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Research Paper

Collocation methods for nonlinear Volterra integral equations with oscillatory kernel

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ABSTRACT

This work is devoted to the numerical solution of second kind nonlinear Volterra integral equations with highly oscillatory kernel. We use a collocation approach by discretizing the oscillatory integrals in the collocation equation using a Filon-type quadrature rule. We investigate the convergence of the numerical method in terms of step length h and frequency ω. As h decreases, the suggested technique converges with order d, while its asymptotic order as the frequency increase, is at least 1 and may reach 2 in some cases. Numerical experiments validate theoretical results.

1. Introduction

Numerous mathematical problems in physics, biology, and engineering can be modeled by integral equations. These equations cannot often be solved analytically. Thus, numerical solution of integral equations has attracted considerable interest [1,6,10,11,17]. Among them, Volterra Integral Equations (VIEs) with highly oscillatory kernels are important for various applications. Due to the oscillation factor, standard techniques may be costly in this situation. Therefore, specific numerical approaches are needed to discretize the highly oscillatory integral of the VIEs. For example, the steepest descent approach [12], the Filon type method [13], the exponential fitting method [14], the Levin type method [15], and other approaches [7,18,21,22] have been considered in the literature.

VIEs with periodic solutions can be treated as oscillatory problems as well. Some research has been reported on numerical approaches for VIEs with periodic solutions. The mixed collocation method, which differs from the polynomial collocation approach, is one of them. It includes additional trigonometric requirements for approximation solutions. The mixed collocation approach was developed by Brunner [2] to solve problems with periodic solutions. Cardone et al. also developed the Exponential Fitting (EF) approach, which solved these VIEs using the EF quadrature formula [5]. Zhao et al. propose the EF collocation method for VIEs based on EF interpolation [26].

In actuality, numerical solutions to integral equations with highly oscillatory kernels are the subject of very few publications. Xiang et al. studied the first form of VIEs with a Bessel kernel [23]. In order to process the oscillatory integration of the solution, they employed analytical expressions and a Filon-type technique to acquire the results. The author in [25] obtained a numerical solution by using a Filon-type method directly to the integral problem. In [16], the authors employed an improved Levin approach for solving Fredholm oscillatory integral equations.

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There exists a category of numerical methods whose error acts asymptotically like a negative frequency power ω. Such methods have asymptotic order α if their error is $O(\omega^{-\alpha})$ for $\omega \gg 1$, where α can be any positive number. Xiang et al. presented linear collocation techniques based on the Filon method for weakly singular VIEs of the second class with Bessel kernel in [24]. They confirmed that the methods have an asymptotic order. Asymptotic order is desirable for highly oscillatory problems. Additionally, they demonstrated that the piecewise ones exhibit classical order. Zhao et al. proposed collocation methods based on the Filon method for the second kind of VIEs with an oscillatory kernel in [27]. Based on an asymptotic solution analysis, they studied the convergence of the technique. For other schemes based on the Filon type technique, please refer to [9,19] and the references therein. Notably, the focus of the abovementioned articles is mainly on the asymptotic order, i.e., on the relationship between the error and the frequency. Less focus has been on the exact relationship between error and step size.

In this study we consider second-kind nonlinear Volterra Integral Equations (VIEs) with an highly oscillatory kernel outlined below:

$$u(t) = f(t) + \int_0^t K(t,s,u(s))e^{i\omega g(t,s)}ds, \quad (\omega \gg 1), t \in I = [0,T]\ (T < \infty) \tag{1.1}$$

where $u(t)$ is the unknown function and $f(t)$ is a given continuous function on I. The function $K = K(t,s,u(s))$ is assumed to be defined and continuous on $\Omega_B := \{(t,s,u) :\ (t,s) \in D,\ u \in \mathbb{R}\ and\ |u - f(t)| \le B, \}$, where $D := \{(t,s) : 0 \le s \le t \le T\}$. The oscillating frequency ω is a real positive fixed parameter. Brunner [3,4] has studied the oscillatory behavior of solutions for separable oscillators, i.e., $g(t,s) = g_0(t) - g_0(s)$. In this paper, we are concerned with the NVIEs for $g(t,s) = s$.

Following the approach of [1] and based on a similar strategy extended to solve VIEs with the oscillatory kernel in [27], we employ the conventional collocation approach with predetermined collocation points for (1.1) to develop approaches for such highly oscillatory cases. The oscillatory integrals in the exact collocation are then discretized using a Filon-type method to produce an utterly discrete scheme. The theoretical part examines the asymptotic property. The error estimates for exact and discrete collocation are then calculated. Our results demonstrate the combined influence of step size h and frequency ω on error. The approach converges with step site h, and adding collocation points enhances the classical order. The approach has an asymptotic order if the frequency is substantial. Numerical tests support the theoretical results.

The remaining sections of the paper are structured as follows: Section 2 describes the collocation approach for NVIEs of the second kind and applies the Filon technique for NVIEs. Section 3 contains theorems and lemmas that are useful for analyzing the solution's asymptotic property and the approach's convergence. Section 4 illustrates numerical experiments. Finally, Section 5 discusses conclusions.

2. Collocation method for second kind of NVIE

The exact collocation method and its fully discrete version are presented in this section. The first description is based on Brunner's classical approach [1]. The integrals in the exact collocation are then discretized using a Filon-type technique. Just for simplicity, we will use a uniform mesh. Discretize the interval $I = [0,T]$ by

$$I_h := \left\{ t_n := nh,\ n = 0, ..., N,\ h \ge 0, Nh = T \right\}. \tag{2.2}$$

Let $\sigma_n := (t_n, t_{n+1}]$. Define the collocation points

$$T_h := \left\{ t = t_{n,j} := t_n + c_j h,\ \ 0 \le c_1 \le ... \le c_m \le 1\ (0 \le n \le N-1) \right\}, \tag{2.3}$$

with c_j being collocation parameters. Now, we aim to identify a collocation solution for (1.1) in the space of piecewise polynomials.

$$S^{(-1)}_{m-1}(I_h) := \left\{ P(s) :\ P(s)|_{\sigma_n} \in \pi_{m-1}\ (0 \le n \le N-1) \right\}, \tag{2.4}$$

where π_{m-1} represents the space of all polynomials of degree less than or equal to $m-1$. Brunner discusses how selecting $c_1 = 0$ and $c_m = 1$ would result in a continuous numerical solution on I [1].

