Research Article

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# Spectral invariance of quasi-Banach algebras of matrices and pseudodifferential operators

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**Abstract:** We extend the stability and spectral invariance of convolution-dominated matrices to the case of quasi-Banach algebras $p < 1$. As an application, we construct a spectrally invariant quasi-Banach algebra of pseudodifferential operators with non-smooth symbols that generalize Sjöstrand's results.

**Keywords:** Pseudodifferential operators, spectral invariance, modulation space, Wiener's lemma, off-diagonal decay matrices, Gabor frame

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## 1 Introduction

Spectral invariance is an important phenomenon for applications in the field of partial differential equations and in the theory of pseudodifferential operators. The first result due to Beals [6] asserts that the inverse of a pseudodifferential operator $T$ that is (i) invertible on $L^2(\mathbb{R}^d)$, and (ii) with a symbol in the Hörmander class $S^0_{0,0}$, is again a pseudodifferential operator with a symbol in the same class. In other words, this class of pseudodifferential operators is inverse-closed (closed under inversion) in the algebra $\mathcal{A}$ of pseudodifferential operators with $S^0_{0,0}$-symbols. As a consequence, the spectrum of $T$ is independent of the weighted $L^p(\mathbb{R}^d)$ space or of the choice of $B^{s,a}_{p,q}(\mathbb{R}^d)$; see [32, 34]. This phenomenon is often referred to as spectral invariance, and the resemblance to Wiener's lemma for absolutely convergent Fourier series has also motivated the terminology of $\mathcal{A}$ being a Wiener algebra.

The next important step in the theory of spectral invariance of pseudodifferential operators was made by Sjöstrand [36] who introduced a class of non-smooth symbols for which the associated algebra of pseudodifferential operators is spectrally invariant in $\mathcal{B}(L^2(\mathbb{R}^d))$. This class, nowadays called the Sjöstrand class, turned out to be an already known function space that is paramount in time-frequency analysis, namely the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$. This connection spawned an intensive investigation of pseudodifferential operators with time-frequency methods [16, 20, 22, 24, 26, 36, 38–42]. The state-of-the-art is presented in the monographs of Benyi and Okoudjou [7] and Cordero and Rodino [10].

Among the new results obtained by time-frequency methods was, firstly, a characterization of both the Hörmander class and the Sjöstrand class by means of the matrix associated to a pseudodifferential operator with respect to a Gabor frame. Secondly, time-frequency analysis established a firm connection between the off-diagonal decay of these matrices and the corresponding properties (boundedness, algebra, inverse-

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closedness) of pseudodifferential operators. In this way, every (solid) inverse-closed subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}^{2d}))$ can be mapped to an algebra of pseudodifferential operators that is inverse-closed in $\mathcal{B}(L^2(\mathbb{R}^d))$ (see [24]). In contrast to the classical hard analysis methods, the time-frequency approach is so flexible that the theory can even be formulated for pseudodifferential operators on locally compact abelian groups [26].

The goal of this paper is the extension of the theory of spectrally invariant algebras of pseudodifferential operators to the realm of quasi-Banach algebras. Quasi-Banach algebras are interesting in their own right, but quasi-Banach spaces and associated operators occur naturally in approximation theory and data compression problems; see, e.g., [13, 25]. Additionally, in time-frequency analysis they occur in the formulation of uncertainty principles [17].

To formulate our main results, we briefly recall the definition of modulation spaces and the Weyl form of pseudodifferential operators. For a fixed non-zero Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$ and a tempered distribution $f$, the *short-time Fourier transform* $V_g f$ is the function on $\mathbb{R}^{2d}$ defined by the formula

$$V_g f(x, \xi) := \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i t\cdot\xi}\, dt, \tag{1.1}$$

with suitable interpretation of the integral. For $0 < p, q \le \infty$, the (unweighted) *modulation space* $M^{p,q}(\mathbb{R}^d)$ is defined by the quasi-norm

$$\|f\|_{M^{p,q}} := \Big( \int_{\mathbb{R}^d} \Big( \int_{\mathbb{R}^d} |V_g f(x,\xi)|^p\, dx\Big)^{q/p} d\xi\Big)^{1/q}, \tag{1.2}$$

with usual modifications in case $p = \infty$ or $q = \infty$, and consists of all tempered distributions $f$ with finite quasi-norm. Modulation spaces on $\mathbb{R}^{2d}$ serve as symbol classes for pseudodifferential operators. Our focus will be on the symbol class $M^{\infty,p_0}(\mathbb{R}^{2d})$ for $p_0 < 1$. This is only a quasi-Banach space.

Given a symbol $\mathfrak{a}$ on $\mathbb{R}^{2d}$, the corresponding pseudodifferential operator in the Weyl calculus is defined formally by

$$\mathfrak{a}^w f(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathfrak{a}\Big(\frac{x+y}{2}, \xi\Big) f(y) e^{2\pi i (x-y)\cdot\xi}\, dy\, d\xi,$$

again with a suitable interpretation of the integral and for $f \in \mathcal{S}(\mathbb{R}^d)$.

With these definitions, our main results can be stated as follows.

**Theorem 1.1** (Spectral invariance). *Let $p_0 \in (0,1]$ and $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$ be such that $\mathfrak{a}^w$ is invertible on $M^p(\mathbb{R}^d)$ for some $p \in [p_0, \infty]$. Then $(\mathfrak{a}^w)^{-1} = \mathfrak{b}^w$ for some $\mathfrak{b} \in M^{\infty,p_0}(\mathbb{R}^{2d})$.*

As a consequence, we obtain that the spectrum of a pseudodifferential operator with a symbol in $M^{\infty,p_0}(\mathbb{R}^{2d})$ is independent of the space on which it acts.

**Theorem 1.2** (Spectral invariance on modulation spaces). *Assume that $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$ for some $p_0 \in (0,1]$. Then the following assertions are equivalent:*
(i) *$\mathfrak{a}^w$ is invertible on $L^2(\mathbb{R}^d)$.*
(ii) *$\mathfrak{a}^w$ is invertible on $M^p(\mathbb{R}^d)$ for some $p \in [p_0, \infty]$.*
(iii) *$\mathfrak{a}^w$ is invertible on $M^q(\mathbb{R}^d)$ for all $q \in [p_0, \infty]$.*

More generally, we show in Theorem 4.12 the invertibility of $\mathfrak{a}^w$ on the more broad class of modulation spaces $M^{r,q}(\mathbb{R}^d)$ with $r, q \in [p_0, \infty)$ under the assumptions of the previous theorem.

Both theorems have already a long history in time-frequency analysis. The Banach algebra case of Theorem 1.1 with $\mathfrak{a} \in M^{\infty,1}(\mathbb{R}^{2d})$ and $\mathfrak{a}^w$ invertible on $L^2(\mathbb{R}^d)$ was already proved by Sjöstrand [36]. For symbols in a weighted symbol class $M^{\infty,1}_{1\otimes v}$, the spectral invariance was established with a new time-frequency method in [24]. Recently, the invertibility results on $L^2$ were further extended to the case of symbols in the weighted quasi-Banach spaces $M^{\infty,p_0}_{1\otimes v}$ for $p_0 < 1$ by Cordero and Giacchi [8].

