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Geodesic complexity of homogeneous Riemannian manifolds

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We study the geodesic motion planning problem for complete Riemannian manifolds and investigate their geodesic complexity, an integer-valued isometry invariant introduced by D Recio-Mitter. Using methods from Riemannian geometry, we establish new lower and upper bounds on geodesic complexity and compute its value for certain classes of examples with a focus on homogeneous Riemannian manifolds. To achieve this, we study properties of stratifications of cut loci and use results on their structures for certain homogeneous manifolds obtained by T Sakai and others.

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1 Introduction

A topological abstraction of the motion planning problem in robotics was introduced by M Farber [12]. The *topological complexity* of a path-connected space X is denoted by $TC(X)$ and intuitively given by the minimal number of open sets needed to cover

$X \times X$ such that, on each of the open sets, there exists a continuous motion planner. Here, a continuous motion planner is a map associating with each pair of points a continuous path from the first point to the second point which varies continuously with the endpoints. Such maps are interpreted as algorithms telling an autonomous robot in the workspace X how it is supposed to move from its position to a desired endpoint. Unfortunately, the topological complexity of a space does not tell us anything about the feasibility or efficiency of the paths taken by motion planners having $TC(X)$ domains of continuity; see the discussion of Z Błaszczyk and J Carrasquel-Vera [3, Introduction]. For example, the explicitly constructed motion planners for configuration spaces of Euclidean spaces by H Mas-Ku and E Torres-Giese [29] and Farber [16, Section 8] require few domains of continuity, but have paths among their values which are far from being length-minimizing. Considering a general metric space, paths taken by the motion planners might become arbitrarily long and thus be unsuited for practical motion planning problems.

Recently, D Recio-Mitter [34] has introduced the notion of *geodesic complexity* of metric spaces. There, the paths taken by motion planners are additionally required to be length-minimizing between their endpoints. Intuitively, this is seen as the complexity of *efficient* motion planning in metric spaces. Recio-Mitter's seminal article has already triggered research in geodesic complexity, especially computations of geodesic complexity for interesting classes of examples; see Davis, Harrison and Recio-Mitter [8; 9; 10].

In this article we study the geodesic complexity of complete Riemannian manifolds and derive new lower and upper bounds for their geodesic complexities by methods from Riemannian geometry.

Before continuing, we recall the definition of geodesic complexity of geodesic spaces from [34, Definition 1.7] for the special case of a complete Riemannian manifold. Let (M, g) be a complete connected Riemannian manifold and let $PM := C^0([0, 1], M)$ be equipped with the compact-open topology. We recall that a geodesic segment $\gamma: [0, 1] \rightarrow M$ is called *minimal* if it minimizes the length compared to all rectifiable paths from $\gamma(0)$ to $\gamma(1)$. For simplicity, we shall call a minimal geodesic segment simply a *minimal geodesic*. Consider

$$GM := \{\gamma \in PM \mid \gamma \text{ is a minimal geodesic in } (M, g)\}$$

as a subspace of PM and let

$$\pi: GM \rightarrow M \times M, \quad \pi(\gamma) = (\gamma(0), \gamma(1)).$$

By standard results from Riemannian geometry, π is surjective since (M, g) is complete; see Petersen [33, Corollary 5.8.5]. The *geodesic complexity* of (M, g) is given by $\text{GC}(M, g) = r$, where $r \in \mathbb{N}$ is the smallest integer with the following property: there are r pairwise disjoint locally compact subsets $E_1, \dots, E_r \subset M \times M$ with $\bigcup_{i=1}^r E_i = M \times M$ such that, for each $i \in \{1, 2, \dots, r\}$, there exists a continuous geodesic motion planner $s_i: E_i \rightarrow GM$, ie a continuous local section of the map π . If there is no such r , we let $\text{GC}(M, g) = +\infty$. Since it is not at all evident how to compute this number explicitly, one is interested in establishing lower and upper bounds for $\text{GC}(M, g)$. This approach is also common in studies of Lusternik–Schnirelmann category or, more generally, sectional categories of fibrations. Given a fibration $p: E \rightarrow B$, the *sectional category* of p is given by $\text{secat}(p) = k$, where $k \in \mathbb{N}$ is the minimal number with the following property: there exists an open cover of B consisting of k open subsets such that p admits a continuous local section over each of these sets. This notion was introduced under the name *genus of a fibration* by A Schwarz [39]. The topological complexity of a topological space X is for example given as the sectional category of the fibration

$$PX \rightarrow X \times X, \quad \gamma \mapsto (\gamma(0), \gamma(1)).$$

Schwarz worked out several ways of obtaining lower and upper bounds for sectional categories which have direct consequences for topological complexity; see eg Farber [14; 15, Chapter 4] for an overview.

However, the restriction $\pi: GM \rightarrow M \times M$ of this fibration to minimal geodesics is in general *not* a fibration. For example, if $M = S^n$ is an n -sphere with $n \in \mathbb{N}$, equipped with a round metric, then $\pi^{-1}(\{(p, q)\})$ consists of one element if $q \neq -p$, while it is homeomorphic to S^{n-1} if $q = -p$. In particular, not all preimages are homotopy-equivalent, so π is not a fibration in this case. Therefore, Schwarz's results are not applicable to the setting of geodesic complexity. Instead we will derive several lower and upper bounds for the geodesic complexity of Riemannian manifolds using methods from Riemannian geometry. By [34, Remark 1.9], every complete Riemannian manifold satisfies $\text{TC}(M) \leq \text{GC}(M, g)$. This formalizes the observation that requiring the paths a robot takes to be as short as possible can increase the complexity of the problem. For example, as shown in [34, Theorem 1.11], for each $n \geq 3$ there exists a Riemannian metric g_n on the sphere S^n for which $\text{GC}(S^n, g_n) - \text{TC}(S^n) \geq n - 3$. In practical applications, a person designing robotic systems that are supposed to move autonomously might not mind a higher complexity. In fact, such a person might accept more instabilities in the motions of robots as a downside if the upside is that the robots move fast and efficiently.

An important observation is that the difficulties of geodesic motion planning lie in the cut loci of (M, g) , as was pointed out by Recio-Mitter [34, page 144] in the more general framework of metric spaces. Let $\text{Cut}_p(M)$ denote the cut locus of $p \in M$ in (M, g) . We refer to Lee [27, page 308], Petersen [33, page 219] or Definition 2.5 below for its definition. If $A \subset M \times M$ satisfies $q \notin \text{Cut}_p(M)$ for each $(p, q) \in A$, then there is a unique minimal geodesic from p to q for each $(p, q) \in A$. The corresponding geodesic motion planner $A \rightarrow GM$ is continuous; see also the observations of Błaszczuk and Carrasquel-Vera [3]. Thus, to compute the geodesic complexity of a manifold, we need to understand its cut loci. While the cut locus of a point in a Riemannian manifold is always closed and of measure zero — see [27, Theorem 10.34(a)] — little else is known about cut loci in general.

In [34, Corollary 3.14], Recio-Mitter establishes a lower bound on the geodesic complexity of metric spaces given in terms of the structure of their cut loci. He considers cut loci which possess stratifications admitting finite coverings. For this purpose, Recio-Mitter introduces the notion of a *levelwise stratified covering* in [34, Definition 3.8]. He then defines a notion of *inconsistency*, which is roughly a condition on the relations between the coverings of the different strata of cut loci by minimal geodesics. It formalizes certain incompatibility properties of families of geodesics connecting a point with points in its cut locus.

Focusing on complete Riemannian manifolds, we will use Riemannian exponential maps to establish a similar inconsistency condition on cut loci, which is more concise than the one from [34]. Given a complete Riemannian manifold M and a point $p \in M$ for which $\text{Cut}_p(M)$ admits a stratification, we study the preimages of the different strata of $\text{Cut}_p(M)$ under the Riemannian exponential map $\exp_p : T_p M \rightarrow M$. Assume that some $x \in M$ lies in the closure of multiple connected components of the same stratum of $\text{Cut}_p(M)$. We then study the closures of the preimages of all of these components under \exp_p as subsets of $T_p M$. The inconsistency condition demands that these closures have no point in common that is mapped to x by \exp_p . We will see that this condition excludes the existence of an open neighborhood U of x with a single continuous geodesic motion planner which connects p to all points of $\text{Cut}_p(M)$ that lie in U .

Note that our definition is only applicable to Riemannian manifolds and not to arbitrary geodesic spaces. One of its benefits in the Riemannian setting is the fact that we can deduce an easier condition than the one introduced by Recio-Mitter. More precisely, we do not require anymore that any point in a cut locus of another point is connected

to that point by only finitely many minimal geodesics. Moreover, our inconsistency condition is explicitly stated as an intersection condition on certain subsets of a tangent cut locus, instead of using the notion of levelwise stratified coverings as in [34].

Our main result on inconsistent stratifications is the following theorem. This result is similar to [34, Corollary 3.14] and our proof is inspired by Recio-Mitter’s proof as well.

Theorem 4.8 *Let (M, g) be a closed Riemannian manifold. Assume that there exists a point $p \in M$ for which $\text{Cut}_p(M)$ admits an inconsistent stratification of depth $N \in \mathbb{N}$. Then*

$$\text{GC}(M) \geq N + 1.$$

There is more to say about cut loci of *homogeneous* Riemannian manifolds, ie Riemannian manifolds (M, g) whose isometry groups act transitively on M . An isometry $\phi: M \rightarrow M$ maps the cut locus of $p \in M$ onto that of $\phi(p)$. Hence, the cut locus of a point is identified with that of another point by an isometry. This translation property of the cut loci allows us to estimate the geodesic complexity of M from above, once we understand how we can decompose one single cut locus into domains of continuous geodesic motion planners. The following result provides an upper bound for geodesic complexity in terms of a sectional category and the subspace geodesic complexities of considerably smaller subsets of $M \times M$. Here, the subspace geodesic complexity of $A \subset M \times M$ is defined in terms of covers of A by domains of continuous geodesic motion planners.

Corollary 5.8 *Let (M, g) be a homogeneous Riemannian manifold with isometry group $\text{Isom}(M, g)$. Let $p \in M$ and assume that $\text{Cut}_p(M)$ has a stratification (S_1, \dots, S_k) of depth k . Then*

$$\text{GC}(M) \leq \text{secat}(\text{ev}_p: \text{Isom}(M, g) \rightarrow M) \cdot \sum_{i=1}^k \max_{Z_i \in \pi_0(S_i)} \text{GC}_p(Z_i) + 1,$$

where $\text{ev}_p(\phi) = \phi(p)$ for all $\phi \in \text{Isom}(M, g)$ and where $\text{GC}_p(Z_i)$ is the subspace geodesic complexity of $\{p\} \times Z_i \subset M \times M$.

In the case of compact, simply connected, irreducible symmetric spaces, we are able to further estimate this upper bound from above in terms of certain sectional categories. This means that for such symmetric spaces we obtain an upper bound on $\text{GC}(M)$ which does not involve any geodesic complexities.

Note that this result produces the first upper bound for geodesic complexity in terms of categorical invariants. Indeed, the only previously known upper bounds were derived by Recio-Mitter [34] either from explicit constructions of geodesic motion planners or from the existence of particularly simple coverings of cut loci. We pick up Recio-Mitter's so-called trivially covered stratifications in this article in the setting of Riemannian manifolds as well.

In addition to establishing new lower and upper bounds for geodesic complexity, we compute the geodesic complexities of some Riemannian manifolds whose cut loci are well understood. We will show that every three-dimensional Berger sphere (S^3, g_α) satisfies $GC(S^3, g_\alpha) = 2$ and that $GC(T^2, g_f) = 3$ for every flat metric g_f on the two-dimensional torus. This extends the two-dimensional case of Recio-Mitter's computation of the geodesic complexity of the standard flat n -torus from [34, Theorem 4.4].

The article is structured as follows: In [Section 2](#) we introduce some additional terminology and recall elementary facts about geodesic complexity and cut loci. [Section 3](#) contains some basic nonexistence results on continuous geodesic motion planners. These results illustrate the difficulties for motion planning that cut loci can create. In [Section 4](#) we establish lower bounds on geodesic complexity by two different approaches. On the one hand, this is done in terms of principal bundles over the manifold and the topological complexities of their total spaces. On the other hand, we study manifolds with stratified cut loci whose stratifications satisfy the above-mentioned inconsistency property. We focus on homogeneous Riemannian manifolds in [Section 5](#). More precisely, we show that their geodesic complexities can be estimated from above in terms of the subspace complexities of a single cut locus. In [Section 6](#) we consider Riemannian manifolds whose cut loci admit trivially covered stratifications. For such stratifications the relations between a cut locus and its corresponding tangent cut locus are particularly simple. [Section 7](#) deals with examples of geodesic complexities. Combining results from the previous sections with new observations, we reobtain Recio-Mitter's computation of geodesic complexity of the standard flat n -torus and determine the geodesic complexity of arbitrary flat 2-tori. As another class of examples, we explicitly compute the geodesic complexity of three-dimensional Berger spheres. In the final [Section 8](#) we consider consequences of the previous results for compact simply connected symmetric spaces. In both situations, the considered cut loci have been studied by T Sakai. Using the estimates from [Section 5](#), we derive an upper bound for geodesic complexity that is given in terms of the Lie groups from which the

symmetric space is built. We further make explicit computations for two examples of symmetric spaces.

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Throughout this article we assume all manifolds to be smooth and connected and all Riemannian metrics to be smooth.

2 Basic notions and definitions

We begin by introducing subspace versions of geodesic complexity for Riemannian manifolds. Afterwards, we recall some basic computations from [34] and several facts about cut loci in Riemannian manifolds.

Definition 2.1 Let (M, g) be a complete Riemannian manifold and $\pi : GM \rightarrow M \times M$, $\pi(\gamma) = (\gamma(0), \gamma(1))$. Let GM be equipped with the subspace topology of $C^0([0, 1], M)$ with the compact–open topology.

- (a) Let $X \subset M \times M$. A *geodesic motion planner on X* is a section $s : X \rightarrow GM$ of π .
- (b) Given $A \subset M \times M$ we let $GC_{(M,g)}(A)$ be the minimum $r \in \mathbb{N}$ for which there are r pairwise disjoint locally compact subsets $E_1, \dots, E_r \subset M \times M$ such that $A \subset \bigcup_{i=1}^r E_i$ and, for each $i \in \{1, 2, \dots, r\}$, there exists a continuous geodesic motion planner $s_i : E_i \rightarrow GM$. If no such r exists, then we put $GC_{(M,g)}(A) := +\infty$. We call $GC_{(M,g)}(A)$ the *subspace geodesic complexity of A* .

We recall that the map π is surjective for complete Riemannian manifolds. This is a consequence of the Hopf–Rinow theorem; see [33, Corollary 5.8.5].

Remarks 2.2 (1) If it is obvious which Riemannian metric we are referring to, we occasionally suppress it from the notation and write

$$GC(M) := GC(M, g) \quad \text{and} \quad GC_M(A) := GC_{(M,g)}(A).$$

Note that, in particular, $GC(M) = GC_M(M \times M)$.

(2) Given $p \in M$ and $B \subset M$, we further put

$$GC_p(B) := GC_{(M,g)}(\{p\} \times B).$$

(3) Our definition differs from Recio-Mitter’s original definition by 1 in the sense that for us $GC(\{pt\}) = 1$, while it would be 0 in the sense of [34, Definition 1.7].

Examples 2.3 (1) As proven in [34, Proposition 4.1], if g_r is a round metric on the sphere S^n with $n \in \mathbb{N}$, then

$$GC(S^n, g_r) = TC(S^n) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

(2) Let g_f be the standard flat metric on T^2 and let g_{emb} denote the metric induced by the standard embedding $T^2 \hookrightarrow \mathbb{R}^3$ and the Euclidean metric on \mathbb{R}^3 . By [34, Theorems 4.4 and 5.1],

$$GC(T^2, g_f) = 3, \quad GC(T^2, g_{\text{emb}}) = 4.$$

(3) It was further shown in [34, Theorem 1.11] that, for each $k \in \mathbb{N}$ with $k \geq 3$, there exists a Riemannian metric g_k on S^k with $GC(S^k, g_k) \geq k$.

Remarks 2.4 Let (M, g) be a complete Riemannian manifold.

(1) For all $A \subset M \times M$, it holds that $TC_M(A) \leq GC_M(A)$, where $TC_M(A)$ is the relative topological complexity of A in $M \times M$; see [15, Section 4.3]. Here, we made use of the characterization of topological complexity by locally compact subsets shown in [15, Proposition 4.9].

(2) It is easy to see that

$$(2-1) \quad GC_M(A \cup B) \leq GC_M(A) + GC_M(B) \quad \text{for all } A, B \subset M \times M.$$

This is shown in analogy with [15, Proposition 4.24].