2.1. The exact collocation scheme

The collocation solution $u_h \in S^{(-1)}_{m-1}(I_h)$, for (1.1) is specifically defined:

$$u_h(t) = f(t) + \int_0^t K(t,s,u_h(s))e^{i\omega s}ds, \quad for\, t \in T_h. \tag{2.5}$$

From another point of view, if we consider $U_{n,i} := u_h(t_{n,i})$, the collocation $u_h(t)$ on σ_n could also be expressed as

$$u_h(t) = u_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) U_{n,j}, \quad s \in (0,1], \tag{2.6}$$

where $L_j(s)$ represents the Lagrange basis functions.

$$L_j(s) := \prod_{k\neq j}^{m} \frac{s-c_k}{c_j-c_k}. \tag{2.7}$$

For $t = t_{n,j}$ the collocation equation (2.5) can be expressed as follows:

$$\begin{aligned} u_h(t) &= f(t) + \int_0^{t_n+c_jh} K(t,s,u_h(s))e^{i\omega s}ds \\ &= f(t) + \sum_{l=0}^{n-1}\int_{t_l}^{t_{l+1}} K(t,s,u_h(s))e^{i\omega s}ds + \int_{t_n}^{t_n+c_jh} K(t,s,u_h(s))e^{i\omega s}ds \\ &= f(t) + \sum_{l=0}^{n-1} h\, e^{i\omega t_l}\int_0^1 K(t,t_l+sh,u_h(t_l+sh))e^{i\omega sh}ds + h\, e^{i\omega t_n}\int_0^{c_j} K(t,t_n+sh,u_h(t_n+sh))e^{i\omega sh}ds \end{aligned} \tag{2.8}$$

Putting the local representation (2.6) of u_h into (2.8) and expressing it in terms of $U_{n,j}$ yields

$$\begin{aligned} U_{n,j} &= f(t_{n,j}) + \sum_{l=0}^{n-1} h\, e^{i\omega t_l}\int_0^1 K(t_{n,j},t_l+sh,\sum_{k=1}^{m} L_k(s)\, U_{l,k})e^{i\omega sh}ds + \\ & h\, e^{i\omega t_n}\int_0^{c_j} K(t_{n,j},t_n+sh,\sum_{k=1}^{m} L_k(s)\, U_{n,k})e^{i\omega sh}ds \end{aligned} \tag{2.9}$$

2.2. The fully discrete scheme

Due to the highly oscillatory integrals, the scheme in the last section could not always applicable in practice. We need a fully discrete system ready to utilize for numerical simulation from a computational viewpoint. In order to deal with the highly oscillatory integral, we use a Filon-type technique. We recommend citing for additional information about oscillatory quadrature [8,12,13,22,26]. The new collocation equation considers

$$\hat{u}_h(t) = f(t) + \int_0^{t_n+c_jh} K(t,s,\hat{u}_h(s))e^{i\omega s}ds, \quad t \in T_h, \tag{2.10}$$

where $\hat{u}_h(t) \in S_{m-1}^{(-1)}(I_h)$ is the fully discrete collocation solution and $\int_0^t K(t,s,\hat{u}_h(s))e^{i\omega s}ds$ is the Filon-type approximation of $u_h(t) = f(t) + \int_0^{t_n+c_jh} K(t,s,u_h(s))e^{i\omega s}ds$. In fact, we consider quadrature rule of the form

$$\int_0^{\nu} K(t,t_n+sh,u_h(t_n+sh))e^{i\omega sh}ds \approx \sum_{j=1}^{m} w_j(\nu)\, K(t,t_n+\nu c_jh,u_h(t_n+\nu c_jh)) \tag{2.11}$$

where $w_j(\nu) := \nu\int_0^1 e^{i\omega\nu sh}ds$ can be computed by using the incomplete Gamma function [13]. Replace the integrals in (2.8) with the previous quadrature approximations and disregard the quadrature errors to obtain the appropriate fully discrete collocation equation. The local representation of $\hat{u}_h$ on σ_h, in analogy to (2.6) is

$$\hat{u}_h(t_n+sh) = \sum_{j=1}^{m} L_j(s)\,\hat{U}_{n,j}, \quad s \in (0,1], \tag{2.12}$$

with $\hat{U}_{n,j} := \hat{u}_h(t_{n,j})$. Therefore, the fully discrete version is

$$\begin{aligned} \hat{U}_{n,j} &= f(t_{n,j}) + \sum_{l=0}^{n-1} h\, e^{i\omega t_l}\sum_{k=1}^{m} w_k(1)K(t_{n,j},t_l+c_kh,\sum_{k=1}^{m} L_k(c_k)\,\hat{U}_{l,k}) + \\ & h\, e^{i\omega t_n}\sum_{l=1}^{m} w_l(c_j)K(t_{n,j},t_n+c_lc_jh,\sum_{k=1}^{m} L_k(c_lc_j)\,\hat{U}_{n,k}) \end{aligned} \tag{2.13}$$

3. Convergence analysis

First of all, the existence and uniqueness of the solution for NVIEs (1.1), are considered by employing some lemmas and theorems.

Lemma 3.1. *[1] Suppose that $z, g \in C(I), k \in C(I)$ and let $I := [0, T]$ with $k(t) \geq 0$. If z satisfies the inequality*

$$z(t) \leq g(t) + \int_0^t k(s)\, z(s)\, ds, \quad t \in I. \tag{3.14}$$

Then

$$z(t) \leq g(t) + \int_0^t k(s)\, g(s) \cdot exp\Big(\int_0^t k(v) dv\Big)\, ds, \quad \textit{for all } t \in I. \tag{3.15}$$

If g is non-decreasing on I the above inequality reduces to

$$z(t) \leq g(t) \cdot exp\Big(\int_0^t k(s) ds\Big), \quad \textit{for all } t \in I. \tag{3.16}$$

Theorem 3.1. *[1] Set $\Omega_B := \{(t,s,u) : (t,s) \in D,\ u \in \mathbb{R}$ and $|u - f(t)| \leq B\}$ and $M_B := \max\left\{|k(t,s,u)| : (t,s,u) \in \Omega_B\right\}$. Assume:*

a) $f \in C(I)$
b) $k \in C(\Omega_B)$
c) *K satisfies the Lipschitz condition for all $(t,s,u), (t,s,z) \in \Omega_B$*

Then

- *The Picard iterates $u_n(t)$ exist for all $n \geq 1$. They are continuous on the interval $I_0 := [0, \sigma_0]$, where*
  $$\sigma_0 := min\{T, \frac{B}{M_B}\}$$
  and they converge uniformly on I_0 to a solution $u \in C(I_0)$ of the NVIEs (1.1).
- *This solution u is the unique continuous solution on I_0.*

Now, we introduce some lemmas that are used to estimate highly oscillating integrals.

Lemma 3.2. *[28] Suppose $q(t)$ is real-valued and smooth in (a,b), and that $|q^{(k)}(t)| \geq 1$ for all $t \in (a,b)$. Then*

$$\left|\int_a^b e^{i\omega q(t)}\right| \leq c(k)\,\omega^{-1/k} \tag{3.17}$$

holds when:

- *1) $k \geq 2$, or*
- *2) $k = 1$ and $q'(t)$ is monotonic.*

The bound $c(k)$ is independent of q and ω and $c(k) = 5 \cdot 2^{k-1} - 2$.

Lemma 3.3. *[28] Under the assumptions on $q(t)$ in Lemma 3.2, we can conclude that*

$$\left|\int_a^b e^{i\omega q(t)} \phi(t) dx\right| \leq c(k)\omega^{-1/k}\left[|\phi(b)| + \int_a^b |\phi'(t)| dt\right]. \tag{3.18}$$

Following the idea of [22], the following theorem is used for oscillating integrals with a specific $\phi(t)$, i.e., $\phi(t)$ has some zero points.

