Convergence of Riemannian manifolds with critical curvature bounds

Dissertation

Zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

von	Dipl.-Math. Norman Zergänge
geb. am	30.04.1988 in Staßfurt

genehmigt durch die Fakultät für Mathematik
der Otto-von-Guericke-Universität Magdeburg

Gutachter:	Prof. Dr. Miles Simon
	Prof. Dr. Mohameden Ahmedou

eingereicht am:	25.04.2017
Verteidung am:	15.08.2017

Acknowledgements:

First, I would like to thank Professor Miles Simon for his support and encouragement over the last years. I am also grateful to all colleagues at the Institut für Analysis und Numerik in Magdeburg for providing a comfortable working atmosphere. In this context, I particularly like to thank Professor Hans-Christoph Grunau and Professor Klaus Deckelnick for supporting my work. I also want to express my thanks to Dr. Bernd Rummler for giving me some hints concerning a part of this work.

Moreover, I would like to thank Professor Peter Topping and Professor Mario Micallef for giving me further inspirations around the first part of this work, during my stay at the University of Warwick in February 2017.

Furthermore, I would like to thank my family and my friends for the support outside of the science.

Abstract

In this work we prove convergence results in Riemannian geometry with scale invariant bounds on the curvature.

The first part of this work (Chapter 2) is about sequences of Riemannian 4-manifolds with almost vanishing L^2-norm of a curvature tensor and a non-collapsing bound on the volume of small balls.

Here, in Theorem 1.1, we consider a sequence of closed Riemannian 4-manifolds, whose L^2-norm of the Riemannian curvature tensor tends to zero. Under the assumption of a uniform non-collapsing bound and a uniform diameter bound, we prove that there exists a subsequence that converges with respect to the Gromov-Hausdorff topology to a flat manifold.

In Theorem 1.2, we consider a sequence of closed Riemannian 4-manifolds, whose L^2-norm of the Riemannian curvature tensor is uniformly bounded from above, and whose L^2-norm of the traceless Ricci-tensor tends to zero. Here, under the assumption of a uniform non-collapsing bound, which is very close to the euclidean situation, and a uniform diameter bound, we show that there exists a subsequence which converges in the Gromov-Hausdorff sense to an Einstein manifold.

In order to prove Theorem 1.1 and Theorem 1.2, we use a smoothing technique, which is called L^2-curvature flow or L^2-flow, introduced by Jeffrey Streets in the series of works [36], [32], [31], [33], [34] and [35]. In particular, we use his "tubular averaging technique", which he has introduced in [35, Section 3], in order to prove distance estimates of the L^2-curvature flow which only depend on significant geometric bounds. This is the content of Theorem 1.3.

In the second part of this work (Chapter 3) we introduce the notion of a harmonic radius which is based on a definite L^n-bound on the first derivative of the metric and a fixed $C^{0,\alpha}$-seminorm bound on the metric, here $n \geq 3$ is the dimension of the manifold. Assuming uniform control of this harmonic radius, we are able to show in Theorem 1.4, that a sequence of open Riemannian manifolds, whose local $L^{\frac{n}{2}}$-norm of the Ricci-tensor tends to zero, contains a subsequence that converges on a smaller domain, in the $W^{2,\frac{n}{2}}$-sense, to an open Ricci-flat manifold.

Zusammenfassung

In der vorliegenden Arbeit werden Konvergenzresultate in der Riemannschen Geometrie bewiesen, welche skalierungsinvariante Krümmungsschranken voraussetzen.

Im ersten Teil der Arbeit, in Kapitel 2, betrachten wir Folgen von Riemannschen Mannigfaltigkeiten der Raumdimension 4, deren L^2-Norm der Krümmung im Unendlichen verschwindet. In diesem Abschnitt setzen wir voraus, dass das Volumen eines hinreichend kleinen Balls in gewisser Hinsicht nicht kollabiert.

In Theorem 1.1 betrachten wir eine Folge von Riemannschen Mannigfaltigkeiten der Raumdimension 4, deren L^2-Norm des Krümmungstensors gegen 0 geht. Unter der Annahme einer geeigneten gleichmäßigen unteren Schranke an das Volumen-Wachstum von Bällen mit kleinem Radius, und einer oberen Schranke an den Durchmesser zeigen wir, dass eine Teilfolge existiert, die, bezüglich der Gromov-Hausdorff Topologie gegen eine flache Mannigfaltigkeit konvergiert.

In Theorem 1.2 betrachten wir eine Folge von Riemannschen Mannigfaltigkeiten der Raumdimension 4, deren L^2-Norm des Krümmungstensors gleichmäßig von oben beschränkt ist, und deren L^2-Norm des spurfreien Ricci-Tensors gegen 0 geht. Unter der Annahme einer Wachstumsbedingung für das Volumen kleiner Bälle, welche sehr nah an der euklidischen Situation ist, zeigen wir, dass eine Teilfolge existiert, die in der Gromov-Hausdorff Topologie gegen eine Einstein Mannigfaltigkeit konvergiert.

Um Theorem 1.1 und Theorem 1.2 zu beweisen, verwenden wir eine Glättungstechnik, die von Jeffrey Streets in den Arbeiten [36], [32], [31], [33], [34] und [35] eingeführt und analysiert wurde, wir nennen diese Methode den sogenannten L^2-Krümmungsfluss. Hierbei verwenden wir die, vom Autor in [35, Section 3] eingeführte, „tubular averaging"-Methode. Diese Methode erlaubt es, Distanz-Abschätzungen des L^2-Krümmungsflusses herzuleiten, die nur von signifikanten geometrischen Größen abhängen. Das ist Gegenstand von Theorem 1.3.

Im zweiten Teil der Arbeit, in Kapitel 3, führen wir eine Notation eines harmonischen Radius ein, die auf einer festen oberen L^n-Schranke an die erste Ableitung der Metrik, und auf einer festen oberen $C^{0,\alpha}$-Seminorm-Schranke

an die Metrik basiert, hierbei ist $n \geq 3$ die Dimension der Mannigfaltigkeit. Unter der Annahme, dass dieser harmonische Radius gleichmäßig nach unten beschränkt ist, zeigen wir in Theorem 1.4, dass eine Folge von offenen Riemannschen Mannigfaltigkeiten, dessen lokale $L^{\frac{n}{2}}$-Norm des Ricci-Tensors gegen 0 geht, eine Teilfolge besitzt, die auf einer kleineren Menge, in der $W^{2,\frac{n}{2}}$-Topologie gegen eine glatte Ricci-flache Mannigfaltigkeit konvergiert.

Contents

Chapter 1

Introduction and statement of results

In order to approach minimization problems in Riemannian geometry, it is often useful to know if a minimizing sequence of smooth Riemannian manifolds contains a subsequence that converges with respect to an appropriate topology to a sufficiently smooth space. Here, in general, the minimization problem refers to a certain geometric functional, for instance the area functional, the total scalar curvature functional, the Willmore functional or the L^p-norm of a specific curvature tensor on a Riemannian manifold, to name just a few. Latter functionals are the main interest in this work. That means that we consider sequences of Riemannian manifolds that have a uniform L^p-bound on the full curvature tensor, the Ricci tensor and the traceless Ricci tensor respectively.

Naturally, the situation is more transparent, if we have more precise information about the L^p-boundedness of curvature tensors of the underlying Riemannian manifolds, that is, that we have a uniform L^p-bound, where $p \in [1, \infty]$ is large. In particular, a uniform L^∞-bound should give the most detailled information about geometric quantities.

One of the basic results in this context is stated in [1, Theorem 2.2, p. 464-466]. Here, for instance, one assumes a uniform L^∞-bound on the full Riemannian curvature tensor, a uniform lower bound on the injectivity radius and a uniform two sided bound on the volume, to show the existence of a subsequence that converges with respect to the $C^{0,\alpha}$-topology to a Riemannian manifold of regularity $C^{1,\alpha}$. The proof uses the fact, that it is possible to find uniform coverings of the underlying manifolds with harmonic charts, which follows from [18].

In [44], Deane Yang has considered sequences of Riemannian manifolds satisfying a suitable uniform L^p-bound on their full Riemannian curvature tensors, where $p > \frac{n}{2}$, and a uniform bound on the Sobolev constant. In order to show compactness and diffeomorphism finiteness results, he examines Hamilton's Ricci flow (cf. [16], [10] and [37]) and he shows curvature decay estimates and existence time estimates that only depend on the significant geometric bounds.

In [42] and [43], Deane Yang has approached a slightly more general problem. Here, he has considered sequences of Riemannian n-manifolds, $n \geq 3$, having a uniform $L^{\frac{n}{2}}$-bound on their full Riemannian curvature tensors and a suitable uniform L^p-bound on their Ricci tensors instead of a uniform L^p-bound on their full Riemannian curvature tensors, where $p > \frac{n}{2}$. Due to the scale invariance of the bound on the Riemannian curvature tensors - we name such bound a "critical curvature bound" - the situation becomes much more difficult, than in the "supercritical" case, that is, when p is bigger than $\frac{n}{2}$. In particular, in general, it is doubtful whether the global Ricci flow is applicable in this situation.

In [42], the author has introduced the idea of a "local Ricci flow" which is, by definition, equal to the Ricci flow weighted with a truncation function that is compactly contained in a local region of a manifold. The author shows that on regions, where the local $L^{\frac{n}{2}}$-norm of the full Riemannian curvature tensor is sufficiently small, the local Ricci flow satisfies curvature decay estimates and existence time estimates that only depend on significant local geometric bounds. So, on these "good" regions one may apply [1, Theorem 2.2, pp. 464-466] to a slightly mollified metric, to obtain local compactness with respect to the $C^{0,\alpha}$-topology. Since the number of local regions having too large $L^{\frac{n}{2}}$-norm of the full Riemannian curvature tensor is uniformly bounded, the author is able to show that each sequence of closed Riemannian manifolds, satisfying a uniform diameter bound, a uniform non-collapsing bound on the volume of small balls, a uniform bound on the $L^{\frac{n}{2}}$-norm of the full Riemannian curvature tensor and a sufficiently small uniform bound on the L^p-norm of the Ricci curvature tensor, where $p > \frac{n}{2}$, contains a subsequence that converges in the Gromov-Hausdorff sense to a metric space, which is, outside of a finite set of points, an open C^1-manifold with a Riemannian metric of regularity C^0.

In [43], the author has used the local Ricci flow to find a suitable harmonic chart around each point in whose neighborhood the local $L^{\frac{n}{2}}$-norm of the full Riemannian curvature tensor and the local L^p-norm of the full Riemannian curvature tensor, where $p > \frac{n}{2}$, is not too large. Using these estimates, the author is able to improve the statements about the convergence behavior in the convergence results in [42] on regions having a sufficiently small curvature concentration.

It seems so, that the reliability of the Ricci flow in [44], and the local Ricci flow in [42] and [43] is based on the appearance of the supercritical curvature bounds. For instance, in order to develop the parabolic Moser iteration in [44] and [42] one uses a well-controlled behavior of the Sobolev constant. As shown in [42, 7, pp. 85-89] this behavior occurs, if one assumes suitable supercritical bounds on the Ricci curvature. The examples in [3] show that the critical case is completely different.

Another important issue is the absence of important comparison geometry results under critical curvature bounds. In order to understand the rough structure of Riemannian manifolds, satisfying a fixed lower bound on the Ricci tensor, one uses the well-known "Bishop-Gromov volume comparison theorem" (cf. [26, 9.1.2., pp. 268-270]) which allows a one-directed volume comparison of balls in Riemannian manifolds satisfying a fixed lower Ricci curvature bound with the volume of balls in a such called "space form" (cf. [23, p. 206]), which is a complete, connected Riemannian manifold with constant sectional curvature. Later, in [27], Peter Petersen and Guofang Wei have shown that it is possible to generalize this result to the situation, in that an L^p-integral of some negative part of the Ricci tensor is sufficiently small. Here the authors assume that p is bigger that $\frac{n}{2}$.

It seems that the treatment of Riemannian manifolds with pure critical curvature bounds needs to be based on methods that are different from the approaches we have just mentioned. Instead of considering the Ricci flow, which is closely related to the gradient flow of the Einstein-Hilbert functional (cf. [10, Chapter 2, Section 4, pp. 104-105]), one could try to deform a Riemannian manifold of dimension 4 into the direction of the negative gradient of the L^2-integral of the full curvature tensor, in order to analyze slightly deformed approximations of the initial metric, having a smaller curvature energy concentration. This evolution equation was examined by Jeffrey Streets in [36], [32], [31], [33], [34], [35]. In this series of works, J. Streets has proved a plenty of properties of this geometric flow and he also shows a couple of applications.

Using J. Streets technique, in Chapter 2, we show compactness results for Riemannian 4-manifolds, that only assume a uniform diameter bound, a uniform non-collapsing bound on the volume of sufficiently small balls and critical curvature bounds.

In the first theorem, we consider a sequence of Riemannian 4-manifolds having almost vanishing Riemannian curvature tensor in some rough sense and we show that a subsequence converges with respect to the Gromov-Hausdorff topology to a flat Riemannian manifold:

Theorem 1.1. *Given* $D, d_0 > 0$, $\delta \in (0,1)$ *and let* $(M_i, g_i)_{i\in\mathbb{N}}$ *be a sequence of closed Riemannian 4-manifolds, satisfying the following assumptions:*

$$\begin{aligned} d_0 \leq diam_{g_i}(M_i) \leq D \qquad & \forall i \in \mathbb{N} \\ Vol_{g_i}(B_{g_i}(x,r)) \geq \delta\omega_n r^n \qquad & \forall i \in \mathbb{N},\, x \in M_i,\, \forall r \in [0,1] \\ \|Rm_{g_i}\|_{L^2(M_i,g_i)} \leq \frac{1}{i} \qquad & \forall i \in \mathbb{N} \end{aligned} \tag{1.1}$$

then, there exists a subsequence $(M_{i_j}, d_{g_{i_j}})_{j\in\mathbb{N}}$ *that converges in the Gromov-Hausdorff sense to a flat manifold* (M, g).

Throughout, a closed Riemannian is defined to be a smooth, compact and connected oriented Riemannian manifold without boundary.

In the second theorem, we consider a sequence of Riemannian 4-manifolds with uniformly bounded curvature energy and almost vanishing traceless Ricci tensor in some rough sense. Under these assumptions, we show that a subsequence converges with respect to the Gromov-Hausdorff topology to an Einstein manifold, provided that the volume of small balls behaves almost euclidean:

Theorem 1.2. *Given* $D, d_0, \Lambda > 0$, *there exists a universal constant* $\delta \in (0,1)$ *close to* 1 *so that if* $(M_i, g_i)_{i\in\mathbb{N}}$ *is a sequence of closed Riemannian 4-manifolds satisfying the following assumptions:*

$$d_0 \leq diam_{g_i}(M_i) \leq D \qquad \forall i \in \mathbb{N}$$

$$
\begin{aligned}
\|Rm_{g_i}\|_{L^2(M_i,g_i)} &\leq \Lambda && \forall i \in \mathbb{N}\\
\|\mathring{R}c_{g_i}\|_{L^2(M_i,g_i)} &\leq \frac{1}{i} && \forall i \in \mathbb{N}\\
Vol_{g_i}(B_{g_i}(x,r)) &\geq \delta\omega_n r^n && \forall i \in \mathbb{N},\, x \in M_i,\, r \in [0,1]
\end{aligned}
$$

then there exists a subsequence $(M_{i_j}, d_{g_{i_j}})_{j\in\mathbb{N}}$ *that converges in the Gromov-Hausdorff sense to a smooth Einstein manifold* (M, g)*.*

As mentioned above, it is our aim to show these results, using the negative gradient flow of the following functional:

$$
\mathcal{F}(g) := \int_M |Rm_g|_g^2\, dV_g \tag{1.2}
$$

That is, on a fixed sequence element (M^4, g_0), we want to evolve the initial metric in the following manner:

$$
\begin{cases}
\frac{\partial}{\partial t} g & = -\text{grad } \mathcal{F} = -2\delta dRc_g + 2\check{R}_g - \frac{1}{2}|Rm_g|_g^2 g\\
g(0) & = g_0
\end{cases} \tag{1.3}
$$

where $\check{R}_{ij} := R_i^{pqr} R_{jpqr}$ in local coordinates and the gradient formula, which appears in (1.3) can be found in [5, Chapter 4, 4.70 Proposition, p. 134]. Here, d denotes the exterior derivative acting on the Ricci tensor and δ denotes the adjoint of d. The gradient of a differentiable Riemannian functional is defined in [5, Chapter 4, 4.10 Definition, p. 119].

In [36, Theorem 3.1, p. 252] J. Streets has proved short time existence of the flow given by (1.3) on closed Riemannian manifolds. The author has also proved the uniqueness of the flow (cf. [36, Theorem 3.1, p. 252]). In this regard, the expression "the" L^2-flow makes sense. In [35, Theorem 1.8, p. 260] J. Streets has proved, that under certain assumptions, the flow given by (1.3) has a solution on a controlled time interval and the solution satisfies certain curvature decay and injectivity radius growth estimates.

In Section 2.1, we use J. Streets ideas, in order to show that, under certain assumptions, the distance between two points does not change too much along the flow. This allows us to bring the convergence behavior of a slightly mollified manifold back to the initial sequence. That means we will prove the following theorem:

Theorem 1.3. *Let* (M^4, g_0) *be a closed Riemannian 4-manifold. Suppose that* $(M, g(t))_{t\in[0,1]}$ *is a solution to* (1.3) *satisfying the following assumptions:*

$$\int_M |Rm_{g_0}|^2_{g_0}\, dV_{g_0} \leq \Lambda \tag{1.4}$$

$$\|Rm_{g(t)}\|_{L^\infty(M,g(t))} \leq Kt^{-\frac{1}{2}} \qquad \forall t \in (0,1] \tag{1.5}$$

$$inj_{g(t)}(M) \geq \iota t^{\frac{1}{4}} \qquad \forall t \in [0,1] \tag{1.6}$$

$$diam_{g(t)}(M) \leq 2(1+D) \tag{1.7}$$

Then we have the following estimate:

$$|d(x,y,t_2) - d(x,y,t_1)| \leq C(K,\iota,D)\Lambda^{\frac{1}{2}} \left(t_2^{\frac{1}{8}} - t_1^{\frac{1}{8}}\right)^{\frac{1}{2}} + C(K,\iota,D)\left(t_2^{\frac{1}{24}} - t_1^{\frac{1}{24}}\right) \tag{1.8}$$

for all $t_1, t_2 \in [0,1]$ *where* $t_1 < t_2$.

These estimates allow one to prove Theorem 1.1 and Theorem 1.2 which are the main goals of Section 2.2 and Section 2.3. Here, in Section 2.2, we may refer to the estimates in [35, 1.3, Theorem 1.8, p. 260]. In Section 2.3, we write down an existence result which allows to apply Theorem 1.3 to the elements of the sequence occurring in Theorem 1.2.

In Chapter 3 we focus our attention on harmonic coordinates. As explained above, in the context of determined L^∞-bounds on the full Riemannian curvature tensor, the proof of [1, Theorem 2.2, pp. 464-466] is based on the existence of appropriate coverings with suitable harmonic charts. Here, in these local charts, the Riemannian metric and their derivatives have fixed bounds with respect to the L^∞-norm.

Using blow-up arguments as in [6, Section 2, pp. 9-14] and [30, Appendix B, pp. 54-64], one may also prove the existence of suitable harmonic charts in the context of integral curvature bounds, where locally, a non-collapsing / non-inflating condition on the volume of small balls, a supercritical bound on the Ricci tensor and a smallness condition on the $L^{\frac{n}{2}}$-norm of the full Riemannian curvature is assumed. In this situation, in harmonic charts, the Riemannian metric has a fixed L^∞-bound and their derivatives have fixed bounds with respect to the L^q-norm, where q depends on the space dimension, the order of the derivative k and the supercritical L^p-bound on the Ricci curvture. The order q is always strictly bigger than $\frac{n}{k}$, and k is at least one or two. Using the theory of Sobolev spaces (cf. [15, Chapter 7, p. 144-173]) one obtains always a fixed $C^{0,\alpha}$-bound on the metric in such a harmonic chart. As p tends to $\frac{n}{2}$ from above, the order q tends to $\frac{n}{k}$ and the Hölder exponent $\alpha > 0$ tends to zero. Since the critical Sobolev spaces $W^{k,\frac{n}{k}}(\Omega)$ are not continuously embedded into the space $C^0(\Omega)$, it is doubtful, if the concept of a harmonic radius with pure critical Sobolev bounds would be a convenient tool, in order to control the C^0-behavior of Riemannian manifolds with scale invariant curvature bounds.

In order to do a step in this direction, in Definition 3.2, we introduce the notation of a harmonic radius r_g which slightly generalizes the notation of the harmonic radius introduced in [6, Section 2, Definition 2.1, p. 9]. Our notation of the harmonic radius requires locally a fixed L^n-bound on the first derivative of the Riemannian metric combined with a $C^{0,\alpha}$-semi norm bound on the metric. Assuming an appropriate behavior of this harmonic radius, we are able to prove the following result, which generalizes [6, Section 2, Theorem 2.3, pp. 13-14] from the view point of the regularity

Theorem 1.4. *Let $n \in \mathbb{N}$, $n \geq 3$, $0 \leq \sigma_1 \leq \sigma_2$ be fixed, let $(M_i^n, g_i, p_i)_{i\in\mathbb{N}}$ be a sequence of smooth complete pointed Riemannian manifolds without boundary such that for all $i \in \mathbb{N}$ the ball $B_{g_i}(p_i, 1) \subseteq M_i$ satisfies the following properties:*

$$\lim_{i\to\infty} \|Rc_{g_i}\|_{L^{\frac{n}{2}}(B_{g_i}(p_i,1),g_i)} = 0 \tag{1.9}$$

$$\omega_n\sigma_1 \leq \frac{Vol_{g_i}(B_{g_i}(x,r))}{r^n} \leq \omega_n\sigma_2 \tag{1.10}$$

for all $x \in B_{g_i}(p_i, 1)$, $r \in (0, 1]$ such that $B_{g_i}(x, r) \subseteq B_{g_i}(p_i, 1)$ and

$$r_g(x) \geq r_0(s) > 0 \quad \forall x \in B_{g_i}(p_i, s) \text{ and } i \in \mathbb{N}, s < 1 \tag{1.11}$$

Then, there exists a smooth Ricci flat manifold (X, g, p) containing the ball $\overline{B}_g(p, 1/10)$ so that, after taking a subsequence, for each $i \in \mathbb{N}$ there exists a diffeomorphism $F_i : B_g(p, 1/10) \longrightarrow F_i(B_g(p, 1/10)) \subseteq B_{g_i}(p_i, 1)$ such that $F_i^* g_i$ converges to g with respect to the $W^{2,\frac{n}{2}}(B_g(p, 1/10))$-topology, as i tends to infinity.

Chapter 2

Convergence of Riemannian 4-manifolds with almost vanishing L^2-integral of the curvature

In this chapter we prove Theorem 1.1 and Theorem 1.2 which are consequences of Theorem 1.3. In order to prove Theorem 1.3 we use the "tubular averaging technique" from [35, Section 3, pp. 269-282]. Our method is derived from [35, Section 3], although it is necessary to make some modifications, see for example Lemma 2.3 here. In Subsection 2.1.3, we apply the "tubular averaging technique" to the time-reversed flow. For the sake of understanding, we give detailed explanations of the steps in the proof, even if the argumentation is based on the content of [35, Section 3]. In order to get a very rough feeling for J. Streets "tubular averaging technique" we recommend to read the first paragraph of [35, p. 270].

2.1 Distance control under the L^2-flow in 4 dimensions (Proof of Theorem 1.3)

In this section we prove Theorem 1.3. The proof is divided in two principal parts:

In the first part of this section we show that, along the flow, the distance between two points in manifold M does not increase too much, i.e.: we derive the estimates of the shape $d(x,y,t) < d(x,y,0)+\epsilon$ for small $t(\epsilon) > 0$. We say that this kind of an estimate is a "forward estimate".

The second part in this section is concerned with the opposite direction, i.e: we show that, along the flow, the distance between two points does not decay too much, which means that we have $d(x,y,t) > d(x,y,0) - \epsilon$ for $t(\epsilon) > 0$ sufficiently small.

We point out that the estimate of the length change of a vector $v \in TM$ along a geometric flow usually requires an integration of the metric change $|g'(t)|_{g(t)}$ from 0 to a later time point T (cf. (A.2)). With a view to (1.3) and (1.5) we note that, on the first view, this would require and integration of the function t^{-1} from 0 to T which is not possible.

In order to overcome this difficulty, we follow the ideas in [35, Section 3], i.e. we introduce some kind of connecting curves which have almost the properties of geodesics. Then we construct an appropriate tube around each of these connecting curves so that the integral $\int_\gamma |\text{grad } \mathcal{F}| \, d\sigma$, which occurs in the estimate of $\left|\frac{d}{dt}L(\gamma,t)\right|$ (cf. (A.1)), can be estimated from above against a well-controlled average integral along the tube plus an error integral which behaves also well with respect to t. We point out that we do not widen J. Streets ideas in [35, Section 3] by fundamental facts, we merely write down detailed information which allow to understand the distance changing behavior of J. Streets L^2-flow in a more detailed way.

2.1.1 Tubular neighborhoods

We quote the following definition from [35, Definition 3.3., pp. 271-272]

Definition 2.1. *Let (M^n, g) be a smooth Riemannian manifold without boundary, and let $\gamma : [a,b] \longrightarrow M$ be an smooth curve. Given $r > 0$, and $s \in [a,b]$ then we define*

$$D(\gamma(s), r) := \exp_{\gamma(s)} \left\{ B(0,r) \cap \langle \dot\gamma(s) \rangle^{\perp} \right\}$$

and

$$D(\gamma, r) := \bigcup_{s\in[a,b]} D(\gamma(s), r)$$

We say "$D(\gamma, r)$ is foliated by $(D(\gamma(s), r))_{s\in[a,b]}$ " if

$$D(\gamma(s_1), r) \cap D(\gamma(s_2), r) = \emptyset$$

for all $a \leq s_1 < s_2 \leq b$.

The following definition is based on [35, Definition 2.2., p. 267].

Definition 2.2. *Let (M^n, g) be a closed Riemannian manifold, $k \in \mathbb{N}$ and $x \in M$, then we define*

$$f_k(x, g) := \sum_{j=0}^{k} |^g\nabla^j Rm_g|_g^{\frac{2}{2+j}}(x)$$

and

$$f_k(M, g) := \sup_{x\in M} f_k(x, g)$$

At this point we refer to the scaling behavior of $f_k(x, g)$ which is outlined in Lemma A.2.

The following result is a slight modification of [35, Lemma 3.4., pp. 272-274]. To be more precise: in this result we allow the considered curve to have a parametrization close to unit-speed, and not alone unit-speed.

Lemma 2.3. *Given $n, D, K, \iota > 0$ there exists a constant $\beta(n, D, K, \iota) > 0$ and a constant $\mu(n) > 0$ so that if (M^n, g) is a complete Riemannian manifold satisfying*

$$diam_g(M) \leq D$$
$$f_3(M^n, g) \leq K$$
$$inj_g(M) \geq \iota$$

and $\gamma : [0, L] \to M$ is an injective smooth curve satisfying

$$L(\gamma) \leq d(\gamma(0), \gamma(L)) + \beta \tag{2.1}$$
$$|\nabla_{\dot\gamma}\dot\gamma| \leq \beta \tag{2.2}$$
$$\frac{1}{1+\beta} \leq |\dot\gamma| \leq 1 + \beta \tag{2.3}$$

then $D(\gamma, R)$ is foliated by $(D(\gamma(s), R))_{s\in[0,L]}$ for $R := \mu \min\left\{\iota, K^{-\frac{1}{2}}\right\}$. Furthermore, if

$$\pi : D(\gamma, R) \longrightarrow \gamma([0, L])$$

is the projection map sending a point $q \in D(p, R)$, where $p \in \gamma([0, L])$, to p, which is well-defined by the foliation property, then

$$|d\pi| \leq 2 \text{ on } D(\gamma, R) \tag{2.4}$$

Here $d\pi$ denotes the differential and $|d\pi|$ denotes the operator norm of the differential of the projection map.

Proof. Above all, we want to point out, that, due to the injectivity of the curve, we can construct a tubular neighborhood around $\gamma([0, L])$. This is a consequence of [24, 26. Proposition, p. 200]. But the size of this neighborhood is not controlled at first. Via radial projection we can ensure that the velocity field of the curve is extendible in the sense of [23, p. 56]. We follow the ideas of the proof of [35, Lemma 3.4, pp. 272-274] with some modifications.

Firstly, we describe how $\mu(n) > 0$ needs to be chosen in order to ensure that the curve has a suitable foliation which can be used to define the projection map.

Secondly, we show that the desired smallness condition of the derivative of the projection map is valid, i.e.: we show (2.4). Here we allow $\mu(n) > 0$ to become smaller.

Let

$$\mu(n) := \min\left\{\widehat{\mu}(n), \frac{1}{20}, \frac{1}{64C_1(n)C_2(n)}\right\} \tag{2.5}$$

where $\widehat{\mu}(n) > 0$ and $C_1(n) > 0$ are taken from Lemma A.8 and $C_2(n) > 0$ will be made explicit below. Let

$$R := \mu \min\left\{\iota, K^{-\frac{1}{2}}\right\}$$

Suppose there exists a point $p \in D(\gamma(s_0), R) \cap D(\gamma(s_1), R)$ where $s_0, s_1 \in [0, L]$, $s_0 < s_1$ and $s_1 - s_0 \leq 10R$ at first. By definition, there exists a normal chart of radius $20R$ around p (cf. [23, pp. 76-81]). In this chart we have the following estimate

$$\sup_{B_g(p,20R)} \mu K^{-\frac{1}{2}} |\Gamma| \leq \frac{1}{64C_2(n)} \tag{2.6}$$

Choosing $\beta \in (0,1)$ small enough compared to R we ensure that $\gamma([s_0,s_1])$ lies in this chart. From [23, Theorem 6.8., pp. 102-103] we obtain

$$\left\langle \frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s_0)} = 0$$

where

$$\frac{\partial}{\partial r}\Big|_{\gamma(s)} := \frac{\gamma^i(s)}{r(\gamma(s))}\,\partial_i|_{\gamma(s)} \tag{2.7}$$

and $\partial_1, ..., \partial_n$ denote the coordinate vector fields and $\gamma^1, ..., \gamma^n$ denote the coordinates of γ in this normal chart and

$$r(\gamma(s)) := \sqrt{\sum_{i=1}^n (\gamma^i(s))^2}$$

(cf. [23, Lemma 5.10, (5.10), p. 77]). We show that it is possible to take $\beta(n,K,\iota) > 0$ small enough to ensure that

$$\left\langle \frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s)} \neq 0 \qquad \forall s \in (s_0, s_1]$$

This would be a contradiction to the fact that [23, Theorem 6.8., pp. 102-103] also implies

$$\left\langle \frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s_1)} = 0 \tag{2.8}$$

From [23, Lemma 5.2 (c), p. 67] we infer on $[s_0, s_1]$

$$\begin{aligned} \frac{\partial}{\partial s}\left\langle \frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s)} &= \left\langle D_s\frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s)} + \left\langle \frac{\partial}{\partial r}, D_s\dot\gamma \right\rangle\Big|_{\gamma(s)} \\ &\geq \left\langle D_s\frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s)} - \left| \left\langle \frac{\partial}{\partial r}, D_s\dot\gamma \right\rangle\Big|_{\gamma(s)} \right| \\ &\geq \left\langle D_s\frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s)} - \left| \frac{\partial}{\partial r}\Big|_{\gamma(s)} \right| \left|\nabla_{\dot\gamma(s)}\dot\gamma(s)\right|_g \\ &\overset{(2.2)}{\geq} \left\langle D_s\frac{\partial}{\partial r}, \dot\gamma \right\rangle\Big|_{\gamma(s)} - \beta \end{aligned} \tag{2.9}$$

Using (2.7) together with [23, Lemma 4.9 (b), p. 57] and [23, p. 56 (4.9)] we calculate

$$D_s\frac{\partial}{\partial r} = \frac{\dot\gamma^i\cdot r - \gamma^i\left\langle \dot\gamma, \frac{\partial}{\partial r}\right\rangle}{r^2}\partial_i + \frac{\gamma^i}{r}D_s\partial_i = \frac{\dot\gamma^i}{r}\partial_i - \frac{\gamma^i\left\langle \dot\gamma, \frac{\partial}{\partial r}\right\rangle}{r^2}\partial_i + \frac{\gamma^i}{r}D_s\partial_i$$

This implies

$$
\begin{aligned}
\left\langle D_s \frac{\partial}{\partial r}, \dot\gamma \right\rangle &= \frac{1}{r}|\dot\gamma|^2 - \frac{1}{r^2}\left\langle \dot\gamma, \frac{\partial}{\partial r}\right\rangle \langle \gamma^i \partial_i, \dot\gamma\rangle + \frac{\gamma^i}{r}\langle D_s \partial_i, \dot\gamma\rangle \\
&\overset{(2.7)}{\geq} \frac{1}{r}|\dot\gamma|^2 - \frac{1}{r^2}\left|\left\langle \dot\gamma, \frac{\partial}{\partial r}\right\rangle\right| \left|\left\langle r\frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - C_2(n)|\Gamma||\dot\gamma|^2 \\
&\geq \frac{1}{r}|\dot\gamma|^2 - \frac{1}{r}|\dot\gamma|\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - C_2(n)|\Gamma||\dot\gamma|^2 \\
&\overset{(2.3)}{\geq} \frac{1}{4r} - \frac{2}{r}\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - 4C_2(n)|\Gamma| \\
&= \frac{1 - 4C_2(n)\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - 16C_2(n)r|\Gamma|}{4r}
\end{aligned}
$$

Here, in order to obtain the first estimate, we refer to Definition C.9 and the fact that

$$\frac{1}{r(\gamma(t))}\sum_{i=1}^{n}|\gamma^i(t)| \leq \widehat{C}(n)$$

Hence, (2.9) implies

$$
\begin{aligned}
\frac{\partial}{\partial s}\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle &\geq \frac{1 - 8\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - 16C_2(n)r|\Gamma| - 4\beta r}{4r} \\
&\geq \frac{1 - 8\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - 16C_2(n)\mu K^{-\frac{1}{2}}|\Gamma| - 4\mu K^{-\frac{1}{2}}\beta}{4r} \\
&\overset{(2.6)}{\geq} \frac{1 - 8\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - \frac{1}{4} - 4\mu K^{-\frac{1}{2}}\beta}{4r} \\
&\geq \frac{1 - 8\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right| - \frac{1}{4} - \frac{1}{4}}{4r} \\
&= \frac{\frac{1}{2} - 8\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right|}{4r} = \frac{1}{8r}\left[1 - 16\left|\left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\right|\right]
\end{aligned}
\tag{2.10}
$$

We show that this differential inequality implies the desired contradiction. Let $w : [s_0, s_1] \longrightarrow \mathbb{R}$, $w(s) := \left\langle \frac{\partial}{\partial r}, \dot\gamma\right\rangle\big|_{\gamma(s)}$. Then (2.10) is equivalent to

$$w' \geq \frac{1}{8r}(1 - 16|w|)$$

on $[s_0, s_1]$. Since $w(s_0) = 0$, there exists $\delta > 0$ such that $w' > 0$ on $[s_0, s_0 + \delta]$. This implies $w > 0$ on $(s_0, s_0 + \delta]$. We show that we have $w > 0$ on $(s_0, s_1]$, which contradicts (2.8). Assumed

$$\widehat{s} := \sup\left\{ s \in (s_0, s_1] \,\middle|\, w|_{(s_0,s]} > 0 \right\} < s_1$$

which implies

$$w(\widehat{s}) = 0 \tag{2.11}$$

Then (2.10) is equivalent to

$$w' \geq \frac{1}{8r}(1 - 16w)$$

on $[s_0, \widehat{s}]$. The function $z : [s_0, s_1] \longrightarrow \mathbb{R}$, $z(s) := \frac{1}{16}(1 - e^{-\frac{2(s-s_0)}{r}})$ satisfies $z' = \frac{1}{8r}(1 - 16z)$ on $[s_0, s_1]$ and $z(s_0) = 0$. Thus we have

$$\begin{cases} (w-z)' \geq -\frac{2}{r}(w-z) & \text{on } [s_0, \widehat{s}] \\ (w-z)(s_0) = 0 \end{cases} \tag{2.12}$$

and we define a new function $\zeta : [s_0, s_1] \longrightarrow \mathbb{R}$ as follows $\zeta(s) := e^{\frac{2}{r}s}(w(s) - z(s))$. Then (2.12) implies

$$\begin{aligned} \zeta'(s) =& \frac{2}{r}e^{\frac{2}{r}s}(w(s) - z(s)) + e^{\frac{2}{r}s}(w'(s) - z'(s)) \\ \geq& \frac{2}{r}e^{\frac{2}{r}s}(w(s) - z(s)) - \frac{2}{r}e^{\frac{2}{r}s}(w(s) - z(s)) = 0 \end{aligned}$$

Hence

$$e^{\frac{2}{r}\widehat{s}}(w(\widehat{s}) - z(\widehat{s})) = \zeta(\widehat{s}) = \int_{s_0}^{\widehat{s}} \zeta'(\tau)\, d\tau \geq 0$$

from this we obtain

$$w(\widehat{s}) \geq z(\widehat{s}) = \frac{1}{16}(1 - e^{-\frac{2(\widehat{s}-s_0)}{r}}) > 0$$

which contradicts (2.11). Consequently, we have $w \geq 0$ on $[s_0, s_1]$. The same argumentation as above, adapted to the interval $[s_0, s_1]$, implies $w(s_1) > 0$ in contradiction to (2.8). This proves that two discs $D(\gamma(s_0), R)$ and $D(\gamma(s_1), R)$ cannot intersect, when $|s_1 - s_0| \leq 10R$.