As pointed out by Recio-Mitter, the crucial ingredients for the discussion of geodesic complexity are the cut loci of points in the space under consideration. The notions of cut loci in metric and in Riemannian geometry are slightly different from each other. While Recio-Mitter used the former notion in his work — see [34, Definition 3.1] — we will use the latter throughout this manuscript. We next recall the notion of cut loci from Riemannian geometry. The relation between the two will be discussed in Remark 2.7(3) below. See also [27, page 308] or [33, page 219] for the following definition:

Definition 2.5 Let (M, g) be a complete Riemannian manifold and let $p \in M$.

- (a) Let $\gamma: [0, +\infty) \rightarrow M$ be a unit-speed geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) \in T_p M$. The *cut time* of γ is given by

$$t_{\text{cut}}(\gamma) = \sup\{t > 0 : \gamma|_{[0,t]} \text{ is minimal}\}.$$

If $t_{\text{cut}}(\gamma)$ is finite, then $t_{\text{cut}}(\gamma)\dot{\gamma}(0) \in T_p M$ is a *tangent cut point* of p and $\gamma(t_{\text{cut}}(\gamma)) \in M$ is a *cut point* of p along γ . Note that

$$\gamma(t_{\text{cut}}(\gamma)) = \exp_p(t_{\text{cut}}(\gamma)\dot{\gamma}(0)).$$

- (b) The set of all cut points of p is called the *cut locus* of p and denoted by $\text{Cut}_p(M)$. The set of all tangent cut points of p is called the *tangent cut locus* of p and denoted by $\widetilde{\text{Cut}}_p(M)$.
- (c) The *total cut locus* of M is given by

$$\text{Cut}(M) := \bigcup_{p \in M} (\{p\} \times \text{Cut}_p(M)) \subset M \times M.$$

Example 2.6 Let $n \in \mathbb{N}$ and let g be a round metric on the sphere S^n . Then, by [27, Example 10.30(a)], $\text{Cut}_p(S^n) = \{-p\}$ for every $p \in S^n$.

Further examples of cut loci will appear in the upcoming sections.

Remarks 2.7 Let (M, g) be a complete Riemannian manifold.

- (1) In general, $\text{Cut}_p(M)$ does not need to be a submanifold of M . H Gluck and D Singer [20, Theorem A] have shown that, if $\dim M \geq 2$, then there exists a Riemannian metric on M and a point $p \in M$ for which $\text{Cut}_p(M)$ is not triangulable.
- (2) By [34, Theorem 3.3], there exists a continuous geodesic motion planner

$$(M \times M) \setminus \text{Cut}(M) \rightarrow GM,$$

from which Recio-Mitter derived that $\text{GC}(M) = 1$ if $\text{Cut}(M) = \emptyset$. By [3, Lemma 4.2], $(M \times M) \setminus \text{Cut}(M)$ is open and therefore locally compact. Using (2-1), this shows that

$$\text{GC}(M, g) \leq \text{GC}_M(\text{Cut}(M)) + 1.$$

- (3) Let $p \in M$. By [2, page 133], the set of points $q \in M$ such that there is more than one minimal geodesic from p to q is a dense subset of $\text{Cut}_p(M)$. This set is also called the *ordinary cut locus* of p . In metric geometry — in particular in

[34, Definition 3.1]—the ordinary cut locus of a point is called its cut locus. The reader should thus keep in mind that the cut locus of a point, as considered in [34], is *not* the cut locus of a point in the sense of this article, but a dense subset of the cut locus.

3 Nonexistence results for geodesic motion planners

We begin our study by discussing two nonexistence results showing that certain subsets of a Riemannian manifold never admit continuous geodesic motion planners. First, we will study complete oriented Riemannian manifolds and see that the Euler class obstructs the existence of some geodesic motion planners. Then we will show that a complete Riemannian manifold (M, g) has the following property: if a subset $A \subset M \times M$ contains an element of the total cut locus in its interior, then there will be no continuous geodesic motion planner on A . Before doing so, we first want to establish a technical proposition that we will make frequent use of throughout the article.

Definition 3.1 Let (M, g) be a complete Riemannian manifold. We call the map

$$v: GM \rightarrow TM, \quad v(\gamma) = \dot{\gamma}(0),$$

the *velocity map of GM*.

Proposition 3.2 Let (M, g) be a complete Riemannian manifold. The velocity map $v: GM \rightarrow TM$ is continuous.

Proof Let $(\gamma_n)_{n \in \mathbb{N}}$ be a convergent sequence in GM and let $\gamma := \lim_{n \rightarrow \infty} \gamma_n \in GM$. By our choice of topology on GM , this means that

$$(3-1) \quad \lim_{n \rightarrow \infty} \gamma_n(t) = \gamma(t) \quad \text{for all } t \in [0, 1].$$

We need to show that $\lim_{n \rightarrow \infty} v(\gamma_n) = v(\gamma)$. Let $\mathcal{L}_g: GM \rightarrow \mathbb{R}$ denote the length of a minimal geodesic with respect to g . From the minimality property of the curves, we derive that

$$\lim_{n \rightarrow \infty} \mathcal{L}_g(\gamma_n) = \lim_{n \rightarrow \infty} d_M(\gamma_n(0), \gamma_n(1)) = d_M(\gamma(0), \gamma(1)) = \mathcal{L}_g(\gamma),$$

where $d_M: M \times M \rightarrow \mathbb{R}$ is the distance function induced by g . Let $|\cdot|: TM \rightarrow \mathbb{R}$ denote the fiberwise norm induced by g . Since $\mathcal{L}_g(\alpha) = |\dot{\alpha}(0)| = |v(\alpha)|$ for each $\alpha \in GM$, it follows that

$$(3-2) \quad \lim_{n \rightarrow \infty} |v(\gamma_n)| = |v(\gamma)|.$$

To show the continuity of v , we need to derive that $\lim_{n \rightarrow \infty} v(\gamma_n) = v(\gamma)$. Let

$$\text{Exp}: TM \rightarrow M \times M, \quad \text{Exp}(p, v) = (p, \exp_p(v)),$$

be the extended exponential map. Let $K \subset M$ be a compact neighborhood of $\gamma(0)$ and let

$$\rho_0 := \sup\{r > 0 : \exp_x|_{B_r(0)} \text{ is injective for all } x \in K\},$$

where $B_r(0)$ denotes the open n -ball around the origin in the respective tangent space. Since K is compact, $\rho_0 > 0$ by [27, Lemma 6.16]. For $r \in (0, \rho_0)$ we put

$$D_r K := \{(p, v) \in TM \mid p \in K, \|v\| \leq r\},$$

ie $D_r K$ is the closed disk bundle over K of radius r . Then Exp maps $D_r K$ diffeomorphically onto its image

$$V_r K := \text{Exp}(D_r K) = \{(p, q) \in K \times M \mid d_M(p, q) \leq r\}.$$

Let $\text{Exp}_K: D_r K \rightarrow V_r K$ be the corresponding restriction of Exp . Since $\text{Exp}_K: D_r K \rightarrow V_r K$ is a diffeomorphism, its inverse $\text{Exp}_K^{-1}: V_r K \rightarrow D_r K$ is a diffeomorphism as well. Thus, if we choose and fix a distance function $d_{TM}: TM \times TM \rightarrow \mathbb{R}$ which induces the topology of TM , then $\text{Exp}_K^{-1}: V_r K \rightarrow D_r K$ is locally Lipschitz-continuous with respect to $d_M \times d_M$ and d_{TM} . We further observe that, for all $\alpha \in GM$ with $\alpha(0) \in K$ and $d_M(\alpha(0), \alpha(1)) \leq r$,

$$\text{Exp}_K^{-1}(\alpha(0), \alpha(1)) = v(\alpha).$$

We consider two different cases:

Case 1 Assume that $|v(\gamma)| < r$. This implies that $(\gamma(0), \gamma(1)) \in V_r K$. Then, by (3-2), there exists $n_0 \in \mathbb{N}$ with

$$\gamma_n(0) \in K \quad \text{and} \quad |v(\gamma_n)| \leq r \quad \text{for all } n \geq n_0.$$

Thus, $(\gamma_n(0), \gamma_n(1)) \in V_r K$ for all $n \geq n_0$. Let C be a local Lipschitz constant for Exp_K^{-1} in a neighborhood of $(\gamma(0), \gamma(1))$. Then, for sufficiently big $n \in \mathbb{N}$,

$$\begin{aligned} d_{TM}(v(\gamma_n), v(\gamma)) &= d_{TM}(\text{Exp}_K^{-1}(\gamma_n(0), \gamma_n(1)), \text{Exp}_K^{-1}(\gamma(0), \gamma(1))) \\ &\leq C (d_M(\gamma_n(0), \gamma(0)) + d_M(\gamma_n(1), \gamma(1))). \end{aligned}$$

By (3-1), this yields $\lim_{n \rightarrow \infty} d_{TM}(v(\gamma_n), v(\gamma)) = 0$, which we wanted to show.

Case 2 Consider the case that $|v(\gamma)| \geq r$. By (3-2), there exists $n_1 \in \mathbb{N}$ such that

$$\gamma_n(0) \in K \quad \text{and} \quad |v(\gamma_n)| < |v(\gamma)| + 1 \quad \text{for all } n \geq n_1.$$

Let $c := r/(|v(\gamma)| + 1) \in (0, 1)$ and put $\xi_n := c \cdot v(\gamma_n)$ for each $n \in \mathbb{N}$ and $\xi := c \cdot v(\gamma)$. Then $\xi \in D_r K$ and $\xi_n \in D_r K$ for all $n \geq n_1$. If we define

$$\tilde{\gamma}_n, \tilde{\gamma}: [0, 1] \rightarrow M, \quad \tilde{\gamma}_n(t) := \gamma_n(ct), \quad \tilde{\gamma}(t) := \gamma(ct) \quad \text{for all } t \in [0, 1],$$

then $\tilde{\gamma}, \tilde{\gamma}_n \in GM$ with $v(\tilde{\gamma}) = \xi$ and $v(\tilde{\gamma}_n) = \xi_n$ for each $n \geq n_1$. Since $(\gamma_n)_{n \in \mathbb{N}}$ converges to γ in the C^0 -topology, it easily follows that $\lim_{n \rightarrow \infty} \tilde{\gamma}_n = \tilde{\gamma}$ in the C^0 -topology as well. Thus, it follows from Case 1 that $\lim_{n \rightarrow \infty} \xi_n = \xi$, which obviously yields $\lim_{n \rightarrow \infty} v(\gamma_n) = v(\gamma)$. □

In the following proposition, we observe that the Euler class of an oriented manifold can obstruct the existence of geodesic motion planners:

Proposition 3.3 *Let (M, g) be a complete, oriented Riemannian manifold whose Euler class is nonvanishing. Let $f: M \rightarrow M$ be a continuous map with $f(p) \neq p$ for all $p \in M$. If $A \subset M \times M$ satisfies $\text{graph } f \subset A$, then there will be no continuous geodesic motion planner on A .*

Proof Assume by contradiction that there exists a continuous geodesic motion planner $s: A \rightarrow GM$. Then, by Proposition 3.2, the map

$$g: M \rightarrow TM, \quad g(p) = (v \circ s)(p, f(p)),$$

is a continuous vector field, where v is the velocity map. Since $f(p) \neq p$ for each p , the geodesic $s(p, f(p))$ is nonconstant for all $p \in M$. Hence, $g(p) \neq 0$ for all $p \in M$. But such a vector field cannot exist since the Euler class of M is nonvanishing. □

Corollary 3.4 *Let (M, g) be a complete, oriented manifold whose Euler class is nonvanishing. Let $f: M \rightarrow M$ be continuous and fixed-point-free. Then, for every Riemannian metric g on M , there exists $p \in M$ with $f(p) \in \text{Cut}_p(M, g)$.*

Proof Assume by contradiction that there is such a metric g for which $f(p) \notin \text{Cut}_p(M, g)$ for all $p \in M$. Then $\text{graph } f$ lies in $(M \times M) \setminus \text{Cut}(M, g)$. But, since there exists a continuous geodesic motion planner on $(M \times M) \setminus \text{Cut}(M, g)$ —see Remark 2.7(3)—this contradicts Proposition 3.3. Hence, such a metric does not exist. □

Corollary 3.5 *Let $n \in \mathbb{N}$. For every Riemannian metric g on S^{2n} there exists $p \in S^{2n}$, such that $-p \in \text{Cut}_p(S^{2n}, g)$.*

Proof Apply Corollary 3.4 to the case of $M = S^{2n}$ and $f(x) = -x$. □

Remark 3.6 Our Corollary 3.4 is complementary to results of M Frumosu and S Rosenberg from [17, page 338], who studied far-point sets, ie sets of points mapped

to their cut loci under self-maps of a Riemannian manifold, in a very general way. Frumosu and Rosenberg focused on self-maps whose far-point sets are infinite and established connections to the Lefschetz numbers of such maps.

In [34, Remark 3.17], Recio-Mitter mentioned that, whenever a subset of $M \times M$ contains a point of the total cut locus in its interior, there is no continuous geodesic motion planner defined on that subset. For the sake of completeness, we report here a proof in the case of Riemannian manifolds.

Proposition 3.7 *Let (M, g) be a complete Riemannian manifold, $p \in M, q \in \text{Cut}_p(M)$ and let $U \subset M$ be an open neighborhood of q . Then there is no continuous geodesic motion planner on $\{p\} \times U$.*

Proof As discussed in Remark 2.7(3), the set of points $r \in M$ for which there is more than one minimal geodesic from p to r is dense in $\text{Cut}_p(M)$. Hence, U contains a point q_0 such that there are at least two minimal geodesics from p to q_0 . In the following, we thus assume without loss of generality that q itself has this property. Assume that a continuous geodesic motion planner $s: \{p\} \times U \rightarrow GM$ existed. By our choice of q , there are $\gamma_1, \gamma_2 \in GM$ with

$$\gamma_1 \neq \gamma_2, \quad \gamma_1(0) = \gamma_2(0) = p \quad \text{and} \quad \gamma_1(1) = \gamma_2(1) = q.$$

Let $(t_n)_{n \in \mathbb{N}}$ be a sequence in $(0, 1)$ with $\lim_{n \rightarrow \infty} t_n = 1$ and $\gamma_1(t_n), \gamma_2(t_n) \in U$ for all $n \in \mathbb{N}$. One checks without difficulties that $\gamma_1(t) \neq \gamma_2(t)$ for all $t \in (0, 1)$, so that, in particular, $\gamma_1(t_n) \neq \gamma_2(t_n)$ for all $n \in \mathbb{N}$.

By definition of a cut locus, it follows for all $r \in (0, 1)$ and $i \in \{1, 2\}$ that

$$\gamma_{i,r} \in GM, \quad \gamma_{i,r}(t) := \gamma_i(rt),$$

is the unique minimal geodesic from p to $\gamma_i(r)$. In particular, this shows that necessarily

$$(3-3) \quad s(p, \gamma_i(t_n)) = \gamma_{i,t_n} \quad \text{for all } n \in \mathbb{N}, i \in \{1, 2\}.$$

Let $v: GM \rightarrow TM$ be the velocity map. It follows from Proposition 3.2 that

$$v \circ s: \{p\} \times U \rightarrow TM$$

is continuous. Since $\gamma_1 \neq \gamma_2$, there are $\xi_1, \xi_2 \in T_p M$ with $\xi_1 \neq \xi_2$ such that $\gamma_1(t) = \exp_p(t\xi_1)$ and $\gamma_2(t) = \exp_p(t\xi_2)$ for all $t \in [0, 1]$. By (3-3) and the fact that the differential of \exp_p in 0 is $\text{id}_{T_p M}$, we thus obtain that

$$\lim_{n \rightarrow \infty} (v \circ s)(p, \gamma_i(t_n)) = \lim_{n \rightarrow \infty} \dot{\gamma}_{i,t_n}(0) = \lim_{n \rightarrow \infty} t_n \xi_i = \xi_i.$$

In particular, $\lim_{n \rightarrow \infty} (v \circ s)(p, \gamma_1(t_n)) \neq \lim_{n \rightarrow \infty} (v \circ s)(p, \gamma_2(t_n))$. This contradicts the continuity of s , since by assumption $(v \circ s)(p, \gamma_1(1)) = (v \circ s)(p, \gamma_2(1))$. Thus, such a continuous s does not exist. \square

This proposition has an immediate consequence in terms of geodesic complexity.

Corollary 3.8 *Let (M, g) be a complete Riemannian manifold and let $A \subset M \times M$ be a locally compact subset with*

$$\text{int}(A) \cap \text{Cut}(M) \neq \emptyset,$$

where $\text{int}(A)$ is the interior of A as a subset of $M \times M$. Then $\text{GC}_M(A) \geq 2$.