Theorem 3.2. *Suppose $\phi(t) \in C^1$, $q(t)$ satisfies the assumptions in Lemma 3.3 and there exists a point $t_0 \in [a,b]$ making $\phi(t_0) = 0$. Then we have*

$$\left|\int_a^b e^{i\omega q(t)} \phi(t) dt\right| \leq 2\,c(k)\,\frac{\|\phi'(t)\|_\infty}{\omega^{1/k}}(b-a). \tag{3.19}$$

Moreover, if $\phi(t) \in C^2$, $q(t) \in C^3$ *with* $k = 1$ *and* $\phi(a) = \phi(b) = 0$, *it can be concluded that*

$$\left| \int_a^b e^{i\omega q(t)} \phi(t) dt \right| \leq \min \left\{ C_1 \frac{b-a}{\omega^2}, C_2 \frac{(b-a)^2}{\omega}, \right\} \tag{3.20}$$

where $C_1 = 6 \left\| \left(\frac{\phi(t)}{u'(t)} \right)'' \right\|_\infty$, $C_2 = 3 \left\| \phi''(t) \right\|_\infty$.

Proof. For the proof refer to [27]. □

3.1. Convergence of collocation solution u_h

Put $1 \leq d \leq m$ and $y \in C^d(I)$. According to Peano's Theorem [1], on σ_n, we can write

$$u(t_n + sh) = \sum_{j=1}^{m} L_j(s) u(t_{n,j}) + h^d R_{d,n}(s), \quad s \in (0, 1], \tag{3.21}$$

where

$$R_{d,n}(s) := \int_0^1 K_d(s, z) u^{(d)}(t_n + zh) dz, \tag{3.22}$$

with

$$K_d(s, z) = \frac{1}{(d-1)!} [(s-z)_+^{d-1} - \sum_{j=1}^{m} L_j(s)(c_j - z)_+^{d-1}], \quad z \in (0, 1], \tag{3.23}$$

$(s-z)_+^p = 0$ for $s < z$ and $(s-z)_+^p = (s-z)^p$ for $s \geq z$. Therefore, the error $e_h := u - u_h$ has the local representation for the exact collocation solution

$$e_h(x_n + sh) = \sum_{j=1}^{m} L_j(s) \varepsilon_{n,j} + h^d R_{d,n}(s). \tag{3.24}$$

By putting $t = t_{n,j}$, in analogy to the proof of Theorem 2.2.3 of [1]

$$\begin{aligned}
e_h(t_{n,j}) &= \int_0^{t_{n,j}} K(t_{n,j}, s, u(s)) e^{i\omega s} ds - \int_0^{t_{n,j}} K(t_{n,j}, s, u_h(s)) e^{i\omega s} ds \\
&= \int_0^{t_n} K(t_{n,j}, s, u(s)) e^{i\omega s} ds + \int_{t_n}^{t_{n,j}} K(t_{n,j}, s, u(s)) e^{i\omega s} ds \\
&- \int_0^{t_n} K(t_{n,j}, s, u_h(s)) e^{i\omega s} ds - \int_{t_n}^{t_{n,j}} K(t_{n,j}, s, u_h(s)) e^{i\omega s} ds \\
&= \sum_{l=0}^{n-1} h_l e^{i\omega t_l} \int_0^1 (K(t_{n,j}, t_l + sh, u(t_l + sh)) - K(t_{n,j}, t_l + sh, u_h(t_l + sh))) e^{i\omega sh} ds \\
&+ h_n e^{i\omega t_n} \int_0^{c_j} (K(t_{n,j}, t_n + sh, u(t_n + sh)) - K(t_{n,j}, t_n + sh, u_h(t_n + sh))) e^{i\omega sh} ds.
\end{aligned} \tag{3.25}$$

Then, as $e_h = u - u_h$, by assuming that $K_u(t, s, .)$ is continuous and bounded and by denoting with z_l $(l \leq n)$ the arguments arising in the Taylor remainder terms, we can write (3.25) in the form

$$\begin{aligned}
e_h(t_{n,j}) &= \sum_{l=0}^{n-1} h_l e^{i\omega t_l} \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) e_h(t_l + sh) e^{i\omega sh} ds \\
&+ h_n e^{i\omega t_n} \int_0^{c_j} (K_u(t_{n,j}, t_n + sh, z_n(s)) e_h(t_n + sh) e^{i\omega sh} ds \\
&= \sum_{l=0}^{n-1} h_l e^{i\omega t_l} \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s))(\sum_{k=1}^{m} L_k(s)\epsilon_{l,k} + h^d R_{d,l}(s)) e^{i\omega sh} ds \\
&+ h_n e^{i\omega t_n} \int_0^{c_j} (K_u(t_{n,j}, t_n + sh, z_n(s))(\sum_{k=1}^{m} L_k(s)\epsilon_{n,k} + h^d R_{d,n}(s)) e^{i\omega sh} ds.
\end{aligned} \tag{3.26}$$

Then $\varepsilon_{n,j} := u(t_{n,j}) - u_h(t_{n,j})$ implies that

$$\begin{aligned}
&\varepsilon_{n,j} - he^{i\omega t_n} \sum_{k=1}^{m} \int_0^{c_j} K_u(t_{n,j}, t_n + sh, z_n(s)) L_k(s) e^{i\omega sh} ds \varepsilon_{n,k} \\
&= \sum_{l=0}^{n-1} he^{i\omega t_l} \sum_{k=1}^{m} \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) L_k(s) e^{i\omega sh} ds \varepsilon_{l,k} \\
&+ \sum_{l=0}^{n-1} h^{d+1} e^{i\omega t_l} \sum_{k=1}^{m} \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) R_{d,l}(s) e^{i\omega sh} ds \\
&+ h^{d+1} e^{i\omega t_n} \int_0^{c_j} K_u(t_{n,j}, t_n + sh, z_n(s)) R_{d,n}(s) e^{i\omega sh} ds
\end{aligned} \tag{3.27}$$

for $j = 1, ..., m$. Define the matrices

$$B_n^l := \left(\sum_{k=1}^{m} \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) L_k(s) e^{i\omega sh} ds \right)_{j,k=1,...,m}, \quad (0 \le l < n \le N-1), \tag{3.28}$$

$$B_n := \left(\sum_{k=1}^{m} \int_0^{c_j} K_u(t_{n,j}, t_n + sh, z_n(s)) L_k(s) e^{i\omega sh} ds \right)_{j,k=1,...,m}, \tag{3.29}$$

$$\rho_n^l := \left(e^{i\omega t_l} \sum_{k=1}^{m} \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) R_{d,l}(s) e^{i\omega sh} ds \right)_{j=1,...,m}^T, \quad (l < n), \tag{3.30}$$

and

$$\rho_n := \left(e^{i\omega t_n} \int_0^{c_j} K_u(t_{n,j}, t_n + sh, z_n(s)) R_{d,n}(s) e^{i\omega sh} ds \right)_{j=1,...,m}^T \tag{3.31}$$

and let $\xi_n := \left(\varepsilon_{n,1}, ..., \varepsilon_{n,m}\right)^T$. The equation (3.27) then assume the form

$$[I_m - he^{i\omega t_n} B_n]\, \xi_n = \sum_{l=0}^{n-1} h\, e^{i\omega t_l} B_n^l \xi_l + \sum_{l=0}^{n-1} h^{d+1} \rho_n^l + h^{d+1} \rho_n, \quad 0 \le n \le N-1 \tag{3.32}$$

Here, I_m denotes the identity matrix.