Now, we show that two discs $D(\gamma(s_0), R)$ and $D(\gamma(s_1), R)$ cannot intersect if we assume $s_0, s_1 \in [0, L]$, $s_0 < s_1$, to be far away from each other, which means that $s_1 - s_0 > 10R$ holds.

We suppose that there exists a point $p \in D(\gamma(s_0), R) \cap D(\gamma(s_1), R)$. As in [35, p. 273] we construct a curve α in the following manner: α follows γ from $\gamma(0)$ to $\gamma(s_0)$, next α connects $\gamma(s_0)$ and p by a minimizing geodesic, then α

connects p and $\gamma(s_1)$ also by a minimizing geodesic, and finally α follows γ again from $\gamma(s_1)$ to $\gamma(L)$. We infer the following estimate:

$$\begin{aligned}
d_g(\gamma(0),\gamma(L)) \le L(\alpha) &\le \int_0^{s_0} |\dot\gamma|\, ds + R + R + \int_{s_1}^{L} |\dot\gamma|\, ds \\
&\overset{(2.3)}{\le} (1+\beta)s_0 + 2R + (1+\beta)(L-s_1) \\
&= (1+\beta)L + 2R - (1+\beta)(s_1-s_0) \\
&= (1+\beta)\int_0^L \frac{|\dot\gamma|}{|\dot\gamma|}\, ds + 2R - (1+\beta)(s_1-s_0) \\
&\overset{(2.3)}{\le} (1+\beta)^2 \int_0^L |\dot\gamma|\, ds + 2R - (1+\beta)10R \\
&\le (1+\beta)^2 L(\gamma) - 8R \\
&\overset{(2.1)}{\le} (1+\beta)^2 (d_g(\gamma(0),\gamma(L)) + \beta) - 8R \\
&\le (1+2\beta+\beta^2)(d_g(\gamma(0),\gamma(L)) + 4\beta - 8R \\
&\le (d_g(\gamma(0),\gamma(L)) + 3\beta D + 4\beta - 8R
\end{aligned}$$

and consequently:

$$0 \le (3D+4)\beta - 8R$$

which yields a contradiction when $\beta(n,D,K,\iota) > 0$ is chosen small enough. Hence, two discs $D(\gamma(s_0),R)$ and $D(\gamma(s_1),R)$ cannot intersect, provided they are not identical. Thus, $D(\gamma,R)$ is foliated by $(D(\gamma(s),R))_{s\in[0,L]}$.

It remains to show the estimate (2.4). We mentioned at the beginning of the proof, that now, we allow μ to become smaller.

As in the proof of [35, Lemma 3.4.] we suppose the assertion would be not true, i.e. there exists a sequence of constants $(\mu_i)_{i\in\mathbb{N}}$, where $\lim_{i\to\infty}\mu_i = 0$, and a sequence of closed Riemannian manifolds $(M_i^n, g_i)_{i\in\mathbb{N}}$ satisfying

$$\begin{aligned}
&f_3(M_i,g_i) \le K_i \quad \text{and} \\
&inj_{g_i}(M_i) \ge \iota_i
\end{aligned}$$

for all $i\in\mathbb{N}$, and curves $\gamma_i : [0,L_i] \longrightarrow M_i$ satisfying

$$\begin{aligned}
L(\gamma_i) &\le d(\gamma_i(0),\gamma_i(L_i)) + \beta_i, \\
|\nabla_{\dot\gamma_i}\dot\gamma_i| &\le \beta_i \quad \text{and} \\
\frac{1}{1+\beta_i} &\le |\dot\gamma_i| \le 1+\beta_i
\end{aligned} \tag{2.13}$$

for all $i \in \mathbb{N}$, where $(\beta_i)_{i\in\mathbb{N}} \subseteq (0,1]$, so that for each $i \in \mathbb{N}$ the tube $D(\gamma_i, R_i)$ is foliated by $(D(\gamma_i(s), R_i))_{s\in[0,L_i]}$, where $R_i := \mu_i \min\left\{\iota_i, K_i^{-\frac{1}{2}}\right\}$, but for each $i \in \mathbb{N}$ there exists a point $p_i = \gamma_i(s_i)$ and $y_i \in D(p_i, R_i)$ such that $|d\pi_i|(y_i) > 2$. From this we construct a blow-up sequence of pointed Riemannian manifolds

$$(M_i, h_i := R_i^{-2} g_i, p_i)_{i\in\mathbb{N}}$$

which satisfies for each $i \in \mathbb{N}$ and $x \in M_i$

$$f_3(x, h_i) = f_3(x, R_i^{-2} g_i) \overset{(A.7)}{=} R_i^2\, f_3(x, g_i) \leq R_i^2\, K_i \leq \mu_i^2 \overset{i\to\infty}{\longrightarrow} 0$$

and

$$inj_{h_i}(M_i) = inj_{R_i^{-2}g_i}(M_i) = R_i^{-1} inj_{g_i}(M_i) \geq R_i^{-1}\iota_i \geq \mu_i^{-1} \overset{i\to\infty}{\longrightarrow} \infty$$

Hence, using Theorem A.11, we may extract a subsequence that converges with respect to the pointed $C^{2,\alpha}$-sense to $(\mathbb{R}^n, g_{can}, 0)$. Next, for each $i \in \mathbb{N}$ we reparametrize the curve γ_i as follows: Let

$$\begin{aligned} \widehat{\gamma}_i : [0, \frac{L_i}{R_i}] &\longrightarrow M_i \\ \widehat{\gamma}_i(s) &:= \gamma(R_i s) \end{aligned}$$

Then for each $i \in \mathbb{N}$ we have for all $s \in [0, \frac{L_i}{R_i}]$

$$\begin{aligned} &|\dot{\widehat{\gamma}}_i(s)|_{h_i} \\ =&|\dot{\gamma}_i(R_i s) \cdot R_i|_{h_i} = R_i \cdot |\dot{\gamma}_i(R_i s)|_{R_i^{-2} g_i} = R_i \cdot R_i^{-1} |\dot{\gamma}_i(R_i s)|_{g_i} \\ =&|\dot{\gamma}_i(R_i s)|_{g_i} \overset{(2.3)}{\in} \left[\frac{1}{1+\beta}, 1+\beta\right] \end{aligned}$$

and, using normal coordinates at $\widehat{\gamma}(s)$

$$\begin{aligned} &|{}^{h_i}\nabla_{\dot{\widehat{\gamma}}(s)} \dot{\widehat{\gamma}}(s)|_{h_i}^2 = (h_i)_{kl} \ddot{\widehat{\gamma}}^k(s) \ddot{\widehat{\gamma}}^l(s) = R_i^{-2} \cdot (g_i)_{kl} \cdot \ddot{\widehat{\gamma}}^k(s) \ddot{\widehat{\gamma}}^l(s) \\ =&R_i^{-2} \cdot (g_i)_{kl} \cdot R_i^2 \cdot \ddot{\gamma}^k(R_i s) \cdot R_i^2 \cdot \ddot{\gamma}^l(R_i s) \\ =&R_i^2 \cdot (g_i)_{kl} \cdot \ddot{\gamma}^k(R_i s) \ddot{\gamma}^l(R_i s) \\ =&R_i^2 \cdot |{}^{g_i}\nabla_{\dot{\gamma}(R_i s)} \dot{\gamma}(R_i s)|_{g_i}^2 \overset{(2.2)}{\leq} R_i^2 \cdot \beta_i^2 \leq R_i^2 \end{aligned}$$

Hence

$$\lim_{i\to\infty} \max_{[0, \frac{L_i}{R_i}]} |{}^{h_i}\nabla_{\dot{\widehat{\gamma}}} \dot{\widehat{\gamma}}|_{h_i} = 0$$

Using the Arzelà-Ascoli Theorem we conclude, that these curves converge with respect to the $C^{1,\alpha}$-sense to a geodesic which goes through the origin. After an eventual rotation, we may assume that $\gamma(t) = (t, 0, ..., 0)$. In the blow-up metric h_i each point y_i has a distance to p_i not bigger than 1. That means, that this point can be considered as a point in $B_{g_{can}}(0, 2) \subseteq \mathbb{R}^n$. This sequence of points will converge to a point $y \in \overline{B}_{g_{can}}(0, 1) \cap \{x \in \mathbb{R}^n : x^1 = 0\}$. We recall that the projection maps $\pi_i : D(\gamma_i, R_i) \longrightarrow \gamma_i([0, L_i])$ are satisfying $|d\pi_i|(y_i) > 2$ by assumption. Due to the scaling invariance, this inequality is also true with respect to the blow-up metric h_i. Since the Riemannian metrics h_i converge in the $C^{2,\alpha}$-sense to the euclidean space and the curves γ_i converge in the $C^{1,\alpha}$-sense, the maps π_i converge in the C^1-sense to a map on the limit space, which will be denoted by π. Here we have used, that each tubular neighborhood is a diffeomorphic image of a neighborhood of the zero section in the normal bundle on the curve γ_i ([24, pp. 199-200, 25. Proposition / 26. Proposition]). Hence, we conclude $|d\pi|(y) \geq 2$, but the map π is explicitly given as $(x^1, ..., x^n) \mapsto (x^1, 0, ..., 0)$ and this map satisfies $|d\pi| \leq 1$, which yields a contradiction. □

We want to point that it is also possible to deduce Lemma 2.3 from the statement of [35, Lemma 3.4, p. 272] by use of unit-speed parametrization. On doing so, it is possible to avoid the dependence of the constant $\beta > 0$ on the diameter $D > 0$.

2.1.2 Forward estimates

In this paragraph we show that, under certain assumptions, distances do not increase too much along the L^2-flow.

Here, we prove the following estimate:

Lemma 2.4. *Let* (M^4, g_0) *be a closed Riemannian 4-manifold and let* $(M^4, g(t))_{t\in[0,1]}$ *be a solution to the flow given in* (1.3) *satisfying* (1.4), (1.5), (1.6) *and* (1.7), *i.e.:*

$$\int_M |Rm_{g_0}|^2_{g_0}\, dV_{g_0} \leq \Lambda$$

$$\|Rm_{g(t)}\|_{L^\infty(M,g(t))} \leq K t^{-\frac{1}{2}}$$

$$inj_{g(t)}(M) \geq \iota t^{\frac{1}{4}}$$

$$diam_{g(t)}(M) \leq 2(1+D)$$

for all $t \in (0,1]$. *Then we have the following estimate:*

$$d(x,y,t_2) - d(x,y,t_1) \leq C(K,\iota,D)\Lambda^{\frac{1}{2}} \left(t_2^{\frac{1}{8}} - t_1^{\frac{1}{8}}\right)^{\frac{1}{2}} + C(K,\iota,D)\left(t_2^{\frac{1}{24}} - t_1^{\frac{1}{24}}\right) \tag{2.14}$$

for all $t_1, t_2 \in [0,1]$ *where* $t_1 < t_2$.

As mentioned at the beginning of this section, we aim to use some kind of connecting curves between two points which are close to geodesics. These curves can be surrounded by a tube such that the projection map has bounded differential (c.f. Lemma 2.3).

The following definition is a modification of [35, Definition 3.1., p. 270]. Our definition is slightly stronger in some sense because we also assume a stability estimate of the length of the velocity vectors along the subintervals. We point out that we call the following objects β-quasi-forward-geodesics and not merely β-quasi-geodesics, as in [35, Definition 3.1., p. 270]. In Subsection 2.1.3 we introduce a time-reversed counterpart to these family of curves.

Definition 2.5. *Let* $(M^n, g(t))_{t\in[t_1,t_2]}$ *be a family of complete Riemannian manifolds. Given* $\beta > 0$ *and* $x, y \in M$ *then we say that a family of curves* $(\gamma_t)_{t\in[t_1,t_2]} : [0,1] \longrightarrow M$ *is a* β*-quasi-forward-geodesic connecting* x *and* y *if there is a constant* $S > 0$ *so that:*

1. *For all* $t \in [t_1, t_2]$ *one has* $\gamma_t(0) = x$ *and* $\gamma_t(1) = y$
2. *For all* $j \in \mathbb{N}_0$ *such that* $t_1 + jS \leq t_2$, γ_{t_1+jS} *is a length minimizing geodesic*
3. *For all* $j \in \mathbb{N}_0$ *such that* $t_1 + jS \leq t_2$, *and all* $t \in [t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2]$ *one has* $\gamma_t = \gamma_{t_1+jS}$
4. *For all* $t \in [t_1, t_2]$ *one has*
$$d(x,y,t) \leq L(\gamma_t, t) \leq d(x,y,t) + \beta \tag{2.15}$$
5. *For all* $j \in \mathbb{N}_0$ *such that* $t_1 + jS \leq t_2$, *and all* $t \in [t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2]$ *one has*
$$\frac{1}{1+\beta} d(x,y,t_1+jS) \leq |\dot\gamma_t|_{g(t)} \leq (1+\beta) d(x,y,t_1+jS) \tag{2.16}$$

$$|^{g(t)}\nabla_{\dot\gamma_t}\dot\gamma_t|_{g(t)} \leq \beta\, d^2(x,y,t_1+jS) \tag{2.17}$$

It is our aim to prove the following existence result:

Lemma 2.6. *Let $(M^n, g(t))_{t\in[t_1,t_2]}$ a smooth family of closed Riemannian manifolds. Given $\beta > 0$ and $x, y \in M$ then there exists a β-quasi-forward-geodesic connecting x and y.*

Remark 2.7. *The interval length $S > 0$ which will be concretized along the following proof has a strong dependency on the given points $x, y \in M$, $\beta > 0$ and the flow itself. As it turns out in the proof of Lemma 2.4, this will not cause problems because estimates on the subintervals will be put together to an estimate on the entire interval $[t_1, t_2]$ via a telescope sum.*

Proof of Lemma 2.6. In order to obtain the desired existence result, we modify the proof of [35, Lemma 3.2., p. 271]. Let

$$A := \max_{t\in[t_1,t_2]} \|g'(t)\|_{L^\infty(M,g(t))} + \max_{t\in[t_1,t_2]} \|\nabla g'(t)\|_{L^\infty(M,g(t))} \tag{2.18}$$

At time t_1+jS we choose a length minimizing geodesic $\gamma_{t_1+jS} : [0,1] \longrightarrow M$ with respect to the metric $g(t_1+jS)$ connecting x and y. This curve satisfies

$$|\nabla_{\dot\gamma_{t_1+jS}}\dot\gamma_{t_1+jS}|_{g(t_1+jS)} \equiv 0 \tag{2.19}$$

and

$$|\dot\gamma_{t_1+jS}|_{g(t_1+jS)} \equiv d(x,y,t_1+jS) \tag{2.20}$$

Firstly, we show that an appropriate choice of $S(\beta,x,y,g) > 0$ implies (2.16). Let $v \in TM$ be an arbitrary vector and $t \in [t_1+jS, t_1+(j+1)S)\cap[t_1,t_2]$ Then, by (A.2), we have

$$\left|\log\left(\frac{|v|^2_{g(t)}}{|v|^2_{g(t_1+jS)}}\right)\right| \leq \int_{t_1+jS}^{t} \|g'(\tau)\|_{(L^\infty(M),g(\tau))}\; d\tau \overset{(2.18)}{\leq} AS \leq \log[(1+\beta)^2] \tag{2.21}$$

Hence, we obtain the estimate

$$\frac{1}{(1+\beta)^2}|\dot\gamma_{t_1+jS}|^2_{g(t_1+jS)} \leq |\dot\gamma_t|^2_{g(t)} \leq (1+\beta)^2|\dot\gamma_{g(t_1+jS)}|^2_{g(t_1+jS)}$$

Using (2.20) we infer (2.16) from this. Next we show (2.15). Using (A.1) we obtain

$$\frac{\partial}{\partial t}L(\gamma_t,t)=\frac{\partial}{\partial t}L(\gamma_{t_1+jS},t)\overset{(2.18)}{\leq}A\cdot L(\gamma_{t_1+jS},t)=A\cdot L(\gamma_t,t) \tag{2.22}$$

on $(t_1+jS,t_1+(j+1)S)\cap[t_1,t_2)$. This implies $\frac{\partial}{\partial t}\log L(\gamma_t,t)\leq A$, and we infer

$$\begin{aligned} d(x,y,t)\leq& L(\gamma_t,t)=\frac{L(\gamma_t,t)}{L(\gamma_{t_1+jS},t_1+jS)}L(\gamma_{t_1+jS},t_1+jS)\\ =&\exp\left(\log\left(\frac{L(\gamma_t,t)}{L(\gamma_{t_1+jS},t_1+jS)}\right)\right)L(\gamma_{t_1+jS},t_1+jS)\\ =&\exp\left(\log\left(L(\gamma_t,t)\right)-\log\left(L(\gamma_{t_1+jS},t_1+jS)\right)\right)L(\gamma_{t_1+jS},t_1+jS)\\ \leq& e^{A(t-(t_1+jS))}L(\gamma_{t_1+jS},t_1+jS)=e^{A(t-(t_1+jS))}d(x,y,t_1+jS) \end{aligned} \tag{2.23}$$

In particular, we have

$$d(x,y,t)\leq e^{A(t_2-t_1)}L(\gamma_{t_1},t_1)=e^{A(t_2-t_1)}d(x,y,t_1) \qquad \forall t\in[t_1,t_2] \tag{2.24}$$

From (2.23) we obtain for all $t\in(t_1+jS,t_1+(j+1)S)\cap[t_1,t_2]$

$$\begin{aligned} L(\gamma_t,t)\leq& d(x,y,t_1+jS)+(e^{AS}-1)d(x,y,t_1+jS)\\ \overset{(2.24)}{\leq}& d(x,y,t_1+jS)+(e^{AS}-1)e^{A(t_2-t_1)}d(x,y,t_1)\\ \leq& d(x,y,t_1+jS)+\frac{\beta}{2} \end{aligned}$$

In order to prove (2.15) it suffices to show that we can choose $S(\beta,x,y,g)>0$ small enough to ensure

$$d(x,y,t_1+jS)\leq d(x,y,t)+\frac{\beta}{2} \qquad \forall t\in(t_1+jS,t_1+(j+1)S)\cap[t_1,t_2] \tag{2.25}$$

From (2.21) we conclude for all $v\in TM$

$$e^{-AS}|v|^2_{g(t_1+jS)}\leq|v|^2_{g(t)}\leq e^{AS}|v|^2_{g(t_1+jS)} \quad \forall t\in(t_1+jS,t_1+(j+1)S)\cap[t_1,t_2] \tag{2.26}$$

At time t, we choose a length minimizing geodesic $\xi:[0,d(x,y,t)]\longrightarrow M$ connecting x and y, then:

$$d(x,y,t_1+jS)\leq L(\xi,t_1+jS)=\int_0^{d(x,y,t)}|\dot{\xi}(s)|_{g(t_1+jS)}\,ds$$

$$
\begin{aligned}
&\overset{(2.26)}{\leq} e^{AS} \int_0^{d(x,y,t)} |\dot{\xi}(s)|_{g(t)}\, ds = e^{AS} L(\xi, t) = e^{AS} d(x,y,t) \\
&= d(x,y,t) + (e^{AS} - 1) d(x,y,t) \\
&\overset{(2.24)}{\leq} d(x,y,t) + (e^{AS} - 1) e^{A(t_2 - t_1)} d(x,y,t_1) \leq d(x,y,t) + \frac{\beta}{2}
\end{aligned}
$$

It remains to show that, under the assumption that $S(\beta, x, y, g) > 0$ is sufficiently small, estimate (2.17) is also valid. From (A.3), (2.18) and (2.16) we conclude for each $t \in (t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2]$

$$
\begin{aligned}
\frac{\partial}{\partial t} |\nabla_{\dot{\gamma}_t} \dot{\gamma}_t|^2_{g(t)} &\leq A\, |\nabla_{\dot{\gamma}_t} \dot{\gamma}_t|^2_{g(t)} + 4AC(n) d^2(x,y,t_1 + jS)\, |\nabla_{\dot{\gamma}_t} \dot{\gamma}_t|_{g(t)} \\
&\overset{(2.24)}{\leq} A\, |\nabla_{\dot{\gamma}_t} \dot{\gamma}_t|^2_{g(t)} + 4AC(n) e^{2A(t_2 - t_1)} d^2(x,y,t_1)\, |\nabla_{\dot{\gamma}_t} \dot{\gamma}_t|_{g(t)}
\end{aligned}
\tag{2.27}
$$

Now let $x \in M$ be arbitrary. We assume that

$$
\begin{aligned}
\widehat{t} := \sup \Big\{ & t \in (t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2) \mid \\
& |\nabla_{\dot{\gamma}_\tau} \dot{\gamma}_\tau|^2_{g(\tau)} (x, \tau) \leq \min\{\beta\, \overline{d}^2, 1\}\ \forall \tau \in [t_1 + jS, t] \Big\} \\
& < \min\{t_1 + (j+1)S, t_2\}
\end{aligned}
$$

where

$$
\overline{d} := \min_{t \in [t_1, t_2]} d(x,y,t) > 0
$$

Then, (2.27) implies

$$
\frac{\partial}{\partial t} |\nabla_{\dot{\gamma}_t} \dot{\gamma}_t|^2_{g(t)} \leq A(1 + 4Ce^{2A(t_2 - t_1)} d^2(x,y,t_1)) \text{ on } \{x\} \times [t_1 + jS, \widehat{t}]
$$

Using this, from (2.19), we conclude:

$$
\begin{aligned}
\min\{\beta\, \overline{d}^2, 1\} = \left|\nabla_{\dot{\gamma}_{\widehat{t}}} \dot{\gamma}_{\widehat{t}}\right|^2_{g(\widehat{t})} (x, \widehat{t}) \\
&\leq A(\widehat{t} - (t_1 + jS))(1 + 4Ce^{2A(t_2 - t_1)} d^2(x,y,t_1)) \\
&\leq AS(1 + 4Ce^{2A(t_2 - t_1)} d^2(x,y,t_1)) \leq \frac{\min\{\beta\, \overline{d}^2, 1\}}{2}
\end{aligned}
$$

which yields a contradiction, if $S(\beta, x, y, g) > 0$ is small enough. $\square$

Now we prove Lemma 2.4. The argumentation is based on [35, pp. 277-280].

Proof of Lemma 2.4. Let $x, y \in M$ be fixed and $t_1, t_2 \in [0,1]$, $t_1 < t_2$. Initially, we construct an appropriate β-quasi-forward geodesic in respect of Lemma 2.3. We choose

$$\beta := \min_{t \in [t_1, t_2]} \beta_t > 0 \tag{2.28}$$

where

$$\beta_t := \beta(n, diam_{g(t)}(M), f_3(M, g(t)), inj_{g(t)}(M))$$

is chosen according to Lemma 2.3 at time t. Next, using Lemma 2.6, we assume the existence of a β-quasi-forward-geodesic

$$(\xi_t)_{t \in [t_1, t_2]} : [0,1] \longrightarrow M$$

connecting x and y. It is our aim to construct an appropriate tubular neighborhood around each ξ_t applying Lemma 2.3, the radii r_t shall be time dependent, where $r_0 = 0$, when $t_1 = 0$. After doing this, we notice that we are able to estimate the integral $\int_{\xi_t} |\text{grad } \mathcal{F}| \, d\sigma$ from above against an average integral of $|\text{grad } \mathcal{F}|^2$ along the tube plus an error term. Each of these terms is controllable.

By construction of the β-quasi-forward-geodesic, we have a finite set of geodesics denoted by $(\xi_{t_1+jS})_{j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor\}}$, where each of these geodesics is parametrized proportional to arc length, i.e.:

$$|\dot{\xi}_{t_1+jS}|_{g(t_1+jS)} \equiv d(x, y, t_1 + jS) \text{ for all } j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor\}$$

we reparametrize these curves with respect to arc length, i.e: for each $j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor\}$ let

$$\begin{aligned} \varphi_{t_1+jS} &: [0, d(x, y, t_1 + jS)] \longrightarrow [0,1] \\ \varphi(s) &:= \frac{s}{d(x, y, t_1 + jS)} \end{aligned}$$

and let

$$\begin{aligned} \gamma_{t_1+jS} &: [0, d(x, y, t_1 + jS)] \longrightarrow M \\ \gamma_{t_1+jS} &:= \xi_{t_1+jS} \circ \varphi_{t_1+jS} \end{aligned}$$

Of course, these curves are satisfying (2.1) (2.2) and (2.3). But we need to get sure that, for each $t \in (t_1 + jS, t_1(j+1)S) \cap [t_1, t_2]$, the curve

$$\gamma_t := \xi_t \circ \varphi_{t_1+jS} : [0, d(x, y, t_1 + jS)] \to M$$

is also satisfying these assumptions. Here $\beta \in (0,1)$ is defined by (2.28). By construction, using (2.16) for each $t \in (t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2]$, we have

$$\frac{1}{1+\beta_t} \leq \frac{1}{1+\beta} \leq |\dot{\gamma}_t|_{g(t)} = \frac{1}{d(x,y,t_1+jS)} |\dot{\xi}_t|_{g(t)} \leq 1 + \beta \leq 1 + \beta_t$$

and, using (2.17)

$$|\nabla_{\dot{\gamma}_t} \dot{\gamma}_t|_{g(t)} = \frac{1}{d(x,y,t_1+jS)^2} |\nabla_{\dot{\xi}_t} \dot{\xi}_t|_{g(t)} \leq \beta \leq \beta_t$$

Thus, by Lemma 2.3, for each time $t \in [jS, (j+1)S) \cap [t_1, t_2]$ the tubular neighborhood $D(\gamma_t, \rho_t)$ is foliated by $(D(\gamma_t(s), \rho_t))_{s \in [0, d(x,y,t_1+jS)]}$ where

$$\rho_t := \mu \min \left\{ inj_{g(t)}(M), f_3(M, g(t))^{-\frac{1}{2}} \right\} \tag{2.29}$$

where $\mu > 0$ is fixed and the differential of the projection map satisfies (2.4). For later considerations, we assume that $\mu > 0$ is also chosen compatible to Lemma A.9. Although we have no control on β_t, we can bound ρ_t from below if we can bound $f_3(M, g(t))^{-\frac{1}{2}}$ from below in the view of (2.29).

Using (A.10) and (1.5) we obtain for each $m \in \{1,2,3\}$:

$$\left\| \nabla^m Rm_{g(t)} \right\|_{L^\infty(M, g(t))} \leq C(m,K) \left(t^{-\frac{1}{2}} \right)^{\frac{2+m}{2}} = C(m,K) t^{-\frac{2+m}{4}} \tag{2.30}$$

and consequently

$$f_3(M, g(t)) \leq C(K) t^{-\frac{1}{2}}$$

Thus, we have for each $t \in [t_1, t_2]$

$$\rho_t \geq \mu \left\{ \iota t^{\frac{1}{4}}, C^{-\frac{1}{2}}(K) t^{\frac{1}{4}} \right\} \geq \mu \min\{\iota, C^{-\frac{1}{2}}(K)\} \cdot t^{\frac{7}{24}} =: R(\iota, K) \cdot t^{\frac{7}{24}} =: r_t(\iota, K) \tag{2.31}$$

Now, we may start to estimate the change of $L(\gamma_t, t)$, where $t \in [t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2)$ and $j \in \left\{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \right\}$. From the explicit formula in (1.3) and (2.30) we conclude $|\nabla \text{ grad } \mathcal{F}_{g(t)}|_{g(t)} \leq C_2(K) t^{-\frac{5}{4}}$. Now let p be an arbitrary point on the curve γ_{t_1+jS} and $q \in D(p, r_t)$ then we obtain

$$|\text{grad } \mathcal{F}_{g(t)}|_{g(t)}(p) \leq |\text{grad } \mathcal{F}_{g(t)}|_{g(t)}(q) + C_3(K) r_t(\iota, K) t^{-\frac{5}{4}} \tag{2.32}$$

In the following, we write r_t instead of $r_t(\iota, K)$ and grad $\mathcal{F}$ instead of grad $\mathcal{F}_{g(t)}$.