Proof Assume that there was a continuous geodesic motion planner $s: A \rightarrow GM$. Let $(p, q) \in \text{int}(A) \cap \text{Cut}(M)$. By definition of the product topology, there are open neighborhoods U of p and V of q with $U \times V \subset \text{int}(A)$, so, in particular, $s|_{\{p\} \times V}$ would be a continuous geodesic motion planner. Since $q \in \text{Cut}_p(M)$, this contradicts Proposition 3.7, so there is no such motion planner. This shows that $\text{GC}_M(A) \geq 2$. \square

Remark 3.9 There is another connection between cut loci and another numerical invariant, namely the Lusternik–Schnirelmann category of a Riemannian manifold M , which we denote by $\text{cat}(M)$. Here, we use the convention that $\text{cat}(X) = 1$ if X is contractible. One observes that $M \setminus \text{Cut}_p(M)$ is contractible for all $p \in M$, which follows from [27, Theorem 10.34(c)]. If $p_1, \dots, p_k \in M$ satisfy $\bigcap_{i=1}^k \text{Cut}_{p_i}(M) = \emptyset$, then

$$\{M \setminus \text{Cut}_{p_1}(M), \dots, M \setminus \text{Cut}_{p_k}(M)\}$$

will be an open cover of M by contractible subsets, and hence $\text{cat}(M) \leq k$. By contraposition this shows that, if $\text{cat}(M) \geq k + 1$ for some $k \in \mathbb{N}$, then, for every choice of $p_1, \dots, p_k \in M$,

$$\bigcap_{i=1}^k \text{Cut}_{p_i}(M) \neq \emptyset.$$

4 Lower bounds for geodesic complexity

Lower bounds on topological complexity are mostly derived from the cohomology rings of a space. In this section, we derive lower bounds on geodesic complexity

from the Riemannian structures of manifolds. We first establish a result involving a principal bundle over the manifold under consideration. By explicitly constructing motion planners, we will establish a lower bound on geodesic complexity in terms of categorical invariants of total space and fiber of the bundle. Afterwards, we will establish the notion of inconsistent stratification that we lined out in the introduction. Then we will go on to prove the second theorem stated in that introduction.

We first establish a technical lemma, whose proof follows that of [14, Theorem 13.1].

Lemma 4.1 *Let E and X be topological spaces, let $p: E \rightarrow X$ be a fibration with $r := \text{secat}(p) < +\infty$ and assume that X is normal. Then there are pairwise disjoint locally compact subsets $A_1, \dots, A_r \subset X$ with $X = \bigcup_{i=1}^r A_i$ such that, for each $i \in \{1, 2, \dots, r\}$, there exists a continuous local section $A_i \rightarrow E$ of p .*

Proof Let $\{U_1, \dots, U_r\}$ be an open cover of X such that, for each $i \in \{1, 2, \dots, r\}$, there exists a continuous local section $s_i: U_i \rightarrow E$ of p . Since X is normal, there exists a partition of unity $\{f_1, \dots, f_r\}$ subordinate to this finite open cover by Theorem 36.1 of [30]. Let $c_1, \dots, c_r \in (0, +\infty)$ with $c_1 + \dots + c_r = 1$. For each $i \in \{1, 2, \dots, r\}$, we put

$$A_i := \{x \in X \mid f_i(x) \geq c_i, f_j(x) < c_j \text{ for all } j < i\}.$$

Each A_i is the intersection of a closed and an open subset of X , and hence is locally compact. One checks without difficulties that the A_i are pairwise disjoint and that $X = \bigcup_{i=1}^r A_i$. Moreover, $A_i \subset U_i$ for each i , so $s_i|_{A_i}: A_i \rightarrow E$ is a continuous local section of p for each $i \in \{1, 2, \dots, r\}$. □

The following proposition establishes a lower bound on $\text{GC}(M, g)$ in terms of a principal G -bundle over M that is a Riemannian submersion. This submersion property will be used in its proof to ensure the existence of horizontal lifts of curves. For each orientable M , its orthonormal frame bundle is an example of such a bundle with $G = \text{SO}(\dim M)$; see eg [25, Example I.5.7].

Proposition 4.2 *Let (M, g) be a complete Riemannian manifold and let $\pi: E \rightarrow M$ be a smooth principal G -bundle, where G is a connected Lie group. Assume that E is equipped with a Riemannian metric for which π is a Riemannian submersion. Then*

$$\text{GC}(M, g) \geq \frac{\text{TC}(E)}{\text{cat}(G)}.$$

Proof Let $GC(M) = k$ and choose pairwise disjoint and locally compact subsets $A_1, \dots, A_k \subset M \times M$ with $\bigcup_{i=1}^k A_i = M \times M$ such that, for each $i \in \{1, 2, \dots, k\}$, there exists a continuous geodesic motion planner $s_i: A_i \rightarrow GM$. Let $v: GM \rightarrow TM$ be the velocity map and put

$$v_i: A_i \rightarrow TM, \quad v_i := v \circ s_i \quad \text{for all } i \in \{1, 2, \dots, k\}.$$

The v_i are continuous by Proposition 3.2. For each i we put

$$B_i := (\pi \times \text{id})^{-1}(A_i) = \{(u, q) \in E \times M \mid (\pi(u), q) \in A_i\}.$$

Clearly the B_i are again pairwise disjoint with $\bigcup_{i=1}^k B_i = E \times M$. Let $\text{Hor}(E) \subset TE$ denote the horizontal subbundle with respect to π . Since $d\pi|_{\text{Hor}(E)}: \text{Hor}(E) \rightarrow TM$ maps $\text{Hor}_u(E)$ isomorphically onto $T_{\pi(u)}M$ for each $u \in E$, we obtain continuous lifts of the v_i by

$$w_i: B_i \rightarrow \text{Hor}(E), \quad w_i(u, q) = (d\pi|_{\text{Hor}_u(E)})^{-1}v_i(\pi(u), q) \quad \text{for all } i \in \{1, 2, \dots, k\}.$$

For each $u \in E$ we let $\exp_u: T_u E \rightarrow E$ be the exponential map of the given Riemannian metric on E . With $PE = C^0([0, 1], E)$, we define continuous maps

$$\eta_i: B_i \rightarrow PE$$

by

$$(\eta_i(u, q))(t) = \exp_u(tw_i(u, q)) \quad \text{for all } (u, q) \in B_i, t \in [0, 1], i \in \{1, 2, \dots, k\}.$$

Each η_i induces a continuous map

$$\alpha_i: B_i \rightarrow E \times E, \quad \alpha_i(u, q) = ((\eta_i(u, q))(0), (\eta_i(u, q))(1)) = (u, \exp_u(w_i(u, q))).$$

Since horizontal geodesics in E project to geodesics in M , we compute that

$$(\text{id} \times \pi)(\alpha_i(u, q)) = (\text{id} \times \pi)(u, \eta_i(u, q)(1)) = (u, (s_i(\pi(u), q))(1)) = (u, q)$$

for all $(u, q) \in B_i$. Here we used that $\pi(\eta_i(u, q)) = s_i(u, q)$ for all $(u, q) \in B_i$. Hence, for each $i \in \{1, 2, \dots, k\}$, the map α_i is a continuous local section of $\text{id} \times \pi: E \times E \rightarrow E \times M$, which is again a principal G -bundle. The right G -action on $E \times E$ is given by $E \times E \times G \rightarrow E \times E, (u, v, h) \mapsto (u, vh)$, where we consider the right G -action on E given by the bundle structure. Thus, we get a local trivialization of $\text{id} \times \pi$ over each B_i , given explicitly by the homeomorphism

$$\Phi_i: B_i \times G \rightarrow E \times E|_{B_i}, \quad \Phi_i(u, q, h) = \alpha_i(u, q)h = (u, \exp_u(w_i(u, q))h).$$

Put $l = \text{cat}(G)$. Let $e \in G$ be the unit, $P_e G = \{\gamma \in PG \mid \gamma(0) = e\}$ and

$$q: P_e G \rightarrow G, \quad q(\gamma) = \gamma(1).$$

Since $P_e G$ is contractible, by [39, Theorem 18], $\text{cat}(G) = \text{secat}(q: P_e G \rightarrow G)$. By Lemma 4.1, there are pairwise disjoint and locally compact subsets $C_1, \dots, C_l \subset G$ with $\bigcup_{j=1}^l C_j = G$ such that, for each $j \in \{1, 2, \dots, l\}$, there is a continuous local section $r_j: C_j \rightarrow P_e G$ of q .

If we put $D_{i,j} := \Phi_i(B_i \times C_j) \subset E \times E$ for all $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, l\}$, then the $D_{i,j}$ are pairwise disjoint, locally compact and satisfy $\bigcup_{i=1}^k \bigcup_{j=1}^l D_{i,j} = E \times E$. For all i and j we further consider the map

$$\sigma_{i,j}: B_i \times C_j \rightarrow PE$$

given by

$$(\sigma_{i,j}(u, q, h))(t) = (\eta_i(u, q))(t) \cdot (r_j(h))(t) \quad \text{for all } (u, q) \in B_i, h \in C_j.$$

Then

$$(\sigma_{i,j}(u, q, h))(0) = (\eta_i(u, q))(0) = u,$$

$$(\sigma_{i,j}(u, q, h))(1) = (\eta_i(u, q))(1)(r_j(h))(1) = \exp_u(w_i(u, q))h$$

and thus

$$(\sigma_{i,j}(u, q, h)(0), \sigma_{i,j}(u, q, h)(1)) = \Phi_i(u, q, h) \quad \text{for all } (u, q) \in B_i, h \in C_j.$$

This shows that $\sigma_{i,j} \circ \Phi_i^{-1}|_{D_{i,j}}: D_{i,j} \rightarrow PE$ is a continuous motion planner for all $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, l\}$. As a smooth manifold, E is a Euclidean neighborhood retract (ENR). Since the $D_{i,j}$ are locally compact subsets of an ENR, they are ENRs themselves. Hence, it follows from [13, Theorem 6.1] that

$$TC(E) \leq k \cdot l = GC(M) \cdot \text{cat}(G),$$

which proves the claimed inequality. □

Remark 4.3 Since $GC(M) \geq TC(M)$ for all complete Riemannian manifolds M , the lower bound from Proposition 4.2 improves this basic inequality if and only if

$$\frac{TC(E)}{\text{cat}(G)} > TC(M) \iff TC(E) > \text{cat}(G) TC(M) = TC(G) TC(M),$$

where we used [13, Lemma 8.2]. Note that the assumption on the bundle to be *principal* in the previous result is necessary, as the following example shows. Consider the Klein

bottle K , which is given as a fiber bundle over S^1 with fiber S^1 and satisfies $\text{TC}(K) = 5$ by [5], while $\text{TC}(S^1) = 2$. Since the round metric g_r on S^1 satisfies

$$\text{GC}(S^1, g_r) = \text{TC}(S^1) = 2 < \frac{5}{2} = \frac{\text{TC}(K)}{\text{cat}(S^1)}$$

by [34, Proposition 4.1], the inequality from Proposition 4.2 would indeed be false in this situation. However, K is not given as a principal S^1 -bundle over S^1 , so Proposition 4.2 is not applicable to this setting. By the classification theorem for principal bundles — see [11, Theorem 14.4.1] — the set of isomorphism classes of principal S^1 -bundles over S^1 is in bijection with the set of homotopy classes $[S^1, BS^1] = [S^1, \mathbb{C}P^\infty]$. But $\mathbb{C}P^\infty$ is simply connected, so it follows that $[S^1, \mathbb{C}P^\infty]$ has only one element. Thus, every principal S^1 -bundle over S^1 is trivial. Since $\pi_1(K) \not\cong \mathbb{Z}^2 = \pi_1(S^1 \times S^1)$, the bundle K is a nontrivial S^1 -bundle. Hence, it cannot be principal.

Our next aim is to derive a lower bound on geodesic complexity from the structure of the cut locus of a point in the manifold. We first introduce the notion of stratification that we are using.

Definition 4.4 Let M be a manifold and let $B \subset M$ be a subset. A stratification of B of depth $N \in \mathbb{N}$ is a family (S_1, \dots, S_N) of locally closed and pairwise disjoint subsets of M such that the following conditions hold:

- (i) $B = \bigcup_{i=1}^N S_i$ and $\bar{S}_i = \bigcup_{j=i}^N S_j$ for all $i \in \{1, 2, \dots, N\}$.
- (ii) Let $i, j \in \{1, 2, \dots, N\}$. If Z_j is a connected component of S_j and Z_i is a connected component of S_i with $Z_j \cap \bar{Z}_i \neq \emptyset$, then $Z_j \subset \bar{Z}_i$.

Example 4.5 Let $M = \mathbb{R}^2$ and let $B = [-1, 1]^2$. Consider

$$\begin{aligned} S_1 &= (-1, 1) \times (-1, 1), \\ S_2 &= ((-1, 1) \times \{-1, 1\}) \cup (\{-1, 1\} \times (-1, 1)), \\ S_3 &= \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}. \end{aligned}$$

One checks without difficulties that (S_1, S_2, S_3) has properties (i) and (ii) from Definition 4.4. Hence, (S_1, S_2, S_3) is a stratification of B .

Given a stratification of the cut locus of a point, we want to introduce an additional condition on those parts of the corresponding tangent cut locus that are mapped to the same stratum. This will be the crucial step for finding a lower bound for geodesic

complexity. The following notion is an analogue of [34, Definition 3.10]; see our introduction and Remark 4.7(2) below for a comparison of the two notions. The terms from Riemannian geometry that are used are to be found, for example, in [27, page 310].

Definition 4.6 Let (M, g) be a complete Riemannian manifold, $p \in M$ and let $S = (S_1, \dots, S_N)$ be a stratification of $\text{Cut}_p(M)$. Let $K \subset T_p M$ denote the union of the tangent cut locus $\widetilde{\text{Cut}}_p(M)$ with the domain of injectivity of \exp_p and let

$$(4-1) \quad \exp_K := \exp_p|_K : K \rightarrow M$$

denote the restriction. We call S *inconsistent* if, for all $i \in \{2, 3, \dots, N\}$ and $x \in S_i$, there exists an open neighborhood $U \subset M$ of x with the following property: Let Z_1, \dots, Z_s be the connected components of $U \cap S_{i-1}$. Then $x \in \bar{Z}_j$ for all $j \in \{1, 2, \dots, s\}$ and

$$\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(\{x\}) \cap \bigcap_{j=1}^s \overline{\exp_K^{-1}(Z_j)} = \emptyset.$$

In Section 7.1, we will encounter explicit examples of inconsistent stratifications when we consider flat tori. Examples for cut loci with nontrivial stratifications which are not inconsistent are Berger spheres, as we shall see in Section 7.2.

Remarks 4.7 Let (M, g) be a complete Riemannian manifold.

(1) If M is a closed manifold, then the set K from Definition 4.6 will be homeomorphic to a closed ball — see [27, Corollary 10.35] — and the map \exp_K from (4-1) is a surjection. As an example, consider the round n -dimensional sphere S^n of radius 1. If $p \in S^n$ is a point, then the domain of injectivity of \exp_p is an open ball of radius π in the tangent space $T_p S^n$. The tangent cut locus $\widetilde{\text{Cut}}_p(S^n)$ is the $(n-1)$ -sphere of radius π in $T_p M$. Consequently, the set K in this example is the closed ball of radius π in $T_p M$.

(2) Recio-Mitter [34, Definition 3.8] introduced the concept of a levelwise stratified covering for arbitrary surjective maps. He then applied this concept to the restriction of the path fibration

$$\pi : GX \rightarrow X \times X,$$

where X is a geodesic space and GX is the space of geodesic paths in X .

To work with this notion, one must study a stratification of the total cut locus of X and explore covering properties of the restrictions of π to its preimage. In contrast, the above Definition 4.6 for Riemannian manifolds only requires a stratification of the

cut locus of a single point p in a Riemannian manifold M as well as properties of the Riemannian exponential map \exp_p . Thus, for complete Riemannian manifolds the above definition seems easier to verify than the corresponding notion from [34].

The following result is an analogue of the corresponding result of Recio-Mitter; see [34, Corollary 3.14]. The proof requires M to be compact, since we will use the property mentioned in Remark 4.7(1). We recall the notation $\text{GC}_p(A) = \text{GC}_M(\{p\} \times A)$ for all $A \subset M$.

Theorem 4.8 *Let (M, g) be a closed Riemannian manifold. Assume that there exists $p \in M$ for which $\text{Cut}_p(M)$ admits an inconsistent stratification of depth $N \in \mathbb{N}$. Then*

$$\text{GC}(M) \geq \text{GC}_p(M) \geq N + 1.$$

Proof Let (S_1, \dots, S_N) be an inconsistent stratification of $\text{Cut}_p(M)$. Assume that there are pairwise disjoint locally compact sets $E_1, E_2, \dots, E_r \subset M$ with $\bigcup_{i=1}^r E_i = M$ such that, for each $i \in \{1, 2, \dots, r\}$, there exists a continuous geodesic motion planner $s_i : \{p\} \times E_i \rightarrow GM$.

We want to show by induction that, for all $k \in \{1, 2, \dots, N\}$ and all $x \in S_k$,

$$(4-2) \quad \#\{i \in \{1, 2, \dots, r\} \mid x \in \bar{E}_i\} \geq k + 1.$$

Consider the base case of $k = 1$ and assume by contradiction that there is an $i \in \{1, 2, \dots, r\}$ with $x \in \bar{E}_i$, but $x \notin \bar{E}_j$ for all $j \neq i$. Then x has an open neighborhood $U \subset M$ such that $U \subset E_i$ and the restriction $s_i|_{\{p\} \times U}$ is a continuous geodesic motion planner on $\{p\} \times U$. But, since $x \in \text{Cut}_p(M)$, this contradicts Proposition 3.7. Hence, $\#\{i \in \{1, 2, \dots, r\} \mid x \in \bar{E}_i\} \geq 2$, which we wanted to show.