If the kernel function $K(t, s, .)$ is continuous, we can ensure that each of the elements in the matrices B_n is bounded. According to the Neumann Lemma [20], the inverse of the matrix $I_m - he^{i\omega t_n} B_n$ exists whenever $h \left\| e^{i\omega t_n} B_n \right\| < 1$ for some matrix norm. Obviously, this holds when h is small enough. In other words, for any mesh I_h with $h \in (0, \bar{h})$ where $\bar{h}$ is suitably small, each matrix $I_m - he^{i\omega t_n} B_n$ has uniformly bounded inverse. Then,

$$\| I_m - he^{i\omega t_n} B_n \|_1 \le D_0 \tag{3.33}$$

for sufficient small h and $0 \le n \le N-1$. Also, we suppose $\left\| B_n^{(l)} \right\|_1 \le D_1$ for $l < n \le N-1$.

Paying attention to $R_{d,l}(c_1) = \dots = R_{d,l}(c_m) = 0$ gives

$$\left| \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) R_{d,l}(s) e^{i\omega sh} ds \right| \leq C \frac{M_d}{\omega h} \tag{3.34}$$

given $a = 0$, $b = 1$, $q(t) = t$ in Theorem 3.2, where $M_d := \left\| u^{(d)}(t) \right\|_\infty$. From now on, we apply C to represent a constant that may have different values in different places, but does not depend on h and ω. In addition, if $d \geq 2$ and $c_1 = 0$, $c_m = 1$ which means that $R_{d,l}(0) = \dots = R_{d,l}(1) = 0$, then we have

$$\left| \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) R_{d,l}(s) e^{i\omega sh} ds \right| \leq C\, M_d \min \left\{ \frac{1}{\omega^2 h^2}, \frac{1}{\omega h} \right\}. \tag{3.35}$$

We can deduce in a similar way

$$\left| \int_0^{c_j} K_u(t_{n,j}, t_n + sh, z_n(s)) R_{d,n}(s) e^{i\omega sh} ds \right| \leq C \frac{M_d}{\omega h} \tag{3.36}$$

and, for $d \geq 2$ and $c_1 = 0$,

$$\left| \int_0^{c_j} K_u(t_{n,j}, t_n + sh, z_n(s)) R_{d,n}(s) e^{i\omega sh} ds \right| \leq C M_d \min \left\{ \frac{1}{\omega^2 h^2}, \frac{1}{\omega h} \right\}. \tag{3.37}$$

Then we have the estimate

$$\begin{aligned} \left\| \rho_n^{(l)} \right\|_1 &\leq \sum_{j=1}^{m} \left| \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) R_{d,l}(s) e^{i\omega sh} ds \right| \\ &\leq C\, M_d \begin{cases} \min \left\{ \frac{1}{\omega^2 h^2}, \frac{1}{\omega h} \right\}, & \text{for } d \geq 2 \text{ and } c_1 = 0,\ c_m = 1, \\ \frac{1}{\omega h} & \text{otherwise} \end{cases} \end{aligned} \tag{3.38}$$

For $\left\| \rho_n \right\|_1$, we have

$$\left\| \rho_n \right\|_1 \leq C M_d \begin{cases} \min \left\{ \frac{1}{\omega^2 h^2}, \frac{1}{\omega h} \right\}, & \text{for } d \geq 2 \text{ and } c_1 = 0, \\ \frac{1}{\omega h} & \text{otherwise} \end{cases} \tag{3.39}$$

Then, (3.32) gives

$$\left\| \varepsilon_n \right\|_1 \leq D_0 D_1 \sum_{l=0}^{n-1} h \left\| \varepsilon_l \right\|_1 + D_0 \left(\sum_{l=0}^{n-1} h^{d+1} \left\| \rho_n^{(l)} \right\|_1 + h^{d+1} \left\| \rho_n \right\|_1 \right). \tag{3.40}$$

With the discrete Gronwall inequality in general [1], we estimate

$$\begin{aligned} \left\| \varepsilon_n \right\|_1 &\leq D_0 \left(\sum_{l=0}^{n-1} h^{d+1} \left\| \rho_n^{(l)} \right\|_1 + h^{d+1} \left\| \rho_n \right\|_1 \right) \exp\left(D_0 D_1 T \right) \\ &\leq C M_d \begin{cases} \frac{h^{d-1}}{\omega} \min \left\{ \frac{1}{\omega h}, 1 \right\}, & \text{for } d \geq 2 \text{ and } c_1 = 0,\ c_m = 1, \\ \frac{h^{d-1}}{\omega} & \text{otherwise} \end{cases} \end{aligned} \tag{3.41}$$

In other words, we have

$$\begin{aligned}\left\|\rho_n^{(l)}\right\|_1 &\le \sum_{j=1}^{m}\left|e^{i\omega t_l}\int_0^1 K_u(t_{n,j},t_l+sh,z_l(s))R_{d,l}(s)e^{i\omega sh}ds\right| \\ &\le \sum_{j=1}^{m}\int_0^1 \left|K_u(t_{n,j},t_l+sh,z_l(s))R_{d,l}(s)\right|ds \\ &\le m\bar{K}k_d M_d,\end{aligned} \tag{3.42}$$

and

$$\left\|\rho_n\right\|_1 \le m\bar{K}k_d M_d, \tag{3.43}$$

where $k_d := \max_{s\in[0,1]} \int_0^1 \left|K_d(s,z)\right| dz$ and $\bar{K} := \max_{t\in I} \int_0^t \left|K_u(t,s,.)\right| ds$. Taking them into (3.40) leads

$$\left\|\varepsilon_n\right\|_1 \le C M_d h^d \tag{3.44}$$

To conclude this subsection, we summarize the above analysis in the following theorem.

Theorem 3.3. *Assume the functions $f(t)$ and $K(t,s,.) \in C^d$ in (1.1) with $1 \le d \le m$. Then error of the numerical method defined by (2.12)-(2.13) is estimated by*

$$\max_{t\in X_h}\left|u(t)-u_h(t)\right| \le C M_d \begin{cases} \min\left\{h^d, \dfrac{h^{d-1}}{\omega}, \dfrac{h^{d-2}}{\omega^2}\right\}, & \text{for } d \ge 2 \text{ and } c_1 = 0,\ c_m = 1, \\ \min\left\{h^d, \dfrac{h^{d-1}}{\omega}\right\} & \text{otherwise} \end{cases} \tag{3.45}$$

with $M_d := \left\|u^{(d)}(t)\right\|_\infty$.

