Ehrhart polynomials, successive minima, and an Ehrhart theory for rational dilates of a rational polytope

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von Dipl.-Math. Eva Linke

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Gutachter: Prof. Dr. Martin Henk

Prof. Dr. Matthias Beck

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Zusammenfassung

Die vorliegende Arbeit untersucht auf zwei verschiedene Weisen, wie sich die Gitterpunktanzahl in Polytopen verhält.

Im ersten Teil der Arbeit wird eine vermutete Ungleichung von Betke, Henk und Wills untersucht, die die Gitterpunktanzahl durch einen Ausdruck in den sukzessiven Minima des Polytops abschätzt. Hier wird die Polytopklasse der Zonotope betrachtet. Zonotope sind Minkowski-Summen von Strecken und daher von einfacher Struktur. Um die Ungleichung anzugehen, benutzen wir den zweiten Satz von Minkowski über sukzessive Minima, der das Volumen eines konvexen Körpers durch einen Ausdruck in den sukzessiven Minima in beide Richtungen beschränkt. Wir verbessern die offensichtliche untere Schranke in Minkowskis zweitem Satz für die Klasse der Zonotope, indem wir das Volumen von Zonotopen, die ein gegebenes Kreuzpolytop enthalten nach unten beschränken.

Im zweiten Teil der Arbeit wird untersucht, wie sich die Gitterpunktanzahl in einem Polytop durch Skalieren mit rationales Faktoren verändert. Dabei wird Ehrharts Theorem über Gitterpunkte in ganzzahligen Vielfachen rationaler Polytope verallgemeinert. Es stellt sich heraus, dass die Anzahl der Gitterpunkte in rP für ein rationales Polytop P und rationales r ein rationales Quasipolynom in r ist, das heißt, eine polynomiale Funktion, deren Koeffizienten periodische Funktionen sind. Diese Koeffizienten sind stückweise polynomial und untereinander durch Ableitungen verknüpft. In einem Spezialfall können wir zeigen, dass die minimalen Perioden der Koeffizienten monoton sind. Dies ist im ganzzahligen Fall nicht korrekt und legt nahe, dass rationale Quasipolynome mehr über die geometrische Struktur aussagen als die ganzzahlige Variante.

Abstract

In this work we study the behavior of the number of lattice points inside a polytope with two different approaches.

In the first part we consider a conjectured inequality of Betke, Henk and Wills, which bounds the number of lattice points inside a polytope by a term involving Minkowski's successive minima. We investigate the class of zonotopes. Zonotopes are Minkowski sums of line segments, and thus they are of easy structure. To approach the above mentioned inequality, we use Minkowski's second theorem, which bounds the volume of a convex body by an expression involving its successive minima. We improve the fairly obvious lower bound for the class of zonotopes by bounding the volume of a zonotope containing a given crosspolytope from below.

In the second part we study the behavior of the number of lattice points inside a polytope, when the polytope is dilated by a rational factor. Thereby, we generalize Ehrhart's theorem on lattice points in integral dilates of rational polytopes. It turns out that the number of lattice points inside rP for a rational polytope P and rational number r is a rational quasi-polynomial in r, that is, a polynomial function whose coefficients are themselves periodic functions. These coefficients are piecewise polynomial and are connected to each other by differentiation. In a special case we show that the minimal periods of these coefficients are monotonically decreasing. This is not true in the integral case, and thus we suspect that the rational quasi-polynomial preserves more of the geometric structure of a polytope than the integral one.

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Introduction

Lattice points occur naturally, whenever a discrete mathematical problem is considered which makes them subject of interest in classic and recent mathematics.

Polytopes naturally show up whenever linear conditions are posed to a problem or when seemingly easy-to-handle geometric objects with nice structure are studied.

Lattice points and polytopes are the central topics this work and what is connecting the two main chapters—counting lattice points in a given polytope P. This is usually done with two different approaches. One way is to connect the number of lattice points inside P to other magnitudes related to the polytope—as in this thesis Minkowski's successive minima—while another possibility is to associate the number of lattice points inside P with the number of lattice points in some transform of P—as in this thesis the dilation of P by a rational factor.

We start this work in Chapter 1 with a short overview of all definitions and notation that are used. Beside introducing the notion of a lattice, we give the essential definition of the lattice point enumerator by

$$\mathrm{G} : M \mapsto \#(M \cap \mathbb{Z}^n)$$

for bounded sets $M \subset \mathbb{R}^n$. Introducing convex bodies, Minkowski's successive minima and polytopes, we mention the main tools and theorems from the literature.

Definition 1.6
The ith successive minimum of a convex body K with $K = -K$, $\lambda_i(K)$ is the minimal positive real number with the property that $\lambda_i(K)K$ contains i linearly independent lattice points.

The most important theorem in this field is Minkowski's second theorem, which gives upper and lower bounds on the volume $\mathrm{vol}(K)$ of a convex body K in terms of the successive minima:

Theorem 1.9 (Minkowski, 1896)

$$\frac{2^n}{n!} \leq \mathrm{vol}(K) \prod_{i=1}^{n} \lambda_i(K) \leq 2^n.$$

We further give a brief overview of the classical integral Ehrhart theory in Section 1.2, with Ehrhart's main theorem:

Theorem 1.16 (Ehrhart, 1962)
For polytopes with integral vertices, $\mathrm{G}(kP)$ *is a polynomial in the integral variable* k *whose degree is the dimension of* P*.*

The coefficients of this polynomial are denoted by $\mathrm{G}_i(P)$ and depend only on P.

Overview and main results—Chapter 2

In Chapter 2 we present results on connections between the lattice point enumerator G and Minkowski's successive minima. The investigations on this topic were motivated by a conjecture of Betke, Henk and Wills that would be a discrete generalization of Minkowski's second theorem, and, somewhat modified from the original version can be formulated as follows:

For which polytopes P *does*

$$\mathrm{G}(P) \leq \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(P)} + 1 \right)$$

hold?

Or, considering the inequality summand-wise:

For which polytopes P *does*

$$\mathrm{G}_i(P) \leq \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ \#J = i}} \prod_{j \in J} \frac{2}{\lambda_j(P, \Lambda)}$$

hold?

The latter is known not to be true for all polytopes. In this work, we investigate the special class of zonotopes, that is, Minkowski sums of line segments. It this case, the Ehrhart polynomial is easily calculated but unfortunately it is difficult to handle the successive minima. To this end, we want to use the easy-to-prove lower bound in Minkowski's second theorem, and so we improve this bound in the first part of the chapter.

Therefore we consider the question, what are maximal crosspolytopes inside a zonotope, or, conversely, what are minimal zonotopes containing a given crosspolytope. Here, minimal is meant with respect to volume. Since this question is of interest as a standalone problem, we also transfer our techniques to the problem of examining minimal zonotopes containing a given simplex. In this question, our main theorem is

THEOREM 2.17

(i) *Let Z be a zonotope of minimal volume containing a crosspolytope C. Then* $\mathrm{vol}(Z) = \mathrm{vol}(C)\,\frac{n!}{\mathrm{maxdet}(n)}$.

(ii) *Let $Z \subseteq \mathbb{R}^n$ be a zonotope of minimal volume containing a simplex T. Then* $\mathrm{vol}(Z) = \mathrm{vol}(T_n)\,\frac{n!\,2^n}{\mathrm{maxdet}(n+1)}$.

Here, $\mathrm{maxdet}(n)$ describes the maximal absolute value of the determinant of an $(n \times n)$-matrix with entries in $\{-1, 1\}$. As the appearance of this determinant suggests, the problem of classifying the minimal zonotopes is deeply connected to classifying those matrices with maximal determinant—a well studied and still widely open problem.

We study in detail the 2-dimensional case for the simplex-problem and the 3-dimensional case for the crosspolytope-problem, and we give a complete classification of the minimal zonotopes in these cases.

The results on minimal zonotopes containing crosspolytopes are published in a joint work with Martin Henk and Jörg M. Wills in [24].

Using the volume bounds on zonotopes containing crosspolytopes, we get the following improvements on the lower bound in Minkowski's second theorem:

COROLLARY 2.33

$$\frac{2^n}{\mathrm{maxdet}(n)} \leq \mathrm{vol}(Z) \prod_{i=1}^{n} \lambda_i(Z).$$

This is an exponential improvement, since $\mathrm{maxdet}(n) \leq n^{n/2}$.
Corollary 2.33 also implies:

COROLLARY 2.37

$$\mathrm{G}_i(Z) \leq \binom{n}{i} \mathrm{maxdet}(n-i) \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ \#J = i}} \prod_{j \in J} \frac{2}{\lambda_j(Z, \Lambda)}.$$

These results, together with some approaches on the Betke–Henk–Wills conjecture for special bodies, are part of a joint work with Christian Bey, Martin Henk, and Matthias Henze in [10].

Overview and main results—Chapter 3

In Chapter 3 we investigate the behavior of G when a polytope is dilated by a rational factor. To this end, we first give an introduction to rational Ehrhart theory, that is, the known results in the theory of integral dilations of rational polytopes, that is, polytopes whose vertices have only rational coordiantes, and their lattice points.

Ehrhart's theorem for rational polytopes reads as follows:

Theorem 3.5 (Ehrhart, 1962)
For a rational polytope P,

$$\mathrm{G}(kP) = \sum_{i=0}^{\dim(P)} \mathrm{G}_i(P,k)k^i, \quad \text{for } k \in \mathbb{Z}_{\geq 0},$$

where $\mathrm{G}_i(P,k)$ depends only on P and k and is periodic with period $\mathrm{den}(P)$.

Here, $\mathrm{den}(P)$ is the denominator of P, which is the smallest positive integral number d such that the vertices of dP are integral. We introduce the i-indices of polytopes, which are magnitudes related to the size of dilations, such that i-faces contain integral points, and present the famous result of McMullen, stating that the i-index is also a period of $\mathrm{G}_i(P,k)$.

We present a generalization of this theory to rational dilation factors. Ehrhart's and McMullen's theorems are generalized to this setting, and we get:

Theorem 3.18
For a rational polytope P,

$$\mathrm{G}(rP) = \sum_{i=0}^{\dim(P)} \mathrm{Q}_i(P,r)r^i, \quad \text{for } r \in \mathbb{Q}_{\geq 0},$$

where $\mathrm{Q}_i(P,r)$ depends only on P and r, and is periodic and the rational i-index is a period of $\mathrm{Q}_i(P,r)$.

The generalization can be used to get more structural results on the coefficients $\mathrm{Q}_i(P,r)$, which are investigated in Section 3.3. Here our main result is:

Theorem 3.30
Let P be an n-dimensional rational polytope. Then $\mathrm{Q}_i(P,\cdot)$ is a piecewise polynomial of degree $n-i$, and

$$\mathrm{Q}_i'(P,r) = -(i+1)\,\mathrm{Q}_{i+1}(P,r), \quad i = 0,\ldots,n-1,$$

for almost all $r \geq 0$.

This immediately implies that, if $\mathbb{0} \in P$, the minimal periods of the ith coefficients are monotonically decreasing when i increases—a fact that is not true in the integral case.

We further work out an explicit formula for the rational Ehrhart quasi-polynomials of 2-dimensional simplices with one integral vertex and deduce the inequality $|\,\mathrm{Q}_1(P,r)| \leq \mathrm{Q}_1(P,0)$ in dimension 2 for all r and all polygons P.

The last two sections of this work address generalizations to several unknowns. We investigate the number of lattice points of the Minkowski sum of several polytopes, each scaled by a different factor. We can generalize most of the univariate statements to a multivariate version using work of McMullen on Minkowski sums. We further take a brief look at the number of integral points in polytopes $\{x \in \mathbb{R}^n : Ax \leq b\}$ with varying rational right-hand sides b, and get a rational quasi-polynomial structure here, too.

The univariate statements of this chapter are published in [27].

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1 Preliminaries

The main aim of this chapter is to give an overview over all definitions and notation used throughout this thesis, as well as to present examples and widely known results.

For proofs we refer the reader to Barvinok [3], Beck and Robins [7], Cassels [15], Gruber [18], Gruber and Lekkerkerker [19], and Ziegler [47].

We denote the n-dimensional Euclidean space by $\mathbb{R}^n = \{(x_1, \ldots, x_n)^\top : x_1, \ldots, x_n \in \mathbb{R}\}$ and equip it with the Euclidean inner product $x^\top y = \sum_{i=1}^n x_i y_i$, $x, y \in \mathbb{R}^n$ and the Euclidean norm $\|x\| = \sqrt{x^\top x}$. The ith *coordinate unit vector* is denoted by e_i for $i = 1 \ldots n$, the origin $(0, \ldots, 0)^\top$ by $\mathbb{0}$ and the vector $(1, \ldots, 1)^\top$ by $\mathbb{1}$. Furthermore, let $\mathbb{R}^{k \times m}$ be the set of all $k \times m$-matrices with entries in $\mathbb{R}$. The $(n \times n)$ identity matrix is denoted by I_n .

We denote the *linear hull* of a set $M \subseteq \mathbb{R}^n$ by

$$\operatorname{lin}(M) = \left\{ \sum_{i=1}^m \alpha_i x_i : m \in \mathbb{N}, \alpha_i \in \mathbb{R}, x_i \in M, \right\},$$

its *affine hull* by

$$\operatorname{aff}(M) = \left\{ \sum_{i=1}^m \alpha_i x_i : m \in \mathbb{N}, \alpha_i \in \mathbb{R}, \sum_{i=1}^m \alpha_i = 1, x_i \in M \right\},$$

its *positive hull* by

$$\operatorname{cone}(M) = \left\{ \sum_{i=1}^m \alpha_i x_i : m \in \mathbb{N}, \alpha_i \in \mathbb{R}_{\geq 0}, x_i \in M \right\},$$

and its *convex hull* by

$$\operatorname{conv}(M) = \left\{ \sum_{i=1}^m \alpha_i x_i : m \in \mathbb{N}, \alpha_i \in \mathbb{R}_{\geq 0}, \sum_{i=1}^m \alpha_i = 1, x_i \in M \right\}.$$

The convex hull $\operatorname{conv}(\{a, b\})$ of two points $a, b \in \mathbb{R}^n$ is abbreviated by $[a, b]$. Furthermore, M is called a linear, affine, or convex set if M equals its own linear, affine, or convex hull, respectively. If M equals its own positive hull, it is called a cone.

For a set $M \subset \mathbb{R}^n$ we denote by $\mathrm{int}(M)$ and $\mathrm{bd}(M)$ its *interior* and *boundary*, respectively, and, in case M is measurable, by $\mathrm{vol}(M)$ its *volume*, which is the usual n-dimensional Lebesgue measure of M. The *dimension* of M is $\dim(M) = \dim(\mathrm{aff}(M))$, if $M \neq \emptyset$, and $\dim(\emptyset) = -1$. Furthermore, by $\mathrm{vol}_{\dim(M)}(M)$ we denote the $\dim(M)$-dimensional volume of M, which is the usual Lebesgue measure of M with respect to $\mathrm{aff}(M)$. In case M is finite, $\#M$ is the number of elements of M.

For two sets $M_1, M_2 \subset \mathbb{R}^n$ we define the *Minkowski addition* $+$ by $M_1 + M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2 \in M_2\}$. We write $m_1 + M_2$ and $M_1 + m_2$ instead of $\{m_1\} + M_2$ and $M_1 + \{m_2\}$, respectively. In addition, we define the dilation $\alpha M = \{\alpha m : m \in M\}$ for $\alpha \in \mathbb{R}$ and a set M, and write $-M$ for $(-1)M$.

1.1 Lattice points and convex bodies

1.1 Definition (Lattice)
Let $k \in \{1, \ldots, n\}$, $b_1, \ldots, b_k \in \mathbb{R}^n$ be linearly independent and let $B = (b_1, \ldots, b_k)$ be the $n \times k$-matrix with columns b_i, $1 \leq i \leq k$. The set

$$\Lambda = \{z_1 b_1 + \ldots + z_k b_k : z_i \in \mathbb{Z}, 1 \leq i \leq k\}$$

is called a lattice with basis B or $\{b_1, \ldots, b_k\}$. k is called the dimension of Λ.

For an affine subspace H of $\mathbb{R}^n$, we say that Λ is a *lattice in H* if $\mathrm{aff}(\Lambda) = H$. Thus by 'lattices in $\mathbb{R}^n$' we mean full-dimensional lattices. The set of all lattices in $\mathbb{R}^n$ is denoted by $\mathcal{L}^n$ and an element of a lattice Λ is referred to as *lattice point*. A lattice point $z \in \Lambda$ is called *primitive*, if $[0, z] \cap \Lambda = \{0, z\}$. For a lattice Λ and an affine subspace H of $\mathbb{R}^n$ we say H is a lattice plane if $H = \mathrm{aff}(\Lambda \cap H)$, that is, H contains $\dim H + 1$ affinely independent lattice points.

For a lattice Λ with given basis B, the set $\{\mu_1 b_1 + \ldots + \mu_k b_k : \mu_i \in [0, 1), 1 \leq i \leq k\}$ is called a *fundamental parallelepiped* with respect to B or simply a *parallelepiped.* Its volume is the value $\sqrt{\det(B^T B)}$. In case that Λ is a lattice in $\mathbb{R}^n$, $\sqrt{\det(B^T B)} = |\det(B)|$. The basis is unique up to unimodular transformations, that is, $\overline{B}$ is a basis as well if and only if there exists a unimodular $(k \times k)$-matrix U with $\overline{B} = BU$. A $(k \times k)$-matrix $U = (u_{ij})$ is called *unimodular* if all entries u_{ij} are integers and $\det(U) \in \{-1, 1\}$. Thus the volume of a fundamental parallelepiped of Λ is independent of the choice of the basis and is referred to as the *determinant* of Λ, $\det(\Lambda)$.

Lattices are additive subgroups of $\mathbb{R}^n$, and they are discrete sets, that is, there exists a positive number λ such that $\|a_1 - a_2\| \geq \lambda$ for all lattice points $a_1 \neq a_2$. The smallest possible such number λ is the *length of a shortest lattice point.* For two lattices Λ, $\overline{\Lambda}$ of same dimension with $\overline{\Lambda} \subset \Lambda$, the *index* of $\overline{\Lambda}$ in Λ is the usual index of subgroups, which is known to be $\frac{\det(\overline{\Lambda})}{\det(\Lambda)}$. For B a basis of Λ and $\overline{B}$ a basis of $\overline{\Lambda}$, there exists a quadratic integral matrix D with $\overline{B} = BD$, and in that case $|\det(D)|$ is the index of $\overline{\Lambda}$ in Λ.

In this work, we will mostly deal with the integral lattice $\mathbb{Z}^n = \{z \in \mathbb{R}^n : z_i \in \mathbb{Z}, 1 \leq i \leq n\}$ but (lower dimensional) sublattices of $\mathbb{Z}^n$ will also occur.

1.2 Definition (Lattice point enumerator)
For a given lattice Λ let

$$\mathrm{G}_\Lambda : \{M \subset \mathbb{R}^n : M \textit{ bounded}\} \to \mathbb{Z}, \quad M \mapsto \#(M \cap \Lambda)$$

be the lattice point enumerator. In the case $\Lambda = \mathbb{Z}^n$ we simply write G instead of $\mathrm{G}_{\mathbb{Z}^n}$.

Numerous problems and results in Combinatorial and Integral Optimization, in Number Theory and especially in Geometry of Numbers can be formulated as statements about the structure or size of $\mathrm{G}(M)$ for some set M.

One classical result about the relation of the volume of sets M and its lattice points is the following theorem of Blichfeldt.

1.3 Theorem (Blichfeldt, 1914, [11])
Let $M \subset \mathbb{R}^n$ be measurable and $\Lambda \in \mathcal{L}^n$. If $\mathrm{vol}(M) > \det(\Lambda)$ then $\mathrm{G}_\Lambda(M - M) \geq 3$. If M is compact, the statement remains true if $\mathrm{vol}(M) = \det(\Lambda)$.

Equivalently, there exists a $t \in \mathbb{R}^n$ such that $\mathrm{G}_\Lambda(t + M) \geq 2$. We cannot expect to get results connecting directly the volume of M with its lattice points, since M can be of arbitrary large volume and nevertheless contained between two adjacent lattice hyperplanes.

Although there is no reasonable chance to get structural results for arbitrary sets M, it is worth analyzing G for certain classes of sets, such as convex bodies or polytopes.

1.4 Definition (Convex body)
A compact, convex, nonempty set $K \subset \mathbb{R}^n$ is called a convex body. *The set of all convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. $K \in \mathcal{K}^n$ is called* $\mathbb{0}$-symmetric *if $K = -K$ and we write $\mathcal{K}_0^n$ for the subset of $\mathcal{K}^n$ of $\mathbb{0}$-symmetric convex bodies.*

More general, a convex body is called *centrally symmetric* if it is symmetric with respect to some point $t \in \mathbb{R}^n$, that is, $K - t$ is $\mathbb{0}$-symmetric:

$$K - t = -(K - t) = -K + t.$$

A special convex body that should be mentioned here is the n-dimensional Euclidean ball $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Its boundary is the unit sphere in $\mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Every $\mathbb{0}$-symmetric convex body $K \in \mathcal{K}_0^n$ is the unit ball of the norm of $\mathbb{R}^n$ defined by

$$\|x\|_K = \min\{t \geq 0 : x \in tK\}.$$

Thus, since for an arbitrary norm $\|\cdot\|$ in $\mathbb{R}^n$, the set $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is a convex body, norms in $\mathbb{R}^n$ and $\mathbb{0}$-symmetric convex bodies $\mathcal{K}_0^n$ are the same classes of objects.

Considering the lattice point enumerator for $\mathbb{0}$-symmetric convex bodies we first state *Minkowski's fundamental theorem*, also known as *Minkowski's first theorem.*

1.5 Theorem (Minkowski, 1891, [35])
Let $\Lambda \in \mathcal{L}^n$ and $K \in \mathcal{K}_0^n$ with $\mathrm{vol}(K) \geq 2^n \det(\Lambda)$. Then K contains a nontrivial lattice point, that is $\mathrm{G}_\Lambda(K) \geq 3$.

This result is best-possible and equality is attained for instance for $\mathbb{0}$-symmetric lattice parallelepipeds, that is, if $B = (b_1, \ldots, b_n)$ is a basis of Λ, then

$$P = \{\mu_1 b_1 + \ldots + \mu_n b_n : \mu_i \in [-1, 1], 1 \leq i \leq n\}$$

has nontrivial lattice points, but λP doesn't for all $\lambda < 1$.

Obviously Minkowski's first theorem follows from the later theorem of Blichfeldt, Theorem 1.3. Both theorems can be generalized to the extent that the set contains at least $2k + 1$ lattice points if the volume is at least $k\ 2^n \det(\Lambda)$. Here we cannot expect that these or some of these lattice points are linearly independent, since K can be of arbitrary large volume with all its lattice points lying on a lattice line. To get the concept of linearly independence into that topic, Minkowski introduced the so-called successive minima:

1.6 Definition (Successive Minima)
Let $\Lambda \in \mathcal{L}^n$ and $K \in \mathcal{K}_0^n$ with non-empty interior. Then the i-th successive minimum of K with respect to Λ, $\lambda_i(K, \Lambda)$, $1 \leq i \leq n$, is defined by

$$\lambda_i(K, \Lambda) = \min\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i\}.$$

If $\Lambda = \mathbb{Z}^n$ we omit the second argument and write $\lambda_i(K)$, $1 \leq i \leq n$.

Thus, if $\lambda_1(K, \Lambda) \leq 1$ then K contains a nontrivial lattice point and hence Minkowski's first theorem can be stated as

$$\mathrm{vol}(K)\lambda_1(K,\Lambda)^n \leq 2^n \det(\Lambda).$$

1.7 Example
Let us consider the integral lattice $\mathbb{Z}^n$ and for $a \in \mathbb{R}^n_{>0}$ with $a_1 \geq a_2 \geq \ldots \geq a_n$ the axis parallel box $R = \{x \in \mathbb{R}^n : |x_i| \leq a_i \quad \text{for all } i\}$ (see Figure 1.1). Then the smallest

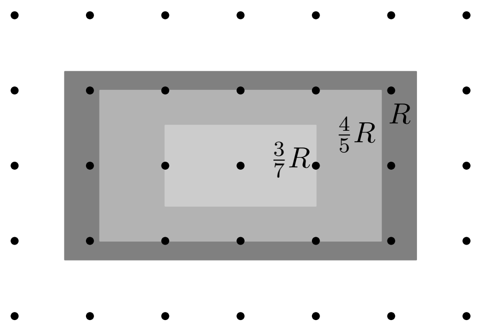

Figure 1.1: R, $\frac{3}{7}R$ and $\frac{4}{5}R$ for $a = (\frac{7}{3}, \frac{5}{4})^\top$.

number λ such that λR contains a nontrivial lattice point is $\lambda_1(R) = \frac{1}{a_1}$ and the lattice point in $\lambda_1(R)R$ is e_1. Furthermore the smallest λ such that λR contains a lattice point outside of $\mathrm{lin}(e_1)$ is $\lambda_2(R) = \frac{1}{a_2}$. This argumentation leads to $\lambda_i(R) = \frac{1}{a_i}$.

To be able to deal with successive minima, we state some basic and well known properties:

1.8 Proposition
Let $\Lambda \in \mathcal{L}^n$ and $K \in \mathcal{K}^n_0$ with non-empty interior.

(i) $\lambda_i(K,\Lambda) \leq \lambda_{i+1}(K,\Lambda)$ *for* $1 \leq i \leq n-1$.

(ii) $\lambda_i(\mu K,\Lambda) = \frac{1}{\mu}\lambda_i(K,\Lambda)$ *for* $1 \leq i \leq n$, $\mu \in \mathbb{R}_{\geq 0}$.

(iii) $\lambda_i(K,\mu\Lambda) = \mu\lambda_i(K,\Lambda)$ *for* $1 \leq i \leq n$, $\mu \in \mathbb{R}_{\geq 0}$.

(iv) $\lambda_1(B^n,\Lambda)$ *is the length of a shortest lattice point in* Λ.

Furthermore, if $a_i \in \lambda_i(K,\Lambda)K \cap \Lambda$, $1 \leq i \leq n$, are linearly independent then $\{a_1,\ldots,a_n\}$ is not necessarily a basis of Λ. Mahler [28] showed that there is a basis $\{b_1\ldots,b_n\}$ of Λ with $b_i \in i\lambda_i(K,\Lambda)K$ for $1 \leq i \leq n$.

With these successive minima we get the following improvement of Minkowski's first theorem, known as *Minkowski's second theorem.* Here, instead of the nth power of the first one, a product of all successive minima is considered and a lower bound can be also obtained. Both bounds are best possible.

1.9 Theorem (Minkowski, 1896, [35, 22])
Let $\Lambda \in \mathcal{L}^n$ and $K \in \mathcal{K}_0^n$ with non-empty interior. Then

$$\frac{2^n}{n!}\det(\Lambda) \le \mathrm{vol}(K)\prod_{i=1}^{n}\lambda_i(K,\Lambda) \le 2^n \det(\Lambda).$$

1.10 Example
We consider again the axis parallel boxes $R = \{x \in \mathbb{R}^n : |x_i| \le a_i, \text{ for all } i\}$ of Example 1.7. Here, $\mathrm{vol}(R) = 2^n \prod_{i=1}^n a_i$ and $\lambda_i(R) = \frac{1}{a_i}$. Therefore,

$$\mathrm{vol}(R)\prod_{i=1}^{n}\lambda_i(R) = 2^n = 2^n \det(\mathbb{Z}^n),$$

which attains the upper bound in Theorem 1.9. On the other hand, for lattice crosspolytopes $X = \mathrm{conv}\{\pm a_i e_i : i = 1,\dots,n\}$ for $a \in \mathbb{R}^n_{>0}$ with $a_1 \ge a_2 \ge \dots \ge a_n$, we have $\mathrm{vol}(X) = \frac{2^n}{n!}\prod_{i=1}^n a_i$ and $\lambda_i(X) = \frac{1}{a_i}$ and thus the lower bound in Theorem 1.9 is attained.

The lower bound in Theorem 1.9 is rather simple to prove and we will discuss the proof as well as improvements for a special class of convex bodies in Section 2.5. The upper bound is a deep result in the geometry of numbers. Its importance is reflected by the number of various proofs and generalizations to others settings. We refer to Bambah, Woods and Zassenhaus [1], Cassels [15], and Gruber and Lekkerkerker [19] for various proofs, and Blichfeldt [12], Bombieri and Vaaler [14], Hlawka [26] and Woods [46] for generalizations.

1.2 Polytopes and integral Ehrhart theory

To get some kind of structural results about the lattice point enumerator G_Λ, a special class of convex bodies is considered: polytopes.

1.11 Definition (Polytope)
A (convex) polytope in $\mathbb{R}^n$ is the convex hull $\mathrm{conv}\{v_1,\dots,v_k\}$ of finitely many points in $\mathbb{R}^n$. The set of all polytopes in $\mathbb{R}^n$ is denoted by $\mathcal{P}^n$.

As for all sets, the *dimension* of a polytope $P \in \mathcal{P}^n$ is the dimension of its affine hull, that is, $\dim(P) = \dim(\mathrm{aff}(P))$. We also consider the empty set $\emptyset = \mathrm{conv}(\emptyset)$ as a polytope and its dimension is set to -1. We call a polytope of dimension $d \in \mathbb{Z}_{\geq -1}$ a d-polytope. An n-dimensional polytope in $\mathbb{R}^n$ is also called *full-dimensional.*

Equivalently, polytopes can be defined as bounded intersections of finitely many closed halfspaces. This is a rather nontrivial statement which is due to Minkowski, 1896 ([34]) and Weyl, 1935 ([45]).

1.12 Definition (Face, Facet, Vertex)
A subset $F \subset P \in \mathcal{P}^n$ is called face of P if there exists $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$a^\top x = b \quad \forall x \in F \quad \textit{and} \quad a^\top x < b \quad \forall x \in P \setminus F.$$

If $\dim(F) = i$, F is called an i-face of P. The $(\dim(P) - 1)$-faces are called facets, 1-faces are called edges and 0-faces are called vertices.

Hence, $\emptyset$ and P itself are faces of dimension -1 and $\dim(P)$, respectively. The faces F with $\dim(F) \in \{0, \ldots, \dim(P) - 1\}$ are called *proper faces.*

Thus faces of a polytope P are polytopes as well, and a face F is the convex hull of all vertices of P that are contained in F. Using that, one easily gets that every face of a face of $P \in \mathcal{P}^n$ is a face of P as well, and if for any two faces F, $\overline{F}$ of P we have that $F \subset \overline{F}$ then F is a face of $\overline{F}$.

1.13 Proposition
Let F_i be an i-face of P. Then there exist j-faces F_j of P for all $j = i+1, \ldots, n-1$ such that

$$F_i \subset F_{i+1} \subset \cdots \subset F_{n-1} \subset P.$$

In this thesis we will mainly consider polytopes whose vertices are contained in a certain set. For a set $M \subset \mathbb{R}^n$ we write $\mathcal{P}(M)$ for the *set of all polytopes with vertices in M*. For a lattice Λ we call a polytope in $\mathcal{P}(\Lambda)$ a *lattice polytope*. Furthermore, polytopes in $\mathcal{P}(\mathbb{Z}^n)$ and $\mathcal{P}(\mathbb{Q}^n)$ are called *integral* and *rational polytopes*, respectively, and we write $\mathcal{P}^n_{\mathbb{Q}}$ and $\mathcal{P}^n_{\mathbb{Z}}$, respectively.

1.14 Example
Here are some typical and well-known polytopes:

(i) As in Examples 1.7 and 1.10, for $a, b \in \mathbb{R}^n$ with $a_i \geq b_i$ for all $i \in \{1, \ldots, n\}$ let

$$R = \{x \in \mathbb{R}^n : b_i \leq x_i \leq a_i\}$$

be the *rectangular axis-parallel box* with edge lengths $a_1 - b_1, \ldots, a_n - b_n$. If $a \in \mathbb{R}^n_{>0}$

and $b=-a$, then R is $\mathbb{0}$-symmetric. For $a=-b=(1,\dots,1)^\top$, R coincides with the n-dimensional $\mathbb{0}$-symmetric standard cube $C_n := [-1,1]^n = \mathrm{conv}\{(\pm 1,\dots,\pm 1)^\top\}$.

(ii) For linearly independent vectors $v_1,\dots,v_n \in \mathbb{R}^n$ let

$$\Delta = \mathrm{conv}(\{v_i : 1 \le i \le n\} \cup \{\mathbb{0}\})$$

be the *simplex* with vertices $\mathbb{0}, v_1,\dots,v_n$. If $v_i = e_i$ for $1 \le i \le n$ then Δ coincides with the n-dimensional standard simplex

$$T_n := \mathrm{conv}(\{e_i : 1 \le i \le n\} \cup \{\mathbb{0}\}) = \{x \in \mathbb{R}^n_{\ge 0} : x_1 + \ldots + x_n \le 1\}.$$

(iii) Similar to Example 1.10, for linearly independent vectors $v_1,\dots,v_n \in \mathbb{R}^n$ let

$$\diamondsuit = \mathrm{conv}\{\pm v_i : 1 \le i \le n\}$$

be the *crosspolytope* with vertices $\pm v_1,\dots,\pm v_n$. If $v_i = e_i$ for $1 \le i \le n$ then $\diamondsuit$ coincides with the n-dimensional standard crosspolytope

$$C_n^\star := \mathrm{conv}\{\pm e_i : 1 \le i \le n\} = \{x \in \mathbb{R}^n : |x_1| + \ldots + |x_n| \le 1\}.$$

For polytopes $P \in \mathcal{P}(\Lambda)$, structural information of the lattice point enumerator $\mathrm{G}_\Lambda(P)$ can be obtained. For 1-dimensional lattices Λ and $P=[a,b] \in \mathcal{P}(\Lambda)$ we get

$$\mathrm{G}_\Lambda(P) = \frac{b-a}{\det(\Lambda)} + 1.$$

For the 2-dimensional case we have the classical result of Pick.