Assume as induction hypothesis that, for some $k \in \{2, 3, \dots, N\}$, we have shown that

$$\#\{i \in \{1, 2, \dots, r\} \mid y \in \bar{E}_i\} \geq k \quad \text{for all } y \in S_{k-1}.$$

Let $x \in S_k$. Assume that (4-2) is false and assume up to reordering that $x \notin \bar{E}_i$ for all $i > k$. Then there exists an open neighborhood U of x with $U \subset \bigcup_{i=1}^k E_i$. By the induction hypothesis, this yields

$$(4-3) \quad U \cap S_{k-1} \subset \bar{E}_i \quad \text{for all } i \in \{1, 2, \dots, k\}.$$

We assume without loss of generality that U is chosen as in Definition 4.6, since this can be achieved by shrinking U . We further assume that $x \in E_1$. Let Z_1, \dots, Z_s be the connected components of $U \cap S_{k-1}$, where $s \in \mathbb{N}$ is suitably chosen.

Let $j \in \{1, 2, \dots, s\}$ and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in Z_j with $\lim_{n \rightarrow \infty} a_n = x$, which exists by our choice of U . For all $n \in \mathbb{N}$ it further holds by (4-3) that $a_n \in \bar{E}_1$. Thus, for each n , there exists a sequence $(b_m^n)_{m \in \mathbb{N}}$ in $U \cap E_1$ with $\lim_{m \rightarrow \infty} b_m^n = a_n$. Put

$$v_1: E_1 \rightarrow T_p M, \quad v_1(y) := (v \circ s_1)(p, y),$$

where v is the velocity map. By Proposition 3.2, v_1 is continuous. Let $\exp_K: K \rightarrow M$ be given as in (4-1). The set K is homeomorphic to a closed ball in $T_p M$; see Remark 4.7(1). By construction, $v_1(y) \in K$ for each $y \in E_1$; hence, $(v_1(b_m^n))_{m \in \mathbb{N}}$ is a sequence in K for each $n \in \mathbb{N}$. Since K is compact, it has a convergent subsequence $(v_1(b_{m_k}^n))_{k \in \mathbb{N}}$ for each $n \in \mathbb{N}$. Put $\xi_n := \lim_{k \rightarrow \infty} v_1(b_{m_k}^n)$ for all $n \in \mathbb{N}$. By continuity of the exponential map,

$$\exp_K(\xi_n) = \exp_p(\xi_n) = \lim_{k \rightarrow \infty} \exp_p(v_1(b_{m_k}^n)) = \lim_{k \rightarrow \infty} b_{m_k}^n = a_n \quad \text{for all } n \in \mathbb{N}.$$

Thus,

$$\xi_n \in K \cap \exp_K^{-1}(\{a_n\}) \subset K \cap \exp_K^{-1}(Z_j) \quad \text{for all } n \in \mathbb{N}.$$

Now $(\xi_n)_{n \in \mathbb{N}}$ is a sequence in K , so it has a convergent subsequence $(\xi_{n_l})_{l \in \mathbb{N}}$. With $\xi_0 := \lim_{l \rightarrow \infty} \xi_{n_l}$, we obtain

$$\exp_p(\xi_0) = \lim_{l \rightarrow \infty} \exp_p(\xi_{n_l}) = \lim_{l \rightarrow \infty} a_{n_l} = x.$$

In particular, it follows from $x \in \text{Cut}_p(M)$ that $\xi_0 \in \widetilde{\text{Cut}}_p(M)$. Since $\xi_n \in \exp_K^{-1}(\{a_n\})$ for each $n \in \mathbb{N}$, we conclude that

$$\xi_0 \in \widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(\{x\}) \cap \overline{\exp_K^{-1}(Z_j)}.$$

Note that ξ_0 depends on the choice of j . To conclude, we still need to show that the same ξ_0 can be chosen for each $j \in \{1, 2, \dots, s\}$. We will do so by showing next that $\xi_0 = v_1(x)$, which does not depend on j .

Let $d_M: M \times M \rightarrow \mathbb{R}$ be the distance function induced by the Riemannian metric. By definition of the ξ_{n_l} , for each $l \in \mathbb{N}$ there exists $k_l \in \mathbb{N}$ such that

$$d_M(a_{n_l}, b_{m_k}^{n_l}) < \frac{1}{l} \quad \text{and} \quad \|\xi_{n_l} - v_1(b_{m_k}^{n_l})\| < \frac{1}{l} \quad \text{for all } k \geq k_l.$$

We can further choose the k_l in such a way that $\lim_{l \rightarrow \infty} k_l = \infty$. By a diagonal argument, $\lim_{l \rightarrow \infty} b_{m_{k_l}}^{n_l} = x$. This in particular shows, by continuity of v_1 , that

$$\xi_0 = \lim_{l \rightarrow \infty} \xi_{n_l} = \lim_{l \rightarrow \infty} v_1(b_{m_{k_l}}^{n_l}) = v_1(x).$$

Thus, $v_1(x) \in \exp_p^{-1}(\{x\}) \cap \overline{\exp_K^{-1}(Z_j)}$. Since j was chosen arbitrarily, it follows that

$$v_1(x) \in \widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(\{x\}) \cap \bigcap_{j=1}^s \overline{\exp_K^{-1}(Z_j)}.$$

This contradicts the inconsistency of the stratification (S_1, \dots, S_N) . Hence, there is no such U , which concludes the proof of the induction step. For $k = N$, it in particular follows from (4-2) that $r \geq N + 1$. Thus, $\text{GC}_p(M) \geq N + 1$. □

We will see in Section 7.1 that flat tori are indeed examples for Riemannian manifolds whose cut loci admit inconsistent stratifications. Next we will discuss a more tangible criterion on a cut locus that implies the existence of an inconsistent stratification. For this purpose, we will use results and constructions of J-I Itoh and Sakai [24]. Large parts of these methods are extensions of those applied by V Ozols [32].

Definition 4.9 [24, page 68 and Definition 2.1] Let (M, g) be a complete Riemannian manifold and let $p \in M$.

- (a) We say that $q \in \text{Cut}_p(M)$ is of order $k + 1$, where $k \in \mathbb{N}$, if there are precisely $k + 1$ minimal geodesics $\gamma_0, \gamma_1, \dots, \gamma_k \in GM$ with $\gamma_i \neq \gamma_j$ if $i \neq j$ and with $\gamma_i(0) = p$ and $\gamma_i(1) = q$ for all $i \in \{0, 1, 2, \dots, k\}$.
- (b) We call q nondegenerate if the vectors $\dot{\gamma}_0(1), \dot{\gamma}_1(1), \dots, \dot{\gamma}_k(1) \in T_qM$ are in general position, ie if $\{\dot{\gamma}_i(1) - \dot{\gamma}_0(1) \mid i \in \{1, 2, \dots, k\}\}$ is linearly independent.

As carried out by Itoh and Sakai [24, Remark 2.2], a large class of two-dimensional flat tori provides an example for manifolds with nondegenerate cut points. However, our study of flat tori in Section 7.1 will not rely on this notion of nondegeneracy, but will employ the above inconsistency condition directly.

We recall that a conjugate point of a point p in a Riemannian manifold (M, g) is a point $q \in M$ such that there is a geodesic segment from p to q along which there exists a nontrivial Jacobi field which vanishes in p and q ; see [27, page 298].

Remarks 4.10 (1) As shown by A Weinstein [40, page 29], every closed manifold M with $\dim M \geq 2$ and not homeomorphic to S^2 admits a Riemannian metric for which there exists $p \in M$ such that $\text{Cut}_p(M)$ does not contain any conjugate points. Itoh and Sakai conjectured in [24, Remark 2.9] that the set of all such metrics on M contains as a dense subset the set of those metrics for which all points in $\text{Cut}_p(M)$ are nondegenerate.

- (2) It is evident from the definition of nondegeneracy that the order of a nondegenerate cut point is at most $\dim M + 1$.

Theorem 4.11 *Let (M, g) be a closed Riemannian manifold and assume that there exists $p \in M$ for which $\text{Cut}_p(M)$ does not contain any conjugate points of p and for which all points in $\text{Cut}_p(M)$ are nondegenerate. Let*

$$N := \max\{k \in \mathbb{N} \mid \text{there is } q \in \text{Cut}_p(M) \text{ of order } k + 1\}.$$

Then $\text{Cut}_p(M)$ admits an inconsistent stratification of depth N .

Proof Let $\mathcal{C} := (C_1, \dots, C_N)$ be given by

$$C_k := \{q \in \text{Cut}_p(M) \mid q \text{ is of order } k + 1\} \quad \text{for all } k \in \{1, 2, \dots, N\}.$$

It is shown in [24, Proposition 2.4] that, under the nondegeneracy assumption on the points in $\text{Cut}_p(M)$, \mathcal{C} is a Whitney stratification of $\text{Cut}_p(M)$, as defined in [21, page 37]. Hence, \mathcal{C} is in particular an \mathcal{S} -decomposition in the sense of Goresky and MacPherson; see [21, page 36]. One checks immediately that the two conditions defining such an \mathcal{S} -decomposition imply that \mathcal{C} is a stratification of $\text{Cut}_p(M)$ in the sense of Definition 4.4. It remains to show that \mathcal{C} is inconsistent. Fix $k \in \{1, 2, \dots, N\}$, let $q \in C_k$ and let $\gamma_0, \gamma_1, \dots, \gamma_k : [0, 1] \rightarrow M$ be geodesics from p to q with $\gamma_i \neq \gamma_j$ whenever $i \neq j$. For each $i \in \{0, 1, \dots, k\}$, put $v_i := \dot{\gamma}_i(0) \in T_p M$, so that

$$\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(\{q\}) = \{v_0, v_1, \dots, v_k\}.$$

Choose an open neighborhood U of q such that $U \cap C_k$ is connected and such that

$$(4-4) \quad \text{Cut}_p(M) \cap U = \bigcup_{i=1}^k C_i \cap U.$$

Such a neighborhood exists by the stratification properties. As discussed in [24, page 68], since q is nondegenerate, we can choose an open neighborhood $V_i \subset T_p M$ of v_i for each $i \in \{0, 1, \dots, k\}$ such that \exp_p maps V_i diffeomorphically onto U . Put $F_i := (\exp|_{V_i})^{-1} : U \rightarrow V_i$. As explained in [32, pages 220–221], up to shrinking U we can assume that every minimal geodesic γ from p to an element of U has $\dot{\gamma}(0) \in \bigcup_{i=0}^k V_i$. We further assume that $\bar{V}_i \cap \bar{V}_j = \emptyset$ whenever $i \neq j$. For $i \in \{1, 2, \dots, k\}$, we define

$$f_i : U \rightarrow \mathbb{R}, \quad f_i(x) = \|F_i(x)\| - \|F_0(x)\|,$$

where $\|\cdot\|$ denotes the norm on T_pM defined by the Riemannian metric. With $f: U \rightarrow \mathbb{R}^k$, $f = (f_1, f_2, \dots, f_k)$, it follows that $f^{-1}(\{0\}) = C_k \cap U$. For $i \in \{1, 2, \dots, k\}$ we further let

$$g_i: U \rightarrow \mathbb{R}^{k-1}, \quad g_i = (f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_k)$$

and put

$$g_0: U \rightarrow \mathbb{R}^{k-1},$$

$$g_0(x) = (\|F_2(x)\| - \|F_1(x)\|, \|F_3(x)\| - \|F_1(x)\|, \dots, \|F_k(x)\| - \|F_1(x)\|).$$

Then, by assumption on U ,

$$C_{k-1} \cap U = \bigcup_{i=0}^k g_i^{-1}(\{0\}) \setminus C_k = \bigcup_{i=0}^k g_i^{-1}(\{0\}) \setminus f^{-1}(\{0\}).$$

The connected components of $C_{k-1} \cap U$ are the sets Z_0, Z_1, \dots, Z_k , where

$$Z_i := g_i^{-1}(\{0\}) \cap f_i^{-1}(0, +\infty) \quad \text{for all } i \in \{1, 2, \dots, k\},$$

$$Z_0 := g_0^{-1}(\{0\}) \cap f_1^{-1}(-\infty, 0).$$

By construction of the sets,

$$\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(Z_i) \subset \bigcup_{j \neq i} V_j \quad \text{for all } i \in \{0, 1, \dots, k\}.$$

A closer investigation, using that $\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(\{q\}) = \{v_0, v_1, \dots, v_k\}$ and that the closures of the V_i are pairwise disjoint, shows that

$$\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(\{q\}) \cap \overline{\exp_p^{-1}(Z_i)} = \{v_0, v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k\}$$

for all $i \in \{0, 1, \dots, k\}$. This implies

$$\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(\{q\}) \cap \bigcap_{i=0}^k \overline{\exp_p^{-1}(Z_i)} = \emptyset.$$

Since k and q were chosen arbitrarily, this shows the inconsistency of \mathcal{C} . □

Combining the previous theorem with our lower bound from [Theorem 4.8](#) yields:

Corollary 4.12 *Let (M, g) be a closed Riemannian manifold and assume that there exists $p \in M$ such that $\text{Cut}_p(M)$ does not contain any conjugate points of p and such that all points in $\text{Cut}_p(M)$ are nondegenerate. If $\text{Cut}_p(M)$ contains a point of order $k + 1$, where $k \in \mathbb{N}$, then*

$$\text{GC}(M, g) \geq k + 1.$$

5 An upper bound for homogeneous Riemannian manifolds

From this section on, we will mostly consider homogeneous Riemannian manifolds and exploit their symmetry properties. Given a Riemannian manifold (M, g) , we let $\text{Isom}(M) := \text{Isom}(M, g)$ denote its group of isometries and consider it as a subspace of $C^0(M, M)$ with the compact–open topology. We recall that (M, g) is called *homogeneous* if $\text{Isom}(M)$ acts transitively on M . Note that every homogeneous Riemannian manifold is necessarily complete; see [25, Theorem IV.4.5].

Having derived lower bounds for geodesic complexity in the previous section, we next want to find upper bounds. After some preparatory lemmas, we will establish an upper bound on $\text{GC}(M)$ for a homogeneous Riemannian manifold M in terms of the subspace complexity $\text{GC}_M(\{p\} \times \text{Cut}_p(M))$ and a categorical invariant determined by its isometry action. Intuitively, the transitivity of the isometry action implies that continuous geodesic motion planners on subsets of cut loci of single points can be continuously extended to larger subsets of the total cut locus. We will then go on to study further upper bounds on $\text{GC}(M)$ in the case that $\text{Cut}_p(M)$ admits a stratification. The following is a folklore result from Riemannian geometry:

Lemma 5.1 *Let (M, g) be a homogeneous Riemannian manifold and let $p \in M$. Then*

$$\text{ev}_p: \text{Isom}(M) \rightarrow M, \quad \text{ev}_p(\phi) = \phi(p),$$

is a principal $\text{Isom}_p(M)$ –bundle, where $\text{Isom}_p(M)$ denotes the isotropy group of the isometry action on M in p .

Proof By [26, Theorem 21.17], ev_p induces an $\text{Isom}(M)$ –equivariant diffeomorphism $f: \text{Isom}(M)/\text{Isom}_p(M) \rightarrow M$. Moreover, the projection $q: \text{Isom}(M) \rightarrow \text{Isom}(M)/\text{Isom}_p(M)$ is a principal $\text{Isom}_p(M)$ –bundle by [25, Example I.5.1]. One easily shows that $\text{ev}_p = f \circ q$, which implies that ev_p is a principal $\text{Isom}_p(M)$ –bundle as well. □

Example 5.2 Given a Lie group G with a left-invariant Riemannian metric, the left multiplication $l_g: G \rightarrow G, l_g(h) = gh$, is an isometry for each $g \in G$. With $e \in G$ denoting the unit, one further derives from $l_g(e) = g$ for each $g \in G$ that the map $s: G \rightarrow \text{Isom}(G), s(g) = l_g$, is a continuous section of the bundle $\text{ev}_e: \text{Isom}(G) \rightarrow G$.

Lemma 5.3 *Let $A, B \subset M$ and $p \in M$. Assume that there are a continuous geodesic motion planner $\sigma_B: \{p\} \times B \rightarrow GM$ and a continuous local section $s: A \rightarrow \text{Isom}(M)$*

of ev_p . Then there exists a continuous geodesic motion planner $\sigma : F \rightarrow GM$, where

$$F := \{(x, y) \in M \times M \mid x \in A, y \in s(x) \cdot B\}.$$

Proof We define $\sigma : F \rightarrow GM$ by

$$\sigma(x, y) = s(x) \circ \sigma_B(p, s(x)^{-1} \cdot y) \quad \text{for all } (x, y) \in F.$$

By construction, $\sigma_B(p, s(x)^{-1} \cdot y)$ is a minimal geodesic from p to $s(x)^{-1} \cdot y$. Since $s(x)$ is an isometry for each x , $\sigma(x, y)$ is indeed a minimal geodesic from

$$s(x) \cdot p = ev_p(s(x)) = x \quad \text{to} \quad s(x) \cdot (s(x)^{-1} \cdot y) = y.$$

So σ is a geodesic motion planner and it only remains to show its continuity.

