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TOPOLOGIES FOR FINITE WORDS: COMPATIBILITY WITH THE CANTOR TOPOLOGY

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Abstract. Infinite words are often considered as limits of finite words. As topological methods have been proved to be useful in the theory of ω-languages it seems to be providing to include finite and infinite words into one (topological) space. In most cases this results in a poor topological structure induced on the subspace of finite words. In the present paper we investigate the possibility to link topologies in the space of finite words with a topology in the space of infinite words *via* a natural mapping. A requirement in this linking of topologies consists in the compatibility of the topological properties (openness, closedness *etc.*) of images with preimages and vice versa. Here, we show that choosing for infinite words the natural topology of the Cantor space and the δ-limit as linking mapping there are several natural topologies on the space of finite words compatible with the topology of the Cantor space. It is interesting to observe that besides the well-known prefix topology there are at least two more whose origin is from language theory – centers and supercenters of languages. We show that several of these topologies on the space of finite words fit into a class of $\mathcal{L}$-topologies and exhibit their special properties w.r.t. to the compatibility with the Cantor topology.

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1. Introduction

Topological methods are useful in the theory of ω-languages in connection with proving hierarchy results (*e.g.* [6, 15, 18]). To this end one considers, for a finite alphabet X, the set of all infinite words (ω-words) over X as the infinite product space of the discrete space X. This topology is also known as the Cantor topology. Infinite words may also be viewed as limits of (infinite) increasing families of finite words w.r.t. the prefix ordering. Thus it seems to be providing to include both into the same space. One attempt into this direction was done by Boasson and Nivat [1]. Redziejowski [11] observed that the limit considered in [1] is different from that one used in the theory of ω-automata. Therefore he proposed another topology including finite and infinite words into one space. Recently, in [3] possibilities to extend right topologies generated by partial orders on X^* to topologies on X^ω or $X^* \cup X^\omega$ were investigated.

Each of these concepts seems to have several drawbacks when considering topology in connection with acceptance of ω-words; the first approach yields a trivial topological structure for finite words whereas the other ones for infinite words give topologies others than the topology of the Cantor space:

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The topology considered by Boasson and Nivat is closely related to the product or CANTOR topology of X^ω; on the one hand its restriction to X^ω is the CANTOR topology and, on the other hand the whole space $X^* \cup X^\omega$ is homeomorphic to a closed subset of the CANTOR space $(X \cup \{\bot\})^\omega$ where $\bot \notin X$ is a new letter. However, all finite word languages are open in this topology, in particular, each finite word is an isolated point in this topology.

Redziejowski's topology has all sets consisting of only infinite words (ω-languages) as closed sets, thus providing no information on the complexity of acceptance by topological means.

Finally, the results of [3] show that unless one uses the prefix order $\sqsubseteq$ the limit process extending the topology on X^* yields ambiguous results, more precisely, the limit set may contain more than one element and the topology induced on the space of infinite words may not coincide with the CANTOR topology.

In this paper we investigate the possibility to link topologies on the countable set X^* to the standard topology [6, 15, 18] of the CANTOR space X^ω *via* a mapping preserving closedness and openness of sets.

As a mapping we use the δ-limit introduced in [4]. Because of the countability of X^* we cannot expect to obtain a full topological correspondence between the spaces X^* and X^ω *via* any mapping. The δ-limit does not go beyond the class $\mathbf{G}_\delta$ of the BOREL hierarchy. For the prefix topology, it provides a correspondence between the open and closed sets in X^* and the CANTOR space, respectively, and furthermore, of the (σ, δ)-subsets of X^* as introduced in [14] and subsets of X^ω being simultaneously of type $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$ (see [15], Sect. 2.4).

Among the topologies on X^* we consider here are the topologies induced by the $\mathsf{Anf}_\mathcal{L}$-operators introduced in [7–9]. Particular cases of these topologies are the topologies induced by the centers [1, 2, 9] and supercenters [17] of languages. It turns out that both topologies play a special rôle: the center-topology being the coarsest one having all finite sets closed, and the supercenter-topology being the finest one for which the prefix set $\mathbf{pref}(F)$ of every closed subset $F \subseteq X^\omega$ is the smallest closed set in X^* corresponding to F *via* the δ-limit.

The paper is organised as follows. After the notation used we introduce some topological background emphasising properties of the derived set operator. Then we deal with several topologies for finite and infinite words. Section 4 describes the linking of topologies for finite words with the CANTOR topology for ω-words. A fundamental property for a topology of finite words is the compatibility with the topology of infinite words. The class of $\mathcal{L}$-topologies introduced by Prodinger and Urbanek [7, 9] is the topic of Section 5. In the subsequent part we deal with topologies fulfilling a strengthened compatibility condition. Some of these results were announced in [16]. Finally we show some limitations of $\mathcal{L}$-topologies in respect to compatibility with the CANTOR topology.

2. PRELIMINARIES

2.1. Notation

We introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. Let X be an alphabet of cardinality $|X| \geq 2$, $a, b \in X, a \neq b$. By X^* we denote the set (monoid) of words on X, including the *empty word* e, and X^ω is the set of infinite sequences (ω-words) over X. For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{i \in \mathbb{N}} W^i$ be the *submonoid* of X^* generated by W, and by $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus \{e\}\}$ we denote the set of infinite strings formed by concatenating words in W. Furthermore $|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ $(\eta \in X^* \cup X^\omega)$ by $w \sqsubseteq \eta$.

Further we denote by $B/w := \{\eta : w \cdot \eta \in B\}$ the *left derivative* (*left quotient*) of the set $B \subseteq X^* \cup X^\omega$ generated by the word w.

In the theory of ω-automata, ω-words are introduced as upper bounds of infinite chains of words ordered by the prefix relation "$\sqsubset$"; this is reflected by the following limit operation (see [3, 4, 11, 14, 15]).

The δ-*limit* of a language $W \subseteq X^*$ is defined as

$$W^\delta := \{\xi : \xi \in X^\omega \wedge |\mathbf{pref}(\xi) \cap W| = \aleph_0\}\,. \tag{2.1}$$

2.2. General topology

Usually, a *topology* $\mathcal{T} = (\mathcal{X}, \mathcal{O})$ on a set (space) $\mathcal{X}$ is given by a family of open sets $\mathcal{O} \subseteq 2^{\mathcal{X}}$. Here $\mathcal{O}$ is a family of subsets of $\mathcal{X}$ containing $\mathcal{X}$ and closed under arbitrary (including empty) union and finite intersection.

Following Kuratowski (*cf.* [5, 10]) one can also define a topology *via* a closure operator. A mapping $\alpha : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ is called a *topological closure operator* provided it satisfies the following conditions.

$$\alpha(\emptyset) = \emptyset \tag{2.2}$$
$$\alpha(M) \supseteq M \tag{2.3}$$
$$\alpha(M_1 \cup M_2) = \alpha(M_1) \cup \alpha(M_2) \text{ , and} \tag{2.4}$$
$$\alpha(\alpha(M)) = \alpha(M) \tag{2.5}$$

In view of equation (2.3) the identity in equation (2.5) can be replaced by $\alpha(\alpha(M)) \subseteq \alpha(M)$.

By $\mathcal{T}_\alpha$ we denote the topology $\mathcal{T}_\alpha = (\mathcal{X}, \mathcal{O}_\alpha)$ where $\mathcal{O}_\alpha := \{\mathcal{X} \setminus \alpha(M) : M \subseteq \mathcal{X}\}$.