Proof. Pursuant to Theorem 3.1, we infer that $u(t) \in C^d$. Thus, the regularity condition for $u(t)$ at the begin of this subsection is satisfied and the above method can be done successfully. Combining (3.41) and (3.44) completes the proof. □

3.2. Convergence of collocation solution $\hat{u}_h$

By taking the quadrature error of (2.11), we have

$$E_n^l(t,v) := \int_0^v K(t,t_n+sh,u_h(t_l+sh))e^{i\omega sh}ds - \sum_{j=1}^{m} w_j(v)\,K(t,t_n+vc_jh,u_h(t_n+vc_jh)) \tag{3.46}$$

By using (2.11) for fixed $v > 0$, we have

$$E_n^l(t,v) := v\int_0^1 \Big(K(t,t_n+vsh,u_h(t_l+vsh)) - \sum_{j=1}^{m} w_j(v)\,K(t,t_n+vc_jh,u_h(t_n+vc_jh))\Big)e^{i\omega sh}ds. \tag{3.47}$$

In (3.47), the expression in the brackets is the interpolation error for $p(s) := K(t,t_n+vsh,u_h(t_n+vsh))$. Thus, by Peano's Theorem, we have

$$E_n^l(t,v) = vh^d \int_0^1 \hat{R}_{d,n}(s)\,e^{i\omega sh}ds, \tag{3.48}$$

with

$$\hat{R}_{d,n}(s) := vh^d \int_0^1 \hat{K}_d(s,z)p^{(d)}(t_n+zh)dz. \tag{3.49}$$

Then, we have $|E_n^l(t,v)| \le Ch^d$. Furthermore, it holds $\hat{R}_{d,n}(c_1) = \dots = \hat{R}_{d,n}(c_m) = 0$. Similar to (3.34) and (3.35), quadrature error (3.48) has the estimate

$$\left|E_n^l(t,v)\right| \le C \begin{cases} \min\left\{h^d, \dfrac{h^{d-1}}{\omega}, \dfrac{h^{d-2}}{\omega^2}\right\}, & \text{for } d \ge 2 \text{ and } c_1 = 0,\ c_m = 1, \\ \min\left\{h^d, \dfrac{h^{d-1}}{\omega}\right\} & \text{otherwise} \end{cases} \tag{3.50}$$

where the same idea of the last subsection is applied. For $\nu = 0$, the above result is obvious.

Theorem 3.4. *Assume that the given functions $f(t)$ and $K(t,s,u) \in C^d$ in (1.1) with $1 \le d \le m$. Then error of the numerical method defined by (2.12)-(2.13) has an estimate*

$$\max_{t\in T_h} \left|u(t) - \hat{u}_h(t)\right| \le C\gamma_d \begin{cases} \min\left\{h^d, \dfrac{h^{d-1}}{\omega}, \dfrac{h^{d-2}}{\omega^2}\right\}, & \textit{for } d \ge 2 \textit{ and } c_1 = 0,\ c_m = 1, \\ \min\left\{h^d, \dfrac{h^{d-1}}{\omega}\right\} & \textit{otherwise} \end{cases} \tag{3.51}$$

with $\gamma_d := max\{M_d, 1\}$.

Proof. Due to

$$\left|u(t) - \hat{u}_h(t)\right| \le \left|u(t) - u_h(t)\right| + \left|u_h(t) - \hat{u}_h(t)\right|. \tag{3.52}$$

The result is conducted in a way that is similar to the last subsection, then we estimate $\left|u_h(t) - \hat{u}_h(t)\right|$.

Let $z_h(t) := u_h(t) - \hat{u}_h(t)$. Then, on σ_n,

$$z_h(t_n + sh) := u_h(t_n + sh) - \hat{u}_h(t_n + sh) = \sum_{j=1}^{m} L_j(s) Z_{n,j}, \tag{3.53}$$

with $Z_{n,j} := U_{n,j} - \hat{U}_{n,j}$. Due to the (2.9) and (2.13), we have

$$\begin{aligned} Z_{n,j} &= \sum_{l=0}^{n-1} h\, e^{i\omega t_l} \sum_{k=1}^{m} \int_0^1 K_u(t_{n,j}, t_l + sh, z_l(s)) L_k(s)\, Z_{l,k} e^{i\omega sh} ds + \\ & h\, e^{i\omega t_n} \sum_{l=1}^{m} \int_0^{c_j} K_u(t_{n,j}, t_n + sh, z_n(s))\, L_k(s)\, Z_{n,k} e^{i\omega sh} ds + \epsilon_n(t_{n,j}), \end{aligned} \tag{3.54}$$

where

$$\epsilon_n(t_{n,j}) := \sum_{l=0}^{n-1} h\, e^{i\omega t_l} E_n^l(t,1) + h\, e^{i\omega t_n} E_n^n(t, c_j)$$

and the functions z_l $(l \le n)$ are the arguments arising in the Taylor remainder terms.

Recalling the definition of B_n^l and B_n in the last section, the system can be written as

$$\left[I_m - h e^{i\omega t_n} B_n\right] \mathbf{Z}_n = \sum_{l=0}^{n-1} h\, e^{i\omega t_l} B_n^l \mathbf{Z}_l + \chi_n \tag{3.55}$$

where $\mathbf{Z} := (Z_{n,1}, ..., Z_{n,m})^T$ and $\chi_n := (\epsilon_{n,1}, ..., \epsilon_{n,m})^T$. This structure is as the same structure (3.32) but different from the term inhomogeneous. It therefore leads to similar inequality as follow (3.40). For χ_n, we have

$$\begin{aligned} \|\chi_n\|_1 &= \sum_{j=1}^{m} \left| \sum_{l=0}^{n-1} h e^{i\omega t_l} E_n^l(t_{n,j}, 1) + h e^{i\omega t_n} E_n^n(t_{n,j}, c_j) \right| \\ &\le \sum_{j=1}^{m} \left(\sum_{l=0}^{n-1} h \left|E_n^l(t_{n,j}, 1)\right| + h \left|E_n^n(t_{n,j}, c_j)\right| \right) \\ &\le C \begin{cases} \min\left\{h^d, \dfrac{h^{d-1}}{\omega}, \dfrac{h^{d-2}}{\omega^2}\right\}, & \text{for } d \ge 2 \text{ and } c_1 = 0,\ c_m = 1, \\ \min\left\{h^d, \dfrac{h^{d-1}}{\omega}\right\}, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.56}$$

Therefore, the following inequality is established

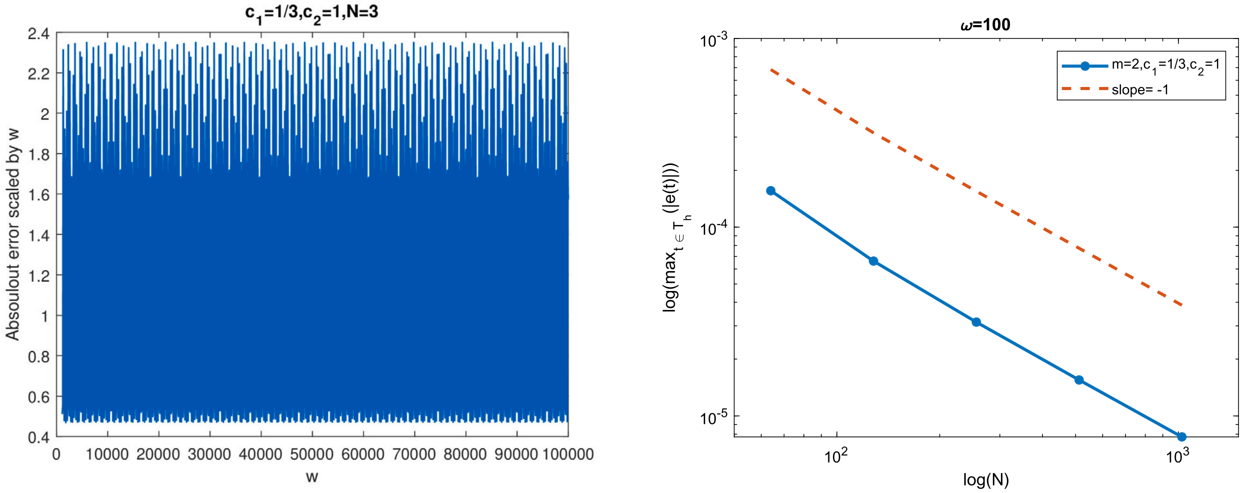

Fig. 1. The asymptotic order and the classical order with $c_1 = \frac{1}{3}, c_2 = 1$ for Example 4.1.