Let $\rho : \text{Isom}(M) \times M \rightarrow M$ denote the action of the isometry group by evaluation and again let $PM = C^0([0, 1], M)$. By [4, Theorem VII.2.10], the composition map

$$\varphi : C^0(M, M) \times PM \rightarrow PM, \quad \varphi(f, \gamma) = f \circ \gamma,$$

is continuous with respect to the compact–open topologies. Thus, the restriction of φ to

$$\text{Isom}(M) \times GM \subset C^0(M, M) \times PM$$

defines a continuous action

$$\tilde{\rho} : \text{Isom}(M) \times GM \rightarrow GM, \quad \tilde{\rho} = \varphi|_{\text{Isom}(M) \times GM}.$$

The inversion $i : \text{Isom}(M) \rightarrow \text{Isom}(M)$, $i(g) = g^{-1}$, is continuous since $\text{Isom}(M)$ is a topological group. We can express σ as

$$\sigma(x, y) = \tilde{\rho}(s(x), \sigma_B(p, \rho(i(s(x)), y))) \quad \text{for all } (x, y) \in F.$$

All maps on the right-hand side are continuous, so σ is continuous as well. □

The previous lemma allows us to make a useful estimate between the subspace geodesic complexity of the total cut locus and that of one single cut locus in the homogeneous case.

Theorem 5.4 *Let (M, g) be a homogeneous Riemannian manifold and let $p \in M$. Then*

$$GC(M) \leq \text{secat}(ev_p : \text{Isom}(M) \rightarrow M) \cdot GC_p(\text{Cut}_p(M)) + 1.$$

Proof As seen in Remark 2.7(2), it holds that $GC(M) \leq GC_M(\text{Cut}(M)) + 1$, so it suffices to show that

$$GC_M(\text{Cut}(M)) \leq \text{secat}(ev_p) \cdot GC_p(\text{Cut}_p(M)).$$

Let $k := \text{secat}(ev_p)$ and $r := GC_p(\text{Cut}_p(M))$. By Lemma 4.1, there are pairwise disjoint locally compact $A_1, \dots, A_k \subset M$ with $M = \bigcup_{i=1}^k A_i$ for which there is a continuous local section $s_i : A_i \rightarrow \text{Isom}(M)$ of ev_p for each $i \in \{1, 2, \dots, k\}$. Let $B_1, \dots, B_r \subset M$ be pairwise disjoint and locally compact with $\text{Cut}_p(M) \subset \bigcup_{j=1}^r B_j$ such that, for each j , there exists a continuous geodesic motion planner $\sigma_j : \{p\} \times B_j \rightarrow GM$. Put

$$F_{i,j} := \{(x, y) \in M \times M \mid x \in A_i, y \in s_i(x) \cdot B_j\}$$

for all $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, r\}$. By construction, the elements of

$$\{F_{i,j} \mid i \in \{1, 2, \dots, k\}, j \in \{1, 2, \dots, r\}\}$$

are pairwise disjoint. Furthermore, for all $i \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, r\}$,

$$\psi_{i,j} : A_i \times B_j \rightarrow F_{i,j}, \quad \psi_{i,j}(x, y) = (x, s_i(x) \cdot y),$$

is a homeomorphism. Consequently, the $F_{i,j}$ are locally compact. If $(x, y) \in \text{Cut}(M)$, then $x \in A_i$ for some $i \in \{1, 2, \dots, k\}$. Since $s_i(x)^{-1}$ is an isometry, it holds that $s_i(x)^{-1} \cdot y \in \text{Cut}_p(M)$. Hence, there is a $j \in \{1, 2, \dots, r\}$ with $s_i(x)^{-1} \cdot y \in B_j$ and therefore $(x, y) \in F_{i,j}$ by definition. This shows that

$$\text{Cut}(M) \subset \bigcup_{i=1}^k \bigcup_{j=1}^r F_{i,j}.$$

Moreover, by Lemma 5.3 we can find a continuous geodesic motion planner $F_{i,j} \rightarrow GM$ of p for all i and j . Thus, $GC_M(\text{Cut}(M)) \leq kr$, which shows the claim. \square

The previous upper bound has a particularly strong consequence for connected Lie groups.

Corollary 5.5 *Let G be a connected Lie group equipped with a left-invariant Riemannian metric and let $e \in G$ denote the unit element. Then*

$$GC(G) \leq GC_e(\text{Cut}_e(G)) + 1.$$

Proof This is an immediate consequence of Theorem 5.4. Since $ev_e : \text{Isom}(G) \rightarrow G$ admits a continuous section — see Example 5.2 — it follows that $\text{secat}(ev_e) = 1$. \square

Sectional categories of fibrations are in general hard to compute. A common way of estimating their values from above is by the Lusternik–Schnirelmann categories of their base spaces using [39, Theorem 18]. In our situation, this leads to the following estimate:

Corollary 5.6 *Let (M, g) be a homogeneous Riemannian manifold and let $p \in M$. Then*

$$GC(M) \leq \text{cat}(M) \cdot GC_p(\text{Cut}_p(M)) + 1.$$

Proof This is an immediate consequence of [Theorem 5.4](#) and the fact that every fibration $p: E \rightarrow B$ satisfies $\text{secat}(p) \leq \text{cat}(B)$ by [[39](#), Theorem 18]. □

We want to further estimate geodesic complexity from above by finding upper bounds for subspace geodesic complexities of cut loci. When $\text{Cut}_p(M)$ admits a stratification, we can compare $GC_p(\text{Cut}_p(M))$ to the subspace geodesic complexities of its strata.

Proposition 5.7 *Let (M, g) be a complete Riemannian manifold, let $p \in M$ and assume that $\text{Cut}_p(M)$ has a stratification (S_1, \dots, S_k) of depth k . Then*

$$GC_p(\text{Cut}_p(M)) \leq \sum_{i=1}^k \max_{Z_i \in \pi_0(S_i)} GC_p(Z_i),$$

where $\pi_0(X)$ denotes the set of connected components of a space X .

Proof Since $\text{Cut}_p(M) = S_1 \cup \dots \cup S_k$, it follows from [Remark 2.2\(3\)](#) that

$$GC_p(\text{Cut}_p(M)) \leq \sum_{i=1}^k GC_p(S_i).$$

Now fix $i \in \{1, 2, \dots, k\}$ and let Z_1, \dots, Z_r be the connected components of S_i . Put

$$s_i := \max_{j \in \{1, 2, \dots, r\}} GC_p(Z_j).$$

For each $j \in \{1, 2, \dots, r\}$, let $A_1^j, \dots, A_{s_i}^j \subset Z_j$ be pairwise disjoint and locally compact, such that, for each $j \in \{1, 2, \dots, r\}$ and $l \in \{1, 2, \dots, s_i\}$, either $A_l^j = \emptyset$ or there exists a continuous geodesic motion planner $\sigma_{j,l}: \{p\} \times A_l^j \rightarrow GM$. Put $A_l := \bigcup_{j=1}^r A_l^j$ for each $l \in \{1, 2, \dots, s_i\}$. Then the A_l are pairwise disjoint and locally compact with $S_i = \bigcup_{l=1}^{s_i} A_l$. Moreover, since, by definition of a stratification, $Z_i \cap \bar{Z}_j = \emptyset$ for all $i \neq j$, the maps

$$\sigma_l: \{p\} \times A_l \rightarrow GM, \quad \sigma_l(p, x) = \sigma_{j,l}(p, x) \quad \text{for all } x \in A_l^j, j \in \{1, 2, \dots, r\},$$

are well-defined continuous geodesic motion planners. This shows $GC_p(S_i) \leq s_i$ for each $i \in \{1, 2, \dots, k\}$, which implies the claim. □

Corollary 5.8 *Let (M, g) be a homogeneous Riemannian manifold, let $p \in M$ and assume that $\text{Cut}_p(M)$ has a stratification (S_1, \dots, S_k) of depth k . Then*

$$\text{GC}(M) \leq \text{secat}(\text{ev}_p : \text{Isom}(M) \rightarrow M) \cdot \sum_{i=1}^k \max_{Z_i \in \pi_0(S_i)} \text{GC}_p(Z_i) + 1.$$

Proof This follows from [Theorem 5.4](#) and [Proposition 5.7](#). □

6 Trivially covered stratifications

Recio-Mitter [\[34\]](#) considered total cut loci with stratifications whose strata are finitely covered by the path fibration. As a part of [\[34, Corollary 3.14\]](#), he showed that, if such a stratification is inconsistent and trivially covered, this knowledge about the total cut locus suffices to compute the geodesic complexity of the space.

In this section, we will revisit the notion of trivially covered stratifications in the setting of Riemannian manifolds, but, in contrast to [\[34\]](#), we will put a covering condition on the cut locus of a single point instead of the total cut locus. We will then derive an upper bound for the numbers $\text{GC}_p(M)$ that we have studied in the previous section. From this estimate we will derive an upper bound for the geodesic complexity of homogeneous Riemannian manifolds for which the cut locus of a point admits a trivially covered stratification.

Definition 6.1 Let M be a complete Riemannian manifold, let $p \in M$ and let $\mathcal{S} = (S_1, \dots, S_N)$ be a stratification of $\text{Cut}_p(M)$. We call \mathcal{S} *trivially covered* if, for all $k \in \{1, 2, \dots, N\}$ and for all connected components Z of S_k , the restriction

$$\exp_p|_{\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(Z)} : \widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(Z) \rightarrow Z$$

is a trivial covering. Here, a trivial covering is understood to be a covering $q: X \rightarrow Y$ for which there is a discrete set D and a homeomorphism $f: X \rightarrow Y \times D$ such that $q = \text{pr} \circ f$, where $\text{pr}: Y \times D \rightarrow Y$ is the projection onto the first factor.

Theorem 6.2 *Let M be a complete Riemannian manifold, let $p \in M$ and assume that $\text{Cut}_p(M)$ admits a trivially covered stratification of depth $N \in \mathbb{N}$. Then*

$$\text{GC}_p(\text{Cut}_p(M)) \leq N.$$

Proof Let $\mathcal{S} = (S_1, \dots, S_N)$ be a trivially covered stratification of $\text{Cut}_p(M)$. We want to show that $\{p\} \times S_k$ admits a continuous geodesic motion planner for each $k \in \{1, 2, \dots, N\}$. For a fixed $k \in \{1, 2, \dots, N\}$, let Z_1, \dots, Z_r be the connected

components of S_k for suitable $r \in \mathbb{N}$. For $i \in \{1, 2, \dots, r\}$, let B_i be an arbitrary sheet of the trivial covering

$$\exp_p|_{\widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(Z_i)} : \widetilde{\text{Cut}}_p(M) \cap \exp_p^{-1}(Z_i) \rightarrow Z_i.$$

Then $\exp_p|_{B_i} : B_i \rightarrow Z_i$ is a homeomorphism. With $\varphi_i := (\exp_p|_{B_i})^{-1} : Z_i \rightarrow B_i$, one checks without difficulties that

$$s_i : \{p\} \times Z_i \rightarrow GM, \quad (s_i(p, q))(t) = \exp_p(t\varphi_i^{-1}(q)),$$

is a continuous geodesic motion planner and thus $\text{GC}_p(Z_i) = 1$. Since $k \in \{1, 2, \dots, N\}$ was chosen arbitrarily, the claim follows from [Proposition 5.7](#). □

With the additional hypotheses that M is compact and that the stratification in [Theorem 6.2](#) is inconsistent, one can derive an equality from [Theorem 6.2](#). The following result is analogous to the corresponding part of [\[34, Corollary 3.14\]](#):

Corollary 6.3 *Let M be a closed Riemannian manifold, let $p \in M$ and assume that $\text{Cut}_p(M)$ admits a trivially covered inconsistent stratification of depth $N \in \mathbb{N}$. Then $\text{GC}_p(M) = N + 1$.*

Proof By restricting the motion planner from [Remark 2.7\(2\)](#), one obtains a continuous geodesic motion planner on $\{p\} \times (M \setminus \text{Cut}_p(M))$. It follows from [Theorem 6.2](#) that

$$\text{GC}_p(M) \leq \text{GC}_p(\text{Cut}_p(M)) + 1 \leq N + 1.$$

But, by [Theorem 4.8](#), it also holds that $\text{GC}_p(M) \geq N + 1$, which proves the equality. □

Corollary 6.4 *Let G be a compact connected Lie group equipped with a left-invariant Riemannian metric and let $e \in G$ denote the unit element. If $\text{Cut}_e(G)$ admits a trivially covered inconsistent stratification of depth N , then*

$$\text{GC}(G) = N + 1.$$

Proof Combining [Theorem 6.2](#) with [Corollary 5.5](#) yields

$$\text{GC}(G) \leq \text{GC}_e(\text{Cut}_e(G)) + 1 \leq N + 1.$$

But, by [Theorem 4.8](#), $\text{GC}(G) \geq N + 1$ as well, so the claim follows. □

7 Examples: flat tori and Berger spheres

We want to use the results of [Sections 5 and 6](#) to compute the geodesic complexities of two classes of examples: two-dimensional flat tori and three-dimensional Berger spheres.

The cut loci of points in such spaces are well understood and admit stratifications of a well-behaved kind.

7.1 Geodesic complexity of flat tori

Recio-Mitter has computed the geodesic complexity of a standard flat n -dimensional torus in [34, Theorem 4.4]. More precisely, he has shown that the standard flat metric g_f on the n -torus T^n satisfies $GC(T^n, g_f) = n + 1$ for each $n \in \mathbb{N}$.

In the course of this subsection, we will extend the two-dimensional case of Recio-Mitter’s result to *arbitrary* flat metrics on two-dimensional tori. The cut loci of such metrics are well understood.

Before we do so, we will reobtain Recio-Mitter’s computation for standard flat tori using the methods of this article. This example is particularly instructive and illustrates the use of inconsistent stratifications. Moreover, in contrast to [34, Theorem 4.4], we only need to consider the cut locus of a single point, while in the proof of [34, Theorem 4.4] a stratification of $T^n \times T^n$ is required and the structure of the space of geodesic paths in T^n needs to be examined.

Example 7.1 Let $n \in \mathbb{N}$ and consider the n -torus T^n with the standard flat metric g_f , ie the quotient metric induced by the standard metric on \mathbb{R}^n and by identifying $T^n = \mathbb{R}^n / (2\mathbb{Z})^n$. Equivalently, T^n is obtained from \mathbb{R}^n by collapsing the lattice defined by an arbitrary family of n pairwise orthogonal vectors of length two. Let $\pi : \mathbb{R}^n \rightarrow T^n$ be the projection and put $o := \pi(0)$ and $M := (T^n, g_f)$. We identify \mathbb{R}^n with T_oM in the obvious way.

Note that T^n is isometric to the Riemannian product $(\mathbb{R}/(2\mathbb{Z}))^n$. For $N := \mathbb{R}/(2\mathbb{Z})$ let $\text{pr} : \mathbb{R} \rightarrow N$ be the obvious Riemannian covering and put $p_0 := \text{pr}(0) \in N$. Then $\text{Cut}_{p_0}(N) = \{\text{pr}(1)\}$ and the tangent cut locus is given by

$$\widetilde{\text{Cut}}_{p_0}(N) = \{-1, 1\}$$

under the identification $T_{p_0}N \cong \mathbb{R}$.

Given the Riemannian product of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) , the cut locus of a point $(p_1, p_2) \in M_1 \times M_2$ is easily seen to be

$$\text{Cut}_{(p_1, p_2)}(M_1 \times M_2) = (\text{Cut}_{p_1}(M_1) \times M_2) \cup (M_1 \times \text{Cut}_{p_2}(M_2));$$

see [7, page 328]. For $i \in \{1, 2\}$, let K_i be the union of the injectivity domain in $T_{p_i}M_i$ with the tangent cut locus $\widetilde{\text{Cut}}_{p_i}(M_i)$. Similar to the cut locus, the tangent cut locus of

(p_1, p_2) is given by

$$\widetilde{\text{Cut}}_{(p_1, p_2)}(M_1 \times M_2) = (\widetilde{\text{Cut}}_{p_1}(M_1) \times K_2) \cup (K_1 \times \widetilde{\text{Cut}}_{p_2}(M_2))$$

under the identification $T_{(p_1, p_2)}(M_1 \times M_2) \cong T_{p_1} M_1 \times T_{p_2} M_2$. For products of finitely many manifolds, one iteratively derives analogous results for cut loci and tangent cut loci.