1.15 Theorem (Pick, 1899 [38])
Let $\Lambda \in \mathcal{L}^2$ and $P = \mathrm{conv}\{v_1,\dots,v_m\}$ with $v_i \in \Lambda$ for all $i=1,\dots,m$. Then

$$\mathrm{G}_\Lambda(P) = \frac{\mathrm{vol}(P)}{\det(\Lambda)} + \frac{1}{2}\,\mathrm{G}_\Lambda(\mathrm{bd}(P)) + 1.$$

If $F_1,\dots,F_m$ are the edges of P, the number of lattice points on the boundary can be written as

$$\mathrm{G}_\Lambda(\mathrm{bd}(P)) = \sum_{i=1}^m (\mathrm{G}_\Lambda(F_i) - 1) = \sum_{i=1}^m \frac{\mathrm{vol}_1(F_i)}{\det(\mathrm{aff}(F_i) \cap \Lambda)}$$

and is homogeneous of degree 1. Thus Pick's Theorem generalizes to

$$\mathrm{G}_\Lambda(k\,P) = \frac{\mathrm{vol}(P)}{\det(\Lambda)} k^2 + \frac{1}{2}\,\mathrm{G}_\Lambda(\mathrm{bd}(P)) k + 1, \quad k \in \mathbb{Z}_{\ge 1}\,.$$

Furthermore, it is also easy to count the number of integral points in the interior of kP using

$$\mathrm{G}_\Lambda(\mathrm{int}(k\,P)) = \mathrm{G}_\Lambda(k\,P) - \mathrm{G}_\Lambda(\mathrm{bd}(kP)) = \frac{\mathrm{vol}(P)}{\det(\Lambda)}k^2 - \frac{1}{2}\,\mathrm{G}_\Lambda(\mathrm{bd}(P))k + 1.$$

Hence, $\mathrm{G}_\Lambda(k\,P)$ is a polynomial of degree 2 in $k \in \mathbb{N}$ with coefficients depending only on P and Λ. This result was generalized by Ehrhart to arbitrary dimensions:

1.16 Theorem (Ehrhart, 1962 [17])
Let $\Lambda \in \mathcal{L}^n$ and $P \in \mathcal{P}(\Lambda)$. Then there exist numbers $\mathrm{G}_i(P, \Lambda)$ depending only on P and Λ such that

$$\mathrm{G}_\Lambda(k\,P) = \sum_{i=0}^{\dim(P)} \mathrm{G}_i(P, \Lambda)k^i, \quad k \in \mathbb{Z}_{\geq 1},$$

is a polynomial of degree at most $\dim(P)$ *in* $k \in \mathbb{Z}_{\geq 1}$.

As a function in k, $\mathrm{G}_\Lambda(k\,P)$ is denoted by $\mathrm{G}_\Lambda(P, k)$ and is called the *Ehrhart polynomial* of P. $\mathrm{G}_i(P, \Lambda)$ are called its coefficients. Again, if $\Lambda = \mathbb{Z}^n$ we omit the lattice and simply write $\mathrm{G}(P, k)$ and $\mathrm{G}_i(P)$.

As in Pick's theorem in dimension 2, we have

$$\mathrm{G}_{\dim(P)}(P, \Lambda) = \frac{\mathrm{vol}_{\dim(P)}(P)}{\det(\mathrm{aff}(P) \cap \Lambda\,\mathrm{aff}(P))} \quad \text{and} \quad \mathrm{G}_0(P, \Lambda) = 1. \tag{1.1}$$

Thus, $\mathrm{G}(P, k)$ is of actual degree $\dim(P)$ in k. We can formally extend $\mathrm{G}_\Lambda(P, \cdot)$ to 0 by setting $\mathrm{G}_\Lambda(P, 0) = 1$ which is the Euler characteristic of P. Furthermore, if $F_1, \ldots, F_k$ are the facets of P, we get, as in the 2-dimensional case,

$$\mathrm{G}_{\dim(P)-1}(P, \Lambda) = \frac{1}{2}\sum_{i=1}^{k} \frac{\mathrm{vol}_{\dim(P)-1}(F_i)}{\det(\mathrm{aff}(F_i) \cap \Lambda)}. \tag{1.2}$$

Furthermore, since for all $k, m \in \mathbb{Z}_{\geq 0}$

$$\sum_{i=0}^{n} \mathrm{G}_i(P, \Lambda)m^i k^i = \mathrm{G}_\Lambda(mk\,P) = \mathrm{G}_\Lambda(k\,m\,P) = \sum_{i=0}^{n} \mathrm{G}_i(mP, \Lambda)k^i$$

and

$$\sum_{i=0}^{n} \mathrm{G}_i(P, \Lambda)m^i k^i = \mathrm{G}_\Lambda(mk\,P) = \mathrm{G}_{\frac{1}{m}\Lambda}(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i\left(P, \frac{1}{m}\Lambda\right)k^i,$$

we have that $\mathrm{G}_i(P, \Lambda)$ is homogeneous of degree i in the first argument and of degree $-i$ in the second argument.

1.17 Example

(i) Let $T \in \mathcal{P}_{\mathbb{Z}}^2$ be the triangle with the vertices $\binom{1}{1}, \binom{1}{-1}, \binom{-3}{0}$. We have $\mathrm{G}(T) = 7$,

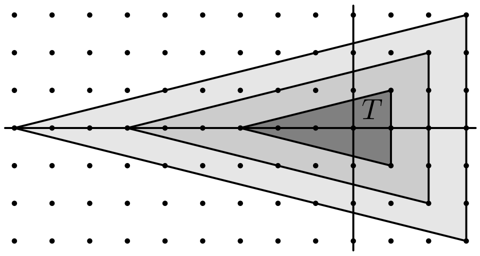

Figure 1.2: Triangles T, $2T$ und $3T$

$\mathrm{G}(2\,T) = 21$, $\mathrm{G}(3\,T) = 43$, and thus the Ehrhart polynomial is

$$\mathrm{G}(T, k) = \mathrm{G}_2(T)k^2 + \mathrm{G}_1(T)k + \mathrm{G}_0(T) = 4k^2 + 2k + 1.$$

(ii) We now consider the general axis-parallel boxes given in Example 1.14. Let $R = \{x \in \mathbb{R}^n : b_i \leq x_i \leq a_i\}$ for $a, b \in \mathbb{Z}^n$ with $a_i \geq b_i$ for all $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned} \mathrm{G}(R, k) &= \mathrm{G}(\{x \in \mathbb{R}^n : kb_i \leq x_i \leq ka_i\}) = \prod_{i=1}^{n} (k(a_i - b_i) + 1) \\ &= \sum_{J \subset \{1,\ldots,n\}} \prod_{j \in J} (k(a_j - b_j)) = \sum_{i=1}^{n} k^i \sum_{\substack{J \subset \{1,\ldots,n\} \\ \#J = i}} \prod_{j \in J} (a_j - b_j). \end{aligned}$$

Thus

$$\mathrm{G}_i(R) = \sum_{\substack{J \subset \{1,\ldots,n\} \\ \#J = i}} \prod_{j \in J} (a_j - b_j).$$

As a special case we get that

$$\mathrm{G}_i(C_n) = 2^i \binom{n}{i}.$$

(iii) For the standard simplex T_n we have

$$\begin{aligned} \mathrm{G}(T_n, k) &= \#\{x \in \mathbb{Z}_{\geq 0}^n : x_1 + \ldots + x_n \leq k\} = \sum_{i=0}^{k} \#\{x \in \mathbb{Z}_{\geq 0}^n : x_1 + \ldots + x_n = i\} \\ &= \sum_{i=0}^{k} \binom{n+i-1}{n-1} = \binom{n+k}{n} = \frac{1}{n!} \sum_{i=0}^{n} k^i \sum_{\substack{J \subset \{1,\ldots,n\} \\ \#J = n-i}} \prod_{j \in J} j. \end{aligned}$$

Thus
$$\mathrm{G}_i(T_n) = \frac{1}{n!} \sum_{\substack{J\subset\{1,\dots,n\}\\ \#J=n-i}} \prod_{j\in J} j.$$

(iv) Unfortunately, the coefficients $\mathrm{G}_i(P,\Lambda)$ are in general not non-negative: For instance, consider the so-called Reeve's simplices $T(m) = \mathrm{conv}\{\mathbb{0}, e_1, e_2, (1,1,m)^\top\} \in \mathcal{P}^3_{\mathbb{Z}}$. Using the formulas in Equations (1.1) and (1.2) for $\mathrm{G}_3(T(m))$, $\mathrm{G}_2(T(m))$, $\mathrm{G}_0(T(m))$ and the fact that $\mathrm{G}(T(m)) = 4$ we get that
$$\mathrm{G}(P,k) = \frac{m}{6}k^3 + k^2 + \frac{12-m}{6}k + 1,$$
which has negative $G_1(T(m))$ for $m \geq 13$.

2 Zonotopes: Minkowski's second theorem, Lattice points, and successive minima

Minkowski's theorems provide a connection between the volume of $\mathbb{0}$-symmetric convex bodies and their successive minima. The successive minima give information about a convex body in terms of the number of linearly independent lattice points it contains. Thus it is reasonable to expect a connection between the lattice point enumerator and the successive minima of a $\mathbb{0}$-symmetric convex body as well. In this chapter, we will explore these connections as well as Minkowski's second theorem in the case of a special class of polytopes, the zonotopes, that is, polytopes that are Minkowski sums of finitely many line segments.

We start with a section containing basics about lattice points, successive minima, and zonotopes. We mention a conjecture of Betke, Henk, and Wills, which can be seen as a discrete generalization of Minkowski's second theorem. The statements in the first section are well known and thus we will not present proofs here.

To get an improvement of the lower bound in Minkowski's second theorem for the special case of zonotopes, we work out bounds on the volume of zonotopes containing a given crosspolytope in Section 2.2. Since bounding the volume of zonotopes containing special convex bodies is a problem of interest in itself, we also apply our strategy to the case of simplices. In both cases, we present some work in the direction of characterizing the equality cases (Section 2.3) and solutions to this question in small dimensions (Section 2.4).

Using the results from Section 2.2 we improve in Section 2.5 the lower bound in Minkowski's second theorem for zonotopes by a factor of, roughly speaking, $n^{n/2}/c^n$. Additionally we present some results into the direction of the discrete version of Minkowski's second theorem, which benefit from the improvements on the lower bound.

2.1 Basics and introduction

The first step in the direction of connecting the lattice point enumerator and successive minima is again due to Minkowski:

2.1 Theorem (Minkowski, 1896, [35])
Let $K \in \mathcal{K}_0^n$ with non-empty interior, $\Lambda \in \mathcal{L}^n$. Then

$$\mathrm{G}_\Lambda(\lambda_1(K,\Lambda)K) \le 3^n.$$

The statement of this theorem is that a $\mathbb{0}$-symmetric convex body without interior lattice points besides $\mathbb{0}$ has at most $3^n - 1$ lattice points on its boundary. This is best-possible, which can again be seen by considering lattice parallelepipeds: If $B = (b_1, \ldots, b_n)$ is a basis of Λ, then $P = \{\mu_1 b_1 + \ldots + \mu_n b_n : \mu_i \in [-1,1], 1 \le i \le n\}$ contains exactly 3^n lattice points, namely $\sum_{i=1}^n \{-1,0,1\} b_i$ but, as mentioned before, exactly one in its interior. This kind of inequality was generalized by relaxing the condition of K having no non-trivial interior lattice points.

2.2 Theorem (Betke, Henk, Wills, 1993, [9])
Let $K \in \mathcal{K}_0^n$ with non-empty interior, $\Lambda \in \mathcal{L}^n$. Then

$$\mathrm{G}_\Lambda(K) \le \left\lfloor \frac{2}{\lambda_1(K,\Lambda)} + 1 \right\rfloor^n.$$

This theorem is a discrete version and generalization of Minkowski's first theorem. Since

$$\lim_{\lambda\to\infty} \frac{\mathrm{G}_\Lambda(\lambda K)}{\lambda^n} = \frac{\mathrm{vol}(K)}{\det(\Lambda)}$$

and

$$\lim_{\lambda\to\infty} \frac{\left\lfloor \frac{2}{\lambda_i(\lambda K,\Lambda)} + 1 \right\rfloor}{\lambda^n} = \frac{2}{\lambda_i(K,\Lambda)},$$

Theorem 2.2 indeed implies Minkowski's first theorem. Thus a natural question is to ask whether this statement can be generalized to a discrete version of Minkowski's second theorem in a way that it implies Minkowski's second theorem. This question was posed in the following conjecture:

2.3 Conjecture (Betke, Henk, Wills, 1993, [9])
Let $K \in \mathcal{K}_0^n$ with non-empty interior, $\Lambda \in \mathcal{L}^n$. Then

$$\mathrm{G}_\Lambda(K) \le \prod_{i=1}^n \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor.$$

The 2-dimensional case of this conjecture was proven by Betke, Henk and Wills [9]. In 2002, Henk [23] proved the conjecture up to a constant that depends exponentially on the dimension:

$$\mathrm{G}_\Lambda(K) \leq 2^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor .$$

Recently this factor was improved by Malikiosis.

2.4 Theorem (Malikiosis, 2010, [30])

$$\mathrm{G}_\Lambda(K) \leq \frac{4}{e} \left(\sqrt[3]{\frac{40}{9}} \right)^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor \approx 1.47 \cdot 1.64^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} + 1 \right\rfloor .$$

Furthermore Malikiosis [29] also settled the 3-dimensional case of Conjecture 2.3.

In fact it is enough to prove Conjecture 2.3 for $\mathbb{0}$-symmetric lattice polytopes. To see that, for an arbitrary convex body K one considers $P_K = \operatorname{conv}\{K \cap \Lambda\}$ and uses monotonicity of the successive minima.

To use integral Ehrhart theory as introduced in Section 1.2, we consider the weaker conjecture for $P \in \mathcal{P}(\Lambda) \cap \mathcal{K}_0^n$:

$$\mathrm{G}_\Lambda(P) \overset{?}{\leq} \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(P,\Lambda)} + 1 \right) . \tag{2.1}$$

In that case, dilating P by a factor of $k \in \mathbb{N}$, both sides of the inequality are polynomials in k:

$$\mathrm{G}_\Lambda(k\,P) = \sum_{i=0}^{n} \mathrm{G}_i(P,\Lambda) k^i$$

and

$$\prod_{i=1}^{n} \left(\frac{2}{\lambda_i(k\,P,\Lambda)} + 1 \right) = \sum_{i=0}^{n} k^i \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ \#J = i}} \prod_{j \in J} \frac{2}{\lambda_j(P,\Lambda)} .$$

Thus a natural question is, for which polytopes does the coefficient-wise inequality

$$\mathrm{G}_i(P,\Lambda) \leq \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ \#J = i}} \prod_{j \in J} \frac{2}{\lambda_j(P,\Lambda)}$$

hold? We denote the right-hand side by $\sigma_i(\frac{2}{\lambda_1(P,\Lambda)},\dots,\frac{2}{\lambda_n(P,\Lambda)})$, which is the i-th elementary symmetric function

$$\sigma_i(a_1,\dots,a_n)=\sum_{\substack{J\subseteq\{1,\dots,n\}\\ \#J=i}}\prod_{j\in J}a_j.$$

Thus the question can be formulated as follows.

2.5 Problem
For which $\mathbb{0}$-symmetric polytopes P with non-empty interior is

$$\mathrm{G}_i(P,\Lambda)\leq\sigma_i\left(\frac{2}{\lambda_1(P,\Lambda)},\dots,\frac{2}{\lambda_n(P,\Lambda)}\right)$$

true?

We will refer to the inequality in Problem 2.5 as a coefficient-wise approach of Equation (2.1). For $i=n$ this inequality coincides with Minkowski's second theorem. Henk, Schürmann, and Wills showed that the inequality is true for $i=n-1$:

2.6 Theorem (Henk, Schürmann, Wills, [25, Corollary 1.5])
For $P\in\mathcal{P}(\Lambda)\cap\mathcal{K}_0^n$ with non-empty interior,

$$\mathrm{G}_{n-1}(P,\Lambda)\leq\sigma_{n-1}\left(\frac{2}{\lambda_1(P,\Lambda)},\dots,\frac{2}{\lambda_n(P,\Lambda)}\right)=\sum_{i=1}^{n}\prod_{j\neq i}\frac{2}{\lambda_j(P,\Lambda)}.$$

Unfortunately, the inequality in Problem 2.5 is not true for all polytopes for $i\leq n-2$, as the following example shows:

2.7 Proposition ([10])
Let $Q_l=\operatorname{conv}\{lC_{n-1}\times\{0\},\pm e_n\}$, where $l\in\mathbb{N}$. Then, for $n\geq 3$ and any constant c there exists an $l\in\mathbb{N}$ such that $\mathrm{G}_{n-2}(Q_l)>c\,\sigma_{n-2}\left(\frac{2}{\lambda_1(Q_l)},\dots,\frac{2}{\lambda_n(Q_l)}\right)$.

In this chapter we will consider the coefficient-wise approach and give improvements of the lower bound in Minkowski's second theorem in the case of zonotopes.

2.8 Definition (Zonotope)
A zonotope $Z\subseteq\mathbb{R}^n$ is the Minkowski sum of finitely many line segments, that is,

$$Z=\sum_{i=1}^{m}[p_i,p_i+z_i]=\sum_{i=1}^{m}p_i+\sum_{i=1}^{m}[0,z_i],$$

with $p_i, z_i \in \mathbb{R}^n$ *for* $1 \le i \le m \in \mathbb{N}$*. The vectors* z_i *are referred to as* generators of the zonotope Z*.*

Zonotopes are centrally symmetric and, in fact, as all faces of a zonotope Z are zonotopes themselves, all faces of Z are centrally symmetric.

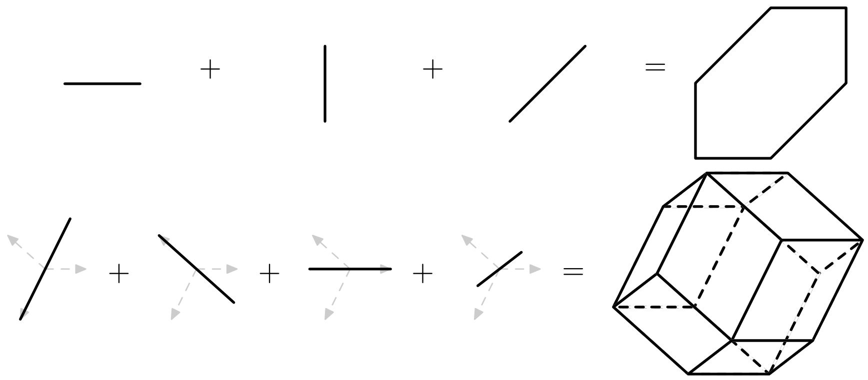

Figure 2.1: Examples of Zonotopes.

In dimension 2, every centrally symmetric polytope is a zonotope. There are also many characterizations of zonotopes for dimensions $n \ge 3$, some of which we state here:

2.9 Proposition (see Bolker [13, Theorem 3.3])
Let $Z \subseteq \mathcal{P}^n$*,* $n \ge 3$*. Then the following statements are equivalent:*

(i) Z *is a zonotope.*

(ii) Z *is an affine image of the cube* C_N *for some* $N \in \mathbb{N}$*.*

(iii) *Every* $(n-1)$*-face of* Z *is a zonotope.*

(iv) *Every* 2*-face is centrally symmetric.*

Zonotopes are built from parallelepipeds, that is, they are union of parallelepipeds intersecting only in lower dimensional faces. Thus, the lattice point enumerator and volume are easily computed:

2.10 Lemma (see Shephard [42, Section 5])
Let $Z = \sum_{i=1}^{m} [p_i, p_i + z_i] \subseteq \mathbb{R}^n$ *be a zonotope,* $p_i, z_i \in \mathbb{R}^n$*,* $1 \le i \le m$*. Then*

$$\operatorname{vol}(Z) = \sum_{\substack{J \subseteq \{1,\dots,m\} \\ \#J = n}} \operatorname{vol}\left(\sum_{j \in J} [0, z_j] \right).$$

For the lattice point enumerator $\mathrm{G}_\Lambda(Z)$ of lattice zonotopes, Stanley showed the following formula:

2.11 Lemma (Stanley [43, p. 272])
Let $\Lambda \in \mathcal{L}^n$ and $Z = \sum_{i=1}^m [p_i, p_i + z_i] \subseteq \mathbb{R}^n$ be a lattice zonotope, that is, $p_i, z_i \in \Lambda$, $1 \leq i \leq m$. Furthermore let $\mathcal{X}_i$ be the set of all $(n \times i)$-matrices whose columns are i linearly independent vectors out of $z_1, \ldots, z_m$. Then

$$\mathrm{G}_\Lambda(Z) = \sum_{i=0}^{n} \sum_{X \in \mathcal{X}_i} \gcd(\{z : z \text{ is } i\text{-minor of } X\}).$$

Here, $\gcd(\{a_1, \ldots, a_k\})$ denotes the greatest common divisior of the $a_1, \ldots, a_k$, and, as usual, a i-minor of X is an $(i \times i)$-subdeterminant of the matrix X.

In order to work with Minkowski's second theorem and successive minima for zonotopes, we first work out volume bounds on special zonotopes in the next section.

2.2 Volume bounds on zonotopes containing crosspolytopes and simplices

Since zonotopes are symmetric with respect to some point $t \in \mathbb{R}^n$, simplices are never zonotopes in dimensions greater than 1. In dimensions greater than 2, crosspolytopes cannot be zonotopes either, since the facets of crosspolytopes are simplices and hence not centrally symmetric (see Proposition 2.9). In this section we will consider those zonotopes that are of minimal volume among all zonotopes containing a given crosspolytope or simplex.

Another approximation of crosspolytopes by zonotopes was studied by Schneider [41]. He considered zonotopes that are contained in a crosspolytope and that have a preferably small dilate containing the crosspolytope itself. We investigate correlations between both approximation problems in dimension 3 in Section 2.4.

First of all, let us mention that it is enough to consider the standard simplex T_n and the standard crosspolytope $C_n^\star$, since arbitrary simplices Δ and crosspolytopes $\diamondsuit$,

$$\Delta = t + \mathrm{conv}\{\mathbb{0}, v_1, \ldots, v_n\}, \quad \diamondsuit = t + \mathrm{conv}\{\pm v_1, \ldots, \pm v_n\},$$

with $v_1, \ldots, v_n$ linearly independent, can be transformed to the standard ones by the affine transformation $x \mapsto \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}^{-1} x - \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}^{-1} t$. Since zonotopes are exactly the affine images of cubes, affine transformation preserves the property of a polytope being a zonotope or not. Hence once we found zonotopes of minimal volume containing T_n or $C_n^\star$, we apply $x \mapsto \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} x + t$ to get the zonotopes of minimal volume containing Δ or $\diamondsuit$, respectively.

Hence we mainly consider T_n and $C_n^\star$ throughout this section. First, we work out some basic observations for T_n, $C_n^\star$, and the zonotopes of minimal volumes containing them.

2.12 PROPOSITION
Let $K \in \mathcal{K}_0^n$. Then every zonotope of minimal volume containing K is $\mathbb{0}$-symmetric as well.

PROOF
Let $Z = s + \sum_{j=1}^m [-a_j, a_j]$, $m \geq n$, $s, a_1, \dots, a_m \in \mathbb{R}^n$, be a zonotope of minimal volume containing K and let $Z_0 := \sum_{j=1}^m [-a_j, a_j]$. So we know that $K \subseteq s + Z_0$, and we have to show that in the case $s \neq 0$ there exists a zonotope of smaller volume containing K. To this end we may assume $s = e_n$ and let $\gamma := \max\{x_n : x \in K\}$, that is, γ is the maximal last coordinate of a point in K. By the $\mathbb{0}$-symmetry of K and Z_0 we know that $\pm e_n + K \subseteq Z_0$. Hence for any point $x \in K$, we have $[x - e_n, x + e_n] \subseteq Z_0$, and thus $M(\gamma)\, x \in Z_0$, where

$$M(\gamma) = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 + \frac{1}{\gamma} \end{pmatrix}.$$

Therefore $K \subseteq M(\gamma)^{-1} Z_0$, and since the right-hand side is a zonotope of smaller volume than Z_0 we have the desired contradiction. □

2.13 PROPOSITION
Let Z be a zonotope of minimal volume containing $C_n^\star$ or T_n. Then all vertices of $C_n^\star$ or T_n, respectively, are contained in the boundary of Z.

PROOF
Let $K \in \{C_n^\star, T_n\}$. First we assume that one of the unit vectors is not contained in the boundary of Z and assume without loss of generality that this is e_1. Then

$$\gamma := \max\{\lambda \in \mathbb{R} : \lambda e_1 \in Z\} > 1.$$

Let $A = (\gamma e_1, e_2, \dots, e_n)^{-1}$ and let $\overline{Z} = AZ$. Then $K \subseteq \overline{Z}$ and

$$\mathrm{vol}(\overline{Z}) = \det(A)\, \mathrm{vol}(Z) = \frac{1}{\gamma}\, \mathrm{vol}(Z) < \mathrm{vol}(Z)$$

which is a contradiction. Now let $K = T_n$ and assume $\mathbb{0}$ is in the interior of Z. Then

$$\gamma := \min\{\lambda \in \mathbb{R} : \lambda(e_1 + \ldots + e_n) \in Z\} < 0.$$

Let $\overline{T} = \operatorname{conv}\{\gamma(e_1 + \ldots + e_n), e_1, \ldots, e_n\}$. Then $\overline{T} \subseteq Z$, $\operatorname{vol}(T_n) < \operatorname{vol}(\overline{T})$ and $\mathbb{0}$ is a vertex of $\overline{T}$. Thus the linear transformation L that transforms $\overline{T}$ into T_n has $\det(L) < 1$ and thus

$$T_n = L\overline{T} \subseteq LZ$$

is contained in a zonotope of smaller volume, which is a contradiction. □

The following Lemmas 2.14 and 2.15 show that we can assume the vertices of simplex and crosspolytopes to be vertices of a zonotope of minimal volume containing them. The statements are rather technical since they also provide a way to abandon this assumption later.

2.14 Lemma
Let Z be a zonotope of minimal volume containing T_n. Then there is an affine transformation $t + L$ with $\det L = 1$ such that $T_n \subseteq t + LZ$ and $\mathbb{0}$ is a vertex of $t + LZ$. In case that Z is a zonotope of minimal volume containing T_n and $\mathbb{0}$ is not a vertex of Z, there is a $v \in \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$ such that $L^{-1} = \begin{pmatrix} e_1 - v & \ldots & e_n - v \end{pmatrix}$ and $t = -Lv$ and furthermore Z and $t + LZ$ have generators in $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$.

Proof
Assume $\mathbb{0}$ is not a vertex of Z. Let F be the face of smallest dimension of Z containing $\mathbb{0}$, that is, $\mathbb{0}$ is contained in the interior of F. Then there is a vertex v of F with $\sum_{i=1}^n v_i \leq 0$. Let $\overline{v}$ be a vertex of F such that

$$\sum_{i=1}^n \overline{v}_i = \min\left\{\sum_{i=1}^n v_i : v \in F\right\}.$$

As above let

$$\overline{T} := -\overline{v} + \operatorname{conv}\{\overline{v}, e_1, \ldots, e_n\}.$$

Then $\overline{T} \subseteq -\overline{v} + Z$, $\operatorname{vol}(\overline{T}) = (1 - \sum_{i=1}^n \overline{v}_i) \operatorname{vol}(T_n)$ and $\mathbb{0}$ is a vertex of $\overline{T}$ and $-\overline{v} + Z$. Thus the linear transformation L that transforms $\overline{T}$ into T_n has $\det(L) = \frac{1}{1 - \sum_{i=1}^n \overline{v}_i}$. We get

$$T_n = L\overline{T} \subseteq L(-\overline{v} + Z) = -L\overline{v} + LZ$$

and $\mathbb{0}$ is a vertex of $-L\overline{v} + LZ$. Since Z was of minimal volume, we get $(1 - \sum_{i=1}^n \overline{v}_i) = 1$ and thus $\sum_{i=1}^n v_i = 0$ for all vertices v of F and thus, Z has a generator in $\{x \in \mathbb{R}^n :$

$\sum_{i=1}^n x_i = 0\}$. Furthermore, since $L(e_i - v) = e_i$ for all i, L preserves the sum of coordinates and thus fixes the hyperplane $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$. □

2.15 Lemma
Let Z be a zonotope of minimal volume containing $C_n^\star$ or T_n. Then there is a linear transformation L with $\det(L) = 1$ such that $C_n^\star \subseteq LZ$ or $T_n \subseteq LZ$, respectively, and all unit vectors are vertices of LZ. In case that $L \neq I_n$, L is a composition

$$L = A_{i_k} \cdots A_{i_1}, \quad i_1, \dots, i_k \in \{1, \dots, n\}$$

of linear transformations A_i with $A_i^{-1} = \begin{pmatrix} e_1 & \dots & e_{i-1} & v & e_{i+1} & \dots & e_n \end{pmatrix}$ with $v_i = 1$. Furthermore for all $1 \leq j \leq k$, $A_{i_j} \dots A_{i_1} Z$ has at least one generator that is orthogonal to e_{i_j}.

Proof
Let $K \in \{C_n^\star, T_n\}$ and let k be the number of unit vectors that are not vertices of Z. We prove the assertion by induction on k. If $k = 0$ we are done. Thus let $k > 0$. Without loss of generality, let e_1 be no vertex of Z. Since Z is of minimal volume and e_1 is a vertex of K, by Proposition 2.13 e_1 is contained in the boundary of Z. Let F be the face of Z of smallest dimension containing e_1. Then $\gamma := \max\{x_1 : x \in F\} \geq 1$. Let v be a vertex of F with $v_1 = \gamma$ and define $\overline{K} := \operatorname{conv}(\pm v, \pm e_2, \dots, \pm e_n)$ if $K = C_n^\star$ or $\overline{K} = \operatorname{conv}(\mathbb{0}, v, e_2, \dots, e_n)$ if $K = T_n$, respectively. Then $\overline{K} \subseteq Z$, and the number of vertices of $\overline{K}$ that are not vertices of Z is smaller than k.

For $\overline{A} := \begin{pmatrix} v & e_2 & \dots & e_n \end{pmatrix}^{-1}$ we get $\overline{A}\,\overline{K} = K$ and $\det\left(\overline{A}\right) = \frac{1}{\gamma}$, and thus $\operatorname{vol}(\overline{A}Z) \leq \operatorname{vol}(Z)$. Since Z is of minimal volume and $K \subseteq \overline{A}Z$, we get $\det\left(\overline{A}\right) = 1$ and thus $\gamma = 1$. It follows that F is contained in $\{x \in \mathbb{R}^n : x_1 = 1\}$ and thus Z has generators that are orthogonal to e_1, namely those that generate the face F. These generators are fixed by $\overline{A}$ and furthermore all unit vectors except for e_1 are fixed by $\overline{A}$ as well.

By construction, e_1 is a new vertex of $\overline{A}Z$ and thus there are at most $k-1$ unit vectors that are not vertices of $\overline{A}Z$. Thus, by induction hypothesis, there exists a matrix A with desired properties such that all unit vectors are vertices of $A\overline{A}Z$. Thus $L := A\overline{A}$ is a matrix of a linear transformation as claimed. □

In both Lemmas 2.14 and 2.15, the constructed linear transformations L need not be unique. We are now able to prove a formula for the minimal volume of a zonotope containing the standard crosspolytope or the standard simplex, which is the main theorem of this section.

In order to simplify the notation, we will denote for a matrix $A \in \mathbb{R}^{n \times m}$ and a subset $I \subseteq \{1, \dots, m\}$ by A_I the $(n \times (\#I))$-submatrix with columns indexed by the elements of I in increasing order. Analogously we denote for $I \subseteq \{1, \dots, n\}$ by A^I the $((\#I) \times m)$-

submatrix with rows indexed by elements of I. We further use the notation $\mathbb{0} = (0, \dots, 0)$ and $\mathbb{1} = (1, \dots, 1)$ for vectors of different dimension. If the dimension is clear from the context, it is not stated explicitly. Another relevant notation is the following:

2.16 Definition (maximal Determinant)
The maximal possible determinant an $(n \times n)$-matrix with entries in $\{-1, +1\}$ can have, is denoted by $\operatorname{maxdet}(n) = \max\{\det(A) : A \in \{-1, +1\}^{n \times n}\}$.

