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Complexity reduction of large-scale stochastic systems using linear quadratic Gaussian balancing

Tobias Damm [a], Martin Redmann [b,*]

[a] RPTU Kaiserslautern-Landau, Department of Mathematics, Gottlieb-Daimler-Straße 48, 67663 Kaiserslautern, Germany
[b] MLU Halle-Wittenberg, Institute of Mathematics, Theodor-Lieser-Str. 5, 06120 Halle (Saale), Germany

A B S T R A C T

In this paper, we consider a model reduction technique for stabilizable and detectable stochastic systems. It is based on a pair of Gramians that we analyze in terms of well-posedness. Subsequently, dominant subspaces of the stochastic systems are identified exploiting these Gramians. An associated balancing related scheme is proposed that removes unimportant information from the stochastic dynamics in order to obtain a reduced system. We show that this reduced model preserves important features like stabilizability and detectability. Additionally, a comprehensive error analysis based on eigenvalues of the Gramian pair product is conducted. This provides an a-priori criterion for the reduction quality which we illustrate in numerical experiments.

1. Introduction

Simulation and optimal control of high-dimensional stochastic processes is extremely challenging but of significant practical interest. Such processes occur for instance as solutions to spatially discretized stochastic partial differential equations. Therefore, it is vital to reduce the computational complexity when solving such large-scale stochastic differential equation (SDE) numerically. Different techniques for model order reduction of deterministic systems have been developed over the years and are well documented, e.g., in [1–3]. A prominent method, balanced truncation, is motivated by energy functionals and appropriate balancing of states. This is achieved via a pair of positive definite matrices, the observability and the reachability Gramian. Given these matrices, a state space transformation can be computed such that both Gramians are equal and diagonal. Then, those states are truncated that correspond to low output and high control energy. Advantages of this approach are the preservation of system properties such as stability and minimality as well as good error bounds, as has been proved for stable systems already in [4,5]. An extension to unstable systems using techniques from linear quadratic (LQ) control theory has been suggested by [6]. Under suitable conditions the reduced unstable system may be used in a low-order compensator to stabilize the original system.

In the current paper, we extend this idea further to stochastic systems. We build upon earlier work, in particular [7–9], where different versions of balanced truncation for asymptotically stable stochastic linear systems have been discussed. The situation is more complicated than in the deterministic setup, since frequency domain considerations are not possible and hence essential tools like transfer functions are not available. Also the duality principle of reachability and observability does not translate literally to the stochastic setup. Therefore, it is not immediate to find an appropriate pair of Gramians, as has been discussed in [8] for the stable case. This seems even more difficult in the unstable case, and we regard it as one of our main contributions in this paper to suggest such a pair. While our observability Gramian is given as the stabilizing solution of the Riccati equation associated to a LQ-state feedback problem of stochastic control, our reachability Gramian solves a modified Riccati type inequality. Both Gramians exist under natural stabilizability and detectability conditions. They can be computed, e.g., by semidefinite programming and yield a balancing state space transformation. Performing a balancing procedure in the usual way, we can show that the reduced system is

* Corresponding author.
E-mail addresses: damm@mathematik.uni-kl.de (T. Damm), martin.redmann@mathematik.uni-halle.de (M. Redmann).

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still stabilizable and detectable and that the truncated closed-loop LQ-controller of the full system also stabilizes the reduced system. Moreover, we prove error bounds for the closed-loop and the open-loop input–output system with respect to the L^2-norm. In both cases, non-trivial technical elaborations are required. Further, these results can be interpreted nicely in the gap metric as we point out in this paper as well.

The concepts used in our approach have been developed over many years. Fundamental results on stochastic stability can be found in [10]. Stochastic linear quadratic control theory and the stochastic Riccati equation were introduced in [11]. For its solution, notions of stochastic stabilizability and detectability are crucial. Different versions appeared in [12–19] and have been adapted for other classes of systems in more recent years. In this paper, we follow the definitions of detectability given in [12,19]. Low-order compensators for stochastic linear systems with multiplicative noise apparently have first been considered in [20] using H^∞-techniques. Model order reduction of discrete-time stochastic systems based on balancing was discussed in [21] using linear matrix inequalities. For continuous-time systems Gramian based methods were suggested in [7,8]. A balancing procedure in a Hilbert space setting can be found in [22]. We refer, e.g., to [23] for a dimension reduction scheme based on an averaging principle. Besides methods relying on Gramians, further recent developments in different directions have been made. In [24], optimization based model reduction was studied, whereas [25,26] focused on techniques based on moment matching and sampling, respectively.

The paper is now organized as follows. We first clarify the notation and provide some tools on stochastic systems, positive operators and system theoretic notions. In Section 3, we introduce the pair of Gramians and characterize them with energy cost functionals. Linear quadratic Gaussian (LQG) balanced truncation is discussed in Section 4, where also essential preservation properties are derived. The more technical results on error bounds are given in Section 5 and the appendix. Some numerical examples that illustrate and support our findings are given in Section 6.

2. Preliminaries

In this section, we introduce the class of stochastic systems for which we want to perform model order reduction by LQG balancing. To define suitable Gramians we consider the well-known Riccati equation of the stochastic linear quadratic control problem, e.g., from [11], and a new Riccati-type inequality which is inspired by the *type II*-Gramian defined in [8]. We also recall notions of stabilizability and detectability that are essential for the existence of the Gramians.

2.1. Basics of stochastic systems

We study the stochastic system

$$dx(t) = [Ax(t) + Bu(t)]\,dt + \sum_{i=1}^{q} N_i x(t)\,dW_i(t), \quad x(0) = x_0 \in \mathbb{R}^n, \tag{1a}$$

$$y(t) = Cx(t), \quad t \geq 0, \tag{1b}$$

where $A, N_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. The vector-valued functions x, u, and y are called state, control input, and measured output respectively. We assume that $W = \left(W_1, \dots, W_q\right)^\top$ is an $\mathbb{R}^q$-valued Wiener process with mean zero and covariance matrix $K = (k_{ij})$, i.e., $\mathbb{E}[W(t)W(t)^\top] = Kt$. All stochastic processes appearing in this paper are defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$[1]. Furthermore, we assume that W is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process with increments $W(t+h) - W(t)$ being independent of $\mathcal{F}_t$ for $t, h \geq 0$. Throughout this paper, suppose that u is an $(\mathcal{F}_t)_{t\geq 0}$-adapted control with $u \in L^2_T$, meaning that

$$\|u\|^2_{L^2_T} := \mathbb{E}\int_0^T \|u(s)\|_2^2\,ds < \infty \tag{2}$$

for all $T > 0$, where $\|\cdot\|_2$ denotes the Euclidean norm with associated inner product $\langle\cdot,\cdot\rangle_2$. If (2) additionally holds for $T = \infty$, we write $u \in L^2$. For given control u and initial state x_0, the corresponding state and output processes are denoted by $x(\cdot, x_0, u)$ and $y(\cdot, x_0, u)$.

Remark 1. There is the potential to extend the results of this paper to square integrable Lévy processes (see, e.g., [27]). We might also consider multiplicative noise at the input terms. This, however, makes many expressions and criteria more complicated, as we demonstrate below in Remark 5.

2.2. Resolvent positive mappings

In our analysis we will consider Lyapunov equations of a generalized type. In this context the following terminology and setup is useful, see [28]. Let H denote a finite dimensional real vector space ordered by a closed, solid, pointed convex cone H_+. A linear mapping $T : H \to H$ is called positive, if $T(H_+) \subset H_+$. It is called *resolvent positive*, if its resolvent $(\alpha I - T)^{-1}$ is positive for all sufficiently large α. The essential property of resolvent positive mappings that we use is a variant of the Perron–Frobenius theorem.

[1] $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and complete.

Proposition 2.1. *Let $T : H \to H$ be resolvent positive with spectrum $\sigma(T)$ and spectral abscissa $\alpha = \max\{\Re(\lambda) \mid \lambda \in \sigma(T)\}$. Then $\alpha \in \sigma(T)$ and there exists $X \in H_+$, $X \neq 0$, such that $T(X) = \alpha X$.*

In our context, we consider the space $\mathbb{S}^n = \{X \in \mathbb{R}^{n\times n} \mid X = X^\top\}$ of symmetric matrices. This space is endowed with the canonical Frobenius scalar product $\langle X_1, X_2\rangle_F = \operatorname{tr}(X_1X_2)$ and ordered by the closed, solid, pointed convex cone of nonnegative definite matrices $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n \mid X \geq 0\}$. We will use the following property.

$$\text{If } X_1, X_2 \in \mathbb{S}^n_+ \text{ then } X_1X_2 = 0 \iff \langle X_1, X_2\rangle_F = 0\,. \tag{3}$$

With the given coefficient matrices $A, N_i \in \mathbb{R}^{n\times n}$, $K = (k_{ij}) \in \mathbb{S}^q_+$ from the previous subsection we define the mappings $\mathcal{L}_A, \Pi_N : \mathbb{S} \to \mathbb{S}$ by

$$\mathcal{L}_A(X) = A^\top X + XA\,, \quad \Pi_N(X) = \sum_{i,j=1}^{q} N_i^\top X N_j\, k_{ij}.$$

Then, Π_N is positive and the sum $\mathcal{L}_A + \Pi_N$ is *resolvent positive*. The same obviously holds for the adjoint mappings

$$\mathcal{L}_A^*(X) = AX + XA^\top\,, \quad \Pi_N^*(X) = \sum_{i,j=1}^{q} N_i X N_j^\top\, k_{ij}.$$

2.3. Stabilizability, observability and detectability

We now introduce notions of stability, stabilizability, and detectability, as they have been considered, e.g., in [12,28].

Definition 2.2. The system (1) is called

- *mean square asymptotically stable*, if there exist constants $M \geq 1, c > 0$ such that for all $x_0 \in \mathbb{R}^n, t \geq 0$, we have $\mathbb{E}\,\|x(t, x_0, 0)\|_2^2 \leq M\,\mathrm{e}^{-ct}\,\|x_0\|_2^2$.
- *stabilizable*, if for all $x_0 \in \mathbb{R}^n$ there exists $u \in L^2$, such that $x(\cdot, x_0, u) \in L^2$.
- *observable*, if the condition that $y(t, x_0, 0) = 0$ almost surely for all $t \geq 0$ implies that $x_0 = 0$.
- *detectable*, if the condition that $y(t, x_0, 0) = 0$ almost surely for all $t \geq 0$ implies that $\lim_{t\to\infty} \mathbb{E}\,\|x(t, x_0, 0)\|_2^2 = 0$.

In these cases, we briefly say that the pair (A, N_i) is (mean square asymptotically) stable, the triple (A, B, N_i) is stabilizable, or the triple (A, C, N_i) is observable or detectable.

Remark 2. In [18] the term *exact observable* is used, where for brevity we just write *observable*. To be more precise, the notion of stability introduced in Definition 2.2 is called mean square exponential stability in general. Since usual mean square asymptotic stability implies exponentially fast decay in the linear case, we do not distinguish between both concepts and omit the term "exponential" in the following.

Unlike in the deterministic case there is no perfect duality between stabilizability and detectability. The following lemma collects known criteria.

Lemma 2.3.

(a) The triple (A, B, N_i) is stabilizable, if and only if there exists a feedback gain matrix F, such that $(A + BF, N_i)$ is stable.
(b) The triple (A, C, N_i) is observable, if and only if the following variant of the Hautus test is satisfied:

$$\text{If } (\mathcal{L}_A + \Pi_N)^*(V) = \lambda V \text{ with } \lambda \in \mathbb{C}, 0 \neq V \geq 0, \text{ then } CV \neq 0\,.$$

(c) The triple (A, C, N_i) is detectable, if and only if the following variant of the Hautus test is satisfied:

$$\text{If } (\mathcal{L}_A + \Pi_N)^*(V) = \lambda V \text{ with } \lambda \geq 0, 0 \neq V \geq 0, \text{ then } CV \neq 0\,.$$

(d) If the triple $(A^\top, C^\top, N_i^\top)$ is stabilizable, then the triple (A, C, N_i) is detectable. The converse does not hold in general.

3. A pair of Gramians

As in the deterministic case, stabilizability, observability and detectability characterize the solvability of Riccati equations.

3.1. An observability Gramian

We first consider the Riccati equation of the stochastic linear quadratic control problem. The following result is a special case of [19, Theorem 4.1] (see also [28, Corollary 5.3.4]).

Theorem 3.1. *Assume that (A, B, N_i) is stabilizable and (A, C, N_i) is detectable. Then, the Riccati equation*

$$\mathcal{R}(Q) := A^\top Q + QA + \sum_{i,j=1}^{q} N_i^\top Q N_j k_{ij} + C^\top C - QBB^\top Q = 0 \tag{4}$$

possesses a stabilizing solution $Q_+ \geq 0$, such that $(A - BB^\top Q_+, N_i)$ is stable.

If (A, C, N_i) is observable, then $Q_+ > 0$.

The stabilizing solution Q_+ of (4) will play the role of an observability Gramian in our LQG balanced truncation approach.

