WILLMORE BOUNDARY VALUE PROBLEMS

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ZUSAMMENFASSUNG

Diese Arbeit widmet sich der Suche nach berandeten Flächen, die sich als kritische Punkte des Willmore-Funktionals unter Dirichlet-Randbedingungen erweisen. Dabei beschränken wir uns auf Flächen, die sich als Graphen einer reellen Funktion im $\mathbb{R}^2$ darstellen lassen. Das bedeutet, dass sie im $\mathbb{R}^3$ eingebettet und gleichzeitig auf $\mathbb{R}^2$ projizierbar sind. Dieser Ansatz hat Vorteile, wie die Kontrolle des Durchmessers und des Flächeninhalts, sowie die Tatsache, dass die explizite Wahl der Koordinaten oft analytische Berechnungen vereinfacht. Es gibt jedoch auch Nachteile, wie zum Beispiel die Möglichkeit, die Projizierbarkeit durch Minimierungsprozesse zu verlieren.

Eine Strategie besteht darin, die Euler-Lagrange-Gleichung, die hier als Willmore-Gleichung bezeichnet wird, für die Graphenfunktion umzuschreiben und als ein elliptisches Randwertproblem zu lösen. In dieser Arbeit wurde die Willmore-Gleichung als ein biharmonischer Operator mit einer rechten Seite in Divergenzform umgeschrieben. Dies ermöglichte es uns, unter Verwendung des Linearisierungsverfahrens und gewichteter Sobolev-Räume die Existenz einer im Inneren glatten Lösung bloß unter einer $C^{1+\alpha}$-Kleinheit an die Randdaten zu zeigen.

Eine andere Möglichkeit besteht darin, den Gradientenfluss des Willmore-Funktionals, den sogenannten Willmore-Fluss, zu betrachten. Wir haben ihn für die Graphenfunktion als eine parabolische Gleichung vierter Ordnung umgeschrieben. Mit Hilfe von zeitgewichteten parabolischen Hölder-Räumen konnten wir die Kurzzeitexistenz für $C^{1+\alpha}$-glatte Anfangsdaten und $C^{4+\alpha}$-Dirichlet-Randdaten ableiten. Darüber hinaus wurde die Langzeitexistenz mit Konvergenz gegen einen kritischen Punkt für ausreichend kleine $C^{1+\alpha}$-Normen der Anfangsdaten bewiesen. Wenn wir die Divergenzstruktur der Willmore-Flussgleichung ausnutzen, können wir sogar Kurz- und Langzeitexistenz mit Zeitkonvergenz gegen einen kritischen Punkt für ausreichend kleine $C^{2+\alpha}$-Normen der Anfangs- und Randdaten in ungewichteten parabolischen Räumen zeigen.

Ein weiteres Werkzeug, das wir verwenden, ist die Untersuchung der Kompaktheitseigenschaften von Willmore-Minimalfolgen. Dabei bauen wir auf den Ergebnissen von Deckelnick, Grunau und Röger auf, die zuerst die $W^{1,1} \cap L^\infty$-Norm durch die Willmore-Energie und die Randdaten beschränkt haben. Anschließend haben sie im Kontext von BV die L^1-Relaxation des Willmore-Funktionals definiert und einen Minimierer gefunden. Um die Frage der Regularität zu klären, versuchen wir, die relaxierte Willmore-Energie zu charakterisieren. Deckelnick, Grunau und Röger haben den Anteil der Energie beschrieben, der aus dem absolut stetigen Anteil von ∇u stammt. In dieser Arbeit gelingt es uns, unter Verwendung von Varifaltigkeiten und Maß-Funktionspaaren einen zusätzlichen Anteil hinzuzufügen, der auch vertikale Komponenten beschreiben kann. Schließlich zeigen wir anhand eines Gegenbeispiels, dass eine endliche relaxierte Willmore-Energie einen Cantor-Anteil nicht ausschließt.

Abstract

This thesis is devoted to the search for surfaces with boundary that serve as critical points of the Willmore functional under Dirichlet boundary conditions. Our focus is on surfaces that can be expressed as graphs of real functions defined on $\mathbb{R}^2$. These surfaces are embedded in $\mathbb{R}^3$ while also projecting onto $\mathbb{R}^2$ simultaneously. This approach offers several advantages, including control over diameter and surface area and the fact that explicit coordinates often simplify analytical calculations. However, it also comes with disadvantages, such as the potential loss of projectability during minimization processes.

One strategy involves reformulating the Euler-Lagrange equation, referred to here as the Willmore equation, for the graph function and solving it as an elliptic boundary value problem. In this work, we express the Willmore equation as a biharmonic operator with a right-hand side in divergence form. This approach allows us to demonstrate the existence of a solution smooth in the interior, provided that the $C^{1+\alpha}$-norm of the boundary data is small enough, using linearization techniques and weighted Sobolev spaces.

Another possibility is to examine the gradient flow of the Willmore functional, known as the Willmore flow. We rewrite it for the graph function as a fourth-order parabolic equation. By employing time-weighted parabolic Hölder spaces, we establish short-term existence for initial data with $C^{1+\alpha}$-smoothness and Dirichlet boundary data with $C^{4+\alpha}$-regularity. Furthermore, we prove long-term existence with convergence toward a critical point for sufficiently small $C^{1+\alpha}$-norms of the initial data. Leveraging the divergence structure of the Willmore flow equation, we can even demonstrate short- and long-term existence with convergence over time to a critical point for sufficiently small $C^{2+\alpha}$-norms of both the initial and boundary data in unweighted parabolic spaces.

Another tool we use is to study the compactness properties of Willmore minimal sequences. We are building upon the results of Deckelnick, Grunau, and Röger, who initially bounded the $W^{1,1} \cap L^\infty$-norm of the boundary data by Willmore energy and boundary data and further investigated the L^1-relaxation of the Willmore functional in the context of BV. We aim to characterize the relaxed Willmore energy, adding contributions not only from the absolutely continuous part of ∇u but also from vertical components. Finally, we provide a counterexample demonstrating that finite relaxed Willmore energy does not exclude the existence of a Cantor component.

Contents

1 Introduction

1.1 State of the Art

In this work, we study boundary value problems for the graphical case within both elliptic and parabolic settings for the first variation of the Willmore energy. This is a particularly interesting and important topic in the field of calculus of variations and partial differential equations with a geometric background. Geometrically, we consider sufficiently smooth two-dimensional surfaces S, with or without boundary, mostly embedded in $\mathbb{R}^3$ and therefore with some normal vector field $N\colon S \to \mathbb{S}^2$ orthogonal to tangent space on each point. Then the Willmore energy of S is defined by

$$\mathcal{W}(S) := \frac{1}{4}\int_S H_S^2 \,\mathrm{d}S \tag{1}$$

with the mean curvature $H_S = \kappa_1 + \kappa_2$ defined as the sum of the principal curvatures. It is worth noticing that regarding the normalization factor $\frac{1}{4}$ in front of the integral, there are several values in the literature. In this setting, the term H_S^2 measures the local density of how the surface is curved from the extrinsic point of view. More precisely, this can be seen as averaged curvature at some point on S with respect to the ambient space, where, for example, curvatures on both sides of a saddle-shaped surface average out. If $H > 0$ at some point p_0 on the surface, then we view the surface as being local bending in average towards $N(p_0)$ the normal vector of S at p_0 in average. In case $H < 0$, the surface is bending more towards $-N(p_0)$. Therefore, $\mathcal{W}(S)$ describes the total bending energy of the surface S. Already introduced by Germain in [Ger13] and Poisson at the beginning of the 19th century, it was considered again early in the 20th century by Thomsen [Tho23] in the conformal geometry framework and then popularized by Willmore.

Mathematically, one can regard the Willmore functional as the second-simplest interesting differential geometric functional, next to the area functional. In particular, the link to minimal surfaces (minimizers of the area functional) is obvious since these have vanishing mean curvature and, therefore, minimize the Willmore energy. Hence, the Willmore surfaces, among other things, generalize the concept of minimal surfaces.

For surfaces with prescribed boundary or even without a boundary at all (1) is equivalent to $\int_S \|A\|^2 \,\mathrm{d}S$ or $\int_S (H_S^2/4 - \mathcal{K}_S)\,\mathrm{d}S$, with Gaussian curvature $\mathcal{K}_S = \kappa_1 \cdot \kappa_2$ and second fundamental form A describing local variance of the normal fields. One of the most characteristic geometric properties of the Willmore functional is the invariance of $\int_S (H_S^2/4 - \mathcal{K}_S)\,\mathrm{d}S$ under conformal transformations $\Phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ of the ambient space, i.e., under Möbius transformations and especially scaling, rotation, and inversions. This was proved by Willmore in [Wil96] and earlier by Blaschke [Bla24].

The Willmore energy for two-dimensional surfaces in $\mathbb{R}^3$ arises not only in a mathematical context. For example, it is also widely used in the modeling of thin elastic plates that resist bending, where it was first studied by Germain and Poisson [Ger13]. Furthermore, the Willmore energy arises in the description of biomembranes as lipid bilayers introduced by Helfrich and Canham in [Hel73] and [Can70] as a term in the Canham-Helfrich energy functional, which also includes area, Gaussian curvature, and spontaneous curvature terms.

One interesting topic to study is the Euler-Lagrange equation for the stationary problem associated with the Willmore energy. For a sufficiently smooth surface, S we introduce the Willmore equation by setting the first variation of the Willmore energy [Dal12, p.7 Remark 2.3, (2.1)] equal to zero

$$\Delta_S H_S + 2H_S \left(\frac{1}{4}H_S^2 - \mathcal{K}_S\right) = 0 \quad \text{on } S \tag{2}$$

with Δ_S the Laplace Beltrami operator on S. One of its basic properties is given by the fact that it defines a quasilinear equation of fourth order that is elliptic, but not uniformly. Namely, when large variations of the tangent planes occur, then a strong degeneration of ellipticity takes place. Also, in [Wil96] Willmore proved that smooth solutions of the Willmore equation, called the Willmore surfaces, are critical points of the Willmore functional. In the present work, we especially want to know whether such critical points exist and, if they exist, how regular they are. Comparing the Willmore energy with the area functional, the analogue to the Willmore equation is $H = 0$, the minimal surface equation which is discussed in detail in [DHS10].

Furthermore, interesting developments occur from studying the L^2-gradient flow of the Willmore energy, called the Willmore flow. To describe the evolution of a surface under such a flow, one considers a sufficiently smooth one-parameter family of immersions $f\colon [0,T) \times \Sigma \to \mathbb{R}^3$ with $T > 0$ and $\Sigma \subset \mathbb{R}^3$ some fixed surface (two-dimensional submanifold of $\mathbb{R}^3$). Therefore, $f(t,\Sigma)$ is a surface with $\nu(t,\,.\,)\colon \Sigma \to \mathbb{S}^2$ some normal vector field on $f(t,\Sigma)$. Then, such a family of immersions solves the Willmore flow equation if

$$\begin{cases} \langle \partial_t f, \nu \rangle = -\left\{\Delta_{f(t,\Sigma)} H_{f(t,\Sigma)} + 2H_{f(t,\Sigma)} \left(\frac{1}{4}H^2_{f(t,\Sigma)} - \mathcal{K}_{f(t,\Sigma)}\right)\right\} & \text{in } [0,T) \times f(\Sigma) \\ f(0,\,.\,) = f_0 \quad \text{in } \Sigma \end{cases} \tag{3}$$

where $\langle \partial_t f, \nu \rangle$ represents the normal velocity and $f_0\colon \Sigma \to \mathbb{R}^3$ is some given immersion. The parameter T plays the role of the lifespan of the solution, whereas in case $T = \infty$ we call such a solution global. In particular, this is a quasilinear (not strictly) parabolic evolution equation of fourth order, related to the Willmore equation. Depending on the setting, we are interested in studying the existence, regularity, and uniqueness of solutions. If we again compare the Willmore case with the area functional, one could relate the Willmore flow equation to the mean curvature flow, which has been widely worked on.

One of the motivations to study this gradient flow is that after proving short-term existence, one hopes to derive some bounds on the local solution. The first one is that the Willmore energy is decreasing. From these bounds and further necessary conditions on data, it may be possible to prove global existence. For $T \to \infty$, one expects a stationary solution that satisfies the Willmore equation. Thus, solving the Willmore flow problem provides an alternative approach to the existence of the Willmore surfaces.

Numerically, due to their smoothing and other properties, the Willmore flow techniques are used in surface restoration [CDD+04]. In this framework, one wants to replace a damaged region of a surface, for example, a broken statue, with a surface patch given by a Willmore surface.

There are basically two most common surface classes to investigate. Namely, 2–dimensional manifolds without boundary, called closed surfaces and surfaces with prescribed boundary ∂S. Additionally, there is a distinction between bounded and unbounded surfaces, where, except in the introduction, we consider only compact surfaces.

1.1.1 Closed Surfaces

In this case, both the elliptic and the parabolic problem corresponding to the Willmore energy are rather well studied. Foremost, considering the elliptic case, in contrast to the minimal surfaces, compact Willmore surfaces without boundaries exist, and the most obvious example is the sphere. In addition, there are some interesting bounds on the Willmore energy for closed surfaces. Already

Willmore proved [Wil96] by using the Gauss-Bonnet-theorem that for any closed two-dimensional surface as immersion $f\colon \Sigma \to \mathbb{R}^3$ one has

$$\mathcal{W}(f(\Sigma)) \geq 4\pi$$

with equality if and only if $f(\Sigma)$ is a round sphere. Based on the divergence theorem on manifolds, Li and Yau [LY82] proved that if $\Sigma \subset \mathbb{R}^3$ is a closed smooth surface and $f\colon \Sigma \to \mathbb{R}^3$ is an immersion of multiplicity $k \in \mathbb{N}$ then $W(f(\Sigma)) \geq k \cdot 4\pi$.

After gaining experience with the topological class of tori, in 1965 Willmore conjectured that any immersion $f\colon T \to \mathbb{R}^3$ of the two-dimensional torus T into $\mathbb{R}^3$ would satisfy

$$\mathcal{W}(f(T)) \geq 2\pi^2$$

with equality, if and only if T is in the conformal class of the Clifford torus. This statement stayed very challenging for the mathematical community for decades until 2012, when Marques and Neves proved this conjecture in [MN14a] by using Almgren-Pitts min-max theory (see [MN14b] for references).

Whether closed compact Willmore minimizers exist in the class of genus-g-surfaces for any given genus $g \in \mathbb{N}_0$ was positively answered by Simon in [Sim93] combined with work of Bauer and Kuwert in [BK03].

The Willmore flow for closed compact two-dimensional immersed surfaces in $\mathbb{R}^3$ and also in higher dimensions was deeply investigated in various works by Simonett, Schätzle, and Kuwert. In [Sim01] Simonett proved that solutions exist globally and converge exponentially fast to a sphere, provided that they start close to spheres with respect to the $C^{2+\alpha}$-topology. Later, Kuwert and Schätzle proved in [KS01], [KS04] in codimension one that for initial energy less than or equal to 8π, the Willmore flow of immersions of the sphere exists for all time and converges to a round sphere. Also in [KS02] they have given a lower bound on the lifespan of a smooth solution, depending solely on how much the curvature of the initial surface is concentrated in space. Later, in [CFS09] Chill, Fasangova and Schätzle proved that in the case that the initial surface $f_0(M)$ is $W^{2,2} \cap C^1$-close to a C^2 local minimizer of the Willmore functional, then there exists a global solution for the Willmore flow with initial data f_0 that converges to a C^2 local Willmore minimizer after some reparametrization.

Even though a lot of research has been done in the area of geometric evolution equations of higher order, the understanding of whether without smallness conditions the Willmore flow develops singularities in finite or infinite time is far from being complete. Furthermore, it is still not clear how to extend the flow after such a singularity. Numerically, Mayer and Simonett provided in [MS02] numerical evidence that the Willmore flow may develop singularities in finite time if a smallness condition is violated. In [Bla09] Blatt gave an example with a singularity to form, which has its Willmore energy arbitrarily close (from above) to 8π and does not converge to a Willmore immersion under the Willmore flow. It happens that either the diameter of the surface becomes unbounded or a small quantum of the curvature concentrates in finite or infinite time.

We should also mention some results considering unbounded surfaces. By using calculations done in [DD06] by Dziuk and Deckelnick, Koch and Lamm showed [KL12] the existence of a globally unique and analytic solution for the Willmore flow (besides other geometric flows) for graphs on $\mathbb{R}^2$ (so-called entire graphs) with Lipschitz initial data and small Lipschitz norm. There, they heavily used the scaling behavior of the Willmore flow and some special structure of the Willmore equation written in the graphical case. We will also intensively use this kind of structure in the present thesis.

1.1.2 Surfaces with Boundary

In order to obtain a potentially well-posed problem, appropriate boundary conditions have to be added to the Willmore equation, which can get quite involved. Structurally, it is a fourth-order equation. Hence, we need two sets of conditions on boundary values. In [Nit93] the author provided a variety of possible choices accompanied by corresponding existence results for small data in strong topologies. Also in [BGN17] various boundary conditions are discussed.

There are two kinds of boundary conditions we want to mention. The first is the so-called Dirichlet problem. In this setting, we are searching for the Willmore surfaces S in $\mathbb{R}^3$, or even the Willmore minimizers, where its boundary ∂S and the corresponding tangential planes along ∂S are both prescribed. It is also called a clamped boundary condition since it fixes the position of the boundary and the angle (relative to ambient space) with which the surface meets its boundary. The second boundary condition is the Navier problem, where we replace the angle condition by prescribing the mean curvature of the surface on the boundary ∂S. Indeed, by setting the vanishing mean curvature of S on the boundary, we get natural conditions arising from the first variation of the Willmore functional.

Boundary value problems for the Willmore surfaces and the Willmore flow evolution become more involved, and much less is known when compared with closed surfaces. One of the reasons is that we cannot directly apply scaling arguments. The other is that, in general, no a-priori bounds are known neither for the solution of the Willmore equation nor for the Willmore energy-minimizing sequences or minimizers. In the general case, only if the Willmore energy is lower than 4π, the diameter and area are both bounded by the Willmore energy and the length of boundary ∂S as provided in Subsection 4.1. In contrast, as shown by Grunau, Deckelnick, and Röger [DGR17, p.5 Theorem 2] for the graphical case, both diameter and area can be bounded for arbitrary fixed Willmore energy.

In general, since the Willmore equation is strongly nonlinear, uniqueness of a solution may not be expected, see [Eic16]. Moreover, for the one-dimensional variant of the Willmore equation with Navier boundary conditions (in that case, the position and the curvature are prescribed on ∂S), Deckelnick and Grunau [DG07, Theorem 1] provided two symmetric solutions, if the boundary conditions lie in some special admissible range. Furthermore, in the framework of the Willmore surfaces of revolution with Navier boundary conditions and vanishing mean curvature at the boundary, Dall'Acqua, Deckelnick, and Wheeler [DDW13] provided the existence of three different solutions to the same data. Despite that, Dall'Acqua [Dal12] showed uniqueness in the case of the boundary of a Willmore surface touching a sphere or a plane tangentially with the condition that the curves bound a strictly star-shaped domain with respect to the corresponding geometry. Then, the Willmore surface is a part of that sphere or the plane, respectively. This is a consequence of invariances for trivial data and does not rule out the possibility of non-uniqueness for non-trivial data.

For the Navier boundary value problem of the higher-dimensional Willmore flow with vanishing mean curvature on the boundary, Menzel proved (see her very interesting thesis [Men21]) short-time existence where the initial data satisfy some regularity and compatibility conditions with the boundary data and the solution is a graph over a reference manifold. This is based on using higher order (fourth order in space and first order in time) anisotropic Sobolev spaces on manifolds as solution space.

To get an idea of which kind of phenomena may be expected, we can restrict ourselves to some special situations imposing different kinds of symmetry or projectability conditions on the surface under consideration and hope that some geometric and analytic information on the Willmore energy, minimizers, or minimizing sequences will be obtained. As the degree of symmetry or projectability decreases, obtaining results gets more and more difficult.

In this work, we will use almost always projectability. More precisely, in the graphical case in $\mathbb{R}^3$ we represent a surface S by a parametrization $\overline{\Omega} \ni (x_1, x_2) \mapsto (x_1, x_2, u(x))^T$ where $\Omega \subset \mathbb{R}^2$ is a sufficiently smooth bounded domain and $u\colon \overline{\Omega} \to \mathbb{R}$ is a sufficiently smooth mapping. We especially write $S = \Gamma(u)$ and call it a graph of u. Then the Willmore equation can be rewritten as a nonlinear biharmonic equation in the form

$$\Delta^2 u = F(D^4u, D^3u, D^2u, \nabla u) \quad \text{in } \Omega, \tag{4}$$

where $\Delta^2 = \Delta\Delta$ is the biharmonic operator and $F(D^4u, D^3u, D^2u, \nabla u)$ is a non-linear polynomial in D^4u, D^3u, D^2u, ∇u and $\sqrt{1 + \|\nabla u\|^2}^{-1}$ that will be linear in D^4u and D^3u. Moreover, all non-vanishing monomials are at least of degree three. The equation (4) can be used for proving the existence in spaces Sobolev space $W^{4,p}(\Omega)$ oder Hölder $C^{4+\alpha}(\Omega)$. Studying the divergence structure of the right side $F(D^4u, D^3u, D^2u, \nabla u)$ like in [KL12] will allow us to use weaker solution spaces of second order instead of fourth order.

Regarding the Dirichlet boundary condition, the elliptic problem is given by prescribed boundary values of the solution and its normal derivative on the boundary $\partial\Omega$.

$$\begin{cases} \Delta_{\Gamma(u)} H_{\Gamma(u)} + 2H_{\Gamma(u)} \left(\dfrac{1}{4} H^2_{\Gamma(u)} - \mathcal{K}_{\Gamma(u)} \right) = 0 & \text{in } \Omega, \\ u = g_0, \quad \dfrac{\partial u}{\partial \nu} = g_1 & \text{on } \partial\Omega \end{cases} \tag{W}$$

with some sufficiently smooth functions g_0 and g_1 on $\partial\Omega$.

In the parabolic graphical case, we can likewise rewrite the Willmore flow equation as a biharmonic heat flow equation with nonlinear right-side

$$\partial_t u + \Delta^2 u = F(D^4u, D^3u, D^2u, \nabla u) \quad \text{in } \Omega, \tag{5}$$

where F up to the factor $\sqrt{1 + |\nabla u|^2}$ plays the same role as in (4). For the Willmore flow equation, we study the immersion mapping $f(t, x) = (x, u(x, t))^T, x \in \Omega$ which in this case is an embedding and consider the parabolic Dirichlet problem with respect to upward normal

$$\begin{cases} \partial_t u + Q\left\{ \Delta_{\Gamma(u)} H_{\Gamma(u)} + 2H_{\Gamma(u)} \left(\dfrac{1}{4} H^2_{\Gamma(u)} - \mathcal{K}_{\Gamma(u)} \right) \right\} = 0 & \text{in } \quad \overline{\Omega} \times (0, T], \\ u(x, t) = g_0(x), \quad \dfrac{\partial u}{\partial \nu}(x, t) = g_1(x), & (x, t) \in \partial\Omega \times [0, T], \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{WF}$$

where $Q = \sqrt{1 + |\nabla u|^2}$. The associated numerical C^1-finite element method for this problem was provided by Deckelnick, Katz, und Schieweck in [DKS15] with quasioptimal error bounds in Sobolev norms for the solution and its time derivative. Additionally, we need some compatibility conditions

$$g_0 = u_0(x), \quad g_1(x) = \frac{\partial u_0}{\partial \nu}(x), \qquad x \in \partial\Omega. \tag{CC}$$

for the solution u to be at least $C^1(\overline{\Omega})$.

1.1.3 Direct Methods of the Calculus of Variations

The basic idea in the direct methods is to consider some (possibly improved) minimizing sequences $\{v_k\}_{k\in\mathbb{N}}$ in a suitable space satisfying the boundary conditions. One idea may be to obtain compactness by modifying this sequence such that some norm of the sequence itself stays bounded. If these bounds turn out to be strong enough, then we can use the usual local-weak-compactness and weakly-lower-semicontinuity reasoning [GO23] in the Sobolev space $W^{2,2}$ to get the existence and regularity of the minimizing solution. In the case of surfaces of revolution, this idea works out, as shown in [DDG08] and [DFGS11] by Dall'Acqua, Deckelnick, Fröhlich, Grunau, and Schieweck. However, the construction of such modifications is not obvious at all and is very subtle. For now, no such construction is known for the graphical general two-dimensional case, and it is not clear how to meaningfully choose or modify the minimizing sequence here since to stay in $W^{2,2}$ one has to match not only the functions but also their derivatives.

Despite that, Deckelnick, Grunau, and Röger [DGR17] developed a new framework for graphs involving a L^1-lower semicontinuous relaxation of the Willmore functional $\overline{\mathcal{W}}$. By assuming graphical projectivity as some sort of obstacle condition, they derived general area and diameter bounds for all possible values of the Willmore energy [DGR17, Theorem 2]. In this way, they obtained bounds in the space $BV(\Omega) \cap L^\infty(\Omega)$, which they have naturally chosen to be the solution class. They also proved that this L^1-lower semicontinuous envelope $\overline{\mathcal{W}}$ is actually the largest possible L^1-lower semicontinuous extension of $\mathcal{W}$ to $L^1(\Omega)$. Finally, by using direct methods, they showed the existence of a minimizer in the class $BV(\Omega) \cap L^\infty(\Omega)$.

To obtain compactness more easily, Schätzle, Kuwert [KS01], Simon [Sim93] and others have chosen the realm of varifolds, that is to say, geometric measure theory. Here, we refer to Menne [Men17]. These are, in general, non-parametric non-smooth surfaces associated with a measure and have generalized mean curvature. This approach allows for separating the existence proof from regularity discussions. As previously mentioned, this often comes at the expense of lacking regularity. Introduced by Almgren as the foundation stone for showing the existence of a generalized minimal surface, it has turned out to be highly influential in geometric analysis; see the proof of the Willmore conjecture by F. Marques and A. Neves [MN14a]. Furthermore, Menne [Men13] was able to show that m-dimensional curvature varifolds, which also have a generalized second fundamental form, can be covered by a countable collection of C^2-regular m-dimensional submanifolds of $\mathbb{R}^n$ up to a null set. Additionally, the second fundamental form of the varifold agrees almost everywhere with the second fundamental form induced by these C^2-submanifolds.

Regarding the Dirichlet problem, Schätzle [Sch10] showed the existence of the Willmore minimizers in $\mathbb{S}^3 \equiv \mathbb{R}^3 \cup \{\infty\}$ in a very general context, making use of geometric measure theory. In fact, the minimizers may be branched, unbounded, not a graph, or even contain ∞. A slightly different classical parametric approach was chosen by Da Lio-Palmurella-Rivière in [DLPR20] with an area constraint, where the authors also obtained the existence of the Willmore surfaces with prescribed boundary and boundary Gauss map.

The case of surfaces with a smooth closed embedded planar curve as boundary and fixed topology was studied by Pozzetta in [Poz21]. Using an approach similar to that presented by Simon in [Sim93], he proved that minimizers do not exist for the minimization problem of the Willmore energy given genus $\mathfrak{g} \geq 1$ and natural Navier boundary condition $H = 0$, where only the position of the boundary curve is given by a fixed circle. Thus, this is a minimization problem without clamped condition, where the conormal is free. Despite the non-existence result, he was able to give the infimum value of the Wilmore energy, which is the energy of the closed minimizing surface of genus $\mathfrak{g}$ minus 4π.

Similar to the area functional, the following phenomenon may occur. If one considers compact minimizers in $\mathbb{R}^3$ with boundaries consisting of a given family of disconnected smooth closed curves, it may happen that direct minimization of the Willmore energy leads to limits that are also disconnected. Thus, it makes sense to impose topological constraints on the minimizer. In [NP20a] Novaga and Pozzetta considered connected compact surfaces. Provided that the infimum of the problem is strictly smaller than 4π, they proved the existence of a connected compact Willmore surface in the class of integer rectifiable curvature varifolds with the assigned boundary conditions.

They used sequences of varifolds with boundary and uniformly bounded Willmore energy and concluded the convergence of their supports in Hausdorff distance.

1.1.4 Higher-order Equations

Instead of considering the Willmore equation and the Willmore flow equation from a geometrical perspective, we can investigate these equations as purely analytical fourth-order boundary value problems with a nonlinear right-hand side. Then it is possible to apply versatile results both on elliptic and parabolic higher-order operators, where much progress has been made in already 60s and 70s by Krasovskiĭ [Kra67a], Belonosov [Bel79] and Solonnikov [Sol65, LSU68] as well as more recently by Maz'ya, Mayboroba and Barton [MM09, MMS10, BM16].

The most famous example of higher-order elliptic operators, namely the polyharmonic operator, is iteratively defined by $\Delta^m = \Delta\Delta^{m-1}, m \in \mathbb{N}$. Corresponding boundary value problems have been intensively investigated. We refer the reader to [GGS10], [ADN59], and the survey [BM16] for non-smooth domains and references therein. It should be noted that the existence and regularity of the solution of the Dirichlet problem strongly depend on the chosen solution space, the regularity of the coefficients, and the boundary data, as well as the regularity of $\partial\Omega$.

First, like in [Gru18], we want to point out the differences to the well-studied elliptic operators of second order, like the Laplace operator, which causes more intricacies in the case of fourth and higher-order operators. One of the main difficulties is the lack of general maximum and comparison principles and Harnack inequalities. Since the Willmore functional also involves second derivatives, one cannot simply pass to u^+ or $|u|$ like in Stampacchia's approach to the maximum principle [MS68]. These phenomena already occur with the biharmonic operator and the corresponding functional $u \mapsto \int_\Omega (\Delta u)^2 \,\mathrm{d}x$. Despite that, for higher-order elliptic operators, there are still some situations [GGS10] where the positivity of solutions is preserved.

For the study of inhomogeneous elliptic boundary problems [ADN59], much work was put into deriving estimates of the fundamental solutions and the corresponding Green function. Basing upon the explicit formula for the polyharmonic operator in a ball by Boggio [Bog05] optimal (from above and below by multiples of the same function) estimates were derived by Grunau, Sweers, and Dall'Acqua [GS97, DAS04]. The Green functions for general higher-order operators were considered by Krasovskiĭ [Kra67a, Kra67b], but at the cost of high regularity imposed on the boundary. Terms modeling the boundary behavior were added by Dall'Acqua and Sweers [DAS04]. In general, the Green function may change sign. Therefore, it is important to show that the negative part is small in the sense that it is bounded by the product of the squared distances to the boundary, which was provided by Grunau, Robert, and Sweers [GRS11]. In particular, they estimated Green functions plus suitable multiples of these boundary terms from above and from below by the same positive prototype function. For the case of non-smooth domains, Green function estimates are due to Mayboroda and Maz'ya [MM09]. These results also allow sharp pointwise and L^p- estimates on derivatives of the solution.

Various works are concerned with the parabolic case. The classic par excellence is the work by Ladyzenskaja, Solonnikov, and Ural'ceva [LSU68]. Later, Dong and Zhang [DZ15] proved Schauder estimates for solutions of $2m$th-order parabolic systems both in divergence and nondivergence case with boundary data in the cylindrical domain over a $C^{m,\alpha}$-smooth base domain. These estimates also hold for systems with time-irregular coefficients for operators. They also showed [DK11] L^p-solvability of higher order parabolic systems with leading coefficients in BMO spaces, see (108) in Subsection 5.4. In general, more rough initial data, where, in particular, certain compatibility conditions on their derivatives are not to be assumed, lead to an initial loss of regularity. This issue was addressed in the framework of weighted parabolic Hölder spaces by Belonosov [Bel79] and further by Solonnikov and Khachatryan [SK80] in a slightly more general situation. For the Willmore flow with rough initial data in Lipschitz class for entire graphs we again refer to [KL12].

For all existence results in this work, we use linearization methods combined with Banach fixed point theorem (contraction mapping principle). This is possible since both in the elliptic and the parabolic case, one can separate a linear elliptic operator and a right-hand side consisting of lower-order terms like in [KL12] with a degree of at least three and term linear D^4u and containing to $|\nabla u|^2$ (see (R), (L). This method requires, as usual, smallness conditions for the data in the elliptic case or short-time existence in the parabolic case. However, exploiting the divergence form of the problem and estimates adapted to this, we can work in much weaker (larger) spaces, like weighted Sobolev and parabolic Hölder spaces, than previous works as [Nit93]. Morally in the present work, it suffices to consider (small) boundary data in $C^{1+\varepsilon}$, see Theorem 1. We describe our results in some detail below in Section 1.2.

1.1.5 Further Research

In this section, we want to briefly mention some further research related to the study of the Willmore energy, but without direct overlap with this thesis. This list is by no means complete.

- ◇ Instead of assuming projectivity, one can take axial symmetry for two-dimensional surfaces embedded in $\mathbb{R}^3$. Consequently, surfaces of revolutions arise. These are then described by their profile curves, which are graphs over one-dimensional intervals. This comes with the huge advantage that the Willmore equation and, in particular, its analysis stay one-dimensional. As already mentioned, much work [DDG08, DFGS11] was done by Grunau, Dall'Acqua, Deckelnick, Fröhlich, and Schieweck. Surprisingly, in this setting, one can rewrite the Willmore functional in the hyperbolic half plane as a simple one-dimensional curvature integral [HJP92] which was introduced by Bryant-Griffith [BG86], then fruitfully used by Langer-Singer [LS84] for curves parameterized by arclength and later by Eichmann [Eic16, Eic17] for, among others, a nonuniqueness result for the Willmore surfaces of revolution with dirichlet data.

- ◇ Another possibility is to consider the one-dimensional Willmore energy, also called elastic energy. Here, one assumes invariance with respect to translations in a chosen direction. For a regular and sufficiently smooth curve $\gamma\colon I \to \mathbb{R}^n, n \geq 2$ it is given by

  $$\mathcal{E}(\gamma) = \int_I |\vec{\kappa}_\gamma|^2(s)\,\mathrm{d}s \tag{6}$$

  where $s = |\partial_x \gamma|\,\mathrm{d}x$ denotes the arclength element and $\vec{\kappa}_\gamma = \partial^2_{ss}\gamma$ denotes the curvature vector of γ with s the arclength. Critical points of $\mathcal{E}(\gamma)$ are called elastic curves that satisfy the one-dimensional Willmore equation

  $$\partial^2_{ss}\kappa + \frac{1}{2}\kappa^3 = 0 \quad \text{on } \gamma(I).$$

  For the graphical case in $\mathbb{R}^2$ it was studied by Deckelnick and Grunau [DG07, DG09] under Dirichlet as well as under Navier boundary conditions. For suitable boundary data, they investigated the symmetry and stability properties of multiple solutions and provided some closed expressions. For the symmetric case, then applied an idea of Euler, see [Eul52, pp. 233-234].

  For the case of general Willmore curves in $\mathbb{R}^2$, Mandel [Man15] solved the Navier problem and the Dirichlet problem. For open curves in Euclidean space subject to clamped boundary conditions and L^2-flow of elastic curves, Lin [Lin12] showed long-time existence of solutions. Moreover, in [DPS16] Dall'Acqua, Pozzi, and Spener proved that the solution to the one-dimensional Willmore flow L^2-converges for large time to a critical point of the functional. They used a Łojasiewicz–Simon gradient inequality for the elastic energy.

◇ One further recent development concerns obstacle problems for Willmore energy, where the admissible functions have to be above the given obstacle. The one-dimensional graphical case with Navier boundary conditions was considered by Dall'Acqua-Deckelnick [DD18]. Müller extended this approach to some larger class of pseudographs and provided in [Mül19] nonexistence results for what he calls large cone obstacles in the case of graph curves. Further, he also studied [Mül20, Mül21] the Willmore gradient flow with obstacles. For surfaces of revolution, the obstacle problem was investigated by Okabe and Grunau [GO23], where they also considered the one-dimensional Willmore equation with clamped boundary conditions and proved the necessity of explicit smallness conditions on general obstacles.

◇ Furthermore, one can study the elastic energy for open and closed curves. Here, Barrett, Garcke, and Nürnberg [BGN12] initiated the study of curve networks meeting in junctions. Suitable boundary conditions at junctions were studied by Garcke, Menzel, and Pluda in [GMP19] where they investigated which boundary conditions are suitable to build a well-posed problem. In the case of Theta–networks, namely planar networks composed of three curves of class H^2, regular and which form 120° at the two junctions, Dall'Aqua, Novaga, and Pluda [DNP20] showed the existence and suitable regularity of minimizers of elastic energy combined plus a length-term. We also refer to the p-elastic flow generalization [NP20b], a survey and lectures on curves and networks under elastic flow [MPP21, MNP19].

◇ We also want to mention the phase field approach for surfaces. Here, one considers a surface as an interface between two phases represented by the auxiliary scalar phase field, which takes values 1 and -1 for each of the phases. In the phase field approximation, the values of the phase field vary smoothly in $(-1, 1)$ in a layer of finite width around the interface. For the limit of infinitesimal width, it leads to the surface as the boundary ∂E of a set E with the role of phase with order $+1$. Especially the phase-field approximates the function $\chi_E - \chi_{E^c}$. This approach is useful for evolution and minimization problems since it can handle topological changes of phases or interfaces.

The initial work in this setting for the Willmore energy was done by De Giorgi, who conjectured a reasonable approximation in [DG91]. Röger and Schätzle then analyzed and proved in [RS06] that in $n = 2, 3$ a modification of De Giorgi's functional Γ-converges (see [Bra06, NDL06]) to the sum of the Willmore and perimeter functional.

◇ One can also consider surfaces confined to a prescribed container. For the unit ball in $\mathbb{R}^3$ as confinement and prescribed surface area, Müller and Röger [MR14] investigated smooth embeddings of the sphere into the unit ball and studied the minimization problem for the Willmore functional by modifying a minimizing sequence. They also estimated the minimal Willmore energy from above and below. In the interesting case, when the prescribed surface area exceeds 4π, the surface area of the unit sphere, the minimizer becomes nonconvex and cannot be a C^2-small perturbation of $\mathbb{S}^2$. Additionally, they showed a sharp increase in the optimal Willmore energy at 4π. Furthermore, Dondl, Lemenant, and Wojtowytsch used a phase field approach to study the minimization of the Willmore energy confined to a given container and with connectedness constraint, see [DLW17].

◇ Moreover, one can study closed surfaces with a prescribed isoperimetric ratio, which is defined for an immersion $f\colon \Sigma \to \mathbb{R}^3$ by

$$\mathcal{I}(f) = 36\pi \frac{\mathcal{V}(f)^2}{\mathcal{A}(f)^3} \tag{7}$$

where $\mathcal{A}(f)$ and $\mathcal{V}(f)$ denote the area and the signed volume enclosed by the immersed surface. For sphere-type surfaces, Schygulla showed in [Sch12] the existence of smooth minimizers of the Willmore functional with a prescribed isoperimetric ratio. For immersions with fixed genus, further studies were done in [KMR14, MS23], which led to the result that the infimum for a given fixed genus is always attained provided the energy is below the threshold 8π. The corresponding non-local L^2-gradient flow of the Willmore functional for the case of immersed surfaces, which preserves the isoperimetric ratio, was introduced and studied by Rupp in [Rup24].

1.2 New Contributions

At this point, we want to compactly present the main results on two-dimensional surfaces with boundary in $\mathbb{R}^3$, which are proved in this thesis. For each of the four research directions, we condense various statements to one or two theorems and briefly mention the other remaining new results.

1.2.1 Willmore Equation

In the elliptic case, we prove various results regarding different regularity assumptions, where all theorems require some kind of smallness of the boundary data and exploit the divergence form of the right-hand side. This is needed to apply a fixed point argument to a linearized problem. One of the novelties is the application of weighted Sobolev spaces $W_p^{2;a}(\Omega)$ to the Willmore problem. With an appropriate parameter choice, these spaces are embedded in $C^1(\overline{\Omega})$, hence have bounded gradient norm $\|\nabla u\|_{L^\infty(\Omega)}$, which turns to be essential for the linearization estimates. Since the weighted Sobolev spaces have quite an elaborate trace theory, we rather use the Hölder boundary data to formulate the following theorem.

1 Main Theorem
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^{1+\alpha}$ for some $\alpha \in (0,1)$. Assume that $\beta \in (0,\alpha)$, $g_0 \in C^{1+\alpha}(\partial\Omega)$ and $g_1 \in C^\alpha(\partial\Omega)$. Additionally, we suppose that $\|g_0\|_{C^{1+\alpha}(\partial\Omega)} + \|g_1\|_{C^\alpha(\partial\Omega)} < K$ for some $K > 0$.

Then there exists a constant $\delta = \delta(\alpha, \beta, K, \Omega) > 0$ such that if $\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)} < \delta$, then there exists a solution $u \in C^{1+\beta}(\overline{\Omega}) \cap C^\infty(\Omega)$ to the Dirichlet problem (W)

This result can be found in Subsection 5.4 in Corollary 33, combined with Theorem 36 in Subsection 5.5. One of the key observations heavily used is rewriting the Willmore equation in the semilinear divergence form done in Lemma 18. There we multiply the geometric Willmore equation (2) with $Q = \sqrt{1 + |\nabla u|^2}$ and get with Einstein summation notation

$$\Delta^2 u = D_i b_1^i[u] + D_{ij}^2 b_2^{ij}[u] \quad \text{in } \Omega, \tag{8}$$

where $b_i[u]$ are polynomials consisting of $D^2u, \nabla u$ and Q^{-1} that contains monomials with polynomial degrees greater than two. Furthermore $b_1[u]$ is quadratic in D^2u and $b_2[u]$ is linear in D^2u and moreover

$$\big|b_1[u]\big| \le C|\nabla u| \cdot |D^2u|^2, \qquad \big|b_2[u]\big| \le C|\nabla u|^2|D^2u|$$

with some algebraic constant C. We want to emphasize that it is important to multiply the Willmore equation with $\sqrt{1+|\nabla u|^2}$ because in another case, like in [KL12], we would get an additional term $b_0[u]$ which is cubic in D^2u.

Lemma 18 allows for defining a notion of variational solution where only its derivatives up to order two are involved. Proved in Lemma 18, it allows us to choose spaces with up to second derivatives instead of using spaces with derivatives of the fourth order. This is natural because of the Willmore energy, which also consists only of derivatives up to the second order.

Also, we want to point out that, similar to the biharmonic equation, due to the elliptic structure of the Willmore equation, the solution is smooth in the interior of Ω, and the regularity up to the boundary is as smooth as consistent with the boundary data. Compared with the $C^{4+\alpha}(\overline{\Omega})$-smallness condition required by Nitsche [Nit93], this is a significant progress. Further improvements are the radically reduced regularity assumptions on the boundary $\partial\Omega$ itself. Applying the same techniques, future research can address cases where the boundary includes Lipschitz pieces and edges with angles approaching π, as discussed in [MMS10, p. 43].

In Subsections 5.2 and 5.3 we use unweighted and hence familiar frameworks: Hölder spaces $C^{2+\alpha}(\overline{\Omega})$ and Sobolev spaces $W^{2,p}(\Omega)$. Again, by linearization, we show the existence of a solution with small boundary data. It is also important to notice that only using weighted Sobolev spaces allows us to work with even weaker boundary Hölder spaces than in the unweighted Sobolev case. In the unweighted case, by trace theorem for $W^{2,p}(\Omega)$ with $p > 2$ one can only work with boundary spaces $C^{1+\alpha}(\partial\Omega)$ such that α greater than $1/2$. In contrast, in the weighted framework, all boundary regularity data with $\alpha \in (0,1)$ are allowed.

Furthermore, we want to emphasize that since all these results are rather analytical than geometric, one can generalize them to other problems that have a similar structure, like the Helfrich equation.

1.2.2 Willmore Flow

In the parabolic case, we study existence, uniqueness, and regularity of the graphical Willmore flow solutions under smallness conditions. We first prove the short-time existence of a solution with initial and boundary data in various regularity classes. One of the main novelties here is the use of the low regularity initial data, which may lie in $C^{m+\alpha}(\overline{\Omega})$, $m \in \{1,2,3\}$ where the Dirichlet boundary data come from Hölder spaces $C^{4+\alpha}(\partial\Omega)$. This can be achieved by using so-called weighted parabolic Hölder spaces $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(\overline{\Omega}\times(0,T])$ for some $T > 0$ as solution space. The Hölder-norms of the spatial derivatives of order higher than $m+\alpha$ are weighted with powers of the time variable $t \in (0,T]$, hence allowing higher derivatives to blow up for $t \to 0$.

As an important consequence, in contrast to the unweighted case ($m = 4$), like in [DKS15], fewer compatibility conditions between initial values and Dirichlet data have to be imposed. Here is the additional condition on boundary data, which is no longer necessary

$$0 = \Delta_{\Gamma(u_0)} H(u_0) + 2H\left(\frac{1}{4}H^2 - \mathcal{K}\right)(u_0), \quad \text{on} \quad \partial\Omega.$$

This equation represents the fact that the boundary of the surface has to stay fixed ($\partial_t u = 0$ by the Willmore equation) in time already at $t = 0$. For references, see [LSvW92]. This reflects the smoothing property of the Willmore flow, similar to the biharmonic heat flow, since for $t > 0$ the initial $C^{m+\alpha}$ surface becomes instantaneously as smooth as the boundary data permit. For the elastic curve flows, the compatibility conditions play an important role in [Men21, GMP19] or [DP14, Appendix D.].

2 Main Theorem
Suppose $m \in \{1,2,3\}$ and Ω is a bounded domain in $\mathbb{R}^2$ with $C^{4+\alpha}$ boundary for some $\alpha \in (0,1)$. Further, let $u_0 \in C^{m+\alpha}(\overline{\Omega})$, $g_0 \in C^{4+\alpha}(\partial\Omega)$ and $g_1 \in C^{3+\alpha}(\partial\Omega)$ with $\|u_0\|_{C^{m+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} < K$ for some $K > 0$.

- ⓐ ***(local)** Then there exists time $T = T(\alpha,m,K,\Omega)$ such that there is a unique solution $u \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(\overline{\Omega}\times(0,T])$ of the Willmore flow problem* (WF).

ⓑ ***(global)*** *There exists $C_{180} = C_{180}(\alpha, \Omega) > 0$ such that if*

$$\|u_0\|_{C^{1+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} < C_{180}$$

then a unique solution for the Willmore-flow (WF) *exists for all times such that for all $T > 0 : u \in C_{1+\alpha}^{4+\alpha,1+\alpha/4}(\overline{\Omega} \times (0, T])$.*

ⓒ ***(subconvergence)*** *Moreover, for the global solution in (b) there exists a time-sequence $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with $\lim_{k\to\infty} t_k = +\infty$ and a critical point of Willmore energy $u_\infty \in C^{4+\beta}(\overline{\Omega})$ for all $\beta \in (0, \alpha)$ such that*

$$u(t_k) \underset{k\to\infty}{\longrightarrow} u_\infty \quad \textit{in} \quad C^{4+\beta}(\overline{\Omega}).$$

The local and global existence results are obtained in Subsection 6.3 in Theorems 47, 49 and the subconvergence to a Willmore surface can be found in Subsection 6.6 in Theorem 59. Here, to use the weighted Schauder estimates by Belonosov [Bel79] we have rewritten the graphical Willmore flow equation in the way that the right-hand side represented by some polynomial $\mathcal{R}$ consisting of derivatives up to the order three (see Lemma 40)

$$u_t = -L_{\nabla u} u + \mathcal{R}(\nabla u, D^2 u, D^3 u) \quad \text{in } \Omega \times (0, T]$$

where $L_{\nabla u}$ is a fourth order elliptic operator in case $\nabla u \in C^1(\overline{\Omega})$. The monomials of $\mathcal{R}$ which contain $D^3 u$ are also linear in $D^3 u$ and moreover, by (R) it holds

$$\big|\mathcal{R}(\nabla u, D^2 u, D^3 u)\big| \leq C|D^3 u| \cdot |D^2 u| \cdot |\nabla u| + C|D^2 u|^3 \tag{9}$$

where C is some algebraic constant. Thus, the degree of all monomials is at least three. It should be noticed that $L_{\nabla u}$ is not Δ^2 and $\mathcal{R}$ is not in divergence form.

We can further weaken the regularity assumptions on the initial surface to $u_0 \in C^1(\overline{\Omega})$ in Subsection 6.4. At the cost of an additional smallness condition on $\|u_0\|_{C^1(\overline{\Omega})}$ we obtain short-time existence in Theorem 54. More precisely, this means that there is a constant $C_{188} = C_{188}(\alpha, \Omega)$ such that if $\|u_0\|_{C^1(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} < C_{188}$ then there is a unique solution in the weighted parabolic class $C_1^{4+\alpha,1+\alpha/4}(\overline{\Omega} \times (0, 1])$ of the Willmore flow problem (WF).

Furthermore, in addition to the weighted framework, we work with unweighted parabolic Hölder spaces of second order $C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{\Omega} \times [0, T])$. In comparison to the weighted Hölder spaces, we reduce the regularity assumptions on the boundary data g_i and the solution itself from $C^{4+\alpha}$ to $C^{2+\alpha}$ at the cost of more regular initial values u_0, which are here in $C^{2+\alpha}(\overline{\Omega})$ instead of being merely in $C^{1+\alpha}(\overline{\Omega})$.

3 Main Theorem
Suppose Ω is a bounded domain in $\mathbb{R}^2$ with $C^{2+\alpha}$ boundary for some $\alpha \in (0, 1)$. Further, let $u_0 \in C^{2+\alpha}(\overline{\Omega})$, $g_0 \in C^{2+\alpha}(\partial\Omega)$ and $g_1 \in C^{1+\alpha}(\partial\Omega)$.

ⓐ ***(local)*** *Then there exists time T depending only on α, the bound $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} \leq C$ and Ω such that there is a solution $u \in C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{\Omega} \times [0, T])$ of the Willmore flow problem* (WF).

ⓑ ***(global)*** *There exists further a constant $C_{205} = C_{205}(\alpha, \Omega)$ such that if*

$$\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} < C_{205}$$

then there exists a solution u of the Willmore flow problem (WF) *for all times, such that for all $T > 0 : u \in C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{\Omega} \times [0, T])$.*

ⓒ ***(convergence)*** *Moreover, there exists a constant* $C_{217} = C_{217}(\Omega, \alpha)$ *such that if* $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} < C_{217}$ *then there exists a critical point of the Willmore energy* $u_\infty \in C^{2+\alpha}(\overline{\Omega})$ *such that*

$$u(t) \underset{t\to\infty}{\longrightarrow} u_\infty \quad \textit{in} \quad C^{2+\beta}(\overline{\Omega})$$

for all $\beta \in (0, \alpha)$.

It is proven in Theorems 56, 58 and 60 in Subsections 6.5 and 6.6.

In order to use the Schauder estimates by Dong and Zhang [DZ15], we have to rewrite the Willmore flow equation once again such that we recover a right-hand side in divergence form, which was done by Koch and Lamm [KL12] for graphs over $\mathbb{R}^2$ (also see (133)

$$\partial_t u + \Delta^2 u = f_0[u] + \nabla_i f_1^i[u] + D^2_{ij} f_2^{ij}[u] \quad \text{in } \Omega \times (0, T].$$

The terms $f_i[u]$ are again polynomials consisting of D^2u, ∇u and Q^{-1} with degree of each monomial at least three and satisfying

$$\big|f_0[u]\big| \le C|D^2u|^3, \quad \big|f_1[u]\big| \le C|D^2u|^2 \cdot |\nabla u|, \quad \big|f_2[u]\big| \le C|D^2u| \cdot |\nabla u|^2$$

with some algebraic constant C.

Since for a solution of the Willmore flow the Willmore energy stays bounded, we use the diameter (i.e. L^∞-)bounds in terms of the initial Willmore energy and the diameter of the boundary derived by Deckelnick, Grunau, and Röger [DGR17, Thm 2]. Actually, it has an advantage over the elliptic case, where we cannot simply use this a-priori estimate. To prove global existence and convergence to a Willmore surface, we use interpolation techniques. Hence, we need some smallness condition for all times, which is not provided by the L^∞-estimate in [DGR17, Thm. 2]. This yields bounds by the Willmore energy and $\operatorname{diam}\Omega$, where the latter is fixed and not assumed to be small. That is the reason to derive L^2-smallness in Theorem 16 in Section 4.

Lastly, we want to emphasize that even when applying the Willmore flow as t approaches infinity, we may still fail to reach a Willmore minimizer. Even in cases where global existence is guaranteed, we can only anticipate a Willmore surface, representing a critical point, as the limit.

1.2.3 Compactness Results

To study minimizing sequences for the Willmore functional of graphs, Deckelnick, Grunau, and Röger considered in [DGR17] sequences with uniformly bounded Willmore energy and the behavior of the Willmore functional regarding L^1-convergence. Due to working in the class of graphs, one has to expect jump discontinuities for the limit functions, which may result in vertical parts and possibly even a highly irregular Cantor part, as explained below.

Since by estimates in [DGR17, Thm. 2] the diameter and area are also bounded, the authors have chosen the space of functions with bounded variation $BV(\Omega) \cap L^\infty(\Omega)$. The gradient of such functions can be decomposed as $\nabla u = \nabla^a u + \nabla^j u + \nabla^c u$ where $\nabla^a u$ is the absolutely continuous part of ∇u in respect to the Lebesgue measure, $\nabla^j u$ is the jump part and $\nabla^c u$ is the Cantor part, all defined in Subsection 7.2. The jump part represents the vertical walls of the function and the remaining singular Cantor part is illustrated by contributions to the Cantor set of the Cantor ternary function.

By considering the absolutely continuous part $\nabla^a u \in L^1(\Omega)$ of $\nabla u \in BV(\Omega)$, Deckelnick, Grunau, and Röger could define the absolutely continuous contribution to the Willmore energy and bound it in a lower semicontinuity estimate for the limit. Based on their work, we want to characterize the missing contribution in their lower semicontinuity estimate in the following theorem. The missing parts are described in the framework of measure-function pairs and curvature varifolds.

4 Main Theorem
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a C^2–boundary, $\varphi \in C_0^2(\mathbb{R}^2)$ and $M > 0$. Furthermore, let $\{u_k\}_{k\in\mathbb{N}}$ be a given sequence in $W^{2,2}(\Omega)$ that satisfies for some given $M > 0$:

$$\forall k \in \mathbb{N}: \; u_k - \varphi \in \mathring{W}^{2,2}(\Omega) \quad \textit{and } \mathcal{W}(u_k) \leq M.$$

Then there exists a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that after passing to a subsequence

$$u_k \to u \quad \textit{in} \quad L^1(\Omega) \quad (k \to \infty).$$

For each graph $\Gamma(u_k)$ we call $Q_k := \sqrt{1 + |\nabla u_k|^2}$ the area element and μ_k the graph area measure. Furthermore, we define the unit upwards pointing normal field $\tilde{N}_k : \Omega \to \mathbb{S}^2$. Then:

(i) *There exist functions $\tilde{N} : \overline{\Omega} \to \mathbb{S}^2$, $\tilde{H} : \overline{\Omega} \to \mathbb{R}$ and a Radon measure μ on $\overline{\Omega}$ such that*

$$\int_\Omega |H^a|^2 Q^a \,\mathrm{d}x + \int_{\overline{\Omega}\setminus A_D(u)} |\tilde{H}|^2 \,\mathrm{d}\mu = \int_{\overline{\Omega}} |\tilde{H}|^2 \,\mathrm{d}\mu \leq \liminf_{k\to\infty} \int_\Omega H_k^2 Q_k \,\mathrm{d}x,$$

where $A_D(u)$ is the set where u is approximately differentiable. For definitions, see (193) *and* (194).

(ii) *Additionally, there is a $W^{2,2} \cap C^1$-surface Σ with boundary and the sequence of oriented varifolds $V^o[\Gamma(u_k) \cup \Sigma, N_k, 1, 0]$ converge in the varifold sense to a curvature varifold $V[\Gamma \cup \Sigma, N, 1, 0]$ with mean curvature vector in the varifold sense $\vec{H} = HN$.*

Here, Γ is the essential boundary of the sublevel set of u and furthermore, we have

$$\int_\Omega |H^a|^2 Q^a \,\mathrm{d}x + \int_{(\overline{\Omega}\setminus A_D(u))\times\mathbb{R}} \|\vec{H}\|^2 \,\mathrm{d}\|V\| = \int_{\overline{\Omega}\times\mathbb{R}} \|\vec{H}\|^2 \,\mathrm{d}\|V\| \leq \liminf_{k\to\infty} \int_\Omega H_k^2 Q_k \,\mathrm{d}x.$$

1.2.4 Finite Relaxed Willmore Energy and Nonzero Cantor Part

While in the general case, the arising of the jump parts for finite "relaxed" Willmore energy $\overline{\mathcal{W}}$ is expected, one would intuitively guess that rather highly irregular Cantor parts vanish for finite relaxed Willmore energy. Moreover, by second-order rectifiability, proved by Menne in [Men13], a graph with finite Willmore energy represented by a varifold is, up to an area null set, a countable collection of C^2-manifolds. This all makes the following main theorem very surprising, since geometrically one would rather expect infinite relaxed Willmore energy by the non-vanishing Cantor part. It seems that by naively applying projection techniques, one can get these irregular sets. Still, it is an open question whether Willmore minimizers may have a nonzero Cantor part.

Here, developing an idea from unpublished notes of Grunau we construct an example for the case of the one-dimensional-Willmore functional. Here $SBV((0,1))$ is the subspace af all $BV((0,1))$ functions with vanishing Cantor part.

5 Main Theorem
There exists a function $u \in BV((0,1))$ with $\overline{\mathcal{W}}(u) < \infty$ so that $|(u')^c|((0,1)) > 0$ and especially $u \notin SBV((0,1))$.

1.3 Outline

In the following, we give a brief outline of the present work. The new results are contained in Sections 4, 5, 6 and 8.

The others provide foundations and recall important definitions and theorems. There is also an Appendix with some supplementary material, a list of references, and a table of notation.

(2) In Section 2 we recall some basic geometric quantities like mean curvature, the Willmore energy, and the second fundamental form in the graphical case and rewrite them as polynomials consisting of derivatives of u as functions over Ω.

(3) In Section 3 we recall the available theory of higher-order elliptic operators and then describe the general scheme of how to apply a fixed point argument by freezing the nonlinear part in both the elliptic and the parabolic case.

(4) In Section 4 we investigate which bounds can be proved in terms of the Willmore energy and boundary values. In particular, we discuss diameter and area bounds, as well as especially L^2-smallness estimates needed later in Section 6 for global existence results.

(5) In Section 5 we show the existence of a smooth solution of the Willmore equation in different settings. First, we consider Hölder spaces and use Schauder estimates to provide existence. Then we use L^p estimates to get the result in the Sobolev framework. Subsequently, the existence is provided in a weighted setting, which allows us to use $C^{1+\alpha}$ data with a small Lipschitz norm. Interior regularity is also shown.

(6) In Section 6 we prove existence and regularity in different parabolic Hölder spaces. We begin by considering different weighted Hölder spaces with initial values in $C^{m+\alpha}(\overline{\Omega})$ or $C^1(\overline{\Omega})$ and investigate smallness conditions for global existence and subconvergence to a Willmore surface. In the unweighted case, we take $u_0 \in C^{2+\alpha}(\overline{\Omega})$ and use the divergence structure of the equation.

(7) In Section 7 we recall the definitions and properties of BV functions, measures-function pairs and varifolds.

(8) In Section 8 we show additional compactness properties of Willmore energy-bounded sequences. We conclude by giving an example of a BV function with finite relaxed Willmore energy and nonzero Cantor part.

2 Geometric Preliminaries

In this chapter, we want to recall some basic geometric definitions and theorems, both considering graphs and, in general, non-embedded, immersed surfaces, which are allowed to have self-intersections. We aim to revisit fundamental concepts that quantify the curvature of the immersed manifold within the ambient Euclidean space. Additionally, to intrinsic curvature, which solely relies on a chosen metric on the manifold without any reference to the ambient geometry, we have to introduce the concept of extrinsic curvature, which characterizes the curvature of the immersed surface in relation to the ambient space. Mean curvature, Gaussian curvature, second fundamental form, and the Willmore energy are concepts we want to introduce. We especially need local representations of curvatures and their derivatives in order to rewrite the Willmore equation and Willmore flow equation as elliptic and parabolic equations for a graph.

2.1 Geometry of Immersions

We consider Σ, a smooth surface with or without boundary $\partial\Sigma$, thus a two-dimensional manifold, for references see [Lee12, Lee97]. Further, we are assuming Σ to be connected without loss of generality, and with $\partial\Sigma \subset \Sigma$ the boundary of Σ, which is diffeomorphic to a disjoint union of copies of $\mathbb{R}$ or $\mathbb{S}^1$. Additionally, for $k > 0$ let Σ be immersed in $\mathbb{R}^{2+k}$ via C^2-class $f\colon \Sigma \hookrightarrow \left(\mathbb{R}^{2+k}, \langle\, .\,,\, .\,\rangle_{\mathbb{R}^{2+k}}\right)$ with $\langle\, .\,,\, .\,\rangle_{\mathbb{R}^{2+k}}$ the Euclidean inner product on $\mathbb{R}^{2+k}$. This means that for all $x \in \Sigma$ the differential $\mathrm{d}f_x\colon T_x\Sigma \to \mathbb{R}^{2+k}$ is injective. We denote for each $x \in \Sigma$

$$\mathrm{d}f_x : T_x\Sigma \to T_{f(x)}\mathbb{R}^{2+k} \cong \mathbb{R}^{2+k}$$

the differential, which we also call push-forward, of the mapping f at point $x \in \Sigma$. Since by [Lee12, p.54] the tangent vectors at $x \in \Sigma$ act as linear maps $D_\tau\colon C^\infty(\Sigma) \to \mathbb{R}$ for each $\tau \in T_x\Sigma$, called *derivations* at $x \in \Sigma$, we can define the differential by

$$\mathrm{d}f_x(\tau)(h) = D_\tau(h \circ f)$$

for all $h \in C^\infty(\mathbb{R}^{2+k})$. In this situation the codimension is given by $\operatorname{codim} f(\Sigma) = 2+k-\dim f(\Sigma) = k$. Next, we equip Σ with $g = f^*\langle\, .\,,\, .\,\rangle_{\mathbb{R}^{2+k}}$, the pull-back Riemannian metric of the standard Euclidean metric along f. In detail, this means

$$g_x(\tau, \xi) := \langle \mathrm{d}f_x(\tau), \mathrm{d}f_x(\xi)\rangle_{\mathbb{R}^{2+k}}.$$

for all $x \in \Sigma$ and $\tau, \xi \in T_x\Sigma$. This means that f is an isometric immersion. If we consider local charts (U, φ) of Σ then by writing $f = f \circ \varphi^{-1}$ we obtain the local representation

$$g_{ij} = g(\partial_i, \partial_j) = \langle \partial_i f, \partial_j f\rangle_{\mathbb{R}^{2+k}}\,, \quad i, j, = 1, 2.$$

where ∂_i denotes the regular partial derivative in $\mathbb{R}^2$ and also the vector fields on Σ. Furthermore, we define the area element factor $\sqrt{\det{(g_{ij})}}$ and the inverse $\left(g^{ij}\right) = (g_{ij})^{-1}$. The integration is carried out using the induced area measure in local representation

$$\mu_g = \sqrt{\det{(g_{ij})}}\mathcal{L}^2$$

in each local chart, where $\mathcal{L}^2$ is the standard Lesbegue measure on $\mathbb{R}^2$ for local coordinates. The measure μ_g is the local representation of μ_f the volume measure induced by f on Σ. It corresponds to the 2-dimensional Hausdorff measure defined by the geodesic distance on Σ (refer to [Lee12, Chapter 16]).

Next, with respect to Riemannian structure, we can define $\nabla^{f(\Sigma)}$ and $\nabla^{\mathbb{R}^{2+k}}$ the corresponding *Levi-Civita connections* on Σ and $\mathbb{R}^{2+k}$ respectively. Acting on functions $\nabla^{\mathbb{R}^{2+k}} = (\partial_1, \dots, \partial_{2+k})^T$ is the ordinary gradient in $\mathbb{R}^{2+k}$. Then for each map $h \in C^1(\Sigma)$ with an extension $\overline{h} \in C^1(\mathbb{R}^{2+k})$ on $\mathbb{R}^{2+k}$, thus $h = \overline{h} \circ f$, we get

$$g_x(\nabla^{f(\Sigma)} h, \xi) = \left\langle \nabla^{\mathbb{R}^{2+k}} \overline{h}, \mathrm{d}f_x(\xi) \right\rangle_{\mathbb{R}^{2+k}}$$

for all $\xi \in T_x\Sigma$. In that case, $\nabla^{f(\Sigma)} h$ is a gradient of h on Σ, which is the orthogonal projection to the tangent space of the ambient space gradient. One quickly convinces oneself that the definition does not depend on any particular choice of the orthonormal basis. In local coordinates, we have

$$(\nabla^{f(\Sigma)} h)^i = g^{ij} \partial_j h, \quad i = 1, 2,$$

where here and in the subsequent content, we will employ the summation convention for repeated indices unless explicitly stated.

To study further geometric properties of Σ immersed in $\mathbb{R}^{2+k}$, we also need the relate the tangent space of Σ to that of $f(\Sigma)$. Hence we want shortly define the orthogonal decomposition of $T_{f(x)}\mathbb{R}^{2+k} \cong \mathbb{R}^{2+k}$ in parts tangential or normal to $T_x\Sigma$. Namely, the inner product $\langle\, .\,,.\,\rangle_{\mathbb{R}^{2+k}}$ signifies orthogonality in $\mathbb{R}^{2+k}$. To be precise, we have the orthogonal decomposition $\mathbb{R}^{2+k} = \mathrm{d}f_x\,(T_x\Sigma) \oplus (\mathrm{d}f_x\,(T_x\Sigma))^{\perp}$ where we use $(\,\cdot\,)^{\top}$ to indicate the projection onto $\mathrm{d}f_x\,(T_x\Sigma)$. Thus, $(\,\cdot\,)^{\perp}$ the projection onto $(\mathrm{d}f_x\,(T_x\Sigma))^{\perp}$ is well defined.

Now, let $\mathfrak{X}(\Sigma)$ denote the space of tangent vector fields on Σ. Each vector field $V \in \mathfrak{X}(\Sigma)$ can be extended to local vector field $\overline{V} \in \mathfrak{X}(\mathbb{R}^{2+k})$, which can be build out of local extensions. In fact, since f is an immersion, then it is also locally an embedding. Hence, for any point $x \in \Sigma \setminus \partial\Sigma$, there is a neighborhood U of x in Σ such that if V is the restriction of $\overline{V}$ to U, then $\mathrm{d}f_x(V)$ can be extended to a vector field on $\mathbb{R}^{2+k}$ that is locally defined in a neighborhood of $f(x)$.

For the later formulation of the Willmore equation, we also need the notation of the divergence and the Laplace operator with respect to the embedding. Let $V \in \mathfrak{X}(\mathbb{R}^{2+k})$ be a C^1 vector field with compact support, and let $\{e_i\}_{i=1}^{2+k}$ be a chosen fixed orthonormal basis of $\mathbb{R}^{2+k}$. We express X as $V^i = \langle V, e_i\rangle, i = 1, \dots, 2+k$ the Cartesian coordinates of V. Based on this, we define the *divergence of V on Σ* as follows:

$$\operatorname{div}_{f(\Sigma)} V := \langle e_i, \nabla^{f(\Sigma)} V^i \rangle, \tag{10}$$

since we can locally extend V to a vectorfield on $\mathbb{R}^{2+k}$. It is important to note that this definition holds for points in $\Sigma \setminus \partial\Sigma$. Additionally, one can verify that the result remains independent of the specific choice of the orthonormal basis. Moving on, we define the *Laplace-Beltrami operator of h on Σ* for a C^2-function $h\colon \Sigma \to \mathbb{R}$ as the divergence of the gradient of h:

$$\Delta_{f(\Sigma)} h := \operatorname{div}_{f(\Sigma)} \nabla^{f(\Sigma)} h.$$

It is shown in [Gul14, Appendix] that this definition is consistent with those given in [Lee12]. The Laplace-Beltrami operator of h can also be calculated locally via the following representation

$$\Delta_{f(\Sigma)} h = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} \cdot g^{ij} \partial_j h \right) \tag{11}$$

1 Definition (Second Fundamental Form)
Let $x \in \Sigma \setminus \partial\Sigma$ then the second fundamental form in $\mathbf{A}_x : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to (\mathrm{d}f_x(T_x\Sigma))^{\perp}$ is the operator

$$\mathbf{A}_x(X,Y) := \left(\nabla^{\mathbb{R}^{2+k}}_{\overline{X}}\overline{Y}\right)^{\perp}_{f(x)}$$

where $\overline{X}, \overline{Y}$ are local extensions of $\mathrm{d}f_x(X), \mathrm{d}f_x(Y)$ in a neighborhood of $f(x)$. In fact, it defines a symmetric tensor that is independent of the specific extensions $\overline{X}, \overline{Y}$, see [Lee97, Lemma 8.1.]. Moreover, we define the squared norm of the second fundamental form by

$$\|\mathbf{A}_x\|_g^2 := \sum_{i,j=1}^{2} \left\|\mathbf{A}_x(\tau_i,\tau_j)\right\|^2_{\mathbb{R}^{2+k}}, \tag{12}$$

where $\|\,.\,\|^2_{\mathbb{R}^{2+k}}$ is the Euclidean norm in $\mathbb{R}^{2+k}$ and $(\tau_i)_{i=1}^2$ is an orthonormal basis of $T_x\Sigma$, which can be chosen arbitrarily.

Next, let assume that the ambient space is $\mathbb{R}^3$, hence $\operatorname{codim} f(\Sigma) = 1$, the normal space $(\mathrm{d}f_x(T_x\Sigma))^{\perp}$ is one-dimensional and additionally there also exists a global continuous *unit normal field N*, thus $N_x \in (\mathrm{d}f_x(T_x\Sigma))^{\perp}$ for all $x \in \Sigma$. Then we may define the *scalar second fundamental form* $A\colon \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathbb{R}$ by:

$$A_x(X,Y) := \left\langle \mathbf{A}_x(X,Y), N_x \right\rangle_{\mathbb{R}^{2+k}} = \left\langle \left(\nabla^{\mathbb{R}^{2+k}}_{\overline{X}}\overline{Y}\right)_{f(x)}, N_x \right\rangle_{\mathbb{R}^3} \tag{13}$$

for all $x \in \Sigma \setminus \partial\Sigma$, where $\overline{X}, \overline{Y}$ are local extensions of $\mathrm{d}f_x(X), \mathrm{d}f_x(Y)$ in a neighborhood of $f(x)$. Then the components of local representation are

$$A_{ij} = A(\partial_i, \partial_j) = \left\langle \partial^2_{ij} f, N \right\rangle_{\mathbb{R}^3}, \quad i, j, = 1, 2.$$

If N is an inward pointing unit vector field normal along the standard round sphere embedded in $\mathbb{R}^3$, then the second fundamental form is positive. Furthermore, in local coordinates, the square norm of the second fundamental form takes the following shape

$$\|A\|_g^2 := \|\mathbf{A}_x\|_g^2 = \sum_{i,j,k,\ell} g^{ij} g^{k\ell} A_{ik} A_{j\ell}. \tag{14}$$

Let us go back to $\mathbb{R}^{2+k}$. For many situations, $\mathbf{A}$ carries more information than needed, so instead we take the mean value over all tangential directions. This is the trace with respect to the metric, which is an invariant of the second fundamental form $\mathbf{A}$.

2 Definition (Mean Curvature)
For each $x \in \Sigma \setminus \partial\Sigma$ we define the vector of mean curvature *by*

$$\mathbf{H}_x := \sum_{i=1}^{2} \mathbf{A}_x(\tau_i,\tau_i) \in (\mathrm{d}f_x(T_x\Sigma))^{\perp}, \tag{15}$$

with τ_1, τ_2 be an orthonormal basis of $T_x\Sigma$. We want to emphasize that we omit the factor $1/2$ before the sum. The squared norm of $\mathbf{H}$ is simply the Euclidean squared vector length $\|\mathbf{H}\|_g^2 = \|\mathbf{H}\|^2_{\mathbb{R}^{2+k}}$. Next, if $\operatorname{codim} f(\Sigma) = 1$, *we take the* scalar mean curvature, *analogous to the scalar second fundamental form so let $H\colon \Sigma \to \mathbb{R}$:*

$$H_x := \langle \mathbf{H}, N\rangle(x) = \sum_{i=1}^{2} A_x(\tau_i,\tau_i). \tag{16}$$

Furthermore, we want to introduce the second invariant of $\mathbf{A}$, which is the determinant with respect to the metric, called the Gaussian curvature. In contrast to the mean curvature, it is independent of the ambient space, thus it depends solely on the induced metric g on Σ.

3 Definition (Gaussian Curvarture)
For $x \in \Sigma \setminus \partial\Sigma$ and τ_1, τ_2 an orthonormal basis of $T_x\Sigma$ the Gaussian curvature $\mathcal{K}\colon \Sigma \setminus \partial\Sigma \to \mathbb{R}$ is pointwise defined by

$$\mathcal{K}_x := \langle \mathbf{A}_x(\tau_1, \tau_1), \mathbf{A}_x(\tau_2, \tau_2)\rangle_{\mathbb{R}^{2+k}} - \langle \mathbf{A}_x(\tau_1, \tau_2), \mathbf{A}_x(\tau_1, \tau_2)\rangle_{\mathbb{R}^{2+k}}.$$

There are some relations between mean curvature, Gaussian curvature, and the second fundamental form that are used later (see [KS12, (1.1.2)-(1.1.7)])

$$\|\mathbf{H}\|_g^2 = \|\mathbf{A}\|_g^2 + 2\mathcal{K}, \qquad \|\mathbf{A}\|_g^2 \geq 2\mathcal{K}, \qquad \|\mathbf{H}\|_g^2 \geq 4\mathcal{K} \quad \text{and} \qquad 2\|\mathbf{A}\|_g^2 \geq \|\mathbf{H}\|_g^2. \tag{17}$$

In case $\operatorname{codim} f(\Sigma) = 1$, the mean and Gaussian curvatures have the following local representation

$$H = \operatorname{trace}_g A = g^{ij} A_{ij}, \quad \mathcal{K} = \det_g A = \det\left(g^{ik} A_{kj}\right).$$

For surfaces immersed into $\mathbb{R}^3$, both curvatures can be conveniently written in terms of the principal curvatures. At any given point $x \in \Sigma \setminus \partial\Sigma$ it is possible to select an orthogonal basis τ_1, τ_2 of $T_x\Sigma$, such that the eigenvalues, denoted as k_1 and k_2, of matrix representation A are displayed along its diagonal. Consequently, we obtain the following expressions

$$H_x = k_1 + k_2, \quad \mathcal{K}_x = k_1 k_2.$$

Particularly, the eigenvalues are called the *principal curvatures*.

Since we want to deal with surfaces with boundary that are described by curves, we introduce *immersed curves*. The immersion property of f implies that the restriction of f on $\partial\Sigma$ is an immersion of each component of $\partial\Sigma$ which are curves diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$. Let us begin by noting that if $\gamma : (0,1) \to \Sigma$ is a parametrization of a smooth curve, then it can be regarded as a smooth curve $f \circ \gamma$ in $\mathbb{R}^{2+k}$ whose image lies within $f(\Sigma) \subset \mathbb{R}^{2+k}$. Furthermore, for any point $x \in \partial\Sigma$ we define the conormal $\mathrm{co}_f : \partial\Sigma \to \mathbb{R}^{2+k}$ to be the unique unit vector $\mathrm{co}_f(x) \in \mathbb{R}^{2+k}$ pointing outwards of Σ that is tangent along $f(\Sigma)$ and normal along $f(\partial\Sigma)$.

The Gaussian curvature induced by g is related to the topology of the surface in a fundamental way by the Gauss-Bonnet Theorem. Namely, $\mathcal{K}$ integrated over Σ with respect to μ_f is equal to $\chi(\Sigma)$ the Euler characteristic of Σ up to a boundary term later discussed. It is a significant result that $\chi(\Sigma)$ is a topological invariant. By [DHS10, p. 38] for compact, orientable, and connected C^2-smooth surfaces with C^2-smooth edges, the Euler characteristic is given by $\chi(\Sigma) = 2 - 2\mathfrak{g} - r$, where $\mathfrak{g}$ represents the genus, defined as the maximum number of cuts along simply closed and disjoint curves that yield a connected cut surface, and r is the number of connected components considered as closed curves.

For the boundary term in the Gauss-Bonnet Theorem, we want to define the orientation of the boundary. Here we assume Σ with $\operatorname{codim} f(\Sigma) = 1$ be oriented with a global continuous unit normal field N, thus $N_x \in (\mathrm{d}f_x(T_x\Sigma))^\perp$ for all $x \in \Sigma$. Then we say the parametrization $\gamma\colon (0,1) \to \partial\Sigma$ *positive oriented*, if at any point $s \in (0,1)$, the determinant of the matrix consisting of the tangent vector $\partial_t(f \circ \gamma)(t)$, the unit co-normal inward pointing $\mathrm{co}_f \circ \gamma(t)$ and $N \circ \gamma(t)$ is positive. If we set $Y : (0,1) \to f(\partial\Sigma)$ by $Y = f \circ \gamma$ we can define the *signed geodesic curvature* by

$$\kappa_g(s) = \frac{1}{|Y'(s)|^3} \det\Big(Y'(s), Y''(s), N(\gamma(s))\Big). \tag{18}$$

4 Theorem (Gauss-Bonnet Theorem)
Let (Σ, g) be oriented smooth Riemannian 2-dimensional compact manifold isometrically immersed in $\mathbb{R}^3$ via $f : \Sigma \hookrightarrow (\mathbb{R}^3, \langle\, . \,,\, . \,\rangle_{\mathbb{R}^3})$ with $C_1, \ldots, C_\ell$ C^2-smooth closed curves forming the immersed boundary $f(\partial\Sigma)$. Then, the following equality holds:

$$\int_\Sigma \mathcal{K}\, \mathrm{d}\mu_f + \sum_{i=1}^{\ell} \int_{C_i} \kappa_g(s)\, \mathrm{d}s = 2\pi\chi(\Sigma), \tag{19}$$

where $\mathcal{K}$ denotes the Gaussian curvature and $\kappa_g(s)$ is the geodesic curvature of the arclength parameterized curves C_i. The boundary components $\{C_i\}_{i=1}^n$ must be parameterized in a positively oriented manner.

Proof: The proof of this theorem can be found in [DC16]. □

There is also an integral geometrical theorem for the mean curvature. Namely, it is linked to tangential divergence in an extension of the Euclidean divergence theorem to nonflat submanifolds of $\mathbb{R}^n$ and vector fields that are non-tangential in general.

5 Theorem (Non-tangential Divergence Theorem)
Let (Σ, g) be oriented smooth Riemannian 2-dimensional compact manifold isometrically immersed in $\mathbb{R}^3$ via $f : \Sigma \hookrightarrow (\mathbb{R}^3, \langle\, . \,,\, . \,\rangle_{\mathbb{R}^3})$ with immersed boundary $f(\partial\Sigma)$ and unit co-normal pointing inward co_f. Then, for every vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ it holds:

$$\int_\Sigma \mathrm{div}_{f(\Sigma)}\, X\, \mathrm{d}\mu_f = -\int_\Sigma \langle X, \mathbf{H}\rangle_{\mathbb{R}^3}\, \mathrm{d}\mu_f - \int_{f(\partial\Sigma)} \langle X, co_f\rangle_{\mathbb{R}^3}\, \mathrm{d}s. \tag{20}$$

Proof: The proof of this theorem can be found in [Sim83, p.45]. □

In case $X(f(x)) \in \mathrm{d}f_x(T_x\Sigma)$ for all $x \in \Sigma$ it follows $\langle X, \mathbf{H}\rangle_{\mathbb{R}^3} \equiv 0$ and we get the *tangential divergence theorem*. Next, we want to introduce the Willmore energy for immersion.

6 Definition (Willmore Energy for Immersions)
If Σ is C^2-surface, $f : \Sigma \hookrightarrow (\mathbb{R}^3, \langle\, . \,,\, . \,\rangle_{\mathbb{R}^3})$ is a C^2-immersion and H is the mean curvature of f then we define the Willmore energy for $f(\Sigma)$ by

$$\mathcal{W}\big(f(\Sigma)\big) = \frac{1}{4}\int_\Sigma H^2\, \mathrm{d}\mu_f,$$

where $\mathrm{d}\mu_f$ is the surface form locally induced by g, i.e. by the pullback of the Euclidean metric.

By using equation (17) and the Gauss-Bonnet Theorem, we can relate the Willmore energy to the integrated square norm of the second fundamental form

$$4\mathcal{W}(f) = \int_\Sigma \|A\|_g^2\, \mathrm{d}\mu_f - 2\sum_{i=1}^{\ell} \int_{C_i} \kappa_g(s)\, \mathrm{d}s + 4\pi\chi(\Sigma). \tag{21}$$

This identity yields that if we fix the topology of Σ and the immersed boundary $f(\partial\Sigma)$ with the signed geodesic curvature, then we can also estimate $\int_\Sigma \|A\|_g^2\, \mathrm{d}\mu_f$, which is the L^2-norm of the second fundamental form by the Willmore energy. In reverse, by (17) we can also estimate $\frac{1}{2}\int_\Sigma \|A\|_g^2\, \mathrm{d}\mu_f \geq \mathcal{W}(f(\Sigma))$.

There is a generalization of the Willmore energy to a functional involving area, Gaussian curvature, and spontaneous mean curvature. It is called *Helfrich functional* [Hel73, Can70], which is defined for a C^2 immersion $f : \Sigma \to \mathbb{R}^3$

$$\mathcal{W}_{\alpha,H_0,\gamma}(f(\Sigma)) = \alpha\int_\Sigma \mathrm{d}\mu_f + \frac{1}{4}\int_\Sigma (H - H_0)^2\, \mathrm{d}\mu_f - \gamma\int_\Sigma \mathcal{K}\, \mathrm{d}\mu_f$$

with parameters α, H_0, γ, where for $\alpha, H_0, \gamma = 0$ we obtain the Willmore functional. The first term describes the surface area of $f(\Sigma)$ and H_0 in the second term stands for spontaneous mean curvature, which is preferred if we minimize $\mathcal{W}_{0,H_0,0}(f(\Sigma))$. The term with Gaussian curvature can be handled by the Gauss-Bonnet Theorem and boundary data. Consequently, bounds for W_{α,γ,H_0} directly yield bounds for the area if $\alpha \neq 0$, and subsequently, for the Willmore energy. If we want a physically meaningful model, then we have to choose a special range of parameter values. By the discussion in [Nit93] we have to assume $\alpha \geq 0$, $0 \leq \gamma \leq 1$, and $\gamma H_0^2 \leq 4\alpha(1-\gamma)$. Especially, these constraints guarantee that the entire integrand $\alpha + \frac{1}{4}\left(H - H_0\right)^2 - \gamma K \geq 0$ pointwise.

Critical points $f : \Sigma \to \mathbb{R}^3$ of the functional $\mathcal{W}_{\alpha,H_0,\gamma}$ solve the Euler-Lagrange equation [Nit93, p. 368 (21)]

$$\Delta_\Sigma H + 2H\left(\frac{1}{4}H^2 - \mathcal{K}\right) - 2(\alpha + H_0^2)H + 2H_0\mathcal{K} = 0$$

which we call the Helfrich equation. For $\alpha, H_0 = 0$, we recover the Willmore equation. If $f : \Sigma \hookrightarrow \left(\mathbb{R}^3, \langle\,.\,,\,.\,\rangle_{\mathbb{R}^3}\right)$ solves the *Helfrich equation* then f is called a *Helfrich surface*.

For closed surfaces, for $\lambda > 0$ one can also add the term $\lambda \operatorname{vol}\Sigma$ which represents the volume enclosed by $f(\Sigma)$. It is set by [MW13])

$$\operatorname{vol}\Sigma = -\frac{1}{3}\int_\Sigma \langle f, N\rangle_{\mathbb{R}^3}\,\mathrm{d}\mu_f, \text{ the signed enclosed volume,}$$

where N denotes the inward-pointing unit normal on Σ. By the divergence theorem, in the case of an embedding, the expression above agrees with the measure of the interior. In this case, we also have to add -2λ to the Helfich equation to get the correct Euler-Lagrange equation.

2.2 Geometry of Graphs

In this subsection, we want to apply the definitions of the previous subsection to graphs. Our goal is to write down the mean and Gaussian curvatures as well as the Willmore energy and the Willmore equation in local coordinates.

Let $\Omega \subset \mathbb{R}^2$ be a *bounded domain* (open, nonempty, and connected subset) with a sufficiently smooth boundary $\partial\Omega$ and the exterior boundary normal $\nu\colon \partial\Omega \to \mathbb{S}^2$ where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. For a C^2-smooth function $u\colon \overline{\Omega} \to \mathbb{R}$ we call its *graph* the surface:

$$\Gamma(u) := \left\{\big(x, u(x)\big) \,\middle|\, x \in \overline{\Omega}\right\}.$$

In the following, we use the notation $\partial_i u = u_{x_i}$. First, we note that the function u parameterizes the surface $\Gamma(u)$ as follows:

$$f\colon \overline{\Omega} \ni (x^1, x^2) \mapsto \begin{pmatrix} x^1 \\ x^2 \\ u(x^1, x^2)\end{pmatrix} \in \Gamma(u) \subset \mathbb{R}^3, \qquad \partial_1 f = \begin{pmatrix} 1 \\ 0 \\ u_{x^1}\end{pmatrix}, \quad \partial_2 f = \begin{pmatrix} 0 \\ 1 \\ u_{x^2}\end{pmatrix}. \tag{22}$$

In fact, in this case it is an immersion $f : \overline{\Omega} \hookrightarrow \left(\mathbb{R}^3, \langle\,.\,,\,.\,\rangle_{\mathbb{R}^3}\right)$ and $\Sigma = \overline{\Omega}$ is a flat surface with boundary $\partial\Omega$. As local coordinates, we use the Cartesian coordinates for $\overline{\Omega}$. For each $x \in \overline{\Omega}$ the vectors $\partial_i f$ span the tangent space of $\Gamma(u)$ which we set by $T_x\Gamma(u) := \mathrm{d}f_x(T_x\overline{\Omega})$. The *first fundamental form* here is the restriction of the Euclidean scalar product of $\mathbb{R}^3$ to $\Gamma(u)$. In local coordinates, it follows

$$(g_{ij}) := \big(\langle\partial_i f, \partial_j f\rangle\big) = \begin{pmatrix} 1 + u_{x^1}^2 & u_{x^1}u_{x^2} \\ u_{x^1}u_{x^2} & 1 + u_{x^2}^2\end{pmatrix}.$$

From now on, we use the following notation for the area element:

$$Q := \sqrt{1+|\nabla u|^2} = \sqrt{\det(g_{ij})}.$$

The inverse of (g_{ij}) by denoted (g^{ij}). In the graphical case, we have

$$(g^{ij}) = \frac{1}{Q^2}\begin{pmatrix} 1+u_{x^2}^2 & -u_{x^1}u_{x^2} \\ -u_{x^1}u_{x^2} & 1+u_{x^1}^2 \end{pmatrix}.$$

According to [Gul14, Theorem A.7 (e)], the *covariant derivative* of a function $F\colon \overline{\Omega} \to \mathbb{R}$ takes the shape

$$\begin{aligned}
\nabla^{\Gamma(u)}F &= g^{k\ell}\partial_k F \partial_\ell f \\
&= \frac{1}{Q^2}\left\{\Big((1+u_{x^2}^2)\partial_1 F - u_{x^1}u_{x^2}\partial_2 F\Big)\begin{pmatrix} 1 \\ 0 \\ u_{x^1}\end{pmatrix} + \Big(-u_{x^1}u_{x^2}\partial_1 F + (1+u_{x^1}^2)\partial_2 F\Big)\begin{pmatrix} 0 \\ 1 \\ u_{x^2}\end{pmatrix}\right\} \\
&= \frac{1}{Q^2}\begin{pmatrix} 1+u_{x^2}^2 & -u_{x^1}u_{x^2} \\ -u_{x^1}u_{x^2} & 1+u_{x^1}^2 \\ u_{x^1} & u_{x^2}\end{pmatrix} \cdot \begin{pmatrix}\partial_1 F \\ \partial_2 F\end{pmatrix}.
\end{aligned}$$

Since, by definition, $\overline{\Omega}$ is an oriented surface, $\Gamma(u)$ is also oriented. Here, we choose the *unit normal vector field* $N\colon \overline{\Omega} \to \mathbb{R}^3$ to be directed upward. Locally, it has the following representation

$$N = \frac{1}{\sqrt{1+|\nabla u|^2}}\begin{pmatrix} -\nabla u \\ 1\end{pmatrix}. \tag{23}$$

Subsequent, since the normal space $(\mathrm{d}f_x(T_x\overline{\Omega}))^\perp$ is only one-dimensional here, that is, it consists of a line, we can restrict ourselves to the scalar second fundamental form $A_x\colon T_x\Omega \times T_x\Omega \to \mathbb{R}$ given in (13). In local coordinates, it has the following representation:

$$A_{ij} = A_x(\partial_i f, \partial_j f) = -\langle \partial_i f, D_{\partial_j f}N\rangle. \tag{24}$$

But since we have $\langle N, \partial_i f\rangle = 0$, the relation $\langle D_{\partial f_i}N, \partial_j f\rangle + \langle N, D_{\partial f_i}\partial_j f\rangle = 0$ holds. Because of [Gul14, Lemma A.4], the relation $D_{\partial f_i}\partial_j f = \partial_{ij} f$ holds. So it results

$$A_{ij} = \langle N, \partial_{ij} f\rangle = \frac{1}{Q}\left\langle \begin{pmatrix} 0 \\ 0 \\ u_{x_ix_j}\end{pmatrix}, \begin{pmatrix} -u_{x^1} \\ -u_{x^2} \\ 1\end{pmatrix}\right\rangle = \frac{u_{x_ix_j}}{Q}. \tag{25}$$

Then we get the local representation of the *Weingarten-mapping* $W = -\,\mathrm{d}N$:

$$\begin{aligned}
(A^i_j) &= \Big(\sum_{\ell=1}^{2} g^{i\ell}A_{\ell j}\Big) = \frac{1}{QQ^2}\begin{pmatrix} 1+u_{x^2}^2 & -u_{x^1}u_{x^2} \\ -u_{x^1}u_{x^2} & 1+u_{x^1}^2\end{pmatrix} \circ \begin{pmatrix} u_{x^1x^1} & u_{x^1x^2} \\ u_{x^2x^1} & u_{x^2x^2}\end{pmatrix} \\
&= \frac{1}{Q^3}\begin{pmatrix} (1+u_{x^2}^2)u_{x^1x^1} - u_{x^1}u_{x^2}u_{x^1x^2} & (1+u_{x^2}^2)u_{x^1x^2} - u_{x^1}u_{x^2}u_{x^2x^2} \\ -u_{x^1}u_{x^2}u_{x^1x^1} + (1+u_{x^1}^2)u_{x^1x^2} & -u_{x^1}u_{x^2}u_{x^1x^2} + (1+u_{x^1}^2)u_{x^2x^2}\end{pmatrix}.
\end{aligned} \tag{26}$$

with the square norm of the second fundamental form

$$\|A\|_g^2 = \sum_{i,j,k,\ell=1}^{2} g^{ij}g^{k\ell}A_{ik}A_{j\ell}.$$

For the square norm of the second fundamental form, Deckelnick, Grunau, and Röger [DGR17, Lemma 1] proved the following estimate with the Euclidean matrix norm of the Hessian of u

$$\frac{|D^2u|^2}{Q^2} = \sum_{i,j=1}^{2} \frac{u_{x^ix^j}^2}{Q^2} \geq \|A\|_g^2 \geq \frac{|D^2u|^2}{Q^6}. \tag{27}$$

Because the codimension equals one, we can take the scalar mean curvature as described in (16). Furthermore, we want to write down the mean curvature in local representation A_{ij}. We also note that H is the trace of the Weingarten mapping (see [Gul14, Theorem A.7 (a)]).

$$\begin{aligned} H &= \operatorname{trace}\big((A^i_j)\big) = \nabla\cdot\left(\frac{\nabla u}{Q}\right) = \frac{\Delta u}{Q} - \frac{\nabla u\cdot\big(\nabla|\nabla u|^2\big)}{2Q^3} = \frac{\Delta u}{Q} - \frac{\nabla u\cdot\big(D^2u\,\nabla u\big)}{Q^3} \\ &= \frac{1}{Q^3}\big(u_{x^1x^1}(1+u_{x^1}^2+u_{x^2}^2) - u_{x^1}(u_{x^1x^1}u_{x^1}+u_{x^1x^2}u_{x^2}) \\ &\qquad + u_{x^2x^2}(1+u_{x^1}^2+u_{x^2}^2) - u_{x^2}(u_{x^1x^2}u_{x^1}+u_{x^2x^2}u_{x^2})\big) \\ &= \frac{1}{Q^3}\Big(u_{x^1x^1}(1+u_{x^2}^2) - 2u_{x^1}u_{x^1x^2}u_{x^2} + u_{x^2x^2}(1+u_{x^1}^2)\Big). \end{aligned} \tag{28}$$

Since the Gaussian curvature is the determinant of (A^i_j), we get

$$\mathcal{K} = \det\big((A^i_j)\big) = \frac{\det D^2u}{Q^4}.$$

In [Gul14] one can see that this definition is equivalent to the one defining $\mathcal{K}$ as the determinant of the Weingarten mapping ($W = -\,\mathrm{d}N$, N unit normal field). Moreover, the *Willmore functional* for the graph of u takes the following shape:

$$\mathcal{W}(u) = \frac{1}{4}\int_\Omega H^2\sqrt{1+|\nabla u|^2}\,\mathrm{d}x = \frac{1}{4}\int_\Omega \left|\nabla\cdot\left(\frac{\nabla u}{Q}\right)\right|^2 Q\,\mathrm{d}x,$$

The Laplacian of mean curvature can be calculated via (11):

$$\begin{aligned} \Delta_{\Gamma(u)}H = \frac{1}{Q}\partial_i\big(Qg^{ij}\partial_jH\big) &= \frac{1}{Q}\frac{\partial}{\partial x^1}\left\{\frac{1}{Q}\Big((1+u_{x^2}^2)\frac{\partial}{\partial x^1}H - u_{x^1}u_{x^2}\frac{\partial}{\partial x^2}H\Big)\right\} \\ &\quad + \frac{1}{Q}\frac{\partial}{\partial x^2}\left\{\frac{1}{Q}\Big(-u_{x^1}u_{x^2}\frac{\partial}{\partial x^1}H + (1+u_{x^1}^2)\frac{\partial}{\partial x^2}H\Big)\right\}. \end{aligned} \tag{29}$$

Finally, we want to mention the first variation of the Willmore functional for graphs. Among other things, it is important for us to obtain the boundary terms and the divergence structure of the Willmore equation.

7 Theorem (First Variation of the Willmore Functional)
Let Ω be a bounded, C^4-smooth bounded domain with the exterior unit normal vector field ν (the negative conormal) and $u \in C^4(\overline{\Omega})$, $\varphi \in C^2(\overline{\Omega})$, then it holds:

$$\begin{aligned} \big\langle \mathcal{W}'(u),\varphi\big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(u+t\varphi)\Big|_{t=0} &= \frac{1}{2}\int_\Omega\left\{\Delta_{\Gamma(u)}H + 2H\left(\frac{1}{4}H^2-\mathcal{K}\right)\right\}\varphi\,\mathrm{d}x \\ &\quad - \frac{1}{2}\int_{\partial\Omega}\left[F\cdot\nu + \left(\frac{\partial}{\partial\tau}\Psi\right)\cdot\tau - \kappa_g\Psi\cdot\nu\right]\varphi\,\mathrm{d}s + \frac{1}{2}\int_{\partial\Omega}\Psi\cdot\nu\frac{\partial\varphi}{\partial\nu}\,\mathrm{d}s, \end{aligned}$$

with $\tau\colon \partial\Omega \to \mathbb{S}^1$ as a tangent unit vector field on $\partial\Omega$, and let κ_g be the geodesic curvature of the boundary with respect to the parametrization such that Ω is on the left side of the parametrization. Here, we use the notation

$$F = \frac{1}{Q}\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right)\nabla\Big(QH\Big) - \frac{H^2}{2Q}\nabla u, \quad \Psi = \sqrt{Q}H\left(\frac{1}{\left(1+|\nabla u|^2\right)^{\frac{1}{4}}}\left(I - \frac{\nabla u \otimes \nabla u}{1+|\nabla u|^2}\right)\nu\right),$$

where I stands for the 2×2-unit martix and $\otimes$ for the matrix-tensor product. In particular, one has the divergence structure:

$$\nabla \cdot F = \Delta_{\Gamma(u)} H + 2H\left(\frac{1}{4}H^2 - \mathcal{K}\right).$$

Proof: See Section "variational formulation and discretization" in [DKS15]. For details, we also refer to [Gul17, Theorem 3.9]. □

3 Analytic Preliminaries

First, we have to recall the most fundamental spaces like the Hölder, Lebesgue, or Sobolev spaces. Foremost, we define the Hölder spaces. Here, we adopt the notation used in the work [LSvW92]. We also use the multi-index-notation, i.e., $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{N}_0^n$, $|\omega| = |\omega_1| + \cdots + |\omega_n|$ and:

$$D^\omega f = \frac{\partial^{|\omega|}}{\partial x_1^{\omega_1} \cdots \partial x_n^{\omega_n}} f, \qquad \partial_i := \frac{\partial}{\partial x_i},$$

for clarity and better readability.

Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ be an open set. In this chapter, we only consider real-valued functions on $\Omega, \overline{\Omega}$ or $\partial\Omega$, and we define a seminorm called the *Hölder coefficient* for $0 < \alpha \leq 1$ as follows:

$$[f]_{C^\alpha(\overline{\Omega})} := \sup \left\{ \frac{|f(x) - f(y)|}{|x-y|^\alpha} \,\middle|\, x, y \in \overline{\Omega},\ x \neq y \right\}.$$

For the special case when $\alpha = 1$, the above definition corresponds to the *Lipschitz condition*. Assume $\Omega \subset \mathbb{R}^n$ is bounded, and $0 < \alpha < \beta \leq 1$. In this case, we can apply the following useful inequality frequently throughout the paper:

$$[f]_{C^\alpha(\overline{\Omega})} \leq \operatorname{diam}(\Omega)^{\beta-\alpha} [f]_{C^\beta(\overline{\Omega})}. \tag{30}$$

For $k \in \mathbb{N}_0$,we use $C^k(\Omega)$ to represent the space of *k-differentiable functions on* $\Omega \subset \mathbb{R}^n$. We define the space $C^k(\overline{\Omega})$ as the set of all functions $f \in C^k(\Omega)$ for which the following holds: f, along with all derivatives of f of order $\leq k$, can be continuously extended to $\overline{\Omega}$.

In the case, where $\overline{\Omega}$ is *compact*, one can define a norm on $C^k(\overline{\Omega})$ as follows:

$$\|f\|_{C^k(\overline{\Omega})} := \sum_{|\ell| \leq k} \sup_{x \in \Omega} |D^\ell f(x)|, \qquad f \in C^k(\overline{\Omega}).$$

We also recall the notation of the *closed support of f*

$$\operatorname{supp} f = \overline{\{x \in \Omega \,|\, f(x) \neq 0\}}.$$

Then by $C_c^k(\Omega)$ we denote the space of functions $f \in C^k(\Omega)$ with compact support in Ω. We can now define the *Hölder spaces* for bounded domains $\Omega \subset \mathbb{R}^n$, where $0 < \alpha < 1$. To avoid confusion with spaces of differentiable functions, we specifically require $k \in \mathbb{N}_0$ so that $k + \alpha \notin \mathbb{N}$:

$$C^{k+\alpha}(\overline{\Omega}) := \left\{ f \in C^k(\overline{\Omega}) \,\middle|\, \forall |\ell| = k \colon [D^\ell f]_{C^\alpha(\overline{\Omega})} < \infty \right\}, \quad \|f\|_{C^{k+\alpha}(\overline{\Omega})} := \|f\|_{C^k(\overline{\Omega})} + \sum_{|\ell|=k} [D^\ell f]_{C^\alpha(\overline{\Omega})},$$

Here, for $f \in C^{k+\alpha}(\overline{\Omega})$, the *Hölder norm* is denoted by $\|f\|_{C^{k+\alpha}(\overline{\Omega})}$. The Hölder spaces are *Banach spaces*. There is a product estimate for Hölder functions, which involves seminorms. Let $u, v \in C^\alpha(\overline{\Omega})$, $\alpha \in (0,1)$:

$$[uv]_{C^\alpha(\overline{\Omega})} \leq \sup_{x \in \Omega} |u(x)| \cdot [v]_{C^\alpha(\overline{\Omega})} + [u]_{C^\alpha(\overline{\Omega})} \cdot \sup_{x \in \Omega} |u(x)|. \tag{31}$$

It follows that:

$$\|gf\|_{C^\alpha(\overline{\Omega})} \leq \|g\|_{C^\alpha(\overline{\Omega})} \cdot \|f\|_{C^\alpha(\overline{\Omega})}.$$

Further, since it is important for the regularity issues, we want to define the *smoothness of boundary*. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For $\ell \in \mathbb{R}_{\geq 0}$ we say that the boundary $\partial\Omega$ is C^ℓ-smooth, if there is a finite number of closed balls $\{B_i\}_{i=1}^N$, whose interior covers the boundary $\partial\Omega \subset \bigcup_{i=1}^N \operatorname{int}(B_i)$, such that there exist C^ℓ-diffeomorphisms $\{\varphi_i\}_{i=1}^N$ from each B_i to the unit ball $B := B_1(0)$ and for each $i = 1, \dots, N$ it holds:

$$\varphi_i(\Omega \cap B_i) = B \cap \{x_n > 0\}, \quad \varphi(\partial\Omega \cap B_i) = B \cap \{x_n = 0\},$$

where $\{x_n > 0\} := \{x \in \mathbb{R}^n | x_n > 0\}$ and $\{x_n = 0\} := \{x \in \mathbb{R}^n | x_n = 0\}$. We call $\{B_i\}_{i=1}^N$ a finite open covering of $\partial\Omega$. In particular, $\partial\Omega$ is a $(n-1)$-dimensional C^ℓ-smooth submanifold.

As in [MMS10, Subsection 6.1] we call Ω a *Lipschitz bounded domain*, if there exists a finite open covering $\{B_i\}_{i=1}^N$ of $\partial\Omega$ such that, for each $i \in \{1, \dots, N\}$, after a rigid motion of $\mathbb{R}^n$ the intersection $B_i \cap \Omega$ coincides with the segment of B_i lying in the over-graph of a Lipschitz function $\psi_i \colon \mathbb{R}^{n-1} \to \mathbb{R}$. Subsequently, we define the Lipschitz constant of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ as:

$$\inf \max\{\|\nabla\psi_i\|_{L^\infty(\mathbb{R}^{n-1})} : 1 \leq i \leq N\} \tag{32}$$

where we take the infimum over all possible finite open coverings of $\partial\Omega$ with corresponding Lipschitz functions. For example, C^1-smooth bounded domains have vanishing Lipschitz constant and a square has the Lipschitz constant equal to one. This definition is based on the notion of minimally smooth domains in [Ste70, Subsection 3.3 p 189]. Also, for each Lipschitz domain, there exists the surface measure on $\partial\Omega$ and outward-pointing normal vector ν almost everywhere with respect to the surface measure on $\partial\Omega$ due to Rademacher's Theorem.

Let us continue with defining the *Lebesgue space* $L^p(\Omega)$ for each $1 \leq p < \infty$. It comprises measurable functions u on Ω that satisfy the condition:

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{1/p} < \infty.$$

By $L^p_{\mathrm{loc}}(\Omega)$ we denote the *space of locally integrable functions*, which means its Lebesgue integral is finite on all compact subsets K of Ω.

Subsequently, we define the *Sobolev space* $W^{m,p}(\Omega)$ as the set of real-valued functions $u \in L^p_{\mathrm{loc}}(\Omega)$ with the property that $D^\alpha u \in L^p_{\mathrm{loc}}(\Omega)$ for all $|\alpha| \leq m$. For u in this space, we have the following norm:

$$\|u\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p} < \infty. \tag{33}$$

It is worth noting that in the case $m = 0$, we recover the Lebesgue space $L^p(\Omega) = W^{0,p}(\Omega)$, and when $m = 1$, it becomes the Sobolev space $W^{1,p}(\Omega)$. For Lipschitz domains the space $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$ with respect to the norm (33), see [GT01, Subsection 7.6 and Problem 7.11]. Consequently, we have $W^{m,p}(\Omega) = \big\{$ closure of $C^\infty(\overline{\Omega})$ in $W^{m,p}(\Omega)\big\}$ with respect to the norm (33). Furthermore, we define the *homogeneous Sobolev space* as follows:

$$\mathring{W}^{m,p}(\Omega) := \big\{ \text{ closure of } C^\infty_c(\Omega) \text{ in } W^{m,p}(\Omega)\big\} \tag{34}$$

Both $W^{m,p}(\Omega)$ and $\mathring{W}^{m,p}(\Omega)$ are separable Banach spaces.

Regarding the embedding results for Sobolev spaces, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. If $p > n$ and $0 < \lambda \leq 1 - \frac{n}{p}$, then it follows that:

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C^\lambda(\overline{\Omega}).$$

Assume $q \geq p$ with $\frac{n}{q} - \frac{n}{p} + 1 > 0$. Then we have the compact embedding:

$$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega).$$

In the context of $\beta \leq \ell$, the Hölder space $C^\beta(\partial\Omega)$ on the boundary is the set of all functions $g\colon \partial\Omega \to \mathbb{R}$ satisfying $\forall i \in \{1, \dots, N\}$ the property $g \circ \varphi_i^{-1} \in C^\beta(B \cap \{x_n = 0\})$, where the norm is defined with respect to parametrizations by arclength (the geodesic distance). The Lebesgue spaces on the boundary are defined similarly. Since for each Lipschitz domain, there exists the surface measure S on $\partial\Omega$, then we can define the *Lebesgue space on boundary* $L^p(\partial\Omega)$ for each $1 \leq p < \infty$ that comprises surface S-measurable functions g on $\partial\Omega$ such that

$$\|g\|_{L^p(\partial\Omega)} := \left(\int_{\partial\Omega} |g(x)|^p \, \mathrm{d}S(x) \right)^{1/p} < \infty.$$

In case $\partial\Omega$ is a curve parametized by a C^1-smooth map $\gamma\colon I \to \mathbb{R}^n$ with $I \subset \mathbb{R}$, then $\|g\|_{L^p(\partial\Omega)} = \|g \circ \gamma\|_{L^p(I)}$ in case $\|\gamma'\| \equiv 1$. The last condition on γ' characterizes parametrizations by arclength.

3.1 Higher-Order Elliptic Operators & Fixed Point Methods

Since we want to rewrite the Willmore equation as an elliptic equation and the Willmore flow equation as a parabolic equation, let us recall the following general definition. Let $m \in \mathbb{N}$ such that $m \geq 2$, then we call L a divergence-form m-order elliptic real scalar operator with variable real coefficients in the case it is acting on *scalar* functions $u\colon \mathbb{R}^n \to \mathbb{R}$

$$Lu = L(x, D_x)u := \sum_{|\alpha|=|\beta|=m} D^\alpha \big(A_{\alpha\beta}(x) D^\beta u \big), \quad x \in \Omega, \tag{35}$$

where $D^\alpha = \partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. Further, we assume that $\Omega \subset \mathbb{R}^n$ is a domain with Lipschitz boundary $\partial\Omega$ and compact closure $\overline{\Omega}$ as well as $A_{\alpha\beta}\colon \Omega \to \mathbb{R}$ are measurable bounded coefficients so that for some $C > 0$

$$\sum_{|\alpha|=|\beta|=m} \|A_{\alpha\beta}\|_{L^\infty(\Omega)} \leq C.$$

Furthermore, we impose the *Legendre–Hadamard ellipticity condition* on coefficients $A_{\alpha\beta}$ which for our real case is defined by the following inequality

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha\beta}(x) \xi^\alpha \xi^\beta \geq \lambda \|\xi\|^{2m} \tag{36}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. We call λ the *ellipticity constant*. In this context, it is worth noting that it is not the most general form of an elliptic operator. Like in [MS11, p.37], one can consider operators with complex coefficients acting on a vector-valued function and satisfying the coercitivity condition.

The most famous classical examples of elliptic operators are *polyharmonic operators*, namely iterations of the Laplace operator, defined inductively by

$$\Delta^m u = \Delta(\Delta^{m-1} u).$$

where in the case $m = 2$ we especially call Δ^2 the *biharmonic operator* or the *bilaplacian*.

It is canonical to search for the solutions of the general elliptic equation consisting of the elliptic operator L and some *right-hand side functions* $f_\alpha \in L_{\text{loc}}(\Omega)$, $|\alpha| \leq m$

$$Lu = \sum_{|\alpha|=0}^{m} D^\alpha f_\alpha \tag{37}$$

which we describe in the following way. Let the function u be a *variational solution* to (37) if the following equation is valid:

$$\forall v \in C_0^\infty(\Omega): \qquad \sum_{|\alpha|=|\beta|=m} \int_\Omega D^\alpha v A_{\alpha\beta}(x) D^\beta u \, \mathrm{d}x = \sum_{|\alpha|=0}^{m} (-1)^{|\alpha|} \int_\Omega f_\alpha \cdot D^\alpha v \, \mathrm{d}x.$$

In case boundary data is considered, one naturally takes the traces of lower-order derivates. Here ones exploits the Lipschitz structure of Ω, namely, there exists the outward unit normal $\nu = (\nu_1, \dots, \nu_n)$. Thus, for every $0 \leq k \leq m-1$ we take *normal derivatives*

$$\frac{\partial^k u}{\partial \nu^k} := \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha \operatorname{Tr}[D^\alpha u], \tag{38}$$

where Tr is the boundary trace operator and for each multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ we define $\nu^\alpha := \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}$. At this point, we can recall the corresponding inhomogeneous Dirichlet problem for such an operator

$$\begin{cases} \displaystyle\sum_{|\alpha|=|\beta|=m} D^\alpha\big(A_{\alpha\beta}(x) D^\beta u\big) = 0 & \text{for } x \in \Omega, \\ \displaystyle\frac{\partial^k u}{\partial \nu^k} = g_k \quad \text{on } \partial\Omega, & 0 \leq k \leq m-1, \end{cases} \tag{39}$$

where $g_k \colon \partial\Omega \to \mathbb{R}$ are some measurable functions with respect to the surface measure, which are traces of some weak differentiable function in the sense of (38) and its regularity class that will be specified later. Especially, it will depend on regularity classes of u and the boundary $\partial\Omega$.

For the second-order elliptic equation, the *maximum modulus theorem* is a well-known classical result. It was extended to the solutions of the biharmonic equation by Miranda and then generalized by Agmon-Douglis-Nirenberg for higher-order elliptic operators for half-space with constant coefficients with no lower-order terms. Finally, Agmon [Agm60] established the weak maximum principle on smooth domains. Which, like pointed out by [BM16], roughly speaking states that for a solution u to the equation $Lu = 0$ in Ω, where L is a $2m$-order elliptic operator with coefficients smooth enough and Ω smooth, it follows

$$\max_{|\alpha| \leq m-1} \|D^\alpha u\|_{L^\infty(\Omega)} \leq C \max_{|\beta| \leq m-1} \|D^\beta u\|_{L^\infty(\partial\Omega)}. \tag{40}$$

3.2 Fixed Point Methods

In this work, we use exclusively *Banach fixed point theorem* out of various fixed point results. Therefore, throughout the subsequent discussions, when we refer to a "fixed point," it is synonymous with the "Banach fixed point."

8 Theorem (Banach Fixed Point Theorem)
Let (X, d) be a complete metric space (X, d) and $\varnothing \neq M \subset X$. Let (X, d) be

$$T \colon M \to M$$

be given an contraction mapping, i.e., $k \in (0,1)$ and for all $x, y \in M$ holds:

$$d(Tx, Ty) \leq k\, d(x,y).$$

Then the following statements hold:

ⓐ *T has exactly one fixed point, i.e., there is exactly one solution $x \in M$ of the equation:*

$$Tx = x.$$

ⓑ *The iterative sequence $\{x_i\}_{i=1}^{\infty}$ defined by the equation below converges to exactly one solution for all initial values $x_0 \in M$:*

$$x_0 \in M, \quad x_{i+1} := Tx_i, i \in \mathbb{N}_0.$$

Proof: [Dei10] Theorem 7.1 page 39. □

The proofs of existence theorems for small data in the elliptic case and short-time existence Theorems in the parabolic case are both based on the same two steps of a *linearization* technique. The first one solves the associated linear problem, and the second uses some fixed point argument to extend the result to the nonlinear case. The first step is done by discussing the biharmonic case, and for the second step, we separate the elliptic linear part $L[u]$ and non-linear part $f[u]$ in the following sense

$$\begin{aligned} &\textit{elliptic:} && L[u] &&= f[u] \text{ in } \Omega, && \operatorname{Tr} u^{\Omega}_{m-1} &&= g, \\ &\textit{parabolic:} && \partial_t u + L[u] &&= f[u] \text{ in } \Omega \times (0,T], && \operatorname{Tr} u^{\Omega}_{m-1} &&= g \text{ and } u|_{t=0} = u_0 \end{aligned} \tag{41}$$

with some trace operators for Ω. Then we freeze the non-linear part $f[u]$ by replacing u with some w from some appropriate space

$$\begin{aligned} &\textit{elliptic:} && L[u] &&= f[w] \text{ in } \Omega, && \operatorname{Tr} u^{\Omega}_{m-1} &&= g, \\ &\textit{parabolic:} && \partial_t u + L[u] &&= f[w] \text{ in } \Omega \times (0,T], && \operatorname{Tr} u^{\Omega}_{m-1} &&= g \text{ and } u|_{t=0} = u_0. \end{aligned} \tag{42}$$

Depending on the regularity assumption on the boundary and Dirichlet's data, we choose some spaces $\mathfrak{X}, \mathfrak{H}$ for u and $f[u]$ which will strongly depend on the setting of Ω and boundary data class. Furthermore, we need boundary spaces $\mathfrak{D}$ for $u|_{\partial\Omega}$ and for parabolic case $\mathfrak{I}$ for initial values $u|_{t=0}$. Then, we want to get existence results and estimates for (42)

$$\begin{aligned} &\textit{elliptic:} && \|u\|_{\mathfrak{X}} &&\leq C\|f[w]\|_{\mathfrak{H}} + C\|g\|_{\mathfrak{D}} \\ &\textit{parabolic:} && \|u\|_{\mathfrak{X}} &&\leq C\|f[w]\|_{\mathfrak{H}} + C\|g\|_{\mathfrak{D}} + \|u_0\|_{\mathfrak{I}}. \end{aligned}$$

So we can define some iteration mappings $G : \mathfrak{X} \to \mathfrak{X}$ by setting for every $w \in \mathfrak{X}$ some $u = G(w)$ as solution of (42).

Now, if we could obtain some fixpoint $u \in \mathfrak{X}$, hence $u = G(u)$, then we could conclude the existence and regularity results for the nonlinear problem (41). Thus, we reformulated our situation as a fixed point problem and we have to show that $G\colon \mathcal{M} \to \mathcal{M}$ is a self-map and contraction on some closed nonempty set $\mathcal{M} \subset \mathfrak{X}$. Both properties are obtained by setting some restrictions on $\mathcal{M}$, for example in the elliptic case, we need some smallness conditions on boundary data, and in the parabolic case, we have to choose existence time small enough.

3.3 Interpolation Spaces

As a technical tool for achieving smallness in some norms, we often use interpolation inequalities. There exists a vast theory of interpolation spaces, including some abstract space-construction techniques. These, roughly speaking, produce some Banach space Y lying between two Banach spaces D and X in the sense that the injections $D \subset Y \subset X$ are continuous and moreover there are a constants $C > 0$ and $\alpha \in [0, 1]$ thus that

$$\forall x \in D\colon \quad \|x\|_Y \leq C\|x\|_D^{1-\alpha} \cdot \|x\|_X^{\alpha}$$

with corresponding norms. We call such estimates *interpolation inequalities*, and these are the results we need from interpolation theorems. We will consequently choose $\|x\|_X$ to be bounded and $\|x\|_D$ small enough to achieve smallness in $\|x\|_Y$.

Some examples are L^p-spaces and Hölder-Spaces. It is a well known fact, that for $1 \leq p_0 \leq p_1 \leq \infty$ and $u \in L^{p_1}(\Omega) \cap L^{p_0}(\Omega)$ it follows with the Hölder's inequality

$$\|u\|_{L^p(\Omega)} \leq C\|u\|_{L^{p_0}(\Omega)}^{1-\theta} \cdot \|u\|_{L^{p_1}(\Omega)}^{\theta}, \quad \text{with} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

9 Theorem (Hölder Interpolation Result)
For $\Omega \subset \mathbb{R}^n$, $0 \leq a \leq b$ and Ω bounded with C^b boundary, $0 < \lambda < 1$ there is a constant $C_1 = C_1(\Omega, a, b)$ such that

$$\|u\|_{C^a(\overline{\Omega})} \leq C_1\|u\|_{C^b(\overline{\Omega})}, \qquad \|u\|_{C^{\lambda a+(1-\lambda)b}(\overline{\Omega})} \leq C_1\|u\|_{C^a(\overline{\Omega})}^{\lambda} \cdot \|u\|_{C^b(\overline{\Omega})}^{1-\lambda}. \tag{43}$$

with $C^k(\overline{\Omega}) = C^{k-1,1}(\overline{\Omega})$ in this Theorem.

Proof: Here we want to use [Hö76, Theorem A.5 p. 50], but since it is only valid for convex sets, we have to extend u to a convex set. In our case, it will be a ball.

Let $b = k + \alpha$, and $R > 0$ with $\overline{\Omega} \subset B_R(0)$. By the proof of [GT01, Lemma 6.37] there exists $C_2 = C_2(\Omega, k, \alpha)$ and an extension operator $T\colon C^{k+\alpha}(\overline{\Omega}) \to C_0^{k+\alpha}(B_R(0))$ such that

$$\forall\, 0 \leq j + \beta \leq k + \alpha : \quad \|Tu\|_{C_0^{j+\beta}(B_R(0))} \leq C_2\|u\|_{C^{j+\beta}(\overline{\Omega})}.$$

Then by [Hö76, Theorem A.5 p. 50] applied in $B_R(0)$ with a constant $C_3 = C_3(a, b, R)$ it follows

$$\begin{aligned}\|u\|_{C^{\lambda a+(1-\lambda)b}(\overline{\Omega})} &\leq \|Tu\|_{C_0^{\lambda a+(1-\lambda)b}(B_R(0))} \leq C_3\|Tu\|_{C_0^a(B_R(0))}^{\lambda} \cdot \|Tu\|_{C_0^b(B_R(0))}^{1-\lambda} \\ &\leq C_3\big(C_2\|u\|_{C^a(\overline{\Omega})}\big)^{\lambda} \cdot \big(C_2\|u\|_{C^b(\overline{\Omega})}\big)^{1-\lambda} \\ &\leq C_3C_2\|u\|_{C^a(\overline{\Omega})}^{\lambda} \cdot \|u\|_{C^b(\overline{\Omega})}^{1-\lambda}.\end{aligned}$$

We finish the proof by setting $C_1 = C_3C_2$. □

For the study of the long-time existence of the Willmore-flow solutions, we also need a Hölder interpolation involving $L^2(\Omega)$-norm, over which we will have more control than over $C^0(\overline{\Omega})$-norm. The idea is the same as in [DPS16, after 5.16]

10 Theorem (Hölder-L^p Interpolation Result)
If $m > 0$, $\partial\Omega \in C^{m+\alpha}$ and $\alpha \in (0, 1)$ then for $u \in C^{m+\alpha}(\overline{\Omega})$, there exist $\theta \in (0, 1)$

$$\|u\|_{C^m(\overline{\Omega})} \leq C\|u\|_{C^{m+\alpha}(\overline{\Omega})}^{1-\theta} \cdot \|u\|_{L^2(\Omega)}^{\theta}.$$

Proof: We use the interpolation results for Besov spaces by [BL76, Thm 6.4.5 (3)] with $\theta = 2(b - a)/(4 + 2b)$:

$$\left(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}\right)_{\theta,p^*} = B^{s^*}_{p^*,q^*}, \qquad (s_0 \neq s_1, p^* = q^*, 1 \leq p_0, p_1, q_0, q_1 \leq \infty)$$

with additional restrictions

$$s^* = (1-\theta)s_0 + \theta s_1, \qquad \frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

With $q_0 = p_0 = 2$ by [Tri78, Theorem 4.6.1 (b) p. 328] we get $B^s_{2,2} = H^s_2$ the generalized Sobolev Space. By [Tri10, Subsection 1.5.1 Theorem (ii) p.29] for $s > 0$, then $B^s_{\infty,\infty} = \mathcal{C}^s$ (Hölder–Zygmund space [Tri10, Subsection 1.2.2]) and by [Tri10, Subsection 1.2.2 Theorem (ii)] if $0 < s \neq$ integer, then $C^s = \mathcal{C}^s$

$$\left(B^0_{2,2}, B^{s_1}_{\infty,\infty}\right)_{\theta,p^*} = B^{s^*}_{p^*,q^*}. \tag{44}$$

To fulfill the above restrictions we set $p^* = q^* = 2/(1-\theta)$ and $s^* = \theta s_0$, thus

$$\left(L^2(\Omega), C^{s_1}(\overline{\Omega})\right)_{\theta,p^*} = B^{\theta s_1}_{\frac{2}{1-\theta},\frac{2}{1-\theta}}. \tag{45}$$

By [Tri78, Theorem 4.6.1. (b) p.327]: For $1 < p < \infty, 1 \leq r \leq \infty, t \geq 0$ and $s > t + n/p$ it follows $B^s(\Omega) \subset C^t(\overline{\Omega})$, and by the proof of (e) and (d) of 2.8.1 on page 205 in [Tri78] the embedding is continuous. In our case: $\theta s_1 > t + (1-\theta)$ so that $s_1/(1 - 1/\theta) > t$, so we can find $\theta \in (0,1)$ for every $t < s_1$. Next, we use the interpolation inequality [Tri78, Thm 1.3.3. p.25]

$$\|u\|_{C^t(\overline{\Omega})} \leq C\|u\|^{1-\theta}_{L^2(\Omega)} \|u\|^{\theta}_{C^{s_1}(\overline{\Omega})}.$$

We used $\partial\Omega \in C^{m+\alpha}$ for smoothering. □

Also, we need an interpolation result for Hölder boundary spaces.

11 Lemma (Hölder Boundary Interpolation Result)
For $\Omega \subset \mathbb{R}^n$ bounded with $C^{2+\alpha}$-smooth boundary, $\gamma, \alpha \in (0,1)$ there exists a constant C_4 such that

$$\|g\|_{C^{1+\gamma}(\partial\Omega)} \leq C_4 \|g\|^{1-\frac{\gamma}{1+\alpha}}_{C^1(\partial\Omega)} \cdot \|g\|^{\frac{\gamma}{1+\alpha}}_{C^{2+\alpha}(\partial\Omega)}$$

Proof: First, we take a glance at the exponents

$$1 + \gamma = \theta \cdot 1 + (1-\theta) \cdot (2 + \alpha) \quad \Leftrightarrow \quad \theta = \frac{1 + \alpha - \gamma}{1 + \alpha}.$$

Let ψ be the solution of the biharmonic problem

$$\begin{cases} \Delta^2 \psi = 0, & \text{in } \Omega, \\ \quad \psi = g, \quad \partial_\nu \psi = \partial_\nu g, & \text{in } \partial\Omega, \end{cases}$$

Then by Hölder-Schauder-estimates (see [GGS10, Theorem 2.19 p.45]), we obtain

$$\|\psi\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_5 \|g\|_{C^{2+\alpha}(\partial\Omega)}$$

Furthermore with the Miranda result

$$\|\psi\|_{C^1(\overline{\Omega})} \leq C_6 \|g\|_{C^1(\partial\Omega)}$$

Combing the results gives

$$\|g\|_{C^{1+\gamma}(\partial\Omega)} \leq \|\psi\|_{C^{1+\gamma}(\overline{\Omega})} \leq C_1 \|\psi\|^{1-\frac{\gamma}{1+\alpha}}_{C^1(\overline{\Omega})} \|\psi\|^{\frac{\gamma}{1+\alpha}}_{C^{2+\alpha}(\overline{\Omega})} \leq C_4 \|g\|^{1-\frac{\gamma}{1+\alpha}}_{C^1(\partial\Omega)} \|g\|^{\frac{\gamma}{1+\alpha}}_{C^{2+\alpha}(\partial\Omega)}$$

□

4 Estimates Involving the Willmore-Energy

This chapter discusses various estimates involving the Willmore energy for surfaces with boundaries. It is essential to ask which quantities can be bounded by the Willmore energy. Here we will recall some diameter and area estimates, shown for immersed surfaces with $\mathcal{W}(f(\Sigma)) < 4\pi$ by Rivière [Riv13], and Pozzetta [Poz21] and for general graphs by Grunau, Röger and Deckelnick [DGR17]. Actually, for the following chapters, we only need the L^2-smallness Theorem 16 from all presented results. It is used to prove the global existence of the graphical Willmore flow for small data in Section 6.

4.1 Immersions

Foremost, we handle the case of immersions, which generally lack projectivity. We want to recall a diameter estimate where Rivière uses Simon's monotonicity formula extended by boundary terms for Lipschitz immersions into $\mathbb{R}^m$ with L^2-bounded second fundamental form [Riv13]. These results use the non-tangential divergence theorem on surfaces, which can be found in [Sim83].

Let Σ be a bounded surface with smooth boundary and let $f\colon \Sigma \hookrightarrow \mathbb{R}^m$ be a C^2-immersion of Σ. Denote by $\mathcal{M} := f(\Sigma)$ the immersed surface with metric g and μ_f which corresponds to the Hausdorff measure $\mathcal{H}^2$ on $f(\mathcal{M})$. Moreover, let $-\operatorname{co}_f = \vec{\nu}$ be the unit limiting tangent vector field to $\mathcal{M}$ on $\partial\mathcal{M}$ orthogonal to it and oriented in the outward direction. In what follows, we write $\|\,.\,\| = \|\,.\,\|_{\mathbb{R}^m}$. Then by [Riv13, pp. 21, Lemma A.3] for any chosen point $p_0 \in \mathbb{R}^m$ and any two radii $0 < t < T < +\infty$ the following identity holds

(46)

$$\begin{aligned}
&\frac{\mathcal{H}^2\left(\mathcal{M}\cap B_T\left(p_0\right)\right)}{T^2} - \frac{\mathcal{H}^2\left(\mathcal{M}\cap B_t\left(p_0\right)\right)}{t^2}\\
&\qquad = \int_{\mathcal{M}\cap B_T(p_0)\setminus B_t(p_0)} \left\| \frac{\vec{H}}{4} + \frac{(p-p_0)^{\perp}}{\|p-p_0\|^2} \right\|^2 \mathrm{d}\mathcal{H}^2(p) - \frac{1}{16}\int_{\mathcal{M}\cap B_T(p_0)\setminus B_t(p_0)} \|\vec{H}\|^2 \,\mathrm{d}\mathcal{H}^2\\
&\qquad\quad - \frac{1}{2T^2}\int_{\mathcal{M}\cap B_T(p_0)} \left\langle p-p_0, \vec{H}\right\rangle \mathrm{d}\mathcal{H}^2(p) + \frac{1}{2t^2}\int_{\mathcal{M}\cap B_t(p_0)} \left\langle p-p_0, \vec{H}\right\rangle \mathrm{d}\mathcal{H}^2(p)\\
&\qquad\quad + \frac{1}{2}\int_{\partial\mathcal{M}\cap B_T(p_0)} \left(\frac{1}{T^2} - \frac{1}{\rho_t^2}\right) \langle p-p_0, \vec{\nu}\rangle \, dl_{\partial\mathcal{M}}(p)
\end{aligned}$$

where $\rho_t := \max\left\{\|p-p_0\|, t\right\}$ and $(p-p_0)^{\perp}$ represents the orthogonal projection of the vector $p-p_0$ onto the normal plane $(T_p\mathcal{M})^{\perp}$ to the surface at point p. We are calling this identity the *monotonicity formula* in the presence of a boundary.

12 Theorem (Lemma 1.2 p.4. Rivière [Riv13])
Let $\Sigma \subset \mathbb{R}^n$ be bounded C^2-surface and $\Sigma \hookrightarrow f(\Sigma) =: \mathcal{M}$ be an immersed surface with boundary. Then the following estimate holds:

(47)
$$4\pi \leq \frac{1}{4}\int_{\mathcal{M}} \|\vec{H}\|^2 \,\mathrm{d}\mathcal{H}^2 + 2\frac{\mathcal{H}^1(\partial\mathcal{M})}{\sup_{x\in\mathcal{M}} \operatorname{dist}(x, \partial\mathcal{M})}$$

where $\mathcal{H}^1(\partial\mathcal{M})$ is the 1-dimensional Hausdorff measure of the boundary of the immersion $\mathcal{M}$.

The above equality is obtained by the flat 2-dimensional disc so that (47) is optimal.

Proof: Theorem 12 is obtained by passing Rivière's monotonicity formula with boundary (46) to the limit the inner radius $t \to 0$ and the outer radius $T \to \infty$. In that case all terms containing T^{-2} vanish and ρ_t becomes $\|p - p_0\|$. Moreover, by the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\left|\frac{1}{t^2}\int_{\mathcal{M}\cap B_t(p_0)}\left\langle p - p_0, \vec{H}\right\rangle \mathrm{d}\mathcal{H}^2(p)\right|^2 &\leq \frac{1}{t^2}\int_{\mathcal{M}\cap B_t(p_0)}\frac{\|p-p_0\|^2}{t^2}\,\mathrm{d}\mathcal{H}^2(p)\int_{\mathcal{M}\cap B_t(p_0)}\|\vec{H}\|^2\,\mathrm{d}\mathcal{H}^2\\ &\leq \frac{\mathcal{H}^2\left(\mathcal{M}\cap B_t\left(p_0\right)\right)}{t^2}\int_{\mathcal{M}\cap B_t(p_0)}\|\vec{H}\|^2\,\mathrm{d}\mathcal{H}^2\end{aligned}$$

so that for a point $p_0 \in \mathcal{M} \setminus \partial\mathcal{M}$ it holds

$$\pi\cdot k + \int_{\mathcal{M}}\left\|\frac{\vec{H}}{4} + \frac{(p-p_0)^{\perp}}{\|p-p_0\|^2}\right\|^2 \mathrm{d}\mathcal{H}^2(p) \leq \frac{1}{2}\int_{\partial\mathcal{M}}\left\langle\frac{p-p_0}{\|p-p_0\|^2}, \vec{\nu}\right\rangle \mathrm{d}\mathcal{H}^1(p) + \frac{1}{16}\int_{\mathcal{M}}\|\vec{H}\|^2\,\mathrm{d}\mathcal{H}^2 \tag{48}$$

since $\mathcal{H}^2\left(\mathcal{M}\cap B_t\left(p_0\right)\right)t^{-2} \to k\cdot\pi$ for $t \to 0$ with $k \in \mathbb{N}$ the $\mathcal{M}$ in point p_0. Subsequently, we take $p_0 \in \mathcal{M}$ as a point where $\sup_{x\in\mathcal{M}}\operatorname{dist}(x, \partial\mathcal{M})$ is attended: $\sup_{x\in\mathcal{M}}\operatorname{dist}(x,\partial\mathcal{M}) = \sup_{p\in\partial\mathcal{M}}\|p-p_0\|$. From (48) we deduce

$$4\pi\cdot k - \frac{1}{4}\int_{\mathcal{M}}\|\vec{H}\|^2\,\mathrm{d}\mu \leq 2\int_{\partial\mathcal{M}}\frac{1}{\|p-p_0\|}\,\mathrm{d}\mathcal{H}^1(p) \leq 2\frac{\mathcal{H}^1(\partial\mathcal{M})}{\sup_{x\in\mathcal{M}}\operatorname{dist}(x,\partial\mathcal{M})}.$$

□

Theorem 12 can be considered as a generalization of the classical Li–Yau inequality [LY82], which yields that for an immersed surface, the Willmore energy is greater than 4π times multiplicity. Especially, an immersion of a compact closed surface with the Willmore energy below 8π has to be an embedding. Thus, it has no self-intersection. In a similar way, by inspecting the proof of Theorem 12, one can state that if $\mathcal{M}$ is an immersion of compact closed surface satisfying

$$\frac{1}{4}\int_{\mathcal{M}}\|\vec{H}\|^2\,\mathrm{d}\mathcal{H}^2 + 2\frac{\mathcal{H}^1(\partial\mathcal{M})}{d(\partial\mathcal{M},\mathcal{M})} < 8\pi$$

than $\mathcal{M}$ is embedded. Next, we want to present a diameter bound by using the inequality (47) for the Willmore energy smaller than 4π, which was already used in the proof of Proposition 4.2. in [Poz21] by Pozzetta.

13 Proposition
Let $\Sigma \subset \mathbb{R}^n$ be a bounded C^2-surface and $\Sigma \hookrightarrow f(\Sigma) =: \mathcal{M}$ be an immersed surface with boundary and $\mathcal{W}(\mathcal{M}) < 4\pi$. Then we have a diameter bound

$$\sup_{x\in\mathcal{M}}\operatorname{dist}(x,\partial\mathcal{M}) \leq \frac{2\mathcal{H}^1(\partial\mathcal{M})}{4\pi - \mathcal{W}(\mathcal{M})}, \quad \textit{thus} \quad \operatorname{diam}\mathcal{M} \leq \frac{4\mathcal{H}^1(\partial\mathcal{M})}{4\pi-\mathcal{W}(\mathcal{M})} + \operatorname{diam}\partial\mathcal{M}.$$

Proof: If we choose $q, p \in \mathcal{M}$ such that $\|p-q\| = \operatorname{diam}\mathcal{M}$ and also $p^*, q^* \in \partial\mathcal{M}$ which satisfy $\|p-p^*\| = \operatorname{dist}(p,\partial\mathcal{M})$ and $\|q-q^*\| = \operatorname{dist}(q,\partial\mathcal{M})$. We conclude

$$\operatorname{diam}\mathcal{M} = \|p-q\| \leq \|p-p^*\| + \|q-q^*\| + \|p^*-q^*\| \leq \frac{4\mathcal{H}^1(\partial\mathcal{M})}{4\pi-\mathcal{W}(\mathcal{M})} + \operatorname{diam}\partial\mathcal{M}.$$

□

By using the monotonicity formula for surfaces without boundary in his classical work [Sim93, Lemma 1.1 p 283], Simon provided a diameter estimate of a smooth surface $\mathcal{M}$ involving its area $\mathcal{H}^2(\mathcal{M})$

$$\sqrt{\mathcal{H}^2(\mathcal{M})/\mathcal{W}(\mathcal{M})} \le \operatorname{diam}(\mathcal{M}) \le \frac{2}{\pi}\sqrt{\mathcal{H}^2(\mathcal{M})\mathcal{W}(\mathcal{M})}$$

with $\operatorname{diam}(\mathcal{M}) := \sup_{x,y\in\mathcal{M}} |x-y|$. For optimal constants, we refer to Topping's work [Top98]. Thus, if we have an area bound by the Willmore energy, we can also estimate the diameter from above and below. In [Sim93, Lemma 1.2 p 283] Simon also proved a celebrated diameter estimate involving the L^1-norm of the second fundamental form A in case $\mathcal{M}$ is connected and compact

$$\operatorname{diam}\mathcal{M} \le C_7 \left(\int_{\mathcal{M}} \|A\|_g \,\mathrm{d}\mathcal{H}^2 + \sum_i \operatorname{diam}\Gamma_i \right) \tag{49}$$

where Γ_i's are the connected components of $\partial\mathcal{M}$ and C_7 some constant only depending on space where $\mathcal{M}$ is embedded.

Then in [Top08] Topping proved that for every connected closed surface $\Sigma \hookrightarrow f(\Sigma) = \mathcal{M} \in \mathbb{R}^3$ it holds $\operatorname{diam}(\mathcal{M}) \le \frac{16}{\pi}\int_{\mathcal{M}} |H| \,\mathrm{d}\mathcal{H}^2$ and further generalized this result to m-dimensional compact manifolds without a boundary for general codimension: $\operatorname{diam}(\mathcal{M}) \le C_8(m)\int_{\mathcal{M}} \|\mathbf{H}\|^{m-1}\,\mathrm{d}\mathcal{H}^m$. For the definition of mean curvature vector for m-dimensional submanifolds of $\mathbb{R}^n$, we refer to [Sim83, eq. 7.4 p.45]. Here, we emphasize that, in contrast, our work primarily considers surfaces with boundary, thus $m = 2$. This result of Topping was then extended to surfaces with boundary by Menne and Scharrer [MS17] in the framework of varifolds: For dimensions $1 < m < n$ the intrinsic diameter $d_{\text{int}}(\mathcal{M})$ of a compact m-dimensional connected submanifold $\mathcal{M} \subset \mathbb{R}^n$ can be bounded

$$d_{\text{int}}(\mathcal{M}) \le C_9(m) \left(\int_{\mathcal{M}} |H_{\mathcal{M}}|^{m-1}\,\mathrm{d}\mathcal{H}^m + \int_{\partial\mathcal{M}} |H_{\partial\mathcal{M}}|^{m-2}\,\mathrm{d}\mathcal{H}^{m-1} \right) \tag{50}$$

where $C_9(m)$ is a constant that does not have a simple form. Since we want to consider only the case $m = 2$, we rather use the result shown by Miura [Miu22] in $\mathbb{R}^3$. His approach provided explicit constants and revealed a direct link to the Topping diameter estimate by constructing a thin closed surface out of a surface with a boundary.

14 Theorem
Let $\Sigma \hookrightarrow f(\Sigma) =: \mathcal{M} \subset \mathbb{R}^3$ denote a compact two-dimensional surface immersed into Euclidean 3-space. Then

$$\operatorname{diam}(\mathcal{M}) \le \frac{16}{\pi} \left(\int_{\mathcal{M}} |H|\,\mathrm{d}\mathcal{H}^2 + \frac{\pi}{2}\mathcal{H}^1(\partial\mathcal{M}) \right) \tag{51}$$

Proof: Here we use [Miu22, Theorem 1.1.] with $n = 3$ and estimate $C_T(3) \ge \pi/16$, where the constant is defined in [Miu22, (1.1)] and the estimate was shown in [Miu22, (A.1)]. □

For the Willmore energy smaller than 4π, we also have an area bound for immersed surfaces. It can be proved by making use of Michael-Simon inequality (for the inequality, we refer to [Whe15, Theorem 11.]) or alternatively like Pozzetta in [Poz21, proof of Proposition 4.2] by rescaling like in the following theorem.

15 Theorem
Let $\Sigma \hookrightarrow f(\Sigma) =: \mathcal{M} \subset \mathbb{R}^3$ denote a compact two-dimensional surface immersed into Euclidean 3-space $\mathcal{W}(\mathcal{M}) < 4\pi$. Then we have an area bound

$$\mathcal{H}^2(\mathcal{M}) \le R^2\mathcal{W}(\mathcal{M}) + R\mathcal{H}^1(\partial\mathcal{M})$$

with the "radius"

$$R := \frac{1}{2}\left(\frac{4\mathcal{H}^1(\partial\mathcal{M})}{4\pi - \mathcal{W}(\mathcal{M})} + \operatorname{diam}\partial\mathcal{M}\right).$$

Proof: By Theorem 12 we have $\operatorname{diam}\mathcal{M} \leq 2R$. Then, we follow the argumentation of Pozzetta in [Poz21, p.16]. So after a translation, we can assume that $\mathcal{M} \subset\subset B_R(0)$. Putting $\mathcal{M}' := \frac{1}{R}\mathcal{M} \subset\subset B_1(0)$ it follows by the divergence theorem with the vector field $\mathbb{R}^3 \ni x \mapsto x \in \mathbb{R}^3$

$$\begin{aligned}
2\mathcal{H}^2(\mathcal{M}') &= \int_{\mathcal{M}'} \operatorname{div}_{\mathcal{M}'} x \,\mathrm{d}\mathcal{H}^2 = -\int_{\mathcal{M}'} \langle \vec{H}', x\rangle \,\mathrm{d}\mathcal{H}^2 + \int_{\partial\mathcal{M}'} \langle x, \vec{\nu}'\rangle \,\mathrm{d}\mathcal{H}^1 \\
&= -\int_{\mathcal{M}'} \langle \vec{H}', x^\perp\rangle \,\mathrm{d}\mathcal{H}^2 - \frac{1}{4}\int_{\mathcal{M}'} \|\vec{H}'\|^2 \,\mathrm{d}\mathcal{H}^2 + \mathcal{W}(\mathcal{M}') - \int_{\mathcal{M}'} \mathrm{d}\mathcal{H}^2 + \mathcal{H}^2(\mathcal{M}') \\
&\quad - \int_{\mathcal{M}'} \|x^\perp\|^2 \,\mathrm{d}\mathcal{H}^2 + \int_{\mathcal{M}'} \|x^\perp\|^2 \,\mathrm{d}\mathcal{H}^2 + \int_{\partial\mathcal{M}'} \langle x, \vec{\nu}'\rangle \,\mathrm{d}\mathcal{H}^1 \\
&= -\int_{\mathcal{M}'} \left(1 - \|x^\perp\|^2\right) \mathrm{d}\mathcal{H}^2 - \int_{\mathcal{M}'} \left\|\frac{\vec{H}'}{2} + x^\perp\right\|^2 \mathrm{d}\mathcal{H}^2 + \mathcal{H}^2(\mathcal{M}') \\
&\quad + \mathcal{W}(\mathcal{M}') + \int_{\partial\mathcal{M}'} \langle x, \vec{\nu}'\rangle \,\mathrm{d}\mathcal{H}^1.
\end{aligned}$$

The first term in the second last line is negative since $\mathcal{M}' \subset\subset B_1(0)$, and the second term is also negative. Since $|\mathcal{M}| = R^2|\mathcal{M}'|$, $\mathcal{W}(\mathcal{M}) = \mathcal{W}(\mathcal{M}')$ and $\mathcal{H}^1(\partial\mathcal{M}) = R\mathcal{H}^1(\partial\mathcal{M}')$ we get

$$|\mathcal{M}| \leq R^2\mathcal{W}(\mathcal{M}) + R\mathcal{H}^1(\partial\mathcal{M})$$

which finishes the proof. □

Finally, we will briefly mention some results for closed surfaces not further used in this work. We begin with the result by Röger and Schätzle [RS12], who proved that for smoothly embedded surfaces $\mathcal{M}$ in $\mathbb{R}^3$ of sphere type with an enclosed inner region one can control the isoperimetric deficit, which is the difference of the isoperimetric ratio (7) from the optimal value given by the round sphere. This estimate was used in [GNR20] by Goldman, Novaga, and Röger where they considered a variational model for charged elastic drops in $\mathbb{R}^3$ with contribution by area, the Willmore energy, and the Riesz interaction energy. Depending on the weights of contributions, they proved that for a small charge, unique minimizers are either balls or centered annuli, and for a large charge, the minimizers do not exist. In [GNR20, Proposition 4.3 p.32] they also showed uniform area and diameter bounds for closed bounded surfaces with volume constraint and the Willmore energy strictly below 8π.

4.2 Graphs

Due to projectivity in the graphical case, there are better estimates available. Here we again consider a smooth domain $\Omega \subset \mathbb{R}^2$ with ν the unit vector field on $\partial\Omega$ orthogonal to it and oriented outwards, as well as $\varphi\colon \overline{\Omega} \to \mathbb{R}$ a C^2-smooth boundary datum and graphs $u\colon \overline{\Omega} \to \mathbb{R}$ in class

$$M := \left\{u \in W^{2,2}(\Omega) \,\middle|\, (u - \varphi) \in \mathring{W}^{2,2}(\Omega)\right\}$$

representing Dirichlet boundary conditions. In the graphical setting Deckelnick, Grunau, and Röger [DGR17, Theorem 2] provided area and diameter bounds by the Willmore energy, $\|\varphi\|_{W^{2,2}(\partial\Omega)}$ and the geometry of the domain. We want to emphasize that their result does not set any assumptions on values of the Willmore energy, like $\mathcal{W}(\mathcal{M}) < 4\pi$ in the immersed setting. As noticed in

[Gru18], such an estimate does not hold for general non-projectable surfaces due to the scaling invariance of the Willmore functional. For example, one sets $\Omega = B_1(0)$ with an arbitrarily large ball above it, then cuts off a disk hole around the south pole and then connects this to the given Dirichlet boundary conditions at the cost of adding only uniformly bounded Willmore energy.

To achieve their area and diameter bounds, Deckelnick, Grunau, and Röger used (49) the diameter estimate with $\|A\|_{L^1(\mathcal{M})}$ proved in [Sim93, Lemma 1.2 p 283] by Simon. A simpler graphical version of this Lemma with explicit constants can be found in [Gul14, Satz 4.2]. In this subsection, we want first to slightly improve this result by instead using Theorem 14. In this way we reduce the assumption $\|\varphi\|_{W^{2,1}(\partial\Omega)}$ to $\|\varphi\|_{W^{1,1}(\partial\Omega)}$ and provide explicit constants. In the second part, we prove a new smallness estimate on $\|u\|_{L^2(\Omega)}$ in contrast to the bound on $\|u\|_{L^\infty(\Omega)}$ in [DGR17, Theorem 2] without smallness statement.

16 Theorem
Suppose that $\varphi \in C_c^2(\mathbb{R}^2)$ and $u \in W^{2,2}(\Omega)$ that satisfies $u - \varphi \in \mathring{W}^{2.2}(\Omega)$.

(a) *Then it follows with $\|\varphi\|_{W^{1,1}(\partial\Omega)} = \|\varphi \circ \gamma\|_{W^{1,1}(I)}$ and $\gamma \colon I \to \mathbb{R}^2$ is a parametrization by arclength of the boundary $\partial\Omega$ that*

$$\sup_{x\in\Omega}|u| + \int_\Omega Q\,\mathrm{d}x \;\le 64\left(\mathcal{H}^2(\Omega) + \mathcal{H}^1(\partial\Omega) + \|\varphi\|_{W^{1,1}(\partial\Omega)} + \frac{16^2}{\pi^2}\mathcal{W}(u)\right)\big(1 + |\Omega|\mathcal{W}(u)\big).$$

(b) *Let $K > 0$, and $\|(\varphi\circ\gamma)'\|_{L^1(I)} < K$. Then for each $\varepsilon > 0$ there exists $\delta(\varepsilon, K, \Omega) > 0$ such that*

$$\mathcal{W}(u) + \|\varphi\|_{L^1(\partial\Omega)} \le \delta \quad \Rightarrow \quad \|u\|_{L^2(\Omega)} < \varepsilon.$$

Proof: We begin with the same crucial integral $\int_\Omega uH\,\mathrm{d}x$ as used in [DGR17, Theorem 2]. Here we apply the two-dimensional divergence Theorem in Ω and $Q^2 = 1 + |\nabla u|^2$ to obtain

$$\int_\Omega uH\,\mathrm{d}x \overset{(28)}{=} \int_\Omega u\,\mathrm{div}\left(\frac{\nabla u}{Q}\right)\mathrm{d}x \underset{\text{thm.}}{\overset{\text{div.}}{=}} -\int_\Omega \frac{|\nabla u|^2}{Q}\,\mathrm{d}x + \int_{\partial\Omega} \frac{u\frac{\partial u}{\partial\nu}}{Q}\,\mathrm{d}s. \tag{52}$$

This identity is used in both statements. Our main goal for (a) and (b) is to estimate the $\int_\Omega |\nabla u|^2/Q\,\mathrm{d}x$ term.

(a) Here we use the proof in [DGR17, Theorem 2] with Theorem 14 and Hölder's inequality to get

$$\begin{aligned}\frac{\pi}{16}\sup_{x\in\Omega}|u| \;&\le \int_{\mathcal{M}} |H|\,\mathrm{d}\mathcal{H}^2 + \frac{\pi}{2}\mathcal{H}^1(\partial\mathcal{M}) \le \sqrt{\int_\Omega |H|^2 Q\,\mathrm{d}x}\sqrt{\int_\Omega Q\,\mathrm{d}x} + \frac{\pi}{2}\int_{\partial\Omega}\sqrt{1+|\varphi'|^2}\,\mathrm{d}s\\ &\le \sqrt{4\mathcal{W}(u)}\sqrt{\int_\Omega Q\,\mathrm{d}x} + \frac{\pi}{2}\big(\mathcal{H}^1(\partial\Omega) + \|\varphi'\|_{L^1(\partial\Omega)}\big).\end{aligned} \tag{53}$$

We conclude with (52) and Hölder's inequality and the notations $|\Omega| = \mathcal{H}^2(\Omega)$ and $|\partial\Omega| = \mathcal{H}^1(\partial\Omega)$

$$\begin{aligned}\int_\Omega Q\,\mathrm{d}x \;&= \int_\Omega \frac{1}{Q}\,\mathrm{d}x + \int_\Omega \frac{|\nabla u|^2}{Q}\,\mathrm{d}x \le |\Omega| + \int_\Omega |uH|\,\mathrm{d}x + \int_{\partial\Omega} \varphi\frac{\frac{\partial u}{\partial \nu}}{\sqrt{1+|\nabla u|^2}}\,\mathrm{d}s\\ &\overset{1\le Q}{\le} |\Omega| + \frac{16}{\pi}\left(\sqrt{4\mathcal{W}(u)}\sqrt{\int_\Omega Q\,\mathrm{d}x} + \frac{\pi}{2}\big(|\partial\Omega| + \|\varphi'\|_{L^1(\partial\Omega)}\big)\right)\sqrt{4|\Omega|\mathcal{W}(u)} + \|\varphi\|_{L^1(\partial\Omega)}\\ &\le |\Omega| + \frac{16^2}{2\pi^2}|\Omega|(4\mathcal{W}(u))^2 + \frac{1}{2}\int_\Omega Q\,\mathrm{d}x + 8\big(|\partial\Omega| + \|\varphi'\|_{L^1(\partial\Omega)}\big)\sqrt{4|\Omega|\mathcal{W}(u)} + \|\varphi\|_{L^1(\partial\Omega)}\end{aligned}$$

$$\leq 2|\Omega| + 2\|\varphi\|_{L^1(\partial\Omega)} + \frac{16^3}{\pi^2}|\Omega|(\mathcal{W}(u))^2 + 32\big(|\partial\Omega| + \|\varphi'\|_{L^1(\partial\Omega)}\big)\sqrt{|\Omega|\mathcal{W}(u)}.$$

Then we use again (53) and Hölder's inequality to show

$$\begin{aligned}
\sup_{x\in\Omega}|u| + \int_\Omega Q\,\mathrm{d}x &\leq \sqrt{2\frac{16^2}{\pi^2}\mathcal{W}(u)}\sqrt{2\int_\Omega Q\,\mathrm{d}x + 8\big(\mathcal{H}^1(\partial\Omega) + \|\varphi'\|_{L^1(\partial\Omega)}\big)} + \int_\Omega Q\,\mathrm{d}x \\
&\leq \frac{16^2}{\pi^2}\mathcal{W}(u) + \int_\Omega Q\,\mathrm{d}x + \int_\Omega Q\,\mathrm{d}x + 8\big(|\partial\Omega| + \|\varphi'\|_{L^1(\partial\Omega)}\big) \\
&\leq 4\big(|\Omega| + \|\varphi\|_{L^1(\partial\Omega)}\big) + 2\frac{16^2}{\pi^2}\big(16|\Omega|\mathcal{W}(u) + 1\big)\mathcal{W}(u) \\
&\quad + 8\big(|\partial\Omega| + \|\varphi'\|_{L^1(\partial\Omega)}\big)\left(1 + 2\cdot 4\sqrt{|\Omega|\mathcal{W}(u)}\right) \\
&\leq 64\left(|\Omega| + |\partial\Omega| + \|\varphi\|_{W^{1,1}(\partial\Omega)} + \frac{16^2}{\pi^2}\mathcal{W}(u)\right)\big(1 + |\Omega|\mathcal{W}(u)\big).
\end{aligned}$$

b To achieve a L^2-smallness condition, we use: boundness of $\|u\|_{L^\infty(\Omega)}$ and $\|\nabla u\|_{L^1(\Omega)}$ in combination with Hölder's inequality as interpolation. Hence, we need smallness of $\|u\|_{W^{1,1}(\Omega)}$ which we deduce by estimating $\int_\Omega |\nabla u|^2/Q\,\mathrm{d}x$ instead of $\int_\Omega Q\,\mathrm{d}x$. In this way, we do not use $\mathcal{H}^2(\Omega)$ that cannot be chosen as small as wanted because it is fixed.

We assume that $\|\varphi\|_{L^1(\partial\Omega)} < K$. By *a* we know that there is a constant C_{10} that depends entirely on $|\partial\Omega|$, $K, \mathcal{W}(u)$ such that

$$\sup_{x\in\Omega}|u| + \int_\Omega Q\,\mathrm{d}x \leq C_{10}.$$

Then, we again use the identity (52)

$$\begin{aligned}
\int_\Omega \frac{|\nabla u|^2}{Q}\,\mathrm{d}x &= \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\frac{u}{Q}\,\mathrm{d}s - \int_\Omega uH\,\mathrm{d}x \leq \|\varphi\|_{L^1(\partial\Omega)} + \sup_{x\in\Omega}\big|u(x)\big|\int_\Omega |H|\,\mathrm{d}x \\
&\leq \|\varphi\|_{L^1(\partial\Omega)} + C_{10}\sqrt{|\Omega|\mathcal{W}(u)}
\end{aligned}$$

thus we can also estimate $\|\nabla u\|_{L^1(\Omega)}$ by

$$\int_\Omega |\nabla u|\,\mathrm{d}x \leq \left(\int_\Omega Q\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega \frac{|\nabla u|^2}{Q}\,\mathrm{d}x\right)^{\frac{1}{2}} \leq (C_{10})^{\frac{1}{2}}\left(\|\varphi\|_{L^1(\partial\Omega)} + C_{10}\sqrt{|\Omega|\mathcal{W}(u)}\right)^{\frac{1}{2}}.$$

By using the Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{2} = 1$, hence the Cauchy-Schwarz inequality, we finally get

$$\begin{aligned}
\left(\int_\Omega u^2\,\mathrm{d}x\right)^2 &= \left(\int_\Omega |u|^{\frac{3}{2}}|u|^{\frac{1}{2}}\,\mathrm{d}x\right)^2 \leq \left(\int_\Omega |u|^3\,\mathrm{d}x\right)\left(\int_\Omega |u|\,\mathrm{d}x\right) \\
&\leq |\Omega|\sup_{x\in\Omega}\big|u(x)\big|^3 C_{11}\left(\int_\Omega |\nabla u|\,\mathrm{d}x + \int_{\partial\Omega}|u|\,\mathrm{d}s\right) \\
&\leq C_{10}^3|\Omega|C_{11}\left((C_{10})^{\frac{1}{2}}\left(\|\varphi\|_{L^1(\partial\Omega)} + C_{10}\sqrt{|\Omega|\mathcal{W}(u)}\right)^{\frac{1}{2}} + \|\varphi\|_{L^1(\partial\Omega)}\right)
\end{aligned}$$

where we additionally used Poincare-Friedrich's inequality with constant $C_{11}(\Omega)$. The proof is finished by choosing the Willmore energy and $\|\varphi\|_{L^1(\partial\Omega)}$ small enough. ☐

Deckelnick, Grunau and Röger also showed, that [DGR17, Lemma 1] with $|D^2u|$ the Euclidean norm of Hessian of u one can estimate $|D^2u|Q^{-3} \leq \|A\|_g$. Additionally, by [DGR17, Lemma 2] we know that

$$\int_\Omega \frac{|D^2u|^2}{Q^5}\,\mathrm{d}x \leq \int_\Omega \|A\|_g^2 Q\,\mathrm{d}x \leq 4\mathcal{W}(u) + 2\big(\|\varphi\|_{W^{2,1}(\partial\Omega)} + \|\kappa\|_{L^1(\partial\Omega)}\big) + 2\pi\chi(\Omega)$$

where κ is the (signed) curvature of $\partial\Omega$, $\chi(\Omega)$ the Euler characteristic [DHS10] and all boundary spaces are defined with parametrizations by arclength. This estimate motivates the search for stronger than the $W^{1,1}(\Omega)$-norm norms, which one can estimate by the boundary data and the Willmore energy. Unfortunately, Deckelnick, Grunau, and Röger presented various counterexamples for such estimates. They showed in [DGR17, Example 2] that for $p > 1$, no $W^{1,p}(\Omega)$-norm may be estimated in terms of the Willmore energy. This example is rotationally symmetric with $|\nabla u| = \infty$ along a circle and thus, via following Theorem 17, has $\mathcal{W}(u) \geq 2\pi$ leaving the possibility for an estimate for functions with $\mathcal{W}(u) < 2\pi$. Furthermore, in [DGR17, Example 1], they constructed a function with unbounded gradients and arbitrarily small Willmore energy simultaneously. Additionally, in [DGR17, Example 3] there is a Willmore finite function $u \notin W^{2,2}(\Omega)$.

For later interpolation techniques, the L^2-smallness is sufficient for proving the global existence of the Willmore flow. A simple question arises in this context: Is there a similar smallness estimate for the L^∞-norm? To our knowledge, this question has not been unanswered yet. Despite that, one may ask for a situation when the Willmore energy gets large. There is a simple bound from below for the Navier boundary conditions that can be deduced. Nevertheless, it is not used in the following chapters. Here by $\operatorname{conv}(\Omega)$ we denote the convex hull of Ω.

17 Theorem

Let $\alpha > 0$ and $u \in C^2(\overline{\Omega})$ so that $u|_{\partial\Omega} = 0$ and $\frac{\partial u}{\partial \nu}|_{\partial\Omega\cap\operatorname{conv}(\overline{\Omega})} < -\alpha < 0$. Then it holds

$$2\pi\left(1 - \frac{1}{\sqrt{1+\alpha^2}}\right) \leq \mathcal{W}(u).$$

Especially, in assumptions we can replace $\partial\Omega \cap \operatorname{conv}(\overline{\Omega})$ by $\{x \in \partial\Omega \mid \kappa(x) > 0\}$.

Proof: This is a modification of the proof for Willmore's estimate $\mathcal{W}(\mathcal{M})) \geq 4\pi$ for a closed surface $\mathcal{M}$. We closely follow the presentation of Proposition 1.1 in [KS12, p.2]. The basic step will be to estimate the measure of the image of the *Gauss map* for points with positive Gaussian curvature. By the pointwise relation $\frac{1}{4}H^2 - \mathcal{K} = \frac{1}{2}\|A^0\|_g^2$, shown in [KS12, (1.1.7)] with $\|A^0\|_g$ the length of the *trace free second fundamental form*, we can show that

$$\mathcal{W}(u) \geq \int_{\Gamma(u),\mathcal{K}>0} \mathcal{K}\,\mathrm{d}\mathcal{H}^2.$$

Next, we use the pointing upward $N\colon \overline{\Omega} \to \mathbb{S}^2$ and obtain like in Proposition 1.1 in [KS12, p.2] that for any unit vector $N_0 \in \mathbb{S}^2$ we can choose a point $x_0 \in \overline{\Omega}$:

$$\left\langle \begin{pmatrix} x_0 \\ u(x_0) \end{pmatrix}, N_0 \right\rangle \overset{!}{=} \max_{x\in\overline{\Omega}} \left\langle \begin{pmatrix} x \\ u(x) \end{pmatrix}, N_0 \right\rangle. \tag{54}$$

Especially, in the case $x_0 \notin \partial\Omega$, we have $N(x_0) = N_0$ and $\mathcal{K}(x_0) \geq 0$. In the case $x_0 \in \partial\Omega$, we know that $x_0 \in \operatorname{conv}\overline{\Omega}$ since (54) is a linear optimization problem. Furthermore, $J_g N = |\mathcal{K}|$ the Jacobian of the Gauss map is the Gaussian curvature, which we will show in the following. Like in [KS12] we consider at some point in Ω the charts with $g_{ij} = \delta_{ij}$ (orthonormal frame). Then we obtain that $\{\partial_1 f, \partial_2 f, N\}$ forms an orthonormal basis of the ambient space $\mathbb{R}^3$. Consequently, since $\|N\|_{\mathbb{R}^3}^2 = 1$ we get $\nabla N \perp N$. Then by (24)

$$A_{ij} = -\langle \partial_i f, D_{\partial_j f} N\rangle = -\langle \partial_i f, \partial_j N\rangle.$$

Thus, since $\{\partial_1 f, \partial_2 f, N\}$ are orthogonal we conclude

$$\partial_i N = -A_{i1}\partial_1 f - A_{i2}\partial_2 f.$$

With symmetry $A_{ij} = A_{ji}$ we calculate the Jacobian

$$\begin{aligned}(J_g N)^2 &= \det\left(\langle \partial_i N, \partial_j N\rangle_{\mathbb{R}^3}\right) = \det\left(A_{i1}A_{j1} + A_{i2}A_{j2}\right) = \det\left((A_{\ell k}A_{km})_{ij}\right) \\ &= \det(A_{ij})^2 = \mathcal{K}^2,\end{aligned}$$

so that using the area formula it follows

$$\mathcal{W}(u) \geq \int_{\Gamma(u),\mathcal{K}>0} J_g N \,\mathrm{d}\mathcal{H}^2 \geq \mathcal{H}^2\big(N(\mathcal{K} \geq 0)\big). \tag{55}$$

Next, we want to show that the following spherical cap is a subset of $N(\mathcal{K} \geq 0)$:

$$C_\alpha := \left\{ N \in \mathbb{S}^2 \;\middle|\; \langle N, (0,0,1)^T\rangle > \frac{1}{\sqrt{1+\alpha^2}} \right\}.$$

We assume that there exists a $N_0 \in C_\alpha$ so that the corresponding maximizing x_0 from (54) lies in $\partial\Omega$ then it is in $\operatorname{conv}\overline{\Omega}$ and for such a point we have $\frac{\partial u}{\partial \nu} < -\alpha$. We denote $\nu_0 := \nu(x_0)$ and $\frac{\partial u}{\partial \nu_0} := \frac{\partial u}{\partial \nu}(x_0)$.

From maximizing property (54) we obtain that $T_{(x_0,0)}\big(\partial\Omega \times \{0\}\big) \perp N_0$ where especially $\partial\Omega \times \{0\} = \partial\Gamma(u)$. Also by definition we have $T_{(x_0,0)}\big(\partial\Omega \times \{0\}\big) \perp (\nu_0^T, 0)$ and $T_{(x_0,0)}\big(\partial\Omega \times \{0\}\big) \perp (0,0,1)^T$. Thus $N_0, (\nu_0^T, 0)$ and $(0,0,1)^T$ lie in the same plane $\big(T_{(x_0,0)}(\partial\Omega \times \{0\})\big)^\perp$. Then we use the two-dimensional Pythagoras theorem to obtain

$$1 = \Big|\big\langle N_0, N_0\big\rangle\Big|^2 = \Big|\big\langle (\nu_0^T, 0)^T, N_0\big\rangle\Big|^2 + \Big|\big\langle (0,0,1)^T, N_0\big\rangle\Big|^2. \tag{56}$$

Next we want to show that the tangent vector in $(x_0, 0)$ showing inward with respect to $\Gamma(u)$ is $\big(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\big)^T$. Firstly, like in [DGR17, Remark 1] we consider a positively oriented parametrization $s \mapsto c(s) \in \partial\Omega$ of (a connected component of) $\partial\Omega$ with respect to its arclength so that $c(0) = x_0$. Then the natural unit tangent vector is $\tau(s) = c'(s)$. In particular we have that $\nu^1 = \tau^2, \nu^2 = -\tau^1$

$$u_\nu(s) = \frac{\partial u}{\partial \nu}\big(c(s)\big), \quad \gamma(s) = \big(c(s)^T, 0\big)^T, \quad N\big(\gamma(s)\big) = \frac{1}{\sqrt{1+u_\nu^2}} \begin{pmatrix} -\nu^1 u_\nu \\ -\nu^2 u_\nu \\ 1 \end{pmatrix} \big(\gamma(s)\big).$$

Then it follows $\big(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\big) \perp N(\gamma(0))$. Furthermore $\big(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\big) \perp \dot\gamma(0)$, thus $\big(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\big) \perp T_{(x_0,0)}\big(\big(\partial\Omega \times \{0\}\big)$.

In the next step, since $(\nu_0^T, 0) \perp (0,0,1)^T$ via orthogonal projections we have:

$$\begin{aligned}\left\langle \left(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\right)^T, N_0 \right\rangle &= -1\big\langle (\nu_0^T, 0)^T, N_0\big\rangle - \frac{\partial u}{\partial \nu_0}\big\langle (0,0,1)^T, N_0\big\rangle \\ &\geq -1\Big|\big\langle (\nu_0^T, 0)^T, N_0\big\rangle\Big| + \alpha\big\langle (0,0,1)^T, N_0\big\rangle.\end{aligned}$$

Then we use the Pythagoras equation (56) and the definition of C_α to show

$$\left\langle \left(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\right)^T, N_0 \right\rangle \overset{(56)}{\geq} -1\sqrt{1 - \big\langle (0,0,1)^T, N_0\big\rangle^2} + \alpha\big\langle (0,0,1)^T, N_0\big\rangle$$

$$
\begin{aligned}
&\overset{N_0\in C_\alpha}{>} -1\sqrt{1-\frac{1}{\sqrt{1+\alpha^2}^2}}+\alpha\frac{1}{\sqrt{1+\alpha^2}} \\
&= \frac{-\sqrt{1+\alpha^2-1}+\alpha}{\sqrt{1+\alpha^2}} = 0.
\end{aligned}
$$

Since the to $\Gamma(u)$ tangent vector $\left(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\right)^T$ is inward showing there exists a curve $\gamma\colon [0,\epsilon] \to \Gamma(u)$ with $\gamma(0) = x_0$ and $\gamma'(0) = \left(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\right)^T$. It follows that this curve can exceed the maximum of (54) for $t > 0$ small enough

$$
\begin{aligned}
\langle \gamma(t), N_0\rangle &= \langle \gamma(0), N_0\rangle + t\langle \gamma'(0), N_0\rangle + \mathcal{O}(t^2) \\
&= \langle x_0, N_0\rangle + t\left(\left\langle \left(-\nu_0^T, -\frac{\partial u}{\partial \nu_0}\right)^T, N_0\right\rangle + \mathcal{O}(|t|)\right) \\
&> \langle x_0, N_0\rangle, \text{ for } t \text{ small enough.}
\end{aligned}
$$

Hence, the condition of x_0 in (54) is violated. This means, that for all $N_0 \in C_\alpha$ there exists a $x_0 \notin \partial\Omega$ so that $N_0 = N(x_0)$ and especially that $C_\alpha \subset N(\mathcal{K} \geq 0))$. Finally, we achieve with (55)

$$
\mathcal{W}\big(\Gamma(u)\big) \geq \mathcal{H}^2(C_\alpha) = 2\pi\left(1-\frac{1}{\sqrt{1+\alpha^2}}\right).
$$

□

5 Elliptic Theory

In his survey article [Nit93] Nitsche investigated $C^{4+\alpha}$-regularity and the existence of solutions for the Willmore equation with the appropriate boundary conditions. He decomposed the problem into a coupled system of second-order elliptic equations complemented by coupled Dirichlet boundary data. One consists of the Dirichlet problem for u with the prescribed mean curvature, and the other considers the Willmore equation as a second-order equation for the mean curvature H itself. In this setting, he used perturbation arguments to show the existence of a unique solution in the regularity class $C^{4+\alpha}(\overline{\Omega})$ at the cost of the smallness of $C^{4+\alpha}(\partial\Omega)$-norm for the boundary data.

From here on, one way to go is to consider weaker boundary spaces, where, due to non-linearity of the Willmore equation, one should expect that some kind of smallness of boundary data norm is necessary for any existence theorem. As a result, if one wants to move away from classical solutions, one has to work with weaker classes like Hölder $C^{m+\alpha}(\overline{\Omega})$ or Sobolev spaces $W^{m,p}(\Omega)$ with $m \leq 4$, even though for the last one the trace theory is more involved.

5.1 Willmore Equation

In this chapter, we want to investigate the Willmore equation in the framework of variational solutions. To work on it effectively, we have to investigate the structure of the Willmore equation in the graphical situation. Already Nitsche recalled in [Nit93] a well-known fact that the principal part of the Willmore equation is effectively the biharmonic operator after some linearization. More precisely, from the work of Dziuk and Deckelnick [DD06, (1.5)-(1.9)] for the Willmore flow of graphs one observes that the Willmore equation is in fact a fourth-order equation, thus involving fourth-order derivatives, if written in the graphical case as

$$0 = \Delta_g H + \frac{1}{2}H^3 - 2H\mathcal{K} = \operatorname{div}\left(\frac{1}{Q}\left(\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right)\nabla(QH)\right) - \frac{H^2}{2Q}\nabla u\right), \tag{57}$$

since the mean curvature consists of second-order derivatives

$$H = \operatorname{div}\left(\frac{\nabla u}{Q}\right) = \frac{\Delta u}{Q} - \frac{\nabla u \cdot (D^2 u \nabla u)}{Q^3}.$$

The main idea here is to separate the biharmonic part $\Delta^2 u$ in (57) from other terms in a new way which is similar to the rewriting done by Koch and Lamm in [KL12, Lemma 3.2 p. 215] in the context of the Willmore flow. As a consequence, we can use various existence and regularity results for inhomogeneous biharmonic problems in weaker spaces.

One of the key points of proving the existence results is that the concrete form of the "nonlinear" terms is not what we want to focus on. To absorb all arising algebraic constants, we have to introduce some notation already used in [KL12, p. 215]. Thus, let the star $\star$ denote an arbitrary linear combination of indices contractions for derivatives of u. For example, consider $|\nabla u|^2 = \nabla u \star \nabla u$ and $\nabla_i u D^2_{ij} u \nabla_j u = \nabla u \star D^2 u \star \nabla u$, that yields

$$H = Q^{-1} \star D^2 u + Q^{-3} \star \nabla u \star D^2 u \star \nabla u. \tag{58}$$

Furthermore, we introduce an abstract notation for gradient polynomials

$$P_\ell(\nabla u) = \underbrace{\nabla u \star \cdots \star \nabla u}_{\ell\text{-times}}$$

which helps us to formulate the non-biharmonic divergence terms in the following Lemma.

18 Lemma
The Willmore equation (57) *can be rewritten as*

$$0 = \operatorname{div}\left(\frac{1}{Q}\left(\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right)\nabla(QH)\right) - \frac{H^2}{2Q}\nabla u\right) = \Delta^2 u - D_i b_1^i[u] - D_{ij}^2 b_2^{ij}[u] \tag{59}$$

where

$$\begin{aligned} b_1[u] &= D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1} P_{2k-1}(\nabla u), \\ b_2[u] &= D^2u \star \sum_{k=1}^{2} Q^{-2k-1} P_{2k}(\nabla u) + D^2u \star P_2(\nabla u) \star (Q(1+Q))^{-1}. \end{aligned} \tag{60}$$

Proof: The reformulation of the Willmore equation will proceed term by term. We begin with calculating the derivation of the surface term

$$\nabla(Q) = \frac{D^2u\,\nabla u}{Q}$$

which we need for the first term that will include the biharmonic part

$$\begin{aligned} \operatorname{div}\left(\frac{1}{Q}\nabla(QH)\right) &= \operatorname{div}\left(\nabla(H) + \frac{D^2u\nabla u}{Q^3}QH\right) \\ &= \Delta\left(\frac{\Delta u}{Q}\right) - \Delta\left(\frac{\nabla u(D^2u\nabla u)}{Q^3}\right) + \operatorname{div}\left(\frac{D^2u\nabla u}{Q^3}QH\right). \end{aligned}$$

Especially since $(1-Q)(1+Q) = 1 - Q^2 = 1 - 1 - |\nabla u|^2 = -|\nabla u|^2$, it follows

$$\Delta\left(\frac{\Delta u}{Q}\right) = \Delta\left(\Delta u + \left(\frac{1}{Q} - 1\right)\Delta u\right) = \Delta^2 u + \Delta\left(\frac{1-Q}{Q}\Delta u\right) = \Delta^2 u - \Delta\left(\frac{|\nabla u|^2}{Q(1+Q)}\Delta u\right).$$

Thus we obtain

$$\operatorname{div}\left(\frac{1}{Q}\nabla(QH)\right) = \Delta^2 u - \Delta\left(\frac{|\nabla u|^2}{Q(1+Q)}\Delta u\right) - \Delta\left(\frac{\nabla u(D^2u\nabla u)}{Q^3}\right) + \operatorname{div}\left(\frac{D^2u\nabla u}{Q^3}QH\right).$$

The second term in (57) we want to rewrite by using the Einstein notation is the following

$$\begin{aligned} \nabla_i\left(\frac{\nabla_i u\nabla_j u}{Q^3}\,\nabla_j(QH)\right) &= \nabla_i\left(\nabla_j\left(\frac{\nabla_i u\nabla_j u}{Q^3}QH\right) - \nabla_j\left(\frac{\nabla_i u\nabla_j u}{Q^3}\right)QH\right) \\ &= \nabla_i\left(\nabla_j\left(\frac{\nabla_i u\nabla_j u}{Q^3}QH\right) - (-3)\nabla_j(Q)\frac{\nabla_i u\nabla_j u}{Q^4}QH - \frac{\nabla_j(\nabla_i u\nabla_j u)}{Q^3}QH\right) \\ &= \nabla_i\left(\nabla_j\left(\frac{\nabla_i u\nabla_j u}{Q^3}QH\right) + 3\frac{\nabla_i u\nabla_j u(D^2u\nabla u)_j}{Q^5}QH - \frac{D_{ij}^2 u\nabla_j u + D_{jj}^2 u\nabla_i u}{Q^3}QH\right) \\ &= D_{ij}^2\left(\frac{\nabla_i u\nabla_j u}{Q^2}H\right) + \operatorname{div}\left(3\frac{\nabla u\cdot(D^2u\nabla u)}{Q^4}H\nabla u - \frac{H}{Q^2}D^2u\nabla u - \frac{\Delta u H}{Q^2}\nabla u\right) \\ &= D_{ij}^2\left(\frac{\nabla_i u\nabla_j u}{Q^2}H\right) + \operatorname{div}\left(-3\frac{H^2}{Q}\nabla u - \frac{H}{Q^2}D^2u\nabla u + 2\frac{\Delta u H}{Q^2}\nabla u\right). \end{aligned}$$

We combine the results and get

$$\Delta_g H + \frac{1}{2}H^3 - 2H\mathcal{K} = \Delta^2 u - \Delta\left(\frac{|\nabla u|^2}{Q(1+Q)}\Delta u + \frac{\nabla u(D^2 u \nabla u)}{Q^3}\right) - D^2_{ij}\left(\frac{\nabla_i u \nabla_j u}{Q^2}H\right)$$
$$+ \operatorname{div}\left(\frac{5}{2}\frac{H^2}{Q}\nabla u + 2\frac{H}{Q^2}D^2 u \nabla u - 2\frac{\Delta u H}{Q^2}\nabla u\right).$$

Since by (58) we have $H = D^2 u \star \sum_{k=1}^{2} Q^{-2k+1} P_{2k-2}(\nabla u)$ we can combine the terms and reformulate the Willmore equation

$$\Delta_g H + \frac{1}{2}H^3 - 2H\mathcal{K} = \Delta^2 u + \Delta\big(D^2 u \star P_2(\nabla u) \star (Q(1+Q))^{-1}\big) + \Delta\big(D^2 u \star Q^{-3} P_2(\nabla u)\big)$$
$$+ D^2\left(D^2 u \star \sum_{k=1}^{2} Q^{-2k-1} P_{2k}(\nabla u)\right)$$
$$+ \operatorname{div}\left(D^2 u \star D^2 u \star \nabla u \star Q^{-1}\left(Q^{-2} + Q^{-4}P_2(\nabla u) + Q^{-6}P_4(\nabla u)\right)\right)$$
$$+ \operatorname{div}\left(D^2 u \star D^2 u \star \sum_{k=1}^{2} Q^{-2k-1} P_{2k-1}(\nabla u)\right).$$

We combine the terms above and arrive at

$$0 = \Delta_g H + \frac{1}{2}H^3 - 2H\mathcal{K} = \Delta^2 u - D_i b_1^i[u] - D^2_{ij} b_2^{ij}[u]$$

with divergence terms on the right-hand side defined above. □

To explain the importance of the preceding lemma, we want to make two remarks about the structure of b_ℓ for $\ell = 1, 2$. The first one is that the lower-order terms are written in divergence form. Especially in difference to the parabolic case in [KL12], the term b_0, defined as a non-divergent term by $0 = \Delta^2 u + b_0[u] + D_i b_1^i[u] + D^2_{ij} b_2^{ij}[u]$, is lacking. Furthermore, since b_1 and b_2 involve at least one ∇u and at most two $D^2 u$ and are at least quadratic in the first and second derivative terms without higher derivatives of u, it will be possible to reduce the regularity assumptions from $C^{4+\alpha}$ to the spaces involving only up to second derivatives.

Now, since we have reformulated the Willmore equation as an inhomogeneous biharmonic equation and the Δ^2 operator is a positive bilinear form and a higher-order elliptic operator, we can use the variational solution's definition from Chapter 3.1. Thus, we say the function u is a variational solution to (59) if the following equation is valid,

$$\forall v \in C_0^\infty(\Omega): \quad 0 = \int_\Omega \Delta u \Delta v \,\mathrm{d}x + \int_\Omega b_1^i[u] D_i v \,\mathrm{d}x - \int_\Omega b_2^{ij}[u] D^2_{ij} v \,\mathrm{d}x, \tag{61}$$

where u should lie at least in $W^{2,2}_{\mathrm{loc}}(\Omega)$ for the above equation to make sense. The term $b_1[u]$ acts like $D^2 u \star D^2 u \star$(something bounded) and $b_2[u]$ operates like $D^2 u \star$(something bounded).

We also note that until now we have not introduced the boundary conditions which are needed for the discussion of existence and uniqueness. The main reason is that the complexity of notation for the traces of the function spaces depends strongly on the spaces themselves and, therefore, will be made clear in each situation.

5.2 Hölder Case

This section is devoted to the existence of the $C^{2+\alpha}(\overline{\Omega})$ Willmore surfaces in the weak sense of (61). Here, we study which conditions on the boundary data are sufficient for a graphical solution to exist. Foremost, we need some appropriate smoothness for the boundary $\partial\Omega$ itself. Since we want to use $C^{2+\alpha}(\partial\Omega)$ and the Schauder estimates from [GGS10, p.45 Theorem 2.19], we assume that $\partial\Omega \in C^{4+\alpha}$.

Then the Willmore equation with prescribed Dirichlet boundary values can be considered a special case of the general biharmonic boundary problem

$$
\begin{cases} \Delta^2 u = b_0 + D_i b_1^i + D_{ij}^2 b_2^{ij} & \text{in } \Omega \subset \mathbb{R}^2, \\ \quad u = g_0, \quad \partial_\nu u = g_1 & \text{on } \partial\Omega, \end{cases} \tag{62}
$$

which is a divergence type fourth-order elliptic boundary value problem with $g_i \in C^{2-i+\alpha}(\partial\Omega)$ for $i = 0, 1$. In the $C^{2+\alpha}$-framework, it is also clear that for $u \in C^{2+\alpha}(\overline{\Omega})$ its trace $u|_{\partial\Omega}$ lies in $C^{2+\alpha}(\partial\Omega)$.

In order to collect some results needed for the proof of the main Theorem 21 we have to modify the Agmon-Miranda maximum modulus estimate and $C^{2+\alpha}$ Schauder estimates from [GGS10] so that the right-hand side terms b_0, b_1, b_2 are admitted to be in weaker spaces.

19 Proposition
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{4+\alpha}$ boundary. Furthermore, assume that $\alpha \in (0,1)$, $s, t \in (1,\infty), p \in (2,\infty)$, $g_0 \in C^{2+\alpha}(\partial\Omega)$, $g_1 \in C^{1+\alpha}(\partial\Omega), b_0 \in L^{\frac{2}{1-\alpha}}(\Omega) \cap L^s(\Omega), b_1 \in L^{\frac{2}{1-\alpha}}(\Omega) \cap L^t(\Omega)$ and $b_2 \in C^\alpha(\overline{\Omega})$. Then (62) admits a unique solution $u \in C^{2+\alpha}(\overline{\Omega})$. Moreover, there exist constants $C_{12} = C_{12}(\alpha, s, \Omega), C_{13} = C_{13}(\alpha, s, t, \Omega)$ and $C_{14} = C_{14}(\alpha, s, t, p, \Omega)$ such that

$$
\|u\|_{C^{2+\alpha}(\overline{\Omega})} \le C_{12}\left(\|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)} + \|b_0\|_{L^{\frac{2}{2-\alpha}}(\Omega)} + \|b_1\|_{L^{\frac{2}{1-\alpha}}(\Omega)} + \|b_2\|_{C^\alpha(\overline{\Omega})} \right), \tag{63}
$$

$$
\|u\|_{C^{1+\alpha}(\overline{\Omega})} \le C_{13}\Big(\|g_0\|_{C^{1+\alpha}(\partial\Omega)} + \|g_1\|_{C^{\alpha}(\partial\Omega)} + \|b_0\|_{L^s(\Omega)} + \|b_1\|_{L^{\frac{2}{2-\alpha}}(\Omega)} + \|b_2\|_{L^{\frac{2}{1-\alpha}}(\Omega)} \Big), \tag{64}
$$

$$
\|u\|_{C^1(\overline{\Omega})} \le C_{14}\Big(\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)} + \|b_0\|_{L^s(\Omega)} + \|b_1\|_{L^t(\Omega)} + \|b_2\|_{L^p(\Omega)} \Big). \tag{65}
$$

Proof: We split problem (62) into four biharmonic subproblems, one containing the Dirichlet boundary data and the others containing each $b_i, i = 0, 1, 2$ on the right-hand side separately.

$$
\begin{cases} \Delta^2 \psi = 0, & \text{in } \Omega, \\ \quad \psi = g_0, \quad \partial_\nu \psi = g_1, & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta^2 \varphi_0 = b_0, & \text{in } \Omega, \\ \quad \varphi_0 = \partial_\nu \varphi_0 = 0, & \text{on } \partial\Omega, \end{cases}
$$

$$
\begin{cases} \Delta^2 \varphi_1 = D_i b_1^i, & \text{in } \Omega, \\ \quad \varphi_1 = \partial_\nu \varphi_1 = 0, & \text{on } \partial\Omega. \end{cases} \qquad \begin{cases} \Delta^2 \varphi_2 = D_{ij}^2 b_2^{ij}, & \text{in } \Omega, \\ \quad \varphi_2 = \partial_\nu \varphi_2 = 0, & \text{on } \partial\Omega. \end{cases}
$$

First, consider ψ as the solution of the biharmonic problem with the same boundary conditions as in (62). We emphasize that by [GGS10] [Theorem 2.19 p.45] a solution to this Dirichlet problem exists. The uniqueness can be shown by multiplying the left side of the equation with ψ and integrating over Ω so that we get $\|D^2\psi\|_{L^2(\Omega)} = 0$. Thus, the associated homogeneous problem only admits the trivial solution. This allows us to use the following Schauder estimates for the classical solutions [GGS10] [Theorem 2.19 p.45]

$$
\begin{aligned}
\|\psi\|_{C^{2+\alpha}(\overline{\Omega})} &\le C_{15}\big(\|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)}\big), \\
\|\psi\|_{C^{1+\alpha}(\overline{\Omega})} &\le C_{16}\big(\|g_0\|_{C^{1+\alpha}(\partial\Omega)} + \|g_1\|_{C^{\alpha}(\partial\Omega)}\big),
\end{aligned}
$$

where C_{15}, C_{16} and all following constants throughout the proof depend only on Ω and α.

Additionally, the uniqueness of the Dirichlet solution justifies the use of the Agmon–Miranda maximum principle without the L^1-part on the right-hand side [Agm60, Theorem 1, p.78]

$$
\|\psi\|_{C^1(\overline{\Omega})} \le C_{17}\big(\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)}\big).
$$

Further on, we consider φ_0 as a solution of the homogeneous subproblem containing only b_0 on the right-hand side. In fact, $b_0 \in L^{\frac{2}{2-\alpha}}(\Omega) \cap L^s(\Omega)$ ensures the existence. Thus we obtain the following a priori L^p-elliptic estimate [GGS10, Theorem 2.20, p.46]

$$\|\varphi_0\|_{W^{4,\frac{2}{2-\alpha}}(\Omega)} \leq C_{18}\|b_0\|_{L^{\frac{2}{2-\alpha}}(\Omega)}, \quad \|\varphi_0\|_{W^{4,s}(\Omega)} \leq C_{19}\|b_0\|_{L^s(\Omega)}.$$

Furthermore, the Sobolev embedding [GGS10, Theorem 2.6] with C^1-boundary yields

$$\|\varphi_0\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{20}\|\varphi_0\|_{W^{4,\frac{2}{2-\alpha}}(\Omega)} \leq C_{21}\|b_0\|_{L^{\frac{2}{2-\alpha}}(\Omega)}$$

as well as the estimate

$$\|\varphi_0\|_{C^1(\overline{\Omega})} \leq \|\varphi_0\|_{C^{1+\alpha}(\overline{\Omega})} \leq C_{22}\|\varphi_0\|_{W^{4,s}(\Omega)} \leq C_{23}\|b_0\|_{L^s(\Omega)}. \tag{66}$$

For divergence-type parabolic equations, there are L^p-estimates even under weaker regularity assumptions of the right-hand side. This fact will prove to be extremely useful in the subproblems involving φ_1 and φ_2. In the case of φ_1 we use $b_1 \in L^{\frac{2}{1-\alpha}}(\Omega)$. Therefore, in the same way as in Lemma 4.26 in [GGS10, p.132] the existence is ensured and

$$\|\varphi_1\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{24}\|\varphi_1\|_{W^{3,\frac{2}{1-\alpha}}(\Omega)} \leq C_{25}\|b_1\|_{L^{\frac{2}{1-\alpha}}(\Omega)} \tag{67}$$

since one further Sobolev embedding is available. Again, we obtain

$$\begin{aligned} \|\varphi_1\|_{C^{1+\alpha}(\overline{\Omega})} &\leq C_{26}\|\varphi_1\|_{W^{3,\frac{2}{2-\alpha}}(\Omega)} \leq C_{27}\|b_1\|_{L^{\frac{2}{2-\alpha}}(\Omega)}, \\ \|\varphi_1\|_{C^1(\overline{\Omega})} &\leq C_{28}\|\varphi_1\|_{C^{1+2-\frac{2}{t}}(\overline{\Omega})} \leq C_{29}\|\varphi_1\|_{W^{3,t}(\Omega)} \leq C_{30}\|b_1\|_{L^t(\Omega)}. \end{aligned}$$

It remains to discuss the problem involving φ_2. To prove the C^1-bound, we stay in the L^p-framework. According to Theorem 2.22 in [GGS10, p.47] the existence of the unique solution $\varphi_2 \in \mathring{W}^{2,p}(\Omega)$ is shown. Combining it with some embedding arguments and the especially important assumption $p > 2$, we conclude

$$\|\varphi_2\|_{C^1(\overline{\Omega})} \leq C_{31}\|\varphi_2\|_{W^{2,p}(\Omega)} \leq C_{32}\|b_2\|_{L^p(\Omega)}.$$

In the same way as (67), we obtain

$$\|\varphi_2\|_{C^{1+\alpha}(\overline{\Omega})} \leq C_{33}\|\varphi_2\|_{W^{2,\frac{2}{1-\alpha}}(\Omega)} \leq C_{34}\|b_2\|_{L^{\frac{2}{1-\alpha}}(\Omega)}.$$

To estimate the $C^{2+\alpha}$-norm of φ_2 we again use the Theorem 2.19 in [GGS10, p.45] for the classical solution. The corresponding equation admits a unique $C^{2+\alpha}$-solution φ_2 and the following estimate holds

$$\|\varphi_2\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{35}\|b_2\|_{C^\alpha(\overline{\Omega})}.$$

Since all the solutions ψ, φ_0, φ_1 and φ_2 are obtained and estimated, the solution u of (62) is determined by addition, $u = \psi + \varphi_0 + \varphi_1 + \varphi_2$. This completes the proof by establishing the $C^{2+\alpha}$- and C^1-estimates. $\square$

If we impose additional regularity assumptions on boundary and b_0, namely $\partial\Omega \in C^{16}$, we can weaken the L^s-norm on b_0 in the $\|u\|_{C^1(\overline{\Omega})}$ estimate (65) to $s = 1$. Actually, by [DAS04, Proposition 17 Remark 19, 21 p. 482] the weighted gradient estimate the inequality (66) can be improved

$$\|\varphi_0\|_{C^0(\overline{\Omega})} + \|\nabla\varphi_0\|_{C^0(\overline{\Omega})} \leq C_{36}\|b_0\|_{L^1(\Omega)}.$$

Nevertheless, since we do not need b_0 to estimate with L^1-norm, we drop this assumption, thus not imposing $\partial\Omega \in C^{16}$.

From here on we shift away from the general problem towards the Willmore equation, thus we prove some simplifying estimates for the $b_i[u], i = 1, 2$ terms in the case of the bounded gradient. Significantly, the difference estimates for the proof of contraction for the fixed point theorem are needed.

20 Lemma

Let $s \in (1, \infty), \alpha \in (0, 1), i, j, \ell, k \in \mathbb{N}_0, i \geq 1, k \geq 2$ then there exist constants $C_{37} = C_{37}(\Omega, s, i, j)$ and $C_{38} = C_{38}(\Omega, s, \alpha, j, \ell, k)$ such that if $u \in C^{2+\alpha}(\overline{\Omega})$ with $\|\nabla u\|_{C^0(\overline{\Omega})} \leq 1$ then we can estimate

$$\begin{aligned} \left\|D^2u \star D^2u \star Q^{-j} P_i(\nabla u)\right\|_{L^s(\Omega)} &\leq C_{37}\|D^2u\|^2_{L^{2s}(\Omega)}\|\nabla u\|_{C^0(\overline{\Omega})}, \\ \left\|D^2u \star Q^{-j}(1+Q)^{-\ell}P_k(\nabla u)\right\|_{C^\alpha(\overline{\Omega})} &\leq C_{38}\|D^2u\|_{C^\alpha(\overline{\Omega})}\|\nabla u\|_{C^\alpha(\overline{\Omega})}. \end{aligned} \tag{68}$$

Furthermore, suppose $w \in C^{2+\alpha}(\overline{\Omega})$ with $\|\nabla w\|_{C^0(\overline{\Omega})} \leq 1$ then it follows

$$\begin{aligned} &\left\|D^2u \star D^2u \star Q^{-j}(u)P_i(\nabla u) - D^2w \star D^2w \star Q^{-j}(w)P_i(\nabla w)\right\|_{L^s(\Omega)} \\ &\quad \leq C_{37}\|u-w\|_{W^{2,2s}(\Omega)}\Big(\|D^2u\|_{L^{2s}(\Omega)} + \|D^2w\|_{L^{2s}(\Omega)}\Big)\Big(\|u\|_{W^{2,2s}(\Omega)} + \|w\|_{W^{2,2s}(\Omega)}\Big) \end{aligned} \tag{69}$$

as well as

$$\begin{aligned} &\left\|D^2u \star Q^{-j}(1+Q)^{-\ell}(u)P_k(\nabla u) - D^2w \star Q^{-j}(1+Q)^{-\ell}(w)P_k(\nabla w)\right\|_{C^\alpha(\overline{\Omega})} \\ &\quad \leq C_{38}\|u-w\|_{C^{2+\alpha}(\overline{\Omega})}\big(\|D^2u\|_{C^\alpha(\overline{\Omega})} + \|D^2w\|_{C^\alpha(\overline{\Omega})}\Big)\Big(\|\nabla u\|_{C^\alpha(\overline{\Omega})} + \|\nabla w\|_{C^\alpha(\overline{\Omega})}\Big). \end{aligned} \tag{70}$$

Proof: If necessary, we write $Q = Q(u)$ and $P_i = P_i(\nabla u)$ for clarity.

1 We begin with the estimate (68). The inequality $Q = \sqrt{1+|\nabla u|^2} \geq 1$ and $\|\nabla u\|_{C^0(\overline{\Omega})} \leq 1$ yields

$$\begin{aligned} \left|D^2u \star D^2u \star Q^{-j}P_i\right| &\leq C_{39}|D^2u|^2 \cdot \left||\nabla u|^i \sqrt{1+|\nabla u|^2}^{(-j)}\right| \leq C_{40}|D^2u|^2 \cdot \|\nabla u\|_{C^0(\overline{\Omega})}, \\ \left|D^2u \star Q^{-j}(1+Q)^{-\ell}P_k\right| &\leq C_{41}|D^2u| \cdot \left||\nabla u|^k \sqrt{1+|\nabla u|^2}^{(-j)}\big(1+\sqrt{1+|\nabla u|^2}^{(-j)}\big)^{-\ell}\right| \\ &\leq C_{42}|D^2u| \cdot \|\nabla u\|_{C^0(\overline{\Omega})} \end{aligned} \tag{71}$$

By using $L^s(\Omega)$ norms on the first inequality, we obtain the first estimate in (68). In order to prove the second estimate in (68) we use $C^0(\overline{\Omega})$ norms for the second inequality in (71). Up to this point, only the Hölder seminorm estimate is missing. As a preparatory step, we observe that for the Hölder seminorm $[Q^{-j}]_{C^\alpha(\overline{\Omega})} + [(1+Q)^{-\ell}]_{C^\alpha(\overline{\Omega})} \leq C_{43}(j, \ell)[\nabla u]_{C^\alpha(\overline{\Omega})}$ and $[gh]_{C^\alpha(\overline{\Omega})} \leq \|g\|_{C^0(\overline{\Omega})}[h]_{C^\alpha(\overline{\Omega})} + [g]_{C^\alpha(\overline{\Omega})}\|h\|_{C^0(\overline{\Omega})}$. Thus, by $\|Q^{-j}\|_{C^0(\overline{\Omega})} + \|(1+Q)^{-\ell}\|_{C^0(\overline{\Omega})} \leq 2$ it follows

$$\begin{aligned} \left[D^2u \star Q^{-j}(1+Q)^{-\ell}P_k\right]_{C^\alpha(\overline{\Omega})} &\leq C_{44}\begin{pmatrix} [D^2u]_{C^\alpha(\overline{\Omega})}\|\nabla u\|^k_{C^0(\overline{\Omega})} \\ + \|D^2u\|_{C^0(\overline{\Omega})}[\nabla u]_{C^\alpha(\overline{\Omega})}\|\nabla u\|^k_{C^0(\overline{\Omega})} \\ + \|D^2u\|_{C^0(\overline{\Omega})}[\nabla u]_{C^\alpha(\overline{\Omega})}\|\nabla u\|^{k-1}_{C^0(\overline{\Omega})} \end{pmatrix} \\ &\leq C_{45}\left(\|D^2u\|_{C^0(\overline{\Omega})}[\nabla u]_{C^\alpha(\overline{\Omega})} + [D^2u]_{C^\alpha(\overline{\Omega})}\|\nabla u\|_{C^0(\overline{\Omega})}\right). \end{aligned}$$

By applying the embedding $C^\alpha(\overline{\Omega}) \hookrightarrow\hookrightarrow C^0(\overline{\Omega})$ the last inequality in (68) is shown.