A topology $\mathcal{T}_1 = (\mathcal{X}, \mathcal{O}_1)$ is *finer*[1] than a topology $\mathcal{T}_2 = (\mathcal{X}, \mathcal{O}_2)$ if $\mathcal{O}_2 \subseteq \mathcal{O}_1$, that is, if every $\mathcal{T}_2$-open set is also $\mathcal{T}_1$-open. This is equivalent to the fact that it holds

$$\alpha_1(M) \subseteq \alpha_2(M) \text{ for all } M \in \mathcal{X} \tag{2.6}$$

where $\alpha_i, i = 1, 2$ is the closure generating the topology $\mathcal{T}_i$.

A closure operator may also be obtained *via* the Cantor-Bendixson *derived set* (set of *accumulation points*) M^d of $M \subseteq \mathcal{X}$ [5], Chapter 1, Section 9, III.

The derived set operator ^{d} has to satisfy the following conditions.

$$\emptyset^d = \emptyset \tag{2.7}$$
$$(M_1 \cup M_2)^d = M_1^d \cup M_2^d \text{ , and} \tag{2.8}$$
$$(M^d)^d \subseteq M^d \tag{2.9}$$

In particular, every closure α is also a derived set operator.

It is readily seen that $\alpha^d(M) := M \cup M^d$ is a topological closure operator. In view of equation (2.9) M^d is closed in $\mathcal{T}_{\alpha^d}$, and more generally, a set M is closed in $\mathcal{T}_{\alpha^d}$ if and only if $M^d \subseteq M$. Via α^d, a derived set operator ^{d} defines a topology on $\mathcal{X}$.

The following properties hold (*cf.* also [5], Ch. 1, Sect. 7, IV and Sect. 9, III).

Property 2.1. *Every M', where $M^d \subseteq M' \subseteq M \cup M^d$, is closed in $\mathcal{T}_{\alpha^d}$.*

Property 2.2. *Let $\gamma : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ be a derived set or closure operator. Then*

$$G \cap \gamma(M) = G \cap \gamma(G \cap M)$$

if G is open in the topology defined by γ.

For the sake of completeness we add a proof.

Proof. The inclusion $\supseteq$ is obvious.

From equation (2.8) we have $\gamma(M) = \gamma(M \cap G) \cup \gamma(M \setminus G) \subseteq \gamma(M \cap G) \cup \gamma(\mathcal{X} \setminus G)$. Since $\mathcal{X} \setminus G$ is closed, $\gamma(\mathcal{X} \setminus G) \subseteq \mathcal{X} \setminus G$, and we have $\gamma(M) \subseteq \gamma(M \cap G) \cup (\mathcal{X} \setminus G)$ whence the reverse inclusion follows *via* intersection with the set G. □

As usual countable unions of closed sets are called $\mathbf{F}_\sigma$*-sets* and countable intersections of open sets are called $\mathbf{G}_\delta$*-sets*.

[1] This includes the case that $\mathcal{T}_1$ and $\mathcal{T}_2$ coincide.

3. Topologies for Words and ω-words

In this section we present some known topologies for finite and infinite words.

3.1. The Cantor topology on X^ω

The first one is the widely investigated Cantor topology.

We consider the set X^ω as a metric space (Cantor space) (X^ω, ρ) of all ω-words over the alphabet X where the metric ρ is defined as follows.

$$\rho(\xi,\eta) := \inf\{|X|^{-|w|} : w \sqsubset \xi \wedge w \sqsubset \eta\} \tag{3.1}$$

Since X is finite, this space is compact, and $\mathcal{C}(F) := \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$ is the *closure* of the set F (smallest closed subset containing F) in (X^ω, ρ), that is, a subset $F \subseteq X^\omega$ is closed if and only if $\mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)$ implies $\xi \in F$. Thus

$$\mathbf{pref}(\mathcal{C}(F)) = \mathbf{pref}(F) \text{ and } \mathcal{C}(F) = \mathbf{pref}(F)^\delta\,. \tag{3.2}$$

The topology can be defined by the metric ρ as in equation (3.1) or, alternatively, by letting $\mathcal{O}_C := \{W \cdot X^\omega : W \subseteq X^*\}$ be the set of open subsets of X^ω.

As (X^ω, ρ) is a metric space the Cantor-Bendixson *derived set* F^{d} of a subset $F \subseteq X^\omega$ can be described as

$$F^{\mathrm{d}} = \big\{\xi : \mathbf{pref}(\xi) \subseteq \{w : w \cdot X^\omega \cap F \text{ is infinite}\}\big\} \tag{3.3}$$

It holds $\mathcal{C}(F) = F \cup F^{\mathrm{d}}$.

We conclude this part with a relation between the δ-limit and $\mathbf{G}_\delta$-sets in Cantor space.

Proposition 3.1 ([4])**.** *A subset $F \subseteq X^\omega$ is a $\mathbf{G}_\delta$-set if and only if there is a language $W \subseteq X^*$ such that $F = W^\delta$.*

3.2. The prefix topology $\mathcal{T}_p$ on X^*

A well-known topology on X^* which resembles the product or Cantor topology is the prefix topology $\mathcal{T}_p$ on X^*, that is, the right topology on X^* derived from the prefix order $\sqsubseteq$ on X^*.

The open subsets in $\mathcal{T}_p$ are of the form $W \cdot X^*$ where $W \subseteq X^*$. As it is easily verified the closure operator defining this topology is the initial word operator $\mathbf{pref}$ assigning to each language $W \subseteq X^*$ its set of prefixes $\mathbf{pref}(W) := \{w : \exists v (v \in W \wedge w \sqsubseteq v)\}$.

It has the following property.

Lemma 3.2. *The prefix topology $\mathcal{T}_p$ is the coarsest topology having all subsets $\mathbf{pref}(F) \subseteq X^*, F \subseteq X^\omega$, closed.*

Proof. This follows from the identity $\mathbf{pref}(W) = \mathbf{pref}(W \cdot a^\omega) \cap \mathbf{pref}(W \cdot b^\omega)$ where $a, b \in X,\ a \neq b$. □

In this topology, however, not every finite set $W \subseteq X^*$ is closed. It holds only the so-called T_0-condition: For every pair $w, v \in X^*, w \neq v$, there is an open set $W \subseteq X^*$ which contains exactly one of the words w or v.

3.3. Topologies on $X^* \cup X^\omega$

In this section we consider several topologies on the space of finite and infinite words $X^* \cup X^\omega$ and its restrictions to the sets X^ω and X^*, respectively.

The first one resembles the Scott topology on $X^* \cup X^\omega$ (*cf.* [13]). Its open sets are of the form $W \cdot (X^* \cup X^\omega)$. The restriction to X^ω is the Cantor topology and the restriction to X^* is the prefix topology $\mathcal{T}_p$.

The topology considered by Boasson and Nivat [1] is closely related to the product or Cantor topology of X^ω; on the one hand its restriction to X^ω is the Cantor topology and, on the other hand the whole space $X^* \cup X^\omega$ is a homeomorphic to a closed subset of the Cantor space $(X \cup \{\bot\})^\omega$ where $\bot \notin X$ is a new letter. However, all finite word languages are open in this topology, in particular, each finite word is an isolated point in this topology. Thus its restriction to X^* is the (trivial) discrete topology.

Redziejowski [11] observed that the limit considered in [1] is different from the one used in the theory of ω-automata. Therefore he proposed another topology on the space of finite and infinite words. Here the closure of a subset $W \cup F \subseteq X^* \cup X^\omega, W \subseteq X^*, F \subseteq X^\omega$, can be described *via* the δ-limit as (*cf.* [11], Property 4.4 (1))

$$\mathrm{cl}(W \cup F) = W \cup W^\delta \cup F .$$

Then subsets of the form $F \subseteq X^\omega$ and $W \cup X^\omega, W \subseteq X^*$, are closed, and, consequently, the restrictions to X^ω and X^*, respectively, are the (trivial) discrete topologies.