3.2. A reachability Gramian

The corresponding reachability Gramian will be chosen as a positive definite solution of the new Riccati-type inequality

$$A^\top P^{-1} + P^{-1}A + \sum_{i,j=1}^{q} N_i^\top P^{-1} N_j k_{ij} - C^\top C + P^{-1}BB^\top P^{-1} \leq 0\,. \tag{5}$$

Lemma 3.2. *The strict inequality*

$$A^\top P^{-1} + P^{-1}A + \sum_{i,j=1}^{q} N_i^\top P^{-1} N_j k_{ij} - C^\top C + P^{-1}BB^\top P^{-1} < 0 \tag{6}$$

possesses a solution $P_+ > 0$ if and only if $(A^\top, C^\top, N_i^\top)$ is stabilizable.

Proof. By [28, Lemma 1.7.3] stabilizability of $(A^\top, C^\top, N_i^\top)$ is equivalent to the existence of a matrix $X > 0$ such that

$$A^\top X + XA + \sum_{i,j=1}^{q} N_i^\top X N_i k_{ij} - C^\top C = -Y < 0\,. \tag{7}$$

Since (6) implies (7) it also implies stabilizability of $(A^\top, C^\top, N_i^\top)$.

For the converse implication, we multiply both sides of (7) by some $0 < \epsilon \leq 1$ and obtain

$$A^\top(\epsilon X) + (\epsilon X)A + \sum_{i,j=1}^{q} N_i^\top(\epsilon X)N_i k_{ij} - \epsilon C^\top C = -\epsilon Y.$$

The choice of $\epsilon \leq 1$ yields $-C^\top C \leq -\epsilon C^\top C$. Moreover, for sufficiently small ϵ, we have $-\epsilon Y < -(\epsilon X)BB^\top(\epsilon X)$. Consequently, the corresponding $P_+ := (\epsilon X)^{-1}$ is a positive definite solution to (6). □

Remark 3. As noted before, stabilizability of $(A^\top, C^\top, N_i^\top)$ is stronger than the more natural detectability of (A, C, N_i) and it is also not implied by observability of (A, C, N_i), [12]. However, in the extreme case, where $BB^\top$ is nonsingular, it is clear that (5) implies (7) and thus stabilizability of $(A^\top, C^\top, N_i^\top)$. In the following, we will make the assumptions that (A, B, N_i) is stabilizable, (A, C, N_i) is observable, and that (5) has a solution $P > 0$.

3.3. State cost estimations

In this section, we measure how much state variables contribute to system (1) based on the proposed Gramians.

Closed-loop dynamics. The relevance of state components with respect to the quadratic cost functional

$$J_T(x_0, u) = \int_0^T \mathbb{E}\left(\|u(t)\|_2^2 + \|y(t)\|_2^2\right)\,dt$$

is investigated. Let us first assume that $x_0 = 0$ and hence $x(t) = x(t, 0, u)$. Moreover, suppose that we have an orthonormal basis (p_i) of eigenvectors $P > 0$ such that we have the representation

$$x(t) = \sum_{i=1}^{n} \langle x(t), p_i\rangle_2\, p_i. \tag{8}$$

In order to tell how much a direction p_i contribute to the state variable, the coefficients $\langle x(t), p_i\rangle_2$ are analyzed below. Secondly, we investigate how much a state variable contributes to the output (and a feedback control). Since a state is fully determined by its initial condition, we focus on $x_0 = \sum_{i=1}^{n} \beta_i q_i$. Here, (q_i) is an orthonormal basis of eigenvectors of Q and $\beta_i \in \mathbb{R}$ and the respective coefficients of the expansion of x_0. We assume that the control has stabilizing feedback structure, i.e., $u(t, x_0) = -B^\top Q x(t, x_0)$. Since the state is linear in x_0 the same is true for the output $y = y(t, x_0)$ and the closed-loop control. Consequently, we have

$$y(t, x_0) = \sum_{i=1}^{n} \beta_i y(t, q_i) \quad \text{and} \quad u(t, x_0) = \sum_{i=1}^{n} \beta_i u(t, q_i). \tag{9}$$

Therefore, it is of interest to investigate how large $u(t, q_i)$ and $y(t, q_i)$ are. We establish the following proposition in order to characterize dominant subspaces.

Proposition 3.3.

(a) If $P > 0$ satisfies (5), and $x(t) = x(t, 0, u)$ is the solution of (1a) on $[0, T]$ with initial value $x(0) = 0$, then we have

$$\sup_{t\in[0,T]} \mathbb{E}\langle x(t), p_i\rangle_2^2 \leq \lambda_{P,i} J_T(0, u)$$

for the coefficients in (8), where $\lambda_{P,i}$ is the eigenvalue of P associated to p_i. If u is stabilizing, then the same holds for $T = \infty$.

(b) Let now $Q \geq 0$ satisfy (4), and consider the initial state $x_0 = q_i$ with stabilizing feedback input $u_F = Fx$ for $F = -B^\top Q$. Then,

$$J_\infty(q_i, u_F) = \|y(\cdot, q_i)\|_{L^2}^2 + \|u_F(\cdot, q_i)\|_{L^2}^2 \leq \lambda_{Q,i} \tag{10}$$

for the coefficients in (9), where $\lambda_{Q,i}$ is the eigenvalue of Q associated to q_i.

Remark 4. Note that each $u \in L^2$ is stabilizing in Proposition 3.3(a) if (1a) is mean square asymptotically stable [20]. We interpret the results of the proposition as follows. Assume that $\lambda_{P,i}$ and $\lambda_{Q,i}$ are very small. Then, from (a) we infer, that the state direction p_i can only be activated at very high cost. From (b) we learn that the initial variable direction q_i has only very small influence on the cost meaning that $u_F(\cdot, q_i)$ and $y(\cdot, q_i)$ are small in L^2. If P and Q are diagonal and equal, we obtain that $p_i = q_i$ are the unit vectors such that unimportant directions can be identified with state components. The ith component might then be neglected in a truncation approach if the associated diagonal entry in $P = Q$ is small. Such a simultaneous diagonalization of the Gramians is discussed in Section 3.4.

Proof of Proposition 3.3. For proving (a), we use (38) with $X = P^{-1}$ and (5) yielding

$$\begin{aligned}
&\mathbb{E}\langle x(t,0,u), p_i\rangle_2^2 \leq \lambda_{P,i}\, \mathbb{E}\left[x(t,0,u)^\top P^{-1} x(t,0,u)\right] \\
&\leq \lambda_{P,i}\left[\int_0^t \mathbb{E}\left[x(s)^\top \left(C^\top C - P^{-1}BB^\top P^{-1}\right) x(s)\right] ds + 2\int_0^t \mathbb{E}\left\langle B^\top P^{-1} x(s), u(s)\right\rangle_2 ds\right] \\
&= \lambda_{P,i}\left[\|y\|_{L_t^2}^2 + \|u\|_{L_t^2}^2 - \left\|B^\top P^{-1}x - u\right\|_{L_t^2}^2\right].
\end{aligned}$$

Consequently, we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\langle x(t,0,u), p_i\rangle_2^2 \leq \lambda_{P,i}\left[\|y\|_{L_T^2}^2 + \|u\|_{L_T^2}^2\right].$$

If $u, y \in L^2$, we can take the supremum over $t \in [0, \infty)$ instead. For (b), we set $X = Q$ in (38) and make use of (4) leading to

$$\begin{aligned}
\mathbb{E}\left[x(t)^\top Q x(t)\right] &= x_0^\top Q x_0 + \int_0^t \mathbb{E}\left[x(s)^\top \left(-C^\top C + QBB^\top Q\right) x(s)\right] ds + 2\int_0^t \mathbb{E}\left\langle B^\top Q x(s), u(s)\right\rangle_2 ds \\
&= x_0^\top Q x_0 + \int_0^t \mathbb{E}\left[-\|y(s)\|_2^2 - \|u(s)\|_2^2 + \left\|B^\top Q x(s) + u(s)\right\|_2^2\right] ds
\end{aligned} \tag{11}$$

for $t \in [0, T]$. Setting $u = u_F$ and $x_0 = q_i$ gives us

$$\|y(\cdot, q_i)\|_{L_T^2}^2 + \|u_F(\cdot, q_i)\|_{L_T^2}^2 \leq \lambda_{Q,i}.$$

Since u_F is a stabilizing control according to Theorem 3.1, the result follows by taking $T \to \infty$. □

Remark 5. Using the Riccati mapping $\mathcal{R}$ from (4), we can write (5) in the form $-\mathcal{R}(-P^{-1}) \leq 0$. This indicates, how the Gramians can be generalized for the case of models with control-dependent noise. Given $M_i \in \mathbb{R}^{n\times m}$, let us consider the following system with controlled diffusion

$$\begin{aligned}
dx(t) &= [Ax(t) + Bu(t)]\,dt + \sum_{i=1}^q [N_i x(t) + M_i u(t)]\,dW_i(t), \quad x(0) = x_0, \\
y(t) &= Cx(t), \quad t \geq 0.
\end{aligned}$$

Then, the Riccati equation of LQ-control with cost functional J_∞ takes the form (see, e.g., [28,29])

$$\begin{aligned}
\mathcal{R}(Q) &= A^\top Q + QA + \sum_{i,j=1}^q N_i^\top Q N_j k_{ij} + C^\top C \\
&\quad - (QB + \sum_{i,j=1}^q N_i^\top Q M_j k_{ij})(I + \sum_{i,j=1}^q M_i^\top Q M_j k_{ij})^{-1}(B^\top Q + \sum_{i,j=1}^q M_i^\top Q N_j k_{ij}) = 0,
\end{aligned}$$

and defines the observability Gramian for this case. A reachability Gramian is given by

$$\begin{aligned}
&0 \geq -\mathcal{R}(-P^{-1}) \\
&= A^\top P^{-1} + P^{-1}A + \sum_{i,j=1}^q N_i^\top P^{-1} N_j k_{ij} - C^\top C
\end{aligned}$$

$$+ (P^{-1}B + \sum_{i,j=1}^{q} N_i^\top P^{-1} M_j k_{ij})(I - \sum_{i,j=1}^{q} M_i^\top P^{-1} M_j k_{ij})^{-1}(B^\top P^{-1} + \sum_{i,j=1}^{q} M_i^\top P^{-1} N_j k_{ij}),$$

if additionally

$$I - \sum_{i,j=1}^{q} M_i^\top P^{-1} M_j k_{ij} > 0. \tag{12}$$

Proposition 3.3 holds accordingly in this setup with $F = -(I + \sum_{i,j=1}^{q} M_i^\top Q M_j k_{ij})^{-1}(B^\top Q + \sum_{i,j=1}^{q} M_i^\top Q N_j k_{ij})$, but the additional constraint (12) on P causes further technical difficulties. Therefore, we prefer to consider only state-dependent noise.

Open-loop dynamics. Below, we discuss that the Gramian Q might not generally be suitable for the dominant subspace characterization of unstable (but stabilizable and detectable) systems. Let u now be an open-loop control. Since u then is independent of the (initial) state, we can neglect it in the considerations below by setting $u \equiv 0$, i.e., $y(t) = y(t, x_0, 0) = Cx(t, x_0, 0)$. Eq. (11) yields

$$\mathbb{E}\left[x(t)^\top Q x(t)\right] = a(t) + \int_0^t \left\|B^\top Q x(s)\right\|_2^2 ds, \quad t \in [0, T],$$

where $a(t) := x_0^\top Q x_0 - \|y\|_{L_t^2}^2$. Using $\|B^\top Q x(s)\|_2^2 \le b\, x(s)^\top Q x(s)$ with $b := \left\|B^\top Q^{\frac{1}{2}}\right\|_2^2$, we can apply Gronwall's lemma, see Lemma A.1. Setting $t = T$ we then obtain

$$\begin{aligned}\mathbb{E}\left[x(T)^\top Q x(T)\right] &= a(T) + \int_0^T a(s) b\, e^{b(T-s)}\, ds = a(T) - \left[a(s)\, e^{b(T-s)}\right]_{s=0}^T + \int_0^T \dot{a}(s)\, e^{b(T-s)}\, ds \\ &= x_0^\top Q x_0\, e^{bT} - \int_0^T \|y(s)\|_2^2\, e^{b(T-s)}\, ds. \end{aligned} \tag{13}$$

We have $y(t, x_0, 0) = \sum_{i=1}^n \beta_i y(t, q_i, 0)$, $t \in [0, T]$. We obtain from (13) that the ith summand of this expansion satisfies

$$\int_0^T e^{-bs} \|y(t, q_i, 0)\|_2^2\, ds \le \lambda_{Q,i}. \tag{14}$$

Inequality (14) is based on a Gronwall estimate that generally is not tight but captures the worst-case scenarios. Therefore, the exponential weight in (14) is an indicator that eigenspaces corresponding to small eigenvalues of Q might generally only be redundant in an unstable open-loop system on a small time scale. However, the kernel of Q remains negligible in any case. We proceed with a strategy to simultaneous diagonalize P and Q in order to be able to remove redundant information in (1a) and (1b) at the same time.