$$\|\mathbf{Z}_n\|_1 \leq C \begin{cases} \min\left\{h^d, \dfrac{h^{d-1}}{\omega}, \dfrac{h^{d-2}}{\omega^2}\right\}, & \text{for } d \geq 2 \text{ and } c_1 = 0,\ c_m = 1, \\ \min\left\{h^d, \dfrac{h^{d-1}}{\omega}\right\}, & \text{otherwise.} \end{cases} \tag{3.57}$$

Combination of inequality (3.52), Theorem 3.3 and (3.57) give us Theorem 3.4. □

Therefore the convergence of the method, for a fixed ω, is proved. Moreover, this theorem shows as our method may produce superior results compared to classical collocation method. As a matter of fact if M_d is bounded independently of ω, our method will have asymptotic order 1, and it may reach 2 if $d \geq 2$ and $c_1 = 0$, $c_m = 1$. Therefore, as ω increases, numerical results will become more accurate under such conditions.

4. Numerical experiments

As demonstrated in the preceding section, Filon's approaches are practical for solving NVIEs with highly oscillatory kernels. According to Theorem 3.4, the errors generated by these approaches decrease significantly as the frequency increases. This section focuses on two examples to illustrate the effectiveness of the strategy. In our experiments, we always take $T = 1$. We fix the parameters $\omega = 100$ and $N = 64, 128, 256, 512, 1024$ and plot figures to show the corresponding classical orders. We can get that the asymptotic order of the error eh is α if the absolute error is scaled by ω^α, i.e., $\omega^\alpha |e_h|$ is bounded as $\omega \longrightarrow \infty$. $N = 3$ is used to plot the asymptotic orders. Moreover, the following notation will be used to denote the numerical approach employed:

- CC = classical collocation method,
- CF = collocation Filon type method.

Example 4.1. Consider the NVIE

$$u(t) = e^t - \frac{1}{(i\omega + 2)}(e^{(i\omega+2)t} - 1) + \int_0^t e^{i\omega s}(u(s))^2\, ds \tag{4.58}$$

such that the exact solution is $u(t) = e^t$.

For this example, we want to illustrate that the classical order is 1 when $m = 2$. For parameters $c_1 = \frac{1}{3}, c_2 = 1$ and $c_1 = 0, c_2 = 1$, the right pictures of Figs. 1 and 2 indicate that the classical order is 1. With fix the parameters $\omega = 100$ and $N = 64, 128, 256, 512, 1024$, these figures show the corresponding classical orders along with slope line. By taking $d = 1$, to demonstrate the asymptotic order behavior for parameters $c_1 = \frac{1}{3}, c_2 = 1$, we investigate the left picture of Fig. 1. In the left picture of Fig. 1, the absolute errors scaled by ω are bounded, confirming that the asymptotic order is 1, which is consistent well with Theorem 3.4. For parameters $c_1 = 0, c_2 = 1$, Theorem 3.4 predicts that the asymptotic order could reach 2 with $d \geq 2$. In this example, f is continuous and dependent on ω, so the solution $u = u(t, \omega)$ behaves like $u(t) - f(t) = O(\omega)$ as $\omega \longrightarrow \infty$, so $\|u^{(2)}(t)\|_\infty = O(\omega)$. In the left picture of Fig. 2, the absolute errors scaled by ω are bounded, confirming that the asymptotic order is 1. This match well with Theorem 3.4. The numerical results from the Figures demonstrate that CC and CF methods are effective and accurate as frequency increases.

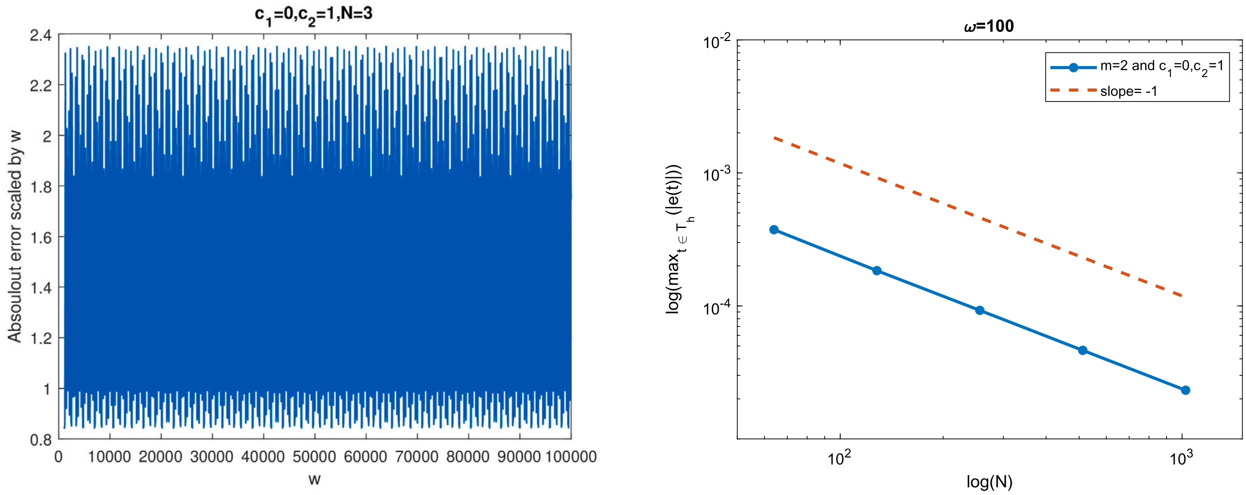

Fig. 2. The asymptotic order and the classical order with $c_1 = 0, c_2 = 1$ for Example 4.1.

Table 1
Comparison of absolute errors with $\omega = 20$ for Example 4.1.

		N				
Parameters	*Methods*	64	128	256	512	1024
$c_1 = \frac{1}{3}, c_2 = 1$	CC	$7.63e-03$	$3.81e-03$	$1.91e-03$	$9.54e-04$	$4.74e-04$
	CF	$7.66e-04$	$3.80e-04$	$1.90e-04$	$9.51e-05$	$4.75e-05$
$c_1 = 0, c_2 = 1$	CC	$2.22e-02$	$1.13e-02$	$5.69e-03$	$2.85e-03$	$1.43e-03$
	CF	$2.23e-03$	$1.13e-03$	$5.67e-04$	$2.84e-04$	$1.42e-04$

Table 2
The absolute errors with $c_1 = \frac{1}{3}, c_2 = 1$ for Example 4.1.