The implication "(i) $\Longrightarrow$ (iii)" in Theorem 1.2 then follows from these results. We note that, in addition to treating weights, [8] also treats the class of generalized metaplectic operators.

Our main contribution is the full characterization of invertibility in Theorem 1.2. To the best of our knowledge, the implication "(ii) $\Longrightarrow$ (i)" is new even for the case of Banach spaces[1]. This implication, where we start with invertibility on $M^p(\mathbb{R}^d)$, $p \neq 2$, is much more involved, as there are no Hilbert space techniques available. For self-adjoint pseudodifferential operators $(\mathfrak{a}^w)^* = \mathfrak{a}^w$, one could argue with duality and interpolation to reduce to the case of invertibility on $L^2(\mathbb{R}^d)$, but there is no cheap trick for non-self-adjoint pseudodifferential operators. When we start with invertibility on a quasi-Banach space $M^p(\mathbb{R}^d)$, $p < 1$, even duality is no longer useful. For this reason, we returned to Sjöstrand's original proof of Wiener's lemma in [36] and added several new elements to his proof.

**Methods.** We follow the outline of proof of [24]. The first step is to study the matrix representation of a pseudodifferential operator with respect to a Gabor frame and then derive a characterization of the symbol class in terms of the off-diagonal decay of the associated matrix.

In the second step, this leads to the study of spectrally invariant matrix algebras. The appropriate class in our context is the class of convolution-dominated matrices, i.e., matrices $A = (a_{\lambda,\rho})_{\lambda,\rho\in\Lambda}$ with an off-diagonal decay of the form

$$|a_{\lambda,\rho}| \leq H(\lambda - \rho)$$

for a (smooth) function $H$ in $L^p(\mathbb{R}^d)$. It turns out that such matrix classes are spectrally invariant in $\mathcal{B}(\ell^p(\Lambda))$. To offer a glimpse of this aspect, we formulate a very special case of our main result on matrices which does not require technical details.

**Theorem 1.3.** *Let $A = (a_{kl})_{k,l\in\mathbb{Z}^d}$ be a matrix over the index set $\mathbb{Z}^d$. Suppose that there exists a sequence $h \in \ell^{p_0}(\mathbb{Z}^d)$ for $0 < p_0 \leq 1$ such that*

$$|a_{kl}| \leq h(k-l) \quad \text{for all } k, l \in \mathbb{Z}^d.$$

(i) *Spectral invariance: If $A$ is invertible on some $\ell^p(\mathbb{Z}^d)$ for $p \in [p_0, \infty]$, then $A$ is invertible on all $\ell^q(\mathbb{Z}^d)$ for $q \in [p_0, \infty]$.*
(ii) *Spectral stability: If, for some $\ell^p(\mathbb{Z}^d)$ with $p \in [p_0, \infty]$, $A$ satisfies the stability condition*

$$\|Ac\|_p \geq C\|c\|_p \quad \text{for all } c \in \ell^p(\mathbb{Z}^d),$$

*then $A$ satisfies $\|Ac\|_q \geq C_q\|c\|_q$ for all $c \in \ell^q(\mathbb{Z}^d)$ with $q \in [p_0, \infty]$.*

In our approach, we follow Sjöstrand ingenious proof of Wiener's lemma for absolutely convergent Fourier series [36] and built on the presentation in [23]. Our ultimate results on spectral invariance and stability (Theorems 3.5 and 3.15) are a significant extension of the above preliminary statement and provide several new facts of spectral invariance of infinite matrices:
(i) They yield both spectral stability and spectral invariance.
(ii) They are formulated with respect to arbitrary operator algebras $\mathcal{B}(\ell^p)$ (not just $\mathcal{B}(\ell^2)$ as is usually done).
(iii) They cover the general case of quasi-Banach algebras.
(iv) In addition, we treat arbitrary index sets and not just $\mathbb{Z}^d$ or a discrete abelian group as in most references.

Technically, the study of quasi-Banach algebras of convolution-dominated matrices is the main part of our paper; its application to pseudodifferential operators is then based on the analysis in [24]. Our proof contains some new features and avoids the functional calculus associated with the pseudo-inverse. These arguments may be useful in other contexts as well.

**Related results.** There are numerous results on the spectral invariance of matrices; we mention here [1, 3, 4, 29, 35–39] for a small sample, and [21] for a survey. As long as the index set is a discrete abelian group, one can use methods from harmonic analysis to establish spectral invariance. This line of thought goes back to Bochner and Philipps and is used in [3, 4, 8, 11] and many others. All these proofs break down, however, when unstructured index sets are considered.

1 To focus on this new feature, we treat only unweighted modulation spaces and leave the weighted version to the reader.

The extension of spectral invariance to quasi-Banach algebras of matrices and operators over $\mathbb{Z}^d$ is the subject of the recent papers [8, 11]. While there is some thematic overlap, not all results are directly comparable. On the one hand, we restrict our attention to unweighted $\ell^p$-spaces, whereas [8, 11] include weights. On the other hand, these papers prove that invertibility of convolution-dominated matrices on the Hilbert space $\ell^2$ implies invertibility on all $\ell^p$ for $p > p_0$. Our results also provide the converse, namely that invertibility on some $\ell^p$ implies invertibility on $\ell^2$.

The paper is organized as follows: In Section 2, we collect the relevant definitions about sequence spaces and amalgam spaces. In Section 3, we treat the spectral invariance of convolution-dominated matrices. We treat both the stability of such matrices on the quasi-Banach spaces $\ell^p$, $p < 1$, and the spectral invariance of the algebra of convolution-dominated matrices. The main results are Theorems 3.5 and 3.15. In Section 4, we first recapitulate the definitions of modulation spaces and Gabor frames and various calculi of pseudodifferential operators, and then prove our main theorems that are already stated in the introduction. For completeness, we have postponed some easy and known proofs to the appendix.

# 2 Preliminaries

For the convenience of the reader, we now list the definitions of some function spaces and their properties needed throughout this paper. We start with the sequence spaces.

## 2.1 Sequence space $\ell^p$

For each $0 < p \le \infty$ and each discrete set $J$, we recall that the set $\ell^p(J)$ consists of all complex-valued sequences $a$ such that

$$\|a\|_p := \Big(\sum_{j\in J}|a_j|^p\Big)^{1/p}, \quad a = (a_j)_{j\in J},$$

is finite (with usual modifications for $p = \infty$). Then $\ell^p(J)$ is a quasi-Banach space with quasi-norm $\|\cdot\|_p$, which is even a norm if $p \ge 1$.

We recall the following properties for $\ell^p(J)$.