We infer:

$$\begin{aligned}|\text{grad } \mathcal{F}|_{g(t)}(p) =&\frac{\int_{D(p,r_t)} |\text{grad } \mathcal{F}|_{g(t)}(p)\, dA}{Area(D(p,r_t))}\\ \leq&\frac{\int_{D(p,r_t)} \left[|\text{grad } \mathcal{F}|_{g(t)}(q) + C_3(K) r_t t^{-\frac{5}{4}}\right]\, dA}{Area(D(p,r_t))}\\ =&\frac{\int_{D(p,r_t)} |\text{grad } \mathcal{F}|_{g(t)}\, dA}{Area(D(p,r_t))} + C_3(K)R(\iota,K)t^{\frac{7}{24}-\frac{5}{4}}\\ \leq&\frac{\left(\int_{D(p,r_t)} |\text{grad } \mathcal{F}|^2_{g(t)}\, dA\right)^{\frac{1}{2}}}{Area^{\frac{1}{2}}(D(p,r_t))} + C_3(K)R(\iota,K)t^{-\frac{23}{24}}\end{aligned} \tag{2.33}$$

From Lemma A.9 we obtain for each $t \in [t_1, t_2]$ that

$$Area(D(\gamma_t(s), r_t)) \geq c r_t^3 = cR^3(\iota,K)t^{\frac{7}{8}} \tag{2.34}$$

Inserting this estimate into (2.33), we infer for each $p \in \gamma_{t_1+jS}$

$$\begin{aligned}|\text{grad } \mathcal{F}|_{g(t)}(p) \leq& c^{-\frac{1}{2}} R^{-\frac{3}{2}}(\iota,K) t^{-\frac{7}{16}} \left(\int_{D(p,r_t)} |\text{grad } \mathcal{F}|^2_{g(t)}\, dA\right)^{\frac{1}{2}}\\ &+ C_3(K)R(\iota,K)t^{-\frac{23}{24}}\end{aligned} \tag{2.35}$$

Hence, on $(t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2)$ we have

$$\begin{aligned}&\frac{d}{dt} L(\gamma_t, t) = \frac{d}{dt} L(\gamma_{t_1+jS}, t) \overset{(A.1)}{\leq} \int_{\gamma_{t_1+jS}} |\text{grad } \mathcal{F}|_{g(t)}\, d\sigma\\ &\overset{(2.35)}{\leq} c^{-\frac{1}{2}} R^{-\frac{3}{2}}(\iota,K) t^{-\frac{7}{16}} \int_{\gamma_{t_1+jS}} \left(\int_{D(p,r_t)} |\text{grad } \mathcal{F}|^2_{g(t)}\, dA\right)^{\frac{1}{2}} d\sigma\\ &\quad + C_3(K)R(\iota,K)t^{-\frac{23}{24}} L(\gamma_{t_1+jS}, t)\\ &\leq\ c^{-\frac{1}{2}} R^{-\frac{3}{2}}(\iota,K) t^{-\frac{7}{16}} \left(\int_{\gamma_{t_1+jS}} \int_{D(p,r_t)} |\text{grad } \mathcal{F}|^2_{g(t)}\, dA\, d\sigma\right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{t_1+jS}, t)\\ &\quad + C_3(K)R(\iota,K)t^{-\frac{23}{24}} L(\gamma_{t_1+jS}, t)\\ &\overset{(A.17)}{\leq} c^{-\frac{1}{2}} R^{-\frac{3}{2}}(\iota,K) t^{-\frac{7}{16}} \sup_{D(\gamma_{t_1+jS}, r_t)} |d\pi|^{\frac{1}{2}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(t)}\, dV_{g(t)}\right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{t_1+jS}, t)\\ &\quad + C_3(K)R(\iota,K)t^{-\frac{23}{24}} L(\gamma_{t_1+jS}, t)\\ &\overset{(2.4)}{\leq} c_2 R^{-\frac{3}{2}}(\iota,K) t^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(t)}\, dV_{g(t)}\right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{t_1+jS}, t)\\ &\quad + C_3(K)R(\iota,K)t^{-\frac{23}{24}} L(\gamma_{t_1+jS}, t)\end{aligned}$$

$$= c_2 R^{-\frac{3}{2}} R(\iota, K) t^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(t)} \, dV_{g(t)} \right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_t, t)$$
$$+ C_3(K) R(\iota, K) t^{-\frac{23}{24}} L(\gamma_t, t)$$

Using

$$L(\gamma_t, t) \overset{(2.1)}{\leq} d(x, y, t_1 + jS) + 1 \overset{(1.7)}{\leq} 2(1 + D) + 1 = 3 + 2D \tag{2.36}$$

we conclude

$$\begin{aligned} \frac{d}{dt} L(\gamma_t, t) \leq & C(D) R^{-\frac{3}{2}}(\iota, K) t^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(t)} \, dV_{g(t)} \right)^{\frac{1}{2}} \\ & + C(K, D) R(\iota, K) t^{-\frac{23}{24}} \end{aligned}$$

on $[t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2)$ where $j \in \left\{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor \right\}$. Integrating this estimate along $[t_1 + jS, t]$ yields:

$$\begin{aligned} & d(x, y, t) - d(x, y, t_1 + jS) = d(x, y, t) - L(\gamma_{t_1 + jS}, t_1 + jS) \\ \leq & L(\gamma_t, t) - L(\gamma_{t_1 + jS}, t_1 + jS) \\ \leq & C(D) R^{-\frac{3}{2}}(\iota, K) \int_{t_1 + jS}^t s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(s)} \, dV_{g(s)} \right)^{\frac{1}{2}} ds \\ & + C(K, D) R(\iota, K) \int_{t_1 + jS}^t s^{-\frac{23}{24}} \, ds \end{aligned}$$

for each $t \in (t_1 + jS, t_1 + (j+1)S) \cap [t_1, t_2]$. In particular, we obtain for each $j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor - 1\}$

$$\begin{aligned} & d(x, y, t_1 + (j+1)S) - d(x, y, t_1 + jS) \\ \leq & C(D) R^{-\frac{3}{2}}(\iota, K) \int_{t_1 + jS}^{t_1 + (j+1)S} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(s)} \, dV_{g(s)} \right)^{\frac{1}{2}} ds \\ & + C(K, D) R(\iota, K) \int_{t_1 + jS}^{t_1 + (j+1)S} s^{-\frac{23}{24}} \, ds \end{aligned}$$

and

$$\begin{aligned} & d(x, y, t_2) - d(x, y, t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S) \\ \leq & C(D) R^{-\frac{3}{2}}(\iota, K) \int_{t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S}^{t_2} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(s)} \, dV_{g(s)} \right)^{\frac{1}{2}} ds \\ & + C(K, D) R(\iota, K) \int_{t_1 + \lfloor \frac{t_2 - t_1}{S} \rfloor S}^{t_2} s^{-\frac{23}{24}} \, ds \end{aligned}$$

and consequently

$$\begin{aligned}
&d(x,y,t_2)-d(x,y,t_1)\\
=&\sum_{j=0}^{\lfloor\frac{t_2-t_1}{S}\rfloor-1}[d(x,y,t_1+(j+1)S)-d(x,y,t_1+jS)]\\
&+d(x,y,t_2)-d(x,y,t_1+\lfloor\frac{t_2-t_1}{S}\rfloor S)\\
\leq& C(D)R^{-\frac{3}{2}}(\iota,K)\int_{t_1}^{t_2}s^{-\frac{7}{16}}\left(\int_M|\text{grad }\mathcal{F}|^2_{g(s)}\,dV_{g(s)}\right)^{\frac{1}{2}}\,ds\\
&+C(K,D)R(\iota,K)\int_{t_1}^{t_2}s^{-\frac{23}{24}}\,ds\\
\leq& C(D)R^{-\frac{3}{2}}(\iota,K)\left(\int_{t_1}^{t_2}s^{-\frac{7}{8}}\,ds\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\int_M|\text{grad }\mathcal{F}|^2_{g(s)}\,dV_{g(s)}\,ds\right)^{\frac{1}{2}}\\
&+C(K,D)R(\iota,K)\int_{t_1}^{t_2}s^{-\frac{23}{24}}\,ds
\end{aligned}$$

Using (1.4) and (A.9) we conclude

$$d(x,y,t_2)-d(x,y,t_1)\leq C(K,\iota,D)\Lambda^{\frac{1}{2}}\left(t_2^{\frac{1}{8}}-t_1^{\frac{1}{8}}\right)^{\frac{1}{2}}+C(K,\iota,D)\left(t_2^{\frac{1}{24}}-t_1^{\frac{1}{24}}\right)$$

□

2.1.3 Backward estimates

In this subsection we reverse the ideas from Subsection 2.1.2 in order to prove that, along the L^2-flow, the distance between two points does not become too small when $t>0$ is small.

Lemma 2.8. *Let* (M^4,g_0) *be a closed Riemannian 4-manifold and let* $(M^4,g(t))_{t\in[0,1]}$ *be a solution to the flow given in* (1.3) *satisfying* (1.4), (1.5), (1.6) *and* (1.7), *i.e.:*

$$\begin{aligned}
\int_M|Rm_{g_0}|^2\,dV_{g_0}&\leq\Lambda\\
\|Rm_{g(t)}\|_{L^\infty(M,g(t))}&\leq Kt^{-\frac{1}{2}}\\
inj_{g(t)}(M)&\geq\iota t^{\frac{1}{4}}\\
diam_{g(t)}(M)&\leq 2(1+D)
\end{aligned}$$

for all $t \in (0,1]$. Then we have the following estimate:

$$d(x,y,t_2) - d(x,y,t_1) \geq -C(K,\iota,D)\Lambda^{\frac{1}{2}} \left(t_2^{\frac{1}{8}} - t_1^{\frac{1}{8}}\right)^{\frac{1}{2}} - C(K,\iota,D)\left(t_2^{\frac{1}{24}} - t_1^{\frac{1}{24}}\right) \tag{2.37}$$

for all $t_1, t_2 \in [0,1]$ where $t_1 < t_2$.

The notion of a β-quasi-backward-geodesic, which is introduced below, is an analogue to the notion of a β-quasi-forward-geodesic, introduced in Subsection 2.1.2. The slight difference is that now, the minimizing geodesics are chosen at the subinterval ends:

Definition 2.9. *Let $(M^n, g(t))_{t\in[t_1,t_2]}$ be a family of complete Riemannian manifolds. Given $\beta > 0$ and $x, y \in M$ then we say that a family of curves $(\gamma_t)_{t\in[t_1,t_2]} : [0,1] \longrightarrow M$ is a β-quasi-backward-geodesic connecting x and y if $(\gamma_t)_{t\in[t_1,t_2]}$ is a β-quasi-forward-geodesic connecting x and y with respect to the time-reversed flow $(M^n, g(t_2+t_1-t))_{t\in[t_1,t_2]}$, i.e.: there is a constant $S > 0$ so that:*

1. *For all $t \in [t_1,t_2]$ one has $\gamma_t(0) = x$ and $\gamma_t(1) = y$*
2. *For all $j \in \mathbb{N}_0$ such that $t_2 - jS \geq t_1$, γ_{t_2-jS} is a minimizing geodesic*
3. *For all $j \in \mathbb{N}_0$ such that $t_2 - jS \geq t_1$, and all $t \in (t_2-(j+1)S, t_2 - jS] \cap [t_1,t_2]$ one has $\gamma_t = \gamma_{t_2-jS}$*
4. *For all $t \in [t_1,t_2]$ one has*
$$d(x,y,t) \leq L(\gamma_t, t) \leq d(x,y,t) + \beta \tag{2.38}$$
5. *For all $j \in \mathbb{N}_0$ such that $t_2 - jS \geq t_1$, and all $t \in (t_2-(j+1)S, t_2 - jS] \cap [t_1,t_2]$ one has*
$$\frac{1}{1+\beta} d(x,y,t_1 - jS) \leq |\dot\gamma_t|_{g(t)} \leq (1+\beta) d(x,y,t_2 - jS) \tag{2.39}$$
$$|\nabla_{\dot\gamma_t}\dot\gamma_t|_{g(t)} \leq \beta\, d^2(x,y,t_2 - jS) \tag{2.40}$$

Applying Lemma 2.6 to $(M^n, g(t_2+t_1-t))_{t\in[t_1,t_2]}$, we infer

Lemma 2.10. *Let $(M^n, g(t))_{t\in[t_1,t_2]}$ a smooth family of closed Riemannian manifolds. Given $\beta > 0$ and $x, y \in \mathbb{N}$ then there exists a β-quasi-backward-geodesic connecting x and y.*

Using this concept, we prove Lemma 2.8:

Proof of Lemma 2.8. The proof is analogous to Lemma 2.4. We choose $x, y \in M$ and $t_1, t_2 \in [0,1]$ where $t_1 < t_2$. It is our aim to construct an appropriate backward-geodesic. As in the proof of Lemma 2.4, let

$$\beta := \min_{t\in[t_1,t_2]} \beta_t > 0 \tag{2.41}$$

where

$$\beta_t := \beta(n, diam_{g(t)}(M), f_3(M, g(t)), inj_{g(t)}(M))$$

is defined in Lemma 2.3, let $(\xi_t)_{t\in[t_1,t_2]}$ be a β-backward-geodesic, connecting x and y, whose existence is ensured by Lemma 2.10. As in the proof of Lemma 2.4 we use Lemma 2.3 to construct an appropriate tubular neighborhood around each ξ_t, where $t \in [t_1, t_2]$, having a time depend radius r_t.

In this situation we have a finite set of geodesics $(\xi_{t_2-jS})_{j\in\{0,...,\lfloor\frac{t_2-t_1}{S}\rfloor\}}$ satisfying

$$|\dot{\xi}_{t_2-jS}|_{g(t_2-jS)} \equiv d(x, y, t_2 - jS) \text{ for all } j \in \{0, ..., \lfloor\frac{t_2-t_1}{S}\rfloor\}$$

Analogous to the proof of Lemma 2.4, we reparametrize these curves with respect to arc length, i.e: for each $j \in \{0, ..., \lfloor\frac{t_2-t_1}{S}\rfloor\}$ we define

$$\begin{aligned}\varphi_{t_2-jS} &: [0, d(x, y, t_2 - jS)] \longrightarrow [0,1]\\ \varphi(s) &:= \frac{s}{d(x, y, t_2 - jS)}\end{aligned}$$

$$\begin{aligned}\gamma_{t_2-jS} &: [0, d(x, y, t_2 - jS)] \longrightarrow M\\ \gamma_{t_2-jS} &:= \xi_{t_2-jS} \circ \varphi_{t_2-jS}\end{aligned}$$

and for each $t \in (t_2 - (j+1)S, t_2 - jS] \cap [t_1, t_2]$ we define

$$\gamma_t := \xi_t \circ \varphi_{t_2-jS} : [0, d(x, y, t_2 - jS)] \to M$$

so that, for each $t \in [t_1, t_2]$ the curve γ_t satisfies (2.1) (2.2) and (2.3) with respect to β_t. Hence, following Lemma 2.3, at each time $t \in (t_2 - (j+1)S, t_2 - jS] \cap [t_1, t_2]$ the tubular neighborhood $D(\gamma_t, \rho_t)$ around γ_t is foliated by $(D(\gamma_t(s), \rho_t))_{s\in[0,d(x,y,t_2-jS)]}$ where $\rho_t := \mu \min\left\{inj_{g(t)}(M), f_3(M, g(t))^{-\frac{1}{2}}\right\}$, again $\mu > 0$ shall also satisfy the requirements of Lemma 2.3. Using the same arguments as in the proof of Lemma 2.4 we also obtain (2.31) and (2.32), i.e.:

$$\rho_t \geq R(\iota, K)t^{\frac{7}{24}} =: r_t(\iota, K) \text{ for each } t \in [t_1, t_2]$$

and

$$|\text{grad } \mathcal{F}|_{g(t)}(p) \leq |\text{grad } \mathcal{F}|_{g(t)}(q) + C_3(K) r_t(\iota, K) t^{-\frac{5}{4}}$$

for each $p \in \gamma_t = \gamma_{t_2 - jS}$ and $q \in D(p, r_t)$ where $t \in (t_2-(j+1)S, t_2-jS] \cap [t_1, t_2]$ and $j \in \{0, ..., \lfloor \frac{t_2-t_1}{S} \rfloor\}$. From this we also obtain (2.33), i.e.:

$$|\text{grad } \mathcal{F}|_{g(t)}(p) \leq \frac{\left(\int_{D(p,r_t)} |\text{grad } \mathcal{F}|^2_{g(t)}(q)\, dA(q)\right)^{\frac{1}{2}}}{Area^{\frac{1}{2}}(D(p, r_t))} + C_3(K)R(\iota, K)t^{-\frac{23}{24}}$$

Using Lemma A.9 we obtain (2.34), i.e.:

$$Area(D(\gamma_t(s)), r_t) \geq c r_t^3 = cR^3 t^{\frac{7}{8}}$$

for all $t \in (t_2 - (j+1)S, t_2 - jS] \cap [t_1, t_2]$. Hence, for each $j \in \{0, ..., \lfloor \frac{t_2-t_1}{S} \rfloor\}$ we infer on $(t_2 - (j+1)S, t_2 - jS) \cap (t_1, t_2]$ the following estimate

$$\begin{aligned}
&\frac{d}{dt} L(\gamma_t, t) = \frac{d}{dt} L(t_2 - jS, t) \overset{(A.1)}{\geq} -\int_{\gamma_{t_2-jS}} |\text{grad } \mathcal{F}|_{g(t)}\, d\sigma \\
&\geq -c^{\frac{1}{2}} R^{-\frac{3}{2}}(\iota, K) t^{-\frac{7}{16}} \int_{\gamma_{t_2-jS}} \left(\int_{D(p,r_t)} |\text{grad } \mathcal{F}|^2_{g(t)}\, dA \right)^{\frac{1}{2}} d\sigma \\
&\quad - C_3(K)R(\iota, K)t^{-\frac{23}{24}} L(\gamma_{t_2-jS}, t) \\
&\geq -c^{\frac{1}{2}} R^{-\frac{3}{2}}(\iota, K) t^{-\frac{7}{16}} \left(\int_{\gamma_{t_2-jS}} \int_{D(p,r_t)} |\text{grad } \mathcal{F}|^2_{g(t)}\, dA\, d\sigma \right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{t_2-jS}, t) \\
&\quad - C_3(K)R(\iota, K)t^{-\frac{23}{24}} L(\gamma_{t_2-jS}, t) \\
&\overset{(A.17)}{\geq} -c^{\frac{1}{2}} R^{-\frac{3}{2}}(\iota, K) t^{-\frac{7}{16}} \sup_{D(\gamma_{t_2-jS}, r_t)} |d\pi|^{\frac{1}{2}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(t)} dV_{g(t)} \right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{t_2-jS}, t) \\
&\quad - C_3(K)R(\iota, K)t^{-\frac{23}{24}} L(\gamma_{t_2-jS}, t) \\
&\overset{(2.4)}{\geq} -c_2 R^{-\frac{3}{2}}(\iota, K) t^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(t)}\, dV_{g(t)} \right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{t_2-jS}, t) \\
&\quad - C_3(K)R(\iota, K)t^{-\frac{23}{24}} L(\gamma_{t_2-jS}, t) \\
&\geq -C(D)R^{-\frac{3}{2}}(\iota, K) t^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(t)}\, dV_{g(t)} \right)^{\frac{1}{2}} - C(K, D)R(\iota, K)t^{-\frac{23}{24}}
\end{aligned}$$

Here we have used the fact that γ_t is nearly length minimizing and that the diameter is bounded (cf. (2.36)). By integration along $[t, t_2 - jS]$ we conclude for each $t \in (t_2 - (j+1)S, t_2 - jS] \cap [t_1, t_2]$

$$\begin{aligned}
&d(x,y,t_2 - jS) - d(x,y,t) = L(\gamma_{t_2 - jS}, t_2 - jS) - d(x,y,t)\\
\geq &L(\gamma_{t_2 - jS}, t_2 - jS) - L(\gamma_t, t)\\
\geq &- C(D) R^{-\frac{3}{2}}(\iota, K) \int_t^{t_2 - jS} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(s)} \, dV_{g(s)} \right)^{\frac{1}{2}} ds\\
&- C(K,D) R(\iota, K) \int_t^{t_2 - jS} s^{-\frac{23}{24}} \, ds
\end{aligned}$$

In particular, we have for each $j \in \{0, ..., \lfloor \frac{t_2 - t_1}{S} \rfloor - 1\}$

$$\begin{aligned}
&d(x,y,t_2 - jS) - d(x,y,t_2 - (j+1)S)\\
\geq &- C(D) R^{-\frac{3}{2}}(\iota, K) \int_{t_2 - (j+1)S}^{t_2 - jS} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(s)} \, dV_{g(s)} \right)^{\frac{1}{2}} ds\\
&- C(K,D) R(\iota, K) \int_{t_2 - (j+1)S}^{t_2 - jS} s^{-\frac{23}{24}} \, ds
\end{aligned}$$

and also

$$\begin{aligned}
&d(x,y,t_2 - \lfloor \frac{t_2 - t_1}{S} \rfloor S) - d(x,y,t_1)\\
\geq &-C(D) R^{-\frac{3}{2}}(\iota, K) \int_{t_1}^{t_2 - \lfloor \frac{t_2 - t_1}{S} \rfloor S} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(s)} \, dV_{g(s)} \right)^{\frac{1}{2}} ds\\
&- C(K,D) R(\iota, K) \int_{t_1}^{t_2 - \lfloor \frac{t_2 - t_1}{S} \rfloor S} s^{-\frac{23}{24}} \, ds
\end{aligned}$$

and finally

$$\begin{aligned}
&d(x,y,t_2) - d(x,y,t_1)\\
= &\sum_{j=0}^{\lfloor \frac{t_2 - t_1}{S} \rfloor - 1} [d(x,y,t_2 - jS) - d(x,y,t_2 - (j+1)S)]\\
&+ d(x,y,t_2 - \lfloor \frac{t_2 - t_1}{S} \rfloor S) - d(x,y,t_1)\\
\geq &- C(D) R^{-\frac{3}{2}}(\iota, K) \int_{t_1}^{t_2} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}|^2_{g(s)} \, dV_{g(s)} \right)^{\frac{1}{2}} ds\\
&- C(K,D) R(\iota, K) \int_{t_1}^{t_2} s^{-\frac{23}{24}} \, ds
\end{aligned}$$

$$\geq - C(K,D)R^{-\frac{3}{2}}(\iota,K)\left(\int_{t_1}^{t_2} s^{-\frac{7}{8}}\,ds\right)^{\frac{1}{2}}\left(\int_{t_1}^{t_2}\int_M |\mathrm{grad}\ \mathcal{F}|^2_{g(s)}\,dV_{g(s)}\,ds\right)^{\frac{1}{2}}$$
$$- C(K,D)R(\iota,K)\int_{t_1}^{t_2} s^{-\frac{23}{24}}\,ds$$

we infer

$$d(x,y,t_2) - d(x,y,t_1) \geq -C(K,\iota,D)\Lambda^{\frac{1}{2}}\left(t_2^{\frac{1}{8}} - t_1^{\frac{1}{8}}\right)^{\frac{1}{2}} - C(K,\iota,D)\left(t_2^{\frac{1}{24}} - t_1^{\frac{1}{24}}\right)$$

□

Finally, (2.14) and (2.37) together imply (1.8), which finishes the proof of Theorem 1.3. Using Theorem 1.3, the following result

Corollary 2.11. *Let $(M^4, g(t))_{t\in[0,1]}$, where M^4 is a closed Riemannian 4-manifold, be a solution to* (1.3) *satisfying the assumptions,* (1.4), (1.5), (1.6) *and* (1.7), *then for each $k \in \mathbb{N}$ there exists $j(k,\Lambda,K,\iota,D) \in \mathbb{N}$ such that*

$$d_{GH}((M,d_g),(M,d_{g(t)})) < \frac{1}{k}$$

for all $t \in [0,1/j]$

is a consequence of the following Lemma

Lemma 2.12. *Let M^n be a closed manifold. Given two metrics g_1 and g_2 on M satisfying*

$$\sup_{x,y\in M} |d_{g_1}(x,y) - d_{g_2}(x,y)| < \epsilon$$

then we have

$$d_{GH}((M,d_{g_1}),(M,d_{g_2})) < \frac{\epsilon}{2}$$

Proof. The set $\mathfrak{R} := \{(x,x) \in M\times M \,|\, x \in M\}$ is a correspondence between M and M itself (cf. Definition C.1) and the distorsion of $\mathfrak{R}$ is (cf. Definition C.2):

$$\mathrm{dis}\,\mathfrak{R} = \sup_{x,y\in M} |d_{g_1}(x,y) - d_{g_2}(x,y)| < \epsilon$$

From [7, Theorem 7.3.25., p. 257] we obtain

$$d_{GH}((M,d_{g_1}),(M,d_{g_2})) \leq \frac{1}{2}\mathrm{dis}\,\mathfrak{R} < \frac{1}{2}\epsilon$$

□

2.2 Proof of Theorem 1.1

In this section we prove Theorem 1.1 using Corollary 2.11. The conditions (1.4), (1.5), (1.6) and (1.7) are ensured by the following result

Theorem 2.13. *(cf. [35, Theorem 1.8, p. 260]) Given $\delta \in (0,1)$, there are constants $\epsilon(\delta), \iota(\delta), A(\delta) > 0$ so that if (M^4, g_0) is a closed Riemannian manifold satisfying the following conditions*

$$\begin{aligned} \mathcal{F}_{g_0} &\leq \epsilon \\ Vol_{g_0}(B_{g_0}(x,r)) &\geq \delta\omega_4 r^4 \qquad \forall x \in M, r \in [0,1] \end{aligned} \tag{2.42}$$

then the flow given in (1.3) with initial metric g_0 has a solution on $[0,1]$ and we have the following estimates:

$$\begin{aligned} \left\|Rm_{g(t)}\right\|_{L^\infty(M,g(t))} &\leq A\mathcal{F}_{g(t)}^{\frac{1}{6}} t^{-\frac{1}{2}} \\ inj_{g(t)}(M) &\geq \iota t^{\frac{1}{4}} \\ diam_{g(t)}(M) &\leq 2(1 + diam_{g(0)}(M)) \end{aligned}$$

for all $t \in (0,1]$.

From these estimates we may conclude the following precompactness result, at first

Corollary 2.14. *Given $D, \delta > 0$. Then there exists $\epsilon(\delta) > 0$ so that the space $\mathcal{M}^4(D,\delta,\epsilon(\delta))$ which consists of the set of all closed Riemannian 4-manifolds (M,g) satisfying*

$$\begin{aligned} diam_g(M) &\leq D \\ Vol_g(B_g(x,r)) &\geq \delta\omega_4 r^4 \qquad \forall x \in M, r \in [0,1] \\ \|Rm_g\|_{L^2(M,g)} &\leq \epsilon^2 \end{aligned}$$

equipped with the Gromov-Hausdorff topology, is precompact.

Proof. Let (M,g) be an element in $\mathcal{M}^4(D,\delta,\epsilon(\delta))$. Using Theorem 2.13 we know that the L^2-flow with initial metric g exists on the time interval $[0,1]$. Together with (A.8) we ensure that the following estimates are valid

$$\left\|Rm_{g(t)}\right\|_{L^\infty(M,g(t))} \leq A\mathcal{F}_{g(t)}^{\frac{1}{6}} t^{-\frac{1}{2}} \overset{Lemma A.3}{\leq} A\mathcal{F}_{g(0)}^{\frac{1}{6}} t^{-\frac{1}{2}} \leq t^{-\frac{1}{2}}$$

$$diam_{g(t)}(M) \leq 2(1+D)$$

Hence, from the Bishop-Gromov comparison principle (cf. [26, Lemma 36. p. 269]) we infer

$$Vol_{g(0)}(M) \overset{(A.12)}{=} Vol_{g(1)}(M) = Vol_{g(1)} B_{g(1)}(x, 2(1+D)) \leq V_0(D) \tag{2.43}$$

Now, let $\{x_1, ..., x_{N(M,g)}\} \subseteq M$ be a maximal r-separated set (cf. Definition C.4), which implies that $\{x_1, ..., x_N\}$ is an r-net (cf. Definition C.3). In this situation the balls

$$B_g(x_1, \frac{r}{2}), ..., B_g(x_N, \frac{r}{2})$$

are mutually disjoint and the balls $B_g(x_1, r), ..., B_g(x_N, r)$ cover M. Using the non-collapsing assumption (cf. (2.42)), we infer

$$\begin{aligned} N\omega_4\delta\left(\frac{r}{2}\right)^n \leq & \sum_{k=1}^{N} Vol_g(B_g(x_k, \frac{r}{2})) \\ =& Vol_g(\bigcup_{k=1}^{N} B_g(x_k, \frac{\epsilon}{2})) \leq Vol_g(M) \overset{(2.43)}{\leq} V_0(D) \end{aligned}$$

This implies that the number of elements in such an r-net is bounded from above by a natural number $N(r, \delta, D)$. The assertion follows from [7, Theorem 7.4.15, p. 264]. $\square$

Proof of Theorem 1.1. As in the proof of Corollary 2.14, we know that for each $i \in \mathbb{N}$ the L^2-flow with initial metric g_i exists on $[0,1]$ and that this flow satisfies the following estimates

$$\begin{aligned} \left\|Rm_{g_i(t)}\right\|_{L^\infty(M,g_i(t))} &\leq A\mathcal{F}^{\frac{1}{6}}_{g_i(t)} t^{-\frac{1}{2}} \overset{Lemma A.3}{\leq} A\left(\frac{1}{i}\right)^{\frac{1}{6}} t^{-\frac{1}{2}} \leq t^{-\frac{1}{2}} \\ inj_{g_i(t)}(M) &\geq \iota t^{\frac{1}{4}} \\ diam_{g_i(t)}(M) &\leq 2(1+D) \end{aligned} \tag{2.44}$$

for all $t \in (0,1]$. Using Corollary 2.11, we may choose a monotone decreasing sequence $(t_j)_{j\in\mathbb{N}} \subseteq (0,1]$ that converges to zero and that satisfies

$$d_{GH}((M_i, g_i), (M_i, g_i(t_j))) < \frac{1}{3j} \qquad \forall i, j \in \mathbb{N}$$

Estimate (A.10) implies, that for each $m \in \mathbb{N}$

$$\left\|\nabla^m Rm_{g_i(t_j)}\right\|_{L^\infty(M_i, g_i(t_j))} \leq C(m) t_j^{-\frac{2+m}{4}} \qquad \forall i, j \in \mathbb{N} \tag{2.45}$$

As in the proof of Corollary 2.14 we also have

$$v_0(D,\delta) \le Vol_{g_i(t_j)}(M_i) = Vol_{g_i(1)}(M_i) \le V_0(D)$$

where we have used the non-collapsing assumption in order to prove the lower bound. Hence, at each time t_j, we are able to apply Theorem A.11 to the sequence of manifolds $(M_i, g_i(t_j))_{i\in\mathbb{N}}$, i.e.: for all $j \in \mathbb{N}$ there exists a subsequence $(M_{i(j,k)}, g_{i(j,k)}(t_j))_{k\in\mathbb{N}}$ converging in the $C^{m,\alpha}$-sense, where $m \in \mathbb{N}$ is arbitrary, to a smooth manifold (N_j, h_j) as k tends to infinity. We may assume that the selection process is organized so that each sequence $(M_{i(j,k)}, g_{i(j,k)}(t_j))_{k\in\mathbb{N}}$ is a subsequence of $(M_{i(j-1,k)}, g_{i(j-1,k)}(t_j))_{k\in\mathbb{N}}$. The smooth convergence together with (2.44) implies $Rm_{h_j} \equiv 0$ for each $j \in \mathbb{N}$.

In order to apply Theorem A.11 to the sequence $(N_j, h_j)_{j\in\mathbb{N}}$, we need an argument for a uniform lower bound on the injectivity radius because the injectivity radius estimate in (2.44) is not convenient. To overcome this issue, we recall that the volume of balls does not decay to quickly along the flow (cf. Lemma A.5) and the convergence is smooth. So, the volume of suitable balls is well-controlled from below. Since (N_j, h_j) is flat, we are able to apply [9, Theorem 4.7, pp. 47-48], which yields a uniform lower bound on the injectivity radius for each (N_j, h_j). Hence, there exists a subsequence of $(N_j, h_j)_{j\in\mathbb{N}}$ that converges in the C^∞-sense, to a flat manifold (M, g). Finally we need to get sure that $(M_i, g_i)_{i\in\mathbb{N}}$ contains a subsequence that also converges to (M, g), at least in the Gromov-Hausdorff sense. For each $m \in \mathbb{N}$, we choose $j(m) \ge m$ so that

$$d_{GH}((M,g),(N_{j(m)},h_{j(m)})) \le \frac{1}{3m}$$

and $k(m) \in \mathbb{N}$ so that

$$d_{GH}((N_{j(m)},h_{j(m)}),(M_{i(j(m),k(m))},g_{i(j(m),k(m))}(t_{j(m)})) \le \frac{1}{3m}$$

This implies

$$\begin{aligned}
&d_{GH}((M,g),(M_{i(j(m),k(m))},g_{i(j(m),k(m))})\\
\le &d_{GH}((M,g),(N_{j(m)},h_{j(m)}))\\
&+ d_{GH}((N_{j(m)},h_{j(m)}),(M_{i(j(m),k(m))},g_{i(j(m),k(m))}(t_{j(m)}))\\
&+ d_{GH}((M_{i(j(m),k(m))},g_{i(j(m),k(m))}(t_{j(m)}),(M_{i(j(m),k(m))},g_{i(j(m),k(m))})))
\end{aligned}$$

$$\leq \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3j(m)} \leq \frac{1}{3m} + \frac{1}{3m} + \frac{1}{3m} = \frac{1}{m}$$

and this implies, that the sequence $(M_{i(j(m),k(m))}, g_{i(j(m),k(m))})_{m\in\mathbb{N}}$ converges with respect to the Gromov-Hausdorff topology to (M, g) as m tends to infinity.

□

2.3 Proof of Theorem 1.2

In order to apply Theorem 1.3 to the situation in Theorem 1.2 we give a proof of the following existence result

Theorem 2.15. *Let $D, \Lambda > 0$. Then there are universal constants $\delta \in (0,1)$, $K > 0$ and constants $\epsilon(\Lambda), T(\Lambda) > 0$ satisfying the following property: Let (M, g) be a closed Riemannian 4-manifold satisfying*

$$\begin{aligned} diam_g(M) &\leq D \\ \|Rm_g\|_{L^2(M,g)} &\leq \Lambda \\ Vol_g(B_g(x,r)) &\geq \delta\omega_n r^n \qquad \forall x \in M,\ r \in [0,1] \\ \|\mathring{Rc}_g\|_{L^2(M,g)} &\leq \epsilon \end{aligned}$$

then the L^2-flow exists on $[0,T]$, and we have the following estimates:

$$\begin{aligned} \|Rm_{g(t)}\|_{L^\infty(M,g(t))} &\leq Kt^{-\frac{1}{2}} \\ inj_{g(t)}(M) &\geq t^{\frac{1}{4}} \end{aligned} \tag{2.46}$$

and

$$diam_{g(t)}(M) \leq 2(1+D) \tag{2.47}$$

for all $t \in (0,T]$.

We point out that J. Streets has proved this result as a part of the proof of [35, Theorem 1.21] (cf. [35, pp. 285-287]). For sake of completeness, we also want to give a proof here, under the viewpoint of the dependence of ϵ and T on given parameters and that (2.47) is also satisfied.

Proof. We follow the lines of [35, pp. 285-286], giving further details. At first, we allow $\delta \in (0,1)$ and $K > 0$ to be arbitrary but fixed. Along the proof, we concretize these constants. We argue by contradiction.