3.4. State space transformation and balancing

A transformation $z = Sx$ in (1) with nonsingular $S \in \mathbb{R}^{n\times n}$ leads to an equivalent stochastic system with the state vector z, where the coefficient matrices undergo the state-space transformation

$$(A, N_i, B, C) \mapsto (SAS^{-1}, SN_iS^{-1}, SB, CS^{-1}).$$

Both systems have the same input and output. Also, none of the properties from Definition 2.2 is affected. Matrices P and Q constitute a pair of Gramians for the original system, if and only if $SPS^\top$ and $S^{-\top}QS^{-1}$ constitute a pair of Gramians for the transformed system. By the spectral transformation theorem for symmetric matrices, there exist orthogonal matrices S_P and S_Q, such that $S_P P S_P^\top = \Sigma_P$ and $S_Q^{-\top} Q S_Q^{-1} = \Sigma_Q$ are diagonal and contain the ordered eigenvalues of P and Q, respectively.

Given that $P, Q > 0$ it is possible to conduct a balancing procedure, where one computes a nonsingular (but not necessarily orthogonal) transformation matrix S_b, so that $S_b P S_b^\top = S_b^{-\top} Q S_b^{-1} = \Sigma_n$ is diagonal, with $\Sigma_n^2 = \mathrm{diag}(\sigma_1^2, \dots, \sigma_n^2) > 0$ containing the ordered eigenvalues of PQ. One way of choosing the matrix S_b is to compute a Cholesky factorization of $P = L_P L_P^\top$ and then a spectral factorization of $L_P^\top Q L_P = U \Sigma_n^2 U^\top$ with orthogonal U, where Σ_n turns out to be the balanced Gramian. According to Remark 4, state components associated to small diagonal entries of Σ_n are less relevant in a balanced system. They can be removed due to their low contribution to the dynamics. This idea is the basis for the reduced model introduced in the next section.

4. LQG balanced truncation

Our standing assumption is that (A, B, N_i) is stabilizable, (A, C, N_i) is observable, and that (5) has a solution $P > 0$. Then, also (4) has a stabilizing solution $Q_+ > 0$. In this case, we can apply the balancing transformation S_b, leading to the balanced realization $(A_n, N_{i,n}, B_n, C_n) = (S_b A S_b^{-1}, S_b N_i S_b^{-1}, S_b B, C S_b^{-1})$ with diagonal Gramians $S_b P S_b^\top = S_b^{-\top} Q_+ S_b^{-1} = \Sigma_n = \mathrm{diag}(\Sigma_r, \Sigma_{2,n-r})$, where $\Sigma_r = \mathrm{diag}(\sigma_1, \dots, \sigma_r)$ contains the large and $\Sigma_{2,n-r} = \mathrm{diag}(\sigma_{r+1}, \dots, \sigma_n)$, $r < n$, the small singular values. The balanced system matrices are partitioned conformingly

$$A_n = \begin{bmatrix} A_r & \star \\ \star & \star \end{bmatrix}, \quad B_n = \begin{bmatrix} B_r \\ \star \end{bmatrix}, \quad C_n = \begin{bmatrix} C_r & \star \end{bmatrix}, \quad N_{i,n} = \begin{bmatrix} N_{i,r} & \star \\ \star & \star \end{bmatrix}. \tag{15}$$

Then, we consider the reduced system

$$dx_r(t) = \left[A_r x_r(t) + B_r u(t)\right] dt + \sum_{i=1}^{q} N_{i,r} x_r(t)\, dW_i(t), \quad x_r(0) = x_{0,r} \in \mathbb{R}^r, \tag{16a}$$

$$y_r(t) = C_r x_r(t), \quad t \geq 0. \tag{16b}$$

By Theorem 3.1 the closed-loop system $(A_n - B_n B_n^\top \Sigma_n, N_{i,n})$ is stable. We will show that the same holds for the reduced closed-loop system $(A_r - B_r B_r^\top \Sigma_r, N_{i,r})$, if $\sigma(\Sigma_r) \cap \sigma(\Sigma_{2,n-r}) = \emptyset$. Moreover, we prove that also detectability is preserved by truncation.

4.1. Preservation of closed-loop stability

We make use of a result in [8,30], which we restate here in a suitable form, see [8, Theorem II.2].

Theorem 4.1. *Let $(\hat{A}, \hat{N}_i, \hat{B}, \hat{C})$ be coefficient matrices with the same partitioning as in (15). Assume that $(\hat{A}, \hat{N}_i)$ is stable and consider the systems*

$$d\hat{x}(t) = \left[\hat{A}\hat{x}(t) + \hat{B}\hat{u}(t)\right] dt + \sum_{i=1}^{q} \hat{N}_i \hat{x}(t) dW_i(t), \quad \hat{y}(t) = \hat{C}\hat{x}(t), \tag{17a}$$

$$d\hat{x}_r(t) = \left[\hat{A}_r \hat{x}_r(t) + \hat{B}_r \hat{u}(t)\right] dt + \sum_{i=1}^{q} \hat{N}_{i,r} \hat{x}_r(t) dW_i(t), \quad \hat{y}_r(t) = \hat{C}_r \hat{x}_r(t). \tag{17b}$$

Let further $\hat{\Sigma} = \operatorname{diag}(\hat{\Sigma}_r, \hat{\Sigma}_{2,n-r})$ with $\sigma(\hat{\Sigma}_r) \cap \sigma(\hat{\Sigma}_{2,n-r}) = \emptyset$ satisfy

$$\left(\mathcal{L}_{\hat{A}} + \Pi_{\hat{N}}\right)(\hat{\Sigma}) \leq -\hat{C}^\top \hat{C} \text{ and } \left(\mathcal{L}_{\hat{A}} + \Pi_{\hat{N}}\right)(\hat{\Sigma}^{-1}) \leq -\hat{\Sigma}^{-1}\hat{B}\hat{B}^\top \hat{\Sigma}^{-1}. \tag{18}$$

Then, $(\hat{A}_r, \hat{N}_{i,r})$ is stable.

Remark 6. Matrices $\hat{\Sigma}$ satisfying (18) have been called *type-II-Gramians* of system $(\hat{A}, \hat{N}_i, \hat{B}, \hat{C})$ and (17b) the reduced model by type-II balancing, see [8].

Theorem 4.2. *Consider the systems (1) and (16) given by the data (15). Then, we have that $(A_r - B_r B_r^\top \Sigma_r, N_{i,r})$ is stable.*

Proof. We balance the system in order to work with the coefficient in (15). We add (4) and (5) with $P = Q = \Sigma_n$ to obtain

$$\begin{aligned}
0 &\geq \mathcal{L}_{A_n}(\Sigma_n + \Sigma_n^{-1}) + \Pi_{N_n}(\Sigma_n + \Sigma_n^{-1}) + \Sigma_n^{-1} B_n B_n^\top \Sigma_n^{-1} - \Sigma_n B_n B_n^\top \Sigma_n \\
&\geq \mathcal{L}_{A_n - B_n B_n^\top \Sigma_n}(\Sigma_n + \Sigma_n^{-1}) + \Pi_{N_n}(\Sigma_n + \Sigma_n^{-1}) + \Sigma_n^{-1} B_n B_n^\top \Sigma_n^{-1} - \Sigma_n B_n B_n^\top \Sigma_n \\
&\quad + \Sigma_n B_n B_n^\top \Sigma_n + \Sigma_n B_n B_n^\top \Sigma_n^{-1} + \Sigma_n^{-1} B_n B_n^\top \Sigma_n + \Sigma_n B_n B_n^\top \Sigma_n \\
&= (\mathcal{L}_{A_n - B_n B_n^\top \Sigma} + \Pi_{N_n})(\Sigma_n + \Sigma_n^{-1}) + (\Sigma_n + \Sigma_n^{-1}) B_n B_n^\top (\Sigma_n + \Sigma_n^{-1}).
\end{aligned}$$

Let us set $Y_n = (\Sigma_n + \Sigma_n^{-1})^{-1}$. Then, we have the two inequalities

$$\begin{aligned}
&(\mathcal{L}_{A_n - B_n B_n^\top \Sigma_n} + \Pi_{N_n})(Y_n^{-1}) \leq -Y_n^{-1} B_n B_n^\top Y_n^{-1}, \\
&(\mathcal{L}_{A_n - B_n B_n^\top \Sigma_n} + \Pi_{N_n})(\Sigma_n) \leq -C_n^\top C_n - \Sigma_n B_n B_n^\top \Sigma_n.
\end{aligned}$$

We recognize Y_n and Σ_n as unbalanced type-II Gramians of the closed-loop system given by $\left(A_n - B_n B_n^\top \Sigma_n, N_{i,n}, B_n, \left[\begin{smallmatrix} -B_n^\top \Sigma_n \\ C_n \end{smallmatrix}\right]\right)$. These are balanced by the similarity transformation with

$$S_n = (Y_n^{-1}\Sigma_n)^{1/4} = \operatorname{diag}(S_r, S_{2,n-r}).$$

For the given Y_n, the balanced type-II Gramian of the closed-loop system then equals

$$\hat{\Sigma}_n = (I + \Sigma_n^{-2})^{-1/2} = \operatorname{diag}\left(\frac{\sigma_j}{(1 + \sigma_j^2)^{1/2}}\right)_{j=1}^{n} = \operatorname{diag}(\hat{\sigma}_j)_{j=1}^{n}. \tag{19}$$

Note that $\hat{\sigma}_j > \hat{\sigma}_k$, if and only if $\sigma_j > \sigma_k$. Hence $\sigma(\Sigma_r) \cap \sigma(\Sigma_{2,n-r}) = \emptyset$ implies $\sigma(\hat{\Sigma}_r) \cap \sigma(\hat{\Sigma}_{2,n-r}) = \emptyset$. Thus, the assumptions of Theorem 4.1 are satisfied with

$$(\hat{A}, \hat{N}_i, \hat{B}, \hat{C}) = (S_n(A_n - B_n B_n^\top \Sigma_n)S_n^{-1}, S_n N_{i,n} S_n^{-1}, S_n B_n, \left[\begin{smallmatrix} -B_n^\top \Sigma \\ C_n \end{smallmatrix}\right] S_n^{-1}), \tag{20a}$$

$$(\hat{A}_r, \hat{N}_{i,r}, \hat{B}_r, \hat{C}_r) = (S_r(A_r - B_r B_r^\top \Sigma_r)S_r^{-1}, S_r N_{i,r} S_r^{-1}, S_r B_r, \left[\begin{smallmatrix} -B_r^\top \Sigma_r \\ C_r \end{smallmatrix}\right] S_r^{-1}). \tag{20b}$$

The stability of $(A_r - B_r B_r^\top \Sigma_r, N_{i,r})$ now follows from Theorem 4.1. □

4.2. Preservation of detectability

Let us now show that the reduced system is also detectable.

Proposition 4.3. *If $\sigma(\Sigma_r) \cap \sigma(\Sigma_{2,n-r}) = \emptyset$, then $(A_r, C_r, N_{i,r})$ given by (15) is detectable.*

Proof. Let us consider the balanced realization with partition in (15), so that $P = Q = \Sigma_n$ in (4) and (5). In more detail, we partition $N_{i,n} = \begin{bmatrix} N_{i,r} & \star \\ M_{i,r} & \star \end{bmatrix}$ and define $\Pi_{M_r} : \mathbb{S}^{n-r} \to \mathbb{S}^r$ in analogy to Π_N by $\Pi_{M_r}(X) = \sum_{i,j=1}^q M_{i,r}^\top X M_{j,r} k_{ij}$. Then,

$$\left(\mathcal{L}_{A_r} + \Pi_{N_r}\right)(\Sigma_r^{-1}) \leq C_r^\top C_r - \Sigma_r^{-1} B_r B_r^\top \Sigma_r^{-1} - \Pi_{M_r}\left(\Sigma_{2,n-r}^{-1}\right), \tag{21}$$

$$\left(\mathcal{L}_{A_r} + \Pi_{N_r}\right)(\Sigma_r) = -C_r^\top C_r + \Sigma_r B_r B_r^\top \Sigma_r - \Pi_{M_r}\left(\Sigma_{2,n-r}\right). \tag{22}$$

Recall that $\langle \cdot, \cdot \rangle_F$ is the Frobenius inner product. Assume that $(A_r, N_{i,r}, C_r)$ is not detectable. Then, according to Lemma 2.3, there exist $\lambda \geq 0$, $V_1 \geq 0$, such that $C_r V_1 = 0$, i.e., $\langle C_r^\top C_r, V_1 \rangle_F = 0$ and

$$\left(\mathcal{L}_{A_r} + \Pi_{N_r}\right)^* (V_1) = \lambda V_1 .$$

The scalar products of (21), (22) with V_1 yield

$$\lambda \langle \Sigma_r^{-1}, V_1 \rangle_F \leq -\langle \Sigma_r^{-1} B_r B_r^\top \Sigma_r^{-1}, V_1 \rangle_F - \langle \Pi_{M_r}\left(\Sigma_{2,n-r}^{-1}\right), V_1 \rangle_F \leq 0, \tag{23}$$

$$\lambda \langle \Sigma_r, V_1 \rangle_F = \langle \Sigma_r B_r B_r^\top \Sigma_r, V_1 \rangle_F - \langle \Pi_{M_r}\left(\Sigma_{2,n-r}\right), V_1 \rangle_F. \tag{24}$$

From the inequality (23) it follows that $\lambda \leq 0$, i.e., $\lambda = 0$.