We conclude that, if $I^n := [-1, 1]^n$, then the tangent cut locus of o in $M = (T^n, g_f)$ is

$$\widetilde{\text{Cut}}_o(M) = \partial I^n.$$

See also [19, page 107] for the case $n = 2$. The boundary ∂I^n admits a stratification $\partial I^n = \bigcup_{k=1}^n \mathcal{A}_k$ of depth n , given as follows: For each $k \in \{1, 2, \dots, n\}$, we put

$$J_k := \{(i_1, \dots, i_k) \in \mathbb{N}^k \mid 1 \leq i_1 < \dots < i_k \leq n\}.$$

Then each \mathcal{A}_k is given as the disjoint union $\mathcal{A}_k = \bigcup_{(i_1, \dots, i_k) \in J_k} A_{i_1, \dots, i_k}$, where $A_{i_1, \dots, i_k} := \{(x_1, \dots, x_n) \in I^n \mid |x_l| = 1 \text{ if } l \in \{i_1, i_2, \dots, i_k\}, |x_l| < 1 \text{ if } l \notin \{i_1, \dots, i_k\}\}$.

For $(i_1, \dots, i_k) \in J_k$ and $j_1, \dots, j_k \in \{-1, 1\}$, we put

$$A_{i_1, \dots, i_k, j_1, \dots, j_k} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_{i_1} = j_1, \dots, x_{i_k} = j_k, |x_l| < 1 \text{ if } l \notin \{i_1, \dots, i_k\}\}.$$

Then the sets $A_{i_1, \dots, i_k, j_1, \dots, j_k}$, where $j_1, \dots, j_k \in \{-1, 1\}$, are precisely the connected components of A_{i_1, \dots, i_k} .

Put $\mathcal{B}_k := \exp_o(\mathcal{A}_k)$. We claim that $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ is a trivially covered stratification of $\text{Cut}_o(M)$. One checks that the connected components of each of the \mathcal{B}_k are precisely the sets

$$B_{i_1, \dots, i_k} := \exp_o(A_{i_1, \dots, i_k}) \quad \text{for } (i_1, \dots, i_k) \in J_k.$$

Moreover, for all $(i_1, \dots, i_k) \in J_k$ and all $j_1, \dots, j_k \in \{-1, 1\}$, the restriction

$$\exp_o|_{A_{i_1, \dots, i_k, j_1, \dots, j_k}} : A_{i_1, \dots, i_k, j_1, \dots, j_k} \rightarrow B_{i_1, \dots, i_k}$$

is a homeomorphism. From the explicit description of the A_{i_1, \dots, i_k} , one derives that $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ is a stratification. It further follows from the above observations that

$$\exp_o|_{A_{i_1, \dots, i_k}} : A_{i_1, \dots, i_k} \rightarrow B_{i_1, \dots, i_k}$$

is a trivial covering map for all $(i_1, \dots, i_k) \in J_k$. Since $k \in \{1, 2, \dots, n\}$ was chosen arbitrarily, this shows that $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ is trivially covered.

We now want to prove that $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ is indeed an inconsistent stratification of $\text{Cut}_o(M)$. For this purpose, let $k \in \{2, 3, \dots, n\}$, $(i_1, \dots, i_k) \in J_k$ and $x \in B_{i_1, \dots, i_k}$. We assume without loss of generality that $(i_1, i_2, \dots, i_k) = (1, 2, \dots, k)$. Then there are $y_1, \dots, y_{n-k} \in (-1, 1)$ such that

$$x = \exp_o(1, 1, \dots, 1, y_1, \dots, y_{n-k}).$$

It further holds that

$$(7-1) \quad \exp_K^{-1}(\{x\}) = \{(j_1, \dots, j_k, y_1, \dots, y_{n-k}) \in T_oM \mid j_1, \dots, j_k \in \{-1, 1\}\},$$

where $K := I^n$ and $\exp_K := \exp_o|_K: K \rightarrow M$, which is a special case of the map defined in (4-1). Let $i \in \{1, 2, \dots, k\}$ and let $\hat{B}_i := B_{1, \dots, i-1, i+1, \dots, k} \subset \mathcal{B}_{k-1}$. Given $\varepsilon > 0$, put

$$U_\varepsilon := \exp_o \left((1 - \varepsilon, 1 + \varepsilon)^k \times \prod_{j=1}^{n-k} (y_j - \varepsilon, y_j + \varepsilon) \right) \subset M.$$

Then U_ε is an open neighborhood of x and, for sufficiently small $\varepsilon > 0$, it holds that $\hat{B}_i \cap U_\varepsilon$ has two components C_i^+ and C_i^- . With $I_+ := (1 - \varepsilon, 1)$ and $I_- := (-1, -1 + \varepsilon)$, we put, for all $j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k \in \{-1, 1\}$,

$$A_{j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k}^\pm := \left\{ (j_1, \dots, j_{i-1}, t, j_{i+1}, \dots, j_k, q) \mid t \in I_\pm, q \in \prod_{l=1}^{n-k} (y_l - \varepsilon, y_l + \varepsilon) \right\}.$$

The two components C_i^+ and C_i^- then satisfy

$$\exp_K^{-1}(C_i^\pm) = \bigcup_{j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k \in \{-1, 1\}} A_{j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k}^\pm.$$

Combining this observation with (7-1) yields

$$\widetilde{\text{Cut}}_o(M) \cap \exp_o^{-1}(\{x\}) \cap \overline{\exp_K^{-1}(C_i^\pm)} = \{(j_1, \dots, j_{i-1}, \pm 1, j_{i+1}, \dots, j_k, y_1, \dots, y_{n-k}) \mid j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k \in \{-1, 1\}\},$$

In particular, $\exp_o^{-1}(\{x\}) \cap \overline{\exp_K^{-1}(C_i^+)} \cap \overline{\exp_K^{-1}(C_i^-)} = \emptyset$, implying that $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ satisfies the inconsistency condition at x . Since $x \in \text{Cut}_o(M)$ was chosen arbitrarily, this shows that $(\mathcal{B}_1, \dots, \mathcal{B}_n)$ is inconsistent. Note that in general $\mathcal{B}_{k-1} \cap U_\varepsilon$ has more connected components than C_i^+ and C_i^- , but considering these two components is sufficient for proving the inconsistency condition.

Since T^n is a Lie group and g_f is left-invariant, it follows from Corollary 6.4 that

$$\text{GC}(T^n, g_f) = n + 1.$$

Next we will compute the geodesic complexity of arbitrary two-dimensional flat tori. The reader should note that, in general, the geodesic complexity of (T^2, g) will vary with the metric g ; see [Example 2.3\(2\)](#). For arbitrary flat tori of higher dimensions, the cut loci of points are not as well understood as in the two-dimensional case. While it might be possible to extend our result to flat tori of higher dimensions, we are not aware of any systematic study of cut loci of flat higher-dimensional tori in the literature.

For $p, q \in \mathbb{R}^2$, we let $[p, q] = \{(1 - t)p + tq \in \mathbb{R}^2 \mid t \in [0, 1]\}$ denote the line segment from p to q .

Theorem 7.2 *Let g be an arbitrary flat metric on T^2 . Then $\text{GC}(T^2, g) = 3$.*

Proof By elementary Riemannian geometry, (T^2, g) is isometric to T^2 with a quotient metric induced by the standard metric on \mathbb{R}^2 and a projection $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 / \Gamma = T^2$, where $\Gamma \subset \mathbb{R}^2$ is a lattice. We thus assume that g itself is such a quotient metric. Put $o := \pi(0, 0)$. We are going to describe $\widetilde{\text{Cut}}_o(T^2)$, following [\[19, page 108\]](#). The case that Γ is generated by two orthogonal vectors is covered in [Example 7.1](#), so we assume in the following that Γ is generated by two vectors $a_1, a_2 \in \mathbb{R}^2$ such that the angle between a_1 and a_2 is acute.

If we identify T_oM with \mathbb{R}^2 , then $\widetilde{\text{Cut}}_o(M)$ is given by a hexagon whose construction we will describe next. Consider the perpendicular bisectors of the line segments

$$[0, a_1], \quad [0, a_2], \quad [0, -a_1], \quad [0, -a_2], \quad [0, a_1 - a_2], \quad [0, a_2 - a_1].$$

These perpendicular bisectors enclose a hexagon in \mathbb{R}^2 ; see [Figure 1](#). The tangent cut locus $\widetilde{\text{Cut}}_o(M)$ consists of the boundary curve of the hexagon, while the domain of injectivity of \exp_o is given by the interior of the hexagon. Let the segments and the corner points of the hexagon be labeled as in [Figure 1](#). Then there are $p, q \in M$ with $p \neq q$ such that $p = \exp_o(p_1) = \exp_o(p_2) = \exp_o(p_3)$ and $q = \exp_o(q_1) = \exp_o(q_2) = \exp_o(q_3)$.

For $x, y \in \mathbb{R}^2$ we put $\llbracket x, y \rrbracket := [x, y] \setminus \{x, y\}$. With p and q as above, the set $\text{Cut}_o(M) \setminus \{p, q\}$ has three connected components

$$\begin{aligned} A_1 &:= \exp_o(\llbracket p_1, q_1 \rrbracket) = \exp_o(\llbracket q_2, p_3 \rrbracket), \\ A_2 &:= \exp_o(\llbracket q_1, p_2 \rrbracket) = \exp_o(\llbracket p_3, q_3 \rrbracket), \\ A_3 &:= \exp_o(\llbracket p_2, q_2 \rrbracket) = \exp_o(\llbracket q_3, p_1 \rrbracket). \end{aligned}$$

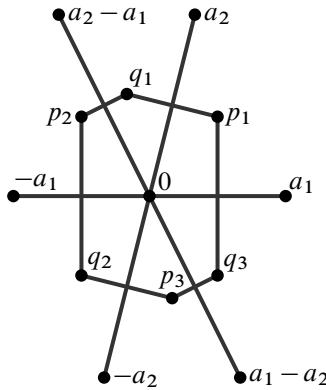


Figure 1: Tangent cut loci of flat two-dimensional tori.

More precisely, \exp_o maps both $\llbracket p_1, q_1 \rrbracket$ and $\llbracket q_2, p_3 \rrbracket$ homeomorphically onto A_1 , both $\llbracket q_1, p_2 \rrbracket$ and $\llbracket p_3, q_3 \rrbracket$ homeomorphically onto A_2 , and both $\llbracket p_2, q_2 \rrbracket$ and $\llbracket q_3, p_1 \rrbracket$ homeomorphically onto A_3 .

Let $S_2 := \{p, q\}$ and $S_1 := A_1 \cup A_2 \cup A_3$. By construction, (S_1, S_2) is a trivially covered stratification of $\text{Cut}_o(T_2, g)$. We want to show that (S_1, S_2) is inconsistent as well. Let $K \subset T_oT^2$ denote the union of $\widetilde{\text{Cut}}_o(T^2, g)$ with the domain of injectivity of \exp_o and let $\exp_K: K \rightarrow T^2$ be the restriction of \exp_o to K . This is again a special case of the map defined in (4-1). Let $U \subset T^2$ be an open neighborhood of p and put $Z_i := A_i \cap U$ for all $i \in \{1, 2, 3\}$. If U is chosen sufficiently small, then, by the above description of $\widetilde{\text{Cut}}_o(T^2, g)$, there are $x_1 \in \llbracket p_1, q_1 \rrbracket$ and $x'_1 \in \llbracket q_2, p_3 \rrbracket$ such that

$$\exp_K^{-1}(Z_1) = \llbracket x'_1, p_3 \rrbracket \cup \llbracket p_1, x_1 \rrbracket.$$

Analogously, one shows that there are

$$x_2 \in \llbracket q_1, p_2 \rrbracket, \quad x'_2 \in \llbracket p_3, q_3 \rrbracket, \quad x_3 \in \llbracket p_2, q_2 \rrbracket \quad \text{and} \quad x'_3 \in \llbracket q_3, p_1 \rrbracket$$

such that

$$\exp_K^{-1}(Z_2) = \llbracket x_2, p_2 \rrbracket \cup \llbracket p_3, x'_2 \rrbracket, \quad \exp_K^{-1}(Z_3) = \llbracket x'_3, p_1 \rrbracket \cup \llbracket p_2, x_3 \rrbracket.$$

Since $\exp_K^{-1}(\{p\}) = \{p_1, p_2, p_3\}$, this shows that

$$\begin{aligned} \exp_K^{-1}(\{p\}) \cap \overline{\exp_K^{-1}(Z_1)} &= \{p_2, p_3\}, \\ \exp_K^{-1}(\{p\}) \cap \overline{\exp_K^{-1}(Z_2)} &= \{p_1, p_3\}, \\ \exp_K^{-1}(\{p\}) \cap \overline{\exp_K^{-1}(Z_3)} &= \{p_1, p_2\}. \end{aligned}$$

Consequently,

$$\widetilde{\text{Cut}}_o(T^2) \cap \exp_o^{-1}(\{p\}) \cap \bigcap_{i=1}^3 \overline{\exp_K^{-1}(Z_i)} = \emptyset,$$

which shows that (S_1, S_2) satisfies the inconsistency condition at p . In complete analogy, one shows that the condition is satisfied at q as well, implying that (S_1, S_2) is inconsistent. Since g is, by construction, left-invariant, it follows from [Corollary 6.4](#) that $\text{GC}(T^2, g) = 3$. □

7.2 Geodesic complexity of Berger spheres

In this subsection we consider a class of homogeneous Riemannian manifolds whose geodesic complexity can be computed explicitly without making use of the upper and lower bounds we previously studied. In [\[1\]](#), M Berger has constructed a one-parameter family of homogeneous metrics g_α for $0 < \alpha \leq \frac{\pi}{2}$ on the three-dimensional sphere S^3 , whose cut loci have been described by Sakai [\[38\]](#).

In the following, we will first recall a particularly interesting class of homogeneous Riemannian manifolds, namely naturally reductive spaces. Berger spheres are special cases of them and we will outline the construction of Berger’s metrics following [\[38\]](#).

Given a Lie group G , we always let $e \in G$ denote its unit element. Let \mathfrak{g} denote the Lie algebra of G and assume that H is a closed subgroup of G . Then the Lie algebra \mathfrak{h} of the Lie group H is a Lie subalgebra of \mathfrak{g} . If there is an Ad_H -invariant subspace \mathfrak{m} of the Lie algebra \mathfrak{g} which is complementary to \mathfrak{h} then there is a bijective correspondence between Ad_H -invariant inner products on \mathfrak{m} and G -invariant metrics on the homogeneous space G/H . See [\[31, Proposition 11.22\(2\)\]](#) for details.

Definition 7.3 [\[31, page 317\]](#) Let G be a Lie group with a closed subgroup H . Let \mathfrak{g} be the Lie algebra of G and \mathfrak{h} be the Lie algebra of H . Assume that there is a subspace $\mathfrak{m} \subset \mathfrak{g}$ which is complementary to \mathfrak{h} and such that $\text{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$, where Ad_H denotes the adjoint representation of H . Suppose there is an Ad_H -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{m} such that

$$\langle [X, Y]_{\mathfrak{m}}, Z \rangle = \langle X, [Y, Z]_{\mathfrak{m}} \rangle$$

for all $X, Y, Z \in \mathfrak{m}$, where the subscript \mathfrak{m} of an element of \mathfrak{g} denotes its projection onto \mathfrak{m} . Then G/H together with the G -invariant Riemannian metric corresponding to this inner product is called a *naturally reductive space*.

Example 7.4 All symmetric spaces are examples of naturally reductive spaces as discussed in [31, page 317]. The real Stiefel manifolds $V_k(\mathbb{R}^n)$ for $n \geq 4$ and $2 \leq k \leq n - 2$ are examples of naturally reductive spaces which are not symmetric spaces; see [18, page 748].

For our purposes, the crucial property of naturally reductive spaces is the observation made in the following proposition. We refer to [31, Proposition 11.25] for its proof.