Suppose, there is a sequence of closed Riemannian 4-manifolds $(M_i, g_i)_{i\in\mathbb{N}}$ so that for all $i \in \mathbb{N}$ we have the following estimates:

$$\begin{aligned}&\int_{M_i} |Rm_{g_i}|^2_{g_i}\, dV_{g_i} \leq \Lambda \\ &Vol_{g_i}(B_{g_i}(x,r)) \geq \delta\omega_n r^n \qquad \forall r \in [0,1]\end{aligned}$$

and

$$\int_{M_i} |\mathring{R}c_{g_i}|^2_{g_i}\, dV_{g_i} \leq \frac{1}{i}$$

but the estimates (2.46) hold on a maximal interval $[0, T_i]$ where $\lim_{i\to\infty} T_i = 0$. We consider the following sequence of rescaled metrics:

$$\overline{g}_i(t) := T_i^{-\frac{1}{2}} g_i(T_i t)$$

Then, for each $i \in \mathbb{N}$ the solution of the L^2-flow exists on $[0,1]$ and satisfies:

$$\begin{aligned}\|Rm_{\overline{g}_i(t)}\|_{L^\infty(M,\overline{g_i}(t))} &= T_i^{\frac{1}{2}} \|Rm_{g_i(T_i t)}\|_{L^\infty(M,g_i(T_i t))} \leq T_i^{\frac{1}{2}} K (T_i t)^{-\frac{1}{2}} = K t^{-\frac{1}{2}} \\ inj_{\overline{g_i}(t)}(M_i) &= T_i^{-\frac{1}{4}} inj_{g_i(T_i t)} \geq T_i^{-\frac{1}{4}} (T_i t)^{\frac{1}{4}} = t^{\frac{1}{4}}\end{aligned} \tag{2.48}$$

on $[0,1]$, which means that the estimates (2.46) are formally preserved under this kind of rescaling.

By assumption, for each $i \in \mathbb{N}$, one of the inequalities in (2.48) is an equality at time $t = 1$. In respect of the generalized Gauss-Bonnet Theorem (cf. [30, Appendix A]), i.e.:

$$\begin{aligned}\int_M |Rm|^2\, dV_g &= c_0\pi^2\chi(M) + 4\int_M |Rc|^2\, dV_g - \int_M R^2\, dV_g \\ &= c_0\pi^2\chi(M) + 4\int_M |\mathring{R}c|^2\, dV_g\end{aligned} \tag{2.49}$$

where we have used

$$\begin{aligned}|\mathring{R}c|^2 &= \left|Rc - \frac{1}{4}Rg\right|^2 = |Rc|^2 - \frac{1}{2}\langle Rc, Rg\rangle + \frac{1}{16}R^2|g|^2 \\ &= |Rc|^2 - \frac{1}{2}R\, tr(Rc) + \frac{1}{4}R^2 = |Rc|^2 - \frac{1}{2}R^2 + \frac{1}{4}R^2 \\ &= |Rc|^2 - \frac{1}{4}R^2\end{aligned}$$

we introduce the following functional

$$\mathcal{G}_g := \int_M |\overset{\circ}{Rc}_g|_g^2 \, dV_g$$

From (2.49) and [5, 4.10 Definition, p. 119] we infer

$$\text{grad } \mathcal{F} \equiv 4 \text{ grad } \mathcal{G}$$

As in the proof of Lemma A.3 we obtain for each $i \in \mathbb{N}$ and $t \in [0, T_i]$

$$\mathcal{G}_{g_i(0)} - \mathcal{G}_{g_i(t)} = \int_0^t \int_{M_i} |\text{grad } \mathcal{G}_{g_i(s)}|^2_{g_i(s)} \, dV_{g_i(s)} \, ds \geq 0$$

which implies $\mathcal{G}_{g_i(t)} \leq \frac{1}{i}$ for each $i \in \mathbb{N}$ and $t \in [0, T_i]$. Due to the scale invariance of the functional $\mathcal{G}$, we have in particular

$$\mathcal{G}_{\overline{g}_i(1)} \leq \frac{1}{i} \quad \text{for all } i \in \mathbb{N}$$

As already stated, (2.48) implies

$$\|Rm_{\overline{g}_i(1)}\|_{L^\infty(M, \overline{g}_i(1))} = K \quad \text{or} \quad inj_{\overline{g}_i}(M_i) = 1$$

for each $i \in \mathbb{N}$.

At first, we assume that there is a subsequence $(M_i, \overline{g}_i)_{i \in \mathbb{N}}$ (we do not change the index) satisfying

$$\begin{cases} \|Rm_{\overline{g}_i(1)}\|_{L^\infty(M_i, \overline{g}_i(1))} = K \\ inj_{\overline{g}_i(1)}(M_i) \geq 1 \end{cases}$$

for each $i \in \mathbb{N}$. Using the compactness, for each $j \in \mathbb{N}$ we may choose a point $p_i \in M_i$ satisfying $|Rm_{\overline{g}_i(1)}(p_i)|_{\overline{g}_i(1)} = K$. From [34, Corollary 1.5, p. 42] we conclude that there exists a subsequence of manifolds, also index by i, and a complete pointed 4-manifold (M_∞, p_∞) together with a 1-parametrized family of Riemannian metrics $(g_\infty(t))_{t \in [1/2, 1]}$ on M_∞ such that for each $t \in [1/2, 1]$

$$(M_i, \overline{g}_i(t), p_i) \overset{i \to \infty}{\longrightarrow} (M_\infty, g_\infty(t), p_\infty)$$

in the sense of C^∞-local submersions (cf. Definition C.13), and

$$\|Rm_{g_\infty(1)}\|_{L^\infty(M_\infty, g_\infty(1))} = |Rm_{g_\infty(1)}(p_\infty)|_{g_\infty(1)} = K$$

as well as, using [28, Theorem]

$$inj_{g_\infty(1)}(M_\infty) \geq 1$$

Since $\lim_{i\to\infty} \mathcal{G}_{\overline{g}_i(1)} = 0$ we conclude that $(M_\infty, g_\infty(1), p_\infty)$ needs to be an Einstein manifold satisfying

$$\int_{M_\infty} |Rm_{g_\infty(1)}|^2_{g_\infty(1)}\, dV_{g_\infty(1)} \leq \Lambda \tag{2.50}$$

In particular, [23, Proposition 7.8, p. 125] implies that the scalar curvature is constant. On the other hand, from the non-collapsing condition and (A.12) we obtain that $Vol_{\overline{g}_i(1)}(M_i)$ tends to infinity as $i \in \mathbb{N}$ tends to infinity. Then, estimate (2.50) implies that the scalar curvature needs to vanish on $(M_\infty, \overline{g}_\infty(1))$, hence $(M_\infty, \overline{g}_\infty(1))$ is a Ricci-flat manifold. From Lemma A.12 we obtain

$$\|Rm_{\overline{g}_\infty(1)}\|_{L^\infty(M_\infty, \overline{g}_\infty(1))} \leq C$$

where C is a universal constant, since the space dimension is fixed and the injectivity radius is bounded from below by 1. Choosing $K = C + 1$ we obtain a contradiction to $|Rm_{\overline{g}_\infty(1)}(p_\infty)|_{\overline{g}_\infty(1)} = K$. This finishes the part of the proof that $\|Rm_{g_i(T_i)}\|_{L^\infty(M, g_i(T_i))} = KT_i^{-\frac{1}{2}}$ can only be valid for a finite number of $i \in \mathbb{N}$.

Now we assume that, after taking a subsequence, we are in the following situation

$$\begin{cases} \|Rm_{\overline{g}_i(1)}\|_{L^\infty(M_i, \overline{g}_i(1))} \leq K \\ inj_{\overline{g}_i(1)}(M_i) = 1 \end{cases}$$

Then, the non-collapsing assumption of the initial sequence implies the following non-collapsing condition concerning the rescaled metrics

$$Vol_{\overline{g}_i(0)}(B_{\overline{g}_i(0)}(x, r)) \geq \delta\omega_n r^n \qquad \forall x \in M_i, r \in [0, T_i^{-\frac{1}{4}}]$$

Hence, for each $\sigma \geq 1$ there exists $i_0(\sigma) \in \mathbb{N}$ so that

$$Vol_{\overline{g}_i(0)}(B_{\overline{g}_i(0)}(x, r)) \geq \delta\omega_n r^n \qquad \forall x \in M_i, r \in (0, \sigma] \tag{2.51}$$

for all $i \geq i_0(\sigma)$. Now let $\lambda \in (0, 1)$ be fixed. This constant will be made explicit below. Using (A.13) we obtain for $i \geq i_0(\sigma, \lambda, \delta)$

$$\begin{aligned} & \left[Vol_{\overline{g}_i(1)}(B_{\overline{g}_i(0)}(x, \lambda\sigma))\right]^{\frac{1}{2}} \geq \left[Vol_{\overline{g}_i(0)}(B_{\overline{g}_i(0)}(x, \lambda\sigma))\right]^{\frac{1}{2}} - C\left(\frac{1}{i}\right)^{\frac{1}{2}} \\ \overset{(2.51)}{\geq} & \left[\delta\omega_4(\lambda\sigma)^4\right]^{\frac{1}{2}} - C\left(\frac{1}{i}\right)^{\frac{1}{2}} = \left[(1-(1-\delta))\omega_4\lambda^4\sigma^4\right]^{\frac{1}{2}} - C\left(\frac{1}{i}\right)^{\frac{1}{2}} \\ \geq & \left[(1-2(1-\delta))\omega_4\lambda^4\sigma^4\right]^{\frac{1}{2}} \end{aligned} \tag{2.52}$$

where the last estimate does not use that i_0 depends on σ, because, in order to choose $i_0 \in \mathbb{N}$ large enough one may fix $\sigma = 1$ at first. Afterwards, one may multiply the inequality by σ^2. Since $\sigma \geq 1$, the desired estimate follows.

It is our intention to prove that

$$B_{\overline{g}_i(0)}(x, \lambda\sigma) \subseteq B_{\overline{g}_i(1)}(x, \sigma) \qquad \forall i \geq i_0(\sigma, \lambda, \delta),\ \forall x \in M_i \tag{2.53}$$

Before proving this, we demonstrate that this fact implies a contradiction.

For each $i \in \mathbb{N}$ we choose a point $p_i \in M_i$ satisfying

$$inj_{\overline{g}_i(1)}(M_i, p_i) = inj_{\overline{g}_i(1)}(M_i) = 1$$

As above, using [34, Corollary 1.5, p. 42], we may assume that there exists a subsequence of manifolds, again indexed by i, and a complete pointed 4-manifold (M_∞, p_∞) as well as a 1-parametrized family of metrics $(g_\infty(t))_{t\in[1/2,1]}$ on M_∞ so that for each $t \in [1/2, 1]$

$$(M_i, \overline{g}_i(t), p_i) \overset{i\to\infty}{\longrightarrow} (M_\infty, g_\infty(t), p_\infty)$$

in the sense of C^∞-local submersions. Using [28, Theorem] we infer

$$inj_{g_\infty(1)}(M_\infty, p_\infty) = 1 \tag{2.54}$$

Let $\zeta > 0$ be equal to the non-collapsing parameter in [2, Gap Lemma 3.1, p. 440] which is denoted by "ϵ" in that work and only depends on the space dimension $n = 4$. We assume $\delta \in (0, 1)$ and $\lambda \in (0, 1)$ to be close enough to 1 so that

$$(1 - 2(1 - \delta))\lambda^4 \geq 1 - \zeta \tag{2.55}$$

Assumed (2.53) is valid, then for each for $i \geq i_0(\sigma, \lambda, \delta)$ we obtain the following estimate

$$Vol_{\overline{g}_i(1)}(B_{\overline{g}_i(1)}(p_i, \sigma)) \overset{(2.53)}{\geq} Vol_{\overline{g}_i(1)}(B_{\overline{g}_i(0)}(p_i, \lambda\sigma)) \overset{(2.52)/(2.55)}{\geq} (1 - \zeta)\omega_4\sigma^4$$

and finally, as $i \in \mathbb{N}$ tends to infinity

$$Vol_{g_\infty(1)}(B_{g_\infty(1)}(p_\infty, \sigma)) \geq (1-\zeta)\omega_4\sigma^4 \quad \forall \sigma \geq 1$$

Then [2, Gap Lemma 3.1, p. 440] implies that $(M_\infty, g_\infty(1))$ is isometric to $(\mathbb{R}^4, g_{can})$ which contradicts (2.54).

Hence, in order to prove the existence result and the validity of (2.46), it remains to prove (2.53). From here on we do not write the subindex $i \in \mathbb{N}$. The following considerations shall be understood with $i \in \mathbb{N}$ fixed. That means that p is one of the points p_i and $\overline{g}(t)$ is the metric $\overline{g}_i(t)$ on $M = M_i$ with the same index. Let

$$y \in B_{\overline{g}(0)}(p, \lambda\sigma) \tag{2.56}$$

be an arbitrary point. As in the proof of Lemma 2.4 we construct a suitable forward-geodesic: Let

$$\beta := \min_{t \in [0,1]} \beta_t > 0$$

where

$$\beta_t := \beta(4, diam_{\overline{g}(t)}(M), f_3(M, \overline{g}(t)), inj_{\overline{g}(t)}(M))$$

is chosen according to Lemma 2.3. Next, using Lemma 2.6, we construct a β-forward-geodesic connecting p and y which is denoted by $(\xi_t)_{t\in[0,1]}$. Hence, we have a finite set of geodesics $(\xi_{jS})_{j\in\{0,\ldots,\lfloor\frac{1}{S}\rfloor\}}$ which are parametrized proportional to arc length, i.e.:

$$|\dot{\xi}_{jS}|_{g(jS)} \equiv d(p, y, jS) \text{ for all } j \in \{0, \ldots, \lfloor\frac{1}{S}\rfloor\}$$

Furthermore, for each $j \in \{0, \ldots, \lfloor\frac{1}{S}\rfloor\}$ let

$$\varphi_j : [0, d(p, y, jS)] \longrightarrow [0, 1]$$
$$\varphi(s) = \frac{s}{d(p, y, jS)}$$

and let

$$\gamma_t := \xi_{jS} \circ \varphi_{jS} \text{ for each } t \in [jS, (j+1)S) \cap [0, 1]$$

Applying the same argumentation as in the proof of Lemma 2.4 we ensure that for each $j \in \{0, \ldots, \lfloor\frac{1}{S}\rfloor\}$ and $t \in [jS, (j+1)S) \cap [0, 1]$ the tubular neighborhood $D(\gamma_t, \rho_t)$ is foliated by $(D(\gamma_t(s), \rho_t))_{s\in[0,d(p,y,jS)]}$ where

$$\rho_t := \mu \min\left\{inj_{\overline{g}(t)}(M), f_3(M,\overline{g}(t))^{-\frac{1}{2}}\right\}$$

and the differential of the projection map satisfies (2.4). Here $\mu > 0$ is chosen fixed but also compatible to Lemma A.9. We want to give a controlled lower bound on ρ_t. The curvature decay estimate from (2.48) together with (A.10) implies for each $m \in \{1,2,3\}$:

$$\left\|\nabla^m Rm_{g(t)}\right\|_{L^\infty(M,g(t))} \leq C(m)t^{-\frac{2+m}{4}} \text{ for all } t \in (0,1] \tag{2.57}$$

From this, we infer

$$f_3(M,g(t)) \leq Ct^{-\frac{1}{2}} \text{ on } (0,1]$$

Combining this estimate with the injectivity radius estimate from (2.48), we obtain, as in the proof of Lemma 2.4

$$\rho_t \geq \mu\left\{t^{\frac{1}{4}}, C^{-\frac{1}{2}}t^{\frac{1}{4}}\right\} \geq \mu\min\{1, C^{-\frac{1}{2}}\}t^{\frac{7}{24}} =: Rt^{\frac{7}{24}} =: r_t$$

we also obtain the estimate

$$\begin{aligned}\frac{d}{dt}L(\gamma_t,t) \leq & C_2R^{-\frac{3}{2}}t^{-\frac{7}{16}}\left(\int_M |\text{grad } \mathcal{F}_{\overline{g}(t)}|^2\, dV_{\overline{g}(t)}\right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{jS},t) \\ & + C_2Rt^{-\frac{23}{24}}L(\gamma_{jS},t)\end{aligned} \tag{2.58}$$

on $(jS,(j+1)S) \cap [0,1)$ where $j \in \{1,...,\lfloor\frac{1}{S}\rfloor\}$. Now we assume that

$$\begin{aligned}&j_0 := \\ &\min\left\{j \in \{1,...,\lfloor\frac{1}{S}\rfloor\} \,|\, \exists t \in [jS,(j+1)S) \cap (0,1] \text{ s. th. } L(\gamma_t,t) = \sigma\right\}\end{aligned}$$

exists, and let

$$t_0 := \sup\left\{t \in [j_0S,(j_0+1)S) \cap (0,1] \,|\, L(\gamma_\tau,\tau) \leq \sigma\ \forall \tau \in [j_0S,t]\right\}$$

Then, for each $j \in \{0,...,j_0\}$ and $t \in (jS,(j+1)S) \cap (0,t_0)$ estimate (2.58) implies

$$\frac{d}{dt}L(\gamma_t,t) \leq \sigma\left[C_2R^{-\frac{3}{2}}t^{-\frac{7}{16}}\left(\int_M |\text{grad } \mathcal{F}_{\overline{g}(t)}|^2\, dV_{\overline{g}(t)}\right)^{\frac{1}{2}} + C_2Rt^{-\frac{23}{24}}\right]$$

and consequently

$$d(p,y,t) - d(p,y,jS)$$

$$
\begin{aligned}
&\leq L(\gamma_t, t) - L(\gamma_{jS}, jS) \\
&\leq \sigma C_2 R^{-\frac{3}{2}} \int_{jS}^{t} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \right)^{\frac{1}{2}} ds + \sigma C_2 R \int_{jS}^{t} s^{-\frac{23}{24}} \, ds
\end{aligned}
$$

In particular, for each $j \in \{0, ..., j_0 - 1\}$ we infer

$$
\begin{aligned}
&d(p, y, (j+1)S) - d(p, y, jS) \\
&\leq \sigma C_2 R^{-\frac{3}{2}} \int_{jS}^{(j+1)S} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \right)^{\frac{1}{2}} ds + \sigma C_2 R \int_{jS}^{(j+1)S} s^{-\frac{23}{24}} \, ds
\end{aligned}
$$

and

$$
\begin{aligned}
&L(\gamma_{t_0}, t_0) - d(p, y, j_0 S) \\
&\leq L(\gamma_{t_0}, t_0) - L(\gamma_{j_0 S}, j_0 S) \\
&\leq \sigma C_2 R^{-\frac{3}{2}} \int_{j_0 S}^{t_0} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \right)^{\frac{1}{2}} ds + \sigma C_2 R \int_{j_0 S}^{t_0} s^{-\frac{23}{24}} \, ds
\end{aligned}
$$

and finally

$$
\begin{aligned}
&L(\gamma_{t_0}, t_0) - d(p, y, 0) \\
&\leq L(\gamma_{t_0}, t_0) - d(p, y, j_0 S) + \sum_{j=0}^{j_0 - 1} [d(p, y, (j+1)S) - d(p, y, jS)] \\
&\leq \sigma \left[C_2 R^{-\frac{3}{2}} \int_0^{t_0} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \right)^{\frac{1}{2}} ds + C_2 R \int_0^{t_0} s^{-\frac{23}{24}} \, ds \right] \\
&\leq \sigma \left[C_2 R^{-\frac{3}{2}} \int_0^{1} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \right)^{\frac{1}{2}} ds + C_2 R \int_0^{1} s^{-\frac{23}{24}} \, ds \right] \\
&\leq \sigma C_2 R^{-\frac{3}{2}} \left(\int_0^1 s^{-\frac{7}{8}} \, ds \right)^{\frac{1}{2}} \left(\int_0^1 \int_M |\text{grad } \mathcal{F}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \, ds \right)^{\frac{1}{2}} \\
&\qquad + \sigma C_2 R \int_0^1 s^{-\frac{23}{24}} \, ds \\
&\leq \sigma C_3 R^{-\frac{3}{2}} \left(\int_0^1 s^{-\frac{7}{8}} \, ds \right)^{\frac{1}{2}} \left(\int_0^1 \int_M |\text{grad } \mathcal{G}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \, ds \right)^{\frac{1}{2}} \\
&\qquad + \sigma C_2 R \int_0^1 s^{-\frac{23}{24}} \, ds \\
&\leq \sigma \left[C_4 R^{-\frac{3}{2}} \left(\int_0^1 \int_M |\text{grad } \mathcal{G}_{\overline{g}(s)}|^2 \, dV_{\overline{g}(s)} \, ds \right)^{\frac{1}{2}} + C_4 R \right] \\
&\leq \sigma C_4 R^{-\frac{3}{2}} \mathcal{G}_{\overline{g}(0)}^{\frac{1}{2}} + \sigma C_4 R
\end{aligned}
$$

$$=\sigma\left[C_4R^{-\frac{3}{2}}\mathcal{G}_{g(0)}^{\frac{1}{2}}+C_4R\right]$$

Together with (2.56) we obtain

$$L(\gamma_{t_0},t_0)<\sigma\left[\lambda+C_4R^{-\frac{3}{2}}\mathcal{G}_{g(0)}^{\frac{1}{2}}+C_4R\right]$$

Throughout, we may assume that $R>0$ is small enough compared to $C_4>0$ and $\lambda>0$ in order to ensure that

$$C_4R\leq\frac{1-\lambda}{2}$$

and we may assume that $i\in\mathbb{N}$ is chosen large enough, so that $\mathcal{G}_{g(0)}=\mathcal{G}_{g_i}\leq\frac{1}{i}$ is small enough compared to $\lambda>0$, $R(\lambda)>0$ and $C_4>0$ so that

$$C_4R^{-\frac{3}{2}}\mathcal{G}_{g(0)}^{\frac{1}{2}}\leq\frac{1-\lambda}{2}$$

Hence, we have $L(\gamma_{t_0},t_0)<\sigma$, which contradicts $L(\gamma_{t_0},t_0)=\sigma$. This implies that $L(\gamma_t,t)<\sigma$ is valid for each $t\in[0,1]$ and consequently $d(p,y,1)<\sigma$. This finishes the proof of (2.53).

We have proved the existence time estimate as well as the curvature decay estimate and the injectivity radius growth estimate. It remains to show the diameter estimate (2.47). The argumentation is based on [35, p. 281] but we are in a different situation. Let $x,y\in M$ so that $d(x,y,1)=diam_{g(1)}(M)$. As above, there exists $\beta>0$, $S>0$ and a family of curves $(\gamma_t)_{t\in[0,T]}$ so that

- for each $j\in\left\{0,...,\left\lfloor\frac{T}{S}\right\rfloor\right\}$

  $$\gamma_{jS}:[0,d(x,y,jS)]\longrightarrow M$$

  is a unit-speed length minimizing geodesic

- for each $j\in\left\{0,...,\left\lfloor\frac{T}{S}\right\rfloor\right\}$ and $t\in[jS,(j+1)S)\cap[0,T]$ the curve

  $$\gamma_t:[0,d(x,y,jS)]\longrightarrow M$$

  satisfies

  $$L(\gamma_t,t)\leq d(x,y,t)+\beta$$

- for each $j \in \left\{0, ..., \lfloor \frac{T}{S} \rfloor\right\}$ and $t \in [jS, (j+1)S) \cap [0, T]$ the tubular neighborhood $D(\gamma_t, r_t)$ is foliated by $(D(\gamma_t(s), r_t)_{s \in [0, d(x,y,jS)]}$ where

$$r_t := Rt^{\frac{7}{24}} := \mu \min\{1, C^{-\frac{1}{2}}\} t^{\frac{7}{24}}$$

Furthermore, the projection map π satisfies (2.4), i.e.

$$|d\pi| \leq 2 \text{ for all } x \in D(\gamma, r_t)$$

Using these conditions we obtain (2.58), i.e.:

$$\begin{aligned} \frac{d}{dt} L(\gamma_t, t) \leq & C_2 R^{-\frac{3}{2}} t^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{g(t)}|^2 \, dV_{\bar{g}(t)} \right)^{\frac{1}{2}} L^{\frac{1}{2}}(\gamma_{jS}, t) \\ & + C_2 R t^{-\frac{23}{24}} L(\gamma_{jS}, t) \end{aligned}$$

on $(jS, (j+1)S) \cap [0, T)$ where $j \in \{1, ..., \lfloor \frac{T}{S} \rfloor\}$. In this situation we assume that

$$\begin{aligned} & j_0 := \\ & \min \left\{ j \in \{1, ..., \lfloor \frac{1}{S} \rfloor\} \,|\, \exists t \in [jS, (j+1)S) \cap (0, T] \text{ s. th. } L(\gamma_t, t) = 2(1+D) \right\} \end{aligned}$$

exists, and we define

$$t_0 := \sup \{ t \in [j_0 S, (j_0+1)S) \cap (0, T] \,|\, L(\gamma_\tau, \tau) \leq 2(1+D) \; \forall \tau \in [j_0 S, t] \}$$

Thus, for each $j \in \{0, ..., j_0\}$ we obtain

$$\begin{aligned} \frac{d}{dt} L(\gamma_t, t) \leq & C_3 R^{-\frac{3}{2}} t^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{g(t)}|^2 \, dV_{g(t)} \right)^{\frac{1}{2}} (1+D)^{\frac{1}{2}} \\ & + C_3 R t^{-\frac{23}{24}} (1+D) \end{aligned}$$

on $(jS, (j+1)S) \cap (0, t_0)$. From this, we infer

$$\begin{aligned} & L(\gamma_{t_0}, t_0) - d(x, y, 0) \\ \leq & L(\gamma_{t_0}, t_0) - d(x, y, j_0 S) + \sum_{j=0}^{j_0 - 1} [d(x, y, (j+1)S) - d(x, y, jS)] \\ \leq & (1+D) C_3 \left[R^{-\frac{3}{2}} \int_0^{t_0} s^{-\frac{7}{16}} \left(\int_M |\text{grad } \mathcal{F}_{g(s)}|^2 \, dV_{g(s)} \right)^{\frac{1}{2}} ds + R \int_0^{t_0} s^{-\frac{23}{24}} \, ds \right] \end{aligned}$$

$$
\begin{aligned}
\leq &(1+D)C_3\left[R^{-\frac{3}{2}}\int_0^1 s^{-\frac{7}{16}}\left(\int_M |\text{grad } \mathcal{F}_{g(s)}|^2\, dV_{g(s)}\right)^{\frac{1}{2}} ds + R\int_0^{t_0} s^{-\frac{23}{24}}\, ds\right] \\
\leq &(1+D)C_3R^{-\frac{3}{2}}\left(\int_0^1 s^{-\frac{7}{8}}\, ds\right)^{\frac{1}{2}}\left(\int_0^1\int_M |\text{grad } \mathcal{F}_{g(s)}|^2\, dV_{g(s)}\, ds\right)^{\frac{1}{2}} \\
&\qquad + (1+D)C_3R\int_0^{t_0} s^{-\frac{23}{24}}\, ds \\
\leq &(1+D)C_4R^{-\frac{3}{2}}\left(\int_0^1 s^{-\frac{7}{8}}\, ds\right)^{\frac{1}{2}}\left(\int_0^1\int_M |\text{grad } \mathcal{G}_{g(s)}|^2\, dV_{g(s)}\, ds\right)^{\frac{1}{2}} \\
&\qquad + (1+D)C_4R\int_0^{t_0} s^{-\frac{23}{24}}\, ds \\
\leq &(1+D)C_4\left[R^{-\frac{3}{2}}\left(\int_0^1 s^{-\frac{7}{8}}\, ds\right)^{\frac{1}{2}}\mathcal{G}^{\frac{1}{2}}_{g(0)} + R\int_0^1 s^{-\frac{23}{24}}\, ds\right] \\
\leq &(1+D)C_5\left[R^{-\frac{3}{2}}\mathcal{G}^{\frac{1}{2}}_{g(0)} + R\right] < 1+D
\end{aligned}
$$

Here, we have assumed that $\mathcal{G}^{\frac{1}{2}}(g(0))$ and $R > 0$ are sufficiently small with respect to universal constants. Finally, we obtain

$$L(\gamma_{t_0}, t_0) < d(x,y,0) + 1 + D = D + 1 + D < 2(1+D)$$

contradicting $L(\gamma_{t_0}, t_0) = 2(1+D)$. This shows, that we have $diam_{g(t)}(M) \leq 2(1+D)$ for all $t \in [0,T]$. $\square$

This existence result allows us to prove the following diffeomorphism finiteness result:

Corollary 2.16. *Let $D, \Lambda > 0$. There exists $\epsilon(\Lambda) > 0$ and a universal constant $\delta \in (0,1)$ so that there are only finitely many diffeomorphism types of closed Riemannian 4-manifolds (M,g) satisfying*

$$
\begin{aligned}
diam_g(M) &\leq D \\
\|Rm_g\|_{L^2(M,g)} &\leq \Lambda \\
Vol_g(B_g(x,r)) &\geq \delta\omega_n r^n \qquad \forall x \in M,\ r \in [0,1] \\
\|\mathring{Rc}_g\|_{L^2(M,g)} &\leq \epsilon
\end{aligned}
$$

Proof. We assume that there exists a sequence of Riemannian 4-manifolds $(M_i, g_i)_{i\in\mathbb{N}}$ satisfying the desired properties but the elements in this sequence are pairwise not diffeomorphic. Using Theorem 2.15 we may smooth out each of these manifolds, then we may apply [1, Theorem 2.2, pp. 464-466] at a fixed later time point which yields a contradiction.

$\square$

Proof of Theorem 1.2. The proof is nearly analogous to the proof of Theorem 1.1 but the argumentation is slightly different. Throughout, using Corollary 2.16, we assume that $M_i = M$ for all $i \in \mathbb{N}$, applying Theorem 2.15, we may assume, that for each $i \in \mathbb{N}$ the L^2-flow on M with initial data g_i exists on $[0, T]$ and satisfies (1.4), (1.5), (1.6) and (1.7). Using Corollary 2.11, we choose a monotone decreasing sequence $(t_j)_{j\in\mathbb{N}} \subseteq (0, 1]$ converging to zero, so that

$$d_{GH}((M, g_i), (M, g_i(t_j))) < \frac{1}{3j} \qquad \forall i, j \in \mathbb{N}$$

(1.5) and (A.10) together imply

$$\left\|\nabla^m Rm_{g_i(t_j)}\right\|_{L^\infty(M,g_i(t_j))} \leq C(m) t_j^{-\frac{2+m}{4}} \qquad \forall i, j \in \mathbb{N} \tag{2.59}$$

for each $m \in \mathbb{N}$, (1.6) implies

$$inj_{g_i(t_j)}(M) \geq t_j^{\frac{1}{4}} \qquad \forall i, j \in \mathbb{N}$$

Applying the same argumentation as in the proof of Theorem 1.1 we infer

$$v_0(\delta) \leq Vol_{g_i(t_j)}(M) \leq V_0(D, \Lambda)$$

for all $i, j \in \mathbb{N}$. We want to point out that $\delta > 0$ only depends on the space dimension which is constant. Using the flow convergence result in Theorem A.6 on each time interval $[t_{j+1}, t_j]$, starting with t_0, we obtain a subsequence $(M_{i(j,k)}, g_{i(j,k)}(t_j))_{k\in\mathbb{N}}$ as well as a family of Riemannian manifolds $(M, g_{\infty,j}(t))_{t\in[t_{j+1},t_j]}$ so that for each $t \in [t_{j+1}, t_j]$ the sequence of Riemannian manifolds $(M, g_{i(j,k)}(t))_{k\in\mathbb{N}}$ converges smoothly to $(M, g_{\infty,j}(t))$ and $(M_{\infty,j}, g_{\infty,j}(t))_{t\in[t_{j+1},t_j]}$ is also a solution to the L^2-flow in the sense of Theorem A.6. Since $\mathcal{G}_{g_i(t)} \leq \mathcal{G}_{g_i} \leq \frac{1}{i}$ for all $i \in \mathbb{N}$, we conclude that $\mathcal{G}_{g_{\infty,j}(t)} = 0$ for all $t \in [t_{j+1}, t_j]$. Hence, at infinity, the metric does not change along the interval $[t_{j+1}, t_j]$, which means that $(M_{\infty,j}, g_{\infty,j}(t_j)) = (M_{\infty,j}, g_{\infty,j}(t_{j+1})) =: (M, g)$ is an Einstein manifold. Inductively, we obtain for each $j \in \mathbb{N}$ a sequence $(M_{i(j,k)}, g_{i(j,k)}(t_j))_{k\in\mathbb{N}}$ that is a subsequence from $(M_{i(j-1,k)}, g_{i(j-1,k)}(t_j))_{k\in\mathbb{N}}$, so that the sequence $(M_{i(j,k)}, g_{i(j,k)}(t_j))_{k\in\mathbb{N}}$ converges to the Einstein manifold

(M, g). Using the same diagonal choice as in the Proof of Theorem 1.1, we infer that there exists a subsequence of $(M_i, g_i)_{i\in\mathbb{N}}$ that also converges in the Gromov-Hausdorff topology to (M, g).

$\square$

Chapter 3

Convergence of a sequence of open Riemannian manifolds having almost vanishing $L^{\frac{n}{2}}$-norm of the Ricci curvature

In this chapter we consider a sequence of open Riemannian manifolds whose $L^{\frac{n}{2}}$-norm of the Ricci curvature converges to zero. Throughout we assume that n is greater than or equal to 3. The considered manifolds shall satisfy

(a) a non-collapsing condition on the volume of small balls

(b) a non-inflating condition on the volume of small balls

(c) a condition on the harmonic radius which consists of a uniform ellipticity condition on the metric, a uniform L^n-bound on the first derivative of the metric and a uniform bound on the modulus of the Hölder-continuity of the metric in a harmonic chart

Under these assumptions, we show that there exists a subsequence that converges with respect to the $W^{2,\frac{n}{2}}$-topology (cf. Definition C.7) to a smooth Ricci-flat manifold. As distinct from [6, Theorem 5] we do not assume any uniform bound on the L^p-norm of the Ricci curvature satisfying $p > \frac{n}{2}$.