4. Linking topologies on X^* to Cantor topology

The open sets in $\mathcal{T}_p$ resemble those in Cantor space. One easily observes, that due to the identity $(W \cdot X^*)^\delta = W \cdot X^\omega$, equation (3.2) and Lemma 3.2 there is a close correspondence between open (closed) subsets in the prefix topology on X^* and Cantor topology, respectively.

4.1. Compatibility of topologies

The connection between the prefix topology on X^* and the Cantor topology on X^ω *via* the δ-limit is shown in [15], Section 2.4. This connection fulfils the following property.

Definition 4.1 (Compatibility)**.** A topology $\mathcal{T} = (X^*, \mathcal{O})$ is *compatible* with the Cantor topology of X^ω provided

1. W^δ is closed (open) if $W \subseteq X^*$ is closed (open, respectively) in $(X^*, \mathcal{O})$.
2. If $F \subseteq X^\omega$ is closed (open) in Cantor space then $F = W^\delta$ for some $W \subseteq X^*$ closed (open, respectively) in $(X^*, \mathcal{O})$.

Therefore we consider topologies on X^* which are linked *via* our δ-limit to the Cantor topology on X^ω. Then every language $W \subseteq X^*$ has as its image the $\mathbf{G}_\delta$-set $W^\delta \subseteq X^\omega$. (In fact, because of the different cardinalities of the spaces X^* and X^ω, we cannot expect to obtain every subset of X^ω as an image.)

Definition 4.1 requires that the image of every open (closed) language $W \subseteq X^*$ is also open (closed), and every open (closed) ω-language $F \subseteq X^\omega$ is the image of an appropriately chosen open (closed) language.

Lemma 4.2. *Let a topology $\mathcal{T}_\alpha$ be compatible with the* Cantor *topology. Then the closure $\alpha : 2^{X^*} \to 2^{X^*}$ satisfies the following inclusions*

$$\mathbf{pref}(W^\delta) \subseteq \mathbf{pref}(\alpha(W)^\delta) \ \subseteq \ \mathbf{pref}(\alpha(W)) \quad \textit{and} \tag{4.1}$$

$$\mathcal{C}(W^\delta) \subseteq \alpha(W)^\delta, \ \textit{for all } W \subseteq X^* . \tag{4.2}$$

Proof. In view of $\mathbf{pref}(V^\delta) \subseteq \mathbf{pref}(V)$ the first inclusion holds for all $\alpha : 2^{X^*} \to 2^{X^*}$ satisfying $W \subseteq \alpha(W)$. The second one follows from the fact that $\alpha(W)^\delta$ is a closed set containing W^δ. □

Moreover, Conditions 1 and 2 of Definition 4.1 are inherited in the following way.

Lemma 4.3. *Let $\hat{\mathcal{T}} = (X^*, \hat{\mathcal{O}})$ be a topology compatible with the* Cantor *topology.*

1. *If the topology $\mathcal{T}_1 = (X^*, \mathcal{O}_1)$ is coarser than $\hat{\mathcal{T}}$ then the δ-image W^δ of every open (closed) subset $W \subseteq X^*$ is open (closed, respectively) in X^ω.*

2. *If the topology $\mathcal{T}_2 = (X^*, \mathcal{O}_2)$ is finer than $\hat{\mathcal{T}}$ then every open (closed) subset $F \subseteq X^\omega$ has an open (closed, respectively) δ-preimage $W \subseteq X^*$.*

Thus, if the topologies $\mathcal{T}_1, \mathcal{T}_2$ are compatible with the Cantor topology then every topology $\mathcal{T}$ which is finer than $\mathcal{T}_1$ and coarser than $\mathcal{T}_2$ is also compatible with the Cantor topology.

Lemma 4.3.2 has the following special instance.

Corollary 4.4. *If a topology $\mathcal{T}$ on X^* is finer than the prefix topology $\mathcal{T}_p$ then every closed subset $F \subseteq X^\omega$ has the closed δ-preimage $\mathbf{pref}(F)$, and every open subset $W \cdot X^\omega$ has the open δ-preimage $W \cdot X^*$.*

This is due to the identity $(W \cdot X^*)^\delta = W \cdot X^\omega$, equation (3.2) and Lemma 3.2 as it was mentioned above.

4.2. (σ, δ)-subsets of X^*

As Proposition 3.1 shows, the δ-limit can map languages only to ω-languages in the Borel-class $\mathbf{G}_\delta$, it is interesting to observe that we can also characterise those $\mathbf{G}_\delta$-sets which are simultaneously $\mathbf{F}_\sigma$-sets via their δ-preimages (see [14]). These subsets of X^* show also some special properties w.r.t. topologies on X^*.

We start with some easily verified properties of the δ-limit defined in equation (2.1).

$$(W \cup V)^\delta = W^\delta \cup V^\delta \tag{4.3}$$

$$\mathbf{pref}(\mathbf{pref}(W^\delta)^\delta) = \mathbf{pref}(W^\delta) \ \subseteq \ \mathbf{pref}(W) \tag{4.4}$$

The identity in equation (4.4) follows from applying equation (3.2) to $F = W^\delta$.

Definition 4.5. A subset $W \subseteq X^*$ is referred to as a (σ, δ)-*subset* of X^* provided for every $\xi \in X^\omega$ one of the sets $\mathbf{pref}(\xi) \cap W$ or $\mathbf{pref}(\xi) \setminus W$ is finite.[2]

Then we have the announced connection to $\mathbf{F}_\sigma$-sets in Cantor space.

Lemma 4.6 ([14], Lem. 12)**.** *A subset $F \subseteq X^\omega$ is simultaneously an $\mathbf{F}_\sigma$- and a $\mathbf{G}_\delta$-set in (X^ω, ρ) if and only if there is a (σ, δ)-subset $W \subseteq X^*$ such that $F = W^\delta$.*

Examples of (σ, δ)-subsets of X^* are languages of the form $\mathbf{pref}(W)$, $W \cdot X^*$, and $W \subseteq X^*$ such that $W^\delta = \emptyset$.

Definition 4.5 is equivalent to the following one

$$W \text{ is a } (\sigma, \delta)\text{-subset of } X^* \text{ if and only if } (X^* \setminus W)^\delta = X^\omega \setminus W^\delta \,. \tag{4.5}$$

We have also the following equivalent property.

Lemma 4.7. *Let $W \subseteq X^*$. Then the following conditions are equivalent.*

1. *A subset $W \subseteq X^*$ is a (σ, δ)-subset of X^*.*
2. *For every $V \subseteq X^*$ the identity $(V \cap W)^\delta = V^\delta \cap W^\delta$ is fulfilled.*
3. *For every $V \subseteq X^*$ the identity $(V \setminus W)^\delta = V^\delta \setminus W^\delta$ is fulfilled.*

Proof. We show first that 1. is equivalent to 2.

The inclusion $(V \cap W)^\delta \subseteq V^\delta \cap W^\delta$ is obvious.

If W is not a (σ, δ)-subset of X^* then there is a $\xi \in W^\delta$ such that $V := \mathbf{pref}(\xi) \setminus W$ is infinite. Then $V^\delta = \{\xi\}$ and, consequently, $V^\delta \cap W^\delta = \{\xi\}$ whereas $(V \cap W)^\delta = \emptyset$.

Let W be a (σ, δ)-subset of X^*, and let $\xi \in W^\delta \cap V^\delta$. Then $\xi \in W^\delta$ implies that $\mathbf{pref}(\xi) \setminus W$ is finite. Since $\mathbf{pref}(\xi) \cap V$ is infinite, $\mathbf{pref}(\xi) \cap V \cap W$ is infinite, too, whence $\xi \in (V \cap W)^\delta$.