2.17 Theorem

(i) *Let $Z \subseteq \mathbb{R}^n$ be a zonotope of minimal volume containing $C_n^\star$. Then*

$$\operatorname{vol}(Z) = \operatorname{vol}(C_n^\star)\,\frac{n!}{\operatorname{maxdet}(n)}.$$

(ii) *Let $Z \subseteq \mathbb{R}^n$ be a zonotope of minimal volume containing T_n. Then*

$$\operatorname{vol}(Z) = \operatorname{vol}(T_n)\,\frac{n!\,2^n}{\operatorname{maxdet}(n+1)}.$$

(iii) *Among all zonotopes of minimal volume containing $C_n^\star$ or T_n there exists always a parallelepiped.*

Proof

(i) We can assume that $Z = \sum_{j=1}^m [-a_j, a_j]$ for $m \geq n$ and $a_1, \dots, a_m \in \mathbb{R}^n$ by Proposition 2.12. Let A be the $(n \times m)$-matrix $\begin{pmatrix} a_1 & \dots & a_m \end{pmatrix}$. By Lemma 2.15 we may assume that $\operatorname{vert}(C_n^\star) \subseteq \operatorname{vert}(Z) \subseteq \{\sum_{j=1}^m \pm a_j\}$, that is, all vertices of $C_n^\star$ can be written as (± 1)-combinations of $a_1, \dots, a_m$. Thus there is a $(m \times n)$-(± 1)-matrix H such that $I_n = A \cdot H$. This yields, together with the Cauchy–Binet formula,

$$\begin{aligned} 1 &= \det(I_n) = \det(A \cdot H) \\ &= \sum_{I \subseteq \{1,\dots,m\}, \#I = n} \det(A_I)\det(H^I) \\ &\leq \sum_{I \subseteq \{1,\dots,m\}, \#I = n} |\det(A_I)|\,|\det(H^I)| \\ &\leq \sum_{I \subseteq \{1,\dots,m\}, \#I = n} |\det(A_I)|\operatorname{maxdet}(n) \\ &= \operatorname{maxdet}(n)\frac{1}{2^n}\operatorname{vol} Z = \frac{\operatorname{maxdet}(n)}{n!}\,\frac{\operatorname{vol}(Z)}{\operatorname{vol}(C_n^\star)}. \end{aligned} \tag{2.2}$$

Obviously, for an $(n \times n)$-(± 1)-matrix H with $\det(H) = \operatorname{maxdet}(n)$ and $A = H^{-1}$ we get equality, which shows part (iii) in the crosspolytope case.

(ii) Next we consider the simplex T_n. Here we assume that $Z = t + \sum_{j=1}^m [-a_j, a_j]$ for $m \geq n$ and $t, a_1, \ldots, a_m \in \mathbb{R}^n$. Let A be the $(n \times m)$-matrix $\begin{pmatrix} a_1 & \ldots & a_m \end{pmatrix}$. Again by Lemmas 2.14 and 2.15 we may assume that $\operatorname{vert}(T_n) \subseteq \operatorname{vert}(Z)$, that is, $\operatorname{vert}(T_n - t) \subseteq \operatorname{vert}(Z - t) \subseteq \{\sum_{j=1}^m \pm a_j\}$. Hence all vertices of $T_n - t$ can be written as (± 1)-combinations of $a_1, \ldots, a_m$. Thus there is an $(m \times (n+1))$-(± 1)-matrix H such that

$$\begin{pmatrix} I_n & \mathbb{0} \end{pmatrix} - \begin{pmatrix} t & \cdots & t \end{pmatrix} = A \cdot H$$

which is equivalent to

$$\begin{pmatrix} I_n & \mathbb{0} \end{pmatrix} = A \cdot H + \begin{pmatrix} t & \cdots & t \end{pmatrix} = \begin{pmatrix} A & t \end{pmatrix} \begin{pmatrix} H \\ \mathbb{1}^\top \end{pmatrix}.$$

To consider determinants again, we add an $(n+1)$st row to the involved matrices:

$$\begin{pmatrix} I_n & \mathbb{0} \\ \mathbb{1}^\top & 1 \end{pmatrix} = \overline{A} \cdot \overline{H},$$

with $\overline{A} = \begin{pmatrix} A & t \\ \mathbb{0}^\top & 1 \end{pmatrix}$ and $\overline{H} = \begin{pmatrix} H \\ \mathbb{1}^\top \end{pmatrix}$. Then, as above, by the Cauchy–Binet formula, we get

$$\begin{aligned} 1 = \det \begin{pmatrix} I_n & \mathbb{0} \\ \mathbb{1}^\top & 1 \end{pmatrix} &= \det\left(\overline{A} \cdot \overline{H}\right) \\ &= \sum_{I \subseteq \{1,\ldots,m+1\}, \#I = n+1} \det(\overline{A}_I) \det(\overline{H}^I) \\ &= \sum_{I \subseteq \{1,\ldots,m\}, \#I = n} \det(A_I) \det \begin{pmatrix} H^I \\ \mathbb{1}^\top \end{pmatrix}. \end{aligned}$$

Here the last equality is true since $\det(\overline{A}_I) = 0$ if $(m+1) \notin I$ and $\det(\overline{A}_I) = \det(A_{I \setminus \{m+1\}})$, otherwise. As above, we can estimate that sum by

$$\begin{aligned} 1 = \det \begin{pmatrix} I_n & \mathbb{0} \\ \mathbb{1}^\top & 1 \end{pmatrix} &\leq \sum_{I \subseteq \{1,\ldots,m\}, \#I = n} |\det(A_I)| \left| \det \begin{pmatrix} H^I \\ \mathbb{1}^\top \end{pmatrix} \right| \\ &\leq \operatorname{maxdet}(n+1) \sum_{I \subseteq \{1,\ldots,m\}, \#I = n} |\det(A_I)| \qquad (2.3) \\ &= \operatorname{maxdet}(n+1) \frac{1}{2^n} \operatorname{vol}(Z) = \frac{\operatorname{maxdet}(n+1)}{n!\, 2^n} \frac{\operatorname{vol}(Z)}{\operatorname{vol}(T_n)}. \end{aligned}$$

Equality holds for an $(n\times(n+1))$-(±1)-matrix H with $\det\begin{pmatrix}H\\ \mathbb{1}^\top\end{pmatrix} = \operatorname{maxdet}(n+1)$ and $\begin{pmatrix}A & t\end{pmatrix} = \begin{pmatrix}I_n & \mathbb{0}\end{pmatrix}\cdot\begin{pmatrix}H\\ \mathbb{1}^\top\end{pmatrix}^{-1}$ which again gives part (iii) in the simplex case. □

In order to simplify the argumentation in the following sections we fix the notation used in the proof above.

NOTATION
For a given $(n\times m)$-matrix $A = \begin{pmatrix}a_1 & \dots & a_m\end{pmatrix}$, $a_i\in\mathbb{R}^n$, $1\le i\le m$, we denote by $\mathrm{Z}(A)$ the zonotope $\sum_{i=0}^{m}[-a_i,a_i]$.
In case that $\mathrm{Z}(A)$ is a zonotope of minimal volume containing the crosspolytope $C_n^\star$ whose vertex set contains the vertices of $C_n^\star$, we denote by $\mathrm{H}(A)$ an $(m\times n)$-(±1)-matrix such that $A\cdot\mathrm{H}(A) = I_n$.
In case that $\mathrm{Z}(A)$ is a zonotope of minimal volume containing the simplex $-t+T_n$, $t\in\mathbb{R}^n$ whose vertex set contains the vertices of T_n, we denote by $\mathrm{H}(A)$ an $(m\times(n+1))$-(±1)-matrix such that $\begin{pmatrix}A & t\end{pmatrix}\cdot\begin{pmatrix}\mathrm{H}(A)\\ \mathbb{1}^\top\end{pmatrix} = \begin{pmatrix}I_n & \mathbb{0}\end{pmatrix}$.

2.3 Results on equality cases in arbitrary dimensions

In this section we study the structure of zontopes of minimal volume containing the crosspolytope $C_n^\star$ or the simplex T_n. More precisely we are interested in the problem to characterize all matrices such that the zonotopes $\mathrm{Z}(A)$ are of minimal volume containing $C_n^\star$ or a translation of T_n, or at least to give some conditions on the number of columns, such a matrix can have.

As discussed in Section 2.2 we may assume that the vertices of $C_n^\star$ or a translate of T_n, respectively, are also vertices of $\mathrm{Z}(A)$ and thus we can assume the existence of $\mathrm{H}(A)$.

Furthermore, we will assume in this section that no two columns of A are linearly dependent, because otherwise we can sum them up to one generator of the zonotope.

2.18 PROPOSITION
Let A be an $(n\times m)$-matrix such that $\mathrm{Z}(A)$ is a zonotope of minimal volume containing $C_n^\star$ or T_n, respectively, and let $H = \mathrm{H}(A)$.

(i) *Let $I\subseteq\{1,\dots,m\}$ with $\#I = n$. If $\det(A_I)\neq 0$ then*

$$|\det(H^I)| = \operatorname{maxdet}(n) \quad \textit{and} \quad \operatorname{sign}(\det(H^I)) = \operatorname{sign}(\det(A_I))$$

or

$$\left|\det\begin{pmatrix} H^I \\ \mathbb{1}^\top \end{pmatrix}\right| = \operatorname{maxdet}(n+1) \quad \textit{and} \quad \operatorname{sign}\left(\det\begin{pmatrix} H^I \\ \mathbb{1}^\top \end{pmatrix}\right) = \operatorname{sign}(\det(A_I)),$$

respectively.

(ii) *If for some $J \subseteq \{1, \ldots, m\}$ the rows of H^J or $\begin{pmatrix} H^J \\ \mathbb{1}^\top \end{pmatrix}$, respectively, are linearly dependent, then the columns of A_J are linearly dependent as well.*

Proof

Equality in the first inequality in Equation (2.2) in the proof of Theorem 2.17 is attained only if $\operatorname{sign}(\det(H^I)) = \operatorname{sign}(\det(A_I))$ for all I, and the equality in the second inequality in Equation (2.2) is attained only if $|\det(H^I)| = \operatorname{maxdet}(n)$ for all I such that $\det(A_I) \neq 0$. Together with the analogous statement in Equation (2.3) we get part (i).

For part (ii) we may assume $J = \{1, \ldots, k\}$ with $k \leq n$. From part (i) we conclude that $\det(A_I) = 0$ for all I such that $\#I = n$, $J \subseteq I$. Now assume that the columns of A_J are linearly independent. Since the rank of A is n we can find an index set $I^\star \supseteq J$, $\#I = n$ such that the columns of $A_{I^\star}$ are linearly independent, which is a contradiction. □

By Proposition 2.18 we may assume that a matrix $\mathrm{H}(A)$ of a zonotope $\mathrm{Z}(A)$ of minimal volume containing $C_n^\star$ or T_n cannot have two linearly dependent rows, because otherwise the two generators of $\mathrm{Z}(A)$ are linearly dependent. Hence the number of rows in the (± 1)-matrix $\mathrm{H}(A)$ can be bounded:

2.19 Corollary

Let $\mathrm{Z}(A)$ be a zonotope of minimal volume containing $C_n^\star$. Then $\mathrm{Z}(A)$ has at most 2^{n-1} pairwise linearly independent generators.

Let $\mathrm{Z}(\overline{A})$ be a zonotope of minimal volume containing T_n. Then $\mathrm{Z}(\overline{A})$ has at most $2^n - 1$ pairwise linearly independent generators.

Next we consider the special case of zonotopes with generators in general position. A set of vectors in $\mathbb{R}^n$ is said to be *in general position*, if any n of them are linearly independent.

2.20 Theorem

In even dimensions, among all zonotopes that contain the crosspolytope and whose generators are in general position, only parallelepipeds have minimal volume.

Proof

Let $m > n$ and let A be an $(n \times m)$-matrix such that $\mathrm{Z}(A)$ is a zonotope of minimal volume containing $C_n^\star$ with columns of A in general position. Furthermore, let

$$\mathrm{H}(A) = \begin{pmatrix} h_1 & \cdots & h_m \end{pmatrix}^\top .$$

By Proposition 2.18 we get $|\det(\mathrm{H}(A)^I)| = \operatorname{maxdet}(n)$ for all $I \subseteq \{1, \ldots, m\}$ with $\#I = n$. Thus h_{n+1} is a (± 1)-combination of $h_1, \ldots, h_n$ since otherwise replacing h_i by h_{n+1}, for $i \in \{1, \ldots, n\}$ in $\mathrm{H}(A)_{\{1,\ldots,n\}}$ would change the absolute value of the determinant. But this is a contradiction, since in even dimensions a (± 1)-combination of $h_1, \ldots, h_n$ is a vector with even entries. □

In particular, this theorem implies that the zonotope of minimal volume containing the 2-dimensional crosspolytope is a parallelepiped, since in dimension 2 all zonotopes have generators in general position. This fact is rather obvious, since the crosspolytope itself is a zonotope in dimension 2. For simplices, an analogous statement is false, since in dimension 2 there exist 6-gons with minimal volume containing T_2. We will work out details and a classification of the 2-dimensional case for simplices and the 3-dimensional case for crosspolytopes in the next Section 2.4.

For $n \geq 4$, the zonotopes of minimal volume containing $C_n^\star$ or T_{n-1} are conjectured to be parallelepipeds. Unfortunately it remains open to classify all zonotopes of minimal volume in arbitrary dimension. In the following we reduce this problem to the problem of classifying $(n \times n)$-(± 1)-matrices with maximal determinant.

The problems of classifying $(n \times n)$-(± 1)-matrices with maximal determinant or of at least finding the maximal determinant of an $(n \times n)$-(± 1)-matrix are well-studied. The latter was posed in 1893 by Jacques Hadamard who also gave the upper bound $n^{n/2}$ in the same article [21]. This bound can only be attained from orthogonal matrices and such matrices do only exist in dimension $n \equiv 0 \bmod 4$. Matrices attaining this bound are called Hadamard matrices and the corresponding dimension n a Hadamard dimension. It is an open problem whether every dimension $n \equiv 0 \bmod 4$ is a Hadamard dimension. To the best of our knowledge, $n = 668$ is the smallest unkown dimension. In Dimensions $n \not\equiv 0 \mod 4$, smaller upper bounds are given by Barba, Ehlich, and Wojtas [37].

The question of classifying $(n \times n)$-(± 1)-matrices with maximal determinant immediately leads to the notion of equivalent matrices: We call two (± 1)-matrices equivalent if one can be obtained from the other by a series of permutations and negations of rows and columns. Thus, two equivalent matrices clearly have the same absolute value of the determinant and need not to be distinguished. Orrick and Solomon [37] present a survey of what is known in the classification problem. Their list of references shows the wide interest and importance of that problem.

To connect the problem of classifying zonotopes of minimal volume containing $C_n^\star$ or T_n to the structure of (± 1)-matrices of maximal determinant, we need the following lemma:

2.21 Lemma

Let $A = \begin{pmatrix} a_1 & \dots & a_{n+1} \end{pmatrix}$ be an $(n \times (n+1))$-matrix of rank n. Furthermore, let $a_j \neq \mathbb{0}$ for all j and $\det(A_{\{1,\dots,n\}}) \neq 0$. Assume that at most k of the $(n \times n)$-subdeterminants of A are not 0. Then a_{n+1} is contained in the linear hull of $k-1$ columns of $A_{\{1,\dots,n\}}$.

Proof

Since $a_1, \dots, a_n$ are linearly independent, we have for $I, J \subseteq \{1, \dots, n\}$:

$$\operatorname{lin}\{a_j : j \in I\} \cap \operatorname{lin}\{a_j : j \in J\} = \operatorname{lin}\{a_j : j \in I \cap J\}. \tag{2.4}$$

To see this, let $x \in \operatorname{lin}\{a_j : j \in I\} \cap \operatorname{lin}\{a_j : j \in J\}$, that is,

$$x = \sum_{j \in I} \alpha_j a_j = \sum_{j \in J} \beta_j a_j.$$

Then

$$0 = \sum_{j \in I \setminus J} \alpha_j a_j + \sum_{j \in I \cap J} (\alpha_j - \beta_j) a_j - \sum_{j \in J \setminus I} \beta_j a_j.$$

Hence $\alpha_j = 0$ for $j \in I \setminus J$ and $\beta_j = 0$ for $j \in J \setminus I$, and thus $x \in \operatorname{lin}\{a_j : j \in I \cap J\}$. By assumption, there exists an index set $S \subseteq \{1, \dots, n\}$, $\#S = n+1-k$, with $\det(A_{\{1,\dots,n+1\} \setminus \{s\}}) = 0$ for all $s \in S$. Again since $a_1, \dots, a_n$ are linearly independent, we get that $a_{n+1} \in \operatorname{lin}\{a_j : j \in \{1, \dots, n\} \setminus \{s\}\}$ for all $s \in S$. Together with (2.4) above, $a_{n+1} \in \operatorname{lin}\{a_j : j \in \{1, \dots, n\} \setminus S\}$, which completes the proof. □

2.22 Corollary

Let A be a matrix as in the lemma above. If $\det(A_I) = 0$ for all $I \neq \{1, \dots, n+1\} \setminus \{l\}, \{1, \dots, n+1\} \setminus \{j\}$ then a_l is a multiple of a_j.

Using this, we can give a rather simple sufficient criterion involving only (± 1)-matrices, which implies that all zonotopes of minimal volume are parallelepipeds.

2.23 Lemma

If every $((n+1) \times n)$-(± 1)-matrix has at most two $(n \times n)$-subdeterminants whose absolute values equal $\operatorname{maxdet}(n)$, then all zonotopes of minimal volume containing the n-dimensional crosspolytope or the $(n-1)$-dimensional simplex are parallelepipeds.

Proof

Assume all $((n+1) \times n)$-(± 1)-matrices have at most two subdeterminants whose absolute values are equal to $\operatorname{maxdet}(n)$. Let $A = \begin{pmatrix} a_1 & \dots & a_m \end{pmatrix}$ be an $(n \times m)$-matrix with $m > n$ such that $\mathrm{Z}(A)$ is a zonotope of minimal volume containing the n-dimensional crosspolytope. Without loss of generality we assume that $\det \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix} \neq 0$. Now we consider $A_{\{1,\dots,n,k\}}$, $k > n$. Since at most two subdeterminants of $\mathrm{H}(A)^{\{1,\dots,n,k\}}$

are maxdet(n), at most two of the subdeterminants of $A_{\{1,\dots,n,k\}}$ are not 0. Thus, by Corollary 2.22, there exists a $j(k) \in \{1,\dots,n\}$ such that a_k is a multiple of $a_{j(k)}$. Since this is true for every $k > n$, A consists of n linearly independent generators and some multiples of them. This means that $\mathrm{Z}(A)$ is a parallelepiped.

The same argument holds true for the case of $(n-1)$-dimensional simplices, since the matrix $\begin{pmatrix} \mathrm{H}(A)^{\{1,\dots,n-1,k\}} \\ \mathbb{1} \end{pmatrix}$ has at most two $(n \times n)$-submatrices that are maxdet(n) and thus at most two of the $((n-1)\times(n-1))$-subdeterminants of A are not 0. □

2.24 Example

Let us consider $n = 5$. In dimension 5, there is no (5×6)-(± 1)-matrix with more than two (5×5)-subdeterminants that are $\pm\,\mathrm{maxdet}(5) = \pm 48$ (See Proposition 2.26). Thus, all zonotopes of minimal volume containing the 5-dimensional crosspolytope or the 4-dimensional simplex are parallelepipeds.

The special character of Hadamard matrices enables us to give a characterization of zonotopes of minimal volume containing $C_n^\star$ or T_{n-1} whenever n is a Hadamard dimension.

2.25 Corollary

Let n be a dimension where Hadamard matrices exist. All zonotopes of minimal volume containing the n-dimensional crosspolytope or the $(n-1)$-dimensional simplex are parallelepipeds.

Proof

Let

$$H = \begin{pmatrix} h_1 & \cdots & h_{n+1} \end{pmatrix}^\top$$

be an $((n+1) \times n)$-(± 1)-matrix with at least two maximal $(n \times n)$-subdeterminants. Without loss of generality let these be $\det\left(H^{\{1,\dots,n\}}\right) = \det\left(H^{\{2,\dots,n+1\}}\right) = \mathrm{maxdet}(n)$. Then $H^{\{1,\dots,n\}}$ and $H^{\{2,\dots,n+1\}}$ are Hadamard matrices, and hence h_1 and h_{n+1} are orthogonal to $\mathrm{lin}\{h_2,\dots,h_n\}$. It follows that h_1 and h_{n+1} are linearly dependent, and thus all other subdeterminants are zero. The claim follows by Lemma 2.23. □

Lemma 2.23 gives us the possibility to use a computational approach to decide whether all zonotopes of minimal volume containing $C_n^\star$ or T_{n-1} are parallelepipeds.

2.26 Proposition

For dimensions 4 to 18, 20 and 21 all $((n+1)\times n)$-(± 1)-matrices have at most two $(n\times n)$-subdeterminants whose absolute values are maxdet(n). *Thus for these n all zonotopes* *of minimal volume containing the n-dimensional crosspolytope or* $(n-1)$*-dimensional simplex are parallelepipeds.*

Proof

The computations in this proof were performed using Maple™. To check whether there exists an $((n+1)\times n)$-(± 1)-matrix that has more than two $(n\times n)$-subdeterminants that are maxdet(n) we used the following algorithm: To every $(n\times n)$-(± 1)-matrix with maximal determinant we append every (± 1)-vector as $(n+1)$st row. Thereby we construct every $((n+1)\times n)$-(± 1)-matrix with at least one subdeterminant being maxdet(n). For all these matrices we calculate how many subdeterminants are equal to maxdet(n). The following Maple™-code does this for a previously given maximal $(n\times n)$-(± 1)-matrix `H`:

```
with(LinearAlgebra):
maxdet:=abs(Determinant(H));
one:=Vector(n,1):
for i from 0 to 2^(n-1)-1 do
  maxdet_set:={}:
  zero_one_vector:=convert(Bits[Split](i,bits=n),Vector);
  H0:=ScalarMultiply(zero_one_vector,2)-one;
  B:=<H0|H>:
  for k from 0 to n do
    if abs(Determinant(DeleteColumn(B,k+1)))=maxdet then
      maxdet_set:=maxdet_set union {k}:
    end if;
  end do;
if nops(maxdet_set)>2 then print(i,maxdet_set); end if;
end do:
```

Getting no output means that the matrix `H` cannot be submatrix of an $((n+1)\times n)$-(± 1)-matrix that has more than two $(n\times n)$-subdeterminants that are equal to maxdet(n). Hence we need to execute the above algorithm for all different (inequivalent) $(n\times n)$-(± 1)-matrices with maximal determinant. For example, in dimension 5 there is only one of these inequivalent matrices, and the input file for this matrix `H` is the following:

```
n:=5:
H:=<<-1,+1,+1,+1,+1>|
<+1,-1,+1,+1,+1>|
<+1,+1,-1,+1,+1>|
<+1,+1,+1,-1,+1>|
<+1,+1,+1,+1,-1>>:
```

Computational results for $n \in \{4, \dots, 18, 20, 21\}$ using the maximal matrices given in [37] show the assertion. For $n = 19$ the maximal determinant is not known. □

This observations motivates the following conjecture:

2.27 Conjecture
Let $n \geq 4$ and let Z be a zonotope of minimal volume containing $C_n^\star$ or T_{n-1}. Then Z is a parallelepiped.

2.4 Equality cases in dimensions 2 and 3

In Section 2.3 we stated a criterion and Conjecture 2.27 about the structure of zonotopes of minimal volume containing crosspolytopes in dimensions greater or equal to 4 or simplices in dimensions greater or equal to 3.

The aim of this section is to work out the remaining cases of crosspolytopes in dimension 3 (since dimensions 1 and 2 are trivial) and simplices in dimension 2.

To characterize all zonotopes of minimal volume in these dimensions, the first step of simplification is to consider only inequivalent (± 1)-matrices H. We recall that two (± 1)-matrices are called equivalent if one can be obtained from the other by a series of permutations and negations of rows and columns.

First we consider the 2-dimensional simplex T_2:

2.28 Theorem
The set of 2-dimensional zonotopes of minimal volume containing the simplex T_2 is

$$\mathcal{Z} = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \mathrm{Z}(A) : \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in T \right\}$$

$$= \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \mathrm{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2t_1 \\ 2t_2 \end{pmatrix}, \begin{pmatrix} 2t_1 - 1 \\ 2t_2 \end{pmatrix}, \begin{pmatrix} 2t_1 \\ 2t_2 - 1 \end{pmatrix} \right\} : \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \in T \right\},$$

where T is the 2-dimensional simplex

$$T = \mathrm{conv} \left\{ \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \text{and} \quad A = \begin{pmatrix} -t_1 & 1/2 - t_1 & 1/2 - t_1 \\ 1/2 - t_2 & -t_2 & 1/2 - t_2 \end{pmatrix}.$$

To get a geometric interpretation of this theorem we will investigate the result before presenting the proof. In fact, a zonotope $t+Z \in \mathcal{Z}$ is of the form $t+\mathrm{conv}\{T_2-t, -(T_2-t)\}$ where the reflection point t is contained in the triangle defined by the three center lines of T_2. A center line is a line defined by the midpoints of two egdes of T_2 and it is parallel to the third edge.

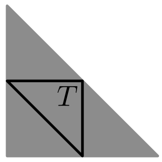

Figure 2.2: The set T of reflection points.

If t is a vertex of T, exactly two of the (2×2)-subdeterminants of the matrix A of generators are 0 and thus one of the columns of A is zero. Thus $\mathrm{Z}(A)$ is a parallelogram. If t is contained in the interior of an edge of T then exactly one of the (2×2)-subdeterminants of A is zero and thus A has two linearly dependent columns. $\mathrm{Z}(A)$ is a parallelogram again. If t is contained in the interior of T all subdeterminants of A are non-zero and thus A is a proper 6-gon.

Figure 2.3 gives some examples of zonotopes of minimal volume. In all cases, the grey filled triangle is T_2 and the black lines are the edges of $t+Z$. t itself is the point inside T_2. Furthermore the small black triangle inside T_2 is the set T of possible reflection points.

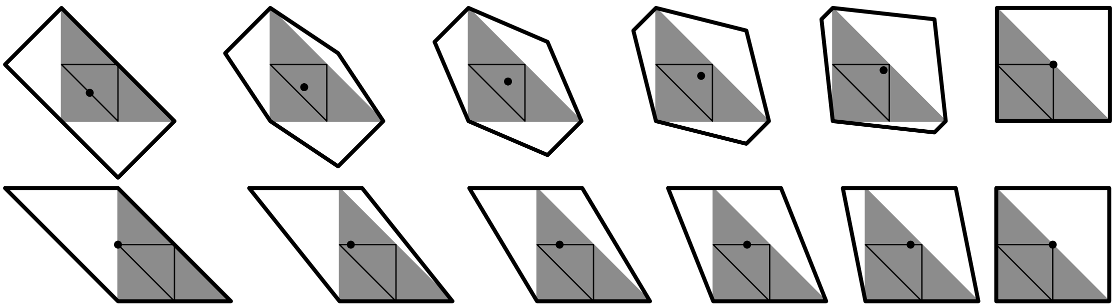

Figure 2.3: Some examples of zonotopes of minimal volume containing T_2.

Proof (of Theorem 2.28)
Let A be a $(2\times m)$-matrix, $t\in\mathbb{R}^2$, and let $t+\mathrm{Z}(A)\subseteq\mathbb{R}^2$ be a zonotope of minimal volume containing T_2. First we assume that all vertices of T_2 are vertices of $t+\mathrm{Z}(A)$. Furthermore, let $\overline{H} = \begin{pmatrix}\mathrm{H}(A)\\ \mathbb{1}\end{pmatrix}$, that is, an $(m\times 3)$-(± 1)-matrix with $\begin{pmatrix}A & t\end{pmatrix}\cdot\overline{H} =$

$\begin{pmatrix} I_2 & 0 \end{pmatrix}$.

We first consider only one single matrix $\overline{H}$ of every equivalence class.

By Lemma 2.19, we can only have $m = 2$ or $m = 3$. First we consider the case $m = 2$. By Proposition 2.18, $\overline{H}$ is a quadratic (± 1)-matrix with maximal determinant. Up to equivalence, the (3×3)-(± 1)-matrix with maximal determinant is unique (see [37]), namely

$$\overline{H} = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then the corresponding matrix of generators A is

$$\begin{pmatrix} A & t \end{pmatrix} = \begin{pmatrix} I_2 & 0 \end{pmatrix} \overline{H}^{-1} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

Now we consider the case $m = 3$ and we assume that A does not have two linearly dependent columns. By Proposition 2.18, $\overline{H}$ cannot have two linearly dependent rows. Thus we can choose 4 of the 8 possible (± 1)-vectors in such a way that no two opposite vectors are chosen. Hence also in this case $\overline{H}$ is unique up to equivalence, namely

$$\overline{H} = \begin{pmatrix} H \\ \mathbb{1} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we can solve the linear system $\begin{pmatrix} A & t \end{pmatrix} \cdot \overline{H} = \begin{pmatrix} I_2 & \mathbb{0} \end{pmatrix}$ and get the solutions

$$A = \begin{pmatrix} -t_1 & 1/2 - t_1 & 1/2 - t_1 \\ 1/2 - t_2 & -t_2 & 1/2 - t_2 \end{pmatrix},$$
$$t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

for all vectors $t = (t_1, t_2)$ of 2 parameters.

One can easily check that all (3×3)-subdeterminants of $\overline{H}$ are ± 4. By Proposition 2.18 $t + \mathrm{Z}(A)$ is of minimal volume if and only if for all I

$$\det(A_I) = 0 \quad \text{or} \quad \operatorname{sign}(\det(A_I)) = \operatorname{sign}\left(\det\begin{pmatrix} H^I \\ \mathbb{1} \end{pmatrix}\right).$$

The following table shows the values of $\det\begin{pmatrix} H^I \\ \mathbb{1} \end{pmatrix}$ and $\det(A_I)$:

I	$\det\begin{pmatrix}H^I\\ \mathbb{1}\end{pmatrix}$	$\det(A_I)$
$\{1,2\}$	4	$\frac{1}{4}(2t_1+2t_2-1)$
$\{1,3\}$	-4	$\frac{1}{4}(2t_2-1)$
$\{2,3\}$	4	$\frac{1}{4}(-2t_1+1)$

These conditions yield the following inequalities for t_i:

$$t_1+t_2\geq\frac{1}{2},\quad t_2\leq\frac{1}{2},\quad t_1\leq\frac{1}{2},$$

which describe the set T in the assertion.

Observe that $t_1=t_2=\frac{1}{2}$ is allowed and coincides with the case $m=2$.

Now let $\begin{pmatrix}H_2\\ \mathbb{1}\end{pmatrix}$ be a (3×4)-(± 1)-matrix that is equivalent to $\begin{pmatrix}H\\ \mathbb{1}\end{pmatrix}$. Then H_2 is obtained from H by negation of rows and interchanging rows and columns. Negating a row of H coincides with negating a column of A, which does not change $\mathrm{Z}(A)$ at all. Interchanging two columns of H coincides with interchanging two rows of A, which is just changing the roles of t_1 and t_2. Interchanging two rows of H coincides with interchanging two columns of A, which again does not change $\mathrm{Z}(A)$ at all. Thus $\begin{pmatrix}H_2\\ \mathbb{1}\end{pmatrix}$ produces the same set $\mathcal{Z}$.

It remains to show that $\mathcal{Z}$ also contains all those zonotopes of minimal volume containing T_2 that do not have all vertices of T_2 as vertices. To this end we use Lemmas 2.14 and 2.15. Whenever $\bar{t}+\overline{Z}$ is a zonotope of minimal volume containing T_2, there is an affine transformation to $t+Z\in\mathcal{Z}$. This transformation is a composition of some of the linear transformations A_i with $A_1^{-1}=\begin{pmatrix}v & e_2\end{pmatrix}$ and $A_2^{-1}=\begin{pmatrix}e_1 & v\end{pmatrix}$ and with $v_i=1$, as well as an affine transformation $x\mapsto -Lv+Lx$ with $L^{-1}=\begin{pmatrix}e_1-v & e_2-v\end{pmatrix}$ and $t=-Lv$ with $v\in\{x\in\mathbb{R}^2: x_1+x_2=0\}$.

Let us first deal with the linear transformations A_i and without loss of generality we just consider A_1: If A_1 is one of the transformations that are necessary to transform $\bar{t}+\overline{Z}$ to some $t+Z\in\mathcal{Z}$ then Z has a generator that is orthogonal to e_1. This is the case only if $t_1=\frac{1}{2}$, and then

$$\begin{aligned}\bar{t}+\overline{Z}=A_1^{-1}(t+Z)&=A^{-1}\left(\operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2t_2\end{pmatrix},\begin{pmatrix}1\\2t_2-1\end{pmatrix}\right\}\right)\\&=\operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2t_2+v_2\end{pmatrix},\begin{pmatrix}1\\2t_2+v_2-1\end{pmatrix}\right\},\end{aligned}$$

which is in $\mathcal{Z}$ with $\overline{t_1}=t_1$ and $\overline{t_2}=t_2+\frac{v_2}{2}$. We remark that $\bar{t}$ is in T since otherwise $\bar{t}+\overline{Z}$ would not be of minimal volume.

Now we consider the affine transformation $x \mapsto -Lv + Lx$. In case that this transformation is needed to transform $\bar{t} + \overline{Z}$ to some $t + Z \in \mathcal{Z}$, Z has a generator in $\{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$. This is the case only if $t_1 + t_2 = \frac{1}{2}$, and then

$$\begin{aligned}
\bar{t} + \overline{Z} = L^{-1}(t+Z) + v &= L^{-1}\left(\operatorname{conv}\left\{\begin{pmatrix}-2t_2\\ 2t_2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1-2t_2\\ 2t_2-1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}\right) + v\\
&= \operatorname{conv}\left\{\begin{pmatrix}-2t_2+v_1\\ 2t_2+v_2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1-2t_2+v_1\\ 2t_2+v_2-1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}
\end{aligned}$$

which is in $\mathcal{Z}$ with $\overline{t_1} = -t_2 + \frac{v_1}{2}$ and $\overline{t_2} = t_2 + \frac{v_2}{2}$. Again, $\bar{t} \in T$ since otherwise $\bar{t} + \overline{Z}$ would not be of minimal volume. $\square$

An analogous statement is true for the zonotopes of minimal volume containing $C_3^\star$:

2.29 Theorem
The set of 3-dimensional zonotopes of minimal volume containing the crosspolytope $C_3^\star$ is

$$\mathcal{Z} = \left\{\mathrm{Z}(A) : (t_1, t_2, t_3)^\top \in T\right\},$$

where T is the 3-dimensional simplex

$$T = \operatorname{conv}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1/2\\1/2\\0\end{pmatrix}, \begin{pmatrix}0\\1/2\\1/2\end{pmatrix}, \begin{pmatrix}1/2\\0\\1/2\end{pmatrix}\right\},$$

and

$$A = \begin{pmatrix} -t_1 & 1/2-t_1 & 1/2-t_1 & t_1\\ 1/2-t_2 & -t_2 & 1/2-t_2 & t_2\\ 1/2-t_3 & 1/2-t_3 & -t_3 & t_3 \end{pmatrix}.$$

We present some examples in Figure 2.4. In all cases, the grey filled polytope is $C_3^\star$ and the black lines are the edges of a zonotope of minimal volume containing $C_3^\star$.