**Lemma 2.1.** *Suppose that $0 < p \le 1$, that $J$ is a discrete set and that $\Lambda$ is countable. Let $a = (a_j)_{j\in J} \in \ell^p(J)$ and $b, b_\lambda \in \ell^p(J)$, $\lambda \in \Lambda$. Then the following assertions hold:*

(i) $\|a\|_1^p \le \|a\|_p^p$ *or*

$$\Big|\sum_{j\in J} a_j\Big|^p \le \sum_{j\in J}|a_j|^p.$$

(ii) *If $\sum_{\lambda\in\Lambda}\|b_\lambda\|_p^p < \infty$, then $\sum_{\lambda\in\Lambda} b_\lambda$ is uniquely defined as an element in $\ell^p(J)$, and*

$$\Big\|\sum_{\lambda\in\Lambda} b_\lambda\Big\|_p^p \le \sum_{\lambda\in\Lambda}\|b_\lambda\|_p^p.$$

(iii) *If $J = \mathbb{Z}^d$, then also $\|a * b\|_p \le \|a\|_p\|b\|_p$.*

## 2.2 Wiener amalgam space

Let $X = L^\infty(\mathbb{R}^d)$ or $X = C_b(\mathbb{R}^d) := L^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, and let $K \subseteq \mathbb{R}^d$ be convex and compact with positive volume. The Wiener amalgam space $W(X, L^{p_0})$ for $0 < p_0 \le \infty$ consists of all $f \in X$ such that

$$\|f\|_{K,W(X,L^{p_0})} := \Big(\int_{\mathbb{R}^d}\|f\|_{L^\infty(x+K)}^{p_0}\,dx\Big)^{1/p_0} < \infty.$$

This is always a quasi-norm, and a norm, if $p_0 \geq 1$ (see, e.g., [27]). For $p_0 = \infty$, we have $W(X, L^\infty) = X$. By compactness, it follows that $W(X, L^{p_0})$ is independent of the choice of $K$, and different $K$ yield equivalent quasi-norms. For convenience, we set

$$\|\cdot\|_{W(X,L^{p_0})} = \|\cdot\|_{B_1(0),W(X,L^{p_0})}.$$

**Remark 2.2.** We observe that every continuous function with compact support is contained in $W(C_b, L^p)$ for all $p > 0$ and that $W(C_b, L^p)$ is translation invariant (see, e.g., [9] and the references therein).

The Wiener amalgam space $W(C_b, L^{p_0})$, $0 < p_0 < \infty$, arises naturally in the formulation of sampling inequalities. We first recall that a set $\Lambda \subseteq \mathbb{R}^d$ is *relatively separated* if

$$\mathrm{rel}(\Lambda) := \sup\{\#(\Lambda \cap B_1(x)) : x \in \mathbb{R}^d\} < \infty. \tag{2.1}$$

**Lemma 2.3.** *Let $0 < p_0 < \infty$, let $\Lambda \subseteq \mathbb{R}^d$ be relatively separated and let $H \in W(C_b, L^{p_0})$. Then $(H(\lambda))_{\lambda\in\Lambda} \in \ell^{p_0}(\Lambda)$.*

Lemma 2.3 follows by straight-forward estimates. In order to be self-contained, a proof of the result is given in Section A.

# 3 Spectral invariance of convolution-dominated matrices

In this section, we prove a spectral invariance result for infinite dimensional convolution-dominated matrices. For this we first list some needed auxiliary tools.

We always denote the conjugate exponent of $p \in [1, \infty]$ by $p' = \frac{p}{p-1}$, so that $p' \in [1, \infty]$ and

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We identify a matrix $A = (a_{\lambda,\rho})_{\lambda\in\Lambda,\rho\in\Pi}$ indexed by $\Lambda$ and $\Pi$ with a linear operator

$$(Ab)_\lambda := \sum_{\rho\in\Pi} a_{\lambda,\rho} b_\rho \quad \text{for } b = (b_\rho)_{\rho\in\Pi}.$$

Then $A$ is always well-defined on finite sequences and $A$ maps finite sequences on $\Pi$ to arbitrary sequences on $\Lambda$. Some boundedness properties of matrices on $\ell^p$-spaces are given by Schur's test.

**Proposition 3.1** (Schur test for $p \geq 1$). *Let $1 \leq p \leq \infty$, let $\Lambda$ and $\Pi$ be countable sets and let*

$$A = (a_{\lambda,\rho})_{\lambda\in\Lambda,\rho\in\Pi} \in \mathbb{C}^{\Lambda\times\Pi}$$

*be a matrix satisfying*

$$\sup_{\rho\in\Pi} \sum_{\lambda\in\Lambda} |a_{\lambda,\rho}| \leq K_1 \quad \text{and} \quad \sup_{\lambda\in\Lambda} \sum_{\rho\in\Pi} |a_{\lambda,\rho}| \leq K_2. \tag{3.1}$$

*Then $A$ is a bounded operator from $\ell^p(\Pi)$ to $\ell^p(\Lambda)$, and*

$$\|A\|_{\mathcal{B}(\ell^p(\Pi),\ell^p(\Lambda))} \leq K_1^{1/p'} K_2^{1/p}.$$

For a proof of Proposition 3.1, see, e.g., [19, Lemma 6.1.2].

Because of (3.1), we let

$$\|A\|_{\mathrm{Schur}} = \sup_{\rho\in\Pi} \sum_{\lambda\in\Lambda} |a_{\lambda,\rho}| + \sup_{\lambda\in\Lambda} \sum_{\rho\in\Pi} |a_{\lambda,\rho}| \tag{3.2}$$

for every $A \in \mathbb{C}^{\Lambda\times\Pi}$.

The following quasi-Banach space version of the Schur test just follows by using the triangle inequality; see [10, Lemma 6.1.7].

**Proposition 3.2** (Schur test for $p \leq 1$). *Fix $0 < p \leq 1$. Let $A = (a_{\lambda,\rho})_{\lambda\in\Lambda,\rho\in\Pi} \in \mathbb{C}^{\Lambda\times\Pi}$ be a matrix satisfying*

$$\|A\|_{S-p} := \sup_{\rho\in\Pi} \sum_{\lambda\in\Lambda} |a_{\lambda,\rho}|^p < \infty. \tag{3.3}$$

*Then A defines a bounded operator from* $\ell^p(\Pi)$ *to* $\ell^p(\Lambda)$*. The operator norm is bounded by* $\|A\|_{S-p}^{1/p}$*. If* $0 < q \le p \le 1$*, then*

$$\|A\|_{S-p}^{q/p} = \Big(\sup_{\rho\in\Pi}\sum_{\lambda\in\Lambda}|a_{\lambda,\rho}|^p\Big)^{q/p} \le \sup_{\rho\in\Pi}\sum_{\lambda\in\Lambda}|a_{\lambda,\rho}|^q = \|A\|_{S-q}. \tag{3.4}$$

Our treatment of the spectral invariance of pseudodifferential operators relies on spectral invariance properties of associated convolution-dominated matrices. Roughly speaking, a matrix $A \in \mathcal{B}(\ell^2(\Lambda))$ is called *convolution-dominated* if there exists a function $H$ such that

$$|a_{\lambda,\rho}| \le H(\lambda - \rho) \quad \text{for all } \lambda \in \Lambda,\ \rho \in \Pi. \tag{3.5}$$

The function $H$ is then called an *envelope* of $A$. By specifying a norm on envelopes, we define a particular class of convolution-dominated matrices as follows.