As, by definition, the complement of a (σ, δ)-subset is also a (σ, δ)-subset, replacing W by $X^* \setminus W$ shows the equivalence of Items 1 and 3. ∎

[2] There are languages $W \subseteq X^*$ such that both sets $\mathbf{pref}(\xi) \cap W$ and $\mathbf{pref}(\xi) \setminus W$ are infinite.

From equation (3.2) we have $W^\delta \subseteq \mathcal{C}(W^\delta) = \mathbf{pref}(W^\delta)^\delta$. Since $\mathbf{pref}(W^\delta)$ is a (σ,δ)-subset, Lemma 4.7 implies the following.

$$\big(W \cap \mathbf{pref}(W^\delta)\big)^\delta = W^\delta \quad \text{and} \quad \big(W \setminus \mathbf{pref}(W^\delta)\big)^\delta = \emptyset\,. \tag{4.6}$$

Moreover, the following is true.

Proposition 4.8 ([14], Prop. 14)**.** *The class of all (σ,δ)-subsets of X^* is a Boolean algebra.*

We finish this section with a sufficient condition for the compatibility of topologies $\mathcal{T} = (X^*, \mathcal{O})$ which is immediate from equation (4.5).

Lemma 4.9. *Let $\mathcal{T} = (X^*, \mathcal{O})$ be a topology on X^* such that every open set $W \in \mathcal{O}$ is a (σ,δ)-subset of X^*. Then $\mathcal{T}$ is compatible with the* Cantor *topology if W^δ is open for every $W \in \mathcal{O}$ and every open subset $E \subseteq X^\omega$ has an open δ-preimage $V \in \mathcal{O}$.*

5. $\mathcal{L}$-topologies

In this section we investigate a method for defining a class of topologies on X^* which are finer that the prefix topology $\mathcal{T}_p$. These so-called *$\mathcal{L}$-topologies* were introduced by Prodinger [7, 8] using the $\mathsf{Anf}_\mathcal{L}$-operator (see also [9]). Under several conditions this $\mathsf{Anf}_\mathcal{L}$-operator is a derived set operator on X^*, thus defines a topology – the $\mathcal{L}$-topology on X^*. The corresponding closure will be denoted by $\alpha_\mathcal{L}$, that is, $\alpha_\mathcal{L}(W) := W \cup \mathsf{Anf}_\mathcal{L}(W)$ for $W \subseteq X^*$.

5.1. The operator Anf

In [7, 9] Prodinger and Urbanek defined a generalisation of the initial word operator **pref** as follows. Let $\mathcal{L} \subseteq 2^{X^*}$ be a family of languages, $W \subseteq X^*$, and define

$$\mathsf{Anf}_\mathcal{L}(W) := \{w : w \in X^* \wedge W/w \in \mathcal{L}\} \tag{5.1}$$

The operator $\mathsf{Anf}_\mathcal{L}$ has the following properties (see [7, 9, 12]):

$$\mathsf{Anf}_\mathcal{L}(W) \subseteq \mathsf{Anf}_\mathcal{M}(W) \text{ if and only if } \mathcal{L} \subseteq \mathcal{M} \tag{5.2}$$

Property 5.1.
1. $\mathsf{Anf}_\mathcal{L}(W/w) = \mathsf{Anf}_\mathcal{L}(W)/w$
2. *$\mathsf{Anf}_\mathcal{L}$ is monotone if and only if $W \in \mathcal{L}$ and $W \subseteq V$ imply $V \in \mathcal{L}$.*
3. *$\mathsf{Anf}_\mathcal{L}(W \cup V) = \mathsf{Anf}_\mathcal{L}W \cup \mathsf{Anf}_\mathcal{L}V$ if and only if $\mathsf{Anf}_\mathcal{L}$ is monotone and $W \cup V \in \mathcal{L}$ implies $W \in \mathcal{L}$ or $V \in \mathcal{L}$.*

Property 5.2. *The following conditions are equivalent.*

1. $\emptyset \notin \mathcal{L}$
2. $\mathsf{Anf}_\mathcal{L}(\emptyset) = \emptyset$, *and*
3. $\forall W(\mathsf{Anf}_\mathcal{L}(W) \subseteq \mathbf{pref}(W))$

Proof. $1 \to 3$. Let $\emptyset \notin \mathcal{L}$ and $v \notin \mathbf{pref}(W)$. Then $W/v = \emptyset \notin \mathcal{L}$ whence $v \notin \mathsf{Anf}_\mathcal{L}(W)$.
$3 \to 2$. is obvious
$2 \to 1$. We have $e \in \mathsf{Anf}_\mathcal{L}(W)$ if and only if $W \in \mathcal{L}$. Thus $e \notin \mathsf{Anf}_\mathcal{L}(\emptyset) = \emptyset$ implies $\emptyset \notin \mathcal{L}$. □

Next we give some relations to the prefix-operator **pref**.

Proposition 5.3. *Let $\mathcal{L} \subseteq 2^{X^*}$. The following conditions are equivalent.*

1. $\forall W\big(\mathsf{Anf}_\mathcal{L}(W) \neq \emptyset \to W \in \mathcal{L}\big)$

2. $\forall W \forall v \big(W/v \in \mathcal{L} \to W \in \mathcal{L}\big)$
3. $\forall W \big(\mathbf{pref}(\mathsf{Anf}_{\mathcal{L}}(W)) = \mathsf{Anf}_{\mathcal{L}}(W)\big)$

Proof. $1 \leftrightarrow 2$. is Theorem 4.18 of [9].

$3 \to 1$. If $W/v \in \mathcal{L}$ then $v \in \mathsf{Anf}_{\mathcal{L}}(W)$. Now 3. implies $e \in \mathbf{pref}(\mathsf{Anf}_{\mathcal{L}}(W)) \subseteq \mathsf{Anf}_{\mathcal{L}}(W)$ which in turn shows $W \in \mathcal{L}$.

$2 \to 3$. Let $w \in \mathbf{pref}(\mathsf{Anf}_{\mathcal{L}}(W))$. Then there is a $v \in X^*$ such that $w \cdot v \in \mathsf{Anf}_{\mathcal{L}}(W)$, that is, $W/w \cdot v = (W/w)/v \in \mathcal{L}$. Now 2. yields $W/w \in \mathcal{L}$ which in turn shows $w \in \mathsf{Anf}_{\mathcal{L}}(W)$. ☐

We continue with the requirement under which the operator $\mathsf{Anf}_{\mathcal{L}}$ satisfies the condition of equation (2.9) of a derived set operator.

Lemma 5.4. *It holds* $\forall W\, (\mathsf{Anf}_{\mathcal{L}}(\mathsf{Anf}_{\mathcal{L}}(W)) \subseteq \mathsf{Anf}_{\mathcal{L}}(W))$ *if and only if* $\forall W (\mathsf{Anf}_{\mathcal{L}}(W) \in \mathcal{L} \to W \in \mathcal{L})$.

Proof. Let $\mathsf{Anf}_{\mathcal{L}}(W) \in \mathcal{L}$. Consequently, $e \in \mathsf{Anf}_{\mathcal{L}}(\mathsf{Anf}_{\mathcal{L}}(W))$. Then $\mathsf{Anf}_{\mathcal{L}}(\mathsf{Anf}_{\mathcal{L}}(W)) \subseteq \mathsf{Anf}_{\mathcal{L}}(W)$ implies $e \in \mathsf{Anf}_{\mathcal{L}}(W)$. Thus $W \in \mathcal{L}$.