Similar statements as in the simplex case are true for different $t \in T$. t being a vertex of T means that all but one of the (3×3)-subdeterminants of A are zero, that is, one of the columns of A equals zero. Thus $\mathrm{Z}(A)$ is a parallelepiped. For t lying in the interior of an edge of T, exactly two of the subdeterminants are 0, and there are two linearly dependent generators. Thus the zonotope is a parallelepiped as well. In the interior of

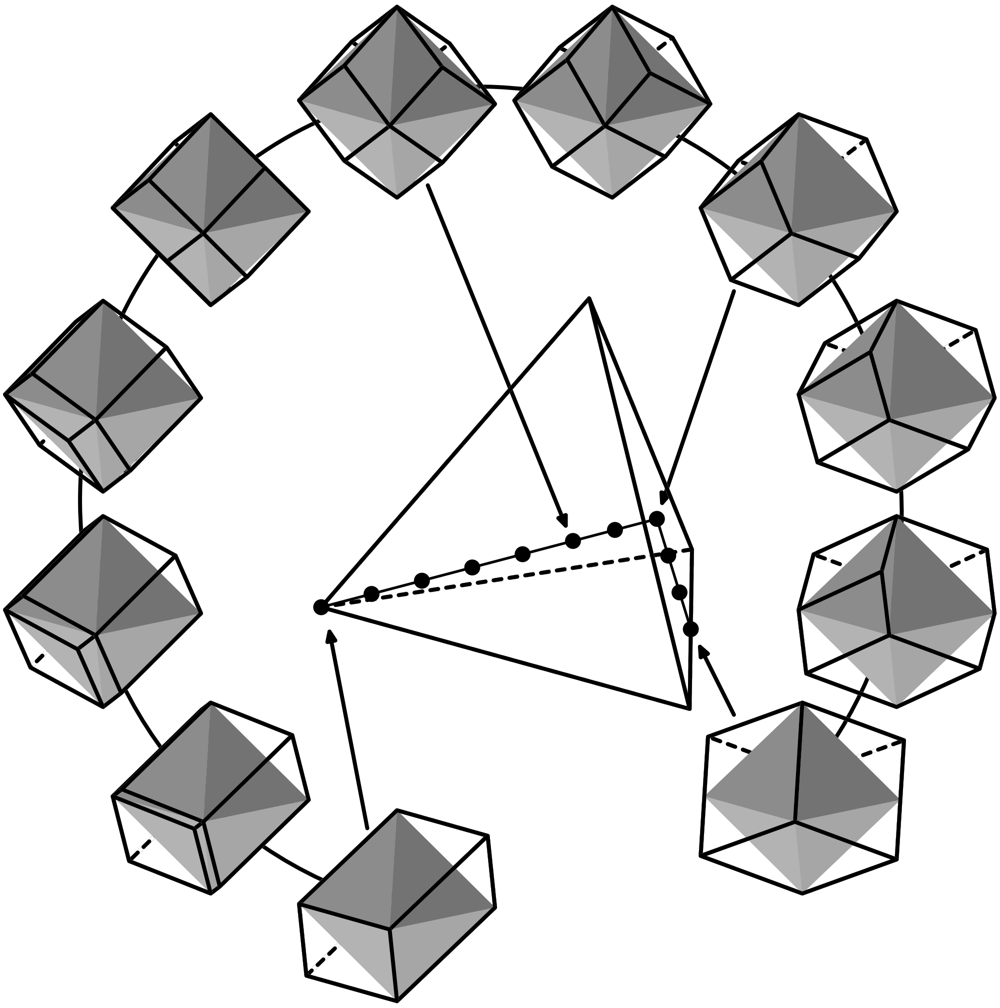

Figure 2.4: The simplex of parameters T and some examples of zonotopes of minimal volume containing $C_3^\star$.

a facet exactly one equality is attained, which implies that the corresponding zonotope is a prism over a 6-gon. Finally, the interior points of T correspond to a zonotope with 4 generators in general position, the rhombic dodecahedron.

PROOF (OF THEOREM 2.29)
Let A be a $(3\times m)$-matrix and let $\mathrm{Z}(A)$ be a zonotope of minimal volume containing $C_3^\star$. First we assume that all vertices of $C_3^\star$ are vertices of $t+Z$. Furthermore, let $H:=\mathrm{H}(A)$, that is, an $(m\times 3)$-(± 1)-matrix with $A\cdot H = I_2$.

We first consider only one single matrix $\overline{H}$ of every equivalence class.

By Lemma 2.19 we can only have $m=3$ or $m=4$. First we consider the case $m=3$. Then, by Proposition 2.18, H is a quadratic (± 1)-matrix with maximal determinant. Up to equivalence, the maximal determinant (3×3)-(± 1)-matrix is unique (see [37]), namely

$$H = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Then the corresponding matrix of generators A is

$$A = H^{-1} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

Now we consider the case $m=4$ and we assume that A does not have two linearly dependent columns. By Proposition 2.18, H cannot have two linearly dependent rows. Thus we can choose 4 of the 8 possible (± 1)-vectors in such a way that no two opposite vectors are chosen. Thus, also in this case H is unique up to equivalence, namely

$$H = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we can solve the linear system $A\cdot H = I_n$ and get the solutions

$$A = \begin{pmatrix} -t_1 & 1/2-t_1 & 1/2-t_1 & t_1 \\ 1/2-t_2 & -t_2 & 1/2-t_2 & t_2 \\ 1/2-t_3 & 1/2-t_3 & -t_3 & t_3 \end{pmatrix}$$

for all vectors $t=(t_1,t_2,t_3)$ of 3 parameters.

One can easily check that all (3×3)-subdeterminants of H are ± 4. By Proposition 2.18, $t+\mathrm{Z}(A)$ is of minimal volume if and only if $\det(A_I)=0$ or $\mathrm{sign}(\det(A_I)) = \mathrm{sign}(\det(H^I))$ for all I:

I	$\det(H^I)$	$\det(A(t)_I)$
$\{1,2,3\}$	4	$\frac{1}{4}(-t_1 - t_2 - t_3 + 1)$
$\{1,2,4\}$	4	$\frac{1}{4}(t_1 + t_2 - t_3)$
$\{1,3,4\}$	-4	$\frac{1}{4}(-t_1 + t_2 - t_3)$
$\{2,3,4\}$	4	$\frac{1}{4}(-t_1 + t_2 + t_3)$

These conditions yield the following inequalities for t_i:

$$\begin{aligned} +t_1 + t_2 + t_3 &\leq 1 \\ +t_1 + t_2 - t_3 &\geq 0 \\ +t_1 - t_2 + t_3 &\geq 0 \\ -t_1 + t_2 + t_3 &\geq 0 \end{aligned}$$

which describes the set T in the assertion.

Observe that $t_1 = t_2 = t_3 = 0$ is allowed and coincides with the case $m = 3$.

Now let H_2 be a (3×4)-(± 1)-matrix that is equivalent to H. Then H_2 is obtained from H by negation of rows and columns and interchanging rows and columns. Negating a row of H coincides with negating a column of A which does not change $\mathrm{Z}(A)$ at all. Negating a column of H coincides with negating a row of A which is a reflection of $\mathrm{Z}(A)$ with respect to a coordinate hyperplane and can be described by a transformation $t_i \mapsto t_i$, $t_j \mapsto \frac{1}{2} - t_j$, $j \neq i$ for each row i. Interchanging two columns of H coincides with interchanging two rows of A, which is just changing the roles of t_i and t_j. Interchanging two rows of H coincides with interchanging two columns of A, which does not change $\mathrm{Z}(A)$ at all. Thus H_2 produces the same set $\mathcal{Z}$.

It remains to show that $\mathcal{Z}$ also contains all those zonotopes of minimal volume containing $C_3^\star$ that do not have all vertices of $C_3^\star$ as vertices. To this end, we use Lemma 2.15. Whenever $\overline{t}+\overline{Z}$ is a zonotope of minimal volume containing $C_3^\star$, there is a linear transformation to $t+Z \in \mathcal{Z}$. This transformation is a composition of some of the linear transformations A_i of the form $A_1^{-1} = \begin{pmatrix} v & e_2 & e_3 \end{pmatrix}$, $A_2^{-1} = \begin{pmatrix} e_1 & v & e_3 \end{pmatrix}$ and $A_3^{-1} = \begin{pmatrix} e_1 & e_2 & v \end{pmatrix}$ with $v_i = 1$.

Without loss of generality we just consider A_1. In case that A_1 is one of the transformations that are necessary to transform $\overline{Z}$ to some $Z \in \mathcal{Z}$, Z has a generator that is orthogonal to e_1. This is the case only if $t_1 = 0$ or $t_1 = \frac{1}{2}$. In both cases, the associated zonotopes depend only on $t_2 \in [0, \frac{1}{2}]$ and for every fixed t_2 they are reflections of each other with respect to the coordinate hyperplane $\{x \in \mathbb{R}^n : x_2 = 0\}$. Since $\mathcal{Z}$ is closed under such reflexions, it is enough to consider only $t_1 = 0$. It this case $t_2 \in [0, \frac{1}{2}]$ and

$t_2 = t_3$:

$$\begin{aligned}\overline{Z} = A_1^{-1}\,\mathrm{Z}(A) &= A_1^{-1} Z \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & -t_2 & 1/2 - t_2 \\ 1/2 & 1/2 - t_2 & -t_2 \end{pmatrix} \\ &= Z \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & v_2/2 - t_2 & v_2/2 + 1/2 - t_2 \\ 1/2 & v_3/2 + 1/2 - t_2 & v_3/2 - t_2 \end{pmatrix}\end{aligned}$$

which is in $\mathcal{Z}$ with $\overline{t_1} = t_1 = 0$, $\overline{t_2} = t_2 - \frac{v_2}{2}$ and $\overline{t_3} = t_2 - \frac{v_3}{2}$. We remark that $\overline{t}$ is in T since otherwise $\overline{Z}$ would not be of minimal volume. If $A_1^{-1}\,\mathrm{Z}(A)$ does still not have all unit vectors as vertices, again by Lemma 2.15, $A_1^{-1}\,\mathrm{Z}(A)$ again has a generator that is orthogonal to a unit vector and we can repeat the argument. $\square$

To conclude this section, we study whether our zonotopes of minimal volume containing $C_3^\star$ are also optimal with respect to another notion of minimality.

For every zonotope $Z \subseteq \mathbb{R}^n$ we denote by $\lambda(Z)$ the smallest $\lambda \geq 1$ with $C_n^\star \subseteq Z \subseteq \lambda C_n^\star$. Then the following problem arises:

2.30 Problem
For which zonotopes Z is $\lambda(Z)$ minimal?

This problem describes the question for which zonotopes the Banach–Mazur distance to $C_n^\star$ is minimal. The Banach–Mazur distance $d_{BM}(K, L)$ of two $\mathbb{0}$-symmetric convex bodies $K, L \in \mathcal{K}_0^n$ is defined as follows:

$$d_{BM}(K, L) = \inf\{\lambda > 0 : \mathcal{T}_L(K, \lambda K) \neq \emptyset\}$$

where $\mathcal{T}_L(K_1, K_2)$ is the set of all invertible linear transformations T such that $K_1 \subset TL \subset K_2$. For an introduction to the Banach–Mazur distance we refer to [18, Chapter 11]. Hence, the Banach–Mazur distance to $C_n^\star$ is minimized exactly by the linear images of all zonotopes that are solutions of Problem 2.30.

Schneider showed in [41] that for any zonotope Z any λ with $C_n^\star \subseteq Z \subseteq \lambda C_n^\star$ satisfies

$$\lambda \geq 2^{-n+1} n \binom{n-1}{\left[\frac{n-1}{2}\right]} \sim \sqrt{\frac{2}{\pi}}\sqrt{n}.$$

For $n = 3$ this reduces to $\lambda(Z) \geq \frac{3}{2}$. Since our zonotopes of minimal volume contain $C_3^\star$, we only need to calculate the factor $\lambda(Z)$ for an arbitrary Z in the set given in Theorem 2.29. To this end, we work out a vertex-description of $Z \in \mathcal{Z}$ and get

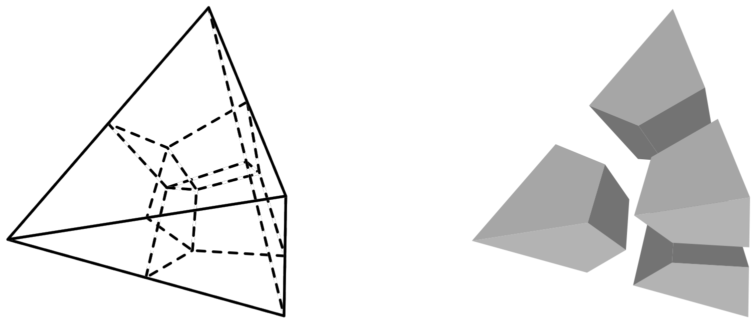

Figure 2.5: Visualization of the partition of T into T_1, T_2, T_3, T_4

$$\mathcal{Z} = \left\{ \operatorname{conv}\left(\pm\begin{pmatrix}1\\0\\0\end{pmatrix}, \pm\begin{pmatrix}0\\1\\0\end{pmatrix}, \pm\begin{pmatrix}0\\0\\1\end{pmatrix}, \pm\begin{pmatrix}1-2t_1\\1-2t_2\\1-2t_3\end{pmatrix}, \pm\begin{pmatrix}1-2t_1\\-2t_2\\-2t_3\end{pmatrix}, \right.\right.$$
$$\left.\left. \pm\begin{pmatrix}-2t_1\\1-2t_2\\-2t_3\end{pmatrix}, \pm\begin{pmatrix}-2t_1\\-2t_2\\1-2t_3\end{pmatrix}\right) : \begin{pmatrix}t_1\\t_2\\t_3\end{pmatrix} \in T \right\}.$$

Calculating the 1-norm of v, namely $|v_1| + |v_2| + |v_3|$ for every vertex v of $Z \in \mathcal{Z}$ yields

$$\lambda(Z) = \begin{cases} 1 + 2(-t_1 + t_2 + t_3), & t_1 \leq t_2,\ t_1 \leq t_3,\ t_2 + t_3 \geq \frac{1}{2}, \\ 1 + 2(t_1 - t_2 + t_3), & t_2 \leq t_3,\ t_2 \leq t_1,\ t_1 + t_3 \geq \frac{1}{2}, \\ 1 + 2(t_1 + t_2 - t_3), & t_3 \leq t_2,\ t_3 \leq t_1,\ t_1 + t_2 \geq \frac{1}{2}, \\ 3 - 2(t_1 + t_2 + t_3), & t_1 + t_2 \leq \frac{1}{2},\ t_1 + t_3 \leq \frac{1}{2},\ t_2 + t_3 \leq \frac{1}{2}. \end{cases}$$

Thus, we consider the barycentric subdivision of the simplex T in 4 parts T_i, that is, the dividing planes are defined by barycenters of edges of T, facets of T, and T itself:

- $T_1 := \{(t_1, t_2, t_3) \in T : t_1 \leq t_2,\ t_1 \leq t_3,\ t_2 + t_3 \geq \frac{1}{2}\}$,
- $T_2 := \{(t_1, t_2, t_3) \in T : t_2 \leq t_3,\ t_2 \leq t_1,\ t_1 + t_3 \geq \frac{1}{2}\}$,
- $T_3 := \{(t_1, t_2, t_3) \in T : t_3 \leq t_2,\ t_3 \leq t_1,\ t_1 + t_2 \geq \frac{1}{2}\}$,
- $T_4 := \{(t_1, t_2, t_3) \in T : t_1 + t_2 \leq \frac{1}{2},\ t_1 + t_3 \leq \frac{1}{2},\ t_2 + t_3 \leq \frac{1}{2}\}$.

Then $\lambda(Z)$ is an affine function in $(t_1, t_2, t_3)^\top$ on each T_i. In particular, we get the following values for the vertices of the subdivision:

vertices of T	Z is a parallelepiped	$\lambda = 3$
midpoints of edges of T	Z is a parallelepiped	$\lambda = 2$
barycenters of facets of T	Z is a prism over a regular 6-gon	$\lambda = 5/3$
barycenter of T	Z is the rhombic dodecahedron	$\lambda = 3/2$.

Thus the minimal value of λ for each T_i is the midpoint of T, $(1/4, 1/4, 1/4)^\top$ and the value of $\lambda(Z)$ at this point is $\frac{3}{2}$. Since this coincides with Schneider's lower bound on λ, we get the following result:

2.31 Proposition
The rhombic dodecahedron is the only zonotope of minimal volume containing $C_3^\star$ that is also a solution to Problem 2.30. Conversely, among all zonotopes for which $\lambda(Z)$ is minimal, the rhombic dodecahedron has minimal volume.

2.5 Minkowski's second theorem and the Betke–Henk–Wills-conjecture for zonotopes

In this section we consider zonotopes with respect to lattice points. The work on volume bounds on zonotopes containing the crosspolytope gives us the possibility to improve the lower bound in Minkowski's second theorem, Theorem 1.9, for zonotopes. First, we will give an equivalent formulation of the bound for zonotopes containing $C_n^\star$:

2.32 Corollary
Let $Z = \mathrm{Z}(A)$, $A = \begin{pmatrix} a_1 & \dots & a_m \end{pmatrix}$, be a zonotope, that is symmetric with respect to $\mathbb{0}$. Then for all $\mathbb{0}$-symmetric crosspolytopes $\diamondsuit = \mathrm{conv}\{\pm v_1, \dots, \pm v_n\}$ with $\diamondsuit \subseteq Z$

$$\mathrm{vol}(Z) \geq \mathrm{vol}(\diamondsuit) \frac{n!}{\mathrm{maxdet}(n)}.$$

Proof
Consider the linear transformation L with $L\diamondsuit = C_n^\star$. Then $C_n^\star = L\diamondsuit \subseteq LZ$, which implies by Theorem 2.17

$$\mathrm{vol}(Z) = \frac{1}{\det(L)} \mathrm{vol}(LZ) \geq \frac{1}{\det(L)} \mathrm{vol}(C_n^\star) \frac{n!}{\mathrm{maxdet}(n)} = \mathrm{vol}(\diamondsuit) \frac{n!}{\mathrm{maxdet}(n)}. \qquad \square$$

2.33 Corollary
Let $\Lambda \in \mathcal{L}^n$ and $Z \subseteq \mathbb{R}^n$ a $\mathbb{0}$-symmetric, n-dimensional zonotope. Then

$$\frac{2^n}{\mathrm{maxdet}(n)} \det(\Lambda) \leq \mathrm{vol}(Z) \prod_{i=1}^{n} \lambda_i(Z, \Lambda).$$

PROOF
Let $z_1, \dots, z_n \in \Lambda$ be linearly independent with $z_i \in \lambda_i(Z,\Lambda)Z$ for $i = 1, \dots n$. Then $\frac{1}{\lambda_i(Z,\Lambda)} z_i \in Z$ for $i = 1, \dots, n$, and

$$\operatorname{conv}\left\{\pm\frac{1}{\lambda_1(Z,\Lambda)}z_1, \dots, \pm\frac{1}{\lambda_n(Z,\Lambda)}z_n\right\} \subseteq Z.$$

By Corollary 2.32 we get

$$\begin{aligned}
\operatorname{vol}(Z) &\geq \frac{n!}{\operatorname{maxdet}(n)} \operatorname{vol}\left(\operatorname{conv}\left\{\pm\frac{1}{\lambda_1(Z,\Lambda)}z_1, \dots, \pm\frac{1}{\lambda_n(Z,\Lambda)}z_n\right\}\right) \\
&= \frac{n!}{\operatorname{maxdet}(n)} \operatorname{vol}\left(\operatorname{conv}\left\{\pm z_1, \dots, \pm z_n\right\}\right) \prod_{i=1}^{n} \frac{1}{\lambda_i(Z,\Lambda)} \\
&= \frac{n!}{\operatorname{maxdet}(n)} \frac{2^n}{n!} \det\begin{pmatrix} z_1 & \dots & z_n \end{pmatrix} \prod_{i=1}^{n} \frac{1}{\lambda_i(Z,\Lambda)} \\
&\geq \frac{2^n}{\operatorname{maxdet}(n)} \det(\Lambda) \prod_{i=1}^{n} \frac{1}{\lambda_i(Z,\Lambda)}
\end{aligned}$$

where the last step follows from $z_1, \dots, z_n \in \Lambda$. □

We remark that, since $\operatorname{maxdet}(n) \leq n^{\frac{n}{2}}$, by Stirling's formula we get

$$\begin{aligned}
\operatorname{vol}(Z) \prod_{i=1}^{n} \lambda_i(Z,\Lambda) &\geq \frac{2^n}{n^{\frac{n}{2}}} \det(\Lambda) \\
&\sim \frac{2^n}{n!} \left(\frac{\sqrt{n}}{e}\right)^n \sqrt{2n\pi} \det(\Lambda).
\end{aligned}$$

Although the inequality in Problem 2.5 is not true for all polytopes, we can present some positive results in the direction of the coefficient-wise approach. For zonotopes, such a coefficient-wise approach is doable, since there are explicit formulas for the Ehrhart coefficients (see Lemma 2.11).

From now on, all results on zonotopes and lattices are only formulated and proven for the integral lattice $\mathbb{Z}^n$ to simplify the notation in statements and proofs. All results can be reformulated for arbitrary lattices, however, by applying a suitable linear transformation to both lattice and zonotope.

2.34 PROPOSITION
Let $Z = \sum_{j=1}^{m}[-z_j, z_j] \subseteq \mathbb{R}^n$ be a $\mathbb{0}$-symmetric integral zonotope, that is, $z_j \in \mathbb{Z}^n$, $1 \leq j \leq m$. We assume that the generators z_i are primitive and in general position.

Then

$$\mathrm{G}_1(Z) = 2m \leq \sum_{i=1}^{n} \frac{2}{\lambda_i(Z)} = \sigma_1\left(\frac{2}{\lambda_1(Z)}, \ldots, \frac{2}{\lambda_n(Z)}\right).$$

Proof

By Lemma 2.11, and since the generators are primitive, we have that

$$\mathrm{G}_1(Z) = 2\sum_{i=1}^{m} \gcd(z_i) = 2m.$$

Furthermore, since the vectors are in general position and any integral $\mathbb{0}$-symmetric parallelepiped has volume at least 2^n, we have $\mathrm{vol}(Z) \geq 2^n\binom{m}{n}$ and together with Minkowski's second theorem (Theorem 1.9) we conclude that

$$2^n \geq \mathrm{vol}(Z)\prod_{i=1}^{n} \lambda_i(Z) \geq 2^n\binom{m}{n}\prod_{i=1}^{n} \lambda_i(Z).$$

Hence,

$$\prod_{i=1}^{n} \frac{1}{\lambda_i(Z)} \geq \binom{m}{n},$$

and the inequality of the arithmetic and geometric mean yields

$$\sum_{i=1}^{n} \frac{1}{\lambda_i(Z)} \geq n\binom{m}{n}^{1/n} \geq m,$$

where the last inequality is an equality only if $m = n$. □

We give another expression of $\mathrm{G}_i(Z)$, which is more adepted to the geomtric structure of zonotopes than the expression of Stanley in Lemma 2.11.

2.35 Lemma

Let $Z = \sum_{j=1}^{m}[0, z_j] \subseteq \mathbb{R}^n$ be an integral zonotope, that is, $z_j \in \mathbb{Z}^n$, $1 \leq j \leq m$ and for $I \subseteq \{1, \ldots, m\}$, $\#I = i$, let $P_I = \sum_{j\in I}[0, z_j]$ be the parallelepiped generated by the vectors z_j, $j \in I$. Then

$$\mathrm{G}_i(Z) = \sum_{\substack{I\subseteq\{1,\ldots,m\}\\ \#I=i}} \frac{\mathrm{vol}_i(P_I)}{\det(\mathrm{lin}(P_I) \cap \mathbb{Z}^n)},$$

for all $i = 1, \ldots, n$.

Proof

If for $I \subseteq \{1, \ldots, m\}$, $\#I = i$, the vectors z_j, $j \in I$, are linearly dependent, then $\mathrm{vol}_i(P_I) = 0$ and so any non-trivial contribution to the sum comes from an i-dimensional parallelepiped. Thus, without loss of generality, let $\{z_j : j \in I\} = \{z_1, \ldots, z_i\} = V_I$ and the vectors be linearly independent. In this case $\mathrm{vol}(P_I)$ is exactly the determinant of the lattice $V_I\,\mathbb{Z}^i$ defined by V_I. Let $\overline{V}_I$ be an $(n \times i)$-matrix whose columns constitute a basis of the lattice $\mathrm{lin}(P_I) \cap \mathbb{Z}^n$. Then there exists an integer matrix $D_I \in \mathbb{Z}^{i\times i}$ with $V_I = \overline{V}_I D_I$ and $|\det(D_I)|$ is the index of the lattice $V_I\,\mathbb{Z}^i$ in the lattice $\mathrm{lin}(P_I) \cap \mathbb{Z}^n$. Hence,

$$|\det(D_I)| = \frac{\mathrm{vol}_i(P_I)}{\det(\mathrm{lin}(P_I) \cap \mathbb{Z}^n)}.$$

Thus, by Lemma 2.11, it remains to show, that

$$|\det(D_I)| = \gcd(\{z : z \text{ is } i\text{-minors of } V_I\}).$$

It is obvious that $|\det(D_I)|$ is a divisior of each i-minor of V_I, since $V_I = \overline{V}_I D_I$. For the converse divisibility we extend the vectors of $\overline{V}_I$ to a basis $\begin{pmatrix}\overline{V}_I & v_{i+1} & \ldots & v_n\end{pmatrix}$ of $\mathbb{Z}^n$ and write

$$\begin{pmatrix}\overline{V}_I & v_{i+1} & \ldots & v_n\end{pmatrix} \cdot \begin{pmatrix} D_I & \mathbb{0} \\ \mathbb{0} & I_{n-i}\end{pmatrix} = \begin{pmatrix}V_I & v_{i+1} & \ldots & v_n\end{pmatrix}.$$

Expanding the determinant of the matrix on the right-hand side using Laplace's formula with respect to the last $n-i$ columns yields together with $\det\begin{pmatrix}\overline{V}_I & v_{i+1} & \ldots & v_n\end{pmatrix} = 1$

$$\det(D_I) = \det\begin{pmatrix}V_I & v_{i+1} & \ldots & v_n\end{pmatrix} = \sum_{i\text{-minors } \mu_k \text{ of } V_I} \rho_k\, \mu_k$$

for some integers ρ_k. Hence, $\det(D_I)$ is a multiple of $\gcd(i\text{-minors of } V_I)$. □

We remark that for $Z = \sum_{j=1}^m [-z_j, z_j] \subseteq \mathbb{R}^n$, Lemma 2.35 implies that for $i = 1, \ldots, n$,

$$\mathrm{G}_i(Z) = 2^i \sum_{\substack{I \subseteq \{1,\ldots,m\} \\ \#I = i}} \frac{\mathrm{vol}_i(P_I)}{\det(\mathrm{lin}(P_I) \cap \mathbb{Z}^n)}. \tag{2.5}$$

Using this, we can give a bound on $\mathrm{G}_i(Z)$ in terms of the $\lambda_i(Z)$.

2.36 Theorem
Let $Z = \sum_{j=1}^m [-z_j, z_j] \subseteq \mathbb{R}^n$ be a $\mathbb{0}$-symmetric integral zonotope, that is, $z_j \in \mathbb{Z}^n$, $1 \le j \le m$. Then

$$\frac{\mathrm{G}_i(Z)}{\mathrm{vol}(Z)} \le \binom{n}{i} \,\mathrm{maxdet}(n-i) \prod_{j=i+1}^{n} \frac{\lambda_j(Z)}{2}, \quad \textit{for } i = 1, \ldots, n.$$

Proof

For $I \subseteq \{1, \dots, m\}$, $\#I = i$, let $P_I = \sum_{j \in I} [-z_j, z_j]$, $L_I = \mathrm{lin}(z_j : j \in I)$ and $L_I^{\perp}$ its orthogonal complement.

For $J \subseteq \{1, \dots, m\}$, $\#J = n$ and $i \in \{1, \dots, n\}$ let $I \subseteq J$, $\#I = i$. Then

$$\mathrm{vol}(P_J) = \mathrm{vol}_i(P_I) \cdot \mathrm{vol}_{n-i}(P_J | L_I^{\perp}), \tag{2.6}$$

where $P_J | L_I^{\perp}$ denotes the orthogonal projection of P_J onto $L_I^{\perp}$. In order to see this, without loss of generality, let $I = \{1, \dots, i\}$ and $L_I = \mathrm{lin}\{e_1, \dots, e_i\}$. Then, the generators of P_J are of the form

$$\begin{pmatrix} Z_i & \overline{Z}_{n-i} \\ \mathbb{0} & Z_{n-1} \end{pmatrix},$$

where the first i columns generate P_I and $P_J | L_I^{\perp}$ is generated by $\begin{pmatrix} \mathbb{0} \\ Z_{n-1} \end{pmatrix}$. Using $\mathrm{vol}_k(P) = \sqrt{\det(A^{\top} A)}$, where A is the $(n \times k)$-matrix whose columns are the generators of the parallelepiped P, we get Equation (2.6).

Equation (2.6) implies, together with Lemma 2.35 and $\mathrm{vol}(Z) = \mathrm{G}_n(Z)$, that

$$\begin{aligned}
\mathrm{vol}(Z) &= 2^n \sum_{\substack{J \subseteq \{1,\dots,n\} \\ \#J = n}} \mathrm{vol}(P_J) \\
&= 2^n \sum_{\substack{J \subseteq \{1,\dots,n\} \\ \#J = n}} \frac{1}{\binom{n}{i}} \sum_{I \subseteq J, \#I = i} \mathrm{vol}_i(P_I) \cdot \mathrm{vol}_{n-i}(P_J | L_I^{\perp}) \\
&= \frac{2^n}{\binom{n}{i}} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I = i}} \mathrm{vol}_i(P_I) \sum_{\substack{I \subseteq J \\ \#J = n}} \mathrm{vol}_{n-i}(P_J | L_I^{\perp}).
\end{aligned}$$

Furthermore, for $I \subseteq \{1, \dots, m\}$, $\#I = i$ we have

$$\sum_{I \subseteq J, \#J = n} \mathrm{vol}_{n-i}(P_J | L_I^{\perp}) = \frac{1}{2^{n-i}} \mathrm{vol}(Z | L_I^{\perp}),$$

because the sum on the left-hand side covers all volumes of $(n-i)$-dimensional parallelepipeds that are spanned by generators of $Z | L_I^{\perp}$. This implies

$$\begin{aligned}
\mathrm{vol}(Z) &= \frac{2^n}{\binom{n}{i}} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I = i}} \mathrm{vol}_i(P_I) \frac{1}{2^{n-i}} \mathrm{vol}(Z | L_I^{\perp}) \\
&= \frac{2^i}{\binom{n}{i}} \sum_{\substack{I \subseteq \{1,\dots,n\} \\ \#I = i}} \frac{\mathrm{vol}_i(P_I)}{\det(\mathbb{Z}^n \cap L_I)} \frac{\mathrm{vol}(Z | L_I^{\perp})}{\det(\mathbb{Z}^n | L_I^{\perp})},
\end{aligned}$$

where for the last step we refer to Martinet [31, Corollary 1.3.5]. Together with Corollary 2.33, we get

$$\mathrm{vol}(Z) \geq \frac{2^i}{\binom{n}{i}} \sum_{\substack{I\subseteq\{1,\dots,n\} \\ \#I=i}} \frac{\mathrm{vol}_i(P_I)}{\det(\mathbb{Z}^n \cap L_I)} \left(\frac{2^{n-i}}{\mathrm{maxdet}(n-i)} \prod_{j=1}^{n-i} \frac{1}{\lambda_j(Z|L_I^\perp, \mathbb{Z}^n\,|L_I^\perp)} \right).$$

Since $\lambda_j(Z|L_I^\perp, \mathbb{Z}^n\,|L_I^\perp) \leq \lambda_{i+j}(Z)$ for $j = 1, \dots, n-i$,

$$\mathrm{vol}(Z) \geq \frac{2^n}{\binom{n}{i}\,\mathrm{maxdet}(n-i)} \sum_{\substack{I\subseteq\{1,\dots,n\} \\ \#I=i}} \frac{\mathrm{vol}_i(P_I)}{\det(\mathbb{Z}^n \cap L_I)} \prod_{j=i+1}^{n} \frac{1}{\lambda_j(Z)}.$$

Together with Equation (2.5), we finally get

$$\mathrm{vol}(Z) \geq \frac{2^{n-i}}{\binom{n}{i}\,\mathrm{maxdet}(n-i)} \mathrm{G}_i(Z) \prod_{j=i+1}^{n} \frac{1}{\lambda_j(Z)} = \frac{\mathrm{G}_i(Z)}{\binom{n}{i}\,\mathrm{maxdet}(n-i)} \prod_{j=i+1}^{n} \frac{2}{\lambda_j(Z)}. \quad \square$$

This immediately implies the following bound:

2.37 Corollary

Let $Z = \sum_{j=1}^m [-z_j, z_j] \subseteq \mathbb{R}^n$ *be a* $\mathbb{0}$*-symmetric integral zonotope, that is,* $z_j \in \mathbb{Z}^n$*,* $1 \leq j \leq m$*. Then*

$$\mathrm{G}_i(Z) \leq \binom{n}{i} \mathrm{maxdet}(n-i)\, \sigma_i \left(\frac{2}{\lambda_1(Z)}, \dots, \frac{2}{\lambda_n(Z)} \right), \quad \textit{for } i = 1, \dots, n.$$

Proof

By Theorem 2.36, we have

$$\mathrm{G}_i(Z) \leq \binom{n}{i} \mathrm{maxdet}(n-i) \prod_{j=i+1}^{n} \frac{\lambda_j(Z)}{2} \mathrm{vol}(Z),$$

which, by Theorem 1.9, implies

$$\begin{aligned} \mathrm{G}_i(Z) &\leq \binom{n}{i} \mathrm{maxdet}(n-i) \prod_{j=i+1}^{n} \frac{\lambda_j(Z)}{2} \prod_{j=1}^{n} \frac{2}{\lambda_j(Z)} \\ &= \binom{n}{i} \mathrm{maxdet}(n-i) \prod_{j=1}^{i} \frac{2}{\lambda_j(Z)}. \end{aligned}$$

The corollary follows, since $\prod_{j=1}^{i} \frac{2}{\lambda_j(Z)}$ is just one summand of $\sigma_i \left(\frac{2}{\lambda_1(Z)}, \dots, \frac{2}{\lambda_n(Z)} \right)$, and all summands are non-negative. $\square$

We remark that, since $\operatorname{maxdet}(n) \le n^{\frac{n}{2}}$, Corollary 2.37 implies

$$\mathrm{G}_i(Z) \le \binom{n}{i}(n-i)^{\frac{n-i}{2}}\sigma_i\left(\frac{2}{\lambda_1(Z)},\ldots,\frac{2}{\lambda_n(Z)}\right), \quad \text{for } i=1,\ldots,n.$$

To finish this section, we state another result on bounds on coefficients of Ehrhart polynomials of zonotopes in terms of their successive minima. The proof of this result is published in [10] together with the other results of this section.