(2) Next we use Lemma 96 and again $\|Q^{-j}\|_{C^0(\overline{\Omega})} \le 1$ as well as $\|\nabla u\|_{C^0(\overline{\Omega})}, \|\nabla w\|_{C^0(\overline{\Omega})} \le 1$ to prove (69)

$$
\begin{aligned}
&\big|D^2u \star D^2u \star Q^{-j}(u)P_i(\nabla u) - \ D^2w \star D^2w \star Q^{-j}(w)P_i(\nabla w)\big| \\
&\qquad \le C_{46}\big|\nabla u \star D^2u \star (D^2u - D^2w)\big| + C_{46}\big|\nabla w \star D^2u \star (D^2u - D^2w)\big| \\
&\qquad\quad + C_{46}\big|\nabla u \star D^2w \star (D^2u - D^2w)\big| + C_{46}\big|\nabla w \star D^2w \star (D^2u - D^2w)\big| \\
&\qquad\quad + C_{46}\big|D^2u \star D^2u \star (\nabla w - \nabla u)\big| + C_{46}\big|D^2u \star D^2w \star (\nabla u - \nabla w)\big| \\
&\qquad\quad + C_{46}\big|D^2w \star D^2w \star (\nabla u - \nabla w)\big|.
\end{aligned} \tag{72}
$$

We emphasize that by the proof of Lemma 96 the terms including $Q^{-j}(w) - Q^{-j}(u)$ are absorbed by $(\nabla w - \nabla u)$. Namely by Lemma 96.

$$
\big|Q^{-j}(w) - Q^{-j}(u)\big| \le \big|\nabla w - \nabla u\big| \left| \frac{(\nabla w + \nabla u)}{\Big(Q(w) + Q(u)\Big)} \sum_{\ell=1}^{k} \frac{Q^{\ell-1}(w)Q^{k-\ell}(u)}{Q^k(w)Q^k(u)} \right| \le k\big|\nabla w - \nabla u\big|.
$$

We conclude by applying the $L^s(\Omega)$-norm that

$$
\begin{aligned}
&\Big\|D^2u \star D^2u \star Q^{-j}(u)P_i(\nabla u) - \ D^2w \star D^2w \star Q^{-j}(w)P_i(\nabla w)\Big\|_{L^s(\Omega)} \\
&\qquad \le C_{47} \begin{pmatrix} \|\nabla u D^2u\|_{L^{2s}(\Omega)} + \|\nabla w D^2u\|_{L^{2s}(\Omega)} \\ + \|\nabla u D^2w\|_{L^{2s}(\Omega)} + \|\nabla w D^2w\|_{L^{2s}(\Omega)} \end{pmatrix} \|D^2w - D^2u\|_{L^{2s}(\Omega)} \\
&\qquad\quad + C_{48}\Big(\|D^2u\|^2_{L^{2s}(\Omega)} + \big\||D^2w||D^2u|\big\|_{L^s(\Omega)} + \|D^2u\|^2_{L^{2s}(\Omega)}\Big)\|\nabla w - \nabla u\|_{C^0(\overline{\Omega})}.
\end{aligned}
$$

By Hölder's inequality $\|D^2wD^2u\|_{L^s(\Omega)} \le \|D^2u\|_{L^{2s}(\Omega)}\|D^2w\|_{L^{2s}(\Omega)}$, Young's inequality and Sobolev embedding $W^{2,2s}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, this finishes the proof of (69). Namely

$$
\|\nabla w - \nabla u\|_{C^0(\overline{\Omega})} \le C_{49}\|w - u\|_{W^{2,2s}(\Omega)}, \quad \text{and} \quad \|\nabla u D^2u\|_{L^{2s}(\Omega)} \le C_{50}\|u\|_{W^{2,2s}(\Omega)} \cdot \|D^2u\|_{L^{2s}(\Omega)}.
$$

(3) The proof of the last estimate (70) also involves $C^0(\overline{\Omega})$ and seminorm inequalities and the assumptions $\|\nabla u\|_{C^0(\overline{\Omega})}, \|\nabla w\|_{C^0(\overline{\Omega})} \le 1$. First, we consider

$$
\begin{aligned}
&\big|D^2u \star Q^{-j}(1+Q)^{-\ell}(u)P_k(\nabla u) - \ D^2w \star Q^{-j}(1+Q)^{-\ell}(w)P_k(\nabla w)\big| \\
&\qquad \le C_{51}\big|\nabla u \star \nabla u \star (D^2u - D^2w)\big| + C_{51}\big|\nabla u \star \nabla w \star (D^2u - D^2w)\big| \\
&\qquad\quad + C_{51}\big|\nabla w \star \nabla w \star (D^2u - D^2w)\big| + C_{51}\big|\nabla u \star D^2u \star (\nabla u - \nabla w)\big| \\
&\qquad\quad + C_{51}\big|\nabla u \star D^2w \star (\nabla u - \nabla w)\big| + C_{51}\big|\nabla w \star D^2u \star (\nabla u - \nabla w)\big| \\
&\qquad\quad + C_{51}\big|\nabla w \star D^2w \star (\nabla w - \nabla u)\big|.
\end{aligned} \tag{73}
$$

It directly follows

$$
\begin{aligned}
&\big\|D^2u \star Q^{-j}(u)(1+Q)^{-\ell}(u)P_k(\nabla u) - \ D^2w \star Q^{-j}(w)(1+Q)^{-\ell}(w)P_k(\nabla w)\big\|_{C^0(\overline{\Omega})} \\
&\qquad \le C_{38}\|u - w\|_{C^{2+\alpha}(\overline{\Omega})}\left(\|D^2u\|_{C^0(\overline{\Omega})} + \|D^2w\|_{C^0(\overline{\Omega})}\right) \cdot \left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right).
\end{aligned}
$$

By applying the $C^0(\overline{\Omega})$-norm on both sides we get the $C^0(\overline{\Omega})$-part in the $C^\alpha(\overline{\Omega})$-estimate (70). For the seminorm part, by using $k \geq 2$ we observe similarly that

$$
\begin{aligned}
&\left[D^2u \star Q^{-j}(1+Q)^{-\ell}(u)P_k(\nabla u) - D^2w \star Q^{-j}(1+Q)^{-\ell}(w)P_k(\nabla w)\right]_{C^\alpha(\overline{\Omega})} \\
&\quad \leq \|\nabla w - \nabla u\|_{C^0(\overline{\Omega})} \begin{pmatrix} \left(\|D^2u\|_{C^0(\overline{\Omega})} + \|D^2w\|_{C^0(\overline{\Omega})}\right)\left([\nabla u]_{C^\alpha(\overline{\Omega})} + [\nabla w]_{C^\alpha(\overline{\Omega})}\right) \\ + \left([D^2u]_{C^\alpha(\overline{\Omega})} + [D^2w]_{C^\alpha(\overline{\Omega})}\right)\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right) \end{pmatrix} \\
&\qquad + [\nabla w - \nabla u]_{C^\alpha(\overline{\Omega})}\left(\|D^2u\|_{C^0(\overline{\Omega})} + \|D^2w\|_{C^0(\overline{\Omega})}\right)\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right) \\
&\qquad + \|D^2u - D^2w\|_{C^0(\overline{\Omega})}\left([\nabla u]_{C^\alpha(\overline{\Omega})} + [\nabla w]_{C^\alpha(\overline{\Omega})}\right)\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right) \\
&\qquad + [D^2u - D^2w]_{C^\alpha(\overline{\Omega})}\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right)\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right)
\end{aligned}
$$

where we again used $[gh]_{C^\alpha(\overline{\Omega})} \leq \|g\|_{C^0(\overline{\Omega})}[h]_{C^\alpha(\overline{\Omega})} + [g]_{C^\alpha(\overline{\Omega})}\|h\|_{C^0(\overline{\Omega})}$. $\square$

At this point, we are prepared for the main theorem of this subsection, namely our goal is to prove the existence of the solution for the Willmore equation rewritten in Lemma 18 in variational form for small Dirichlet data

$$
\begin{cases} \Delta^2 u = D_i b_1^i[u] + D_{ij}^2 b_2^{ij}[u] & \text{in } \Omega, \\ u = g_0, \quad \partial_\nu u = g_1 & \text{on } \Omega. \end{cases} \tag{74}
$$

with the following conditions (60) imposed on the structure of the right-hand side

$$
\begin{aligned}
b_1[u] &= D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1}P_{2k-1}(\nabla u), \\
b_2[u] &= D^2u \star \sum_{k=1}^{2} Q^{-2k-1}P_{2k}(\nabla u) + D^2u \star P_2(\nabla u) \star (Q(1+Q))^{-1}.
\end{aligned}
$$

The principal significance of the Theorem below is that for a given $C^{2+\alpha}$-bound on the boundary data, only smallness in the C^1-norm is required, even though some special structural properties of b_1 and b_2 are used.

21 Theorem
Assume that $\alpha \in (0,1)$, $\partial\Omega \in C^{4+\alpha}$, $g_0 \in C^{2+\alpha}(\partial\Omega)$ and $g_1 \in C^{1+\alpha}(\partial\Omega)$. Additionally, we suppose that $\|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)} < K$ for some $K > 0$.

Then there is a constant $\delta = \delta(K, \Omega, \alpha) > 0$ such that if $\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)} < \delta$, then there exists a variational solution $u \in C^{2+\alpha}(\overline{\Omega})$ to the Willmore-type Dirichlet problem, thus u solves (74) *with the right-hand side* (60).

Proof: The main idea is to apply a fixed point argument, which is divided into four steps for the sake of clarity. Without the restriction of generality, we can also assume, by increasing K if necessary, that $K \geq 1$.

(1) Definition of the iteration map and set

We begin by defining the iteration mapping $G\colon C^{2+\alpha}(\overline{\Omega}) \to C^{2+\alpha}(\overline{\Omega})$. For each $w \in C^{2+\alpha}(\overline{\Omega})$ we set Gw as the solution v of the following problem

$$
\begin{cases} \Delta^2 v = D_i b_1^i[w] + D_{ij}^2 b_2^{ij}[w] & \text{in } \Omega, \\ v = g_0, \quad \partial_\nu v = g_1 & \text{on } \partial\Omega. \end{cases} \tag{75}
$$

By Theorem 2.19 in [GGS10, p.45] the existence and the $C^{2+\alpha}(\overline{\Omega})$-regularity of Gw is ensured, thus the mapping G is well-defined.

Furthermore, we introduce the constant δ, which bounds the $C^1(\overline{\Omega})$-norm of the boundary data. For the proof-making work, we will impose four conditions on δ, where the first one is

$$\text{(C1)} \qquad 2C_{14}\delta \in (0,1)$$

and the other conditions (C2), (C3) and (C4) which we specify in the following. Next, we define the corresponding non-empty set

$$\mathcal{M}_\delta^K := \left\{ u \in C^{2+\alpha}(\overline{\Omega}) \,\middle|\, \|u\|_{C^1(\overline{\Omega})} \le 2C_{14}\delta, \quad \|u\|_{C^{2+\alpha}(\overline{\Omega})} \le 2C_{12}K \right\},$$

where C_{12} and C_{14} are constants in (63) and (65).

2 G **is a self-map**

In this paragraph, we show that G maps $\mathcal{M}_\delta^K$ to $\mathcal{M}_\delta^K$. Let $w \in \mathcal{M}_\delta^K$ then $Gw \in C^{2+\alpha}(\overline{\Omega})$ solves problem (75). In the first part of the proof, we consider the estimate of the $C^{2+\alpha}(\overline{\Omega})$-norm of Gw. Since the first condition (C1) yields $\|w\|_{C^1(\overline{\Omega})} < 1$, we can incorporate the Schauder estimate (63) from Proposition 19 with the preliminary estimates (68) for the $b_i's$ in Lemma 20. Combining these results yields

$$\text{(76)} \qquad \begin{aligned} \|Gw\|_{C^{2+\alpha}(\overline{\Omega})} &\le C_{12}\left(\|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)} + \|b_1[w]\|_{L^{\frac{2}{1-\alpha}}(\Omega)} + \|b_2[w]\|_{C^\alpha(\overline{\Omega})} \right) \\ &\le C_{12}\left(K + C_{52}\|D^2w\|^2_{L^{\frac{2}{1-\alpha}}(\Omega)} \|\nabla w\|_{C^0(\overline{\Omega})} + C_{53}\|D^2 w\|_{C^\alpha(\overline{\Omega})}\|\nabla w\|_{C^\alpha(\overline{\Omega})} \right) \\ &\le C_{12}K + C_{54}\Big(\|w\|^2_{C^{2+\alpha}(\overline{\Omega})}\|w\|_{C^1(\overline{\Omega})} + \|w\|_{C^{2+\alpha}(\overline{\Omega})}\|w\|_{C^2(\overline{\Omega})} \Big), \end{aligned}$$

where C_{54} and all other constants in this proof depend only on α, Ω and the algebraic structure of b_1 and b_2. What is still lacking is the bound on $C^2(\overline{\Omega})$-norm of w in terms of δ. That is obtained by the real interpolation inequality Theorem 9 with $\lambda = \frac{\alpha}{1+\alpha}, a = 1, b = 2+\alpha$

$$\text{(77)} \qquad \|w\|_{C^2(\overline{\Omega})} \le C_1 \|w\|^{\frac{\alpha}{1+\alpha}}_{C^1(\overline{\Omega})} \cdot \|w\|^{\frac{1}{1+\alpha}}_{C^{2+\alpha}(\overline{\Omega})}.$$

Applying this to the result (76) yields

$$\text{(78)} \qquad \begin{aligned} \|Gw\|_{C^{2+\alpha}(\overline{\Omega})} &\le C_{12}K + C_{54}\left((2C_{12}K)^2 2C_{14}\delta + C_1\|w\|^{1+\frac{1}{1+\alpha}}_{C^{2+\alpha}(\overline{\Omega})} \cdot \|w\|^{\frac{\alpha}{1+\alpha}}_{C^1(\overline{\Omega})} \right) \\ &\le C_{12}K + C_{54}\left((2C_{12}K)^2 2C_{14}\delta + C_1 (2C_{12}K)^{\frac{2+\alpha}{1+\alpha}} \cdot (2C_{14}\delta)^{\frac{\alpha}{1+\alpha}} \right). \end{aligned}$$

At this point, we want to impose the second condition on δ in the way that the right-hand side of the equation above is small enough. Hence, we choose $\delta_0 = \delta_0(\alpha, \Omega, K)$ depending on K such that

$$\text{(C2)} \qquad C_{54}\left((2C_{12}K)^2 2C_{14}\delta_0 + C_1(2C_{12}K)^{\frac{2+\alpha}{1+\alpha}} \cdot (2C_{14}\delta_0)^{\frac{\alpha}{1+\alpha}} \right) \le C_{12}K.$$

Then by monotonicity, we get

$$\text{(79)} \qquad \forall \delta \in (0, \delta_0] \quad \forall w \in \mathcal{M}_\delta^K : \quad \|Gw\|_{C^{2+\alpha}(\overline{\Omega})} \le 2C_{12}K.$$

In the second part, we consider the $C^1(\overline{\Omega})$-norm of Gw. Here we again use the Schauder estimate (65) from Proposition 19 thereby employing estimates (68) for the b_i terms where we can set $t = p \in (2, \infty)$ arbitrarily since $w \in C^{2+\alpha}(\overline{\Omega})$

$$\|Gw\|_{C^1(\overline{\Omega})} \le C_{14}\left(\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)} + \|b_1[w]\|_{L^p(\Omega)} + \|b_2[w]\|_{L^p(\Omega)} \right)$$

$$\le C_{14}\left(\delta + C_{55}\left(\|D^2w\|^2_{L^{2p}(\Omega)}\|\nabla w\|_{C^0(\overline{\Omega})} + \|D^2w\|_{L^p(\Omega)}\|\nabla w\|_{C^0(\overline{\Omega})}\right)\right)$$
$$\le C_{14}\left(\delta + C_{56}\left(\|w\|^2_{C^2(\overline{\Omega})}\|\nabla w\|_{C^0(\overline{\Omega})} + \|w\|_{C^2(\overline{\Omega})}\|\nabla w\|_{C^0(\overline{\Omega})}\right)\right).$$

In the above inequality, we want to factor out δ so that the corresponding coefficient is smaller than $2C_{14}$. This can be achieved by using interpolation (77) for the $C^2(\overline{\Omega})$ norm

$$(80)\qquad \begin{aligned}\|Gw\|_{C^1(\overline{\Omega})} &\le C_{14}\left(\delta + C_{56}\left(C_1^2\|w\|^{\frac{2}{1+\alpha}}_{C^{2+\alpha}(\overline{\Omega})}\|w\|^{1+\frac{2\alpha}{1+\alpha}}_{C^1(\overline{\Omega})} + C_1\|w\|^{\frac{1}{1+\alpha}}_{C^{2+\alpha}(\overline{\Omega})}\|w\|^{1+\frac{\alpha}{1+\alpha}}_{C^1(\overline{\Omega})}\right)\right)\\ &\le C_{14}\left(\delta + C_{57}\left(K^{\frac{2}{1+\alpha}}\delta^{1+\frac{2\alpha}{1+\alpha}} + K^{\frac{1}{1+\alpha}}\delta^{1+\frac{\alpha}{1+\alpha}}\right)\right)\\ &\le C_{14}\delta\left(1 + C_{57}\left(K^{\frac{2}{1+\alpha}}\delta^{\frac{2\alpha}{1+\alpha}} + K^{\frac{1}{1+\alpha}}\delta^{\frac{\alpha}{1+\alpha}}\right)\right).\end{aligned}$$

Now, to get a self-map we choose some $\delta_1 = \delta_1(\alpha, \Omega, K)$ by

$$(C3)\qquad C_{57}\left(K^{\frac{2}{1+\alpha}}\delta_1^{\frac{2\alpha}{1+\alpha}} + K^{\frac{1}{1+\alpha}}\delta_1^{\frac{\alpha}{1+\alpha}}\right) \le 1.$$

Together with δ_0 defined by (C2) the third condition on δ is imposed by considering only the case

$$\delta \le \delta_1,$$

which gives us $\|Gw\|_{C^1(\overline{\Omega})} \le 2C_{14}\delta$. Combining the results in this paragraph shows that

$$0 < \delta \le \min\left\{(2C_{14})^{-1}, \delta_0, \delta_1\right\}: \qquad w \in \mathcal{M}_\delta^K \Rightarrow Gw \in \mathcal{M}_\delta^K.$$

3 **G is a contraction**

The last property we have to check is contraction, thus for all $u, w \in \mathcal{M}_\delta^K$ the difference between Gw and Gu has to be estimated. Note that $Gu - Gw$ is a solution of the following problem

$$(81)\qquad \begin{cases}\Delta^2(Gu - Gw) = D_i\big(b_1^i[u] - b_1^i[w]\big) + D^2_{ij}\big(b_2^{ij}[u] - b_2^{ij}[w]\big) & \text{in } \Omega,\\ Gu - Gw = 0, \quad \partial_\nu Gu - \partial_\nu Gw = 0 & \text{on } \partial\Omega,\end{cases}$$

thus similarly to the previous steps by Proposition 19 it follows together with (69) and (70) that

$$\begin{aligned}&\|Gu - Gw\|_{C^{2+\alpha}(\overline{\Omega})}\\ &\quad\le C_{12}\left(\|b_1[u] - b_1[w]\|_{L^{\frac{2}{1-\alpha}}(\Omega)} + \|b_2[u] - b_2[w]\|_{C^\alpha(\overline{\Omega})}\right)\\ &\quad\le C_{58}\|D^2u - D^2w\|_{L^{\frac{4}{1-\alpha}}(\Omega)}\Big(\|D^2u\|_{L^{2s}(\Omega)} + \|D^2w\|_{L^{2s}(\Omega)}\Big)\\ &\qquad + C_{58}\|u - w\|_{C^{2+\alpha}(\overline{\Omega})}\big(\|D^2u\|_{C^\alpha(\overline{\Omega})} + \|D^2w\|_{C^\alpha(\overline{\Omega})}\big)\Big(\|\nabla u\|_{C^\alpha(\overline{\Omega})} + \|\nabla w\|_{C^\alpha(\overline{\Omega})}\Big)\\ &\quad\le C_{59}\|u - w\|_{C^{2+\alpha}(\overline{\Omega})}\Big(K\big(\|\nabla u\|_{C^\alpha(\overline{\Omega})} + \|\nabla w\|_{C^\alpha(\overline{\Omega})}\big)\Big)\end{aligned}$$

by applying the interpolation inequality (77). Further, we need to estimate the $C^{1+\alpha}(\overline{\Omega})$-norms. It is possible since $C^{1+\alpha}(\overline{\Omega})$ is an interpolation space between $C^{2+\alpha}(\overline{\Omega})$ and $C^1(\overline{\Omega})$. Thus, by the real interpolation inequality Theorem 9 with $\lambda = \frac{1}{1+\alpha}, a = 1, b = 2 + \alpha$

$$(82)\qquad \|w\|_{C^{1+\alpha}(\overline{\Omega})} \le C_1\|w\|^{\frac{1}{1+\alpha}}_{C^1(\overline{\Omega})} \cdot \|w\|^{\frac{\alpha}{1+\alpha}}_{C^{2+\alpha}(\overline{\Omega})}.$$

Applying this to the last inequality yields

$$\|Gu - Gw\|_{C^{2+\alpha}(\overline{\Omega})} \le C_{60}\Big(K^{1+\frac{\alpha}{1+\alpha}}\delta^{\frac{1}{1+\alpha}}\Big)\|u - w\|_{C^{2+\alpha}(\overline{\Omega})}.$$

To obtain a contraction, we need to impose the last condition on δ. Hence, we choose some δ_2 such that

$$C_{60} K^{1+\frac{\alpha}{1+\alpha}} \delta_2^{\frac{1}{1+\alpha}} < \frac{1}{2}. \tag{C4}$$

Then, by monotonicity, we can assert that

$$0 < \delta \leq \delta_2 \quad \Rightarrow \quad \forall u, w \in \mathcal{M}_\delta^K: \quad \|Gu - Gw\|_{C^{2+\alpha}(\overline{\Omega})} \leq \frac{1}{2}\|u - w\|_{C^{2+\alpha}(\overline{\Omega})}. \tag{83}$$

4 Applying the fixed point theorem

By combining all four conditions on δ

$$0 < \delta \leq \min\Big\{1/(2C_{14}), \delta_0, \delta_1, \delta_2\Big\}$$

all assumptions of the fixed point theorem are satisfied. Thus, there exists a unique fixed point $u \in \mathcal{M}_\delta^K \subset C^{2+\alpha}(\overline{\Omega})$ such that $u = Gu$. This finally means that u solves the original problem (74) in the space $\mathcal{M}_\delta^K$. □

At this point, we want to remark on elliptic regularity. Namely, if the boundary data are more regular $g_0 \in C^{4+\alpha}(\partial\Omega), g_1 \in C^{3+\alpha}(\partial\Omega)$, then the solution is also more regular $u \in C^{4+\alpha}(\overline{\Omega})$. Despite that, the $C^\alpha(\partial\Omega)$ and $C^1(\partial\Omega)$-norm smallness condition for the existence of such u $C^\alpha(\partial\Omega)$ and $C^1(\partial\Omega)$-norm remains the same.

The preceding observation, when looked at more general right-hand side terms b_i, leads to a similar existence result, thus this proof idea is not unique to the Willmore equation. The main point is to observe that we use some kind of non-linearity, namely b_i's must be at least a $D^2u, \nabla u$-polynomial of the second order with some Hölder interpolation inequalities. This is needed to extract δ like in $C^1(\overline{\Omega})$ estimate (80) as well as to achieve the contraction inequality (83). Thus, we generalize our results to the following problem

$$\begin{cases} \Delta^2 u = f_0[u] + D_i f_1^i[u] + D_{ij}^2 f_2^{ij}[u] & \text{in } \Omega, \\ u = g_0, \quad \partial_\nu u = g_1 & \text{on } \partial\Omega. \end{cases} \tag{84}$$

22 Theorem

Assume that $\alpha \in (0,1)$, $\partial\Omega \in C^{4+\alpha}$, $g_0 \in C^{2+\alpha}(\partial\Omega)$ and $g_1 \in C^{1+\alpha}(\partial\Omega)$. Additionally, we suppose that $\|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)} < K$ for some $K > 0$. We impose the following conditions on the structure of the right-hand side

$$\begin{gathered} f_0[u] = P_{k_0}(D^2u) P_{\ell_0}(\nabla u) Q^{-m_0}(\nabla u), \\ f_1[u] = P_{k_1}(D^2u) P_{\ell_1}(\nabla u) Q^{-m_1}(\nabla u), \qquad f_2[u] = P_{k_2}(D^2u) P_{\ell_2}(\nabla u) Q^{-m_2}(\nabla u) \end{gathered} \tag{85}$$

for some $i = 0,1,2 : k_i, \ell_i, m_i \in \mathbb{N}_0$, with $k_i + \ell_i \geq 2$.

Then there exists a constant $\mu = \mu(K, \alpha, \Omega) > 0$ such that if $\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)} < \mu$, then there exists a variational solution $u \in C^{2+\alpha}(\overline{\Omega})$ to (84) *with the right-hand side* (85).

Proof: Since we do not want to replicate the complete proof of the previous theorem, we will only highlight the changes to be made.

First, we change the iteration mapping $G : C^{2+\alpha}(\overline{\Omega}) \to C^{2+\alpha}(\overline{\Omega})$. In order to incorporate f_0 we set Gw as the solution v of the following problem

$$\begin{cases} \Delta^2 v = f_0[w] + Df_1[w] + D^2 f_2[w] & \text{in } \Omega, \\ v = g_0, \quad \partial_\nu v = g_1 & \text{on } \partial\Omega. \end{cases} \tag{86}$$

The $C^{2+\alpha}$-regularity and existence of Gw is ensured by [GGS10, Theorem 2.19 p.45]. Since we are not assuming $\ell_i \geq 2$, we might not have enough ∇u-terms in some estimates. Thus contrary to the proof of Theorem 21 in (80) we cannot extract δ^1 in $\|Gw\|_{C^1(\overline{\Omega})} \leq C_{61}\delta^1 \cdot \delta^\varepsilon$ estimate via interpolation, and let $C_{61}\delta^\varepsilon$ small enough. Therefore, we replace the $C^1(\overline{\Omega})$-norm with a Hölder norm $C^{1+\gamma}(\overline{\Omega})$, where $\gamma > 1-\alpha$ and define

$$\mathcal{M}_\delta^K := \left\{ u \in C^{2+\alpha}(\overline{\Omega}) \;\middle|\; \|u\|_{C^{1+\gamma}(\overline{\Omega})} \leq 2C_{13}\delta, \quad \|u\|_{C^{2+\alpha}(\overline{\Omega})} \leq 2C_{12}K \right\}$$

which we consider as a closed subspace of complete space $C^{2+\beta}(\overline{\Omega})$ with $\beta < \alpha$, which is the main trick here. Let us assume that $\|g_0\|_{C^{1+\gamma}(\partial\Omega)} + \|g_1\|_{C^\gamma(\partial\Omega)} \leq C_{13}\delta$ and

$$2C_{13}\delta \leq 1 \tag{C1}$$

Since we will need to estimate $\|u\|_{C^2(\overline{\Omega})}$, we conclude with $\lambda \in (0,1)$ such that $2 = \lambda(1+\gamma) + (1-\lambda)(2+\alpha)$ the interpolation inequality

$$\|u\|_{C^2(\overline{\Omega})} \leq C_{62}\|u\|^\lambda_{C^{1+\gamma}(\overline{\Omega})} \cdot \|u\|^{1-\lambda}_{C^{2+\alpha}(\overline{\Omega})}. \tag{87}$$

To show that G is a self-map, we estimate the $C^{2+\alpha}$-norm of Gw. Since $\|w\|_{C^{2+\alpha}(\overline{\Omega})} < K$, one can show in the same way as in Lemma 20 that with constants $C_{63}, C_{64}, C_{65}, C_{66}$ depending only on K, Ω, k_i, ℓ_i and α

$$\begin{aligned}\big\|f_0[w]\big\|_{C^0(\overline{\Omega})} + \big\|f_1[w]\big\|_{C^0(\overline{\Omega})} + \big\|f_2[w]\big\|_{C^0(\overline{\Omega})} &\leq C_{63}\|w\|^2_{C^2(\overline{\Omega})} \leq C_{64}\|u\|^{2\lambda}_{C^{1+\gamma}(\overline{\Omega})} \cdot \|u\|^{2-2\lambda}_{C^{2+\alpha}(\overline{\Omega})},\\ \big\|f_2[w]\big\|_{C^\alpha(\overline{\Omega})} \leq C_{65}\|w\|_{C^{2+\alpha}(\overline{\Omega})}\|w\|_{C^2(\overline{\Omega})} &\leq C_{66}\|u\|^\lambda_{C^{1+\gamma}(\overline{\Omega})} \cdot \|u\|^{2-\lambda}_{C^{2+\alpha}(\overline{\Omega})}.\end{aligned}$$

We combine this with the Schauder estimate (63) and obtain

$$\begin{aligned}\|Gw\|_{C^{2+\alpha}(\overline{\Omega})} &\leq C_{12}\begin{pmatrix} \|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)} + \|f_0[w]\|_{L^{\frac{2}{2-\alpha}}(\Omega)} \\ + \|f_1[w]\|_{L^{\frac{2}{1-\alpha}}(\Omega)} + \|f_2[w]\|_{C^\alpha(\overline{\Omega})}\end{pmatrix}\\ &\leq C_{12}K + C_{67}\left(\delta^{2\lambda}\cdot K^{2-2\lambda} + \delta^\lambda \cdot K^{2-\lambda}\right)\end{aligned}$$

where C_{67} depends only on on K, Ω and α. Further, choosing δ such that

$$C_{77}\left(\delta^{2\lambda}\cdot K^{2-2\lambda} + \delta^\lambda\cdot K^{2-\lambda}\right) \leq C_{12}K \tag{C2}$$

we get $\|Gw\|_{C^{2+\alpha}(\overline{\Omega})} \leq 2C_{12}K$. For the $C^{1+\gamma}(\overline{\Omega})$-estimate we use the Schauder results from Proposition 19 and especially (87) for some $s > 0$

$$\begin{aligned}\|Gw\|_{C^{1+\gamma}(\overline{\Omega})} &\leq C_{13}\begin{pmatrix}\|g_0\|_{C^{1+\gamma}(\partial\Omega)} + \|g_1\|_{C^\gamma(\partial\Omega)} + \|f_0[w]\|_{L^s(\Omega)}\\ + \|f_1[w]\|_{L^{\frac{2}{2-\gamma}}(\Omega)} + \|f_2[w]\|_{L^{\frac{2}{1-\gamma}}(\Omega)}\end{pmatrix}\\ &\leq C_{13}\delta + C_{68}\left(\delta^{2\lambda}\cdot K^{2-2\lambda}\right) \leq C_{13}\delta + \left(C_{69}\delta^{2\lambda-1}\cdot K^{2-2\lambda}\right)\delta.\end{aligned}$$

In fact $2\lambda - 1 > 0$ since by $\gamma > 1-\alpha$ and $2 = \lambda(1+\gamma) + (1-\lambda)(2+\alpha) = \lambda(1+\gamma-2-\alpha) + 2 + \alpha$ it follows

$$\lambda = \frac{\alpha}{\alpha+1-\gamma} > \frac{\alpha}{\alpha+\alpha} = \frac{1}{2}.$$

This shows that

$$\text{(C3)} \qquad \Big(C_{69}\delta^{2\lambda-1}\cdot K^{2-2\lambda}\Big) < C_{13}$$

can be achieved. Thus, we add this condition to the previous assumptions on δ. As a consequence, we get $\|Gw\|_{C^{1+\gamma}(\overline{\Omega})} < 2 \leq C_{13}\delta$ and moreover $Gw \in \mathcal{M}_\delta^K$. This means that for δ small enough, G is a self-map.

For the discussion of the contraction property, we also change some details. Here, we consider $Gu - Gw$ as a solution to a modified problem

$$\begin{cases} \Delta^2(Gu-Gw) = f_0[u]-f_0[w] + D_i\big(f_1^i[u]-f_1^i[w]\big) + D_{ij}^2\big(f_2^{ij}[u]-f_2^{ij}[w]\big) & \text{in } \Omega, \\ Gu-Gw = 0, \quad \partial_\nu Gu - \partial_\nu Gw = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Since we consider $\mathcal{M}_\delta^K$ as a subspace of $C^{2+\beta}(\overline{\Omega})$ we estimate $Gu-Gw$ in the $C^{2+\beta}(\overline{\Omega})$-norm. By Proposition 19, which shows existence and Schauder estimates of solutions and $\|u\|_{C^\beta(\overline{\Omega})}, \|w\|_{C^\beta(\overline{\Omega})} \leq C_{70}K$ it follows with $C_{71} = C_{71}(\beta,K,\Omega)$, $C_{72} = C_{72}(\beta,K,\Omega)$

$$\begin{aligned} \|Gu-Gw\|_{C^{2+\beta}(\overline{\Omega})} &\leq C_{12}(\beta)\begin{pmatrix} \|f_0[u]-f_0[w]\|_{L^{\frac{2}{2-\beta}}(\Omega)} + \|f_1[u]-f_1[w]\|_{L^{\frac{2}{1-\beta}}(\Omega)} \\ + \|f_2[u]-f_2[w]\|_{C^\beta(\overline{\Omega})} \end{pmatrix} \\ &\leq C_{71}\big(\|u\|_{C^{2+\beta}(\overline{\Omega})} + \|w\|_{C^{2+\beta}(\overline{\Omega})}\big)\cdot\|u-w\|_{C^2(\overline{\Omega})} \\ &\quad + C_{72}\big(\|u\|_{C^2(\overline{\Omega})} + \|w\|_{C^2(\overline{\Omega})}\big)\cdot\|u-w\|_{C^{2+\beta}(\overline{\Omega})} \\ &\leq C_{73}\begin{pmatrix} \|u\|^\lambda_{C^{1+\gamma}(\overline{\Omega})}\|u\|^{1-\lambda}_{C^{2+\alpha}(\overline{\Omega})} + \|w\|^\lambda_{C^{1+\gamma}(\overline{\Omega})}\|w\|^{1-\lambda}_{C^{2+\alpha}(\overline{\Omega})} \\ + \|u\|^\xi_{C^{1+\gamma}(\overline{\Omega})}\|u\|^{1-\xi}_{C^{2+\alpha}(\overline{\Omega})} + \|w\|^\xi_{C^{1+\gamma}(\overline{\Omega})}\|w\|^{1-\xi}_{C^{2+\alpha}(\overline{\Omega})} \end{pmatrix}\cdot\|u-w\|_{C^{2+\beta}(\overline{\Omega})} \\ &\leq C_{74}\Big(\delta^\lambda K^{1-\lambda} + \delta^\xi K^{1-\xi}\Big)\cdot\|u-w\|_{C^{2+\beta}(\overline{\Omega})} \end{aligned}$$

where $\xi = (\alpha-\beta)/(1+\alpha-\gamma)$ since by interpolation $\|u\|_{C^{2+\beta}(\overline{\Omega})} \leq C_{75}\|u\|^\xi_{C^{1+\gamma}(\overline{\Omega})}\|u\|^{1-\xi}_{C^{2+\alpha}(\overline{\Omega})}$. We emphasize that we used $\beta > \alpha$. Next, we get the contraction by imposing the condition

$$\text{(C4)} \qquad C_{74}\Big(\delta^\lambda K^{1-\lambda} + \delta^\xi K^{1-\xi}\Big) < 1/2.$$

Hence, it follows the contraction

$$\forall u,w \in \mathcal{M}_\delta^K: \quad \|Gu-Gw\|_{C^{2+\beta}(\overline{\Omega})} \leq \frac{1}{2}\|u-w\|_{C^{2+\beta}(\overline{\Omega})}.$$

By contraction mapping principle, we get the solution of problem (84). In the last step, we observe that by Lemma 11, we can use the interpolation inequality on boundary

$$\begin{aligned} \|g_0\|_{C^{1+\gamma}(\partial\Omega)} &+ \|g_1\|_{C^\gamma(\partial\Omega)} \\ &\leq C_4\left(\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)}\right)^{1-\frac{\gamma}{1+\alpha}}\cdot\left(\|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)}\right)^{\frac{\gamma}{1+\alpha}} \\ &\leq C_{76}\mu^{1-\frac{\gamma}{1+\alpha}}K^{\frac{\gamma}{1+\alpha}} \overset{!}{\leq} C_{13}\delta \end{aligned}$$

to achieve δ small enough by choosing μ sufficient smaller such that the conditions (C1), (C2), (C3), (C4) are satisfied, and thus finishing the proof. □

5.3 Sobolev Case

This chapter treats the Willmore equation employing its biharmonic expansion under even weaker regularity assumptions than in the Hölder class. Here, we pursue the derivation of existence theorems in the Sobolev framework. This means that we consider second derivatives in the weak sense.

Even though there exist trace theorems for Sobolev spaces $W^{2,p}(\Omega)$ such that we can define Dirichlet boundary conditions in the same way as for the Hölder case, we omit this description and consider only the class of functions $u \in W^{2,p}(\Omega)$ such that $u - \varphi \in \mathring{W}^{2,p}(\Omega)$ for a given $W^{2,p}(\Omega)$ representing the boundary conditions.

For the general existence result, we will need some biharmonic preliminary estimates in the L^p-framework analogous to Proposition 19 in the Hölder case.

23 Proposition
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\partial\Omega \in C^3$. Assume that $s \in (1,\infty)$, $p \in (2,\infty)$, $t > \frac{2p}{2+p} > 1$ and $\varphi \in W^{2,p}(\Omega), b_0 \in L^s(\Omega), b_1 \in L^t(\Omega)$ and $b_2 \in L^p(\Omega)$. Then the following problem

$$\Delta^2 u = b_0 + D_i b_1^i + D_{ij}^2 b_2^{ij} \quad \text{in } \Omega \subset \mathbb{R}^2,$$

admits a weak solution $u \in W^{2,p}(\Omega)$ with $u - \varphi \in \mathring{W}^{2,p}(\Omega)$. Moreover, there exist constants $C_{77} = C_{77}(p,\Omega), C_{78} = C_{78}(p,s,t,\Omega)$ and $C_{79} = C_{79}(\Omega), C_{80} = C_{80}(p,s,\Omega)$ such that

$$\begin{aligned} \|u\|_{W^{2,p}(\Omega)} &\le C_{77}\|\varphi\|_{W^{2,p}(\Omega)} + C_{78}\left(\|b_0\|_{L^s(\Omega)} + \|b_1\|_{L^t(\Omega)} + \|b_2\|_{L^p(\Omega)}\right), \\ \|u\|_{C^1(\overline{\Omega})} &\le C_{79}\|\varphi\|_{C^1(\overline{\Omega})} + C_{80}\left(\|b_0\|_{L^s(\Omega)} + \|b_1\|_{L^s(\Omega)} + \|b_2\|_{L^p(\Omega)}\right). \end{aligned}$$

Proof: We begin with the observation that from $t > \frac{2p}{2+p} > 1$ it follows $p > 2$. In this situation, we should split the problem (62) into three parts. Thus, we are investigating the solvability of the following problems in the homogeneous space $\mathring{W}^{2,p}(\Omega)$

$$\Delta^2\psi_0 = b_0 \quad \text{in } \Omega, \quad \Delta^2\psi_1 = D_i b_1^i \quad \text{in } \Omega, \quad \Delta^2\psi_2 = D_{ij}^2 b_2^{ij} - \Delta^2\varphi \quad \text{in } \Omega,$$

so that we could recombine $u = \varphi + \psi_0 + \psi_1 + \psi_2$. Since $\varphi \in W^{2,p}(\Omega)$, the last problem including ψ_2 is solvable by [GGS10, Thm. 2.20 p. 46] for the case $k = 2$. Moreover, we obtain

$$\|\psi_2\|_{W^{2,p}(\Omega)} \le C_{81}(\Omega,p)\big(\|\varphi\|_{W^{2,p}(\Omega)} + \|b_2\|_{L^p(\Omega)}\big).$$

With the same theorem in the case $k = 4$ we get with the Sobolev embedding for all $s \in (1,\infty)$

$$\|\psi_0\|_{W^{2,p}(\Omega)} \le C_{82}\|\psi_0\|_{W^{4,s}(\Omega)} \le C_{83}(\Omega,p,s)\|b_0\|_{L^s(\Omega)}.$$

Furthermore, we use [GGS10, Lemma 4.2 p. 132] and the Sobolev embedding with $t > \frac{2p}{2+p} > 1$ to obtain

$$\|\psi_1\|_{W^{2,p}(\Omega)} \le C_{84}\|\psi_1\|_{W^{3,t}(\Omega)} \le C_{85}(\Omega,p,s,t)\|b_1\|_{L^t(\Omega)}.$$

Finally, the $C^1(\overline{\Omega})$ estimate was already proved in Proposition 19 for $p > 2$ in the Hölder framework. By combining the results above, the desired estimate follows. □

With regard to the following existence Theorem 25, we again consider the $b_i[u]$'s as defined in (60), which are specified by the Willmore equation. Since $b_i[u]$'s include the terms $D^2u \star D^2u$, it is important to notice that in the lemma below, we require $p > 2$! At this point, we want furthermore to prove some preparatory estimates. Here, in the Sobolev case, we can take advantage by partly reusing Lemma 20 already proved in the Hölder case.

24 Lemma
Let $p \in (2,\infty), i,j,\ell,k \in \mathbb{N}_0, i \geq 1, k \geq 2$ then there exists a constant $C_{86} = C_{86}(\Omega,p)$ such that if $u \in W^{2,p}(\Omega)$ with $\|\nabla u\|_{C^0(\overline{\Omega})} \leq 1$ then we can estimate

$$\begin{aligned} \left\|D^2u \star D^2u \star Q^{-j} P_i(\nabla u)\right\|_{L^{p/2}(\Omega)} &\leq C_{86}\|D^2u\|^2_{L^p(\Omega)}\|\nabla u\|_{C^0(\overline{\Omega})}, \\ \left\|D^2u \star Q^{-j}(1+Q)^{-\ell} P_k(\nabla u)\right\|_{L^p(\Omega)} &\leq C_{86}\|D^2u\|_{L^p(\Omega)}\|\nabla u\|_{C^0(\overline{\Omega})}. \end{aligned} \tag{88}$$

Furthermore suppose $w \in W^{2,p}(\overline{\Omega})$ with $\|\nabla w\|_{C^0(\overline{\Omega})} \leq 1$ then it follows

$$\begin{aligned} &\left\|D^2u \star D^2u \star Q^{-j}(u)P_i(\nabla u) \ - D^2w \star D^2w \star Q^{-j}(w)P_i(\nabla w)\right\|_{L^{p/2}(\Omega)} \\ &\quad\leq C_{86}\|u-w\|_{W^{2,p}(\Omega)}\Big(\|D^2u\|_{L^p(\Omega)} + \|D^2w\|_{L^p(\Omega)}\Big)\Big(\|u\|_{W^{2,p}(\Omega)} + \|w\|_{W^{2,p}(\Omega)}\Big) \end{aligned} \tag{89}$$

as well as

$$\begin{aligned} &\left\|D^2u \star Q^{-j}(1+Q)^{-\ell}(u)P_k(\nabla u) - D^2w \star Q^{-j}(1+Q)^{-\ell}(w)P_k(\nabla w)\right\|_{L^p(\Omega)} \\ &\quad\leq C_{86}\|u-w\|_{W^{2,p}(\Omega)}\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right)\left(\|u\|_{W^{2,p}(\Omega)} + \|w\|_{W^{2,p}(\Omega)}\right). \end{aligned} \tag{90}$$

Proof: The estimates in (88) had been proven in the same manner as (68) in Lemma 20 by using the inequalities (71). Analogously, by setting $s = p/2$ in the proof of (69) we obtain (89).

For the proof of the last estimate we apply a L^p-norm on the inequality (73)

$$\begin{aligned} &\left\|D^2u\star Q^{-j}(1+Q)^{-\ell}(u)P_k(\nabla u) - \ D^2w \star Q^{-j}(1+Q)^{-\ell}(w)P_k(\nabla w)\right\|_{L^p(\Omega)} \\ &\quad\leq C_{87}\left(\|\nabla u\|^2_{C^0(\overline{\Omega})} + \|\nabla u\|_{C^0(\overline{\Omega})}\|\nabla w\|_{C^0(\overline{\Omega})} + \|\nabla w\|^2_{C^0(\overline{\Omega})}\right)\|D^2u - D^2w\|_{L^p(\Omega)} \\ &\quad\leq C_{88}\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right)\left(\|D^2u\|_{L^p(\Omega)} + \|D^2w\|_{L^p(\Omega)}\right)\|\nabla u - \nabla w\|_{C^0(\overline{\Omega})}. \end{aligned}$$

We finish the proof by applying the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow\hookrightarrow C^1(\overline{\Omega})$ for $p > 2$. $\square$

At this point, we want to prove the existence of the $W^{2,p}(\Omega)$-solution for the rewritten Willmore equation in biharmonic expansion

$$\Delta^2 u = D_i b_1^i[u] + D^2_{ij} b_2^{ij}[u] \quad \text{in } \Omega. \tag{91}$$

with the right-hand side terms (60)

$$\begin{aligned} b_1[u] &= D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1}P_{2k-1}(\nabla u), \\ b_2[u] &= D^2u \star \sum_{k=1}^{2} Q^{-2k-1}P_{2k}(\nabla u) + D^2u \star P_2(\nabla u) \star (Q(1+Q))^{-1} \end{aligned}$$

in the class $\left\{v \in W^{2,p}(\Omega) \,\middle|\, v - \varphi \in \mathring{W}^{2,p}(\Omega)\right\}$ where $\varphi \in W^{2,p}(\Omega)$ is some small given function representing the boundary data. In contrast to the Hölder case, here we will need smallness not only in the $C^1(\overline{\Omega})$ norm but also in a Sobolev norm like $\|\,.\,\|_{W^{2,p-\varepsilon}(\Omega)}$. The reason is that we have to apply some interpolation argument for the space $W^{2,q}(\Omega)$ for some $q \in (1,p)$ and $W^{2,q}(\Omega)$ is not an interpolation space between $W^{2,p}(\Omega)$ and $C^1(\overline{\Omega})$.

25 Theorem
Assume that $p \in (2,\infty)$ and $K \in (0,\infty), \partial\Omega \in C^3$, and $\varphi \in W^{2,p}(\Omega)$. Additionally, we suppose that $\|\varphi\|_{W^{2,p}(\Omega)} < K$ for some $K > 0$.

Then there is a constant $\mu = \mu(K,p,\Omega) > 0$ such that if $\|\varphi\|_{W^{2,1}(\Omega)} < \mu$, then there exists a weak solution $u \in W^{2,p}(\Omega)$ to the Willmore-type Dirichlet problem, thus u solves (91) *in the class $\left\{v \in W^{2,p}(\Omega) \,\middle|\, v - \varphi \in \mathring{W}^{2,p}(\Omega)\right\}$ with the right-hand side* (60).

Proof: Again, we split the proof into several steps to use the fixed point argument.

① **Definition of the iteration map & set**

We define the iteration map $G\colon W^{2,p}(\Omega) \to W^{2,p}(\Omega)$. Let $G(u)$ be defined as the solution w of the following problem in the class $\{v \in W^{2,p}(\Omega) \mid v - \varphi \in \mathring{W}^{2,p}(\Omega)\}$

$$\Delta^2 w = D_i b_1^i[u] + D_{ij}^2 b_2^{ij}[u] \quad \text{in } \Omega. \tag{92}$$

Let $q \in (2,p)$. To introduce the iteration set, we need a constant $\delta > 0$ which will be specified depending on Ω, p and q in four inequalities (C1),(C2),(C3) and (C4) below. First, let us revisit the constant of the Sobolev embedding into the $C^1(\overline{\Omega})$ space. For any $t > 2$ let $C_{89}(\Omega, t)$ be the embedding constant of the $W^{2,t} \hookrightarrow C^1(\overline{\Omega})$, so that we have for all $u \in W^{2,t}(\Omega)$

$$\|u\|_{C^1(\overline{\Omega})} \le C_{89}(t,\Omega)\|u\|_{W^{2,t}(\Omega)}.$$

Further, we define the iteration set

$$\mathcal{M}_\delta^K := \left\{u \in W^{2,p}(\Omega) \mid \|u\|_{W^{2,q}(\Omega)} \le 2C_{77}(q,\Omega)\delta,\ \|u\|_{W^{2,p}(\Omega)} \le 2C_{77}(p,\Omega)K\right\}$$

where δ is some constant that has to fulfill several conditions that we will establish in the course of the proof. The first condition is

$$2C_{77}(q,\Omega)C_{89}(q,\Omega)\delta < 1, \tag{C1}$$

therefore for all $u \in \mathcal{M}_\delta^K$ it follows $\|u\|_{C^1(\overline{\Omega})} < 1$. Also, it is worth noting that we choose $q > 2$ because in the case $q \le 2$ we cannot abandon either smallness in the $C^1(\overline{\Omega})$ or $W^{2,q}(\Omega)$-norm, since only in the case $q > 2$ one has the Sobolev embedding $W^{2,q}(\Omega) \hookrightarrow\hookrightarrow C^1(\overline{\Omega})$.

We also observe that $W^{2,q}(\Omega)$ is actually an interpolation space between $W^{2,1}(\Omega)$ and $W^{2,p}(\Omega)$. This means for $\varphi \in W^{2,p}(\Omega)$

$$\|\varphi\|_{W^{2,q}(\Omega)} \le C_{90}\|\varphi\|_{W^{2,1}(\Omega)}^{\alpha}\|\varphi\|_{W^{2,p}(\Omega)}^{1-\alpha} \le C_{91}\mu^\alpha K^{1-\alpha} \tag{93}$$

with $\alpha = (p/q - 1/)/(p-1) \in (0,1)$. It follows, that by choosing $\mu > 0$ small enough, we can get

$$\|u\|_{W^{2,q}(\Omega)} \le 2C_{77}(q,\Omega)\delta < C_{91}\mu^\alpha K^{1-\alpha}$$

for all $u \in \mathcal{M}_\delta^K$.

② G **is a self-map**

Let $w \in \mathcal{M}_\delta^K$. Since we want to apply Lemma 24 with the $W^{2,p}(\Omega)$-estimate of Proposition 23 we need some t for the b_1-term. We can choose some $t = \frac{p}{2}$ since $\frac{2p}{2+p} < \frac{p}{2}$. It follows by the first condition $\|w\|_{C^1(\overline{\Omega})} < 1$. This yields

$$\begin{aligned}
\|Gw\|_{W^{2,p}(\Omega)} &\le C_{77}(p,\Omega)\|\varphi\|_{W^{2,p}(\Omega)} + C_{78}(p,\Omega)\left(\big\|b_1[w]\big\|_{L^{p/2}(\Omega)} + \|b_2[w]\|_{L^p(\Omega)}\right)\\
&\le C_{77}(p,\Omega)K + C_{92}\left(\big\|D^2 w\big\|_{L^p(\Omega)}^2\big\|\nabla w\|_{C^0(\overline{\Omega})} + \big\|D^2 w\big\|_{L^p(\Omega)}\big\|\nabla w\|_{C^0(\overline{\Omega})}\right)\\
&\le C_{77}(p,\Omega)K + C_{93}(\Omega,p)(K^2\delta + K\delta).
\end{aligned}$$

By choosing δ_0 such that

$$C_{93}(\Omega,p)(\delta_0 K^2 + K\delta_0) \le C_{77}(p,\Omega)K, \tag{C2}$$

we impose the second constraint on δ, hence considering only $\delta \in (0,\delta_0)$. In this case, we get $\|Gw\|_{W^{2,p}(\Omega)} \le 2C_{77}(p,\Omega)K$.

Next we have to estimate the $W^{2,q}(\Omega)$-norm. Since $q > 2$, in the same way as in $W^{2,p}(\Omega)$-estimate, we conclude

$$\begin{aligned}\|Gw\|_{W^{2,q}(\Omega)} &\le C_{77}(q,\Omega)\|\varphi\|_{W^{2,q}(\Omega)} + C_{78}(q,\Omega)\left(\left\|b_1[w]\right\|_{L^{q/2}(\Omega)} + \|b_2[w]\|_{L^q(\Omega)}\right)\\ &\le C_{77}(q,\Omega)\delta + C_{94}\left(\left\|D^2 w\right\|^2_{L^q(\Omega)}\|\nabla w\|_{C^0(\overline{\Omega})} + \left\|D^2 w\right\|_{L^q(\Omega)}\|\nabla w\|_{C^0(\overline{\Omega})}\right)\\ &\le C_{77}(q,\Omega)\delta + C_{95}(\Omega,q)(\delta^2+\delta)\delta.\end{aligned}$$

We formulate the third constraint by choosing δ_1 such that

$$\text{(C3)} \qquad C_{95}(\Omega,q)(\delta_1^2+\delta_1) \le C_{77}(q,\Omega).$$

For all $\delta' s$ smaller than δ_0, δ_1 and satisfying (C1) the map $G\colon \mathcal{M}^K_\delta \to \mathcal{M}^K_\delta$ is a self map.

3 G **is a contraction**

For the last property, we have to check the contraction property, thus for all $u, w \in \mathcal{M}^K_\delta$ the difference between Gw and Gu has to be estimated. It can be noticed that $Gu - Gw$ is a solution to the problem

$$\text{(94)} \qquad \Delta^2(Gu - Gw) = D_i\big(b_1^i[u] - b_1^i[w]\big) + D^2_{ij}\big(b_2^{ij}[u] - b_2^{ij}[w]\big) \quad \text{in } \Omega,$$

in the class $\mathring{W}^{2,p}(\Omega)$. Similarly to the previous step, we again use Proposition 23. Next, we choose some $t > 1$ such that $\frac{2p}{2+p} < t < \frac{p}{2}$. We put $t = \frac{p}{2+p} + \frac{p}{4}$, as the arithmetic mean, then we have

$$\|Gu - Gw\|_{W^{2,p}(\Omega)} \le C_{78}(p,t,\Omega)\left(\left\|b_1[u] - b_1[w]\right\|_{L^t(\Omega)} + \left\|b_2[u] - b_2[w]\right\|_{L^p(\Omega)}\right).$$

Now, we can estimate each of the parts with estimates with the results from Lemma 24. First, we use (89) and show by $2t < p$ that

$$\begin{aligned}\left\|b_1[u] - b_1[w]\right\|_{L^t(\Omega)} &\le C_{86}(\Omega,t)\|u-w\|_{W^{2,2t}(\Omega)}\left(\|D^2u\|^2_{L^{2t}(\Omega)} + \|D^2w\|^2_{L^{2t}(\Omega)}\right)\\ &\le C_{96}\|u-w\|_{W^{2,p}(\Omega)}\left(\|D^2u\|^2_{L^{2t}(\Omega)} + \|D^2w\|^2_{L^{2t}(\Omega)}\right)\\ &\le C_{97}\|u-w\|_{W^{2,p}(\Omega)}\left(\|D^2u\|^\alpha_{L^1(\Omega)}\|D^2u\|^{1-\alpha}_{L^p(\Omega)} + \|D^2u\|^\alpha_{L^1(\Omega)}\|D^2w\|^{1-\alpha}_{L^p(\Omega)}\right)^2\\ &\le C_{98}\|u-w\|_{W^{2,p}(\Omega)}(\delta^\alpha K^{1-\alpha})^2\end{aligned}$$

by the L^p-interpolation with $\alpha = (p/2t-1)/(p-1) \in (0,1)$. Furthermore, by using (90) we get

$$\begin{aligned}\left\|b_2[u] - b_2[w]\right\|_{L^p(\Omega)} &\le C_{86}\|u-w\|_{W^{2,p}(\Omega)}\left(\|\nabla u\|_{C^0(\overline{\Omega})} + \|\nabla w\|_{C^0(\overline{\Omega})}\right)\left(\|u\|_{W^{2,p}(\Omega)} + \|w\|_{W^{2,p}(\Omega)}\right)\\ &\le C_{99}\|u-w\|_{W^{2,p}(\Omega)}\delta K.\end{aligned}$$

Subsequently, we combine the results above and get for some $C_{100} = C_{100}(p,\Omega)$

$$\|Gu - Gw\|_{W^{2,p}(\Omega)} \le C_{100}\big((\delta^\alpha K^{1-\alpha})^2 + \delta K\big)\|u-w\|_{W^{2,p}(\Omega)}.$$

By choosing $\delta_2 = \delta_2(p,K,\Omega)$ small enough we can achieve

$$\text{(C4)} \qquad C_{100}(\delta_2^{2\alpha}K^{2-2\alpha} + \delta_2 K) \le \frac{1}{2}$$

which is the last condition on δ. It follows for $\delta \le \delta_2$ satisfying all the previous constraints (C1),(C2),(C3)

$$\|Gu - Gw\|_{W^{2,p}(\Omega)} \le \frac{1}{2}\|u-w\|_{W^{2,p}(\Omega)},$$

therefore G is a contraction on $\mathcal{M}_\delta^K$.

④ Using the fixed point theorem

In the last step we combine all necessary conditions (C1),(C2),(C3) and (C4) on δ

$$0 < \delta \le \min\Big\{1/\big(2C_{77}C_{89}(q,\Omega)\big), \delta_0, \delta_1, \delta_2\Big\} \tag{95}$$

The fixed point theorem yields the existence of a unique fixed point $u \in \mathcal{M}_\delta^K \subset W^{2,p}(\Omega)$ such that $u = Gu$. Thus $u \in W^{2,p}(\Omega)$ is a solution of the Willmore equation in $\big\{v \in W^{2,p}(\Omega) \,\big|\, v - \varphi \in \mathring{W}^{2,p}(\Omega)\big\}$.

To finish the proof, we again observe the estimate (93) and conclude that by choosing $\mu > 0$ small enough, we can fulfill all δ conditions (95). □

We can generalize the results to more general right-hand side terms b_i by using ideas from the proof corresponding to the Willmore equation. To apply L^p-interpolation inequalities, we require a non-linearity in b_i's as a $D^2u, \nabla u, u$-polynomial. Thus, we consider the following differential equation

$$\Delta^2 u = f_0[u] + D_i f_1^i[u] + D_{ij}^2 f_2^{ij}[u] \quad \text{in } \Omega. \tag{96}$$

Unlike in the Hölder case, here, due to Hölder's inequality, $f_2[u]$ may contain only one D^2u at most. Otherwise, we can not use Proposition 23 properly, since $\|u\|_{W^{2,p}(\Omega)}$ is estimated by $\|f_2[u]\|_{L^p(\Omega)}$. Moreover, the $L^p(\Omega)$-power $p \in (1,\infty)$ has to be chosen big enough, depending on the number of D^2u-terms in $f_0[u]$ and $f_1[u]$.

26 Theorem

We impose the following conditions on the structure of the right-hand side

$$\begin{gathered} f_0[u] = P_{k_0}(D^2u)P_{\ell_0}(\nabla u)Q^{-m_0}(\nabla u), \\ f_1[u] = P_{k_1}(D^2u)P_{\ell_1}(\nabla u)Q^{-m_1}(\nabla u), \qquad f_2[u] = D^2u \star P_{\ell_2}(\nabla u)Q^{-m_2}(\nabla u) \end{gathered} \tag{97}$$

for some $i = 0,1 : k_i, \ell_i, m_i \in \mathbb{N}_0$, with $k_i + \ell_i \ge 2$ and $\ell_2, m_i \in \mathbb{N}_0$, with $\ell_2 \ge 1$.

Assume that $p \in (2,\infty)$ such that $p > k_0$ and $p > 2(k_1 - 1)$. Let $\partial\Omega \in C^3$ and $\varphi \in W^{2,p}(\Omega)$. Additionally, we suppose that $\|\varphi\|_{W^{2,p}(\Omega)} < K$ for some $K > 0$.

Then there exists a constant $\mu = \mu(K,p,\Omega) > 0$ such that if $\|\varphi\|_{W^{2,1}(\Omega)} < \mu$, then there exists a weak solution $u \in W^{2,p}(\Omega)$ to the generalized Willmore-type Dirichlet problem, thus u solves (96) *in the class $\big\{v \in W^{2,p}(\Omega) \,\big|\, v - \varphi \in \mathring{W}^{2,p}(\Omega)\big\}$ with the right-hand side* (97).

Proof: We only consider the case $k_i > 0$ for all $i = 1,2,3$. Thus each $f_i[u]$ contains at least one D^2u. The other case is an easy matter to check.

Since we are familiar with the proof of Theorem 25, we only highlight important changes to be made. First, we change the iteration map $G\colon W^{2,p}(\Omega) \to W^{2,p}(\Omega)$. Let $G(u)$ be defined as the solution w

$$\Delta^2 w = f_0[u] + D_i f_1^i[u] + D_{ij}^2 f_2^{ij}[u] \quad \text{in } \Omega. \tag{98}$$

in the class $\big\{v \in W^{2,p}(\Omega) \,\big|\, v - \varphi \in \mathring{W}^{2,p}(\Omega)\big\}$. Let $q \in (2,p)$ such that $q > k_0$ and $q > 2(k_1 - 1)$. Furthermore, suppose δ and $\mathcal{M}_\delta^K$ play the same role as in the proof of Theorem 25 only with one difference that we consider the same $\varnothing \ne \mathcal{M}_\delta^K$ with

$$\mathcal{M}_\delta^K = \big\{u \in W^{2,p}(\Omega) \,\big|\, \|u\|_{W^{2,q}(\Omega)} \le 2C_{77}(q,\Omega)\delta,\ \|u\|_{W^{2,p}(\Omega)} \le 2C_{77}(p,\Omega)K\big\}$$

as a closed subset of $W^{2,q}(\Omega)$ instead of $W^{2,p}(\Omega)$ (by Fatou's Lemma), which is one of the main tricks in this proof. This will prove useful in showing the property contraction. We will also use the interpolation result (93) for boundary data φ.

In order to prove that $G\colon \mathcal{M}_\delta^K \to \mathcal{M}_\delta^K$ is a self map, we choose the first condition on δ like in (C1) in the proof of Theorem 25 such that

$$\forall\, u \in \mathcal{M}_\delta^K: \qquad \|u\|_{C^1(\overline{\Omega})} < 1. \tag{99}$$

In order to apply Lemma 24 with the $W^{2,p}(\Omega)$-estimate of Proposition 23 we choose some $t_0 \in \left(1, \frac{p}{k_0}\right)$ and $t_1 \in \left(\frac{2p}{2+p}, \frac{p}{k_1}\right)$ since $\frac{2p}{2+p} < \frac{2p}{2+2(k_1-1)} = \frac{p}{k_1}$ by assumptions imposed on p. Let $w \in \mathcal{M}_\delta^K$. Then it follows by interpolation with some $\alpha_0, \alpha_1, \alpha_2 \in (0,1)$

$$\begin{aligned}
\|Gw\|_{W^{2,p}(\Omega)} &\le C_{77}(p,\Omega)\|\varphi\|_{W^{2,p}(\Omega)} + C_{78}(p,\Omega)\left(\big\|f_0[w]\big\|_{L^{t_0}(\Omega)} + \big\|f_1[w]\big\|_{L^{t_1}(\Omega)} + \big\|f_2[w]\big\|_{L^p(\Omega)}\right)\\
&\le C_{77}(p,\Omega)K + C_{101}\begin{pmatrix} \big\|D^2 w\big\|_{L^{k_0 t_0}(\Omega)}^{k_0}\big\|\nabla w\big\|_{C^0(\overline{\Omega})}^{\ell_0} + \big\|D^2 w\big\|_{L^{k_1 t_1}(\Omega)}^{k_1}\big\|\nabla w\big\|_{C^0(\overline{\Omega})}^{\ell_1}\\ +\ \big\|D^2 w\big\|_{L^p(\Omega)}\big\|\nabla w\big\|_{C^0(\overline{\Omega})}^{\ell_2}\end{pmatrix}\\
&\le C_{77}(p,\Omega)K + C_{102}\begin{pmatrix} \|w\|_{W^{2,1}(\Omega)}^{k_0\alpha_0}\|w\|_{W^{2,p}(\Omega)}^{k_0(1-\alpha_0)}\delta^{\ell_0} + \|w\|_{W^{2,1}(\Omega)}^{k_1\alpha_1}\|w\|_{W^{2,p}(\Omega)}^{k_1(1-\alpha_1)}\delta^{\ell_0}\\ + \|w\|_{W^{2,p}(\Omega)}\delta^{\ell_2}\end{pmatrix}\\
&\le C_{77}(p,\Omega)K + C_{103}(p,\Omega)\left(\delta^{k_0\alpha_0+\ell_0}K^{k_0(1-\alpha_0)} + \delta^{k_1\alpha_1+\ell_1}K^{k_1(1-\alpha_1)} + K\delta^{\ell_2}\right).
\end{aligned}$$

From here on, we consider only δ satisfying

$$C_{103}(p,\Omega)\left(\delta^{k_0\alpha_0+\ell_0}K^{k_0(1-\alpha_0)} + \delta^{k_1\alpha_1+\ell_1}K^{k_1(1-\alpha_1)} + K\delta^{\ell_2}\right) \le C_{77}(p,\Omega)K. \tag{C2}$$

Thus, we obtain $\|Gw\|_{W^{2,p}(\Omega)} \le 2C_{77}(p,\Omega)K$. To estimate the $W^{2,q}(\Omega)$-norm we observe, that $\frac{2q}{2+q} < \frac{q}{k_1}$ and $1 < \frac{q}{k_0}$. It follows by Proposition 23 that,

$$\begin{aligned}
\|Gw\|_{W^{2,q}(\Omega)} &\le C_{77}(q,\Omega)\|\varphi\|_{W^{2,q}(\Omega)} + C_{78}(q,\Omega)\left(\big\|f_0[w]\big\|_{L^{q/k_0}(\Omega)} + \big\|f_1[w]\big\|_{L^{q/k_1}(\Omega)} + \big\|f_2[w]\big\|_{L^q(\Omega)}\right)\\
&\le C_{77}(q,\Omega)\delta + C_{104}\begin{pmatrix} \big\|D^2 w\big\|_{L^q(\Omega)}^{k_0}\big\|\nabla w\big\|_{C^0(\overline{\Omega})}^{\ell_0} + \big\|D^2 w\big\|_{L^q(\Omega)}^{k_1}\big\|\nabla w\big\|_{C^0(\overline{\Omega})}^{\ell_1}\\ +\ \big\|D^2 w\big\|_{L^q(\Omega)}\big\|\nabla w\big\|_{C^0(\overline{\Omega})}^{\ell_2}\end{pmatrix}\\
&\le C_{77}(q,\Omega)\delta + C_{105}(\Omega,q)\left(\delta^{k_0+\ell_0} + \delta^{k_1+\ell_1} + \delta^{1+\ell_2}\right).
\end{aligned}$$

This means that we have only to consider δ such that

$$C_{105}(\Omega,q)\left(\delta^{k_0+\ell_0-1} + \delta^{k_1+\ell_1-1} + \delta^{\ell_2}\right) \le C_{77}(q,\Omega). \tag{C3}$$

So that we get $\|Gw\|_{W^{2,q}(\Omega)} \le 2C_{77}(q,\Omega)\delta$. With all constraints imposed on δ above, the map $G\colon \mathcal{M}_\delta^K \to \mathcal{M}_\delta^K$ is a self-map.

To check the contraction property, suppose $u, w \in \mathcal{M}_\delta^K$ and observe that $Gu - Gw$ is a solution of the modified problem

$$\Delta^2(Gu - Gw) = \big(f_0[u] - f_0[w]\big) + D_i\big(f_1^i[u] - f_1^i[w]\big) + D_{ij}^2\big(f_2^{ij}[u] - f_2^{ij}[w]\big) \quad \text{in } \Omega, \tag{100}$$

in the class $W_0^{2,q}(\Omega)$. Note here that we have chosen q instead of p. In the same way as in the previous step, we use Proposition 23 and get

$$\|Gu - Gw\|_{W^{2,q}(\Omega)} \le C_{78}(q,\Omega)\begin{pmatrix} \big\|f_0[u] - f_0[w]\big\|_{L^{q/k_0}(\Omega)} + \big\|f_1[u] - f_1[w]\big\|_{L^{q/k_1}(\Omega)}\\ + \|f_2[u] - f_2[w]\|_{L^q(\Omega)}\end{pmatrix}.$$

Furthermore, we estimate the previous inequality piecewise. In a similar way as in the results from Lemma 24 by using Sobolev embedding and Hölder inequality in the case $k_0, k_1 \geq 2$ we can show that

$$\begin{aligned}
\big\|f_0[u] - f_0[w]\big\|_{L^{q/k_0}(\Omega)} &\leq C_{106}(q,\Omega)\|u-w\|_{W^{2,q}(\Omega)}\Big(\|u\|^{k_0-1+\ell_0}_{W^{2,q}(\Omega)} + \|w\|^{k_0-1+\ell_0}_{W^{2,q}(\Omega)}\Big) \\
&\leq C_{107}\|u-w\|_{W^{2,q}(\Omega)}\delta^{k_0+\ell_0-1}, \\
\big\|f_1[u] - f_1[w]\big\|_{L^{q/k_1}(\Omega)} &\leq C_{108}(q,\Omega)\|u-w\|_{W^{2,q}(\Omega)}\Big(\|u\|^{k_1-1+\ell_1}_{W^{2,q}(\Omega)} + \|w\|^{k_1-1+\ell_1}_{W^{2,q}(\Omega)}\Big) \\
&\leq C_{107}\|u-w\|_{W^{2,q}(\Omega)}\delta^{k_1+\ell_1-1}, \\
\|f_2[u] - f_2[w]\|_{L^{q}(\Omega)} &\leq C_{109}(q,\Omega)\|u-w\|_{W^{2,q}(\Omega)}\Big(\|u\|^{\ell_2}_{W^{2,q}(\Omega)} + \|w\|^{\ell_2}_{W^{2,q}(\Omega)}\Big) \\
&\leq C_{107}\|u-w\|_{W^{2,q}(\Omega)}\delta^{\ell_2}.
\end{aligned}$$

By combining the estimates above we get for some constant $C_{110} = C_{110}(q,\Omega)$

$$\|Gu - Gw\|_{W^{2,q}(\Omega)} \leq C_{110}(\delta^{k_0+\ell_0-1} + \delta^{k_1+\ell_1-1} + \delta^{\ell_2})\|u-w\|_{W^{2,q\cdot}(\Omega)}$$

As the last constraint, we consider only δ satisfying

(C4) $$C_{110}(\delta^{k_0+\ell_0-1} + \delta^{k_1+\ell_1-1} + \delta^{\ell_2}) \leq \frac{1}{2}$$

which yields $\|Gu - Gw\|_{W^{2,q}(\Omega)} \leq \frac{1}{2}\|u-w\|_{W^{2,q}(\Omega)}$, therefore G is a contraction on $\mathcal{M}^K_\delta$.

With all δ's conditions (C1), (C2),(C3) and (C4) in mind and the fact that $\mathcal{M}^T_\delta$ is a closed subset of $W^{2,q}(\Omega)$ we obtain by the fixed point Theorem the existence of a unique fixed point $u \in \mathcal{M}^K_\delta$ such that $u = Gu$. Thus, $u \in W^{2,p}(\Omega)$ is a weak solution of the equation (96) in the class $\big\{v \in W^{2,p}(\Omega) \,\big|\, v - \varphi \in \mathring{W}^{2,p}(\Omega)\big\}$ with the right-hand side (97). As the last step, we again use the interpolation (93) to observe that we only need smallness in the $W^{2,1}(\Omega)$-norm of φ. $\square$

5.4 Weighted Sobolev Case

We can generalize the results from the previous subsection to the framework of weighted Sobolev spaces where some additional positive a.e. measurable functions are multiplied to the Lebesgue measure as *weights*. For our purposes, we choose as weights the powers of $d(x) := \operatorname{dist}(x, \partial\Omega)$, that means the distance function to the boundary. Such weights are not only the canonical choice, but they also have the advantage of requiring less regularity on boundary and Dirichlet boundary data if the power is positive. The reason is that while approaching the boundary, the distance function decreases, and therefore, the integral contributions will play less and less of a role.

In the beginning, we have to recall the definition of weighted Lebesgue, Sobolev, and Besov spaces. In this context, up to embedding theorems, we mostly follow the description given in [MS11] and [MMS10]. First, we define for each $1 \leq p \leq \infty$ and $\beta \in \mathbb{R}$ the *weighted Lebesgue space* $L^p(\Omega; d^\beta)$ as a set of all measurable functions u on Ω such that

$$\|u\|_{L^p(\Omega;d^\beta)} := \left(\int_\Omega |u(x)|^p d(x)^\beta \,\mathrm{d}x\right)^{1/p} < \infty.$$

In our applications, we want to consider various powers of $d(x)$, but not all values for powers are allowed. Especially we consider only

(pa) $$p \in (1,\infty), \qquad a \in \left(-\frac{1}{p}, 1 - \frac{1}{p}\right),$$

although we will mainly consider $a \geq 0$ cases. Then we can define the *weighted Sobolev space* $W_p^{m,a}(\Omega)$ as the space of real-valued functions $u \in L^p_{\text{loc}}(\Omega)$ with the property that $D^\alpha u \in L^p_{\text{loc}}(\Omega)$ for all $|\alpha| \leq m$, for which

$$\|u\|_{W_p^{m,a}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega; d^{ap})} \right)^{1/p} < \infty. \tag{101}$$

It is important to notice that in the case $\partial\Omega$ is a Lipschitz boundary and $a = 0$ we recover the unweighted Sobolev space $W^{m,p}(\Omega)$ and in the case $m = 0$ it becomes the weighted Lebesgue space $L^p(\Omega; d^{ap}) = W_p^{0,a}(\Omega)$. It is also known that in case Ω is a bounded Lipschitz domain $C^\infty(\overline{\Omega})$ is dense in $W_p^{m,a}(\Omega)$ like in the unweighted case

$$W_p^{m,a}(\Omega) = \big\{ \text{ closure of } C^\infty(\overline{\Omega}) \text{ in } W_p^{m,a}(\Omega) \big\}$$

with respect to the norm (101). For the proofs, we refer the reader for $a \geq 0$ to [Kuf80, p.55 Theorem 7.2] and for $a < 0$ to [Kuf80, p.119 Remarks 11.12 (iii)].

Further, we need the *homogeneous weighted space*, see [MMS10, Subsection 6.1], and its dual space defined by

$$\mathring{W}_p^{m,a}(\Omega) := \big\{ \text{ closure of } C_c^\infty(\Omega) \text{ in } W_p^{m,a}(\Omega) \big\}, \qquad W_{p'}^{-m,-a}(\Omega) := \Big(\mathring{W}_p^{m,a}(\Omega) \Big)^* \tag{102}$$

with *dual exponent* $p' = p/(p-1)$. By [Kuf80, p.18 Theorem 3.6] the space $W_p^{m,a}(\Omega)$ and both $\mathring{W}_p^{m,a}(\Omega), W_{p'}^{-m,-a}(\Omega)$ are separable Banach spaces.

Analogously to the unweighted case, for weighted Sobolev spaces there exists a variety of embedding theorems into spaces of continuous functions, Hölder functions, or into other weighted Sobolev spaces. Moreover, since we have more parameters, like powers of weights, even more combinations of weighted embeddings arise than for the unweighted case. For bounded Lipschitz domains the most work was done by Kufner, Brown, and Opic in [Kuf80], [OK90], [BO92] and [Bro98]. We only mention the embeddings we need in what follows.

27 Lemma
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. Let p and a satisfy the conditions (pa).

(a) *Then we have the continuous embedding*

$$\mathring{W}_p^{1,a}(\Omega) \hookrightarrow L^p(\Omega; d^{ap-p}).$$

(b) *Furthermore suppose $q \geq p$ with $\frac{n}{q} - \frac{n}{p} + 1 > 0$, $a \leq 0, \gamma \in \mathbb{R}$ satisfying*

$$\frac{\gamma + n}{q} - \frac{ap + n}{p} + 1 > 0, \qquad \textit{then} \qquad \mathring{W}_p^{1,a}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\gamma),$$

thus, this embedding is compact.

(c) *If $p > n$, $a \leq 1 - \frac{n}{p}$ and $0 < \lambda < 1 - a - \frac{n}{p}$ then it follows*

$$\mathring{W}_p^{1,a}(\Omega) \hookrightarrow\hookrightarrow C^\lambda(\overline{\Omega}),$$

thus this embedding is compact.

(d) *Assume $q \geq p$ with $\frac{n}{q} - \frac{n}{p} + 1 > 0, \gamma \in \mathbb{R}$ satisfying*

$$\frac{\gamma + n}{q} - \frac{n-1}{p} > 0, \qquad \textit{then} \qquad W_p^{1,a}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega; d^\gamma),$$

thus, this embedding is compact.

Proof: At the beginning, let us recall some notation used in the literature cited for the embedding theorem. We set for $\alpha, \beta \in \mathbb{R}$ a weighted Sobolev space with different power of weight for the first derivative

$$W^{1,p}(\Omega; d^\beta, d^\alpha) := \left\{ u \in L^p(\Omega; d^\beta) \;\middle|\; \|u\|_{W^{1,p}(\Omega;d^\beta,d^\alpha)} = \|u\|_{L^p(\Omega;d^\beta)} + \|Du\|_{L^p(\Omega;d^\alpha)} < \infty \right\}$$

with $W_0^{1,p}(\Omega; d^\beta, d^\alpha)$ the closure of $C_c^\infty(\Omega)$ under $\| \,.\, \|_{W^{1,p}(\Omega;d^\beta,d^\alpha)}$.

(a) Here we use [OK90, p.274 Theorem 19.10] with $\beta = ap, \alpha = ap - p$ and $q = p, \kappa = 1$ and get the continuous embedding

$$\mathring{W}_p^{1,a}(\Omega) = W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow L^p(\Omega; d^\alpha) = L^p(\Omega; d^{ap-p}).$$

(b) In the work of Opic and Kufner [OK90, p.275 Theorem 19.12] we set $\kappa = 1$, since $\Omega \in C^{0,1}$ has a Lipschitz boundary, and put $\beta = ap$ and $\alpha = \gamma$ thus $\beta \le 0$.

(c) The embeddings of the weighted Sobolev spaces into the spaces of continuous functions have been proved by R. C. Brown and B. Opic in [BO92]. To use these results, we set $A = \partial\Omega$ as a singular set and $\Omega_a = \Omega$ in [BO92] on pages 282 and 283. Then the condition (A1) in chapter 3 in [BO92] is satisfied with $r(t) = d(t)/2$ since

$$\forall t \in \Omega : \quad \overline{B}(t, d(t)/2) \neq \varnothing \quad \text{and also} \quad \bigcup_{t\in\Omega} B(t, d(t)/2) = \Omega.$$

For the Hölder embedding we want to show [BO92, p. 292 (5.8)-(5.9)] with definitions

$$C(\Omega; d^{\beta_0}) = \left\{ u \in C(\Omega) \;\middle|\; \sup_{s\in\Omega} \left(d^{\beta_1}(s)|u(s)| \right) < \infty \right\},$$

$$C^{0,\lambda}(\Omega; d^{\beta_0}, d^{\beta_1}, \{\mathcal{U}(t)\}) = \left\{ u \in C(\Omega; d^{\beta_0}) \;\middle|\; \sup_{s\in\mathcal{U}(t), s\neq t} \left(d^{\beta_1}(s) \frac{|u(s) - u(t)|}{|s-t|^\lambda} \right) < \infty \right\}$$

By setting $\{\mathcal{U}(t)\} = \{B(t, \frac{1}{2}d(t))\}$, $\beta_0 = 0 = \beta_1, \gamma = \alpha - p$ we obtain the inequality

$$0 = \beta_1 p > \gamma \left(1 - \frac{n}{p} - \lambda \right) + \alpha \left(\frac{n}{p} + \lambda \right) = ap - p + n + \lambda p$$

which is satisfied with $\lambda < 1 - a - \frac{n}{p}$. Hence the inequality [BO92, p. 292 (5.8)] is strict and by [BO92, p. 292 (5.9) and p.295 Remark 5.2] the embedding is compact

$$W^{1,p}(\Omega; d^{ap-p}, d^{ap}) = W^{1,p}(\Omega; d^{\alpha-p}, d^\alpha) \hookrightarrow\hookrightarrow C^\lambda \left(\Omega; 1, 1, \left\{ B\left(t, \tfrac{1}{2}d(t)\right) \right\} \right)$$

and especially

$$\mathring{W}_p^{1,a}(\Omega) \hookrightarrow\hookrightarrow C^\lambda(\overline{\Omega})$$

since $\overline{\Omega}$ is compact with Lipschitz boundary and $u = 0$ on $\partial\Omega$ for $u \in \mathring{W}_p^{1,a}(\Omega)$.

(d) Again, in the work of Opic and Kufner [OK90, p.275 (19.39) Theorem 19.11] we set $\kappa = 1$, since $\Omega \in C^{0,1}$ has a Lipschitz boundary, and put $\beta = ap$ and $\alpha = \gamma$. □

From these embedding Theorems, we can draw some conclusions. First, from Lemma 27 *(d)* for $q = p$ we get some simpler embedding

(103) $$\forall \gamma > -1 : \qquad W_p^{1,a}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega; d^\gamma),$$

and especially $W_p^{1,a}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$. Furthermore, by taking $u \equiv 1$ with $a = 0$, we observe that the distance to boundary function d satisfies $d^\gamma \in L^1(\Omega)$ for all $\gamma > -1$ while Ω has a Lipschitz boundary. Likewise, as it is pointed out in [MMS10, p.208 (7.1)] from (103) we can deduce that the following norm in

$$W_p^{m,a}(\Omega) \ni u \mapsto \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega;d^{ap})} + \|u\|_{L^p(\omega)}$$

where $\omega \neq \varnothing$ is an open domain with $\overline{\omega} \subset \Omega$, is equivalent to norm (101) on $W_p^{m,a}(\Omega)$. It is shown in Lemma 99 by using a weighted Poincare inequality proved in Lemma 98.

In a similar spirit, one can investigate norm equivalence on spaces $\mathring{W}^{m,a}(\Omega)$. By using some Hardy-type inequalities [OK90] Kufner has shown in [Kuf80, p. 91 Theorem 9.2] that the following norm lacking all lower-order terms

$$\mathring{W}_p^{m,a}(\Omega) \ni u \mapsto \left(\sum_{|\alpha|=m} \|D^\alpha u\|^p_{L^p(\Omega;d^{ap})} \right)^{1/p} \tag{104}$$

is equivalent to (101) on $\mathring{W}_p^{m,a}(\Omega)$. Furthermore, by using among other things the embedding theorem in Lemma 27 (a), one can obtain one more equivalent norm. Especially, by [Kuf80, p. 94 Theorem 9.7] the norm (101) is equivalent to

$$\mathring{W}_p^{m,a}(\Omega) \ni u \mapsto \left(\sum_{|\alpha|\le m} \|D^\alpha u(x)\|^p_{L^p(\Omega;d^{(a-m+|\alpha|)p})} \right)^{1/p}$$

on $\mathring{W}_p^{1,a}(\Omega)$. Moreover, the same Theorem [Kuf80, p. 94 Theorem 9.7] states that one can characterize $\mathring{W}_p^{m,a}(\Omega)$ not only as closure of $C_c^\infty(\Omega)$ but also as

$$\mathring{W}_p^{m,a}(\Omega) = \left\{ u \in W^{m,p}_{\mathrm{loc}}(\Omega) \;\middle|\; \sum_{|\alpha|\le m} \|D^\alpha u(x)\|^p_{L^p(\Omega;d^{(a-m+|\alpha|)p})} < \infty \right\}.$$

Regarding the trace theory of the weighted Sobolev spaces in Lipschitz domains, we use the well-developed results in [MMS10, p. 208 Chapter 7]. In fact, we will first define spaces for traces $\mathrm{Tr}[D^\alpha u]$ and then define spaces for Dirichlet data $\frac{\partial^k u}{\partial \nu^k} = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha \,\mathrm{Tr}[D^\alpha u]$ in the sense of (38) for higher-order boundary value problems. Initially, we need to recall a definition of the *Besov space on boundary* $B_p^s(\partial\Omega)$ with $p \in (1,\infty)$ and $s \in (0,1)$ depending on a

$$s := 1 - a - \frac{1}{p} \in (0,1), \tag{S}$$

consisting of functions $f \in L^p(\partial\Omega)$ satisfying the following condition

$$\|f\|_{B_p^s(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n-1+sp}} \,\mathrm{d}S_x \,\mathrm{d}S_y \right)^{\frac{1}{p}} < \infty$$

where by $\mathrm{d}S$ we denote the surface element on $\partial\Omega$. In fact, by [MS11, Lemma 1 p.39] $B_p^s(\partial\Omega)$ is a trace space of $W^{1,a}(\Omega)$ in the sense that for a satisfying (pa) and s defined by (S) the operator $\mathrm{Tr} : W_p^{1,a}(\Omega) \to B_p^s(\partial\Omega)$ is well-defined, bounded, onto, linear, and has the homogeneous space $\mathring{W}_p^{1,a}(\Omega)$ as its null space. Moreover, there exists an extension operator $\mathrm{E} : B_p^s(\partial\Omega) \to W_p^{1,a}(\Omega)$, which is also linear and continuous and satisfies $\mathrm{Tr} \circ \mathrm{E} = \mathrm{Id}$.

Now, we want to introduce *higher-order Besov spaces on the boundary* of a Lipschitz domain. Let $p \in (1,\infty)$, $s \in (0,1)$ and $m \in \mathbb{N}$ then

$$\dot{B}_p^{m-1+s}(\partial\Omega) := \text{ closure of } \left\{ (D^\alpha v|_{\partial\Omega})_{|\alpha|\le m-1} \,\middle|\, v \in C_c^\infty(\mathbb{R}^n) \right\} \text{ in } B_p^s(\partial\Omega).$$

The trace and extension theorem for $\dot{B}_p^{m-1+s}(\partial\Omega)$ and $W_p^{m,a}(\Omega)$ is proven in [MMS10, Proposition 7.3]. It states for parameters a, s like in (pa), (S) and $\mathrm{Tr} : W_p^{1,a}(\Omega) \to B_p^s(\partial\Omega)$ above, that there ist a higher-order trace operator $\mathrm{tr}_{m-1} : W_p^{m,a}(\Omega) \to \dot{B}_p^{m-1+s}(\partial\Omega)$ which is a well-defined, linear, bounded operator, onto setting $\mathrm{tr}_{m-1}\, u := \{\mathrm{Tr}[D^\alpha u]\}_{|\alpha|\le m-1}$. Moreover, $\mathring{W}_p^{m,a}(\Omega)$ is its null-space, and there exists a linear continuous extension operator $\mathrm{E} : \dot{B}_p^{m-1+s}(\partial\Omega) \to W_p^{m,a}(\Omega)$ such that $\mathrm{tr}_{m-1} \circ \mathrm{E} = \mathrm{Id}$.

Next, we have to discuss the space for the Dirichlet data, which differs from trace spaces like $B_p^{m-1+s}(\partial\Omega)$. So we understand the boundary data $\{g_k\}_{0\le k\le m-1}$ in (39) in the following sense [MMS10, p.229 Theorem 7.8 (7.107)]: there exists an array (vector-valued function)

$$\dot{f} = \{f_\alpha\}_{|\alpha|\le m-1} \in \dot{B}_p^{m-1+s}(\partial\Omega) \quad \text{ such that } \quad g_k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu^\alpha f_\alpha \text{ for each } 0 \le k \le m-1.$$

Then by $\dot{W}_p^{m-1+s}(\partial\Omega)$ we denote the set of families $\{g_k\}_{0\le k\le m-1}$. We also set the corresponding norm [MMS10, p.223 Theorem 7.8 (7.78)] by

$$\|g\|_{\dot{W}_p^{m-1+s}(\partial\Omega)} := \sum_{|\alpha|\le m-1} \|f_\alpha\|_{B_p^s(\partial\Omega)}.$$

There are corresponding trace and extension results for this space shown in [MMS10, p.223 Theorem 7.8]. Therefore, there is a well-defined, bounded $\mathrm{Tr}_{m-1} : W_p^{m,a}(\Omega) \ni u \mapsto \left\{\frac{\partial^k u}{\partial \nu^k}\right\}_{0\le k\le m-1} \in \dot{W}_p^{m-1+s}(\partial\Omega)$ with right-inverse $\mathrm{Ext}_{m-1} : \dot{W}_p^{m-1+s}(\partial\Omega) \to W_p^{m,a}(\Omega)$ that is also well-defined and bounded linear operator that $\mathrm{Tr}_{m-1} \circ \mathrm{Ext}_{m-1} = \mathrm{Id}$. In this context, the most important fact for us is that the null-space of Tr_{m-1} consists precisely of functions in $\mathring{W}_p^{m,a}(\Omega)$. Hence

$$\mathring{W}_p^{m,a}(\Omega) = \left\{ u \in W_p^{m,a}(\Omega) \,\middle|\, \frac{\partial^k u}{\partial \nu^k} = 0 \text{ on } \Omega \text{ for } 0 \le k \le m-1 \right\}. \tag{105}$$

By [MMS10, corollary 7.11] the space $\dot{W}_p^{1+s}(\partial\Omega)$, which we only need to in our case $m = 2$, has actually a rather simple form. For its description, we introduce the *tangential derivative*

$$\frac{\partial}{\partial \tau_{jk}} := \nu_j \frac{\partial}{\partial x_k} - \nu_k \frac{\partial}{\partial x_j}, \quad 1 \le j,k \le n.$$

Then the *tangential gradient* on the surface $\partial\Omega$ is given by

$$\nabla_{\tan} := \left(\sum_{j=1}^n \nu_j \frac{\partial}{\partial \tau_{jk}} \right)_{1\le k\le n} = \nabla - \frac{\partial}{\partial \nu}. \tag{106}$$

This enables us to define Sobolev spaces of order one on $\partial\Omega$ that is important for the case $m = 2$, which we will use later. Let $\varphi \colon I \to \partial\Omega$ parametrization by arclength, then for $1 < p < \infty$ we define

$$L_p^1(\partial\Omega) := \left\{ f \circ \varphi \in W^{1,p}(I) \,\middle|\, \|f\|_{L_p^1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \|\nabla_{\tan} f\|_{L^p(\partial\Omega)} < \infty \right\}.$$

Finally, by [MMS10, p. 232 Corollary 7.11] the space $\dot{W}_p^{1+s}(\partial\Omega)$ takes the shape

$$\dot{W}_p^{1+s}(\partial\Omega) = \left\{(g_0, g_1) \in L_p^1(\partial\Omega) \oplus L^p(\partial\Omega) \;\middle|\; \nu g_1 + \nabla_{\tan} g_0 \in B_p^s(\partial\Omega)\right\} \tag{107}$$

with the corresponding norm

$$\|g\|_{\dot{W}_p^{1+s}(\partial\Omega)} := \|g_0\|_{B_p^s(\partial\Omega)} + \|\nu g_1 + \nabla_{\tan} g_0\|_{B_p^s(\partial\Omega)}$$

where $p \in (1, \infty)$ and $s \in (0, 1)$.

Since we have defined weighted Sobolev spaces and spaces for Dirichlet data, let us revisit the inhomogeneous Dirichlet problem (39) for an elliptic operator A

$$\begin{cases} \displaystyle\sum_{|\alpha|=|\beta|=m} D^\alpha\big(A_{\alpha\beta}(x) D^\beta u\big) = \mathcal{F} \quad \text{for } x \in \Omega, \\ \dfrac{\partial^k u}{\partial \nu^k} = g_k \quad \text{on } \partial\Omega, \quad 0 \le k \le m-1. \end{cases} \tag{inD}$$

with right-hand side $\mathcal{F} \in W_p^{-m,a}(\Omega)$, and boundary data $g := \{g_k\}_{0 \le k \le m-q} \in \dot{W}_p^{m-1+s}(\partial\Omega)$. Solvability and uniqueness for (inD) in the weighted Sobolev-Besov setting has been shown in [MMS10, p. 169 Theorem 1.1] under some further assumptions on the Lipschitz bounded Ω and the elliptic bounded coefficients $A_{\alpha\beta}$. It can be found in Proposition 28. In order to formulate the last condition, we have to define the *BMO modulo VMO character of a function* $f \in L^1(\Omega)$ by the quantity

$$\{f\}_{*,\Omega} := \lim_{\varepsilon \to 0} \left(\sup_{t \in \Omega} \fint_{B_\varepsilon^n(t) \cap \Omega} \fint_{B_\varepsilon^n(t) \cap \Omega} \big|f(x) - f(y)\big| \,\mathrm{d}x \,\mathrm{d}y \right)$$

where $B_\varepsilon^n(t)$ stands for n-dimensional open ball with the center t and radius ε. Similarly, we define

$$\{f\}_{*,\partial\Omega} := \lim_{\varepsilon \to 0} \left(\sup_{t \in \partial\Omega} \fint_{B_\varepsilon^n(t) \cap \partial\Omega} \fint_{B_\varepsilon^n(t) \cap \partial\Omega} \big|f(x) - f(y)\big| \,\mathrm{d}S_x \,\mathrm{d}S_y \right).$$

Now, we can state the existence and regularity result in the weighted Sobolev setting.

28 Proposition
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with the exterior normal vector ν and $A_{\alpha\beta}, |\alpha| = |\beta| = m$ bounded measurable coefficients of an elliptic operator L in the sense (36). *Further, suppose $p \in (1, \infty)$ and $s \in (0, 1)$ with $a := 1 - s - 1/p$ according to* (pa) *as well as $\mathcal{F} \in W_p^{-m,a}(\Omega)$, and Dirichlet boundary data $g := \{g_k\}_{0 \le k \le m-q} \in \dot{W}_p^{m-1+s}(\partial\Omega)$. Then there exist a constant $C_{111} > 0$ depending only on Ω and the ellipticity constant of L such that if*

$$\{\nu\}_{*,\partial\Omega} + \sum_{|\alpha|=|\beta|=m} \{A_{\alpha\beta}\}_{*,\Omega} \le C_{111}\, s(1-s)\Big(p^2(p-1)^{-1} + s^{-1}(1-s)^{-1}\Big)^{-1} \tag{nuA}$$

then there exists a unique solution $u \in W_p^{m,a}(\Omega)$ to the inhomogeneous Dirichlet problem (inD). *Moreover, there exists a constant $C_{112} = C_{112}(\partial\Omega, A_{\alpha\beta}, p, s)$ such that*

$$\|u\|_{W_p^{m,a}(\Omega)} \le C_{112} \left(\|g\|_{\dot{W}_p^{m-1+s}(\partial\Omega)} + \|\mathcal{F}\|_{W_p^{-m,a}(\Omega)} \right).$$

Proof: Theorem 8.1. on page 233 in [MMS10]. □

It is possible to replace the technical condition (nuA) with some directly involving BMO and VMO spaces. Which stand, respectively, for the space of *functions of bounded mean oscillations* and its subspace of *functions of vanishing mean oscillations* considered either on $\partial\Omega$ for ν or on Ω for $A_{\alpha\beta}$. More precisely, we call $f \in L^1_{\mathrm{loc}}(\Omega)$ a $\mathrm{BMO}(\Omega)$-function if

$$\|f\|_* := \sup_{t\in\Omega} \fint_{B^n_\varepsilon(t)\cap\Omega} \left| f(x) - \fint_{B^n_\varepsilon(t)\cap\Omega} f(y)\,\mathrm{d}y \right| \mathrm{d}x < \infty \tag{108}$$

which becomes a norm on this space after dividing out constant functions. We define the space $\mathrm{VMO}(\Omega)$ consisting of functions $f \in \mathrm{BMO}(\Omega)$ satisfying

$$\lim_{\varepsilon\to 0} \left(\sup_{t\in\Omega} \fint_{B^n_\varepsilon(t)\cap\Omega} \left| f(x) - \fint_{B^n_\varepsilon(t)\cap\Omega} f(y)\,\mathrm{d}y \right| \mathrm{d}x \right) = 0.$$

This space is discussed in detail by [Sar75]. The spaces $\mathrm{BMO}(\partial\Omega)$ and $\mathrm{VMO}(\partial\Omega)$ are defined in the same way by replacing in the formula above Ω with $\partial\Omega$. One of the basic properties of $\mathrm{VMO}(\Omega)$ or $\mathrm{VMO}(\partial\Omega)$ is that it includes uniformly continuous functions that are bounded in Ω or $\partial\Omega$.