Proposition 7.5 *Let G be a Lie group and $H \subset G$ be a closed subgroup. If $M = G/H$ is a naturally reductive space and $\pi : G \rightarrow M$ is the projection, then the geodesics starting at $o = \pi(e)$ are precisely the curves of the form $\gamma(t) = \pi(\exp(t\xi))$ for $\xi \in \mathfrak{m}$, where $\exp : \mathfrak{g} \rightarrow G$ is the Lie group exponential of G .*

We proceed by constructing Berger spheres as naturally reductive spaces following the exposition of [38]. Let $G = \text{SU}(2) \times \mathbb{R}$ and let $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}$ be its Lie algebra. We consider the Ad_G -invariant inner product on \mathfrak{g} given by

$$\langle (A, x), (B, y) \rangle = -\frac{1}{2} \text{Tr}(AB) + xy \quad \text{for all } (A, x), (B, y) \in \mathfrak{g}.$$

For $\alpha \in (0, \frac{\pi}{2}]$, we define a linear subspace of \mathfrak{g} as

$$\mathfrak{h}_\alpha = \left\{ \left(\begin{pmatrix} il \cos \alpha & 0 \\ 0 & -il \cos \alpha \end{pmatrix}, l \sin \alpha \right) \in \mathfrak{g} \mid l \in \mathbb{R} \right\}.$$

Consider the closed subgroup $H_\alpha \subset G$, $H_\alpha = \exp(\mathfrak{h}_\alpha)$, where \exp again denotes the Lie group exponential of G . Explicitly, H_α is given as

$$H_\alpha = \left\{ \left(\begin{pmatrix} e^{il \cos \alpha} & 0 \\ 0 & e^{-il \cos \alpha} \end{pmatrix}, l \sin \alpha \right) \mid l \in \mathbb{R} \right\}.$$

One checks that G/H_α is diffeomorphic to S^3 . The orthogonal complement to \mathfrak{h}_α in \mathfrak{g} with respect to $\langle \cdot, \cdot \rangle$ is the space

$$\mathfrak{m}_\alpha = \left\{ \left(\begin{pmatrix} ir \sin \alpha & a + ib \\ -a + ib & -ir \sin \alpha \end{pmatrix}, -r \cos \alpha \right) \in \mathfrak{g} \mid a, b, r \in \mathbb{R} \right\}.$$

A direct computation shows that \mathfrak{m}_α is Ad_{H_α} -invariant. The restriction of the inner product $\langle \cdot, \cdot \rangle$ to $\mathfrak{m}_\alpha \times \mathfrak{m}_\alpha$ defines an Ad_{H_α} -invariant inner product on \mathfrak{m}_α . We equip the homogeneous space G/H_α with the G -invariant metric that is defined by this inner product and the abovementioned correspondence between G -invariant Riemannian metrics on G/H_α and Ad_{H_α} -invariant inner products on \mathfrak{m}_α .

Since $\mathfrak{m}_\alpha \perp \mathfrak{h}_\alpha$ by construction, the space $M_\alpha = G/H_\alpha$ equipped with the described homogeneous metric is a naturally reductive space; see [18, Proposition 23.29]. Thus, by Proposition 7.5, the geodesics in M_α emanating from o are precisely the images of the one-parameter groups in G under π of elements of \mathfrak{m}_α . For $\alpha = \frac{\pi}{2}$, one further observes that G/H_α is isometric to the round sphere S^3 of sectional curvature one; see [28, page 77].

The following observation gives us a strong upper bound on $GC(M_\alpha)$. We refer to [28, Section 3] for its proof.

Proposition 7.6 *For each $\alpha \in (0, \frac{\pi}{2})$ the Berger sphere M_α is isometric to $SU(2)$ equipped with a left-invariant metric.*

Combining Proposition 7.6 with Corollary 5.5 yields

$$(7-2) \quad GC(M_\alpha) \leq GC_o(\text{Cut}_o(M_\alpha)) + 1,$$

where $o = \pi(1)$. To compute $GC_o(\text{Cut}_o(M_\alpha))$, we will outline the results from [38] about the cut loci of M_α . For $\alpha = \frac{\pi}{2}$, we already know that $GC(M_{\pi/2}) = 2$; see Examples 2.3(1). Thus, in the following we fix $\alpha \in (0, \frac{\pi}{2})$.

Let $S \subset \mathfrak{m}_\alpha$ denote the unit sphere in \mathfrak{m}_α with respect to the norm induced by $\langle \cdot, \cdot \rangle$ and let $D\pi_e: \mathfrak{g} \rightarrow T_oM_\alpha$ denote the differential of π in the unit $e \in G$. Consider the isometric isomorphism of vector spaces

$$\varphi := D\pi_e|_{\mathfrak{m}_\alpha} : \mathfrak{m}_\alpha \xrightarrow{\cong} T_oM_\alpha.$$

Then φ maps S to the unit sphere in T_oM_α . Let $\psi: T_oM_\alpha \rightarrow T_oM_\alpha$ be the radial homeomorphism which maps the unit sphere homeomorphically onto the tangent cut locus $\widetilde{\text{Cut}}_o(M_\alpha)$ of o in M_α . Then the map $F: S \rightarrow \widetilde{\text{Cut}}_o(M_\alpha)$, $F := \psi \circ \varphi|_S$ is a homeomorphism. We consider $e_1, e_2 \in \mathfrak{m}_\alpha$ given by

$$e_1 = \left(\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right), \quad e_2 = \left(\left(\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, 0 \right) \right).$$

Let $U := \text{span}_{\mathbb{R}}(\{e_1, e_2\})$ and let $r_U: \mathfrak{m}_\alpha \rightarrow \mathfrak{m}_\alpha$ denote the reflection through U . Let \mathfrak{m}_1 and \mathfrak{m}_2 denote the two components of $\mathfrak{m}_\alpha \setminus U$ and put $D_i := S \cap \mathfrak{m}_i$ for $i \in \{1, 2\}$. By the results of [38, page 151]:

- $\text{Cut}_o(M_\alpha) = (\exp_o \circ F)(S)$.
- $\exp_o \circ F|_{D_1}$ and $\exp_o \circ F|_{D_2}$ are injective.
- $(\exp_o \circ F)(v) = (\exp_o \circ F)(r_U(v))$ for all $v \in D_1 \cup D_2$.

Hence, the map $\exp_o \circ F|_{\bar{D}_1}$ is a bijective continuous map from a closed disk onto the cut locus $\text{Cut}_o(M_\alpha)$. Since the disk is compact and $\text{Cut}_o(M_\alpha)$ is a Hausdorff space, this shows that $\text{Cut}_o(M_\alpha)$ is homeomorphic to a closed disk. Moreover, $\exp_o \circ F|_{\bar{D}_i} : \bar{D}_i \rightarrow M_\alpha$ is an embedding of \bar{D}_i onto $\text{Cut}_o(M_\alpha)$ for $i \in \{1, 2\}$.

Theorem 7.7 For all $\alpha \in (0, \frac{\pi}{2}]$, it holds that $\text{GC}(M_\alpha) = 2$.

Proof For $\alpha = \frac{\pi}{2}$, ie the case of a round metric, this is observed in [34, Proposition 4.1], so we will only consider the case of $\alpha \in (0, \frac{\pi}{2})$. In the notation from above, we put $E := F(\bar{D}_1)$ and let $f : \text{Cut}_o(M_\alpha) \rightarrow E$, $f := (\exp_o \circ F|_E)^{-1}$. Define

$$s : \{o\} \times \text{Cut}_o(M_\alpha) \rightarrow GM_\alpha, \quad (s(o, q))(t) = \exp_o(t \cdot f(q)) \quad \text{for all } t \in [0, 1].$$

By Proposition 7.5, the map s is a continuous geodesic motion planner, which shows

$$\text{GC}_o(\text{Cut}_o(M_\alpha)) = 1$$

and thus $\text{GC}(M_\alpha) \leq 2$ by (7-2). Since $\text{GC}(M_\alpha) \geq \text{TC}(S^3) = 2$, this shows the claim. \square

Remarks 7.8 (1) As Recio-Mitter has shown [34, Example 2.4], there exists a Riemannian metric g_m on S^3 for which $\text{GC}(S^3, g_m) = 3$. This shows that also in the case of S^3 , the value of GC depends on the chosen metric.

(2) The cut locus of a point in the Berger sphere M_α for $0 < \alpha < \frac{\pi}{2}$ is a closed disk. It therefore seems tempting to determine the geodesic complexity of M_α via a stratification of this cut locus similarly to what we have done in previous sections. More precisely, an obvious stratification of a closed disk is given by taking one stratum as its interior and another stratum as its boundary. However, this is not an inconsistent stratification as in Definition 4.6 since we would then obtain $\text{GC}(M_\alpha) \geq 3$, whereas we have shown that $\text{GC}(M_\alpha) = 2$.

(3) As this example is particularly instructive, we want to sketch briefly how to show directly that the stratification from the previous paragraph is not inconsistent. Let $K \subset T_o M_\alpha$ be the union of the injectivity domain with the tangent cut locus $\widetilde{\text{Cut}}_o(M_\alpha)$. Using the same notation as in the exposition above, put

$$S_1 := (\exp_o \circ F)(\bar{D}_1) \subset \text{Cut}_o(M_\alpha) \quad \text{and} \quad S_2 := (\exp_o \circ F)(\partial D_1) \subset \text{Cut}_o(M_\alpha).$$

Under the identification of $\text{Cut}_o(M_\alpha)$ and a closed disk, this is the decomposition from part (2) of this remark. Evidently, this is a stratification in the sense of Definition 4.4.

Let $x \in S_2$ and let $U \subset M_\alpha$ be a neighborhood of x . For sufficiently small U , the intersection $U \cap S_1$ has only one connected component, which we call Z . We claim that

$$\widetilde{\text{Cut}}_o(M_\alpha) \cap \exp_o^{-1}(\{x\}) \cap \overline{\exp_K^{-1}(Z)} \neq \emptyset.$$

By the above discussion of $\text{Cut}_o(M_\alpha)$, the intersection $\widetilde{\text{Cut}}_o(M_\alpha) \cap \exp_o^{-1}(\{x\})$ consists of a single point $v \in T_o M_\alpha$. By choosing a sequence in Z converging to x and recalling that $\exp_o \circ F|_{\overline{D}_i}$ is a homeomorphism for $i \in \{1, 2\}$, we see that

$$v \in \overline{\exp_K^{-1}(Z)}.$$

This shows that

$$\widetilde{\text{Cut}}_o(M_\alpha) \cap \exp_o^{-1}(\{x\}) \cap \overline{\exp_K^{-1}(Z)} = \{v\} \neq \emptyset.$$

Hence, the stratification (S_1, S_2) of $\text{Cut}_o(M_\alpha)$ is not inconsistent.

8 Explicit upper bounds for symmetric spaces

In [36, Theorem 5.3], Sakai has determined the cut loci of compact simply connected irreducible symmetric spaces. He showed that their cut loci always allow for stratifications for which each stratum is a submanifold. Since every symmetric space is a Riemannian product of irreducible symmetric spaces, Sakai’s results are enough to determine the cut loci of compact simply connected symmetric spaces in general; see our explanations on cut loci of product manifolds in [Example 7.1](#).

In this section, we will first apply the results from [Section 5](#) to find an upper bound for the geodesic complexity of a compact, simply connected, irreducible symmetric space. From Sakai’s results, in particular [36, Proposition 4.10], we will further derive estimates on the subspace geodesic complexities of the strata of a cut locus. These numbers appeared on the right-hand side of the inequality in [Corollary 5.8](#) and we will show that they can be estimated from above by certain sectional categories. As a result, we will obtain an upper bound for the geodesic complexity of compact, simply connected, irreducible symmetric spaces given purely in terms of categorical invariants.

We begin by summarizing the main results of [36], stated here in the form of [37, Section 4]. We assume basic knowledge on symmetric spaces that is provided by textbooks in Riemannian geometry like [23] or [33]. In the following, we always let Df_x denote the differential of a differentiable map f in the point x .

Let $M = G/K$ be a compact, simply connected and irreducible symmetric space, where (G, K) is a Riemannian symmetric pair. Explicitly, G is a compact, connected Lie group, K is a closed, connected Lie subgroup of G and G admits an involutive automorphism $\sigma: G \rightarrow G$ whose fixed-point set satisfies $(\text{Fix}(\sigma))_0 \subset K \subset \text{Fix}(\sigma)$, where $(\text{Fix}(\sigma))_0$ is the identity component of $\text{Fix}(\sigma)$.

Let $\pi: G \rightarrow M$ denote the orbit space projection, let $e \in G$ denote the unit element and put $o := \pi(e) \in M$. Let \mathfrak{g} and \mathfrak{k} denote the Lie algebras of G and K , respectively, and let $\mathfrak{m} \subset \mathfrak{g}$ be the -1 eigenspace of $D\sigma_e$. Then, since \mathfrak{k} is the $+1$ eigenspace of $D\sigma_e$, there is a vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Furthermore, the restriction $D\pi_e|_{\mathfrak{m}}: \mathfrak{m} \rightarrow T_oM$ is a linear isomorphism; see [23, Theorem IV.3.3].

In the following we give a concise overview of the most important notions related to root systems of symmetric spaces:

- Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of \mathfrak{g} . By [23, page 284], there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$. We recall that a *root* α of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is an element of the dual space \mathfrak{h}^* such that there is a nonzero vector $X \in \mathfrak{g}_{\mathbb{C}}$ satisfying

$$[H, X] = \alpha(H)X \quad \text{for all } H \in \mathfrak{h}.$$

The set of nonzero roots of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ will be called R .

- Let \mathfrak{a} be a maximal abelian subalgebra of \mathfrak{m} , which again exists by [23, page 284]. We will call \mathfrak{a} a *Cartan subalgebra of (G, K)* .

A root $\alpha \in R$ with $\alpha|_{\mathfrak{a}} \neq 0$ will be called a *root of the symmetric pair (G, K)* . The set of roots of the symmetric pair (G, K) will be denoted by $R(G, K)$.

- By choosing a certain real subspace $\mathfrak{h}_{\mathbb{R}}$ of the Cartan subalgebra \mathfrak{h} and defining a lexicographic ordering on $\mathfrak{h}_{\mathbb{R}}$, one defines an ordering on the set of roots R ; see [23, page 173]. This defines a set of positive roots $R^+ \subset R$ of $\mathfrak{g}_{\mathbb{C}}$. The set of *positive roots of (G, K)* is then defined as

$$R(G, K)^+ := R^+ \cap R(G, K).$$

There is a maximal element of $R(G, K)^+$ with respect to this ordering, which we denote by δ and call the *highest root of (G, K)* .

- Let k be the rank of the symmetric space $M = G/K$. A *simple root of (G, K)* is a positive root α which cannot be written as a sum $\alpha = \beta + \gamma$ with $\beta, \gamma \in R(G, K)^+$. There are precisely k simple roots and one finds that every positive root can be written

as a linear combination of the simple roots with nonnegative integer coefficients; see [23, Theorem VII.2.19]. Denote the system of simple roots of (G, K) by $\pi(G, K)$.

- By virtue of the chosen Ad_K -invariant inner product on \mathfrak{g} , we will from now on consider the roots to be vectors in \mathfrak{a} in order to follow [36, Section 2].

Based on this terminology, we next recall Sakai’s results on the structure of cut loci of symmetric spaces. In the case that there are two or more positive roots of (G, K) , we define a subset \mathcal{D} of the power set of $\pi(G, K)$ by

$$\mathcal{D} := \{\Delta \subset \pi(G, K) \mid \Delta \neq \emptyset, \delta \notin \Delta\}.$$

If there is only one positive root γ , which is therefore also the only simple root and also the highest root, define

$$\mathcal{D} := \{\{\gamma\}\}.$$

Let $\langle \cdot, \cdot \rangle$ denote the chosen Ad_K -invariant inner product on \mathfrak{g} and consider the Weyl chamber of $\pi(G, K)$ that is given by

$$W := \{X \in \mathfrak{a} \mid \langle \gamma, X \rangle > 0 \text{ for all } \gamma \in \pi(G, K)\}.$$

See [23, Section VII.2] for further details on Weyl chambers and their connection to root systems. If there is more than one positive root, let

$$S_\Delta :=$$

$$\{X \in \overline{W} \mid \langle \gamma, X \rangle > 0 \text{ for all } \gamma \in \Delta, \langle \gamma, X \rangle = 0 \text{ for all } \gamma \in \pi(G, K) \setminus \Delta, 2\langle \delta, X \rangle = 1\}$$

for each $\Delta \in \mathcal{D}$. If there is just one positive root γ , then set

$$S_{\{\gamma\}} := \{X \in \mathfrak{a} \mid 2\langle \gamma, X \rangle = 1\}.$$

Since \mathfrak{a} is one-dimensional in that case, $S_{\{\gamma\}}$ consists of a single point.