**Definition 3.3.** Let $\Lambda, \Pi \subseteq \mathbb{R}^d$ be relatively separated and let $0 < p_0 \le 1$. The set $\mathcal{C}^{p_0} = \mathcal{C}^{p_0}(\Lambda,\Pi)$ consists of all convolution-dominated matrices $A$ such that (3.5) holds for an envelope $H \in W(C_b, L^{p_0})$. For $A = (a_{\lambda,\rho})_{\lambda\in\Lambda,\rho\in\Pi}$, we set

$$\|A\|_{\mathcal{C}^{p_0}} = \inf\{\|H\|_{W(C_b,L^{p_0})} : |a_{\lambda,\rho}| \le H(\lambda-\rho) \text{ for all } \lambda\in\Lambda,\ \rho\in\Pi\}.$$

We set $\mathcal{C}^{p_0}(\Lambda) = \mathcal{C}^{p_0}(\Lambda,\Lambda)$. We observe that if $\Lambda = \Pi \subseteq \mathbb{R}^d$ is a lattice, then the restriction of $H \in W(C_b, L^{p_0})$ to $\Lambda$ belongs to $\ell^{p_0}(\Lambda)$ due to Lemma 2.3. In this case, $A \in \mathcal{C}^{p_0}(\Lambda)$ if and only if there is a sequence $H \in \ell^{p_0}(\Lambda)$ such that $|A_{\lambda,\rho}| \le H(\lambda - \rho)$ for all $\lambda, \rho \in \Lambda$.

We note that $\|A\|_{\mathcal{C}^{p_0}}$ is a quasi-norm for $p_0 < 1$, and $\mathcal{C}^{p_0}$ is a quasi-Banach $*$-algebra (sometimes called a $p$-algebra) with respect to addition and multiplication of matrices. For $p_0 = 1$, $\mathcal{C}^1$ is a Banach $*$-algebra with norm $\|\cdot\|_{\mathcal{C}^1}$.

For convolution-dominated operators in Definition 3.3, we state the following boundedness result. Here we set $q' = \infty$ when $q \le 1$.

**Proposition 3.4.** *Let* $0 < p_0 \le 1$*, let* $\Lambda$ *and* $\Pi$ *be as in Definition 3.3 and let* $A \in \mathcal{C}^{p_0}(\Lambda,\Pi)$*. Then A is bounded from* $\ell^q(\Pi)$ *to* $\ell^q(\Lambda)$ *for every* $q \in [p_0,\infty]$*, and*

$$\|A\|_{\mathcal{B}(\ell^q(\Pi),\ell^q(\Lambda))} \le C\,\mathrm{rel}(\Lambda)^{1/q}\,\mathrm{rel}(\Pi)^{1/q'}\|A\|_{\mathcal{C}^{p_0}(\Lambda,\Pi)}, \tag{3.6}$$

*where the constant* $C > 0$ *only depends on d.*

The result follows by suitable combinations of Hölder's and Young's inequalities. In order to be self-contained, we present a proof in Section A.

## 3.1 Invariance of the lower bound property on $\ell^p$ of convolution-dominated matrices

Our main technical contribution is the so-called stability of convolution-dominated matrices. By this we mean the invariance of the lower bound property of such matrices on $\ell^p$.

**Theorem 3.5.** *Let* $p_0 \le 1$*, let* $q \in [p_0,\infty]$*, let* $\Lambda, \Pi \subseteq \mathbb{R}^d$ *be relatively separated, and let* $A \in \mathcal{C}^{p_0}(\Lambda,\Pi)$*. Assume that there exist* $p \in [p_0,\infty]$ *and* $C_0 > 0$ *such that*

$$\|Ac\|_p \ge C_0\|c\|_p, \quad c \in \ell^p(\Pi). \tag{3.7}$$

*Then there exists a constant* $C > 0$*, which is independent of q, such that*

$$\|Ac\|_q \ge C\|c\|_q \quad \text{for all } c \in \ell^q(\Pi). \tag{3.8}$$

In other words, if $A$ in Theorem 3.5 is bounded from below on some $\ell^p$ with $p \ge p_0$, then $A$ is bounded from below on $\ell^q$ for all $q \in [p_0,\infty]$. Note that (3.7) is equivalent to saying that $A$ is one-to-one on $\ell^p(\Lambda)$ with closed range in $\ell^p(\Pi)$. Thus if $A$ is one-to-one with closed range for *some* $p \in [p_0,\infty]$, then it is one-to-one with closed range for *all* $p \in [p_0,\infty]$.

Note that, by (3.6), $A$ in Theorem 3.5 is bounded from $\ell^p(\Pi)$ to $\ell^p(\Lambda)$ for all $p \geq p_0$, and hence estimates (3.7) and (3.8) really make sense.

The proof of Theorem 3.5 is modelled on Sjöstrand's treatment of Wiener's lemma for convolution-dominated matrices. It exploits the flexibility of Sjöstrand's methods to transfer lower bounds for a matrix from one value of $p$ to all others.

**Remark 3.6.** The proof of the previous proposition for $p_0 = 1$ can be found in [23, Proposition 8.1]. Hence we can restrict ourselves to the case $p_0 < 1$. The following cases have to be considered:
(i) From $p \geq 1$ to $q \geq 1$.
(ii) From $p \leq 1$ to $q < p$.
(iii) From $p \geq 1$ to $q < 1$.
(iv) From $p < 1$ to $p < q$.
Case (iii) is a consequence of cases (i) and (ii). If $p \geq 1$, we can assume $p = 1$ on account of case (i). For $p = 1$, the statement of case (iii) is included in case (ii).

We need some preparations for the proof. First, let $\varphi \in C^\infty(\mathbb{R}^d)$ with $0 \leq \varphi \leq 1$, and let $\operatorname{supp}(\varphi) \subseteq B_2(0)$ be such that $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}^d}$ is a partition of unity. Set $\varphi_k^\varepsilon(x) := \varphi(\varepsilon x - k)$, $x, y \in \mathbb{R}^d$. Then

$$\begin{cases} \sum_{k\in\mathbb{Z}^d} \varphi_k^\varepsilon = 1, \quad |\varphi_k^\varepsilon(x) - \varphi_k^\varepsilon(y)| \lesssim \varepsilon|x-y|, \quad 0 \leq \varphi_k^\varepsilon \leq 1, \\ \Phi^\varepsilon := \sum_{k\in\mathbb{Z}^d} (\varphi_k^\varepsilon)^2 \asymp 1. \end{cases} \tag{3.9}$$

By combining these properties, we obtain

$$|\varphi_k^\varepsilon(x) - \varphi_k^\varepsilon(y)| \lesssim \min\{1, \varepsilon|x-y|\}, \quad x, y \in \mathbb{R}^d. \tag{3.10}$$

For $k \in \mathbb{Z}^d$ and $\varepsilon > 0$ let

$$\varphi_k^\varepsilon c := \varphi_k^\varepsilon\big|_\Pi \cdot c, \quad c = (c_\rho)_{\rho\in\Pi},$$

denote the multiplication operator $\varphi_k^\varepsilon$. This multiplication operator enables us to get equivalent norms for sequence spaces.