If we now denote by $\operatorname{dist}(f, \mathrm{VMO}(\Omega))$ for an arbitrary function $f \in \mathrm{BMO}(\Omega)$ its distance to $\mathrm{VMO}(\Omega)$ measured in the BMO norm $\|\,.\|_*$ then it follows the equivalence $\operatorname{dist}(f, \mathrm{VMO}(\Omega)) \sim \{f\}_{*,\Omega}$. With the same results for $\partial\Omega$ we can reformulate the condition (nuA) as

$$\operatorname{dist}(\nu, \mathrm{VMO}(\partial\Omega)) + \sum_{|\alpha|=|\beta|=m} \operatorname{dist}(A_{\alpha\beta}, \mathrm{VMO}(\Omega)) \le C\, s(1-s)\Big(p^2(p-1) + s^{-1}(1-s)^{-1}\Big)^{-1}.$$

Consequently, if $\nu \in \mathrm{VMO}(\partial\Omega)$ and $A_{\alpha\beta} \in \mathrm{VMO}(\Omega)$ for $|\alpha| = |\beta| = m$ the condition (nuA) is fulfilled for all values $p \in (1,\infty)$, $s \in (0,1)$. This is especially the case when $\partial\Omega \in C^1$ and $A_{\alpha\beta} \in C^0(\overline{\Omega})$ for $|\alpha| = |\beta| = m$. The same inequality (nuA) is also valid if $\|\nu\|_* + \sum_{|\alpha|=|\beta|=m} \|A_{\alpha\beta}\|_*$ is small enough. Regarding the regularity of the boundary, we notice that depending on p, s one can achieve $\|\nu\|_*$ small enough by allowing only a sufficiently small Lipschitz constant for the boundary. As noticed in [MS11, p.43], some examples with $\|\nu\|_*$ small enough are Lipschitz graph polyhedral domains with dihedral angles chosen sufficiently close to π, depending on p and s. It means that the Lipschitz boundary makes only small kinks.

In the case of the Lipschitz class boundary, there is also a generalization of Miranda's maximum modulus Theorem proved by Pipher and Verchota in [PV93].

29 Proposition (Pipher, Verchota)
Suppose $\Omega \subset \mathbb{R}^n$, $n = 2,3$ is a bounded Lipschitz graph domain. Moreover, let u be the unique L^2-solution of (35) *for $\Delta^2 u = 0$ with the given Dirichlet boundary data $g_0 \in L^1_2(\partial\Omega)$, $g_1 \in L^2(\partial\Omega)$ and $|\nabla u| \in L^\infty(\partial\Omega)$. Then*

$$\sup_{x\in\Omega} \big|\nabla u(x)\big| \le C_{113} \|\nabla u\|_{L^\infty(\partial\Omega)}$$

with C_{113} depending only on the Lipschitz structure of $\partial\Omega$.

Proof: [PV93, p. 387 Theorem 1.2]. □

Like in the unweighted Sobolev setting, we need some preliminary estimates. The following one concerns a general biharmonic problem with a divergence right-hand side. Furthermore, in order to be able to use weighted embedding result $\mathring{W}^{2,a}_p(\Omega) \hookrightarrow\hookrightarrow C^1(\overline{\Omega})$ by Lemma 27 ⓒ, we have to restrain the distance weight powers to $a < 1 - \frac{2}{p}$.

30 Proposition
Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with exterior normal vector ν satisfying (nuA). *Furthermore, assume that $p > 2$, $p \geq t > \frac{2p}{2+p-ap}$, $0 < a < 1 - \frac{2}{p}$ and $h_1 \in L^t(\Omega; d^{2at})$, $h_2 \in L^p(\Omega; d^{ap})$ as well as $g := \{g_0, g_1\} \in \dot{W}^{1+s}_p(\partial\Omega)$ such that $g_0, \nu g_1 + \nabla_{\tan} g_0 \in L^\infty(\partial\Omega)$ with $s := 1 - a - \frac{1}{p}$. Then the following Dirichlet problem*

$$
\begin{cases} \Delta^2 u = D_i h_1^i + D^2_{ij} h_2^{ij} & \text{in } \Omega, \\ \dfrac{\partial^k u}{\partial \nu^k} = g_k & \text{on } \partial\Omega, \quad 0 \leq k \leq 1, \end{cases} \tag{109}
$$

admits a unique weak solution $u \in W^{2,a}_p(\Omega) \cap C^1(\overline{\Omega})$. Moreover, there exist constants $C_{114} = C_{114}(p, t, a, \partial\Omega)$, $C_{115} = C_{115}(a, \partial\Omega)$ such that

$$
\begin{aligned}
\|u\|_{W^{2,a}_p(\Omega)} &\leq C_{112} \|g\|_{\dot{W}^{1+s}_p(\partial\Omega)} + C_{114} \left(\|h_1\|_{L^t(\Omega; d^{2at})} + \|h_2\|_{L^p(\Omega; d^{ap})} \right), \\
\|D^2 u\|_{L^1(\Omega; d^a)} &\leq C_{115} \|\nu g_1 + \nabla_{\tan} g_0\|_{L^\infty(\partial\Omega)} + C_{114} \left(\|h_1\|_{L^t(\Omega; d^{2at})} + \|h_2\|_{L^p(\Omega; d^{ap})} \right), \\
\|\nabla u\|_{L^\infty(\Omega)} + &\leq C_{113} \|\nu g_1 + \nabla_{\tan} g_0\|_{L^\infty(\partial\Omega)} + C_{114} \left(\|h_1\|_{L^t(\Omega; d^{2at})} + \|h_2\|_{L^p(\Omega; d^{ap})} \right).
\end{aligned}
$$

Proof: Here, we want to incorporate the embedding lemma 27 and Proposition 28 where we set $n = 2$. In order to prove the $C^1(\overline{\Omega})$ estimate, we split (109) in two parts

$$
\begin{cases} \Delta^2 w = 0 & \text{in } \Omega, \\ \dfrac{\partial^k w}{\partial \nu^k} = g_k & \text{on } \partial\Omega, \quad 0 \leq k \leq 1, \end{cases} \qquad
\begin{cases} \Delta^2 v = D_i h_1^i + D^2_{ij} h_2^{ij} & \text{in } \Omega, \\ \dfrac{\partial^k v}{\partial \nu^k} = 0 & \text{on } \partial\Omega, \quad 0 \leq k \leq 1, \end{cases}
$$

1 We begin with considering v. In this situation, by definition (102) we have to check $D_i h_1^i + D^2_{ij} h_2^{ij} \in W^{-2,a}_p(\Omega) = \big(\mathring{W}^{2,-a}_{p'}(\Omega)\big)^*$. Therefore, let $\varphi \in C^\infty_c(\Omega)$ and $t' = t/(t-1)$ the dual exponent of t then we obtain

$$
\left| \int_\Omega \varphi D h_1 \, \mathrm{d}x \right| = \left| \int_\Omega D\varphi h_1 \, \mathrm{d}x \right| \leq \int_\Omega |D\varphi| d^{-2a} d^{2a} |h_1| \, \mathrm{d}x \leq \|D\varphi\|_{L^{t'}(\Omega; d^{-2at'})} \|h_1\|_{L^t(\Omega; d^{2at})}
$$

by Hölder inequality. Now, what we have to show is the following estimate

$$
\|D\varphi\|_{L^{t'}(\Omega; d^{-2at'})} \leq C_{116} \|D\varphi\|_{\mathring{W}^{1,-a}_{p'}(\Omega)}. \tag{110}
$$

Hence we have to check that the assumptions of Lemma 27 *b* are satisfied. First, we observe by putting $t' \to q$ and $p' \to p$ in Lemma 27 *b* (here one hast to check $q \geq p$, i.e. $t' \geq p'$) that we get the first assumption

$$
\frac{2}{t'} - \frac{2}{p'} + 1 = \frac{2}{p} - \frac{2}{t} + 1 > \frac{2}{p} - \frac{p+2}{p} + 1 = 0
$$

since $t > \frac{2p}{2+p-ap} \geq \frac{2p}{2+p}$. The second condition follows by additionally setting $\gamma \to -2at'$ in Lemma 27 *b*

$$
\frac{-2at'}{t'} - \frac{-ap'}{p'} + \frac{2}{p} - \frac{2}{t} + 1 \ = -a + \frac{2}{p} - \frac{2}{t} + 1 > 0.
$$

Therefore, we obtain the compact embedding $\mathring{W}^{1,-a}_{p'}(\Omega) \hookrightarrow\hookrightarrow L^{t'}(\Omega; d^{-2at'})$ and the estimate (110) follows. Next, we consider the h_2 part. Again with some $\varphi \in C^\infty_c(\Omega)$ we conclude

$$
\left| \int_\Omega \varphi D^2 h_1 \, \mathrm{d}x \right| = \left| \int_\Omega D^2\varphi h_1 \, \mathrm{d}x \right| \leq \int_\Omega |D^2\varphi| d^{-a} d^{a} |h_2| \, \mathrm{d}x \leq \|D^2\varphi\|_{L^{p'}(\Omega; d^{-ap'})} \|h_2\|_{L^p(\Omega; d^{ap})}.
$$

Combining two results, we get

$$\left|\int_\Omega \varphi(D_i h_1^i + D_{ij}^2 h_2^{ij})\,\mathrm{d}x\right| \le \|\varphi\|_{\mathring{W}_{p'}^{2,-a}(\Omega)} \cdot C_{117}\left(\|h_1\|_{L^t(\Omega;d^{2at})} + \|h_2\|_{L^p(\Omega;d^{ap})}\right). \tag{111}$$

Since $C_c^\infty(\Omega)$ is dense in $\mathring{W}_{p'}^{2,-a}(\Omega)$ the mapping

$$\mathring{W}_{p'}^{2,-a}(\Omega) \ni u \mapsto \int_\Omega u(D_i h_1^i + D_{ij}^2 h_2^{ij})\,\mathrm{d}x$$

lies in $W_p^{-2,a}(\Omega)$ with its norm bounded by a multiple of $\left(\|h_1\|_{L^t(\Omega;d^{2at})} + \|h_2\|_{L^p(\Omega;d^{ap})}\right)$. Finally, we use Proposition 28 and obtain existence, uniqueness of a solution $v \in W_p^{2,a}(\Omega)$ to the homogeneous Dirichlet problem as well as the $W_p^{2,a}(\Omega)$-a-priori estimate. Moreover, since $\frac{\partial^k v}{\partial \nu^k} = 0$ on $\partial\Omega$ for $0 \le k \le 1$ it follows by (105) that $v \in \mathring{W}_p^{2,a}(\Omega)$ and we get the following estimate

$$\|v\|_{\mathring{W}_p^{2,a}(\Omega)} \le C_{112}C_{118}\left(\|h_1\|_{L^t(\Omega;d^{2at})} + \|h_2\|_{L^p(\Omega;d^{ap})}\right).$$

By Hölder's inequality and boundedness of Ω, we also get

$$\|D^2 v\|_{L^1(\Omega;d^a)} \le C_{119}\left(\|h_1\|_{L^t(\Omega;d^{2at})} + \|h_2\|_{L^p(\Omega;d^{ap})}\right).$$

Further on, since $p > 2$ and $a < 1 - \frac{2}{p}$ we have the weighted embedding $\mathring{W}_p^{2,a}(\Omega) \hookrightarrow\hookrightarrow C^1(\overline{\Omega})$ by Lemma 27 c. Thus, we conclude

$$\|v\|_{C^1(\overline{\Omega})} \le C_{120}\left(\|h_1\|_{L^t(\Omega;d^{2at})} + \|h_2\|_{L^p(\Omega;d^{ap})}\right).$$

2 Now, let us turn to the biharmonic Dirichlet problem corresponding to w with inhomogeneous boundary data. Here, we again are making use of Proposition 28 and get existence and uniqueness in $W_p^{2,a}(\Omega)$ as well as

$$\|w\|_{W_p^{2,a}(\Omega)} \le C_{112}\|g\|_{\dot{W}_p^{1+s}(\partial\Omega)}.$$

The following L^∞-gradient estimate involves Proposition 29 (the Agmon-Miranda maximum modulus estimate) proved by Pipher and Verchota for Lipschitz domains. To be able to use this result, we have to relate Dirichlet boundary data $g \in \dot{W}_p^{1+s}(\partial\Omega)$ to the space $WA^{1,\infty}(\partial\Omega)$ (for notation see [PV93]). First, we observe, that by [MMS10, p.230 Corollary 7.10] there exists an array in non-Dirichlet trace space $f \in \dot{B}_p^{1+s}(\partial\Omega)$ such that

$$g_0 = f_0, \quad g_1 = \nu f_1 \quad \text{and} \quad f_1 = \nu g_1 + \nabla_{\tan} g_0$$

and by trace Theorems $\mathrm{Tr}(u) = g_0, \mathrm{Tr}(\nabla u) = f_1 = \nu g_1 + \nabla_{\tan} g_0$. Now since $g_0 \in L_2^1(\partial\Omega), g_1 \in L_2(\partial\Omega)$ then Proposition 29 yields

$$\|\nabla w\|_{L^\infty(\Omega)} \le C_{113}\|\nu g_1 + \nabla_{\tan} g_0\|_{L^\infty(\partial\Omega)}.$$

3 Next, we use the result that the maximal function dominates square functions in L^2 [DKPV97, p.1455 Theorem 2] and get

$$\int_\Omega |D^2 w(x)|^2 d(x)\,\mathrm{d}x \le C_{121}\int_{\partial\Omega} |N(\nabla w)|^2\,\mathrm{d}s \le C_{122}\|\nabla w\|_{L^\infty(\partial\Omega)}^2$$

where $N(\nabla u)$ is the non-tangential maximal function of a function ∇u on boundary.

$$\left|\int_\Omega |D^2 w(x)| d^{\frac{1}{2}}(x) d^{a-\frac{1}{2}}(x)\,\mathrm{d}x\right| \le C_{123}\|\nabla w\|_{L^\infty(\partial\Omega)} \cdot \int_\Omega d^{2a-1}\,\mathrm{d}x < \infty$$

since due to $a \ge 0$ we have $2a - 1 > -1$. Finally by combining the above results, the proof is complete. □

Similar to the unweighted case, in the following Lemma, we take some preparatory steps needed to applying a fixed point argument.

31 Lemma
Let $p \in (2,\infty), a \in \big(-\frac{1}{p}, 1-\frac{1}{p}\big)$ and $i,j,\ell,k \in \mathbb{N}_0, i \geq 1, k \geq 2$ then there exists a constant $C_{124} = C_{124}(\Omega,p,a)$ such that if $u \in W_p^{2,a}(\Omega)$ with $\|\nabla u\|_{L^\infty(\Omega)} \leq 1$ then it follows

$$\begin{aligned} \big\|D^2u \star D^2u \star Q^{-j}P_i(\nabla u)\big\|_{L^{p/2}(\Omega;d^{ap})} &\leq C_{124}\|D^2u\|^2_{L^p(\Omega;d^{ap})}\|\nabla u\|_{L^\infty(\Omega)}, \\ \big\|D^2u \star Q^{-j}(1+Q)^{-\ell}P_k(\nabla u)\big\|_{L^p(\Omega;d^{ap})} &\leq C_{124}\|D^2u\|_{L^p(\Omega;d^{ap})}\|\nabla u\|_{L^\infty(\Omega)}. \end{aligned} \tag{112}$$

Furthermore suppose $w \in W_p^{2,a}(\Omega)$ with $\|\nabla w\|_{L^\infty(\Omega)} \leq 1$ then it follows

$$\begin{aligned} &\Big\|D^2u\star D^2u \star Q^{-j}(u)P_i(\nabla u) \, - D^2w \star D^2w \star Q^{-j}(w)P_i(\nabla w)\Big\|_{L^{p/2}(\Omega;d^{ap})} \\ &\qquad \leq C_{124}\|u-w\|_{W_p^{2,a}(\Omega)}\Big(\|D^2u\|^2_{L^p(\Omega;d^{ap})} + \|D^2w\|^2_{L^p(\Omega;d^{ap})}\Big)) \end{aligned} \tag{113}$$

as well as

$$\begin{aligned} &\big\|D^2u \star Q^{-j}(1+Q)^{-\ell}(u)P_k(\nabla u) - D^2w \star Q^{-j}(1+Q)^{-\ell}(w)P_k(\nabla w)\big\|_{L^p(\Omega;d^{ap})} \\ &\qquad \leq C_{124}\|u-w\|_{W_p^{2,a}(\Omega)}\left(\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)}\right)\Big(\|u\|_{W_p^{2,a}(\Omega)} + \|w\|_{W_p^{2,a}(\Omega)}\Big). \end{aligned} \tag{114}$$

Proof: In the same way as in Lemma 24. □

We are now equipped to prove the existence of the weighted $W_a^{2,p}(\Omega)$ solution for the rewritten Willmore equation

$$\begin{cases} \Delta^2 u = D_i b_1^i[u] + D_{ij}^2 b_2^{ij}[u] & \text{for } x \in \Omega, \\ \dfrac{\partial^k u}{\partial \nu^k} = g_k \quad \text{on } \partial\Omega, \quad 0 \leq k \leq 1, \end{cases} \tag{115}$$

with the right-hand side given by (60) and some boundary conditions $g := \{g_0, g_1\}$ lying in the Dirichlet boundary space $\dot{W}_p^{1+s}(\partial\Omega)$ characterized by (107)

$$\dot{W}_p^{1+s}(\partial\Omega) = \big\{(g_0,g_1) \in L_p^1(\partial\Omega) \oplus L^p(\partial\Omega) \;\big|\; \nu g_1 + \nabla_{\tan} g_0 \in B_p^s(\partial\Omega)\big\}$$

with boundary functions space

$$L_p^1(\partial\Omega) := \big\{f \circ \varphi \in W^{1,p}(I) \;\big|\; \|f\|_{L_p^1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \|\nabla_{\tan} f\|_{L^p(\partial\Omega)} < \infty\big\}$$

and tangent gradient (106) $\nabla_{\tan} = \nabla - \frac{\partial}{\partial\nu}$. Moreover the term $\nu g_1 + \nabla_{\tan} g_0$ will play the role of ∇u at the boundary. Subsequently, in Corollary 33, it will be shown that boundary condition data belonging to the $C^{1+\alpha}$ class can be accommodated within the more general space $\dot{W}_p^{1+s}(\partial\Omega)$.

32 Theorem
Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain with exterior normal vector ν satisfying (nuA). *Furthermore, assume that $p \in (2,\infty), a \in \Big(0, 1-\frac{2}{p}\Big)$, as well as $g := \{g_0,g_1\} \in \dot{W}_p^{1+s}(\partial\Omega)$ such that $g_0, \nu g_1 + \nabla_{\tan} g_0 \in L^\infty(\partial\Omega)$ with $s := 1-a-\frac{1}{p}$. Additionally, we suppose that $\|g\|_{\dot{W}_p^{1+s}(\partial\Omega)} < K$ for some $K > 0$.*
Then there exists a constant $\delta = \delta(\Omega,K,p,a) > 0$ such that if

$$\|\nu g_1 + \nabla_{\tan} g_0\|_{L^\infty(\Omega)} < \delta$$

then there exists a weak solution $u \in W_p^{2,a}(\Omega)$ to the Willmore-type Dirichlet problem, thus u solves (115) *with the right-hand side* (60).

Proof: We use the usual steps to apply a fixed point argument.

(1) Definition of the iteration map & set

We define the iteration map G by setting Gv for each $v \in W_p^{2,a}(\Omega)$ as the solution $w \in W_p^{2,a}(\Omega)$ to the boundary problem

$$\begin{cases} \Delta^2 w = D_i b_1^i[v] + D_{ij}^2 b_2^{ij}[v] & \text{in } \Omega, \\ \dfrac{\partial^k u}{\partial \nu^k} = g_k \quad \text{on } \partial\Omega, & 0 \leq k \leq 1. \end{cases} \tag{116}$$

Existence, regularity, and uniqueness are provided by Proposition 28.

Since $a < 1 - \frac{2}{p}$, we set $q \in \left(\frac{2}{1-a}, p\right)$ by $q := \frac{1}{2}\left(\frac{2}{1-a} + p\right)$ Then we observe that by $q > \frac{2}{1-a} > 2$ it follows $a < 1 - \frac{2}{q}$ and thus the space $W_q^{2,a}(\Omega)$ (with the same a as in $W_p^{2,a}(\Omega)$!) is also well-defined. Moreover, since Ω is bounded $W_p^{2,a}(\Omega) \hookrightarrow\hookrightarrow W_q^{2,a}(\Omega)$ by Hölder's inequality. Further on, we define the iteration set

$$\mathcal{M}_\delta^K := \left\{ u \in W_p^{2,a}(\Omega) \;\middle|\; \begin{array}{r} \|\nabla u\|_{L^\infty(\Omega)} \leq 2C_{113}\delta, \; \|D^2 u\|_{L^1(\Omega;d^a)} \leq 2C_{115}(a,\Omega)\delta, \\ \|u\|_{W_p^{2,a}(\Omega)} \leq 2C_{112}(p,a,\Omega)K \end{array} \right\}$$

with $\delta > 0$ some constant, which will be specified by several conditions that we will state in the following.

First we observe, that there exists a power $\gamma := \frac{p}{q}\frac{q-1}{p-1} \in (0,1)$ such that we can interpolate

$$\|D^2 u\|_{L^q(\Omega;d^{aq})} \leq \|D^2 u\|_{L^1(\Omega;d^a)}^{1-\gamma} \|D^2 u\|_{L^p(\Omega;d^{ap})}^{\gamma}. \tag{117}$$

We formulate the first condition by

$$2C_{113}\delta_1 \leq 1, \tag{WC1}$$

hence for all $\delta \leq \delta_1$ we get $\|\nabla u\|_{L^\infty\Omega)} \leq 1$ for all $u \in \mathcal{M}_\delta^K$ thus we can use the results in Lemma 31.

(2) G is a self-map

Suppose $w \in \mathcal{M}_\delta^K$. In order to prepare us for the application of Proposition 30, we need to discuss the possible t values for the b_1 term. Actually, we want to justify that we can put $t = \frac{p}{2}$. Therefore, while $0 < a \leq 1 - \frac{2}{p}$ it follows $1 < \frac{p(1-a)}{2}$. We deduce

$$1 < \frac{2p}{2 - ap + p} = \frac{p}{1 + \frac{p(1-a)}{2}} < \frac{p}{2} = t. \tag{118}$$

By Proposition 30 and $\|w\|_{L^\infty(\Omega)} \leq 1$ as well as the condition (WC1) for Lemma 31 we conclude

$$\begin{aligned} \|Gw\|_{W_p^{2,a}(\Omega)} &\leq C_{112}\|g\|_{\dot{W}_p^{1+s}(\partial\Omega)} + C_{114}\left(\|b_1[w]\|_{L^{p/2}(\Omega;d^{ap})} + \|b_2[w]\|_{L^p(\Omega;d^{ap})}\right) \\ &\leq C_{112}(p,a,\Omega)K + C_{125}\left(\left\|D^2 w\right\|_{L^p(\Omega;d^{ap})}^2 \|\nabla w\|_{L^\infty(\Omega)} + \left\|D^2 w\right\|_{L^p(\Omega;d^{ap})} \|\nabla w\|_{L^\infty(\Omega)}\right) \\ &\leq C_{112}(p,a,\Omega)K + C_{126}(\Omega,p,a)(K^2\delta + K\delta). \end{aligned}$$

Thus, we impose the second constraint on δ by choosing δ_2 such that

$$C_{126}(\Omega,p)(\delta_2 K^2 + K\delta_2) \leq C_{112}(p,a,\Omega)K, \tag{WC2}$$

and from here on consider only $\delta \in (0,\delta_2)$. It follows that $\|Gw\|_{W_p^{2,a}(\Omega)} \leq 2C_{112}(p,a,\Omega)K$.

The corresponding $W_q^{2,a}(\Omega)$-estimate is similar. Hence, we conclude

$$\|D^2 Gw\|_{L^1(\Omega;d^a)} \leq C_{115}(a,\Omega)\delta + C_{127}\left(\left\|D^2 w\right\|_{L^q(\Omega;d^{aq})}^2 \|\nabla w\|_{L^\infty(\Omega)} + \left\|D^2 w\right\|_{L^q(\Omega;d^{aq})} \|\nabla w\|_{L^\infty(\Omega)}\right)$$

$$\leq C_{115}(a,\Omega)\delta + C_{128}(\Omega,a)((\delta^{1-\gamma}K^\gamma)^2 + \delta^{1-\gamma}K^\gamma)\delta.$$

We now state the third constraint by choosing δ_3 by

$$\text{(WC3)} \qquad C_{128}(\Omega,a)(\delta_3^{1-\gamma}K^{\gamma^2} + \delta_3^{1-\gamma}K^\gamma) \leq C_{112}(q,a,\Omega).$$

It remains to consider the gradient estimate. Again, by Proposition 30 and Lemma 31 it follows

$$\begin{aligned}\|\nabla(Gw)\|_{L^\infty(\Omega)} &\leq C_{113}\|\nu g_1 + \nabla_{\tan} g_0\|_{L^\infty(\partial\Omega)} + C_{114}\left(\|b_1[w]\|_{L^{q/2}(\Omega;d^{aq})} + \|b_2[w]\|_{L^q(\Omega;d^{aq})}\right)\\ &\leq C_{113}\delta + C_{128}(\Omega,q,a)((\delta^{1-\gamma}K^\gamma)^2 + \delta^{1-\gamma}K^\gamma)\delta.\end{aligned}$$

Therefore, we set the fourth condition by

$$\text{(WC4)} \qquad C_{128}(\Omega,q,a)((\delta_4^{1-\gamma}K^\gamma)^2 + \delta_4^{1-\gamma}K^\gamma) \leq C_{113}.$$

Combining the above results yields that the map $G\colon \mathcal{M}_\delta^K \to \mathcal{M}_\delta^K$ is a self map for all δ smaller than δ_1, δ_2, δ_3 and δ_4.

3 **G is a contraction**

The last property we want to verify is the contraction property, thus for all $u,w \in \mathcal{M}_\delta^K$ the difference between Gw and Gu has to be bounded. First, we notice that $Gu - Gw$ is a solution to the following problem

$$\Delta^2(Gu - Gw) = D_i\big(b_1^i[u] - b_1^i[w]\big) + D_{ij}^2\big(b_2^{ij}[u] - b_2^{ij}[w]\big) \quad \text{in } \Omega,$$

in the class $\mathring{W}_p^{2,a}(\Omega)$. In the same way as in the previous step, we want to use again the Proposition 30 and $\|w\|_{L^\infty(\Omega)} \leq 1$ with Lemma 31. In preparation, we observe that by (118) we are able to choose some $t > 0$ such that $\max\left\{\frac{2p}{2-ap+p}, \frac{q}{2}\right\} < t < \frac{p}{2}$. Especially, we put $t := \frac{1}{2}\max\left\{\frac{2p}{2-ap+p}, \frac{q}{2}\right\} + \frac{p}{4}$ as the arithmetic mean. Therefore

$$\|Gu - Gw\|_{W_p^{2,a}(\Omega)} \leq C_{114}(p,t,\Omega)\left(\|b_1[u] - b_1[w]\|_{L^t(\Omega;d^{2at})} + \|b_2[u] - b_2[w]\|_{L^p(\Omega;d^{ap})}\right).$$

At this point, it makes sense to estimate each part separately. We begin with the b_1 terms

$$\begin{aligned}&\big\|b_1[u] - b_1[w]\big\|_{L^t(\Omega;d^{2at})}\\ &\quad\leq C_{124}\|u-w\|_{W_{2t}^{2,a}(\Omega)}\left(\|D^2u\|^2_{L^{2t}(\Omega;d^{2at})} + \|D^2w\|^2_{L^{2t}(\Omega;d^{2at})}\right)\\ &\quad\leq C_{129}\|u-w\|_{W_{2t}^{2,a}(\Omega)}\left(\|D^2u\|^\alpha_{L^q(\Omega;d^{qa})}\|D^2u\|^{1-\alpha}_{L^p(\Omega;d^{pa})} + \|D^2u\|^\alpha_{L^q(\Omega;d^{qa})}\|D^2u\|^{1-\alpha}_{L^p(\Omega;d^{pa})}\right)\end{aligned}$$

with $\alpha > 0$ by L^p-interpolation, since we have chosen $2t = \alpha q + (1-\alpha)p$ to lie strictly between q and p. While $(u-w) \in \mathring{W}_p^{2,a}(\Omega) \hookrightarrow\hookrightarrow C^1(\overline{\Omega})$ and for Ω bounded $\mathring{W}_p^{2,a}(\Omega) \hookrightarrow\hookrightarrow \mathring{W}_{2t}^{2,a}(\Omega)$ we get

$$\big\|b_1[u] - b_1[w]\big\|_{L^t(\Omega;d^{2at})} \leq C_{130}\|u-w\|_{W_p^{2,a}(\Omega)}(\delta^{1-\gamma}K^\gamma)^{2\alpha}K^{2(1-\alpha)}.$$

We continue with the b_2-term

$$\begin{aligned}&\big\|b_2[u] - b_2[w]\big\|_{L^p(\Omega;d^{ap})}\\ &\quad\leq C_{124}\|u-w\|_{W_p^{2,a}(\Omega)}\left(\|\nabla u\|_{L^\infty(\Omega)} + \|\nabla w\|_{L^\infty(\Omega)}\right)\left(\|u\|_{W_p^{2,a}(\Omega)} + \|w\|_{W_p^{2,a}(\Omega)}\right)\\ &\quad\leq C_{131}\|u-w\|_{W_p^{2,a}(\Omega)}\delta K.\end{aligned}$$

We combine the both estimates and obtain for some constant $C_{132} = C_{132}(p,a,\Omega)$

$$\|Gu - Gw\|_{W_p^{2,a}(\Omega)} \leq C_{132}((\delta^{1-\gamma}K^\gamma)^{2\alpha}K^{2(1-\alpha)} + \delta K)\|u-w\|_{W_p^{2,a}(\Omega)}.$$

Hence, by setting $\delta_5 = \delta_5(p, q, K, \Omega)$ such that

$$\text{(WC5)} \qquad C_{132}\big((\delta_5^{1-\gamma}K^{\gamma})^{2\alpha}K^{2(1-\alpha)} + \delta_5 K\big)\|u-w\|_{W^{2,p}(\Omega)} \leq \frac{1}{2}$$

we get the final condition on δ. For each $\delta \leq \delta_5$ satisfying all the previous constraints (WC1), (WC2), (WC3), (WC4) we obtain for all $u, w \in \mathcal{M}_\delta^K$

$$\|Gu - Gw\|_{W_p^{2,a}(\Omega)} \leq \frac{1}{2}\|u-w\|_{W_p^{2,a}(\Omega)}.$$

Therefore if we put $\delta \leq \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ then G is a contraction on $\mathcal{M}_\delta^K$.

4 Making use of the fixed point theorem

Finally, we combine all conditions (WC1),(WC2),(WC3),(WC4) and (WC5) on δ

$$0 < \delta \leq \delta_0 := \min\Big\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\Big\} \tag{119}$$

and use Banach fixed point theorem to get the existence of a unique fixed point $u^* \in \mathcal{M}_\delta^K \subset W_p^{2,a}(\Omega)$ such that $u^* = Gu^*$. Therefore $u^* \in W_p^{2,a}(\Omega)$ is a solution of the Willmore equation in $W_p^{2,a}(\Omega)$. □

33 Corollary

Assume that $\Omega \subset \mathbb{R}^2, \alpha \in (0,1), \beta \in (0,\alpha), \partial\Omega \in C^{1+\alpha}, g_0 \in C^{1+\alpha}(\partial\Omega)$ and $g_1 \in C^{\alpha}(\partial\Omega)$. Additionally, we suppose that $\|g_0\|_{C^{1+\alpha}(\partial\Omega)} + \|g_1\|_{C^{\alpha}(\partial\Omega)} < K$ for some $K > 0$.

Then there exists a constant $\delta = \delta(\alpha, \beta, K, \Omega) > 0$ such that if $\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)} < \delta$, then there exists a weak solution $u \in C^{1+\beta}(\overline{\Omega}) \cap W_{loc}^{2,2/(\alpha-\beta)}(\Omega)$ to the Willmore-type Dirichlet problem, therefore u solves (74) *with the right-hand side* (60).

Proof: Let $s \in (\beta, \alpha)$ that will be specified later. At the beginning, we want to prove that $C^{\alpha}(\partial\Omega) \hookrightarrow\hookrightarrow B_p^s(\partial\Omega)$ for each $p > 2$. Let $f \in C^{\alpha}(\partial\Omega)$ then we make a use of Besov norm definition

$$\begin{aligned}\|f\|_{B_p^s(\partial\Omega)} &= \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{\left|f(x)-f(y)\right|^p}{|x-y|^{2-1+sp}}\,\mathrm{d}S_x\,\mathrm{d}S_y\right)^{\frac{1}{p}} \\ &\leq |\partial\Omega|\|f\|_{C^0(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} |x-y|^{-1+(\alpha-s)p}\,\mathrm{d}S_x\,\mathrm{d}S_y\right)^{\frac{1}{p}} \|f\|_{C^{\alpha}(\partial\Omega)} < \infty\end{aligned}$$

since $\alpha - s > 0$ and $\partial\Omega \in C^{1+\alpha}$. Furthermore, $\nu \in C^{\alpha}(\partial\Omega)$ yields $\nabla_{\tan}g_0 \in C^{\alpha}(\partial\Omega)$ and moreover $g_0 \in L_p^1(\partial\Omega)$. Hence, with $g_1 \in C^{\alpha}(\partial\Omega)$ it follows $\nu g_1 + \nabla_{\tan}g_0 \in B_p^s(\partial\Omega)$ with

$$\|\nu g_1 + \nabla_{\tan}g_0\|_{L^\infty(\partial\Omega)} \leq C_{133}(\partial\Omega)\big(\|g_0\|_{C^1(\partial\Omega)} + \|g_1\|_{C^0(\partial\Omega)}\big)$$

and we conclude that $g := \{g_0, g_1\} \in \dot{W}_p^{1+s}(\partial\Omega)$.

In order to use Theorem 32 we recall the possible values of $p > 2$ and $0 \leq a := 1 - s - \frac{1}{p} < 1 - \frac{2}{p}$. Thus we have to choose p later in the range of

$$\frac{1}{p} < \min(s, 1-s)\,. \tag{120}$$

Under this condition and boundary data constraints, we obtain $u \in W_p^{2,a}(\Omega)$, the unique solution to the Willmore problem (74) with the right-hand side (60). It follows in particular that $u \in W_{\mathrm{loc}}^{2,2}(\Omega)$.

Now, let $w \in W_p^{2,a}(\Omega)$ be solution to the biharmonic equation with Dirichlet boundary data

$$\begin{cases} \Delta^2 w = 0, & \text{in } \Omega, \\ w = g_0, \quad \partial_\nu u = g_1 & \text{on } \partial\Omega. \end{cases}$$

then by [GGS10, Theorem 2.19 p. 45] we have the following Schauder estimate

$$\|w\|_{C^{1+\alpha}(\partial\Omega)} \leq C_{134}\|g_0\|_{C^{1+\alpha}(\partial\Omega)} + C_{135}\|g_0\|_{C^{\alpha}(\partial\Omega)}.$$

Furthermore, by Lemma 27 (c) with $\lambda = 1 - a - \frac{2}{p} = s - \frac{1}{p}$ we can use the compact embedding

$$\mathring{W}_p^{2,a}(\Omega) \hookrightarrow\hookrightarrow C^{1+s-\frac{1}{p}}(\overline{\Omega}).$$

For the purpose of the right Hölder power, we set the parameters

$$s := \frac{1}{2}(\alpha + \beta) \quad \text{and} \quad p := \frac{1}{s - \beta} = \frac{2}{\alpha - \beta}.$$

Then we check that the condition for p in (120) is fulfilled

$$\frac{1}{p} = s - \beta < s \quad \text{and} \quad \frac{1}{p} = \frac{1}{2}(\alpha - \beta) = \alpha - s < 1 - s.$$

Therefore, with $w - u \in \mathring{W}_p^{2,a}(\Omega)$ we obtain $w - u \in C^{1+\beta}(\overline{\Omega})$ and

$$\|w - u\|_{C^{1+\beta}(\overline{\Omega})} \leq C_{136}\|w - u\|_{\mathring{W}_p^{2,a}(\Omega)}.$$

Consequently, we get $u \in C^{1+\beta}(\overline{\Omega}) \cap W^{2,1-\alpha}_{2/(\alpha-\beta)}(\Omega)$. Therefore $u \in W^{2,2/(\alpha-\beta)}_{\text{loc}}(\Omega)$. □

5.5 Higher Regularity

Due to some structural properties of the Willmore-type equation (57), via bootstrapping it is actually possible to show that, despite using lower regularity spaces $C^{\alpha+2}(\overline{\Omega}), W^{2,p}(\Omega)$ and $W_p^{2,a}(\Omega)$, the solution u is smooth in Ω. Moreover, if the Dirichlet data has more regularity than $\dot{W}_p^{1+s}(\partial\Omega)$ then the solution u has as much regularity as allowed by trace theorems, thus up to the boundary.

At this point, let us recall the Willmore equation (57) described in Lemma 18 as

$$\begin{aligned} \Delta_g H + \frac{1}{2}H^3 - 2H\mathcal{K} &= \Delta^2 u - \Delta\left(\frac{|\nabla u|^2}{Q(1+Q)}\Delta u + \frac{\nabla u(D^2 u \nabla u)}{Q^3}\right) - D^2_{ij}\left(\frac{\nabla_i u \nabla_j u}{Q^2}H\right) \\ &\quad + \operatorname{div}\left(\frac{5}{2}\frac{H^2}{Q}\nabla u + 2\frac{H}{Q^2}D^2 u \nabla u - 2\frac{\Delta u H}{Q^2}\nabla u\right) \\ &= \Delta^2 u - D_i b_1^i[u] - D^2_{ij} b_2^{ij}[u] \end{aligned}$$

with $Q = \sqrt{1 + |\nabla u|^2}$ and the divergence structure

$$\begin{aligned} b_1[u] &= D^2 u \star D^2 u \star \sum_{k=1}^{3} Q^{-2k-1} P_{2k-1}(\nabla u), \\ b_2[u] &= D^2 u \star \sum_{k=1}^{2} Q^{-2k-1} P_{2k}(\nabla u) + D^2 u \star P_2(\nabla u) \star (Q(1+Q))^{-1}. \end{aligned}$$

The basic idea of how to gain more regularity is to combine the biharmonic operator with the b_2 terms to a new elliptic operator $L_{(\nabla u)}$ depending only on ∇u defined later such that we deal with an equation of the form

$$L_{(\nabla u)} w = \nabla \cdot P.$$

where for some miltiindex α we will mostly put $D^\alpha u$ in the place of w and P corresponds to some right-hand side in divergence form. Moreover, in the case of w with vanishing trace, the $W^{3,p}$ norm of w can be bounded by the $L^p(\Omega)$-norm of P, as observed in Proposition 23 for ψ_1. This will improve interior regularity from $W^{2,p}$ to $W^{3,p}$, the idea that we can successively use for derivatives of any order.

Now let us define

$$\begin{aligned} L_{(\nabla u)} w &= \Delta^2 w - \Delta \left(\frac{|\nabla u|^2}{Q(1+Q)} \Delta w + \frac{\nabla u (D^2 w \nabla u)}{Q^3} \right) - D^2_{ij} \left(\frac{\nabla_i u \nabla_j u}{Q^2} \left(\frac{\Delta w}{Q} - \frac{\nabla u (D^2 w \nabla u)}{Q^3} \right) \right) \\ &= \sum_{|\alpha|,|\beta|=2} D^\alpha \big(A_{\alpha\beta}(\nabla u) D^\beta u \big) \end{aligned}$$

Especially, in case $w = u$ we get $L_{(\nabla u)} u = \Delta^2 u - D^2_{ij} b_2^{ij}[u]$. Moreover, let $NB_{(\nabla u)}$ be the non-biharmonical part of $L_{(\nabla u)}$ such that $L_{(\nabla u)} w - \Delta^2 w = D^2 NB_{(\nabla u)}[w]$. In particular, it has the form

$$\begin{aligned} D^2_{ij} \big(NB^{ij}_{(\nabla u)}[w] \big) &= -\Delta \left(\frac{|\nabla u|^2}{Q(1+Q)} \Delta^2 w + \frac{\nabla u (D^2 w \nabla u)}{Q^3} \right) \\ &\quad - D^2_{ij} \left(\frac{\nabla_i u \nabla_j u}{Q^2} \left(\frac{\Delta w}{Q} - \frac{\nabla u (D^2 w \nabla u)}{Q^3} \right) \right). \end{aligned}$$

Again in the case $w = u$ we get $NB_{(\nabla u)}[u] = -b_2[u]$.

Next, let us discuss ellipticity. For each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ it follows

$$\begin{aligned} \sum_{|\alpha|,|\beta|=2} \xi_1^{\alpha_1} \xi_2^{\alpha_2} \big(A_{\alpha\beta}(\nabla u) \xi_1^{\beta_1} \xi_2^{\beta_2} \big) &= \left(1 - \frac{|\nabla u|^2}{Q(1+Q)} \right) (\|\xi\|^2)^2 - 2 \frac{\|\xi\|^2 (\nabla u \cdot \xi)^2}{Q^3} + \frac{(\nabla u \cdot \xi)^4}{Q^5} \\ &= \frac{1}{Q} \left(\|\xi\|^2 - \frac{(\nabla u \cdot \xi)^2}{Q^2} \right)^2 \geq \frac{\|\xi\|^4}{Q^5} \end{aligned}$$

thus for each $u \in C^1(\overline{\Omega})$ the operator $L_{(\nabla u)}$ is strongly elliptic.

In more detail, the main idea is to formally rewrite the Willmore-type equation such that for each multi-index γ with $|\gamma| \in \mathbb{N}$ order derivative

$$L_{(\nabla u)} D^\gamma u = D_i \big(P^i_\gamma[u] \big) \tag{121}$$

with $P_\gamma[u] = P_\gamma(D^{|\gamma|+2} u, D^{|\gamma|+1} u, \ldots, \nabla u)$ a polynomial consisting of derivatives of order up to $|\gamma| + 2$. Especially, $P^i_0[u] = b^i_1[u]$. Hence for each γ, the structure of such polynomial has to be described more precisely. Such P_γ involves higher derivatives of b_1 and b_2, where we have to subtract some terms associated with $L_{(\nabla u)}$.

We can reconstruct the structure of $P_\gamma[u]$ in the following way. We formally m-times differentiate the both sides of the Willmore-type equation $\Delta^2 u = D_i b^i_1[u] + D^2_{ij} b^{ij}_2[u]$ and obtain

$$\Delta^2 D^\gamma u = D_i D^\gamma b^i_1[u] + D^2_{ij} D^\gamma b^{ij}_2[u]. \tag{122}$$

Therefore, we conclude by definition of $L_{(\nabla u)}$

$$L_{(\nabla u)} D^\gamma u = \Delta^2 D^\gamma u + D^2_{ij} \big(NB^{ij}_{(\nabla u)}[D^\gamma u] \big) \overset{(122)}{=} D_i D^\gamma b^i_1[u] + D^2_{ij} \big(D^\gamma b^{ij}_2[u] + NB^{ij}_{(\nabla u)}[D^\gamma u] \big)$$

$$= D_i\Big(D^\gamma b_1^i[u] + D_j\big(D^\gamma b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[D^\gamma u]\big)\Big).$$

Thus we set

$$P^i_\gamma[u] := D^\gamma b_1^i[u] + D_j\big(D^\gamma b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[D^\gamma u]\big). \tag{123}$$

In the following lemma, we will observe that in $D^\gamma b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[D^\gamma u]$ all terms that include $D^2 D^\gamma u$ cancel out. The reason is that by product rule the only possibility to get the $D^\gamma D^2 u$-terms by differentiating $b_2^{ij}[u]$ $|\gamma|$-times is to differentiate all D^2u-terms in $b_2^{ij}[u]$ $|\gamma|$-times (all other terms have lower order). This is equivalent to replacing all D^2u-terms in $b_2^{ij}[u]$ by $D^\gamma D^2 u$-terms. In fact, the resulting term is by definition equal to $-NB^{ij}_{(\nabla u)}[D^\gamma u]$. This is important because otherwise, we would have to move these terms from the right-hand side to the operator $L_{(\nabla u)}$. The remaining terms give us $P_\gamma[u]$ for which we derive some preparatory estimates.

34 Lemma
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{m-1,1}$ boundary, $m \in \mathbb{N}, m > 2, p > 2, u \in W^{m,a}_p(\Omega)$ and γ a multiindex with $|\gamma| = m-2$.

Then there exits a constant $C_{137} = C_{137}(m)$ depending only on the algebraic structure of b_1 and b_2 and m such that

$$\begin{aligned}
&\textit{for } m=3: \quad \big|P_\gamma[u]\big| \le C_{137}(3)\left(|D^3u|\cdot|D^2u| + |D^2u|^3\right),\\
&\textit{for } m=4: \quad \big|P_\gamma[u]\big| \le C_{137}(4)\left(|D^4u|\cdot|D^2u| + |D^3u|^2 + |D^3u|\cdot|D^2u|^2 + |D^2u|^4\right).
\end{aligned}$$

If $m \ge 5$ then it follows with some constant C_{138} depending on a, p, m, Ω and the algebraic structure of b_1 and b_2

$$\big\|P_\gamma[u]\big\|_{L^p(\Omega;d^{ap})} < C_{138}\left(\|u\|^2_{W^{m,a}_p\Omega)} + \|u\|^m_{W^{m,a}_p\Omega)}\right).$$

Proof: 1 We begin with $m=3$. Thus, we have to describe P_γ corresponding to ∇u. Then for $|\gamma| = 1$ we use $\nabla Q^{-1} = Q^{-3}D^2u \star \nabla u$ and get.

$$\begin{aligned}
\nabla b_1[u] &= D^3u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1}P_{2k-1}(\nabla u) + D^2u \star D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1}P_{2k-2}(\nabla u)\\
&\quad + D^2u \star D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-3}P_{2k}(\nabla u)\\
&= D^3u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1}P_{2k-1}(\nabla u) + P_3(D^2u) \star \sum_{k=1}^{4} Q^{-2k-1}P_{2k-2}(\nabla u).
\end{aligned}$$

Furthermore, we have to differentiate the corresponding $b_2[u]$ twice. First, since $\|\nabla u\|Q^{-1} < 1$ and $Q^{-1} < 1$ it follows

$$\big|\nabla b_1[u]\big| \le C_{139}\left(|D^3u|\cdot|D^2u| + |D^2u|^3\right).$$

In a similar way, for $b_2[u]$ we first calculate

$$\begin{aligned}
\nabla b_2^{ij}[u] &= D^2\nabla u \star \sum_{k=1}^{2} Q^{-2k-1}P_{2k}(\nabla u) + D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1}P_{2k-1}(\nabla u)\\
&\quad + D^2\nabla u \star P_2(\nabla u) \star (Q(1+Q))^{-1} + D^2u \star D^2u \star \nabla u \star (Q(1+Q))^{-1}
\end{aligned}$$

$$+ D^2u \star D^2u \star P_3(\nabla u) \star \big(Q^{-3} \star (1+Q)^{-1} + Q^{-2} \star (1+Q)^{-2}\big).$$

At this point, we observe that the terms corresponding to $NB_{(\nabla u)}[\nabla u]$ are those with $D^2\nabla u$-term. Thus we conclude

$$\begin{aligned}\nabla b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[\nabla u] \;&= D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k-1} P_{2k-1}(\nabla u)\\ &\quad + D^2u \star D^2u \star \nabla u \star (Q(1+Q))^{-1}\\ &\quad + D^2u \star D^2u \star P_3(\nabla u) \star \big(Q^{-3} \star (1+Q)^{-1} + Q^{-2} \star (1+Q)^{-2}\big).\end{aligned} \tag{124}$$

Similar to $b_1[u]$ we get

$$\big|\nabla\big(\nabla b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[\nabla u]\big)\big| \le C_{140}\left(|D^3u| \cdot |D^2u| + |D^2u|^3\right).$$

(2) For the case $m > 3$, we observe that $|\nabla Q^{-1}| = |Q^{-3}D^2u \star \nabla u| \le |D^2u|Q^{-2}$. Since in each term of $D^\gamma b_i$ there are always more $Q's$ than $\nabla u's$, by applying the absolute value later on, we can skip the parts involving ∇u and Q. Furthermore, since all terms we want to derive have structure given by a $\star$-chain of derivatives

$$D^{|\gamma|+2}u \star D^{|\gamma|+1}u \star \cdots \star D^2u \star \nabla u \star Q^{-1},$$

only by deriving Q^{-1} the chain gets an additional link D^2u. Thus after applying the absolute value, we get at most m-product of derivatives of order greater than 1. Hence, by skipping all ∇u and Q we get schematically,

$$\begin{aligned}D^2b_1[u] \;&\sim D^4u \star D^2u + D^3u \star D^3u + D^3 \star P_2(D^2u) + P_4(D^2u),\\ D^\gamma b_1[u] \;&\sim D^mu \star D^2u + D^{m-1}u \star Q_1(D^{m-2}u, \ldots, D^2u) + Q_2(D^{m-2}u, \ldots, D^2u)\end{aligned}$$

for $m = |\gamma| + 2 > 4$ and $Q_i(D^\gamma u, \ldots, D^2u)$ stands for some polynomial consisting of derivatives up to order γ. Moreover, Q_2 has a polynomial degree of m and that of Q_1 is lower than $m-1$.

Furthermore, by embedding results in Lemma 27 (d) and especially (103) for $p > n$ and $C^{m-1,1}$ boundary of Ω, it follows

$$W_p^{m,a}(\Omega) \hookrightarrow\hookrightarrow W^{m-1,p}(\Omega) \hookrightarrow\hookrightarrow C^{m-2}(\overline{\Omega}).$$

Hence, we can estimate $\|u\|_{C^{m-2}(\overline{\Omega})} \le C_{141}\|u\|_{W_p^{m,a}(\Omega)}$. By combing the results we get

$$\|D^\gamma b_1[u]\|_{L^p(\Omega;d^{ap})} \le C\left(\|u\|^2_{W_p^{m,a}(\Omega)} + \|u\|^m_{W_p^{m,a}(\Omega)}\right). \tag{125}$$

For the $b_2[u]$ estimate we first observe that $D^\gamma b_2^{ij}[u] + NB_{(\nabla u)}[\nabla u] \sim D^{\gamma-1}b_1^{ij}[u]$ again by skipping ∇u and Q. It follows

$$\begin{aligned}&\nabla\big(D^\gamma b_2^{ij}[u] \;+ NB^{ij}_{(D^\gamma u)}[\nabla u]\big)\\ &\qquad\sim D^mu \star D^2u \;+\; D^{m-1}u \star P_1(D^{m-2}u, \ldots, D^2u) \;+\; P_2(D^{m-2}u, \ldots, D^2u)\end{aligned}$$

and we can conclude

$$\left\|\nabla\big(D^\gamma b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[D^\gamma u]\big)\right\|_{L^p(\Omega;d^{ap})} \le C\left(\|u\|^2_{W_p^{m,a}(\Omega)} + \|u\|^m_{W_p^{m,a}(\Omega)}\right). \tag{126}$$

We finish the proof by combining (125) and (126). □

5.5.1 Interior Regularity

In this subsection, we want to show that each solution of the Willmore-type equation is smooth in the interior of Ω. Our strategy is to demonstrate that if $u \in W^{k,p}_{\text{loc}}(\Omega)$ then $u \in W^{k+1,p}_{\text{loc}}(\Omega)$ by using a bootstrap argument for the formulation (121) with (123). We consider for each point in Ω a sequence of shrinking open balls in which we step-wise gain one order of regularity. One of the main tools is the next lemma, where we show more regularity in a smaller inner open ball for the operator $L_{(\nabla u)}w$ with divergence right-hand side. Especially, the assumption $u \in C^1$ is important.

35 Lemma
Let $t, s > 1$, $x_0 \in \mathbb{R}^n$, $u \in C^1(\overline{B_{2R}(x_0)}), h \in L^t(B_{2R}(x_0))$. Furthermore suppose $w \in W^{2,s}(B_{2R}(x_0))$ with $w \cdot |D^2u| \in L^t(B_{2R}(x_0))$ satisfying

$$L_{(\nabla u)}w = Dh \quad \text{in } B_{2R}(x_0) \tag{127}$$

then $w \in W^{3,t}(B_R(x_0))$ and the following local estimate holds

$$\|w\|_{W^{3,t}(B_R(x_0))} \leq C_{142}\Big(\|h\|_{L^t(B_{2R}(x_0))} + \big\|w \cdot |D^2u|\big\|_{L^t(B_{2R}(x_0))} + \|w\|_{L^t(B_{2R}(x_0))}\Big). \tag{128}$$

with $C_{142} = C_{142}(\nabla u, t, R)$.

Proof: Here we want to use the result in [ADN59, Theorem 15.1″ p. 707] for $\ell = 3$ in order to obtain inner $W^{3,t}$ regularity. Thus we have to rewrite the operator L and (127)

$$L_{(\nabla u)}w = \sum_{|\alpha|,|\beta|=2} D^{\alpha_1}\big(A_{\alpha\beta}(\nabla u)D^{\alpha_2}D^\beta w\big) + \sum_{|\alpha|,|\beta|=2} D^{\alpha_1}\Big(D^{\alpha_2}\big(A_{\alpha\beta}(\nabla u)\big)D^\beta w\Big) = Dh.$$

with $|\alpha_1| = 1 = 4 - \ell$. Next we observe that the estimate $\big|\nabla A_{\alpha\beta}(\nabla u)\big| \leq C_{143}\big|D^2u\big|$ holds. Since the coefficients $A_{\alpha\beta}(\nabla u)$ are continuous, by [ADN59, Theorem 15.1″ p. 707] we conclude that $w \in W^{3,t}(B_R(x_0))$ with

$$\|w\|_{W^{3,t}(B_R(x_0))} \leq C_{142}\Big(\|h\|_{L^t(B_{2R}(x_0))} + \big\||D^2u|w\big\|_{L^t(B_{2R}(x_0))} + \|w\|_{L^t(B_{2R}(x_0))}\Big)$$

so that the proof is finished. □

36 Theorem
Assume $p > 2, 0 \leq a \leq 1 - \frac{1}{p}$ and $u \in C^1(\overline{\Omega}) \cap W^{2,a}_p(\Omega)$ is a solution to the Willmore-type Dirichlet problem (115) *with the right-hand side* (60).
Then $u \in C^\infty(\Omega)$.

Proof: Here, we want to use Lemma 35 locally in Ω to show more internal regularity iterativly. We begin by choosing an open ball in Ω. More precisely, let $x_0 \in \Omega$ then there exists $d(x_0) > 0$ and we can consider $B_{d(x_0)}(x_0) \subset \Omega$.

(1) Our first goal is to show that

$$\forall s > 2\ \exists\, \rho > 0: \quad u \in W^{2,s}(B_\rho(x_0)) \cap W^{3,s/2}(B_{\rho/2}(x_0))$$

by considering an elliptic equation for u, thus (121) with $|\alpha| = 0$. We begin by observing that since $u \in C^1(\overline{\Omega}) \cap W^{2,a}_p(\Omega)$ solves (115) with the right-hand side (60), it follows $u \in C^1(\overline{B_{d(x_0)}(x_0)}) \cap W^{2,p}(B_{d(x_0)}(x_0))$ with

$$L_{(\nabla u)}u = D_i b^i_1[u] \quad \text{in } B_{d(x_0)}(x_0).$$

In order to use Lemma 35 for this equation we chose $2R = d(x_0)$, $t = p/2$ then $b_1 \in L^{p/2}(B_{d(x_0)}(x_0))$ with

$$\|b_1\|_{L^{p/2}(B_{d(x_0)}(x_0))} \leq \|D^2 u\|_{L^p(B_{d(x_0)}(x_0))} < \infty.$$

Then, by $u \in C^1(\overline{B_{d(x_0)}(x_0)})$ it follows $u \in W^{3,p/2}(B_{d(x_0)/2}(x_0))$ with

$$\begin{aligned}\|u\|_{W^{3,p/2}(B_{d(x_0)/2}(x_0))} &\leq C_{144}\big(\|b_1\|_{L^{p/2}(B_{d(x_0)}(x_0))} + \|D^2 u\|_{L^p(B_{d(x_0)}(x_0))} + \|u\|_{L^{p/2}(B_{d(x_0)}(x_0))}\big) \\ &\leq C_{145}\big(\|u\|_{W^{2,p}(B_{d(x_0)}(x_0))}\big) < \infty\end{aligned} \tag{129}$$

where C_{145} and most of the following constants are also depending on ∇u.

(2) In this step, we want to discuss how to raise the integrability power to arbitrary powers. First, we observe that by Sobolev's embedding theorem it follows

$$\begin{aligned}&\text{for } p \geq 4: \quad u \in W^{3,p/2}(B_{d(x_0)/2}(x_0)) \hookrightarrow W^{2,s}(B_{d(x_0)/2}(x_0)) \text{ for all } s > 1, \\ &\text{for } p < 4: \quad u \in W^{3,p/2}(B_{d(x_0)/2}(x_0)) \hookrightarrow W^{2,2p/(4-p)}(B_{d(x_0)/2}(x_0)).\end{aligned}$$

where we observe that $\frac{2}{4-p}p > p$ for $p > 2$. Especially in case $\frac{2}{4-p}p \geq 4$ we can again apply the arguments in step (1) with radius $d(x_0)/2$ and get

$$u \in W^{3,\frac{p}{4-p}}(B_{d(x_0)/4}(x_0)) \hookrightarrow W^{2,s}(B_{d(x_0)/4}(x_0)) \text{ for all } s > 1.$$

Hence, we only have to discuss the case $p < 4$ and $\frac{2}{4-p}p < 4$ where we can again use Lemma 35 like in step (1) with radius $d(x_0)/2$, $t = \frac{p}{4-p}$ and obtain

$$u \in W^{2,\ell_0}(B_{d(x_0)/4}(x_0)) \quad \text{for } \ell_0 := \frac{2}{4 - \frac{2p}{4-p}}\frac{2p}{4-p} > \left(\frac{2}{4-p}\right)^2 p$$

since $\frac{2p}{4-p} > p > 2$. In case $\ell_0 < 4$ we repeat the same procedure and get $\ell_1 > \left(\frac{2}{4-p}\right)^3 p$ with $u \in W^{2,\ell_1}(B_{d(x_0)/8}(x_0))$. Thus, by applying Lemma 35 with on $B_{d(x_0)/2^{j+1}}(x_0)$

$$\ell_j := \frac{2}{4 - \ell_{j-1}}\ell_{j-1}, \quad \text{for } j = 1, 2, \ldots$$

at most $\ln\left(\frac{4}{p}\right) / \ln\left(\frac{2}{4-p}\right)$ times one obtains $\ell_k \geq 4$ for some $k = k(p) \in \mathbb{N}$. Hence

$$\forall s > 2 : u \in W^{2,s}(B_{d(x_0)/2^{k+3}}(x_0)) \cap W^{3,s/2}(B_{d(x_0)/2^{k+4}}(x_0)).$$

(3) In the following, we want to show that u is C^2-smooth in some small ball, so that the estimates in Lemma 34 for $P_\alpha[u]$ involving $D^2 u$ in some nonlinear manner become much easier to handle. We set $s = 2p$ and get with some $\rho = \rho(s) < d(x_0)$

$$u \in W^{2,2p}(B_\rho(x_0)) \cap W^{3,p}(B_{\rho/2}(x_0)).$$

Therefore, by Sobolev's embedding it follows $u \in C^2(\overline{B_\rho(x_0)})$ and we conclude for the results of Lemma 34 in the case $m = 3$

$$\begin{aligned}\big\|\nabla b_1[u] \; &+ \nabla\big(\nabla b_2^{ij}[u] + N B^{ij}_{(\nabla u)}[\nabla u]\big)\big\|_{L^p(B_\rho(x_0)))} \\ &\leq C_{137}\left(\|D^3 u\|_{L^p(B_\rho(x_0))} \cdot \|D^2 u\|_{C^0(\overline{B_\rho(x_0)})} + \|D^2 u\|^3_{L^p(B_\rho(x_0))}\right) \\ &\leq C_{146}\big(\|u\|^2_{W^{3,p}(B_\rho(x_0))} + \|u\|^3_{W^{3,p}(B_\rho(x_0))}\big).\end{aligned}$$

Actually, in the next step 4 we will also show that $u \in W^{4,p}(B_{\rho/2}(x_0))$ by iteration argument. For preparation, we assume only for the next estimate $u \in W^{4,p}(B_{\rho/2}(x_0)) \hookrightarrow\hookrightarrow W^{3,p}(B_{\rho/2}(x_0))$ since $p > 2$ and show by Lemma 34 for $m = 4$

$$\begin{aligned}
\big\|D^2 b_1[u] \ &+ D^2\big(\nabla b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[\nabla u]\big)\big\|_{L^p(B_{\rho/2}(x_0))} \\
&\leq C_{147}\begin{pmatrix} \|u\|_{W^{4,p}(B_{\rho/2}(x_0))}\cdot\|D^2u\|_{C^0(\overline{B_\rho(x_0)})} + \|u\|^2_{W^{3,2p}(B_{\rho/2}(x_0))} \\ + \|u\|^3_{W^{3,p}(B_{\rho/2}(x_0))}\cdot\|D^2u\|^2_{C^0(\overline{B_\rho(x_0)})} + \|D^2u\|^4_{C^0(\overline{B_\rho(x_0)})}\end{pmatrix} \\
&\leq C_{148}\left(\|u\|^2_{W^{4,p}(B_{\rho/2}(x_0))} + \|u\|^3_{W^{4,p}(B_{\rho/2}(x_0))} + \|u\|^4_{W^{4,p}(B_{\rho/2}(x_0))}\right).
\end{aligned}$$

4 By induction on $m \geq 3$, we want to establish $u \in W^{m,p}(B_{\rho/2^{m-2}}(x_0))$ by using step 3 and Lemma 34 for the unweighted case $a = 0$ on $B_{\rho/2^{m-2}}(x_0)$ and Lemma 35. From now on, for any fixed $m \geq 3$ assume that $u \in W^{3,2p}(B_{\rho/2}(x_0))$ for $m = 3$ and $u \in W^{m,p}(B_{\rho/2^{m-2}}(x_0))$ for $m \geq 4$ and let γ be any multiindex with

$$|\gamma| = m - 2.$$

Our goal is to prove that $D^\gamma u \in W^{2,p}(B_{\rho/2^{m-2}}(x_0))$ satisfy the corresponding elliptic equation (121) with (123). Therefore we can choose any test function $\tilde{v} \in C_c^\infty(B_{\rho/2^{m-2}}(x_0))$ and set

$$v := (-1)^{|\gamma|}D^\gamma\tilde{v}$$

and use it for the Willmore-type equation $L_{(\nabla u)}u = D_i b_1^i[u]$. Since by Lemma 34 $D^\gamma b_1[u] \in L^p(B_{\rho/2^{m-2}}(x_0))$ we get with $A_{\alpha\beta}$ as coefficients of $L_{(\nabla u)}$ and integration by parts

$$\begin{aligned}
\sum_{|\alpha|,|\beta|=2}\int_\Omega A_{\alpha\beta}(\nabla u)D^\alpha u D^\beta\big((-1)^{|\gamma|}D^\gamma\tilde{v}\big)\,\mathrm{d}x \ &= -\int_\Omega b_1[u]\cdot\nabla(-1)^{|\gamma|}D^\gamma\tilde{v}\,\mathrm{d}x \\
&= \int_\Omega \nabla_i D^\gamma b_1^i[u]\tilde{v}\,\mathrm{d}x.
\end{aligned}$$

For the part with elliptic coefficients it follows

$$\begin{aligned}
\sum_{|\alpha|,|\beta|=2}\int_\Omega A_{\alpha\beta}(\nabla u)D^\alpha u D^\beta\big((-1)^{|\gamma|}D^\gamma\tilde{v}\big)\,\mathrm{d}x &= \sum_{|\alpha|,|\beta|=2}\int_\Omega D^\gamma\big(A_{\alpha\beta}(\nabla u)D^\alpha u\big)D^\beta\tilde{v}\,\mathrm{d}x \\
= \sum_{|\alpha|,|\beta|=2}\int_\Omega A_{\alpha\beta}(\nabla u)D^\alpha D^\gamma u)D^\beta\tilde{v}\,\mathrm{d}x &+ \sum_{i,j=1}^2\int_\Omega\big(D^\gamma b_2^{ij}[u] + NB^{ij}(\nabla u)[D^\gamma u]\big)D^2_{ij}\tilde{v}\,\mathrm{d}x \\
= \sum_{|\alpha|,|\beta|=2}\int_\Omega A_{\alpha\beta}(\nabla u)(D^\alpha D^\gamma u)D^\beta\tilde{v}\,\mathrm{d}x &- \sum_{i,j=1}^2\int_\Omega\nabla_i\big(D^\gamma b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[D^\gamma u]\big)\nabla_j\tilde{v}\,\mathrm{d}x
\end{aligned}$$

where $NB_{(\nabla u)}$ represents the non-biharmonical elements of $L_{(\nabla u)}$ like defined in Lemma 35 and $\nabla\big(D^\gamma b_2[u] + NB_{(\nabla u)}[D^\gamma u]\big) \in L^p(B_{\rho/2^{m-2}}(x_0))$. Therefore, $D^\gamma u \in W_p^{m-2}(B_{\rho/2^{m-2}}(x_0))$ solves

$$L_{(\nabla u)}D^\gamma u = D_i\Big(D^\gamma(b_1^i[u]) + \nabla_j\big(D^\gamma b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[D^\gamma u]\big)\Big) = D_i(P^i_\gamma[u]) \quad \text{in } B_{\rho/2^{m-2}}(x_0).$$

Then again by Lemma 34 and step 3 we get $P_\alpha[u] \in L^p(B_{\rho/2^{m-2}}(x_0))$ with

$$\big\|P_\alpha[u]\big\|_{L^p(B_{\rho/2^{m-2}}(x_0))} \leq C_{149}\left(\|u\|^2_{W^{m,p}(B_{\rho/2^{m-2}}(x_0))} + \|u\|^m_{W^{m,p}(B_{\rho/2^{m-2}}(x_0))}\right)$$

and by Lemma 35 we obtain that $D^\gamma u \in W^{3,p}(B_{\rho/(2^{m-1})}(x_0))$ with

$$\|D^\gamma u\|_{W^{3,p}(B_{\rho/2^{m-1}}(x_0))} \le C_{142} \begin{pmatrix} \|u\|^2_{W^{m,a}_p(B_{\rho/2^{m-2}}(x_0))} + \|u\|^m_{W^{m,a}_p(B_{\rho/2^{m-2}}(x_0))} \\ + \big\||D^2u| \cdot |D^\gamma u|\big\|_{L^p(B_{\rho/2^{m-2}}(x_0))} + \|D^\gamma u\|_{L^p(B_{\rho/2^{m-2}}(x_0))} \end{pmatrix}.$$

Hence $u \in W^{m+1}_p(B_{\rho/2^{m-1}}(x_0))$. By applying the same scheme to all $m = 3, 4, \ldots$ we deduce infinite differentiability in x_0 since $W^m_p(B_{\rho/2^{m-2}}(x_0)) \hookrightarrow\hookrightarrow C^{m-1}(\overline{B_{\rho/2^{m-1}}(x_0)})$. □

5.5.2 Boundary Regularity

With a slightly different technique, we can also obtain higher boundary regularity in the case Dirichlet data provides enough regularity.

37 Theorem
Let $m \in \mathbb{N}, m \ge 2$, and $\partial\Omega \in C^m$. Furthermore, assume $p, q > 2$, $0 \le a \le 1-\frac{1}{p}$ and $u \in C^1(\overline{\Omega}) \cap W^{2,a}_p(\Omega)$ is a solution to the Willmore-type Dirichlet problem (115) *with the right-hand side* (60) *and $u-\varphi \in \mathring{W}^{2,a}_p(\Omega)$ with some $\varphi \in W^{m,q}(\Omega) \cap W^{2,a}_p(\Omega)$.*

Then $u \in W^{m,q}(\Omega)$.

Proof: (1) At the beginning, we observe that by interior regularity Theorem 36 $u \in C^\infty(\Omega)$ and moreover for all multi-index γ we get

$$L_{(\nabla u)} D^\gamma u = D_i\big(P^i_\gamma[u]\big) \quad \text{in } \Omega.$$

Subsequently, consider a biharmonical problem with boundary data given by φ. Thus, let $u_0 \in W^{m,q}(\Omega)$ be the solution of the following problem

$$L_{(\nabla u)} u_0 = 0 \quad \text{in } \Omega$$

such that $u_0 - \varphi \in \mathring{W}^{m,q}(\Omega)$. Then we get the estimate

$$\|u_0\|_{W^{m,q}(\Omega)} \le C_{150} \|\varphi\|_{W^{m,q}(\Omega)}.$$

(2) Consider first the case $|\gamma| = 1$. Especially, since in Ω:

$$\Delta^2 \nabla u_0 + D^2 \nabla(NB_{(\nabla u)}[u_0]) = \nabla L_{(\nabla u)}(u_0) = 0 \tag{130}$$

it follows that

$$L_{(\nabla u)}(\nabla u_0) = \Delta^2 \nabla u_0 + D^2 NB_{(\nabla u)}[\nabla u_0] \overset{(130)}{=} D^2 NB_{(\nabla u)}[\nabla u_0] - D^2 \nabla\big(NB_{(\nabla u)}[u_0]\big),$$

where in the right side the terms with $D^2\nabla u_0$ cancel out

$$L_{(\nabla u)}(\nabla u) = D_i\Big(\nabla(b^i_1[u]) + \nabla_j\big(\nabla b^{ij}_2[u] + NB^{ij}_{(\nabla u)}[\nabla u]\big)\Big),$$

where on the right-hand side the terms with $D^2\nabla u$ cancel out. Thus we get

$$\begin{aligned} L_{(\nabla u)}(\nabla(u-u_0)) = D_i\Big(\nabla(b^i_1[u]) + \nabla_j\big(\nabla b^{ij}_2[u] + NB^{ij}_{(\nabla u)}[\nabla u]\big)\Big) \\ - D^2\Big(NB_{(\nabla u)}[\nabla u_0] - \nabla\big(NB_{(\nabla u)}[u_0]\big)\Big) \end{aligned}$$

By (111) from Proposition 30 and general weighted Sobolev estimate 28, as well as (124) we get similar to the arguments given in interior regularity Theorem 36 step 1 that $\nabla(u-u_0) \in W^{2,a}_{p/2}(\Omega)$ with

$$\|\nabla(u-u_0)\|_{W^{2,a}_{p/2}(\Omega)} \le C_{151} \begin{pmatrix} \|b_1[u]\|_{L^{p/2}(\Omega;d^{ap/2})} + \left\| |D^2u|^2 \right\|_{L^{p/2}(\Omega;d^{ap/2})} \\ + \left\| |D^2u| \cdot |D^2u_0| \right\|_{L^{p/2}(\Omega;d^{ap/2})} \end{pmatrix} < \infty.$$

Then by weighted embedding Lemma 27 *d* we obtain

$$\forall s < p\colon \quad W^{1,a}_{p/2}(\Omega) \hookrightarrow\hookrightarrow L^s(\Omega).$$

Hence it follows $\nabla(u-u_0) \in W^{2,s}(\Omega)$.

3 In case $q < p$ then we have it follows $u \in W^{2,q}(\Omega)$. Let us now assume that $q \ge p$. Then we can choose any $s = \frac{1}{2}(2+p) \in (2,p)$. Let us assume that $s > 4$, then it follows $s/2 > 2 > \frac{2}{2/q+1} = \frac{2q}{2+q}$ and we can use unweighted Proposition 23 from Sobolev theory with $t = s$ and $p = q$ and get for

$$L_{(\nabla u)}(u-u_0) = D_i b_1^i[u] \quad \text{in } \Omega$$

with $u - u_0 \in \mathring{W}^{2,q}(\Omega)$ the L^q-estimate

$$\begin{aligned} \|u-u_0\|_{W^{2,q}(\Omega)} &\le C_{152}\|b_1[u]\|_{L^{s/2}(\Omega)} \le C_{153}\big(\|u\|_{W^{2,s}(\Omega)} + \|u-u_0\|_{W^{2,s}(\Omega)}\big) \\ &\le C_{154}\big(\|u\|_{W^{2,a}_p(\Omega)} + \|u\|_{W^{2,a}_p(\Omega)} + \|\varphi\|_{W^{2,q}(\Omega)}\big). \end{aligned}$$

That means $u \in W^{2,q}(\Omega)$.

In case $s \le 4$ by using the same iteration technique as in interior regularity Theorem 36 step 1 to get with some $k \in \mathbb{N}$

$$\|u-u_0\|_{W^{2,\ell_i}(\Omega)} \le C\|u-u_0\|_{W^{2,\ell_{i-1}}(\Omega)} \quad \text{with } \ell_0 = s,\ \ell_i = \frac{2}{4-\ell_{i-1}}\ell_{i-1}, \quad i = 1,\dots,k$$

and either $\ell_k > 4$ or $\ell \ge q$. In both cases, we obtain $u \in W^{2,q}(\Omega)$. In the case $m > 2$, we can conclude like in Theorem 36 steps 2 and 3 iterative for each $2,\dots,m$. ☐

38 Theorem
Let $m \in \mathbb{N}, m \ge 2$, $\ell = \max\{4,m\}$ and $\partial\Omega \in C^{\ell+\alpha}$. Further, assume $u \in C^1(\overline{\Omega}) \cap W^{2,a}_p(\Omega)$ is a solution to the Willmore-type Dirichlet problem (115) *with the right-hand side* (60) *and Dirichlet data $g = \{g_0, g_1\}$ such that $g_0 \in C^{m+\alpha}(\partial\Omega)$ and $g_1 \in C^{m-1+\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$.*

Then $u \in C^{m+\alpha}(\overline{\Omega})$.

Proof: First, we observe that by Theorem 37 $u \in C^\infty(\Omega) \cap W^{m,p}(\Omega)$ for all $p > 2$. This means $u \in C^{m-1+\beta}(\overline{\Omega})$ for all $\beta \in (0,1)$.

We continue with a biharmonical problem with g boundary data. Thus, let $u_0 \in C^{m+\alpha}(\overline{\Omega})$ be the solution of the following problem

$$\begin{cases} L_{(\nabla u)} u_0 = 0 & \text{in } \Omega \\ u_0 = g_0, \quad \frac{\partial u_0}{\partial \nu} = g_1 & \text{on } \partial\Omega. \end{cases}$$

Then since $u \in C^{1+\alpha}(\overline{\Omega})$ by [ADN59, p. 680 Theorem 9.3] we get existence and Hölder-estimate

$$\|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \le C_{155}\left(\|g_0\|_{C^{m+\alpha}(\partial\Omega)} + \|g_1\|_{C^{m-1+\alpha}(\partial\Omega)}\right).$$

To obtain $C^{m+\alpha}$ regularity, with each multiindex γ such that $|\gamma| = m-2$ we get

$$L_{(\nabla u)} D^\gamma u = D_i\Big(D^\gamma(b_1^i[u]) + \nabla_j\big(D^\gamma b_2^{ij}[u] + NB^{ij}_{(\nabla u)}[D^\gamma u]\big)\Big) \quad \text{in } \Omega.$$

Let us first consider the case $m = 2$. Then, with the same calculation as in Theorem 37 2 we get for all $p > 2$ by Proposition 23 from Sobolev theory

$$\begin{aligned}\|\nabla(u_0 - u)\|_{W^{2,p/2}(\Omega)} &\le C_{156}\|b_1[u]\|_{L^{p/2}(\Omega)}\\ &\quad + C_{157}\left\|\nabla(b_2^{ij}[u] - b_2^{ij}[u_0]) + NB^{ij}_{(\nabla u)}[\nabla u] - NB^{ij}_{(\nabla u)}[\nabla u_0]\right\|_{L^{p/2}(\Omega)}\\ &\le C_{158}\big(\|u\|_{W^{2,p}(\Omega)} + \|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)}\big)\end{aligned}$$

with C_{158} depending on $\nabla u, \Omega, p$. Thus for some $s = 2p > 2$ big enough we get

$$\begin{aligned}\|\nabla(u_0 - u)\|_{C^{1+\alpha}(\overline{\Omega})} &\le C_{159}\|\nabla(u_0 - u)\|_{W^{2,s}(\Omega)} \le C_{160}\big(\|u\|_{W^{2,s/2}(\Omega)} + \|g_0\|_{C^{2+\alpha}(\partial\Omega)} + \|g_1\|_{C^{1+\alpha}(\partial\Omega)}\big)\\ &< \infty.\end{aligned}$$

This means $u \in C^{2+\alpha}(\overline{\Omega})$.

Next, assume $m > 2$ and $u \in C^{m-1+\alpha}(\overline{\Omega})$. Therefore using results from the proof in Lemma 34 we get for each multiindex γ with $|\gamma| = m-2$

$$\begin{aligned}\|D^\gamma u - D^\gamma u_0\|_{C^{2+\alpha}(\overline{\Omega})} &\le C_{161}\|D^{|\gamma|-1} b_1[u]\|_{C^\alpha(\overline{\Omega})}\\ &\quad + C_{162}\left\|D^{|\gamma|-1}\big(\nabla(b_2^{ij}[u] - b_2^{ij}[u_0]) + NB^{ij}_{(\nabla u)}[\nabla u] - NB^{ij}_{(\nabla u)}[\nabla u_0]\big)\right\|_{C^\alpha(\overline{\Omega})}\\ &\le C_{163}\big(\|u\|_{C^{m-1+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{m+\alpha}(\partial\Omega)} + \|g_1\|_{C^{m-1+\alpha}(\partial\Omega)}\big)\end{aligned}$$

hence $u \in C^{m+\alpha}(\overline{\Omega})$. □

6 Parabolic Theory

In this chapter, we study the Willmore flow initial boundary value problem with irregular initial data by using time-weighted and unweighted parabolic Hölder spaces (see [Bel79]). For graphs on $\mathbb{R}^2$, it was already done in [KL12], where initial data was merely Lipschitz. Since we have prescribed boundary values, to get continuity for $t \searrow 0$ we have to make sure that the initial and boundary values are consistent with each other. With $C^{4+\alpha}$-regularity of initial data u_0, only if u_0 satisfies the condition

$$0 = \Delta_{\Gamma(u_0)} H(u_0) + 2H\left(\frac{1}{4}H^2 - \mathcal{K}\right)(u_0), \quad \text{on } \partial\Omega \tag{uCC}$$

the solution of the Willmore flow will stay $C^{4+\alpha}$-bounded for $t \searrow 0$. In fact, it is preferable to avoid (uCC), since the *compatibility condition* cannot be realized well numerically, and is also not physically relevant, since like for biharmonic heat flow, irregularities are expected to smooth out. First, we have to formulate the Willmore flow equation for the graphical case such that the divergence structure is usable. Then, we linearize the parabolic problem by freezing coefficients. Thus, one can use weighted and unweighted parabolic Hölder spaces estimates of the linear theory. Then, one moves to the spaces with smaller Hölder power and formulates a fixed point problem to obtain local existence. The case of this $C^{3+\alpha}$ smooth initial data was already handled in [Gul17]. In this work, we want to use the same methods for $C^{m+\alpha}, m = 1, 2$ class u_0.

6.1 Willmore Flow

Like in the elliptic case, we again use the work of Dziuk and Deckelnick [DD06, (1.5)-(1.9)] and Koch and Lamm [KL12] for the Willmore flow of graphs. There, they have written the Willmore-flow equation (3) for the graphical case as

$$-\frac{u_t}{Q} = \Delta_g H + \frac{1}{2}H^3 - 2H\mathcal{K} = \operatorname{div}\left(\frac{1}{Q}\left(\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right)\nabla(QH)\right) - \frac{H^2}{2Q}\nabla u\right), \tag{131}$$

with the mean curvature in the form

$$H = \operatorname{div}\left(\frac{\nabla u}{Q}\right) = \frac{\Delta u}{Q} - \frac{\nabla u \cdot (D^2 u \nabla u)}{Q^3}.$$

In this chapter, we are searching for the maximal existence time of the *Willmore flow problem*:

$$\begin{cases} \partial_t u + Q\left\{\Delta_{\Gamma(u)} H + 2H\left(\frac{1}{4}H^2 - \mathcal{K}\right)\right\} = 0 & \text{in } \overline{\Omega} \times (0, T], \\ u(x,t) = g_0(x), & (x,t) \in \partial\Omega \times [0,T], \\ \frac{\partial u}{\partial \nu}(x,t) = g_1(x), & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), & x \in \overline{\Omega} \end{cases} \tag{WF}$$

where we have to mention the corresponding compatibility conditions (CC)

$$g_0 = u_0(x), \quad g_1(x) = \frac{\partial u_0}{\partial \nu}(x), \qquad x \in \partial\Omega \tag{CC}$$

and (uCC) for $u_0 \in C^{4+\alpha}(\overline{\Omega})$ and $u(t,.) \in C^{4+\alpha}(\overline{\Omega})$ for $t \searrow 0$. Further, we want to split the right-hand side of the Willmore flow equation in two different ways. To present these parts more clearly, we again use $\star$ notation from [KS01] and [KL12]. These denote a linear combination of tensor contractions for derivatives of u. With the results out of [KL12, Lemma 3.2 p.215] we denote

$$P_i(u) = \underbrace{\nabla u \star \nabla u \star \cdots \star \nabla u}_{i\text{-times}}. \tag{132}$$

Especially in this notation, the Willmore flow equation takes the shape

$$\partial_t u + \Delta^2 u = f_0[u] + \nabla_i f_1^i[u] + D^2_{ij} f_2^{ij}[u] =: f[u] \tag{133}$$

with right-hand side terms

$$\begin{aligned} f_0[u] &= D^2u \star D^2u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-2}(\nabla u) \\ f_1[u] &= D^2u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-1}(\nabla u) \\ f_2[u] &= D^2u \star \sum_{k=1}^{2} Q^{-2k} P_{2k}(\nabla u). \end{aligned} \tag{134}$$

Next, we want to split the Willmore flow equation in another way into two parts. One is linear in the fourth-order derivatives, and the other is a polynomial of derivatives up to the third order. For the linear part, we have to prove uniform ellipticity. It can be shown that

$$Q\Delta_{\Gamma(u)} H + 2QH\left(\frac{1}{4}H^2 - \mathcal{K}\right) = L(\nabla u)D^4u + \mathcal{R}(\nabla u, D^2u, D^3u). \tag{A}$$

with the operator

$$L(\nabla u)D^4u = \sum_{k+\ell=4} L_{k\ell}(\nabla u)\partial^k_{x_1}\partial^\ell_{x_2}u,$$

where the coefficients are

$$L_{k\ell}(\nabla u) = \begin{pmatrix} L_{40}(\nabla u) \\ L_{31}(\nabla u) \\ L_{22}(\nabla u) \\ L_{13}(\nabla u) \\ L_{04}(\nabla u) \end{pmatrix}^T = \frac{1}{Q^4}\begin{pmatrix} (1+u_{x^2}^2)^2 \\ -4(1+u_{x^2}^2)u_{x^1}u_{x^2} \\ 2(1+u_{x^2}^2)(1+u_{x^1}^2) + 4u_{x^1}^2u_{x^2}^2 \\ -4(1+u_{x^1}^2)u_{x^1}u_{x^2} \\ (1+u_{x^1}^2)^2 \end{pmatrix}^T. \tag{L}$$

With this explicit representation, it can be proven that $L(\nabla u)$ is uniformly elliptic provided its gradient is bounded.

39 Lemma (Ellipticity)
Let $u \in C^1(\overline{\Omega})$, then with $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$:

$$\frac{1}{\left(1 + \|\nabla u\|^2_{C^0(\overline{\Omega})}\right)^2}|\xi|^4 \leq \sum_{k+\ell=4} L_{k\ell}(\nabla u)\xi_1^k\xi_2^\ell \leq 4|\xi|^4.$$

Proof: [Gul17, Lemma 6.6] □

The polynomial term $\mathcal{R}$ contains monomes of derivatives and $1/Q^2$. If these contain D^3u, then only linearly and in combination with up to one D^2u. If a monom does not contain any D^3u, then it includes up to three D^2u. With (134) and $D(Q^{-2\ell}) = D^2u \star \nabla u \star Q^{-2(\ell+1)}$ it results

$$
\begin{aligned}
&\mathcal{R}(\nabla\, u, D^2u, D^3u)\\
&= -\, f_0[u] - \nabla_i f_1^i[u] - D^2_{ij} f_2^{ij}[u] + \Delta^2 u - L(\nabla u)D^4u\\
&= D^2u \star D^2u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-2}(\nabla u)\\
&\quad + D\left[D^2u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-1}(\nabla u)\right]\\
&\quad + D^2\left[D^2u \star \sum_{k=1}^{2} Q^{-2k} P_{2k}(\nabla u)\right] + \Delta^2u - L(\nabla u)D^4u\\
&= D^2u \star D^2u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-2}(\nabla u)\\
&\quad + D^3u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-1}(\nabla u) + D^2u \star D^2u \star D^2u \star \sum_{k=1}^{4} Q^{-2(k+1)} P_{2k}(\nabla u)\\
&\quad + D\left[D^3u \star \sum_{k=1}^{2} Q^{-2k} P_{2k}(\nabla u) + D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2k} P_{2k-1}(\nabla u)\right] + \Delta^2u - L(\nabla u)D^4u\\
&= D^2u \star D^2u \star D^2u \star \sum_{k=1}^{5} Q^{-2k} P_{2k-2}(\nabla u) + D^3u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-1}(\nabla u)\\
&\quad + D^3u \star D^2u \star \sum_{k=1}^{2} Q^{-2(k+1)} P_{2k+1}(\nabla u) + D^3u \star D^2u \star \sum_{k=1}^{3} Q^{-2k} P_{2k-1}(\nabla u)\\
&\quad + D^2u \star D^2u \star D^2u \star \sum_{k=1}^{3} Q^{-2(k+1)} P_{2k}(\nabla u)\\
&= D^3u \star D^2u \star \sum_{k=1}^{4} Q^{-2k} P_{2k-1}(\nabla u) + D^2u \star D^2u \star D^2u \star \sum_{k=0}^{4} Q^{-2(k+1)} P_{2k}(\nabla u).
\end{aligned}
\tag{R}
$$

We collect the above results to the following lemma.

40 Lemma
The Willmore flow equation (131) *can be rewritten as*

$$
\begin{aligned}
-u_t &= Q\,\mathrm{div}\left(\frac{1}{Q}\left(\left(I - \frac{\nabla u \otimes \nabla u}{Q^2}\right)\nabla(QH)\right) - \frac{H^2}{2Q}\nabla u\right)\\
&= L(\nabla u)D^4u + \mathcal{R}(\nabla u, D^2u, D^3u) \quad \textit{in}\ \ \Omega \times (0,T]
\end{aligned}
\tag{135}
$$

where L and $\mathcal{R}$ are given by (L) *and* (R).

6.2 Parabolic Hölder Spaces

In this chapter, we treat the theory of classical solutions of parabolic differential equations of fourth order. For this purpose, we will deal with parabolic Hölder spaces. Therefore, for these spaces, we recall the existence and uniqueness theorem for fourth-order parabolic differential equations with the Schauder estimates.

Just as in the case of the elliptic differential equations, we have to switch to the Hölder spaces for optimal regularity. There, however, one should treat the time and place components separately because the problem behaves asymmetrically with respect to the components. The definition of parabolic Hölder spaces for higher orders with their properties can be found in works [Sol65] and [LSvW92]. Instead of working in $\overline{\Omega}$ we work in the parabolic case in the closed time cylinder over $\overline{\Omega}$

$$\overline{Q}_T := \overline{\Omega} \times [0, T].$$

Since the Willmore equation, with which we will deal later, is a fourth-order parabolic differential equation, we will always stick to the fourth order in the following discussion (in Belonosov's notation in [Bel79] that means $m = 2$).

Let $\Omega \subset \mathbb{R}^n$ be bounded with $C^{4+\alpha}$-smooth boundary $\partial\Omega$, $T > 0$. We define for any positive $\ell \notin \mathbb{N}$ the *parabolic Hölder norm*

$$\begin{aligned} \|u\|_{C^{\ell,\frac{\ell}{4}}_{x,t}(\overline{Q}_T)} := & \sum_{4k+|\beta|\le\lfloor\ell\rfloor} \sup_{(x,t)\in\overline{Q}_T} \left|D^k_t D^\beta_x u(x,t)\right| + \sum_{4k+|\beta|=\lfloor\ell\rfloor} \sup_{t\in[0,T]} \left[D^k_t D^\beta_x u(\,.\,,t)\right]_{C^{\ell-\lfloor\ell\rfloor}(\overline{\Omega})} \\ & + \sum_{\ell-4<4k+|\beta|<\ell} \sup_{x\in\overline{\Omega}} \left[D^k_t D^\beta_x u(x,.\,)\right]_{C^{\frac{\ell-4k-|\beta|}{4}}\left([0,T]\right)}, \end{aligned}$$

and for $\ell \in \mathbb{N}$ we set

$$\|u\|_{C^{\ell,\frac{\ell}{4}}_{x,t}(\overline{Q}_T)} := \sum_{4k+|\beta|\le\lfloor\ell\rfloor} \sup_{(x,t)\in\overline{Q}_T} \left|D^k_t D^\beta_x u(x,t)\right| + \sum_{\ell-4<4k+|\beta|<\ell} \sup_{x\in\overline{\Omega}} \left[D^k_t D^\beta_x u(x,.\,)\right]_{C^{\frac{\ell-4k-|\beta|}{4}}\left([0,T]\right)},$$

as well as for $s \le \ell$ the norm:

$$\|u\|_{C^{\ell,\frac{\ell}{4}}_s(Q_T)} = \sup_{t<T} t^{\frac{\ell-s}{4}} [u]^\ell_{Q'_t} + \sum_{s<4k+|\beta|<\ell} \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{4k+|\beta|-s}{4}} \left|D^k_t D^\beta_x u(x,t)\right| + \begin{cases} \|u\|_{C^{s,\frac{s}{4}}_{x,t}(\overline{Q}_T)}, & s \ge 0, \\ 0, & s < 0, \end{cases}$$

where $Q'_t = \overline{\Omega} \times [t/2, t]$ and:

$$[u]^\ell_{Q'_t} = \sum_{4k+|\beta|=\lfloor\ell\rfloor} \sup_{t'\in[t/2,t]} \left[D^k_t D^\beta_x u(\,.\,,t')\right]_{C^{\ell-\lfloor\ell\rfloor}(\overline{\Omega})} + \sum_{\ell-4<4k+|\beta|<\ell} \sup_{x\in\overline{\Omega}} \left[D^k_t D^\beta_x u(x,.\,)\right]_{C^{\frac{\ell-4k-|\beta|}{4}}\left([t/2,t]\right)}.$$

In fact, this notation is from [SK80]. In Lemma 91 in Appendix, we show that it is equivalent to the definitions of norms by Belonosov in [Bel79]. Next we define the *weighted parabolic Hölder spaces for* $\ell > s \notin \mathbb{N}$ on $Q_T = \overline{\Omega} \times (0, T]$

$$s > 0: \qquad C^{\ell,\frac{\ell}{4}}_s(Q_T) := \left\{ u \in C^0(\overline{Q}_T) \;\middle|\; \begin{array}{l} \text{for } \; 0 \; < 4k + |\beta| \le \lfloor s\rfloor : \exists\, D^k_t D^\beta_x u \text{ in } \overline{Q}_T, \\ \text{for } \lfloor s\rfloor < 4k + |\beta| \le \lfloor\ell\rfloor : \exists\, D^k_t D^\beta_x u \text{ in } Q_T, \\ \text{and } \|u\|_{C^{\ell,\frac{\ell}{4}}_s(Q_T)} < \infty \end{array} \right\},$$

$$s < 0: \qquad C^{\ell,\frac{\ell}{4}}_s(Q_T) := \left\{ u \in C^0(Q_T) \;\middle|\; \begin{array}{l} \text{for } \lfloor 0\rfloor < 4k + |\beta| \le \lfloor\ell\rfloor : \exists\, D^k_t D^\beta_x u \text{ in } Q_T, \\ \text{and } \|u\|_{C^{\ell,\frac{\ell}{4}}_s(Q_T)} < \infty \end{array} \right\}.$$

In case $s \in \mathbb{N}$ we define $C^{\ell,\frac{\ell}{4}}_s(Q_T)$ to be the the closure of the $C^{\ell,\frac{\ell}{4}}_{x,t}(\overline{Q}_T)$-functions with respect to the $C^{\ell,\frac{\ell}{4}}_s(Q_T)$-norm. Additionally, we define $C^{\ell,\frac{\ell}{4}}_{x,t}(\overline{Q}_T)$, the *unweighted parabolic Hölder spaces* on $\overline{Q}_T$

by setting $s = \ell$ in the definition of $C_s^{\ell,\frac{\ell}{4}}(Q_T)$. In the same way we can define $C_s^{\ell,\frac{\ell}{4}}(\partial\Omega \times (0,T])$ for $\ell \notin \mathbb{N}$ with $s \le \ell$ on *the weighted parabolic Hölder space on boundary* $\partial\Omega \times (0,T]$ by replacing $\overline{\Omega}$ with $\partial\Omega$ in the upper definition.

For a $u \in C_s^{\ell,\frac{\ell}{4}}(Q_T)$ its derivatives $D_t^k D_x^\beta u$ in $\overline{Q}_T$ are continuous for $k \in \mathbb{N}_0, \beta \in \mathbb{N}_0^n$, $4k + |\beta| \le \lfloor s \rfloor$. By results presented in [Bel79, p.154] i $C_r^{s,\frac{s}{4}}(Q_T)$ is a Banach space, ii $C_{x,t}^{s,\frac{s}{4}}(Q_T) \subset C_r^{s,\frac{s}{4}}(Q_T)$ and $C_s^{s,\frac{s}{4}}(Q_T) = C_{x,t}^{s,\frac{s}{4}}(Q_T)$ and by iii for $4k + |\beta| \le \ell$

$$\|D_t^k D^\beta u\|_{C_{r-4k+|\beta|}^{\ell-4k+|\beta|,\frac{\ell-4k+|\beta|}{4}}(Q_T)} \le \|u\|_{C_r^{\ell,\ell/4}(Q_T)} \tag{136}$$

and by v with $C_{164}(T)$ bounded for $T \to 0$

$$\|uw\|_{C_r^{\ell,\ell/4}(Q_T)} \le C_{164}(T)\|u\|_{C_{r_1}^{\ell,\ell/4}(Q_T)}\|w\|_{C_{r_2}^{\ell,\ell/4}(Q_T)} \tag{137}$$

where $r = \min(r_1, r_2, r_1 + r_2)$.

One of the fundamental theorems we use excessively for the linearization method is Schauder existence and estimates results for weighted parabolic Hölder spaces. In the following, we will assume that the Dirichlet boundary data is in $C^{4+\alpha}$-regularity class with $\alpha \in (0,1)$. Therefore we set $\ell = 4 + \alpha$.

41 Theorem (Weighted Existence)
Let $\Omega \subset \mathbb{R}^n$ bounded, $C^{4+\alpha}$-smooth domain, $0 < \alpha < 1$, $s \in [0, 4+\alpha]$, $\nu(x)$ the exterior normal in $x \in \partial\Omega$, $f \in C_{s-4}^{\alpha,\frac{\alpha}{4}}(Q_T)$, for all $\beta \in \mathbb{N}^n$ such that $|\beta| \le 4$: $a_\beta \in C_{\max\{0,s-4\}}^{\alpha,\alpha/4}(Q_T)$. Moreover, we assume, that the uniform ellipticity condition is fulfilled

$$\lambda|\xi|^4 \le \sum_{|\beta|=4} a_\beta(x,t)\xi^\beta \le \Lambda|\xi|^4, \quad \forall \xi \in \mathbb{R}^n, (x,t) \in \overline{Q}_T.$$

with constants $0 < \lambda \le \Lambda$. Consider the following initial value problem with $u_0 \in C^s(\overline{\Omega})$, $\varphi \in C_s^{4+\alpha,1+\alpha/4}(\partial\Omega \times (0,T])$, $h \in C_{s-1}^{3+\alpha,\frac{3+\alpha}{4}}(\partial\Omega \times (0,T])$:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) + \sum_{|\beta|\le 4} a_\beta(x,t) D_x^\beta u(x,t) = f(x,t), \quad (x,t) \in \overline{\Omega} \times (0,T],\\
&u(x,0) = u_0(x), \quad x \in \overline{\Omega},\\
&u(x,t) = \varphi(x,t), \quad \frac{\partial u}{\partial \nu}(x,t) = h(x,t), \quad (x,t) \in \partial\Omega \times [0,T],
\end{aligned}$$

with the compatibility conditions

$$\begin{aligned}
&\varphi(x,0) = u_0(x), \quad h(x,0) = \frac{\partial u_0}{\partial \nu}(x) \quad x \in \partial\Omega,\\
\textit{only for } s \ge 4: \quad &\frac{\partial \varphi}{\partial t}(x,0) = -\sum_{|\beta|\le 4} a_\beta(x,0) D_x^\beta u_0(x) + f(x,0) \quad x \in \partial\Omega.
\end{aligned}$$

Then there is a unique solution $u \in C_s^{4+\alpha,1+\alpha/4}(Q_T)$ for the initial value problem.

Proof: Combined results from [Bel79, S. 185] theorem 4 with Dirichlet boundary values and [LSvW92]. □

42 Theorem (Weighted Schauder Estimate)
For the initial value problem above, the following Schauder estimate holds:

$$\|u\|_{C_s^{4+\alpha,1+\alpha/4}(Q_T)} \leq C_{165}(T) \begin{pmatrix} \|f\|_{C_{s-4}^{\alpha,\alpha/4}(Q_T)} + \|\varphi\|_{C_s^{4+\alpha,1+\alpha/4}(\partial\Omega\times(0,T])} \\ + \|h\|_{C_{s-1}^{3+\alpha,\frac{3+\alpha}{4}}(\partial\Omega\times(0,T])} + \|u_0\|_{C^s(\overline{\Omega})} \end{pmatrix}$$

where $C_{165} \colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a monotone function that depends on* λ, Λ, Ω *and* $c > 0$ *such that for all* $|\beta| \leq 4$*:* $\|a_\beta\|_{C^{\alpha,\alpha/4}_{\max\{0,s-4\}}(Q_T)} < c$.