	N				
ω	64	128	256	512	1024
50	$2.57e-04$	$1.23e-04$	$6.09e-05$	$3.04e-05$	$1.52e-05$
100	$1.56e-04$	$6.61e-05$	$3.14e-05$	$1.55e-05$	$7.74e-06$
200	$2.24e-04$	$4.22e-05$	$1.77e-05$	$8.39e-06$	$4.14e-06$
400	$5.04e-03$	$6.27e-05$	$1.23e-05$	$5.11e-06$	$2.42e-06$

Table 3
The absolute errors with $c_1 = 0, c_2 = 1$ for Example 4.1.

	N				
ω	64	128	256	512	1024
50	$7.16e-04$	$3.61e-04$	$1.81e-04$	$9.08e-05$	$4.55e-05$
100	$3.74e-04$	$1.84e-04$	$9.25e-05$	$4.63e-05$	$2.32e-05$
200	$3.87e-04$	$1.01e-04$	$4.96e-05$	$2.47e-05$	$1.23e-05$
400	$5.04e-03$	$1.05e-04$	$2.95e-05$	$1.44e-05$	$7.13e-06$

The boundedness of the errors scaled by ω in the figures indicates that it is 1 for the asymptotic order. To confirm the effectiveness, we compare CF with the CC method in [1] and the numerical findings are compared in Table 1 for $\omega = 20$. One can observe that the method converges concerning h. Fig. 3 shows the superiority of the CF method in comparison with the CC method. However, our strategy improves the conventional one with the same settings for collocation. The absolute errors are displayed in Tables 2 and 3. The data in Tables 2 and 3 indicate that our technique is convergent with respect to h. The numerical results show the scheme's effectiveness, and our theoretical analysis is precise. They inform us that the approach proposed here is suited for oscillatory NVIEs, particularly $\omega \gg 1$.

To test performance of the proposed method, we also report the following examples.

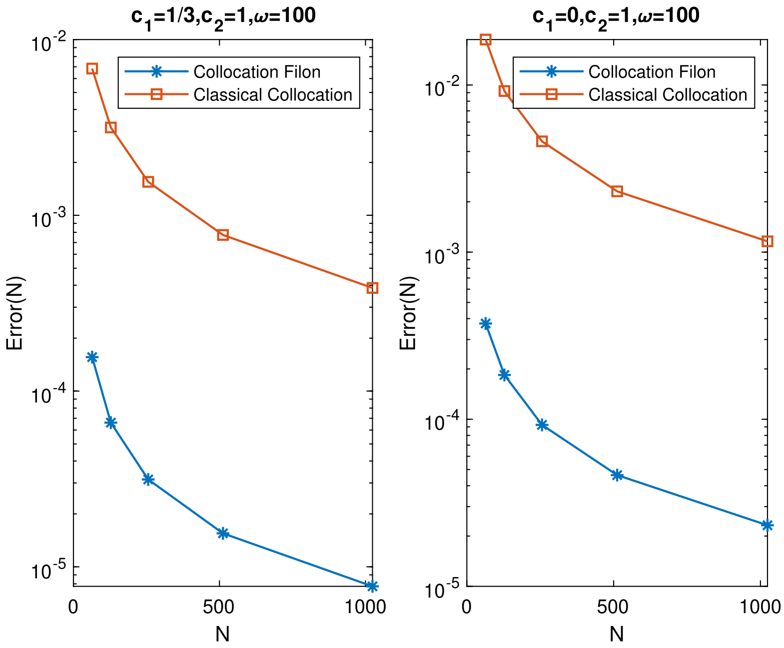

Fig. 3. Comparison of absolute errors with $\omega = 100$ for Example 4.1.

Table 4
Comparison of absolute errors with $\omega = 20$ for Example 4.2.

		N				
Parameters	*Methods*	64	128	256	512	1024
$c_1 = \frac{1}{3}, c_2 = 1$	CC	$2.73e-03$	$1.36e-03$	$6.82e-04$	$3.40e-04$	$1.70e-04$
	CF	$1.53e-04$	$7.62e-05$	$3.81e-04$	$1.90e-05$	$9.52e-06$
$c_1 = 0, c_2 = 1$	CC	$8.11e-03$	$4.07e-03$	$2.04e-03$	$1.02e-03$	$5.11e-04$
	CF	$4.54e-04$	$2.28e-04$	$1.14e-04$	$5.71e-05$	$2.85e-05$

Example 4.2. Consider the NVIE

$$u(t) = \sqrt{t} - e^{i\omega t}\left(\frac{t}{i\omega} + \frac{1}{\omega^2}\right) + \frac{1}{\omega^2} + \int_0^t e^{i\omega s}(u(s))^2\, ds \tag{4.59}$$

such that the exact solution is $u(t) = \sqrt{t}$.

For this example, we want to demonstrate that the classical order is 1 when $m = 2$. For parameters $c_1 = \frac{1}{3}, c_2 = 1$ and $c_1 = 0, c_2 = 1$, the right pictures of Figs. 1 and 2 tell that the classical order is 1. These figures show the corresponding classical orders along with slope line for fix the parameters $\omega = 100$ and $N = 64, 128, 256, 512, 1024$. By taking $d = 1$ in Theorem 3.4, the asymptotic order of the method is 1 for $c_1 = \frac{1}{3}$ and $c_2 = 1$ as well, and it is confirmed by the left picture of Fig. 4. Noticing $M_2 := \|u^{(2)}(t)\|_\infty = O(\omega)$, Theorem 3.4 predicts that the asymptotic order is 1 with $c_1 = 0$ and $c_2 = 1$ and it is confirmed by the left picture of Fig. 5. The numerical results confirm that the method is more efficient and accurate as of the frequency increases. We also compare the CF and CC approaches to investigate superiority. The numerical results are given in Table 4 for $\omega = 20$. It can be seen that the method converges with respect to h. To verify the superiority, Fig. 6 shows the CF method's superiority in comparison with the CC method. Proposed method outperforms the classical one with the same collocation parameters. The absolute errors are presented in Tables 5 and 6. Tables 5 and 6 describe the convergence of the procedure with respect to h and the order 2 is verified by the right pictures of Figs. 4 and 5. The numerical results demonstrate the scheme's effectiveness, and our theoretical analysis is incisive. They inform us that the approach shown here is suited for oscillatory NVIEs, particularly $\omega \gg 1$.