In [6] the authors make use of the following result:

Theorem 3.1. *(cf. [6, Theorem 5]) Given* $n \in \mathbb{N}$, $p \in \left(\frac{n}{2}, \infty\right)$ *and* $0 \leq \sigma_1 \leq \sigma_2$, *there exists a constant* $\epsilon(n, p, \sigma_1, \sigma_2) > 0$ *such that the following holds. Let* $(M_i^n, g_i, p_i)_{i \in \mathbb{N}}$ *be a sequence of smooth complete pointed Riemannian manifolds without boundary such that* $B_{g_i}(p_i, 1) \subseteq M_i$ *satisfies the following properties for all* $i \in \mathbb{N}$:

$$\lim_{i \to \infty} \|Rc_{g_i}\|_{L^p(B_{g_i}(p_i,1),g_i)} = 0$$

$$\omega_n \sigma_1 \leq \frac{Vol_{g_i}(B_{g_i}(x,r))}{r^n} \leq \omega_n \sigma_2$$

for all $x \in B_{g_i}(p_i, 1)$, $r \in (0, 1]$ *such that* $B_{g_i}(x, r) \subseteq B_{g_i}(p_i, 1)$ *and*

$$\|Rm_{g_i}\|_{L^{\frac{n}{2}}(B_{g_i}(p_i,1),g_i)} \leq \epsilon(n, p, \sigma_1, \sigma_2)$$

Then, for all $s \in (0, 1)$, *the sequence* $(B_{g_i}(p_i, s), g_i, p_i)_{i \in \mathbb{N}}$ *subconverges in the pointed* $W^{2,p}$*-topology to a smooth pointed Ricci-flat Riemannian manifold* $(B_{g_\infty}(p_\infty, s), g_\infty, p_\infty)$, *i.e: for all* $s \in (0, 1)$, *after taking a subsequence, for each* $i \in \mathbb{N}$ *there exists a diffeomorphism* $F_i : B_{g_\infty}(p_\infty, s) \longrightarrow F_i(B_{g_\infty}(p_\infty, s)) \subseteq B_{g_i}(p_i, 1)$ *with* $F_i(p_\infty) = p_i$ *such that* $F_i^* g_i$ *converges to* g_∞ *with respect to the* $W^{2,p}(B_{g_\infty}(p_\infty, s))$*-topology, as* i *tends to infinity.*

The proof of this result uses the fact that L^p-bounds on the Ricci curvature ($p > \frac{n}{2}$) imply suitable estimates on the $W^{1,q(p)}$-harmonic radius ($q(p) > n$), provided that the $L^{\frac{n}{2}}$-norm of the full Riemannian curvature is sufficiently small on regions of interest, whereas the proof of these desired estimates can be adapted from [2, Section 2], (cf. [6, Section 2] and [29, Appendix B]). These approaches are based on suitable L^∞-bounds or L^p-bounds on the local integral of the Ricci curvature, where $p > \frac{n}{2}$. In general, the situation becomes more difficult when we consider scale invariant integral bounds on the Ricci curvature, which means a bound on the $L^{\frac{n}{2}}$-norm of the Ricci curvature in this case. In this context, we introduce the following notation of a harmonic radius, which separates the L^q-bound on the first derivative of the metric from the Hölder-bound of the metric:

Definition 3.2. *Let* (M^n, g) *be a complete Riemannian manifold,* $B_g(q, R_0) \subseteq M$ *be a reference ball and* $x \in B_g(q, R_0)$. *Given* $\alpha > 0$ *and* $K_1, K_2, K_3 \geq 0$ *then we define the harmonic radius* $r_g(x)$ *as the supremum over all* $r > 0$ *such that there exists a smooth chart* $\varphi : U \longrightarrow B(0, r)$, *where* $x \in U \subseteq B_g(q, R_0)$ *and* $\varphi(x) = 0$, *satisfying the following properties:*

(i)

$$(1+K_1)^{-1}|\xi|^2_{\mathbb{R}^n} \leq g_{ij}(x)\xi^i\xi^j \leq (1+K_1)|\xi|^2_{\mathbb{R}^n}$$

for all $x \in B(0,r)$ and $\xi = (\xi^1,...,\xi^n) \in \mathbb{R}^n$

(ii)

$$\|\partial g\|_{L^n(B(0,r))} := \max_{1\leq i\leq j\leq n} \max_{1\leq k\leq n} \|\partial_k g_{ij}\|_{L^n(B(0,r))} \leq K_2$$

(iii)

$$r^\alpha [g]_{C^\alpha(B(0,r))} := r^\alpha \max_{1\leq i\leq j\leq n} \sup_{x,y\in B(0,r),\, x\neq y} \frac{|g_{ij}(x)-g_{ij}(y)|}{|x-y|^\alpha} \leq K_3$$

(iv)

$$\Delta_g \varphi^m = 0$$

i.e.: $\varphi^m : U \longrightarrow \mathbb{R}$ is harmonic for each $m \in \{1,...,n\}$.

Here Δ_g is the Laplacian, introduced in [23, p. 44, 3-4.] and g_{ij} denotes the local representation of the metric g with respect to the chart φ.

Remark 3.3. *Of course the definition of $r_g(x)$ depends on R_0, α, K_1, K_2 and K_3, but for the sake of simplicity, we suppress this explicit dependence in the notation because these parameters are fixed.*

3.1 Proof of Theorem 1.4

Proof. The structure of the proof is inspired by [25, Theorem 2.2 (Fundamental Theorem of Convergence Theory), p. 173]. We also subdivide the proof in different "Facts" but these Facts are not identical to the Facts in [25, Theorem 2.2, p. 173]. Moreover, our assumptions are different from those assumptions in [25, Theorem 2.2, p. 173]. Fact 1 proves the precompactness with respect to the Gromov-Hausdorff distance. This part of the proof, together with Facts 2,3 and 4 are related to Fact 1-5 in [25, Theorem 2.2, p. 173]. Fact 4 in this proof is also closely related to the argumentation in [29, p. 58]. The crucial part of this proof is the proof of the regularity of the boundary space, which is contained in Fact 5. The argumentation is inspired by the interpolation argument in [19, pp. 18-19]. Our proof uses a part of the interpolation theory of Sobolev spaces from [38], [39], [40] and [41]. An overview of the used results is given in Appendix B. We point out that Fact 5 has a substantial effect on the rest of the proof, since the Riemannian metric on the limit manifold is smooth as in the indirect proof of [29, Theorem B.7, pp. 56-64]. The proof of Fact 6 and Fact 7 are elementary, the content is closely related to the C^∞-regularity discussion around [29, p. 60, (B.19)] and the flatness argument in [29, p. 63, (B.25)], but we want to emphasize, that we are in a different situation than in [29, Theorem B.7]. Finally, the construction of the desired diffeomorphisms (cf. Fact 8) coincides with the construction in the proof of [29, Theorem B.7]. This construction is explained in [29, pp. 60-62]. Since we are in a different situation, we give details here. Due to the fact that, in this chapter we are working mainly locally, we write $g(i)$ instead of g_i.

3.1.1 Gromov-Hausdorff precompactness

Fact 1. *There exists a metric space $(\Omega_\infty, d_\infty, p_\infty)$ so that, after taking a subsequence, the sequence of metric spaces $(B_{g(i)}(p_i, 3/4), d_{g(i)}, p_i)$ converges to $(\Omega_\infty, d_\infty, p_\infty)$ in the Gromov-Hausdorff sense.*

According to [7, Theorem 8.1.10., p. 274] it suffices to prove that for each $\epsilon \in (0, 1/10)$ there exists a number $N(\epsilon) \in \mathbb{N}$ such that the metric space $(B_{g(i)}(p_i, 3/4), d_{g(i)}, p_i)$ admits an ϵ-net of no more than $N(\epsilon)$ points (cf. Definition C.3).

Let $i \in \mathbb{N}$ be fixed, $\epsilon \in (0, 1/10)$ and let $\{x_1, ..., x_N\} \subseteq \overline{B}_{d_{g(i)}}(p_i, 3/4)$ be a maximal ϵ-separated set, which implies that $\{x_1, ..., x_N\}$ is an ϵ-net, then the balls

$$B_{d_{g(i)}}(x_1, \frac{\epsilon}{2}), ..., B_{d_{g(i)}}(x_N, \frac{\epsilon}{2})$$

are mutually disjoint. This yields

$$\begin{aligned} N \cdot \omega_n \sigma_1 \left(\frac{\epsilon}{2}\right)^n &\overset{(1.10)}{\leq} \sum_{k=1}^{N} Vol_{g(i)}(B_{d_{g(i)}}(x_k, \frac{\epsilon}{2})) \\ &= Vol_{g(i)}(\bigcup_{k=1}^{N} B_{d_{g(i)}}(x_k, \frac{\epsilon}{2})) \leq Vol_{g(i)}(B_{d_{g(i)}}(p_i, 1)) \overset{(1.10)}{\leq} \omega_n \sigma_2 \end{aligned}$$

Hence, the number of elements in such an ϵ-net is bounded from above by a natural number $N(\epsilon, n, \sigma_1, \sigma_2)$. In particular, this number does not depend on the index $i \in \mathbb{N}$ and we write $N(\epsilon)$ instead of $N(\epsilon, n, \sigma_1, \sigma_2)$ because the other parameters are fixed. Hence, using [7, Theorem 8.1.10., p. 274], after extracting a subsequence, the sequence $(B_{g(i)}(p_i, 3/4), d_{g(i)}, p_i)_{i\in\mathbb{N}}$ converges to a metric space $(\Omega_\infty, d_\infty, p_\infty)$ in the Gromov-Hausdorff sense.

As explained in [26, pp. 296-297] there exists a metric $d_{\mathbb{B}}$ on

$$\mathbb{B} := \Omega_\infty \amalg \coprod_{i=1}^{\infty} \overline{B}_{g(i)}(p_i, 3/4)$$

which is an extension, so that the sequence $\overline{B}_{g(i)}(p_i, 3/4)$ converges with respect to the Hausdorff topology, concerning $d_{\mathbb{B}}$, to Ω_∞.

3.1.2 Compactness of the ambient space $\mathbb{B}$

Fact 2. *The metric space* $(\mathbb{B}, d_{\mathbb{B}})$ *is compact.*

Let $(b_j)_{j\in\mathbb{N}} \subseteq \mathbb{B}$ an arbitrary sequence. If a subsequence of $(b_j)_{j\in\mathbb{N}}$ is contained in one of the spaces $\overline{B}_{g(i)}(p_i, 3/4)$ or contained in Ω_∞, then this subsequence needs to contain a converging subsequence because the considered spaces are compact. Thus, after possibly extracting a subsequence, we can assume that for each $j \in \mathbb{N}$ there exists a $i(j) \in \mathbb{N}$ so that $b_j \in \overline{B}_{g(i)}(p_i, 3/4)$, where $\lim_{j\to\infty} i(j) = \infty$. Since $(\overline{B}_{g(i)}(p_i, 3/4), d_{g(i)})_{i\in\mathbb{N}}$ converges in the Hausdorff topology to $(\Omega_\infty, d_\infty)$ we can assume that for each $j \in \mathbb{N}$ there exists an element $\widehat{b}_j \in \Omega_\infty$ satisfying $d_{\mathbb{B}}(b_j, \widehat{b}_j) < \frac{1}{j}$. Since $(\Omega_\infty, d_\infty)$ is compact, the sequence $(\widehat{b}_j)_{j\in\mathbb{N}}$ needs to contain a subsequence that converges with respect to d_∞, and consequently with respect to $d_{\mathbb{B}}$, to an element b_∞. This element is also the limit of the sequence $(b_j)_{j\in\mathbb{N}} \subseteq \mathbb{B}$ with respect to the metric $d_{\mathbb{B}}$.

Let

$$\delta := \frac{1}{10000} \cdot \min\left\{\frac{1}{10(1+K_1)}, r_0(3/4)\right\} \tag{3.1}$$

where $r_0(3/4)$ is taken from (1.11) and let

$$N_\infty := \{x_{\infty,1}, ..., x_{\infty,N(\delta)}\} \subseteq B_{d_\infty}(p_\infty, 1/2)$$

be a maximal δ-separated set. Using [21, 3.5. Proposition (a), p. 73] we can choose for each $i \in \mathbb{N}$ a 2δ-net

$$N_i := \{x_{i,1}, ..., x_{i,N(\delta)}\} \subseteq B_{g(i)}(p_i, 1/2)$$

such that N_i converges to N_∞ with respect to the Lipschitz distance (cf. [21, 3.1. Definition, p. 71]). Using (3.1), for each $i \in \mathbb{N}$ and $j \in \{1, ..., N(\delta)\}$ we may choose a smooth chart $\varphi_{i,j} : U_{i,j} \longrightarrow B(0, 100\delta)$, centered at $x_{i,j} \in U_{i,j} \subseteq \overline{B}_{g(i)}(p_i, 3/4)$, i.e.: $\varphi_{i,j}(x_{i,j}) = 0$, satisfying the requirements (i) to (iv) from Definition 3.2, i.e.:

(i) $(1 + K_1)^{-1}\delta_{kl} \leq g(i)_{kl}(x) \leq (1 + K_1)\delta_{kl} \quad \forall x \in B(0, 100\delta)$

(ii) $\|\partial g(i)\|_{L^n(B(0,100\delta))} \leq K_2$

(iii) $(100\delta)^\alpha[g(i)]_{C^\alpha(B(0,100\delta))} \leq K_3$

(iv) $\Delta_{g(i)}\varphi^m_{i,j} = 0$ for all $m \in \{1, ..., n\}$

3.1.3 Distance distorsion of a coordinate chart

Fact 3. *For each $i \in \mathbb{N}$ and $j \in \{1, ..., N(\delta)\}$ we have the following estimates:* $\forall y_1, y_2 \in B(0, 50\delta))$

$$d_{g(i)}(\varphi^{-1}_{i,j}(y_1), \varphi^{-1}_{i,j}(y_2)) \leq (1 + K_1)|y_1 - y_2| \tag{3.2}$$

and

$$d_{g(i)}(\varphi^{-1}_{i,j}(y_1), \varphi^{-1}_{i,j}(y_2)) \geq \frac{1}{1 + K_1}|y_1 - y_2| \tag{3.3}$$

We fix $i \in \mathbb{N}$ and $j \in \{1, ..., N(\delta)\}$, i.e.: $\varphi = \varphi_{i,j}$ and $g = g(i)$. Let $y_1, y_2 \in B(0, 50\delta)$ and $\gamma : [0, 1] \to B(0, 50\delta)$ be defined as

$$\gamma(t) := y_1 + s(y_2 - y_1)$$

then, using [22, p. 70: Proposition 3.24] and [22, p. 60: (3.8)] we obtain from property (i)

$$d_g(\varphi^{-1}(y_1), \varphi^{-1}(y_2)) \leq \int_0^1 |(\varphi^{-1} \circ \gamma)'(s))|_g \, ds \leq (1 + K_1)|y_1 - y_2|$$

which proves estimate (3.2). Now let $\gamma : [0, 1] \to M_i$ be a length minimizing geodesic connecting $\varphi^{-1}(y_1)$ and $\varphi^{-1}(y_2)$, i.e.: $\gamma(0) = \varphi^{-1}(y_1)$, $\gamma(1) = \varphi^{-1}(y_2)$ and $|\dot{\gamma}| \equiv d_g(\varphi^{-1}(y_1), \varphi^{-1}(y_2))$. We need to differentiate between two cases: At first we assume that $\gamma([0, 1]) \subseteq U$. Then

$$|y_1 - y_2| \leq \int_0^1 |(\varphi \circ \gamma)'(s)| \, ds \leq (1 + K_1) \cdot d_g(\varphi^{-1}(y_1), \varphi^{-1}(y_2))$$

If the assumption $\gamma([0,1]) \subseteq U$ does not hold, then there exist $s_1, s_2 \in (0,1)$ so that $\gamma(s_1) \notin U$ and $\gamma(s_2) \notin U$ but $\gamma([0,s_1) \subseteq U$ and $\gamma((s_2,1]) \subseteq U$. Then $(\varphi\circ\gamma)(s_1)$ and $(\varphi\circ\gamma)(s_2)$ are contained in the boundary of the set $B(0,100\delta)$. Hence

$$\begin{aligned}
&d_g(\varphi^{-1}(y_1), \varphi^{-1}(y_2))\\
\geq &\int_0^{s_1} |\dot{\gamma}(s)|_g \, ds + \int_{s_2}^1 |\dot{\gamma}(s)|_g \, ds\\
\overset{\text{(i)}}{\geq} &\frac{1}{1+K_1}|(\varphi\circ\gamma)(s_1) - y_1| + \frac{1}{1+K_1}|y_2 - (\varphi\circ\gamma)(s_2)|\\
\geq &\frac{1}{1+K_1}\left(|(\varphi\circ\gamma)(s_1)| - |y_1|\right) + \frac{1}{1+K_1}\left(|(\varphi\circ\gamma)(s_2)| - |y_2|\right)\\
= &\frac{1}{1+K_1}\left(100\delta - |y_1|\right) + \frac{1}{1+K_1}\left(100\delta - |y_2|\right)\\
= &\frac{1}{1+K_1}(200\delta - |y_1| - |y_2|) \geq \frac{1}{1+K_1} \cdot 100\delta \geq \frac{1}{1+K_1}|y_1 - y_2|
\end{aligned}$$

This proves the estimate (3.3).

3.1.4 $C^{2,\beta}$-regularity of the limit space

Fact 4. *$(B_{d_\infty}(p_\infty, 1/2), d_\infty)$ is a $C^{2,\beta}$-manifold, where $\beta \in (0,\alpha)$.*

From (3.2) and (3.3) we infer that for each $j \in \{1, ..., N(\delta)\}$ the sequence

$$\{\varphi_{i,j}^{-1} : \overline{B}(0, 50\delta) \longrightarrow \mathbb{B}\}_{i=1}^{\infty}$$

is an equicontinuous sequence of functions between compact spaces. Thus [26, 10.1.3., Lemma 45, p. 299] implies that, for each $j \in \{1, ..., N(\delta)\}$ there exists a subsequence that converges uniformly to a function

$$\varphi_{\infty,j}^{-1} : \overline{B}(0, 50\delta) \longrightarrow \mathbb{B}$$

where $\varphi_{\infty,j}^{-1}$ is a formal notation at first. Without loss of generality the sequence itself satisfies the desired property. (3.2) and (3.3) together imply that for each $j \in \{1, ..., N(\delta)\}$ the function

$$\varphi_{\infty,j}^{-1} : B(0, 50\delta) \longrightarrow U_{\infty,j} := \varphi_{\infty,j}^{-1}(B(0, 50\delta))$$

is continuous and there exists an inverse function $\varphi_{\infty,j} : U_{\infty,j} \longrightarrow B(0, 50\delta)$ that is also continuous, thus $\varphi_{\infty,j}$ is a homeomorphism. Furthermore, from the uniform convergence of these mappings, we obtain that the sequence of metric spaces $\varphi_{i,j}^{-1}(B(0,50\delta))$ converges to $\varphi_{\infty,j}^{-1}(B(0,50\delta))$ with respect to the Hausdorff topology as i tends to infinity. Due to the fact that for each $i \in \mathbb{N}$ the set $\varphi_{i,j}^{-1}(B(0,50\delta))$ is contained in $B_{g(i)}(p_i, 3/4)$, and that the sequence of balls $(B_{g(i)}(p_i, 3/4), d_{g(i)})$ converges to $(\Omega_\infty, d_\infty)$ with respect to the Hausdorff topology concerning $(\mathbb{B}, d_{\mathbb{B}})$ as i tends to infinity, we obtain that the domain of definition $U_{\infty,j}$ must be contained in Ω_∞. This procedure may be done for each $j \in \{1, ..., N(\delta)\}$. Whenever we need to extract a subsequence we do not change the notation. Hence, by choice of N_i, we obtain the following covering property

$$B_{g(i)}(p_i, 1/2) \subseteq \bigcup_{j=1}^{N(\delta)} \varphi_{i,j}^{-1}(B(0,10\delta)) \subseteq \bigcup_{j=1}^{N(\delta)} \varphi_{i,j}^{-1}(B(0,50\delta)) \subseteq B_{g(i)}(p_i, 3/4) \tag{3.4}$$

for each $i \in \mathbb{N}$, and we conclude that

$$B_{d_\infty}(p_\infty, 1/2) \subseteq \bigcup_{j=1}^{N(\delta)} \varphi_{\infty,j}^{-1}(B(0,10\delta)) \subseteq \bigcup_{j=1}^{N(\delta)} \varphi_{\infty,j}^{-1}(B(0,50\delta)) \subseteq \Omega_\infty \tag{3.5}$$

This shows that $B_{d_\infty}(p_\infty, 1/2)$ can be covered by an appropriate system of coordinate charts. Hence, the metric space $(B_{d_\infty}(p_\infty, 1/2), d_\infty)$ is a topological manifold.

It remains to prove, that for each fixed choice $s, t \in \{1, ..., N(\delta)\}$ satisfying $U_{\infty,s} \cap U_{\infty,t} \neq \emptyset$ the transition map

$$T_{s,t} : \varphi_{\infty,t}(U_{\infty,s} \cap U_{\infty,t}) \longrightarrow \varphi_{\infty,s}(U_{\infty,s} \cap U_{\infty,t}) \tag{3.6}$$

$$T_{s,t} := \varphi_{\infty,s} \circ \varphi_{\infty,t}^{-1} \tag{3.7}$$

is a $C^{2,\beta}$-diffeomorphism, where $\beta \in (0, \alpha)$. Let $y \in \varphi_{\infty,t}(U_{\infty,s} \cap U_{\infty,t}) \subseteq B(0,50\delta)$ and $\epsilon(y,s,t) > 0$ so that both $\overline{B}(y, 2\epsilon) \subseteq \varphi_{\infty,t}(U_{\infty,s} \cap U_{\infty,t})$ and $\overline{B}(y, 2\epsilon) \subseteq \varphi_{i,t}(U_{i,s} \cap U_{i,t})$ for each $i \geq i_0(y,s,t)$. Then

$$T_{i,s,t} := \varphi_{i,s} \circ \varphi_{i,t}^{-1} : B(y, 2\epsilon) \longrightarrow B(0, 50\delta) \tag{3.8}$$

converges uniformly to

$$T_{s,t}|_{B(y,2\epsilon)} : B(y, 2\epsilon) \longrightarrow B(0, 50\delta)$$

The property (iv) from Definition 3.2 and [11, Lemma 1.1] together imply

$$g^{kl}(i)\partial_k\partial_l T^m_{i,s,t} = 0 \quad \forall m \in \{1,...,n\} \tag{3.9}$$

for all $m \in \{1,...,n\}$, where $g^{kl}(i)$ is the inverse of the local representation of the metric in a chart. These equations are linear elliptic equations of second order in the fashion of [15, Chapter 6, (6.1), p. 87]. Using [15, Theorem 6.2, p. 90] we obtain

$$\left\|T^m_{i,s,t}\right\|_{C^{2,\alpha}(B(y,\epsilon))} \leq C_3(n,\alpha,K_1,K_3,\epsilon)\cdot\left\|T^m_{i,s,t}\right\|_{C^0(B(y,2\epsilon))} \leq C_4(n,\alpha,K_1,K_3,\epsilon,\delta)$$

and using the Arzelà-Ascoli theorem we can extract a subsequence from the sequence $(T_{i,s,t})_{i\in\mathbb{N}}$ that converges with respect to the $C^{2,\beta}(\overline{B}(y,\epsilon),\mathbb{R}^n)$-topology to $T_{s,t}|_{B(y,\epsilon)}$, where $\beta \in (0,\alpha)$. This implies the desired regularity.

3.1.5 Local $W^{2,\frac{n}{2}}$-convergence to a smooth metric

Fact 5. *There exists a smooth Riemannian metric g on $B_{d_\infty}(p_\infty,1/2)$ so that the sequence of metrics $g(i)$ converges to g locally, in the harmonic coordinates from above with respect to the $W^{2,\frac{n}{2}}$-topology.*

From [11, Lemma 4.1] we infer for each $i \in \mathbb{N}$, $j \in \{1,...,N(\delta)\}$ and $k,l \in \{1,...,n\}$

$$g(i)^{\alpha\beta}\partial_\alpha\partial_\beta g(i)_{kl} = -2Rc(i)_{kl} + (g(i)^{-1} * g(i)^{-1} * \partial g(i) * \partial g(i))_{kl} \tag{3.10}$$

on $B(0,50\delta)$, where $*$ denotes a sum of contractions with a certain rule which is not written down here. Using the $C^{0,\alpha}$-bound from (i) and (iii) in Definition 3.2. and the Arzelà-Ascoli theorem we conclude that for each $j \in \{1,...,N(\delta)\}$ there exists a system of functions

$$\{g_{kl} : B(0,50\delta) \longrightarrow \mathbb{R}\}_{k,l\in\{1,...n\}} \subseteq C^{0,\beta}(B(0,50\delta)) \text{ where } \beta < \alpha$$

so that, after extracting a subsequence, for each $k,l \in \{1,...n\}$ the sequence $g(i)_{kl}$ converges to g_{kl} with respect to the $C^{0,\beta}(B(0,50\delta))$-topology as i tends to infinity. For the sake of readability we do not write the index j here. Our aim is to show that for each $i \in \mathbb{N}$, $j \in \{1,...,N(\delta)\}$ and $k,l \in \{1,...,n\}$ the sequence $g(i)_{kl}$ converges to g_{kl} with respect to the $W^{2,\frac{n}{2}}(B(0,40\delta))$-topology as i tends to infinity. Firstly, using Theorem B.19, we want to prove that for each $p \in [n, C(\alpha)n)$, where $C(\alpha) > 1$ is fixed, the sequence $g(i)_{kl}$ converges to g_{kl} with respect to the $W^{1,p}(B(0,45\delta))$-topology. From (3.10) and [15, Theorem 9.11, p. 235-236] we obtain

$$\begin{aligned}
&\|g(i)_{kl}\|_{W^{2,\frac{n}{2}}(B(0,45\delta))}\\
\leq&C(n,\alpha,K_1,K_3,\delta)\cdot\\
&\quad\left[\|Rc(i)_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}+\|\partial g(i)\|^2_{L^n(B(0,50\delta))}+\|g(i)_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}\right] \qquad (3.11)\\
\leq&C(n,\alpha,K_1,K_3,\delta)\left[\|Rc(i)_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}+K_2^2+\|g(i)_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}\right]\\
\leq&C(n,\alpha,K_1,K_2,K_3,\delta)
\end{aligned}$$

Now we choose a cutoff function $\psi \in C_0^\infty(B(0,50\delta))$ satisfying $\psi|_{B(0,45\delta)} \equiv 1$ and $\psi(x) \in [0,1]$ for all $x \in B(0,50\delta)$. For the sake of readability, we set $u := g(i)_{kl} - g(j)_{kl}$. Let $q \in (1,\infty)$ and $\beta \in (0,\alpha)$ then

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|(\psi u)(x)-(\psi u)(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
=&\int_{B(0,50\delta)}\int_{B(0,50\delta)}\frac{|\psi(x)u(x)-\psi(y)u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
\leq&2^q\int_{B(0,50\delta)}\int_{B(0,50\delta)}\frac{|\psi(x)u(x)-\psi(x)u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
&+2^q\int_{B(0,50\delta)}\int_{B(0,50\delta)}\frac{|\psi(x)u(y)-\psi(y)u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
\leq&2^q\int_{B(0,50\delta)}\int_{B(0,50\delta)}\frac{|u(x)-u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
&+2^q\int_{B(0,50\delta)}\int_{B(0,50\delta)}\frac{|\psi(x)-\psi(y)|^q|u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy=:I+II
\end{aligned}$$

On the one hand we obtain

$$\begin{aligned}
I=&\int_{B(0,50\delta)}\int_{B(0,50\delta)}\frac{1}{|x-y|^{n+\frac{\beta-\alpha}{2}q}}\cdot\frac{|u(x)-u(y)|^q}{|x-y|^{\frac{\beta+\alpha}{2}q}}\,dx\,dy\\
\leq&C(n,q,\delta,\alpha,\beta)\;[u]^q_{\frac{\beta+\alpha}{2},B(0,50\delta)}
\end{aligned}$$

and on the other hand we obtain

$$\begin{aligned}
II=&\int_{B(0,50\delta)}\int_{B(0,50\delta)\cap\overline{B}(y,1)}\frac{|\psi(x)-\psi(y)|^q|u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
&+\int_{B(0,50\delta)}\int_{B(0,50\delta)\cap(\mathbb{R}^n\setminus\overline{B}(y,1))}\frac{|\psi(x)-\psi(y)|^q|u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
\leq & C(n,q,\psi)\int_{B(0,50\delta)}\int_{\overline{B}(y,1)}\frac{|u(y)|^q|x-y|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
&+\int_{B(0,50\delta)}\int_{B(0,50\delta)\cap(\mathbb{R}^n\setminus\overline{B}(y,1))}\frac{|u(y)|^q}{|x-y|^{n+\beta q}}\,dx\,dy\\
= & C(n,q,\psi)\int_{B(0,50\delta)}|u(y)|^q\int_{B(0,50\delta)\cap\overline{B}(y,1)}\frac{1}{|x-y|^{n+(\beta-1)q}}\,dx\,dy\\
&+\int_{B(0,50\delta)}|u(y)|^q\int_{B(0,50\delta)\cap(\mathbb{R}^n\setminus\overline{B}(y,1))}\frac{1}{|x-y|^{n+\beta q}}\,dx\,dy\\
\leq & C_2(n,q,\delta,\beta,\psi)\|u\|^q_{C^0(B(0,50\delta))}
\end{aligned}
$$

Hence, from Theorem B.19 and (3.11) we obtain the desired $W^{1,p}(B(0,40\delta))$-convergence of the sequence $g_{kl}(i)$, where $p\in[n,C(\alpha)n)$. Using this fact, we can show, that for each $k,l\in\{1,...,n\}$ the sequence $g(i)_{kl}$ is a $W^{2,\frac{n}{2}}(B(0,40\delta))$-Cauchy-sequence. In order to establish this, we need the following consideration which follows from (3.10)

$$
\begin{aligned}
&g(i)^{\alpha\beta}\partial_\alpha\partial_\beta(g(i)_{kl}-g(j)_{kl})+(g(i)^{\alpha\beta}-g(j)^{\alpha\beta})\partial_\alpha\partial_\beta g(j)_{kl}\\
=&g(i)^{\alpha\beta}\partial_\alpha\partial_\beta g(i)_{kl}-g(j)^{\alpha\beta}\partial_\alpha\partial_\beta g(j)_{kl}\\
=&-2Rc(i)_{kl}+2Rc(j)_{kl}+(g(i)^{-1}*_{1,i}g(i)^{-1}*_{2,i}\partial g(i)*_{3,i}\partial g(i))_{kl}\\
&\qquad-(g(j)^{-1}*_{1,j}g(j)^{-1}*_{2,j}\partial g(j)*_{3,j}\partial g(j))_{kl}
\end{aligned}
\tag{3.12}
$$

where $*_{m,i}$ denotes a sum of contractions of a rule m with respect to the metric $g(i)$ where $m\in\{1,...,3\}$ (cf. [11, Lemma 4.1]). Now, we rearrange the first order term

$$
\begin{aligned}
&g(i)^{-1}*_{1,i}g(i)^{-1}*_{2,i}\partial g(i)*_{3,i}\partial g(i)-g(j)^{-1}*_{1,j}g(j)^{-1}*_{2,j}\partial g(j)*_{3,j}\partial g(j)\\
&=g(i)^{-1}*_{1,i}g(i)^{-1}*_{2,i}\partial g(i)*_{3,i}\partial g(i)-g(j)^{-1}*_{1,i}g(j)^{-1}*_{2,i}\partial g(i)*_{3,i}\partial g(i)\\
&+g(j)^{-1}*_{1,i}g(j)^{-1}*_{2,i}\partial g(i)*_{3,i}\partial g(i)-g(j)^{-1}*_{1,j}g(j)^{-1}*_{2,j}\partial g(j)*_{3,j}\partial g(j)\\
&=[g(i)^{-1}*_{1,i}g(i)^{-1}-g(j)^{-1}*_{1,i}g(j)^{-1}]*_{2,i}\partial g(i)*_{3,i}\partial g(i)\\
&+g(j)^{-1}*_{1,i}g(j)^{-1}*_{2,i}[\partial g(i)*_{3,i}\partial g(i)-\partial g(j)*_{3,i}\partial g(j)]+f^{(1)}\\
&=\\
&[g(i)^{-1}*_{1,i}(g(i)^{-1}-g(j)^{-1})+(g(i)^{-1}-g(j)^{-1})*_{1,i}g(j)^{-1}]*_{2,i}\partial g(i)*_{3,i}\partial g(i)\\
&+g(j)^{-1}*_{1,i}g(j)^{-1}*_{2,i}[\partial g(i)*_{3,i}(\partial g(i)-\partial g(j))+(\partial g(i)-\partial g(j))*_{3,i}\partial g(j)]\\
&+f^{(1)}
\end{aligned}
$$

where $f^{(1)}$ arises from the change of the metric which is involved in the contraction. Due to Hölder's inequality, after possibly taking a subsequence, this expression satisfies the estimate

$$\max_{1\le k,l\le n} \|f^{(1)}_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))} \le \left(\frac{1}{i}+\frac{1}{j}\right)$$

Hence, from (3.12) we obtain

$$\begin{aligned}
&g(i)^{\alpha\beta}\partial_\alpha\partial_\beta(g(i)_{kl}-g(j)_{kl})\\
&= \underbrace{-2Rc(i)_{kl}+2Rc(j)_{kl}}_{=:f^{(2)}_{kl}}\underbrace{-(g(i)^{\alpha\beta}-g(j)^{\alpha\beta})\partial_\alpha\partial_\beta g(j)_{kl}}_{=:f^{(3)}_{kl}}\\
&+\underbrace{[G_1(i,j)*_{2,i}\partial g(i)*_{3,i}\partial g(i))]_{kl}}_{=:f^{(4)}_{kl}}\\
&+\underbrace{[G_2(i,j)*_{2,i}[\partial g(i)*_{3,i}(\partial g(i)-\partial g(j))+(\partial g(i)-\partial g(j))*_{3,i}\partial g(j)]]_{kl}}_{=:f^{(5)}_{kl}}\\
&+f^{(1)}_{kl}
\end{aligned}$$

where

$$\max_{1\le k,l\le n}\|G_1(i,j)_{kl}\|_{C^0(B(0,50\delta))}\le\left(\frac{1}{i}+\frac{1}{j}\right) \tag{3.13}$$

and

$$\max_{1\le k,l\le n}\|G_2(i,j)_{kl}\|_{C^0(B(0,50\delta))}\le C(n,K_1) \tag{3.14}$$

for all $i,j\in\mathbb{N}$. Now, we show that each of the expressions $f^{(m)}$, where $m\in\{2,...,5\}$, is sufficiently small with respect to the $L^{\frac{n}{2}}$-norm. Here, we allow that the considered ball becomes smaller. We have

$$\begin{aligned}
&\|f^{(2)}_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}\\
&=\|-2Rc(i)_{kl}+2Rc(j)_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}\\
&\le 2\|Rc(i)_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}+2\|Rc(j)_{kl}\|_{L^{\frac{n}{2}}(B(0,50\delta))}\overset{(1.9)}{\le}\left(\frac{1}{i}+\frac{1}{j}\right)
\end{aligned}$$

$$\begin{aligned}
&\|f^{(3)}_{kl}\|_{L^{\frac{n}{2}}(B(0,45\delta)}\\
&=\|(g(i)^{\alpha\beta}-g(j)^{\alpha\beta})\partial_\alpha\partial_\beta g(j)_{kl}\|_{L^{\frac{n}{2}}(B(0,45\delta))}\\
&\overset{(3.11)}{\le}\sum_{1\le\alpha,\beta\le n}\|(g(i)^{\alpha\beta}-g(j)^{\alpha\beta})\|_{C^0(B(0,45\delta))}\|\partial_\alpha\partial_\beta g(j)_{kl}\|_{L^{\frac{n}{2}}(B(0,45\delta))}\\
&\le\left(\frac{1}{i}+\frac{1}{j}\right)
\end{aligned}$$

$$\begin{aligned}
&\|f_{kl}^{(4)}\|_{L^{\frac{n}{2}}(B(0,50\delta))} \\
&\overset{(3.13)}{\leq} C(n,K_1,\delta)\cdot\left(\frac{1}{i}+\frac{1}{j}\right)\cdot\|\partial g(i)\|^2_{L^n(B(0,50\delta))} \\
&\leq\ C(n,K_1,K_2,\delta)\cdot\left(\frac{1}{i}+\frac{1}{j}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\|f_{kl}^{(5)}\|_{L^{\frac{n}{2}}(B(0,45\delta))} \\
&\overset{(3.14)}{\leq} C(n,K_1,\delta)\,\|\partial g(i)\|_{L^n(B(0,45\delta))}\|\partial(g(i)-g(j))\|_{L^n(B(0,45\delta))} \\
&\leq\ C(n,K_1,K_2,\delta)\,\|\partial(g(i)-g(j))\|_{L^n(B(0,45\delta))} \\
&\leq\ \left(\frac{1}{i}+\frac{1}{j}\right)
\end{aligned}$$

Where in the last line, we have used that $(g(i)_{kl})_{i\in\mathbb{N}}$ is a $W^{1,n}(B(0,45\delta))$-Cauchy sequence which was proved above. Hence, [15, Theorem 9.11, pp. 235-236] implies that $g(i)_{kl}$ is a $W^{2,\frac{n}{2}}(B(0,40\delta))$-Cauchy sequence which converges to g_{kl} in this topology. From (1.9) we infer

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{kl}=(g^{-1}*g^{-1}*\partial g*\partial g)_{kl}\qquad\forall 1\leq k\leq l\leq n \tag{3.15}$$

in the limit. Using this system of equations, we show that g is smooth. First we show that g_{kl} is contained in the space $W^{1,p}(B(0,r))$ for each $p\in(1,\infty)$ and $r\in(0,40\delta)$.