Proof: [Bel79, p. 184] Corollary, (4.10) for $m = 2, r = s, s = 4+\alpha$ with Dirichlet boundary conditions. □

6.3 Time-Weighted $C^{m+\alpha}$-$C^{4+\alpha}$-case

In this subsection, we consider the case where the initial data is $u_0 \in C^{m+\alpha}, m = 1,2,3,4$. Therefore, we set $s = m+\alpha$, $f = \mathcal{R}$ and $(a_\beta) \cong L(\nabla u)$ with time-constant Dirichlet boundary values $\varphi = g_0$ and $h = g_1$. Let us first recall the weighted norm for our case. We begin with the unweighted term

$$\|u\|_{C_{x,t}^{m+\alpha,\frac{m+\alpha}{4}}(\overline{Q}_T)} = \sum_{4k+|\beta|\leq m} \sup_{(x,t)\in\overline{Q}_T} \left|D_t^k D_x^\beta u(x,t)\right| + \sum_{4k+|\beta|=m} \sup_{t\in[0,T]} \left[D_t^k D_x^\beta u(\,.\,,t)\right]_{C^\alpha(\overline{\Omega})}$$
$$+ \sum_{m-4+\alpha<4k+|\beta|\leq m} \sup_{x\in\overline{\Omega}} \left[D_t^k D_x^\beta u(x,.\,)\right]_{C^{\frac{m+\alpha-4k-|\beta|}{4}}\left([0,T]\right)}.$$

Then, we continue with the weighted norm

$$\|u\|_{C_{m+\alpha}^{4+\alpha,1+\frac{\alpha}{4}}(Q_T)} = \sup_{t<T} t^{\frac{4-m}{4}} [u]^{4+\alpha}_{Q'_t} + \sum_{m<4k+|\beta|\leq 4} \sup_{(x,t)\in Q_T} t^{\frac{4k+|\beta|-m-\alpha}{4}} \left|D_t^k D_x^\beta u(x,t)\right| + \|u\|_{C_{x,t}^{m+\alpha,\frac{m+\alpha}{4}}(\overline{Q}_T)}$$

with $Q'_t = \overline{\Omega} \times [t/2, t]$ and

$$[u]^{4+\alpha}_{Q'_t} = \sum_{4k+|\beta|=4} \sup_{t'\in[t/2,t]} \left[D_t^k D_x^\beta u(\,.\,,t')\right]_{C^\alpha(\overline{\Omega})} + \sum_{1\leq 4k+|\beta|\leq 4} \sup_{x\in\overline{\Omega}} \left[D_t^k D_x^\beta u(x,.\,)\right]_{C^{\frac{4+\alpha-4k-|\beta|}{4}}\left([t/2,t]\right)}.$$

To show the short-time existence in the $C_{m+\alpha}^{4+\alpha,1+\frac{\alpha}{4}}(Q_T)$-space via a linerization method, we have to verify various technical properties, which for better readability are moved to Appendix. One of the tricks is to change the Hölder power to $\gamma \leq \alpha$ so that for vanishing initial values we have control via $T^{\frac{\alpha-\gamma}{4}}$. This will allow us to choose a time small enough for us to apply a fixed-point argument.

43 Lemma
Let $m = 1,2,3,4$. *If* $u \in C_{m+\alpha}^{4+\alpha,1+\frac{\alpha}{4}}(Q_T)$ *and* $\forall x \in \overline{\Omega}$

$$\forall 4k + |\beta| \leq m: \qquad D_t^k D_x^\beta u(x,0) = 0$$

then there is a constant $C_{166} = C_{166}(\alpha,\gamma)$ *such that for all* $0 < \gamma < \alpha$ *and* $T \leq 1$

$$\|u\|_{C_{m+\gamma}^{4+\gamma,1+\frac{\gamma}{4}}(Q_T)} \leq C_{166} T^{\frac{\alpha-\gamma}{4}} \|u\|_{C_{m+\alpha}^{4+\alpha,1+\frac{\alpha}{4}}(Q_T)}.$$

Proof: The proof is presented in Appendix Lemma 92. □

Also we have to estimate the right hand side term $\mathcal{R}$ in the $C^{4+\alpha,1+\frac{\alpha}{4}}_{m+\alpha-4}(Q_T)$-norm by the $C^{4+\gamma,1+\frac{\gamma}{4}}_{m+\gamma}(Q_T)$-norm of u. Since $\mathcal{R}$ consists of products of derivatives u, we need the following fundamental product estimates.

44 Lemma (Product Rule)
Let $m = 1,2,3,4$ and $0 < \gamma \le \alpha$, $\alpha/2 \le \gamma$. If $u, v, w \in C^{4+\gamma,1+\frac{\gamma}{4}}_{m+\gamma-4}(Q_T)$ and $T \le 1$, then there is a constant $C_{167} = C_{167}(\alpha,\gamma,\Omega)$ depending on algebraic structure of $\mathcal{R}$ and L such that

$$\|\nabla u\|_{C^{\alpha,\frac{\alpha}{4}}_{\max\{0,m+\alpha-4\}}(Q_T)} \le C_{167}\|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)}, \tag{138}$$

$$\|D^3 w D^2 u\|_{C^{\alpha,\frac{\alpha}{4}}_{m+\alpha-4}(Q_T)} \le C_{167}\|D^3 w\|_{C^{1+\gamma,\frac{1+\gamma}{4}}_{m+\gamma-3}(Q_T)} \cdot \|D^2 u\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)}, \tag{139}$$

$$\begin{aligned}&\|D^2 u D^2\, w D^2 v\|_{C^{\alpha,\frac{\alpha}{4}}_{m+\alpha-4}(Q_T)}\\ &\qquad \le C_{167}\|D^2 u\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)} \cdot \|D^2 w\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)} \cdot \|D^2 v\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)}.\end{aligned} \tag{140}$$

Proof: See Appendix Lemma 93. □

45 Lemma (Hölder Estimates I)
Let $m = 1,2,3,4$ and $0 < \gamma, \alpha < 1$, $T \le 1$ then there exist constants $C_{168} = C_{168}(\Omega,\alpha,\gamma)$ and $k_H \in \mathbb{N}$ depending on algebraic structure of $\mathcal{R}$ and L, so that:

$$\begin{aligned}\left\|\mathcal{R}(\nabla u, D^2 u, D^3 u)\right\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} &\le C_{168}\Big(1 + \|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)}\Big)^{k_H} \|\nabla u\|^3_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)},\\ \sum_{k+\ell=4} \|L_{k\ell}(\nabla u)\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)} &\le C_{168}\Big(1 + \|\nabla u\|^4_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)}\Big).\end{aligned}$$

Proof: We refer to Appendix Lemma 94. □

For applying the fixed point theorem, we also have to verify the contraction property. Hence, we need to estimate the differences of the right-hand sides and also the differences of the elliptic components.

46 Lemma (Hölder Estimates II)
Let $m = 1,2,3,4$ and $0 < \gamma, \alpha < 1$, $T \le 1$ then there exist constants $C_{169} = C_{169}(\alpha,\gamma,\Omega)$ and $k'_H \in \mathbb{N}$ depending on algebraic structure of $\mathcal{R}$ and L so that for $u, w \in C^{4+\gamma,1+\gamma/4}_{m+\gamma-4}(Q_T)$ it holds

$$\begin{aligned}&\left\|\mathcal{R}(\nabla u, D^2 u, D^3 u) - \mathcal{R}(\nabla w, D^2 w, D^3 w)\right\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)}\\ &\quad + \sum_{k+\ell=4} \|L_{k\ell}(\nabla u) - L_{k\ell}(\nabla w)\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)}\\ &\qquad\qquad \le C_{169}\Big(1 + \max\Big\{\|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)}, \|w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)}\Big\}\Big)^{k'_H}\\ &\qquad\qquad\qquad \cdot \max\Big\{\|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)}, \|w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)}\Big\}^2 \cdot \|\nabla(u-w)\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)}.\end{aligned}$$

Proof: The proof is similar to 94, where additionally we need to consider how to rewrite a difference of polynomials as in Lemma 96. □

47 Theorem (Existence of a Unique Solution for Small Times)
Let $\Omega \subset \mathbb{R}^2$ bounded with a $C^{4+\alpha}$-smooth boundary and the exterior normal ν as well as $m = 1,2,3,4$.

Then there exists $T \in (0,1)$, so that there is a unique solution $u \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$ of the Willmore flow problem (WF) *where $u_0 \in C^{m+\alpha}(\overline{\Omega})$, $g_0 \in C^{4+\alpha}(\partial\Omega)$ and $g_1 \in C^{3+\alpha}(\partial\Omega)$ with*

$$g_0(x) = u_0(x), \quad g_1(x) = \frac{\partial u_0}{\partial \nu}(x) \quad x \in \partial\Omega$$

and additionally, in the case $m = 4$ we need the compatibility condition

$$0 = \Delta_{\Gamma(u_0)} H(u_0) + 2H\left(\frac{1}{4}H^2 - \mathcal{K}\right)(u_0), \quad x \in \partial\Omega.$$

Proof: The main idea of this proof is to modify the quasilinear problem to a linear problem by freezing all derivatives of order smaller than four. To do this, we define a fixed point problem for an iterative solution of a linear parabolic differential equation whose fixed point then coincides with the desired solution of the Willmore flow equation. To ensure the validity of this approach, we need to check that the assumptions of the Banach fixed point theorem (see Theorem 8) are satisfied. The proof structure can be outlined as follows:

1. *Adaptation of the time-independent boundary values* to the parabolic Hölder spaces by the constant continuation in time to apply Schauder existence theorem and estimates,
2. *Definition of the iteration mapping G as well as the iteration set $\mathcal{M}$* for the fixed point problem,
3. *The iteration mapping is a self-mapping*: $G\colon \mathcal{M} \to \mathcal{M}$,
4. *The iteration mapping is a contraction*: $u, w \in \mathcal{M} : \|G(u) - G(w)\| \leq q\|u - w\|, q \in (0,1)$,
5. *Application of the fixed point theorem* to infer the existence of the fixed point,
6. *Uniqueness of solution* in the parabolic Hölder space $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$.

In the first two steps $T \in (0,1)$ is not chosen, then in 3 and 4 we choose the time small enough for using the Banach fixed point Theorem.

1 Boundary Values Discussion

Since the boundary values $u|_{\partial\Omega}(\,.\,,t) = g_0(\,.\,)$, $\partial_\nu u|_{\partial\Omega}(\,.\,,t) = g_1(\,.\,)$ are fixed for all $t \in [0,1]$, we can extend these on $\overline{\Omega} \times (0,T]$. Thus we set $t \in (0,1]$:

$$\overline{g}_0(x,t) := g_0(x), \quad \overline{g}_1(x,t) := g_1(x), \quad \forall x \in \partial\Omega.$$

Then all time derivatives of $\overline{g}_0$ vanish, in particular for all time Hölder seminorms are zero

$$\|\overline{g}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(\partial\Omega\times(0,1])} = \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \qquad \|\overline{g}_1\|_{C^{3+\alpha,\frac{3+\alpha}{4}}_{m-1+\alpha}(\partial\Omega\times(0,1])} = \|g_1\|_{C^{3+\alpha}(\partial\Omega)}.$$

Additionally, we have to extend $u_0 \in C^{m+\alpha}(\overline{\Omega})$, but not in the trivial constant way, because in the case $m < 4$ the extension would not be $C^{4+\alpha}(\overline{\Omega})$ for $t > 0$, and thus also not in $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_1)$. We extend by solving the following biharmonic heat equation

$$\text{(A)} \qquad \begin{cases} \partial_t v = -\Delta^2 v = -\partial_1^4 v - 2\partial_1^2\partial_2^2 v - \partial_2^4 v, & \text{in } \overline{\Omega} \times (0,1], \\ v(x,0) = u_0(x), & x \in \overline{\Omega}, \\ v(x,t) = \overline{g}_0(x,t), & (x,t) \in \partial\Omega \times [0,1], \\ \dfrac{\partial v}{\partial \nu}(x,t) = \overline{g}_1(x,t), & (x,t) \in \partial\Omega \times [0,1]. \end{cases}$$

It is an easy matter to check that (A) fulfills the requirements of Theorem 41. So there exists $\overline{u}_0 \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_1)$ that solves (A). Also we get the Schauder estimate in Theorem 42 with a constant $C_{165}(1)$ depending only on Ω, m, α and the structure of (A)

$$
\begin{aligned}
\|\overline{u}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_1)} &\leq C_{165}(1)\Big[\|\overline{g}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(\partial\Omega\times(0,1])} + \|\overline{g}_1\|_{C^{4+\alpha,\frac{3+\alpha}{4}}_{m-1+\alpha}(\partial\Omega\times(0,1])} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\Big] \\
&\leq C_{170}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\big).
\end{aligned} \tag{141}
$$

The compatibility conditions are satisfied, since $v(\,.\,,0) = u_0$:

$$
\overline{g}_0(x,0) = g_0(x) = u_0(x), \quad \overline{g}_1(x,0) = g_1(x) = \frac{\partial u_0}{\partial \nu}(x) \quad x \in \partial\Omega.
$$

In case $m = 4$, we simply take extend $\overline{u}_0(x,t) := u_0(x), x \in \Omega, t \in [0,1]$. Then $\overline{u}_0 \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_1)$ and (141) is also fulfilled.

2 Definition of the Iteration Map and Set

We define iteration map $G_T\colon C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T) \to C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$ by freezing the coefficients of L and $\mathcal{R}$. For each $w \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$ we set $v = G_T w$ as a solution of

$$
\begin{cases}
\partial_t v = -L(\nabla w)D^4 v - \mathcal{R}\big(\nabla w, D^2 w, D^3 w\big), & \text{in } \overline{\Omega}\times(0,T], \\
v(x,0) = u_0(x), & x \in \overline{\Omega}, \\
v(x,t) = \overline{g}_0(x,t), & (x,t) \in \partial\Omega\times[0,T], \\
\dfrac{\partial v}{\partial \nu}(x,t) = \overline{g}_1(x,t), & (x,t) \in \partial\Omega\times[0,T].
\end{cases} \tag{G}
$$

Since $w \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$ then due to Lemma 45 $L(\nabla w) \in C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)$, $\mathcal{R}\big(\nabla w, D^2 w, D^3 w\big) \in C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)$. Additionally, by Lemma 39 one obtains uniform ellipticity, and by Theorem 41 there exists $v = G_T w \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$. Therefore, this mapping is well defined.

For the case $m = 4$, we also need the existence of $G_T\overline{u_0}$ with the estimates to get later the same derivatives in $t = 0$ as for the fixed point solution, see (146). For $G_T\overline{u_0}$ by Theorem 45 it holds for $|\beta| = 4$ that

$$
\|L_{\beta_1,\beta_2}(\nabla\overline{u}_0)\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)} \leq C_{168}\Big(1 + \|\overline{u}_0\|^4_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_1)}\Big),
$$

and one also has the uniform ellipticity by (141) with

$$
\frac{|\xi|^4}{\Big(1 + \|\nabla\overline{u}_0\|^2_{C^0(\overline{Q}_T)}\Big)^2} \leq \sum_{k+\ell=4} L_{k\ell}(\nabla w)\xi_1^k\xi_2^\ell \leq 4|\xi|^4.
$$

Using the Schauder estimate in Theorem 42 for the boundary problem (G) with constant C_{165} it results

$$
\begin{aligned}
\|G_T\overline{u}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} &\overset{\text{Thm. 42}}{\leq} C_{165}(1)\begin{bmatrix} \|\mathcal{R}(\nabla\overline{u}_0, D^2\overline{u}_0, D^3\overline{u}_0)\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} \\ + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \end{bmatrix}, \\
&\underset{(143)}{\overset{\text{Lem. 45}}{\leq}} C_{171}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\big),
\end{aligned} \tag{142}
$$

We choose $\gamma \in (0, \alpha)$ and define the closed subset of $C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)$ as the iteration-set

$$\text{(M)}\quad \mathcal{M}_T := \left\{ w \in C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T) \;\middle|\; \begin{array}{l} \|w - G_T\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \le 1, \quad v(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ v(x,t) = g_0(x), \quad \partial_\nu v(x,t) = g_1(x), \quad (x,t) \in \partial\Omega \times [0,T] \end{array} \right\}.$$

which is non-trivial since $G_T\overline{u}_0 \in \mathcal{M}_T$. For all $w \in \mathcal{M}_T$ we also obtain an useful estimate

$$\text{(143)}\qquad \|w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \le \|w - G_T\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} + \|G_T\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \overset{(142)}{\le} 1 + C_{171}.$$

3 **G is a self-map**

Here we want to show that $G_T\colon \mathcal{M}_T \to \mathcal{M}_T$ by using the Schauder estimates in Theorem 42. Let $w \in \mathcal{M}_T$, then by Theorem 45 we obtain for $|\beta| = 4$

$$\|L_{\beta_1,\beta_2}(\nabla w)\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)} \le C_{168}\Big(1 + \|w\|^4_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_1)}\Big) \overset{(143)}{\le} C_{168}\Big(1 + (1 + C_{171})^4\Big),$$

which is bounded by a constant non depending on T.

Also by (143) we have an uniform ellipticity on $\mathcal{M}_T$ with constants λ and Λ for all $w \in \mathcal{M}_T$, $T \in (0,1)$ depending only on Ω, $\|g_0\|_{C^{4+\alpha}(\partial\Omega)}$, $\|g_1\|_{C^{3+\alpha}(\partial\Omega)}$ and $\|u_0\|_{C^{m+\alpha}(\overline{\Omega})}$

$$\text{(144)}\qquad \lambda := \frac{1}{\Big(1 + \big(1 + C_{171}\big)^2\Big)^2}, \quad \Lambda := 4, \qquad \overset{\text{Lem. } 39}{\Rightarrow} \qquad \lambda|\xi|^4 \le \sum_{k+\ell=4} L_{k\ell}(\nabla w)\xi_1^k\xi_2^\ell \le \Lambda|\xi|^4.$$

By the Schauder estimate in Theorem 42 for the boundary problem (G) with constant C_{165} for all $w \in \mathcal{M}_T$ it follows that for $v = G_T w$

$$\text{(145)}\qquad \begin{aligned} \|v\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} &\overset{\text{Thm. } 42}{\le} C_{165}(1) \begin{bmatrix} \|\mathcal{R}(\nabla w, D^2 w, D^3 w)\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} + \|\overline{g}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(\partial\Omega\times(0,T])} \\ + \|\overline{g}_1\|_{C^{3+\alpha,\frac{3+\alpha}{4}}_{m-1+\alpha}(\partial\Omega\times(0,T])} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \end{bmatrix}, \\ &\underset{(143)}{\overset{\text{Lem. } 45}{\le}} C_{172}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\big), \end{aligned}$$

where by Theorem 42, previous work in 1 and (143) the T-independent constant $C_{165}(1)$ depends only on C_{171} and Ω.

Further, we have to show that by choosing T small enough, we obtain $v \in \mathcal{M}_T$. First, we consider the difference $v - G_T\overline{u}_0$

$$\begin{aligned} \|v - G_T\overline{u}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} &\le \|v\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} + \|G_T\overline{u}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} \le C_{172} + C_{171} \\ &=: C_{173}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\big). \end{aligned}$$

So the difference is $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$-bounded, because $G_T\overline{u}_0$ and v are $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$-bounded. The $v - G_T\overline{u}_0$ *derivatives* of the order smaller than three vanish at time $t = 0$. The reason is that $v(x,0) = u(x) = G_T\overline{u}_0(x,0)$ for all $x \in \overline{\Omega}$ fulfill the same initial conditions

$$\forall|\beta| \le m\ \forall x \in \overline{\Omega}\colon \qquad D^\beta_x v(x,0) - D^\beta_x G_T\overline{u}_0(x,0) = D^\beta_x u_0(x) - D^\beta_x u_0(x) = 0.$$

In the case $m = 4$, one can use the Willmore flow equation and derive

$$\text{(146)}\qquad \forall x \in \overline{\Omega}\colon \qquad D_t v(x,0) - D_t G_T\overline{u}_0(x,0) = 0$$

where only for this step we need $G_T\overline{u}_0$. For $m < 4$ we could take $\overline{u}_0$ instead.

Next, we consider the parabolic Hölder space with a smaller Hölder power $C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)$ and apply Lemma 43 for $v - G_T\overline{u}_0$. In this case, the initial conditions vanish (see also (146), so that

$$\|v - G_T\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \leq C_{166}\|v - G_T\overline{u}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} \cdot T^{\frac{\alpha-\gamma}{4}} \leq C_{166}C_{173} \cdot T^{\frac{\alpha-\gamma}{4}}.$$

By choosing $T < 1/(C_{166}C_{173})^{\frac{4}{\alpha-\gamma}}$, we get $v \in \mathcal{M}_T$ and thus $G_T\colon \mathcal{M}_T \to \mathcal{M}_T$, because:

$$\|v - G_T\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \leq 1.$$

4 G **is a contraction**

In this paragraph, we want to show that for T small enough, $G_T\colon \mathcal{M}_T \to \mathcal{M}_T$ is a contraction. Let $u, w \in \mathcal{M}_T$, then because $G_T w$ and $G_T u$ have the same initial values it holds

$$\begin{aligned}
G_T u(x,0) - G_T w(x,0) &= u_0(x) - u_0(x) = 0, && x \in \overline{\Omega},\\
G_T u(x,t) - G_T w(x,t) &= g_0(x,t) - g_0(x,t) = 0, && (x,t) \in \partial\Omega \times [0,T],\\
\frac{\partial(G_T u - G_T w)}{\partial\nu}(x,t) &= g_1(x,t) - g_1(x,t) = 0, && (x,t) \in \partial\Omega \times [0,T].
\end{aligned}$$

Thus $G_T u - G_T w$ solves the following linear initial value problem (Z). Let $v = G_T u - G_T w$ be the solution of the following problem

$$\text{(Z)}\qquad \begin{cases}
\partial_t v = -\dfrac{1}{2}\big(L(\nabla u) + L(\nabla w)\big)\big(D^4 v\big) - \dfrac{1}{2}\big(L(\nabla u) - L(\nabla w)\big)\big(D^4(G_T u + G_T w)\big)\\
\qquad\quad - \mathcal{R}\big(\nabla u, D^2 u, D^3 u\big) + \mathcal{R}\big(\nabla w, D^2 w, D^3 w\big) \qquad \text{in } \overline{\Omega} \times (0,T],\\
v(x,0) = 0, \qquad x \in \overline{\Omega},\\
v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T],\\
\partial_\nu v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T].
\end{cases}$$

The operator $\frac{1}{2}\big(L(\nabla u) + L(\nabla w)\big) \in C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)$ is again uniformly elliptic, with the same time-independent constants λ and Λ as in (144) for $L(\nabla u)$ and $L(\nabla w)$. Moreover, by the Schauder estimates for $G(u)$ and $G(w)$ as solutions of (G) it follows that

$$\begin{aligned}
\|D^4(G_T u + G_T w)\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} &\leq \|G_T u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} + \|G_T w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)}\\
&\leq 2C_{172}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\big).
\end{aligned} \tag{147}$$

Again, by Theorem 42 one obtains a Schauder estimate for $G_T u - G_T w$ as solution of (Z) with a constant $C_{165}(1)$ depending only on $\|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}}(\overline{\Omega})$ and Ω

$$\begin{aligned}
&\|G_T u - G_T w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)}\\
&\quad \leq C_{165}(1) \cdot \left\| \begin{aligned} &-\frac{1}{2}\big(L(\nabla u) - L(\nabla w)\big)\big(D^4(G_T u + G_T w)\big)\\ &- \mathcal{R}\big(\nabla u, D^2 u, D^3 u\big) + \mathcal{R}\big(\nabla w, D^2 w, D^3 w\big)\end{aligned} \right\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)}\\
&\quad \overset{(137)}{\leq} \frac{C_{165}(1)}{2} \cdot \big\|\big(L(\nabla u) - L(\nabla w)\big)\big\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)} \cdot \big\|\big(D^4(G_T u + G_T w)\big)\big\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)}\\
&\qquad + C_{165}(1)\,\big\|\mathcal{R}(\nabla u, D^2 u, D^3 u) - \mathcal{R}(\nabla w, D^2 w, D^3 w)\big\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)}.
\end{aligned}$$

Further, let us estimate by 46 and (147) for $D^4(G_T u + G_T w)$

$$\|G_T u \ - G_T w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} \le C_{174} \cdot \|u-w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)},$$

for all $u, w \in \mathcal{M}_T$ with a new constant C_{174} depending only on Ω, $\|g_0\|_{C^{4+\alpha}(\partial\Omega)}$, $\|g_1\|_{C^{3+\alpha}(\partial\Omega)}$, $\|u_0\|_{C^{m+\alpha}(\overline{\Omega})}$. Once again we apply Lemma 43 to estimate $G_T u - G_T w$ with vanishing initial data

$$\|G_T u - G_T w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \le C_{166}\, C_{174} T^{\frac{\alpha-\gamma}{4}} \cdot \|u-w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)}.$$

Finally, we obtain G_T a contraction on $\mathcal{M}_T$ by choosing $T < 1/\big(C_{166}\, C_{174}\big)^{\frac{4}{\alpha-\gamma}}$.

5 Applying the Fixed Point Theorem

For a time T small enough, we can use the fixed point theorem and get a fixed point $v^* \in \mathcal{M}_T \subset C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)$ with $v^* = G_T v^*$. This v^* solves the original Willmore-flow problem (WF). Actually, we obtain even stronger regularity $v^* \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$ by using the Schauder estimate in Theorem 42

$$\|v^*\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} \le C_{172}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\big).$$

6 Uniqueness

Until now, we only obtained the uniqueness in $\mathcal{M}_T$. To show that there exists only one solution in $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$, we have to define a new initial value problem. Like in the step 4 one has to choose time small enough and use Lemma 43 to relate the norms $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$ and $C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)$.

With T time as in the step 4 and $u \in \mathcal{M}_T$ the solution in the step 5 and $w \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})$ another solution, we consider only $T' < T$ without loss of generality. Additionally, let $0 < T_0 < T'$ then $u - w$ is a solution of the following initial value problem

$$(\mathrm{E}_w) \qquad \begin{cases} \partial_t v = \ -\dfrac{1}{2}\big(L(\nabla u) + L(\nabla w)\big)\big(D^4 v\big) - \dfrac{1}{2}\big(L(\nabla u) - L(\nabla w)\big)\big(D^4(u+w)\big) \\ \qquad\qquad - \mathcal{R}\big(\nabla u, D^2 u, D^3 u\big) + \mathcal{R}\big(\nabla w, D^2 w, D^3 w\big) \qquad \text{in } \overline{\Omega} \times (0, T_0], \\ v(x,0) = 0, \qquad x \in \overline{\Omega}, \\ v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0, T_0], \\ \partial_\nu v(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0, T_0]. \end{cases}$$

It follows that $\frac{1}{2}\big(L(\nabla u) + L(\nabla w)\big) \in C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_{T_0})$ is again an uniform elliptic operator with time-independent ellipticity constants

$$\lambda^* := \frac{1}{2\Big(1 + \|u\|^2_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})} + \|w\|^2_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}\Big)}, \ \Lambda^* := 4.$$

Moreover, we obtain the estimates

$$\|D^4(u+w)\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_{T_0})} \le \|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})} \ + \|w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})},$$

$$\|L(\nabla u) + L(\nabla w)\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_{T_0})} \overset{(45)}{\le} C_{168} \begin{pmatrix} \Big(1 + \|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}\Big)^{k_H} \\ + \Big(1 + \|w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}\Big)^{k_H} \end{pmatrix}.$$

Once again, by Theorem 42 there exists a Schauder estimate with a constant $C_{165}(1)$ for the problem (E_w) which depends only on $\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}$, $\|w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}$ and Ω. Like in step 4:

$$\begin{aligned}
&\|u- w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_0})}\\
&\quad\le \frac{C_{165}(1)}{2}\cdot \big\|\big(L(\nabla u)-L(\nabla w)\big)\big\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_{T_0})}\cdot \big\|\big(D^4(u+w)\big)\big\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_{T_0})}\\
&\qquad + C_{165}(1)\,\big\|\mathcal{R}(\nabla u, D^2u, D^3u)-\mathcal{R}(\nabla w, D^2w, D^3w)\big\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_{T_0})}\\
&\quad\le C_{174}\cdot \|u-w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma-4}(Q_{T_0})},
\end{aligned}$$

where C_{174} depends only on $\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}$, $\|w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}$ and Ω. By Lemma 43 it holds

$$\|u-w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_0})}\le C_{166}\, C_{174} T_0^{\frac{\alpha-\gamma}{4}}\cdot \|u-w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_0})}.$$

In the following, we choose $T_0 < 1/\big(C_{166}\, C_{174}\big)^{\frac{4}{\alpha-\gamma}}$ so that

$$\|u-w\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_0})}=0.$$

Therefore, u and w are identical in $C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_0})$. To show the equality in $t\in(T_0,T']$ we first consider the unweighted case $m=4$ and use the fact, that the time T_0 depends only on Ω and the bounds on $C^{4+\alpha,1+\alpha/4}_{x,t}(Q_{T'})$-norm of u and w. Namely, since $\forall x\in\overline{\Omega}: D^\beta_x u(x,T_0) = D^\beta_x w(x,T_0)$ and by choosing the same uniqueness time T_0, we obtain uniqueness on the time interval $[0,\min\{2T_0,T'\}]$. In the same way, we can repeat this procedure until we reach T'.

By the definition of the weighted norms, in the cases $m=1,2,3$ for all times greater than $T_0>0$, the solution of the initial problem is actually in the unweighted Hölder space for times between T_0 and T'

$$\begin{aligned}
\|u\|_{C^{4+\alpha,1+\alpha/4}_{x,t}(\overline{\Omega}\times[T_0,T'])} &\le C(T_0,T')\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})},\\
\|w\|_{C^{4+\alpha,1+\alpha/4}_{x,t}(\overline{\Omega}\times[T_0,T'])} &\le C(T_0,T')\|w\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})}.
\end{aligned}$$

Furthermore, the compatibility condition (CC) is fulfilled for all times in $[T_0,T']$. Thus one can take $u(x,T')\equiv w(x,T')$ as the new initial value for the corresponding initial value problem on $C^{4+\alpha,1+\alpha/4}_{x,t}(\overline{\Omega}\times[T_0,T'])$. Hence, it results $u=w$ in $C^{4+\alpha,1+\alpha/4}_{x,t}(\overline{\Omega}\times[T_0,T'])$ and thus $u=w$ in $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T'})$. Finally, we extend w with u up to T and get the uniqueness in $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$. □

In the next part of the chapter, we want to investigate global existence of the graphical Willmore flow solutions. Especially, we will need a lower bound on short existence time in order to be able to extend the local solution by this fixed time. In this way, we will prevent blow-ups. In what follows we always assume the compatibility condition (CC) for u_0, g_0 and g_1.

48 Lemma
Suppose $m=1,2,3,4$, then these exist constants $C_{175}=C_{175}(m,\alpha,\Omega)$, $C_{176}=C_{176}(m,\alpha,\Omega)$ and a time $0<T_1(m,\alpha,\Omega)<1$ such that if

$$\|u_0\|_{C^{m+\alpha}(\overline{\Omega})}+\|g_0\|_{C^{4+\alpha}(\partial\Omega)}+\|g_1\|_{C^{3+\alpha}(\partial\Omega)}\le C_{175} \tag{148}$$

then there exists a unique solution $u\in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})$ of the Willmore flow problem (WF) *and*

$$\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})}\le C_{176}\left(\|g_0\|_{C^{4+\alpha}(\partial\Omega)}+\|g_1\|_{C^{3+\alpha}(\partial\Omega)}+\|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\right). \tag{149}$$

Proof: Let us begin by assuming that $C_{175} \leq 1$. Then the short time existence Theorem 47 states that there exists the time $T_0 = T_0(m, \alpha, \Omega)$ and the solution $u \in C^{4+\alpha, 1+\alpha/4}_{m+\alpha}(Q_{T_0})$ of (WF) and especially there is an estimate

$$\begin{aligned}\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_0})} &\leq C_{172}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^{m+\alpha}(\overline{\Omega})}\big) \\ &\leq C_{177}\end{aligned}$$

where $C_{177} = C_{177}(m, \alpha, \Omega)$ since $C_{175} \leq 1$. Let $T \in (0, T_0)$, then u solves the following problem (G)

$$\text{(G)} \qquad \begin{cases} \qquad \partial_t u = \, - L(\nabla u) D^4 u - \mathcal{R}\big(\nabla u, D^2 u, D^3 u\big), & \text{in } \overline{\Omega} \times (0, T], \\ u(x, 0) = u_0(x), & x \in \overline{\Omega}, \\ u(x, t) = g_0(x), & (x, t) \in \partial\Omega \times [0, T], \\ \dfrac{\partial u}{\partial \nu}(x, t) = g_1(x), & (x, t) \in \partial\Omega \times [0, T]. \end{cases}$$

Since $u \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_0})$ it follows that according to Lemma 45 $\mathcal{R}\big(\nabla u, D^2 u, D^3 u\big) \in C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)$. Combining this result with Lemma 39 we infer by Theorem 42 that there is a Schauder estimate for $T \in (0, T_0]$

$$\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} \leq C_{165}(T) \begin{pmatrix} \|\mathcal{R}\big(\nabla u, D^2 u, D^3 u\big)\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_{T_0})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} \\ + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \end{pmatrix}$$

where $C_{165} \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone function depending only on (G), Ω and the $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_0})$-norm of u since we can bound the ellipticity condition constant and the leading coefficients by Lemma 45

$$\lambda \leq \frac{1}{\big(1 + \|u\|^2_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_0})}\big)^2}, \qquad \|L(\nabla u)\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)} \leq C_{168}(1 + \|u\|^4_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_0})}).$$

It should be particularly noticed that as in the proof of Theorem 47, we can select $\gamma = \alpha/2 < \alpha$ and obtain

$$\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)} \leq C_{165}(T) \begin{pmatrix} C_{168}\Big(1 + \|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)}\Big)^{k_H} \|u\|^3_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \\ + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \end{pmatrix}.$$

Evidently, since $\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_0})} \leq C_{177}(m, \alpha, \Omega)$ we emphasize that there is an estimate on the Schauder constants

$$\text{(150)} \qquad \forall T \leq T_0\colon \qquad C_{165}(T) \leq C_{165}(T_0) := C_{178}(m, \alpha, \Omega).$$

Furthermore, there exists a constant $C_{179}(\alpha, \gamma, m, \Omega)$ such that $\forall T \leq T_0$:

$$\text{(151)} \quad \|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \leq C_{179} \;\Rightarrow\; C_{178} C_{168}\Big(1 + \|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)}\Big)^{k_H} \|u\|^2_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \leq \frac{1}{2C_{166}}$$

where constant C_{166} is from Lemma 43 and depends on α and γ

Next, we consider the proof of the short-time existence Theorem 47 and modify the set $\mathcal{M}_T$ defined in (M) by

$$\mathcal{M}_T := \left\{ w \in C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T) \;\middle|\; \begin{aligned} \|w - G_T\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_T)} \le C_{179}/2, \quad v(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ v(x,t) = g_0(x), \quad \partial_\nu v(x,t) = g_1(x), \quad (x,t) \in \partial\Omega \times [0,T] \end{aligned} \right\}.$$

Similar to the Theorem 47 we find a fixed point u of $G_T\colon \mathcal{M}_T \to \mathcal{M}_T$ whenever $T = T_1(\le T_0)$ is small enough. If we set $\gamma = \alpha/2$, then with the help of $C_{175} \le 1$ we see that T_1 depends only on m, α and Ω. Furthermore we notice that $u \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})$ and u is the solution of (G) for $T = T_1$. According to the above definition of $\mathcal{M}_T$ we conclude

$$\|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})} \le \frac{C_{179}}{2} + \|G_T\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})}. \tag{152}$$

In this step we choose $C_{175} = C_{175}(C_{179}, m, \Omega)$ small enough for the first time to achieve the $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(\overline{Q}_{T_1})$ norm of $G_T\overline{u}_0$ smaller than $C_{179}/2$:

$$\begin{aligned} &\|G_T\overline{u}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})} \\ &\quad \le C_{165}(T_1) \begin{pmatrix} C_{168}\Big(1 + \|\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})}\Big)^{k_H} \|\overline{u}_0\|^2_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})} \|\overline{u}_0\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})} \\ + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \end{pmatrix} \\ &\quad \le C_{179}/2 \end{aligned}$$

where $C_{165}(T_1)$ is a constant depending on $\|\overline{u}_0\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})}$ in a similar way to $C_{165}(T_1)$ of (G) depending on $\|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})}$. Finally, combining this result with (152) we infer

$$\|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})} \le C_{179}.$$

We can therefore apply the above statement (151). Namely, by using the $C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})$-Schauder estimate for the solution u of (G) in Theorem 47 and $T_1 \le T_0$ we then get

$$\begin{aligned} &\|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})} \\ &\quad \le C_{165}(T_1) \begin{pmatrix} C_{168}\Big(1 + \|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})}\Big)^{k_H} \|u\|^3_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})} \\ + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \end{pmatrix} \\ &\quad \overset{(150)}{\le} C_{178} \begin{pmatrix} C_{168}\Big(1 + \|u\|_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})}\Big)^{k_H} \|u\|^2_{C^{4+\gamma,1+\gamma/4}_{m+\gamma}(Q_{T_1})} C_{166} T_1^{\frac{\alpha-\gamma}{4}} \|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})} \\ + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \end{pmatrix} \\ &\quad \overset{(151)}{\le} C_{178} \left(\frac{1}{2C_{178}} \|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_1})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \right). \end{aligned}$$

Collecting terms, we conclude

$$\begin{aligned} \|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_{T_s})} &\le 2C_{178} \Big(\|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \Big) \\ &\le C_{176}(\Omega, m, \alpha) \Big(\|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^{m+\alpha}(\overline{\Omega})} \Big). \end{aligned}$$

$\square$

49 Theorem (Global Existence)
Suppose Ω is a domain in $\mathbb{R}^2$ with $C^{4+\alpha}$-smooth boundary then there exists $C_{180} = C_{180}(\alpha, \Omega) > 0$ such that if

$$\|u_0\|_{C^{1+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} < C_{180}$$

then the solution for the Willmore-flow (WF) *exists for all times $t \in (0, \infty)$*

Proof: We first assume, that $C_{180} \leq C_{175}(1, \alpha, \Omega)/2$ with C_{175} introduced in the previous Lemma. In fact, by Lemma 48 there is $T_1 = T_1(1, \alpha, \Omega)$ such that $u \in C^{4+\alpha, 1+\alpha/4}_{1+\alpha}(Q_{T_1})$ solves (WF) uniquely. Notice that we can estimate the space Hölder norm by its parabolic Hölder norm

$$\begin{aligned} \|u\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{1+\alpha}(Q_{T_1})} &\geq \sum_{|\beta|\leq 1} \sup_{(x,t)\in\overline{\Omega}\times(0,T_1]} \left|D^\beta_x u(x,t)\right| + \sum_{|\beta|=1} \sup_{t\in[0,T_1]} \left[D^\beta_x u(\,.\,,t)\right]_{C^\alpha(\overline{\Omega})} \\ &= \sup_{t\in[0,T_1]} \|u(t)\|_{C^{1+\alpha}(\overline{\Omega})}. \end{aligned} \tag{153}$$

Moreover we choose $C_{180} \leq C_{175}/(2C_{176})$, where especially $C_{176}(1, \alpha, \Omega)$ does not depend on C_{180}. Then by Lemma 48 we get

$$\begin{aligned} \sup_{t\in[0,T_1]} \|u(t)\|_{C^{1+\alpha}(\overline{\Omega})} &\overset{(153)}{\leq} \|u\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{1+\alpha}(Q_{T_1})} \leq C_{176}\left(\|u_0\|_{C^{1+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)}\right) \\ &\leq C_{175}/2. \end{aligned} \tag{154}$$

Due to the $C^{1+\alpha}(\overline{\Omega})$-bound in the previous equation, we can continue the Willmore flow for $t > T_1$ but for a short time.

Suppose, contrary to our claim, that there is a finite maximal existence time $T_2 > 0$ such that $\forall T \in (T_1, T_2)$ for the following boundary value problem

$$\begin{cases} \partial_t v = -L(\nabla v)D^4 v - \mathcal{R}\big(\nabla v, D^2 v, D^3 v\big), & \text{in } \Omega \times [T_1, T], \\ v(x, 0) = u(x, T_1), & x \in \overline{\Omega}, \\ v(x,t) = g_0(x), & (x,t) \in \partial\Omega \times [T_1, T], \\ \dfrac{\partial v}{\partial \nu}(x,t) = g_1(x), & (x,t) \in \partial\Omega \times [T_1, T]. \end{cases} \tag{155}$$

there exists a unique $C^{4+\alpha,1+\alpha/4}_{x,t}(\overline{\Omega}\times[T_1, T])$-solution. Therefore, we can set $\forall t \in [T_1, T_2) : \, u(x,t) := v(x,t)$. Further, we will consider the maximal time $T_3 \in [T_1, T_2)$ such that

$$\forall t \in [0, T_3]: \qquad \|u(t)\|_{C^{1+\alpha}(\overline{\Omega})} \leq C_{175}/2. \tag{156}$$

Obviously, by (154) we obtain $T_3 \geq T_1$. Combining $C_{180} \leq C_{175}/2$ with the bound (156) we have

$$\forall t \in [0, T_3]: \qquad \|u(t)\|_{C^{1+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} \leq C_{175}/2 + C_{175}/2 = C_{175}. \tag{157}$$

By virtue of the previous Lemma 48 with the same time $T_1(1, \alpha, \Omega)$ it follows that $\forall t \in [T_1, T_3]$:

$$\begin{aligned} \|u\|_{C^{4+\alpha,1+\alpha/4}_{1+\alpha}(\overline{\Omega}\times(t-T_1,t])} &\overset{(157)}{\leq} C_{176}\left(\|u(\,.\,,t-T_1)\|_{C^{1+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)}\right) \\ &\overset{(157)}{\leq} C_{176}C_{175}. \end{aligned}$$

The definition of the weighted Hölder spaces leads to the estimate $\forall t \in [T_1, T_3]$:

$$
\begin{aligned}
\|u\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{1+\alpha}(\overline{\Omega}\times(t-T_1,t])} \geq\ & \sum_{|\beta|\leq 1} \sup_{(x,t')\in\overline{\Omega}\times(t-T_1,t]} \left|D_x^\beta u(x,t')\right| \\
&+ \sup_{(x,t')\in\overline{\Omega}\times(t-T_1,t]} (t'-t+T_1)^{\frac{2-\alpha}{4}} \left|D_x^3 u(x,t')\right| \\
&+ \sup_{(x,t')\in\overline{\Omega}\times(t-T_1,t]} (t'-t+T_1)^{\frac{1-\alpha}{4}} \left|D_x^2 u(x,t')\right| \\
\geq\ & \sup_{x\in\overline{\Omega}} T_1^{\frac{2-\alpha}{4}} \left|D_x^3 u(x,t)\right| + \sup_{x\in\overline{\Omega}} T_1^{\frac{1-\alpha}{4}} \left|D_x^2 u(x,t)\right| + \sum_{|\beta|\leq 1} \sup_{x\in\overline{\Omega}} \left|D_x^\beta u(x,t)\right|.
\end{aligned}
$$

Since the minimal existence time T_1 depends only on α and Ω we conclude

$$
\forall t \in [T_1, T_3]\colon \qquad \|u(t)\|_{C^3(\overline{\Omega})} \leq \left(1 + T_1^{\frac{-2+\alpha}{4}} + T_1^{\frac{-1+\alpha}{4}}\right) C_{176} C_{175} \leq C_{181}
$$

where C_{181} depends only on α and Ω. By the interpolation Theorem 10 there exists $0 < \beta < 1$ such that

$$
\begin{aligned}
\forall t \in [T_1, T_3]\colon \qquad \|u(t)\|_{C^{1+\alpha}(\overline{\Omega})} &\leq C_{182}(\alpha,\Omega) \|u(t)\|_{C^2(\overline{\Omega})} \leq C_{183}(\alpha,\Omega) \|u(t)\|^\beta_{L^2(\Omega)} \cdot \|u(t)\|^{1-\beta}_{C^{2+\alpha}(\overline{\Omega})} \\
&\leq C_{183} \|u(t)\|^\beta_{L^2(\Omega)} \big(C_{182}(\alpha,\Omega)\|u(t)\|_{C^3(\overline{\Omega})}\big)^{1-\beta} \\
&\leq C_{183} \|u(t)\|^\beta_{L^2(\Omega)} \big(C_{182} C_{181}\big)^{1-\beta}
\end{aligned}
$$

with C_{181}, C_{182} and C_{183} constants depending only on Ω and α, and especially independent of C_{180}. Let us suppose for the moment that we could bound $\|u(t)\|_{L^2(\Omega)}$ for all times $t \in [T_1, T_3]$ small enough so that

$$
\forall t \in [T_1, T_3]\colon \qquad \|u(t)\|_{C^{1+\alpha}(\overline{\Omega})} < \frac{C_{175}}{4}
$$

which would be a contradiction to (156). Thus, we could conclude $T_3 = T_2$ so especially in the maximal time T_2 one still would have $C^{1+\alpha}$-regularity for $u(\,.\,, T_2)$. Then, it would be possible to continue the solution, which contradicts the maximality of T_2. This would complete the proof of $T_2 = \infty$.

So our goal is to show, that with smaller C_{180} we can achieve $\|u(t)\|_{L^2(\Omega)}$ small enough. We first observe, that from (153)

$$
\begin{aligned}
&\sup_{x\in\overline{\Omega}} T_1^{\frac{1-\alpha}{4}} \left|D_x^2 u(x,T_1)\right| + \sum_{|\beta|\leq 1} \sup_{x\in\overline{\Omega}} \left|D_x^\beta u(x,T_1)\right| \leq \|u\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{1+\alpha}(Q_{T_1})} \\
\Rightarrow\quad & \|u(T_1)\|_{C^2(\overline{\Omega})} \leq (1 + T_1^{-\frac{1-\alpha}{4}}) \|u\|_{C^{4+\alpha,1+\frac{\alpha}{4}}_{1+\alpha}(Q_{T_1})} \\
&\overset{(154)}{\leq} C_{183} C_{176} \left(\|u_0\|_{C^{1+\alpha}(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)}\right) \\
&\leq C_{183} C_{176} C_{180}
\end{aligned}
$$

where C_{183} depends on T_1 and T_1 still depends only on α and Ω. By choosing C_{180} small enough we can get $\|u(T_1)\|_{C^2(\overline{\Omega})}$ even smaller. Moreover, $u(T_1)$ has finite Willmore energy and it is decreasing for all $t > T_1$.

$$
\forall t > T_1\colon \qquad \mathcal{W}\big(u(t)\big) \leq \mathcal{W}\big(u(T_1)\big) \leq \|u(T_1)\|^3_{C^2(\overline{\Omega})} |\Omega|.
$$

By choosing C_{180} small enough with Theorem 16 b we achieve $\|u(t)\|_{L^2(\Omega)}$ so small enough. $\square$

50 Corollary
Suppose Ω is a domain in $\mathbb{R}^2$ with $C^{4+\alpha}$ boundary, $\|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)} = 0$ then there exists $C_{184} = C_{184}(\alpha, \Omega) > 0$ such that if

$$\|u_0\|_{C^{1+\alpha}(\overline{\Omega})} < C_{184}$$

then the solution for the Willmore-flow (WF) *exists for all times $t \in (0, \infty)$.*

6.4 Time-Weighted C^1-$C^{4+\alpha}$-case

In this subsection, we weaken the regularity assumptions on the initial data to $u_0 \in C^1(\overline{\Omega})$ on the cost of also imposing a smallness condition on the data. That means we set $s = 1$, $f = \mathcal{R}$ and $(a_\beta) \cong L(\nabla u)$ with time-constant Dirichlet boundary values $\varphi = g_0$ and $h = g_1$. Like in the previous case, for proving the short-time existence in the $C_1^{4+\alpha,1+\frac{\alpha}{4}}(Q_T)$-space, we have to establish some auxiliary results. Let $\Omega \subset \mathbb{R}^n$ be bounded with $C^{4+\alpha}$-boundary $\partial\Omega$ and $T > 0$. We consider the norm

$$\|u\|_{C^{1,\frac{1}{4}}_{x,t}(\overline{Q}_T)} = \sum_{|\beta|\le 1} \sup_{(x,t)\in\overline{Q}_T} \left|D_x^\beta u(x,t)\right| + \sup_{x\in\overline{\Omega}} \left[u(x,.)\right]_{C^{\frac{1}{4}}\left([0,T]\right)}$$

and for parameter $s \le \ell$ the norm:

$$\|u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)} = \sup_{t<T} t^{\frac{3+\alpha}{4}} [u]^{4+\alpha}_{Q'_t} + \sum_{2\le 4k+|\beta|\le 4} \sup_{(x,t)\in Q_T} t^{\frac{4k+|\beta|-1}{4}} \left|D_t^k D_x^\beta u(x,t)\right| + \|u\|_{C^{1,\frac{1}{4}}_{x,t}(\overline{Q}_T)}$$

where $Q'_t = \overline{\Omega} \times [t/2, t]$ and:

$$[u]^{4+\alpha}_{Q'_t} = \sum_{4k+|\beta|=4} \sup_{t'\in[t/2,t]} \left[D_t^k D_x^\beta u(\,.\,,t')\right]_{C^\alpha(\overline{\Omega})} + \sum_{1\le 4k+|\beta|\le 4} \sup_{x\in\overline{\Omega}} \left[D_t^k D_x^\beta u(x,.)\right]_{C^{\frac{4+\alpha-4k-|\beta|}{4}}\left([t/2,t]\right)}.$$

Let us compare this norm with that of $C^{4+\alpha,1+\alpha/4}_{1+\alpha}(Q_T)$. The term $[u]^{4+\alpha}_{Q'_t}$ remained the same, but gets more weight $t^{\alpha/4}$ more weight as well as the $|D_t^k D_x^\beta u(x,t)|$-terms. Moreover, in comparison to the $C^{1+\alpha,(1+\alpha)/4}_{x,t}(\overline{Q}_T)$-case the $C^{1,1/4}_{x,t}(\overline{Q}_T)$-norm is missing the spatial Hölder seminorm $[\nabla u(\,.\,,t)]_{C^\alpha(\overline{\Omega})}$ and the temporal Hölder seminorm $[\nabla u(x,.)]_{C^{\alpha/4}([0,T])}$. Also, it is not clear how to use the trick with changing the Hölder power to $\gamma \le \alpha$. Thus, we will also have to choose the parabolic boundary norms small enough for us to apply a fixed point argument. Therefore, we drop γ in the following lemmas.

51 Lemma
Let $0 < \alpha < 1$. If $u, v, w \in C_1^{4+\alpha,1+\frac{\alpha}{4}}(Q_T)$ and $T \le 1$. Then there exists a constant $C_{185} = C_{185}(\Omega)$ such that

(158) $$\|D^3 w D^2 u\|_{C^{\alpha,\frac{\alpha}{4}}_{-3}(Q_T)} \le C_{185} \|D^3 w\|_{C^{1+\alpha,\frac{1+\alpha}{4}}_{-2}(Q_T)} \cdot \|D^2 u\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)},$$

(159) $$\|D^2 u D^2 w D^2 v\|_{C^{\alpha,\frac{\alpha}{4}}_{-3}(Q_T)} \le C_{185} \|D^2 u\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)} \cdot \|D^2 w\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)} \cdot \|D^2 v\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)},$$

(160) $$\|\nabla u\|_{C^{\alpha,\frac{\alpha}{4}}_{0}(Q_T)} \le C_{185} \|\nabla u\|_{C^{3+\alpha,\frac{3+\alpha}{4}}_{0}(Q_T)}.$$

Proof: See Appendix Lemma 95. □

Further, we again show preliminary results.

52 Lemma (Hölder Estimates I)
Let $0 < \alpha < 1, T \leq 1$. Then there exist constants $C_{186} = C_{186}(\Omega, \alpha)$ and $k_H \in \mathbb{N}$ depending on algebraic structure of $\mathcal{R}$ and L, so that it holds for all $u \in C_1^{4+\alpha,1+\alpha/4}(Q_T)$

$$\begin{aligned}\big\|\mathcal{R}(\nabla u, D^2u, D^3u)\big\|_{C_{-3}^{\alpha,\alpha/4}(Q_T)} &\leq C_{186}\Big(1 + \|\nabla u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}\Big)^{k_H}\|\nabla u\|^3_{C_1^{4+\alpha,1+\alpha/4}(Q_T)},\\ \sum_{k+\ell=4}\|L_{k\ell}(\nabla u)\|_{C_0^{\alpha,\alpha/4}(Q_T)} &\leq C_{186}\Big(1 + \|\nabla u\|^4_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}\Big).\end{aligned}$$

Proof: The proof is similar to Appendix Lemma 94. □

53 Lemma (Hölder Estimates II)
Let $0 < \alpha < 1, T \leq 1$. Then there exist constants $C_{187} = C_{187}(\Omega, \alpha)$ and $k'_H \in \mathbb{N}$ depending on algebraic structure of $\mathcal{R}$ and L, so that it holds for all $u, w \in C_1^{4+\alpha,1+\alpha/4}(Q_T)$

$$\begin{aligned}&\big\|\mathcal{R}(\nabla u, D^2u, D^3u) - \mathcal{R}(\nabla w, D^2w, D^3w)\big\|_{C_{-3}^{\alpha,\alpha/4}(Q_T)}\\ &\qquad\leq C_{187}\Big(1 + \max\Big\{\|u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}, \|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}\Big\}\Big)^{k'_H}\\ &\qquad\qquad\cdot \max\Big\{\|u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}, \|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}\Big\}^2 \cdot \|u-w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)},\\ &\sum_{k+\ell=4}\|L_{k\ell}(\nabla u) - L_{k\ell}(\nabla w)\|_{C_0^{\alpha,\alpha/4}(Q_T)}\\ &\qquad\leq C_{187}\Big(1 + \max\Big\{\|u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}, \|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}\Big\}\Big)^{k'_H} \cdot \|u-w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_T)}.\end{aligned}$$

Proof: The proof is similar to 94, where additionally we need to consider how to rewrite a difference of polynomials as in Lemma 96. □

54 Theorem (Short Time Existence)
Let $0 < \alpha < 1, T = 1$. Then there is a constant $C_{188} = C_{188}(\Omega, \alpha)$ such that if $u_0 \in C^1(\overline{\Omega})$, $g_0 \in C^{4+\alpha}(\partial\Omega)$ and $g_1 \in C^{3+\alpha}(\partial\Omega)$ as well as

$$\|u_0\|_{C^1(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} < C_{188}$$

then there exists a solution $u \in C_1^{4+\alpha,1+\alpha/4}(Q_1)$ of the initial Willmore flow problem (WF).

Proof: As in the proof of Theorem 47, we split the proof into several steps.

(1) Definition of the iteration map and set
In the same manner as in (1) in the proof of Theorem 47 we extend the boundary data and get $\overline{u}_0 \in C_1^{4+\alpha,1+\alpha/4}(Q_1)$ that solves (A). Also, we obtain with the Schauder estimate in Theorem 42 that

$$\|\overline{u}_0\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \leq C_{189}\big(\Omega, \|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^1(\overline{\Omega})}\big). \tag{161}$$

For the Banach fixed point theorem we have to define iteration map $G_1\colon C_1^{4+\alpha,1+\alpha/4}(Q_1) \to C_1^{4+\alpha,1+\alpha/4}(Q_1)$ by setting $v = G_1 w$ as a solution of (G) Then since $w \in C_1^{4+\alpha,1+\alpha/4}(Q_1)$, by to Lemma 52 $L(\nabla w) \in C_0^{\alpha,\alpha/4}(Q_1)$, $\mathcal{R}\big(\nabla w, D^2w, D^3w\big) \in C_{-3}^{\alpha,\alpha/4}(Q_1)$. The uniform ellipticity is obtained by Lemma 39, and by Theorem 41 there exists $v = G_1 w \in C_1^{4+\alpha,1+\alpha/4}(Q_1)$. Hence, the mapping is well-defined.

Since we assume that $T=1$, we also have the Schauder estimate

$$\|G_1\overline{w}\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \overset{\text{Thm. 42}}{\leq} C_{165}(1)\left[\begin{array}{l}\|\mathcal{R}(\nabla w, D^2 w, D^3 w)\|_{C_{-3}^{\alpha,\alpha/4}(Q_1)} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} \\ \quad + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^1(\overline{\Omega})}\end{array}\right]. \tag{162}$$

Let us define a non-trivial set which is characterized by boundary conditions and the smallness of the $C_1^{4+\alpha,1+\alpha/4}(Q_1)$-norm a

$$\mathcal{M}_1 := \left\{ w \in C_1^{4+\alpha,1+\alpha/4}(Q_1) \,\middle|\, \begin{array}{l} \|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \leq \hat{C}, \quad w(\,.\,,0) = u_0, \\ w(x,t) = g_0(x), \quad \dfrac{\partial w}{\partial \nu}(x,t) = g_1(x), \quad (x,t) \in \partial\Omega \times [0,1]. \end{array} \right\} \tag{163}$$

where $\hat{C}$ is a constant that we specify in the following. The set $\mathcal{M}_1$ is non-empty because it contains in 1 constructed $\overline{u}_0$ for C_{188} small enough.

2 G **is a self-map**

Now we observe that for $w \in \mathcal{M}_1$ it holds

$$\|L_{\beta_1,\beta_2}(\nabla \overline{u}_0)\|_{C_0^{\alpha,\alpha/4}(Q_1)} \leq C_{168}\Big(1+\hat{C}^4\Big),$$

and we also obtain a uniform ellipticity by (141) with

$$\frac{|\xi|^4}{\Big(1+\hat{C}^2\Big)^2} \leq \sum_{k+\ell=4} L_{k\ell}(\nabla w)\xi_1^k \xi_2^\ell \leq \; 4|\xi|^4.$$

So by (162) and $T=1$ we conclude by Lemma 52 that it holds

$$\begin{aligned}\|G_1 w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} &\leq C_{165}(1)\left[\begin{array}{l}C_{186}(1+\|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)})^{k_H}\|w\|^3_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} \\ + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^1(\overline{\Omega})}\end{array}\right] \\ &\leq C_{190}\left[\begin{array}{l}C_{186}(1+\hat{C})^{k_H}\hat{C}^3 + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} \\ + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^1(\overline{\Omega})}\end{array}\right].\end{aligned}$$

From now on, we always assume that $\hat{C}$ satisfies the following condition

$$C_{190}C_{186}(1+\hat{C})^{k_H}\hat{C}^2 \leq \frac{1}{2}. \tag{C1}$$

We also have to set a condition on the parabolic boundary values

$$\|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} + \|u_0\|_{C^1(\overline{\Omega})} \leq \frac{\hat{C}}{2C_{190}}. \tag{C2}$$

Consequently, by (C1) and (C2) $\|G_1 w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \leq \hat{C}$ it follows that G_1 is a self-map.

3 G **is a contraction**

Let $u, w \in \mathcal{M}_1$, then as in Theorem 41 $G_1 u - G_1 w$ solves the linear initial value problem (Z). The operator $\frac{1}{2}\big(L(\nabla u) + L(\nabla w)\big) \in C_0^{\alpha,\alpha/4}(Q_1)$ is again uniformly elliptic, with the same time-independent constants λ and Λ as for $L(\nabla u)$ and $L(\nabla w)$. Furthermore, it follows that

$$\begin{aligned}\|D^4(G_1 u + \, G_1 w)\|_{C_0^{\alpha,\alpha/4}(Q_1)} &\leq \|G_1 u\|_{C_0^{4+\alpha,1+\alpha/4}(Q_1)} \; + \|G_1 w\|_{C_0^{4+\alpha,1+\alpha/4}(Q_1)} \\ &\leq 2\hat{C}.\end{aligned} \tag{164}$$

Once again, by Theorem 41 we get the Schauder estimate $G_1 u - G_1 w$ as solution of (Z) with a constant $C_{165}(1)$ depending only on $\|g_0\|_{C^{4+\alpha}(\partial\Omega)}, \|g_1\|_{C^{3+\alpha}(\partial\Omega)}, \|u_0\|_{C^1}(\overline{\Omega})$ and Ω such that

$$\begin{aligned}
&\|G_1 u - G_1 w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \\
&\quad\leq C_{165}(1)\cdot \left\| \begin{matrix} -\frac{1}{2}\big(L(\nabla u)-L(\nabla w)\big)\big(D^4(G_1u+G_1w)\big) \\ -\mathcal{R}\big(\nabla u, D^2u, D^3u\big)+\mathcal{R}\big(\nabla w, D^2w, D^3w\big) \end{matrix} \right\|_{C_{-3}^{\alpha,\alpha/4}(Q_1)} \\
&\quad\overset{(137)}{\leq} \frac{C_{165}(1)}{2}\cdot \big\|\big(L(\nabla u)-L(\nabla w)\big)\big\|_{C_0^{\alpha,\alpha/4}(Q_1)}\cdot \big\|\big(D^4(G_1u+G_1w)\big)\big\|_{C_{-3}^{\alpha,\alpha/4}(Q_1)} \\
&\qquad + C_{165}(1)\,\big\|\mathcal{R}(\nabla u, D^2u, D^3u)-\mathcal{R}(\nabla w, D^2w, D^3w)\big\|_{C_{-3}^{\alpha,\alpha/4}(Q_1)} \\
&\quad\overset{(164)}{\leq} C_{165}(1)C_{187}\hat{C}\Big(1+\max\Big\{\|u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}, \|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}\Big\}\Big)^{k'_H} \\
&\qquad\qquad\qquad\qquad \cdot \|u-w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \\
&\qquad + C_{165}(1)C_{187}\Big(1+\max\Big\{\|u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}, \|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}\Big\}\Big)^{k'_H} \\
&\qquad\qquad\qquad\qquad \cdot \max\Big\{\|u\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}, \|w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}\Big\}^2 \cdot \|u-w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \\
&\quad\leq C_{165}(1)C_{187}(1+\hat{C})^{k'_H}(\hat{C}+\hat{C}^2)\|u-w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}.
\end{aligned}$$

Therefore, we add another condition on $\hat{C}$

$$\text{(C3)} \qquad C_{165}(1)C_{187}(1+\hat{C})^{k'_H}(\hat{C}+\hat{C}^2) \leq \frac{1}{2}.$$

It follows that by conditions (C1), (C2), (C3) we have

$$\|G_1 u - G_1 w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)} \leq \frac{1}{2}\cdot \|u-w\|_{C_1^{4+\alpha,1+\alpha/4}(Q_1)}.$$

That means that G_1 is a contraction on $\mathcal{M}_1$.

(4) **Applying the fixed point Theorem**

In this step for $\|u_0\|_{C^1(\overline{\Omega})} + \|g_0\|_{C^{4+\alpha}(\partial\Omega)} + \|g_1\|_{C^{3+\alpha}(\partial\Omega)} \leq \frac{\hat{C}}{2C_{190}} =: C_{188}$ (see (C2)) small enough one uses the fixed point theorem and obtains a fixed point $v^* \in \mathcal{M}_1 \subset C_1^{4+\alpha,1+\alpha/4}(Q_1)$ with $v^* = H_1 v^*$. This v^* solves the original Willmore-flow problem (WF) in the space $\mathcal{M}_1$. □

6.5 Time-Unweighted $C^{2+\alpha}$-$C^{2+\alpha}$-case

Until now, unlike in the elliptic case, we did not use the divergence structure of the Willmore-flow equation (133) derived in [KL12]. In this section, we want to use this structure and incorporate the results from [DZ15] for Schauder's estimate and solvability for divergence-type higher-order systems in cylindrical domains. Here, we will deal with weaker parabolic Hölder spaces in $C^{2+\alpha}$-initial value and $C^{2+\alpha}$-boundary values framework, which will allow us to work with the Willmore-flow without weighted derivatives.

In this case we consider the norm from Subsection 6.2 from with $\ell = 2+\alpha$

$$\begin{aligned}
\|u\|_{C_{x,t}^{2+\alpha,\frac{2+\alpha}{4}}(\overline{Q}_T)} &= \sum_{|\beta|\leq 2}\sup_{(x,t)\in\overline{Q}_T}\Big|D_x^\beta u(x,t)\Big| + \sum_{|\beta|=2}\sup_{t\in[0,T]}\Big[D_x^\beta u(\,.\,,t)\Big]_{C^\alpha(\overline{\Omega})} \\
&\quad + \sum_{0\leq|\beta|\leq 2}\sup_{x\in\overline{\Omega}}\Big[D_x^\beta u(x,.)\Big]_{C^{\frac{2+\alpha-|\beta|}{4}}\left([0,T]\right)},
\end{aligned}$$

Next, we want to recall some notation used in from [DZ15] for the parabolic Hölder spaces (there they used reversed order of space and time). For a function u we define semi norms

$$\text{"parabolic"} \qquad [u]_{a,b,\overline{Q}_T} := \sup\left\{ \frac{|u(x,t)-u(y,s)|}{|t-s|^a+|x-y|^b} \,\middle|\, (x,t),(y,s)\in\overline{Q}_T, (t,x)\neq(s,y)\right\},$$

$$\text{"temporal"} \qquad \langle u\rangle_{a,\overline{Q}_T} := \sup\left\{ \frac{|u(x,t)-u(x,s)|}{|t-s|^a} \,\middle|\, (x,t),(x,s)\in\overline{Q}_T, t\neq s\right\},$$

$$\text{"spatial"} \qquad [u]^*_{b,\overline{Q}_T} := \sup\left\{ \frac{|u(x,t)-u(y,t)|}{|x-y|^b} \,\middle|\, (x,t),(y,t)\in\overline{Q}_T, x\neq y\right\}$$

and the spatial norm

$$\|u\|^*_{\alpha,\overline{Q}_T} := \|u\|_{L^\infty(\overline{Q}_T)} + [u]^*_{\alpha,\overline{Q}_T} \cong \sup_{(x,t)\in\overline{Q}_T} \big|u(x,t)\big| + \sup_{t\in[0,T]} \big[u(\,.\,,t)\big]_{C^\alpha(\overline{\Omega})},$$

where $\alpha\in(0,1)$. By $C^{\alpha*}(\overline{Q}_T)$ we denote the space corresponding to $\|\,.\,\|^*_{\alpha,\overline{Q}_T}$. Dong and Zhang also defined an equivalent norm of $C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)$

$$\|u\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)} \cong \|u\|_{\frac{2+\alpha}{4},2+\alpha} := \|u\|_{L^\infty(\overline{Q}_T)} + \sum_{|\beta|\le 2} [D^\beta u]_{\frac{2+\alpha-|\beta|}{4},\alpha,\overline{Q}_T}.$$

Their fourth-order operators are of the form

$$\mathcal{L}u = \sum_{|k|,|\ell|\le 2} D^k(A^{k\ell}D^\ell u)$$

where for each $k\le 2$ and $\ell\le 2$, $A^{k\ell}$ is an real-valued measurable function with

$$\forall |k|,|\ell|\le 2: \quad |A^{k\ell}|\le K$$

for some $K>0$. They also impose the ellipticity condition on the leading coefficients

$$\sum_{|k|=|\ell|=2} A^{k\ell}\xi^k\xi^\ell \ge \lambda|\xi|^2 \tag{165}$$

with some constant $\lambda>0$. Their main result was the following Schauder estimate.

55 Theorem (Schauder Estimate and Existence)
Let $\alpha\in(0,1)$ and $T\in(0,\infty]$. Assume $f_\beta\in C^{\alpha*}(\overline{Q}_T)$ for $|\beta|=2$ and $f_\beta\in L_\infty(\overline{Q}_T)$ for $|\beta|<2$. Suppose that the the operator $\mathcal{L}$ satisfy the Legendre-Hadamard condition (165)*, and $A^{k\ell}\in C^{\alpha*}(\overline{Q}_T)$. Let g be a smooth function in $\mathbb{R}^3$ and $\Omega\in C^{2+\alpha}$. Then*

$$\begin{cases} \partial_t u + \mathcal{L}u = \sum_{|\beta|\le 2} D^\beta f \quad \text{in} \quad \Omega\times(0,T), \\ u = g, Du = Dg \quad \text{on} \quad \partial\Omega\times[0,T), \\ u = g \quad \text{on} \quad \Omega\times\{0\} \end{cases}$$

has a unique variational solution u such that $u\in C^{2+\alpha,\frac{2+\alpha}{4}}_{x,t}\big(\overline{\Omega}\times\big[0,\min(T,k)\big)\big)$ for any $k>0$, and it satisfies the Schauder estimate

$$\|u\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)} \le C_{191}\big(\|u\|_{L^2(\Omega\times(0,T))} + F + G\big), \tag{166}$$

where

$$F=\sum_{|\beta|=2}[f_\beta]^*_{\alpha,\overline{Q}_T}+\sum_{|\beta|<2}\|f_\beta\|_{L^\infty(\overline{Q}_T)},$$
$$G=\sum_{|\beta|=2}\|D^\alpha g\|^*_{\alpha,\overline{Q}_T}+\sum_{|\beta|<2}\|D^\beta g\|_{L^\infty(\overline{Q}_T)}+\|g_t\|_{L^\infty(\overline{Q}_T)}$$

and $C_{191}>0$ is a constant depending only on λ,K,Ω,α and a bound on $\|A^{k\ell}\|^*_\alpha$. Moreover, for any constant $k>0$, we have

$$\|u\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{x,t}(\overline{\Omega}\times[0,\min(T,k)))}\leq C_{192}e^{C_{193}k}\big(F_k+G_k\big),$$

where

$$F_k=\sum_{|\beta|=2}[f_\beta]^*_{\alpha,\overline{\Omega}\times[0,\min(T,k))}+\sum_{|\beta|<2}\|f_\beta\|_{L^\infty(\overline{\Omega}\times[0,\min(T,k)))}$$
$$G_k=\sum_{|\beta|=2}\|D^\beta g\|^*_{\alpha,\overline{\Omega}\times[0,\min(T,k))}+\sum_{|\beta|<2}\|D^\beta g\|_{L^\infty(\overline{\Omega}\times[0,\min(T,k)))}+\|\partial_t g\|_{L^\infty(\overline{\Omega}\times[0,\min(T,k)))},$$

$C_{192}>0$ is a constant depending only on λ,K,Ω,α and a bound on $\|A^{k\ell}\|^*_\alpha$, as well as $C_{193}>0$ is a constant depending only on λ, and K.

Proof: [DZ15] Theorem 2.1 p. 5. □

Let us now recall the Willmore flow graphical representation by Koch and Lamm (133) with (134)

$$\partial_t u+\Delta^2u=f_0[u]+\nabla_i f_1^i[u]+D^2_{ij}f_2^{ij}[u]$$

with the right-hand side (see [KL12, Lemma 3.2 p. 215])

$$\begin{aligned}f_0[u]&=D^2u\star D^2u\star D^2u\star\sum_{k=1}^4 Q^{-2k}P_{2k-2}(\nabla u),\\ f_1[u]&=D^2u\star D^2u\star\sum_{k=1}^4 Q^{-2k}P_{2k-1}(\nabla u),\\ f_2[u]&=D^2u\star\sum_{k=1}^2 Q^{-2k}P_{2k}(\nabla u).\end{aligned}\tag{167}$$

Now, if we combine $\Delta^2 u$ with $f_2[u]$ to an $\mathcal{L}_{\nabla u}u$ then we obtain

$$\mathcal{L}_{\nabla u}u=\Delta^2u-D^2_{ij}f_2^{ij}[u]=\Delta^2u-\Delta\left(\frac{\nabla Q\cdot\nabla u}{Q}\right)-D^2_{ij}\left(\frac{\nabla_i u\nabla_j u}{Q}H\right).$$

Therefore, for each $u\in C^1(\overline{\Omega})$ we can define the elliptical operator $\mathcal{L}_{\nabla u}$ acting on $w\in W^{2,2}_{\mathrm{loc}}(\Omega)$ with

$$\mathcal{L}_{\nabla u}w=\Delta^2w-\Delta\left(\frac{(\nabla u)^T\cdot D^2w\circ\nabla u}{Q^2[u]}\right)-D^2_{ij}\left(\frac{\nabla_i u\nabla_j u}{Q[u]}\left(\frac{\Delta w}{Q[u]}-\frac{(\nabla u)^T\cdot D^2w\circ\nabla u}{Q^3[u]}\right)\right)\tag{168}$$

where $Q[u]=\sqrt{1+|\nabla u|^2}$ depends only on u. For this operator, we have the ellipticity condition

$$\sum_{|k|,|\ell|=2}\mathcal{L}_{\nabla u,k\ell}\xi^k\xi^\ell=\left(|\xi|^2-\left(\xi\cdot\frac{1}{Q[u]}\nabla u\right)^2\right)^2\geq|\xi|^4\left(1-\frac{|\nabla u|^2}{Q^2[u]}\right)^2\geq\frac{|\xi|^4}{Q^4[u]}.\tag{169}$$

It follows that we have the Willmore flow equation in the form

$$\partial_t u + \mathcal{L}_{\nabla u} u = f_0[u] + \nabla_i f_1^i[u].$$

With these preparatory results, we can prove the short-time existence.

56 Theorem (Short Time Existence)
Let $\alpha \in (0,1)$, $\Omega \in C^{2+\alpha}$ and $u_0 \in C^{2+\alpha}(\overline{\Omega})$, then there exists time T depending only on α, the bound on $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} \le C$ and Ω such that there exists a unique variational solution $u \in C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)$ of the initial Willmore flow problem

$$\begin{cases} \partial_t u + \Delta^2 u = f_0[u] + \nabla_i f_1^i[u] + D^2_{ij} f_2^{ij}[u] \quad \textit{in} \quad \Omega \times (0,T), \\ u = u_0, Du = Du_0 \quad \textit{on} \quad \partial\Omega \times [0,T), \\ u = u_0 \quad \textit{on} \quad \Omega \times \{0\} \end{cases}$$

with the right-hand side (167).

Proof: We split the proof into six steps.
(1) **Definition of the iteration map and set**
Let $0 < \gamma = \alpha/2 < \alpha$. As in the proof of Theorem 47 we extend u_0 to $\overline{u}_0$ in time by setting $\overline{u}_0(x,t) := u_0(x)$ for all $x \in \overline{\Omega}$. Then, we define the iteration map $H_T\colon C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T) \to C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)$ in the following way. For each $w \in C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)$ we set $v = H_T w$ as a solution of

$$\begin{cases} \partial_t v = -\mathcal{L}_{\nabla w} v + f_0[w] + \nabla_i f_1^i[w], \quad \text{in } \Omega \times (0,T), \\ v(x,0) = u_0(x), \qquad x \in \overline{\Omega}, \\ v(x,t) = u_0(x), \qquad (x,t) \in \partial\Omega \times [0,T), \\ Dv(x,t) = Du_0(x), \qquad (x,t) \in \partial\Omega \times [0,T). \end{cases} \tag{170}$$

Since $w \in C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)$ it follows the coefficients of $\mathcal{L}_{\nabla w}$ are in $C^{\alpha *}(\overline{Q}_T)$, $f_1^i[w] \in L_\infty(\overline{Q}_T)$, $f_0[w] \in L_\infty(\overline{Q}_T)$ and by Theorem 55 there exists $H_T w \in C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)$. Thus, this mapping is well defined. Also, we assume that $T \le 1$, then we have the Schauder estimate

$$\|H_T w\|_{C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)} \le C_{192} e^{C_{193}} \left(\sum_{|\beta|<2} \|f_\beta[w]\|_{L^\infty(\overline{Q}_T)} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})} \right) \tag{171}$$

with C_{192}, C_{193} depending only on a bound on $\|w\|_{C^{\alpha *}(\overline{Q}_T)}$.

(2) **H is a self-map**
Let us define a non-trivial set

$$\mathcal{M}_T := \left\{ w \in C^{2+\alpha,\frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T) \;\middle|\; \|w - \overline{u}_0\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{x,t}(\overline{Q}_T)} \le 1 \right\}. \tag{172}$$

For $T \le 1$ it holds for all $w \in \mathcal{M}_T$

$$\begin{aligned} \|\text{the coefficients of } \mathcal{L}_{\nabla w}\|_{C^{\alpha *}(\overline{Q}_T)} + \|\nabla w\|_{C^{\alpha *}(\overline{Q}_T)} &\le C_{194} \|\nabla w\|_{C^{\alpha *}(\overline{Q}_T)} \le C_{195} \|w\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{x,t}(\overline{Q}_T)} \\ &\overset{w \in \mathcal{M}_T}{\le} C_{196}(\alpha, \gamma, \Omega, \|\overline{u}_0\|_{C^{2+\alpha}(\overline{\Omega})}) \end{aligned}$$

where C_{196} depends only on α, γ, Ω and the $C^{2+\gamma}(\overline{\Omega})$-norm of u_0 since $\gamma < \alpha$. That also means that there is an ellipticity constant $\lambda = \lambda(\alpha, \gamma, \Omega, \|\overline{u}_0\|_{C^{2+\alpha}(\overline{\Omega})})$. Now we observe that for $w \in \mathcal{M}_T$ by (171) and $T \leq 1$ with C_{192}, C_{193} depending on C_{196} it holds

$$
\begin{aligned}
\|H_T w\|_{C^{2+\alpha, \frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)} &\leq C_{192} e^{C_{193}} \left(\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + \sup_{t \in (0,T)} \left(\|D^2 w\|^3_{C^0(\overline{\Omega})} + \|D^2 w\|^2_{C^0(\overline{\Omega})} \right) \right) \\
&\leq C_{192} e^{C_{193}} \sup_{t \in (0,T)} \left(\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|w\|^3_{C^{2+\gamma, \frac{2+\gamma}{4}}_{x,t}(\overline{Q}_T)} + \|w\|^2_{C^{2+\gamma, \frac{2+\gamma}{4}}_{x,t}(\overline{Q}_T)} \right) \\
&\leq C_{197}(\alpha, \gamma, \Omega, \|u_0\|_{C^{2+\alpha}(\overline{\Omega})})
\end{aligned}
$$

by the definition of $\mathcal{M}_T$. Then we have for $w \in \mathcal{M}_t$ the estimate

$$
\begin{aligned}
\|H_T w - \overline{u}_0\|_{C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)} \leq \|H_T w\|_{C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})} &\leq C_{197} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})} \\
&\leq C_{198}(\alpha, \gamma, \Omega, \|u_0\|_{C^{2+\alpha}(\overline{\Omega})})
\end{aligned}
$$

and with the same technique as in Lemma 92 we conclude

$$
\|H_T w - \overline{u}_0\|_{C^{2+\gamma, \frac{2+\gamma}{4}}_{x,t}(\overline{Q}_T)} \leq C_{199} T^{\frac{\alpha-\gamma}{4}} \|H_T w - \overline{u}_0\|_{C^{2+\alpha, \frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)} \leq C_{199} C_{198} T^{\frac{\alpha-\gamma}{4}}.
$$

By choosing T small enough we can achieve $\|H_T w\|_{C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)} \leq 1$ and H_T is a self-map.

3 H_T **is a contraction**

Let $u, w \in \mathcal{M}_T$, then because $H_T w$ and $H_T u$ have the same initial values

$$
\begin{aligned}
H_T u(x,0) - H_T w(x,0) &= u_0(x) - u_0(x) = 0, \qquad x \in \overline{\Omega}, \\
H_T u(x,t) - H_T w(x,t) &= u_0(x) - u_0(x) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \\
D(H_T u - H_T w)(x,t) &= Du_0(x) - Du_0(x) = 0, \qquad (x,t) \in \partial\Omega \times [0,T].
\end{aligned}
$$

This means that $v := H_T u - H_T w$ solves the following linear initial value problem

$$
(\mathrm{Y}_w) \qquad \begin{cases} \partial_t v = -\dfrac{1}{2}\big(\mathcal{L}_{\nabla u} + \mathcal{L}_{\nabla w}\big)(v) - \dfrac{1}{2}\big(\mathcal{L}_{\nabla u} - \mathcal{L}_{\nabla w}\big)(H_T u + H_T w) \\ \qquad\qquad + f_0[u] - f_0[w] + \nabla_i(f_1^i[u] - f_1^i[w])), \ \text{ in } \Omega \times (0,T], \\ v(x,0) = 0, \qquad x \in \overline{\Omega}, \\ v(x,t) = 0, \quad Dv(x) = 0, \qquad (x,t) \in \partial\Omega \times [0,T]. \end{cases}
$$

Analogously by Theorem 55 one obtains the Schauder estimate $H_T u - H_T w$ as solution of (Y_w) with the same λ and constant C_{196}

$$
\begin{aligned}
&\|H_T u - H_T w\|_{C^{2+\alpha, \frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)} \\
&\quad \leq C_{192} e^{C_{193}} \cdot \left(\begin{aligned} &\|f_0[u] - f_0[w]\|_{L^\infty(\overline{Q}_T)} + \big\|f_1[u] - f_1[w]\big\|_{L^\infty(\overline{Q}_T)} \\ &+ C_{200} \|\nabla w - \nabla u\|_{C^{\alpha*}(\overline{Q}_T)} \cdot \Big(\|D^2 H_T u\|_{C^{\alpha*}(\overline{Q}_T)} + \|D^2 H_T w\|_{C^{\alpha*}(\overline{Q}_T)} \Big) \end{aligned} \right) \\
&\quad \overset{w \in \mathcal{M}_T}{\leq} C_{201}(\alpha, \gamma, \Omega, \|\overline{u}_0\|_{C^{2+\alpha}(\overline{\Omega})}) \cdot \|u - w\|_{C^{2+\gamma, \frac{2+\gamma}{4}}_{x,t}(\overline{Q}_T)}.
\end{aligned}
$$

Therefore, we conclude

$$
\|H_T w - H_T u\|_{C^{2+\gamma, \frac{2+\gamma}{4}}_{x,t}(\overline{Q}_T)} \leq C_{199} T^{\frac{\alpha-\gamma}{4}} \|H_T w - H_T u\|_{C^{2+\alpha, \frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)}
$$

$$\leq C_{199} C_{201} T^{\frac{\alpha-\gamma}{4}} \cdot \|u-w\|_{C_{x,t}^{2+\gamma, \frac{2+\gamma}{4}}(\overline{Q}_T)}.$$

Next, by choosing time small enough, we can achieve

$$\|H_T u - H_T w\|_{C_{x,t}^{2+\gamma, \frac{2+\gamma}{4}}(\overline{Q}_T)} \leq q \cdot \|u-w\|_{C_{x,t}^{2+\gamma, \frac{2+\gamma}{4}}(\overline{Q}_T)},$$

with $q < 1$ for all $u, w \in \mathcal{M}_T$. Thus, the mapping H_T is a contraction.

4 Applying the fixed point theorem

In this step for a time T small enough one uses the fixed point Theorem 8 in $C_{x,t}^{2+\gamma,(2+\gamma)/4}(\overline{Q}_T)$ and gets a fixed point $v^* \in \mathcal{M}_T \subset C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{Q}_T)$ with $v^* = H_T v^*$. This v^* solves the original Willmore flow problem (WF) in the space $\mathcal{M}_T$. There is still uniqueness in the space $C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{Q}_T)$ to show.

5 Uniqueness

Here, we will use the same initial value problem (Y_w). Let $v^* \in \mathcal{M}_T$ be the fixed point solution in 4. Furthermore we assume there is an another solution $w \in C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{Q}_{T'})$ where we consider only $T' < T$ without loss of generality. Additionally, let the time $0 < T_0 < T' < 1$, which we will take like in 3 small enough. Then $v^* - w$ is a solution of the (Y_w) and we have a similar Schauder estimate

$$\begin{aligned}
&\|v^* - w\|_{C_{x,t}^{2+\alpha, \frac{2+\alpha}{4}}(\overline{Q}_{T_0})} \\
&\leq C_{192} e^{C_{193}} \cdot \left(\begin{aligned} &\|f_0[u] - f_0[w]\|_{L^\infty(\overline{Q}_{T_0})} + \left\|f_1[u] - f_1[w]\right\|_{L^\infty(\overline{Q}_{T_0})} \\ &+ C_{202} \|\nabla w - \nabla u\|_{C^{\alpha*}(\overline{Q}_{T_0})} \cdot \Big(\|D^2 H_{T_0} u\|_{C^{\alpha*}(\overline{Q}_{T_0})} + \|D^2 H_{T_0} w\|_{C^{\alpha*}(\overline{Q}_{T_0})}\Big) \end{aligned} \right) \\
&\leq C_{203} \|v^* - w\|_{C_{x,t}^{2+\gamma, \frac{2+\gamma}{4}}(\overline{Q}_{T_0})}
\end{aligned}$$

with constant C_{203} depending only on α, γ, Ω and $C_{x,t}^{2+\gamma,(2+\gamma)/4}(\overline{Q}_{T'})$-norms of v^* and u. Since these norms are fixed we can choose T_0 and $\|\nabla u_0\|_{C^0(\overline{\Omega})}$ small enough so that

$$\|v^* - w\|_{C_{x,t}^{2+\alpha, \frac{2+\alpha}{4}}(\overline{Q}_{T_0})} \leq \frac{1}{2} \|v^* - w\|_{C_{x,t}^{2+\alpha, \frac{2+\alpha}{4}}(\overline{Q}_{T_0})}.$$

Thus we get:

$$\|v^* - w\|_{C_{x,t}^{2+\alpha, \frac{2+\alpha}{4}}(\overline{Q}_{T_0})} = 0.$$

This means that v^* and w are identical in $C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{Q}_{T_0})$. To end the proof, we also have to show the equality in $t \in (T_0, T']$. Here we observe, that the time T_0 depends only on Ω and the bounds of $C_{x,t}^{2+\alpha,(2+\alpha)/4}(\overline{Q}_{T_0})$-norms of v^* and w. Since $\|v^*(\,.\,, T_0) - w(\,.\,, T_0)\|_{C^{2+\alpha}(\overline{\Omega})} = 0$ by choosing the same uniqueness time T_0, one obtains uniqueness on $[0, \min\{2T_0, T'\}]$. Finally, we repeat this procedure until one reaches T'. We emphasize that $\gamma = \alpha/2$ depends only on α. □

In order to show global existence similar to Theorem 49, we need an a-priori estimate for the following problem

$$\begin{cases} \partial_t v = -\Delta^2 v + f_0[w] + \nabla_i f_1^i[w] + D_{ij}^2 f_2^{ij}[w], \quad \text{in } \Omega \times (0,T), \\ v = 0, Dv = 0 \quad \text{on} \quad \partial\Omega \times [0,T), \\ v = 0 \quad \text{on} \quad \Omega \times \{0\}. \end{cases} \tag{173}$$

In the next Lemma, for the global existence we will need to replace in the Schauder estimate (166) the norm $\|u\|_{L^2(\Omega\times(0,T))}$ in Theorem (55) by $\|u\|_{L^\infty(\Omega\times(0,T))}$. In contrast to the former, the latter can be controlled by the diameter estimate (a) in Theorem 16 by Grunau, Deckelnick, and Röger.

57 Lemma
Let $v, w \in C^{2+\alpha,\frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)$ such that v is a solution of (173) then

$$\langle v\rangle_{\frac{2+\alpha}{4},\overline{Q}_T} + [D^2 v]_{\frac{\alpha}{4},\alpha,\overline{Q}_T} \leq C_{204}\left(\|w\|_{L^\infty(\overline{Q}_T)} + \sum_{|\beta|<2}\|f_\beta[w]\|_{L^\infty(\overline{Q}_T)} + \sup_{t\in(0,T)}\|f_2[w](\,.\,,t)\|_{C^\alpha(\overline{\Omega})}\right)$$

where C_{204} depends on Ω and α.

Proof: See the proof of Proposition 5.2 in [DZ15]. $\square$

58 Theorem (Global Existence)
There exist a constant $C_{205} = C_{205}(\alpha,\Omega)$ such that if

$$\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} < C_{205} \tag{174}$$

then there exists a solution u of the Willmore flow problem

$$\begin{cases} \partial_t u + \Delta^2 u = f_0[u] + \nabla_i f_1^i[u] + D^2_{ij} f_2^{ij}[u] & \text{in} \quad \Omega\times(0,T),\\ u = u_0, Du = Du_0 \quad \text{on} \quad \partial\Omega\times[0,T), \\ u = u_0 \quad \text{on} \quad \Omega\times\{0\} \end{cases}$$

with the right-hand side (167) for all times, such that $\forall T\in(0,\infty): u\in C^{2+\alpha,\frac{2+\alpha}{4}}_{x,t}(\overline{Q}_T)$. Furthermore there exists a constant $C_{206} = C_{206}(\alpha,\Omega)$ such that

$$\forall t\in(0,\infty): \quad \|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{206}. \tag{175}$$

Proof: First, we take $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} < C_{205}$ and specify C_{205} later. By the short time existence in Theorem 56 we obtain a solution $u\in C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_{\tilde{T}})$ with some $\tilde{T}$ depending only on α, Ω and C_{205}. We can local extend the solution in time, if $u(\,.\,,\tilde{T})\in C^{2+\alpha}(\overline{\Omega})$, so that we can work with the maximum existence time $T_{\max} > 0$.

We assume $T_{\max} < \infty$ and consider constant $\hat{C}$ that we specify later. We also consider the maximal time $T^{\hat{C}}_{\max}$ that

$$\forall t\in[0,T^{\hat{C}}_{\max}): \quad \|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})} < \hat{C}.$$

Especially in this case $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} < \hat{C}$. By Lemma 57 we know that for all $T\in(0,T_{\max})$ it holds

$$\langle u-u_0\rangle_{\frac{2+\alpha}{4}\overline{Q}_T} + [D^2(u-u_0)]_{\frac{\alpha}{4},\alpha,\overline{Q}_T} \leq C_{204}\left(\|u-u_0\|_{L^\infty(\overline{Q}_T)} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + \sum_{|\beta|<2}\|f_\beta[u]\|_{L^\infty(\overline{Q}_T)} + \sup_{t\in(0,T)}\|f_2[u](\,.\,,t)\|_{C^\alpha(\overline{\Omega})}\right).$$

First, since $Q\geq|\nabla u|$ it follows $|Q^{-\ell}P_\ell(\nabla u)|\leq C_{207}$ with some constant C_{207} depending only on algebraic structure of $P_\ell(\nabla u)$. We conclude (where all constants are algebraic)

$$\|f_0[u]\|_{L^\infty(\overline{Q}_T)} \leq C_{208}\|D^2u\|^3_{L^\infty(\overline{Q}_T)}\sum_{k=1}^4\left\|Q^{-2k}P_{2k-2}(\nabla u)\right\|_{L^\infty(\overline{Q}_T)} \leq C_{209}\|D^2u\|^3_{L^\infty(\overline{Q}_T)},$$

$$
\begin{aligned}
\|f_1[u]\|_{L^\infty(\overline{Q}_T)} &\le C_{210}\|D^2u\|^2_{L^\infty(\overline{Q}_T)} \sum_{k=1}^{4} \left\|Q^{-2k}P_{2k-1}(\nabla u)\right\|_{L^\infty(\overline{Q}_T)} \le C_{211}\|D^2u\|^2_{L^\infty(\overline{Q}_T)}\|\nabla u\|_{C^0(\overline{\Omega})},\\
[f_2[u]]_{C^\alpha(\overline{\Omega})} &\le \|D^2u\|_{C^0(\overline{\Omega})} \sum_{k=1}^{2}\left[Q^{-2k}P_{2k}(\nabla u)\right]_{C^\alpha(\overline{\Omega})} + [D^2u]_{C^\alpha(\overline{\Omega})}\sum_{k=1}^{2}\left\|Q^{-2k}P_{2k}(\nabla u)\right\|_{C^0(\overline{\Omega})}\\
&\le \|D^2u\|_{C^0(\overline{\Omega})}[\nabla u]_{C^\alpha(\overline{\Omega})}\|\nabla u\|_{C^0(\overline{\Omega})} + [D^2u]_{C^\alpha(\overline{\Omega})}\|\nabla u\|^2_{C^0(\overline{\Omega})}
\end{aligned}
$$

where we used $Q^{-1} \le 1$ and the Hölder seminorm product estimate (31) as well as $[Q^{-1}]_{C^\alpha(\overline{\Omega})} = [\nabla u]_{C^\alpha(\overline{\Omega})}$. We deduce

$$
\begin{aligned}
\sup_{t\in(0,T)} \|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})} &\le \|u\|_{C^{2+\alpha,(2+\alpha)/4}_{x,t}(\overline{Q}_T)}\\
&\overset{(134)}{\le} C_{212}\left(\|u\|_{L^\infty(\overline{Q}_T)} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + \sup_{t\in(0,T)}\|u(\,.\,,t)\|^3_{C^{2+\alpha}(\overline{\Omega})}\right).
\end{aligned}
$$

The solution of the Willmore flow u has a $L^\infty(\Omega)$-bound [DGR17] (or Theorem 16 a). By this bound, we can not choose the $L^\infty(\Omega)$-norm small enough in contrast to the $L^2(\Omega)$-norm. For preparing a $L^2(\Omega)$-bound in Theorem 16 b we use a interpolation inequality in Theorem 10 for all $T \in (0, T_{\max})$

$$
\|u(\,.\,,t)\|_{L^\infty(\Omega)} \le \varepsilon\|D^2u(\,.\,,t)\|_{C^\alpha(\overline{\Omega})} + C(\varepsilon)\|u(\,.\,,t)\|_{L^2(\Omega)}.
$$

Consequently, we obtain for all $T \in (0, T_{\max})$

$$
\sup_{t\in(0,T)} \|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})} \le C_{213}\begin{pmatrix} \sup_{t\in(0,T)}\|u(\,.\,,t)\|_{L^2(\Omega)} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})}\\ + \sup_{t\in(0,T)}\|u(\,.\,,t)\|^3_{C^{2+\alpha}(\overline{\Omega})}\end{pmatrix}.
$$

By choosing $\hat{C}$ small enough and from now on fixed: $C_{213}(\hat{C}^2) \le \frac{1}{2}$ we get

$$
\forall t \in (0, T^{\hat{C}}_{\max}): \quad \|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})} \le 2C_{213}\left(\sup_{t\in(0,T)}\|u(\,.\,,t)\|_{L^2(\Omega)} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})}\right) \tag{176}
$$

Since the Willmore energy stays bounded for all $t \ge 0$ we get

$$
\mathcal{W}(u) + \|\varphi\|_{L^1(\partial\Omega)} \le \mathcal{W}(u_0) + \|\varphi\|_{L^1(\partial\Omega)} \le C\|u_0\|_{C^{2+\alpha}(\overline{\Omega})}.
$$

Next by choosing by C_{205} (depending on $\hat{C}$ and C_{213}) smaller we get $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})}$ smaller, therefore we can also achieve $\|u(\,.\,,t)\|_{L^2(\Omega)}$ small enough by Theorem 16 b so that

$$
\forall t \in (0, T^{\hat{C}}_{\max}): \quad \|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})} \le \frac{\hat{C}}{2}.
$$

which gives a contradiction to the maximality property of $T^{\hat{C}}_{\max}$. Therefore, the existence is global in time. □

6.6 Subconvergence/Convergence to a Critical Point

59 Theorem (Subconvergence)
Let $g_0 \in C^{4+\alpha}(\partial\Omega)$, $g_1 \in C^{3+\alpha}(\partial\Omega)$, $u_0 \in C^1(\overline{\Omega})$ and $(u(t))_{t\in\mathbb{R}_+}$ a global solution of the Willmore flow equation with $u(0) = u_0$.

Let us assume that $\max_{t\in\mathbb{R}_+}\|u(\,.\,,t)\|_{C^1(\overline{\Omega})} < C_{188}$*, where* C_{188} *is a constant from the short time existence Theorem 54, or* u *is the global solution in Theorem 49, then there exists a timesequence* $(t_k)_{k\in\mathbb{N}}\subset\mathbb{R}_+$ *with* $\lim_{k\to\infty}t_k=+\infty$ *and a critical point of the Willmore energy* u_∞ *such that* $\forall\beta\in(0,\alpha)$:

$$u(t_k)\underset{k\to\infty}{\longrightarrow}u_\infty\quad\textit{in}\quad C^{4+\beta}(\overline{\Omega}).$$

Proof: In this proof, we want to discuss the subconvergence.