Without loss of generality, let us assume that $\sigma_{r+1} = \max\{\sigma_{r+1}, \ldots, \sigma_n\}$. Then, $\frac{\sigma_{r+1}^2}{\sigma_j} \geq \sigma_{r+1} \geq \sigma_j$ for $j = r+1, \ldots, n$, i.e., $\Upsilon = \sigma_{r+1}^2 \Sigma_{2,n-r}^{-1} - \Sigma_{2,n-r} \geq 0$. Subtracting (23) multiplied with σ_{r+1}^2 from (24) we obtain

$$0 \geq \langle \Sigma_r B_r B_r^\top \Sigma_r + \sigma_{r+1}^2 \Sigma_r^{-1} B_r B_r^\top \Sigma_r^{-1}, V_1 \rangle_F + \langle \Pi_{M_r}(\Upsilon), V_1 \rangle_F \geq 0 .$$

In particular, it holds that $B_r B_r^\top \Sigma_r V_1 = 0$ and therefore

$$0 = A_r V_1 + V_1 A_r^\top + \Pi_{N_r}(V_1) = (A_r - B_r B_r^\top \Sigma_r) V_1 + V_1 (A_r - B_r B_r^\top \Sigma_r)^\top + \Pi_{N_r}(V_1) ,$$

contradicting the stability of the reduced closed-loop system by Theorem 4.2. □

4.3. Reduced order controller

Given a reduced model of an unstable system, it is a natural idea to use it for stabilization. This has been discussed in [6] for deterministic systems. In the stochastic setup, the problem is even more involved, and we just sketch some questions.

Consider again the systems (1) and (16) given by the data (15). We partition the balancing transformation matrix as $S_b = \begin{bmatrix} S_{b,r}^\top \\ \star \end{bmatrix}$, where $S_{b,r}^\top$ contains the first r rows. The state x_r of the reduced system (16) approximately satisfies $x_r = S_{b,r}^\top x$. If a state feedback control $u = F_r x_r$ stabilizes (16), i.e., $(A_r - B_r F_r, N_{i,r})$ is stable, then we may choose $u = F_r S_{b,r}^\top x$ as a candidate to stabilize the original system. By Theorem 4.2 we can try $F_r = -B_r^\top \Sigma_r$. This choice is also natural as the LQG reduced systems is designed based on neglecting unimportant information in the original stabilizing feedback control, see Proposition 3.3(b). For that reason, the truncated singular values $\sigma_{r+1}, \ldots, \sigma_n$ are a good indicator for the stabilization by the reduced feedback. Unfortunately, we cannot give detailed a-priori estimates for suitable r, such that $u = -B_r^\top \Sigma_r S_{b,r}^\top x$ stabilizes (1). But, of course, we can check the closed-loop a posteriori for stability. This will be done in an example in Section 6.

Pursuing the idea further, we may also try to design a reduced dynamic compensator for (1). In our setup, this could proceed via the reduced observer system

$$dx_r(t) = \left[A_r x_r(t) + B_r u(t) + K_r (C_r x_r(t) - y(t))\right] dt + \sum_{i=1}^q N_{i,r} x_r(t)\, dW_i(t). \tag{25}$$

Setting $K_r = -\Sigma_r C_r^\top$ and $u = -B_r^\top \Sigma_r x_r$, the closed-loop system can be shown to be stable for $r = n$. For smaller r, stability may be checked a-posteriori. But there is a more serious problem with this approach, since the noise terms dW_i usually cannot be reproduced in the observer. Therefore, a thorough analysis would have to consider only the deterministic part of (25). We have not carried out any such work yet which is part of future studies.

5. Error analysis and its discussion

In this section, we begin with an overview on how the error analysis of LQG balancing is conducted in the deterministic case and address difficulties in using the same techniques in the stochastic setting. Subsequently, we provide error bounds for stochastic LQG balancing and show links to the deterministic gap metric analysis.

Deterministic case ($N_i = 0$ and deterministic control u). Given that $N_i = 0$, the error analysis between (1) and (16) is often conducted in the frequency domain. To do so, one applies the Laplace transformation to (1) and hence obtains $\mathbf{y} = \mathbf{G}\mathbf{u}$, where $\mathbf{u}$, $\mathbf{y}$ are the Laplace transforms of the input and the output, respectively, and $\mathbf{G}$ is the matrix-valued transfer function of the system. The difference between the full and the reduced system can now be measured based on $\mathbf{G} - \mathbf{G}_r$, where $\mathbf{G}_r$ is the reduced transfer function. A possible error norm can be the $\mathcal{H}_\infty$-norm defined by

$$\|\mathbf{G}\|_{\mathcal{H}_\infty} := \sup_{w\in\mathbb{R}} \|\mathbf{G}(\mathrm{i}w)\|_2 = \sup_{u\neq 0} \frac{\|y\|_{L^2}}{\|u\|_{L^2}}. \tag{26}$$

However, a more suitable error measure in the LQG balancing context is the so-called gap metric. An error analysis for different types of deterministic settings in this metric can be found in [31–33]. A possible definition of the gap metric relies on a normalized (right) coprime factorization of the transfer function, i.e., $\mathbf{G}(s) = \mathbf{N}(s)\mathbf{M}(s)^{-1}$. We refer to [31,33,34] for more details on this factorization. The normalized coprime factors $\mathbf{M}, \mathbf{N}$ can now be used to define the gap metric [35]:

$$\delta(\mathbf{G}, \mathbf{G}_r) := \max\left\{ \inf_{\Pi\in\mathcal{H}_\infty} \left\| \begin{bmatrix}\mathbf{M}_r\\ \mathbf{N}_r\end{bmatrix} - \begin{bmatrix}\mathbf{M}\\ \mathbf{N}\end{bmatrix}\Pi \right\|_{\mathcal{H}_\infty}, \inf_{\Pi\in\mathcal{H}_\infty} \left\| \begin{bmatrix}\mathbf{M}\\ \mathbf{N}\end{bmatrix} - \begin{bmatrix}\mathbf{M}_r\\ \mathbf{N}_r\end{bmatrix}\Pi \right\|_{\mathcal{H}_\infty} \right\}.$$

A time-domain interpretation of this distance is, e.g., discussed in [33,34,36]. Given and L^2-input–output pair u and y, the gap metric guarantees the existence of a reduced L^2-pair u_r and y_r, so that we have

$$\left\| \begin{bmatrix} u - u_r \\ y - y_r \end{bmatrix} \right\|_{L^2} \leq \delta(\mathbf{G}, \mathbf{G}_r) \left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_{L^2}. \tag{27}$$

A bound for the gap metric is often found using the following estimate

$$\delta(\mathbf{G}, \mathbf{G}_r) \leq \left\| \begin{bmatrix}\mathbf{M}\\ \mathbf{N}\end{bmatrix} - \begin{bmatrix}\mathbf{M}_r\\ \mathbf{N}_r\end{bmatrix} \right\|_{\mathcal{H}_\infty}. \tag{28}$$

The $\mathcal{H}_\infty$-error in (28) can be determined based on the time-domain representation of this norm given in (26). This means, that we can work with system realizations of the transfer functions $\begin{bmatrix}\mathbf{M}\\ \mathbf{N}\end{bmatrix}$, $\begin{bmatrix}\mathbf{M}_r\\ \mathbf{N}_r\end{bmatrix}$ and compute the L^2-distance of two associated systems in order to find a bound for the gap metric. However, working with stochastic systems causes various issues since frequency-domain considerations cannot be applied. This is due to the fact that the "derivatives" in (1) are no longer classical functions not allowing for a Laplace transformation. Therefore, a gap metric study is not possible but our error analysis will rely on generalized system realizations of normalized coprime factorizations. In particular, a reduced input–output pair is supposed to be constructed, so that we find an estimate of the form given in (27).

Stochastic error analysis. In order to conduct a gap-metric type error analysis, we construct a pair u_r, y_r that is supposed to well approximate u, y. In order to show the error between both vectors, system (1) is rewritten. To be more precise, its input–output pair can be parameterized as

$$\begin{aligned} dx(t) &= [\bar{A}x(t) + Bv(t)]dt + \sum_{i=1}^{q} N_i x(t) dW_i(t), \\ \bar{y}(t) &:= \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \bar{C}x(t) + \begin{bmatrix} v(t) \\ 0 \end{bmatrix}, \quad t \geq 0, \end{aligned} \tag{29}$$

where $\bar{A} = A - BB^\top Q$, $\bar{C} = \begin{bmatrix} -B^\top Q \\ C \end{bmatrix}$ and $v(t) = B^\top Q x(t) + u(t)$. We can interpret (29) as a generalized realization (additional dW_i terms) of the coprime factors $\begin{bmatrix}\mathbf{M}\\ \mathbf{N}\end{bmatrix}$. In some way, (29) mimics an asymptotically mean square stable control system since an open-loop system with coefficients $(\bar{A}, N_i)$ is asymptotically mean square stable due to Theorem 3.1. However, v depends on the solution itself besides depending on u. On the other hand, $\bar{y}$ represents and input–output pair rather than an output. If $N_i = 0$, v and (29) are called driving-variable and driving-variable system, respectively. The relation between such driving-variable and input–output systems are nicely described in [37].

We investigate a particular input–output pair of the reduced system fixing control $u_r(t) = B_r^\top \Sigma_r x_r(t) + B^\top Q x(t) + u(t)$ ($\Sigma_r = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$), since this allows to rewrite the reduced model as

$$\begin{aligned} dx_r(t) &= [\bar{A}_r x_r(t) + B_r v(t)]dt + \sum_{i=1}^{q} N_{i,r} x_r(t) dW_i(t), \\ \bar{y}_r(t) &:= \begin{bmatrix} u_r(t) \\ y_r(t) \end{bmatrix} = \bar{C}_r x_r(t) + \begin{bmatrix} v(t) \\ 0 \end{bmatrix}, \quad t \geq 0, \end{aligned} \tag{30}$$

setting $\bar{A}_r = A_r - B_r B_r^\top \Sigma_r$ and $\bar{C}_r = \begin{bmatrix} -B_r^\top \Sigma_r \\ C_r \end{bmatrix}$. Again, (30) can be interpreted as generalized driving variable system or system realization of the reduced coprime factorization. The following theorem establishes an error between the original pair $\begin{bmatrix} u \\ y \end{bmatrix}$ and the chosen reduced pair $\begin{bmatrix} u_r \\ y_r \end{bmatrix}$. The result relies on L^2_T-error estimates between (29) and (30).

Theorem 5.1. *Let $u_r(t) = -B_r^\top \Sigma_r x_r(t) + B^\top Q x(t) + u(t)$ and y_r the reduced order output associated to this input. Given $x_0 = 0$ and $x_{0,r} = 0$, we have*

$$\left\| \begin{bmatrix} u - u_r \\ y - y_r \end{bmatrix} \right\|_{L^2_T} \leq 2 \sum_{k=r+1}^{n} \frac{\sigma_k}{\sqrt{1+\sigma_k^2}} \left(\left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_{L^2_T}^2 + \mathbb{E}\left[x(T)^\top Q x(T) \right] \right)^{\frac{1}{2}}. \tag{31}$$

If it, furthermore, holds that the input and the state are square integrable on $\Omega \times [0, \infty)$, i.e., $u, x \in L^2$, then we have

$$\left\| \begin{bmatrix} u - u_r \\ y - y_r \end{bmatrix} \right\|_{L^2} \leq 2 \sum_{k=r+1}^{n} \frac{\sigma_k}{\sqrt{1+\sigma_k^2}} \left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_{L^2}. \tag{32}$$

Proof. We improve the readability of this paper by moving the proof to Appendix B. □

As a consequence of Theorem 5.1, we observe that the singular values σ_k deliver a good a-priori criterion for the choice of r because removing only small singular values leads to a small bound for the error between the original and the reduced input–output pair. However, this argument is only valid if the (finite time) cost functional and, in case of (31), the terminal value $x(T)$ is not too large. The result in (32) is a gap-metric type estimate in the sense of (27). We formulate a special case of Theorem 5.1 for u being a stabilizing feedback control.