**Lemma 3.7.** *Let $\varepsilon > 0$ and let $\Lambda \subseteq \mathbb{R}^{2d}$ be relatively separated. Then, for $0 < q \leq \infty$, we get*

$$\Big(\sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon a\|_q^q\Big)^{1/q} \asymp \|a\|_q, \quad a \in \ell^q(\Pi), \tag{3.11}$$

*with the usual modifications in case $q = \infty$.*

For $1 \leq q \leq \infty$, Lemma 3.7 was already proved in [23], and the other cases are obtained by similar arguments. For completeness, we present a proof for $q < 1$ in Section A.

**Lemma 3.8.** *Let $0 < p_0 \leq 1$, let $p, q \in [p_0, \infty]$, let $\varepsilon > 0$, and let $\Lambda \subseteq \mathbb{R}^{2d}$ be relatively separated. Then*

$$\sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon a\|_p^q \asymp \|a\|_q^q, \quad a \in \ell^q(\Pi), \tag{3.12}$$

*with the usual modifications in case $p = \infty$ or $q = \infty$. The constants in* (3.12) *are independent of $p, q$, but depend on $\varepsilon$.*

In the case $p, q \geq 1$, the claim was already shown in [23].

*Proof.* Let $p, q \in [p_0, \infty]$ be arbitrary. For fixed $\varepsilon > 0$, we get

$$N := \sup_{k\in\mathbb{Z}^d} \#\operatorname{supp}\Big(\varphi_k^\varepsilon\big|_\Pi\Big) = \sup_{k\in\mathbb{Z}^d} \#\{\rho \in \Pi : \varphi_k^\varepsilon(\rho) \neq 0\} < \infty.$$

Since $q \geq p_0$, we have

$$\|\varphi_k^\varepsilon a\|_q \leq \|\varphi_k^\varepsilon a\|_{p_0} \leq N^{1/p_0}\|\varphi_k^\varepsilon a\|_\infty \leq N^{1/p_0}\|\varphi_k^\varepsilon a\|_p, \quad a \in \ell^\infty(\Pi),$$

and similarly

$$\|\varphi_k^\varepsilon a\|_p \le N^{1/p_0}\|\varphi_k^\varepsilon a\|_q, \quad a \in \ell^\infty(\Pi).$$

As a consequence, we obtain, for $q \neq \infty$,

$$\Big(\sum_{k\in\mathbb{Z}^d}\|\varphi_k^\varepsilon a\|_p^q\Big)^{1/q} \asymp \Big(\sum_{k\in\mathbb{Z}^d}\|\varphi_k^\varepsilon a\|_q^q\Big)^{1/q}, \quad a \in \ell^q(\Pi). \tag{3.13}$$

with constants depending only on the minimal index $p_0$ and $\varepsilon$, but not on $p$ and $q$.

An application of Lemma 3.7 on (3.13) yields the claim. The corresponding statement for $q = \infty$ follows similarly. □

The technical part of the proof consists of precise estimates for the Schur-type norms

$$V_{j,k}^{\varepsilon,p} := \|[A, \varphi_k^\varepsilon]\varphi_j^\varepsilon\|_{S-p}, \quad 0 < p \le 1,\ j, k \in \mathbb{Z}^d, \tag{3.14}$$

and

$$V_{j,k}^{\varepsilon} := \|[A, \varphi_k^\varepsilon]\varphi_j^\varepsilon\|_{\mathrm{Schur}}, \quad j, k \in \mathbb{Z}^d, \tag{3.15}$$

of the commutator $[A, \varphi_k^\varepsilon] = A\varphi_k^\varepsilon - \varphi_k^\varepsilon A$, when $A \in \mathcal{C}^{p_0}(\Lambda, \Pi)$ and $\varphi_k^\varepsilon$ is considered as a multiplication operator.

**Lemma 3.9.** *Let $\varepsilon > 0$, let $p_0 \in (0, 1]$, let $p, q \in (p_0, \infty]$ be such that $q \le p$, let $\Lambda, \Pi \subset \mathbb{R}^d$ be relatively separated, and suppose that*

$$A = (a_{\lambda,\rho})_{\lambda\in\Lambda,\rho\in\Pi} \in C^{p_0}(\Lambda, \Pi).$$

*Also, let $V_{j,k}^{\varepsilon,p}$ and $V_{j,k}^{\varepsilon}$ be the Schur norms given by* (3.14) *and* (3.15)*, respectively, and let $K := \max_x \Phi^\varepsilon(x)^{-\min(1,p)}$. Assume that*

$$\|c\|_p \le \|Ac\|_p, \quad c \in \ell^p(\Pi).$$

*Then the following assertions hold:*

(i) *If $p \le 1$, then*

$$\|\varphi_k^\varepsilon c\|_p^q \le \|\varphi_k^\varepsilon Ac\|_p^q + K^{q/p}\sum_{j\in\mathbb{Z}^d}(V_{j,k}^{\varepsilon,p})^{q/p}\|\varphi_j^\varepsilon c\|_p^q, \quad c \in \ell^p(\Pi). \tag{3.16}$$

(ii) *If $p > 1$, then*

$$\|\varphi_k^\varepsilon c\|_p \le \|\varphi_k^\varepsilon Ac\|_p + K\sum_{j\in\mathbb{Z}^d}V_{j,k}^{\varepsilon}\|\varphi_j^\varepsilon c\|_p, \quad c \in \ell^p(\Pi). \tag{3.17}$$

*Proof.* (i) We apply the triangle inequality for $p \le 1$ (Lemma 2.1 (ii)) and obtain

$$\begin{aligned}\|\varphi_k^\varepsilon c\|_p^p &\le \|A\varphi_k^\varepsilon c\|_p^p\\ &\le \|\varphi_k^\varepsilon Ac\|_p^p + \|[A, \varphi_k^\varepsilon]c\|_p^p\\ &\le \|\varphi_k^\varepsilon Ac\|_p^p + \sum_{j\in\mathbb{Z}^d}\|[A, \varphi_k^\varepsilon]\varphi_j^\varepsilon(\Phi^\varepsilon)^{-1}\varphi_j^\varepsilon c\|_p^p\\ &\le \|\varphi_k^\varepsilon Ac\|_p^p + K\sum_{j\in\mathbb{Z}^d}V_{j,k}^{\varepsilon,p}\|\varphi_j^\varepsilon c\|_p^p.\end{aligned}$$

Claim (i) now follows by raising this inequality to the power $q/p \le 1$ and applying Lemma 2.1 (i).

Assertion (ii) was proved in [23, (36)] with the same argument. □

Next, we consider the matrix $V^\varepsilon$ with entries $(V_{j,k}^{\varepsilon,p})^{q/p}$, $j, k \in \mathbb{Z}^d$, in case $p \le 1$ and estimate its $q/p$-Schur norm as $\varepsilon \to 0+$. First, we prove the convergence of the entries of $V^\varepsilon$.