First, let us consider the case $\max_{t\in\mathbb{R}_+}\|u(\,.\,,t)\|_{C^1(\overline{\Omega})} < C_{188}$. Even though u_0 is allowed to be merely C^1, here we are assuming $C^{4+\alpha}$ regularity on the Dirichlet boundary data. Hence it is possible to show a global $C^{4+\alpha}(\overline{\Omega})$-bound for u in case $t>1$. Since for all $t\geq 0$: $\|u(\,.\,,t)\|_{C^1(\overline{\Omega})} < C_{188}$, by the short time existence result Theorem 54 there exists short existence time $\tilde{T}=1$ independent of t so that

$$\forall t>0:\quad \|u\|_{C_1^{4+\alpha,1+\alpha/4}(\overline{\Omega}\times(t,t+1])}\leq C_{214}(\Omega,\alpha).$$

In other case when u is the global solution in Theorem 49, we can use (156)

$$\forall t>0:\qquad \|u(t)\|_{C^{1+\alpha}(\overline{\Omega})}\leq C_{175}/2\quad\Rightarrow\quad \|u\|_{C_{1+\alpha}^{4+\alpha,1+\alpha/4}(\overline{\Omega}\times(t,t+T_1])}\leq C_{215}(\Omega,\alpha)$$

with some $T_1\in(0,1)$ from Lemma 48. Thus in both cases $\forall t>0: u(\,.\,,t)\in C^{4+\alpha}(\overline{\Omega})$ and moreover $\forall t>1: \|u(\,.\,,t)\|_{C^{4+\alpha}(\overline{\Omega})} < C_{216}$. Furthermore, since the Dirichlet boundary data is constant with respect to time, one can show that

$$\begin{aligned}\forall t_0>0:\quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(u)\Big|_{t=t_0} &= -\frac{1}{2}\int_\Omega \left|\Delta_{\Gamma(u)}H+2H\left(\frac{1}{2}H^2-\mathcal{K}\right)\right|^2 Q\,\mathrm{d}x\Big|_{t=t_0}\\ &= -\frac{1}{2}\int_\Omega |\partial_t u|^2 Q\,\mathrm{d}x\Big|_{t=t_0}\leq 0,\end{aligned}$$

hence the Willmore energy decreases monotonically. It follows $\forall t_1,t_2\in(0,\infty)$

$$\mathcal{W}\big(u(\,.\,,t_2)\big)-\mathcal{W}\big(u(\,.\,,t_1)\big)=-\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega|\partial_t u|^2Q\,\mathrm{d}x\,\mathrm{d}t.$$

We let $t_1\to 0$ and $t_2\to\infty$. In fact, since $\mathcal{W}\geq 0$ and $Q\geq 1$ we obtain the estimate

$$\|\partial_t u\|_{L^2(\Omega\times\mathbb{R}_+)}\leq\sqrt{2\mathcal{W}(u_0)}$$

concluding $(t\mapsto\|\partial_t u(\,.,\,,t)\|^2_{L^2(\Omega)})\in L^1(\mathbb{R}_+)$. Thus there exists $\{t_k\}_{k\in\mathbb{N}}\subset\mathbb{R}$ a time-sequence with $\lim_{k\to\infty}t_k=+\infty$ and

$$\|\partial_t u(\,.,\,,t_k)\|^2_{L^2(\Omega)}\underset{k\to\infty}{\longrightarrow}0. \tag{177}$$

We can assume, that $t_1>1$, then the sequence $\big\{u(t_k)\big\}_{k\in\mathbb{N}}$ is uniformly bounded in $C^{4+\alpha}(\overline{\Omega})$. Thus, there exists a subsequence $\{t_{k_\ell}\}_{\ell\in\mathbb{N}}$ and $u_\infty\in C^{4+\beta}(\overline{\Omega})$ for each $0<\beta<1$ such that $u(t_{k_\ell})\to u_\infty$ in any $C^{4+\beta}(\overline{\Omega})$. Moreover by the equation (177) and uniform $C^{4+\alpha}(\overline{\Omega})$-bounds on the subsequence it follows

$$\Delta_{\Gamma\left(u(\,.,\,,t_k)\right)}H+2H\left(\frac{1}{2}H^2-\mathcal{K}\right)\big(u(\,.,\,,t_k)\big)\underset{k\to\infty}{\longrightarrow}0\quad\text{in}\quad C^0(\overline{\Omega}).$$

Thus, we conclude that u_∞ is indeed a critical point satisfying the graphical Willmore equation:

$$\Delta_{\Gamma(u_\infty)}H+2H\left(\frac{1}{2}H^2-\mathcal{K}\right)(u_\infty)=0.$$

$\square$

In the subsequent theorem, we establish that as $t \to \infty$, there is convergence towards a unique critical point. This critical point is derived from the elliptic solution to the Willmore equation, as detailed in Theorem 21 in Section 5. At that point, it was necessary to impose a smallness condition on the $C^{2+\alpha}$-boundary data, which we also assume for the temporal limit for the Willmore flow solution.

60 Theorem (Convergence)
In the case $u_0 \in C^{2+\alpha}(\overline{\Omega})$ there exists a constant $C_{217} = C_{217}(\Omega, \alpha)$ such that if $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} < C_{217}$ then there exists a unique $u_\infty \in C^{2+\alpha}(\overline{\Omega})$, which is also a critical point of the Willmore energy, such that

$$u(t) \underset{t\to\infty}{\longrightarrow} u_\infty \quad in \quad C^{2+\beta}(\overline{\Omega})$$

for all $\beta \in (0, \alpha)$.

Proof: In this proof, we want to show the convergence to the critical limit for a small $C^{2+\alpha}(\overline{\Omega})$ norm. To begin, let us recall the results from Lemma 57 and the proof of Theorem 58. There in (176) we could show that we can make the global $C^{2+\alpha}(\overline{\Omega})$ bound of u as small as desired, if we choose $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})}$ small enough

(178)

$$\forall t \in \mathbb{R}_+ : \ \langle u \rangle_{\frac{2+\alpha}{4}, \Omega\times(0,\infty)} + \|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq 2C_{213} \left(\sup_{t\in(0,\infty)} \|u(\,.\,,t)\|_{L^2(\Omega)} + \|u_0\|_{C^{2+\alpha}(\overline{\Omega})} \right).$$

The Willmore energy stays bounded for all times, therefore, we get

$$\mathcal{W}(u) + \|\varphi\|_{L^1(\partial\Omega)} \leq C\|u_0\|_{C^{2+\alpha}(\overline{\Omega})}.$$

Hence by Theorem 16 $\|u(\,.\,,t)\|_{L^2(\Omega)}$ can be achieved small enough by choosing $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})}$ small enough.

For $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} < \delta$ small enough we obtain the solution of the Willmore equation u_∞ with the same boundary values as u_0 by Theorem 21. We emphasize that in (178) we also obtained a temporal Hölder estimate on u. Thus, if we can bound a L^p norm over $\Omega \times [\tilde{T}, \infty)$ of the difference between solution u and the limit u_∞, the convergence of $u(\,.\,,t)$ to $u_\infty(\,.\,)$ for all $t \to +\infty$ follows.

For this purpose, we write $u - u_\infty$ as the solution to the following problem

$$\begin{cases} \partial_t(u - u_\infty) + \Delta^2(u - u_\infty) = f_0[u] - f_0[u_\infty] + \nabla_i\big(f_1^i[u] - f_1^i[u_\infty]\big) + D^2_{ij}\big(f_2^{ij}[u] - f_2^{ij}[u_\infty]\big) & \text{in } \Omega, \\ u - u_\infty = 0, \quad Du - Du_\infty = 0 \quad \text{on } \partial\Omega. \end{cases}$$

By multiplying with $u - u_\infty$ and integrating by parts as well as using the Poincare and the Cauchy-Schwarz inequality, we obtain for all $t \geq 0$

$$\begin{aligned} &\frac{1}{2}\int_\Omega \partial_t |u - u_\infty|^2 \,\mathrm{d}x + \int_\Omega |D^2(u - u_\infty)|^2 \,\mathrm{d}x \\ &\quad \leq C_{218}\left(\int_\Omega |f_0[u] - f_0[u_\infty]|^2 \,\mathrm{d}x + \int_\Omega |f_1[u] - f_1[u_\infty]|^2 \,\mathrm{d}x + \int_\Omega |f_2[u] - f_2[u_\infty]|^2 \,\mathrm{d}x \right) \\ &\quad \leq C_{219} \max\left\{ \|u\|_{C^2(\overline{\Omega})}, \|u_\infty\|_{C^2(\overline{\Omega})} \right\}^2 \cdot \|\nabla u - \nabla u_\infty\|^2_{L^2(\Omega)} \\ &\qquad + C_{220} \max\left\{ \|u\|_{C^2(\overline{\Omega})}, \|u_\infty\|_{C^2(\overline{\Omega})} \right\}^2 \cdot \|D^2 u - D^2 u_\infty\|^2_{L^2(\Omega)} \\ &\quad \leq C_{221} \sup_{t'\geq 0} \|u(\,.\,,t')\|^2_{C^2(\overline{\Omega})} \int_\Omega |D^2(u - u_\infty)|^2 \,\mathrm{d}x. \end{aligned}$$

For $\sup_{t\geq 0} \left\|u(\,.\,,t)\right\|^2_{C^2(\overline{\Omega})}$ small enough it follows by integrating over time and Poincare inequality

$$\|u-u_\infty\|^2_{L^2(\Omega\times\mathbb{R}_+)} \leq C_{222}(\Omega)\int_0^\infty\int_\Omega |D^2(u-u_\infty)|^2\,\mathrm{d}x\,\mathrm{d}t \leq 2C_{222}\int_\Omega |u_0-u_\infty|^2\,\mathrm{d}x$$

thus we conclude

$$\lim_{t\to\infty}\int_t^{t+1}\int_\Omega |u-u_\infty|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since we could show that $\langle u\rangle_{\frac{2+\alpha}{4},\overline{Q}_t}$ and $\|u(\,.\,,t)\|_{C^{2+\alpha}(\overline{\Omega})}$ are bounded, we use interpolation result to obtain the following

$$\lim_{t\to\infty}\|u-u_\infty\|_{C^0(\overline{\Omega}\times[t,t+1])}\to 0.$$

In remains to show that $\lim_{t\to\infty} u = u_\infty$ in $C^{2+\beta}(\overline{\Omega})$ which follows by interpolation and global $C^{2+\alpha}(\overline{\Omega})$ bounds. The uniqueness of u_∞ follows from the uniqueness of the $C^0(\overline{\Omega})$-limit for $\lim_{t\to\infty} u$ convergence. $\square$

7 Varifolds, Measures & BV

This section will recall basic definitions and theorems considering measures, varifolds, and BV-functions. These results are needed for the next Section 8 in which compactness results for the Willmore functional for graphs in the framework of varifolds, measure-functions pairs, and BV-functions are presented.
Notation: Whenever $0 < \rho < \infty$ and $x \in \mathbb{R}^n$, we define an open ball and a sphere:

$$B_\rho(x) = \{y \in \mathbb{R}^n \mid \|y - x\| < \rho\} \quad \text{and} \quad \mathbb{S}^{n-1} = \{y \in \mathbb{R}^n \mid \|y\| = 1\}$$

Let $\mathcal{B}(X)$ be the set of all Borel subsets of a space X. Let $C^0_c(X)$ be the space of continuous functions on X with compact support.

7.1 Radon Measures

To obtain compactness results we will work with Radon measures. They form the backbone of the definition of varifolds and measure-function pairs and help to characterize the different contributions of the gradient of a BV-function. This subsection follows the presentation from [All72, 2.3 Measures p.424]. We call $\mu : X \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ a *Borel regular measure* on a locally compact Hausdorff space X if

$$\forall A \subset X : \qquad \mu(A) = \inf\{\mu(B) \mid A \subset B \text{ and } B \text{ is a Borel set }\},$$

and whenever $B_1, B_2, \dots$ is a disjoint sequence of Borel sets of X

$$\forall A \subset X : \qquad \mu\left(A \cap \bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \mu(A \cap B_k).$$

The support of a Borel regular measure μ is

$$\operatorname{supp} \mu = X \setminus \bigcup \{G \mid G \text{ is open and } \mu(G) = 0\}.$$

For each subset B of X, we set the *restriction of measure* μ

$$\forall A \subset X : \qquad (\mu \llcorner B)(A) = \mu(B \cap A). \tag{179}$$

Next, we want to recall the *μ-measurability of sets and functions*. We say a set $E \subset X$ μ-measurable if there exist a set Z with $\mu(Z) = 0$ and a Borel set B that $E = B \cup Z$. Whenever Y is a topological space and $f : X \to Y$ we say a function f with values in Y is μ-measurable if the domain of f is μ-almost equal X and for all open $U \subset Y$ the set $f^{-1}(U) \cap X$ is μ-measurable.
The push-forward of a measure by mapping $\pi : X \to Y$ is defined by

$$\forall K \in \mathcal{B}(Y) : \qquad \pi_\# \mu(K) = \mu\big(\pi^{-1}(K)\big).$$

We call μ a *Radon measure* on X if μ is a Borel regular measure on X and finite on each compact subset of X which means

$$\forall K \subset X : \quad K \text{ compact} \Rightarrow \mu(K) < \infty.$$

Subsequently, we define integration by introducing simple functions $g\colon X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ which are defined by having just a countable image. It is called μ-integrable simple if one of the following terms is finite

$$\int g^+ \,\mathrm{d}\mu := \sum_{0\leq y\leq\infty} y \cdot \mu\big((g^+)^{-1}(y)\big), \qquad \int g^- \,\mathrm{d}\mu := \sum_{0\leq y\leq\infty} y \cdot \mu\big((g^-)^{-1}(y)\big)$$

with $g^+ = \max(g, 0)$ and $g^- = (-g)^+$. For a function $f\colon X \to \overline{\mathbb{R}}$, we define its upper and lower integral by

$$\begin{aligned} \int^* f \,\mathrm{d}\mu \; &:= \inf\left\{ \int g \,\mathrm{d}\mu \;\middle|\; g \text{ simple, } \mu\text{-integrable and } g \geq f \ \mu\text{-a.e} \right\} \\ \int_* f \,\mathrm{d}\mu \; &:= \sup\left\{ \int g \,\mathrm{d}\mu \;\middle|\; g \text{ simple, } \mu\text{-integrable and } g \leq f \ \mu\text{-a.e} \right\}. \end{aligned}$$

Further, we call a μ-measurable function $f\colon X \to \overline{\mathbb{R}}$ μ-integrable if

$$\int_X f \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu := \int^* f \,\mathrm{d}\mu = \int_* f \,\mathrm{d}\mu < +\infty.$$

Next, we recall the restriction of a Borel regular measure μ for a nonnegative extended real-valued function f that domain is X up to μ-null set

$$\forall A \subset X: \qquad (\mu \llcorner f)(A) = \int_A^* f \,\mathrm{d}\mu = \int_X^* f\chi_A \,\mathrm{d}\mu. \tag{180}$$

Whenever X is a locally compact space, due to Riesz representation theorem, each linear functional in the form $\mathcal{F}\colon C_c^0(X) \to \mathbb{R}$ that is nonnegative on the nonnegative members of $C_c^0(X)$ can be uniquely represented by a Radon measure. In this sense, we write

$$\forall f \in C_c^0(X): \qquad \mu(f) := \int_X f \,\mathrm{d}\mu.$$

Then we can rewrite μ as a variation measure for every open $A \subset X$

$$\mu(A) = \sup\left\{ \int_A f \,\mathrm{d}\mu \;\middle|\; f \in C_c^0(X), |f| \leq 1, \operatorname{supp}(f) \subset V \right\}.$$

The *convergence for the Radon measures* is defined in the dual sense. For the sequence $\{\mu_i\}_{i\in\mathbb{N}}$ and μ the Radon measures in X and $Y \subset X$ we denote

$$\mu_i \to \mu \text{ in } Y, \quad \text{if} \quad \forall \varphi \in C_c^0(Y)\colon\ \mu_k(\varphi) \to \mu(\varphi). \tag{181}$$

Further, we recall the definition of L^p-spaces. Let μ be a Radon measure on X and $f\colon X \to \overline{\mathbb{R}}$ a μ-measurable function. Like in [AFP00, Definition 1.16 p.9], we set for each $p \in [1, \infty)$

$$\|f\|_{L^p(\mu)} := \left(\int_X |f|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}$$

and for $p = \infty$ we define

$$\|f\|_{L^\infty(\mu)} := \inf\big\{ C \in [0, \infty] \,\big|\, |f| \leq C \ \mu\text{-a.e in } X \big\}.$$

By $L^p(\mu)$ or $L^p(X, \mu)$ we denote the real vector space of functions $f\colon X \to \overline{\mathbb{R}}$ satisfying $\|f\|_{L^p(\mu)} < \infty$. The semi-norm $\|\,.\,\|_{L^p(\mu)}$ becomes a norm if one considers functions that are equal μ-a.e. as identical. By $L^p(\mu;\mathbb{R}^m)$ we denote the space of $\mathbb{R}^m$-valued functions with finite $L^p(\mu;\mathbb{R}^m)$-norm, which is defined in the same way as the $L^p(\mu)$-norm where the absolute value is replaced by euclidean length of a vector in $\mathbb{R}^m$.

The next convergence theorems for integrals with respect to a Radon measure are among the most used tools in the framework of $L^p(\mu)$-spaces.

61 Theorem (Convergence Theorems)
Assume $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $L^1(\mu)$.

1. *(Fatou's Lemma) Then* $\displaystyle\int_X \liminf_{k\to\infty} f_k \,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_X f_k \,\mathrm{d}\mu$, *provided that all $f_k \ge 0$.*

2. *(Monotone Convergence Theorem) Let $f_1 \le f_2 \le \dots \le f_k \le f_{k+1} \le \dots$. Then*

$$\lim_{k\to\infty} \int_X f_k \,\mathrm{d}\mu = \int_X \lim_{k\to\infty} f_k \,\mathrm{d}\mu.$$

3. *(Dominated Convergence Theorem) Assume $g \ge 0$, $g, f \in L^1(\mu)$, $f_k \to f, k \to \infty$ μ-a.e. and $\forall k \in \mathbb{N} : |f_k| \le g$. Then*

$$\lim_{k\to\infty} \int_X |f_k - f| \,\mathrm{d}\mu = 0.$$

4. *(A.e. convergent subsequence) Assume $f \in L^1(\mu)$ with $\displaystyle\int_X |f_k - f| \,\mathrm{d}\mu = 0$. Then there exists a subsequence $\{f_{k_\ell}\}_{\ell=1}^{\infty}$ such that*

$$f_{k_\ell} \to f, \quad \text{for } \ell \to \infty \quad \mu\text{-a.e.}.$$

Proof: [EG15, Theorems 1.17-1.21] □

In the following, we always consider $X \subset \mathbb{R}^n$. Further, we want to recall how to derive one Radon measure with respect to another, which is provided in the next theorem for $X = \Omega \subset \mathbb{R}^n$.

62 Theorem (Besicovitch Derivation Theorem)
Let μ be a positive Radon measure in an open set $\Omega \subset \mathbb{R}^n$ and η a $\mathbb{R}^m$-valued Radon measure. Then, for μ-a.e. x in the support of μ the limit

$$f(x) := \lim_{\rho \searrow 0} \frac{\eta\big(B_\rho(x)\big)}{\mu\big(B_\rho(x)\big)}$$

exists in $\mathbb{R}^m$ and moreover the unique Radon-Nykodym decomposition of η is given by $\eta = f\mu + \eta^s$, where $\eta^s = \eta \llcorner E$ and E is the μ-negligible set

$$E = (\Omega \setminus \operatorname{supp}\mu) \cup \left\{ x \in \operatorname{supp}\mu \;\middle|\; \lim_{\rho \searrow 0} \frac{|\eta|\big(B_\rho(x)\big)}{\mu\big(B_\rho(x)\big)} = \infty \right\}.$$

Proof: By [AFP00, Theorem 2.22 (Besicovitch derivation Theorem) p.54]. □

We will also need the following result, which allows identifying functions μ-a.e..

63 Theorem (Fundamental Lemma of Calculus of Variations for Radon measures)
Let μ be a positive Radon measure on an open set $\Omega \subset \mathbb{R}^n$, $f \in L^1(\mu;\mathbb{R}^m)$. Suppose that

$$\forall g \in C_c^0(\Omega)\colon \qquad \int_\Omega gf \,\mathrm{d}\mu = 0$$

then for μ-a.e. x in the support of μ:

$$f(x) = 0.$$

Proof: Here, we want to use the Besicovitch derivation theorem. So we define a Radon measure $\nu := f\mu$ which is $\nu(A) = \int_A f \,\mathrm{d}\mu$ for any measurable set $A \subset \Omega$. This means $\nu \ll \mu$. From the assumption we get with [EG15, Theorem 1.8 p.13]

$$A \subset \Omega \text{ open} \qquad \Longrightarrow \qquad \nu(A) = \sup\left\{ \int_\Omega uf \,\mathrm{d}\mu \;\middle|\; u \in C_c^0(\Omega),\ |u| \leq \chi_A \right\} = 0.$$

Then by Theorem 62 it follows that for μ a.e. x in the support of μ:

$$f(x) = \lim_{\rho \searrow 0} \frac{\nu\big(B_\rho(x)\big)}{\mu\big(B_\rho(x)\big)} = 0.$$

. $\square$

One important class of outer measures, which defines k-dimensional analogies of the area without using parametrizations are Hausdorff measures. This intrinsic approach is helpful in geometric measure theory. For $k \geq 0$ and $A \subset \mathbb{R}^n$ we define the *$k-$dimensional Hausdorff measure of A* by

$$\mathcal{H}^k(A) := \lim_{\delta \searrow 0} \frac{\omega_k}{2^k} \inf\left\{ \sum_{i\in I} \big[\operatorname{diam}(A_i)\big]^k \;\middle|\; \operatorname{diam}(A_i) < \delta, A \subset \bigcup_{i\in I} A_i \right\}$$

where ω_k is the $\mathcal{L}^k$ measure of k-dimensional unit ball. For all $k \in [0,\infty)$ $\mathcal{H}^k$ is a Borel regular measure, see [EG15, Theorem 2.1 p.82].

Consequently, we can define the notion of a rectifiable set [ABG98], Definition 2.1 p.6]. Let $E \subset \mathbb{R}^n$, then we call A *countably $\mathcal{H}^k$-rectifiable* if A can be covered with the sequence of C^1-hypersurfaces Γ_i up to a $\mathcal{H}^k$-null set. This means

$$\mathcal{H}^k\left(A \setminus \bigcup_{i=1}^\infty \Gamma_i \right) = 0.$$

Furthermore, we call A *$\mathcal{H}^k$-rectifiable* if A is countably $\mathcal{H}^k$-rectifiable and it holds $\mathcal{H}^k(A) < +\infty$. By Besicovitch-Marstrand-Mattila Theorem in [AFP00, Theorem 2.63 p. 83] we know that every $A \in \mathcal{B}(\mathbb{R}^n)$ with $\mathcal{H}^k(A) < \infty$ is $\mathcal{H}^k$-rectifiable if and only if the *upper and lower k-dimensional densities of $\mathcal{H}^k \llcorner A$* are equal 1:

$$\Theta_k^*(\mathcal{H}^k \llcorner A, x) := \limsup_{\rho\searrow 0} \frac{\mathcal{H}^k\big(A \cap B_\rho(x)\big)}{\omega_k \rho^k} = 1 = \liminf_{\rho\searrow 0} \frac{\mathcal{H}^k\big(A \cap B_\rho(x)\big)}{\omega_k \rho^k} =: \Theta_{*k}(\mathcal{H}^k \llcorner A, x)$$

for $\mathcal{H}^k$-a.e. $x \in A$.

One can also study the density properties of k-rectifiable measures and show that they are asymptotically concentrated near to x on an affine k-plane for $\mathcal{H}^k$-a.e. x. This k-plane generalizes the classical tangent space in differential geometry. In the same way as in [ABG98], we define the *approximate tangent space* $\mathrm{Tan}^k(\mathcal{H}^k \llcorner A, x)$ of a $\mathcal{H}^k$-rectifiable set A at x by the k-plane $P \subset \mathbb{R}^n$ (the set of all k-planes will be defined later in Subsection 7.4 such that, by using the dilations of A around x denoted by $A_\rho = \rho^{-1}(A - x)$, we have with multiplicity $\theta \in \mathbb{R}$

$$\forall \phi \in C_c^1(\mathbb{R}^n): \qquad \lim_{\rho \searrow 0} \int_{A_\rho} \phi(y) \,\mathrm{d}\mathcal{H}^k(y) = \int_P \theta\phi(y) \,\mathrm{d}\mathcal{H}^k(y).$$

Then, by [Fed69, 3.2.25] the mapping $x \mapsto \mathrm{Tan}^k(\mathcal{H}^k \llcorner A, x)$ is defined $\mathcal{H}^k$-a.e. on A and is $\mathcal{H}^k$-measurable. Also, by [Sim83, Remark 11.5] we have the locality result

$$\mathrm{Tan}^k(\mathcal{H}^k \llcorner A, x) = \mathrm{Tan}^k(\mathcal{H}^k \llcorner B, x) \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in A \cap B$$

for any $\mathcal{H}^k$-rectifiable sets A, B. By [AFP00, Theorem 2.83 (Rectifiability criterion for measures) p.93] the *multiplicity* is given for $\mathcal{H}^k$-a.e. x by

$$\theta(x) = \Theta_k(\mathcal{H}^k \llcorner A, x) := \lim_{\rho \searrow 0} \frac{\mathcal{H}^k\big(A \cap B_\rho(x)\big)}{\omega_k \rho^k}.$$

Next, we want to present the area and coarea formula. The first describes how to compute $\mathcal{H}^k$-measure of image $f(B)$ of a Lipschitz map $f\colon \mathbb{R}^k \to \mathbb{R}^N, N \geq k$ generalizing parameterized surface area formula. The second generalizes Fubini's Theorem in the following way. For a Lipschitz map $f\colon \mathbb{R}^M \to \mathbb{R}^k$ and a $\mathcal{H}^N$-rectifiable $E \subset \mathbb{R}^M$ we can slice E into level sets $\{f = t\} \cap E$ such that the $\mathcal{H}^N$-integral over E is given by $\mathcal{L}^k$-integral over slices-volume $\mathcal{H}^{N-k}(E \cap \{f = t\})$.

Further, $M \subset \mathbb{R}^{n+\ell}$ is supposed to be $\mathcal{H}^n$-measurable, so that we can express $M = \bigcup_{j=0}^\infty M_j$ with $\mathcal{H}^n(M_0) = 0$ and disjoint M_j is $\mathcal{H}^n$-measurable, $\mathcal{H}^n(M_j) < \infty$, moreover $M_j \subset N_j, j \leq 1$, where N_j are embedded n-dimensional C^1-submanifolds of $\mathbb{R}^{n+\ell}$. Let $f\colon U \to \mathbb{R}^m$ be locally Lipschitz with $U \subset \mathbb{R}^{n+\ell}$ an open subset, then by definitions presented in Subsection 2.1 the gradient of f is given $\mathcal{H}^n$-a.e locally by $\nabla^M f(y) = \nabla^{N_j} f(y), y \in M_j$. Corresponding, the differential $\mathrm{d}^M f_x\colon T_x M \to \mathbb{R}^m$ is $\mathrm{d}^M f_x(\tau) = D_\tau f(x) = \langle \tau, \nabla^M f(x)\rangle, \tau \in T_x M$ and the *adjoint* of $\mathrm{d}^M f_x$ by $(\mathrm{d}^M f_x)^*\colon \mathbb{R}^m \to T_x M$ which is characterized by

$$\forall v \in \mathbb{R}^m\ \forall u \in T_x M: \qquad \langle (\mathrm{d}^M f_x)^*(v), u\rangle_{T_x M} = \langle v, \mathrm{d}^M f_x(u)\rangle_{\mathbb{R}^m}. \tag{182}$$

64 Theorem (Area Formula for Rectifiable Sets)
Let $U \subset \mathbb{R}^{n+\ell}$ an open subset, $M \subset U$ a $\mathcal{H}^n$-measurable, n-rectifiable set and $f\colon U \to \mathbb{R}^m$ locally Lipschitz, $m = n + \ell \geq n, \ell \geq 0$ and h is any non-negative $\mathcal{H}^n$-measurable function on M and in the case $f|_M$ is $1:1$. Then it holds

$$\int_M h J_f^M \,\mathrm{d}\mathcal{H}^n = \int_{f(M)} h \circ f^{-1} \,\mathrm{d}\mathcal{H}^n$$

with the Jacobian $J_f^M(x)$ for $\mathcal{H}^n$-a.e. $x \in M$ given by

$$J_f^M(x) = \sqrt{\det J(x)} = \sqrt{\det\big((\mathrm{d}^M f_x)^* \circ (\mathrm{d}^M f_x)\big)} \tag{183}$$

where $J(x)$ is the matrix with $D_{\tau_p} f(x) \cdot D_{\tau_q} f(x)$ in the p-th row and q-th column for $\tau_1, \dots, \tau_n$ any orthonormal basis of $T_x M$.

Proof: [Sim83, 2.6 p. 77]. □

65 Theorem (Co-Area Formula for Rectifiable Sets)
Let $U \subset \mathbb{R}^{n+\ell}$ an open subset, $M \subset U$ a $\mathcal{H}^n$-measurable, n-rectifiable set and $f\colon U \to \mathbb{R}^m$ locally Lipschitz, $m \leq n = m + k, k \geq 0$, g is a given non-negative $\mathcal{H}^n$-measurable function and $A \subset M$ any $\mathcal{H}^n$-measurable. Then it holds

$$\int_A g J_f^{M*} \,\mathrm{d}\mathcal{H}^n = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(y) \cap A} g \,\mathrm{d}\mathcal{H}^k \right) \mathrm{d}\mathcal{L}^m(y)$$

where J_f^{M*} is the adjoint Jacobian

$$J_f^{M*}(x) = \sqrt{\det\left((\mathrm{d}^M f_x) \circ (\mathrm{d}^M f_x)^* \right)}. \tag{184}$$

Proof: [Sim83, 2.9 p. 77]. □

For, in our case the most important, case $\pi\colon \mathbb{R}^3 \ni (x^1, x^2, x^3) \mapsto (x^1, x^2) \in \mathbb{R}^2$ the orthogonal projection in the first two components we want to calculate the adjoint Jacobian. For that let φ be a local representation of surface M at p. We represent the tangential space by $T_x M \ni w = w^1 \partial_1 \varphi + w^2 \partial_2 \varphi$. In local representation we have

$$\left(\mathrm{d}^M \pi_\varphi \right) w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \circ \left(w^1 \partial_1 \varphi + w^2 \partial_2 \varphi \right) = \begin{pmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 \\ \partial_1 \varphi_2 & \partial_2 \varphi_2 \end{pmatrix} \circ \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$$

So we can locally rewrite (182)

$$\left\langle \left(\mathrm{d}^M \pi_\varphi \right)^* v, (g_{ij})\, w \right\rangle_{\mathbb{R}^2} = \left\langle v, \begin{pmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 \\ \partial_1 \varphi_2 & \partial_2 \varphi_2 \end{pmatrix} \circ w \right\rangle_{\mathbb{R}^2} \quad \Rightarrow \quad \left(\mathrm{d}^M \pi_\varphi \right)^* = (g^{ij}) \circ \begin{pmatrix} \partial_1 \varphi_1 & \partial_1 \varphi_2 \\ \partial_2 \varphi_1 & \partial_2 \varphi_2 \end{pmatrix}.$$

Finally, we can deduce

$$\det \left(\begin{pmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 \\ \partial_1 \varphi_2 & \partial_2 \varphi_2 \end{pmatrix} \circ (g^{ij}) \circ \begin{pmatrix} \partial_1 \varphi_1 & \partial_1 \varphi_2 \\ \partial_2 \varphi_1 & \partial_2 \varphi_2 \end{pmatrix} \right) = \det \left(\begin{pmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 \\ \partial_1 \varphi_2 & \partial_2 \varphi_2 \end{pmatrix} \right)^2 \frac{1}{\det\left((g_{ij}) \right)}.$$

Next, we want to compare these terms with the last component of the normal. By Lagrange's identity, we get

$$N = \frac{1}{\sqrt{\det \begin{pmatrix} \|\partial_1 \varphi\|^2 & \langle \partial_1 \varphi, \partial_2 \varphi \rangle \\ \langle \partial_2 \varphi, \partial_1 \varphi & \|\partial_2 \varphi\|^2 \end{pmatrix}}} \left(\partial_1 \varphi \times \partial_2 \varphi \right) = \frac{1}{\sqrt{\det\left((g_{ij}) \right)}} \begin{pmatrix} \partial_1 \varphi_2 \partial_2 \varphi_3 - \partial_1 \varphi_3 \partial_2 \varphi_2 \\ \partial_1 \varphi_3 \partial_2 \varphi_1 - \partial_1 \varphi_1 \partial_2 \varphi_3 \\ \partial_1 \varphi_1 \partial_2 \varphi_2 - \partial_1 \varphi_2 \partial_2 \varphi_1 \end{pmatrix}.$$

Therefore, the adjoint Jacobian is equal to the absolute value of the third component of the normal

$$J_f^{M*}(p) = \sqrt{\det\left((\mathrm{d}^M \pi_p) \circ (\mathrm{d}^M \pi_p)^* \right)} = |N_3|(p). \tag{185}$$

7.2 Functions of Bounded Variation and Fine Properties of Functions

In this subsection, we want to recall the definition of spaces of bounded variation and some of their fine properties. Since we want to consider Willmore bounded graph sequences in Subsection 8.2, by Theorem 16 the $L^1(\Omega)$-norm of ∇u, also called variation, will stay bounded. Especially the $W^{1,1}$-Sobolev functions have bounded variation $\int_\Omega |\nabla u| \,\mathrm{d}x$, which plays a role in variational problems like least area problems. The justification for the inclusion of BV-spaces instead of $W^{1,1}$-spaces in the field of calculus of variations is that the $W^{1,1}$-spaces do not exhibit useful compactness property, like Theorem 66 for BV-functions.

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, $u \in L^1(\Omega)$. We say that u belongs to the *space of functions of bounded variation* if its distributional derivatives $\nabla_i u$ for $i = 1, \dots, n$ are given by finite signed Radon measures on Ω in the following sense

$$\forall g \in C_c^1(\Omega; \mathbb{R}^n): \quad \int_\Omega \langle \nabla u, g \rangle_{\mathbb{R}^n} \,\mathrm{d}x = \int_\Omega u \operatorname{div} g \,\mathrm{d}x.$$

The total variation of ∇u of a function $u \in L^1(\Omega)$ is given by

$$\int_\Omega |\nabla u| = \sup \left\{ \int_\Omega u \operatorname{div} g \,\mathrm{d}x \;\middle|\; g \in C_c^1(\Omega; \mathbb{R}^n) \text{ and } \|g\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

Then, the space of L^1-functions with bounded variation is denoted by

$$BV(\Omega) := \left\{ u \in L^1(\Omega) \;\middle|\; \int_\Omega |\nabla u| < \infty \right\}$$

For each $u \in BV(\Omega)$ the ∇u and $|\nabla u|$ are vector valued and scalar Radon-measures respectively on $\mathbb{R}^n$ and $\mathbb{R}$. These can be decomposed in a *absolutely continuous part* of ∇u with respect to $\mathcal{L}^n$ and a *singular part* $\nabla^s u$ which consists out of a *jump part* $\nabla^j u$ and a *Cantor part* $\nabla^c u$

$$\nabla u = \nabla^a u \mathcal{L}^n + \nabla^s u = \nabla^a u \mathcal{L}^n + \nabla^j u + \nabla^c u.$$

Furthermore, it is possible to characterize the singular part more precisely as restrictions on the following sets with the help of the following sets

$$\Sigma_u := \left\{ x \in \Omega \;\middle|\; \lim_{\rho \searrow 0} \frac{|\nabla u|\big(B_\rho(x)\big)}{\rho^n} = \infty \right\}, \quad \Theta_u := \left\{ x \in \Omega \;\middle|\; \lim_{\rho \searrow 0} \frac{|\nabla u|\big(B_\rho(x)\big)}{\rho^n} \rho > 0 \right\}.$$

Then we have the restrictions of ∇u as Radon measures [AFP00, Prop. 3.92 p. 184]

$$\nabla^a u \mathcal{L}^n = \nabla u \llcorner (\Omega \setminus \Sigma_u), \quad \nabla u^j = \nabla u \llcorner \Theta_u, \quad \nabla^c u = \nabla u \llcorner (\Sigma_u \setminus \Theta_u)$$

where by Besicovitch derivation theorem [AFP00, Theorem 2.22 p.54] the absolutely continuous part is computed for all $x \in \Omega \setminus \Sigma_u$ by

$$\nabla^a u(x) = \lim_{\rho \searrow 0} \frac{\nabla u\big(B_\rho(0)\big)}{\omega_n \rho^n}$$

with $\nabla^a u \in L_{\text{loc}}(\Omega; \mathbb{R}^n)$ and moreover $\mathcal{L}^n(\Sigma_u) = 0$.

It is also possible to describe the jump part more directly. To do so, we define the approximate jump points [AFP00, p.163 Def. 3.67]. Let $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$ and $x \in \Omega$. We say that x is an *approximate jump point of* u if there exist $a, b \in \mathbb{R}^m$ and $\nu \in \mathbb{S}^{N-1}$ such that $a \neq b$ and

$$\lim_{\rho \searrow 0} \fint_{B_\rho^+(x,\nu)} \big|u(y) - a\big| \,\mathrm{d}y = 0, \qquad \lim_{\rho \searrow 0} \fint_{B_\rho^-(x,\nu)} \big|u(y) - b\big| \,\mathrm{d}y = 0. \tag{186}$$

with the convenient notation

$$B_\rho^+(x,\nu) := \Big\{ y \in B_\rho(x) \mid \big\langle y - x, \nu \big\rangle > 0 \Big\}, \qquad B_\rho^{-}(x,\nu) := \Big\{ y \in B_\rho(x) \mid \big\langle y - x, \nu \big\rangle < 0 \Big\}.$$

Up to a permutation $a \leftrightarrow b$ and a change of sign of ν the triplet (a, b, ν) is uniquely determined by (186) and is denoted by $(u^+(x), u^-(x), \nu_u(x))$. We also denote J_u as the set of approximate jump points. Moreover, by [AFP00, Definition 3.67] $J_u \subset \Theta_u$ is $\mathcal{H}^{n-1}$-rectifiable and $\mathcal{H}^{n-1}(\Theta_u \setminus J_u) = 0$

[AFP00, Proposition 3.92]. Also, by [AFP00, Thm 3.78 p. 173] we know that with ν_u, which denotes a Borel unit normal vector field to J_u, it follows

$$\nabla^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u$$

where u^+ and u^- are the traces of u on J_u.

The main justification for the usage of the BV-spaces is the following compactness theorem. We again emphasize that the Sobolev space $W^{1,1}(\Omega)$ does not have such a Theorem.

66 Theorem (Compactness for BV functions)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with Lipschitz boundary $\partial\Omega$. Assume $\{u_k\}_{k=1}^{\infty}$ is a sequence in $BV(\Omega)$ satisfying

$$\sup_{k\in\mathbb{N}} \left\{ \|u_k\|_{L^1(\Omega)} + |\nabla u_k|(\Omega) \right\} < \infty.$$

Then there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and a function $u \in BV(\Omega)$ such that

$$u_{k_j} \to u \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \lim_{k\to\infty} \int_\Omega \varphi \nabla u_k = \int_\Omega \varphi \nabla u, \text{ for all } \varphi \in C_c^0(\Omega).$$

We call the last property the weak convergence of ∇u_k to ∇u.*

Proof: [AFP00, Theorem 3.23 Definition 3.11] □

We want also to consider the sublevelsets of graphs. Then, such a set E has a finite perimeter, meaning that the indicator function χ_E lies in BV-space. Hence, theoretically, its boundary has a bounded surface area and an inner normal measure. We recall a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ to be a set of *finite perimeter* if the *characteristic functions* χ_E have a finite perimeter of E relative to open $U \subset \mathbb{R}^n$

$$\mathcal{P}(E, U) := \int_U |\nabla \chi_E| < \infty$$

so that χ_E lies in $BV(U)$. We use here the notation U to avoid confusion with Ω because later we will set $U = B_R(0) \times \mathbb{R}$ where $\Omega \subset B_R(0) \subset \mathbb{R}^2$. This total variation of $\nabla\chi_E$ measures the $(n-1)$-dimensional Hausdorff area of $\partial E \cap U$. Like in [AFP00, Definition 3.54 p.154] we further call the *reduced boundary* $\mathcal{F}E$ the collection of all points $x \in \operatorname{supp} |\nabla\chi_E| \cap U$ such that the limit

$$\nu_E(x) := \lim_{\rho\searrow 0} \frac{\nabla\chi_E\big(B_\rho(x)\big)}{|\nabla\chi_E|\big(B_\rho(x)\big)}$$

exists in $\mathbb{R}^N$ and satisfies $|\nu_E(x)| = 1$. The function $\nu_E \colon \mathcal{F}E \to \mathbb{S}^{n-1}$ is called *generalized inner normal to E*. The motivation for introducing the reduced boundary is to have a set of boundary points where we can define an inner normal in a measure-theoretical sense. From [AFP00, p.154] we know that $\mathcal{F}E$ is a countably $\mathcal{H}^{n-1}$-rectifiable. Also by the Besicovitch derivation Theorem [AFP00, Theorem 2.22, p.54] $|\nabla\chi_E|$ is concentrated on $\mathcal{F}E$ and furthermore [AFP00, Theorem 3.59 (De Giorgi) p.157] yields

$$|\nabla\chi_E| = \mathcal{H}^{N-1} \llcorner \mathcal{F}E. \tag{187}$$

Moreover, the polar decomposition $\nabla\chi_E = \nu_E|\nabla\chi_E|$ holds. Like in [AFP00, Definition 3.60 p.158], for every $t \in [0,1]$ and every $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ we denote:

$$E^t := \left\{ x \in R^n \,\middle|\, \lim_{\rho\searrow 0} \frac{|B_\rho(x) \cap E|}{|B_\rho(x)|} = t \right\}.$$

While one can consider E^0 and E^1 a the measure theoretic exterior and interior of the set E, the *essential boundary of* E is defined as:

(188) $$\partial^* E := \mathbb{R}^n \setminus (E^0 \cup E^1).$$

By [AFP00, Theorem 3.61. (Federer) p.158] we know that for E a set of finite perimeter in U and it holds

(189) $$\mathcal{F}E \cap U \subset E^{1/2} \subset \partial^* E \quad \text{and} \quad \mathcal{H}^{n-1}\big(U \setminus (\mathcal{F}E \cup E^1 \cup E^0)\big) = 0.$$

Hence, it follows that up to $\mathcal{H}^{n-1}$-null set the essential and reduced boundaries are the same

(190) $$\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F}E) = 0.$$

The rather abstract measure theoretical notion of sets of finite perimeter still allows the usual Gauss-Green Theorem provided the usage of the measure-theoretic boundary $\mathcal{F}E$.

67 Theorem (Gauss-Green Theorem)
For any set E of finite perimeter in U the distributional derivative $\nabla\chi_E$ is a $\mathbb{R}^n$-valued finite Radon measure in U and it holds

(191) $$\forall \varphi \in C^1_c(U;\mathbb{R}^n)\colon \qquad \int_E \operatorname{div}\varphi \,\mathrm{d}x = -\int_{\mathcal{F}E} \langle \nu_E, \varphi\rangle \,\mathrm{d}\mathcal{H}^{n-1}.$$

Proof: [AFP00] thm. 3.36 p.143 and equation (3.47) p. 154 and (189). □

Not only that, but it is also possible to introduce the notion of the mean curvature of $\mathcal{F}E$ in the sense of Luckhaus-Sturzenhecker [LS95]. They introduced it for a parabolic mean curvature problem in a time cylinder $\Omega \times (0,T), T > 0$. In contrast, we define the mean curvature for the reduced boundary of sets in $U \subset \mathbb{R}^n$.

68 Definition (Mean Curvature of the Reduced Boundary, [LS95] (0.4))
Let $U \subset \mathbb{R}^n$ be open and bounded, and let E be a set of finite perimeter in U. Then we say that a reduced boundary $\mathcal{F}E$ has mean curvature, if there exists such a function $H \in L^1(|\nabla\chi_E|;\mathbb{R})$ that satisfies

$$\forall \zeta \in C^\infty_c(U;\mathbb{R}^n)\colon \qquad \int_U \left(\operatorname{div}_{\mathbb{R}^n}\zeta - \left(\frac{\nabla\chi_E}{|\nabla\chi_E|}\right)^T \circ D\zeta \circ \frac{\nabla\chi_E}{|\nabla\chi_E|}\right)|\nabla\chi_E| = \int_U H\zeta\nabla\chi_E,$$

It can be checked that this definition is consistent with the classical notion of mean curvature for graphs. Namely, in case $\partial E \cap U$ is a C^1-class two-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^3$ then $|\nabla\chi_E| = \mathcal{H}^2 \llcorner (\partial E \cap U)$. Furthermore, one can check that $N := -\nabla\chi_E/|\nabla\chi_E|$ is a normal vector field to $\mathcal{M}$ by the classical Gaussian theorem (see step 4 in proof of Theorem 89). Then, by the tangential divergence theorem, it follows with test functions $\zeta \in C^\infty_c(U;\mathbb{R}^3)$ that

$$\begin{aligned}
&\int_U \left(\operatorname{div}_{\mathbb{R}^3}\zeta - \left(\frac{\nabla\chi_E}{|\nabla\chi_E|}\right)^T \circ D\zeta \circ \frac{\nabla\chi_E}{|\nabla\chi_E|}\right)|\nabla\chi_E| \\
&\quad = \int_{\mathcal{M}} \big(\operatorname{div}_{\mathbb{R}^3}\zeta - N^T \circ D\zeta \circ N\big)\,\mathrm{d}\mathcal{H}^2 = \int_{\mathcal{M}} \operatorname{div}_{T_x\mathcal{M}}\zeta(x)\,\mathrm{d}\mathcal{H}^2 \overset{\text{tang. Div}}{\underset{\text{Thm.}}{=}} -\int_{\mathcal{M}} \vec{H}\cdot\zeta\,\mathrm{d}\mathcal{H}^2 \\
&\quad = -\int_{\mathcal{M}} H\zeta\cdot N\,\mathrm{d}\mathcal{H}^2 = \int_U H\zeta\nabla\chi_E.
\end{aligned}$$

Next, we want to introduce some fine properties of functions. Let $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$, by *Lebesgue set of* u, denoted by $\mathcal{L}_u$, we call the set of all points $x \in \Omega$ such that [AFP00, Definition 3.63 p.160]

(192) $$\exists z = z_u(x) \in \mathbb{R}^m\colon \qquad \lim_{\rho\searrow 0} \fint_{B_\rho(x)} \big|u(y) - z\big|\,\mathrm{d}y = 0.$$

Like in [GMS92, p. 54] by Giaquinta, Modica and Souček, we call for each $x \in \mathcal{L}_u$

$$\tilde{u}(x) := z_u(x)$$

the *Lebesgue value of* u at $x \in \mathcal{L}_u$ (Ambrosio, Fusco and Pallara [AFP00, Defintion 3.63 p. 160] call it the approximate limit of u at x). The *complement of the Lebesgue set* is denoted by $S_u := \Omega \setminus \mathcal{L}_u$. It is known that S_u is $\mathcal{H}^{n-1}$-rectifiable [Fed69, 4.5.9(16)]. Furthermore, by *Lebesgue point*, we call each point such that

$$\lim_{\rho \searrow 0} \fint_{B_\rho(x)} \big|u(y) - u(x)\big| \,\mathrm{d}y = 0. \tag{193}$$

Furthermore, the limit function $u^* \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$

$$u^*(x) := \begin{cases} \lim_{r \searrow 0} \fint_{B_\rho(x)} u \,\mathrm{d}y, & \text{if limit exists} \\ 0, & \text{otherwise} \end{cases}$$

is called the *precise representative* on u in $x \in \Omega$. It is possible [EG15, Theorem 5.19 p. 241] that the above limit exists even if x is not a Lebesgue point, if f has a "measure theoretic jump" across some hyperplane, thus approximate jump points. Next, we say that a function $u\colon \Omega \to \mathbb{R}^m$ has an *approximate limit at* $x \in \mathbb{R}^n$ if there exists $z \in \mathbb{R}^m$ such that

$$\forall \varepsilon > 0 : \qquad \lim_{r \searrow 0} \frac{\mathcal{L}^n\Big(B_r(x) \cap \big\{|u - z| \geq \varepsilon\big\}\Big)}{\mathcal{L}^n\big(B_r(x)\big)} = 0.$$

Then z is uniquely determined and we set the *approximate limit of* u *at* $x \in \Omega$ by

$$\operatorname*{aplim}_{y \to x} u(y) := z.$$

Moreover, we call u *approximately continuous at* $x \in \mathbb{R}^n$ if the condition $\operatorname*{aplim}_{y \to x} u(y) = u(x)$ is satisfied. If $u \in L^1(\Omega)$, then the Lebesgue set is also the set of approximate continuity since, like in [EG15, Remark p. 59], it holds

$$\frac{\mathcal{L}^n\Big(B_r(x) \cap \big\{|u - z| \geq \varepsilon\big\}\Big)}{\mathcal{L}^n\big(B_r(x)\big)} \leq \frac{1}{\varepsilon} \fint_{B_r(x)} |u - z| \,\mathrm{d}y.$$

Therefore, for all $x \in \mathcal{L}_u$ it holds $u^*(x) = \operatorname*{aplim}_{y \to x} u(y) = \tilde{u}(x)$ (see [GMS92, Proposition 4 p. 62]) .

Now like in [EG15, Definition 6.1 p. 262], we can define a general notion of differentiability. Let $u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and $x \in \Omega \setminus S_u$. We say u is *approximately differentiable at* x if there exists a $m \times n$ matrix L such that

$$\operatorname*{aplim}_{y \to x} \frac{\big|u(y) - \tilde{u}(x) - L(y - x)\big|}{|y - x|} = 0. \tag{194}$$

If u is approximately differentiable at x the matrix L, uniquely determined by (194), is called the *approximate differential of* u at x and denoted by $\operatorname{ap} Du$. We also denote the set of points where u is approximately differentiable by $A_D(u) \subset \Omega \setminus S_u$. Actually, in [GMS92, Definition 5 p. 63] we can find a slightly weaker definition which uses $\operatorname{aplimsup}$ instead of aplim. Since we will use BV-functions, which have stronger differentiability properties, we better stick to (194).

69 Theorem (Differentiability for BV functions)
Any function $u \in BV(\Omega;\mathbb{R}^m)$ is approximately differentiable at $\mathcal{L}^n$-almost every point of Ω. Moreover, if $x \in \Omega \setminus (S_u \cup S_{\nabla^a u} \cup \Sigma_u)$ then it follows

$$\lim_{\rho \searrow 0} \fint_{B_\rho(x)} \frac{\big|u(y) - \tilde{u}(x) - \langle \nabla^a u, (y-x)\rangle\big|}{\rho}\,\mathrm{d}y = 0 \quad \textit{and} \quad \operatorname{ap} Du(x) = \nabla^a u(x)$$

Therefore, the approximate differential $\operatorname{ap} Du$ is the density of the absolutely continuous part of ∇u with respect to $\mathcal{L}^n$ almost $\mathcal{L}^n$-everywhere. Also it follows that

$$\Omega \setminus (S_u \cup S_{\nabla^a u} \cup \Sigma_u) \subset A_D(u).$$

Proof: [EG15, Proof of Theorem 6.1 p.258, conditions (a),(b) and (c)] and [AFP00, 3.83 p.176]. □

Moreover, by the theorem of Federer-Vol'pert [AFP00, Thm. 3.78 p. 173] it follows for $u \in BV(\Omega;\mathbb{R}^m)$ that the discontinuity set S_u is countably $\mathcal{H}^{n-1}$-rectifiable and $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ with the jump set J_u. This means that $\mathcal{H}^{n-1}(A_D(u) \cap J_u) = 0$.

Following the results from [ABG98] and [GMS92], we want to connect the approximate differentiability and the set $A_D(u)$ to the rectifiability properties of its graph. We especially want to describe the $\mathcal{H}^n$-integration on a graph over the set $A_D(u)$. First, we need the following approximation theorem.

70 Theorem (Federer)
Let A be a measurable set in $\mathbb{R}^n$ and let $u\colon A \to \mathbb{R}$ be a measurable $\mathcal{L}^n$-a.e. approximately differentiable function. Then there exists a non decreasing sequence $\{C_j\}_{j\in\mathbb{N}}$ of measurable sets and a sequence $\{u_j\}_{j\in\mathbb{N}}$ of Lipschitz functions in $\mathbb{R}^n$ such that

$$u = u_j \textit{ on } C_j, \quad \mathcal{L}^n\Big(A \setminus \bigcup_{j=1}^{\infty} C_j\Big) = 0 \quad \textit{and } \nabla^a u(x) = \nabla u_j(x)\ \mathcal{L}^n\textit{-a.e } x \in C_j.$$

Proof: [GMS92, Theorem 7 of part 1]. □

The next theorem is an analogy to Theorem 2.11 from [ABG98], where one uses the classical area formula for Lipschitz functions and the exhaustion from Theorem 70.

71 Theorem
Let $M \subset \mathbb{R}^n$ of dimension n with $\mathcal{L}^n(M) < +\infty$, let $u\colon M \to \mathbb{R}$ be $\mathcal{L}^n$-differentiable at every point in M, and let $\Gamma_M := \big\{x, u(x) \;\big|\; x \in M\big\} \subset \mathbb{R}^n \times \mathbb{R}$ be the graph of u on M and $\mathcal{H}^n(\Gamma_M) < +\infty$. Then

(i) $\Gamma_{M\cap A_D(u)}$ *is $\mathcal{H}^n$-measurable and*

$$\mathcal{H}^n(\Gamma_{M\cap A_D(u)}) = \int_{M\cap A_D(u)} Q_a\,\mathrm{d}x,$$

where $Q_a := \sqrt{1 + |\nabla^a u|^2}$

(ii) *Furthermore, then we have for any bounded Borel function η*

$$\int_{\Gamma_{M\cap A_D(u)}} \eta(x,y)\,\mathrm{d}\mathcal{H}^n(x,y) = \int_{M\cap A_D(u)} \eta\big(x, u(x)\big) Q_a(x)\,\mathrm{d}x. \tag{195}$$

Proof: For the first part (i), we define the function

$$\Phi\colon M \to \mathbb{R}^{n+1}\colon \quad \Phi(x) = \big(x, u(x)\big),$$

Then Φ is $\mathcal{L}^n$-a.e. approximately differentiable in M. By Theorem 70 we can exhaust $A_D(u)$ by a non decreasing sequence of measurable sets $\{F_k\}_{k=1}^{\infty}$ and find Lipschitz functions $\Phi_k \in \mathrm{Lip}(\mathbb{R}^n), k \in \mathbb{N}$ such that

$$\Big\{\Phi_k(x) = \big(x, u_k(x)\big) = \Phi(x), \quad \text{on } F_k, \quad D\Phi_k = D\Phi \text{ a.e. on } F_k, \quad A_D(u) \subset \textstyle\bigcup_{k=1}^{\infty} F_k.$$

With the area formula in Theorem 64 for Lipschitz graphs we get for each $k \in \mathbb{N}$ the Jacobian [GMS92, Section 3 Theorem 2 (Area Formula) p. 79]

$$J_{\Phi_k}^{\mathbb{R}^n} = \sqrt{\det\big(D_i\Phi_k \cdot D_j\Phi_k\big)_{i,j=1,\ldots,n}} = \sqrt{1 + |\nabla u_k|^2}.$$

With these results, we conclude

$$\begin{aligned}
\int_{M \cap A_D(u) \cap F_k} \sqrt{1 + |\nabla^a u|^2}\,\mathrm{d}x &= \int_{M \cap A_D(u) \cap F_k} \sqrt{1 + |\nabla^a u_k|^2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^{n+1}} \mathcal{H}^0\left(M \cap A_D(u) \cap F_k \cap \Phi^{-1}(y)\right) \mathrm{d}\mathcal{H}^n(y) \\
&= \int_{\mathbb{R}^{n+1}} \chi_{\Phi(M \cap A_D(u) \cap F_k)}\,\mathrm{d}\mathcal{H}^n(y).
\end{aligned}$$

Now we observe that the characteristic functions $\chi_{\Phi(M \cap A_D(u) \cap F_k)}$ are non-negative and $\mathcal{H}^n$–measurable [Sim83, Thm 1.8]. Moreover the sequence $\{\chi_{\Phi(M \cap A_D(u) \cap F_k)}\}_{k \in \mathbb{N}}$ increases pointwise to $\chi_{\Phi(M \cap A_D(u))}$, so that $\chi_{\Phi(M \cap A_D(u))}$ is also $\mathcal{H}^n$-measurable. By monotone convergence Theorem we obtain

$$\int_{\mathbb{R}^{n+1}} \chi_{\Phi(M \cap A_D(u) \cap F_k)}\,\mathrm{d}\mathcal{H}^n(y) \underset{k \to \infty}{\to} \int_{\Phi(M \cap A_D(u))} \mathrm{d}\mathcal{H}^n(y).$$

Analogously, since we have by Theorem 69 (Calderon Zygmund) $\mathcal{L}^n(M \setminus A_D(u)) = 0$ and $A_D(u) \cap F_k \nearrow A_D(u)$ as well as $\sqrt{1 + |\nabla u_a|^2}\chi_{A_D(u) \cap F_k}$ is positive and increasing pointwise to $\sqrt{1 + |\nabla u_a|^2}$ it follows

$$\int_{M \cap A_D(u) \cap F_k} \sqrt{1 + |\nabla^a u|^2}\,\mathrm{d}x \underset{k \to \infty}{\to} \int_{M \cap A_D(u)} \sqrt{1 + |\nabla^a u|^2}\,\mathrm{d}x.$$

The second part (ii) follows by an analogous calculation by approximation with simple functions.

□

7.3 Measure-Function Pairs & Disintegration Theorem

In this subsection, we recall the definition of measure-function pairs and corresponding theorems from [Mos01] and [Hut86]. This approach allows us to investigate the subconvergence of a sequence of Radon measures combined with a sequence of L^1_{loc}-functions. Later, in the case of graphs, we will take a sequence of two-dimensional Lebesgue measures with area elements Q and as functions the mean curvature or the normal vector field.

Let $E \subset \mathbb{R}^n$ be an open subset or some manifold embedded in $\mathbb{R}^n$. Suppose μ is a Radon measure on E and $f\colon E \to \mathbb{R}^m$ is well defined μ-almost everywhere, $f \in L^1_{\mathrm{loc}}(\mu; \mathbb{R}^m)$ where $L^1_{\mathrm{loc}}(\mu; \mathbb{R}^m)$ is the space of locally μ-integrable functions on E with values in $\mathbb{R}^m$. Then we say (μ, f) is a *measure-function pair* over E with values in $\mathbb{R}^m$.

72 Definition (Measure-Function Pair Weak Convergence)
Suppose $\{(\mu_k, f_k)\}_{k=1}^{\infty}$ *and* (μ, f) *are measure-function pairs over* E *with values in* $\mathbb{R}^m$*. Additionally, suppose* $\mu_k \to \mu$ *in* E *as* $k \to \infty$ *(see* (181)*). Then we say* (μ_k, f_k) *converges to* (μ, f) *in the weak sense in* E *and write*

$$(\mu_k, f_k) \rightharpoonup (\mu, f) \quad \textit{if} \quad \mu_k \llcorner f_k \to \mu \llcorner f$$

in the sense of vector-valued measures. In other words, if

$$\forall \varphi \in C_c^0(E; \mathbb{R}^m): \qquad \int \langle f_k, \varphi \rangle \, \mathrm{d}\mu_k \underset{k\to\infty}{\to} \int \langle f, \varphi \rangle \, \mathrm{d}\mu.$$

In the same way as in [Mos01] we say that (μ_k, f_k) *converge in the weak* L^p*-sense to* (μ, f)*, denoted as*

$$(\mu_k, f_k) \overset{L^p}{\rightharpoonup} (\mu, f), \quad \textit{if} \quad (\mu_k, f_k) \rightharpoonup (\mu, f) \quad \wedge \quad \|f_k\|_{L^p(\mu_k)} \textit{ uniformly bounded}.$$

Mostly, we want to study measure-function pair convergence with respect to some functional, which will stay bounded. In our case, the Willmore energy or the area functional can play such a role.

73 Assumption (Hutchinson [Hut86] 4.1.2)
Suppose $F\colon E \times \mathbb{R}^m \to \mathbb{R}$*. We denote variables in* $E \times \mathbb{R}^m$ *by* (y, q)*.* F *shall always satisfy the following conditions:*

1. F *is continuous.*
2. F *is non-negative:* $\forall (y, q) \in E \times \mathbb{R}^m : F(y, q) \geq 0$.
3. F *is convex in the* q *variables:*

   $$\forall \lambda \in (0,1), y \in E, p \in \mathbb{R}^m, q \in \mathbb{R}^m: \quad F(y, \lambda p + (1-\lambda)q) \leq \lambda F(y,p) + (1-\lambda)F(y,q).$$

   If the above equation holds strictly, then we call F *strictly convex.*
4. F *has non-linear growth in the q variables, i.e. there exists a continuous function* $\varphi\colon E \times [0, \infty) \to [0, \infty)$*,* $0 \leq \varphi(y, s) \leq \varphi(y, t)$*, for* $0 \leq s \leq t$ *and* $y \in E$*,* $\varphi(y,t) \to \infty$ *locally uniformly in* y *as* $t \to \infty$*, and*

   $$\forall (y, q) \in E \times \mathbb{R}^m: \quad \varphi(y, |q|)|q| \leq F(y, q).$$

74 Definition (F-strong Convergence, Hutchinson [Hut86] 4.2.2 p.54)
Suppose $\{(\mu_k, f_k)\}_{k \in \mathbb{N}}$ *and* (μ, f) *are measure-function pairs over* E *with values in* $\mathbb{R}^m$*. We write*

$$(\mu, f) \in F, \quad \textit{if} \quad \int_E F\big(y, f(y)\big) \, \mathrm{d}\mu(y) < \infty,$$
$$(\mu_k, f_k) \in F \textit{ uniformly}, \quad \textit{if} \quad \int_E F\big(y, f_k(y)\big) \, \mathrm{d}\mu_k(y) \leq C_0.$$

for some $C_0 > 0$*. Furthermore, suppose* $\mu_k \to \mu$ *in* E*, we say* (μ_k, f_k) *converges to* (μ, f) *in the* F*-strong sense (in* E*), and we write*

$$(\mu_k, f_k) \overset{F}{\to} (\mu, f),$$

if the following holds:

(i) $\forall k \in \mathbb{N}\colon (\mu_k, f_k) \in F$,

(ii) *If* $E_{kj} := \Big\{ y \in E \;\Big|\; |y| \geq j \;\vee\; \big|f_k(y)\big| \geq j \Big\}$, *then:*

$$\lim_{j\to\infty} \int_{E_{kj}} F\big(y, f_k(y)\big)\, \mathrm{d}\mu_k = 0, \text{ uniformly in } k,$$

(iii) $\forall \varphi \in C^0_c(E \times \mathbb{R}^m)\colon \quad \lim_{k\to\infty} \int_E \varphi\big(y, f_k(y)\big)\, \mathrm{d}\mu_k(y) = \int_E \varphi\big(y, f(y)\big)\, \mathrm{d}\mu(y).$

Next, we need to know how a product of two sequences of function-products acts if the sequences of measures are the same. This will allow us to combine the limits of normal vector fields and mean curvatures.

75 Theorem (Product Rule)
Let $p, r \in (1,\infty)$, *such that* $\frac{1}{r} + \frac{1}{p} = 1$. *Suppose that* μ_k, μ *are Radon measures on* E *and that* $f_k \in L^p(\mu_k;\mathbb{R}^m)$, $f \in L^p(\mu;\mathbb{R}^m)$, $g_k \in L^r(\mu_k;\mathbb{R}^m)$, $g \in L^r(\mu;\mathbb{R}^m)$. *It follows*

$$(\mu_k, f_k) \stackrel{L^p}{\to} (\mu, f) \quad \wedge \quad (\mu_k, g_k) \stackrel{L^r}{\rightharpoonup} (\mu, g) \quad \Rightarrow \quad (\mu_k, f_k \cdot g_k) \stackrel{L^1}{\rightharpoonup} (\mu, f\cdot g) \tag{196}$$

where by $(\mu_k, f_k) \stackrel{L^p}{\to} (\mu, f)$ *we mean the* F*-strong convergence with* $F(y,q) = |q|^p$.

Proof: Proposition 3.2 p 6. in [Mos01]. □

The next theorem provides further convergence properties.

76 Theorem (Convergence Theorem)
Suppose $\{(\mu_k, f_k)\}_{k=1}^\infty$ *is a sequence of measure-function pairs over* $E \subset \mathbb{R}^n$ *with values in* $\mathbb{R}^m$. *Further, suppose* μ *is a Radon measure on* E *and* $\mu_k \to \mu$ *in* E *as* $k \to \infty$. *Then the following is true:*

(i) *If* $(\mu_k, f_k) \in F$ *uniformly then some subsequence of* $\{(\mu_k, f_k)\}_{k=1}^\infty$ *converges in the weak sense to a measure-function pair* (μ, f) *for some* f.

(ii) *If* $(\mu_k, f_k) \in F$ *uniformly and* $(\mu_k, f_k) \rightharpoonup (\mu, f)$ *then:*

$$\int_E F(y, f(y))\, \mathrm{d}\mu \leq \liminf_{k\to\infty} \int_E F(y, f_k(y))\, \mathrm{d}\mu_k.$$

(iii) *If* F *is strictly convex (see Assumption 73) and* $\{(\mu_k, f_k)\}_{k=1}^\infty \subset F$ *then the following are equivalent:*

  (a) $(\mu_k, f_k) \stackrel{F}{\to} (\mu, f)$,

  (b) $(\mu_k, f_k) \rightharpoonup (\mu, f)$ *and*

$$\int_E F(y, f_k(y))\, \mathrm{d}\mu_k \to \int_E F(y, f(y))\, \mathrm{d}\mu.$$

Proof: [Hut86, Theorem 4.4.2 p. 58] □

Since we will work with graphs, we will work with Radon measures on some domain $\Omega \subset \mathbb{R}^2$ with a limit μ from area measure sequence. Also, we will obtain a Hausdorff measure from a varifold limit. Hence, it makes sense to split an integral over the Hausdorff measure into an integration over μ and some other auxiliary measure. This splitting will be called disintegration. In preparing for the disintegration theorem, we need a definition for *measurable measure-valued* *maps* from [AFP00, Definition 2.25 p.56]. Let $E \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^m$ be open sets, μ a positive Radon measure on E as well as $x \mapsto \nu_x$ a function which assigns to each $x \in E$ a $\mathbb{R}^m$-valued Radon measure ν_x on G. We say that this map is μ-measurable if

$$\forall B \in \mathcal{B}(G) \colon \quad x \mapsto \nu_x(B) \text{ is } \mu\text{-measurable}$$

where μ-measurability was defined in the beginning of Subsection 7.1

77 Theorem (Disintegration Theorem)
Let $m \geq 1, E \subset \mathbb{R}^n$ and $G \subset \mathbb{R}^m$ open sets, ν an $\mathbb{R}^m$-valued Radon measure on $E \times G, \pi \colon E \times G \to E$ the projection on the first factor and $\mu = \pi_{\#}|\nu|$ is a push-forward, which means $\forall K \in \mathcal{B}(E) : \mu(K) = \mu\big(\pi^{-1}(K)\big)$.

Let us assume that μ us a Radon measure, i.e. that $|\nu|(K \times G) < \infty$ for any compact set $K \subset E$.

Then there exist $\mathbb{R}^m$-valued finite Radon measures ν_x on G such that $x \mapsto \nu_x$ is μ-measurable,

$$|\nu_x|(F) = 1, \quad \mu\text{-a.e. in } E$$

and for any $f \in L^1\big(E \times G, |\nu|\big)$ it holds

$$f(x, .\,) \in L^1\big(G, |\nu_x|\big) \quad \mu\text{-a.e. } x \in E, \tag{197}$$

$$x \mapsto \int_G f(x,y)\,\mathrm{d}\nu_x(y) \in L^1(E,\mu), \quad \int_{E\times G} f(x,y)\,\mathrm{d}\nu(x,y) = \int_E \left(\int_G f(x,y)\,\mathrm{d}\nu_x(y)\right)\mathrm{d}\mu(x). \tag{198}$$

Moreover, if ν'_x is any other μ-measurable map satisfying (198) *for every bounded Borel function with compact support and such that $\nu'_x(G) \in L^1_{loc}(E,\mu)$, then $\nu_x = \nu'_x$ for μ-a.e. $x \in E$.*

Proof: [AFP00, Theorem 2.28, p.57] □

7.4 Varifolds

In this subsection, we introduce varifolds, their curvature properties, and compactness results. Like in [Hut86, Chapter 3], assume N be a smooth p-dimensional Riemannian manifold isometrically imbedded in $\mathbb{R}^n$ $(n \geq p)$. Further, let $G_{m,n}\big(G^o_{m,n}\big)$ be the *Grassmannian manifold* of all unoriented (oriented) m-dimensional subspaces of $\mathbb{R}^n$. We can consider each given unoriented m-subspace $P \subset \mathbb{R}^n$ as the *projection matrix*

$$[P_{ij}] \in \mathbb{R}^{n\times n}$$

of the orthogonal projection over $P = B \circ B^T$ with $B = (b_1, b_2, \dots, b_m)$ filled with an orthonormal basis of P. Hence, as noted in [Man96] we can consider $G_{m,n}$ as a compact subset of $\mathbb{R}^{n\times n}$ endowed with relative metric. For example, one can show that in the oriented case $G^o_{2,3} \cong \mathbb{S}^2$ and in the unoriented case $G_{2,3} \cong \mathbb{R}P^2 = \mathbb{S}^2/\{\mathrm{Id}, -\mathrm{Id}\}$.

If we denote by q the standard 2-fold covering map $q : G^o_{m,n} \to G_{m,n}$ then we can set

$$G_m(N) = \big(N \times G_{m,n}\big) \cap \{(x,P) \,|\, P \subset T_xN\}, \quad G^o_m(N) = \big(N \times G^o_{m,n}\big) \cap \{(x,P) \,|\, q(P) \subset T_xN\}$$

An *unoriented (oriented) m-varifold V on N* is a Radon measure on $G_m(N)$ $(G^o_m(N))$. The sets of such varifolds are denoted by

$$\mathbf{V}_m(N) \text{ and } \mathbf{V}^o_m(N).$$

By *varifold convergence*, we understand the convergence in the sense of Radon measures. For $\{V_k\}_{k\in\mathbb{N}} \subset \mathbf{V}_m(N)(\mathbf{V}^o_m(N))$ and $V \in \mathbf{V}_m(N)(\mathbf{V}^o_m(N))$ we write

$$V_k \to V \quad \text{if} \quad \forall \varphi \in C^0_c(G^{(o)}_m(\mathbb{R}^n)) \colon \; V_k(\varphi) \to V(\varphi).$$

For each oriented varifold $V \in \mathbf{V}^o_m(N)$ we define

$$q_\# V \in \mathbf{V}_m(N)$$

the corresponding *unoriented varifold associated to V by projection onto $G_m(N)$.*

Next we define the *associated Radon measure on N obtained by projection* $\pi\colon N \times G_{m,n} \to N$:

$$\|V\| = \mu = \mu_V = \pi_\# V, \quad \mu(A) = V(A \times G_{m,n}) \tag{199}$$

Suppose $E \subset \mathbb{R}^n$ is a countably m-rectifiable, $\mathcal{H}^m$-measurable and $\theta, \theta_i > 0$ are locally $\mathcal{H}^m$-measurable functions on E. Additionally, let us assume that an *orientation function* $\xi\colon E \to G^o_{m,n}$ is $\mathcal{H}^m$-measurable and suppose that $\xi(x)$ is one of the two oriented approximate tangent planes $T_x E := \operatorname{Tan}^m(\mathcal{H}^m \llcorner E, x)$ to E at x for $\mathcal{H}^m \llcorner E$. Then we define *rectifiable varifolds*, $\forall \varphi \in C^0_c(G_m(\mathbb{R}^n))$ or $\varphi \in C^0_c(G^o_m(\mathbb{R}^n))$ respectively:

$$\begin{aligned}
V_1 &= v(E, \theta), \quad V_2 = v(E, \xi, \theta_1) + v(E, -\xi, \theta_2),\\
V_1(\varphi) &= \int_E \theta(x) \varphi(x, T_x E)\, \mathrm{d}\mathcal{H}^m(x),\\
V_2(\varphi) &= \int_E [\theta_1(x)\varphi(x, \xi(x)) + \theta_2(x)\varphi(x, -\xi(x))]\, \mathrm{d}\mathcal{H}^m(x).
\end{aligned}$$

We denote the corresponding sets of such rectifiable varifolds by

$$\mathbf{RV}_m(\mathbb{R}^n) \text{ and } \mathbf{RV}^o_m(\mathbb{R}^n).$$

In the case the functions $\theta, \theta_1, \theta_2$ are integer-valued, we call V_1, V_2 *integral varifolds* which belong to the corresponding classes

$$\mathbf{IV}_m(\mathbb{R}^n) \text{ and } \mathbf{IV}^o_m(\mathbb{R}^n).$$

Let us shortly discuss how to reformulate a regular and oriented surface S in $\mathbb{R}^3$ with normal vector field $N(\,.\,)$ to the rectifiable varifold setting. First, we set the surface measure $\mu := \mathcal{H}^2 \llcorner S$. Then for each $p \in S$ we set Radon measures by delta-distributions on $G^o_{2,3} \cong \mathbb{S}^2$ or $G_{2,3} \cong \mathbb{R}P^2$ respectively

$$\begin{cases} \nu_p := \delta_{N(p)}(\,.\,) & \text{in oriented varifold case,}\\ \nu_p := \delta_{(N(p))^\perp}(\,.\,) & \text{in unoriented varifold case.} \end{cases}$$

and $\nu_p = 0$ for $p \notin S$. In oriented varifold case, we can set for all $B \in \mathcal{B}(\mathbb{R}^3 \times \mathbb{S}^2)$

$$V_{(S,N(\,.\,))}(B) = \int_{\mathbb{R}^3} \left(\int_{\mathbb{S}^2} \chi_B(p, x)\, \mathrm{d}\nu_p(x) \right) \mathrm{d}\mu(p).$$

For the unoriented varifold case, we simply replace $\mathbb{S}^2$ with $\mathbb{R}P^2$ in the above formula and consider $B \in \mathcal{B}(\mathbb{R}^3 \times \mathbb{R}P^2)$. Additionally, we get for all $A \subset \mathbb{R}^3$

$$\|V_{(S,N(\,.\,))}\|(A) = V_{(S,N(\,.\,))}(A \times \mathbb{S}^2) = \int_{\mathbb{R}^3} \chi_A(p)\, \mathrm{d}(\mathcal{H}^2 \llcorner S)(p) = \mu(A \cap S) = \mathcal{H}^2(A \cap S).$$

We continue on with the *first variation of m-varifold V*, that we define as the linear functional on $C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ vector fields:

$$\forall X \in C^1_c(\mathbb{R}^n; \mathbb{R}^n): \quad \delta V(X) := \int_{G_m(\mathbb{R}^n)} \operatorname{div}_P X(x)\, \mathrm{d}V(x, P),$$

where we define

$$\forall P \in G_{m,n}: \quad \operatorname{div}_P X := \sum_{i=1}^{n} \nabla_i^P X^i = \sum_{i,j=1}^{n} P_{ij} D_j X^i$$

with $\nabla^P \varphi = P(\nabla\varphi)$ is the projection on P and for $\{e_i\}_{i=1}^n$ the orthonormal basis of $\mathbb{R}^n$ $\nabla_i^P := e_i \cdot \nabla^P$. For $V \in \mathbf{V}_m(\mathbb{R}^n)$ we recall the *total variation* of δV to be the largest Borel regular measure on $\mathbb{R}^n$ such that

$$\|\delta V\|(G) = \sup\{\delta V(g) \mid g \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \operatorname{spt} g \subset G \text{ and } |g| \leq 1\}$$

whenever G is an open subset of $\mathbb{R}^n$, see [All72, p. 435 with $M = \mathbb{R}^n$].

In [Men13, Thm. 1] Menne has shown that if $V \in \mathbf{IV}_m(U)$ and $\|\delta V\|$ is a Radon measure or V is a curvature varifold then V is countable C^2-rectifiable. That means there exists a countable collection $\{C_k\}_{k\in\mathbb{N}}$ of m-dimensional submanifolds of $\mathbb{R}^n$ of class 2 such that $\|V\|(U \setminus \cup_{k\in\mathbb{N}} C_k) = 0$ and each member C_k of the collection has $\|V\|$-a.e. the same mean curvature vector as V in $U \cap M$.

78 Definition (Weak Mean Curvature, [Mon14] Def 2.10)
Let V be an unoriented m-varifold on $\mathbb{R}^n$ and $\vec{H}\colon G_m(\mathbb{R}^n) \to \mathbb{R}^n$ a $L^1_{loc}(V)$-function, then we say that V has weak curvature $\vec{H}$ if one has

$$\forall X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)\colon \quad \delta V(X) := \int_{G_m(\mathbb{R}^n)} \operatorname{div}_P X(x)\,\mathrm{d}V(x,P) = -\int_{G_m(\mathbb{R}^n)} \vec{H}\cdot X\,\mathrm{d}V(x,P). \tag{200}$$

For $V = v(E,\theta) \in \mathbf{RV}_m(\mathbb{R}^n)$ with weak mean curvature we write $\vec{H}(x) = \vec{H}(x, T_x E)$, so we get:

$$\int_E \operatorname{div}_E X\,\mathrm{d}\mu_V = -\int_E \vec{H}(x)\cdot X\,\mathrm{d}\mu_V, \tag{201}$$

where $\operatorname{div}_E X(x) := \operatorname{div}_{T_xE} X(x)$ is the tangential divergence, and T_xE is the μ_V-a.e. existing approximate tangent vector space to E at x. From [All72, 4.1] it follows that $\delta V \llcorner E = -\mu_v \llcorner \vec{H}$. For the oriented case, we define

$$\delta V = \delta(q_\# V).$$

Suppose that the first variation of V is locally bounded, then according to [Men17] there exists a $\|V\|$-almost unique locally $\|V\|$-summable, $\mathbb{R}^n$-valued function mean curvature $\vec{H}(\cdot)$ satisfying the equation (200). Moreover [Hut86, Remark 5.2.3.]

$$\forall B \subset \mathbb{R}^n: \quad \|\delta V\|(B) = \int_B \|\vec{H}\|\,\mathrm{d}\mu_V.$$

Next we introduce the notation: for a given $\varphi = \varphi(x,P) \in C^1(\mathbb{R}^n \times \mathbb{R}^{n\times n})$ we denote the partial derivatives with respect to the variables x_i and P_{ij} respectively by:

$$D_i\varphi \quad \text{and} \quad D^*_{jk}\varphi \quad \text{for} \quad i,j,k \in \{1,\dots,n\}.$$

79 Definition (Curvature Varifold)
Suppose U is an open subset of $\mathbb{R}^n$. We say a integral varifold $V \in \mathbf{IV}_m(\mathbb{R}^n)$ has generalized curvature and generalized second fundamental form in U if there exist $\mathbb{R}$-valued functions B_{ijk} for $1 \leq i,j,k \leq n$, defined $V \llcorner U$-almost everywhere in $G_m(\mathbb{R}^n)$ such that the following is true :

(i) *$(V,[B_{ijk}])$ is a measure-function pair on $G_m(\mathbb{R}^n)$ with values in $\mathbb{R}^{n\times n\times n}$*

(ii) *One has $\forall i=1,\dots,n$ with $[P_{ij}]\in G_{m,n}$ the following*

$$\forall\varphi=\varphi(x,P)\in C_c^1(\mathbb{R}^n\times\mathbb{R}^{n\times n})\colon\quad 0=\int_{G_m(N)}\big(P_{ij}D_j\varphi+B_{ijk}D^*_{jk}\varphi+B_{jij}\varphi\big)\,\mathrm{d}V,$$

where we sum over all the values of the index if the index variable appears twice in a single term.

We call the function B generalized curvature and write

$$\mathcal{S}_U V\quad\text{or}\quad \mathcal{S}_U V(x,P)=[B_{ijk}(x,P)].$$

In this case, we say $V\in\mathbf{IV}_m(\mathbb{R}^n)$ is a curvature varifold on U. Furthermore we call an oriented varifold $V\in\mathbf{IV}^o_m(\mathbb{R}^n)$ an oriented curvature varifold on U if $q_\# V\in\mathbf{CV}_m(\mathbb{R}^n)$ and denote the corresponding unoriented and oriented classes by

$$\mathbf{CV}_m(\mathbb{R}^n)\quad\text{and}\quad\mathbf{CV}^o_m(\mathbb{R}^n).$$

Let us return to more a general case were $V\in\mathbf{IV}_m(\mathbb{R}^n)$ is a m-varifold on $N\hookrightarrow\mathbb{R}^n, m<p\le n$, ($m=\dim V, p=\dim N$) we can define an orthogonal projection matrix Q onto the tangent space of N:

$$T_xN=Q(x):=[Q_{ij}]\in\mathbb{R}^{n\times n}.$$

80 Definition (Generalized Second Fundamental Form)
Let $V\in\mathbf{CV}_m(\mathbb{R}^n)\cap\mathbf{V}_m(N)$. We define the generalized second fundamental form of V in N as $L^1_{loc}(V)$ function:

$$A\colon G_m(N)\to\mathbb{R}^{n\times n\times n},\quad A^k_{ij}(x,P):=P_{\ell j}B_{ik\ell}-P_{\ell j}P_{iq}\frac{\partial Q_{k\ell}}{\partial x_q}(x)\tag{202}$$

where we sum over all the values of the index if the index variable appears twice in a single term.

In the Appendix, in Lemma 102 it is shown that this definition is consistent with that of the classical second fundamental form introduced in Definition 1. At this point, it is important to notice that here we follow the notation of Mondino in [Mon14, p.7 bottom], instead of that of Hutchinson, who calls A the curvature and B the second fundamental form. The Hutchinson also showed that B can be expressed in terms of A:

$$B_{ijk}=A^k_{ij}+A^j_{ik}+P_{j\ell}P_{iq}\frac{\partial Q_{\ell k}}{\partial x_q}(x)+P_{k\ell}P_{iq}\frac{\partial Q_{\ell j}}{\partial x_q}(x).\tag{203}$$

Furthermore according to [Hut86, 5.2.3 p. 62] we have $P=\operatorname{Tan}^m(\mu_V,x)$ for V a.e. (x,P) and

$$\vec{H}_i(V,x)=B_{jij}(x,\operatorname{Tan}^m(\mu_V,x))\text{ for }\mu_V\text{ a.e. }x\in U.\tag{204}$$

The weak curvature vector can also be described with A-terms by (203).

$$\vec{H}_i=A^j_{ji}+A^i_{jj}+P_{i\ell}P_{jq}\frac{\partial Q_{\ell j}}{\partial x_q}(x)+P_{j\ell}P_{jq}\frac{\partial Q_{\ell i}}{\partial x_q}(x).\tag{205}$$

In the case where the varifold is considered directly on $\mathbb{R}^n$ ($N=\mathbb{R}^n$), we obtain $T_x\mathbb{R}^n\cong\mathbb{R}^n$, so that Q is in this case a unit matrix, those derivatives vanish.

As shown by Brakke, if V is an integral varifold and $\|\delta V\|$ is a Radon measure, then the mean curvature vector is perpendicular to the varifold at V-almost everywhere.

81 Theorem (Perpendicularity Theorem, Brakke)
If $V = v(E,1) \in \mathbf{IV}_k(\mathbb{R}^n)$ and $\|\delta V\|$ is a Radon measure then V-a.e.:

$$\vec{H} \perp T_x E.$$

Proof: [Bra78] Brakke Thm 5.8 p.157. □

Now, we can use the fact that by Theorem 81 the tangential projection of $\vec{H}$ is zero. Therefore, we obtain

$$\vec{H}_i \overset{205}{=} A^j_{ji} + A^i_{jj} \overset{202}{=} P_{\ell i} B_{jj\ell} + A^i_{jj} = P_{\ell i}\vec{H}_\ell + A^i_{jj} \overset{81}{=} A^i_{jj}, \tag{206}$$

by the symmetry $B_{jj\ell} = B_{j\ell j}$, see [Hut86, 5.2.4. (i)].

We want to finish the subsection by presenting varifold compactness results. These strong theorems are the justification for using the (curvature) varifold framework. They provide existence, thus the remaining difficulty is showing higher regularity.

82 Theorem (Allard Compactness Theorem, Not-Oriented)
Let $(V_j)_{k\in\mathbb{N}} \subset \mathbf{IV}^o_m(\mathbb{R}^n)$ be a sequence of m-rectifiable varifolds with locally bounded first variation on an open set $\Omega \subset \mathbb{R}^n$. If

$$\sup_{k\in\mathbb{N}} \Big\{ \big\|V_k(W)\big\| + \|\delta V_k\|(W) \Big\} \le c(W) < \infty$$

for every open set $W \subset\subset \Omega$, then there exists a subsequence $\{V_{k_\ell}\}_{\ell\in\mathbb{N}}$ converging weakly-$*$ to a m-rectifiable varifold $V \in \mathbf{IV}^o_m(\mathbb{R}^n)$ with locally bounded first variation on Ω and:

$$\|\delta V\|(W) \le \liminf_{\ell\to\infty} \|\delta V_{k_\ell}\|(W), \quad \forall W \subset\subset \Omega.$$

Proof: [All72, Thm. 5.6. p. 452] □

83 Theorem (Compactness Theorem for Oriented Integral Varifolds, Hutchinson)
Let $\mathbb{R}^n = \bigcup_{i=1}^\infty A_i$ where A_i are open. Then for any sequence $\{M_i\}_{i=1}^\infty$ of positive constants the following is sequentially compact w.r.t. oriented varifold convergence:

$$\big\{ V \in \mathbf{IV}^o_m(\mathbb{R}^n) \,\big|\, \forall i \in \mathbb{N} : (\mu_V + \|\delta V\|)(A_i) \le M_i \big\}.$$

Proof: [Hut86, Thm. 3.1 p. 49] □

84 Assumption (Notation Hutchinson [Hut86] 5.2.8. p. 65)
Suppose $F\colon G_m(N) \times \mathbb{R}^{n\times n\times n} \to \mathbb{R}$. We denote variables in $G_m(N) \times \mathbb{R}^{n\times n\times n}$ by $((x,P),B)$. F shall always satisfy the following conditions:

1. *F is continuous.*
2. *F is non-negative: $\forall ((x,P),B) \in G_m(N) \times \mathbb{R}^{n\times n\times n} : F((x,P),B) \ge 0$.*
3. *F is convex in the B variables: $\forall \lambda \in (0,1), (x,P) \in G_m(N), \overline{B}, B \in \mathbb{R}^{n\times n\times n}$:*

   $$F((x,P), \lambda\overline{B} + (1-\lambda)B) \le \lambda F((x,P),\overline{B}) + (1-\lambda)F((x,P),B).$$

   If the above equation holds strictly, then we call F strictly convex.
4. *F has non-linear growth in the B variables, i.e. there exists a continuous function $\varphi\colon G_m(N) \times [0,\infty) \to [0,\infty)$, $0 \le \varphi((x,P),s) \le \varphi((x,P),t)$, for $0 \le s \le t$ and $(x,P) \in G_m(N)$, $\varphi((x,P),t) \to \infty$ locally uniformly in (x,P) as $t \to \infty$, and*

   $$\forall ((x,P),B) \in G_m(N) \times \mathbb{R}^{n\times n\times n}\colon \quad \varphi((x,P),|B|)|B| \le F((x,P),B).$$

Let F satisfy Assumption 84 then for each $V \in \mathbf{CV}_m(E)$, we define

$$\mathcal{F}_E[V] = \int_{G_m(E)} F\big((x,P), \mathcal{S}_E V(x,P)\big)\, \mathrm{d}V(x,P). \tag{207}$$

Suppose $\{V_k\}_{k=1}^{\infty} \subset \mathbf{CV}_m(E)$ and $V \in \mathbf{CV}_m(E)$. Then we say V_k converges to V in the weak sense in E, and write

$$V_k \overset{C,E}{\rightharpoonup} V \quad \text{if} \quad (V_k \llcorner E, \mathcal{S}_E V_k) \rightharpoonup (V \llcorner E, \mathcal{S}_E V)$$

in $G_m(E)$ in the sense of measure-function pair weak convergence (72)

85 Theorem (Compactness, Lower Semicontinuity)
Let $\mathcal{F}_E$ be like in the above Definition 84. Suppose $\{V_k\}_{k=1}^{\infty} \subset \mathbf{CV}_m(E), V \in \mathbf{IV}_m(N), V_k \to V$ in $G_m(E)$ and $\mathcal{F}_E[V_k]$ is bounded uniformly in k. Then

$$V \in \mathbf{CV}_m(E), \quad V_k \overset{C,E}{\rightharpoonup} V, \quad \textit{and} \quad \mathcal{F}_E[V] \le \liminf_{k\to\infty} \mathcal{F}_E[V_k].$$

Proof: [Hut86, Thm 5.3.2. p.66] □

8 Compactness Results

This chapter provides additional results to the work done in [DGR17] by Deckelnick, Grunau, and Röger. In their work, they demonstrated several key results. First, in [DGR17, Theorem 2] they proved the L^∞ and area bounds like in Theorem 16 (a) for the surfaces as graphs on a bounded smooth domain with sufficiently regular Dirichlet or Navier boundary data which allows working in the $L^\infty \cap BV$-setting. Then, they introduced the L^1-lower semicontinuous relaxation of the Willmore functional and showed lower-bound and compactness estimates for Willmore energy-bounded sequences, which allowed them to show the existence of a minimizer in the $L^\infty \cap BV$-space for the relaxed energy. Also importantly, since the graph is described by a BV-function they were able to characterize the lower bound for a suitable subset of BV by a term originating from the mean curvature of the absolutely continuous part of the gradient as Radon measure.

In this chapter in Subsection 8.1 the results from [DGR17], based on which we will construct new contributions, are presented. Then, in Subsection 8.2, we will add terms to the lower-bound mentioned above. Namely, by rewriting a $W^{2,2}$- sequence as a sequence of measure-function pairs or varifolds, we can recover an until now missing irregular term for the lower-bound. For example, it can represent the curvature supported on the vertical jump part. This work is strongly based on unpublished notes of Deckelnick, Grunau, and Röger done subsequently to [DGR17]. Finally, in Subsection 8.3 based on unpublished notes of Grunau, we will construct a one-dimensional example of a BV-function with Cantor part and finite relaxed Willmore energy.

8.1 Preliminaries

As mentioned above, considering sequences with uniformly bounded Willmore energy subject to appropriate boundary conditions, by Theorem 16 the $L^\infty(\Omega)\cap BV(\Omega)$-space forms a natural framework to work with. The reason is that such $L^\infty(\Omega) \cap BV(\Omega)$-bounded sequences are precompact in $L^1(\Omega)$. Additionally, as shown by counterexamples in [DGR17, Examples 1 and 2], no a-priori bounds in $W^{1,p}(\Omega)$, $1 < p \leq \infty$ can be achieved in terms of the Willmore energy. In general, the subsequence limit point of a Willmore energy bounded sequence in $W^{2,2}(\Omega)$ will instead lie not better than in $L^\infty(\Omega) \cap BV(\Omega)$. Consequently, as discussed in Subsection 7.2 the limit point may have jump discontinuities and a highly irregular Cantor part. Also, it is not absolutely clear how to characterize the Willmore energy for $BV(\Omega)$-functions.

Furthermore, we have to define the boundary conditions. In what follows, let us always assume that $\Omega \subset \mathbb{R}^2$ is a bounded C^2-smooth domain with exterior normal vector $\nu\colon \partial\Omega \to \mathbb{S}^1$. The boundary data is then represented by a given function $\varphi \in C^2_c(\mathbb{R}^2)$. Furthermore, we can define the set of functions satisfying the Dirichlet boundary conditions represented by φ by

$$\mathcal{M} := \left\{ v \in W^{2,2}(\Omega) \,\middle|\, v - \varphi \in \mathring{W}^{2,2}(\Omega) \right\}.$$

In [DGR17] it was showed that uniformly Willmore energy bounded sequences $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ have $L^1(\Omega)$-convergent subsequences. Therefore, it makes sense to investigate the behavior of the Willmore energy under $L^1(\Omega)$-convergence, whereby the limit point may not lie in $\mathcal{M}$. Deckelnick, Grunau and Röger has chosen the *L^1-lower semicontinuous relaxation of the Willmore functional*:

$$\overline{\mathcal{W}} : L^1(\Omega) \to [0,\infty], \quad \overline{\mathcal{W}}(u) := \inf\left\{ \liminf_{k\to\infty} \mathcal{W}(u_k) \,\middle|\, \mathcal{M} \ni u_k \to u \text{ in } L^1(\Omega) \right\},$$

which they showed is the lower-semicontinuous extension of the classical Willmore functional on $\mathcal{M}$ [DGR17, Theorem 4]. One of the main difficulties is that a more explicit characterization of the relaxation has not been obtained, yet. For the one-dimensional case, this was achieved in [DMFLM09] (also see Subsection 8.3). However, Deckelnick, Grunau, and Röger were able to show some (mild) regularity properties for the limit point and also give a lower bound of the relaxed Willmore energy given by the absolutely continuous part of ∇u. It is defined as an absolutely continuous contribution to the Willmore energy in the following.

Let $u \in BV(\Omega)$ with the $\mathbb{R}^2$-valued measure ∇u and its the absolutely continuous part $\nabla^a u \in L^1(\Omega)$. Furthermore, we define $Q^a := \sqrt{1+|\nabla^a u|^2}$ and the *absolutely continuous contribution to the Willmore energy* as

$$\mathcal{W}^a(u) := \begin{cases} \frac{1}{4}\int_\Omega \left(\nabla \cdot \frac{\nabla^a u}{Q^a}\right)^2 Q^a \,\mathrm{d}x & \text{if } \frac{\nabla^a u}{Q^a} \in H(\mathrm{div},\Omega) \text{ and the integral is finite} \\ \infty & \text{else ,} \end{cases}$$

where $H(\mathrm{div},\Omega) := \{u \in L^2(\Omega) \mid \mathrm{div}\, u \in L^2(\Omega)\}$ is a Hilbert space [Tem01, Chapter 1, Section 1.2].

The main results in [DGR17] are presented in the next theorem. It states that each energy-bounded sequence has a L^1-convergent subsequence, such that the $BV(\Omega) \cap L^\infty(\Omega)$ limit point has some regularity $\frac{\nabla^a u}{Q^a} \in H(\mathrm{div},\Omega)$ and has finite the absolutely continuous contribution to the Willmore energy $\mathcal{W}^a$ with some estimate from above.

86 Theorem (Theorem 3 in [DGR17])
Let $\{u_k\}_{k\in\mathbb{N}}$ be a given sequence in $W^{2,2}(\Omega)$ that satisfies $u_k - \varphi \in \mathring{W}^{2,2}(\Omega)$ for all $k \in \mathbb{N}$ and

$$\liminf_{k\to\infty} \mathcal{W}(u_k) < \infty.$$

Then there exists a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ with $\frac{\nabla^a u}{Q^a} \in H(\mathrm{div},\Omega)$ such that after passing to a subsequence

$$u_k \to u \text{ in } L^1(\Omega) \quad (k \to \infty) \quad \text{and} \quad \mathcal{W}^a(u) \le \liminf_{k\to\infty} \mathcal{W}(u_k).$$

If in addition $u \in W^{1,1}(\Omega)$ then the mean curvature $H = \nabla \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in L^2(\Omega)$ exists in the weak sense and it holds

$$\mathcal{W}(u) = \frac{1}{4}\int_\Omega H^2 \sqrt{1+|\nabla u|^2} dx \le \liminf_{k\to\infty} \mathcal{W}(u_k). \tag{208}$$

Proof: [DGR17, Theorem 3] □

This means $\mathcal{W}(u) = \overline{\mathcal{W}}(u)$ for $u \in \mathcal{M}$. Regarding boundary data for the limit point u, in [DGR17] it was also proved that the trace on $\partial\Omega$ of u satisfies he boundary condition $u = \varphi$ $\mathcal{H}^1$-almost everywhere on $\{(Q^a)^{-1} > 0\} \cap \partial\Omega$ [DGR17, Proposition 2] and $\nabla^a u \cdot \nu = \nabla\varphi \cdot \nu$ $\mathcal{H}^1$-almost everywhere on $\partial\Omega$.

At this point, it should be noticed that (208) can also be deduced from the corresponding lower semicontinuity theorem in the framework of integral currents proved by Schätzle in [Sch09, Theorem 5.1] with some additional work.

Next, we want to list some convergence results which were also proved [DGR17] and will be used in the next subsection. These are some auxiliary sequences with some compactness properties provided by Theorem 16 in the proof of Theorem 86. It was shown that [DGR17, (35)]

$$q_k := \left(1+|\nabla u_k|^2\right)^{-5/4} \to q \in W^{1,2}(\Omega) \text{ in } L^p(\Omega), p \in (1,\infty), \text{ and almost everywhere in } \Omega.$$

Moreover, by [DGR17, (39)] $q > 0$ $\mathcal{L}^2$-almost everywhere in Ω. Additionally, the set $\{q = 0\}$ represents the points where the graph of u may become vertical. Furthermore, by [DGR17, (47),(48)] with some set $E \subset \Omega$ such that $\mathcal{H}^1(E) = 0$ it follows

$$\nabla u_k \to \nabla^a u \qquad \text{in } \{q > 0\}\backslash E, \tag{209}$$

$$Q_k := \sqrt{1 + |\nabla u_k|^2} \to \sqrt{1 + |\nabla^a u|^2} = Q^a \qquad \text{in } \{q > 0\}\backslash E \tag{210}$$

therefore $\nabla u_k \to \nabla^a u$ and $Q_k \to Q^a$ almost everywhere in Ω.

There are also convergence properties of the mean curvatures so that after passing to a subsequence

$$H_k := \operatorname{div}\left(\frac{\nabla u_k}{Q_k}\right) \rightharpoonup \operatorname{div}\left(\frac{\nabla^a u}{Q^a}\right) = H^a \text{ in } L^2(\Omega). \tag{211}$$

By [DGR17, (53)] it even holds

$$H_k \sqrt{Q_k} \rightharpoonup H^a \sqrt{Q^a} \text{ in } L^2(\Omega).$$

In the next theorem, which is a corollary to Theorem 86 we want to recall the existence result of the extended functional $\overline{\mathcal{W}}$. There, a minimizing sequence for the Willmore functional was considered. Then it followed that the regularity property stated in Theorem 86 is satisfied, and the minimizer attains the boundary conditions in a sense explained above. Also, we want to emphasize that the Dirichlet boundary conditions encoded by φ are assumed by restricting approximating sequences to be in $\mathcal{M}$, thus to the set of functions that satisfy in $W^{2,2}(\Omega)$ the boundary conditions fixed by φ.