2.38 Theorem ([10])
Let $Z = \sum_{j=1}^{m}[-z_j, z_j] \subseteq \mathbb{R}^n$ be a $\mathbb{0}$-symmetric integral zonotope, that is $z_j \in \mathbb{Z}^n$, $1 \le j \le m$, and we assume that the generators z_i are in general position. Then for all $1 \le i \le n$,

$$\mathrm{G}_i(Z) \le \frac{\binom{m}{i}}{\binom{n}{i}}\sigma_i\left(\frac{2}{\lambda_1(Z)},\ldots,\frac{2}{\lambda_n(Z)}\right).$$

We further mention that in case of parallelepipeds, Theorem 2.38 for $m = n$ solves Problem 2.5 positively.

3 Rational Ehrhart quasi-polynomials

The statements in the introduction on Ehrhart theory in Chapter 1 are only true if the polytope of consideration is a lattice polytope. In this chapter, we consider the case of rational polytopes. In this case, the lattice point enumerator of kP is a quasi-polynomial in the integral dilation factor k, that is, a polynomial-type function whose coefficients are themselves periodic functions in the argument. The periods of these coefficients are subject to active research and show a behavior that is not well understood so far.

In the first section of the chapter, we give an introduction into integral Ehrhart theory of rational polytopes. The statements in this section are basic information and known results, and thus we will not give any proofs here.

As a generalization, we consider rational dilation factors. Since a rational polytope remains rational after a dilation with a rational factor, this is a natural generalization. In Section 3.2, we will generalize most of the basic results stated before to the rational case. Since the rational version of the Ehrhart quasi-polynomial is a function on a dense set of numbers, we get more structural information about the coefficients, now as functions in rational arguments. The coefficients are piecewise-defined polynomials and derivatives of each other. We state and proof these results in Section 3.3. We further explicitly work out an example in dimension 2 (Section 3.4).

Finally, we give a generalization to the setting of investigating the number of integral points in polytopes $P_A(b) = \{x \in \mathbb{R}^n : Ax \leq b\}$ with varying rational right-hand sides b. Here, we also get a quasi-polynomial structure with polynomial coefficients (Section 3.6). To this end, we first consider the case of Minkowski sums of several polytopes, each dilated with a different rational factor. Since these results are of independent interest, we state and prove them in a seperate Section 3.5.

3.1 Basics on Ehrhart quasi-polynomials

To give an introduction to Ehrhart theory of rational polytopes, we restrict ourselves to the integral lattice. The results can be translated to arbitrary lattices by applying a suitable linear transformation on both lattice and polytope.

The basic definition is that of a quasi-polynomial.

3.1 Definition (Quasi-polynomial)
A function $p : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ *is called a* quasi-polynomial *with period* d *of degree at most* n *if there exist periodic functions* $p_i : \mathbb{Z}_{\geq 0} \to \mathbb{R}$, $i = 1, \ldots, n$, *with period* d *such that* $p(k) = \sum_{i=0}^{n} p_i(k) k^i$.

Roughly speaking, a quasi-polynomial looks like a usual polynomial whose coefficients are not constants, but periodic functions. In particular, the coefficients can take finitely many different values.

3.2 Example
Let $p : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ with

$$p(k) = \sum_{i=0}^{2} p_i(k) k^i = \begin{cases} \frac{9}{4}k^2 + \frac{3}{2}k + 1, & k \equiv 0 \bmod 4, \\ \frac{9}{4}k^2 \qquad\quad + \frac{3}{4}, & k \equiv 1 \bmod 4, \\ \frac{9}{4}k^2 \qquad\quad + 1, & k \equiv 2 \bmod 4, \\ \frac{9}{4}k^2 \qquad\quad + \frac{3}{4}, & k \equiv 3 \bmod 4. \end{cases}$$

p is a quasi-polynomial with period 4 of degree 2. Here $p_2(k) = 9/4$,

$$p_1(k) = \begin{cases} \frac{3}{2}, & k \equiv 0 \bmod 4 \\ 0, & \text{otherwise}, \end{cases} \qquad p_0(k) = \begin{cases} 1, & k \equiv 0 \bmod 2, \\ \frac{3}{4}, & k \equiv 1 \bmod 2. \end{cases}$$

Quasi-polynomials behave like polynomials, in the sense that two quasi-polynomials are equal if and only if their coefficients are equal.

3.3 Lemma (Barvinok, 2006, [2, Section 4.3.])
Let

$$p(k) = \sum_{i=0}^{n} p_i(k) k^i, \quad q(k) = \sum_{i=0}^{n} q_i(k) k^i, \quad k \in \mathbb{Z}_{\geq 0}$$

be quasi-polynomials with period d *of degree at most* n. *If* $p(k) = q(k)$ *for all* $k \in \mathbb{Z}_{\geq 0}$ *then* $p_i(k) = q_i(k)$ *for all* $i = 0, \ldots, n$.

Ehrhart's theorem, Theorem 1.16, states that the lattice point enumerator of kP is a polynomial in $k \in \mathbb{Z}_{>0}$ if P is an integral polytope. To give an analogous statement if P is rational we define the notion of the denominator of a rational polytope.

3.4 Definition
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ *be a rational polytope. We call the smallest positive integral number* d *such that* dP *is an integral polytope the* denominator *of* P *and denote it by* $\mathrm{den}(P)$.

It is obvious that the denominator of a rational polytope P is the lowest common multiple of all denominators of all coordiantes of all vertices of P. Ehrhart's theorem for rational polytopes now can be stated as follows.

3.5 Theorem (Ehrhart, 1962, [17])
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then

$$\mathrm{G}(kP) = \sum_{i=0}^{\dim(P)} \mathrm{G}_i(P,k)k^i, \quad \textit{for } k \in \mathbb{Z}_{\geq 1}$$

is a quasi-polynomial with period $\mathrm{den}(P)$ *of degree* $\dim(P)$*.*

As a function in k, $\mathrm{G}(kP)$ is denoted by $\mathrm{G}(P,k)$ and is called the *Ehrhart quasi-polynomial* of P and $\mathrm{G}_i(P,\cdot)$ are called its coefficients.

3.6 Example
We revisit some of the polytopes from Example 1.17.

(i) First let $\widehat{T} \in \mathcal{P}^n_{\mathbb{Q}}$ be the triangle with vertices $\binom{3/4}{3/4}, \binom{3/4}{-3/4}, \binom{-9/4}{0}$. This is the triangle T from Example 1.17, scaled by a factor of $3/4$. Then for the Ehrhart

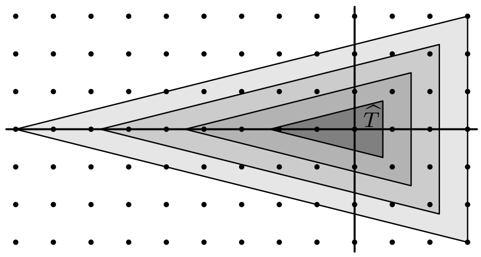

Figure 3.1: $\widehat{T}$, $2\widehat{T}$, $3\widehat{T}$ and $4\widehat{T}$.

quasi-polynomial we get

$$\mathrm{G}_{\widehat{T}}(k) = \mathrm{G}_2(\widehat{T},k)k^2 + \mathrm{G}_1(\widehat{T},k)k + \mathrm{G}_0(\widehat{T},k) = \begin{cases} \frac{9}{4}k^2 + \frac{3}{2}k + 1, & k \equiv 0 \bmod 4, \\ \frac{9}{4}k^2 \qquad\quad + \frac{3}{4}, & k \equiv 1 \bmod 4, \\ \frac{9}{4}k^2 \qquad\quad + 1, & k \equiv 2 \bmod 4, \\ \frac{9}{4}k^2 \qquad\quad + \frac{3}{4}, & k \equiv 3 \bmod 4. \end{cases}$$

with the coefficients $\mathrm{G}_2(\widehat{T},k) = 9/4$,

$$\mathrm{G}_1(\widehat{T},k) = \begin{cases} \frac{3}{2}, & k \equiv 0 \bmod 4, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathrm{G}_0(\widehat{T},k) = \begin{cases} 1, & k \equiv 0 \bmod 2, \\ \frac{3}{4}, & k \equiv 1 \bmod 2. \end{cases}$$

which is the quasi-polynomial of Example 3.2

(ii) We consider again the axis-parallel boxes given in Example 1.14, that is, let $R := \{x \in \mathbb{R}^n : b_i \leq x_i \leq a_i\}$ for $a, b \in \mathbb{Q}^n$ with $a_i \geq b_i$ for all $i \in \{1, \dots, n\}$. Then

$$\mathrm{G}(R,k) = \prod_{i=1}^{n} (\lfloor k a_i \rfloor - \lceil k b_i \rceil + 1).$$

We write $a_i = \frac{l_i s_i + r_i}{s_i}$, $b_i = \frac{\bar{l}_i t_i + \bar{r}_i}{t_i}$ with $l_i, \overline{l_i} \in \mathbb{Z}$, $r_i, \overline{r_i}, s_i, t_i \in \mathbb{Z}_{\geq 0}$, $r_i < s_i$, $\overline{r_i} < t_i$. Then

$$\mathrm{G}_i(R,k) = \sum_{\substack{J \subset \{1,\dots,n\} \\ \#J = i}} \prod_{j \in J} (l_j - \overline{l_j}) \prod_{j \notin J} \left(\left\lfloor \frac{k r_j}{s_j} \right\rfloor - \left\lceil \frac{k \overline{r_j}}{t_j} \right\rceil + 1 \right)$$

which is a periodic function with period $\mathrm{lcm}\left(\{s_i, t_i : i = 1, \dots, n\}\right)$. As a special case, we get for $i = 0, \dots, n$ that

$$\mathrm{G}_i\left(\frac{1}{2} C_n, k\right) = \begin{cases} \binom{n}{i}, & \text{if } k \equiv 0 \bmod 2 \text{ or } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

3.7 Definition (Minimal period of Ehrhart quasi-polynomial)
For $i = 1, \dots, \dim(P)$, we denote by $\mathrm{g}_i(P)$ the minimal period of $\mathrm{G}_i(P, \cdot)$*, that is, the smallest positive integral number g such that $\mathrm{G}_i(P,k) = \mathrm{G}_i(P, k+g)$ for all $k \in \mathbb{Z}_{\geq 0}$. The least common multiple of all $\mathrm{g}_i(P)$ is the* minimal period of $\mathrm{G}(P, \cdot)$ *and is denoted by $\mathrm{g}(P)$.*

As in the integral case, the coefficients fulfill some kind of homogeneity: Let $m, k \in \mathbb{Z}_{\geq 0}$, then

$$\sum_{i=0}^{\dim(P)} \mathrm{G}_i(mP, k) k^i = \mathrm{G}(mP, k) = \mathrm{G}(P, mk) = \sum_{i=0}^{\dim(P)} \mathrm{G}_i(P, mk) m^i k^i.$$

That yields the following:

3.8 Lemma
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope and $m, k \in \mathbb{Z}_{\geq 0}$. Then

$$\mathrm{G}_i(mP, k) = m^i\, \mathrm{G}_i(P, mk), \quad \textit{for all } i = 0, \dots, \dim(P).$$

Using that we get $\mathrm{G}_0(P, 0) = \mathrm{den}(P)^0\, \mathrm{G}_0(P, \mathrm{den}(P) \cdot 0) = \mathrm{G}_0(\mathrm{den}(P) P, 0) = 1$. Thus, Ehrhart's formula can again be extendend to $k = 0$ such that $\mathrm{G}(P, 0)$ is the Euler-characteristic 1 of polytopes.

Ehrhart quasi-polynomials yield a relation between the number of lattice points in P and in its interior $\operatorname{int}(P)$:

3.9 THEOREM (EHRHART–MACDONALD–RECIPROCITY, see [7, Chapter 4])
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then

$$\mathrm{G}(k(\operatorname{int}(P)) = (-1)^{\dim(P)}\,\mathrm{G}(P, -k).$$

McMullen [33] refined the statement in Theorem 3.5 concerning the periods of the coefficients $\mathrm{G}_i(P, \cdot)$ by introducing further numbers associated with a polytope, the so-called indices:

3.10 DEFINITION (INDEX OF A POLYTOPE)
*Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then for all $i = 0, \ldots, \dim(P)$, the i-*index of P*, $\mathrm{d}_i(P)$, is the smallest positive integral number d, such that for each i-face F of P the affine space $d \operatorname{aff}(F)$ contains integral points.*

Obviously, $\mathrm{d}_0(P) = \operatorname{den}(P)$ and $\mathrm{d}_{i+1}(P)$ is a divisior of $\mathrm{d}_i(P)$ which is denoted by $\mathrm{d}_{i+1}(P) | \mathrm{d}_i(P)$ for all $i = 0, \ldots, \dim(P) - 1$.

The following theorem is due to McMullen and connects the minimal period of the coefficients $\mathrm{G}_i(P, \cdot)$ to the geometric properties of the polytope P.

3.11 THEOREM (MCMULLEN, 1978, [33])
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then $\mathrm{g}_i(P)$ is a divisor of $\mathrm{d}_i(P)$ for all $i = 0, \ldots, \dim(P)$.

If P is full-dimensional then $\mathrm{d}_n(P) = 1$ and thus $\mathrm{G}_n(P, \cdot)$ is constant. Together with Lemma 3.8 it follows that

$$\mathrm{G}_n(P, k) = \mathrm{G}_n(P, \operatorname{den}(P)k) = \frac{1}{\operatorname{den}(P)^n}\,\mathrm{G}_n(\operatorname{den}(P)P, k) = \operatorname{vol}(P).$$

If P is not full-dimensional, then $\mathrm{G}_{\dim(P)}(P, k) = \operatorname{vol}_{\dim(P)}(P)$, whenever k is a multiple of $\mathrm{d}_{\dim(P)}(P)$, that is, $\operatorname{aff}(kP)$ contains integral points, and $\mathrm{G}_{\dim(P)}(P, k) = 0$ otherwise.

3.12 EXAMPLE
Again, let $\widehat{T} \in \mathcal{P}^n_{\mathbb{Q}}$ be the triangle with the vertices $\binom{3/4}{3/4}$, $\binom{3/4}{-3/4}$, $\binom{-9/4}{0}$ (see Figure 3.1).

We have $\mathrm{d}_0(\widehat{T}) = \mathrm{d}_1(\widehat{T}) = 4$, and the Ehrhart coefficients

$$\mathrm{G}_1(\widehat{T}, k) = \begin{cases} \frac{3}{2}, & k \equiv 0 \bmod 4, \\ 0, & \text{otherwise}, \end{cases} \qquad \mathrm{G}_0(\widehat{T}, k) = \begin{cases} 1, & k \equiv 0 \bmod 2, \\ \frac{3}{4}, & k \equiv 1 \bmod 2, \end{cases}$$

have periods $\mathrm{g}_1(\widehat{T}) = 4$, $\mathrm{g}_0(\widehat{T}) = 2$, and $\mathrm{g}(\widehat{T}) = 4$.

The minimal periods have been subject to active research recently. By McMullen's theorem, $\mathrm{g}_i(P)$ is bounded between 1 and $\mathrm{d}_i(P)$. We say a coefficient $\mathrm{G}_i(P, \cdot)$ is *of full period* if $\mathrm{g}_i(P) = \mathrm{d}_i(P)$. If, conversely, $\mathrm{g}_i(P) < \mathrm{d}_i(P)$, we say *the period of* $\mathrm{G}_i(P, \cdot)$ *collapses*.

McAllister and Woods [32] studied the 1- and 2-dimensional case with the result that period collapse does not occur in dimension 1, and they gave a characterization of those rational polygons in dimension 2 whose Ehrhart quasi-polynomial is indeed a polynomial. All those that are not integral are examples for period collapse.

3.13 Theorem (McAllister, Woods, 2005 [32])
Let $P \in \mathcal{P}^2_{\mathbb{Q}}$ be a 2-dimensional rational polytope. Then the Ehrhart quasi-polynomial of P is a polynomial if and only if the following holds:

(i) $\mathrm{G}(P, k) = \mathrm{vol}(kP) + \frac{1}{2}\#(\mathrm{bd}(kP) \cap \mathbb{Z}^2) + 1$ *for* $k = 1, \ldots, \mathrm{den}(P)$, *and*

(ii) $\#(\mathrm{bd}(kP) \cap \mathbb{Z}^2) = \#(\mathrm{bd}(P) \cap \mathbb{Z}^2)k$ *for* $k = 1, \ldots, \mathrm{den}(P)$.

In this case, $\mathrm{G}(P, k) = \mathrm{vol}(P)k^2 + \frac{1}{2}\#(\mathrm{bd}\, P \cap \mathbb{Z}^2)k + 1$ *for all* $k \in \mathbb{Z}_{\geq 0}$.

They also showed that, in contrast to the indices of P, the minimal periods of $\mathrm{G}_i(P, \cdot)$ are not necessarily decreasing with i, which can also be seen in Example 3.12.

3.14 Theorem (McAllister, Woods, 2005 [32])
Given $p, d \in \mathbb{Z}_{\geq 1}$ such that p divides d, there exists an n-dimensional rational polytope P with $\mathrm{den}(P) = d$ *and* $\mathrm{g}(P) = p$.

Beck, Sam and Woods [8], showed that McMullens bound is best-possible, that is, they constructed polytopes with arbitrary indices and full period for every $i = 0, \ldots, \dim(P)$. Furthermore, they showed that period collapse never occurs for $\mathrm{G}_{\dim(P)-1}(P, \cdot)$:

3.15 Theorem (Beck, Sam, Woods [8])
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then

$$\mathrm{g}_{\dim(P)-1}(P) = \mathrm{d}_{\dim(P)-1}(P).$$

Haase and McAllister [20] gave a conjectural explanation of period collapses to polynomials involving splitting the polytope into pieces and applying unimodular transformations onto these pieces. To the best of our knowledge, the conjecture is still open.

3.16 Conjecture (Haase, McAllister, 2008[20])
The Ehrhart quasi-polynomial of a rational polytope $P \in \mathcal{P}^n_{\mathbb{Q}}$ is a polynomial if and only if there exist a set $\mathcal{Q}$ of integral open simplices, pairwise disjoint open simplices $S_1, \ldots, S_n$, unimodular transformations $U_1, \ldots, U_n$ and translation vectors $t_1, \ldots, t_n \in \mathbb{Z}^d$ such that

$$P = \bigcup_{i=1}^{n} S_i \qquad \text{and} \qquad \bigcup_{i=1}^{n} U_i(S_i) + t_i = \bigcup_{Q \in \mathcal{Q}} Q.$$

3.2 Rational dilations

In this section we give a generalization of Ehrhart quasi-polynomials of rational polytopes to rational dilation factors. To this end, we first transform the definitions from Section 3.1 in a way that respects rationality.

3.17 Definition (Rational Quasi-polynomial)
A function $p : \mathbb{Q}_{\geq 0} \to \mathbb{R}$ is called a rational quasi-polynomial *with period d of degree at most n if there exist periodic functions $p_i : \mathbb{Q}_{\geq 0} \to \mathbb{R}$, $i = 0, \ldots, n$, with period d such that $p(k) = \sum_{i=0}^{n} p_i(k) k^i$.*

Now we generalize Ehrhart's theorem in a way that allows rational dilation factors.

3.18 Theorem
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then

$$\mathrm{G}(rP) = \sum_{i=0}^{\dim(P)} \mathrm{Q}_i(P, r) r^i, \quad \text{for } r \in \mathbb{Q}_{\geq 0}$$

is a quasi-polynomial with period $\mathrm{den}(P)$ *of degree* $\dim(P)$.

As a function in r, $\mathrm{G}(rP)$ is denoted by $\mathrm{Q}(P, r)$ and is called *rational Ehrhart quasi-polynomial* of P, $\mathrm{Q}_i(P, \cdot)$ are called its coefficients. We remark that $\mathrm{Q}(P, \cdot)$ is an extension of $\mathrm{G}(P, \cdot)$ to rational numbers, that is, $\mathrm{Q}(P, k) = \mathrm{G}(P, k)$ for $k \in \mathbb{Z}_{\geq 0}$. As a special case, this implies that $\mathrm{Q}(P, 0) = 1$.

Before proving Theorem 3.18 we present the following example.

3.19 Example

We consider again the triangle $\widehat{T}$ from Example 3.6. For $r \in \mathbb{Q}_{\geq 0}$,

$$\mathrm{Q}(\widehat{T}, r) = \mathrm{Q}_2(\widehat{T}, r)r^2 + \mathrm{Q}_1(\widehat{T}, r)r + \mathrm{Q}_0(\widehat{T}, r),$$

where the functions $\mathrm{Q}_i(\widehat{T}, r)$ are as in Table 3.1. To get a better impression about what these functions look like, Figure 3.3 shows the graphs of the functions $\mathrm{Q}_i(\widehat{T}, r)$ for $i = 0, 1$. Here it can be seen, that the minimal periods of $\mathrm{G}_1(P, \cdot)$ and $\mathrm{G}_1(P, \cdot)$ are $\frac{4}{3}$.

We further remark that Theorem 3.18 follows from McMullen's proof of Ehrhart's Theorem 3.5 in [33] although not stated explicitly there, since the integrality of the dilation factor was never used. Nevertheless we show that this also follows directly from Ehrhart's Theorem 3.5:

Proof (of Theorem 3.18)

Let $\mathrm{G}(P, k) = \sum_{i=0}^{\dim(P)} \mathrm{G}_i(P, k)k^i$ for $k \in \mathbb{Z}_{\geq 0}$ be the Ehrhart quasi-polynomial of P. We define

$$\mathrm{Q}_i\left(P, \frac{a}{b}\right) := \mathrm{G}_i\left(\frac{1}{b}P, a\right) b^i.$$

$\mathrm{Q}_i\left(P, \frac{a}{b}\right)$ is well-defined, since for $\frac{ka}{kb} = \frac{a}{b}$ we get $\mathrm{Q}_i\left(P, \frac{ka}{kb}\right) = \mathrm{G}_i(\frac{1}{kb}P, ka)k^i b^i = \mathrm{G}_i(\frac{1}{b}P, a)b^i = \mathrm{Q}_i\left(P, \frac{a}{b}\right)$ by Lemma 3.8. Then

$$\mathrm{Q}\left(P, \frac{a}{b}\right) = \mathrm{G}\left(\frac{1}{b}P, a\right) = \sum_{i=0}^{\dim(P)} \mathrm{G}_i\left(\frac{1}{b}P, a\right) a^i = \sum_{i=0}^{\dim(P)} \mathrm{Q}_i\left(P, \frac{a}{b}\right)\left(\frac{a}{b}\right)^i.$$

It remains to show that $\mathrm{Q}_i\left(P, \frac{a}{b}\right)$ is periodic with period $\mathrm{den}(P)$. Since $b\ \mathrm{den}(P)$ is a multiple of $\mathrm{den}\left(\frac{1}{b}P\right)$, we get

$$\mathrm{Q}_i\left(P, \frac{a}{b} + \mathrm{den}(P)\right) = \mathrm{G}_i\left(\frac{1}{b}P, a + b\ \mathrm{den}(P)\right) b^i = \mathrm{G}_i\left(\frac{1}{b}P, a\right) b^i = \mathrm{Q}_i\left(P, \frac{a}{b}\right). \qquad \square$$

From the proof it follows that $\mathrm{Q}_{\dim(P)}\left(P, \frac{a}{b}\right) = \mathrm{G}_{\dim(P)}\left(\frac{1}{b}P, a\right) b^i = \mathrm{vol}(P)$ for all $\frac{a}{b} \in \mathbb{Q}_{\geq 0}$ such that $\mathrm{aff}\left(\frac{a}{b}P\right)$ contains integral points. Furthermore the proof implies that knowing the classical Ehrhart quasi-polynomial of $\frac{1}{b}P$ for all positive integers b is equivalent to knowing the rational Ehrhart quasi-polynomial of P. However, as the next remark shows, it is not enough to know the Ehrhart quasi-polynomial of a polytope to recover the rational version:

3.20 Remark

$\mathrm{Q}(P, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Z}$ is not invariant under translations of P with respect to integral vectors. Furthermore, $\mathrm{Q}(P, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Z}$ is not necessarily monotonically increasing, if $\mathbb{0} \notin P$. For instance, let $\widehat{T}$ as in Example 3.6 and let $\widehat{T}_2 = \widehat{T} + \binom{0}{2}$ (see Figure 3.4). Then

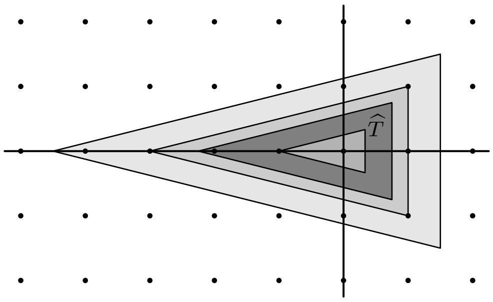

Figure 3.2: $\widehat{T}$, $\frac{4}{9}\widehat{T}$, $\frac{4}{3}\widehat{T}$ and $2\widehat{T}$.

r	$\mathrm{Q}_2(\widehat{T},r)$	$\mathrm{Q}_1(\widehat{T},r)$	$\mathrm{Q}_0(\widehat{T},r)$
$r \in \left[0, \frac{4}{9}\right)$	$\frac{9}{4}$	$-\frac{9}{2}r + \frac{3}{2}$	$\frac{9}{4}r^2 - \frac{3}{2}r + 1$
$r \in \left[\frac{4}{9}, \frac{8}{9}\right)$	$\frac{9}{4}$	$-\frac{9}{2}r + 3$	$\frac{9}{4}r^2 - 3r + 2$
$r \in \left[\frac{8}{9}, \frac{4}{3}\right)$	$\frac{9}{4}$	$-\frac{9}{2}r + \frac{9}{2}$	$\frac{9}{4}r^2 - \frac{9}{2}r + 3$
$r \in \left[\frac{4}{3}, \infty\right)$	$\mathrm{Q}_2(\widehat{T}, r - \frac{4}{3})$	$\mathrm{Q}_1(\widehat{T}, r - \frac{4}{3})$	$\mathrm{Q}_0(\widehat{T}, r - \frac{4}{3})$

Table 3.1: $\mathrm{Q}_i(\widehat{T},r)$.

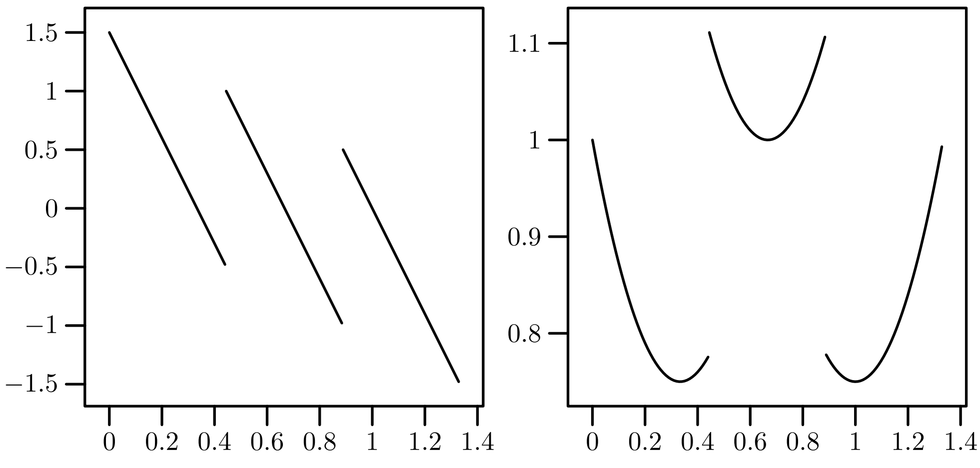

Figure 3.3: $\mathrm{Q}_1(\widehat{T},r)$ (left) and $\mathrm{Q}_0(\widehat{T},r)$ (right) on $[0, \frac{4}{3}]$.

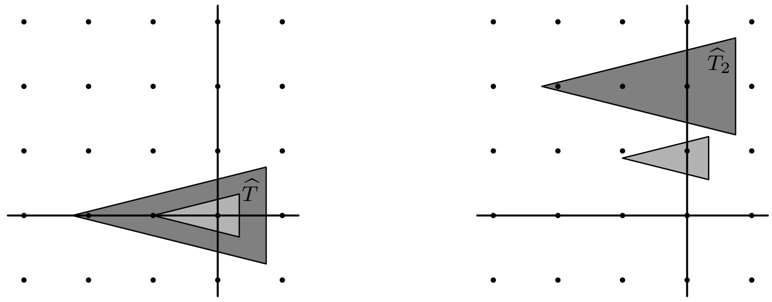

Figure 3.4: $\widetilde{T}$ and $\widetilde{T}_2$.

we have $\mathrm{Q}(\widehat{T}, 4/9) = 2$ and $\mathrm{Q}(\widehat{T}_2, 4/9) = 1$ while both of them have the same Ehrhart quasi-polynomial.

We can also generalize the homogeneity:

3.21 Lemma
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope, $r, s \in \mathbb{Q}_{\geq 0}$. Then

$$\mathrm{Q}_i(sP, r) = \mathrm{Q}_i(P, rs)s^i, \quad \textit{for all } i = 0, \dots, \dim(P).$$

Proof
Let $r = \frac{a}{b}$, $s = \frac{c}{d}$. By the definition of $\mathrm{Q}_i(P, r)$ in the proof of Theorem 3.18, we get, together with Lemma 3.8,

$$\mathrm{Q}_i(sP, r) = \mathrm{Q}_i\left(\frac{c}{d}P, \frac{a}{b}\right) = \mathrm{G}_i\left(\frac{c}{db}P, a\right) b^i, \quad \text{and}$$
$$\mathrm{Q}_i(P, sr)s^i = \mathrm{Q}_i\left(P, \frac{ac}{bd}\right)\frac{c^i}{d^i} = \mathrm{G}_i\left(\frac{c}{bd}P, a\right) b^i. \qquad \square$$

Like the classical Ehrhart quasi-polynomials, the rational versions also fulfill Ehrhart-Macdonald-reciprocity.

3.22 Theorem (Ehrhart–Macdonald–reciprocity)
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then

$$\mathrm{G}(r \operatorname{int}(P)) = (-1)^{\dim(P)} \mathrm{Q}(P, -r), \quad \textit{for all } r \in \mathbb{Q}_{\geq 0}.$$

Proof
Let $r = \frac{a}{b}$ with $a, b \in \mathbb{Z}_{\geq 0}$. Then, by the Ehrhart–Macdonald–reciprocity law (Theo-

rem 3.9) and Lemma 3.21, we get

$$\begin{aligned}
\#\left(\frac{a}{b}\operatorname{int}(P)\cap\mathbb{Z}^n\right) &= (-1)^{\dim(P)}\sum_{i=0}^{\dim(P)} \mathrm{G}_i\left(\frac{1}{b}P,-a\right)(-a)^i \\
&= (-1)^{\dim(P)}\sum_{i=0}^{\dim(P)} \mathrm{Q}_i\left(P,-\frac{a}{b}\right)\left(\frac{1}{b}\right)^i(-a)^i \\
&= (-1)^{\dim(P)}\,\mathrm{Q}\left(P,-\frac{a}{b}\right).
\end{aligned}$$

□

To formulate a result corresponding to McMullen's Theorem 3.11, we also need a rational equivalent to the index of a polytope and the minimal periods of rational Ehrhart quasi-polynomials.

The *least common multiple* of rational numbers $q_1,\dots,q_m$, $m\in\mathbb{N}$ is the smallest positiv rational number q such that there exists integral numbers $k_1,\dots,k_m$ with $q_ik_i=q$ for all $i=1,\dots,m$ and is denoted by $\operatorname{lcm}(q_1,\dots,q_m)$.