**Lemma 3.10.** *Suppose that the hypothesis of Lemma 3.9 hold true. Then, for $\varepsilon \to 0^+$,*

$$\sup_{j,k\in\mathbb{Z}^d} V_{j,k}^{\varepsilon,p} \to 0 \quad \text{if } p \le 1, \qquad \text{and} \qquad \sup_{j,k\in\mathbb{Z}^d} V_{j,k}^{\varepsilon} \to 0 \quad \text{if } p > 1. \tag{3.18}$$

*Proof.* The case $p \ge 1$ of (3.18) was proved in [23, (38)]. The necessary adaptions for the proof of the case $p \le 1$ are as follows. We first note that the matrix entries of $[A, \varphi_k^\varepsilon]\varphi_j^\varepsilon$, for $j, k \in \mathbb{Z}^d$, are

$$([A, \varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\rho} = -a_{\lambda,\rho}\varphi_j^\varepsilon(\rho)(\varphi_k^\varepsilon(\lambda) - \varphi_k^\varepsilon(\rho)), \quad \rho \in \Pi,\ \lambda \in \Lambda.$$

Using an envelope $H \in W(C_b, L^{p_0})$ of $A$ and estimate (3.10) for $\varphi_k^\varepsilon$, we bound the entries of the commutator by

$$\big|([A, \varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\rho}\big|^p \lesssim H(\lambda-\rho)^p \min\{1, \varepsilon|\lambda-\rho|\}^p.$$

Hence, if we define

$$H^{\varepsilon,p}(x) := H(x)^p \min\{1, \varepsilon|x|\}^p,$$

then, by the choice of $H$,

$$V_{j,k}^{\varepsilon,p} = \big\|[A, \varphi_k^\varepsilon]\varphi_j^\varepsilon\big\|_{S-p} \lesssim \mathrm{rel}(\Lambda)^{p/p_0}\|H^{\varepsilon,p}\|_{W(C_b, L^{p_0/p})}.$$

Since $H \in W(C_b, L^{p_0})$ and $p_0 \le p \le 1$, it follows with dominated convergence that

$$\|H^{\varepsilon,p}\|_{W(C_b, L^{p_0/p})} \to 0 \quad \text{as } \varepsilon \to 0^+.$$

This proves (3.18) for the case $p \le 1$. □

Next, we shall estimate $V_{j,k}^{\varepsilon,p}$ in terms of

$$\Delta^{\varepsilon,q}(s) := \sum_{t\in\mathbb{Z}^d : |\varepsilon t - s|_\infty \le 5} \sup_{z\in[0,1]^d+t} |H(z)|^q, \quad s \in \mathbb{Z}^d. \tag{3.19}$$

First, we have the following lemma.

**Lemma 3.11.** *Let $0 < p_0 \le q \le 1$, let $H \in W(C_b, L^{p_0})$ and let $\Delta^{\varepsilon,q}(s)$ be given by* (3.19). *Then*

$$\sum_{s\in\mathbb{Z}^d, |s|>6\sqrt{d}} \Delta^{\varepsilon,q}(s) \to 0 \quad \textit{as } \varepsilon \to 0^+. \tag{3.20}$$

*Proof.* Since $H \in W(L^\infty, L^{p_0})$, we obtain

$$\sum_{s\in\mathbb{Z}^d, |s|>6\sqrt{d}} \Delta^{\varepsilon,q}(s) \le \sum_{s\in\mathbb{Z}^d, |s|_\infty>6} \Delta^{\varepsilon,q}(s) \lesssim \sum_{t\in\mathbb{Z}^d, |t|_\infty > 1/\varepsilon} \sup_{z\in[0,1]^d+t} |H(z)^q| \to 0$$

as $\varepsilon \to 0^+$. □

**Lemma 3.12.** *Suppose that the hypotheses of Lemma 3.9 hold true with $q \le 1$ and $\varepsilon \le 1$, and let $\Delta^{\varepsilon,q}(s)$ be given by* (3.19). *Then, for $|k-j| > 4$, we have*

$$(V_{j,k}^{\varepsilon,p})^{q/p} \le \sup_{\rho\in\Pi} \sum_{\lambda\in\Lambda} |([A, \varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\rho}|^q \lesssim \Delta^{\varepsilon,q}(k-j) \quad \textit{if } p \le 1, \tag{3.21}$$

$$(V_{j,k}^{\varepsilon})^q \lesssim \Delta^{\varepsilon,q}(k-j) \quad \textit{if } p > 1. \tag{3.22}$$

*Proof.* Suppose that $|k-j| > 4$. Since $\varphi$ is supported in $B_2(0)$, it follows that $\varphi_j^\varepsilon(\rho)\varphi_k^\varepsilon(\rho) = 0$.

As a consequence, the matrix entries of $[A, \varphi_k^\varepsilon]\varphi_j^\varepsilon$ simplify into

$$([A, \varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\rho} = -a_{\lambda,\rho}\varphi_j^\varepsilon(\rho)\varphi_k^\varepsilon(\lambda), \quad \rho \in \Pi,\ \lambda \in \Lambda.$$

This gives

$$|([A, \varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\rho}|^q \le |H(\lambda-\rho)|^q \varphi_j^\varepsilon(\rho)^q \varphi_k^\varepsilon(\lambda)^q.$$

Consequently, using (3.4) for $|k-j| > 4$ and $q/p \le 1$, we have

$$(V_{j,k}^{\varepsilon,p})^{q/p} = \big\|[A, \varphi_k^\varepsilon]\varphi_j^\varepsilon\big\|_{S-p}^{q/p} \le \sup_{\rho\in\Pi} \sum_{\lambda\in\Lambda} \big|([A, \varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\rho}\big|^q \le \sup_{\rho\in\Pi} \sum_{\lambda\in\Lambda} |H(\lambda-\rho)|^q \varphi_j^\varepsilon(\rho)^q \varphi_k^\varepsilon(\lambda)^q.$$

If $\varphi_j^\varepsilon(\rho)\varphi_k^\varepsilon(\lambda) \ne 0$, then $|\varepsilon\rho - j| \le 2$ and $|\varepsilon\lambda - k| \le 2$, whence

$$|\varepsilon(\lambda-\rho) + (j-k)| \le 4. \tag{3.23}$$

Hence,

$$\sup_{\rho\in\Pi} \sum_{\lambda\in\Lambda} \big|([A, \varphi_k^\varepsilon]\varphi_j^\varepsilon)_{\lambda,\rho}\big|^q \lesssim \sup_{\rho\in\Pi} \sum_{\substack{\lambda\in\Lambda \\ |\varepsilon(\lambda-\rho)+(j-k)|\le 4}} |H(\rho-\lambda)|^q. \tag{3.24}$$

For fixed $\varepsilon \le 1$, we bound the sum in (3.24) by

$$\begin{aligned}\sum_{\lambda\in\Lambda:|\varepsilon(\lambda-\rho)+(j-k)|\le 4} |H(\rho-\lambda)|^q &\le \sum_{t\in\mathbb{Z}^d} \sum_{\substack{\lambda\in\Lambda:|\varepsilon(\lambda-\rho)+(j-k)|\le 4\\ (\lambda-\rho)\in[0,1]^d+t}} |H(\rho-\lambda)|^q\\ &\lesssim \operatorname{rel}(\rho-\Lambda) \sum_{t\in\mathbb{Z}^d:|\varepsilon t+(j-k)|_\infty\le 5} \sup_{z\in[0,1]^d+t} |H(z)|^q\\ &\lesssim \Delta^{\varepsilon,q}(k-j),\end{aligned}$$

since $\operatorname{rel}(\Lambda)$ is translation-invariant. By substituting this bound in (3.24), we obtain (3.21). Analogously, we obtain (3.22). □