5. Conclusions

This paper presented efficient collocation methods using the Filon type method for NVIEs with an oscillatory kernel. Based on the solution's asymptotic analysis, the method's convergence has been achieved. The theorem demonstrates that the approach has a classical order and, for high-frequency values, an asymptotic order. In addition, by increasing the number of collocation parameters,

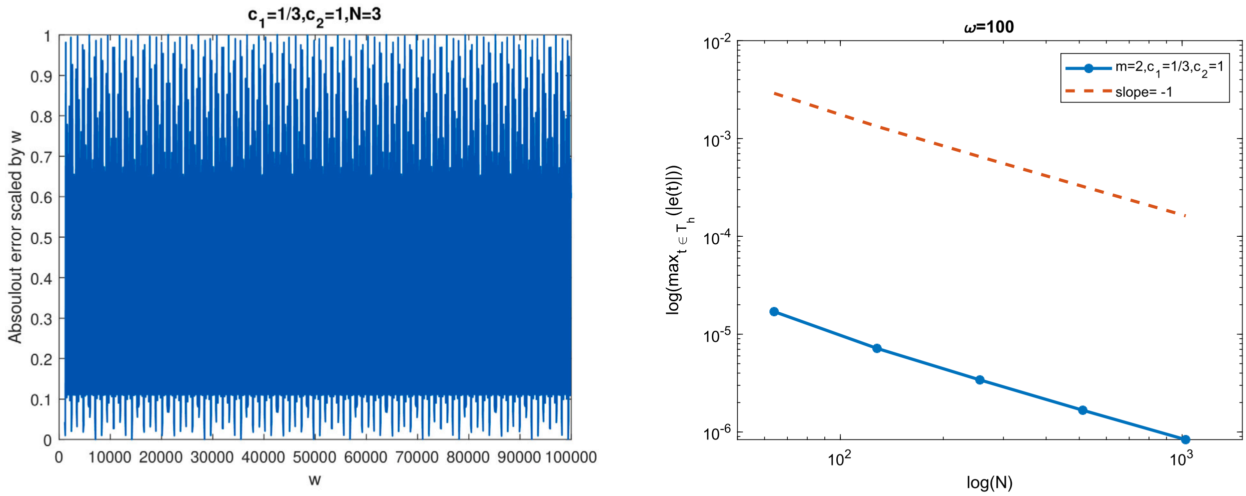

Fig. 4. The asymptotic order and the classical order with $c_1 = \frac{1}{3}, c_2 = 1$ for Example 4.2.

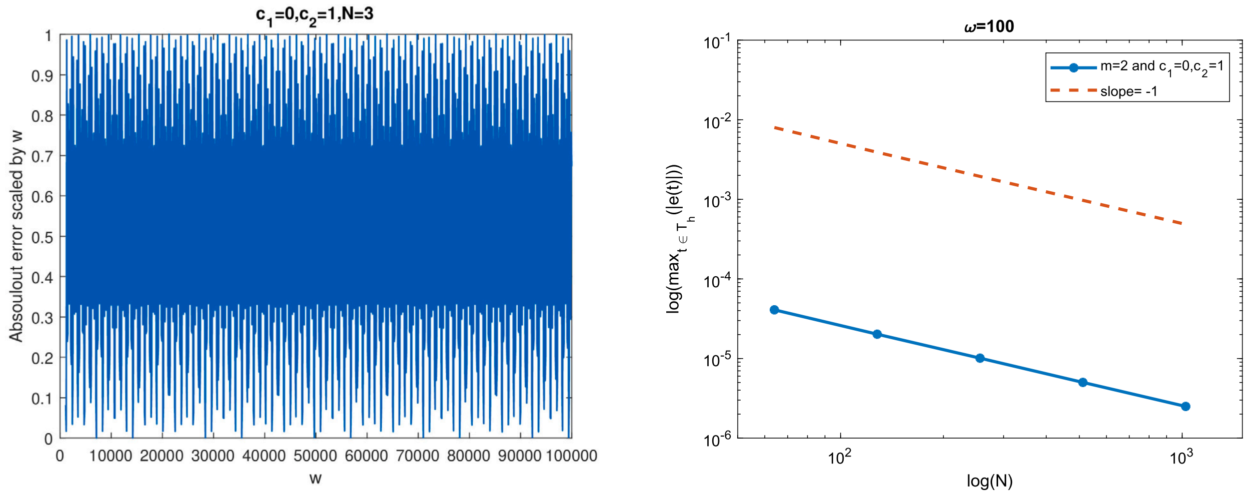

Fig. 5. The asymptotic order and the classical order with $c_1 = 0, c_2 = 1$ for Example 4.2.

Table 5
The absolute errors with $c_1 = \frac{1}{3}$, $c_2 = 1$ for Example 4.2.

	N				
ω	64	128	256	512	1024
50	$1.36e-05$	$4.49e-06$	$3.21e-06$	$1.60e-06$	$7.99e-07$
100	$1.70e-05$	$7.16e-06$	$3.41e-06$	$1.67e-06$	$8.36e-07$
200	$1.58e-06$	$8.40e-06$	$3.51e-06$	$1.66e-06$	$8.21e-07$
400	$2.06e-03$	$1.21e-05$	$3.66e-06$	$1.52e-06$	$7.22e-07$

Table 6
The absolute errors with $c_1 = 0$, $c_2 = 1$ for Example 4.2.

	N				
ω	64	128	256	512	1024
50	$3.81e-05$	$1.90e-05$	$9.56e-06$	$4.79e-06$	$2.41e-06$
100	$4.09e-05$	$2.02e-05$	$1.01e-05$	$5.01e-06$	$2.50e-06$
200	$3.19e-05$	$2.02e-05$	$9.87e-06$	$4.91e-06$	$2.45e-06$
400	$2.06e-03$	$2.08e-05$	$8.81e-06$	$4.29e-06$	$2.14e-06$

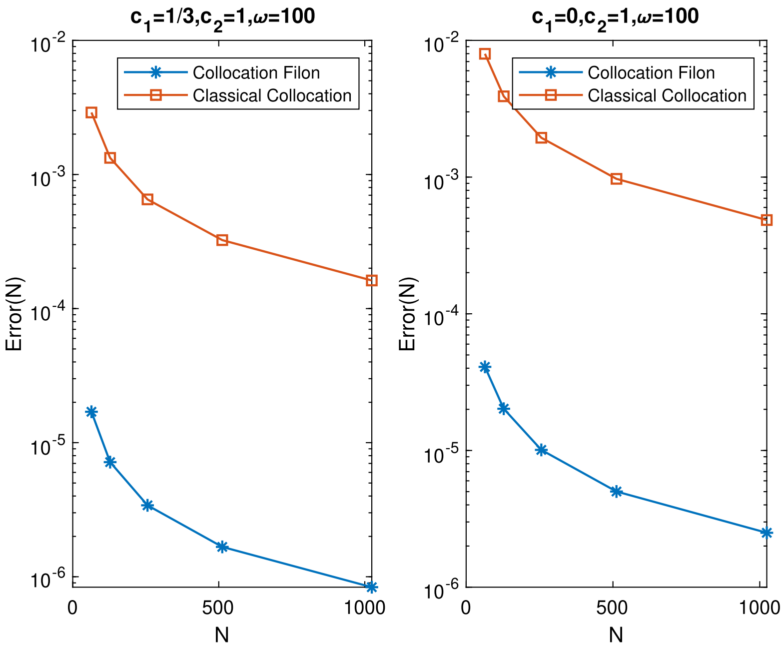

Fig. 6. Comparison of absolute errors with $\omega = 100$ for Example 4.2.

the classical order could be raised, and in some cases, the asymptotic order 2 can be reached. The theoretical analysis and numerical tests confirmed that these methods are efficient and become more accurate as of the frequency increases.

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