Conversely, let $\mathsf{Anf}_{\mathcal{L}}(W) \in \mathcal{L}$ imply $W \in \mathcal{L}$, and let $v \in \mathsf{Anf}_{\mathcal{L}}(\mathsf{Anf}_{\mathcal{L}}(W))$. Then $\mathsf{Anf}_{\mathcal{L}}(W)/v \in \mathcal{L}$. Since $\mathsf{Anf}_{\mathcal{L}}(W)/v = \mathsf{Anf}_{\mathcal{L}}(W/v)$, we have $\mathsf{Anf}_{\mathcal{L}}(W/v) \in \mathcal{L}$ and, consequently, $W/v \in \mathcal{L}$, that is $v \in \mathsf{Anf}_{\mathcal{L}}(W)$. ☐

5.2. $\mathcal{L}$-topologies

Here we investigate under which conditions an $\mathsf{Anf}_{\mathcal{L}}$-operator has the properties of a derived set operator on X^* (*cf.* Prop. 5.1, 5.2 and Lem. 5.4). The following theorem is an analogue to Theorem 2.3 of [7] for derived set operators.

Theorem 5.5. *A mapping* $\mathsf{Anf}_{\mathcal{L}}$ *is a derived set operator on* X^* *if and only if the following conditions are satisfied.*

1. $\emptyset \notin \mathcal{L}$,
2. $W \in \mathcal{L}$ *and* $W \subseteq V$ *imply* $V \in \mathcal{L}$, *and* $W \cup V \in \mathcal{L}$ *implies* $W \in \mathcal{L}$ *or* $V \in \mathcal{L}$, *and*
3. $\mathsf{Anf}_{\mathcal{L}}(W) \in \mathcal{L}$ *implies* $W \in \mathcal{L}$.

Proof. 1. First $\emptyset \in \mathcal{L}$ if and only if $\mathsf{Anf}_{\mathcal{L}}(\emptyset) = \emptyset$. Thus equation (2.7) holds.

2. Theorem 4.13 of [9] shows that this is equivalent to $\mathsf{Anf}_{\mathcal{L}}(V \cup W) = \mathsf{Anf}_{\mathcal{L}}(V) \cup \mathsf{Anf}_{\mathcal{L}}(W)$ which in turn is equation (2.8).

3. Lemma 5.4 shows that Item 3 is equivalent to equation (2.9). ☐

Simple examples of $\mathcal{L}$-topologies are the prefix topology $\mathcal{T}_p$ and the discrete topology $\mathcal{D} = (X^*, 2^{X^*})$. Here one can choose $\mathcal{L}_p = \{W : W \neq \emptyset\}$, and $\mathcal{L}_{\mathcal{D}} = \emptyset$, respectively. This yields $\mathsf{Anf}_{\mathcal{L}_p} = \mathbf{pref}$ and $\mathsf{Anf}_{\mathcal{L}_{\mathcal{D}}}(W) = \emptyset$, for all $W \subseteq X^*$.

The following shows a sufficient condition for Item 3 of Theorem 5.5.

Proposition 5.6. *If* $\emptyset \neq \mathcal{L}$ *and* $\forall W \forall w (W/w \in \mathcal{L} \to W \in \mathcal{L})$ *then* $\forall W (\mathsf{Anf}_{\mathcal{L}}(W) \in \mathcal{L} \to W \in \mathcal{L})$.

Proof. Let $\mathsf{Anf}_{\mathcal{L}}(W) \in \mathcal{L}$. Then $\mathsf{Anf}_{\mathcal{L}}(W) \neq \emptyset$. Now in view of the equivalence $1 \leftrightarrow 2$ of Proposition 5.3 we obtain $W \in \mathcal{L}$. ☐

Next, we give an example that the condition $\forall W \forall w (W/w \in \mathcal{L} \to W \in \mathcal{L})$ is not necessary.

Example 5.7. Define $\mathcal{L} := \{V : V \cap (X^2)^* \neq \oslash\}$. Then $a \cdot (X^2)^* \notin \mathcal{L}$. Conditions 1 and 2 of Theorem 5.5 are trivially satisfied. To verify Condition 3 we observe that, if $\mathsf{Anf}_{\mathcal{L}} W \in \mathcal{L}$ it contains a word v of even length. Then $W/v \in \mathcal{L}$. Thus W/v contains also a word w of even length, whence $wv \in W$, that is, $W \in \mathcal{L}$.

Since every $\mathcal{L}$-topology is finer than the prefix topology $\mathcal{T}_p$, Lemma 4.3.2 and Corollary 4.4 yield the following.

Proposition 5.8. An $\mathcal{L}$-topology is *compatible* with the CANTOR topology on X^ω if and only if for every $W \subseteq X^*$, the set $\alpha_{\mathcal{L}}(W)^\delta$ is closed and the set $(X^* \setminus \alpha_{\mathcal{L}}(W))^\delta$ is open in CANTOR space.

Finally, we mention that $\mathcal{L}$-spaces can be characterised also *via* their open sets.

Theorem 5.9 ([7], Thm. 2.16)**.** *A topological space $\mathcal{T} = (X^*, \mathcal{O})$ is an $\mathcal{L}$-space if and only if $W \in \mathcal{O}$ implies $w \cdot W, W/w \in \mathcal{O}$ for $w \in X^*$.*

6. Strongly compatible topologies on X^*

In this section we consider topologies on X^* compatible with the CANTOR topology. Several of them can be defined using the apparatus introduced in the preceding section.

6.1. Center and supercenter topologies

Special $\mathsf{Anf}_{\mathcal{L}}$-operators were considered in connection with language-theoretic questions. These were referred to as centers [2] or supercenters [17], respectively, of languages.

Definition 6.1 (Center)**.**

$$\mathsf{center}(W) := \mathsf{Anf}_{\mathcal{L}_c}(W) \text{ , where } \mathcal{L}_c = \{V : V \subseteq X^* \text{ is infinite}\}$$

Definition 6.2 (Supercenter)**.**

$$\mathsf{sctr}(W) := \mathsf{Anf}_{\mathcal{L}_{sc}}(W) \text{ , where } \mathcal{L}_{sc} = \{V : V \subseteq X^* \wedge V^\delta \neq \emptyset\}$$

Definition 6.2 is equivalent to $\mathsf{sctr}(W) = \mathbf{pref}(W^\delta) = \mathbf{pref}(\mathcal{C}(W^\delta))$.

Since the alphabet X is finite, König's infinity lemma shows that $W \subseteq X^*$ is infinite if and only if $\mathbf{pref}(W)$ is infinite which is equivalent to $\mathbf{pref}(W)^\delta \neq \emptyset$. Thus we have

$$\mathsf{center}(W) = \mathsf{sctr}(\mathbf{pref}(W)) = \mathbf{pref}(\mathbf{pref}(W)^\delta)\,. \tag{6.1}$$

Both families $\mathcal{L}_c$ and $\mathcal{L}_{sc}$ satisfy the conditions of Theorem 5.5 and Proposition 5.6. So center and sctr are derived set operators and define topologies $\mathcal{T}_c$ and $\mathcal{T}_{sc}$, respectively, on X^*. As $\mathcal{L}$-topologies the center or supercenter topologies are finer than the prefix topology $\mathcal{T}_p$. Moreover, equation (5.2) shows that $\mathcal{T}_{sc}$ is finer than $\mathcal{T}_c$.

In $\mathcal{T}_{sc}$ every W with $W^\delta = \emptyset$ is closed. The language $a^* \cdot b$ is infinite, $(a^* \cdot b)^\delta = \emptyset$ and $\mathsf{center}(a^* \cdot b) = a^* \not\subseteq a^* \cdot b$. Therefore it is not closed in $\mathcal{T}_c$. This shows that $\mathcal{T}_{sc}$ does not coincide with $\mathcal{T}_c$.

