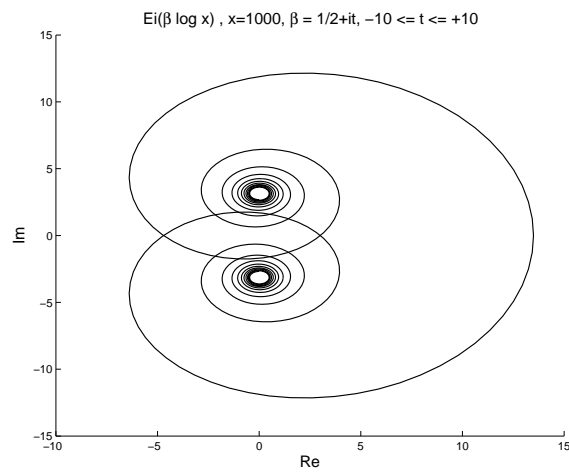


Figure 1: The correct Ei-curve

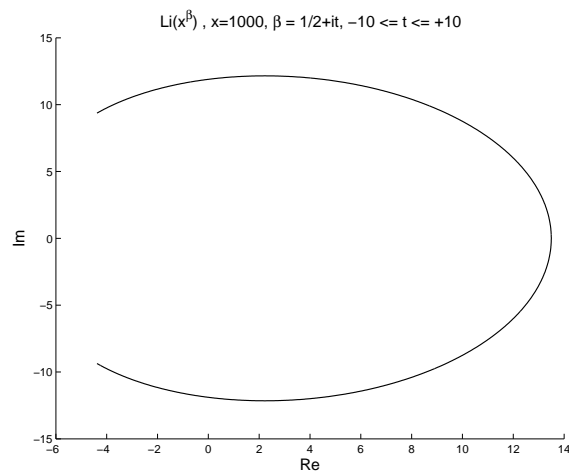


Figure 2: The incorrect Li-curve

The Ei-curve in Figure 1 indicates the correct convergence to the limits  $\pm i\pi$ . On the other hand, the Li-curve in Figure 2 has a jump at its boundary



points when  $x^\beta$  crosses the negative real axis for  $\Im\beta = \pm \frac{(2k+1)\pi}{\log x}$  and an integer number  $k \in \mathbb{Z}$ , and  $x^\beta$  describes a circle with radius  $\sqrt{x}$ . This shortcoming is independent on the branch of the logarithm, and we conclude that the analytic continuation principle cannot be applied here.

This may be disregarded for the following reasons. First of all numerical calculations of  $\pi_*(x)$  are performed due to the correct expression in (6.12) or due to correct approximations of (6.12). For numerical aspects of explicit prime number formulas and their relations to  $\pi(x)$  and  $\pi_*(x)$  we also refer to the book of Riesel [12] as well as to Lagarias, Miller, Odlyzko [9]. A second reason may be that Riemann's formulas for  $\pi_*(x)$  or  $\pi(x)$  are often neglected nowadays since they are equivalent to the simpler von Mangoldt formula for  $\psi(x)$ . We have used Hadamard's theorem for entire functions, Theorem 5.2 for the distribution of the zeros of  $\xi(s)$  and von Mangoldt's formula for  $\psi(x)$ , which are derived in the textbooks of Edwards [6] and Ingham [7].

## References

- [1] M. ABRAMOWITZ, I. STEGUN, *Handbook of Mathematical Functions*.
- [2] G. ANDREWS, R. ASKEY, R. ROY, *Special Functions*, Cambridge University Press (2000).
- [3] R. BACKLUND, *Sur les zéros de la fonction  $\zeta(s)$  de Riemann*, C.R. Acad. Sci. Paris **158**, 1979-1982 (1914).
- [4] C.K. CHUI, *An introduction to wavelets*, Academic Press (1992).
- [5] J. B. CONREY, *The Riemann Hypothesis*, Notices of the AMS, March 2003.
- [6] H. M. EDWARDS, *Riemann's Zeta Function*, Dover Publications, New York (2001).
- [7] A. E. INGHAM, *The distribution of prime numbers*, Cambridge University Press (2000).
- [8] M. KUNIK, *Logarithmic Fourier integrals for the Riemann zeta function*, Preprint No. 01/2008, Otto-von-Guericke University of Magdeburg, Faculty of Mathematics (2008). To appear in *New Directions in the Theory of universal Zeta- and L-Functions*, Proceeding of the Würzburg Conference, October 2008. Editors: J. Steuding, R. Steuding, Shaker Publishing (2009).
- [9] J.C. LAGARIAS, V.S. MILLER, AND A.M. ODLYZKO, *Computing  $\pi(x)$ : The Meissel-Lehmer method*, Math. Comp. **44**, 537-560 (1985).
- [10] H. VON MANGOLDT, *Zur Verteilung der Nullstellen der Riemannschen Funktion  $\xi(t)$* . Math. Ann **60**, 1-19 (1905).
- [11] B. RIEMANN, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse* (1859), in "Gesammelte Werke", Teubner, Leipzig (1892), reissued in Dover Books, New York (1953).
- [12] H. RIESEL, *Prime numbers and computer methods for factorization*. Birkhäuser, 2nd edition (1994).
- [13] E. C. TITCHMARSH, *The Theory of the Riemann Zeta-function*, Oxford University Press, New York (2001).