From the considerations above, we already know that

$$g_{kl}\in W^{1,C(\alpha)n}(B(0,40\delta))$$

where $C(\alpha)>1$. We proceed inductively: we assume, that for each $r\in(0,40\delta)$ we have $g_{kl}\in W^{1,q}(B(0,\frac{r+40\delta}{2}))$ where $q\in[C(\alpha)n,2n)$ is fixed, then Hölder's inequality implies:

$$\|(g^{-1}*g^{-1}*\partial g*\partial g)_{kl}\|_{L^{\frac{q}{2}}(B(0,\frac{r+40\delta}{2}))}\leq C_1$$

where $C_1\geq 0$ depends on the L^q-norm of ∂g. Then (3.15) and [14, Theorem 7.3, p. 140] together imply

$$\|\partial^2 g_{kl}\|_{L^{\frac{q}{2}}(B(0,r))}:=\max_{1\leq\lambda\leq\beta\leq n}\|\partial_\lambda\partial_\beta g_{kl}\|_{L^{\frac{q}{2}}(B(0,r))}\leq C_2(C_1)$$

Using the Sobolev embedding theorem (cf. [13, 5.6.3., Theorem 6, pp. 284-285]) we obtain

$$\|\partial g_{kl}\|_{L^{\frac{n\frac{q}{2}}{n-\frac{q}{2}}}(B(0,r/2))} \leq C_3(C_2)$$

and the assumption $q \in [C(\alpha)n, 2n)$ implies that

$$\frac{n\frac{q}{2}}{n-\frac{q}{2}} = \frac{\frac{n}{2}}{n-\frac{q}{2}} \cdot q \geq \frac{\frac{n}{2}}{n-\frac{C(\alpha)n}{2}} \cdot q > q$$

Hence, we have more regularity than assumed and the ratio of the increase of the regularity is bounded away from zero. This argument may be iterated, where q shall tend to $2n$. This shows that $g_{kl} \in W^{1,p}(B(0,r))$ for all $p \in (1,\infty)$.

Applying [14, Theorem 7.3, p. 140] to (3.15) again, we obtain $g_{kl} \in W^{2,p}(B(0,r))$ for all $p \in (1,\infty)$ and $r \in (0,40\delta)$. In this situation we may apply Lemma A.7 to (3.15) and we infer that g_{kl} is also contained in $\in W^{3,p}(B(0,r))$ for all $p > 1$ and $r \in (0,40\delta)$.

Now, it is possible to apply the argumentation in the proof of Lemma A.7 iteratively so that the L^p-regularity from a higher order derivative of the right hand side in (3.15) carries over to the iterated higher order derivative of the considered function on the left hand side, which is the metric g in that case. This means, that we obtain $g_{kl} \in W^{k,p}(B(0,r))$ for all $k \in \mathbb{N}$, $p \in (1,\infty)$ and $r \in (0,40\delta)$ and finally, using [15, Corollary 7.11., p. 158], we infer that g_{kl} is smooth.

3.1.6 C^∞-regularity of the limit space

Fact 6. *$(B_{d_\infty}(p_\infty, 1/2), d_\infty)$ is a C^∞-manifold.*

We continue the argumentation from Fact 4, using the fact that the limit metric is smooth. From (3.9) we obtain

$$g^{kl}(i)\partial_k\partial_l(T^m_{s,t} - T^m_{i,s,t}) = g^{kl}(i)\partial_k\partial_l T^m_{s,t} = (g^{kl}(i) - g^{kl})\partial_k\partial_l T^m_{s,t} + g^{kl}\partial_k\partial_l T^m_{s,t} \tag{3.16}$$

for all $m \in \{1,...,n\}$. Using the fact that $T^m_{i,s,t}$ converges with respect to the $C^2(\overline{B}(y,\epsilon))$-topology to $T_{s,t}|_{B(y,\epsilon)}$ and the fact, that $g_{kl}(i)$ converges with respect to the $C^0(\overline{B}(y,\epsilon))$-topology to g_{kl}, we obtain the following equation

$$g^{kl}\partial_k\partial_l T^m_{s,t} = 0 \tag{3.17}$$

in the limit. Since the coefficients are smooth, [13, 6.3.1, Theorem 3, p. 334] implies that each component of the transition function $T_{s,t}$ is also smooth. In what follows, we show that the Ricci tensor vanishes on $(B_{d_\infty}(p_\infty, 1/2), g)$

3.1.7 Ricci flatness of the limit manifold

Fact 7. *$(B_{d_\infty}(p_\infty, 1/2), g)$ is Ricci-flat.*

We have

$$\left(\int_{B_{d_\infty}(p_\infty,1/2)} |Ric_g|_g^{\frac{n}{2}}\, dV_g\right)^{\frac{2}{n}} \leq \sum_{j=1}^{N(\delta)} \left(\int_{\varphi_{\infty,j}^{-1}(B(0,40\delta))} |Ric_g|_g^{\frac{n}{2}}\, dV_g\right)^{\frac{2}{n}}$$

and from Fact 5 and (3.10) we infer

$$\left(\int_{\varphi_{\infty,j}^{-1}(B(0,40\delta))} |Ric_g|_g^{\frac{n}{2}}\, dV_g\right)^{\frac{2}{n}} = \lim_{i\to\infty} \left(\int_{\varphi_{i,j}^{-1}(B(0,40\delta))} |Ric_{g_i}|_g^{\frac{n}{2}}\, dV_g\right)^{\frac{2}{n}} = 0$$

Here, we have also used, that $g(i)$ converges to g with respect to the C^0-topology in the local charts.

3.1.8 Construction of the diffeomorphisms

Fact 8. *For each $i \in \mathbb{N}$ there exists a diffeomorphism*

$$F_i : B_g(p_\infty, 1/10) \longrightarrow F_i(B_g(p_\infty, 1/10)) \subseteq B_{g(i)}(p_i, 1)$$

such that $F_i^* g_i$ converges to the metric g with respect to the $W^{2,\frac{n}{2}}$-topology, as i tends to infinity.

We introduce a set of radii $(\delta_k)_{k\in\{1,...,7\}}$ which is defined as follows:

$$\delta_k := \left(\frac{10}{9}\right)^{k-1} \cdot 10\delta \tag{3.18}$$

For each $i \in \mathbb{N} \cup \{\infty\}$ and $k \in \{1, ..., 7\}$ we define

$$V_i^{(k)} := \bigcup_{j=1}^{N(\delta)} \varphi_{i,j}^{-1}(B(0, \delta_k))$$

so that, regarding (3.4), we have the following inclusions

$$B_{g(i)}(p_i,1/2)\subseteq V_i^{(1)}\subseteq ...\subseteq V_i^{(7)}\subseteq B_{g(i)}(p_i,3/4)$$

Now, let $\xi:[0,\infty)\longrightarrow[0,1]$ be a smooth cut-off function satisfying

$$\xi(s)\begin{cases}=1 & \text{if } s\in[0,\delta_5]\\ >\frac{95}{100} & \text{if } s\in[\delta_5,\delta_6]\\ \leq\frac{95}{100} & \text{if } s\in(\delta_6,\delta_7]\\ =0 & \text{if } s\in[\delta_7,\infty)\end{cases}\tag{3.19}$$

then for all $i\in\mathbb{N}$ and $k\in\{1,...,7\}$ we define a smooth map

$$E_{V_i^{(k)}}:V_i^{(k)}\longrightarrow\mathbb{R}^{N_0}$$

where $N_0=N_0(n,\delta)=n\cdot N(\delta)+N(\delta)$, as follows:

$$E_{V_i^{(k)}}:=[\xi(|\varphi_{i,1}|)\cdot\varphi_{i,1},...,\xi(|\varphi_{i,N(\delta)}|)\cdot\varphi_{i,N(\delta)},\xi(|\varphi_{i,1}|),...,\xi(|\varphi_{i,N(\delta)}|)]\tag{3.20}$$

where $\xi(|\varphi_{i,j}|)$ is considered as a global function, which is equal to zero whenever $\varphi_{i,j}$ becomes undefined on the respective manifold.

Let $i\in\mathbb{N}\cup\{\infty\}$, $j\in\{1,...,N(\delta)\}$ and $k\in\{1,...,7\}$, then

$$E_{V_i^{(k)}}\circ\varphi_{i,j}^{-1}\Big|_{B(0,\delta_k)}:B(0,\delta_k)\longrightarrow\mathbb{R}^{N_0}$$

has the following shape

$$\begin{aligned}&E_{V_i^{(k)}}\circ\varphi_{i,j}^{-1}\\ =&[\xi(|F_{1,j}^{(i)}|)\cdot F_{1,j}^{(i)},...,\xi(|F_{j-1,j}^{(i)}|)\cdot F_{j-1,j}^{(i)},\\ &\qquad\qquad\xi(|Id|)Id,\\ &\qquad\xi(|F_{j+1,j}^{(i)}|)\cdot F_{j+1,j}^{(i)},...,\xi(|F_{N(\delta),j}^{(i)}|)\cdot F_{N(\delta),j}^{(i)},\\ &\xi(|F_{1,j}^{(i)}|),...,\xi(|F_{1,j-1}^{(i)}|),\xi(|Id|),\xi(|F_{1,j+1}^{(i)}|),...,\xi(|F_{N(\delta),j}^{(i)}|)]\end{aligned}\tag{3.21}$$

where, for all $j,l\in\{1,...,N(\delta)\}$ the transition function

$$F_{l,j}^{(i)}:=\varphi_{i,l}\circ\varphi_{i,j}^{-1}:B(0,\delta_k)\longrightarrow\mathbb{R}^n$$

is always combined with a suitable truncation function, so in this context, the component functions of the function $E_{V_i^{(k)}}$ are always well-defined.

Let $j \in \{1, ..., N(\delta)\}$, for the sake of simplicity, we introduce the following swapping map $T_j : \mathbb{R}^{N_0} \longrightarrow \mathbb{R}^{N_0}$

$$\begin{aligned}
&T_j(v_1, v_2, ..., v_{j-1}, v_j, v_{j+1}, ..., v_{N(\delta)}, r_1, ..., r_{N(\delta)}) \\
=&(v_j, v_2, ..., v_{j-1}, v_1, v_{j+1}, ..., v_{N(\delta)}, r_1, ..., r_{N(\delta)})
\end{aligned}$$

Clearly this function satisfies $T_j \circ T_j = Id_{\mathbb{R}^{N_0}}$. From (3.21) we infer

$$\begin{aligned}
&T_j \circ E_{V_i^{(k)}} \circ \varphi_{i,j}^{-1} \\
=&[\xi(|Id|)Id, ..., \xi(|F^{(i)}_{j-1,j}|) \cdot F^{(i)}_{j-1,j}, \\
&\qquad \xi(|F^{(i)}_{1,j}|) \cdot F^{(i)}_{1,j}, \\
&\qquad \xi(|F^{(i)}_{j+1,j}|) \cdot F^{(i)}_{j+1,j}, ..., \xi(|F^{(i)}_{N(\delta),j}|) \cdot F^{(i)}_{N(\delta),j}, \\
&\xi(|F^{(i)}_{1,j}|), ..., \xi(|F^{(i)}_{1,j-1}|), \xi(|Id|), \xi(|F^{(i)}_{1,j+1}|), ..., \xi(|F^{(i)}_{N(\delta),j}|)] \\
=:&[\xi(|Id|)Id, u_{i,j}]
\end{aligned} \tag{3.22}$$

where $k \in \{1, ..., 7\}$ and for each $i \in \mathbb{N} \cup \{\infty\}$, $j \in \{1, ..., N(\delta)\}$, the function $u_{i,j}$ is a well-defined map from $B(0, \delta_7)$ to $\mathbb{R}^{N_0 - n}$.

From Fact 4 and Fact 6 we already know that for each $j \in \{1, ..., N(\delta)\}$ and $k \in \{1, ..., 7\}$ the sequence $(u_{i,j})_{i \in \mathbb{N}} \subseteq C^{2,\beta}(\overline{B(0, \delta_k)}, \mathbb{R}^{N_0-n})$, where $\beta \in (0, \alpha)$, converges with respect to the $C^{2,\beta}(\overline{B(0, \delta_k)}, \mathbb{R}^{N_0-n})$-topology to the smooth function

$$u_{\infty,j} : B(0, \delta_k) \longrightarrow \mathbb{R}^{N_0-n}$$

as i tends to infinity. For each $i \in \mathbb{N} \cup \{\infty\}$ and $k \in \{1, ..., 5\}$ we define

$$\begin{aligned}
M_i^{(k)} :=&E_{V_i^{(k)}}(V_i^{(k)}) = E_{V_i^{(k)}}\left(\bigcup_{j=1}^{N(\delta)} \varphi_{i,j}^{-1}(B(0, \delta_k))\right) \\
=&\bigcup_{j=1}^{N(\delta)} E_{V_i^{(k)}}\left(\varphi_{i,j}^{-1}(B(0, \delta_k))\right) \\
=&\bigcup_{j=1}^{N(\delta)} \{T_j(x, u_{i,j}(x)) |\, x \in B(0, \delta_k)\} \subseteq \mathbb{R}^{N_0}
\end{aligned} \tag{3.23}$$

It is our aim to prove that for each $i \in \mathbb{N} \cup \{\infty\}$ and $k \in \{1, ..., 4\}$ the mapping $E_{V_i^{(k)}}$ is a smooth embedding. This would imply that $M_i^{(k)}$ is an n-dimensional smooth embedded manifold (cf. [22, Proposition 5.2, p. 99]).

The appearance of the identity in (3.22) implies that for each $k \in \{1, ..., 5\}$ the mapping $E_{V_i^{(k)}}$ is an immersion. Furthermore, from the last $N(\delta)$ components in the definition of (3.20), we infer that this mapping is also injective because one of these components always needs to be equal to one and if the functional values of two points in $V_i^{(k)}$ coincide, then they are contained in a common domain of a coordinate chart. In this situation, the corresponding part in the first $N(\delta) \cdot n$ components contains the information that these two points need to coincide. Moreover, from the fact that for each $j \in \{1, ..., N(\delta)\}$ the function $u_{\infty,j}$ has a bounded first derivative and from the choice of the truncation function in (3.19), analogous to [29, p. 61, ll. 27-40], we infer that for each $y_0 \in V_i^{(4)}$ there exists a sufficiently small $s_0(x_0 = \varphi_{i,j}(y_0)) > 0$, where $j \in \{1, ..., N(\delta)\}$ such that $y_0 \in \varphi_{i,j}^{-1}(B(0, \delta_4))$, so that for each $s \leq s_0$ we have

$$\{T_j(x, u_{i,j}(x)) | \, x \in B(x_0, s)\} = (T_j(B(x_0, s) \times B(u_{i,j}(x_0), O(s)))) \cap M_i^{(k)}$$

where $O(s) \subseteq \mathbb{R}^{N_0 - n}$ is an open set. Since the swapping map T_j and the chart $\varphi_{i,j}$ are homeomorphisms, this would imply that for each $i \in \mathbb{N} \cup \{\infty\}$ and $k \in \{1, ..., 4\}$ the mapping $E_{V_i^{(k)}}$ is also an open map and consequently an embedding.

Using [24, Proposition. 26., p. 200], there exists an open set

$$\mathcal{N} \subseteq \coprod_{p \in M_\infty^{(4)}} N_p(M_\infty^{(4)})$$

where $N_p(M_\infty^{(4)})$ is the orthogonal complement of the tangent space $T_p(M_\infty^{(4)})$, containing the zero section

$$\mathcal{Z} := \coprod_{p \in M_\infty^{(4)}} 0_p$$

and a set $\mathcal{O} \subseteq \mathbb{R}^{N_0}$, containing $M_\infty^{(4)}$, such that

$$exp_\perp : \mathcal{N} \longrightarrow \mathcal{O}$$

is a diffeomorphism. Since $\overline{M_\infty^{(3)}} \subseteq M_\infty^{(4)} \subseteq \mathcal{O}$ there exists $\sigma > 0$ so that

$$B(M_\infty^{(3)}, \sigma) := \bigcup_{p \in M_\infty^{(3)}} B(p, \sigma) \subseteq \mathcal{O}$$

Thus the projection mapping

$$\pi := \pi_{\mathcal{N}} \circ exp_\perp^{-1} : B(M_\infty^{(3)}, \sigma) \longrightarrow M_\infty^{(4)} \tag{3.24}$$

where

$$\pi_{\mathcal{N}} : \mathcal{N} \longrightarrow M_\infty^{(4)}$$

is the natural projection, mapping each element $v_p \in N_p$ to p, is well-defined. Since $exp_\perp(0_p) = p$ for all $p \in M_\infty^{(4)} \supseteq M_\infty^{(3)}$ the map π satisfies $\pi|_{M_\infty^{(3)}} = Id|_{M_\infty^{(3)}}$. Taking $\sigma > 0$ sufficiently small, we may assume that

$$\sup_{z \in B(M_\infty^{(3)}, \sigma)} |\pi(z) - z| \leq \epsilon \tag{3.25}$$

It is our aim to show, that, for sufficiently large $i \in \mathbb{N}$, the mapping

$$E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(2)}}\Big|_{V_i^{(2)}} : V_i^{(2)} \longrightarrow (E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(2)}})(V_i^{(2)}) \subseteq V_\infty^{(4)} \tag{3.26}$$

is a diffeomorphism, satisfying

$$(E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(2)}})(V_i^{(2)}) \supseteq V_\infty^{(1)}$$

The first part of the following argumentation shows, that the map in (3.26) is well-defined.

From Fact 4 and Fact 6 we already know, that for each $j \in \{1, ..., N(\delta)\}$ the sequence

$$(u_{i,j})_{i\in\mathbb{N}} \subseteq C^{2,\beta}(\overline{B(0,\delta_3)}, \mathbb{R}^{N_0-n})$$

converges to the mapping

$$u_{\infty,j} \in C^\infty(\overline{B(0,\delta_3)}, \mathbb{R}^{N_0-n})$$

with respect to the $C^{2,\beta}(\overline{B(0,\delta_3)}, \mathbb{R}^{N_0-n})$-topology as i tends to infinity. In particular, (3.23) and (3.20) imply, that $M_i^{(3)}$ converges to $M_\infty^{(3)}$ with respect to the Hausdorff distance in $\mathbb{R}^{N_0}$. This allows to assume that $E_{V_i^{(3)}}(V_i^{(3)}) = M_i^{(3)} \subseteq B(M_\infty^{(3)}, \sigma)$ holds for all $i \in \mathbb{N}$. Hence, regarding (3.24),

$$\pi \circ E_{V_i^{(3)}} : V_i^{(3)} \longrightarrow M_\infty^{(4)}$$

is a well-defined mapping. Now, let $x_0 \in B(0,\delta_3)$, $j \in \{1, ..., N(\delta)\}$, $y_0 = \varphi^{-1}_{\infty,j}(x_0) \in V_\infty^{(3)}$ and $z_0 = u_{\infty,j}(x_0)$, then

$$T_j(x_0, z_0) = E_{V_\infty^{(3)}}(y_0) \in E_{V_\infty^{(3)}}(\varphi^{-1}_{\infty,j}(B(0,\delta_3)))$$

Since $E_{V_\infty^{(3)}}$ is a diffeomorphism onto his image, the set $E_{V_\infty^{(3)}}(\varphi^{-1}_{\infty,j}(B(0,\delta_3)))$ is relatively open in $M_\infty^{(3)}$. Thus, there exists $r_0 > 0$ so that

$$T_j(B_{r_0}(x_0) \times B_{r_0}(z_0)) \cap M_\infty^{(3)} \subseteq E_{V_\infty^{(3)}}(\varphi^{-1}_{\infty,j}(B(0,\delta_3)))$$

Here, we may assume that $r_0 > 0$ is chosen to be small enough so that we also have

$$\pi(T_j(B_{r_0}(x_0) \times B_{r_0}(z_0))) \subseteq E_{V_\infty^{(3)}}(\varphi_{\infty,j}^{-1}(B(0,\delta_3))) \tag{3.27}$$

Let $s_0 \in (0, r_0]$ so that

$$T_j(E_{V_\infty^{(3)}} \circ \varphi_{\infty,j}^{-1}(x)) = (x, u_{\infty,j}(x)) \in B_{r_0}(x_0) \times B_{r_0}(z_0) \quad \forall x \in B_{s_0}(x_0)$$

Due to the convergence of the sequence $(u_{i,j})_{i\in\mathbb{N}} \subseteq C^{2,\beta}(\overline{B(0,\delta_3)}, \mathbb{R}^{N_0-n})$ we may also assume that

$$T_j(E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1}(x)) = (x, u_{i,j}(x)) \in B_{r_0}(x_0) \times B_{r_0}(z_0) \quad \forall x \in B_{s_0}(x_0)$$

respectively

$$E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1}(x) \in T_j(B_{r_0}(x_0) \times B_{r_0}(z_0)) \cap M_i^{(3)} \quad \forall x \in B_{s_0}(x_0)$$

for all $i \in \mathbb{N}$. In this situation, (3.27) implies

$$\pi \circ E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1}(x) \in E_{V_\infty^{(3)}}(\varphi_{\infty,j}^{-1}(B(0,\delta_3))) \quad \forall x \in B_{s_0}(x_0)$$

Thus, the mapping

$$\varphi_{\infty,j} \circ E_{V_\infty^{(4)}}^{-1} \circ \pi \circ E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1}\Big|_{B_{s_0}(x_0)} : B_{s_0}(x_0) \longrightarrow B(0,\delta_4) \tag{3.28}$$

is well-defined. Now, we show that the map in (3.28) defines a diffeomorphism, where the range needs to be restricted. This implies that the map, that is defined in (3.26), is a local diffeomorphism. We have

$$\begin{aligned} &\varphi_{\infty,j} \circ E_{V_\infty^{(4)}}^{-1} \circ \pi \circ E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1} \\ =&(E_{V_\infty^{(4)}} \circ \varphi_{\infty,j}^{-1})^{-1} \circ \pi \circ T_j \circ (\cdot, u_{i,j}(\cdot)) \\ =&(T_j \circ (\cdot, u_{\infty,j}(\cdot))^{-1} \circ \pi \circ T_j \circ (\cdot, u_{i,j}(\cdot)) \end{aligned}$$

Hence, using the $C^{2,\beta}$-convergence of the sequence $(u_{i,j})_{i\in\mathbb{N}}$, we conclude that the sequence $(\varphi_{\infty,j} \circ E_{V_\infty^{(4)}}^{-1} \circ \pi \circ E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1})_{i\in\mathbb{N}}$ converges with respect to the $C^1(B_{s_0}(x_0), B(0,\delta_4))$-topology to

$$\begin{aligned} &(T_j \circ (\cdot, u_{\infty,j}(\cdot))^{-1} \circ \pi \circ T_j \circ (\cdot, u_{\infty,j}(\cdot)) \\ =&(E_{V_\infty^{(4)}} \circ \varphi_{\infty,j}^{-1})^{-1} \circ \pi \circ E_{V_\infty^{(3)}} \circ \varphi_{\infty,j}^{-1} \end{aligned}$$

$$=(E_{V_\infty^{(4)}} \circ \varphi_{\infty,j}^{-1})^{-1} \circ E_{V_\infty^{(3)}} \circ \varphi_{\infty,j}^{-1} = Id_{B_{s_0}(x_0)}$$

as $i \in \mathbb{N}$ tends to infinity. Consequently, since each immersion is a local diffeomorphism, we may assume that

$$\varphi_{\infty,j} \circ E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1}\Big|_{B_{s_0}(x_0)} :$$
$$B_{s_0}(x_0) \longrightarrow (\varphi_{\infty,j} \circ E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(3)}} \circ \varphi_{i,j}^{-1})(B_{s_0}(x_0))$$

is a diffeomorphism, provided that $s_0 > 0$ is small enough. From this, we conclude, that

$$\varphi_{\infty,j} \circ E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(2)}} \circ \varphi_{i,j}^{-1}\Big|_{B(0,\frac{\delta_2+\delta_3}{2})} : B(0, \frac{\delta_2+\delta_3}{2}) \longrightarrow B(0, \delta_4)$$

is well-defined and converges with respect to the $C^2(B(0, \frac{\delta_2+\delta_3}{2}), B(0,\delta_4))$-topology to $Id_{B(0,\frac{\delta_2+\delta_3}{2})}$.

Finally, we show that the mapping, which is defined in (3.26), is also a global diffeomorphism. It remains to show the global injectivity (cf. [29, ll. 36-43]): Let $y_1, y_2 \in V_i^{(2)}$ so that

$$(E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(2)}})(y_1) = (E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(2)}})(y_2)$$

Using the fact that $E_{V_\infty^{(4)}} : V_\infty^{(4)} \longrightarrow M_\infty^{(4)}$ is bijective, we obtain

$$(\pi \circ E_{V_i^{(2)}})(y_1) = (\pi \circ E_{V_i^{(2)}})(y_2)$$

Then (3.25) implies

$$\begin{aligned}
&|E_{V_i^{(2)}}(y_1) - E_{V_i^{(2)}}(y_2)| \\
=&|E_{V_i^{(2)}}(y_1) - (\pi \circ E_{V_i^{(2)}})(y_1) + (\pi \circ E_{V_i^{(2)}})(y_2) - E_{V_i^{(2)}}(y_2)| \\
\leq&|E_{V_i^{(2)}}(y_1) - (\pi \circ E_{V_i^{(2)}})(y_1)| + |(\pi \circ E_{V_i^{(2)}})(y_2) - E_{V_i^{(2)}}(y_2)| \\
\leq&2\epsilon
\end{aligned}$$

Now, let $j \in \{1, ..., N(\delta)\}$ such that $y_1 \in \varphi_{i,j}^{-1}(B(0,\delta_2))$, then, from the definition of $E_{V_i^{(2)}}$ (cf. (3.20)) we infer

$$|\varphi_{i,j}(y_1) - \xi(|\varphi_{i,j}(y_2)|) \cdot \varphi_{i,j}(y_2))| \leq 2\epsilon \leq \frac{\delta_3 - \delta_2}{4} \tag{3.29}$$

and

$$|1 - \xi(|\varphi_{i,j}(y_2)|)| \leq 2\epsilon \leq \frac{1}{100} \tag{3.30}$$

From (3.30) and (3.19), we infer $y_2 \in \varphi_{i,j}^{-1}(B(0,\delta_6))$. Suppose that $|\varphi_{i,j}(y_2)| \in [\delta_5, \delta_6)$, then

$$|\xi(|\varphi_{i,j}(y_2)|)\varphi_{i,j}(y_2)| = |\xi(|\varphi_{i,j}(y_2)|)| \cdot |\varphi_{i,j}(y_2)| \geq \frac{95}{100}\delta_5 \overset{(3.18)}{=} \frac{95}{100} \cdot \frac{10}{9}\delta_4 > \delta_4$$

which yields a contradiction because (3.29) and $\varphi_{i,j}(y_1) \in B(0,\delta_2)$ together imply

$$|\xi(|\varphi_{i,j}(y_2)|)\varphi_{i,j}(y_2))| < \delta_4$$

Hence, we have $y_2 \in \varphi_{i,j}^{-1}(B(0, \frac{\delta_2+\delta_3}{2}))$. This shows, that the mapping in (3.26) is a diffeomorphism. Finally, we are interested in the inverse mappings. These mappings shall satisfy the desired properties from the statement of the result.

For each $i \in \mathbb{N}$ let

$$\phi_i := (E^{-1}_{V_\infty^{(4)}} \circ \pi \circ E_{V_i^{(2)}})^{-1}\Big|_{V_\infty^{(1)}} : V_\infty^{(1)} \longrightarrow B_{g(i)}(p_i, 1)$$

and let

$$F_i := \phi_i|_{B_{d_\infty}(p_\infty, 1/10)} : B_{d_\infty}(p_\infty, 1/10) \longrightarrow B_{g(i)}(p_i, 1) \tag{3.31}$$

It remains to show that $F_i^* g(i)$ converges to g in the $W^{2,\frac{n}{2}}$-topology, as i tends to infinity. In local coordinates we have

$$\begin{aligned}
(F_i^* g(i))_{kl} &= \partial_k F_i^{j_1} g(i)_{j_1 j_2} \partial_l F_i^{j_2} \\
\partial_{m_1}(F_i^* g(i))_{kl} &= \partial_{m_1}\partial_k F_i^{j_1} g(i)_{j_1 j_2} \partial_l F_i^{j_2} + \partial_k F_i^{j_1} \partial_{m_1} g(i)_{j_1 j_2} \partial_l F_i^{j_2} \\
&\quad + \partial_k F_i^{j_1} g(i)_{j_1 j_2} \partial_{m_1}\partial_l F_i^{j_2}
\end{aligned}$$

and

$$\begin{aligned}
\partial_{m_2}\partial_{m_1}(F_i^* g(i))_{kl} = {} & \partial_{m_2}\partial_{m_1}\partial_k F_i^{j_1} g(i)_{j_1 j_2} \partial_l F_i^{j_2} + \partial_{m_1}\partial_k F_i^{j_1} \partial_{m_2} g(i)_{j_1 j_2} \partial_l F_i^{j_2} \\
& \quad + \partial_{m_1}\partial_k F_i^{j_1} g(i)_{j_1 j_2} \partial_{m_2}\partial_l F_i^{j_2} \\
& + \partial_{m_2}\partial_k F_i^{j_1} \partial_{m_1} g(i)_{j_1 j_2} \partial_l F_i^{j_2} + \partial_k F_i^{j_1} \partial_{m_2}\partial_{m_1} g(i)_{j_1 j_2} \partial_l F_i^{j_2} \\
& \quad + \partial_k F_i^{j_1} \partial_{m_1} g(i)_{j_1 j_2} \partial_{m_2}\partial_l F_i^{j_2} \\
& + \partial_{m_2}\partial_k F_i^{j_1} g(i)_{j_1 j_2} \partial_{m_1}\partial_l F_i^{j_2} + \partial_k F_i^{j_1} \partial_{m_2} g(i)_{j_1 j_2} \partial_{m_1}\partial_l F_i^{j_2} \\
& \quad + \partial_k F_i^{j_1} g(i)_{j_1 j_2} \partial_{m_2}\partial_{m_1}\partial_l F_i^{j_2}
\end{aligned} \tag{3.32}$$

for all $m_1, m_2 \in \{1, ..., n\}$. We recall that for each $i \in \mathbb{N}$ the diffeomorphism F_i is, by definition (cf. (3.31)), the inverse mapping of $E^{-1}_{V^{(4)}_\infty} \circ \pi \circ E_{V^{(2)}_i}$ and the mappings $E_{V^{(k)}_i}$ (cf. (3.20) / (3.21)), are constructed from the transition maps in (3.8). So, in order to analyze the convergence behavior of the sequence $(F_i)_{i \in \mathbb{N}}$, we need to consider the transition maps $T_{i,s,t}$ in (3.8), keeping Cramer's rule for the Jacobian of an inverse mapping in mind.