Corollary 5.2. *Let $u(t) = -B^\top Q x(t) + u^{(1)}(t)$ and $u_r(t) = -B_r^\top \Sigma_r x_r(t) + u^{(1)}(t)$ with $u^{(1)} \in L^2_T$. Given $x_0 = 0$ and $x_{0,r} = 0$, we have*

$$\left\| \begin{bmatrix} u - u_r \\ y - y_r \end{bmatrix} \right\|_{L^2_T} \leq 2 \sum_{k=r+1}^{n} \frac{\sigma_k}{\sqrt{1+\sigma_k^2}} \left\| u^{(1)} \right\|_{L^2_T}.$$

Proof. By (11), we have $\left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_{L^2_T}^2 + \mathbb{E}\left[x(T)^\top Q x(T) \right] = \left\| B^\top Q x + u \right\|_{L^2_T}^2 = \left\| u^{(1)} \right\|_{L^2_T}^2$. For that reason, this result is a direct consequence of Theorem 5.1. □

Corollary 5.2 tells that the stabilizing feedback control $u = -B^\top Q x + u^{(1)}$ and its output can be well-approximated by the reduced feedback $u_r = -B_r^\top \Sigma_r x_r + u^{(1)}$ and the associated output in case the truncated singular values are small.

We draw our attention back to open-loop controls and discuss the benefit of Theorem 5.1 in this context since this might not be obvious seeing that u_r depends on the original state x. Therefore, it seems that we did not gain much from the practical point of view although we found a good candidate for an approximating input–output pair. However, there is a fundamental difference between stochastic and deterministic settings since in the context of stochastic differential equations, there are many problems that cannot be solved in moderate high dimensions n even though one is willing to simulate the original system (1). To be more precise, one often needs to compute conditional expectations of the form

$$g(x) := \mathbb{E}[f(y(t)) | x(s) = x], \quad x \in \mathbb{R}^n, \quad s < t,$$

which is the expectation of some quantity of interest $f(y)$ at time t given that the state at time s is x. Such objects occur in stochastic optimal stopping problems, e.g., in the context of pricing (Bermudan) options in finance. In order to find an approximation $g(\cdot) \approx \sum_{k=1}^{K} \hat{\beta}_k \psi_k(\cdot)$ of the unknown function g, where $\psi_1, \ldots, \psi_K$ is some suitable (polynomial) basis, we have to solve the least squares problem

$$\hat{\beta} := \arg\min_{\beta \in \mathbb{R}^K} \sum_{i=1}^{M} \left| f(y(t)^i) - \sum_{k=1}^{K} \beta_k \psi_k(x(s)^i) \right|^2, \tag{33}$$

where $y(t)^i$ and $x(s)^i$ i.i.d. samples of the random variables $y(t)$ and $x(s)$, respectively. Notice that (33) is the discretized version by Monte Carlo of the original continuous problem $\min_{\beta \in \mathbb{R}^K} \mathbb{E} \left| f(y(t)) - \sum_{k=1}^{K} \beta_k \psi_k(x(s)) \right|^2$. Now, solving the regression problem in (33) requires a huge computational effort already in moderate high dimensions since regression suffers from the curse of dimensionality. This often makes this procedure infeasible for dimensions $n \geq 10$. Therefore, a possible strategy can be to simulate the original system (1) in order to determine the reduced order input u_r defined in Theorem 5.1 that gives a good approximation y_r of y. If r is sufficiently small, one can then solve (33) in the reduced system, in which the impact of the curse of dimensionality is drastically decreased. This leads to a good estimate g_r (defined on $\mathbb{R}^r$) of the original g.

We finally investigate the scenario, in which we do not intent to simulate the original system (1) but an open-loop control u is used. Fortunately, Theorem 5.1 also provides a bound for the distance between u_r (defined within this theorem) and the original input u. For that reason, we know that u and u_r must be close if the truncated singular values of the system are small. Subsequently, we can use that the (reduced) output is Lipschitz continuous in the control term. This is proved in the following lemma.

Lemma 5.3. *Given the reduced order model (16) with $x_{0,r} = 0$, then there exists a constant $\gamma_T > 0$ such that*

$$\left\| y_r(\cdot, 0, u_r) \right\|_{L^2_T} \leq \gamma_T \left\| u_r \right\|_{L^2_T} \tag{34}$$

for all $u_r \in L^2_T$.

Proof. We use Eq. (4) associated to the balanced realization with diagonal solution Σ_n. We can now exploit the partition in (15) and evaluate the left upper block of the balanced version of the matrix Eq. (4). This yields the following inequality

$$A_r^\top \Sigma_r + \Sigma_r A_r + \sum_{i,j=1}^{q} N_{i,r}^\top \Sigma_r N_{j,r} k_{ij} + C_r^\top C_r - \Sigma_r B_r B_r^\top \Sigma_r \leq 0, \tag{35}$$

where $\Sigma_r = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ contains the first r singular values of the system. Applying (38) to the reduced system with initial state zero and setting $X = \Sigma_r$, we obtain

$$\begin{aligned}\mathbb{E}\left[x_r(t)^\top \Sigma_r x_r(t)\right] &= \int_0^t \mathbb{E}\left[x_r(s)^\top \left(A_r^\top \Sigma_r + \Sigma_r A_r + \sum_{i,j=1}^{q} N_{i,r}^\top \Sigma_r N_{j,r} k_{ij}\right) x_r(s)\right] ds \\ &\quad + 2\int_0^t \mathbb{E}\left\langle B_r^\top \Sigma_r x_r(s), u_r(s)\right\rangle_2 ds.\end{aligned} \tag{36}$$

With $2\left\langle B_r^\top \Sigma_r x_r(s), u_r(s)\right\rangle_2 \leq \|B_r^\top \Sigma_r x_r(s)\|_2^2 + \|u_r(s)\|_2^2$ and (35), identity (36) becomes

$$\begin{aligned}\mathbb{E}\left[x_r(t)^\top \Sigma_r x_r(t)\right] &\leq \|u_r\|_{L_t^2}^2 - \|y_r\|_{L_t^2}^2 + 2\int_0^t \mathbb{E}\left\|B_r^\top \Sigma_r x_r(s)\right\|_2^2 ds \\ &\leq \|u_r\|_{L_t^2}^2 - \|y_r\|_{L_t^2}^2 + 2b_r \int_0^t \mathbb{E}\left[x_r(s)^\top \Sigma_r x_r(s)\right] ds,\end{aligned}$$

where $b_r := \left\|B_r^\top \Sigma_r^{\frac{1}{2}}\right\|_2^2$. Gronwall's Lemma A.1 for $t = T$ leads to

$$\begin{aligned}\mathbb{E}\left[x_r(T)^\top \Sigma_r x_r(T)\right] &\leq \|u_r\|_{L_T^2}^2 - \|y_r\|_{L_T^2}^2 + \int_0^T (\|u_r\|_{L_s^2}^2 - \|y_r\|_{L_s^2}^2) 2b_r \,\mathrm{e}^{2b_r(T-s)}\, ds \\ &= \int_0^T (\|u_r(s)\|_2^2 - \|y_r(s)\|_2^2)\, \mathrm{e}^{2b_r(T-s)}\, ds\end{aligned}$$

using integration by parts in the last step. Therefore, we have

$$\|y_r\|_{L_T^2}^2 \leq \int_0^T \|y_r(s)\|_2^2 \,\mathrm{e}^{2b_r(T-s)}\, ds \leq \int_0^T \|u_r(s)\|_2^2 \,\mathrm{e}^{2b_r(T-s)}\, ds \leq \mathrm{e}^{2b_r T} \|u_r\|_{L_T^2}^2 .$$

This concludes the proof. □

By the linearity of y_r in u_r, (34) means that controls being close to each other lead to similar outputs. Therefore, only a slight deviation between $y_r(\cdot, 0, u_r)$ and $y_r(\cdot, 0, u)$ is expected. The smallest constant in (34) is $\gamma_T = \sup_{u_r \in L_T^2 \setminus \{0\}} \frac{\|y_r(\cdot,0,u_r)\|_{L_T^2}}{\|u_r\|_{L_T^2}}$. If there is a Lipschitz constant independent of T, we can consider $\gamma_T = \gamma = \sup_{u_r \in L^2 \setminus \{0\}} \frac{\|y_r(\cdot,0,u_r)\|_{L^2}}{\|u_r\|_{L^2}}$ in (34) which is the norm of the input–output operator on the entire positive real line. This holds, e.g., if (16) is asymptotically stable [20]. We can now formulate the result when $u_r = u$ is chosen in Theorem 5.1.

Corollary 5.4. *Let $x_0 = 0$, $x_{0,r} = 0$, $u \in L_T^2$ and $y_r = y_r(\cdot, 0, u)$. If $\gamma_T > 0$ is a constant satisfying (34), we have*

$$\|y - y_r\|_{L_T^2} \leq 2(1 + \gamma_T) \sum_{k=r+1}^{n} \frac{\sigma_k}{\sqrt{1 + \sigma_k^2}} \left(\left\|\begin{bmatrix} u \\ y \end{bmatrix}\right\|_{L_T^2}^2 + \mathbb{E}\left[x(T)^\top Q x(T)\right]\right)^{\frac{1}{2}}.$$

If additionally holds that $u, x \in L^2$, then we have

$$\|y - y_r\|_{L_T^2} \leq 2(1 + \gamma_T) \sum_{k=r+1}^{n} \frac{\sigma_k}{\sqrt{1 + \sigma_k^2}} \left\|\begin{bmatrix} u \\ y \end{bmatrix}\right\|_{L^2}.$$

Proof. It holds that

$$\begin{aligned}\|y(\cdot, 0, u) - y_r(\cdot, 0, u)\|_{L_T^2} &\leq \|y(\cdot, 0, u) - y_r(\cdot, 0, u_r)\|_{L_T^2} + \|y_r(\cdot, 0, u_r) - y_r(\cdot, 0, u)\|_{L_T^2} \\ &\leq \|y(\cdot, 0, u) - y_r(\cdot, 0, u_r)\|_{L_T^2} + \gamma_T \|u_r - u\|_{L_T^2} ,\end{aligned} \tag{37}$$

where u_r is defined as in Theorem 5.1. Applying (31) to both terms in (37) yields the first estimate. If $u, x \in L^2$ holds, we can use (32) instead and obtain the second inequality. This concludes the proof. □

According to Theorem 5.1 the singular values of (1) can be used a-priori to find a suitable dimension r of an accurate reduced system (16) using a control that is possibly not available. Corollary 5.4 now additionally tells us that this unavailable control can be replaced by the original system control if the norm of the input–output operator is not too large. Computing this norm γ_T is feasible in small dimensions r without causing large computation cost. Hence, γ_T provides an a-posteriori criterion for a good approximation of $y(\cdot, 0, u)$ by $y_r(\cdot, 0, u)$. We finally provide a bound that neither contains the state x nor the output y of the original system.

Theorem 5.5. *Let $y = y(\cdot, 0, u)$ and $y_r = y_r(\cdot, 0, u)$ and $u \in L^2_T$. Then,*

$$\left(\mathbb{E}\int_0^T \mathrm{e}^{-\beta t}\left\|y(t) - y_r(t)\right\|_2^2 dt\right)^{\frac{1}{2}} \le 2\sum_{k=r+1}^{n}\sigma_k\left(\mathbb{E}\int_0^T \mathrm{e}^{-\beta t}\left\|u(t)\right\|_2^2 dt\right)^{\frac{1}{2}},$$

where $\beta = \max\{\left\|B^\top Q^{\frac{1}{2}}\right\|_2^2, \left\|CP^{\frac{1}{2}}\right\|_2^2\}$.

Proof. We present the proof in Appendix C. □

The bound of Theorem 5.5 is practically computable since it does not involve variables of the original system (1). However, a high accuracy cannot be expected since it is a worst-case bound (based on Gronwall's lemma) that also captures systems with exponentially growing states that might not be approximated well with the underlying dimension reduction scheme. Therefore, the result of Theorem 5.5 can also be read as a warning that LQG balancing is not working well for all types of unstable open-loop systems (satisfying our assumptions) even though the truncated singular values are small.

6. Numerical examples

For $t \in [0, T]$, we consider the following 2D stochastic heat equation with Neumann boundary conditions and scalar noise ($q = 1$):

$$\frac{\partial X(t,\zeta)}{\partial t} = \alpha\Delta X(t,\zeta) + f(\zeta)u(t) + \nu g(\zeta)X(t,\zeta)\frac{\partial W(t)}{\partial t}, \quad \zeta \in [0,\pi]^2,$$
$$\frac{\partial X(t,\zeta)}{\partial \mathbf{n}} = 0, \quad \zeta \in \partial[0,\pi]^2, \quad X(0,\zeta) \equiv 0,$$

where $\alpha, \nu > 0$ and f, g are bounded functions on $[0,\pi]^2$. We set $\alpha = 0.2$, $\nu = 2$, $f(\zeta) = 1_{[\frac{\pi}{4},\frac{3\pi}{4}]^2}(\zeta)$, $g(\zeta) = \mathrm{e}^{-\left|\zeta_1 - \frac{\pi}{2}\right| - \zeta_2}$ and $H = L^2([0,\pi]^2)$ to be the solution space for the mild solution of the stochastic partial differential equation (SPDE). In this context, let $\langle\cdot,\cdot\rangle_H$ denote the inner product in H and $\|\cdot\|_H$ the corresponding norm. The output is the mean temperature on the uncontrolled area,

$$Y(t) = \mathfrak{C}X(t,\zeta) := \frac{4}{3\pi^2}\int_{[0,\pi]^2\setminus[\frac{\pi}{4},\frac{3\pi}{4}]^2} X(t,\zeta)d\zeta.$$

We discretize this SPDE by a spectral Galerkin method according to [9]. The eigenvalues of the Neumann Laplacian on $[0,\pi]^2$ are given by $\lambda_{ij} = -(i^2 + j^2)$ and the corresponding eigenvectors representing an ONB of H are $h_{ij} = \frac{f_{ij}}{\left\|f_{ij}\right\|_H}$, where $f_{ij} = \cos(i\cdot)\cos(j\cdot)$. We order these eigenvalues and write λ_k and h_k for the kth largest eigenvalue and the associated eigenvector, respectively. We obtain a system of the form (1) with matrices $C^\top = \left(\mathfrak{C}h_k\right)_{k=1,\dots,n}$, $A = \alpha\,\mathrm{diag}(\lambda_1, \lambda_2, \dots) = \alpha\,\mathrm{diag}(0, -1, \dots)$, $N_1 = \nu\left(\langle gh_i, h_k\rangle_H\right)_{k,i=1,\dots,n}$, $B = \left(\langle f, h_k\rangle_H\right)_{k=1,\dots,n}$. We observe that this spatial discretization is unstable but the requirements for applying LQG balancing are fulfilled.