3.23 Definition (Minimal period of rational Ehrhart quasi-polynomial)
For $i=1,\dots,\dim(P)$, we denote by $\mathrm{q}_i(P)$ the minimal period *of $\mathrm{Q}_i(P,\cdot)$, that is, the smallest positive rational number q such that $\mathrm{Q}_i(P,r)=\mathrm{Q}_i(P,r+q)$ for all $r\in\mathbb{Q}_{\geq 0}$. The least common multiple of all $\mathrm{q}_i(P)$ is the* minimal period *of $\mathrm{Q}(P,\cdot)$ and is denoted by $\mathrm{q}(P)$.*

3.24 Definition (Rational index of a polytope)
Let $P\in\mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. For $i=0,\dots,\dim(P)$, the rational i-index *of P, denoted by $\widehat{\mathrm{d}}_i(P)$, is the smallest positive rational number d, such that for each i-face F of P the affine space $\operatorname{aff}(dF)$ contains integral points.*
Furthermore, we call $\widehat{\mathrm{d}}_0(P)$ the rational denominator*, which is the smallest positive rational number d such that dP is an integral polytope, and denote it by $\widehat{\operatorname{den}}(P)$.*

We remark that, if the affine hull of dP contains integral points for all $d\in\mathbb{Q}_{>0}$ (for example if P is full-dimensional), we set $\widehat{\mathrm{d}}_{\dim(P)}(P)=0$.

As in the integral case, the rational indices are divisors of each other. Here, a rational number r is a divisor of a rational number s, if $\frac{s}{r}\in\mathbb{Z}$, and we denote this, as in the integral case, with $r|s$.

3.25 Lemma
Let P be a rational polytope. Then $\widehat{\mathrm{d}}_{i+1}(P)|\,\widehat{\mathrm{d}}_i(P)$ for $i=0,\dots,\dim(P)-1$.

Proof
Let $H^i_1, \dots, H^i_{f_i}$ be the respective affine hulls of the $f_i \in \mathbb{Z}_{\geq 1}$ i-faces of P and let r^i_j be the smallest positive rational number such that H^i_j contains integral points. Then rH^i_j contains integral points if and only if r is an integral multiple of r^i_j, for $j = 1, \dots, f_j$. Thus, $\widehat{\mathrm{d}}_i(P)$ is the smallest positive rational number that is an integral multiple of all r^i_j. Furthermore, since $H^{i-1}_j \subset H^i_{\tilde{j}}$ for some $\tilde{j}$, we have that r^{i-1}_j is an integral multiple of $r^i_{\tilde{j}}$, and thus $\widehat{\mathrm{d}}_{i-1}(P)$ is an integral multiple of $\widehat{\mathrm{d}}_i(P)$. □

Now we are able to prove that the rational indices are periods of the coefficients of the rational Ehrhart quasi-polynomials:

3.26 Theorem
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then $\mathrm{q}_i(P)$ is a divisor of $\widehat{\mathrm{d}}_i(P)$, for $i = 0, \dots, \dim(P)$. As a special case, $\mathrm{q}(P)$ is a divisor of $\widehat{\mathrm{den}}(P)$.

Proof
Since $\mathrm{d}_i(\widehat{\mathrm{d}}_i(P)P) = 1$ for all i, we know that

$$\mathrm{Q}_i(\widehat{\mathrm{d}}_i(P)P, r + k) = \mathrm{Q}_i(\widehat{\mathrm{d}}_i(P)P, r), \quad \forall r \in \mathbb{Q}_{\geq 0}, k \in \mathbb{Z}_{\geq 0} .$$

This implies, together with Lemma 3.21,

$$\mathrm{Q}_i(P, r\ \widehat{\mathrm{d}}_i(P) + k\ \widehat{\mathrm{d}}_i(P)) = \mathrm{Q}_i(P, r\ \widehat{\mathrm{d}}_i(P)), \quad \forall r \in \mathbb{Q}_{\geq 0}, k \in \mathbb{Z}_{\geq 0},$$

and thus

$$\mathrm{Q}_i(P, \tilde{r} + k\ \widehat{\mathrm{d}}_i(P)) = \mathrm{Q}_i(P, \tilde{r}), \quad \forall \tilde{r} \in \mathbb{Q}_{\geq 0}, k \in \mathbb{Z}_{\geq 0} .$$

Furthermore, together with Lemma 3.25, we get that $\widehat{\mathrm{d}}_0(P)/\widehat{\mathrm{d}}_i(P) \in \mathbb{Z}$ for $i = 1, \dots, n$, and thus $\widehat{\mathrm{d}}_0(P) = \widehat{\mathrm{den}}(P)$ is a period of $\mathrm{Q}(P, \cdot)$. □

3.27 Example
We refer to Example 3.19. It is easy to see that $\widehat{\mathrm{d}}_1(\widehat{T}) = \widehat{\mathrm{d}}_0(\widehat{T}) = 4/3$. Furthermore, the coefficients of the rational Ehrhart quasi-polynomial have the periods $\mathrm{q}_1(\widehat{T}) = \mathrm{q}_0(\widehat{T}) = 4/3$, which can be seen from Table 3.1.

3.3 Results on the coefficients of rational Ehrhart quasi-polynomials

As argued above, $\mathrm{Q}_{\dim(P)}(P, \cdot)$ is constant and equals $\mathrm{vol}(P)$ if P is full-dimensional. In case that P is not full-dimensional, we have that $\mathrm{G}(rP) = 0$ for all $r \in \mathbb{Q}_{\geq 0}$ that are not integral multiples of $\widehat{\mathrm{d}}_{\dim(P)}(P)$. This implies that $\mathrm{Q}_{\dim(P)}(P,r) = \mathrm{vol}_{\dim(P)}(P)$ only if $r = k\,\widehat{\mathrm{d}}_{\dim(P)}(P)$ for some $k \in \mathbb{Z}_{\geq 0}$, and 0 otherwise.

After having clarified the leading coefficient, we show that the period of the second leading coefficient is known, too.

3.28 Corollary
Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be a rational polytope. Then $\mathrm{q}_{\dim(P)-1}(P) = \widehat{\mathrm{d}}_{\dim(P)-1}(P)$.

Proof
By Lemma 3.21 and since $\widehat{\mathrm{d}}_{\dim(P)-1}$ is homogeneous, it suffices to show the statement for all P with $\widehat{\mathrm{d}}_{\dim(P)-1}(P) = 1$. Thus we assume that $\frac{s}{t} < 1$ is a period of $\mathrm{Q}_{\dim(P)-1}(P,\cdot)$ with $s,t \in \mathbb{Z}_{\geq 1}$, that is, $\mathrm{Q}_{\dim(P)-1}(P,r) = \mathrm{Q}_{\dim(P)-1}\left(P, r+\frac{s}{t}\right)$ for all $r \in \mathbb{Q}_{\geq 0}$. Again by Lemma 3.21, we get

$$\mathrm{Q}_{\dim(P)-1}\left(\frac{1}{t}P, rt\right) = \mathrm{Q}_{\dim(P)-1}\left(\frac{1}{t}P, rt+s\right) \quad \text{for all } r \in \mathbb{Q}_{\geq 0}\,.$$

In particular, this is true if $rt \in \mathbb{Z}_{\geq 0}$, and hence, s is a period of $\mathrm{G}_{\dim(P)-1}\left(\frac{1}{t}P,\cdot\right)$. This is a contradiction, since

$$\mathrm{d}_{\dim(P)-1}\left(\frac{1}{t}P\right) \geq \widehat{\mathrm{d}}_{\dim(P)-1}\left(\frac{1}{t}P\right) = t\,\widehat{\mathrm{d}}_{\dim(P)-1}(P) = t > s,$$

but by Theorem 3.15 we know that no period collapse occurs for $\mathrm{G}_{\dim(P)-1}(\frac{1}{t}P,\cdot)$. □

To get some more structural results on the coefficients, we need some general statements concerning rational quasi-polynomials. By f' we denote the first derivative of a differentiable function f.

3.29 Lemma
Let $p : \mathbb{Q} \to \mathbb{Q}$ be a rational quasi-polynomial of degree $n \in \mathbb{Z}_{\geq 1}$ with period $d \in \mathbb{Q}_{>0}$ and constant leading coefficient, that is,

$$p(r) = p_n r^n + p_{n-1}(r)r^{n-1} + p_{n-2}(r)r^{n-2} + \ldots + p_1(r)r + p_0(r),$$

where $0 \neq p_n \in \mathbb{Q}$ and $p_i : \mathbb{Q} \to \mathbb{Q}$ are periodic functions with period d for $i = 0,\ldots,n-1$. Furthermore, suppose there exist an interval $(r_1,r_2) \subset \mathbb{Q}$ and $c_k \in \mathbb{Q}$ for $k \in \mathbb{Z}_{\geq 0}$ such that

$$p(r+kd) = c_k, \quad \forall r \in (r_1,r_2), \forall k \in \mathbb{Z}_{\geq 0}\,.$$

Then $p_i : (r_1,r_2) \to \mathbb{Q}$ is a polynomial of degree $n-i$ and $p_i'(r) = -(i+1)p_{i+1}(r)$ for $i = 0,\ldots,n-1$. Furthermore, if $p_n > 0$ then $p_{n-1}(r)$ has negative leading coefficient.

PROOF
We prove the polynomiality result by induction on n. For $n = 1$ we have $c_0 = p(r) = p_1 r + p_0(r)$ for all $r \in (r_1, r_2)$. Thus, $p_0(r) = c_0 - p_1 r$ for $r \in (r_1, r_2)$, which is a polynomial of degree $n - 0 = 1$ with negative leading coefficient.

Now let $n > 1$. We have

$$c_k = p_n \cdot (r + kd)^n + \sum_{i=0}^{n-1} p_i(r)(r + kd)^i, \quad \forall r \in (r_1, r_2), \forall k \in \mathbb{Z}_{\geq 0}\,.$$

Then $q : \mathbb{Q} \to \mathbb{Q}$ with $q(r) := p_n \cdot ((r + d)^n - r^n) + \sum_{i=0}^{n-1} p_i(r) \left((r + d)^i - r^i\right)$ is a rational quasi-polynomial of degree $n - 1$ with period d and constant leading coefficient, and

$$\begin{aligned} q(r + md) &= p_n \cdot ((r + (m + 1)d)^n - (r + md)^n) \\ &\qquad + \sum_{i=0}^{n-1} p_i(r) \left((r + (m + 1)d)^i - (r + md)^i\right) \\ &= c_{m+1} - c_m \quad \forall r \in (r_1, r_2), \forall m \in \mathbb{Z}_{\geq 0}\,. \end{aligned}$$

Thus, we can use the induction hypothesis for q, and together with

$$\begin{aligned} q(r) &= p_n \cdot ((r + d)^n - r^n) + \sum_{i=0}^{n-1} p_i(r) \left((r + d)^i - r^i\right) \\ &= p_n n d r^{n-1} + \sum_{j=0}^{n-2} \left(p_n \binom{n}{j} d^{n-j} + \sum_{i=j+1}^{n-1} p_i(r) \binom{i}{j} d^{i-j} \right) r^j \end{aligned}$$

we get that

$$q_j(r) := p_n \binom{n}{j} d^{n-j} + \sum_{i=j+1}^{n-1} p_i(r) \binom{i}{j} d^{i-j}$$

is a polynomial of degree $n-1-j$ for $r \in (r_1, r_2)$, for all $j = 0, \dots, n-2$. Since $p_n n d > 0$ we get, also by induction, that $q_{n-2}(r) = p_n \binom{n}{2} d^2 + p_{n-1}(r) \binom{n-1}{n-2} d$ has a negative leading coefficient.
Now we use induction again to show that p_{j+1} is a polynomial of degree $n - j - 1$ for $r \in (r_1, r_2)$. For $j = n - 2$ we have that $q_{n-2}(r) = p_n \binom{n}{2} d^2 + p_{n-1}(r) \binom{n-1}{n-2} d$ is a polynomial of degree 1 with negative leading coefficient, hence the same is true for p_{n-1}. For $j < n - 2$ write

$$q_j(r) = p_n \binom{n}{j} d^{n-j} + \sum_{i=j+1}^{n-1} p_i(r) \binom{i}{j} d^{i-j} = \sum_{i=0}^{n-j-1} \alpha_i r^i, \quad \forall r \in (r_1, r_2).$$

Then for $r \in (r_1, r_2)$,

$$p_{j+1}(r)(j+1)d = \sum_{i=0}^{n-j-1} \alpha_i r^i - \sum_{i=j+2}^{n-1} p_i(r)\binom{i}{j} d^{i-j} - p_n \binom{n}{j} d^{n-j},$$

which is a polynomial of degree $n-j-1$, since $p_i(r)$ is a polynomial of degree $n-i$ for $i \geq j+2$ by induction hypothesis. We conclude that $p_i(r)$ is a polynomial of degree $n-i$ for $r \in (r_1, r_2)$ and $i = 1, \ldots, n-1$. That $p_0(r)$ is a polynomial follows immediately from $p_0(r) = c_0 - p_n r^n - \sum_{i=1}^{n-1} p_i(r) r^i$.

It remains to show that $p_i'(r) = -(i+1)p_{i+1}(r)$. Since p_i is a polynomial of degree $n-i$, we can write it as

$$p_i(r) = \sum_{j=0}^{n-i} p_{i,j} r^j. \tag{3.1}$$

Since p_i is a periodic function with period d we can write $r = \tilde{r} + kd$ with $k \in \mathbb{Z}$ and get

$$c_k = \sum_{i=0}^{n} p_i(r) r^i = \sum_{i=0}^{n} p_i(\tilde{r})(\tilde{r} + kd)^i,$$

which expands, using Equation (3.1), to

$$c_k = \sum_{i=0}^{n} \sum_{j=0}^{n-i} p_{i,j} \tilde{r}^j (\tilde{r} + kd)^i = \sum_{i=0}^{n} \sum_{j=0}^{n-i} p_{i,j} \tilde{r}^j \sum_{h=0}^{i} \binom{i}{h} \tilde{r}^h (kd)^{i-h}.$$

Exchanging the summation order yields

$$c_k = \sum_{h=0}^{n} \sum_{i=0}^{n} \sum_{j=\max(0,h-i)}^{\min(h,n-i)} \binom{i}{h-j} p_{i,j} (kd)^{i-h+j} \tilde{r}^h,$$

which is a constant polynomial in $\tilde{r}$. Thus for $h \neq 0$,

$$\sum_{i=0}^{n} \sum_{j=\max(0,h-i)}^{\min(h,n-i)} \binom{i}{h-j} p_{i,j} (kd)^{i-h+j} = 0,$$

and therefore

$$p(\tilde{r} + kd) = c_k = \sum_{i=0}^{n} p_{i,0} (kd)^i = \sum_{i=0}^{n} p_{i,0} (r - \tilde{r})^i.$$

Expanding to the quasi-polynomial form yields

$$p(\tilde{r} + kd) = \sum_{i=0}^{n} p_{i,0} \sum_{j=0}^{i} \binom{i}{j} r^j \tilde{r}^{i-j} (-1)^{i-j} = \sum_{j=0}^{n} \left(\sum_{i=0}^{n-j} \binom{i+j}{j} p_{i+j,0} (-1)^i \tilde{r}^i \right) r^j.$$

This implies that for all $\tilde r \in (r_1, r_2)$,

$$p_j(\tilde r) = \sum_{i=0}^{n-j} \binom{i+j}{j} p_{i+j,0} (-1)^i \tilde r^i,$$

and thus by differentiation

$$\begin{aligned} p_j'(\tilde r) &= \sum_{i=1}^{n-j} \binom{i+j}{j} (-1)^i i\, p_{i+j,0}\, \tilde r^{i-1} = \sum_{i=0}^{n-j-1} \binom{i+j+1}{j} (-1)^{i+1} (i+1)\, p_{i+j+1,0}\, \tilde r^i \\ &= -(j+1) \sum_{i=0}^{n-j-1} \binom{i+j+1}{j+1} (-1)^i\, p_{i+j+1,0}\, \tilde r^i = -(j+1)\, p_{j+1}(\tilde r), \end{aligned}$$

which finishes the proof. □

Next we show that Ehrhart quasi-polynomials of full-dimensional rational polytopes satisfy the conditions in Lemma 3.29.

As explained at the beginning of this section, in case P is not full-dimensional, we have that $\mathrm{G}(rP) = 0$ for all $r \in \mathbb{Q}_{\geq 0}$ that are not integral multiples of $\widehat{\mathrm{d}}_{\dim(P)}(P)$, which implies that $\mathrm{Q}_i(P, r) \neq 0$ only if $r = k\, \widehat{\mathrm{d}}_{\dim(P)}(P)$ for some $k \in \mathbb{Z}_{\geq 0}$. Therefore, for these values the rational Ehrhart quasi-polynomial coincides with $\mathrm{G}(\widehat{\mathrm{d}}_{\dim(P)}(P)P, k)$. Thus, in that case the approach using rational dilations cannot give new results, since the rational Ehrhart quasi-polynomial does not contain more information than one integral Ehrhart quasi-polynomial. Because of that, we will restrict ourselves to full-dimensional polytopes in the main theorem of this section. This provides a structural result about the coefficients of the rational Ehrhart quasi-polynomials of a polytope P.

3.30 Theorem

Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be an n-dimensional rational polytope. Then $\mathrm{Q}_i(P, \cdot)$ is a piecewise-defined polynomial of degree $n - i$, and

$$\mathrm{Q}_i'(P, r) = -(i+1)\, \mathrm{Q}_{i+1}(P, r), \quad i = 0, \ldots, n-1,$$

for all $r \geq 0$ such that $\mathrm{Q}(P, \cdot)$ is (one-sided) continuous at $r + k\, \widehat{\mathrm{den}}(P)$ for all $k \in \mathbb{Z}_{\geq 0}$.

Proof

By Theorem 3.18, $\mathrm{Q}(P, r)$ is a rational quasi-polynomial of degree n with period $\widehat{\mathrm{den}}(P)$ and constant, nonzero leading coefficient. To apply Lemma 3.29 it remains to show that there exist $0 = r_0 < r_1 < \ldots < r_l = \widehat{\operatorname{den}}(P)$, $l \in \mathbb{Z}_{>0}$, such that $\mathrm{Q}(P,r)$ is constant for $r \in (r_i + k\,\widehat{\operatorname{den}}(P), r_{i+1} + k\,\widehat{\operatorname{den}}(P))$ and $i = 0,\ldots,l-1$, $k \in \mathbb{Z}_{\geq 0}$. To this end, we consider $\mathrm{Q}(P,r)$ as a function $\mathbb{Q}_{\geq 0} \to \mathbb{Q}$. $\mathrm{Q}(P,r)$ is certainly piecewise constant, and it jumps only if integral points leave or enter rP, which can only happen if one of the facets of rP lies in a hyperplane containing integral points. Thus, for every facet F of P let α_F be the smallest positive rational number such that $\alpha_F F$ lies in a hyperplane containing integral points. Then $\{k\alpha_F : F \text{ facet of } P, k \in \mathbb{Z}_{\geq 0}\}$ are the only possible discontinuities of $\mathrm{Q}(P,r)$. By the definition of $\widehat{\operatorname{den}}(P)$, for a facet F of P there exists a $k_F \in \mathbb{Z}$ such that $k_F\alpha_F = \widehat{\operatorname{den}}(P)$. Thus for $\{r_0,\ldots,r_l\} = \{k\alpha_F : k = 0,\ldots,k_F, F \text{ facet of } P\}$ we can apply Lemma 3.29. This implies the claim for all open intervalls (r_i, r_{i+1}).

Now we consider the jump points of $\mathrm{Q}(P,\cdot)$. Let $\tilde{r}$ be fixed and assume that $\mathrm{Q}(P,\cdot)$ is one-sided continuous at $r_k = \tilde{r} + k\,\widehat{\operatorname{den}}\,P \in \mathbb{Q}_{\geq 0}$ for all $k \in \mathbb{Z}_{\geq 0}$, say

$$\mathrm{Q}(P,r_k) = \lim_{\substack{r\to r_k\\ r>r_k}} \mathrm{Q}(P,r).$$

Then

$$\mathrm{Q}(P,r_k) = \lim_{\substack{r\to r_k\\ r>r_k}} \sum_{i=0}^{n} \mathrm{Q}_i(P,r)r^i = \sum_{i=0}^{n} \lim_{\substack{r\to r_k\\ r>r_k}} \mathrm{Q}_i(P,r)r^i.$$

Since $\mathrm{Q}(P,r_k)$ is a polynomial in k, its coefficients are uniquely determined by its values, and thus,

$$\mathrm{Q}_i(P,r_k) = \lim_{\substack{r\to r_k\\ r>r_k}} \mathrm{Q}_i(P,r),$$

since these limits are one possible choice for the coefficients $\mathrm{Q}_i(P,r_k)$. We remark that these limits exist, since $\mathrm{Q}_i(P,\cdot)$ is a polynomial for $r > r_k$ small. Thus $\mathrm{Q}_i(P,\cdot)$ is one-sided continuous at r_k as well and the polynomial can be extended to r_k. Hence the claim follows for r_k as well. □

3.31 Example

We revisit the triangle $\widehat{T}$ from Example 3.19. For $r \in \left[0, \frac{4}{9}\right)$ the coefficients $\mathrm{Q}_i(\widehat{T},r)$ are $\mathrm{Q}_2(\widehat{T},r) = \frac{9}{4}$, $\mathrm{Q}_1(\widehat{T},r) = -\frac{9}{2}r + \frac{3}{2}$ and $\mathrm{Q}_0(\widehat{T},r) = \frac{9}{4}r^2 - \frac{3}{2}r + 1$, and we have $\mathrm{Q}_0'(\widehat{T},r) = \frac{9}{2}r - \frac{3}{2} = -\mathrm{Q}_1(\widehat{T},r)$ and $\mathrm{Q}_1'(\widehat{T},r) = -\frac{9}{2} = -2\,\mathrm{Q}_2(\widehat{T},r)$.

We can use Theorem 3.30 to give a result on the periods $\mathrm{q}_i(P)$ if $\mathbb{0} \in P$, which is not true in the integral case:

3.32 Corollary

Let $P \in \mathcal{P}^n_{\mathbb{Q}}$ be an n-dimensional rational polytope and $\mathbb{0} \in P$. Then $\mathrm{q}_{i+1}(P) \mid \mathrm{q}_i(P)$, for $i = 0,\ldots,n-1$.

Proof

Since $\mathbb{0} \in P$, only lattice points enter rP, as r increases. Thus $\mathrm{Q}(P, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Z}$ is a piecewise constant function that is continuous from above. By Theorem 3.30,

$$\mathrm{Q}_i'(P, r) = -(i+1)\,\mathrm{Q}_{i+1}(P, r), \quad i = 0, \ldots, n-1,$$

for all $r \in \mathbb{Q}_{\geq 0}$. Then for $i = 0, \ldots, n-1$ and $r \in \mathbb{Q}_{\geq 0}$,

$$\begin{aligned} -(i+1)\,\mathrm{Q}_{i+1}(P, r + \mathrm{q}_i(P)) &= \mathrm{Q}_i'(P, r + \mathrm{q}_i(P)) \\ &= \mathrm{Q}_i'(P, r) \\ &= -(i+1)\,\mathrm{Q}_{i+1}(P, r), \end{aligned}$$

which implies that $\mathrm{q}_i(P)$ is a period of $\mathrm{Q}_{i+1}(P)$, and thus $\mathrm{q}_{i+1}(P) \mid \mathrm{q}_i(P)$. $\square$

3.4 Results in dimension 2

In the following, we denote by $\lfloor\cdot\rfloor$ the floor function, that is, $\lfloor x\rfloor$ is the largest integer not greater than x, by $\lceil\cdot\rceil$ the ceiling function, that is, $\lceil x\rceil$ is the smallest integer not smaller than x, and by $\{.\}$ the fractional part, that is, $\{x\} = x - \lfloor x\rfloor$. For many of the calculations, we make use of the following fact: If $n, m, t, r \in \mathbb{Z}$, $m > 0$ and $t \equiv r \bmod m$, then $\left\lfloor \frac{nt}{m} \right\rfloor = \frac{nt}{m} - \left\{\frac{nr}{m}\right\}$ and $\left\lceil \frac{nt}{m} \right\rceil = \frac{nt}{m} + \left\{-\frac{nr}{m}\right\}$.

First, we consider 2-dimensional triangles of the form $T = \operatorname{conv}\left\{\binom{0}{0}, \binom{x_1}{y}, \binom{x_2}{y}\right\}$, where $x_1 < x_2 \in \mathbb{Q}$ and $y \in \mathbb{Q}_{>0}$.

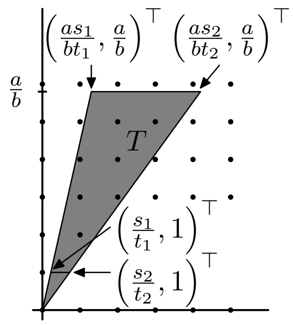

Figure 3.5: Triangle T.

3.33 Theorem

Let T be the traingle above, that is,

$$T = \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{s_1}{t_1}\frac{a}{b} \\ \frac{a}{b} \end{pmatrix}, \begin{pmatrix} \frac{s_2}{t_2}\frac{a}{b} \\ \frac{a}{b} \end{pmatrix} \right\},$$

with $a, b, t_1, t_2 \in \mathbb{Z}_{\geq 1}$, $s_1, s_2 \in \mathbb{Z}$, $\frac{s_2}{t_2} > \frac{s_1}{t_1}$, *and* $\gcd(a,b) = \gcd(s_1,t_1) = \gcd(s_2,t_2) = 1$.

Then for $r \in \mathbb{Q}_{\geq 0}$ *the following hold:*

(i) $\mathrm{Q}_2(T,r) = \frac{1}{2}\frac{a^2}{b^2}\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)$.

(ii) $\mathrm{Q}_1(T,r) = \frac{a}{b}\left(\frac{t_1+t_2}{2t_1t_2} - \left(\left\{\frac{ar}{b}\right\} - \frac{1}{2}\right)\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)\right)$.

(iii)
$$\begin{aligned}\mathrm{Q}_0(T,r) = 1 &- \frac{1}{2}\left\{\frac{ar}{b}\right\}\left(\frac{s_2}{t_2} - \frac{s_1}{t_1} + 2\right) + \frac{1}{2}\left\{\frac{ar}{b}\right\}^2\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)\\ &+ \left\{\frac{ar}{b\,\mathrm{lcm}\,(t_1,t_2)}\right\}\mathrm{lcm}\,(t_1,t_2)\left(\frac{t_2-1}{2t_2} + \frac{t_1-1}{2t_1}\right)\\ &- \sum_{i=0}^{\left\{\left\lfloor \frac{ar}{b}\right\rfloor/\mathrm{lcm}(t_1,t_2)\right\}\mathrm{lcm}(t_1,t_2)}\left(\frac{s_2 i}{t_2} - \left\lfloor\frac{s_2 i}{t_2}\right\rfloor + \left\lceil\frac{s_1 i}{t_1}\right\rceil - \frac{s_1 i}{t_1}\right).\end{aligned}$$

Proof

We determine $\mathrm{Q}(T,t) = \sum_{i=0}^{\lfloor at/b\rfloor}\mathrm{Q}(Q,i)$, $t \in \mathbb{Q}_{\geq 0}$, $Q = \mathrm{conv}\left(\begin{pmatrix}\frac{s_1}{t_1}\\1\end{pmatrix}, \begin{pmatrix}\frac{s_2}{t_2}\\1\end{pmatrix}\right)$, (see Figure 3.5, [39]). For abbreviation let $l := \mathrm{lcm}\,(t_1,t_2)$ and r an arbitrary integer with $r \equiv t \mod lb$. Then

$$\begin{aligned}\mathrm{Q}(Q,t) = \#(tQ \cap \mathbb{Z}^2) &= \left\lfloor\frac{s_2 t}{t_2}\right\rfloor - \left\lceil\frac{s_1 t}{t_1}\right\rceil + 1\\ &= \left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)t - \left(\left\{\frac{s_2 r}{t_2}\right\} + \left\{-\frac{s_1 r}{t_1}\right\}\right) + 1.\end{aligned}$$

This implies

$$\mathrm{Q}(T,t) = \sum_{i=0}^{\lfloor at/b\rfloor}\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)i - \left(\left\{\frac{s_2 i}{t_2}\right\} + \left\{1 - \frac{s_1 i}{t_1}\right\}\right) + 1. \tag{3.2}$$

Since $\lfloor at/b\rfloor = \frac{at}{b} - \left\{\frac{ar}{b}\right\}$, the first part can be written as

$$\sum_{i=0}^{\lfloor at/b\rfloor} i = t^2\frac{a^2}{2b^2} + t\frac{a}{b}\left(\frac{1}{2} - \left\{\frac{ar}{b}\right\}\right) + \frac{1}{2}\left\{\frac{ar}{b}\right\}^2 - \frac{1}{2}\left\{\frac{ar}{b}\right\}.$$

For the second part, we remark that $\left\{\frac{s_2 i}{t_2}\right\}$ is periodic with period t_2, and

$$\sum_{i=0}^{l-1}\left\{\frac{s_2 i}{t_2}\right\} = \frac{l}{t_2}\sum_{i=0}^{t_2-1}\left\{\frac{s_2 i}{t_2}\right\} = \frac{l(t_2-1)}{2t_2}.$$

Thus, we get

$$\begin{aligned}\sum_{i=0}^{\lfloor at/b\rfloor}\left\{\frac{s_2 i}{t_2}\right\} &= \left\lfloor\frac{\lfloor\frac{at}{b}\rfloor}{l}\right\rfloor\sum_{i=0}^{l-1}\left\{\frac{s_2 i}{t_2}\right\} + \sum_{i=0}^{\{\lfloor\frac{ar}{b}\rfloor/l\}l}\left\{\frac{s_2 i}{t_2}\right\}\\ &= \left(\frac{at}{bl}-\left\{\frac{ar}{bl}\right\}\right)\frac{l(t_2-1)}{2t_2} + \sum_{i=0}^{\{\lfloor\frac{ar}{b}\rfloor/l\}l}\left\{\frac{s_2 i}{t_2}\right\},\end{aligned}$$

and similarly

$$\sum_{i=0}^{\lfloor at/b\rfloor}\left\{-\frac{s_1 i}{t_1}\right\} = \left(\frac{at}{bl}-\left\{\frac{ar}{bl}\right\}\right)\frac{l(t_1-1)}{2t_1} + \sum_{i=0}^{\{\lfloor\frac{ar}{b}\rfloor/l\}l}\left\{1-\frac{s_1 i}{t_1}\right\}.$$

After some elementary algebra, Equation (3.2) expands to the claim. □

In particular, as shown in Section 3.3, $\frac{b}{a}$ is a period of $\mathrm{Q}_1(T,\cdot)$ which is piecewise linear, and $\frac{b\,\mathrm{lcm}(t_1,t_2)}{a}$ is a period of $\mathrm{Q}_0(T,r)$ which is piecewise quadratic. Furthermore, the pieces differ only by a constant depending only on k (see Figure 3.6).

We remark that Beck and Robins [6] gave an explicit formula for the classical Ehrhart quasi-polynomial for rectangular triangles and deduced an algorithm for efficient calculation of the number of integral points in rational polygons.

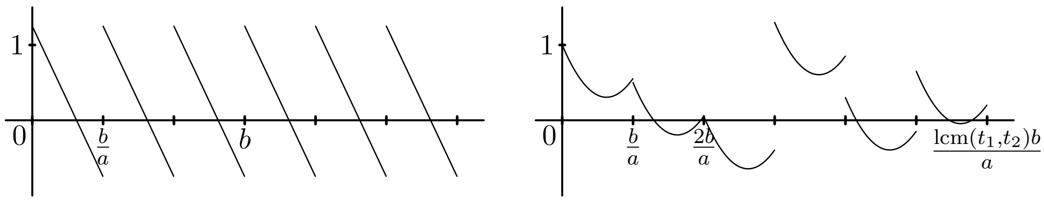

Figure 3.6: $\mathrm{Q}_1(T,r)$ (left) and $\mathrm{Q}_0(T,r)$ (right).

3.34 Corollary

Let T be as in Theorem 3.33, $l_i = \mathrm{conv}\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\frac{a}{b}\frac{s_i}{t_i}\\ \frac{a}{b}\end{pmatrix}\right)$, for $i=1,2$. Then $|\,\mathrm{Q}_1(T,r)| \le \mathrm{Q}_1(T,0)$ for all $0 \le r < b\,\mathrm{lcm}\,(t_1,t_2)$. Additionally, $\mathrm{Q}_1(T,r) \ge -\mathrm{Q}_1(T,0) + \mathrm{Q}_1(l_1,r) + \mathrm{Q}_1(l_2,r)$.