The prefix topology has the property that not every finite subset of X^* is closed. The center topology proves to be the coarsest topology refining $\mathcal{T}_p$ and having all finite sets closed. To this end we show the following.

Theorem 6.3. *If $\alpha : 2^{X^*} \to 2^{X^*}$ is a topological closure such that all $\mathbf{pref}(F), F \subseteq X^\omega$, and all finite sets are closed then*

$$\alpha(W) \subseteq W \cup \mathsf{center}(W)\,.$$

Proof. If all $\mathbf{pref}(F), F \subseteq X^\omega$, are α-closed, then according to Lemma 3.2 all $\mathbf{pref}(W), W \subseteq X^*$, are α-closed, whence $\alpha(W) \subseteq \mathbf{pref}(W)$; moreover, all $u \cdot X^*$ are α-open.

We show $\alpha(W) \setminus \mathsf{center}(W) \subseteq W$.

Let $u \in \alpha(W) \setminus \mathsf{center}(W) \subseteq \mathbf{pref}(W) \setminus \mathsf{center}(W)$. Then $W \cap u \cdot X^*$ is finite. Consequently, $\alpha(W \cap u \cdot X^*) = W \cap u \cdot X^*$. Since $u \cdot X^*$ is α-open, Property 2.2 shows $\alpha(W) \cap u \cdot X^* = \alpha(W \cap u \cdot X^*) \cap u \cdot X^* = W \cap u \cdot X^*$ whence $u \in W$. Thus, $\alpha(W) \setminus \mathsf{center}(W) \subseteq W$. □

Corollary 6.4. *The topology $\mathcal{T}_c$ is the coarsest topology having all subsets* $\mathbf{pref}(F) \subseteq X^*$ *and all finite subsets closed.*

Proof. This follows from equation (2.6) and Theorem 6.3. □

The next lemma shows a connection between the supercenter topology and (σ, δ)-subsets of X^*.

Lemma 6.5. *If a topology $\mathcal{T}$ on X^* is coarser than the* supercenter *topology $\mathcal{T}_{sc}$ then every open or closed set is a (σ, δ)-subset X^*.*

Proof. If $\mathcal{T}$ is coarser than $\mathcal{T}_{sc}$, then every $V \subseteq X^*$ closed in $\mathcal{T}$ is also closed in $\mathcal{T}_{sc}$, that is, is of the form $V = W \cup \mathbf{pref}(W^\delta) = (W \setminus \mathbf{pref}(W^\delta)) \cup \mathbf{pref}(W^\delta)$. Along with all prefix-closed languages, all languages W with $\mathsf{sctr}(W) = \mathbf{pref}(W^\delta) = \emptyset$ are (σ, δ)-subsets of X^*. Now the assertion follows with equation (4.6) and Proposition 4.8. □

The proof of Lemma 6.5 shows also that every set open in the supercenter topology has the form $W \cdot X^* \setminus V$ where $V^\delta = \emptyset$.

Then the identity $(W \cdot X^* \setminus V)^\delta = W \cdot X^\omega$, for $V^\delta = \emptyset$, in connection with Lemma 4.9 shows that $\mathcal{T}_{sc}$ is compatible with the Cantor topology.

6.2. Strong compatibility - characterisation

In every topology finer than the prefix topology $\mathcal{T}_p$ all languages $\mathbf{pref}(F)$ where $F \subseteq X^\omega$ are closed. Moreover, $\mathbf{pref}(F)^\delta = \mathcal{C}(F)$. In this section we are investigating which topologies have the languages $\mathbf{pref}(F)$ as smallest closed sets V yielding $V^\delta = \mathcal{C}(F)$. It turns out that supercenters of languages play a crucial rôle in this respect.

Definition 6.6 (Strong compatibility)**.** A topology $\mathcal{T} = (X^*, \mathcal{O})$ is *strongly compatible* provided $\mathcal{T}$ satisfies

$$\mathbf{pref}(F) = \min_{\subseteq}\left\{\alpha_{\mathcal{T}}(W) : F \subseteq W^\delta\right\} \text{ for all } F \subseteq X^\omega . \tag{6.2}$$

In particular, every $\mathbf{pref}(F), F \subseteq X^\omega$, is closed in $\mathcal{T}$.

Theorem 6.7. *A topology $\mathcal{T}$ on X^* is strongly compatible if and only if the corresponding closure operator $\alpha_{\mathcal{T}}$ satisfies*

$$\mathbf{pref}(W^\delta) \subseteq \alpha_{\mathcal{T}}(W) \subseteq \mathbf{pref}(W) \textit{ for all } W \subseteq X^*. \tag{6.3}$$

Proof. If $\mathcal{T}$ is strongly compatible then Lemma 3.2 shows that $\mathcal{T}$ refines the prefix topology $\mathcal{T}_p$. Hence $\alpha_{\mathcal{T}}(W) \subseteq \mathbf{pref}(W)$; and equation (6.2), for $F = W^\delta$, yields $\mathbf{pref}(W^\delta) \subseteq \alpha_{\mathcal{T}}(W)$.

To prove the converse, we refer to equations (3.2) and (6.3). This shows $\mathbf{pref}(\mathbf{pref}(F)^\delta) \subseteq \alpha_{\mathcal{T}}(\mathbf{pref}(F)) \subseteq \mathbf{pref}(\mathbf{pref}(F)) = \mathbf{pref}(F)$, that is, $\alpha_{\mathcal{T}}(\mathbf{pref}(F))$ is prefix closed. Then $F \subseteq W^\delta$ yields $\mathbf{pref}(F) \subseteq \mathbf{pref}(W^\delta) \subseteq \alpha_{\mathcal{T}}(W)$. Consequently, $\mathbf{pref}(F)$ is the minimum w.r.t. $\subseteq$ of the set $\{\alpha_{\mathcal{T}}(W) : F \subseteq W^\delta\}$. □

In view of equation (2.6) the inequality of Theorem 6.7 shows that a topology $\mathcal{T}$ on X^* is strongly compatible if and only if $\mathcal{T}$ is finer than $\mathcal{T}_p$ and coarser than $\mathcal{T}_{sc}$. Then, since both topologies $\mathcal{T}_p$ and $\mathcal{T}_{sc}$ are compatible with the Cantor topology, Lemma 4.3 yields the following.

Corollary 6.8. *Every strongly compatible topology on X^* is compatible with the* Cantor *topology.*

6.3. Strong compatibility and $\mathcal{L}$-topologies

The so far considered strongly compatible topologies $\mathcal{T}_p, \mathcal{T}_c$ and $\mathcal{T}_{sc}$ are $\mathcal{L}$-topologies. It arises the question whether all strongly compatible topologies are $\mathcal{L}$-topologies. In view of Theorem 6.7 this is equivalent to whether all topologies between the prefix and the supercenter topology are $\mathcal{L}$-topologies.

A further observation is that as a consequence of Lemma 6.5 and Theorem 6.7, the closed and the open sets in every topology strongly compatible with the CANTOR topology are always (σ,δ)-subsets of X^*. We also address the question whether all compatible $\mathcal{L}$-topologies having as open sets (σ,δ)-subsets of X^* are strongly compatible.

In this section we will show that for both instances we find counter-examples. To this end we use the well-known possibility to define topologies *via* their bases (*e.g.* [10], Ch. I, Sects. 2, 2.1):

Property 6.9. *Let $\mathcal{X}$ be a set and $\mathcal{B} \subseteq \mathcal{X}$ be closed under intersection. Then for $\mathcal{O} := \{\bigcup_{M\in\mathcal{A}} M : \mathcal{A} \subseteq \mathcal{B}\}$ the pair $\mathcal{T} = (\mathcal{X},\mathcal{O})$ is a topological space with open sets $\mathcal{O}$.*

Remark 6.10. The set $\mathcal{B}$ in Property 6.9 has the properties of a base of the topological space $\mathcal{T}$, as it generates all its open sets. In topology, however, it is not required that a base be closed under intersection [5, 10].