Since $g_{kl}(i)$ converges locally to g_{kl} with respect to the $W^{2,\frac{n}{2}}$-topology (cf. Fact 5) and the transition maps converge locally with respect to the $C^{2,\beta}$-topology to the transition map in the limit space (cf. Fact 4), it remains to consider the terms in (3.32) which contain a third order derivative of F_i. In order to get information about these derivatives we derive (3.16) having (3.17) in mind, i.e.:

$$\begin{aligned} g^{kl}(i)\partial_k\partial_l\partial_j(T^m_{s,t} - T^m_{i,s,t}) = &-(\partial_j g^{kl}(i))\partial_k\partial_l(T^m_{s,t} - T^m_{i,s,t}) \\ &+ \partial_j(g^{kl}(i) - g^{kl})\partial_k\partial_l T^m_{s,t} + (g^{kl}(i) - g^{kl})\partial_k\partial_l\partial_j T^m_{s,t} \end{aligned} \tag{3.33}$$

Furthermore, deriving (3.9), i.e.:

$$g^{kl}(i)\partial_k\partial_l\partial_j T^m_{i,s,t} = -(\partial_j g^{kl}(i))\partial_k\partial_l T^m_{i,s,t} \quad \forall m \in \{1, ..., n\}$$

yields a uniform local L^n-bound on $\partial_k\partial_l\partial_j T^m_{i,s,t}$, where we have used [15, Theorem 9.11, p. 235-236]. Hence, also using [15, Theorem 9.11, p. 235-236], (3.33) implies that $T^m_{i,s,t}$ converges locally with respect to the $W^{3,n}$-topology to $T^m_{s,t}$ for each $m \in \{1, ..., n\}$. Consequently, (3.32) implies the desired $W^{2,\frac{n}{2}}$-convergence of the sequence $F_i^* g(i)$ to the limit.

$\square$

Appendix A

Auxilary Results and Results from Riemannian geometry

A.1 Auxilary Results

Lemma A.1. *Let* $(M^n, g(t))_{t\in[t_1,t_2]}$ *be a smooth family of Riemannian manifolds and let* $\gamma : [0, L] \longrightarrow M$ *be a smooth curve. Then we have the estimates:*

$$\left|\frac{d}{dt}L(\gamma, t)\right| \leq \int_\gamma |g'(t)|_{g(t)}\, d\sigma_t \tag{A.1}$$

$$\left|\log\left(\frac{|v|^2_{g(t_2)}}{|v|^2_{g(t_1)}}\right)\right| \leq \int_{t_1}^{t_2} \|g'(t)\|_{L^\infty(M,g(t))}\, dt \qquad \forall v \in TM \tag{A.2}$$

$$\left|\frac{\partial}{\partial t}\left|\nabla_{\dot\gamma}\dot\gamma\right|^2_{g(t)}\right| \leq |g'|_{g(t)}\left|\nabla_{\dot\gamma}\dot\gamma\right|^2_{g(t)} + C(n)|\dot\gamma|^2_{g(t)}\left|\nabla_{\dot\gamma}\dot\gamma\right|_{g(t)}\left|\nabla g'\right|_{g(t)} \tag{A.3}$$

on $M \times (t_1, t_2)$.

Proof. Using a unit-speed-parametrization of γ we infer (A.1). Estimate (A.2) is proven in [16, 14.2 Lemma, p. 279]. In order to prove (A.3), we fix $x \in M$ and $t \in (t_1, t_2)$ and use normal coordinates around x (cf. [23, pp. 76-81]). In this point we have:

$$|\dot\gamma|^2_{g(t)} = (\dot\gamma^k\partial_k, \dot\gamma^l\partial_l)_{g(t)} = \sum_{k=1}^n (\dot\gamma^k)^2 \tag{A.4}$$

and, using [23, Lemma 4.3., p. 51]

$$
\begin{aligned}
|\nabla_{\dot\gamma}\dot\gamma|^2_{g(t)} =&(\ddot\gamma^k\partial_k + \Gamma^k_{ij}(g(t))\dot\gamma^i\dot\gamma^j\partial_k, \ddot\gamma^p\partial_p + \Gamma^p_{lm}(g(t))\dot\gamma^l\dot\gamma^m\partial_p)_{g(t)} \\
=&(\ddot\gamma^k\partial_k, \ddot\gamma^n\partial_n)_{g(t)} = \sum_{k=1}^n (\ddot\gamma^k)^2
\end{aligned} \tag{A.5}
$$

We also need the variation of the Christoffel symbols from (cf. [10, Lemma 2.27, p. 108]), i.e.:

$$
\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\nabla_i g'_{jl} + \nabla_j g'_{il} - \nabla_l g'_{ij}) \tag{A.6}
$$

Here, we have suppressed the time dependency in the notation. Using (A.4), (A.5), (A.6) and $\Gamma^k_{ij} = 0$ in x, we obtain:

$$
\begin{aligned}
&\left|\frac{\partial}{\partial t}|\nabla_{\dot\gamma}\dot\gamma|^2_g\right| = \left|\frac{\partial}{\partial t}(\ddot\gamma^k\partial_k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j\partial_k, \ddot\gamma^p\partial_p + \Gamma^p_{lm}\dot\gamma^l\dot\gamma^m\partial_p)_g\right| \\
\le&\left|\frac{\partial}{\partial t}(\ddot\gamma^k\partial_k, \ddot\gamma^p\partial_p)_g\right| + 2\left|\frac{\partial}{\partial t}(\ddot\gamma^k\partial_k, \Gamma^p_{lm}\dot\gamma^l\dot\gamma^m\partial_p)_g\right| + \left|\frac{\partial}{\partial t}(\Gamma^k_{ij}\dot\gamma^i\dot\gamma^j\partial_k, \Gamma^p_{lm}\dot\gamma^l\dot\gamma^m\partial_p)_g\right| \\
\le&\left|\frac{\partial}{\partial t}\sum_{k=1}^n(\ddot\gamma^k)^2(\partial_k,\partial_k)_g\right| + 2\left|\frac{\partial}{\partial t}\left[\ddot\gamma^k\Gamma^p_{lm}\dot\gamma^l\dot\gamma^m(\partial_k,\partial_p)_g\right]\right| \\
=&\left|\sum_{k=1}^n(\ddot\gamma^k)^2(\partial_k,\partial_k)_{g'}\right| + 2\left|\frac{\partial}{\partial t}\left[\ddot\gamma^k\Gamma^p_{lm}\dot\gamma^l\dot\gamma^m(\partial_k,\partial_p)_g\right]\right| \\
=&\left|\sum_{k=1}^n(\ddot\gamma^k)^2(\partial_k,\partial_k)_{g'}\right| + 2\left|\sum_{k,l,m,p=1}^n\ddot\gamma^k\left(\frac{\partial}{\partial t}\Gamma^p_{lm}\right)\dot\gamma^l\dot\gamma^m(\partial_k,\partial_p)_g\right| \\
=&\left|\sum_{k=1}^n(\ddot\gamma^k)^2(\partial_k,\partial_k)_{g'}\right| + 2\left|\sum_{k,l,m=1}^n\ddot\gamma^k\left(\frac{\partial}{\partial t}\Gamma^k_{lm}\right)\dot\gamma^l\dot\gamma^m\right| \\
\le&|g'|_g\,|\nabla_{\dot\gamma}\dot\gamma|^2_t + C_1(n)|\dot\gamma|^2_t|\nabla_{\dot\gamma}\dot\gamma|_g\left|\sum_{k,l,m=1}^n\left(\frac{\partial}{\partial t}\Gamma^k_{lm}\right)\right| \\
\le&|g'|_g\,|\nabla_{\dot\gamma}\dot\gamma|^2_g + C_2(n)|\dot\gamma|^2_t|\nabla_{\dot\gamma}\dot\gamma|_g|\nabla g'|_g
\end{aligned}
$$

□

Lemma A.2. *Let (M^n, g) be a closed Riemannian manifold, $k \in \mathbb{N}$, $x \in M$ and $c > 0$. Then we have the following equality*

$$
f_k(x, cg) = c^{-1}f_k(x, g) \tag{A.7}
$$

Proof. For each $j \in \{0, ..., k\}$ we obtain in local coordinates

$$\begin{aligned}
&|^{cg}\nabla^j Rm_{cg}|^2_{cg}\\
=&(c^{-1}g^{\alpha_1\beta_1}) \cdot ... \cdot (c^{-1}g^{\alpha_j\beta_j}) \cdot (c^{-1}g^{ip})(c^{-1}g^{kq})(c^{-1}g^{lr})(c^{-1}g^{ms})\\
&\cdot \ {}^{cg}\nabla_{\alpha_1,...,\alpha_j} R_{iklm}(cg)^{cg}\nabla_{\beta_1,...,\beta_j} R_{pqrs}(cg)\\
=&c^{-j} \cdot g^{\alpha_1\beta_1} \cdot ... \cdot g^{\alpha_j\beta_j} \cdot c^{-4} \cdot g^{ip}g^{kq}g^{lr}g^{ms}\\
&\cdot c^2 \cdot \ {}^{g}\nabla_{\alpha_1,...,\alpha_j} R_{iklm}(g)^{g}\nabla_{\beta_1,...,\beta_j} R_{pqrs}(g)\\
=&c^{-j-2} \cdot g^{\alpha_1\beta_1} \cdot ... \cdot g^{\alpha_j\beta_j} \cdot g^{ip}g^{kq}g^{lr}g^{ms} \cdot \ {}^{g}\nabla_{\alpha_1,...,\alpha_j} R_{iklm}(g)^{g}\nabla_{\beta_1,...,\beta_j} R_{pqrs}(g)\\
=&c^{-(j+2)}|^{g}\nabla^j Rm_g|^2_g
\end{aligned}$$

Here, we have used the fact that, the covariant derivative is invariant under rescaling (cf. [10, p. 3, Exercise 1.2]) and the scaling behavior of the Riemannian curvature tensor (cf. [10, p. 6, Exercise 1.11]).

□

Lemma A.3. *Let* $(M^n, g(t))_{t\in[0,T]}$ *be a smooth solution to the flow given in* (1.3) *then we have:*

$$\int_0^t \int_M |grad\ \mathcal{F}_{g(s)}|^2\, dV_{g(s)} ds = \mathcal{F}(g(0)) - \mathcal{F}(g(t)) \tag{A.8}$$

for all $t \in [0, T]$.

Proof. This follows from [5, 4.10 Definition, p. 119]. □

In particular, we can see that the energy $\mathcal{F}(g(t))$ is monotone decreasing under the flow given in (1.3), and

$$\int_0^t \int_M |\text{grad } \mathcal{F}_{g(s)}|^2\, dV_{g(s)}\, ds \leq \epsilon \tag{A.9}$$

for all $t \in [0, T]$ under the assumption that $\mathcal{F}(g_0) \leq \epsilon$

Theorem A.4. *([35, Lemma 2.11, p. 269]) Fix* $m, n \geq 0$. *There exists a constant* $C(n, m) > 0$ *so that if* $(M^n, g(t))_{t\in[0,T]}$ *is a complete solution to the* L^2*-flow satisfying*

$$\sup_{t\in[0,T]} t^{\frac{1}{2}} \left\|Rm_{g(t)}\right\|_{L^\infty(M,g(t))} \leq A \tag{A.10}$$

then for all $t \in (0, T]$,

$$\left\|\nabla^m Rm_{g(t)}\right\|_{L^\infty(M,g(t))} \leq C\left((A+1)t^{-\frac{1}{2}}\right)^{1+\frac{m}{2}} \tag{A.11}$$

Lemma A.5. *Let M^4 be a closed Riemannian manifold and $(M,g(t))_{t\in[0,T]}$ be a solution to the L^2-flow. We have the following estimates*

$$Vol_{g(t)}(M) = Vol_{g(0)}(M) \text{ for all } t \in (0,T] \tag{A.12}$$

and

$$Vol_{g(t)}(U)^{\frac{1}{2}} = Vol_{g(0)}(U)^{\frac{1}{2}} - Ct^{\frac{1}{2}} \left(\int_0^t \int_U |grad\ \mathcal{F}_{g(s)}|^2_{g(s)}\, dV_{g(s)}\, ds \right)^{\frac{1}{2}} \tag{A.13}$$
$$\text{for all } t \in (0,T] \text{ and } U \subseteq M \text{ open}$$

Proof. The equation (A.12) is a special case of the first equation in [34, p. 44]. Furthermore

$$\begin{aligned}
&\left[Vol_{g(t)}(U)\right]^{\frac{1}{2}} - \left[Vol_{g(0)}(U)\right]^{\frac{1}{2}} \\
=& \int_0^t \frac{d}{ds}\left[Vol_{g(s)}(U)\right]^{\frac{1}{2}} ds = \frac{1}{2}\int_0^t \frac{\frac{d}{ds}Vol_{g(s)}(U)}{\left[Vol_{g(s)}(U)\right]^{\frac{1}{2}}}\, dt \\
=& -\frac{1}{4}\int_0^t \frac{\int_U \mathrm{tr}_{g(s)}\ \mathrm{grad}\ \mathcal{F}_{g(s)}\, dV_{g(s)}}{\left[Vol_{g(s)}(U)\right]^{\frac{1}{2}}}\, ds \\
\geq& -\frac{1}{4}\int_0^t \frac{\left(\int_U |\mathrm{tr}_{g(s)}\ \mathrm{grad}\ \mathcal{F}_{g(s)}|^2\, dV_{g(s)}\right)^{\frac{1}{2}}}{\left[Vol_{g(s)}(U)\right]^{\frac{1}{2}}}\left[Vol_{g(s)}(U)\right]^{\frac{1}{2}} ds \\
\geq& -C\int_0^t \left(\int_U |\mathrm{grad}\ \mathcal{F}_{g(s)}|^2_{g(s)}\, dV_{g(s)}\right)^{\frac{1}{2}} ds \\
\geq& -Ct^{\frac{1}{2}}\left(\int_0^t\int_U |\mathrm{grad}\ \mathcal{F}_{g(s)}|^2_{g(s)}\, dV_{g(s)}\, ds\right)^{\frac{1}{2}}
\end{aligned}$$

□

Lemma A.6. *(cf. [34, Corollary 1.5]) Let $(M_i^n,(g_i(t))_{t\in(t_1,t_2)},p_i)$ be a sequence of complete solutions to the flow given in* (1.3)*. Suppose there exists a constant $K>0$ such that*

$$\sup_{M_i\times(t_1,t_2)} |Rm_{g_i}|_{g_i} \leq K$$

Then there exists a subsequence $(M_{i_j}^n,(g_{i_j}(t))_{t\in[t_1,t_2]},p_{i_j})$ and a one-parameter familiy of complete pointed metric spaces $(X,(d(t))_{t\in[t_1,t_2]},x)$ such that for each $t\in(t_1,t_2)$ the sequence $(M_{i_j}^n,d_{g_{i_j}(t)},p_{i_j})$ converges to $(X,d(t),x)$ in the sense *of C^∞-local submersions (cf. Definition C.12). The local lifted metrics $h_y(t)$ are solutions to* (1.3). *If there exists a constant $\delta > 0$ so that*

$$inj_{g_i(t)}(M_i, p_i) \geq \delta$$

then the limit space $(X, d(t), x)$ is a smooth n-dimensional Riemannian manifold, and the limiting metric is the C^∞-limit of the metrics $g_i(t)$.

Lemma A.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and $u \in W^{2,2p}(\Omega)$, where $p \geq p_0(n)$, satisfying*

$$a^{kl}\partial_k\partial_l u = f \tag{A.14}$$

where $(a^{kl})_{1\leq k,l\leq n} \subseteq C^{0,\alpha}(\Omega)$ satisfies

$$a^{kl} \equiv a^{lk} \quad \forall k, l \in \{1, ..., n\}$$
$$c_1|\xi|^2 \leq a^{kl}(x)\xi_k\xi_l \leq c_2|\xi|^2 \quad \forall x \in \Omega \text{ and } \xi \in \mathbb{R}^n$$
$$\max_{1\leq k,l\leq n} \sup_{x,y\in\Omega, x\neq y} \frac{|a^{kl}(x) - a^{kl}(y)|}{|x-y|^\alpha} \leq c_3$$
$$a^{kl} \in W^{1,2p}(\Omega) \quad \forall k, l \in \{1, ..., n\}$$

and $f \in W^{1,p}(\Omega)$, then $u \in W^{3,p}(\Omega_0)$ for each $\Omega_0 \subset\subset \Omega$. Furthermore we have the following estimate

$$\begin{aligned} &\|u\|_{W^{3,p}(\Omega_0)} \\ \leq &C\left(n, p, c_1, c_2, c_3, \alpha, \Omega_0, \Omega, \max_{1\leq k,l\leq n}\left\|a^{kl}\right\|_{W^{1,2p}(\Omega)}, \|f\|_{W^{1,p}(\Omega)}, \|u\|_{W^{2,2p}(\Omega)}\right) \end{aligned} \tag{A.15}$$

Proof. Let $i \in \{1, ..., n\}$, $h \in \left(0, \frac{1}{2}dist(\Omega_1, \partial\Omega)\right)$ where $\Omega_1 \subseteq \Omega$ is a domain satisfying $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. As in [15, 7.11, (7.53), p. 168] we define for each $x \in \Omega_1$ the difference quotient in the direction e_i of size $h \neq 0$ as follows

$$\Delta_i^h u(x) := \frac{u(x+h) - u(x)}{h}$$

An application of this operation to (A.14) implies

$$a^{kl}(\cdot + h)\ (\partial_k\partial_l\Delta_i^h u) = \Delta_i^h f - (\Delta_i^h a^{kl})\partial_k\partial_l u \tag{A.16}$$

on Ω_1, [15, 9.5, Theorem 9.11., pp. 235-236] and [15, 7.11, Lemma 7.23., p. 168] imply

$$\left\|\Delta_i^h u\right\|_{W^{2,p}(\Omega_0)} \leq C(n, p, c_1, c_2, c_3, \alpha, \Omega_0, \Omega)\cdot$$

$$
\begin{aligned}
&\left[\left\|\Delta_i^h u\right\|_{L^p(\Omega_1)} + \left\|\Delta_i^h f\right\|_{L^p(\Omega_1)} + \left\|(\Delta_i^h a^{kl})\partial_k\partial_l u\right\|_{L^p(\Omega_1)}\right] \\
&\leq C(n,p,c_1,c_2,c_3,\alpha,\Omega_0,\Omega)\left[\|u\|_{W^{1,p}(\Omega)} + \|f\|_{W^{1,p}(\Omega)}\right] \\
&\quad + C(n,p,c_1,c_2,c_3,\alpha,\Omega_0,\Omega)\sum_{1\leq k,l\leq n}\left\|\Delta_i^h a^{kl}\right\|_{L^{2p}(\Omega_1)}\|\partial_k\partial_l u\|_{L^{2p}(\Omega)} \\
&\leq C(n,p,c_1,c_2,c_3,\alpha,\Omega_0,\Omega)\left[\|u\|_{W^{1,p}(\Omega)} + \|f\|_{W^{1,p}(\Omega)}\right] \\
&\quad + C(n,p,c_1,c_2,c_3,\alpha,\Omega_0,\Omega)\max_{1\leq k\leq l\leq n}\left\|a^{kl}\right\|_{W^{1,2p}(\Omega)}\|u\|_{W^{2,2p}(\Omega)}
\end{aligned}
$$

This implies (A.15).

□

A.2 Results from Riemannian geometry

Lemma A.8. *([35, Lemma 2.9, p. 268]) Let (M^n, g) be a complete Riemannian manifold, satisfying $f_1(M,g) \leq K$ then there are constants $C(n), \mu(n) > 0$ such that in any normal coordinate chart around p one has*

$$\sup_{B_g(p,\mu K^{-\frac{1}{2}})} |\Gamma| \leq CK^{\frac{1}{2}}$$

Here, Γ is introduced in Definition C.9.

Lemma A.9. *([35, Lemma 2.7, p. 268]) Let (M^n, g) be a complete Riemannian manifold with $\|Rm_g\|_{L^\infty(M,g)} \leq K$ and $inj_g(M) \geq \iota$, then there exists $\mu(n) > 0$ and $c(n) > 0$ so that for all $r < \mu \min\{i, K^{-\frac{1}{2}}\}$ and all $p \in M$ and $v \in T_pM$ one has the following estimate*

$$Area\left[\exp_p(B_r(0) \cap \langle v\rangle^\perp)\right] \geq cr^{n-1}$$

Lemma A.10. *([35, Lemma 2.8, p. 268]) Let (M,g) and (N,h) be smooth Riemannian manifolds and let $F : M \longrightarrow N$ be a smooth submersion. Furthermore, let $\phi : M \longrightarrow [0,\infty)$ be a smooth function, then one has:*

$$\int_M \phi\, dV_g = \int_{y\in N}\int_{x\in F^{-1}(y)} \frac{\phi(x)}{NJac\,F(x)}\, dF^{-1}(y)\, dV_h \tag{A.17}$$

where $NJac\,F(x)$ is the determinant of the derivative restricted to the orthogonal complement of its kernel. This quantity is also called "normal Jacobian".

Theorem A.11. *([1, Theorem 2.2]) Let $(N_i^n, h_i)_{i\in\mathbb{N}}$ be a sequence of closed Riemannian manifolds with the following properties: There exists $k \in \mathbb{N}$ and $\{\Gamma_l\}_{l\in\{1,...,k\}} \subseteq \mathbb{R}_{>0}$, $\iota, v_0, V_0 > 0$ such that:*

$$\begin{aligned} \left\|\nabla^l Rm_{g_i}\right\|_{L^\infty(M_i,g_i)} &\leq \Gamma_l \quad \forall i \in \mathbb{N},\ \forall l \in \{0,...,k\} \qquad (A.18) \\ inj_{g_i}(M_i) &\geq \iota \quad \forall i \in \mathbb{N} \qquad (A.19) \\ v_0 \leq Vol_{g_i} &\leq V_0 \quad \forall i \in \mathbb{N} \qquad (A.20) \end{aligned}$$

then there exists a subsequence $(N_{i_j}^n, h_{i_j})_{j\in\mathbb{N}}$ converging in the $C^{k,\alpha}$-sense to a $C^{k+1,\alpha}$-Riemannian manifold.

Lemma A.12. *Let $n \in \mathbb{N}$, $\iota > 0$ and let (M^n, g) be a complete n-dimensional Riemannian manifold such that the following is true*

$$\begin{aligned} Rc_g &\equiv 0 \\ \|Rm_g\|_{L^\infty(M^n,g)} &< \infty \\ inj_g(M) &\geq \iota \end{aligned}$$

then

$$\|Rm_g\|_{L^\infty(M^n,g)} \leq C(n, \iota).$$

Proof. We argue by contradiction. Suppose this statement would be wrong, then we could find a sequence of complete n-dimensional Ricci-flat manifolds $(M_i, g_i)_{i\in\mathbb{N}}$ so that

$$inj_{g_i}(M_i) \geq \iota$$

and

$$\|Rm_{g_i}\|_{L^\infty(M_i,g_i)} = C_i$$

where

$$\lim_{i\to\infty} C_i = \infty$$

We construct a blow-up sequence as follows: for each $i \in \mathbb{N}$ let

$$h_i := C_i \cdot g_i$$

so that

$$inj_{h_i}(M_i) \geq \sqrt{C_i}\iota$$

and

$$\|Rm_{h_i}\|_{L^\infty(M_i,h_i)} = 1$$

For each $i \in \mathbb{N}$ we choose a fixed point $p_i \in M_i$, so that $|Rm_{h_i}(p_i)|_{h_i} \geq \frac{1}{2}$. Using $Rc_{h_i} \equiv 0$, the first equation on [1, p. 461] or [16, 7., 7.1. Theorem, p. 274] implies

$$\Delta_{h_i} Rm_{h_i} = Rm_{h_i} * Rm_{h_i} \tag{A.21}$$

and consequently

$$\|\Delta_{h_i} Rm_{h_i}\|_{L^\infty(M_i,h_i)} \leq K(n)$$

Furthermore, from [17, Lemma 1], we obtain uniform C^0-bounds on the metrics $(h_i)_{i\in\mathbb{N}}$ in normal coordinates. Hence, an iterative application of the theory of linear elliptic equations of second order to (A.21), following the arguments of [1, p. 478, second paragraph], we obtain uniform higher order estimates, i.e.:

$$\left\|\nabla^k_{h_i} Rm_{h_i}\right\|_{L^\infty(M_i,h_i)} \leq K(n,k)$$

for all $i, k \in \mathbb{N}$. Hence, [1, Theorem 2.2, pp. 464-466] implies that there exists a subsequence $(M_i, g_i, p_i)_{i\in\mathbb{N}}$ that converges in the pointed $C^{k,\alpha}$-sense, where $k \in \mathbb{N}$ is arbitrary, to a smooth manifold (X, h, p) satisfying

$$|Rm_h(p)|_h \geq \frac{1}{2}$$

and, using [28, Theorem]

$$inj_h(X,p) = \infty$$

An iterative application of [8, Theorem 2] implies that $(X, h, p) = (\mathbb{R}^n, g_{euc}, 0)$ which yields a contradiction.

$\square$

Appendix B

Interpolation, Sobolev spaces and Besov spaces

B.1 Interpolation theory

The following definition of an *interpolation couple* is a standard definition in the interpolation theory. We refer to [41, 1.2.1., p. 18].

Definition B.1. *An interpolation couple $\{A_0, A_1\}$ is a couple of complex Banach spaces A_0 and A_1 which are linear subspaces of a linear complex Hausdorff space A and continuously embedded in A.*

Lemma B.2. *([41, 1.2.1., Lemma, p. 18]) Let $\{A_0, A_1\}$ be an interpolation couple, then the space $A_0 \cap A_1$ endowed with the norm*

$$\|a\|_{A_0\cap A_1} := \max(\|a_0\|_{A_0}, \|a_1\|_{A_1})$$

and the space $A_0 + A_1$ endowed with the norm

$$\|a\|_{A_0+A_1} := \inf_{\substack{a=a_0+a_1\\ a_j\in A_j}} (\|a_0\|_{A_0} + \|a_1\|_{A_1})$$

are Banach spaces.

In the following, we introduce the concept of *complex interpolation* (cf. [41, 1.9., pp. 55-61]). A part of the theory of analytic functions with values in a Banach space is explained in [12, III.14., pp. 224-232]. Throughout let $S := \{z \in \mathbb{C} : \; Re(z) \in (0, 1)\}$.

Definition B.3. *([41, 1.9.1., Definition., p. 56]) Let* $\{A_0, A_1\}$ *be an interpolation couple and let* $\gamma \in \mathbb{R}$*. Then by definition* $F(A_0, A_1, \gamma)$ *is the set of all functions* $f : \overline{S} \longrightarrow A_0 + A_1$ *satisfying the following properties*

- f *is continuous with respect to the* $\|\cdot\|_{A_0+A_1}$*-norm*
- $f|_S$ *is analytic with respect to the* $\|\cdot\|_{A_0+A_1}$*-norm, i.e.: for each* $z_0 \in S$

  $$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

  exists, where convergence is understood to be as convergence with respect to the $\|\cdot\|_{A_0+A_1}$*-norm.*
- $\sup_{z \in \overline{S}} e^{-|\gamma| \cdot |Im(z)|} \|f(z)\|_{A_0+A_1} < \infty$
- *the map* $f(i\cdot) : \mathbb{R} \longrightarrow A_0$ *(i.e.* $t \mapsto f(it)$*) is well-defined and continuous with respect to the* $\|\cdot\|_{A_0}$*-norm*
- *the map* $f(1 + i\cdot) : \mathbb{R} \longrightarrow A_1$ *(i.e.* $t \mapsto f(1 + it)$*) is well-defined and continuous with respect to the* $\|\cdot\|_{A_1}$*-norm*
- $\|f\|_{F(\gamma)} := \max\{\sup_{t \in \mathbb{R}} e^{-\gamma|t|} \|f(it)\|_{A_0}, \sup_{t \in \mathbb{R}} e^{-\gamma|t|} \|f(1 + it)\|_{A_1}\} < \infty$

Theorem B.4. *([41, 1.9.1., Theorem. (a), p. 56]) Let* $\{A_0, A_1\}$ *be an interpolation couple and let* $\gamma \in \mathbb{R}$ *then* $F(A_0, A_1, \gamma)$ *endowed with the norm* $\|\cdot\|_{F(\gamma)}$ *is a Banach space.*

Definition B.5. *([41, 1.9.2., Definition., p. 58]) Let* $\{A_0, A_1\}$ *be an interpolation couple,* $\theta \in (0, 1)$ *and* $\gamma \in \mathbb{R}$*. Then we define*

$$[A_0, A_1]_{\theta,\gamma} := \{a \in A_0 + A_1 : \ \exists f \in F(A_0, A_1, \gamma) \ s. \ th. \ f(\theta) = a\}$$

and

$$\|a\|_{[A_0,A_1]_{\theta,\gamma}} := \inf \left\{ \|a\|_{F(\gamma)} : \ f \in F(A_0, A_1, \gamma) \ s. \ th. \ f(\theta) = a \right\}$$

Theorem B.6. *([41, 1.9.2., Theorem., pp. 58-59]) Let* $\{A_0, A_1\}$ *be an interpolation couple,* $\theta \in (0, 1)$ *and* $\gamma \in \mathbb{R}$*. Then* $[A_0, A_1]_{\theta,\gamma}$ *endowed with the norm* $\|\cdot\|_{[A_0,A_1]_{\theta,\gamma}}$ *is a Banach space.*

Definition B.7. *([41, 1.9.2., Convention., p. 59]) Let $\{A_0, A_1\}$ be an interpolation couple and $\theta \in (0,1)$. Then we define*

$$[A_0, A_1]_\theta := [A_0, A_1]_{\theta,0}$$

There is a wide range of standard properties of the spaces $[A_0, A_1]_\theta$, where $\theta \in (0,1)$. Some of them are listed in [41, 1.9.3., Theorem., p. 59]. In our context, the property that is stated in [41, 1.9.3., p. 59 (3)] is crucial:

Lemma B.8. *([41, 1.9.3., Theorem. (f), p. 59]) Let $\{A_0, A_1\}$ be an interpolation couple and $\theta \in (0,1)$ then the following estimate holds:*

$$\|a\|_{[A_0,A_1]_\theta} \leq C(\theta)\, \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta} \quad \forall a \in A_0 \cap A_1 \tag{B.1}$$

B.2 Sobolev spaces and Besov spaces

The following two definitions are deduced from [39, 1.2.1]

Definition B.9. *([39, 1.2.1, pp. 12-13]) We define the Schwartz space $\mathcal{S}$ as follows*

$$\mathcal{S} := \left\{ u \in C^\infty(\mathbb{R}^n, \mathbb{C}) :\, p_k(u) := \sup_{x\in\mathbb{R}^n} (1+|x|)^k \sum_{|\alpha|\leq k} |D^\alpha u(x)| < \infty \,\forall k \in \mathbb{N} \right\}$$

and we introduce the following metric on $S(\mathbb{R}^n, \mathbb{C})$

$$d_{\mathcal{S}}(u,v) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(u-v)}{1+p_k(u-v)}$$

The topological dual space (in the distributional sense) $\mathcal{S}'$, equipped with the strong topology, is called space of all tempered distibutions.

Definition B.10. *([39, 1.2.1, p. 13 (2) / (3)]) The mapping*

$$F : \mathcal{S} \longrightarrow \mathcal{S}$$
$$[Fu](x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\, u(\xi)\, d\xi, \quad x \in \mathbb{R}^n$$

is called Fourier transform. The inverse Fourier transform is given by

$$F^{-1} : \mathcal{S} \longrightarrow \mathcal{S}$$
$$[F^{-1}u](x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle}\, u(\xi)\, d\xi, \quad x \in \mathbb{R}^n$$

The induced dual mapping on $\mathcal{S}'$ is also denoted by F and F^{-1}

The following remark can be found in [41, 2.2.1, p. 152]

Remark B.11. *([41, 2.2.1, p. 152]) The mapping F an isomorphism from $\mathcal{S}$ to $\mathcal{S}$ and from $\mathcal{S}'$ to $\mathcal{S}'$.*

We also need the following consequence of the Paley-Wiener-Schwartz theorem:

Lemma B.12. *([41, 2.2.1, p. 152]) Let $u \in \mathcal{S}'$ such that Fu has compact support, then u is a regular distribution that is induced by an analytic function which shall be also denoted by u and we have the following estimate*

$$|u(x)| \leq C(1+|x|^2)^N \quad \forall x \in \mathbb{R}^n$$

where C and N do not depend on x.