Gramians Q and P according to their definitions in Theorem 3.1 and in (5) can now be computed. For Q we have used a fixed point iteration with $Q_0 = I$ and Q_{k+1} being the stabilizing solution of the Riccati equation

$$\mathcal{L}_A(Q_{k+1}) + \Pi_N(Q_k) + C^\top C - Q_{k+1}BB^\top Q_{k+1} = 0.$$

This converges quite fast to the Gramian Q, e.g., [28, Sec. 5.4.3]. The Gramian P is computationally more involved. By our error analysis in Section 5, it is natural to seek for a P with a large number of small eigenvalues, so that we aim to find the Gramian with minimal trace subject to (5). However, we do not have a linear matrix inequality (LMI) formulation for P but rather for its inverse. Therefore, we have rewritten (5) as the LMI

$$\begin{bmatrix} A^\top P^{-1} + P^{-1}A + N_1^\top P^{-1}N_1 - C^\top C & P^{-1}B \\ B^\top P^{-1} & -I \end{bmatrix} \le 0\,, \quad P^{-1} \ge 0$$

and maximized the trace of P^{-1} using the solver Mosek [38] with the Matlab package Yalmip [39]. However, this might not ensure the same approximation quality as when being able to solve for P directly. For the computation of P^{-1} in dimension $n = 100$, it took about 5 to 6 min, where the empirical complexity is about n^6. Therefore, we did not consider larger systems.

For $n = 100$, we choose the reduced order $r = 10$. The decay of the singular values σ_j is shown in Fig. 1. As pointed out in Section 5, the truncated singular values determine the error of LQG balanced truncation. We observe that $r = 10$ provides very small $\sigma_{r+1}, \dots, \sigma_n$ relative to σ_1. We combine (1) and (16) ($u_r = u$) with zero initial states and define $\xi = \left[\begin{smallmatrix} x \\ x_r \end{smallmatrix}\right]$ leading the open-loop error system

$$d\xi = \left(\begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}\xi + \begin{bmatrix} B \\ B_r \end{bmatrix}u\right)dt + \begin{bmatrix} N_1 & 0 \\ 0 & N_{1,r} \end{bmatrix}\xi\,dW, \quad y_\xi = \begin{bmatrix} C & -C_r \end{bmatrix}\xi,$$

where $y_\xi = y - y_r$. Secondly, we introduce a closed-loop version by

$$d\xi = \left(\begin{bmatrix} A - BB^\top Q & 0 \\ 0 & A_r - B_rB_r^\top\Sigma_r \end{bmatrix}\xi + \begin{bmatrix} B \\ B_r \end{bmatrix}u\right)dt + \begin{bmatrix} N_1 & 0 \\ 0 & N_{1,r} \end{bmatrix}\xi\,dW, \quad y_\xi = \begin{bmatrix} C & -C_r \end{bmatrix}\xi,$$

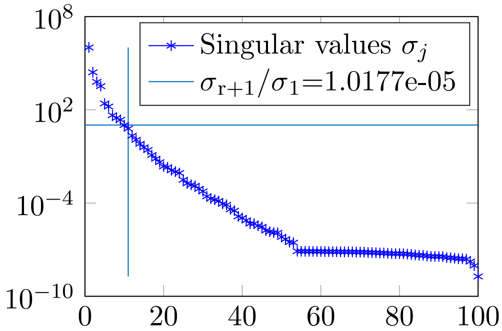

Fig. 1. Decay singular values of discretized heat equation for $n = 100$.

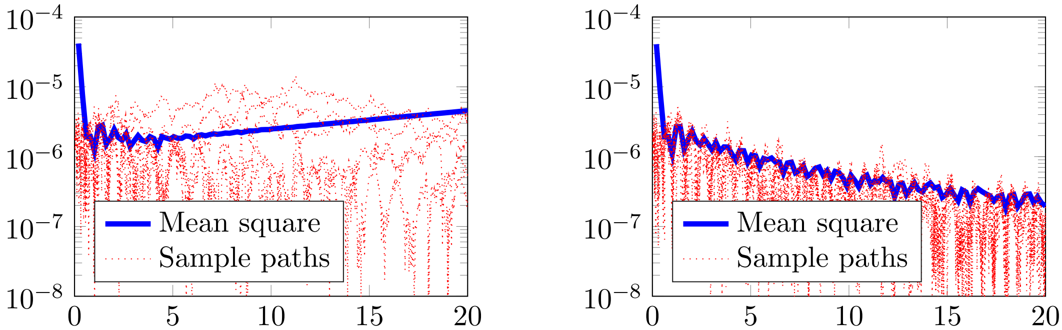

Fig. 2. Open-loop and closed-loop error systems driven by L^2-input u.

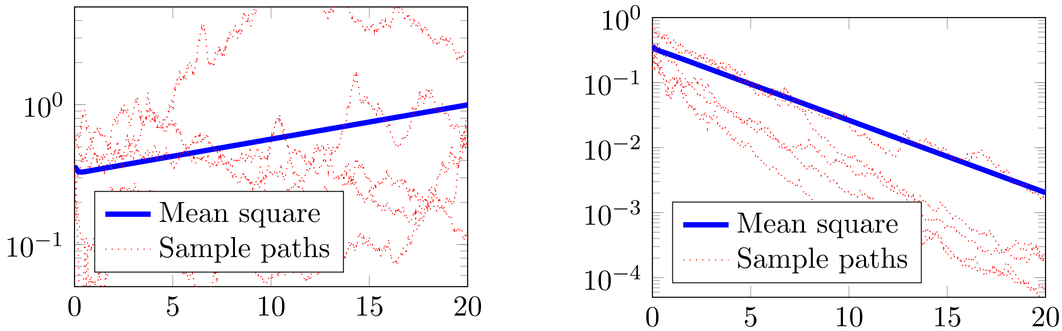

Fig. 3. No control vs. reduced controller in (1) with random initial state. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

i.e., a stabilizing feedback control is used. We have computed $t \mapsto \sqrt{\mathbb{E}\left|y_\xi(t)\right|^2}$ (blue graphs) and five trajectories $t \mapsto \left|y_\xi(t,\omega)\right|$ (red dotted graphs) in Fig. 2. Notice that the open-loop case is depicted left and the closed-loop scenario is given in the right picture. In both cases, we have used zero initial states and the L^2-input $u = \frac{\cos(5t)}{t+1}$. The mean square error in blue has been computed from a deterministic Lyapunov type ordinary differential equation and the five sample output paths from a drift implicit Euler–Maruyama method.

We can see that the error is small in both cases, but by stability it decays only in the closed-loop case. Furthermore, we observed in this example that the reduced feedback controller also stabilizes the full system. To visualize this effect, we have computed $t \mapsto \sqrt{\mathbb{E}\,|y(t)|^2}$ (blue graphs) for system (1) with $u \equiv 0$ (Fig. 3 left), $u(t) = B_r^\top \Sigma_r S_{b,r}^\top x(t)$ (Fig. 3 right) and a randomly generated initial state x_0, where $S_{b,r}^\top$ are the first r rows of the balancing transformation S_b in Section 4. As mean square stability is stronger than path-wise stability in the linear case, we see the same effect for the trajectories $t \mapsto |y(t,\omega)|$ (red dotted graphs) in Fig. 3.

7. Conclusions

In this paper, we considered dimension reduction techniques for large-scale stochastic systems. Such schemes are vital in both control and probabilistic settings as many system evaluations are required. In this context, one can think of aiming to investigate statistical properties by sampling methods or the optimal control of spatially discretized stochastic partial differential equations. These fit into our framework, in which we have studied potentially unstable stochastic differential equations. They occur, for instance if the driving noise is large. Therefore, we have made an essential contribution since many existing model reduction schemes require certain stability conditions. We have introduced a pair of Gramians that are designed in order to characterize dominant subspaces of the underlying stochastic system. In particular, unnecessary direction in closed-loop dynamics have been identified by these Gramians but we have also pointed out their relevance for open-loop controls. These considerations led to a reduced order model that captures many important features of the original one. We proved that, e.g., stabilizability and detectability are preserved. Our dimension reduction procedure further allowed for a detailed error analysis. Based on the error estimates provided in this work, algebraic a-priori criteria for the approximation quality have been found. These error bounds therefore give a clear guidance on how to fix the reduced order dimension. The effectiveness of our method has been demonstrated by applying it to an unstable stochastic heat equation. This means that an infinite dimensional state dynamics could be approximated by a low-order stochastic system.

CRediT authorship contribution statement

Tobias Damm: Methodology, Software, Visualization, Writing – review & editing, Formal analysis. **Martin Redmann:** Methodology, Writing – original draft, Writing – review & editing, Formal analysis, Conceptualization.

Declaration of competing interest

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Appendix A. Supporting lemmas

Lemma A.1 (*Gronwall Lemma*)**.** *Given $T > 0$ let $z, \alpha : [0, T] \to \mathbb{R}$ and $\beta : [0, T] \to [0, \infty)$ be continuous functions. If*

$$z(t) \leq \alpha(t) + \int_0^t \beta(s) z(s) ds, \quad t \in [0, T],$$

then for all $t \in [0, T]$, it holds that

$$z(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(w) dw\right) ds.$$

Proof. The result can be shown following the steps in [40, Proposition 2.1]. □

Lemma A.2. *Let $a, b_1, \dots, b_q$ be $\mathbb{R}^d$-valued processes, where a is $\left(\mathcal{F}_t\right)_{t\in[0,T]}$-adapted and almost surely Lebesgue integrable and the functions b_i are integrable with respect to the mean zero Wiener process $W = (W_1, \dots, W_q)^\top$ with covariance matrix $K = \left(k_{ij}\right)_{i,j=1,\dots,q}$. If the process x is given by*

$$dx(t) = a(t)dt + \sum_{i=1}^{v} b_i(t) dW_i, \quad t \in [0, T].$$

Then, we have

$$\frac{d}{dt}\mathbb{E}\left[x(t)^\top x(t)\right] = 2\mathbb{E}\left[x(t)^\top a(t)\right] + \sum_{i,j=1}^{v} \mathbb{E}\left[b_i(t)^\top b_j(t)\right] k_{ij}.$$

Proof. We refer to [41, Lemma 5.2] for a proof of this lemma. □

Let x now be the solution to (1a). As a direct consequence, we obtain the following identity:

$$\begin{aligned}
\mathbb{E}\left[x(t)^\top X x(t)\right] &= x_0^\top X x_0 + 2\int_0^t \mathbb{E}\left[x(s)^\top X\left(Ax(s)+Bu(s)\right)\right] ds \\
&\quad + \int_0^t \sum_{i,j=1}^q \mathbb{E}\left[x(s)^\top N_i^\top X N_j x(s)\right] k_{ij}\, ds \\
&= x_0^\top X x_0 + \int_0^t \mathbb{E}\left[x(s)^\top \left(A^\top X + XA + \sum_{i,j=1}^q N_i^\top X N_j k_{ij}\right) x(s)\right] ds \\
&\quad + 2\int_0^t \mathbb{E}\left\langle B^\top X x(s), u(s)\right\rangle_2 ds,
\end{aligned} \tag{38}$$

where $X \geq 0$ is a semidefinite matrix.