87 Theorem (Theorem 5 in [DGR17])
There exists a function $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that

$$\forall v \in L^1(\Omega): \quad \overline{\mathcal{W}}(u) \le \overline{\mathcal{W}}(v).$$

Proof: [DGR17, Theorem 5]. □

8.2 Additional Compactness Results

In this subsection, we want to present additional compactness results based on unpublished notices of Deckelnick, Grunau, and Röger. They extend the statements of Theorem 86, such that the lower bound $\mathcal{W}^a(u)$ gets an additional term not originating from $\nabla^a u$.

Especially, we want to investigate the convergence of area measures and normal vectors. Hence, we make some new definitions. For each graph $\Gamma(u_k)$, we call Q_k the area element and

$$\mu_k = Q_k \mathcal{L}^2 \llcorner \overline{\Omega}$$

the graph area measure. Furthermore, we define the unit upwards pointing normal fields $N_k : \Gamma(u_k) \to \mathbb{S}^2$ and associate functions on $\overline{\Omega}$ $\tilde{N}_k : \overline{\Omega} \to \mathbb{S}^2$ by

$$N_k\left(x, u_k(x)\right) := \frac{1}{Q_k}\begin{pmatrix} -\nabla u_k \\ 1 \end{pmatrix}, \qquad \tilde{N}_k(x) = N_k\left(x, u_k(x)\right).$$

We also define the unit upward pointing normal fields of the absolutely continuous part $\nabla^a u$ by

$$N^a := \frac{1}{Q^a}\begin{pmatrix} -\nabla^a u \\ 1 \end{pmatrix}.$$

Since we also want to incorporate the boundary into our discussion, we have to extend the sequence of functions and measures from Ω to a bigger open ball $B_R(0) \subset \mathbb{R}^2$. Otherwise, the test functions in Definition 72 introducing the measure-function pair weak convergence would be merely in $C_c^0(\Omega)$.

Furthermore, to be able to work with varifolds (without boundary), we also want to extend the surfaces with boundary $\Gamma(u_k), k \in \mathbb{N}$ to closed $C^1 \cap W^{2,2}$ surfaces $\Gamma(u_k) \cup \Sigma$ with some auxiliary surface Σ with boundary depending only on φ Dirichlet boundary data. Σ in some sense "closes" $\Gamma(u_k)$ geometrically while still having a uniformly bound on the Willmore energy. In detail, the additional parts in $\overline{\Omega} \times \mathbb{R}$ will consist of constant graphs over Ω that are strictly separated from all $\Gamma(u_k)$, which is possible by L^∞-bound on u_k by the Willmore energy (Theorem 16). The closing of graphs will be done similarly to the technique presented by Miura in [Miu22].

88 Lemma (Extensions)
For given $\varphi \in C_c^2(\mathbb{R}^2)$ and $M > 0$, let $\{u_k\}_{k\in\mathbb{N}}$ be a given sequence in $W^{2,2}(\Omega)$ that satisfies

$$\forall k \in \mathbb{N}\colon\ u - \varphi \in \mathring{W}^{2,2}(\Omega) \quad \textit{and} \quad \forall k \in \mathbb{N}\colon\ \mathcal{W}(u_k) \leq M.$$

Then there is a radius $R = R(\operatorname{diam}(\operatorname{supp}\varphi), \|\varphi\|_{W^{1,1}(\partial\Omega)}, \Omega, M)$ such that:

(a) *For each $k \in \mathbb{N}$ there is an extension $\overline{u}_k \in \mathring{W}^{2,2}(B_R(0))$*

$$\forall x \in B_R(0) \setminus \overline{\Omega}\colon \quad \overline{u}_k(x) = \varphi(x) \quad \textit{and} \quad \forall x \in \overline{\Omega}\colon \quad \overline{u}_k(x) = u_k(x).$$

We also extend the values of the limit u to $\overline{u}$, as well as normal vector fields $N_k, \tilde{N}_k$, the mean curvature H_k, and the surface measure μ_k by the values corresponding to the graph of φ outside of $\overline{\Omega}$ without changing the notation.

(b) *There is a constant $C_{223} > 0$ and a surface Σ with boundary $\partial\Sigma$, constructed only based on $\operatorname{diam}(\operatorname{supp}\varphi), \|\varphi\|_{W^{2,2}(B_R(0))}, \|\varphi\|_{W^{1,1}(\partial\Omega)}, \Omega, M$ such that $\Gamma(\overline{u}_k) \cup \Sigma$ is a closed embedded $C^1 \cap W^{2,2}$ surface with*

$$\Gamma(\overline{u}_k) \cup \Sigma \subset B_{2R}(0) \times \mathbb{R}, \quad \mathcal{H}^2(\Gamma(\overline{u}_k) \cup \Sigma) + \operatorname{diam}(\Gamma(\overline{u}_k) \cup \Sigma) \leq C_{223}.$$

We also extend the values of the normal vector fields $N_k, \tilde{N}_k$, and the mean curvature H_k outside of $\Gamma(\overline{u}_k)$ by choosing the values of that on the surface Σ without changing the notation.

Proof: **(a)** First, let assume, that $R > \operatorname{diam}(\operatorname{supp}(\varphi))$. We extend each function u_k by φ on $B_R(0) \setminus \overline{\Omega}$. Let $R > 0$ such that $\overline{\Omega} \subset \operatorname{supp}(\varphi) \subset B_R(0)$. For each $k \in \mathbb{N}$ we define

$$\forall x \in B_R(0) \setminus \overline{\Omega}\colon \quad \overline{u}_k(x) := \varphi(x) \quad \text{and} \quad \forall x \in \overline{\Omega}\colon \quad \overline{u}_k(x) := u_k(x)$$

then $\overline{u}_k \in \mathring{W}^{2,2}(B_R(0))$ with $\operatorname{supp}(\overline{u}_k) \subset \big(\operatorname{supp}(\varphi) \cup \overline{\Omega}\big) \subset B_R(0)$.

(b) Here in addition to the condition on R in (a) we assume that $R > \|\varphi\|_{L^\infty(B_R(0))} + \mathcal{H}^2(\Gamma(u_k))$ and

$$R > 64 \left(\mathcal{H}^2(\Omega) + \mathcal{H}^1(\partial\Omega) + \|\varphi\|_{W^{1,1}(\partial\Omega)} + \frac{16^2}{\pi^2} M\right) \big(1 + |\Omega| M\big).$$

Then by Theorem 16 (a) we get $\|u\|_{L^\infty(\Omega)} < R$.

In order to construct Σ, we observe that we can glue $\Gamma(\overline{u}_k)$ on top of the convex hull of a horn torus which has the radius of the tube R and distance R from the center of the tube to the axis of revolution. Hence, the parametrization of the horn torus is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} (\theta, \varphi) = \begin{pmatrix} (R + R\cos\theta)\cos\varphi \\ (R + R\cos\theta)\sin\varphi \\ R(\sin\theta - 1) \end{pmatrix}$$

for $\theta, \varphi \in [0, 2\pi)$. We denote this horn torus as $\mathbb{T}^2_{R,R}$. Then we remove the upper flat circle $B_R(0) \times \{0\}$ from the convex hull of $\mathbb{T}^2_{R,R}$ denoted by $\operatorname{conv}(\mathbb{T}^2_{R,R})$ and glue it to $\Gamma(\overline{u}_k)$. It follows that for each $k \in \mathbb{N}$

$$\Sigma_k := \Gamma(\overline{u}_k) \cup \big(\operatorname{conv}(\mathbb{T}^2_{R,R}) \setminus (B_R(0) \times \{0\})\big)$$

is a $C^1 \cap W^{2,2}$ surface without boundary. We can estimate its Willmore energy

$$\begin{aligned}
\mathcal{W}(\Sigma_k) &\leq \mathcal{W}(u_k) + \mathcal{W}(\varphi) + \mathcal{W}(\operatorname{conv}(\mathbb{T}^2_{R,R})) \leq M + \mathcal{W}(\varphi) + \mathcal{W}(\mathbb{T}^2_{R,R}),\\
\mathcal{H}^2(\Sigma_k) &\leq \mathcal{H}^2(\Gamma(u_k)) + \mathcal{H}^2(\Gamma(\varphi|_{B_R(0)})) + \mathcal{H}^2(\mathbb{T}^2_{R,R}) + \mathcal{H}^2(B_R(0)))\\
&\leq R + \mathcal{H}^2(\Gamma(\varphi|_{B_R(0)})) + \mathcal{H}^2(\mathbb{T}^2_{R,R}) + \pi R^2,\\
\operatorname{diam}(\Sigma_k) &\leq \operatorname{diam}(\Gamma(u_k)) + \operatorname{diam}(\Gamma(\varphi|_{B_R(0)})) + \operatorname{diam}(\mathbb{T}^2_{R,R})\\
&\underset{R \geq |u|, |\varphi|}{\leq} 4R + \operatorname{diam}(\mathbb{T}^2_{R,R}).
\end{aligned}$$

Therefore, all quantities on the right-hand sides can estimated by $R, \operatorname{diam}(\operatorname{supp}\varphi), \|\varphi\|_{W^{2,2}(B_R(0))}$, $\|\varphi\|_{W^{1,1}(\partial\Omega)}, \Omega$ and M independent of $k \in \mathbb{N}$. We finish the proof by defining

$$\Sigma := \operatorname{conv}(\mathbb{T}^2_{R,R}) \setminus (B_R(0) \times \{0\})$$

with the property $\Sigma \cup \Gamma(\overline{u}_k) = \Sigma_k$ for all $k \in \mathbb{N}$. The embedding property is due to $R > \|\varphi\|_{L^\infty(B_R(0))}, \|u\|_{L^\infty(\Omega)}$. □

Now, we are ready to state the main result of this subsection.

89 Theorem (Additional Compactness Results)
For given $\varphi \in C^2_c(\mathbb{R}^2)$ and $M > 0$, let $\{u_k\}_{k\in\mathbb{N}}$ be a given sequence in $W^{2,2}(\Omega)$ that satisfies

$$\forall k \in \mathbb{N}\colon\ u - \varphi \in \mathring{W}^{2,2}(\Omega) \quad \textit{and} \quad \forall k \in \mathbb{N}\colon\ \mathcal{W}(u_k) \leq M.$$

Then there exists a subsequence $\{u_{k_\ell}\}_{\ell\in\mathbb{N}}$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$ with

$$u_{k_\ell} \to u \quad \textit{in} \quad L^1(\Omega) \quad (\ell \to \infty).$$

Moreover, the following holds:

ⓐ *There exist functions $\tilde{N}\colon \overline{\Omega} \to \mathbb{S}^2$, $\tilde{H}\colon \overline{\Omega} \to \mathbb{R}$ and a Radon measure μ on $\overline{\Omega}$ such that for all $p \in (1, \infty)$ it follows*

$$(\mu_k, \tilde{N}_k) \xrightarrow{L^p} (\mu, \tilde{N}), \quad (\mu_k, H_k) \overset{L^2}{\rightharpoonup} (\mu, \tilde{H}), \quad (\mu_k, H_k N_k) \overset{L^1}{\rightharpoonup} (\mu, \tilde{H}\tilde{N}).$$

Additionally, we have

$$\mathcal{W}^a(u) + \int_{\overline{\Omega}\setminus A_D(u)} |\tilde{H}|^2 \,\mathrm{d}\mu = \int_{\overline{\Omega}} |\tilde{H}|^2 \,\mathrm{d}\mu \leq \liminf_{k\to\infty} \int_\Omega H_k^2 Q_k \,\mathrm{d}x.$$

ⓑ *Let $\{\overline{u}_k\}_{k\in\mathbb{N}}, \overline{u}, R$ and Σ as in Lemma 88. Then the sequence of oriented varifolds $\{V^o[\Gamma(\overline{u}_k) \cup \Sigma, N_k, 1, 0]\}_{k\in\mathbb{N}}$ converges in varifold sense to the curvature varifold $V^o[\Gamma \cup \Sigma, N, 1, 0]$ with mean curvature vector in varifold sense $\vec{H} = HN$ and Γ the reduced boundary of the sublevel set of $\overline{u}$*

$$V^o[\Gamma(\overline{u}_k) \cup \Sigma, N_k, 1, 0] \to V^o[\Gamma \cup \Sigma, N, 1, 0] \in \mathbf{CV}^0(B_{2R}(0) \times \mathbb{R})$$

and furthermore

$$\mathcal{W}^a(u) + \int_{(\overline{\Omega}\setminus A_D(u))\times\mathbb{R}} |\vec{H}|^2 \,\mathrm{d}\|V\| = \int_{\overline{\Omega}\times\mathbb{R}} |\vec{H}|^2 \,\mathrm{d}\|V\| \leq \liminf_{k\to\infty} \int_\Omega H_k^2 Q_k \,\mathrm{d}x.$$

ⓒ *Also, we can relate the different mean curvature to each other*

$$H^a(x) = H(x,u(x)) \quad \textit{and} \quad H^a(x) = \tilde{H}(x) \quad \mathcal{L}^2\text{-a.e..}$$

There exist $\mathbb{R}$-valued finite Radon measures ν_x on $\mathbb{R}$ with $\nu_x(\mathbb{R}) = 1$ μ-a.e. such that $x \to \nu_x$ is μ-measurable and

$$\tilde{H}(x) = \int_{\mathbb{R}} H(x,r)\,\mathrm{d}\nu_x \quad \mu\text{-a.e in } \Omega \quad \textit{and} \quad \nu_x = \delta_{u(x)} \quad \mathcal{L}^2\text{-a.e..}$$

Proof: We split this proof into six parts. In (1) we investigate the measure-function pair convergence of the normal field or mean curvature with the area measure and combine its limits. Then, in (2) we reformulate the graphs as varifolds and compare the normal vector field limit with a normal field of the varifold sequence limit. Next, in (3) we look at mean curvature vector convergence. In (4) we again rewrite graphs as characteristic functions of sublevelsets in BV-framework and show that the mean curvature of the varifold limit coincides with the generalized mean curvature of $\mathcal{F}E$ in the sense of Luckhaus-Sturzenbecker. In Step (5), we, via disintegration theorem, relate the H^a to that of the varifold limit. Finally, in the last step (6) we look at lower semicontinuites of the curvature varifold convergence and measure-pair convergence and separate the part corresponding to the absolutely continuous part of ∇u.

(1) Due to Theorem 86 it follows after passing to a subsequence there exists $u \in BV(\Omega) \cap L^\infty(\Omega)$ with $\frac{\nabla^a u}{Q^a} \in H(\mathrm{div}, \Omega)$ such that

$$\exists u \in BV(\Omega)\, \forall p \in [1,\infty)\colon u_k \to u \text{ in } L^p(\Omega) \text{ and a.e. in } \Omega.$$

Moreover, due to Theorem 16 (a), we possess uniform bounds on the area measures defined on $\Omega \subset \mathbb{R}^2$. Since we have $\nabla u_k \to \nabla^a u$ as well as $Q_k \to Q^a$ in $\{q > 0\} \setminus E, \mathcal{H}^1(E) = 0$ (209) and in particular $q > 0$ almost everywhere [DGR17, (39)], we obtain that $\tilde{N}_\ell \to N^a$ almost everywhere in Ω.

From now on, we use the extension results from Lemma 88 (a). Henceforth, it follows that after passing to a subsequence the measures μ_k converge to a Radon measure μ with $\mathrm{spt}(\mu - \mu_k) \subset \overline{B_R(0)}$, since $\mu \equiv \mu_k$ for arbitrary k outside of $\overline{\Omega}$. We observe that $\forall p \in [1,\infty)$:

(212)
$$\begin{aligned}
\sup_k \int_{B_R(0)} \Big(\|\tilde{N}_k\|^p + |H_k|^2\Big)\,\mathrm{d}\mu_k &= \sup_k \int_{\mathbb{R}^2} \Big(1 + |H_k|^2\Big)\,\mathrm{d}\mu_k \\
&\le \sup_k \Big(\int_\Omega Q_k\,\mathrm{d}x + \mathcal{W}(u_k)\Big) + \int_{B_R(0)} \sqrt{1+|\nabla\varphi|^2}\,\mathrm{d}x + \mathcal{W}(\varphi) \\
&\underset{\text{Thm.16}}{\le} C < \infty.
\end{aligned}$$

Now we consider the measure-function pairs $(\mu_k, \tilde{N}_k)$ and (μ_k, H_k) over $B_R(0)$ with the functions $F_{N,p}\colon \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}$ for $p \in (1,\infty)$ and $F_H\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$:

$$F_{N,p}(x,N) := \|N\|^p, \quad F_H(x,H) := |H|^2. \tag{213}$$

These functions are (i) nonnegative, (ii) continuous (iii) convex, and (iv) have nonlinear growth in the variable N or H. The measure-function pair $\{(\mu_k, \tilde{N}_k)\}_{k\in\mathbb{N}}$ is then $F_{N,p}$-bounded and $\{(\mu_k, H_k)\}_{k\in\mathbb{N}}$ F_H-bounded by (212).

Since $\mu_k \to \mu$ (see definition (181)) we obtain by Theorem 76 (i) subsequences of $\{(\mu_k, \tilde{N}_k)\}_{k\in\mathbb{N}}$ and $\{(\mu_k, H_k)\}_{k\in\mathbb{N}}$ that converge in the weak sense to a measure-function pair $\big(\mu, \tilde{N}\big)$ and $\big(\mu, \tilde{H}\big)$:

$$(\mu_k, \tilde{N}_k) \overset{L^p}{\rightharpoonup} (\mu, \tilde{N}), \tag{214}$$

$$(\mu_k, H_k) \overset{L^2}{\rightharpoonup} (\mu, \tilde{H}). \tag{215}$$

This means that especially for any $\xi \in C_c^0(B_R(0); \mathbb{R}^3)$ and $\nu \in C_c^0(B_R(0))$ it holds

$$\int_{B_R(0)} \langle \tilde{N}_k, \xi\rangle \,\mathrm{d}\mu_k \to \int_{B_R(0)} \langle \tilde{N}, \xi\rangle \,\mathrm{d}\mu, \qquad \int_{B_R(0)} \nu \tilde{H}_k \,\mathrm{d}\mu_k \to \int_{B_R(0)} \nu \tilde{H} \,\mathrm{d}\mu. \tag{216}$$

Next, we use a special calculation to control the L^2-differences in the sequence of the normal fields:

$$\begin{aligned}
\int_{B_R(0)} \frac{1}{2}\|\tilde{N}_k - \tilde{N}_\ell\|^2 (Q_k + Q_\ell)\,\mathrm{d}x &= \int_{B_R(0)} \frac{1}{2}\left\| \frac{1}{Q_k}\begin{pmatrix} -\nabla u_k \\ 1\end{pmatrix} - \frac{1}{Q_\ell}\begin{pmatrix} -\nabla u_\ell \\ 1\end{pmatrix}\right\|^2 (Q_k + Q_\ell)\,\mathrm{d}x \\
&= \int_{B_R(0)} \left[1 - \frac{\langle \nabla u_k, \nabla u_\ell\rangle + 1}{Q_k Q_\ell}\right](Q_k + Q_\ell)\,\mathrm{d}x \\
&= \int_{B_R(0)} \left[\frac{(Q_k Q_\ell)(Q_k+Q_\ell)}{Q_kQ_\ell} - \frac{\langle \nabla u_k, \nabla u_\ell\rangle + 1}{Q_k} - \frac{\langle \nabla u_k, \nabla u_\ell\rangle + 1}{Q_\ell}\right]\mathrm{d}x \\
&= \int_{B_R(0)} \left[\frac{(\|\nabla u_k\|^2+1)Q_\ell + (\|\nabla u_\ell\|^2+1)Q_k}{Q_kQ_\ell} - \frac{\langle \nabla u_k, \nabla u_\ell\rangle + 1}{Q_k} - \frac{\langle \nabla u_k, \nabla u_\ell\rangle + 1}{Q_\ell}\right]\mathrm{d}x \\
&= \int_{B_R(0)} \left[\frac{\|\nabla u_k\|^2}{Q_k} + \frac{\|\nabla u_\ell\|^2}{Q_\ell} - \frac{\langle \nabla u_k, \nabla u_\ell\rangle}{Q_k} - \frac{\langle \nabla u_k, \nabla u_\ell\rangle}{Q_\ell}\right]\mathrm{d}x \\
&= \int_{B_R(0)} \left\langle \nabla u_k - \nabla u_\ell, \frac{\nabla u_k}{Q_k} - \frac{\nabla u_\ell}{Q_\ell}\right\rangle \mathrm{d}x \\
&= -\int_{B_R(0)} (u_k - u_\ell)\left(\operatorname{div}\frac{\nabla u_k}{Q_k} - \operatorname{div}\frac{\nabla u_\ell}{Q_\ell}\right)\mathrm{d}x \\
&= -\int_{B_R(0)} (u_k - u_\ell)\left(\tilde{H}_k - \tilde{H}_\ell\right)\mathrm{d}x \to 0 \quad (k,\ell\to\infty),
\end{aligned} \tag{217}$$

because we have $L^2(B_R(0))$-bounds on $\tilde{H}_k$ and $u_k \to u$ in $L^2(B_R(0))$. Especially, since $Q_k \geq 1$ this implies that $\{\tilde{N}_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in L^2, hence $\tilde{N}_k \to \tilde{N}$ in L^2 and up to a subsequence $\tilde{N}_k \to \tilde{N}$ a.e.. Furthermore, it follows by (217), the weak convergence (216) with $\tilde{N}_k \in C_c^0(B_R(0))$ and Lebesgue dominated convergence heorem that

$$\begin{aligned}
0 &= \lim_{k,\ell\to\infty} \int_{B_R(0)} \frac{1}{2}\|\tilde{N}_k - \tilde{N}_\ell\|^2 (Q_k+Q_\ell)\,\mathrm{d}x \\
&= 2\lim_{k,\ell\to\infty}\int_{B_R(0)} (1 - \langle \tilde{N}_k, \tilde{N}_\ell\rangle) Q_\ell\,\mathrm{d}x \\
&\qquad \left(= 2\lim_{k,\ell\to\infty}\int_{B_R(0)} \frac{1}{2}\|\tilde{N}_k - \tilde{N}_\ell\|^2 Q_k\,\mathrm{d}x = \lim_{k\to\infty}\int_{B_R(0)} \|\tilde{N}_k - N^a\|^2 Q_k\,\mathrm{d}x\right) \\
&\overset{(214)}{=} 2\lim_{k\to\infty}\int_{B_R(0)} (1 - \langle \tilde{N}_k, \tilde{N}\rangle)\,\mathrm{d}\mu \\
&= 2\int_{B_R(0)} \left(\frac{1}{2} - \frac{1}{2}\|\tilde{N}\|^2\right)\mathrm{d}\mu + 2\lim_{k\to\infty}\int_{B_R(0)}\left(\frac{1}{2} - \langle \tilde{N}_k, \tilde{N}\rangle + \frac{1}{2}\|\tilde{N}\|^2\right)\mathrm{d}\mu
\end{aligned}$$

$$= \int_{B_R(0)} (1 - \|\tilde{N}\|^2)\, \mathrm{d}\mu + \lim_{k\to\infty} \int_{B_R(0)} \left\| \tilde{N}_k - \tilde{N} \right\|^2 \mathrm{d}\mu.$$

There is a way to see that $\int_{B_R(0)} (1 - \|\tilde{N}\|^2)\, \mathrm{d}\mu \geq 0$ due to Theorem 76 (ii) because (μ_k, N_k) is $F_{N,p}$ bounded for $p = 2$:

$$\int_{B_R(0)} \|\tilde{N}\|^2\, \mathrm{d}\mu \leq \liminf_{k\to\infty} \int_{B_R(0)} \|\tilde{N}_k\|^2\, \mathrm{d}\mu_k = \liminf_{k\to\infty} \int_{B_R(0)} 1\, \mathrm{d}\mu_k = \int_{B_R(0)} 1\, \mathrm{d}\mu.$$

As a result we have that $\tilde{N}_k \to \tilde{N}$ in $L^2(\mu)$ in $B_R(0)$ and up to a subsequence $\tilde{N}_k(x) \to \tilde{N}(x)$ for μ-a.e. $x \in B_R(0)$. Also, we get

$$\lim_{k\to\infty} \int_{B_R(0)} \left\| \tilde{N}_k - \tilde{N} \right\|^2 \mathrm{d}\mu = 0. \tag{218}$$

Moreover, we observe that due to $\|\tilde{N}_k\| \equiv 1$

$$0 \leq (1 - \|\tilde{N}\|)^2 = 1 - 2\|\tilde{N}\| + \|\tilde{N}\|^2 \leq 1 - 2\langle \tilde{N}_k, \tilde{N}\rangle + \|N\|^2 = \left\| \tilde{N}_k - \tilde{N} \right\|^2.$$

With (218) it follows that $\|\tilde{N}\| = 1$ μ-almost everywhere.

Therefore, it follows by the measure convergence $\mu_k \to \mu$ in $B_R(0)$ for each $p \in (1, \infty)$:

$$\lim_{k\to\infty} \int_{B_R(0)} \|\tilde{N}_k\|^p\, \mathrm{d}\mu_k = \lim_{k\to\infty} \int_{B_R(0)} 1\, \mathrm{d}\mu_k = \lim_{k\to\infty} \int_{B_R(0)} 1\, \mathrm{d}\mu = \int_{\mathbb{R}^2} \|\tilde{N}\|^p\, \mathrm{d}\mu.$$

By Theorem 76 (iii) we obtain $(\mu_k, \tilde{N}_k) \overset{F_{N,p}}{\to} (\mu, \tilde{N})$, written as

$$(\mu_k, \tilde{N}_k) \overset{L^p}{\to} (\mu, \tilde{N}). \tag{219}$$

From that it follows with the weak convergence $(\mu_k, H_k) \overset{L^2}{\rightharpoonup} (\mu, \tilde{H})$ and the product-rule Theorem 75, that:

$$(\mu_k, H_K \tilde{N}_k) \overset{L^1}{\rightharpoonup} (\mu, \tilde{H}\tilde{N}). \tag{220}$$

(2) Now, we translate the situation into a varifold setting. Here, we use the extensions from Lemma 88 (b). So we associate to each $u_k, k \in \mathbb{N}$ the unoriented and oriented integral 2-varifolds $V_k^o = V^o[\Gamma(\overline{u}_k) \cup \Sigma, N_k, 1, 0] \in \mathbf{IV}_2^o(\mathbb{R}^3)$, $V_k = V[\Gamma(\overline{u}_k) \cup \Sigma, 1] \in \mathbf{IV}_2(\mathbb{R}^3)$ defined by:

$$\forall \psi \in C_c^0\big(B_{2R}(0) \times \mathbb{R} \times G_{2,3}\big): \quad V_k(\psi) = \int_{\Gamma(\overline{u}_k)\cup\Sigma} \psi\big(z, N_k(z)^\perp\big) 1\, \mathrm{d}\mathcal{H}^2(z),$$

$$\forall \psi \in C_c^0\big(B_{2R}(0) \times \mathbb{R} \times \mathbb{S}^2\big): \quad V_k^o(\psi) = \int_{\Gamma(\overline{u}_k)\cup\Sigma} \Big[1 \cdot \psi\big(z, N_k(z)\big) + 0 \cdot \psi\big(z, -N_k(z)\big)\Big]\, \mathrm{d}\mathcal{H}^2(z).$$

It follows that $\|V_k\| = \mathcal{H}^2 \llcorner (\Gamma(\overline{u}_k) \cup \Sigma)$ is the mass measure of V_k.

The compactness theorems by Allard 82 and Hutchinson 83 yield after passing to a subsequence nonoriented $V[\Gamma \cup \Sigma, \theta]$ and oriented $V^o[\Gamma \cup \Sigma, N, \theta^+, \theta^-]$ limit integer 2-varifolds. Here $\Gamma \cup \Sigma \subset B_{2R}(0) \times \mathbb{R}$ is a 2-rectifiable set without boundary, $N\colon \Gamma \cup \Sigma \to \mathbb{S}^2$ (we choose an orientation arbitrarily) is an $\mathcal{H}^2$-measurable unit normal field and $\theta, \theta^+, \theta^-\colon \Gamma \cup \Sigma \to \mathbb{N}$ are $\mathcal{H}^2$-integrable functions, such that

$$V_k^o \overset{*}{\rightharpoonup} V^o \qquad \text{in } C_c^0\big((B_{2R}(0) \times \mathbb{R}) \times \mathbb{S}^2\big)^*, \tag{221}$$

$$V_k \overset{*}{\rightharpoonup} V \qquad \text{in } C_c^0\big((B_{2R}(0)\times\mathbb{R})\times G_{2,3}\big)^*, \tag{222}$$

$$\|V_k\| \overset{*}{\rightharpoonup} \|V\| \qquad \text{in } C_c^0(B_{2R}(0)\times\mathbb{R})^*. \tag{223}$$

The set Γ should be glued on the boundary of Σ in a sense $\Gamma\cap\Sigma\subset\partial\Sigma$. Additionally, $\theta=\theta^++\theta^-$ and $\|V\|=\theta\mathcal{H}^2 \llcorner (\Gamma\cup\Sigma)$ so that:

$$\forall\psi\in C_c^0\big(B_{2R}(0)\times\mathbb{R}\times G_{2,3}\big): \qquad V(\psi)=\int_{\Gamma\cup\Sigma}\psi\big(z,N(z)^\perp\big)\theta\,\mathrm{d}\mathcal{H}^2(z),$$

$$\forall\psi\in C_c^0\big(B_{2R}(0)\times\mathbb{R}\times\mathbb{S}^2\big): \qquad V^o(\psi)=\int_{\Gamma\cup\Sigma}\Big[\theta^+\cdot\psi\big(z,N(z)\big)+\theta^-\cdot\psi\big(z,-N(z)\big)\Big]\,\mathrm{d}\mathcal{H}^2(z),$$

where only the part concentrated on Γ should attract our attention.

Furthermore, the compactness Theorem by Allard 82 implies that V has locally bounded first variation on $B_{2R}(0)\times\mathbb{R}$ and also generalized mean curvature $\vec{H}$.

By the weakly-$*$ convergence (221) of oriented varifolds, we conclude

$$\begin{aligned}
&\forall g\in C_c^0(B_{2R}(0)\times\mathbb{R}\times\mathbb{S}^2):\\
&\quad V^o(g)=\lim_{k\to\infty}V_k^o(g)=\lim_{k\to\infty}\int_{\Gamma(\overline{u}_k)}g\big(z,N_k(z)\big)\,\mathrm{d}\mathcal{H}^2(z)+\int_{\Sigma\setminus\Gamma(\overline{u}_k)}g\big(z,N_k(z)\big)\,\mathrm{d}\mathcal{H}^2(z)\\
&\qquad=\lim_{k\to\infty}\int_{B_R(0)}g\Big(\big(x,u_k(x)\big),\tilde{N}_k(x)\Big)Q_k(x)\,\mathrm{d}x+\int_{\Sigma\setminus\Gamma(\overline{u}_k)}g\big(z,N_k(z)\big)\,\mathrm{d}\mathcal{H}^2(z).
\end{aligned}$$

Then, we can conclude

$$\begin{aligned}
\forall g\in C_c^0(B_{2R}(0)\times\mathbb{R}\times\mathbb{S}^2):\quad &\int_\Gamma\theta^+(z)g\big(z,N(z)\big)+\theta^-(z)g\big(z,-N(z)\big)\,\mathrm{d}\mathcal{H}^2(z)\\
&=\lim_{k\to\infty}\int_{\Gamma(\overline{u}_k)}g\big(z,N_k(z)\big)\,\mathrm{d}\mathcal{H}^2(z)\\
&=\lim_{k\to\infty}\int_{B_R(0)}g\Big(\big(x,u_k(x)\big),\tilde{N}_k(x)\Big)Q_k(x)\,\mathrm{d}x.
\end{aligned}$$

Next, we want to relate the varifold measure $\|V\|$ to the Radon measure μ from step 1. By (219) we know that $(\mu_k,\tilde{N}_k)\overset{L^p}{\to}(\mu,\tilde{N})$ for which we choose the test function $\tilde{g}(x,v)=g\big((x,r),v\big)\in C_c^0(B_R(0)\times\mathbb{S}^2)$ independent of the u-variable r. Then, by the strong convergence Definition 74 iii we obtain

$$\int_\Gamma\theta^+(z)\tilde{g}\big((z_1,z_2)^T,N(z)\big)+\theta^-(z)\tilde{g}\big((z_1,z_2)^T,-N(z)\big)\,\mathrm{d}\mathcal{H}^2(z)=\int_{B_R(0)}\tilde{g}\big(x,\tilde{N}(x)\big)\,\mathrm{d}\mu(x). \tag{224}$$

If we insert the test function $g\big((x,r),v\big)=\hat{g}(x)\in C_c^0(B_R(0))$, then we get with the same arguments as above that it holds

$$\int_\Gamma\big(\theta^+(z)+\theta^-(z)\big)\hat{g}(z_1,z_2)\,\mathrm{d}\mathcal{H}^2(z)=\int_{B_R(0)}\hat{g}(x)\,\mathrm{d}\mu(x).$$

By uniqueness result in the Riesz representation theorem, this means that the projected measure of $\|V\|$ and μ are the same:

$$\pi_\#^{\mathbb{R}^2}(\|V\|\llcorner\Gamma)=\mu$$

while $\pi^{\mathbb{R}^2}:\mathbb{R}^3\to\mathbb{R}^2$ denotes the orthogonal projection onto $\mathbb{R}^2\cong\mathbb{R}^2\times\{0\}$ and $\pi_\#^{\mathbb{R}^2}(\|V\|\llcorner\Gamma)$ denotes the corresponding push-forward.

By the disintegration theorem (Theorem 77) there exist $\mathbb{R}$-valued finite Radon measures $x \mapsto \nu_x$ on $B_R(0)$ that are $\pi_{\#}^{\mathbb{R}^2}(\|V\| \llcorner \Gamma) = \mu$-measurable and for any $f \in L^1(B_R(0) \times \mathbb{R}, \|V\|)$ we conclude

$$\begin{aligned} \int_{B_R(0)\times\mathbb{R}} f(z)\,\mathrm{d}(\|V\| \llcorner \Gamma)(z) &\overset{(198)}{=} \int_{B_R(0)} \left(\int_{\mathbb{R}} f(x,r)\,\mathrm{d}\nu_x(r) \right) \mathrm{d}\big(\pi_{\#}^{\mathbb{R}^2}(\|V\| \llcorner \Gamma)\big)(x) \\ &= \int_{B_R(0)} \left(\int_{\mathbb{R}} f(x,r)\,\mathrm{d}\nu_x(r) \right) \mathrm{d}\mu(x). \end{aligned} \tag{225}$$

Since $\|V\| \llcorner \Gamma = \theta \mathcal{H}^2 \llcorner \Gamma, \theta = \theta^+ + \theta^-$ it follows that

$$\begin{aligned} &\int_{B_R(0)} \tilde{g}\big(x, \tilde{N}(x)\big)\,\mathrm{d}\mu \\ &\quad= \int_{B_R(0)\times\mathbb{R}} \frac{\theta^+(z)\tilde{g}\big((z_1,z_2)^T, N(z)\big) + \theta^-(z)\tilde{g}\big((z_1,z_2)^T, -N(z)\big)}{\theta(z)}\,\mathrm{d}(\|V\| \llcorner \Gamma)(z) \\ &\quad\overset{(225)}{=} \int_{B_R(0)} \left[\int_{\mathbb{R}} \left(\frac{\theta^+(x,r)}{\theta(x,r)} \tilde{g}\big(x, N(x,r)\big) + \frac{\theta^-(x,r)}{\theta(x,r)} \tilde{g}\big(x, -N(x,r)\big) \right) \mathrm{d}\nu_x(r) \right] \mathrm{d}\mu(x). \end{aligned}$$

If we now again take the test functions $\tilde{g}(x,v) = \eta(x)v_i, i \in \{1,2,3\}, \eta \in C_c^0(B_R(0))$, then we can deduce that

$$\int_{B_R(0)} \eta(x) \left(\int_{\mathbb{R}} \frac{\theta^+(x,r) - \theta^-(x,r)}{\theta(x,r)} N(x,r)\,\mathrm{d}\nu_x(r) \right) \mathrm{d}\mu(x) = \int_{B_R(0)} \eta(x)\tilde{N}(x)\,\mathrm{d}\mu(x).$$

Since we can choose any $\eta \in C_c^0(B_R(0))$ it follows by the fundamental lemma of calculus of variations for Radon measures (Theorem 63 that for μ-almost all $x \in B_R(0)$

$$\int_{\mathbb{R}} \frac{\theta^+(x,r) - \theta^-(x,r)}{\theta(x,r)} N(x,r)\,\mathrm{d}\nu_x(r) = \tilde{N}(x). \tag{226}$$

Due to $\|\tilde{N}\| = 1$ μ-almost everywhere and $\|N\| = 1$ by the existence statement from the Hutchinson compactness Theorem 83, and by the disintegration theorem 77, ν_x is a probability measure $\nu_x(\mathbb{R})$=1 μ-a.e. we obtain μ-a.e.

$$\begin{aligned} 1 = \|\tilde{N}(x)\| &\le \int_{\mathbb{R}} \left| \frac{\theta^+(x,r) - \theta^-(x,r)}{\theta(x,r)} \right| \big\|N(x,r)\big\|\,\mathrm{d}\nu_x(r) \le \int_{\mathbb{R}} \left| \frac{\theta^+(x,r) - \theta^-(x,r)}{\theta(x,r)} \right| \mathrm{d}\nu_x(r) \\ &\le \max_{r\in\mathbb{R}} \left| \frac{\theta^+(x,r) - \theta^-(x,r)}{\theta^+(x,r) + \theta^-(x,r)} \right| 1 \le 1. \end{aligned}$$

It follows that for μ-almost all $x \in B_R(0) : \big|\theta^+(x,r) - \theta^-(x,r)\big| = \big|\theta^+(x,r) + \theta^-(x,r)\big|$ ν_x-almost everywhere. Thus, for μ-almost all x for ν_x-almost all r either $\theta^+(x,r) = 0$ or $\theta^-(x,r) = 0$. Further, we can multiply (226) by $\tilde{N}(x)$ then μ-a.e. it follows

$$\begin{aligned} 1 = \langle \tilde{N}(x), \tilde{N}(x) \rangle &= \int_{\mathbb{R}} \frac{\theta^+(x,r) - \theta^-(x,r)}{\theta(x,r)} \langle N(x,r), \tilde{N}(x) \rangle\,\mathrm{d}\nu_x(r) \\ &\le \int_{\mathbb{R}} \big\|N(x,r)\big\| \cdot \big\|\tilde{N}(x)\big\|\,\mathrm{d}\nu_x(r) = 1. \end{aligned}$$

Since $\frac{\theta^+(x,r)-\theta^-(x,r)}{\theta(x,r)} \langle N(x,r), \tilde{N}(x) \rangle \le 1$ which means for μ-almost all $x \in B_R(0)$:

$$\frac{\theta^+(x,r) - \theta^-(x,r)}{\theta(x,r)} \langle N(x,r), \tilde{N}(x) \rangle = 1 \quad \nu_x\text{-almost everywhere.}$$

In the case $\theta^- = 0$, we have ν_x-almost everywhere $\langle N(x,r), \tilde{N}(x)\rangle = 1$ so that by $\|N\|, \|\tilde{N}\| = 1$ we obtain $N(x,r) = \tilde{N}(x)$. Analogously in the case $\theta^+ = 0$, we have ν_x-almost everywhere $N(x,r) = -\tilde{N}(x)$. This shows that for μ-almost all x for ν_x-almost all $r \in \mathbb{R}$ one of the following two properties holds

$$\begin{cases} \theta^- = 0, & \text{and } N(x,r) = \tilde{N}(x), \\ \theta^+ = 0, & \text{and } N(x,r) = -\tilde{N}(x). \end{cases} \tag{227}$$

So we can redefine N in such a way that $\theta^- = 0$, $\theta = \theta^+$ and moreover $N(x,r) = \tilde{N}(x)$ holds $\|V\| \llcorner \Gamma$-almost everywhere.

3 Next, we want to show that V has a generalized second fundamental form $A \in L^2(V)$. From (14) and (17) it follows $\|A\|_g^2 = H^2 - 2\mathcal{K}$ [DGR17, (11) p. 6]. Hence, by [DGR17, Lemma 2 (17)] and extension Lemma 88 b we know that

$$\int_{B_{2R}(0)\times\mathbb{R}} \|(A)_k\|_g^2 \,\mathrm{d}\|V_k\| \quad \text{is uniformly bounded} \tag{228}$$

since the topology of $\Gamma(\overline{u}_k) \cup \Sigma$ is fixed and we can use the Gauss-Bonnet theorem. Further. we define a functional in the sense of the compactness Theorem 85

$$\forall V' \in \mathbf{CV}_2(B_{2R}(0)\times\mathbb{R})\colon \qquad \mathcal{F}^A_{B_{2R}(0)\times\mathbb{R}}[V'] := \int_{G_2(\mathbb{R}^3)} \left|\mathcal{S}_{B_{2R}(0)\times\mathbb{R}} V'(x,P)\right|^2 \mathrm{d}V'(x,P).$$

Relative to the generalized curvature $\mathcal{S}_{B_{2R}(0)\times\mathbb{R}}$ the integrand is continuous, convex, non-negative, and has non-linear growth. Thus it follows with (228) and Theorem 85 that $V \in \mathbf{CV}_2(B_{2R}(0)\times\mathbb{R})$ and:

$$V_k \overset{C,B_{2R}(0)\times\mathbb{R}}{\rightharpoonup} V \overset{\text{Def. after Thm 84}}{\Rightarrow} (V_k \llcorner B_{2R}(0)\times\mathbb{R}, \mathcal{S}_{B_{2R}(0)\times\mathbb{R}} V_k) \rightharpoonup (V \llcorner B_{2R}(0)\times\mathbb{R}, \mathcal{S}_{B_{2R}(0)\times\mathbb{R}} V).$$

Here, we use generalized curvature instead of generalized second fundamental form in order to use compactness Theorem 85, which is formulated for generalized curvature. By the definition of the weak measure-function pair convergence Definition 72 this means

$$\forall \psi \in C^0_c(B_{2R}(0)\times\mathbb{R}\times\mathbb{R}^{3\times3};\mathbb{R}^{3\times3\times3})\colon$$
$$\int_{G_2(\mathbb{R}^3)} \langle \mathcal{S}_{B_{2R}(0)\times\mathbb{R}} V_k, \psi\rangle \,\mathrm{d}V_k \underset{k\to\infty}{\to} \int_{G_2(\mathbb{R}^3)} \langle \mathcal{S}_{B_{2R}(0)\times\mathbb{R}} V, \psi\rangle \,\mathrm{d}V.$$

Moreover, from (206) we know that the generalized mean curvature vector is

$$\vec{H} = \left(\sum_{i=1}^3 (\mathcal{S}_{B_{2R}(0)\times\mathbb{R}} V)^j_{ii}\right)^3_{j=1}$$

So we can choose the test function ψ, so that:

$$\forall \zeta \in C^0_c(B_{2R}(0)\times\mathbb{R};\mathbb{R}^3)\colon \quad \int_{B_{2R}(0)\times\mathbb{R}} \langle \vec{H}_k, \zeta\rangle \,\mathrm{d}\|V_k\| \underset{k\to\infty}{\to} \int_{B_{2R}(0)\times\mathbb{R}} \langle \vec{H}, \zeta\rangle \,\mathrm{d}\|V\|. \tag{229}$$

Now, the mean curvature vector $\vec{H}$ is V-a.e. orthogonal to corresponding tangential space by the Brakke perpendicularity Theorem 81. Hence, being in $\mathbb{R}^3$ implies $\vec{H}||N$ V-a.e. we can define

$$H := \langle \vec{H}, N\rangle \tag{230}$$

and write $\vec{H} = HN$ for the oriented limit varifold V.

(4) Let us define the sublevel set of u and the sublevel set of u_k

$$E := \{(x,h) \in B_R(0) \times \mathbb{R} \mid h < \overline{u}(x)\}$$
$$E_k := \{(x,h) \in B_R(0) \times \mathbb{R} \mid h < \overline{u}_k(x)\}$$

Since $\Gamma(\overline{u}_k)$ is a $W^{2,2}$-graph one obtains $\partial E_k = \Gamma(\overline{u}_k)$. There exist the characteristic functions $\chi_{E_k}, \chi_E \colon B_R(0) \times \mathbb{R} \to 0, 1$ of the sublevelsets of functions $\overline{u}_k$ and $\overline{u}$:

$$\chi_k := \chi_{E_k} = \begin{pmatrix} x \\ h \end{pmatrix} = \begin{cases} 1, & \text{if } h < \overline{u}_k(x), \\ 0, & \text{else,} \end{cases} \qquad \chi := \chi_E = \begin{pmatrix} x \\ h \end{pmatrix} = \begin{cases} 1, & \text{if } h < \overline{u}(x), \\ 0, & \text{else.} \end{cases}$$

Since $\overline{u}_k \to \overline{u}$ in $L^1(B_R(0))$ we obtain

$$\begin{aligned}
\int_{B_R(0)\times[-R,R]} |\chi_k - \chi| \,\mathrm{d}\mathcal{L}^3 &= \int_{B_R(0)} \left(\int_{-R}^{R} \left| \chi_k \begin{pmatrix} x \\ h \end{pmatrix} - \chi \begin{pmatrix} x \\ h \end{pmatrix} \right| \mathrm{d}h \right) \mathrm{d}x \\
&= \int_{B_R(0)} \left(\int_{-R}^{R} \left(\chi_{\{\overline{u}_k(x) \le h < \overline{u}(x)\}} + \chi_{\{\overline{u}_k(x) > h \ge \overline{u}(x)\}} \right) \mathrm{d}h \right) \mathrm{d}x \\
&= \int_{B_R(0)} |u_k - u| \,\mathrm{d}x \to 0, \text{ for } k \to \infty.
\end{aligned}$$

Therefore, it follows $\chi_k \to \chi$ in $L^1(B_R(0) \times (-R,R))$. The uniform area bound in terms of the Willmore energy in Theorem 16 implies that the sequence of perimeters of the sublevel sets $P(E_k, B_R(0) \times (-R,R)) = \mathcal{H}^2(\Gamma(\overline{u}_k))$ is uniformly bounded. Especially, by BV-definition this means that $\chi_k \in BV(B_R(0) \times (-R,R))$. By compactness Theorem 66 for BV-functions there exist a subsequence and a function $\hat{\chi} \in BV(B_R(0) \times (-R,R))$ such that $\chi_k \to \hat{\chi}$ in $L^1(B_R(0) \times (-R,R))$. With $\chi_k \to \chi$ in $L^1(B_R(0) \times (-R,R))$ we then deduce that $\chi \in BV(B_R(0) \times (-R,R))$.

Furthermore, it follows by (187) and (189) that $|\nabla\chi| = \mathcal{H}^2 \llcorner \mathcal{F}E$ is a Radon measure on $B_R(0) \times \mathbb{R}$ with the support on the reduced boundary of E in the sense of definition in (188) ([AFP00, Thm 3.36 (3.62) p. 159]). Moreover, $\mathcal{F}E$ may not be contained in the graph over u in the Lebesgue points. The former can also contain additional vertical parts.

We define

$$\frac{\nabla\chi}{|\nabla\chi|}(x) := \nu_E(x) = \lim_{\rho\to 0} \frac{\nabla\chi(B_\rho(x))}{|\nabla\chi|(B_\rho(x))}.$$

Next, we use the Gauss-Green Theorem 67 for BV characteristic functions. Again by the oriented varifold convergence (221) we obtain $\forall \psi \in C_c^1(B_R(0) \times \mathbb{R}; \mathbb{R}^3)$:

$$\begin{aligned}
\int_{\mathcal{F}E} \psi^T \circ \frac{-\nabla\chi}{|\nabla\chi|} \,\mathrm{d}\mathcal{H}^2 &= -\int_{\mathcal{F}E} \langle \psi, \nu_E \rangle \,\mathrm{d}\mathcal{H}^2 \overset{(191)}{=} \int_{(B_R(0)\times\mathbb{R})\cap E} \operatorname{div}\psi \,\mathrm{d}x \\
&= \int_{B_R(0)\times\mathbb{R}} \chi \operatorname{div}\psi \,\mathrm{d}x = \lim_{k\to\infty} \int_{B_R(0)\times\mathbb{R}} \chi_k \operatorname{div}\psi \,\mathrm{d}x \overset{\text{classical}}{\underset{\text{Gauss}}{=}} \lim_{k\to\infty} \int_{\Gamma_k} \langle \psi, N_k \rangle \,\mathrm{d}\mathcal{H}^2 \\
&\overset{(221)}{=} \int_{\Gamma} \langle \psi, N \rangle \theta \,\mathrm{d}\mathcal{H}^2.
\end{aligned}$$

We conclude that up to an $\mathcal{H}^2$-null set

$$\mathcal{F}E = \Gamma, \quad \theta = 1, \quad N = -\frac{\nabla\chi}{|\nabla\chi|}$$

holds. Especially, the varifold V has unit density, $\|V\| = |\nabla\chi|$ and the support $\operatorname{supp}(V)$ coincides in $B_R(0)\times\mathbb{R}$ with the reduced boundary $\mathcal{F}E$ of E the sublevel set of u up to a $\mathcal{H}^2$-null set. Since N is normal to $T_x\Gamma$ $\mathcal{H}^2$-a.e. it follows $\sum P_{k\ell}N^\ell = 0$. Therefore, since the situation in embedded in $\mathbb{R}^3$ we get $\delta_{k\ell} = P_{k\ell} + N^k N^\ell$ which means $Px = x - \langle x, N\rangle N$. We conclude $\forall\psi \in C_c^\infty(B_R(0);\mathbb{R}^3)$:

$$\begin{aligned}
&\int_{B_R(0)\times\mathbb{R}} \left(\operatorname{div}_{\mathbb{R}^3}\psi - \left(\frac{\nabla\chi_E}{|\nabla\chi_E|}\right)^T \circ D\psi \circ \frac{\nabla\chi}{|\nabla\chi|}\right)|\nabla\chi| \\
&\quad = \int_\Gamma \left(\operatorname{div}_{\mathbb{R}^3}\psi - N^T\circ D\psi\circ N\right)\mathrm{d}\mathcal{H}^2 = \int_\Gamma\left(\sum_i D_i\psi^i - \sum_{k\ell} N^k D_k\psi^\ell N^\ell\right)\mathrm{d}\mathcal{H}^2 \\
&\quad = \int_\Gamma\left(\sum_{k\ell} P_{k\ell}D_k\psi^\ell\right)\mathrm{d}\mathcal{H}^2 = \int_\Gamma \operatorname{div}_{T_x\Gamma}\psi(x)\,\mathrm{d}\mu_V(x) \overset{(201)}{=} -\int_\Gamma \langle\vec{H},\psi\rangle\,\mathrm{d}\mu_V \\
&\quad \overset{(230)}{=} -\int_\Gamma H\langle\psi, N\rangle\,\mathrm{d}\mathcal{H}^2 = \int_{B_R(0)\times\mathbb{R}} H\psi\nabla\chi.
\end{aligned}$$

Thus H coincides with the generalized mean curvature of $\mathcal{F}E$ in the sense of Luckhaus-Sturzenbecker [LS95] written in Definition 68.

5 By (211) the weak convergence $H_k \rightharpoonup H^a = \operatorname{div}\left(\frac{\nabla^a u}{Q^a}\right)$ in $L^2(B_R(0))$ follows. Additionally, since $\langle\vec{e}_3, \tilde{N}_k\rangle Q_k = 1$ by (229) we get for any $\eta\in C_c^0(B_R(0))$

$$\begin{aligned}
&\int_{B_R(0)} \eta H^a\,\mathrm{d}x \underset{(211)}{=} \lim_{k\to\infty}\int_{B_R(0)}\eta H_k\,\mathrm{d}x = \lim_{k\to\infty}\int_{B_R(0)}\eta H_k\langle\vec{e}_3,\tilde{N}_k\rangle Q_k\,\mathrm{d}x \\
&\quad = \lim_{k\to\infty}\int_{B_R(0)\times\mathbb{R}}\langle\vec{H}_k,\eta\vec{e}_3\rangle\,\mathrm{d}\|V_k\| \overset{(229)}{=} \int_{B_R(0)\times\mathbb{R}}\eta\langle\vec{H},\vec{e}_3\rangle\,\mathrm{d}\|V\| \\
&\quad = \int_{B_R(0)\times\mathbb{R}} H\eta N_3\,\mathrm{d}\|V\|.
\end{aligned}\tag{231}$$

In the next step, we want to relate different limits of mean curvature H, H^a and $\tilde{H}$ of the sequence of graphs $\{\Gamma(\overline{u}_k)\}_{k\in\mathbb{N}}$. To use the co-area formula Theorem 65, we first obtain for the projection $\pi^{\mathbb{R}^2}: B_R(0)\times\mathbb{R}\to B_R(0), \pi^{\mathbb{R}^2}(x,z)\mapsto x$ that by (185) its Jacobian is $J_\pi^V = N_3$ $\mathcal{H}^2$-a.e.. We conclude

$$\begin{aligned}
\int_{B_R(0)\times\mathbb{R}} H\eta N_3\,\mathrm{d}\|V\| &= \int_{\mathcal{F}E} H\eta N_3\,\mathrm{d}\mathcal{H}^2 = \int_{\mathbb{R}^2}\int_{(\pi^{\mathbb{R}^2})^{-1}(x)\cap\mathcal{F}E} H\eta\,\mathrm{d}\mathcal{H}^0\,\mathrm{d}\mathcal{L}^2(x) \\
&= \int_{B_R(0)} H\big(x,u(x)\big)\eta(x)\,\mathrm{d}x
\end{aligned}\tag{232}$$

because $\|V\|\llcorner\Gamma = \mathcal{H}^2\llcorner\mathcal{F}E$. Because of the varifold convergence, we get

$$\begin{aligned}
\int_{B_R(0)}\eta\,\mathrm{d}x &= \lim_{k\to\infty}\int_{B_R(0)}\eta\langle\vec{e}_3,\tilde{N}_k\rangle Q_k\,\mathrm{d}x \overset{(222)}{=} \int_{B_R(0)\times\mathbb{R}}\eta\langle N,\vec{e}_3\rangle\,\mathrm{d}\|V\| = \int_{\mathcal{F}E}\eta N_3\,\mathrm{d}\mathcal{H}^2 \\
&= \int_{\mathbb{R}^2}\int_{(\pi^{\mathbb{R}^2})^{-1}(x)}\eta\,\mathrm{d}\mathcal{H}^0\,\mathrm{d}\mathcal{L}^2(x)
\end{aligned}$$

so that $\nu_x = \delta_{u(x)}$ $\mathcal{L}^2$-almost everywhere. Furthermore it follows that the set $(\pi^{\mathbb{R}^2})^{-1}(x)\cap\mathcal{F}E$ contains $\mathcal{L}^2$-a.e. exactly one element $\big(x,u(x)\big)$. Since (232) and (231) are valid for any $\eta\in C_c^0(B_R(0))$ we get

$$H^a(x) = H\big(x,u(x)\big)\quad \mathcal{L}^2\text{-a.e..}\tag{233}$$

Also, it holds

$$H^a(x) = \tilde{H}(x) \quad \mathcal{L}^2\text{-a.e..} \tag{234}$$

Next, we want to relate H to $\tilde{H}$. By the measure-function pair convergence (220) of the $(\mu_k, \tilde{H}_k \tilde{N}_k)$ and again using the disintegration result (225) we get $\forall \zeta \in C_c^0(B_R(0); \mathbb{R}^3)$ and some $\eta \in C_0^\infty(\mathbb{R})$ with $\eta|_{[-R,R]} \equiv 1$

$$\begin{aligned}
\int_{B_R(0)} \langle \zeta, \tilde{N} \rangle \tilde{H} \, \mathrm{d}\mu &\overset{(220)}{=} \lim_{k\to\infty} \int_{B_R(0)} \langle \zeta, \tilde{N}_k \tilde{H}_k \rangle \, \mathrm{d}\mu_k = \lim_{k\to\infty} \int_{B_R(0)\times\mathbb{R}} \langle \vec{H}_k, \zeta \rangle \eta \, \mathrm{d}(\|V_k\| \llcorner \Gamma(\overline{u}_k)) \\
&\overset{(229)}{=} \int_{B_R(0)\times\mathbb{R}} \langle \vec{H}, \zeta \rangle \eta \, \mathrm{d}(\|V\| \llcorner \Gamma) = \int_{B_R(0)\times\mathbb{R}} \langle \zeta, N \rangle \eta H \, \mathrm{d}(\|V\| \llcorner \Gamma) \\
&\overset{(198)}{=} \int_{B_R(0)} \langle \zeta(x), \tilde{N}(x) \rangle \left(\int_{\mathbb{R}} H(x,r) \, \mathrm{d}\nu_x(r) \right) \mathrm{d}\mu(x).
\end{aligned}$$

By Theorem 63, the fundamental lemma of calculus of variations for Radon measures, we obtain

$$\tilde{H}(x) = \int_{\mathbb{R}} H(x,r) \, \mathrm{d}\nu_x(r) \quad \text{ for } \mu\text{-almost all } x \in B_R(0). \tag{235}$$

In contrast to the case (233) the formula (235) also contains information about vertical parts.

6 Let $A_D(u)$ be the set of points where u is approximately differentiable. We observe that u is approximately differentiable by Calderon-Zygmund Theorem 69 on the Borel set $A_D(u) \subset \Omega$ with $\mathcal{L}^2(\Omega \setminus A_D(u)) = 0$. By using the general area formula in Theorem 71 and disintegration Theorem 77 we have for all $\eta \in C_c^\infty(\Omega \times \mathbb{R})$

$$\begin{aligned}
\int_{A_D(u)} \eta\big(x, u(x)\big) Q^a(x) \, \mathrm{d}x &\overset{\text{Thm. } 71}{=} \int_{\Gamma \cap (A_D(u)\times\mathbb{R})} \eta(z) \, \mathrm{d}\mathcal{H}^2(z) = \int_{A_D(u)\times\mathbb{R}} \eta(z) \, \mathrm{d}(\|V\| \llcorner \Gamma)(z) \\
&= \int_{\mathbb{R}^3} \eta(z) \, \mathrm{d}\big(\|V\| \llcorner (\Gamma \cap (A_D(u) \times \mathbb{R}))\big)(z) \\
&\overset{\text{Thm. } 77}{=} \int_{A_D(u)} \int_{\mathbb{R}} \eta(x,r) \, \mathrm{d}\nu_x \, \mathrm{d}\mu(x).
\end{aligned}$$

This implies that $\mu \llcorner A_D(u) = Q^a \mathcal{L}^2 \llcorner A_D(u)$ and $\nu_x = \delta_{u(x)}$ for $\mathcal{L}^2$-almost all $x \in A_D(u)$.

Finally, we characterize some parts of the missing contribution in the semicontinuity estimate by Grunau-Deckelnick-Röger. To do so, we again redefine the Willmore-functional in the sense of the compactness Theorem 85:

$$\forall V' \in \mathbf{CV}_2(\Omega \times \mathbb{R}): \qquad \mathcal{F}^H_{\Omega\times\mathbb{R}}[V'] := \int_{G_2(\mathbb{R}^3)} \sum_{j=1}^3 \left(\sum_{i=1}^3 (\mathcal{S}_{\Omega\times\mathbb{R}} V')^j_{ii} \right)^2 \mathrm{d}V'(x,P).$$

Relative to the generalized curvature $\mathcal{S}_{\Omega\times\mathbb{R}}$ the integrand is continuous, convex, non-negative, and has non-linear growth. While the Willmore energy of the sequence is uniformly bounded, by Theorem 85 we also get the lower semicontinuity

$$\int_{\Omega\times\mathbb{R}} |\vec{H}|^2 \, \mathrm{d}\|V\| \overset{\text{Thm.85}}{\leq} \liminf_{k\to\infty} \int_\Omega |\vec{H}_k|^2 \, \mathrm{d}\|V_k\| = \liminf_{k\to\infty} \int_\Omega H_k^2 Q_k \, \mathrm{d}x.$$

So, we obtain the missing Willmore varifold part:

$$\int_\Omega |H^a|^2 Q^a \, \mathrm{d}x + \int_{(\Omega\setminus A_D(u))\times\mathbb{R}} |\vec{H}|^2 \, \mathrm{d}\|V\| = \int_{\Omega\times\mathbb{R}} |\vec{H}|^2 \, \mathrm{d}\|V\| \leq \liminf_{k\to\infty} \int_\Omega H_k^2 Q_k \, \mathrm{d}x.$$

Furthermore, by the F_H functional from (213), we also have the lower semicontinuity

$$\int_\Omega |H^a|^2 Q^a \,\mathrm{d}x + \int_{\Omega\setminus A_D(u)} |\tilde{H}|^2 \,\mathrm{d}\mu = \int_\Omega |\tilde{H}|^2 \,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_\Omega H_k^2 Q_k \,\mathrm{d}x.$$

□

8.3 Finiteness of the Relaxed Willmore Energy Does Not Imply SBV

The Cantor part of a BV-function is difficult to handle. Therefore, it would be important to characterize the situations when the singular part of the gradient consists only of a jump part. In the context of the Willmore functional, one would expect that a non-vanishing Cantor part would cause the relaxed Willmore energy to blow up since it is highly irregular. Surprisingly, finite relaxed energy does not completely exclude a Cantor part in the derivative as shown in the following example. It is based on unpublished notes of H.-Ch. Grunau and concerns the one-dimensional Willmore functional. This, in turn, heavily relies upon [DMFLM09, Prop. 2.3 & Thm 3.4. pp. 2356 ff and 2373].

First, we shortly recall the one-dimensional Willmore energy, also called elastic energy, defined in (6). If we consider a regular and sufficiently smooth curve $\gamma\colon I \to \mathbb{R}^n, n \ge 2$ it is given by the total squared curvature functional

$$\mathcal{E}(\gamma) = \int_I |\vec{\kappa}_\gamma|^2(s) \,\mathrm{d}s$$

with s the arclength and $\vec{\kappa}_\gamma = \partial_{ss}^2\gamma$ the curvature vector of γ. Actually, here we are only interested in projectable curves. Therefore, in the same way as in [DG07] we consider curves as graphs over the unit interval $[0,1]$ instead of arclength parametrization. For each function $u\colon [0,1] \to \mathbb{R}$ we define the arclength curve $\gamma : I \to \mathbb{R}^2$ obtained by reparametrizing the curve $[0,1] \ni x \mapsto (x, u(x))$ to the arclenght. Then according to [DG07] for graph$[u] = \gamma(I)$ the Willmore functional takes the shape

$$\mathcal{W}(u) = \int_{\text{graph}[u]} \kappa^2(x) \,\mathrm{d}s(x) = \int_0^1 \kappa^2(x)\sqrt{1+u'(x)^2} \,\mathrm{d}x \tag{236}$$

with the curvature κ

$$\kappa(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) = \frac{u''(x)}{\sqrt{1+u'(x)^2}^3}.$$

If $u\colon [0,1] \to \mathbb{R}$ has finite one-dimensional Willmore energy as a graph, then we can extend it to a function $\overline{u}\colon [0,1]^2 \to \mathbb{R}$ with finite two-dimensional Willmore energy as a graph. We simple set for all $x, y \in [0,1]^2\colon \overline{u}(x,y) = u(x)$. Then it follows

$$\mathcal{W}(\overline{u}) = \frac{1}{4}\mathcal{W}(u) \qquad \text{and} \qquad \int_{[0,1]^2} \|\nabla \overline{u}\|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_{[0,1]} |u'|^2 \,\mathrm{d}x$$

Next, we want to recall the space of BV functions with one variable as in [DMFLM09, Subsection 2.1]. As in Subsection 7.2 a function $u \in L^1((a,b))$ belongs to $BV((a,b))$ if and only if its *total variation* $\mathrm{V}(u,(a,b))$ is finite

$$\mathrm{V}(u,(a,b)) := \sup\left\{ \int_a^b u\varphi' \,\mathrm{d}x \;\middle|\; \varphi \in C_0^1((a,b)) \text{ and } \|\varphi\|_\infty \le 1 \right\} < +\infty.$$

Then, u', called the distributional derivative of u, is a bounded scalar Radon-measure on (a,b) with the total variation measure $|u'|$. Especially, by [AFP00, Proposition 3.6] it holds $|u'|((a,b)) = V(u,(a,b))$. Further, due to Lebesgue decomposition, we can split the distributional derivative into its absolutely continuous part $(u')^a$ and singular part $(u')^s$ with respect to $\mathcal{L}^1$ on (a,b)

$$u' = (u')^a \mathcal{L}^1 + (u')^s = (u')^a \mathcal{L}^1 + (u')^j + (u')^c$$

where $(u')^j$ is its jump part and $(u')^c$ is its Cantor part characterized later. It can be shown that every function u in $BV((a,b))$ is $\mathcal{L}^1$-a.e. differentiable in (a,b) and for $\mathcal{L}^1$-a.e. x in (a,b) the derivative is given by $(u')^a(x)$. Also, for every function $u \in BV((a,b))$ there are left and right approximate limits

$$u^\ell(y) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{y-\varepsilon}^{y} u(x)\,\mathrm{d}x \quad \text{and} \quad u^r(y) := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{y}^{y+\varepsilon} u(x)\,\mathrm{d}x$$

which are well defined at every point $y \in (a,b)$ and left and right continuous, respectively. In fact, the functions u^r and u^ℓ coincide a.e with respect to $\mathcal{L}^1$. The complementary null set where the functions u_- and u_+ differ is called the set of essential discontinuities or jump points of u

$$S_u := \left\{ y \in (a,b) \,\middle|\, u^\ell(y) \neq u^r(y) \right\}.$$

This set is at most countable. Considering the singular part $(u')^s$ in more detail, we can rewrite the jump part sum with the counting measure $\mathcal{H}^0$ concentrated on S_u so that

$$(u')^s = (u^r - u^\ell)\mathcal{H}^0 \llcorner S_u + (u')^c.$$

The jump part is often referred to as an atomic measure and the Cantor part as a singular diffuse measure. There is also a notion of the total variation for a function defined everywhere. We recall that $u : (a,b) \to \mathbb{R}$ has bounded pointwise variation $\mathrm{pV}(u,(c,d))$ over the interval $(c,d) \subset (a,b)$ if

$$\mathrm{pV}(u,(c,d)) := \sup \sum_{i=1}^{k} |u(y_i) - u(y_{i-1})| < +\infty,$$

where we take the supremum over all finite families of points $(y_0, y_1, \ldots, y_k)$ such that $c < y_0 < y_1 < \cdots < y_k < d$ and $k \in \mathbb{N}$. The above-defined left and right approximate limits are in fact precise and good representatives of $u \in BV((a,b))$. This means for every interval $(c,d) \subset (a,b)$

$$\left|u'\right|((c,d)) = \mathrm{pV}(u^\ell,(c,d)) = \mathrm{pV}(u^r,(c,d))$$

If we now consider a general function $u \in L^1((a,b))$ that has bounded pointwise total variation in (a,b), then it belongs to $BV((a,b))$, with $|u'|((c,d)) \leq \mathrm{pV}(u,(c,d))$ for every interval $(c,d) \subset (a,b)$.

Finally, we want to recall Theorem 66 stating a compactness result for BV spaces. If $\{u_k\}_{k=1}^\infty$ is a sequence in $BV((a,b))$ satisfying the bounds

$$\sup_k \left\{ \|u_k\|_{L^1((a,b))} + |u_k'|((a,b)) \right\} < \infty$$

then there is a subsequence $\{u_{k_j}\}_{j=1}^\infty$ and a limit function $u \in BV((a,b))$ with the following convergence properties

$$u_{k_j} \to u \quad \text{in } L^1((a,b)) \text{ as } j \to \infty \quad \text{and} \quad \lim_{j\to\infty} \int_a^b \varphi \nabla u_j = \int_a^b \varphi \nabla u, \text{ for all } \varphi \in C_0((a,b)).$$

Now, let us focus on some results presented in [DMFLM09]. There the authors consider the following functional for a total variation-based model for image restoration involving a second-order derivative term that eliminates the staircase effect. For a exponent $p \in (1, +\infty)$ let $\mathcal{F}_p : L^1((a,b)) \to [0, +\infty]$ be defined by

$$\mathcal{F}_p(u) := \begin{cases} \displaystyle\int_a^b |u'| \, \mathrm{d}x + \int_a^b \psi(u') |u''|^p \, \mathrm{d}x & \text{if } u \in W^{2,p}((a,b)), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\psi : \mathbb{R} \to (0, +\infty)$ is a bounded Borel function to be specified, jet. In this definiton, we extend $\mathcal{F}_p$ to $L^1((a,b))$ by setting $\mathcal{F}_p(u) := +\infty$ if $u \in L^1((a,b)) \setminus W^{2,p}((a,b))$. Then one uses the theory of relaxation and identifies its lower semicontinuous envelope with respect to the strong L^1-convergence. For every $u \in L^1((a,b))$ we set

$$\overline{\mathcal{F}}_p(u) := \inf \left\{ \liminf_{k \to \infty} \mathcal{F}_p(u_k) \,\middle|\, u_k \to u \in L^1((a,b)) \right\}. \tag{237}$$

where we take the infimum over all sequences $\{u_k\}_{k \in \mathbb{N}}$ in $L^1((a,b))$ with $u_k \to u \in L^1((a,b))$. At this point, let us very briefly compare $\mathcal{F}_p$ with the Willmore energy. We take $p = 2$ and get

$$\mathcal{F}_2(u) = \mathcal{W}(u) + \int_a^b |u'| \, \mathrm{d}x, \quad \text{with} \quad \psi(\tau) := \frac{1}{(1+\tau^2)^{5/2}} \quad \text{for all } \tau \in \mathbb{R}. \tag{238}$$

It means that we can apply arguments from [DMFLM09] on the one-dimensional Willmore functional. But first, let us present two additional necessary conditions on a bounded Borel function $\psi : \mathbb{R} \to (0, +\infty)$. It must satisfy

$$M := \int_{-\infty}^{+\infty} (\psi(t))^{1/p} \, \mathrm{d}t < +\infty \quad \text{and} \quad \inf_{t \in K} \psi(t) > 0 \text{ for every compact set } K \subset \mathbb{R}. \tag{239}$$

If we now define $\Psi_p : \overline{\mathbb{R}} \to [0, M]$ as the antiderivative of $\psi^{1/p}$ by

$$\Psi_p(t) := \int_{-\infty}^{t} (\psi(s))^{1/p} \, \mathrm{d}s,$$

and the function $\Psi_p^{-1} : [0, M] \to \overline{\mathbb{R}}$ as the inverse function of Ψ_p, then for every $u \in W^{1,p}((a,b))$ we obtain

$$\mathcal{F}_p(u) = \int_a^b |u'| \, \mathrm{d}x + \int_a^b \left| \frac{\mathrm{d}}{\mathrm{d}x} (\Psi_p \circ u'(x)) \right|^p \mathrm{d}x. \tag{240}$$

In [DMFLM09, Theorem 3.4] the authors also identify the relaxation of the functional $\mathcal{F}_p$ with respect to strong convergence in $L^1((a,b))$. In detail, they define the subspace of L^1-functions that can be approximated by $\mathcal{F}_p$-bounded sequences, that they call $X_\psi^p((a,b))$ [DMFLM09, Definition 3.1 and Remark 3.2] which we will recall in a moment. With regard to (240), one of the properties of a functions $u \in X_\psi^p((a,b))$ has to be $v := \Psi_p \circ (u')^a \in W^{1,p}((a,b))$. Since Ψ_p^{-1} is continuous and $v \in C^0([a,b])$ by Sobolev embedding it follows that $(u')^a = \Psi_p^{-1}(v)$ is continuous on $[a,b]$ with values in $\overline{\mathbb{R}}$. Next, we denote the sets where the absolutely continuous part $(u')^a$ blows up by

$$Z^+ \left[(u')^a \right] := \left\{ x \in (a,b) \,\middle|\, (u')^a = +\infty \right\}, \quad Z^- \left[(u')^a \right] := \left\{ x \in (a,b) \,\middle|\, (u')^a = -\infty \right\}.$$

Then, we can define the set

$$X_\psi^p((a,b)) := \left\{ u \in BV((a,b)) \,\middle|\, \Psi_p \circ (u')^a \in W^{1,p}((a,b)), \left((u')^s\right)^\pm \text{ is concentrated on } Z^\pm \left[(u')^a \right] \right\}.$$

With the set X^p_ψ the authors in [DMFLM09, Theorem 3.4] were able to identify the relaxation of $\mathcal{F}_p$ with respect to strong convergence in $L^1((a,b))$ defined in (237) by

$$\overline{\mathcal{F}}_p(u) = \begin{cases} \left|u'\right|((a,b)) + \displaystyle\int_a^b \left|v'\right|^p \mathrm{d}x & \text{if } u \in X^p_\psi((a,b)), \\ +\infty & \text{otherwise} \end{cases}$$

where $v := \Psi_p \circ (u')^a$ in the the higher-order term depends only on $(u')^a$.

Here we want to discuss some properties of functions $u \in X_\psi((a,b))$. Namely, for every jump point x_0 with $u^r(x_0) - u_\ell(x_0) > 0$ it holds $\lim_{x\to x_0} (u')^a(x) = +\infty$ and for every jump point x_0 with $u^r(x_0) - u^\ell(x_0) < 0$ we have $\lim_{x\to x_0} (u')^a(x) = -\infty$. Furthermore, if u' has a non-vanishing Cantor or jump part, then u' cannot have a bounded absolutely continuous part. This means that piece-wise constant functions with jumps and the Cantor function are excluded from $X^p_\psi((a,b))$ since $Z^\pm[(u')^a] = \varnothing$. The same applies to polygons. For the case $p = 2$, this means that the Cantor function (for definition, see the proof of Theorem 90) or piece-wise smooth graph with corners has infinite relaxed one-dimensional Willmore energy.

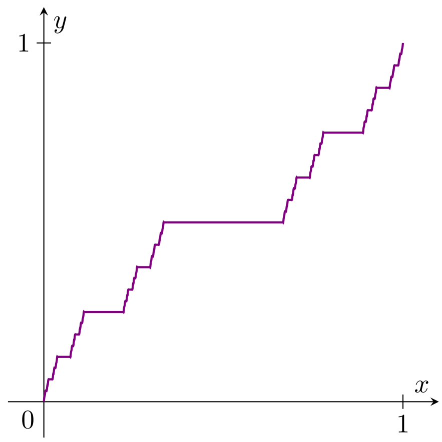

Figure 1: a Cantor function.

From the above Cantor function example, one may expect that all functions with Cantor part can be excluded from $X^p_\psi((a,b))$. Surprisingly, in [DMFLM09, Remark 3.2 (iv)] Maso, Fonseca, Leoni, and Morini constructed functions with nontrivial Cantor part in $X^p_\psi((a,b))$ for the case $2 > p > 1$ near 1 provided ψ satisfies $\psi(t) \leq ct^{-\alpha}$ for all $t \geq 1$ and for some $c > 0, \alpha > 1$. The Willmore functional as a part in $\mathcal{F}_2$ defined in (238) satisfy this condition with $\alpha = 5$ and condition (239) like shown in [DG07, Lemma 1.]. In the next theorem, we want to extend this result to $p = 2$, hence the relaxed Willmore energy

$$\overline{\mathcal{W}}(u) := \inf \left\{ \liminf_{k\to\infty} \mathcal{W}(u_k) \,\middle|\, u_k \to u \in L^1((a,b)) \right\}$$

by constructing a function with $\overline{\mathcal{W}}(u) < +\infty$ and nonvanishing Cantor part. This is in fact a surprising result since $p = 2$ is not near $p = 1$ and therefore is not covered by [DMFLM09, Remark 3.2 (iv)].

90 Theorem
There exists a function $u \in BV((0,1))$ with $\overline{\mathcal{W}}(u) < +\infty$ so that $|(u')^c|((0,1)) > 0$ and especially $u \notin SBV((0,1))$.

Proof: By the results presented in [DMFLM09], we know that Cantor function $f_\delta \colon \to [0,1]$ defined below in the step 2 has $\mathcal{W}(f_\delta) = +\infty$. Therefore, it is not a suitable example. One of the main problems is that the measure $((u')^s)^+$ is concentrated on a Cantor-set and not on $Z^+ [(u')^a] = \varnothing$. We can correct this if we add to the Cantor function a continuous function $U \colon [0,1] \to (-\infty, \infty)$ (3 - 6) that obeys $(u')^a = +\infty$ on the Cantor-set $\mathbb{D}_\delta$. Additionally, U is supposed to have finite one-dimensional Willmore energy. The added-up function $f_\delta + U$ then has finite relaxed Willmore energy, and interestingly f_δ does contribute the Cantor part but not any Willmore energy. To show the finite value of the Willmore energy, in step 7 we will approximate $f_\delta + U$ by a Willmore equibounded sequence of functions $u_k \in W^{2,2}$ for $k \in \mathbb{N}$. These are constructed by locally replacing the Cantor part with the same-length jump part and then replacing a jump part with a linear non-vertical slope.

1 First, for clarity we define the functions

$$\Psi_2(t) := \int_{-\infty}^{t} \frac{1}{(1+\tau^2)^{\frac{5}{4}}} \,\mathrm{d}\tau, \quad M := \Psi_2(\infty) = \int_{\mathbb{R}} \frac{1}{(1+\tau^2)^{\frac{5}{4}}} \,\mathrm{d}\tau = \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}. \tag{241}$$

by [DG07, Lemma 1.] with Γ the gamma function and

$$\Psi_2 \colon \overline{\mathbb{R}} \to [0, M], \qquad \Psi_2^{-1} \colon [0, M] \to \overline{\mathbb{R}}.$$

Further, we denote by ψ the following function for all $\tau \in \mathbb{R}$

$$\psi(\tau) := \frac{1}{(1+\tau^2)^{5/2}}.$$

For in $u \in W^{2,2}((0,1))$, by slightly abusing the notation in [DMFLM09], we define the Willmore functional

$$\begin{aligned} \mathcal{F}_2(u) &:= \int_0^1 \psi(u')|u''|^2 \,\mathrm{d}x = \int_0^1 \frac{u''(x)^2}{\left(1+u'(x)^2\right)^{5/2}} = \int_0^1 \kappa(x)^2 \sqrt{1+u'(x)^2} \,\mathrm{d}x \\ &= \int_{\text{graph}[u]} \kappa(x)^2 \,\mathrm{d}s(x) = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}x} \Psi_2\big(u'(x)\big) \right)^2 \mathrm{d}x. \end{aligned}$$

Here we abandon the term $\int_0^1 |u'| \,\mathrm{d}x$. In this context, we want to emphasize that by Lemma 101 only small values of $\mathcal{W}(u)$ (along with the boundary condition) imply a $\|u'\|_{L^1((0,1))}$-bound and thus also a length bound for the graph of u. Conversely, in cases where the smallness criteria are not fulfilled, the graph could potentially feature arbitrarily long vertical parts that are not penalized by $\mathcal{W}(u)$ at all.

2 In the next step following [DMFLM09, pp.2356 ff] we construct the generalized Cantor set $\mathbb{D}_\delta$ and function f_δ. With

$$\delta \in (0, 1/2) \tag{C1}$$

which will be specified later. We cut out the open interval of length $(1-2\delta)$

$$I_{11} := (\delta, 1-\delta)$$

from the closed interval $[0,1]$ so that two intervals remain, each of length δ :

$$[0,1] \setminus I_{11} = [0, \delta] \cup [1-\delta, 1].$$

We again cut out of $[0,\delta] \cup [1-\delta, 1]$ the open intervals each of the length $\delta(1-2\delta)$

$$I_{12} := \big(\delta^2,\ \delta(1-\delta)\big), \quad I_{22} := \big((1-\delta)+\delta^2,\ (1-\delta)+\delta(1-\delta)\big) = (1-\delta+\delta^2,\ 1-\delta^2).$$

We repeat the procedure on $[0,1] \setminus (I_{11} \cup I_{12} \cup I_{22})$ and remove four open intervals $I_{13}, I_{23}, I_{33}, I_{44}$, and repeat it successively of the resulting closed set. After $(j-1)$ steps, only 2^{j-1} intervals remain, each of them of length δ^{j-1}. Then, in the current j-th step we cut out from each of these 2^{j-1} intervals the open central part of length $\delta^{j-1}(1-2\delta)$

$$I_{kj}, \quad k = 1, \ldots, 2^{j-1}$$

so that in particular $I_{1j} = \big(\delta^j, \delta^{j-1}(1-\delta)\big)$. The length of all I_{jk} for fixed j is $\delta^{j-1}(1-2\delta)$. Further, we define the generalized closed Cantor set by

$$\mathbb{D}_\delta := [0,1] \setminus \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{2^{(j-1)}} I_{kj}. \tag{242}$$

with the corresponding approximative Cantor function

$$g_\ell := \frac{1}{(2\delta)^\ell} \left(1 - \sum_{j=1}^{\ell} \sum_{k=1}^{2^{(j-1)}} \mathbb{1}_{I_{kj}}(x) \right), \qquad f_\ell := \int_0^x g_\ell(\xi)\, \mathrm{d}\xi.$$

In fact, by Lemma 100 (a) the generalized Cantor set has zero Lebesgue measure. For each step ℓ the the function f_ℓ takes constant value $(2k-1)/2^j$ on I_{kj}, $j = 1, \ldots, \ell$, $k = 1, \ldots 2^{j-1}$. On the intermediate 2^ℓ intervals of length δ^ℓ the function f_ℓ increases linearly with slope $1/(2\delta)^\ell$, which is the value of g_ℓ. As mentioned above, there are only 2^ℓ intervals of length δ^ℓ where g_ℓ is non-vanishing. Then

$$f_\ell(1) = \frac{1}{(2\delta)^\ell} 2^\ell \delta^\ell = 1.$$

Since $g_\ell \geq 0$, each function f_ℓ is monotonically increasing. It follows $f_\ell \colon [0,1] \to [0,1]$.

By Lemma 100 (b) for $\ell \to \infty$ we have uniform convergence $f_\ell \to f =: f_\delta$ to the continuous Cantor function with $(f_\delta')^a(x) \equiv 0$ and $(f_\delta')^c(x)$ is supported on the Cantor set $\mathbb{D}_\delta$. The function f_δ is monotonically increasing but also constant on $[0,1] \setminus \mathbb{D}_\delta$. Moreover $f_\delta(0) = 0$ and $f_\delta(1) = 1$.

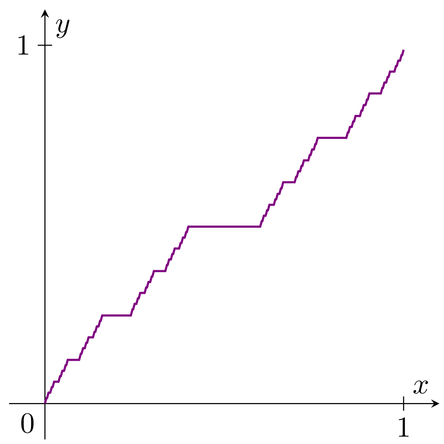

Figure 2: f_δ for $\delta = 0.2$

③ In the next step, we construct the derivative of the function U we want to add to the Cantor function f_δ so that the derivative of $f_\delta + U$ has a Cantor part and finite relaxed Willmore energy. We call the derivative $w := w_\delta \in L^1((0,1))$ and want it to map $[0,1]$ continuously on $[0,\infty]$ and to obey $\Psi_2 \circ w \in W^{1,2}((0,1))$ which means that the desired forthcoming example should have integrable mean curvature and $w(x) = +\infty \Leftrightarrow x \in \mathbb{D}_\delta$. In comparison to [DMFLM09] the choice of the singular function Φ below is a new contribution of H-Ch. Grunau. We choose

$$\beta \in \left(\frac{2}{3}, 1\right), \tag{C2}$$

that will later guarantee finite Willmore energy in each interval $I_{k\ell}$. Also, we define

$$\Phi\colon [0,1] \to [0,\infty], \quad \Phi(x) = C_\beta \left(\frac{1}{x^\beta (1-x)^\beta} - 4^\beta \right),$$

with normalization constant C_β such that

$$\int_0^1 \Phi(x)\,\mathrm{d}x = 1.$$

We can calculate the normalization constant by

$$C_\beta^{-1} = \int_0^1 \left(\frac{1}{x^\beta (1-x)^\beta} - 4^\beta \right) \mathrm{d}x = \mathrm{B}(1-\beta, 1-\beta) - 4^\beta = \frac{\Gamma(1-\beta)^2}{\Gamma(2-2\beta)} - 4^\beta$$

where B is the Euler's Beta function [DLMF, (5.12.1)]. Up to some scaling and shift Φ will play the role as a port of the absolutely continuous part. Thus we consider for $x \in [0,1]$ with [DLMF, (8.17.7)]

$$C_\beta^{-1} \int_0^x \Phi(s)\,\mathrm{d}s = \mathrm{B}_x(1-\beta, 1-\beta) - 4^\beta x = \frac{x^{1-\beta}}{\beta+1} {}_2F_1(1-\beta, \beta, 2-\beta; x) - 4^\beta x$$

where B_x is the incomplete Beta function and ${}_2F_1$ is the hypergeometric function.

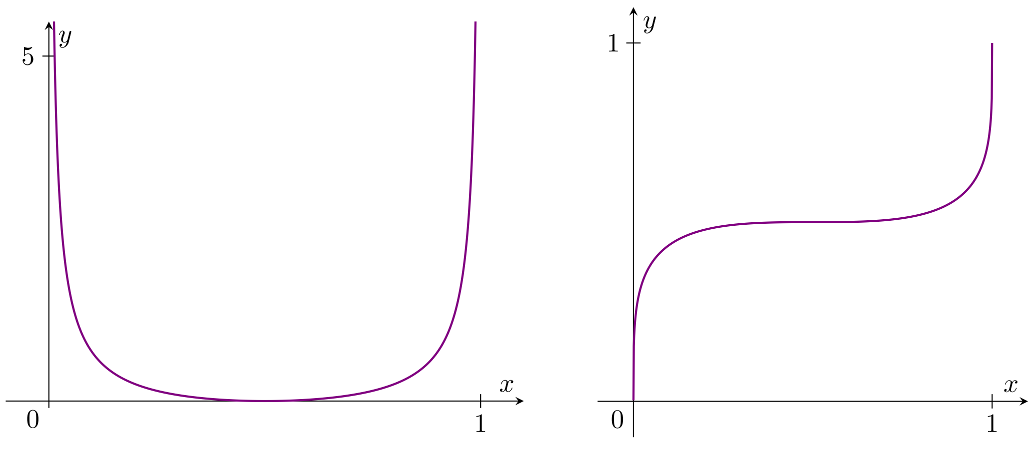

Figure 3: left: $\Phi(x)$ for $\beta = \frac{3}{4}$; right: $\int_0^x \Phi(s)\,\mathrm{d}s$ for $\beta = \frac{3}{4}$.

The term -4^β in the definition of Φ is chosen so that Φ stays positive, thus later we have monotonicity for its primitive function. It follows that $\Phi(1/2) = 0$ and moreover Φ is convex since the second derivative is strictly positive.

$$\frac{\Phi'(x)}{C_\beta} = \frac{-\beta}{x^{\beta+1}(1-x)^\beta} + \frac{\beta}{x^\beta (1-x)^{\beta+1}} = \frac{\beta\big[-(1-x)+x\big]}{x^{\beta+1}(1-x)^{\beta+1}} = \frac{\beta\big[2x-1\big]}{x^{\beta+1}(1-x)^{\beta+1}},$$

$$\frac{\Phi''(x)}{C_\beta} = \beta\frac{2}{x^{\beta+1}(1-x)^{\beta+1}} - \beta(\beta+1)\frac{(2x-1)(1-2x)}{x^{\beta+2}(1-x)^{\beta+2}} = \beta\frac{2x(1-x)+(1-2x)^2+\beta(1-2x)^2}{x^{\beta+2}(1-x)^{\beta+2}}$$
$$= \beta\frac{(1-x)^2+x^2+\beta(1-2x)^2}{x^{\beta+2}(1-x)^{\beta+2}} > 0.$$

Next, we choose a parameter $s > 4$ such that

$$2^{-s-1} > 2^{1-\frac{3}{2}s} \tag{C3}$$

so that later we can find a number δ between 2^{-s-1} and $2^{1-3s/2}$. Let a_{kj} be the centre of the interval I_{kj}, i.e.

$$I_{kj} = \left[-\frac{1}{2}\delta^{j-1}(1-2\delta) + a_{kj}, a_{kj} + \frac{1}{2}\delta^{j-1}(1-2\delta)\right].$$

Consequently, in each I_{kj}, we put a scaled-down version Φ such that the resulting function stays integrable. We set $x \in I_{kj}$:

$$\Phi_{kj} := 2^{sj} + \Phi\left(\frac{x-a_{kj}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right), \quad w(x) := w_\delta(x) := \begin{cases}\Phi_{kj}(x), & \text{if } x \in I_{kj}\\ \infty, & \text{if } x \in \mathbb{D}_\delta.\end{cases} \tag{243}$$

The term 2^{sj} in Φ_{kj} is an extra added constant slope since Φ_{kj} will be a derivative. It will play an important role in achieving finite Willmore energy because it will appear in the denominator. Now we integrate w_δ on I_{kj}. Since $|I_{kj}| = \delta^{j-1}(1-2\delta)$ we get

$$\int_{I_{kj}} w_\delta(x)\,\mathrm{d}x = \int_{I_{kj}} \Phi_{kj}(x)\,\mathrm{d}x = 2^{sj}(1-2\delta)\delta^{j-1} + (1-2\delta)\delta^{j-1}\int_0^1 \Phi(x)\,\mathrm{d}x$$
$$= (1-2\delta)\delta^{j-1}(2^{sj}+1),$$
$$\Rightarrow \int_0^1 w_\delta(x)\,\mathrm{d}x = \sum_{j=1}^\infty 2^{j-1}(1-2\delta)\delta^{j-1}(2^{sj}+1) = \frac{1-2\delta}{2\delta}\sum_{j=1}^\infty (2\delta)^j(2^{sj}+1).$$

With an extra condition

$$\delta < 2^{-s-1} \Leftrightarrow 2^{s+1}\delta < 1 \tag{C4}$$

it follows by using results for geometric series that

$$\int_0^1 w_\delta(x)\,\mathrm{d}x = \frac{1-2\delta}{2\delta}\left(\frac{2^{s+1}\delta}{1-2^{s+1}\delta} + \frac{2\delta}{1-2\delta}\right) = 2^s\frac{1-2\delta}{1-2^{s+1}\delta} + 1 < \infty.$$

4 The function $\Psi_2 \circ w_\delta$ is continuous and therefore in $L^\infty((0,1)) \subset L^2((0,1))$. Moreover, we can calculate the classical derivative on each interval I_{kj}

$$(\Psi_2 \circ w_\delta)'(x) = (\Psi_2 \circ \Phi_{kj})'(x) = \Psi_2'\big(\Phi_{kj}(x)\big)\cdot \Phi_{kj}'(x)$$
$$= \frac{1}{(1+\Phi_{kj}(x)^2)^{5/4}} \cdot \Phi'\left(\frac{x-a_{kj}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right)\cdot \frac{1}{(1-2\delta)\delta^{j-1}}.$$

On the boundaries of the intervals I_{kj} it follows that $w_\delta = \infty$ thus we extend $\Psi_2 \circ w_\delta$ by $M = \Psi_2(\infty)$ so that we get a pointwise absolutely continuous function that has a.e. a derivative whose primitive function equals $\Psi_2 \circ w_\delta$ itself as will be shown in the following (also see [Nat61, Chapter 7]).

By definition, the function $\Psi_2 \circ w_\delta$ is continuously differentiable in the interior of each interval I_{kj}. Moreover, in the next step 5 we prove that $(\Psi_2 \circ w_\delta)' \in L^2((0,1))$ therefore also $(\Psi_2 \circ w_\delta)' \in L^1((0,1))$. We want to show that

$$\forall x \in [0,1]: \qquad (\Psi_2 \circ w_\delta)(x) = \int_0^x (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi + \underbrace{M}_{=(\Psi_2 \circ w_\delta)(0)} \tag{244}$$

where M was defined in (241). For this purpose, consider

$$\forall j \in \mathbb{N}, k = 1, \ldots, 2^{j-1}: \qquad \int_{I_{kj}} (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi = M - M = 0.$$

For each $x \in \mathbb{D}_\delta$, we then check (244)

$$\begin{aligned}\int_0^x (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi + M &= \sum_{I_{kj} \text{ left of } x} \int_{I_{kj}} (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi + M \\ &= 0 + M = (\Psi_2 \circ w_\delta)(x). \quad \checkmark\end{aligned}$$

In case $x \notin \mathbb{D}_\delta$, thus $x \in I_{jk}$ for some $j \in \mathbb{N}, jk \in \{1, \ldots, 2^{j-1}\}$, there exists the left boundary point of I_{jk}, which we denote by $x_0 \in \mathbb{D}_\delta$. It follows

$$\begin{aligned}\int_0^x (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi + M &= \int_0^{x_0} (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi + \int_{x_0}^x (\Psi_2 \circ w_\delta)'(\xi)\,\mathrm{d}\xi + M \\ &= 0 + (\Psi_2 \circ w_\delta)(x) - M + M = (\Psi_2 \circ w_\delta)(x). \quad \checkmark\end{aligned}$$

Therefore we proved (244).

5 Our next aim is to show that

$$A_{kj} := \int_{I_{kj}} \big((\Psi_2 \circ w_\delta)'(x)\big)^2\,\mathrm{d}x < \infty.$$

In what follows, we use the notation $\preceq$ meaning an inequality up to a constant which is then not written. It is important to observe that the following estimates are uniform with respect to k and j. Below, in the second inequality in the third line, the same singular behavior of the function Φ close to 0 and close to 1 is important.

$$\begin{aligned}
A_{kj} &= \frac{\delta^2}{(1-2\delta)^2\delta^{2j}} \int_{I_{kj}} \frac{\left[\Phi'\left(\frac{x-a_{kj}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right)\right]^2}{\left(1 + \left[2^{sj} + \Phi\left(\frac{x-a_{kj}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right)\right]^2\right)^{5/2}}\,\mathrm{d}x \\
&\preceq \frac{\delta}{(1-2\delta)\delta^j} \int_0^1 \frac{[\Phi'(y)]^2}{\left(1 + [2^{sj} + \Phi(y)]^2\right)^{5/2}}\,\mathrm{d}y \overset{2\delta<2^{-s}}{\underset{\text{(C4)}}{\preceq}} \delta^{-j} \int_0^1 \frac{[\Phi'(y)]^2}{(2^{sj} + \Phi(y))^5}\,\mathrm{d}y \\
&\preceq \delta^{-j} \int_0^1 \frac{y^{-2(\beta+1)}(1-y)^{-2(\beta+1)}}{(2^{sj} + C_\beta y^{-\beta}(1-y)^{-\beta})^5}\,\mathrm{d}y \preceq \delta^{-j} \int_0^1 \frac{y^{-2(\beta+1)}}{(2^{sj} + C_\beta y^{-\beta})^5}\,\mathrm{d}y \\
&\preceq \delta^{-j} \int_0^1 \frac{y^{-2(\beta+1)}}{\underbrace{(2^{sj} + C_\beta y^{-\beta})^2}_{\succeq y^{-2\beta}} (2^{sj} + C_\beta y^{-\beta})^3}\,\mathrm{d}y \preceq \delta^{-j} \int_0^1 \frac{y^{-2}}{\underbrace{(2^{sj} + C_\beta y^{-\beta})^{3/2}}_{\succeq y^{-3\beta/2}} \underbrace{(2^{sj} + C_\beta y^{-\beta})^{3/2}}_{\succeq 2^{3sj/2}}}\,\mathrm{d}y \\
&\preceq \frac{1}{(\delta 2^{\frac{3}{2}s})^j} \int_0^1 y^{-2+\frac{3}{2}\beta}\,\mathrm{d}y.
\end{aligned}$$

It is also important to notice, that due to the $\frac{|u''|^2}{(1+u'^2)^{5/2}}$-structure the term 2^{sj} in the definition of Φ appears here in the denominator resulting in $2^{3sj/2}$ in the last line. Since we have chosen $\beta \in (\frac{2}{3}, 1)$ the last integral is finite, hence we get

$$A_{kj} \preceq \frac{1}{\left(\delta 2^{\frac{3}{2}s}\right)^j} < \infty.$$

Observe that this estimate does not depend on k. Now, we add the integrals A_{kj} over all the intervals I_{kj} and obtain

$$\int_{[0,1]\setminus\mathbb{D}_\delta} \left[(\Psi_2 \circ w_\delta)'(x)\right]^2 \mathrm{d}x = \sum_{j=1}^{\infty}\sum_{k=1}^{2^{j-1}} A_{kj} \preceq \sum_{j=1}^{\infty}\sum_{k=1}^{2^{j-1}} \left(\delta^{-1}2^{-\frac{3}{2}s}\right)^j \leq \sum_{j=1}^{\infty} \left(\delta^{-1}2^{1-\frac{3}{2}s}\right)^j < \infty$$

in the case that δ and s satisfy an extra condition

(C5) $$\delta^{-1}2^{1-\frac{3}{2}s} < 1 \quad \Leftrightarrow \quad \delta > 2^{1-\frac{3}{2}s}.$$

which is compatible with the first condition (C4) because by (C3) one can choose $s > 4$ with $2^{-s-1} > \delta > 2^{1-3s/2}$. For example, one admissible choice is $s = 5$ and $\delta = 1/80$.