The first example is a topology on X^* which is strongly compatible with the CANTOR topology but not an $\mathcal{L}$-topology.

Example 6.11. Let $X = \{a, b\}$ and define $\mathcal{B} := \{w \cdot X^* \setminus V : w \in X^* \wedge V \subseteq a^* \cdot b\}$ and the family of open sets $\mathcal{O}$ as in Property 6.9. Then $\mathcal{T} = (X^*,\mathcal{O})$ is a topology finer than the prefix topology $\mathcal{T}_p$ and, since $V^\delta = \emptyset$ for $V \subseteq a^* \cdot b$, coarser than $\mathcal{T}_{sc}$.

Consider $W := b \cdot (X^* \setminus a^* \cdot b)$. Then $b \in W$ and $bb \notin W$. The smallest open set containing the word b is $b \cdot X^* \setminus a \cdot a^* \cdot b$ which contains the word bb. Thus W is not open, and according to Theorem 5.9 the topology $\mathcal{T}$ is no $\mathcal{L}$-topology.

Since the topology $\mathcal{T} = (X^*,\mathcal{O})$ of Example 6.11 is coarser than $\mathcal{T}_{sc}$, its open sets are (σ,δ)-subsets of X^*. Next we provide an example of a compatible but not strongly compatible $\mathcal{L}$-topology having all open sets as (σ,δ)-subsets of X^*.

Lemma 6.12. *There are compatible $\mathcal{L}$-topologies on X^* such that all its open sets are (σ,δ)-subsets of X^* which are not strongly compatible with the* CANTOR *topology.*

Proof. We construct an $\mathcal{L}$-topology $\mathcal{T} = (X^*,\mathcal{O})$ such that every open set $W \in \mathcal{O}$ and hence also every closed set is a (σ,δ)-subset. To this end we use Theorem 5.9 and Property 6.9.

We let $X = \{a, b\}$ and $\mathcal{B} = \big\{w \cdot X^* \setminus U \cdot b \cdot a^* : w \in X^* \wedge U \subseteq X^* \text{ finite}\big\}$ be a base of $\mathcal{T}$.

Then every open set has the form[3]

$$W = \bigcup\nolimits_{i\in M} (w_i \cdot X^* \setminus U_i \cdot b \cdot a^*),\ M \subseteq \mathbb{N}\,. \tag{6.4}$$

First we show that $\mathcal{O}$ is closed under the operations $w\,\cdot$ and $/w$. Closure under premultiplication with a word is trivial. It remains to show that $(w \cdot X^* \setminus U \cdot b \cdot a^*)/v = (w \cdot X^*/v) \setminus (U \cdot b \cdot a^*)/v$ is a union of sets of the given shape. If $v \notin \mathbf{pref}(w \cdot X^*)$, $(w \cdot X^* \setminus U \cdot b \cdot a^*)/v = \emptyset$. Otherwise, $w \cdot X^*/v = w' \cdot X^*$ for a suffix w' of w. Moreover, observe that

$$(U \cdot b \cdot a^*)/v = \begin{cases} (U/v) \cdot b \cdot a^*\ , & \text{if } v \notin U \cdot b \cdot a^* \text{ and} \\ (U/v) \cdot b \cdot a^* \cup a^*\ , & \text{otherwise.} \end{cases}$$

In the former case $(w \cdot X^* \setminus U \cdot b \cdot a^*)/v = w' \cdot X^* \setminus (U/v) \cdot b \cdot a^*$ is of the required form. In the latter case we get $w' \cdot X^* \setminus a^* = \bigcup_{v'\in w'\cdot a^*\cdot b} v' \cdot X^*$ and, consequently, $(w \cdot X^* \setminus U \cdot b \cdot a^*)/v = \bigcup_{v'\in w'\cdot a^*\cdot b} (v' \cdot X^* \setminus (U/v) \cdot b \cdot a^*)$ is also in $\mathcal{O}$.

Next, we show that every open set $W \subseteq X^*$ is a (σ,δ)-subset of X^*, that is, for $\xi \in W^\delta$ we have to show that $\mathbf{pref}(\xi) \setminus W$ is finite. If $\xi \in W^\delta$ in view of equation (6.4) there is an $i \in \mathbb{N}$ such that $\xi \in w_i \cdot X^\omega$. Since U_i is finite, $\xi \notin U_i \cdot b \cdot a^\omega$ implies that $\mathbf{pref}(w_i) \cup (\mathbf{pref}(\xi) \setminus W) \subseteq \mathbf{pref}(\xi) \cap U_i \cdot b \cdot a^*$ is also finite.

[3]Observe that the union in equation (6.4) is always a countable one.

If $\xi \in U_i \cdot b \cdot a^\omega$ then $\xi = u \cdot b \cdot a^\omega$ for some $u \in U_i$. Since $\mathbf{pref}(\xi) \cap W$ is infinite, there is a $j \in \mathbb{N}$ such that $\mathbf{pref}(\xi) \cap (w_j \cdot X^* \setminus U_j \cdot b \cdot a^*)$ contains a word $|w| \geq |u| + 2$. Thus $u \cdot b \cdot a \sqsubseteq w \sqsubset \xi$.

For $w \in u \cdot b \cdot a^*$ and $w \notin U_j \cdot b \cdot a^*$ it holds $u \notin U_j$. Thus $u \cdot b \cdot a^* \cap U_j \cdot b \cdot a^* = \emptyset$ whence $\mathbf{pref}(u) \cup (\mathbf{pref}(\xi) \cap U_j \cdot b \cdot a^*) \supseteq \mathbf{pref}(\xi) \setminus W$ is finite.

For the compatibility of $\mathcal{T} = (X^*, \mathcal{O})$ with the Cantor topology, in view of Lemma 4.9, it suffices to show that W^δ is open if W is given by equation (6.4). First observe that $W^\delta \subseteq \bigcup_{i\in\mathbb{N}} w_i \cdot X^\omega$ and $w_i \cdot X^\omega \supseteq (W^\delta \cap w_i \cdot X^\omega) \supseteq w_i \cdot X^\omega \setminus U_i \cdot b \cdot a^\omega$. Thus $w_i \cdot X^\omega \setminus W^\delta$ is finite, hence $w_i \cdot X^\omega \cap W^\delta$ is open. Consequently $W^\delta = \bigcup_{i\in\mathbb{N}}(w_i \cdot X^\omega \cap W^\delta)$ is also open.

Finally, the set $b \cdot a^*$ is closed in $\mathcal{T}$ and $(b \cdot a^*)^\delta = \{b \cdot a^\omega\}$ but $\mathbf{pref}(\{b \cdot a^\omega\}) \not\subseteq b \cdot a^*$. Thus $\mathcal{T} = (X^*, \mathcal{O})$ is not strongly compatible with the Cantor topology. □

7. Miscellaneous

In this section we first present examples of topologies which are compatible with the Cantor topology but not strongly compatible. Then we consider $\mathcal{L}$-topologies related to the Cantor-Bendixson derived set in Cantor space.

7.1. Some examples

The first one is coarser than the prefix topology, and the second one is a refinement of the first one incomparable with the prefix topology and the center topology. For both topologies the open (and closed) sets are (σ, δ)-subsets.

Example 7.1. Define the topology $\mathcal{T}_{2p}$ by $\mathcal{O}_{2p} := \{W \cdot X^* : W \subseteq (X^2)^*\}$. Then every open set is also a (σ, δ)-subset of X^*.