Lemma A.3. *Let W be as in Lemma A.2 and $A, N_i \in \mathbb{R}^{k\times k}$ be generic matrices. Suppose that b is an $\mathbb{R}^k$-valued and $c_0, \ldots, c_q$ are scalar $(\mathcal{F}_t)_{t\in[0,T]}$-adapted processes in L^2_T. If x is given by*

$$dx(t) = [Ax(t) + b(t) \pm \left[\begin{smallmatrix} 0 \\ c_0(t)\end{smallmatrix}\right]]dt + \sum_{i=1}^q [N_i x(t) \pm \left[\begin{smallmatrix} 0 \\ c_i(t)\end{smallmatrix}\right]]dW_i(t), \quad x(0) = 0. \tag{39}$$

Then, for $t \in [0, T]$, we have

$$\begin{aligned}
\mathbb{E}\left[x(t)^\top D x(t)\right] &= \mathbb{E}\int_0^t x(s)^\top \left(A^\top D + DA^\top + \sum_{i,j=1}^q N_i^\top D N_j k_{ij}\right) x(s) + 2x(s)^\top D b(s) ds \\
&\quad \pm d_k \mathbb{E}\int_0^t 2x_2(s)c_0(s) + \sum_{i,j=1}^q \left(2n_i x(s) \pm c_i(s)\right) c_j(s) k_{ij}\, ds,
\end{aligned} \tag{40}$$

where $D = \operatorname{diag}(d_1, \ldots, d_k) \geq 0$, n_i is the last row of N_i and x_2 the last entry of x.

Proof. Applying Lemma A.2, we find

$$\begin{aligned}
\mathbb{E}\left[x(t)^\top D x(t)\right] &= 2\int_0^t \mathbb{E}\left[x(s)^\top D\left(Ax(s) + b(s) \pm \left[\begin{smallmatrix} 0 \\ c_0(s)\end{smallmatrix}\right]\right)\right] ds \\
&\quad + \int_0^t \sum_{i,j=1}^q \mathbb{E}\left[\left(N_i x(s) \pm \left[\begin{smallmatrix} 0 \\ c_i(s)\end{smallmatrix}\right]\right)^\top D\left(N_j x(s) \pm \left[\begin{smallmatrix} 0 \\ c_j(s)\end{smallmatrix}\right]\right)\right] k_{ij}\, ds \\
&= \mathbb{E}\int_0^t x(s)^\top \left(A^\top D + DA^\top + \sum_{i,j=1}^q N_i^\top D N_j k_{ij}\right) x(s) + 2x(s)^\top D b(s) ds \\
&\quad \pm \mathbb{E}\int_0^t 2x(s)^\top D\left[\begin{smallmatrix} 0 \\ c_0(s)\end{smallmatrix}\right] + \sum_{i,j=1}^q \left(2N_i x(s) \pm \left[\begin{smallmatrix} 0 \\ c_i(s)\end{smallmatrix}\right]\right)^\top D\left[\begin{smallmatrix} 0 \\ c_j(s)\end{smallmatrix}\right] k_{ij}\, ds.
\end{aligned}$$

We observe that $x(s)^\top D\left[\begin{smallmatrix} 0 \\ c_0(s)\end{smallmatrix}\right] = d_k x_2(s) c_0(s)$ and

$$\left(2N_i x(s) \pm \left[\begin{smallmatrix} 0 \\ c_i(s)\end{smallmatrix}\right]\right)^\top D\left[\begin{smallmatrix} 0 \\ c_j(s)\end{smallmatrix}\right] = d_k\left(2n_i x(s) \pm c_i(s)\right) c_j(s),$$

so that the result follows. □

Appendix B. Proof of Theorem 5.1

Proof of Theorem 5.1. Let $(A_n, B_n, C_n, N_{i,n})$ be the balanced realization of (1) with state variable x_n. Let us further introduce A_k and $N_{i,k}$ as the left upper $k\times k$ blocks of A_n and $N_{i,n}$. Moreover, suppose that B_k and C_k are the first k rows of B_n and first k columns of C_n, $k = r, \ldots, n-1$. We define

$$\begin{aligned}
dx_k(t) &= [\bar{A}_k x_k(t) + B_k v(t)]dt + \sum_{i=1}^q N_{i,k} x_k(t) dW_i(t), \\
\bar{y}_k(t) &= \bar{C}_k x_k(t) + \begin{bmatrix} v(t) \\ 0 \end{bmatrix}, \quad t \geq 0,
\end{aligned} \tag{41}$$

where $\bar{A}_k = A_k - B_k B_k^\top \Sigma_k$, $\bar{C}_k = \begin{bmatrix} -B_k^\top \Sigma_k \\ C_k \end{bmatrix}$ and $k = r, \ldots, n$. Clearly, $k = r$ yields (30). On the other hand, $k = n$ provides the input–output parameterization of the balanced version of (1) which can be seen by exploiting $B^\top Q x(t) = B_n^\top \Sigma_n x_n(t)$. Consequently,

$\bar{y}_n$ coincides with $\bar{y}$ in (29). Therefore, we have

$$\left\| \begin{bmatrix} u-u_r \\ y-y_r \end{bmatrix} \right\|_{L_T^2} = \|\bar{y}-\bar{y}_r\|_{L_T^2} \leq \sum_{i=r+1}^{n} \|\bar{y}_k - \bar{y}_{k-1}\|_{L_T^2}, \tag{42}$$

for which we investigate every summand $\|\bar{y}_k - \bar{y}_{k-1}\|_{L_T^2}$ in the following. Exploiting the definitions of $\bar{A}_n$ and $\bar{C}_n$ the balanced version of (4) becomes

$$\bar{A}_k^\top \Sigma_k + \Sigma_k \bar{A}_k + \sum_{i,j=1}^{q} N_{i,k}^\top \Sigma_k N_{i,k} k_{ij} + \bar{C}_k^\top \bar{C}_k \leq 0 \tag{43}$$

for $k = n$ (here even the equality holds in (43)). The inequalities in (43) for $k = r, \dots, n-1$ follow by evaluating the $k \times k$ left upper block of inequality with $k = n$. We now define $L_k = \Sigma_k + \Sigma_k^{-1}$. Then, adding the balanced versions of both (in)equalities (4) and (5) yields

$$\bar{A}_k^\top L_k + L_k \bar{A}_k + \sum_{i,j=1}^{q} N_{i,k}^\top L_k N_{i,k} k_{ij} + L_k B_k B_k^\top L_k \leq 0 \tag{44}$$

for $k = n$. The evaluation of the left upper blocks then provides the results for $k = r, \dots, n-1$. We partition

$$x_k = \begin{bmatrix} x_{k,1} \\ x_{k,2} \end{bmatrix}, \quad \bar{A}_k = \begin{bmatrix} \bar{A}_{k-1} & \star \\ a_{21} & \star \end{bmatrix}, \quad B_k = \begin{bmatrix} B_{k-1} \\ b_2 \end{bmatrix}, \quad N_{i,k} = \begin{bmatrix} N_{i,k-1} & \star \\ n_{i,21} & n_{i,22} \end{bmatrix}. \tag{45}$$

The variable $x_{k,2}$ is scalar and we omit the index k in $a_{21}, n_{i,21} \in \mathbb{R}^{1\times(k-1)}$, $b_2 \in \mathbb{R}^{1\times m}$, $n_{i,22} \in \mathbb{R}$ in order to simplify the notation. We set

$$x_- = \begin{bmatrix} x_{k,1} - x_{k-1} \\ x_{k,2} \end{bmatrix}, \quad x_+ = \begin{bmatrix} x_{k,1} + x_{k-1} \\ x_{k,2} \end{bmatrix} \tag{46}$$

and obtain from (41) that

$$dx_-(t) = [\bar{A}_k x_-(t) + \left[\begin{smallmatrix} 0 \\ c_0(t) \end{smallmatrix}\right]]dt + \sum_{i=1}^{q} [N_{i,k} x_-(t) + \left[\begin{smallmatrix} 0 \\ c_i(t) \end{smallmatrix}\right]]dW_i(t), \tag{47a}$$

$$dx_+(t) = [\bar{A}_k x_+(t) + 2B_k v(t) - \left[\begin{smallmatrix} 0 \\ c_0(t) \end{smallmatrix}\right]]dt + \sum_{i=1}^{q} [N_{i,k} x_+(t) - \left[\begin{smallmatrix} 0 \\ c_i(t) \end{smallmatrix}\right]]dW_i(t), \tag{47b}$$

where $c_0(t) := a_{21} x_{k-1}(t) + b_2 v(t)$ and $c_i(t) := n_{i,21} x_{k-1}(t)$. We apply Lemma A.3 to (47a) with $b(t) = 0$ and $D = \Sigma_k$. Moreover, we immediately insert (43) into the result of this lemma resulting in

$$\begin{aligned} \mathbb{E}\left[x_-(T)^\top \Sigma_k x_-(T)\right] \leq{} & -\mathbb{E}\int_0^T x_-(t)^\top \bar{C}_k^\top \bar{C}_k x_-(t) dt \\ & + \sigma_k \mathbb{E}\int_0^T 2x_{k,2}(t) c_0(t) + \sum_{i,j=1}^{q} \left(2\left[\begin{smallmatrix} n_{i,21} & n_{i,22} \end{smallmatrix}\right] x_-(t) + c_i(t)\right) c_j(t) k_{ij} dt. \end{aligned} \tag{48}$$

Using the definitions of c_i and x_-, we find an upper bound by replacing x_- by x_k in the last term, i.e.,

$$\sum_{i,j=1}^{q} \left(2\left[\begin{smallmatrix} n_{i,21} & n_{i,22} \end{smallmatrix}\right] x_-(t) + c_i(t)\right) c_j(t) k_{ij} \leq \sum_{i,j=1}^{q} \left(2\left[\begin{smallmatrix} n_{i,21} & n_{i,22} \end{smallmatrix}\right] x_k(t) + c_i(t)\right) c_j(t) k_{ij} \tag{49}$$

exploiting that $\sum_{i,j=1}^q c_i(t) c_j(t) k_{ij} \geq 0$ because $K = (k_{ij})$ is positive semidefinite. Secondly, we see that $\bar{C}_k x_- = \bar{C}_k x_k - \bar{C}_{k-1} x_{k-1} = \bar{y}_k - \bar{y}_{k-1}$ since $\bar{C}_k = \begin{bmatrix} \bar{C}_{k-1} & \star \end{bmatrix}$. Now, inserting these estimates into (48) implies

$$\|\bar{y}_k - \bar{y}_{k-1}\|_{L_T^2}^2 \leq \sigma_k \mathbb{E}\int_0^T 2x_{k,2}(t) c_0(t) + \sum_{i,j=1}^{q} \left(2\left[\begin{smallmatrix} n_{i,21} & n_{i,22} \end{smallmatrix}\right] x_k(t) + c_i(t)\right) c_j(t) k_{ij} dt. \tag{50}$$

We apply Lemma A.3 to (47b) with $b(t) = 2Bv(t)$, $D = L_k$ and directly make use of (44) providing

$$\begin{aligned} \mathbb{E}\left[x_+(T)^\top L_k x_+(T)\right] \leq{} & \mathbb{E}\int_0^T -x_+(t)^\top L_k B_k B_k^\top L_k x_+(t) + 4x_+(t)^\top L_k B_k v(t) dt \\ & - (\sigma_k + \sigma_k^{-1}) \mathbb{E}\int_0^T 2x_{k,2}(t) c_0(t) + \sum_{i,j=1}^{q} \left(2\left[\begin{smallmatrix} n_{i,21} & n_{i,22} \end{smallmatrix}\right] x_+(t) - c_i(t)\right) c_j(t) k_{ij} dt. \end{aligned} \tag{51}$$

We observe that

$$\left(2\left[\begin{smallmatrix} n_{i,21} & n_{i,22} \end{smallmatrix}\right] x_+(t) - c_i(t)\right) c_j(t) = \left(2\left[\begin{smallmatrix} n_{i,21} & n_{i,22} \end{smallmatrix}\right] x_k(t) + c_i(t)\right) c_j(t) \tag{52}$$

based on the definitions of c_i and x_+ and furthermore find

$$4\|v(t)\|_2^2 \geq \|2v(t)\|_2^2 - \left\|B_k^\top L_k x_+(t) - 2v(t)\right\|_2^2 \tag{53}$$

$$= -x_+(t)^\top L_k B_k B_k^\top L_k x_+(t) + 4x_+(t)^\top L_k B_k v(t).$$

Combining (51) with (52) and (53) leads to

$$\mathbb{E}\int_0^T 2x_{k,2}(t)c_0(t) + \sum_{i,j=1}^{q} \left(2\left[\, n_{i,21}\, n_{i,22}\,\right] x_k(t) + c_i(t)\right) c_j(t) k_{ij}\, dt \leq \frac{4}{\sigma_k + \sigma_k^{-1}} \|v\|_{L_T^2}^2 .$$

This together with (50) gives us

$$\|\bar{y}_k - \bar{y}_{k-1}\|_{L_T^2}^2 \leq 4\frac{\sigma_k}{\sigma_k + \sigma_k^{-1}} \|v\|_{L_T^2}^2 = 4\frac{\sigma_k^2}{\sigma_k^2 + 1} \|v\|_{L_T^2}^2 .$$