Proof

We have that $\mathrm{Q}_2(l_i,r)=0$, $\mathrm{Q}_1(l_i,r)=\frac{a}{bt_i}$, and $\mathrm{Q}_0(l_i,r) = 1-\left(\frac{ar}{bt_i}-\left\lfloor\frac{ar}{bt_i}\right\rfloor\right)$ for all $r\in\mathbb{Q}$,

$i = 1, 2$. Since $\frac{s_2}{t_2} - \frac{s_1}{t_1} > 0$, we get

$$\begin{aligned} \mathrm{Q}_1(T, r) &= \frac{a}{b}\left(\frac{t_1 + t_2}{2t_1t_2} - \left(\left\{\frac{ar}{b}\right\} - \frac{1}{2}\right)\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)\right) \\ &\leq \frac{a}{b}\left(\frac{t_1 + t_2}{2t_1t_2} + \frac{1}{2}\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)\right) = \mathrm{Q}_1(T, 0). \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{Q}_1(T, r) &= \frac{a}{b}\left(\frac{t_1 + t_2}{2t_1t_2} - \left\{\frac{ar}{b}\right\}\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right) + \frac{1}{2}\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)\right) \\ &\geq \frac{a}{b}\left(-\frac{t_1 + t_2}{2t_1t_2} - \frac{1}{2}\left(\frac{s_2}{t_2} - \frac{s_1}{t_1}\right)\right) + \frac{a}{b}\left(\frac{t_1 + t_2}{t_1t_2}\right) \\ &= -\mathrm{Q}_1(T, 0) + \mathrm{Q}_1(l_1, r) + \mathrm{Q}_1(l_2, r). \end{aligned}$$

□

3.35 Remark

An analogous statement to Corollary 3.34 for $\mathrm{Q}_0(T, \cdot)$ is not true. To see this, we consider the triangles

$$T_\alpha = \operatorname{conv}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{\alpha-1}{\alpha}\\1\end{pmatrix}, \begin{pmatrix}\frac{\alpha+1}{\alpha}\\1\end{pmatrix}\right), \quad \alpha \in \mathbb{Z}.$$

Together with Theorem 3.33, we get, for $m \in \mathbb{Z}_{\geq 0}$, $0 \leq k < \alpha$ and $0 \leq \tilde{r} < 1$,

$$\mathrm{Q}_0(T_\alpha, m\alpha + k + \tilde{r}) = \frac{1}{\alpha}\left(k(\alpha - k - 2) + \tilde{r}^2 - 2\tilde{r}\right) + 1.$$

Then for $k = \begin{cases} \frac{\alpha}{2}, & \alpha \equiv 0 \bmod 2, \\ \frac{\alpha-1}{2}, & \alpha \equiv 1 \bmod 2, \end{cases}$

$$\begin{aligned} \mathrm{Q}_0(T_\alpha, m\alpha + k + \tilde{r}) &= \begin{cases} \frac{\alpha}{4} + \frac{1}{\alpha}\left(\tilde{r}^2 - 2\tilde{r}\right), & \text{if } \alpha \equiv 0 \bmod 2, \\ \frac{\alpha}{4} + \frac{1}{\alpha}\left(\tilde{r}^2 - 2\tilde{r} + \frac{3}{4}\right), & \text{if } \alpha \equiv 1 \bmod 2, \end{cases} \\ &\geq \frac{\alpha}{4} - \frac{1}{\alpha} \end{aligned}$$

which tends to infinity, as α goes to infinity, but $\mathrm{Q}_0(T, 0) = 1$.

Now we can deduce the inequality $|\mathrm{Q}_1(P, r)| \leq \mathrm{Q}_1(P, 0)$ for arbitrary rational polygons:

3.36 Theorem

Let P be an arbitrary 2-dimensional rational polygon. Then $|\mathrm{Q}_1(P, r)| \leq \mathrm{Q}_1(P, 0)$ for all $0 \leq r < \widehat{\mathrm{den}}(P)$.

Proof

We first consider only integral dilation factors, that is, we show that

$$|\operatorname{G}_1(P,k)| \le \operatorname{G}_1(P,0) \quad \text{for all } k \in \mathbb{Z}\,. \tag{3.3}$$

This is done in tree steps:

1. An integral version of Corollary 3.34 holds true for $\operatorname{G}_1(P,k)$ *for arbitrary triangles* P *with at least one integral vertex.*

This is true, since $\operatorname{G}_1(P,k)$ is invariant under translations and unimodular transformations.

2. Equation (3.3) *is true for every rational polygon* P *with one integral point in its interior.*

Let P be an arbitrary 2-dimensional polygon with m vertices and let z be an integral point in the interior of P. We consider the triangulation of P into triangles $T_1, \dots, T_m$ as given in Figure 3.7.

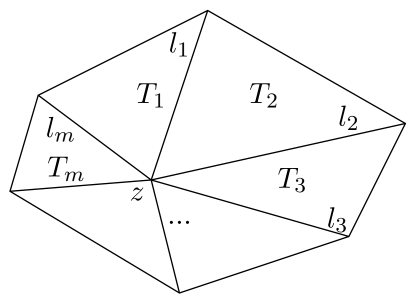

Figure 3.7: Triangulation of P.

Here, l_i is the edge between T_i and T_{i+1} for $i = 1, \dots, m-1$ and $l_0 = l_m = T_m \cap T_1$. Then for all $k \in \mathbb{Z}$,

$$\operatorname{G}(P,k) = \sum_{i=1}^{m} \operatorname{G}(T_i,k) - \sum_{i=1}^{m} \operatorname{G}(l_i,k) + 1.$$

Thus, expanding all Ehrhart polynomials and Step 1 yields

$$\begin{aligned}
\operatorname{G}_1(P,k) &= \sum_{i=1}^{m} \left(\operatorname{G}_1(T_i,k) - \frac{1}{2} \operatorname{G}_1(l_{i-1},k) - \frac{1}{2} \operatorname{G}_1(l_i,k) \right) \\
&\ge \sum_{i=1}^{m} \left(- \operatorname{G}_1(T_i,0) + \frac{1}{2} \operatorname{G}_1(l_{i-1},k) + \frac{1}{2} \operatorname{G}_1(l_i,k) \right) \\
&= \sum_{i=1}^{m} \left(- \operatorname{G}_1(T_i,0) + \frac{1}{2} \operatorname{G}_1(l_{i-1},0) + \frac{1}{2} \operatorname{G}_1(l_i,0) \right) = - \operatorname{G}_1(P,0).
\end{aligned}$$

3. Equation (3.3) *is true for arbitrary rational polygons P.*
Let $m \in \mathbb{Z}_{\geq 0}$ such that $m(\operatorname{den}(P)+1)P$ contains at least one integral point in its interior. Then for all $k \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{G}_1((m \operatorname{den}(P)+1)P, k) = \mathrm{G}_1(P, k)(m \operatorname{den}(P)+1).$$

Thus, Step 2 implies that $|\,\mathrm{G}_1(P, k)| \leq \mathrm{G}_1(P, 0)$.

Finally, we consider arbitrary rational dilation factors. Let P be an arbitrary rational polygon and let $r = \frac{p}{q} \in \mathbb{Q}_{\geq 0}$ with $r \leq \operatorname{den}(P)$, where $p \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{Z}_{\geq 1}$. Then, by Lemma 3.21, $\mathrm{Q}_1(P, r) = \mathrm{G}_1\left(\frac{1}{q}P, p\right) q^1$. Hence, Equation (3.3) implies that $|\,\mathrm{Q}_1(P, r)| \leq \mathrm{Q}_1(P, 0)$. □

Using Theorem 3.36, we can deduce a generalization of Theorem 2.6 for rational Ehrhart quasi-coefficients in dimension 2:

3.37 Corollary
For $P \in \mathcal{P}^2_{\mathbb{Q}} \cap \mathcal{K}^2_0$,

$$\mathrm{Q}_1(P, r) \leq \frac{2}{\lambda_1(P)} + \frac{2}{\lambda_2(P)}, \quad \textit{for all } r \in \mathbb{Q}_{\geq 0}\,.$$

Proof
We have by Theorem 3.36 and Lemma 3.21 for all $r \in \mathbb{Q}_{\geq 0}$,

$$\mathrm{Q}_1(P, r) \leq \mathrm{Q}_1(P, 0) = \mathrm{Q}_1(\operatorname{den}(P)P, 0)\frac{1}{\operatorname{den}(P)},$$

which is $\mathrm{G}_1(\operatorname{den}(P)P)\frac{1}{\operatorname{den}(P)}$, since $\operatorname{den}(P)P$ is an integral polytope. Then by Theorem 2.6,

$$\begin{aligned}\mathrm{Q}_1(P, r) &\leq \left(\frac{2}{\lambda_1(\operatorname{den}(P)P)} + \frac{2}{\lambda_2(\operatorname{den}(P)P)}\right)\frac{1}{\operatorname{den}(P)} \\ &= \left(\frac{2\operatorname{den}(P)}{\lambda_1(P)} + \frac{2\operatorname{den}(P)}{\lambda_2(P)}\right)\frac{1}{\operatorname{den}(P)} \\ &= \frac{2}{\lambda_1(P)} + \frac{2}{\lambda_2(P)},\end{aligned}$$

for all $r \in \mathbb{Q}_{\geq 0}$. □

3.5 Rational Ehrhart quasi-polynomials of Minkowski sums

In this section, we generalize the results from Sections 3.2 and 3.3 to Minkowski sums of rational polytopes. That is, for $P_1, \ldots, P_k \in \mathcal{P}^n_{\mathbb{Q}}$, $k \in \mathbb{Z}_{\geq 1}$, we consider the function $\mathrm{Q}(P_1, \ldots, P_k, \cdot) : \mathbb{Q}^k_{\geq 0} \to \mathbb{Z}_{\geq 0}$ given by

$$\mathrm{Q}(P_1, \ldots, P_k, r) := \mathrm{G}\left(\sum_{i=1}^{k} r_i P_i\right), \quad \text{for } r = (r_1, \ldots, r_k) \in \mathbb{Q}^k_{\geq 0}\,.$$

Most of the results in this section are straightforward generalizations of the work in Section 3.3.

To state the results in a comprehensive way, we fix some abbreviatory notation: For $x \in \mathbb{R}^n$, $y \in \mathbb{Z}^n_{\geq 0}$ we write $x^y := \prod_{j=1}^n x_j^{y_j}$. For a k-tupel $I = (i_1, \ldots, i_k) \in \{0, \ldots, n\}^k$ we denote by $|I|_1 = \sum_{j=1}^k i_j$ the usual 1-norm.

3.38 Definition (Rational Quasi-polynomial in several unknowns)
A function $p : \mathbb{Q}^k \to \mathbb{Q}$ is called a rational quasi-polynomial of total degree n with period $d = (d_1, \ldots, d_k) \in \mathbb{Q}^k$ *if there exist periodic functions $p_I : \mathbb{Q}^k_{\geq 0} \to \mathbb{Q}$ for all $I \in \{0, \ldots, n\}^k$ with period d such that*

$$p(r) = \sum_{\substack{I \in \{0,\ldots,n\}^k \\ |I|_1 \leq n}} p_I(r) r^I.$$

We call $p_I(\cdot)$ the Ith coefficient of p.

The first theorem is a generalization of Theorem 3.18.

3.39 Theorem
Let $P_1, \ldots, P_k \in \mathcal{P}^n_{\mathbb{Q}}$ be rational polytopes. Then $\mathrm{Q}(P_1, \ldots, P_k, \cdot)$ is a rational quasi-polynomial of total degree $\dim(P_1 + \ldots + P_k)$ *with period* $d = (\widehat{\mathrm{den}}(P_1), \ldots, \widehat{\mathrm{den}}(P_k))$. $\mathrm{Q}(P_1, \ldots, P_k, \cdot)$ *is called the* rational Ehrhart quasi-polynomial of $P_1, \ldots, P_k$. *The Ith coefficient of* $\mathrm{Q}(P_1, \ldots, P_k, \cdot)$ *is denoted by* $\mathrm{Q}_I(P_1, \ldots, P_k, \cdot)$.

As in the univariate case, this theorem follows from McMullens proof of Theorem 7 in [33] although not stated explicitely there. Nevertheless, the straightforward generalization of the integral case stated in [33, Theorem 7] works with the same arguments as in the proof of Theorem 3.18.

In fact, McMullen's work implies something stronger. Recall that for a facet F of a rational polytope P, α_F denotes the smallest positive rational number such that $\alpha_F\,\mathrm{aff}(F)$

contains integral points.

3.40 Remark
For rational polytopes $P_1, \ldots, P_k \in \mathcal{P}^n_{\mathbb{Q}}$ the combinatorial structure of $r_1 P_1 + \ldots + r_k P_k$ does not depend on r. Hence we call the corresponding facets $F_1(r), \ldots, F_l(r)$, $l \in \mathbb{Z}_{>0}$. Then $\mathrm{Q}_I(P_1, \ldots, P_k, r)$ as a function in r depends only on the values $\left\{\frac{1}{\alpha_{F_i(r)}}\right\}$ for $i = 1, \ldots, l$. $\left\{\frac{1}{\alpha_{F_i(r)}}\right\}$ is the fractional part of a linear function in $r_1, \ldots, r_k$, since the right-hand side of $F_i(r)$ in the facet description of $r_1 P_1 + \ldots + r_k P_k$ depends linearly on r.

To work out the leading coefficients in the multivariate case, we introduce the so-called mixed volumes of convex bodies. Let $K_1, \ldots, K_k \in \mathcal{K}^n$ be convex bodies. Then the volume of the Minkowski sum of dilated bodies, $\mathrm{vol}(r_1 K_1 + \ldots + r_k K_k)$, is a homogeneous polynomial of degree n in $r_1, \ldots, r_k$ (see for example [18, Chapter 6] and [40, Chapter 5]), that is,

$$\mathrm{vol}(r_1 K_1 + \ldots + r_k K_k) = \sum_{\substack{I \in \{0,\ldots,n\}^k \\ |I|_1 = n}} \frac{n!}{i_1! \cdots i_k!} \mathrm{V}_I(P_1, \ldots, P_k)\, r^I.$$

The coefficients $\mathrm{V}_I(P_1, \ldots, P_k)$ are called mixed volumes. Mixed volumes are non-negative and depend only on the bodies that are involved.

3.41 Lemma
Let $I = (i_1, \ldots, i_k) \in \{0, \ldots, n\}^k$, *with* $i_1 + \ldots + i_k = \dim(P_1 + \ldots + P_k) =: m$. *Then*

$$\mathrm{Q}_I(P_1, \ldots, P_k, r) = \frac{n!}{i_1! \cdots i_k!} \mathrm{V}_I(P_1, \ldots, P_k)$$

for all $r \in \mathbb{Q}^k_{\geq 0}$ *such that* $\mathrm{aff}(r_1 P_1 + \ldots + r_k P_k)$ *contains integral points.*

Proof
For $s \in \mathbb{Q}_{\geq 0}$ and $r \in \mathbb{Q}^k_{\geq 0}$,

$$\begin{aligned}\sum_{\substack{I \in \{0,\ldots,n\}^k \\ |I|_1 \leq m}} \mathrm{Q}_I(P_1, \ldots, P_k, sr)(sr)^I &= \#((sr_1 P_1 + \ldots + sr_k P_k) \cap \mathbb{Z}^n) \\ &= \sum_{i=0}^{m} \mathrm{Q}_i(r_1 P_1 + \ldots + r_k P_k, s) s^i.\end{aligned}$$

Thus, by sorting the left-hand side by powers of s and comparing coefficients for $|I|_1 = m$, we have

$$\sum_{\substack{I\in\{0,\dots,n\}^k\\ |I|_1=m}} \mathrm{Q}_I(P_1,\dots,P_k,sr)r^I = \mathrm{Q}_m(r_1P_1+\dots+r_kP_k,s),$$

that is, if $\operatorname{aff}(s(r_1P_1+\dots+r_kP_k))\cap\mathbb{Z}^n\neq\emptyset$

$$\operatorname{vol}_m(r_1P_1+\dots+r_kP_k) = \sum_{\substack{I\in\{0,\dots,n\}^k\\ |I|_1=m}} \frac{m!}{i_1!\cdots i_k!}\,\mathrm{V}_I(P_1,\dots,P_k)r^I,$$

and 0 otherwise. This implies the assertion, again by comparing coefficients. $\square$

In what follows, we denote by $r\odot s := (r_1s_1,\dots,r_ks_k)$ the componentwise multiplication of $r,s\in\mathbb{Q}^k$. Furthermore, when considering k-tuples in $\{0,\dots,n\}^k$, we allow them to be added componentwise.

To show a result corresponding to Theorem 3.30 we prove the following lemma.

3.42 Lemma
Let $p:\mathbb{Q}^k\to\mathbb{Q}$ be a rational quasi-polynomial of total degree $n\in\mathbb{Z}_{\geq 1}$ with period $d\in\mathbb{Q}^k_{>0}$ and constant leading coefficients, that is,

$$p(r) = \sum_{\substack{I\in\{0,\dots,n\}^k\\ |I|_1\leq n}} p_I(r)r^I, \quad r=(r_1,\dots,r_k)\in\mathbb{Q}^k,$$

where $p_I(r) =: p_I\in\mathbb{Q}$ for $|I|_1=n$, and $p_I:\mathbb{Q}^k\to\mathbb{Q}$ are periodic functions with period d for $I\in\{0,\dots,n\}^k$, $|I|_1<n$. Furthermore, suppose there exist a set $S\subset\mathbb{Q}^k$ and $c_U\in\mathbb{Q}$ for all $U\in\mathbb{Z}^k_{\geq 0}$ such that

$$p(r+U\odot d)=c_U, \quad \textit{for all } r\in S,\ U\in\mathbb{Z}^k_{\geq 0}\,.$$

Then $p_I:S\to\mathbb{Q}$ is a polynomial of total degree $n-|I|_1$ and of degree at most $n-i_j$ in r_j, and the partial derivative

$$\frac{\partial}{\partial r_j}p_I(r) = -(i_j+1)p_{I+e_j}(r),$$

for all $I=(i_1,\dots,i_k)\subset\{0,\dots,n\}^k, |I|_1<n$ and for all $r\in S$.

As in the univariate case the proof is by induction on the total degree. For simplification, we subdivide a part of the induction step which gives the following lemma:

3.43 Lemma

For all $I = (i_1, \dots, i_k) \in \{0, \dots, n\}^k$ with $|I|_1 \leq n-1$ and $l \in \{1, \dots, k\}$ and for a fixed subset $S \subset \mathbb{Q}^k$ let $q_I^l : S \to \mathbb{Q}$ and $p_I : S \to \mathbb{Q}$ with

$$q_I^l(r) = \sum_{j=i_l+1}^{n-|I|_1+i_l} p_{(i_1,\dots,i_{l-1},j,i_{l+1},\dots,i_k)}(r)\, c_I^l(j),$$

for constants $c_I^l(j)$. Furthermore suppose q_I^l is a polynomial of total degree $n-1-|I|_1$ in r and of degree $n-1-i_h$ in r_h for all $h \in \{1, \dots, k\}$. Then for $J = (j_1, \dots, j_k) \in \{0, \dots, n\}^k$ with $1 \leq |J|_1 \leq n$, p_J is a polynomial of total degree $n - |J|_1$ in r and of degree at most $n - j_h$ in r_h for all $h \in \{1, \dots, k\}$.

Proof

We show this by induction on $|J|_1$.

For $|J|_1 = n$ the statement is clear, since for $j_k \neq 0$, say, we have that

$$q^k_{(j_1,\dots,j_{k-1},j_k-1)}(r) = p_J(r)\, c^k_{(j_1,\dots,j_{k-1},j_k-1)}(j_k).$$

For $|J|_1 < n$ and again, without loss of generality, $j_k \neq 0$, consider $q^k_{J-e_k}$, which is

$$\begin{aligned} q^k_{J-e_k}(r) &= \sum_{i=j_k}^{n-|J|_1+j_k} p_{J+(i-j_k)e_k}(r)\, c^k_{J-e_k}(i) \\ &= p_J(r)\, c^k_{J-e_k}(j_k) + \sum_{i=j_k+1}^{n-|J|_1+j_k} p_{J+(i-j_k)e_k}(r)\, c^k_{J-e_k}(i). \end{aligned}$$

Thus

$$p_J(r)\, c^k_{J-e_k}(j_k) = q^k_{J-e_k}(r) - \sum_{i=j_k+1}^{n-|J|_1+j_k} p_{J+(i-j_k)e_k}(r)\, c^k_{J-e_k}(i).$$

By induction hypothesis for $i > j_k$, $p_{J+(i-j_k)e_k}(r)$ is a polynomial of total degree $n - |J|_1 - i + j_k \leq n - |J|_1 - 1$ in r, of degree $n - i \leq n - j_k - 1$ in r_k, and of degree $n - j_h$ in r_h for all $h \in \{1, \dots, k-1\}$ and $r \in S$. Furthermore, $q^k_{J-e_k}(r)$ is a polynomial of total degree $n - |J|_1$ in r, of degree $n - j_k$ in r_k, and of degree $n - 1 - j_h$ in r_h for all $h \in \{1, \dots, k-1\}$ and $r \in S$.

Thus $p_J(r)$ is a polynomial of total degree $n - |J|_1$ in r, of degree $n - j_k$ in r_k, and of degree $n - j_h$ in r_h for all $h \in \{1, \dots, k-1\}$ and $r \in S$. □

Proof (of Lemma 3.42)

We prove the polynomiality result by induction on n.

For $n = 1$, we have $c_{\mathbb{0}} = p(r) = \sum_{|I|_1=1} p_I \cdot r^I + p_{\mathbb{0}}(r)$ for all $r \in S$. Thus $p_{\mathbb{0}}(r) = c_{\mathbb{0}} - \sum_{|I|_1=1} p_I \cdot r^I$ for $r \in S$, which is a polynomial of total degree $n - 0 = 1$ and degree $n - 0 = 1$ in r_h for all $h \in \{1, \dots, k\}$ and $r \in S$.

Now let $n > 1$. Consider $q(r) := p(r + d_k e_k) - p(r)$.
Then $q(r + U \odot d) = p(r + (U + e_k) \odot d) - p(r + U \odot d) = c_{U+e_k} - c_U$ for all $r \in S$, $U \in Z^k_{\geq 0}$.

To shorten the notation in this proof, $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$, and $H = (h_1, \dots, h_k)$ are always vectors in $\{0, \dots, n\}^k$, and we write $I \leq J$ if $i_l \leq j_l$ for all $l = 1, \dots, k$. Furthermore, for $r = (r_1, \dots, r_k) \in S$ and $I = (i_1, \dots, i_k)$ we denote by $\bar{r} = (r_1, \dots, r_{k-1})$ and $\bar{I} = (i_1, \dots, i_{k-1})$, respectively, the vector with the last coordinate removed.

Then we get

$$\begin{aligned}
q(r) = p(\bar{r}, r_k + d_k) - p(r) &= \sum_{|I|_1 \leq n} p_I(r) \left(\bar{r}^{\bar{I}} (r_k + d_k)^{i_k} - r^I \right) \\
&= \sum_{|I|_1 \leq n} p_I(r) \left(\sum_{j=0}^{i_k} \binom{i_k}{j} d_k^{i_k - j} \bar{r}^{\bar{I}} r_k^j - r^I \right) \\
&= \sum_{|I|_1 \leq n} p_I(r) \left(\sum_{j=0}^{i_k - 1} \binom{i_k}{j} d_k^{i_k - j} \bar{r}^{\bar{I}} r_k^j \right) \\
&= \sum_{|\bar{I}|_1 \leq n} \sum_{i_k=0}^{n - |\bar{I}|_1} \sum_{j=0}^{i_k - 1} p_{(\bar{I}, i_k)}(r) \binom{i_k}{j} d_k^{i_k - j} \bar{r}^{\bar{I}} r_k^j \\
&= \sum_{|\bar{I}|_1 \leq n} \sum_{j=0}^{n - |\bar{I}|_1 - 1} \left(\sum_{i_k = j+1}^{n - |\bar{I}|_1} p_{(\bar{I}, i_k)}(r) \binom{i_k}{j} d_k^{i_k - j} \right) \bar{r}^{\bar{I}} r_k^j.
\end{aligned}$$

Interchanging the names of i_k and j yields

$$\begin{aligned}
q(r) &= \sum_{|\bar{I}|_1 \leq n} \sum_{i_k=0}^{n - |\bar{I}|_1 - 1} \left(\sum_{j = i_k + 1}^{n - |\bar{I}|_1} p_{(\bar{I}, j)}(r) \binom{j}{i_k} d_k^{j - i_k} \right) r^I \\
&= \sum_{|I|_1 \leq n-1} \left(\sum_{j = i_k + 1}^{n - |I|_1 + i_k} p_{(\bar{I}, j)}(r) \binom{j}{i_k} d_k^{j - i_k} \right) r^I.
\end{aligned}$$

Thus $q : \mathbb{Q}^k \to \mathbb{Q}$ is a rational quasi-polynomial of total degree $n - 1$ with period d and coefficients

$$q_I(r) := \sum_{j=i_k+1}^{n-|I|_1+i_k} p_{(\bar{I},j)}(r)\binom{j}{i_k} d_k^{j-i_k}.$$

The leading coefficient for $|I| = n-1$ is

$$q_I(r) = \sum_{j=i_k+1}^{n-|I|_1+i_k} p_{(\bar{I},j)}(r)\binom{j}{i_k} d_k^{j-i_k} = p_{I+e_k}(r)\cdot(i_k+1)d_k,$$

which is constant in r. Thus by induction hypothesis we get that

$$q_I(r) = \sum_{j=i_k+1}^{n-|I|_1+i_k} p_{(\bar{I},j)}(r)\binom{j}{i_k} d_k^{j-i_k}$$

is a polynomial of total degree $n-1-|I|_1$ in r and of degree $n-1-i_h$ in r_h for all $h \in \{1,\dots,k\}$ and $r \in S$. By renaming variables, this is also true if we replace r_k and i_k by arbitrary r_j and i_j: For all $I = (i_1,\dots,i_k)$,

$$q_I^l(r) := \sum_{j=i_l+1}^{n-|I|_1+i_l} p_{(i_1,\dots,i_{l-1},j,i_{l+1},\dots,i_k)}(r)\binom{j}{i_l} d_l^{j-i_l}$$

is a polynomial of total degree $n-1-|I|_1$ in r and of degree $n-1-i_h$ in r_h for all $h \in \{1,\dots,k\}$ and $r \in S$. Thus $p_I(r)$ is a polynomial of total degree $n-|I|_1$ in r and of degree $n-i_h$ in r_h for all $h \in \{1,\dots,k\}$ and $r \in S$ by Lemma 3.43. For finishing the inductive step it remains to show that $p_{\mathbb{0}}(r)$ is a polynomial of total degree n in r and of degree n in r_h for all $h \in \{1,\dots,k\}$ and $r \in S$, which follows since

$$c_{\mathbb{0}} = p_{\mathbb{0}}(r) + \sum_{0<|I|_1\le n} p_I(r)r^I.$$

It remains to show that

$$\frac{\partial}{\partial r_j} p_I(r) = -(i_j+1)p_{I+e_j}(r),$$

for all $I = (i_1,\dots,i_k) \in \{0,\dots,n\}^k, |I|_1 < n, i_j < n$, and for all $r \in S$. To this end, since $p_I(r)$ is a polynomial of total degree $n-|I|_1$ in r, we can write it as

$$p_I(r) = \sum_{\substack{J\in\{0,\dots,n\}^k \\ |J|_1\le n-|I|_1}} p_{I,J} r^J,$$

for some coefficients $p_{I,J}$. Since $p_I(r)$ is periodic with period $d \in \mathbb{Q}^k$, we can write $r = \tilde{r} + d \odot U$ with $U = (u_1,\dots,u_k) \in \mathbb{Z}_{\ge 0}^k$ and $\tilde{r} \in S$ and get

$$c_U = p(\tilde{r} + d\odot U) = \sum_{|I|_1\le n} p_I(\tilde{r})(\tilde{r} + d\odot U)^I.$$

Then we get

$$\begin{aligned} c_U = p(\tilde{r} + d \odot U) &= \sum_{|I|_1 \le n} \left(\sum_{|J|_1 \le n - |I|_1} p_{I,J} \tilde{r}^J \right) (\tilde{r} + d \odot U)^I \\ &= \sum_{|I|_1 \le n} \left(\sum_{|J|_1 \le n - |I|_1} p_{I,J} \tilde{r}^J \right) \prod_{m=1}^{k} (\tilde{r}_m + d_m u_m)^{i_m}. \end{aligned}$$

Expanding all binomial powers yields

$$\begin{aligned} c_U &= \sum_{|I|_1 \le n} \left(\sum_{|J|_1 \le n - |I|_1} p_{I,J} \tilde{r}^J \right) \prod_{m=1}^{k} \sum_{h_m=0}^{i_m} \binom{i_m}{h_m} \tilde{r}_m^{h_m} (d_m u_m)^{i_m - h_m} \\ &= \sum_{|I|_1 \le n} \sum_{|J|_1 \le n - |I|_1} \sum_{H \le I} p_{I,J} \left(\prod_{m=1}^{k} \binom{i_m}{h_m} (d_m u_m)^{i_m - h_m} \right) \tilde{r}^{J+H} \\ &= \sum_{|I|_1 \le n} \sum_{|J|_1 \le n - |I|_1} \sum_{J \le H \le I+J} p_{I,J} \left(\prod_{m=1}^{k} \binom{i_m}{h_m - j_m} (d_m u_m)^{i_m - h_m + j_m} \right) \tilde{r}^{H} \\ &= \sum_{|H|_1 \le n} \sum_{|I|_1 \le n} \sum_{\substack{H - I \le J \le H \\ |J|_1 \le n - |I|_1}} p_{I,J} \left(\prod_{m=1}^{k} \binom{i_m}{h_m - j_m} (d_m u_m)^{i_m - h_m + j_m} \right) \tilde{r}^{H}, \end{aligned}$$

which is a constant polynomial c_U in $\tilde{r}$. Thus for $|H|_1 \neq 0$,

$$\sum_{|I|_1 \le n} \sum_{\substack{H - I \le J \le H \\ |J|_1 \le n - |I|_1}} p_{I,J} \left(\prod_{m=1}^{k} \binom{i_m}{h_m - j_m} (d_m u_m)^{i_m - h_m + j_m} \right) = 0,$$

and therefore

$$\begin{aligned} p(\tilde{r} + d \odot U) &= \sum_{|I|_1 \le n} \sum_{\substack{\mathbb{0} - I \le J \le \mathbb{0} \\ |J|_1 \le n - |I|_1}} p_{I,J} \left(\prod_{m=1}^{k} \binom{i_m}{-j_m} (d_m u_m)^{i_m + j_m} \right) \\ &= \sum_{|I|_1 \le n} p_{I,\mathbb{0}} (d \odot U)^I = \sum_{|I|_1 \le n} p_{I,\mathbb{0}} (r - \tilde{r})^I. \end{aligned}$$

Expanding to the quasi-polynomial form yields

$$\begin{aligned}
p(\tilde{r} + d \odot U) &= \sum_{|I|_1 \le n} p_{I,\mathbb{0}} (r - \tilde{r})^I = \sum_{|I|_1 \le n} p_{I,\mathbb{0}} \prod_{m=1}^{k} (r_m - \tilde{r}_m)^{i_m} \\
&= \sum_{|I|_1 \le n} p_{I,\mathbb{0}} \sum_{J \le I} \prod_{m=1}^{k} \binom{i_m}{j_m} r_m^{j_m} (-\tilde{r}_m)^{i_m - j_m} \\
&= \sum_{|J|_1 \le n} \left(\sum_{\substack{|I|_1 \le n \\ I \ge J}} \left(\prod_{m=1}^{k} \binom{i_m}{j_m} (-1)^{i_m - j_m} \right) p_{I,\mathbb{0}} \tilde{r}^{I-J} \right) r^J,
\end{aligned}$$

which, by redefinition of the index set I, is

$$\sum_{|J|_1 \le n} \left(\sum_{|I|_1 \le n - |J|_1} \left(\prod_{m=1}^{k} \binom{i_m + j_m}{j_m} (-1)^{i_m} \right) p_{I+J,\mathbb{0}} \tilde{r}^I \right) r^J.$$

This implies, for all $\tilde{r} \in S$,

$$p_J(\tilde{r}) = \sum_{|I|_1 \le n - |J|_1} \left(\prod_{m=1}^{k} \binom{i_m + j_m}{j_m} (-1)^{i_m} \right) p_{I+J,\mathbb{0}} \tilde{r}^I.$$

Together with $\alpha(i, j) := \binom{i+j}{j}(-1)^i$, for $|J| < n$ and $h \in \{1, \ldots, n\}$ we get

$$\begin{aligned}
\frac{\partial}{\partial r_h} p'_J(\tilde{r}) &= \sum_{\substack{|I|_1 \le n - |J|_1 \\ i_h \ge 1}} \left(\prod_{m=1}^{k} \alpha(i_m, j_m) \right) p_{I+J,\mathbb{0}} i_h \tilde{r}^{I - e_h} \\
&= \sum_{|I|_1 \le n - |J|_1 - 1} \left(\prod_{\substack{m=1 \\ m \ne h}}^{k} \alpha(i_m, j_m) \right) (-1)^{i_h + 1} \tbinom{i_h + j_h + 1}{j_h} p_{I+J+e_h,\mathbb{0}} (i_h + 1) \tilde{r}^I \\
&= -(j_h + 1) \sum_{|I|_1 \le n - |J|_1 - 1} \left(\prod_{\substack{m=1 \\ m \ne h}}^{k} \alpha(i_m, j_m) \right) (-1)^{i_h} \tbinom{i_h + j_h + 1}{j_h + 1} p_{I+J+e_h,\mathbb{0}} \tilde{r}^I \\
&= -(j_h + 1) p_{J + e_h}(\tilde{r}),
\end{aligned}$$

which finishes the proof. □

Now $\mathrm{Q}(P_1, \ldots, P_k, r)$ is piecewise constant and satisfies the conditions of Lemma 3.42, if the sum $\sum_{i=1}^{k} P_i$ is full-dimensional. Thus, we get the following result:

3.44 THEOREM
Let $P_1, \ldots, P_k \in \mathcal{P}^n_{\mathbb{Q}}$ *be rational polytopes and let* $\dim(P_1 + \ldots + P_k) = n$*. Then for* $I = (i_1, \ldots, i_k) \in \{0, \ldots, n\}^k$, $\mathrm{Q}_I(P_1, \ldots, P_k, \cdot)$ *is a piecewise-defined polynomial function of total degree* $n - |I|_1$ *and of degree* $n - i_j$ *in* r_j*, and*

$$\frac{\partial}{\partial r_j} \mathrm{Q}_I(P_1, \ldots, P_k, r) = -(i_j + 1) Q_{I+e_j}(P_1, \ldots, P_k, r),$$

for all $|I|_1 < n$ *and for all* $r \in \mathbb{Q}_{\geq 0}$ *such that* $\mathrm{Q}(P_1, \ldots, P_k, \cdot)$ *is continuous at* $r + U \odot (\widehat{\mathrm{den}}(P_1), \ldots, \widehat{\mathrm{den}}(P_k))^\top$ *for all* $U \in \mathbb{Z}^k_{\geq 0}$*.*

PROOF
By Theorem 3.39, it is enough to show the statement for

$$r \in S = [0, q(P_1)] \times \ldots \times [0, q(P_k)].$$

Since for $z \in \mathbb{Z}^n$ we have that $z \in \sum_{i=1}^k r_i P_i$ if and only if $\sum_{i=1}^k r_i h(P_i, z) \geq 1$, $\mathrm{Q}(P_1, \ldots, P_k, r)$ is constant on components of the hyperplane arrangement defined by the hyperplanes

$$\left\{ \left\{ x \in \mathbb{R}^n : \sum_{i=1}^k x_i h(P_i, z) = 1 \right\} : z \in \mathbb{Z}^n \right\}.$$

For $r \in S$ it is enough to consider only $z \in \sum_{i=1}^k q(P_i) P_i \cap \mathbb{Z}^n$ since other integral points are never in $r_1 P_1 + \ldots + r_k P_k$ for $r \in S$. Thus $\mathrm{Q}(P_1, \ldots, P_k, r)$ satisfies the conditions of Lemma 3.42 for every maximal cell of the arrangement

$$\left\{ \left\{ x \in S : \sum_{i=1}^k x_i h(P_i, z) = 1 \right\} : z \in \sum_{i=1}^k q(P_i) P_i \, \mathbb{Z}^n \right\}. \qquad \square$$

3.45 EXAMPLE
As an example, we consider $C_2 = \mathrm{conv}\{\binom{1}{1}, \binom{-1}{1}, \binom{-1}{-1}, \binom{1}{-1}\}$ and the triangle $T = \{\binom{0}{1}, \binom{1}{-1}, \binom{-1}{-1}\}$ (see Figure 3.8). For $r, s \in \mathbb{Q}_{\geq 0}$, the sum $rC_2 + sT$ has the following structure:

$$\begin{aligned} rC_2 + sT &= \mathrm{conv}\left\{ \begin{pmatrix} r \\ r+s \end{pmatrix}, \begin{pmatrix} -r \\ r+s \end{pmatrix}, \begin{pmatrix} -(r+s) \\ r-s \end{pmatrix}, \begin{pmatrix} -(r+s) \\ -(r+s) \end{pmatrix}, \begin{pmatrix} (r+s) \\ -(r+s) \end{pmatrix}, \begin{pmatrix} r+s \\ r-s \end{pmatrix} \right\} \\ &= \{x \in \mathbb{R}^2 : \quad \begin{aligned} -(r+s) \leq x_1 &\leq r+s, \\ -(r+s) \leq x_2 &\leq r+s, \\ \pm 2x_1 + x_2 &\leq 3r+s\}. \end{aligned} \end{aligned}$$

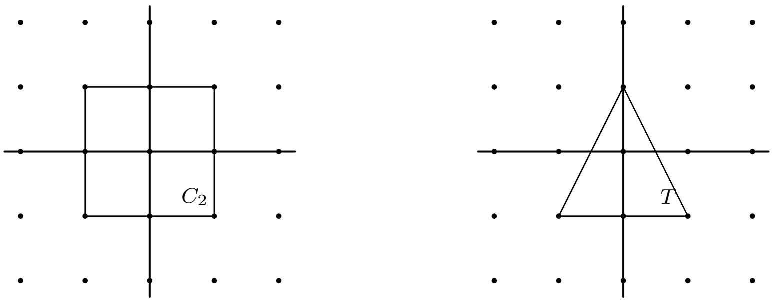

Figure 3.8: C_2 and T.