**Lemma 3.13.** *Suppose that the hypotheses of Lemma 3.9 hold for $p \le 1$. Then*

$$\sup_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} (V_{j,k}^{\varepsilon,p})^{q/p} + \sup_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (V_{j,k}^{\varepsilon,p})^{q/p} \to 0 \quad \textit{as } \varepsilon \to 0^+. \tag{3.25}$$

*Proof.* Let $\varepsilon \le 1$ and $\Delta^{\varepsilon,q}$ be defined as in (3.19). Fix $j \in \mathbb{Z}^d$ and use Lemma 3.12 to estimate

$$\sum_{k:|k-j|>6\sqrt{d}} (V_{j,k}^{\varepsilon,p})^{q/p} \lesssim \sum_{k:|k-j|>6\sqrt{d}} \Delta^{\varepsilon,q}(k-j).$$

For the sum over $\{k \in \mathbb{Z}^d : |j-k| \le 6\sqrt{d}\}$, we use the bound

$$\sum_{k:|j-k|\le 6\sqrt{d}} (V_{j,k}^{\varepsilon,p})^{q/p} \le \#\{k : |j-k| \le 6\sqrt{d}\} \sup_{s,t} (V_{s,t}^{\varepsilon,p})^{q/p} \lesssim \sup_{s,t} (V_{s,t}^{\varepsilon,p})^{q/p}.$$

Hence,

$$\sum_{k\in\mathbb{Z}^d} (V_{j,k}^{\varepsilon,p})^{q/p} \lesssim \sup_{s,t} (V_{s,t}^{\varepsilon,p})^{q/p} + \sum_{|s|>6\sqrt{d}} \Delta^{\varepsilon,q}(s),$$

which tends to 0 uniformly in $j$ as $\varepsilon \to 0^+$ by Lemmas 3.10 and 3.11. The convergence of the first term in (3.25) follows in exactly the same way by interchanging the roles of $j$ and $k$. □

With the auxiliary results at hand, we now prove Theorem 3.5.

*Proof of Theorem 3.5.* As already mentioned, we restrict ourselves to the case $p_0 < 1$; cf. Remark 3.6. Since

$$W(C^0; L^{p_0})(\mathbb{R}^d) \subseteq W(L^\infty; L^1)(\mathbb{R}^d),$$

an application of [23, Proposition 8.1] provides the claim in the Banach space case $p, q \ge 1$.

Case $p \le 1$ and $q < p$. We have $q/p < 1$. After multiplying $A$ with a constant, we may assume that

$$\|c\|_p \le \|Ac\|_p, \quad c \in \ell^p(\Pi),$$

since $A$ is bounded from below on $\ell^p(\Pi)$ by assumption (3.7). By (3.16), we obtain

$$\sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon c\|_p^q \le \sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon Ac\|_p^q + K^{q/p} \sum_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} (V_{j,k}^{\varepsilon,p})^{q/p} \|\varphi_j^\varepsilon c\|_p^q, \quad c \in \ell^p(\Pi), \tag{3.26}$$

for some $K > 0$. According to Lemma 3.13, we may choose $\varepsilon > 0$ such that

$$K^{q/p} \sup_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} (V_{j,k}^{\varepsilon,p})^{q/p} < \frac{1}{2}.$$

Using this bound in (3.26) and Proposition 3.2, we obtain that

$$\sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon c\|_p^q \le \sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon Ac\|_p^q + \frac{1}{2} \sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon c\|_p^q.$$

Hence,

$$\Big(\sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon c\|_p^q\Big)^{1/q} \le 2^{1/q} \Big(\sum_{k\in\mathbb{Z}^d} \|\varphi_k^\varepsilon Ac\|_p^q\Big)^{1/q}. \tag{3.27}$$

Using the equivalent norm of Lemma 3.8 in (3.27), we deduce that, for all $p_0 \le q < p$,

$$\|c\|_q \lesssim \|Ac\|_q,$$

with a constant independent of $q$ (since $2^{1/q} \le 2^{1/p_0}$). This completes the proof of the case $p_0 \le q \le p$.

Case $p < 1$ and $p < q$. Again we may assume that

$$\|c\|_p \le \|Ac\|_p, \quad c \in \ell^p(\Pi).$$

According to Lemma 3.13, in case $q = p$ we may choose $\varepsilon > 0$ such that

$$K \sup_{j\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}^d} V^{\varepsilon,p}_{j,k} < \frac{1}{2} \quad \text{and} \quad K \sup_{k\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} V^{\varepsilon,p}_{j,k} < \frac{1}{2}. \tag{3.28}$$

Due to Lemma 3.9, we have, for $c \in \ell^p(\Pi)$,

$$\|\varphi^\varepsilon_k c\|^p_p \le \|\varphi^\varepsilon_k Ac\|^p_p + K \sum_{j\in\mathbb{Z}^d} V^{\varepsilon,p}_{j,k} \|\varphi^\varepsilon_j c\|^p_p.$$

We set $a^\varepsilon_k = \|\varphi^\varepsilon_k c\|^p_p$, $b^\varepsilon_k = \|\varphi^\varepsilon_k Ac\|^p_p$ and let $V^\varepsilon$ be the operator associated to the matrix $V^{\varepsilon,p}_{j,k}$. Then the previous inequality can be written as

$$a^\varepsilon_k \le b^\varepsilon_k + K(V^\varepsilon a^\varepsilon)_k.$$

We take the $q/p$–norm of the previous estimate, first applying the triangle inequality. Next, we apply Proposition 3.1, use (3.28) and get

$$\|a^\varepsilon\|_{q/p} \le \|b^\varepsilon\|_{q/p} + K\|V^\varepsilon a^\varepsilon\|_{q/p} \le \|b^\varepsilon\|_{q/p} + \frac{1}{2}\|a^\varepsilon\|_{q/p}.$$

In other words, we have $\|a^\varepsilon\|_{q/p} \le 2\|b^\varepsilon\|_{q/p}$. Reversing the abbreviations, this means that

$$\Big(\sum_{k\in\mathbb{Z}^d} \|\varphi^\varepsilon_k c\|^q_p\Big)^{1/q} = \|a^\varepsilon\|^{1/p}_p \le 2^{1/p}\|b^\varepsilon\|^{1/p}_p = 2^{1/p}\Big(\sum_{k\in\mathbb{Z}^d} \|\varphi^\varepsilon_k Ac\|^q_p\Big)^{1/q}.$$

An application of the norm equivalence of Lemma 3.8 provides the claim

$$\|c\|_q \lesssim \|Ac\|_q.$$

This concludes the proof. □

As observed in [23, Remark A.2], the lower bound guaranteed by Theorem 3.5 is uniform for all $p$.

Now, as an immediate consequence of Theorem 3.5, we get a first spectral invariance result of a matrix $A \in \mathcal{C}^{p_0}(\Lambda, \Pi)$ by means of standard functional analytical arguments.