The following definitions are introduced in [41, 2.3.1, pp. 168-169]

Definition B.13. *We define the following system of sets*

$$M_j := \begin{cases} \{x \in \mathbb{R}^n : \ |x| \leq 2\} & \text{if } j = 0 \\ \{x \in \mathbb{R}^n : \ |x| \in [2^{j-1}, 2^{j+1}]\} & \text{if } j \in \mathbb{N}\backslash\{0\} \end{cases}$$

Definition B.14. *(a) For $s \in (-\infty, \infty)$, $p \in (1, \infty)$ and $q \in [1, \infty)$ we set*

$$B^s_{p,q}(\mathbb{R}^n, \mathbb{C}) := \Big\{ u \in \mathcal{S}' : \ u \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} u_j \ \text{s.th. } supp(Fu_j) \subseteq M_j \\ \text{for all } j \in \mathbb{N} \text{ and } \left(\sum_{j=0}^{\infty} (2^{sj} \|u_j\|_{L^p(\mathbb{R}^n,\mathbb{C})})^q \right)^{\frac{1}{q}} < \infty \Big\}$$

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n,\mathbb{C})} := \inf \left\{ \left(\sum_{j=0}^{\infty} (2^{sj} \|u_j\|_{L^p(\mathbb{R}^n,\mathbb{C})})^q \right)^{\frac{1}{q}} : \ u \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} u_j \text{ and} \\ supp(Fu_j) \subseteq M_j \ \text{ for all } j \in \mathbb{N} \right\}$$

and for $s \in (-\infty, \infty)$, $p \in (1, \infty)$ and $q = \infty$ we set

$$B^s_{p,\infty}(\mathbb{R}^n, \mathbb{C}) := \Big\{ u \in \mathcal{S}' : \ u \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} u_j \ \text{s.th. } supp(Fu_j) \subseteq M_j \\ \text{for all } j \in \mathbb{N} \text{ and } \sup_{j \in \mathbb{N}} 2^{sj} \|u_j\|_{L^p(\mathbb{R}^n,\mathbb{C})} < \infty \Big\}$$

$$\|u\|_{B^s_{p,\infty}(\mathbb{R}^n,\mathbb{C})} := \inf\Bigg\{ \sup_{j\in\mathbb{N}} 2^{sj}\|u_j\|_{L^p(\mathbb{R}^n,\mathbb{C})} : \ u \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} u_j \ \textit{and}$$
$$\textit{supp}(Fu_j) \subseteq M_j \ \textit{for all} \ j \in \mathbb{N}\Bigg\}$$

(b) For $s \in (-\infty, \infty)$, $p \in (1,\infty)$ and $q \in (1,\infty)$ we set

$$F^s_{p,q}(\mathbb{R}^n,\mathbb{C}) :=\Bigg\{u \in \mathcal{S}' : \ u \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} u_j \ \textit{s.th. supp}(Fu_j) \subseteq M_j \ \textit{and}$$
$$\left[\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{sjq}|u_j(x)|^q\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} < \infty\Bigg\}$$
$$\|u\|_{F^s_{p,q}(\mathbb{R}^n,\mathbb{C})} :=\inf \left\{ \left[\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{sjq}|u_j(x)|^q\right)^{\frac{p}{q}} dx\right]^{\frac{1}{p}} : \ u \underset{\mathcal{S}'}{=} \sum_{j=0}^{\infty} u_j \ \textit{and}\right.$$
$$\left.\textit{supp}(Fu_j) \subseteq M_j \ \textit{for all} \ j \in \mathbb{N}\right\}$$

(c) For $s \in (-\infty, \infty)$, $p \in (1,\infty)$ we set

$$H^s_p(\mathbb{R}^n,\mathbb{C}) := F^s_{p,2}(\mathbb{R}^n,\mathbb{C})$$

(d) For $p \in (1,\infty)$ we set

$$W^s_p(\mathbb{R}^n,\mathbb{C}) := \begin{cases} H^s_p(\mathbb{R}^n,\mathbb{C}) & \textit{if } s \in \mathbb{N}\cup\{0\} \\ B^s_{p,p}(\mathbb{R}^n,\mathbb{C}) & \textit{if } s \in \mathbb{R}_{>0}\backslash\mathbb{N} \end{cases}$$

The following result is a special case of a result which was suggested in [41, pp. 179-180: Remark 4]. In order to clarify the continuous embeddedness we give a proof.

Lemma B.15. *For each $s \in (0,1)$, $p \in (1,\infty)$ and $\epsilon > 0$ such that $s-\epsilon > 0$ we have*

$$B^s_{p,p}(\mathbb{R}^n,\mathbb{C}) \hookrightarrow H^{s-\epsilon}_p(\mathbb{R}^n,\mathbb{C}) \tag{B.2}$$

Proof. Using [41, p. 172 (3)] or [39, p. 47 (7)] we obtain

$$B^s_{p,p}(\mathbb{R}^n,\mathbb{C}) \hookrightarrow B^{s-\frac{\epsilon}{2}}_{p,p}(\mathbb{R}^n,\mathbb{C})$$

[41, p. 172 (4a) and (4b)] imply that

$$B^{s-\frac{\epsilon}{2}}_{p,p}(\mathbb{R}^n, \mathbb{C}) = F^{s-\frac{\epsilon}{2}}_{p,p}(\mathbb{R}^n, \mathbb{C})$$

where the corresponding norms are equivalent. Finally [41, p. 172 (3)] or [39, p. 47 (8)] implies

$$F^{s-\frac{\epsilon}{2}}_{p,p}(\mathbb{R}^n, \mathbb{C}) \hookrightarrow F^{s-\epsilon}_{p,2}(\mathbb{R}^n, \mathbb{C})$$

By Definition B.14 (c), the space $F^{s-\epsilon}_{p,2}(\mathbb{R}^n, \mathbb{C})$ coincides with $H^{s-\epsilon}_p(\mathbb{R}^n, \mathbb{C})$ which finishes the proof. $\square$

The first part of the following Lemma yields a characterization of the space $H^s_p(\mathbb{R}^n, \mathbb{C})$ by means of Fourier transformations. The second part shows that, in the case of $s \in \mathbb{N}$, the definition of the space $W^s_p(\mathbb{R}^n, \mathbb{C})$ is equivalent to the requirement that suitable weak derivatives exist and are bound in $L^p(\mathbb{R}^n, \mathbb{C})$. This establishes a connection to the "classical analysis". The content of the following Lemma is stated in [41, 2.3.3, Theorem., p. 177]

Lemma B.16. *(a) Let $s \in (-\infty, \infty)$ and $p \in (1, \infty)$. Then*

$$H^s_p(\mathbb{R}^n, \mathbb{C}) = \{u \in \mathcal{S}' : \ \|u\|_{H^s_p(\mathbb{R}^n,\mathbb{C})} := \|F^{-1}(1 + |x|^2)^{\frac{s}{2}} Fu\|_{L^p(\mathbb{R}^n,\mathbb{C})} < \infty\}$$

and the norms $\|\cdot\|_{H^s_p(\mathbb{R}^n,\mathbb{C})}$ and $\|\cdot\|_{F^s_{p,2}(\mathbb{R}^n,\mathbb{C})}$ are equivalent.

(b) If $s \in \mathbb{N}_{>0}$ and $p \in (1, \infty)$ then

$$W^s_p(\mathbb{R}^n, \mathbb{C}) = \left\{u \in \mathcal{S}' : \ \|u\|_{W^s_p(\mathbb{R}^n,\mathbb{C})} = \left(\sum_{|\alpha| \le s} \|D^\alpha u\|^p_{L^p(\mathbb{R}^n,\mathbb{C})}\right)^{\frac{1}{p}} < \infty\right\}$$

and the norms $\|\cdot\|_{W^s_p(\mathbb{R}^n,\mathbb{C})}$ and $\|\cdot\|_{H^s_p(\mathbb{R}^n,\mathbb{C})}$ are equivalent.

We also want to give an appropriate characterization of the space $W^s_p(\mathbb{R}^n, \mathbb{C})$ where $s > 0$ is not a natural number. The following result, whose content can be deduced from [40, 1.2.5, Theorem., p. 8] together with [40, 1.5.1, Definition., p. 28] shows, that for each $s \in (0, 1)$ and $p \in (1, \infty)$ the definition of the the space $B^s_{pp}(\mathbb{R}^n, \mathbb{C})$ is equivalent to the definition of the *Slobodeckij space* $W^s_p(\mathbb{R}^n, \mathbb{C})$, introduced in [39, 2.2.2., p. 36 (8)]. We also refer to [38, p. 60: 9.2.11. Remark.] where the equivalence of the corresponding norms is stated explicitly.

Lemma B.17. *Given $s \in (0,1)$ and $p \in (1,\infty)$, then*

$$\begin{aligned} &W_p^s(\mathbb{R}^n,\mathbb{C}) = B_{pp}^s(\mathbb{R}^n,\mathbb{C}) \\ =&\Bigg\{ u \in W_p^s(\mathbb{R}^n,\mathbb{C}) :\ \|u\|_{W_p^s(\mathbb{R}^n,\mathbb{C})} := \left(\int_{\mathbb{R}^n} |u(x)|^p\,dx\right)^{\frac{1}{p}} \\ &\qquad + \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}} < \infty \Bigg\} \end{aligned}$$

and the norm $\|\cdot\|_{W_p^s(\mathbb{R}^n,\mathbb{C})}$ is equivalent to the norm $\|\cdot\|_{B_{pp}^s(\mathbb{R}^n,\mathbb{C})}$ which was introduced in Definition B.14 (a).

B.3 Interpolation of Sobolev spaces and Besov spaces

There are a lot of interpolation results concerning $F_{p,q}^s$-spaces. Some of them are listed in [41, 2.4.2, Theorem 1., pp. 184-185]. We are interested in a special case which is stated in [41, 2.4.2, Remark 2., p. 185], i.e.:

Lemma B.18. *([41, 2.4.2, p. 185 (11)]) If $s_0, s_1 \in (-\infty,\infty)$, $p_0, p_1 \in (1,\infty)$ and $\theta \in (0,1)$ then*

$$[H_{p_0}^{s_0}(\mathbb{R}^n,\mathbb{C}), H_{p_1}^{s_1}(\mathbb{R}^n,\mathbb{C})]_\theta = H_p^s(\mathbb{R}^n,\mathbb{C})$$

where

$$s = (1-\theta)s_0 + \theta s_1 \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

It is our aim to use this theory to treat the following problem: Given a function $u \in W_{n/2}^2(\mathbb{R}^n,\mathbb{C})$, then the Sobolev embedding theorem tells us that $u \in W_n^1(\mathbb{R}^n,\mathbb{C})$ and we have the following estimate:

$$\|u\|_{W_n^1(\mathbb{R}^n,\mathbb{C})} \leq C(n)\ \|u\|_{W_{n/2}^2(\mathbb{R}^n,\mathbb{C})}$$

Under the additional assumption that u is also contained in $W_p^\epsilon(\mathbb{R}^n,\mathbb{C})$ where $\epsilon > 0$ and $p \in (1,\infty)$, we want to know if we have more regularity than $W_n^1(\mathbb{R}^n,\mathbb{C})$, i.e.: we are interested in the question if u is also contained in $W_q^1(\mathbb{R}^n,\mathbb{C})$ where $q > n$. This question will be answered in the following theorem:

Theorem B.19. *Given $n \in \mathbb{N}$ and $\epsilon \in (0,1)$. Then for each $p \in (n, \frac{4-\epsilon}{4-2\epsilon}n)$ there exists $q(n,\epsilon,p) \in (1,\infty)$ such that if $u \in W^2_{n/2}(\mathbb{R}^n,\mathbb{C}) \cap W^\epsilon_q(\mathbb{R}^n,\mathbb{C})$, then $u \in W^1_p(\mathbb{R}^n,\mathbb{C})$ and we have the following estimate*

$$\|u\|_{W^1_p(\mathbb{R}^n,\mathbb{C})} \leq C(n,q,\epsilon)\,\|u\|_{W^2_{n/2}(\mathbb{R}^n,\mathbb{C})}\,\|u\|_{W^\epsilon_q(\mathbb{R}^n,\mathbb{C})}$$

Proof. By Definition B.14 (d)

$$W^2_{n/2}(\mathbb{R}^n,\mathbb{C}) = H^2_{n/2}(\mathbb{R}^n,\mathbb{C})$$

and

$$W^\epsilon_q(\mathbb{R}^n,\mathbb{C}) = B^\epsilon_{qq}(\mathbb{R}^n,\mathbb{C}) \overset{(B.2)}{\hookrightarrow} H^{\epsilon/2}_q(\mathbb{R}^n,\mathbb{C}) \tag{B.3}$$

Now, we apply Lemma B.18 to the spaces $H^2_{n/2}(\mathbb{R}^n,\mathbb{C})$ and $H^{\epsilon/2}_q(\mathbb{R}^n,\mathbb{C})$ i.e.: we set

$$\begin{aligned} s_0 &= 2 \qquad & s_1 &= \frac{\epsilon}{2} \\ p_0 &= \frac{n}{2} \qquad & p_1 &= q \end{aligned}$$

and we choose $\theta \in (0,1)$ s.th. $1 = (1-\theta)2 + \theta s_1 = 1$, i.e.: $\theta = \frac{1}{2-s_1}$, then

$$\begin{aligned} \frac{1}{p} &= (1-\theta)\frac{2}{n} + \theta\frac{1}{q} = \frac{1-s_1}{2-s_1}\cdot\frac{2}{n} + \frac{1}{2-s_1}\cdot\frac{1}{q} = \frac{2-2s_1}{2-s_1}\cdot\frac{1}{n} + \frac{1}{2-s_1}\cdot\frac{1}{q} \\ &= \frac{2-\epsilon}{2-\frac{\epsilon}{2}}\cdot\frac{1}{n} + \frac{1}{2-\frac{\epsilon}{2}}\cdot\frac{1}{q} = \frac{4-2\epsilon}{4-\epsilon}\cdot\frac{1}{n} + \frac{2}{4-\epsilon}\cdot\frac{1}{q} \end{aligned}$$

Lemma B.8 implies

$$\begin{aligned} \|u\|_{F^1_{p,2}(\mathbb{R}^n,\mathbb{C})} &\leq C(\epsilon)\,\|u\|_{F^2_{n/2,2}(\mathbb{R}^n,\mathbb{C})}\,\|u\|_{F^{\epsilon/2}_{q,2}(\mathbb{R}^n,\mathbb{C})} \\ &\overset{(B.3)}{\leq} C(n,p,\epsilon)\,\|u\|_{F^2_{n/2,2}(\mathbb{R}^n,\mathbb{C})}\,\|u\|_{B^\epsilon_{q,q}(\mathbb{R}^n,\mathbb{C})} \end{aligned}$$

The claim follows from the equivalence of the corresponding norms, stated in Lemma B.16 (b) and Lemma B.17.

□

Appendix C

Notation and Definitions

C.1 Notation

Here, we give an overview of a huge part of the notation we are using in this work. Sometimes it is clear that a quantity depends on a certain metric. In this situation we often omit the dependency in the notation, i.e. $Rm_g = Rm$ for instance.

- For $i \in \{1, ..., n\}$ $\partial_i = \frac{\partial}{\partial x^i}$ denotes a coordinate vector in a local coordinate system

- g_{ij} is a Riemannian metric in a local coordinate system and g^{ij} is the inverse of the Riemannian metric

- $dV_g = dV$ is the volume form induced by a Riemannian metric g

- $Vol_g(\cdot) = Vol(\cdot)$ is the n-dimensional volume of a set in a Riemannian manifold (M, g)

- $dA_g = dA$ is the $n-1$-dimensional volume form induced by a Riemannian metric g

- $Area_g(\cdot) = Area(\cdot)$ is the $n-1$-dimensional volume of a set in a Riemannian manifold (M, g)

- ω_n is the euclidean volume of a euclidean unit ball

- $Rm_g = Rm$ is the Riemannian curvature tensor. As in [36], in local coordinates, the sign convention is consistent with [10, p. 5], i.e. $R_{ijkl} = R_{ijk}^m g_{ml}$.
- $Rc_g = Rc$ is the Ricci tensor
- $R_g = R$ is the scalar curvature
- $\frac{\partial}{\partial t} g = g'$ is the time derivative of the metric
- grad $\mathcal{F}_g$ is the gradient of the functional $\mathcal{F}_g$ with respect to g (cf. [5, Chapter 4, 4.10 Definition, p. 119])
- $\overset{\circ}{Rc}_g = \overset{\circ}{Rc}$ is the traceless Ricci tensor, i.e.: $\overset{\circ}{Rc}_g = Rc_g - \frac{1}{n} R g$
- ${}^g\nabla T = \nabla T$ is the covariant derivative of a tensor T with respect to g
- ${}^g\nabla^m T = \nabla^m T$ is the covariant derivative of order m
- Δ_g is the Laplacian, introduced in [23, p. 44, 3-4.]
- $\langle T, S\rangle_g = \langle T, S\rangle$ is the inner product of two tensors (cf. [23, Exercise 3.8, p. 29])
- $|T|_g = |T|$ is the norm of a tensor, i.e. $|T|_g := \sqrt{\langle T, T\rangle}_g$
- $diam_g(\cdot) = diam(\cdot)$ is the diameter of a set in a Riemannian manifold
- $inj_g(M, x)$ is the injectivity radius in a point of a Riemannian manifold
- $inj_g(M)$ is the injectivity radius of a Riemannian manifold
- $d_g(x, y) = d(x, y)$ is the distance between the points x and y in a Riemannian manifold
- $B_d(x, r) = B(x, r)$ is the ball of radius $r > 0$ around x in a metric space
- d_g is the metric which is induced by a Riemannian metric g
- $B_g(x, r) = B_{d_g}(x, r)$ is a metric ball in a Riemannian manifold
- $d(x, y, t)$ is the distance between the points x and y in a Riemannian manifold $(M, g(t))$

- $L(\gamma, t)$ is the length of a curve γ in a Riemannian manifold $(M, g(t))$
- The notation $d\sigma$, which occurs in an integral like $\int_\gamma |\text{grad } \mathcal{F}| \, d\sigma$, refers to the integration with respect to arc length
- $D(\gamma(t), r)$ / $D(\gamma, r)$ is a normal disc around a point in a curve γ / a (normal) tube around a curve γ with radius r (cf. Definition 2.1)
- $f_k(x, g)$ / $f_k(M, g)$ is introduced in Definition 2.2
- $d\pi$ denotes the push forward and $|d\pi|$ denotes the operator norm of the push forward of the projection map in the context of Theorem 2.3
- Γ denotes the local bilinear form in Definition C.9, $|\Gamma|$ is the norm of this bilinear form which is also introduced in Definition C.9
- $W^{k,p}(\Omega)$ is / are the Sobolev space/-es defined in [15, Chapter 7, pp. 144-176] we point out, that in Definition B.14, we have also introduced spaces which are denoted by $W_p^s(\mathbb{R}^n, \mathbb{C})$.
- $r_g(x)$ is the harmonic radius, which is introduced in Definition 3.2. The dependency of the constants R_0, α, K_1, K_2 and K_3 does not appear in the notation, because these constants are assumed to be fixed along the considered sequence
- The expressions $\|\partial g\|_{L^n(B(0,r))}$ and $[g]_{C^\alpha(B(0,r))}$ appear in the definition of the harmonic radius in Definition 3.2. Other L^p-norms of the derivative of g are defined similar.

C.2 Definitions

Definition C.1. *([7, Definition 7.3.17., p. 256]) Let X and Y be two sets. A correspondence between X and Y is a set $\mathfrak{R} \subseteq X \times Y$ satisfying the following condition: for each $x \in X$ there exists at least one $y \in Y$ such that $(x, y) \in \mathfrak{R}$, and for each $y \in Y$ there exists an element $x \in X$ such that $(x, y) \in \mathfrak{R}$.*

Definition C.2. *([7, Definition 7.3.21., p. 257]) Let $\mathfrak{R}$ be a correspondence between metric spaces (X, d_X) and (Y, d_Y). The distorsion of $\mathfrak{R}$ is defined by*

$$dis\ \mathfrak{R} := \sup \{ |d_X(x_1, y_1) - d_Y(x_2, y_2)| \mid (x_1, x_2), (y_1, y_2) \in \mathfrak{R} \}$$

Definition C.3. *([21, p. 35: 2.14. Definition]) Let (X, d) be a metric space and $\epsilon > 0$. A set $S \subseteq X$ is called an ϵ-net if*

$$dist_d(x, S) := \inf_{y \in S} d(x, y) < \epsilon \quad \forall x \in X$$

Definition C.4. *([7, Exercise 1.6.4., p. 14]) Let (X, d) be a metric space and $\epsilon > 0$. A subset $S \subseteq X$ is called ϵ-separated if $d(x, y) \geq \epsilon$ for all $x, y \in S$ satisfying $x \neq y$.*

We want to mention that this definition is stronger than the definition of an r-net in [7, Definition 1.6.1., p. 13], i.e.: an r-net in the sense of [21] is also an r-net in the sense of [7].

Definition C.5. *(cf. [7, Definition 8.1.1., p. 272]) A sequence $(X_i, d_i, p_i)_{i \in \mathbb{N}}$ of pointed metric spaces converges in the pointed Gromov-Hausdorff sense to a pointed metric space (X, d, p) if the following holds: For every $r > 0$ and $\epsilon > 0$ theres exists a number i_0 such that for every $i \geq i_0$ there exists a map $f_i : B_{d_i}(p_i, r) \longrightarrow X$ satisfying the following properties:*

- *$f(p_i) = p$ for all $i \geq i_0$*
- *$dis(f_i) := \sup_{x_1, x_2 \in B_{d_i}(p_i, r)} |d(f(x_1), f(x_2)) - d_i(x_1, x_2)| < \epsilon$ for all $i \geq i_0$*
- *the ϵ-neighborhood of the set $f(B_{d_i}(p_i, r))$ contains the ball $B_d(p, r - \epsilon)$*

Definition C.6. *A sequence of pointed Riemannian manifolds $(M_i, g_i, p_i)_{i \in \mathbb{N}}$ converges to (M, g, p) with in the $C^{k,\alpha}$-sense, if for each $R > 0$ there exists a domain $\Omega_R \subseteq M$ with $B_g(p, R) \subseteq \Omega_R$ and embeddings $f_i : \Omega_R \longrightarrow M_i$, where $i \geq I_0(R) \in \mathbb{N}$ so that $f_i(\Omega_R) \supseteq B_{g_i}(p_i, R)$ and $f_i^* g_i$ converges to g in the $C^{k,\alpha}$-sense on Ω_R.*

Definition C.7. *Let (M^n, g) be a Riemannian manifold. A sequence of tensors $(T_i)_{i \in \mathbb{N}}$ on M converges to a tensor T with respect to the $W^{2,\frac{n}{2}}$-topology, if there exists a covering of charts $(\varphi_s : U_s \longrightarrow \mathbb{R}^n)_{s \in \{1,\ldots,N\}}$ so that the overlap is smooth and the components of T_i, considered as functions on $\varphi_s(U_s)$, converge to the components of T with respect to the $W^{2,\frac{n}{2}}$-topology.*

Definition C.8. *Let (M^n, g) be a smooth Riemannian manifold, and let T be a k-tensor field, then for each $q \in [1, \infty)$ we define*

$$\|T\|_{L^q(M,g)} := \left(\int_M |T|_g^q \, dV_g \right)^{\frac{1}{q}}$$

and

$$\|T\|_{L^\infty(M,g)} := \operatorname*{ess\,sup}_M |T|_g$$

Here we assume, that the desired expressions exist.

The following definition is based on [20, (1), p. 261]

Definition C.9. *Let (M^n, g) be a smooth Riemannian manifold $p \in M$, $U \subseteq M$ a star-shaped neighborhood around p, and $\varphi : U \longrightarrow V$ a normal chart centered at p, then for each $q \in U$ we define a symmetric, bilinear map Γ as follows:*

$$\begin{aligned} \Gamma : T_qM \times T_qM &\longrightarrow T_qM \\ (u, v) &\mapsto \Gamma^k_{ij} u^i v^i \partial_k \end{aligned}$$

and $|\Gamma|$ is defined to be the smallest value $C > 0$ so that

$$|\Gamma(u, v)|_g \leq C|u|_g|v|_g$$

for all $u, v \in T_pM$.

In the following, we introduce the concept of convergence *in the sense of C^k-local submersions* which is needed in the proof of Theorem 2.15 and in the proof of Theorem 1.2. Here, we quote [34, Definition 2.1-Definition 2.4, p. 45]

Definition C.10. *A topological space G is a pseudogroup if there exist pairs $(a, b) \in G \times G$ such that the product $ab \in G$ is defined and satisfies*

(1) If ab, bc, $(ab)c$ and $a(bc)$ are all defined, then $(ab)c = a(bc)$

(2) If ab is defined, then for every neighborhood W of ab, there are neighborhoods $U \ni a$ and $V \ni b$ such that for all $x \in U$, $y \in V$, xy is defined and $xy \in W$

(3) There exists an element $e \in G$ such that for all $a \in G$, ae is defined and $ae = a$

(4) If for $(a, b) \in G \times G$, ab is defined and $ab = e$, then a is a left-inverse for b and we write $a = b^{-1}$. If b has a left inverse, then for every neighborhood U of b^{-1} there is a neighborhood V of b such that every $y \in V$ has a left inverse $y^{-1} \in G$

Definition C.11. *A pseudogroup G is a Lie group germ if a neighborhood of the identity element $e \in G$ can be given a differentiable structure such that the operations of multiplication and inversion are differentiable maps when defined.*

Definition C.12. *Fix $k \in (0,\infty]\backslash\mathbb{N}$. A sequence of pointed n-dimensional Riemannian manifolds $(M_i, g_i, p_i)_{i\in\mathbb{N}}$ locally converges to a pointed metric space (X, d, x) in the sense of C^k-local submersions at x, if there is a Riemannian metric h on an open neighborhood of $0 \in V \subseteq \mathbb{R}^n$, a pseudo group Γ of local isometries of (V, h) such that the quotient is well-defined, an open set $U \subseteq X$ and maps*

$$\Phi_i : (V, 0) \longrightarrow (M_i, p_i)$$

so that

(1) $(M_i, d_{g_i}, p_i)_{i\in\mathbb{N}}$ *converges to* (X, d, x) *in the pointed Gromov-Hausdorff topology*

(2) the identity component of Γ *is a Lie group germ*

(3) $(V/\Gamma, \overline{d}_h) \cong (U, d)$ *where* $\overline{d}_h$ *is the induced distance function on the quotient*

(4) $(\Phi_i)_*$ *is nonsingular on* V *for all* $i \in \mathbb{N}$

(5) h *is the* C^k*-limit of* $\Phi_i^* g_i$ *in the sense of uniform convergence on compact sets of the first* k *derivatives. Here,* $k \in (0,\infty)\backslash\mathbb{N}$ *is meant in the usual Hölder space.*

Definition C.13. *Fix $k \in (0,\infty]\backslash\mathbb{N}$. A sequence of pointed n-dimensional Riemannian manifolds $(M_i, g_i, p_i)_{i\in\mathbb{N}}$ converges to a pointed metric space (X, d, x) in the sense of C^k-local submersions if for every $y \in X$ there are points $q_i \in M_i$ such that $(M_i, g_i, q_i)_{i\in\mathbb{N}}$ converges to (X, d, y) in the sense of C^k-local submersions at y.*

Bibliography

[1] M. T. Anderson. Ricci curvature bounds and Einstein metrics on compact manifolds. *Journal of the American Mathematical Society*, pages 455–490, 1989.

[2] M. T. Anderson. Convergence and rigidity of manifolds under Ricci curvature bounds. *Inventiones mathematicae*, 102(1):429–445, 1990.

[3] E. Aubry. Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature. *Annales Scientifiques de l'École Normale Supérieure*, 40(4):675–695, 2007.

[4] Y. Bernard. Fundamentals of Interpolation theory, WS 2009/10. Albert-Ludwigs-Universität Freiburg, AG Geometrische Analysis.

[5] A. L. Besse. *Einstein manifolds.* Springer Science & Business Media, 1987. First Reprint 2002.

[6] C. Böhm, R. Lafuente, and M. Simon. Optimal curvature estimates for homogeneous Ricci flows. *arXiv preprint arXiv:1604.02625*, 2016. version 2.

[7] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33. American Mathematical Society Providence, 2001.

[8] J. Cheeger, D. Gromoll, et al. The splitting theorem for manifolds of nonnegative Ricci curvature. *Journal of Differential Geometry*, 6(1):119–128, 1971.

[9] J. Cheeger, M. Gromov, M. Taylor, et al. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry

of complete Riemannian manifolds. *Journal of Differential Geometry*, 17(1):15–53, 1982.

[10] B. Chow, P. Lu, and L. Ni. *Hamilton's Ricci flow*, volume 77. American Mathematical Soc., 2006.

[11] D. M. DeTurck and J. L. Kazdan. Some regularity theorems in Riemannian geometry. *Annales scientifiques de lÉcole Normale Supérieure*, 14(3):249–260, 1981.

[12] N. Dunford and J. T. Schwartz. *Linear operators. Part I. General theory. With the assistance of William G. Bade and Robert G. Bartle. Reprint of the 1958 original.* Wiley Classics Lib. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.

[13] L. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.

[14] M. Giaquinta and L. Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*. Springer Science & Business Media, 2012.

[15] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*, volume 224. Springer Science & Business Media, 2001.

[16] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, 17(2):255–306, 1982.

[17] S. Hildebrandt, H. Kaul, and K.-O. Widman. An existence theorem for harmonic mappings of Riemannian manifolds. *Acta Mathematica*, 138(1):1–16, 1977.

[18] J. Jost and H. Karcher. Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen. *manuscripta mathematica*, 40(1):27–77, 1982.

[19] V. Julin, T. Liimatainen, and M. Salo. p-harmonic coordinates for Hölder metrics and applications. *arXiv preprint arXiv:1507.03874*, 2015. version 1.

[20] H. Kaul. Schranken für die Christoffelsymbole. *manuscripta mathematica*, 19(3):261–273, 1976.

[21] J. LaFontaine, M. Katz, M. Gromov, S. M. Bates, P. Pansu, P. Pansu, and S. Semmes. *Metric structures for Riemannian and non-Riemannian spaces*. Springer, 2007.

[22] J. Lee. *Introduction to smooth manifolds*, volume 218. Springer, 2012.

[23] J. M. Lee. *Riemannian manifolds: An Introduction to Curvature*, volume 176. Springer Science & Business Media, 1997.

[24] B. O'Neill. *Semi-Riemannian Geometry With Applications to Relativity, 103*, volume 103. Academic press, 1983.

[25] P. Petersen. Convergence theorems in Riemannian geometry. *MSRI Publications*, 30:167–202, 1997.

[26] P. Petersen. *Riemannian geometry*, volume 171. Springer, 2006.

[27] P. Petersen and G. Wei. Relative volume comparison with integral curvature bounds. *Geometric & Functional Analysis GAFA*, 7(6):1031–1045, 1997.

[28] T. Sakai. On continuity of injectivity radius function. *Math. J. Okayama Univ*, 25(1):91–97, 1983.

[29] M. Simon. Extending four dimensional Ricci flows with bounded scalar curvature. *arXiv preprint arXiv:1504.02910*, 2015. version 1.

[30] M. Simon. Some integral curvature estimates for the Ricci flow in four dimensions. *arXiv preprint arXiv:1504.02623*, 2015. version 1.

[31] J. Streets. The gradient flow of the L^2 curvature energy near the round sphere. *Advances in Mathematics*, 231(1):328–356, 2012.

[32] J. Streets. The gradient flow of the L^2 curvature functional with small initial energy. *Journal of Geometric Analysis*, 22(3):691–725, 2012.

[33] J. Streets. Collapsing in the L^2 Curvature Flow. *Communications in Partial Differential Equations*, 38(6):985–1014, 2013.

[34] J. Streets. The long time behavior of fourth order curvature flows. *Calculus of Variations and Partial Differential Equations*, 46(1-2):39–54, 2013.

[35] J. Streets. A Concentration-Collapse Decomposition for L^2 Flow Singularities. *Communications on Pure and Applied Mathematics*, 69(2):257–322, 2016.

[36] J. D. Streets. The gradient flow of $\int_M |Rm|^2$. *Journal of Geometric Analysis*, 18(1):249–271, 2008.

[37] P. Topping. *Lectures on the Ricci flow*, volume 325. Cambridge University Press, 2006.

[38] H. Triebel. Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation. *Arkiv för Matematik*, 11(1):13–64, 1973.

[39] H. Triebel. *Theory of Function Spaces.* Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.

[40] H. Triebel. *Theory of Function Spaces II.* Monographs in Mathematics. Springer Basel, 1992.

[41] H. Triebel. *Interpolation Theory, Function spaces, Differential operators.* Johann Ambrosius Barth, 1995.

[42] D. Yang. Convergence of Riemannian manifolds with integral bounds on curvature. I. *Annales scientifiques de l'École normale supérieure*, 25(1):77–105, 1992.

[43] D. Yang. Convergence of Riemannian manifolds with integral bounds on curvature. II. *Annales scientifiques de l'École normale supérieure*, 25(2):179–199, 1992.

[44] D. Yang. L^p pinching and compactness theorems for compact Riemannian manifolds. *Forum Mathematicum*, 4(4):323–334, 1992.

Ehrenerklärung

Ich versichere hiermit, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; verwendete fremde und eigene Quellen sind als solche kenntlich gemacht.
Ich habe insbesondere nicht wissentlich:

- Ergebnisse erfunden oder widersprüchliche Ergebnisse verschwiegen,
- statistische Verfahren absichtlich missbraucht, um Daten in ungerechtfertigter Weise zu interpretieren,
- fremde Ergebnisse oder Veröffentlichungen plagiiert oder verzerrt wiedergegeben.

Mir ist bekannt, dass Verstöße gegen das Urheberrecht Unterlassungs- und Schadenersatzansprüche des Urhebers sowie eine strafrechtliche Ahndung durch die Strafverfolgungsbehörden begründen kann.
Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form als Dissertation eingereicht und ist als Ganzes auch noch nicht veröffentlicht.

(Ort, Datum)

(Unterschrift)