Thus for the edges F of $rC_2 + sT$ it holds that $\frac{1}{\alpha_F} \in \{r+s, 3r+s\}$, and hence the coefficients of the rational Ehrhart quasi-polynomial depend only on $\{r+s\}$ and $\{3r+s\}$ (see Remark 3.40).

To give a formula for the number of integral points in rC_2+sT we consider the subdivision of $rC_2 + sT$ given in Figure 3.9.

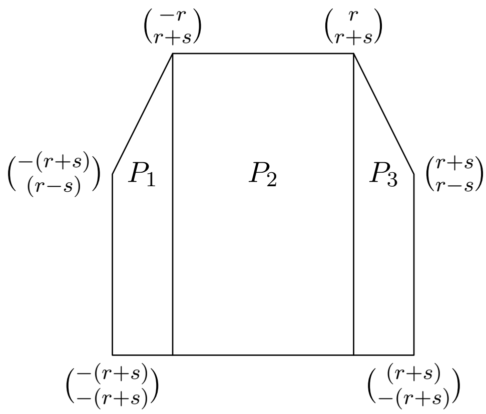

Figure 3.9: $rC_2 + sT$.

The number of integral points in

$$P_2 = \operatorname{conv}\left\{ \begin{pmatrix} r \\ r+s \end{pmatrix}, \begin{pmatrix} -r \\ r+s \end{pmatrix}, \begin{pmatrix} -r \\ -(r+s) \end{pmatrix}, \begin{pmatrix} r \\ -(r+s) \end{pmatrix} \right\}$$

is $(2\lfloor r \rfloor + 1)(2\lfloor r+s \rfloor + 1)$. P_1 and P_3 are reflections of each other with respect to the line $x = 0$ and thus they contain exactly the same number of integral points. Besides the points already counted for P_2, this number is

$$\sum_{x_1=\lfloor r\rfloor+1}^{\lfloor r+s\rfloor} \lfloor 3r+s-2x_1\rfloor - \lceil -(r+s)\rceil + 1 = \sum_{x_1=\lfloor r\rfloor+1}^{\lfloor r+s\rfloor} \lfloor 3r+s\rfloor - 2x_1 + \lfloor r+s\rfloor + 1.$$

Alltogether and writing $k := \lfloor r+s\rfloor$ and $l := \lfloor 3r+s\rfloor$, the number of integral points in $rC_2 + sT$ is

$$\begin{aligned}
\#((rC_2+sT)\cap\mathbb{Z}^2) &= (2\lfloor r\rfloor+1)(2k+1) + 2\sum_{x_1=\lfloor r\rfloor+1}^{k} k+l+1-2x_1 \\
&= (2\lfloor r\rfloor+1)(2k+1) + 2(k+l+1)(k-\lfloor r\rfloor) - 4\sum_{x_1=\lfloor r\rfloor+1}^{k} x_1 \\
&= 2k+2lk+1+2(\lfloor r\rfloor^2 + \lfloor r\rfloor(k-l+1)).
\end{aligned}$$

With $k=\lfloor r+s\rfloor$ and $l=\lfloor 3r+s\rfloor$, we get that $\frac{l-k-1}{2} < r < \frac{l-k-1}{2}+1$. Thus, if $k-l$ is odd then $\lfloor r\rfloor = \frac{l-k-1}{2}$, and if $k-l$ is even then $\lfloor r\rfloor \in \{\frac{l-k}{2}-1, \frac{l-k}{2}\}$. Hence, if $k-l$ is odd, we get

$$\#((rC_2+sT)\cap\mathbb{Z}^2) = -\frac{1}{2}(l^2+k^2+1) + 3kl + k + l + 1,$$

and if $k-l$ is even, we get in both cases

$$\#((rC_2+sT)\cap\mathbb{Z}^2) = -\frac{1}{2}(l^2+k^2) + 3kl + k + l + 1.$$

Now using $k = r+s-\{r+s\}$ and $l = 3r+s-\{3r+s\}$,

$$\begin{aligned}
\#((rC_2+sT)\cap\mathbb{Z}^2) = 4r^2 + 8rs + 2s^2 + r(-8\{r+s\}+4) \\
&+ s(-2\{3r+s\} - 2\{r+s\} + 2) - \frac{1}{2}\{3r+s\}^2 - \frac{1}{2}\{r+s\}^2 \\
&+ 3\{3r+s\}\{r+s\} - \{r+s\} - \{3r+s\} + 1 \\
&- \begin{cases} \frac{1}{2}, & \text{if } \{3r+s\}-\{r+s\}-2r \text{ odd}, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

which gives the coefficients as functions in $\{r+s\}$ and $\{3r+s\}$:

$$\begin{aligned}
\mathrm{Q}_{(2,0)}(C_2,T,r,s) &= 4,\\
\mathrm{Q}_{(1,1)}(C_2,T,r,s) &= 8,\\
\mathrm{Q}_{(0,2)}(C_2,T,r,s) &= 2,\\
\mathrm{Q}_{(1,0)}(C_2,T,r,s) &= -8\{r+s\}+4,\\
\mathrm{Q}_{(0,1)}(C_2,T,r,s) &= -2\{3r+s\}-2\{r+s\}+2,\\
\mathrm{Q}_{(0,0)}(C_2,T,r,s) &= -\frac{1}{2}\left(\{3r+s\}^2+\{r+s\}^2\right)+3\{3r+s\}\{r+s\}-\{r+s\}\\
&\quad -\{3r+s\}+1-\begin{cases}\frac{1}{2}, & \text{if } \{3r+s\}-\{r+s\}-2r \text{ odd},\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

To write the coefficients as functions in $\{r\}$ and $\{s\}$, we use the division of $[0,1)^2 \cap \mathbb{Q}$ as in Figure 3.10. For $\{r\},\{s\} \in [0,1)$ we have $3\{r\}+\{s\} \in [0,4)$ and $\{r\}+\{s\} \in [0,2)$.

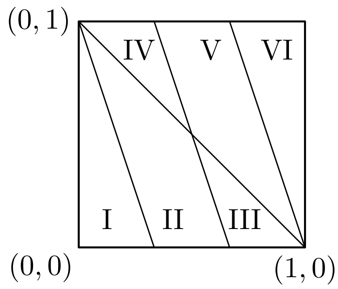

Figure 3.10: The space of parameters in $(\{r\},\{s\})$.

The situation in every section I to VI is listed in Table 3.2. Using this, we get for the

	$3\{r\}+\{s\}$	$\{r\}+\{s\}$	$\{3r+s\}$	$\{r+s\}$
I	$\in [0,1)$	$\in [0,1)$	$3\{r\}+\{s\}$	$\{r\}+\{s\}$
II	$\in [1,2)$	$\in [0,1)$	$3\{r\}+\{s\}-1$	$\{r\}+\{s\}$
III	$\in [2,3)$	$\in [0,1)$	$3\{r\}+\{s\}-2$	$\{r\}+\{s\}$
IV	$\in [1,2)$	$\in [1,2)$	$3\{r\}+\{s\}-1$	$\{r\}+\{s\}-1$
V	$\in [2,3)$	$\in [1,2)$	$3\{r\}+\{s\}-2$	$\{r\}+\{s\}-1$
VI	$\in [3,4)$	$\in [1,2)$	$3\{r\}+\{s\}-3$	$\{r\}+\{s\}-1$

Table 3.2: Situation in Sections I to VI.

coefficients as functions in $\{r\}$ and $\{s\}$:

$$
\begin{aligned}
&\mathrm{Q}_{(2,0)}(C_2,T,r,s) = 4,\\
&\mathrm{Q}_{(1,1)}(C_2,T,r,s) = 8,\\
&\mathrm{Q}_{(0,2)}(C_2,T,r,s) = 2,\\
&\mathrm{Q}_{(1,0)}(C_2,T,r,s) = \begin{cases} -8\{r\} - 8\{s\} + 4, & \text{if } (\{r\},\{s\}) \in I, II, III,\\ -8\{r\} - 8\{s\} + 12, & \text{if } (\{r\},\{s\}) \in IV, V, VI, \end{cases}\\
&\mathrm{Q}_{(0,1)}(C_2,T,r,s) = \begin{cases} -8\{r\} - 4\{s\} + 2, & \text{if } (\{r\},\{s\}) \in I,\\ -8\{r\} - 4\{s\} + 4, & \text{if } (\{r\},\{s\}) \in II,\\ -8\{r\} - 4\{s\} + 6, & \text{if } (\{r\},\{s\}) \in III, IV,\\ -8\{r\} - 4\{s\} + 8, & \text{if } (\{r\},\{s\}) \in V,\\ -8\{r\} - 4\{s\} + 10, & \text{if } (\{r\},\{s\}) \in VI, \end{cases}\\
&\mathrm{Q}_{(0,0)}(C_2,T,r,s) = \begin{cases} 4\{r\}^2 + 2\{s\}^2 + 8\{r\}\{s\} - 4\{r\} - 2\{s\} + 1, & \text{if } (\{r\},\{s\}) \in I,\\ 4\{r\}^2 + 2\{s\}^2 + 8\{r\}\{s\} - 4\{r\} - 4\{s\} + 1, & \text{if } (\{r\},\{s\}) \in II,\\ 4\{r\}^2 + 2\{s\}^2 + 8\{r\}\{s\} - 4\{r\} - 6\{s\} + 1, & \text{if } (\{r\},\{s\}) \in III,\\ 4\{r\}^2 + 2\{s\}^2 + 8\{r\}\{s\} - 12\{r\} - 6\{s\} + 5, & \text{if } (\{r\},\{s\}) \in IV,\\ 4\{r\}^2 + 2\{s\}^2 + 8\{r\}\{s\} - 12\{r\} - 8\{s\} + 7, & \text{if } (\{r\},\{s\}) \in V,\\ 4\{r\}^2 + 2\{s\}^2 + 8\{r\}\{s\} - 12\{r\} - 10\{s\} + 9, & \text{if } (\{r\},\{s\}) \in VI. \end{cases}
\end{aligned}
$$

3.6 A generalization

For this section let $P_A(b) := \{x \in \mathbb{R}^n : Ax \le b\}$ for an integral $(m \times n)$-matrix A and a rational vector $b \in \mathbb{Q}^m$. We want to generalize the statements in Sections 3.2 and 3.3 to the problem of counting the number of lattice points in $P_A(b)$ as a function in b for a given matrix A.

Let $A \in \mathbb{Z}^{m\times n}$ such that $P_A(b)$ is bounded for all $b \in \mathbb{Q}^m$, that is, $\operatorname{cone}(A^\top) = \mathbb{R}^n$. We denote the number of lattice points by

$$\Phi(A,b) := \#(P_A(b) \cap \mathbb{Z}^n), \quad b \in \mathbb{Q}^m\,.$$

Since we cannot expect uniform behavior of $\Phi(A,b)$ when the polytope $P_A(b)$ changes combinatorically, we consider subsets of $\mathbb{Q}^m$ on which the combinatorial structure of $P_A(\cdot)$ is constant. For this, we consider the possible normal fans of $P_A(b)$.

For a fixed vertex v of a polytope $P_A(b)$, the *normal cone* τ_v of v is the set of all directions $u \in \mathbb{R}^n$, such that the function $x \mapsto u^\top x$ is maximized on $P_A(b)$ by v. That is, τ_v is the positive hull of all those rows i of A, denoted by A^i, such that $A^i v = b_i$. If useful, we identify τ_v with the set of all indices $\{i = 1, \dots, m : A^i v = b_i\}$. By the definition of vertices as 0-faces, the normal cone τ_v of a vertex v is full-dimensional.

We call the set of the normal cones of all vertices of $P_A(b)$ the *normal fan*, denoted by $N_A(b)$. Observe that this differs from the usual notion of the normal fan, which is a polyhedral subdvision of $\mathbb{R}^n$ and hence contains also lower dimensional normal cones. In our case it is enough to consider only the maximal cells. Still, the union of all normal cones in $N_A(b)$ is $\mathbb{R}^n$, and the interiors of two normal cones in $N_A(b)$ do not intersect. We refer to Ziegler [47, Chapter 7] for an introduction to polyhedral fans.

For a given matrix A, there are only finitely many possible normal fans and the normal fan fixes the combinatorial structure of $P_A(b)$. Hence, in the following we will always fix the normal fan N. For a fixed normal fan N, let $C_N \in \mathbb{Q}^m$ be the set of all vectors b such that $N_A(b) = N$. The set $\{C_N : N = N_A(b) \text{ for some } b \in \mathbb{Q}^m\}$ is denoted by $\mathcal{C}_A$.

As in Chapter 2 we denote by A^I the submatrix with rows indexed by elements of I. If $I = \{i\}$ consists of one single element, A^I will also be denoted by A^i.

3.46 Lemma
Every $C \in \mathcal{C}_A$ is a polyhedral cone.

Proof
Let $N = \{\tau_1, \dots, \tau_k\}$, such that $C = C_N$, with $\tau_j = \{i_{(j,1)}, \dots, i_{(j,l_j)}\}$, and let $\sigma_j \subset \tau_j$, such that A^{σ_j} is invertible. We claim that

$$\begin{aligned} C = \{b \in \mathbb{Q}^m : {} & A\,(A^{\sigma_j})^{-1}\, b_{\sigma_j} \le b, \\ & A^{i_{(j,h)}}\,(A^{\sigma_j})^{-1}\, b_{\sigma_j} = b_{i_{(j,h)}}, \\ & \text{for all } h = 1, \dots, l_j,\ j = 1, \dots, k\} =: M, \end{aligned}$$

which immediately implies the assertion, since the right-hand side is a set of vectors fulfilling some homogeneous linear inequalities.

Let $b \in C$. Since the normal fan of C is N, there are vertices $v_1, \dots, v_k$ such that the normal cone $\tau_{v_j} = \tau_j$ and thus $A^{i_{(j,h)}} v_j = b_{i_{(j,h)}}$. Since A^{σ_j} is invertible, v_j can be written as $(A^{\sigma_j})^{-1}\, b_{\sigma_j}$, which shows that $b \in M$.

Conversely, let $b \in M$ and let $v_j := (A^{\sigma_j})^{-1}\, b_{\sigma_j}$. Then, since v_j is the intersection of n linear independent inequalities on $P_A(b)$ and since $Av_j \le b$, v_j is a vertex of $P_A(b)$. By $A^{i_{(j,h)}} v_j = b_{i_{(j,h)}}$ we get that $\tau_{v_j} \subset \tau_j$, and since the union of all τ_j is $\mathbb{R}^n$ and their interiors do not intersect, τ_{v_j} must be equal to τ_j. Therefore, the normal fan of $P_A(b)$ cannot contain other normal cones than $\tau_1, \dots, \tau_k$. Hence $N_A(b) = N$, and thus $b \in C$. $\square$

In general $P_A(b) + P_A(c) \subset P_A(b + c)$ for $b, c \in \mathbb{Q}^m$, but for $C \in \mathcal{C}_A$ we have more:

3.47 Lemma (see Mount, [36, Theorem 2])
Let $C \in \mathcal{C}_A$. For $b, c \in C$ we have that $P_A(b) + P_A(c) = P_A(b+c)$.

Proof
Let $b, c \in C$. Then, by Lemma 3.46, the normal fans of $P_A(b)$, $P_A(c)$ and $P_A(b+c)$ are equal. Let $N = \{\tau_1, \ldots, \tau_k\}$ be the normal fan of these three polytopes and τ_j as in the proof of Lemma 3.46. Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ be the vertices of $P_A(b)$ and $P_A(c)$, respectively. Then $A^{i_{(j,h)}} v_j = b_{i_{(j,h)}}$ and $A^{i_{(j,h)}} w_j = c_{i_{(j,h)}}$ and thus $A^{i_{(j,h)}}(v_j + w_j) = b_{i_{(j,h)}} + c_{i_{(j,h)}}$ for all $h = 1, \ldots, l_j$, $j = 1, \ldots, k$. This implies that $v_j + w_j$ are vertices of $P_A(b+c)$. Furthermore $P_A(b+c) = \operatorname{conv}\{v_j + w_j : j = 1, \ldots, k\} \subset P_A(b) + P_A(c)$, since $P_A(b+c)$ has exactly k vertices with normal cones $\tau_1, \ldots, \tau_k$. $\square$

Using Lemma 3.47, Theorem 3.39, and Theorem 3.44 we get a structural result for $\Phi_A(b)$.

Dahmen and Micchelli, 1988, [16, Theorem 3.1] gave a structural result for

$$\Phi_A^{=}(b) := \#(\{x \in \mathbb{R}^n : Ax = b\} \cap \mathbb{Z}^n)$$

for a fixed matrix A and suitable b also inside certain cones of $\mathbb{R}^m$ if b is integral. As a corollary [16, Corollary 3.1], they get that $\Phi_A^{=}(\cdot)$ is a polynomial in these cones, if A is a unimodular matrix. Sturmfels, 1995, [44] gave a formula on how the polynomials differ inside these cones, if A is not unimodular. The works make use of the theory of polyhedral splines and representation techniques of groups.

Mount, 1998, [36] described methods for actually calculating the polynomials and cones, if A is unimodular and b integral. To this end, Mount gave an alternative argument for [16, Corollary 3.1] which we also follow in the proof of the next theorem.

Beck [4, 5] gave an elementary proof of the quasi-polynomiality of $\Phi_A(b)$. He also gave an Ehrhart–Macdonald reciprocity law for multi-dimensional Ehrhart quasi-polynomials.

3.48 Theorem
Let $C \in \mathcal{C}_A$ and let $\operatorname{cone}\{h_1, \ldots, h_m\}$ be a cone in a triangulation of C with $H = (h_1, \ldots, h_m)$ invertible. Then $\Phi(A, b)$ is a quasi-polynomial function in b, that is,

$$\Phi(A, b) = \sum_{\substack{I \in \{0, \ldots, n\}^k \\ |I|_1 \leq n}} \Phi_I(A, b) b^I,$$

where $\Phi_I(A, b) = \Phi_I(A, b + \widehat{\operatorname{den}}(P_A(h_i)) h_i)$ for all I. Furthermore, $\Phi_I(A, b)$ is a piecewise-defined polynomial of total degree $n - |I|_1$ in b.

PROOF
Using Lemma 3.47 and Theorem 3.39, we get the number of lattice points in $P_A(b)$:

$$\begin{aligned}\Phi(A,b) &= \#(P_A(b)\cap\mathbb{Z}^n) = \#\left(P_A\left(\sum_{i=1}^m r_i h_i\right)\cap\mathbb{Z}^n\right) = \#\left(\sum_{i=1}^m r_i P_A(h_i)\cap\mathbb{Z}^n\right)\\
&= \mathrm{Q}(P_A(h_1),\ldots,P_A(h_n),r) = \sum_{\substack{I\in\{0,\ldots,n\}^k\\ |I|_1\le n}} \mathrm{Q}_I(P_A(h_1),\ldots,P_A(h_m),r)r^I\\
&= \sum_{\substack{I\in\{0,\ldots,n\}^k\\ |I|_1\le n}} \mathrm{Q}_I(P_A(h_1),\ldots,P_A(h_m),H^{-1}b)(H^{-1}b)^I\\
&= \sum_{\substack{I\in\{0,\ldots,n\}^k\\ |I|_1\le n}} \Phi_I(A,b)b^I,\end{aligned}$$

where $\Phi_I(A,b)$ is a linear combination of $\mathrm{Q}_J(P_A(h_1),\ldots,P_A(h_m),H^{-1}b)$ for $|J|_1=|I|_1$. Thus there are $\lambda_J\in\mathbb{Q}$ for $J\in\{0,\ldots,n\}^k$ such that

$$\begin{aligned}\Phi_I(A,b) &= \sum_{\substack{J\in\{0,\ldots,n\}^k\\ n\ge|J|_1\ge|I|_1}} \lambda_J\,\mathrm{Q}_J(P_A(h_1),\ldots,P_A(h_m),H^{-1}b)\\
&= \sum_{\substack{J\in\{0,\ldots,n\}^k\\ n\ge|J|\ge|I|_1}} \lambda_J\,\mathrm{Q}_J(P_A(h_1),\ldots,P_A(h_m),H^{-1}b+\widehat{\mathrm{den}}(P_A(h_i))e_i)\\
&= \sum_{\substack{J\in\{0,\ldots,n\}^k\\ n\ge|J|_1\ge|I|_1}} \lambda_J\,\mathrm{Q}_J(P_A(h_1),\ldots,P_A(h_m),H^{-1}(b+\widehat{\mathrm{den}}(P_A(h_i))h_i))\\
&= \Phi_I(A,b+\widehat{\mathrm{den}}(P_A(h_i))h_i).\end{aligned}$$

Since $\mathrm{Q}_J(P_A(h_1),\ldots,P_A(h_m),H^{-1}b)$ is a polynomial of total degree $n-|J|_1=n-|I|_1$ in $H^{-1}b$, $\Phi_I(A,b)$ is a polynomial of total degree $n-|I|_1$ in b. □

3.49 EXAMPLE
Consider the simplex

$$T(r,s,t)=\{x\in\mathbb{R}^2:\quad \begin{array}{rrl} -x_1 & & \le r,\\ & -x_2 & \le s,\\ x_1 & +x_2 & \le t\} \end{array}$$

If $r+s+t<0$ then $T(r,s,t)$ is empty, and if $r+s+t=\epsilon>0$ then $(-r,-s)^\top$, $(-r+\epsilon,-s)^\top$ and $(-r,-s+\epsilon)^\top$ are contained in $T(r,s,t)$ and thus $T(r,s,t)$ is a triangle. The only closed cone of right-hand sides, where the combinatorial structure of $T(r,s,t)$ does not change and $T(r,s,t)$ is not empty, is $C = \{(r,s,t)^\top \in \mathbb{Q}^3 : r+s+t \geq 0\}$. For $(r,t,s)^\top \in C$ the triangle looks like in Figure 3.11.

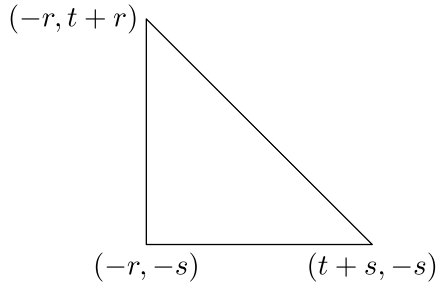

Figure 3.11: $T(r,s,t)$.

The number of integral points in $T(r,s,t)$ is the following:

$$\#(T(r,s,t)\cap\mathbb{Z}^2) = \sum_{x_1=\lceil -r\rceil}^{\lfloor t+s\rfloor} \lfloor t-x_1\rfloor - \lceil -s\rceil + 1 = \sum_{x_1=-\lfloor r\rfloor}^{\lfloor t+s\rfloor} \lfloor t\rfloor + \lfloor s\rfloor + 1 - x_1,$$

which expands to

$$\begin{aligned}\#(T(r,s,t)\cap\mathbb{Z}^2) =& \frac{1}{2}r^2+\frac{1}{2}s^2+\frac{1}{2}t^2+rs+rt+st+r\left(\frac{3}{2}-\{r\}-\{s\}-\{t\}\right)\\ &+s\left(\frac{3}{2}-\{r\}-\{s\}-\{t\}\right)+t\left(\frac{3}{2}-\{r\}-\{s\}-\{t\}\right)\\ &+\Bigg(1-\frac{3}{2}\{r\}-\{s\}-\{t\}+\{r\}\{s\}+\{r\}\{t\}+\frac{1}{2}\{r\}^2\\ &\quad+\{t+s\}\left(-\frac{1}{2}+\{s\}+\{t\}\right)-\frac{1}{2}\{t+s\}^2\Bigg).\end{aligned}$$

Furthermore, $\{t+s\}\in\{\{s\}+\{t\},\{s\}+\{t\}+1\}$, and in both cases

$$\{t+s\}\left(-\frac{1}{2}+\{s\}+\{t\}\right)-\frac{1}{2}\{t+s\}^2=\frac{1}{2}\{s\}^2+\frac{1}{2}\{t\}^2+\{s\}\{t\}-\frac{1}{2}\{s\}-\frac{1}{2}\{t\}$$

and thus

$$\begin{aligned}\#(T(r,s,t)\cap\mathbb{Z}^2) =& \frac{1}{2}r^2+\frac{1}{2}s^2+\frac{1}{2}t^2+rs+rt+st+r\left(\frac{3}{2}-\{r\}-\{s\}-\{t\}\right)\\ &+s\left(\frac{3}{2}-\{r\}-\{s\}-\{t\}\right)+t\left(\frac{3}{2}-\{r\}-\{s\}-\{t\}\right)\\ &+\Bigg(1-\frac{3}{2}\{r\}-\frac{3}{2}\{s\}-\frac{3}{2}\{t\}+\{r\}\{s\}+\{r\}\{t\}+\{s\}\{t\}\\ &\quad+\frac{1}{2}\{r\}^2+\frac{1}{2}\{s\}^2+\frac{1}{2}\{t\}^2\Bigg).\end{aligned}$$

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List of Symbols

$[a, b]$	convex hull of a and b	9
$\{\cdot\}$	fractional part function	72
$\lfloor\cdot\rfloor$	floor function	72
$\lceil\cdot\rceil$	ceiling function	72
$\odot$	componentwise multiplication	80
$\mathbb{0}$	origin	9
$\mathbb{1}$	$(1, \ldots, 1)^\top$	9
$\mathrm{aff}(M)$	affine hull of M	9
$\mathrm{cone}(M)$	positive hull of M	9
A^I	submatrix with rows I	29, 91
A_I	submatrix with columns I	29
$\mathrm{bd}(M)$	boundary	10
B^n	n-dimensional Euclidean unit ball	12
$\mathrm{conv}(M)$	convex hull of M	9
$C_n^\star$	n-dimensional crosspolytope	16
$\mathrm{den}(P)$	denominator of P	56
$\widehat{\mathrm{den}}(P)$	rational denominator of polytope P	65
$\dim(M)$	dimension	10
$\mathrm{d}_i(P)$	index of polytope P	59
$\widehat{\mathrm{d}}_i(P)$	rational index of polytope P	65
e_1	ith coordinate unit vector	9
f'	first derivative of a differentiable function f	67
$\mathrm{g}(P)$	minimal period of Ehrhart quasi-polynomial	58
$\mathrm{g}_i(P)$	minimal period of i-th Ehrhart coefficient	58
G	lattice point enumerator for integral lattice	11
G_Λ	lattice point enumerator	11
$\mathrm{G}(P, \cdot)$	Ehrhart (quasi-)polynomial of P w.r.t. lattice $\mathbb{Z}^n$	17, 57
$\mathrm{G}_\Lambda(P, \cdot)$	Ehrhart (quasi-)polynomial of P w.r.t. lattice Λ	17
$\mathrm{G}_i(P)$, $\mathrm{G}_i(P, \cdot)$	i-th Ehrhart coefficient of P w.r.t. lattice $\mathbb{Z}^n$	17, 57
$\mathrm{G}_i(P, \Lambda)$	i-th Ehrhart coefficient of P w.r.t. lattice Λ	17
$\mathrm{H}(A)$	matrix such that $A \cdot \mathrm{H}(A) = I_n$	32
$\mathrm{int}(M)$	interior	10
I_n	$(n \times n)$ identity matrix	9

$\mathcal{K}^n$	set of convex bodies in $\mathbb{R}^n$	11
$\mathcal{K}_0^n$	set of $\mathbb{0}$-symmetric convex bodies in $\mathbb{R}^n$	11
$\lambda(Z)$	smallest $\lambda \geq 1$ with $C_n^\star \subseteq Z \subseteq \lambda C_n^\star$	46
$\lambda_i(K,)$	i-thsuccessive minimum of K with respect to $\mathbb{Z}^n$	12
$\lambda_i(K, \Lambda)$	i-thsuccessive minimum of K with respect to Λ	12
$\mathrm{lin}(M)$	linear hull of M	9
$\mathrm{maxdet}(n)$	maximal determinant of an $(n \times n)$-(± 1)-matrix	30
$\Phi(A, b)$	$\#(P_A(b) \cap \mathbb{Z}^n)$	90
$\Phi_I(A, b)$	coefficients of $\Phi(A, b)$	92
$\mathcal{P}(M)$	set of all polytopes with vertices in M	15
$\mathcal{P}^n$	set of polytopes in $\mathbb{R}^n$	14
$\mathcal{P}_{\mathbb{Q}}^n$	set of all polytopes with vertices in $\mathbb{Q}^n$	15
$\mathcal{P}_{\mathbb{Z}}^n$	set of all polytopes with vertices in $\mathbb{Z}^n$	15
$P_A(b)$	$\{x \in \mathbb{R}^n : Ax \leq b\}$	90
$\mathrm{Q}(P, \cdot)$	rational Ehrhart (quasi-)polynomial of P w.r.t. lattice $\mathbb{Z}^n$	61
$\mathrm{Q}_i(P, \cdot)$	i-th rational Ehrhart coefficient of P w.r.t. lattice $\mathbb{Z}^n$	61
$\mathrm{q}(P)$	minimal period of rational Ehrhart quasi-polynomial	65
$\mathrm{q}_i(P)$	minimal period of i-th rational Ehrhart coefficient	65
$\mathrm{Q}(P_1, \ldots, P_k, r)$	rational Ehrhart quasi-polynomial of several polytopes	78
$\mathrm{Q}_I(P_1, \ldots, P_k, r)$	rational Ehrhart coefficient of several polytopes	78
$\mathbb{R}^n$	n-dimensional Euclidean space	9
$\mathbb{R}^{k \times m}$	set of all $k \times m$-matrices with entries in $\mathbb{R}$	9
$\sigma_i(a_1, \ldots, a_n)$	i-th elementary symmetric function of $a_1, \ldots, a_n$	24
S^n	n-dimensional Euclidean unit sphere	12
T	simplex of parameters for set $\mathcal{Z}$	38, 42
T_n	n-dimensional standard simplex	16
$\mathrm{V}_I(P_1, \ldots, P_k)$	mixed volume of polytopes	79
$\mathrm{vol}(M)$	volume	10
x^y	multiindex monomial	78
$\mathcal{Z}$	set of zonotopes of minimal volume containing T_2 or $C_3^\star$	38, 42
$\mathbb{Z}^n$	integral lattice	11
$\mathrm{Z}(A)$	zonotope generated by A	32