Inserting this into (42), it follows that $\left\|\begin{bmatrix} u - u_r \\ y - y_r \end{bmatrix}\right\|_{L_T^2} \leq 2\sum_{k=r+1}^{n} \frac{\sigma_k}{\sqrt{\sigma_k^2+1}} \|v\|_{L_T^2}$. It remains to calculate $\|v\|_{L_T^2}$ with $v(t) = B^\top Q x(t) + u(t)$. Based on (11), we obtain

$$\mathbb{E}\left[x(T)^\top Q x(T)\right] = \int_0^T \mathbb{E}\left[-\|y(t)\|_2^2 - \|u(t)\|_2^2 + \left\|B^\top Q x(t) + u(t)\right\|_2^2\right] dt,$$

which provides the first claim of this theorem. If $u, x \in L^2$, then the limit as $T \to \infty$ of the above right hand side exists. Therefore, $\lim_{T\to\infty} \mathbb{E}\left[x(T)^\top Q x(T)\right]$ exists. Hence it is zero, otherwise it contradicts $x \in L^2$. Now, taking the limit as $T \to \infty$ in (31) yields the second claim. □

Appendix C. Proof of Theorem 5.5

Proof of Theorem 5.5. Showing this result is more complex than the proof given in Appendix B. However, some basic steps are identical such that a similar notation will be used below. As before, let $(A_n, B_n, C_n, N_{i,n})$ be the balanced realization of (1), i.e., the associated Gramians are identical and equal to Σ_n. Again, $A_k, B_k, C_k, N_{i,k}$ for $k = r, \ldots, n-1$ are the respective submatrices of the balanced realization. They define the reduced system of dimension k given by

$$\begin{aligned} dx_k(t) &= [A_k x_k(t) + B_k u(t)]dt + \sum_{i=1}^{q} N_{i,k} x_k(t) dW_i(t), \\ y_k(t) &= C_k x_k(t), \quad t \geq 0. \end{aligned} \tag{54}$$

Setting $k = r$ now yields the reduced system (16) with $u_r = u$. Given that $k = n$, we obtain the balanced realization of (1) and hence $y_n = y$. The inequality of Theorem 5.5 involves a scaled L^2-norm for which we can apply triangle inequality leading to

$$\left(\mathbb{E}\int_0^T \mathrm{e}^{-\beta t} \|y(t) - y_r(t)\|_2^2\, dt\right)^{\frac{1}{2}} \leq \sum_{k=r+1}^{n} \left(\mathbb{E}\int_0^T \mathrm{e}^{-\beta t} \|y_k(t) - y_{k-1}(t)\|_2^2\, dt\right)^{\frac{1}{2}} . \tag{55}$$

In order to proceed further, the error between y_k and y_{k-1} is analyzed. The associated matrix inequalities are derived from the ones for the balanced realization which are obtained by replacing (A, B, C, N_i, P, Q) by $(A_n, B_n, C_n, N_{i,n}, \Sigma_n, \Sigma_n)$ in (4) and (5). Evaluating the left upper $k \times k$ blocks of these (in)equalities yields

$$A_k^\top \Sigma_k^{-1} + \Sigma_k^{-1} A_k + \sum_{i,j=1}^{q} N_{i,k}^\top \Sigma_k^{-1} N_{j,k} k_{ij} - C_k^\top C_k + \Sigma_k^{-1} B_k B_k^\top \Sigma_k^{-1} \leq 0, \tag{56a}$$

$$A_k^\top \Sigma_k + \Sigma_k A_k + \sum_{i,j=1}^{q} N_{i,k}^\top \Sigma_k N_{j,k} k_{ij} + C_k^\top C_k - \Sigma_k B_k B_k^\top \Sigma_k \leq 0 \tag{56b}$$

for $k = r, \ldots, n$. We partition x_k, $N_{i,k}$ and B_k like in (45) and set $A_k = \begin{bmatrix} A_{k-1} & \star \\ a_{21} & \star \end{bmatrix}$. Below, the variables x_- and x_+ are defined analogously to (46). Based on (54), we find the respective equations by

$$dx_-(t) = [A_k x_-(t) + \left[\begin{smallmatrix} 0 \\ c_0(t) \end{smallmatrix}\right]]dt + \sum_{i=1}^{q} [N_{i,k} x_-(t) + \left[\begin{smallmatrix} 0 \\ c_i(t) \end{smallmatrix}\right]] dW_i(t), \tag{57a}$$

$$dx_+(t) = [A_k x_+(t) + 2B_k u(t) - \left[\begin{smallmatrix} 0 \\ c_0(t) \end{smallmatrix}\right]]dt + \sum_{i=1}^{q} [N_{i,k} x_+(t) - \left[\begin{smallmatrix} 0 \\ c_i(t) \end{smallmatrix}\right]] dW_i(t), \tag{57b}$$

where $c_0(t) := a_{21} x_{k-1}(t) + b_2 u(t)$ and $c_i(t) := n_{i,21} x_{k-1}(t)$. We apply Lemma A.3 to $\mathbb{E}\left[x_-(t)^\top \Sigma_k x_-(t)\right]$, $t \in [0, T]$, using (57a) and exploit (56b) giving us

$$\begin{aligned} \mathbb{E}\left[x_-(t)^\top \Sigma_k x_-(t)\right] \leq{}& \mathbb{E}\int_0^t x_-(s)^\top \Sigma_k B_k B_k^\top \Sigma_k x_-(s) ds - \mathbb{E}\int_0^t x_-(s)^\top C_k^\top C_k x_-(s) ds \\ &+ \sigma_k \mathbb{E}\int_0^t 2x_{k,2}(s)c_0(s) + \sum_{i,j=1}^{q} \left(2\left[\, n_{i,21}\;\; n_{i,22}\,\right] x_-(s) + c_i(s)\right) c_j(s) k_{ij}\, ds. \end{aligned} \tag{58}$$

We obtain that

$$x_-(s)^\top \Sigma_k B_k B_k^\top \Sigma_k x_-(s) = \left\| B_k^\top \Sigma_k^{\frac{1}{2}} \Sigma_k^{\frac{1}{2}} x_-(s) \right\|_2^2 \leq \left\| B_k^\top \Sigma_k^{\frac{1}{2}} \right\|_2^2 x_-(s)^\top \Sigma_k x_-(s).$$

Since $B_k^\top \Sigma_k^{\frac{1}{2}} = B_n^\top \Sigma_n^{\frac{1}{2}} \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, where I_k is a $k \times k$ identity matrix, we have $\left\| B_k^\top \Sigma_k^{\frac{1}{2}} \right\|_2^2 \leq \left\| B_n^\top \Sigma_n^{\frac{1}{2}} \right\|_2^2 = \left\| B^\top Q^{\frac{1}{2}} \right\|_2^2 \leq \beta$. Therefore, we have

$$x_-(s)^\top \Sigma_k B_k B_k^\top \Sigma_k x_-(s) \leq \beta x_-(s)^\top \Sigma_k x_-(s).$$

Moreover, we define $\alpha_k(t) = \mathbb{E} \int_0^t 2x_{k,2}(s)c_0(s) + \sum_{i,j=1}^q \left(2 \left[\, n_{i,21} \;\; n_{i,22} \,\right] x_k(s) + c_i(s)\right) c_j(s) k_{ij} ds$ and see that α_k is an upper bound for the last integral in (58) taking (49) into account. We further observe that $C_k x_- = y_k - y_{k-1}$ such that (58) becomes

$$\mathbb{E}\left[x_-(t)^\top \Sigma_k x_-(t)\right] \leq \sigma_k \alpha_k(t) - \|y_k - y_{k-1}\|_{L_t^2}^2 + \beta \int_0^t \mathbb{E}\left[x_-(s)^\top \Sigma_k x_-(s)\right] ds.$$

We apply Lemma A.1 resulting in

$$\mathbb{E}\left[x_-(t)^\top \Sigma_k x_-(t)\right] \leq \sigma_k \alpha_k(t) - \|y_k - y_{k-1}\|_{L_t^2}^2 + \beta \int_0^t [\sigma_k \alpha_k(s) - \|y_k - y_{k-1}\|_{L_s^2}^2] \,\mathrm{e}^{\beta(t-s)} ds.$$

Using integration by parts, we obtain

$$\begin{aligned} &\beta \int_0^t [\sigma_k \alpha_k(s) - \|y_k - y_{k-1}\|_{L_s^2}^2] \,\mathrm{e}^{\beta(t-s)} ds \\ &= \left[-(\sigma_k \alpha_k(s) - \|y_k - y_{k-1}\|_{L_s^2}^2) \,\mathrm{e}^{\beta(t-s)}\right]_{s=0}^t + \int_0^t [\sigma_k \dot{\alpha}_k(s) - \mathbb{E}\,\|y_k(s) - y_{k-1}(s)\|_2^2] \,\mathrm{e}^{\beta(t-s)} ds. \end{aligned}$$

Therefore, we have

$$\mathbb{E}\left[x_-(t)^\top \Sigma_k x_-(t)\right] \leq \int_0^t [\sigma_k \dot{\alpha}_k(s) - \mathbb{E}\,\|y_k(s) - y_{k-1}(s)\|_2^2] \,\mathrm{e}^{\beta(t-s)} ds$$

and hence, by setting $t = T$, we obtain

$$\mathbb{E} \int_0^T \|y_k(s) - y_{k-1}(s)\|_2^2 \,\mathrm{e}^{-\beta s} ds \leq \sigma_k \int_0^T \dot{\alpha}_k(s) \,\mathrm{e}^{-\beta s} ds. \tag{59}$$

In the following, an upper bound of the above right-hand side is found that depends on the control u. For that reason, we exploit (56a) after applying Lemma A.3 to find an expression for $\mathbb{E}\left[x_+(t)^\top \Sigma_k^{-1} x_+(t)\right]$ based on (57b). Consequently,

$$\begin{aligned} \mathbb{E}\left[x_+(t)^\top \Sigma_k^{-1} x_+(t)\right] \leq{}& \mathbb{E} \int_0^t x_+(s)^\top C_k^\top C_k x_+(s) ds \\ &+ \mathbb{E} \int_0^t -x_+(s)^\top \Sigma_k^{-1} B_k B_k^\top \Sigma_k^{-1} x_+(s) + 4x_+(s)^\top \Sigma_k^{-1} B_k u(s) ds \\ &- \sigma_k^{-1} \mathbb{E} \int_0^t 2x_{k,2}(s)c_0(s) + \sum_{i,j=1}^q \left(2 \left[\, n_{i,21} \;\; n_{i,22} \,\right] x_+(s) - c_i(s)\right) c_j(s) k_{ij} ds. \end{aligned} \tag{60}$$

With the same argument like in (53), it can be shown that

$$4\,\|u(s)\|_2^2 \geq -x_+(s)^\top \Sigma_k^{-1} B_k B_k^\top \Sigma_k^{-1} x_+(s) + 4x_+(s)^\top \Sigma_k^{-1} B_k u(s).$$

Using the definitions of x_+ and c_i, it immediately follows that $\mathbb{E} \int_0^t 2x_{k,2}(s)c_0(s) + \sum_{i,j=1}^q \left(2 \left[\, n_{i,21} \;\; n_{i,22} \,\right] x_+(s) - c_i(s)\right) c_j(s) k_{ij} ds = \alpha_k(t)$. Inserting these insights into (60), we obtain

$$\mathbb{E}\left[x_+(t)^\top \Sigma_k^{-1} x_+(t)\right] \leq \mathbb{E} \int_0^t x_+(s)^\top C_k^\top C_k x_+(s) ds + 4\,\|u\|_{L_t^2}^2 - \sigma_k^{-1} \alpha_k(t).$$

Since it holds that $\left\| C_k \Sigma_k^{\frac{1}{2}} \right\|_2^2 \leq \left\| C_n \Sigma_n^{\frac{1}{2}} \right\|_2^2 = \left\| C P^{\frac{1}{2}} \right\|_2^2 \leq \beta$, we have

$$\mathbb{E}\left[x_+(t)^\top \Sigma_k^{-1} x_+(t)\right] \leq \beta \mathbb{E} \int_0^t x_+(s)^\top \Sigma_k^{-1} x_+(s) ds + 4\,\|u\|_{L_t^2}^2 - \sigma_k^{-1} \alpha_k(t).$$

Lemma A.1 now delivers

$$\mathbb{E}\left[x_+(t)^\top \Sigma_k^{-1} x_+(t)\right] \leq \beta \int_0^t [4\,\|u\|_{L_s^2}^2 - \sigma_k^{-1} \alpha_k(s)] \,\mathrm{e}^{\beta(t-s)} ds + 4\,\|u\|_{L_t^2}^2 - \sigma_k^{-1} \alpha_k(t).$$

Again, integration by parts leads to

$$\mathbb{E}\left[x_+(t)^\top \Sigma_k^{-1} x_+(t)\right] \leq \int_0^t [4\mathbb{E}\,\|u(s)\|_2^2 - \sigma_k^{-1} \dot{\alpha}_k(s)] \,\mathrm{e}^{\beta(t-s)} ds.$$

Consequently, we find that

$$\int_0^T \dot{\alpha}_k(s)\, \mathrm{e}^{-\beta s}\, ds \le 4\sigma_k \mathbb{E} \int_0^T \|u(s)\|_2^2\, \mathrm{e}^{-\beta s}\, ds.$$

Using this estimate for (59), the result follows from (55). □

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