6 For this step, we choose some fixed δ and s according to (C4), (C5), (C3), (C2) and (C1). Finally, we can define the actual function $u \in BV((0,1))$ as counterexample

$$u(x) := \int_0^x w_\delta(\xi)\,\mathrm{d}\xi + f_\delta(x).$$

where f_δ is the Cantor function defined in step 2 and u is continuous on $[0,1]$. Furthermore, we can decompose the derivative of u into absolutely continuous and a Cantor part

$$(u')^a = w_\delta, \qquad (u')^c = (f_\delta')^c.$$

Additionally, we define the absolutely differentiable part U of u by

$$U(x) := \int_0^x w_\delta(\xi)\,\mathrm{d}\xi = \int_0^x (u')^a(\xi)\,\mathrm{d}\xi$$

without the Cantor part of u. It can be shown that U has a weak curvature (similar to bowler example [DGR17, Example 2]) but no integrable second derivative. In detail, in $[0,1]\setminus\mathbb{D}_\delta$ the function U has the classical second derivative in each $x \in I_{kj}$

$$w_\delta'(x) = \Phi'\left(\frac{x - a_{kj}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right) \cdot \frac{1}{(1-2\delta)\delta^{j-1}}.$$

Thus U has a locally integrable weak second derivative in $[0,1]\setminus\mathbb{D}_\delta$. In order to investigate whether the weak second derivative of U is also L^1-integrable, we consider

$$\begin{aligned}
\int_{I_{kj}} \left|w_\delta'(x)\right| \mathrm{d}x &= \frac{\delta}{(1-2\delta)\delta^j} \int_{I_{kj}} \left|\Phi'\left(\frac{x-a_{kj}}{(1-2\delta)\delta^{j-1}} + \frac{1}{2}\right)\right| \mathrm{d}x \\
&\succeq \int_0^1 \frac{\beta\left|2y-1\right|}{y^{\beta+1}(1-y)^{\beta+1}}\,\mathrm{d}y \\
&\succeq \int_0^1 \frac{1}{y^{\beta+1}(1-y)^{\beta+1}}\,\mathrm{d}y \succeq \int_0^1 \frac{1}{y^{5/3}(1-y)^{5/3}}\,\mathrm{d}y = \infty
\end{aligned}$$

since $\beta \in \left(\frac{2}{3},1\right)$. Therefore, the last integral diverges independently of k and j, and therefore $w_\delta' \notin L^1((0,1))$ and $U \notin W^{2,1}((0,1))$. Despite that U has integrable curvature (also its geodesic curvature in $\mathbb{R}^2$), especially κ_U is integrable since by step 5

$$\int_0^1 \kappa_U(x)^2\sqrt{1+U'(x)^2}\,\mathrm{d}x = \int_{\text{graph}[U]} \kappa_U(x)^2\,\mathrm{d}s_U(x) = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}x}\Psi_2\big(\underbrace{U'(x)}_{=w_\delta(x)}\big)\right)^2 \mathrm{d}x < \infty.$$

7 Our next claim is

$$\overline{\mathcal{F}}_2(u) \le \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}x}(\Psi_2\circ w_\delta)\right)^2 \mathrm{d}x = \int_0^1 \frac{\big(w_\delta'(x)\big)^2}{\big(1+w_\delta(x)^2\big)^{5/2}}\,\mathrm{d}x = \mathcal{F}_2(U) = \mathcal{F}_2^a(u).$$

Here, we want to successively improve the regularity of the above constructed u, simultaneously controlling the increase of the Willmore energy.

a First, we replace the total variation of the Cantor part with a jump part.
Claim: There is a sequence $\{u_k\}_{k\in\mathbb{N}} \subset BV((0,1))$ with $(u_k')^c = 0$ such that $u_k \to u$ in $L^1((0,1))$ and it holds

$$\liminf_{k\to\infty} \mathcal{F}_2^a(u_k) := \liminf_{k\to\infty} \int_0^1 \frac{\Big(\big((u_k')^a\big)'(x)\Big)^2}{\Big(1+\big((u_k')^a\big)^2(x)\Big)^{5/2}}\,\mathrm{d}x \le \mathcal{F}_2^a(u).$$

Proof: In fact, the generalized Cantor set

$$\mathbb{D}_\delta = \left\{x\in[0,1]\;\middle|\;\left|(u')^a(x)\right| = \infty\right\} = \left\{x\in[0,1]\;\middle|\;(u')^a(x) = +\infty\right\}$$

is closed and, especially, of measure zero. For each $k \in \mathbb{N}$ we define the open $1/k$-neighborhoods of $\mathbb{D}_\delta$

$$\mathcal{A}_k := \left\{x\in[0,1]\;\middle|\;\operatorname{dist}(x,\mathbb{D}_\delta) < \frac{1}{k}\right\},\quad \mathcal{A}_{k+1}\subset\mathcal{A}_k,\quad \bigcap_{k\in\mathbb{N}}\mathcal{A}_k = \mathbb{D}_\delta,\quad \lim_{k\to\infty}|\mathcal{A}_k| = 0.$$

Furthermore, for each $k \in \mathbb{N}$ we denote the family of the connected components of $\mathcal{A}_k$ having non-empty intersection with $\mathbb{D}_\delta$ by $\{I_j^k\}_{j\in J_k}$ with J_k a suitable index set. Since by definition the helenght of each of the connected components of $\mathcal{A}_k$ is greater than $\frac{1}{k}$, then each of the indexes sets J_k contain less than k elements. Moreover, we choose for each $j \in J_k$ a point $x_j^k \in I_j^k \cap \mathbb{D}_\delta \cap (0,1)$ lying in the generalized Cantor set. It follows that $(u')^a(x_j^k) = +\infty$, which is one of the main points of the whole construction, and $\lim_{x\searrow 0} u(x) = 0$. Additionally, we choose

$$c_j^k := (u')^s\big(I_j^k\big)$$

and define

$$u_k(x) := \int_0^x (u')^a(\xi)\,\mathrm{d}\xi + \sum_{\{j\in J_k\,|\,x_j^k\le x\}} c_j^k.$$

Hence, we replaced the total variation of the Cantor part with a jump part. Next, observe that

$$(u_k')^a = (u')^a \quad\Longrightarrow\quad \mathcal{F}_2^a(u_k) = \mathcal{F}_2^a(u).$$

According to the definition of u_k, for $x \in [0,1] \setminus \mathcal{A}_k$ we have $u_k(x) = u(x)$, thus point-wise

$$\forall x \in [0,1] \setminus \bigcap_{k\in\mathbb{N}} \mathcal{A}_k = [0,1] \setminus \mathbb{D}_\delta: \qquad u(x) = \lim_{k\to\infty} u_k(x).$$

Since $u(x)$ and $u_k(x)$ are monotonically increasing and $\forall k \in \mathbb{N}\ \forall x \in [0,1]$: $0 \leq u_k(x) \leq u(1)$, the Lebesgue's theorem yields that

$$u_k \to u \quad \text{in } L^1((0,1)) \quad \text{for} \quad k \to \infty.$$

□

b Here we use the replacement technique like in [DMFLM09, Theorem 3.4]. Namely, the vertical jump part will be replaced by a linear non-vertical linear slope, such that the curve remains C^1. For the next calculations, we keep a sufficiently large $k \in \mathbb{N}$ fixed and consider

$$\hat{u} := u_k$$

For each k there is only a finite number $|J_k|$ of intervals I_j^k, and so only a finite number of jump points x_j^k. Therefore we assume w.l.o.g. that $(\hat{u}')^s$ consists of a single jump point $(x_0) \in (0,1)$. Because of the way we constructed $\hat{u}$, we can find sequences of points $\{x_\ell\}_{\ell=1}^\infty$ and $\{y_\ell\}_{\ell=1}^\infty$ such that $x_\ell \nearrow x_0$ and $y_\ell \searrow x_0$ and

$$\hat{u}'(x_\ell) = \hat{u}'(y_\ell) \nearrow +\infty, \quad \hat{u}(x_\ell) \nearrow \hat{u}_-(x_0) = \lim_{x\nearrow x_0} \hat{u}(x), \quad \hat{u}(y_\ell) \searrow \hat{u}_+(x_0) = \lim_{x\searrow x_0} \hat{u}(x).$$

Since the function $x \mapsto \hat{u}(x_\ell) + (\hat{u}')(x_\ell)(x - x_\ell)$ is affine, there exists for every sufficiently large ℓ a $z_\ell \in (x_0, 1)$ such that

$$\hat{u}_+(x_0) = \hat{u}(x_\ell) + (\hat{u}')(x_\ell)(z_\ell - x_\ell).$$

Furthermore, the sequence $\{z_\ell\}_{\ell=1}^\infty$ is such that $z_\ell \searrow x_0$ because $(u')^a \nearrow +\infty$ as $x_\ell \nearrow x_0$ which is also an important fact. Then we define for $\ell \in \mathbb{N}$ large enough the function

$$\hat{u}_\ell(x) := \begin{cases} \hat{u}(x), & \text{if } 0 \leq x \leq x_\ell, \\ \hat{u}(x_\ell) + (\hat{u}')(x_\ell)(x - x_\ell), & \text{if } x_\ell \leq x \leq z_\ell, \\ \hat{u}(x + y_\ell - z_\ell) + \hat{u}_+(x_0) - \hat{u}(y_\ell), & \text{if } z_\ell \leq x \leq 1. \end{cases}$$

It follows that between x_ℓ and z_ℓ the function is affine with the constant slope $(\hat{u}')(x_\ell)$, and thus does now carry any curvature because of $(\hat{u}'')|_{(x_\ell, z_\ell)} = 0$. Moreover,

$$\lim_{x\nearrow x_\ell} \hat{u}_\ell'(x) = (\hat{u}')(x_\ell) = (\hat{u}')(y_\ell) = \lim_{x\searrow z_\ell} \hat{u}_\ell'(x)$$

thus $\hat{u}_\ell \in W^{1,1}((0,1))$, $\hat{u}_\ell \to \hat{u}$ in $L^1((0,1))$. Since by the construction, we in general replace a curved part with a straight line, we can only reduce the Willmore energy

$$\mathcal{F}_2(\hat{u}_\ell) \overset{2}{=} \mathcal{F}_2^a(\hat{u}_\ell) \leq \mathcal{F}_2^a(\hat{u}).$$

Notice that until now, we replaced one jump with one finite slope in each component I_j^k of A_k. Since $U \notin W^{2,2}((0,1))$ we may still not be in $W^{2,2}((0,1))$.

c In this step, we keep fixed ℓ sufficiently large and consider

$$\tilde{u} := \hat{u}_\ell.$$

Claim. We can find a sequence $\{\tilde{u}_j\}_{j\in\mathbb{N}} \subset W^{2,2}((0,1))$ such that $\tilde{u}_j \to \tilde{u}$ in $L^1((0,1))$ and

$$\mathcal{F}_2(\tilde{u}_j) = \mathcal{F}_2^a(\tilde{u}_j) \leq \mathcal{F}_2^a(\tilde{u}) = \mathcal{F}_2(\tilde{u}).$$

Proof: What we want to do now is to bound the second derivative by cutting off the first derivative, especially near the points where still $(u')^a = +\infty$.

We have $0 \leq \tilde{u}' = (\tilde{u}')^a$, thus for each $j \in \mathbb{N}$ we define $\tilde{w}_j(x) := \min\{j, \tilde{u}'(x)\}$ so that $|\tilde{w}_j| \leq j$. In points where we did any replacing in this or the previous step *b* (by linear slope), we have $\tilde{w}_j' \equiv 0$. For the other points, we want to show that for each j the derivative $\tilde{w}_j'$ is bounded. Thus, for each interval $I_{km}, k = 1, \ldots, 2^{m-1}$ from the generalized Cantor set construction we only consider the points

$$x \in I_{km} \quad \text{with} \quad \left|2^{sm} + \Phi\left(\frac{x - a_{km}}{(1-2\delta)\delta^{m-1}} + \frac{1}{2}\right)\right| \leq j. \tag{245}$$

Since $\Phi \geq 0$ we only have to work with intervals such that $2^{sm} \leq j$. Hence for each $j \in \mathbb{N}$ we have only finite many intervals I_{km} with $k = 1, \ldots, j^{1/s}/2$ and $m \leq \log(j)/(s\log(2))$. We want to show that there is a constant C_j independent of k and m so that for x in (245)

$$\underbrace{\left|\frac{1}{(1-2\delta)\delta^{(m-1)}}\Phi'\left(\frac{x - a_{km}}{(1-2\delta)\delta^{m-1}} + \frac{1}{2}\right)\right|}_{=:\left|\tilde{w}_j'(x)\right|} \leq C_j. \tag{246}$$

Namely, from (245) and $\Phi \geq 0$ we conclude

$$0 \leq \Phi\left(\frac{x - a_{km}}{(1-2\delta)\delta^{m-1}} + \frac{1}{2}\right) \leq j.$$

Since $\Phi \in C^2((0,1))$, there exists $\varepsilon = \varepsilon(j)$ also depending on Φ such that for $y \in (0,1)$ it holds

$$\Phi(y) \leq j \quad \Rightarrow \quad y \in [\varepsilon(j), 1-\varepsilon(j)] \quad \Rightarrow \quad \left|\Phi'(y)\right| \leq C_{224}(\varepsilon(j)) := \max_{[\varepsilon(j),1-\varepsilon(j)]} \left|\Phi'(x)\right|.$$

For x in (245) we get the estimate

$$\left|\frac{1}{(1-2\delta)\delta^{(m-1)}}\Phi'\left(\frac{x - a_{km}}{(1-2\delta)\delta^{m-1}} + \frac{1}{2}\right)\right| \leq \frac{C_{224}(\varepsilon(j))}{(1-2\delta)\delta^{(\log(j)/(s\log(2))-1)}} =: C_j.$$

Thus, for each j the absolute value of the derivative $|\tilde{w}_j'|$ is bounded by C_j.

Next, it follows that $\tilde{w}_j \in W^{1,\infty}((0,1)) = C^{0,1}((0,1))$ and

$$\tilde{u}_j(x) := \int_0^x \tilde{w}_j(\xi)\,\mathrm{d}\xi \to \tilde{u}(x) \quad \text{a.e.},$$

monotonically, hence in L^1 and since $\tilde{u}$ is continuous. Now, we obtain

$$\mathcal{F}_2(\tilde{u}_j) = \mathcal{F}_2^a(\tilde{u}_j) = \int_0^1 \frac{\left(((\tilde{u}_j')^a)'(x)\right)^2}{\left(1 + ((\tilde{u}_j')^a)^2(x)\right)^{5/2}}\,\mathrm{d}x = \int_0^1 \frac{\left(\tilde{w}_j'(x)\right)^2}{\left(1 + \left(\tilde{w}_j(x)\right)^2\right)^{5/2}}\,\mathrm{d}x \leq \mathcal{F}_2^a(\tilde{u}).$$

□

Finally, we combine the steps (a), (b), (c) and find that:

$$\overline{\mathcal{F}_2}(u) \leq \mathcal{F}_2^a(u).$$

□

It is essential to notice that the Cantor part of the function is due to special projection. If one rotates the Cantor function, that is monotonically increasing, clockwise, then it immediately becomes a Lipschitz function, which by [AFP00, Theorem 3.16 p 127] lacks a singular part. Thus, the Cantor part vanishes by rotation. The relaxed one-dimensional Willmore energy of the Cantor function stays infinite, reflecting its geometric nature. The reason is that $(u')^a$ is non-continuous.

Similar arguments apply likewise to the example function constructed in Theorem 90 above. Since it is monotonically increasing, a clockwise rotation instantly lets the Cantor part vanish. Interestingly, in the same example, the Cantor part does not contribute any Willmore energy.

9 Appendix

91 Lemma
For $4\mu + |\nu| \le s$, we have (see notation in [Bel79])

$$[u]^{Q_T}_{r,s,t} = \sup_{\overline{\Omega}\times(0,T]} \left(\theta^{\frac{s-r}{4}} \left|\Delta^t_\tau D^\mu_t D^\nu_x u\right| \cdot |t-\tau|^{\frac{s-4\mu-|\nu|}{4}}\right) \cong \sup_{t<T} t^{\frac{s-r}{4}} \sup_{x\in\overline{\Omega}} \left[D^\mu_t D^\mu_x u(x,.)\right]_{C^{\frac{s-4\mu-|\nu|}{4}}\left([t/2,t]\right)}$$

where $\theta = \min(t,\tau)$. Also for $4\mu + |\nu| = \lfloor s \rfloor$ it holds

$$[u]^{Q_T}_{r,s,x} = \sup_{\overline{\Omega}\times(0,T]} \left(t^{\frac{s-r}{2m}} \left|\Delta^x_y D^\mu_t D^\nu_x u\right| \cdot |x-y|^{\lfloor s\rfloor - s}\right) \cong \sup_{t<T} t^{\frac{s-r}{4}} \sup_{t'\in[t/2,t]} \left[D^\mu_t D^\nu_x u(\,.\,,t')\right]_{C^{s-\lfloor s\rfloor}(\overline{\Omega})}.$$

Proof: Here, we prove only the first equivalence. W.l.o.g. let us assume that $w \in C^\infty(Q_T)$, then we have to show

$$\sup_{\overline{\Omega}\times(0,T]} \left(\theta^{\frac{s-r}{4}} \left|\Delta^t_\tau w\right| \cdot |t-\tau|^{\frac{s-4\mu-|\nu|}{4}}\right) \cong \sup_{t<T} t^{\frac{s-r}{4}} \sup_{x\in\overline{\Omega}} \left[w(x,.)\right]_{C^{\frac{s-4\mu-|\nu|}{4}}\left([t/2,t]\right)}.$$

For the sake of simplicity, we define $A = \frac{s-r}{4}$ and $B = \frac{s-4\mu-|\nu|}{4}$. Let $\tau < t$ then there exists $m \in \mathbb{N}_0$ such that $t = 2^m\tau + \gamma$ with $\gamma < 2^m\tau$. Also, W.l.o.g. let $m > 0$, since otherwise all sums in the following estimates vanish. We conclude that

$$\begin{aligned}
\left|w(t) - w(\tau)\right| &\le \sum_{\ell=1}^m \left|w(2^\ell\tau) - w(2^{\ell-1}\tau)\right| + \left|w(t) - w(2^m\tau)\right| \\
&\le \sum_{\ell=1}^m C_{225}[w]_{C^B([2^{\ell-1}\tau,2^\ell\tau])} \cdot \left|2^\ell\tau - 2^{\ell-1}\tau\right|^B + C_{225}[w]_{C^B([t/2,t])} \cdot \left|t - 2^m\tau\right|^B \\
&\le C_{225} \sum_{\ell=1}^m [w]_{C^B([2^{\ell-1}\tau,2^\ell\tau])} \cdot 2^{(\ell-1)B}|\tau|^B + C_{225}[w]_{C^B([t/2,t])} \cdot \left|\gamma\right|^B \\
&\le C_{225} \sum_{\ell=1}^m \sup_{t<T} \left(t^A[w]_{C^B([t/2,t])}\right) \cdot (2^\ell|\tau|)^{-A} 2^{(\ell-1)B}|\tau|^B \\
&\quad + C_{225} \sup_{t<T} \left(t^A[w]_{C^B([t/2,t])}\right) \cdot |t|^{-A}\left|\gamma\right|^B \\
&\le C_{225} \sup_{t<T} \left(t^A[w]_{C^B([t/2,t])}\right) \cdot \left(|t|^{-A}\left|\gamma\right|^B + \sum_{\ell=1}^m |\tau|^{-A} 2^{(\ell-1)B}|\tau|^B\right)
\end{aligned}$$

since $2^\ell > 1$. Next, we multiply it with some term and get

$$\begin{aligned}
|\tau|^A \frac{\left|w(t) - w(\tau)\right|}{|t-\tau|^B} &\le C_{225} \sup_{t<T} \left(t^A[w]_{C^B([t/2,t])}\right) \cdot \left(\left|\frac{\tau}{t}\right|^A \left|\frac{\gamma}{t-\tau}\right|^B + \sum_{\ell=1}^m 2^{(\ell-1)B} \left|\frac{\tau}{t-\tau}\right|^B\right) \\
&\le C_{225} \sup_{t<T} \left(t^A[w]_{C^B([t/2,t])}\right) \cdot \left(1 + \frac{1}{(2^m-1)^B} \sum_{\ell=1}^m 2^{B-1\ell}\right)
\end{aligned}$$

since, by definition, it holds

$$\left|\frac{\tau}{t}\right| < 1, \quad \left|\frac{\gamma}{t-\tau}\right| = \left|\frac{\gamma}{(2^m-1)\tau + \gamma}\right| \le 1, \quad \left|\frac{\tau}{t-\tau}\right| = \left|\frac{\tau}{(2^m-1)\tau + \gamma}\right| \le \frac{1}{2^m-1}.$$

Further, we use the geometric series and derive

$$|\tau|^A \frac{|w(t)-w(\tau)|}{|t-\tau|^B} \le C_{225} \sup_{t<T}\left(t^A [w]_{C^B([t/2,t])}\right)\cdot\left(1+\frac{1}{(2^m-1)^B}\frac{|1-2^{Bm}|}{|1-2^B|}\right).$$

Finally, we can estimate $|1-2^{Bm}|/(2^m-1)^B$ by a constant which is independent of m. Hence we obtain

$$|\tau|^A \frac{|w(t)-w(\tau)|}{|t-\tau|^B} \le C_{226} \sup_{t<T}\left(t^A [w]_{C^B([t/2,t])}\right)$$

with some constant C_{226} that depends only on A and B. □

92 Lemma
Let $m = 1,2,3,4$. If $u \in C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)$ and $\forall x \in \overline{\Omega}$

$$\forall 4k+|\beta| \le m: \qquad D_t^k D_x^\beta u(x,0) = 0$$

then there is a constant $C_{166} = C_{166}(\alpha,\gamma)$ such that for all $0<\gamma<\alpha$ and $T \le 1$ it holds

$$\|u\|_{C^{4+\gamma,1+\frac{\gamma}{4}}_{m+\gamma}(Q_T)} \le C_{166} T^{\frac{\alpha-\gamma}{4}} \|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)}$$

Proof: The case $m=4$ was already considered in [Gul17]. For the following proof we write

$$C_{227} := \|u\|_{C^{4+\alpha,1+\alpha/4}_{m+\alpha}(Q_T)}.$$

Then, we split the $C^{4+\gamma,1+\frac{\gamma}{4}}_{m+\gamma}(Q_T)$-norm into different parts.

$$\begin{aligned}\|u\|_{C^{4+\gamma,1+\frac{\gamma}{4}}_{m+\gamma}(Q_T)} = &\underbrace{\sum_{m<4k+|\beta|\le 4} \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{4k+|\beta|-m-\gamma}{4}} \left|D_t^k D_x^\beta u(x,t)\right|}_{①} \\ &+ \|u\|_{C^{m+\gamma,\frac{m+\gamma}{4}}_{x,t}(\overline{Q}_T)} + \sup_{t\in(0,T]} t^{\frac{4-m}{4}} [u]^{4+\gamma}_{Q'_t},\end{aligned}$$

where we also separate the last two terms into and use $\alpha > 0$

$$\begin{aligned}\|u\|_{C^{m+\gamma,\frac{m+\gamma}{4}}_{x,t}(\overline{Q}_T)} = &\underbrace{\sum_{m-3\le 4k+|\beta|\le m} \sup_{x\in\overline{\Omega}} \left[D_t^k D_x^\beta u(x,.)\right]_{C^{\frac{m+\gamma-4k-|\beta|}{4}}([0,T])}}_{②} \\ &+ \underbrace{\sum_{4k+|\beta|\le m} \sup_{(x,t)\in\overline{Q}_T} \left|D_t^k D_x^\beta u(x,t)\right|}_{③} + \underbrace{\sum_{4k+|\beta|=m} \sup_{t\in[0,T]} \left[D_t^k D_x^\beta u(\,.\,,t)\right]_{C^\gamma(\overline{\Omega})}}_{④}.\end{aligned}$$

$$[u]^{4+\gamma}_{Q'_t} = \underbrace{\sum_{1\le 4k+|\beta|\le 4} \sup_{x\in\overline{\Omega}} \left[D_t^k D_x^\beta u(x,.)\right]_{C^{\frac{4-4k-|\beta|+\gamma}{4}}([t/2,t])}}_{⑤} + \underbrace{\sum_{4k+|\beta|=4} \sup_{t\in[t/2,t]} \left[D_t^k D_x^\beta u(\,.\,,t)\right]_{C^\gamma(\overline{\Omega})}}_{⑥}.$$

1 Since $t < T \leq 1$ we estimate for $m < 4k + |\beta| \leq 4$

$$\sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{4k+|\beta|-m-\gamma}{4}} \left|D_t^k D_x^\beta u(x,t)\right| \leq \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{4k+|\beta|-m-\alpha}{4}} \left|D_t^k D_x^\beta u(x,t)\right| t^{\frac{\alpha-\gamma}{4}} \leq C_{227} T^{\frac{\alpha-\gamma}{4}}.$$

2 This term is the first part of the unweighted parabolic Hölder norm $C_{x,t}^{m+\gamma,(m+\gamma)/4}(\overline{Q}_T)$. For all $m-3 \leq 4k+|\beta| \leq m$ and for all $x \in \overline{\Omega}$ we conclude

$$\begin{aligned}\left[D_t^k D_x^\beta u(x,.)\right]_{C^{\frac{m+\gamma-4k-|\beta|}{4}}\left([0,T]\right)} &\leq \left[D_t^k D_x^\beta u(x,.)\right]_{C^{\frac{m+\alpha-4k-|\beta|}{4}}\left([0,T]\right)} \cdot |T|^{\frac{m-4k-|\beta|+\alpha-(m-4k-|\beta|+\gamma)}{4}} \\ &\leq C_{227} T^{\frac{\alpha-\gamma}{4}}.\end{aligned}$$

3 Since the temporal Hölder seminorms are bounded, we obtain for all $m-3 \leq 4k+|\beta| \leq m$ and for all $x \in \overline{\Omega}$

$$\begin{aligned}&\left[D_t^k D_x^\beta u(x,.)\ \right]_{C^{\frac{m+\alpha-4k-|\beta|}{4}}\left([0,T]\right)} \leq C_{227} \\ &\qquad \Rightarrow \sup_{t\in(0,T]} \left|D_t^k D_x^\beta u(x,t)\right| \leq C_{227} T^{\frac{m+\alpha-4k-|\beta|}{4}} + \underbrace{\sup_{x\in\overline{\Omega}} \left|D_t^k D_x^\beta u(x,0)\right|}_{=0}.\end{aligned}$$

4 As the last part of the $C_{x,t}^{m+\gamma,(m+\gamma)/4}(\overline{Q}_T)$-norm we estimate the spatial Hölder seminorm for $4k+|\beta| = m$. Thus we consider $x, y \in \overline{\Omega}$ and $t \in [0,T]$ then it holds

$$\begin{aligned}&\frac{\left|D_t^k D_x^\beta u(x,t) - D_t^k D_x^\beta u(y,t)\right|}{|x-y|^\gamma} \\ &\quad \leq \left(\frac{\left|D_t^k D_x^\beta u(x,t) - D_t^k D_x^\beta u(y,t)\right|}{|x-y|^\alpha}\right)^{\frac{\gamma}{\alpha}} \cdot \left|D_t^k D_x^\beta u(x,t) - D_t^k D_x^\beta u(y,t)\right|^{1-\frac{\gamma}{\alpha}} \\ &\quad \overset{3}{\leq} \left(\frac{\left|D_x^\beta u(x,t) - D_x^\beta u(y,t)\right|}{|x-y|^\alpha}\right)^{\frac{\gamma}{\alpha}} \cdot \left|2C_{227} T^{\alpha/4} + 2\sup_{x\in\overline{\Omega}} \left|D_x^\beta u(x,0)\right|\right|^{1-\frac{\gamma}{\alpha}}.\end{aligned}$$

Therefore, due to initial values, we conclude

$$\sup_{t\in[0,T]} \left[D_t^k D_x^\beta u(\,.\,,t)\right]_{C^\gamma(\overline{\Omega})} \leq 2^{1-\frac{\gamma}{\alpha}} C_{227} T^{\frac{\alpha-\gamma}{4}}.$$

5 With the temporal seminorm estimate and $1 \leq 4k+|\beta| \leq 4$, $x \in \overline{\Omega}$ and $t \in (0,T]$ like in 2 we get

$$t^{\frac{4-m}{4}} \left[D_t^k D_x^\beta u(x,.)\right]_{C^{\frac{4-4k-|\beta|+\gamma}{4}}\left([t/2,t]\right)} \leq t^{\frac{4-m}{4}} \left[D_t^k D_x^\beta u(x,.)\right]_{C^{\frac{4-4k-|\beta|+\alpha}{4}}\left([t/2,t]\right)} \cdot t^{\frac{\alpha-\gamma}{4}} \overset{t\leq T}{\leq} C_{227} T^{\frac{\alpha-\gamma}{4}}.$$

6 Next, we estimate the spacial Hölder seminorm of the time derivative, $4k+|\beta| = 4$:

$$\sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{4-m-\alpha}{4}} \left|D_t^k D_x^\beta u(x,t)\right| \leq C_{227} \quad \Rightarrow \quad \forall (x,t) \in \overline{\Omega}\times(0,T]: \ \left|D_t^k D_x^\beta u(x,t)\right| \leq C_{227} t^{\frac{\alpha+m-4}{4}}.$$

So we see that for $x, y \in \overline{\Omega}$ and $t' \in [t/2,t]$, $4k+|\beta| = 4$ like in 4:

$$t^{\frac{4-m}{4}} \frac{\left|D_t^k D_x^\beta u(x,t') - D_t^k D_x^\beta (y,t')\right|}{|x-y|^\gamma}$$

$$\leq \left(\frac{t^{\frac{4-m}{4}} \left| D_t^k D_x^\beta u(x,t') - D_t^k D_x^\beta u(y,t') \right|}{|x-y|^\alpha} \right)^{\frac{\gamma}{\alpha}} \cdot \left| 2C_{227} t'^{\frac{\alpha+m-4}{4}} t^{\frac{4-m}{4}} \right|^{1-\frac{\gamma}{\alpha}}.$$

Hence, we conclude for $4k + |\beta| = 4$

$$\sup_{t\in(0,T]} t^{\frac{4-m}{4}} \sup_{t'\in[t/2,t]} \left[D_t^k D_x^\beta u(\,.\,,t') \right]_{C^\gamma(\overline{\Omega})} \leq 2^{1-\frac{\gamma}{\alpha}} C_{227} t^{\frac{\alpha-\gamma}{4}} \leq 2^{1-\frac{\gamma}{\alpha}} C_{227} T^{\frac{\alpha-\gamma}{4}}.$$

$\square$

93 Lemma
Let $m = 1,2,3,4$ and $0 < \gamma \leq \alpha$, $\alpha/2 \leq \gamma$. If $u, v, w \in C^{4+\gamma, 1+\frac{\gamma}{4}}_{m+\gamma-4}(Q_T)$ and $T \leq 1$, then there is a constant $C_{167} = C_{167}(\alpha, \gamma, \Omega)$ depending on algebraic structure of $\mathcal{R}$ and L such that

$$\|\nabla u\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)} \leq C_{167} \|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)} \tag{138}$$

$$\|D^3 w D^2 u\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} \leq C_{167} \|D^3 w\|_{C^{1+\gamma,\frac{1+\gamma}{4}}_{m+\gamma-3}(Q_T)} \cdot \|D^2 u\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)} \tag{139}$$

$$\begin{aligned} &\|D^2 u D^2\, w D^2 v\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} \\ &\qquad \leq C_{167} \|D^2 u\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)} \cdot \|D^2 w\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)} \cdot \|D^2 v\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)}. \end{aligned} \tag{140}$$

Proof: For $m = 4$, we refer to [Gul17]. By definition, it follows

$$\begin{aligned} \|h\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} = {} & \sup_{t<T} t^{\frac{4-m}{4}} \left(\sup_{t'\in[t/2,t]} \left[h(\,,\,,t') \right]_{C^\alpha(\overline{\Omega})} + \sup_{x\in\overline{\Omega}} \left[h(x,\,.\,) \right]_{C^{\alpha/4}\left([t/2,t]\right)} \right) \\ & + \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{4-m-\alpha}{4}} \left| h(x,t) \right|. \end{aligned}$$

First, in preparation for the later estimates, we omit the weights and conclude for $t' \in [t/2, t]$

$$\begin{aligned} \left[D^3 w \cdot D^2 u(\,.\,,t') \right]_{C^\alpha(\overline{\Omega})} = {} & \sup_{x\in\overline{\Omega}} \left| D^3 w(x,t') \right| \cdot \left[D^2 u(\,.\,,t') \right]_{C^\alpha(\overline{\Omega})} + \left[D^3 w(\,.\,,t') \right]_{C^\alpha(\overline{\Omega})} \cdot \sup_{x\in\overline{\Omega}} \left| D^2 u(x,t') \right| \\ \leq {} & \sup_{x\in\overline{\Omega}} \left| D^3 w(x,t') \right| \cdot \sup_{x\in\overline{\Omega}} \left| D^3 u(x,t') \right|^\alpha \cdot \sup_{x\in\overline{\Omega}} \left| D^2 u(x,t') \right|^{1-\alpha} \\ & + \sup_{x\in\overline{\Omega}} \left| D^4 w(x,t') \right|^\alpha \cdot \sup_{x\in\overline{\Omega}} \left| D^3 u(x,t') \right|^{1-\alpha} \cdot \sup_{x\in\overline{\Omega}} \left| D^2 u(x,t') \right| \end{aligned}$$

and for the temporal seminorm, it holds

$$\begin{aligned} & \left[D^3 w \cdot D^2 u(x,.) \right]_{C^{\alpha/4}\left([t/2,t]\right)} \\ & \quad \leq \sup_{t'\in[t/2,t]} \left| D^3 w(x,t') \right| \cdot \left[D^2 u(x,.) \right]_{C^{\alpha/4}\left([t/2,t]\right)} + \sup_{t'\in[t/2,t]} \left| D^2 u(x,t') \right| \cdot \left[D^3 w(x,.) \right]_{C^{\alpha/4}\left([t/2,t]\right)} \\ & \quad \leq \sup_{t'\in[t/2,t]} \left| D^3 w(x,t') \right| \cdot \left(\frac{t}{2} \right)^{\frac{2+\gamma-\alpha}{4}} \left[D^2 u(x,.) \right]_{C^{\frac{2+\gamma}{4}}\left([t/2,t]\right)} \\ & \qquad + \sup_{t'\in[t/2,t]} \left| D^2 u(x,t') \right| \cdot \left(\frac{t}{2} \right)^{\frac{1+\gamma-\alpha}{4}} \left[D^3 w(x,.) \right]_{C^{\frac{1+\gamma}{4}}\left([t/2,t]\right)} \\ & \quad \leq t^{\frac{2+\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]} \left| D^3 w(x,t') \right| \cdot \left[D^2 u(x,.) \right]_{C^{\frac{2+\gamma}{4}}\left([t/2,t]\right)} \end{aligned}$$

$$+\, t^{\frac{1+\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]} \left|D^2u(x,t')\right| \cdot \left[D^3w(x,\,.\,)\right]_{C^{\frac{1+\gamma}{4}}\left([t/2,t]\right)}.$$

Then we estimate each part by itself. The first case is $m=1$ here it holds

$$\begin{aligned}\sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{3-\alpha}{4}}\left|D^3w\cdot D^2u(x,t)\right| &\le \sup_{(x,t)\in\overline{\Omega}\times(0,T]} T^{\frac{2\gamma-\alpha}{4}} t^{\frac{3-2\gamma}{4}}\left|D^3w\cdot D^2u(x,t)\right|\\ &\le T^{\frac{2\gamma-\alpha}{4}} \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{2-\gamma}{4}}\left|D^3w(x,t)\right| \cdot \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{1-\gamma}{4}}\left|D^2u(x,t)\right|.\end{aligned}$$

Furthermore, for the spatial seminorm, we derive for $t'\in[t/2,t]$

$$\begin{aligned}&t^{\frac{3}{4}}\left[D^3w\cdot D^2u(\,.\,,t')\right]_{C^\alpha(\overline{\Omega})}\\ &\quad\le T^{\frac{2\gamma-\alpha}{4}} t^{\frac{3-2\gamma+\alpha}{4}}\left[D^3w\cdot D^2u(\,.\,,t')\right]_{C^\alpha(\overline{\Omega})}\\ &\quad\le T^{\frac{2\gamma-\alpha}{4}} \sup_{x\in\overline{\Omega}}\left|t^{\frac{2-\gamma}{4}}D^3w(x,t')\right| \cdot \sup_{x\in\overline{\Omega}}\left|t^{\frac{2-\gamma}{4}}D^3u(x,t')\right|^{\alpha} \cdot \sup_{x\in\overline{\Omega}}\left|t^{\frac{1-\gamma}{4}}D^2u(x,t')\right|^{1-\alpha}\\ &\qquad+ T^{\frac{2\gamma-\alpha}{4}} \sup_{x\in\overline{\Omega}}\left|t^{\frac{3-\gamma}{4}}D^4w(x,t')\right|^{\alpha} \cdot \sup_{x\in\overline{\Omega}}\left|t^{\frac{2-\gamma}{4}}D^3w(x,t')\right|^{1-\alpha} \cdot \sup_{x\in\overline{\Omega}}\left|t^{\frac{1-\gamma}{4}}D^2u(x,t')\right|.\end{aligned}$$

Analogously, we conclude

$$\begin{aligned}&t^{\frac{3}{4}}\left[D^3w\cdot D^2u(x,\,.\,)\right]_{C^{\alpha/4}\left([t/2,t]\right)}\\ &\qquad\le T^{\frac{2\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]}\left|t^{\frac{2-\gamma}{4}}D^3w(x,t')\right| \cdot t^{\frac{3}{4}}\left[D^2u(x,\,.\,)\right]_{C^{\frac{2+\gamma}{4}}\left([t/2,t]\right)}\\ &\qquad\quad+ T^{\frac{2\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]}\left|t^{\frac{1-\gamma}{4}}D^2u(x,t')\right| \cdot t^{\frac{3}{4}}\left[D^3w(x,\,.\,)\right]_{C^{\frac{1+\gamma}{4}}\left([t/2,t]\right)}.\end{aligned}$$

The proof for $\|D^2uD^2wD^2v\|_{C^{\alpha,\alpha/4}_{\alpha-3}(Q_T)}$ is similar. For the case $m=2$, it results

$$\sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{2-\alpha}{4}}\left|D^3w\cdot D^2u(x,t)\right| \le T^{\frac{1+\gamma-\alpha}{4}} \cdot \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{1-\gamma}{4}}\left|D^3w(x,t)\right| \cdot \sup_{(x,t)\in\overline{\Omega}\times(0,T]}\left|D^2u(x,t)\right|.$$

Next, for the spacial seminorm with weight, we get for $t'\in[t/2,t]$

$$\begin{aligned}&t^{\frac{2}{4}}\left[D^3w\;\cdot D^2u(\,.\,,t')\right]_{C^\alpha(\overline{\Omega})}\\ &\quad\le T^{\frac{1+\gamma-(1-\gamma)\alpha}{4}} \sup_{x\in\overline{\Omega}}\left|t^{\frac{1-\gamma}{4}}D^3w(x,t')\right| \cdot \left|t^{\frac{1-\gamma}{4}}D^3u(x,t')\right|^{\alpha} \cdot \sup_{x\in\overline{\Omega}}\left|D^2u(x,t')\right|^{1-\alpha}\\ &\qquad+ T^{\frac{1+\gamma-\alpha}{4}} \sup_{x\in\overline{\Omega}}\left|t^{\frac{2-\gamma}{4}}D^4w(x,t')\right|^{\alpha} \cdot \sup_{x\in\overline{\Omega}}\left|t^{\frac{1-\gamma}{4}}D^3w(x,t')\right|^{1-\alpha} \cdot \sup_{x\in\overline{\Omega}}\left|D^2u(x,t')\right|\end{aligned}$$

and for the temporal seminorm, we derive

$$\begin{aligned}t^{\frac{2}{4}}\left[D^3w\cdot D^2u(x,\,.\,)\right]_{C^{\alpha/4}\left([t/2,t]\right)} &\le T^{\frac{1+2\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]}\left|t^{\frac{1-\gamma}{4}}D^3w(x,t')\right| \cdot t^{\frac{2}{4}}\left[D^2u(x,\,.\,)\right]_{C^{\frac{2+\gamma}{4}}\left([t/2,t]\right)}\\ &\quad+ T^{\frac{1+\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]}\left|D^2u(x,t')\right| \cdot t^{\frac{2}{4}}\left[D^3w(x,\,.\,)\right]_{C^{\frac{1+\gamma}{4}}\left([t/2,t]\right)}.\end{aligned}$$

For the case $m=3$, we compute

$$\sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{1-\alpha}{4}}\left|D^3w\cdot D^2u(x,t)\right| \le \sup_{(x,t)\in\overline{\Omega}\times(0,T]}\left|D^3w(x,t)\right| \cdot \sup_{(x,t)\in\overline{\Omega}\times(0,T]}\left|D^2u(x,t)\right|$$

as well as the following estimate for $t' \in [t/2, t]$

$$
\begin{aligned}
t^{\frac{1}{4}} &\Big[D^3 w \cdot D^2 u(\,.\,, t')\Big]_{C^\alpha(\overline{\Omega})} \\
&\qquad \le T^{\frac{1}{4}} \sup_{x\in\overline{\Omega}} \big|D^3 w(x,t')\big| \cdot \big|D^3 u(x,t')\big|^\alpha \cdot \sup_{x\in\overline{\Omega}} \big|D^2 u(x,t')\big|^{1-\alpha} \\
&\qquad\quad + T^{\frac{1-(1-\gamma)\alpha}{4}} \sup_{x\in\overline{\Omega}} \big|t^{\frac{1-\gamma}{4}} D^4 w(x,t')\big|^\alpha \cdot \sup_{x\in\overline{\Omega}} \big|D^3 w(x,t')\big|^{1-\alpha} \cdot \sup_{x\in\overline{\Omega}} \big|D^2 u(x,t')\big|.
\end{aligned}
$$

Further, we work with the temporal seminorm on $[t/2, t]$ and conclude

$$
\begin{aligned}
t^{\frac{1}{4}} &\Big[D^3 w \cdot D^2 u(x,\,.\,)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} \\
&\qquad \le T^{\frac{2+\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]} \big|D^3 w(x,t')\big| \cdot t^{\frac{1}{4}} \Big[D^2 u(x,\,.\,)\Big]_{C^{\frac{2+\gamma}{4}}\left([t/2,t]\right)} \\
&\qquad\quad + T^{\frac{1+\gamma-\alpha}{4}} \sup_{t'\in[t/2,t]} \big|D^2 u(x,t')\big| \cdot t^{\frac{1}{4}} \Big[D^3 w(x,\,.\,)\Big]_{C^{\frac{1+\gamma}{4}}\left([t/2,t]\right)}.
\end{aligned}
$$

At last, we consider the gradient estimate

$$
\begin{aligned}
\|\nabla u\|_{C_0^{\alpha,\alpha/4}(Q_T)} = &\sup_{t<T} t^{\alpha/4} \left(\sup_{t'\in[t/2,t]} \Big[\nabla u(\,,\,,t')\Big]_{C^\alpha(\overline{\Omega})} + \sup_{x\in\overline{\Omega}} \Big[\nabla u(x,\,.\,)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} \right) \\
&+ \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \big|\nabla u(x,t)\big|.
\end{aligned}
$$

In preparation, we derive

$$
\sup_{x\in\overline{\Omega}} \Big[\nabla u(x,\,.\,)\Big]_{C^{\frac{m+\alpha-1}{4}}\left([0,T]\right)} \le C \quad \Rightarrow \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \Big|\nabla u(x,t)\Big| \le C T^{\frac{m+\alpha-1}{4}} + \sup_{x\in\overline{\Omega}} \Big|\nabla u(x,0)\Big|.
$$

Therefore, now we can estimate

$$
\begin{aligned}
t^{\alpha/4} \sup_{x\in\overline{\Omega}} \Big[\nabla u(x,\,.\,)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} &\le t^{\alpha/4} \left(\frac{t}{2}\right)^{\frac{3+\gamma-\alpha}{4}} \Big[\nabla u(x,\,.\,)\Big]_{C^{\frac{3+\gamma}{4}}\left([t/2,t]\right)} \\
&\le T^{\frac{m-1+\gamma}{4}} t^{\frac{4-m}{4}} \Big[\nabla u(x,\,.\,)\Big]_{C^{\frac{3+\gamma}{4}}\left([t/2,t]\right)}.
\end{aligned}
$$

For $m = 2, 3$ it follows

$$
t^{\alpha/4} \sup_{t'\in[t/2,t]} \Big[\nabla u(\,,\,,t')\Big]_{C^\alpha(\overline{\Omega})} \le T^{\alpha/4} \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \Big|D_x^2 u(x,t)\Big|^\alpha \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \Big|2\nabla u(x,t)\Big|^{1-\alpha}.
$$

For $m = 1$ we conclude

$$
t^{\alpha/4} \sup_{t'\in[t/2,t]} \Big[\nabla u(\,,\,,t')\Big]_{C^\alpha(\overline{\Omega})} \le T^{\frac{\alpha\gamma}{4}} \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \Big|t^{\frac{1-\gamma}{4}} D_x^2 u(x,t)\Big|^\alpha \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \Big|2\nabla u(x,t)\Big|^{1-\alpha}.
$$

□

94 Lemma (Hölder Estimates I)
Let $m = 1, 2, 3, 4$ and $0 < \gamma, \alpha < 1$, $T \le 1$ then there exist constants $C_{168} = C_{168}(\Omega, \alpha, \gamma)$ and $k_H \in \mathbb{N}$ depending on algebraic structure of $\mathcal{R}$ and L, so that:

$$
\big\|\mathcal{R}(\nabla u, D^2 u, D^3 u)\big\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} \le C_{168} \Big(1 + \|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)}\Big)^{k_H} \|\nabla u\|^3_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)},
$$

$$
\sum_{k+\ell=4} \|L_{k\ell}(\nabla u)\|_{C^{\alpha,\alpha/4}_{\max\{0,m+\alpha-4\}}(Q_T)} \le C_{168} \Big(1 + \|\nabla u\|^4_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)}\Big).
$$

Proof: Here, we use the formula (R). Then, we can estimate

$$\begin{aligned}
&\left\|\mathcal{R}(\nabla u, D^2 u, D^3 u)\right\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} \\
&\quad\overset{(137)}{\le} C_{164}\left\|D^3 u \star D^2 u\right\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} \sum_{k=1}^{4}\left\|Q^{-2k}P_{2k-1}(\nabla u)\right\|_{C^{\alpha,\alpha/4}_{\max\{0,m-\alpha-4\}}(Q_T)} \\
&\qquad + C_{164}\left\|D^2 u \star D^2 u \star D^2 u\right\|_{C^{\alpha,\alpha/4}_{m-\alpha-4}(Q_T)} \sum_{k=0}^{4}\left\|Q^{-2(k+1)}P_{2k}(\nabla u)\right\|_{C^{\alpha,\alpha/4}_{\max\{0,m-\alpha-4\}}(Q_T)}.
\end{aligned}$$

By Lemma 44 we conclude that for all $\ell, b \in \mathbb{N}_0$

$$\begin{aligned}
\left\|Q^{-\ell}P_b(\nabla u)\right\|_{C^{\alpha,\alpha/4}_{\max\{0,m-\alpha-4\}}(Q_T)} &\le C_{164}\left\|Q^{-\ell}(\nabla u)\right\|_{C^{\alpha,\alpha/4}_{\max\{0,m-\alpha-4\}}(Q_T)}\left\|P_b(\nabla u)\right\|_{C^{\alpha,\alpha/4}_{\max\{0,m-\alpha-4\}}(Q_T)} \\
&\overset{(160)}{\le} C_{164}(1+C_{167}\|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)})^{\ell}(C_{167}\|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)})^{b}.
\end{aligned}$$

Again by Lemma 44: (139) and (140) we obtain

$$\begin{aligned}
&\left\|\mathcal{R}(\nabla u, D^2 u, D^3 u)\right\|_{C^{\alpha,\alpha/4}_{m+\alpha-4}(Q_T)} \\
&\quad \le C_{228}\|D^3 u\|_{C^{1+\gamma,\frac{1+\gamma}{4}}_{m+\gamma-3}(Q_T)}\cdot\|D^2 u\|_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)}\sum_{k=1}^{4}(1+\|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)})^{2k}\|\nabla u\|^{2k-1}_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)} \\
&\qquad + C_{229}\|D^2 u\|^3_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)}\sum_{k=0}^{4}(1+\|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)})^{2(k+1)}\|\nabla u\|^{2k}_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)} \\
&\quad \le C_{168}\|D^2 u\|^3_{C^{2+\gamma,\frac{2+\gamma}{4}}_{m+\gamma-2}(Q_T)}\left(1+\|\nabla u\|_{C^{3+\gamma,\frac{3+\gamma}{4}}_{m+\gamma-1}(Q_T)}\right)^{k_H}.
\end{aligned}$$

□

95 Lemma
Let $0 < \alpha < 1$. If $u, v, w \in C^{4+\alpha,1+\frac{\alpha}{4}}_1(Q_T)$ and $T \le 1$. Then there exists a constant $C_{185} = C_{185}(\Omega)$ such that

(247) $$\|D^3 w D^2 u\|_{C^{\alpha,\alpha/4}_{-3}(Q_T)} \le C_{185}\|D^3 w\|_{C^{1+\alpha,\frac{1+\alpha}{4}}_{-2}(Q_T)}\cdot\|D^2 u\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)}$$

(248)
$$\|D^2 u D^2 w D^2 v\|_{C^{\alpha,\alpha/4}_{-3}(Q_T)} \le C_{185}\|D^2 u\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)}\cdot\|D^2 w\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)}\cdot\|D^2 v\|_{C^{2+\alpha,\frac{2+\alpha}{4}}_{-1}(Q_T)}$$

(249) $$\|\nabla u\|_{C^{\alpha,\alpha/4}_{0}(Q_T)} \le C_{185}\|\nabla u\|_{C^{3+\alpha,\frac{3+\alpha}{4}}_{0}(Q_T)} + \sup_{x\in\overline{\Omega}}\left|\nabla u(x,0)\right|$$

Proof: By definition, it holds

$$\begin{aligned}
\|h\|_{C^{\alpha,\alpha/4}_{-3}(Q_T)} = \sup_{t<T} t^{\frac{3+\alpha}{4}}&\left(\sup_{t'\in[t/2,t]}\left[h(\,,\,,t')\right]_{C^{\alpha}(\overline{\Omega})} + \sup_{x\in\overline{\Omega}}\left[h(x,\,.\,)\right]_{C^{\alpha/4}\left([t/2,t]\right)}\right) \\
&+ \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{3}{4}}\left|h(x,t)\right|.
\end{aligned}$$

First, we prepare for the later estimation for $t' \in [t/2, t]$

$$\left[D^3 w\cdot D^2 u(\,.\,,t')\right]_{C^{\alpha}(\overline{\Omega})} = \sup_{x\in\overline{\Omega}}\left|D^3 w(x,t')\right|\cdot\left[D^2 u(\,.\,,t')\right]_{C^{\alpha}(\overline{\Omega})}$$

$$
\begin{aligned}
&+ \Big[D^3 w(\, . \, , t')\Big]_{C^\alpha(\overline{\Omega})} \cdot \sup_{x\in\overline{\Omega}} \big|D^2 u(x,t')\big| \\
&\le C_{230}(\Omega) \sup_{x\in\overline{\Omega}} \big|D^3 w(x,t')\big| \cdot \sup_{x\in\overline{\Omega}} \big|D^3 u(x,t')\big|^\alpha \cdot \sup_{x\in\overline{\Omega}} \big|D^2 u(x,t')\big|^{1-\alpha} \\
&+ C_{231}(\Omega) \sup_{x\in\overline{\Omega}} \big|D^4 w(x,t')\big|^\alpha \cdot \sup_{x\in\overline{\Omega}} \big|D^3 u(x,t')\big|^{1-\alpha} \cdot \sup_{x\in\overline{\Omega}} \big|D^2 u(x,t')\big|
\end{aligned}
$$

and also we conclude

$$
\begin{aligned}
\Big[D^3 w \cdot D^2 u(x, .)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} &\le \sup_{t'\in[t/2,t]} \big|D^3 w(x,t')\big| \cdot \Big[D^2 u(x, .)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} \\
&\quad + \sup_{t'\in[t/2,t]} \big|D^2 u(x,t')\big| \cdot \Big[D^3 w(x, .)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} \\
&\le t^{\frac{2}{4}} \sup_{t'\in[t/2,t]} \big|D^3 u(x,t')\big| \cdot \Big[D^2 u(x, .)\Big]_{C^{\frac{2+\alpha}{4}}\left([t/2,t]\right)} \\
&\quad + t^{\frac{1}{4}} \sup_{t'\in[t/2,t]} \big|D^2 u(x,t')\big| \cdot \Big[D^3 w(x, .)\Big]_{C^{\frac{1+\alpha}{4}}\left([t/2,t]\right)}.
\end{aligned}
$$

Furthermore, we can estimate

$$
\sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{3}{4}} \big|D^3 w \cdot D^2 u(x,t)\big| \le \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{2}{4}} \big|D^3 w(x,t)\big| \cdot \sup_{(x,t)\in\overline{\Omega}\times(0,T]} t^{\frac{1}{4}} \big|D^2 u(x,t)\big|.
$$

Moreover, we get for $t' \in [t/2, t]$ that it holds

$$
\begin{aligned}
&t^{\frac{3+\alpha}{4}} \Big[D^3 w \;\cdot D^2 u(\, . \, , t')\Big]_{C^\alpha(\overline{\Omega})} \\
&\quad\le C_{232}(\Omega) \sup_{x\in\overline{\Omega}} \big|t^{\frac{2}{4}} D^3 w(x,t')\big| \cdot \big|t^{\frac{2}{4}} D^3 u(x,t')\big|^\alpha \cdot \sup_{x\in\overline{\Omega}} \big|t^{\frac{1}{4}} D^2 u(x,t')\big|^{1-\alpha} \\
&\quad\quad + C_{233}(\Omega) \sup_{x\in\overline{\Omega}} \big|t^{\frac{3}{4}} D^4 w(x,t')\big|^\alpha \cdot \sup_{x\in\overline{\Omega}} \big|t^{\frac{2}{4}} D^3 w(x,t')\big|^{1-\alpha} \cdot \sup_{x\in\overline{\Omega}} \big|t^{\frac{1}{4}} D^2 u(x,t')\big|
\end{aligned}
$$

and in the same way, it follows

$$
\begin{aligned}
t^{\frac{3+\alpha}{4}} \Big[D^3 w \cdot D^2 u(x, .)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} &\le \sup_{t'\in[t/2,t]} \big|t^{\frac{2}{4}} D^3 w(x,t')\big| \cdot t^{\frac{3+\alpha}{4}} \Big[D^2 u(x, .)\Big]_{C^{\frac{2+\alpha}{4}}\left([t/2,t]\right)} \\
&\quad + \sup_{t'\in[t/2,t]} \big|t^{\frac{1}{4}} D^2 u(x,t')\big| \cdot t^{\frac{3+\alpha}{4}} \Big[D^3 w(x, .)\Big]_{C^{\frac{1+\alpha}{4}}\left([t/2,t]\right)}.
\end{aligned}
$$

The proof for $\|D^2 u D^2 w D^2 v\|_{C^{\alpha,\alpha/4}_{-3}(Q_T)}$ is similar. At last, we consider

$$
\begin{aligned}
\|\nabla u\|_{C^{\alpha,\alpha/4}_0(Q_T)} = \sup_{t<T} t^{\alpha/4} &\left(\sup_{t'\in[t/2,t]} \Big[\nabla u(\, , t')\Big]_{C^\alpha(\overline{\Omega})} + \sup_{x\in\overline{\Omega}} \Big[\nabla u(x, \, .)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} \right) \\
&+ \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \big|\nabla u(x,t)\big|.
\end{aligned}
$$

as well as temporal Hölder seminorm term

$$
t^{\alpha/4} \sup_{x\in\overline{\Omega}} \Big[\nabla u(x, \, .)\Big]_{C^{\alpha/4}\left([t/2,t]\right)} \le t^{\alpha/4} \left(\frac{t}{2}\right)^{\frac{3}{4}} \Big[\nabla u(x, .)\Big]_{C^{\frac{3+\alpha}{4}}\left([t/2,t]\right)} \le t^{\frac{3+\alpha}{4}} \Big[\nabla u(x, .)\Big]_{C^{\frac{3+\alpha}{4}}\left([t/2,t]\right)}
$$

and the spatial Hölder seminorm term

$$
t^{\alpha/4} \sup_{t'\in[t/2,t]} \Big[\nabla u(\, , t')\Big]_{C^\alpha(\overline{\Omega})} \le C_{234}(\Omega) \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \Big|t^{\frac{1}{4}} D_x^2 u(x,t)\Big|^\alpha \sup_{(x,t)\in\overline{\Omega}\times(0,T]} \Big|\nabla u(x,t)\Big|^{1-\alpha}.
$$

□

96 Lemma
Let $\beta_i \in \mathbb{N}_0^2$ for each $i = 1, \dots, m$ then

$$\frac{\prod_{i=1}^m D_x^{\beta_i} u}{Q^k(u)} - \frac{\prod_{i=1}^m D_x^{\beta_i} w}{Q^k(w)} = \frac{(\nabla w + \nabla u)}{\Big(Q(w) + Q(u)\Big)} \sum_{\ell=1}^{k} \prod_{i=1}^{m} Q^{\ell-k-1}(w) Q^{-\ell}(u)) D_x^{\beta_i} u (\nabla w - \nabla u)$$
$$- \sum_{\ell=1}^{m} \prod_{i=1}^{\ell-1} \prod_{i=\ell+1}^{m} \frac{D_x^{\beta_i} w D_x^{\beta_i} u}{Q^k(w)} \Big(D_x^{\beta_\ell} w - D_x^{\beta_\ell} u \Big).$$

Proof: It is the same proof as in [Gul17, step 2 in the proof of Lemma 6.8 p. 73] First, consider

$$\begin{aligned}
\frac{\prod_{i=1}^m D_x^{\beta_i} u}{Q^k(u)} - \frac{\prod_{i=1}^m D_x^{\beta_i} w}{Q^k(w)} &= \frac{\prod_{i=1}^m D_x^{\beta_i} u Q^k(w) - \prod_{i=1}^m D_x^{\beta_i} w Q^k(u)}{Q^k(u) Q^k(w)} \\
&= \frac{\big(Q^k(w) - Q^k(u)\big) \prod_{i=1}^m D_x^{\beta_i} u - Q^k(u) \Big(\prod_{i=1}^m D_x^{\beta_i} w - \prod_{i=1}^m D^{\beta_i} u \Big)}{Q^k(u) Q^k(w)}.
\end{aligned}$$

Next, we work with the second part and derive

$$\begin{aligned}
\prod_{i=1}^m D_x^{\beta_i} w - \prod_{i=1}^m D^{\beta_i} u &= \prod_{i=1}^m D_x^{\beta_i} w - \prod_{i=1}^m D^{\beta_i} u + \sum_{\ell=0}^{m} \prod_{i=1}^{\ell} D_x^{\beta_i} w \prod_{i=\ell+1}^{m} D_x^{\beta_i} u - \sum_{\ell=0}^{m} \prod_{i=1}^{\ell} D_x^{\beta_i} w \prod_{i=\ell+1}^{m} D_x^{\beta_i} u \\
&= \sum_{\ell=1}^{m} \prod_{i=1}^{\ell} D_x^{\beta_i} w \prod_{i=\ell+1}^{m} D_x^{\beta_i} u - \sum_{\ell=0}^{m-1} \prod_{i=1}^{\ell} D_x^{\beta_i} w \prod_{i=\ell+1}^{m} D_x^{\beta_i} u \\
&= \sum_{\ell=1}^{m} \left[\prod_{i=1}^{\ell} D_x^{\beta_i} w \prod_{i=\ell+1}^{m} D_x^{\beta_i} u - \prod_{i=1}^{\ell-1} D_x^{\beta_i} w \prod_{i=\ell}^{m} D_x^{\beta_i} u \right] \\
&= \sum_{\ell=1}^{m} \left[\prod_{i=1}^{\ell-1} D_x^{\beta_i} w \prod_{i=\ell+1}^{m} D_x^{\beta_i} u \Big(D_x^{\beta_\ell} w - D_x^{\beta_\ell} u \Big) \right].
\end{aligned}$$

The difference terms can be calculated in the same way

$$\begin{aligned}
Q^k(w) - Q^k(u) &= \sum_{\ell=1}^{k} \left[Q^{\ell-1}(w) Q^{k-\ell}(u) \Big(Q(w) - Q(u) \Big) \right] \\
&= \frac{\Big(Q(w) - Q(u)\Big)\Big(Q(w) + Q(u)\Big)}{\Big(Q(w) + Q(u)\Big)} \sum_{\ell=1}^{k} Q^{\ell-1}(w) Q^{k-\ell}(u) \\
&= \frac{\Big((\nabla w)^2 - (\nabla u)^2\Big)}{\Big(Q(w) + Q(u)\Big)} \sum_{\ell=1}^{k} Q^{\ell-1}(w) Q^{k-\ell}(u) \\
&= \frac{\big(\nabla w - \nabla u\big)\big(\nabla w + \nabla u\big)}{\Big(Q(w) + Q(u)\Big)} \sum_{\ell=1}^{k} Q^{\ell-1}(w) Q^{k-\ell}(u).
\end{aligned}$$

By combining the results, we obtain

$$\frac{\prod_{i=1}^m D_x^{\beta_i} u}{Q^k(u)} - \frac{\prod_{i=1}^m D_x^{\beta_i} w}{Q^k(w)} = \frac{(\nabla w + \nabla u)}{\Big(Q(w) + Q(u)\Big)} \sum_{\ell=1}^{k} \prod_{i=1}^{m} Q^{\ell-k-1}(w) Q^{-\ell}(u)) D_x^{\beta_i} u (\nabla w - \nabla u)$$

$$
-\sum_{\ell=1}^{m}\prod_{i=1}^{\ell-1}\prod_{i=\ell+1}^{m}\frac{D_x^{\beta_i}wD_x^{\beta_i}u}{Q^k(w)}\Big(D_x^{\beta_\ell}w-D_x^{\beta_\ell}u\Big).
$$

□

The next Lemma explains how with embedding, one can get better weight powers for lower derivatives.

97 Lemma
Let $1\le q,p<\infty$, $\alpha,\beta,\gamma\in\mathbb{R}$, $\Omega\subset\mathbb{R}^n$ is quasibounded, $\alpha-\gamma\ge p$ and $q>p$ and $p>n$ or $p<n$ and $p\le q<\frac{pn}{n-p}$ and

$$
\frac{\beta}{q}>\frac{\gamma}{p}\left(1+\frac{n}{q}-\frac{n}{p}\right)+\frac{\alpha}{p}\left(\frac{n}{p}-\frac{n}{q}\right)
$$

Then, the following embedding is compact

$$
W^{1,p}(\Omega;d^\gamma,d^\alpha)\hookrightarrow\hookrightarrow L^q(\Omega;d^\beta)
$$

Proof: [Bro98, p.338 corollary 3.1]. □

98 Lemma (Poincare's Inequality)
Let $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain and $\omega\neq\varnothing$ an open subset of Ω. Further, assume $1<p<\infty$. Then there exists a constant $C_{235}=C_{235}(n,\omega,\Omega,p,a)$ such that

$$
\left\|u-\frac{1}{|\omega|}\int_\omega u(x)\,\mathrm{d}x\right\|_{L^p(\Omega;d^{ap})}\le C_{235}\|Du\|_{L^p(\Omega;d^{ap})}
$$

for each function $u\in W_p^{1,a}(\Omega)$.

Proof: We will use the basic idea of proving unweighted Poincare inequality from [Eva10, p.290 Chapter 5.8.1 Theorem 1], namely arguing by contradiction. First of all, we recall the notation

$$
(u)_\omega=\frac{1}{|\omega|}\int_\omega u(x)\,\mathrm{d}x
$$

for the average of u over ω. By contradiction, we assume there exists for each $k\in\mathbb{N}$ a function $u_k\in W_p^{1,a}(\Omega)$ with

$$
\|u_k-(u_k)_\omega\|_{L^p(\Omega;d^{ap})}>k\|Du_k\|_{L^p(\Omega;d^{ap})}\ge 0.
$$

If we now renormalize by defining

$$
v_k:=\frac{u_k-(u_k)_\omega}{\|u_k-(u_k)_\omega\|_{L^p(\Omega;d^{ap})}},
$$

therefore, we obtain the following properties

$$
(v_k)_\omega=0,\quad \|v_k\|_{L^p(\Omega;d^{ap})}=1,\quad \forall k\in\mathbb{N}:\ \|Dv_k\|_{L^p(\Omega;d^{ap})}<\frac{1}{k}. \tag{250}
$$

Especially, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $W_p^{1,a}(\Omega)$. Since by (103) we have compact embedding $W_p^{1,a}(\Omega)\hookrightarrow\hookrightarrow L^p(\Omega;d^{pa})$, there exist a subsequence $\{v_{k_\ell}\}_{\ell\in\mathbb{N}}$ and a function $v\in L^p(\Omega;d^{ap})$ such that

$$
v_{k_\ell}\to v\quad\text{in } L^p(\Omega;d^{ap}),\qquad\text{thus}\qquad (v)_\omega=0,\quad \|v\|_{L^p(\Omega;d^{ap})}=1. \tag{251}
$$

It follows that for each $\varphi \in C_c^\infty(\Omega)$ and each component $j \in \{1, \dots, n\}$ the condition in (250) implies

$$\int_\Omega \varphi_{x_j} v \,\mathrm{d}x = \lim_{\ell\to\infty} \int_\Omega \varphi_{x_j} v_{k_\ell} \,\mathrm{d}x = -\lim_{\ell\to\infty} \int_\Omega \varphi v_{k_\ell, x_j} \,\mathrm{d}x = 0.$$

Then we get $Dv \equiv 0$ a.e. and thus $v \in W_p^{1,a}(\Omega)$. Since Ω is connected, we conclude that v is constant. Consequently, by $(v)_\omega$ we obtain $v \equiv 0$ a.e., since ω is a open set. Hence $\|v\|_{L^p(\Omega;d^{ap})} = 0$, a contradiction to (251). □

99 Lemma (Equivalence of Weighted Norms)
Let $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain, $1 < p < \infty$ and $\omega \neq \varnothing$ is an open subset of Ω. Then

$$W_p^{m,a}(\Omega) \ni u \mapsto \sum_{|\alpha|=m} \|D^\alpha u\|_{L^p(\Omega;d^{ap})} + \|u\|_{L^p(\omega)}$$

is equivalent to (101) *on $W_p^{m,a}(\Omega)$.*

Proof: Since by (103) the embedding $W_p^{m,a}(\Omega) \hookrightarrow\hookrightarrow W_p^{m-1,a}(\Omega)$ is compact, we can use the Ehrling's Lemma and Young's inequality to show that for all $\varepsilon > 0$ there exists a constant C_ε such that

$$\begin{aligned}\|u\|^p_{W_p^{m-1,a}(\Omega)} &\le \varepsilon \|u\|^p_{W_p^{m,a}(\Omega)} + C_\varepsilon \|u\|^p_{L^p(\Omega;d^{ap})} \\ &\le \varepsilon \|D^m u\|^p_{L^p(\Omega;d^{ap})} + \varepsilon \|u\|^p_{W_p^{m-1,a}(\Omega)} + C_\varepsilon \|u\|^p_{L^p(\Omega;d^{ap})}.\end{aligned}$$

Consequently, we set $\varepsilon = \frac{1}{2}$ and use again Young's inequality to derive

$$\|u\|_{W_p^{m,a}(\Omega)} \le C_{236} \left(\|D^m u\|_{L^p(\Omega;d^{ap})} + \|u\|_{L^p(\Omega;d^{ap})} \right).$$

Next, we observe that by weighted Poincare's inequality formulated in Lemma 98 it follows

$$\begin{aligned}\|u\|_{L^p(\Omega;d^{ap})} &\le C_{235} \left(\|Du\|_{L^p(\Omega;d^{ap})} + \frac{1}{|\omega|} \int_\omega |u|(x) \,\mathrm{d}x \cdot \|1\|_{L^p(\Omega;d^{ap})} \right) \\ &\le C_{237} \left(\|Du\|_{L^p(\Omega;d^{ap})} + \|u\|_{L^p(\omega)} \right)\end{aligned}$$

since ω is also bounded. □

100 Lemma
Let $\mathbb{D}_\delta$ and f_δ be the Cantor set and function defined in Theorem 90. Furthermore, let f_ℓ the functions defined in step 2 of Theorem 90. Then

- *a* $\mathcal{L}^1(\mathbb{D}_\delta) = 0$,
- *b* *Moreover, there exists a function $f_\delta \in C^0([0,1])$ with $f_\ell \to f_\delta$ in $C^0([0,1])$ for $\ell \to \infty$ with $f_\delta(0) = 0$ and $f_\delta(1) = 1$. Also it follows that $f'_\delta = (f'_\delta)^c$, which is supported on $\mathbb{D}_\delta$.*

Proof: ***a*** Since the intervals I_{kj} are non-overlapping, we conclude by countable additivity and $2\delta < 1$ that it holds

$$\begin{aligned}\mathcal{L}^1(\mathbb{D}_\delta) &= 1 - \sum_{j=1}^\infty \sum_{k=1}^{2^{(j-1)}} \mathcal{L}^1(I_{kj}) = 1 - \sum_{j=1}^\infty \sum_{k=1}^{2^{(j-1)}} (1-2\delta)\delta^{j-1} \\ &= 1 - \sum_{j=1}^\infty (2\delta)^{(j-1)}(1-2\delta) = 1 - (1-2\delta)\sum_{j=0}^\infty (2\delta)^j = 1 - \frac{1-2\delta}{1-2\delta} = 0.\end{aligned}$$

$\textcircled{b}$ First, we want to observe the self-similarity property of the construction of f_ℓ. Foremost, let $m \geq \ell$ then it follows that $f_\ell = f_m$ in sets where f_ℓ is constant: $\bigcup_{j=1}^{\ell} \bigcup_{k=1}^{2(j-1)} I_{kj}$ which we will show in the following.

The complementary set $[0,1] \setminus \bigcup_{j=1}^{\ell} \bigcup_{k=1}^{2(j-1)} I_{kj}$ consists of 2^ℓ intervals $J_{i\ell}, i = 1, \dots 2^\ell$ of length δ^ℓ. We want to prove that both f_ℓ and f_m rise to the same level in each $J_{i\ell}$. By the construction of f_m, there are intervals where f_m is constant, which are cut out of in each interval $J_{i\ell}$. The remaining $2^{m-\ell}$ intervals in $J_{i\ell}$ are of length $\delta^{m-\ell} \cdot |J_{i\ell}| = \delta^m$. Here g_m is non-vanishing. We conclude

$$\int_{J_{i\ell}} f_m' \,\mathrm{d}x = \int_{J_{i\ell}} g_m \,\mathrm{d}x = \frac{1}{(2\delta)^m} 2^{m-\ell}\delta^m = \frac{1}{2^\ell} = \frac{\delta^\ell}{(2\delta)^\ell} = \int_{J_{i\ell}} g_\ell \,\mathrm{d}x = \int_{J_{i\ell}} f_\ell' \,\mathrm{d}x.$$

Since f_ℓ and f_m are both constant on $\bigcup_{j=1}^{\ell} \bigcup_{k=1}^{2(j-1)} I_{kj}$, it follows that $f_\ell = f_m$ on $\bigcup_{j=1}^{\ell} \bigcup_{k=1}^{2(j-1)} I_{kj}$.

Furthermore, with the same arguments, we observe that for all $x \in J_{i\ell}, i = 1, \dots 2^\ell$ (in complement the situation is trivial $f_\ell = f_m$)

$$\begin{aligned}
|f_\ell - f_m|(x) &\leq \int_0^x \left| \frac{1}{(2\delta)^\ell} \left(1 - \sum_{j=1}^{\ell} \sum_{k=1}^{2(j-1)} \mathbb{1}_{I_{kj}}(x) \right) - \frac{1}{(2\delta)^m} \left(1 - \sum_{j=1}^{m} \sum_{k=1}^{2(j-1)} \mathbb{1}_{I_{kj}}(x) \right) \right| \mathrm{d}x \\
&\leq \int_{J_{i\ell}} \left| \frac{1}{(2\delta)^\ell} - \frac{1}{(2\delta)^m} \left(1 - \sum_{j=1}^{m} \sum_{k=1}^{2(j-1)} \mathbb{1}_{I_{kj}}(x) \right) \right| \mathrm{d}x \\
&\leq \left| \frac{1}{(2\delta)^\ell} - \frac{1}{(2\delta)^m} \right| 2^{m-\ell}\delta^m + \frac{1}{(2\delta)^\ell} (\delta^\ell - 2^{m-\ell}\delta^m) \\
&< \frac{\left|(2\delta)^m - (2\delta)^\ell\right|}{(2\delta)^{m+\ell}} 2^{m-\ell}\delta^m + \sum_{j=m+1}^{\ell} \sum_{k=1}^{2^{j-1}} \frac{1}{(2\delta)^m} \delta^{j-1}(1-2\delta) \\
&= \left|(2\delta)^{m-\ell} - 1\right| \frac{1}{2^\ell} + \frac{1}{2^\ell} < \frac{3}{2^\ell}
\end{aligned}$$

thus $\forall m \geq \ell : \|f_\ell - f_m\|_{C^0([0,1])} < \frac{3}{2^\ell}$. The sequence $\{f_\ell\}_{\ell \in \mathbb{N}}$ is a Cauchy sequence in $C^0([0,1])$. Therefore there exists a function $f_\delta \in C^0([0,1])$ with $f_\ell \to f_\delta$ in $C^0([0,1])$ for $\ell \to \infty$. Also since $\forall \ell \in \mathbb{N} : f_\ell(0) = 0$ and $f_\ell(1) = 1$ we have $f_\delta(0) = 0$ and $f_\delta(1) = 1$.

On the open set $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{2(j-1)} I_{kj}$ the function f_δ is constant and hence $f_\delta' \equiv 0$ there. Since $\mathcal{L}^1(\mathbb{D}_\delta) = 0$ then $(f_\delta')^a \equiv 0$. Moreover, since f_δ is continuous, there are not any jump parts. Furthermore since $|Df|((0,1)) = f_\delta(1) - f_\delta(0) = 1 > 0$, it follows that

$$f_\delta' = (f_\delta')^c,$$

where the later is supported on $\mathbb{D}_\delta$.

$\square$

101 Lemma

Let $u\colon [0,1] \to \mathbb{R}$ be sufficiently smooth, with associated curvature κ. We assume that

$$\|\kappa\|_{L^1((0,1))}(u) < 1 - \left| \frac{u(1) - u(0)}{\sqrt{1 + \big(u(1) - u(0)\big)^2}} \right|.$$

Then we have the following L^∞-estimate for the derivative u'

$$\|u'\|_{L^\infty} \leq \frac{\sqrt{1 + \big(u(1) - u(0)\big)^2} \|\kappa\|_{L^1((0,1))}(u) + |u(1) - u(0)|}{\sqrt{1 + \big(u(1) - u(0)\big)^2} \big(1 - \|\kappa\|_{L^1((0,1))}(u)\big) - |u(1) - u(0)|}.$$

Proof: This proof is based on unpublished notes of Grunau and Deckelnick. There exists $\xi \in (0,1)$ such that $(u(1)-u(0))/1 = u'(\xi)$. Then

$$\left|\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right| = \left|\int_0^1 \kappa(\tau)\,\mathrm{d}\tau\right| + \left|\frac{u'(\xi)}{\sqrt{1+u'(\xi)^2}}\right|.$$

It follows

$$\left|u'(x)\right| \le \left(\|\kappa\|_{L^1((0,1))} + \left|\frac{u'(\xi)}{\sqrt{1+u'(\xi)^2}}\right|\right)\left(1+\left|u'(x)\right|\right)$$

which finishes the proof. $\square$

102 Lemma (Generalized and Classical Second Fundamental Form)
Let M and N be the isometrical embedded Riemannian submanifolds of $\mathbb{R}^n$ such that M with $m = \dim M$ is embedded in N. Then, the classical second fundamental form of M with respect to N is also the generalized second fundamental form defined in Definition 80. That means for $1 \le i,j,k \le n$

$$\big\langle \mathbf{A}(e_i,e_j), e_k\big\rangle = P_{\ell j}(P_{it}D_tP_{k\ell}) - P_{\ell j}P_{it}D_tQ_{k\ell} = P_{\ell j}B_{ik\ell} - P_{\ell j}P_{iq}D_qQ_{k\ell} \tag{252}$$

Proof: We recall the classical second fundamental form with respect to N at $x \in M$

$$\mathbf{A}\colon T_xM \times T_xM \to N_xM, \qquad \mathbf{A}(v,w) = (D_vw)^\perp,$$

where $z^\perp$ is the projection component into N_xM and the covariant differentiation in $\mathbb{R}^n$ is denoted by D_vw. Next we extend the situation into $\mathbb{R}^n$ by projections of $T_x\mathbb{R}^n$ into T_xM denoted by $T_x\mathbb{R}^n \ni v \mapsto v^T \in T_xM$

$$\mathbf{A}\colon T_x\mathbb{R}^n \times T_x\mathbb{R}^n \to T_x\mathbb{R}^n, \qquad \mathbf{A}(v,w) = \mathbf{A}(v^T,w^T).$$

Now let $\{e_i\}_{i=1}^n$ be the canonical orthonormal basis of $\mathbb{R}^n$. We define the components of A by

$$A_{ij}^k = \big\langle \mathbf{A}(e_i,e_j), e_k\big\rangle, \qquad 1 \le i,j,k \le n.$$

In the case of $\operatorname{codim}_N M = 1$ (codimension relative to N), the space N_xM is one-dimensional and let N be a locally defined normal vector field. Then we can compare with the more convenient definition

$$\mathbf{A}(\tau,\eta) := -\langle \eta, D_\tau N\rangle N, \quad \tau,\eta \in T_x\mathbb{R}^n.$$

Also, in the $\operatorname{codim}_N M = 1$ case, we have the scalar second fundamental form

$$A(\tau,\eta) := -\langle \eta, D_\tau N\rangle = \big\langle \mathbf{A}(\tau,\eta), N\big\rangle. \quad \tau,\eta \in T_x\mathbb{R}^n.$$

Let $x \in M$ and φ a local parametrization of M near x so that $\varphi(0) = x$. In the local formulation, we then have the coordinates

$$A_{ij} = A(D_i\varphi, D_j\varphi) = -\langle D_i\varphi, D_{D_j\varphi}N\rangle = \langle N, D_{ij}\varphi\rangle, \qquad 1 \le i,j \le m.$$

It is important to notice that A_{ij}^k are coordinates relative to the orthogonal base of $\mathbb{R}^n$ defined extrinsically, unlike the local coordinates A_{ij} defined with respect to local parametrization. Nevertheless, we can relate each form to each other over the relation between the Riemannian metric on T_xM and the Euclidian metric on $T_x\mathbb{R}^n$. Let $W, V \in T_xM$ with $W = \sum_i W^i D_i\varphi(0)$ and $V = \sum_j V^j D_j\varphi(0)$, since M is isometrically embedded in $\mathbb{R}^n$ then

$$\begin{aligned}
\langle V, W\rangle_{\mathbb{R}^n} &= \sum_{i,j=1}^m V^i W^j g_{ij}\Big|_0 = \sum_{i,j=1}^m V^i W^j \sum_{k,\ell=1}^m g_{ik} g^{k\ell} g_{\ell j}\Big|_0 = \sum_{k,\ell=1}^m g^{k\ell} \sum_{i=1}^m V^i g_{ik} \sum_{j=1}^m W^j g_{\ell j}\Big|_0 \\
&= \sum_{k,\ell=1}^m g^{k\ell} \Big\langle \sum_{i=1}^m V^i D_i\varphi, D_k\varphi\Big\rangle \Big\langle \sum_{j=1}^m W^j D_j\varphi, D_\ell\varphi\Big\rangle\Big|_0 \qquad (253) \\
&= \sum_{k,\ell=1}^m g^{k\ell} \langle V, \partial_k\varphi\rangle \langle W, D_\ell\varphi\rangle\Big|_0.
\end{aligned}$$

From which one can directly deduce $\ell = 1, \ldots, n : e_\ell^T = \sum_{i,j=1}^m g^{ij}\langle e_\ell, D_i\varphi\rangle D_j\varphi$. Next, we reformulate A_{ik} in terms of A_{st}^ℓ and $D_k\varphi$. For $1 \le i, k \le m$ we get

$$\begin{aligned}
A_{ik} = A(D_i\varphi, D_k\varphi) &= \sum_{t,s=1}^n \langle D_i\varphi, e_t\rangle \langle D_k\varphi, e_s\rangle A(e_t, e_s) \\
&= \sum_{t,s,\ell=1}^n \langle D_i\varphi, e_t\rangle \langle D_k\varphi, e_s\rangle \big\langle \mathbf{A}(e_t, e_s), N^\ell e_\ell\big\rangle \\
&= \sum_{t,s,\ell=1}^n \langle D_i\varphi, e_t\rangle \langle D_k\varphi, e_s\rangle A_{ts}^\ell N^\ell
\end{aligned}$$

since one can define locally $N = D_1\varphi \wedge \cdots \wedge D_m\varphi / \|D_1\varphi \wedge \cdots \wedge D_m\varphi\|$, where wedge product is intrinsically in T_xN. Analogously, we obtain for $1 \le s, t, \ell \le n$

$$\begin{aligned}
A_{st}^\ell = \big\langle A(e_s, e_t)N, e_\ell\big\rangle &= A(e_s^T, e_t^T)N^\ell \\
&= \sum_{i,j,k,r=1}^m g^{ij}\langle e_t, D_i\varphi\rangle A(D_j\varphi, D_k\varphi) g^{kr}\langle e_s, D_r\varphi\rangle N^\ell \\
&= \sum_{i,j,k,r=1}^m g^{ij}\langle e_t, D_i\varphi\rangle g^{kr}\langle e_s, D_r\varphi\rangle A_{jk} N^\ell.
\end{aligned}$$

We can also compare the mean curvature definitions with the norm of the second fundamental form. Let $\{\tau_i\}_{i=1}^n$ a orthonormal basis of $T_x\mathbb{R}^n$ such that $\{\tau_i\}_{i=1}^m$ a orthonormal basis of T_xM then

$$\begin{aligned}
\sum_{i,j=1}^m g^{ij} A_{ji} &= \sum_{i,j=1}^m g^{ij} A\big(D_j\varphi, D_i\varphi\big) = \sum_{i,j=1}^m g^{ij} A\Big(\sum_{k=1}^m \langle D_j\varphi, \tau_k\rangle \tau_k, \sum_{\ell=1}^m \langle D_i\varphi, \tau_\ell\rangle \tau_\ell\Big) \\
&= \sum_{k,\ell=1}^m A(\tau_k, \tau_\ell) \sum_{i,j=1}^m g^{ij}\langle D_j\varphi, \tau_k\rangle \langle D_i\varphi, \tau_\ell\rangle \\
&\overset{(253)}{=} \sum_{k,\ell=1}^m A(\tau_k, \tau_\ell)\langle \tau_k, \tau_\ell\rangle = \sum_{k=1}^m A(\tau_k, \tau_k) = \sum_{k=1}^n A(\tau_k, \tau_k) \\
&= \sum_{k=1}^n A(e_\ell, e_t) \sum_{\ell=1}^n \langle \tau_k, e_\ell\rangle \sum_{t=1}^n \langle \tau_k, e_t\rangle = \sum_{\ell,t=1}^n A(e_\ell, e_t) \sum_{k=1}^n \langle \tau_k, e_\ell\rangle \langle \tau_k, e_t\rangle \\
&= \sum_{\ell=1}^n A(e_\ell, e_\ell) = \sum_{\ell,k=1}^n A_{\ell\ell}^k N^k = H(x).
\end{aligned}$$

Moreover, we check the absolute value $\|A\|_g$. We use again $\{\tau_i\}_{i=1}^n$ orthonormal basis of $T_x\mathbb{R}^n$ such that $\{\tau_i\}_{i=1}^m$ a orthonormal basis of T_xM

$$\begin{aligned}
\sum_{i,j,k,\ell=1}^m g^{ij}g^{k\ell}A_{ik}A_{j\ell} &= \sum_{i,j,k,\ell=1}^m g^{ij}g^{k\ell}A(D_i\varphi, D_k\varphi)A(D_j\varphi, D_\ell\varphi) \\
&= \sum_{i,j,k,\ell=1}^m g^{ij}g^{k\ell}\sum_{r,q=1}^m \langle D_i\varphi, \tau_r\rangle\langle D_k\varphi, \tau_q\rangle A(\tau_r, \tau_q)\sum_{s,t=1}^m \langle D_j\varphi, \tau_s\rangle\langle D_\ell\varphi, \tau_t\rangle A(\tau_s, \tau_t) \\
&= \sum_{r,q,s,t=1}^m A(\tau_r, \tau_q)A(\tau_s, \tau_t)\sum_{i,j=1}^m g^{ij}\langle D_i\varphi, \tau_r\rangle\langle D_j\varphi, \tau_s\rangle\sum_{k,\ell=1}^m g^{k\ell}\langle D_k\varphi, \tau_q\rangle\langle D_\ell\varphi, \tau_t\rangle \\
&= \sum_{r,q,s,t=1}^m A(\tau_r, \tau_q)A(\tau_s, \tau_t)\langle \tau_r, \tau_s\rangle\langle \tau_q, \tau_t\rangle \\
&= \sum_{r,q=1}^m A(\tau_r, \tau_q)A(\tau_r, \tau_q) = \sum_{r,q=1}^n A(\tau_r, \tau_q)A(\tau_r, \tau_q) \\
&= \sum_{i,j,\ell,k=1}^n A(e_i, e_j)A(e_\ell, e_k)\sum_{r=1}^n\langle e_i, \tau_r\rangle\langle \tau_r, e_\ell\rangle\sum_{q=1}^n\langle e_j, \tau_q\rangle\langle \tau_q, e_k\rangle \\
&= \sum_{i,j=1}^n A(e_i, e_j)A(e_i, e_j) = \sum_{i,j,\ell=1}^n (A_{ij}^\ell N^\ell)^2 = \sum_{i,j,\ell=1}^n (A_{ij}^\ell)^2 = \|A\|_g^2(x)
\end{aligned}$$

since $\mathbf{A}$ is parallel to N.

Hutchinson defined the generalized curvature as weak $[P_{i\ell}D_\ell P_{jk}]$, which in classical case is related to $\mathbf{A}$ by [Hut86, 5.1.1. Proposition p.60]

(i) $$A_{ij}^k = P_{\ell j}P_{it}D_t(P_{k\ell} - Q_{k\ell})$$

(ii) $$P_{i\ell}D_\ell P_{jk} = A_{ij}^k + A_{ik}^j + P_{j\ell}P_{it}D_tQ_{\ell k} + P_{k\ell}P_{it}D_tQ_{\ell j}$$

As motivation for his Definition 79 of curvature varifold, Hitchinson used the tangential divergence theorem. We want to recall his calculations [Hut86, p. 61]. First, suppose $U \subset \mathbb{R}^n$ open and $\partial M \cap U = \varnothing$ and $i \in \{1, \dots, n\}$. Additionally, we need a test function

$$\varphi = \varphi(x, P) \in C^1(U \times \mathbb{R}^{n\times n}) \qquad \text{and} \qquad \forall P \in \mathbb{R}^{n\times n}\colon \varphi(\,.\,, P) \in C_0^1(U).$$

Finally, we define the test vector field

$$X(x) = \varphi\big((x, P(x)\big)e_i.$$

By the tangential divergence theorem and $\partial M \cap U = \varnothing$ it follows

$$\begin{aligned}
0 &= \int_M \operatorname{div}_{T_xM}(X^T)\,\mathrm{d}\mathcal{H}^2(x) = \int_M P_{rs}D_s(X^T)^r\,\mathrm{d}\mathcal{H}^2 = \int_M P_{rs}D_s(P_{ir}\varphi)\,\mathrm{d}\mathcal{H}^2 \\
&= \int_M \big[P_{rs}D_s(P_{ir})\varphi + P_{rs}P_{ir}D_s\varphi + P_{rs}P_{ir}D_{jk}^*\varphi D_sP_{jk}\big]\,\mathrm{d}\mathcal{H}^2 \\
&= \int_M \big[P_{is}D_s\varphi + (P_{is}D_sP_{jk})D_{jk}^*\varphi + (P_{rs}D_sP_{ir})\varphi\big]\,\mathrm{d}\mathcal{H}^2
\end{aligned}$$

thus with some relabeling indices we notice that $[P_{i\ell}D_\ell P_{jk}]$ is fulfilling the equation for generalized curvature in Definition 79 (ii) since $P = \operatorname{Tan}^m(\mu_V, x)$ for a.e. (x, P). So that we can set $B_{ijk} =$

$P_{i\ell} D_\ell P_{jk}$ which by [Hut86, 5.2.2. Proposition, p. 62] is V-a.e. unique generalized curvature. Then the classical second fundamental form

$$A_{ij}^k \overset{(i)}{=} P_{\ell j}(P_{it} D_t P_{k\ell}) - P_{\ell j} P_{it} D_t Q_{k\ell} = P_{\ell j} B_{ik\ell} - P_{\ell j} P_{iq} D_q Q_{k\ell}$$

which by Definition 80 is equal to the generalized second fundamental form. □

Glossary

$\|f\|_{C^k(\overline{\Omega})}$ norm on space $C^k(\overline{\Omega})$ 33

$[f]_{C^\alpha(\Omega)}$ Hölder coefficient on Ω 33

$\|f\|_{C^{k+\alpha}(\overline{\Omega})}$ Hölder norm on space $C^{k+\alpha}(\overline{\Omega})$ 33

$\{f\}_{*,\Omega}$ BMO modulo VMO character of a function $f \in L^1(\Omega)$ 73

$\|f\|_*$ $\mathrm{BMO}(\Omega)$ norm of $f \in \mathrm{BMO}(\Omega)$ 74

$\{f\}_{*,\partial\Omega}$ BMO modulo VMO character of a function $f \in L^1(\Omega)$ 73

$[u]_{a,b,\overline{Q}_T}$ "parabolic" semi-norm 112

$\langle u\rangle_{a,\overline{Q}_T}$ "temporal" semi-norm 112

$[u]^*_{b,\overline{Q}_T}$ "spatial" semi-norm 112

$\|u\|^*_{\alpha,\overline{Q}_T}$ "spatial" norm 112

$\|f\|_{L^p(\mu)}$ $:= \left(\int_X |f|^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}$ 123

$(\mu \llcorner B)$ restriction of measure μ on a set B 122

$(\mu \llcorner f)$ restriction of μ on a function f 123

$\hookrightarrow\hookrightarrow$ compact embedding 69

$\preceq$ inequality up to a constant which is then not written 164

$(\,\cdot\,)^\top$ projection onto $\mathrm{d}f_x\,(T_x\Sigma)$ 25

$(\,\cdot\,)^\perp$ projection onto $(\mathrm{d}fx\,(T_x\Sigma))^\perp$ 25

$\star$ arbitrary linear combination of indices contractions for derivatives of u 49

$\langle\,.\,,\,.\,\rangle_{\mathbb{R}^{2+k}}$ inner Euclidean product on $\mathbb{R}^{2+k}$ 25

$\mathbf{A}_p$ second fundamental form of $\mathcal{M}$ in p 26

$\|A\|_g$ amount of the second fundamental form 26

$A_D(u)$ set of points where u is approximately differentiable 131

$\operatorname{ap}\lim_{y\to x} u(y)$ approximate limit 131

A_p scalar second fundamental form in p 26

A_{ij} local representation of the scalar second fundamental form 30

$B^n_\varepsilon(t)$ n-dimensional open ball with the center t and radius ε 73

$B_p^s(\partial\Omega)$ Besov space on boundary 71

$\dot{B}_p^{m-1+s}(\partial\Omega)$ higher-order Besov space on boundary 72

$\mathrm{BMO}(\Omega)$ space of functions of bounded mean oscillations 74

$\mathcal{B}(X)$ set of all Borel subsets of a space X 122

$BV(\Omega)$ space of functions of bounded variation 128

$C^\beta(\partial\Omega)$ Hölder space on boundary 35

$C^k(\Omega)$ space of k-fold differentiable functions on $\Omega \subset \mathbb{R}^n$ 33

$C_c^k(\Omega)$ space of functions $f \in C^k(\Omega)$ with compact support in Ω 33

$C^k(\overline{\Omega})$ functions from $C^k(\Omega)$: all derivatives are continuous continuable on $\overline{\Omega}$ 33

$C^{k+\alpha}(\overline{\Omega})$ space of $k+\alpha$-Hölder functions on $\overline{\Omega} \subset \mathbb{R}^n$ 33

$C_c^0(X)$ space of continuous functions on X with compact support 122

co_f conormal on $f(\partial\Sigma)$ 27

$\mathrm{conv}(A)$ convex hull of a set A 46, 146

$\mathbf{CV}_m(\mathbb{R}^n)$ set of unoriented curvature varifolds 139

$\mathbf{CV}_m^o(\mathbb{R}^n)$ set of oriented curvature varifolds 139

$C_\beta^{\ell,\frac{\ell}{4}}(Q_T)$ weighted parabolic Hölder space of order ℓ and β on Q_T 94

$C_{x,t}^{\ell,\frac{\ell}{4}}(\overline{Q}_T)$ parabolic Hölder space of order ℓ on $\overline{Q}_T$ 94

$\partial^\alpha f \; = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_1}\right)^{\alpha_n} f$ 35

$D_{jk}^*\varphi$ partial derivatives of φ with respect to the variables P_{ij} 138

$\nabla_{\mathbf{tan}}$ tangential gradient 72

$\Delta_{f(\Sigma)}h$ Laplace-Beltrami operator of h 25

$\Delta_{\Gamma(u)}H$ Laplace-Beltrami operator of H on the graph 31

$\Delta_{\mathcal{M}}f$ Laplace-Beltrami operator from f to $\mathcal{M}$ 25

$\nabla^a u$ absolutely continuous part of ∇u with respect to $\mathcal{L}^n$ 128

$\nabla^c u$ Cantor part of ∇u with respect to $\mathcal{L}^n$ 128

$\nabla^j u$ jump part of ∇u with respect to $\mathcal{L}^n$ 128

$\nabla^s u$ singular part of ∇u with respect to $\mathcal{L}^n$ 128

$\partial\Sigma$ boundary of Σ 24

∇^M Levi-Civita connection on M 25

$L^p(\Omega)$ Lebesgue space 34

$L^p(\partial\Omega)$ Lebesgue space on boundary 35

$L^p_{\mathbf{loc}}(\Omega)$ space of locally integrable functions 34

$L^p(X,\mu)$ real vector space of functions $f\colon X \to \overline{\mathbb{R}}$ satisfying $\|f\|_{L^p(\mu)} < \infty$ 123

$L^p(\mu)$ real vector space of functions $f\colon X \to \overline{\mathbb{R}}$ satisfying $\|f\|_{L^p(\mu)} < \infty$ 123

$L^p(\mu;\mathbb{R}^m)$ space of $\mathbb{R}^m$-valued functions with finite $L^p(\mu;\mathbb{R}^m)$-norm 124

$\mathcal{L}_u$ Lebesgue set of u 130

N^a unit upward pointing normal fields of the absolutely continuous part $\nabla^a u$ 145

ν_E generalized inner normal to E 129

ν $: \partial\Omega \to \mathbb{S}^2$ exterior boundary normal 29

Ω $\subset \mathbb{R}^2$ a bounded domain (open, nonempty, and connected subset) 29

ω_k $\mathcal{L}^k$ measure of k-dimensional unit ball 125

$\mathcal{P}(E,\Omega)$ perimeter of E relative to Ω 129

$\pi_\#\mu(K)$ push-forward of a measure 122

$P_\ell(\nabla u)$ $= \nabla u \star \cdots \star \nabla u \quad \ell$-times 49

p' $= p/(p-1)$ the dual exponent of p 69

Q $= \sqrt{1+|\nabla u|^2}$, Jacobian of the area formula for graphs 30

Q^a $= \sqrt{1+|\nabla^a u|^2}$ 143

$\overline{Q}_T$ $= \overline{\Omega} \times [0,T]$, closed time cylinder 94

Q_T $= \overline{\Omega} \times (0,T]$ 94

$q_\# V$ unoriented varifold associated to V by projection onto $G_m(N)$ 137

$\overline{\mathbb{R}}$ $= \mathbb{R} \cup \{\pm\infty\}$ 123

$\mathbb{R}P^2$ $= \mathbb{S}^2/\{\mathrm{Id}, -\mathrm{Id}\}$ real projective plane 136

$\mathbf{RV}_m(\mathbb{R}^n)$ set of unoriented rectifiable varifolds 137

$\mathbf{RV}^o_m(\mathbb{R}^n)$ set of oriented rectifiable varifolds 137

$SBV((a,b))$ subspace af all $BV((a,b))$ functions with vanishing Cantor part 22

Σ smooth surface with or without boundary $\partial\Sigma$ 24

Σ_u $= \{x \in \Omega \mid \lim_{\rho\searrow 0} |\nabla u|\big(B_\rho(x)\big)/\rho^n = \infty\}$ 128

$\mathbb{S}^{n-1}$ $= \big\{y \in \mathbb{R}^n \mid \|y\| = 1\big\}$ 122

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