Let $\exp: \mathfrak{g} \rightarrow G$ be the exponential map of G and put

$$\text{Exp}: \mathfrak{m} \rightarrow M, \quad \text{Exp} := \pi \circ \exp|_{\mathfrak{m}}.$$

For $\Delta \in \mathcal{D}$, we let

$$\tilde{\Phi}_\Delta: K \times S_\Delta \rightarrow M, \quad \tilde{\Phi}_\Delta(k, X) = \text{Exp}(\text{Ad}(k)(X)),$$

and put $Z_\Delta := \{k \in K \mid \text{Exp}(\text{Ad}(k)(X)) = \text{Exp}(X) \text{ for all } X \in S_\Delta\}$. One checks without difficulties that Z_Δ is a closed subgroup of K . As shown in [36, Proposition 4.10], each $\tilde{\Phi}_\Delta$ induces a differentiable embedding

$$\Phi_\Delta: K/Z_\Delta \times S_\Delta \rightarrow M.$$

Put $C_\Delta := \text{im } \Phi_\Delta$ for each $\Delta \in \mathcal{D}$. By [36, Theorem 5.3], the cut locus of M at o is then given by

$$\text{Cut}_o(M) = \bigcup_{\Delta \in \mathcal{D}} C_\Delta$$

and the C_Δ satisfy

$$(8-1) \quad \begin{aligned} C_\Delta \cap C_{\Delta'} &= \emptyset && \text{for all } \Delta, \Delta' \in \mathcal{D}, \Delta \neq \Delta', \\ \bar{C}_\Delta &= \bigcup_{\Delta' \subset \Delta} C_{\Delta'} && \text{for all } \Delta \in \mathcal{D}. \end{aligned}$$

Let k be the rank of M . For $i \in \{1, 2, \dots, k\}$, we put

$$\mathcal{D}_i := \{\Delta \in \mathcal{D} \mid \#\Delta = i\} \quad \text{and} \quad C_i := \bigcup_{\Delta \in \mathcal{D}_i} C_\Delta.$$

It follows from (8-1) that $(C_k, C_{k-1}, \dots, C_1)$ is a stratification of $\text{Cut}_o(M)$ and that the C_Δ for $\Delta \in \mathcal{D}_i$ are precisely the connected components of C_i . Since M is a homogeneous Riemannian manifold, we thus obtain from Corollary 5.8 that

$$(8-2) \quad \text{GC}(M) \leq \text{secat}(\text{ev}_o : \text{Isom}(M) \rightarrow M) \cdot \sum_{i=1}^k \max_{\Delta \in \mathcal{D}_i} \text{GC}_o(C_\Delta) + 1.$$

It remains to find upper bounds on the numbers $\text{GC}_o(C_\Delta)$.

Proposition 8.1 *For each $\Delta \in \mathcal{D}$, it holds that*

$$\text{GC}_o(C_\Delta) \leq \text{secat}(q_\Delta : K \rightarrow K/Z_\Delta),$$

where q_Δ denotes the orbit space projection.

Proof Let $r := \text{secat}(q_\Delta)$. Then, by Lemma 4.1, there are pairwise disjoint and locally compact subsets $B_1, \dots, B_r \subset K/Z_\Delta$ such that, for each $i \in \{1, 2, \dots, r\}$, there is a continuous local section $s_i : B_i \rightarrow K$ of q_Δ . Using these s_i , we define

$$\sigma_i : \{o\} \times \Phi_\Delta(B_i \times S_\Delta) \rightarrow GM, \quad (\sigma_i(o, \Phi_\Delta(x, X)))(t) = \text{Exp}(t \cdot \text{Ad}(s_i(x))(X)),$$

for every $i \in \{1, 2, \dots, r\}$. By construction, each σ_i is continuous and $\sigma_i(o, \Phi_\Delta(x, X))$ is a geodesic segment for all $(x, X) \in B_i \times S_\Delta$ and each $i \in \{1, 2, \dots, r\}$. Moreover,

$$\begin{aligned} (\sigma_i(o, \Phi_\Delta(x, X)))(0) &= \text{Exp}(0) = o, \\ (\sigma_i(o, \Phi_\Delta(x, X)))(1) &= \text{Exp}(\text{Ad}(s_i(x))(X)) = \Phi_\Delta(x, X), \end{aligned}$$

by definition of Φ_Δ . Thus, the σ_i are continuous geodesic motion planners. Since the sets $\Phi_\Delta(B_1 \times S_\Delta), \dots, \Phi_\Delta(B_r \times S_\Delta)$ are pairwise disjoint, locally compact and cover $\Phi(K/Z_\Delta \times S_\Delta) = C_\Delta$, this shows that $\text{GC}_o(C_\Delta) \leq r$. □

Combining Proposition 8.1 with (8-2) yields the following upper bound:

Theorem 8.2 *Let (G, K) be a Riemannian symmetric pair and let $M = G/K$ be the corresponding symmetric space. Assume that M is compact, simply connected and irreducible. Then, with \mathcal{D}_i and Z_Δ given as above,*

$$GC(M) \leq \text{secat}(\text{ev}_o: \text{Isom}(M) \rightarrow M) \cdot \sum_{i=1}^{\text{rk}(M)} \max_{\Delta \in \mathcal{D}_i} \text{secat}(q_\Delta: K \rightarrow K/Z_\Delta) + 1,$$

where $\text{rk}(M)$ denotes the rank of M .

Corollary 8.3 *Let (G, K) be a Riemannian symmetric pair and let $M = G/K$ be the corresponding symmetric space. Assume that M is compact, simply connected and irreducible. Then, with \mathcal{D}_i and Z_Δ given as above,*

$$GC(M) \leq \text{cat}(M) \cdot \sum_{i=1}^{\text{rk}(M)} \max_{\Delta \in \mathcal{D}_i} \text{cat}(K/Z_\Delta) + 1.$$

Proof This is an immediate consequence of Theorem 8.2 and [39, Theorem 18]. \square

We want to conclude by applying the upper bounds to two examples of compact symmetric spaces whose cut loci have already been discussed in the works of Sakai, more precisely in [36, Example 5.4; 35, Section 4.2].

Example 8.4 Consider the complex projective space $\mathbb{C}P^n = U(n+1)/(U(1) \times U(n))$ with the Fubini–Study metric. This is a compact and simply connected symmetric space of rank one. Its cut locus is studied in detail in [36, Example 5.4]. Let $G = U(n+1)$, let $\mathfrak{g} = \mathfrak{u}(n+1)$ be its Lie algebra and let $K = U(1) \times U(n)$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ denote the decomposition of \mathfrak{g} with respect to the symmetric pair (G, K) . By the same methods as in [23, page 452], which treats the Lie algebra of $SU(n)$, one computes that

$$\mathfrak{k} = \left\{ \begin{pmatrix} ia & 0 \\ 0 & \xi \end{pmatrix} \mid a \in \mathbb{R}, \xi \in \mathfrak{u}(n) \right\} \quad \text{and} \quad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & \bar{u}^T \\ -u & 0 \end{pmatrix} \mid u \in \mathbb{C}^n \right\}.$$

Then $\mathfrak{a} = \text{span}_{\mathbb{R}}(\{H_0\})$ is a Cartan subalgebra of (G, K) , where $H_0 = (h_{ij}) \in \mathfrak{g}$ is given by

$$h_{ij} = \begin{cases} \frac{\pi}{2} & \text{if } (i, j) = (1, n+1), \\ -\frac{\pi}{2} & \text{if } (i, j) = (n+1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, every system of simple roots of (G, K) consists of a unique element. Let o be the equivalence class of the neutral element of G in $G/K = \mathbb{C}P^n$. Then $\text{Cut}_o(M)$

consists of a unique submanifold, given by

$$\text{Cut}_o(M) = \{\text{Exp}(\text{Ad}(k)(H_0)) \mid k \in U(1) \times U(n)\}.$$

Sakai further showed that

$$Z_0 := \{k \in U(1) \times U(n) \mid \text{Exp}(\text{Ad}(k)(H_0)) = \text{Exp}(H_0)\}$$

can be identified with $Z_0 = U(1) \times U(n-1) \times U(1)$. Hence, by [Proposition 8.1](#),

$$(8-3) \quad \text{GC}_M(\text{Cut}_o(M)) \leq \text{secat}(U(1) \times U(n) \rightarrow (U(1) \times U(n))/(U(1) \times U(n-1) \times U(1))).$$

One easily checks that the map

$$\begin{aligned} \varphi: (U(1) \times U(n))/(U(1) \times U(n-1) \times U(1)) &\rightarrow U(n)/(U(n-1) \times U(1)) = \mathbb{C}P^{n-1}, \\ \varphi([z, A]) &= [A], \end{aligned}$$

where $(z, A) \in U(1) \times U(n)$, is a well-defined homeomorphism. Let $p: U(n) \rightarrow \mathbb{C}P^{n-1}$ denote the principal $U(1)$ -bundle over the homogeneous space $\mathbb{C}P^{n-1}$. Assume that $s: V \rightarrow U(n)$ is a continuous local section of p over a subset $V \subset \mathbb{C}P^{n-1}$. Then we obtain a continuous local section $\tilde{s}: V \rightarrow U(1) \times U(n)$ of the principal fiber bundle

$$U(1) \times U(n) \rightarrow (U(1) \times U(n))/(U(1) \times U(n-1) \times U(1))$$

by setting $\tilde{s}(\varphi^{-1}(p)) = (z_0, s(p))$ for $p \in V$, where $z_0 \in U(1)$ is a fixed element. This shows that

$$\text{secat}(U(1) \times U(n) \rightarrow (U(1) \times U(n))/(U(1) \times U(n-1) \times U(1))) \leq \text{secat}(U(n) \rightarrow \mathbb{C}P^{n-1}).$$

Hence, we derive from (8-3) that

$$\text{GC}_o(\text{Cut}_o(M)) \leq \text{secat}(U(n) \rightarrow \mathbb{C}P^{n-1}) \leq \text{cat}(\mathbb{C}P^{n-1}) = n,$$

where we used [39, Theorem 18] for the second inequality. The fact that $\text{cat}(\mathbb{C}P^{n-1}) = n$ is shown in [6, Example 1.51]. Eventually, by [Theorem 5.4](#) and the same references,

$$\begin{aligned} \text{GC}(\mathbb{C}P^n) &\leq \text{secat}(U(n+1) \rightarrow \mathbb{C}P^n) \text{GC}_o(\text{Cut}_o(M)) + 1 \\ &\leq \text{cat}(\mathbb{C}P^n) \text{cat}(\mathbb{C}P^{n-1}) + 1 \\ &= (n+1)n + 1. \end{aligned}$$

Since $\text{TC}(\mathbb{C}P^n) = 2n + 1$, as computed in [14, Lemma 28.1], we derive using [Remark 2.4\(1\)](#) that

$$2n + 1 \leq \text{GC}(\mathbb{C}P^n) \leq (n+1)n + 1 \quad \text{for all } n \in \mathbb{N}.$$

For $n = 2$, this shows that $\text{GC}(\mathbb{C}P^2) \in \{5, 6, 7\}$.

Example 8.5 Consider the complex Grassmann manifold

$$G_2(\mathbb{C}^4) = U(4)/(U(2) \times U(2)).$$

As a quotient of a compact Lie group by a closed subgroup, $G_2(\mathbb{C}^4)$ is compact. Let $V_2(\mathbb{C}^4)$ be the corresponding complex Stiefel manifold. As shown in Example 4.54 of [22], $V_2(\mathbb{C}^4)$ is simply connected. Since the fiber $U(2)$ of the fibration $V_2(\mathbb{C}^4) \rightarrow G_2(\mathbb{C}^4)$ is connected, it follows from the long exact homotopy sequence of this fibration that $G_2(\mathbb{C}^4)$ is simply connected as well. The cut loci of $G_2(\mathbb{C}^4)$ are discussed in [35, Section 4.2; 36, Example 5.5]. The corresponding decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{u}(4)$ of $G = U(4)$ is given by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in \mathfrak{u}(2) \right\} \quad \text{and} \quad \mathfrak{m} = \left\{ \begin{pmatrix} 0 & \xi \\ -\xi^T & 0 \end{pmatrix} \mid \xi \in M_2(\mathbb{C}) \right\}.$$

Here, \mathfrak{k} is the Lie algebra of $K = U(2) \times U(2)$. A Cartan subalgebra $\mathfrak{a} \subset \mathfrak{m}$ is spanned by

$$e_1 := \frac{1}{2\pi} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_2 := \frac{1}{2\pi} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

By [36, page 143], one can define positive roots and simple roots of $(U(4), U(2) \times U(2))$ in such a way that $2e_1 = \delta$ is the highest root and that a system of simple roots is given by

$$\pi(G, K) = \{\gamma_1, \gamma_2\}, \quad \text{where } \gamma_1 := 2e_2, \gamma_2 := e_1 - e_2.$$

Thus, in the notation from above, $\mathcal{D} = \{\Delta_0, \Delta_1, \Delta_2\}$, where $\Delta_0 = \{\gamma_1, \gamma_2\}$, $\Delta_1 = \{\gamma_1\}$ and $\Delta_2 = \{\gamma_2\}$. With $S_i := S_{\Delta_i}$ for $i \in \{0, 1, 2\}$, one computes that

$$S_0 = \{\pi^2 e_1 + \lambda e_2 \in \mathfrak{a} \mid \lambda \in (0, \pi^2)\}, \quad S_1 = \{\pi^2 e_1\}, \quad S_2 = \{\pi^2(e_1 + e_2)\}.$$

We further put $Z_i := Z_{\Delta_i}$ for each i . By computing the corresponding matrix exponentials, we obtain

$$Z_1 = \{\text{diag}(a, b, c, d) \in U(2) \times U(2) \mid a, b, c, d \in U(1)\} \cong U(1)^4.$$

Since $U(2)/(U(1) \times U(1))$ is diffeomorphic to $\mathbb{C}P^1 \cong S^2$, it follows that $K/Z_1 \cong S^2 \times S^2$. Hence, $\text{cat}(K/Z_1) = \text{cat}(S^2 \times S^2) \leq 3$ by the product inequality for cat ; see [6, Theorem 1.37]. One further computes by matrix exponentials that $Z_2 = K$, so K/Z_2 consists of a single point, which yields $\text{cat}(K/Z_2) = 1$. By [36, Lemma 4.9],

for every fixed $X \in S_0$ we obtain $Z_0 = \{k \in K \mid \text{Exp}(\text{Ad}(k)(X)) = \text{Exp}(X)\}$. We choose $X = \pi^2 e_1 + \frac{1}{2}\pi^2 e_2$ and claim that

$$Z_0 = \{\text{diag}(a, b, c, b) \in U(2) \times U(2) \mid a, b, c \in U(1)\}.$$

To see this, we compute that

$$\exp(X) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The condition $\text{Exp}(\text{Ad}_k(X)) = \text{Exp}(X)$ is equivalent to $\exp(-X)k \exp(X) \in K$. One then checks by an explicit computation that $k \in K$ satisfies this condition if and only if

$$k = \text{diag}(a, b, c, b) \quad \text{with } a, b, c \in U(1).$$

Hence, $K \rightarrow K/Z_0$ is a bundle with typical fiber $U(1)^3$, where an inclusion of the fiber is given by

$$f: U(1)^3 \rightarrow U(2) \times U(2), \quad f(a, b, c) = \text{diag}(a, b, c, b).$$

We want to show that K/Z_0 is simply connected. By the long exact sequence of homotopy groups of that bundle, it suffices to show that $f_*: \pi_1(U(1)^3) \rightarrow \pi_1(U(2)^2)$ is surjective. Let $\gamma: [0, 1] \rightarrow U(1)$, $\gamma(t) = e^{2\pi i t}$. We observe that $\pi_1(U(1)^3) \cong \mathbb{Z}^3$. A set of generators of $\pi_1(U(1)^3)$ is given by the homotopy classes of the loops $\gamma_1, \gamma_2, \gamma_3: [0, 1] \rightarrow U(1)^3$ defined as

$$\gamma_1(t) = (\gamma(t), 1, 1), \quad \gamma_2(t) = (1, \gamma(t), 1), \quad \gamma_3(t) = (1, 1, \gamma(t)).$$

We further observe that $\pi_1(U(2)^2) \cong \mathbb{Z}^2$, where a set of generators is given by the homotopy classes of

$$\beta_1, \beta_2: [0, 1] \rightarrow U(2) \times U(2), \quad \beta_1(t) = \text{diag}(\gamma(t), 1, 1, 1), \quad \beta_2(t) = \text{diag}(1, 1, \gamma(t), 1).$$

Here we used [4, Example VII.8.1]. One immediately sees that $f \circ \gamma_1 = \beta_1$ and $f \circ \gamma_3 = \beta_2$. This shows that the image f_* contains a set of generators, and hence f_* is surjective. Thus, $\pi_1(K/Z_0)$ is the trivial group, which implies by [6, Theorem 1.50] that

$$\text{cat}(K/Z_0) \leq \frac{1}{2} \dim(K/Z_0) + 1 = \frac{5}{2} + 1 = \frac{7}{2}.$$

Since cat is integer-valued, we obtain $\text{cat}(K/Z_0) \leq 3$. To employ Corollary 8.3, we still need to estimate $\text{cat}(G_2(\mathbb{C}^4))$ from above. Another use of [6, Theorem 1.50] shows that

$$\text{cat}(G_2(\mathbb{C}^4)) \leq \frac{1}{2} \dim(G_2(\mathbb{C}^4)) + 1 = 5.$$

Inserting the results of our computations into [Corollary 8.3](#), we derive

$$\begin{aligned} \text{GC}(G_2(\mathbb{C}^4)) &\leq \text{cat}(G_2(\mathbb{C}^4))(\text{cat}(K/Z_0) + \max\{\text{cat}(K/Z_1), \text{cat}(K/Z_2)\}) + 1 \\ &\leq 5(3 + 3) + 1 = 31. \end{aligned}$$

By [\[14, Lemma 28.1\]](#), it further holds that $\text{TC}(G_2(\mathbb{C}^4)) = \dim(G_2(\mathbb{C}^4)) + 1 = 9$. Thus, by the previous inequality and [Remarks 2.4](#), we obtain

$$9 \leq \text{GC}(G_2(\mathbb{C}^4)) \leq 31.$$

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