The set $a \cdot X^* = \{a\} \cup (a \cdot X) \cdot X^*$ is not open. Thus $\mathcal{T}_{2p}$ is strictly coarser than $\mathcal{T}_p$. Since $W \cdot X^\omega = (W \cdot X^* \cap (X^2)^*) \cdot X^\omega = ((W \cdot X^* \cap (X^2)^*) \cdot X^*)^\delta$, Lemma 4.3.1 shows that $\mathcal{T}_{2p}$ is compatible with the Cantor topology.

Lemma 7.2. *Let $\mathcal{B}_{2sc} := \{W_i \cdot X^* \setminus V_i : W_i \subseteq (X^2)^* \wedge V_i^\delta = \emptyset\}$ and define $\mathcal{O}_{2sc}$ as in Property 6.9. Then $\mathcal{B}_{2sc}$ is closed under finite intersection and $\mathcal{O}_{2sc}$ consists solely of (σ, δ)-subsets of X^*.*

Proof. Closure under intersection follows from the identities $(W_i \cdot X^* \setminus V_i) \cap (W_j \cdot X^* \setminus V_j) = (W_i \cdot (X^2)^* \cap W_j \cdot (X^2)^*) \cdot X^* \setminus (V_i \cup V_j)$ and $(V_i \cup V_j)^\delta = \emptyset$.

Finally, since $\mathcal{O}_{2sc} \subseteq \mathcal{O}_{sc}$, all open sets are (σ, δ)-subsets of X^*. □

Example 7.3. Lemma 7.2 shows that $\mathcal{O}_{2sc}$ is a family of open sets consisting solely of (σ, δ)-subsets. Assume $a \cdot X^* = \{a\} \cup (a \cdot X) \cdot X^*$ to be open. Then $a \cdot X^* = \bigcup_{i\in I}(W_i \cdot X^* \setminus V_i)$ implies that $e \in W_i \subseteq (X^2)^*$ for some $i \in I$, that is $a \cdot X^* \supseteq X^* \setminus V_i$. This contradicts $a \cdot X^\omega = (a \cdot X^*)^\delta \subsetneq (X^* \setminus V_i)^\delta = X^\omega$.

Thus the topology $\mathcal{T}_{2sc}$ is not finer than the prefix topology $\mathcal{T}_p$. Since every U with $U^\delta = \emptyset$ is closed in $\mathcal{T}_{2sc}$, in particular, every finite set is closed in $\mathcal{T}_{2sc}$, the topology is not coarser than $\mathcal{T}_c$.

7.2. Cantor-Bendixson-topology

In this section we first present two examples of $\mathcal{L}$-topologies which are strictly finer than the supercenter topology. Thus they are not strongly compatible. The first example is related to the Cantor-Bendixson derived set in Cantor space and compatible with the Cantor topology. The second one refers to the Cantor-Bendixson Theorem as it is concerned with condensation points in W^δ, that is, points $\xi \in X^\omega$ for which every $w \cdot X^\omega \cap W^\delta, w \sqsubset \xi$, is uncountable.

Theorem 7.4. *Let $\mathcal{L}_\infty := \{W : W^\delta$ is infinite$\}$. Then $\mathsf{Anf}_{\mathcal{L}_\infty}$ is a derived set operator on X^* and the topology defined by $\mathsf{Anf}_{\mathcal{L}_\infty}$ is compatible with the Cantor topology.*

Moreover, $\mathsf{Anf}_{\mathcal{L}_\infty}(W)^\delta = \emptyset$ whenever W^δ is finite.

Proof. It is readily seen that $\mathsf{Anf}_{\mathcal{L}_\infty}$ satisfies Conditions 1 and 2 of Theorem 5.5 and the hypotheses of Proposition 5.6. Thus $\mathsf{Anf}_{\mathcal{L}_\infty}$ is a derived set operator which, according to Proposition 5.3, is prefix closed.

Now, in view of Proposition 5.8 it suffices to show that, for the closure $\alpha_{\mathcal{L}_\infty}$ of the topology generated by $\mathsf{Anf}_{\mathcal{L}_\infty}$, the ω-languages $\alpha_{\mathcal{L}_\infty}(W)^\delta$ are closed, and $(X^* \setminus \alpha_{\mathcal{L}_\infty}(W))^\delta$ are open in Cantor space.

Observe that $w \in \mathsf{Anf}_{\mathcal{L}_\infty}(W)$ is equivalent to W^δ/w is infinite which in turn is equivalent to $W^\delta \cap w \cdot X^\omega$ is infinite. Thus, $\xi \in \mathsf{Anf}_{\mathcal{L}_\infty}(W)^\delta$ if and only if $W^\delta \cap w \cdot X^\omega$ is infinite for all $w \in \mathbf{pref}(\xi)$, that is, in view of equation (3.3) $\mathsf{Anf}_{\mathcal{L}_\infty}(W)^\delta$ is the Cantor-Bendixson derived set of W^δ. This shows that $\alpha_{\mathcal{L}_\infty}(W)^\delta = W^\delta \cup \mathsf{Anf}_{\mathcal{L}_\infty}(W)^\delta = \mathcal{C}(W^\delta)$ is closed.

It remains to prove that $(X^\omega \setminus \alpha_{\mathcal{L}_\infty}(W))^\delta$ is open. To this end we use again the fact that $\mathsf{Anf}_{\mathcal{L}_\infty}(W)^\delta$ is the Cantor-Bendixson derived set of W^δ and $\mathsf{Anf}_{\mathcal{L}_\infty}(W) = \mathbf{pref}(\mathsf{Anf}_{\mathcal{L}_\infty}(W))$ is a (σ, δ)-subset of X^*.

The latter implies $\mathsf{Anf}_{\mathcal{L}_\infty}(W)^\delta = X^\omega \setminus (X^* \setminus \mathsf{Anf}_{\mathcal{L}_\infty}(W))^\delta$. Then $\mathsf{Anf}_{\mathcal{L}_\infty}(W)^\delta \subseteq X^\omega \setminus (X^* \setminus \alpha_{\mathcal{L}_\infty}(W))^\delta \subseteq \alpha_{\mathcal{L}_\infty}(W)^\delta = \mathcal{C}(W^\delta)$, and according to Property 2.1 the set $X^\omega \setminus (X^* \setminus \alpha_{\mathcal{L}_\infty}(W))^\delta$ is closed.

The second assertion is obvious. □

Example 7.5. To see that the $\mathcal{L}_\infty$-topology is not strongly compatible we remark that the $\mathcal{L}_\infty$-closed set $\{ab\}^*$ is not a (σ, δ)-subset of X^*.

Indeed, since $(\{ab\}^*)^\delta$ is finite, we have $\alpha_{\mathcal{L}_\infty}(\{ab\}^*) = \{ab\}^*$.

Finally, we give a non-trivial example[4] of an $\mathcal{L}$-topology not compatible with the Cantor topology.

Example 7.6. Let $\mathcal{L}_{CB} := \{W : W^\delta \text{ is uncountable}\}$. As in the case of $\mathcal{L}_\infty$ we prove that $\mathsf{Anf}_{\mathcal{L}_{CB}}$ is a derived set operator. Here $(\mathsf{Anf}_{\mathcal{L}_{CB}}(W))^\delta = \{\xi : \forall w(w \sqsubset \xi \to w \cdot X^\omega \cap W^\delta \text{ is uncountable})\}$ is the set of condensation points of W^δ in Cantor-space (*cf.* [5], Ch. 2, Sect. 23, III).

As $\mathsf{Anf}_{\mathcal{L}_{CB}}(a^*ba^*) = \emptyset$ the language a^*ba^* is closed, but $(a^*ba^*)^\delta = a^*ba^\omega$ is not closed in Cantor space.

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