**Corollary 3.14.** *Let $p_0 \in (0, 1]$, let $\Lambda, \Pi \subseteq \mathbb{R}^d$ be relatively separated and suppose that $A \in C^{p_0}(\Lambda, \Pi)$ is invertible from $\ell^p(\Pi)$ to $\ell^p(\Lambda)$ for some $p \in [p_0, \infty]$. Then the following assertions hold:*
(i) *$A$ is injective from $\ell^q(\Pi)$ to $\ell^q(\Lambda)$ for all $q \in [p_0, \infty]$.*
(ii) *$A$ is invertible from $\ell^q(\Pi)$ to $\ell^q(\Lambda)$ for all $q \in [p, \infty)$.*

*Proof.* As already observed, $A : \ell^p(\Pi) \to \ell^p(\Lambda)$ satisfies the stability condition $\|c\|_p \lesssim \|Ac\|_p$ if and only if $A$ is one-to-one on $\ell^p(\Pi)$ and has closed range in $\ell^p(\Lambda)$. Thus, if $A$ is invertible on some $\ell^p(\Pi)$, then, by Theorem 3.5 $A$ satisfies the stability condition (3.7) for all $q \ge p_0$. Consequently, $A$ is one-to-one on all $\ell^q(\Pi)$ for $q \ge p_0$, which is (i).

If $q \ge p$, then $\ell^p(\Lambda)$ is dense in $\ell^q(\Lambda)$. Since $A$ is onto $\ell^p(\Lambda)$, we obtain $\ell^p(\Lambda) = A\ell^p(\Pi) \subseteq \ell^q(\Lambda)$, and $A$ has also dense range in $\ell^q(\Lambda)$. Consequently, $A$ is onto $\ell^q(\Lambda)$, and thus invertible on $\ell^q(\Lambda)$. □

In the case, when $A$ is invertible on the Hilbert space $\ell^2$, the above results are already contained in [33, Theorem 4.6 and Theorem 8.5], in [8, Theorem 3.9] and in [11].

Note that Corollary 3.14 asserts only that the invertibility of $A$ on $\ell^p$ implies the invertibility on the larger space $\ell^q$, $q > p$. To obtain the same conclusion for the smaller spaces $\ell^q$, $q < p$, we need to refine our arguments. We will show that the inverse of $A$ has an envelope belonging to the same Wiener amalgam space as the envelope of $A$. The main idea of the proof of the following theorem is taken from [36].

**Theorem 3.15.** *Let $0 < p_0 \leq 1$ and let $\Lambda, \Pi \subseteq \mathbb{R}^d$ be relatively separated. Suppose that $A \in \mathcal{C}^{p_0}(\Lambda, \Pi)$ is invertible from $\ell^p(\Pi)$ to $\ell^p(\Lambda)$ for some $p \in [p_0, \infty]$. Then $A^{-1} \in \mathcal{C}^{p_0}(\Pi, \Lambda)$.*

*Proof.* By assumption, the matrix $A = (a_{\lambda,\rho})_{\lambda\in\Lambda,\rho\in\Pi}$ has an envelope $H_1 \in W(C_b, L^{p_0})$ so that $|a_{\lambda,\rho}| \leq H_1(\lambda - \rho)$. We need to prove that the matrix $A^{-1}$ also has an envelope $H_2 \in W(C_b, L^{p_0})$. For this we recall the notation used in the previous lemmas: $K = \max_x \Phi^\varepsilon(x)^{-\min(1,p)}$ from Lemma 3.9, and

$$\widetilde{V}^\varepsilon = \begin{cases} ((KV^{\varepsilon,p}_{j,k})^{p_0/p})_{j,k\in\mathbb{Z}^d}, & p \leq 1, \\ ((KV^{\varepsilon}_{j,k})^{p_0})_{j,k\in\mathbb{Z}^d}, & p > 1, \end{cases}$$

where $V^{\varepsilon,p}_{j,k}$ and $V^{\varepsilon}_{j,k}$ are defined as in (3.14) and (3.15), respectively.

Given $c \in \ell^p(\Pi)$, the sequences $a$ and $a_A$ are defined by

$$a_k = \|\varphi^\varepsilon_k c\|^{p_0}_p \quad \text{and} \quad a_{A,k} = \|\varphi^\varepsilon_k Ac\|^{p_0}_p, \qquad k \in \mathbb{Z}^d.$$

With this notation, Lemma 3.9 (and additionally Lemma 2.1 if $p > 1$) says that

$$a \leq a_A + \widetilde{V}^\varepsilon a. \tag{3.29}$$

Furthermore, Lemma 3.12 can be reformulated as saying that $\widetilde{V}^\varepsilon$ is convolution-dominated. In fact, recall from (3.19) that

$$\Delta^{\varepsilon,p_0}(s) := \sum_{t\in\mathbb{Z}^d:|\varepsilon t - s|_\infty \leq 5} \sup_{z\in[0,1]^d+t} |H_1(z)|^{p_0},$$

and define $\Psi^\varepsilon$ by $\Psi^\varepsilon(s) := \Delta^{\varepsilon,p_0}(s)$ for all $s \in \mathbb{Z}^d$ with $|s| > 6\sqrt{d}$, and for all $s \in \mathbb{Z}^d$ with $|s| \leq 6\sqrt{d}$, by

$$\Psi^\varepsilon(s) := \big(\sup_{j,k\in\mathbb{Z}^d} |V^{\varepsilon,p}_{j,k}|\big)^{p_0/p}$$

if $p \leq 1$, and by

$$\Psi^\varepsilon(s) := \big(\sup_{j,k\in\mathbb{Z}^d} |V^{\varepsilon}_{j,k}|\big)^{p_0}$$

if $p \geq 1$. In view of the normalization $K$, $\tilde{V}^\varepsilon$ possesses the envelope $K^{p_0/\min(1,p)}\Psi^\varepsilon$. This means that

$$(V^{\varepsilon,p}_{j,k})^{p_0/p} \lesssim \Psi^\varepsilon(k-j), \quad p \leq 1,\ j,k \in \mathbb{Z}^d, \tag{3.30}$$

and

$$V^{\varepsilon}_{j,k} \lesssim \Psi^\varepsilon(k-j), \quad p > 1,\ j,k \in \mathbb{Z}^d. \tag{3.31}$$

Our next goal is to represent $(I - \widetilde{V}^\varepsilon)^{-1}$ as a Neumann series. For this we choose $\varepsilon > 0$ such that

$$\|\widetilde{V}^\varepsilon\|_{S-1} \leq K^{p_0/\min(1,p)}\|\Psi^\varepsilon\|_1 \leq \frac{1}{2}. \tag{3.32}$$

This is possible due to Lemma 3.11 and Lemma 3.10, since $\sharp\{s \in \mathbb{Z}^d : |s| \leq 6\sqrt{d}\}$ is finite and depends only on the dimension $d$.

As a consequence, the geometric series $\widetilde{W} := \sum_{k=1}^\infty (\widetilde{V}^\varepsilon)^k$ converges in the $\|\cdot\|_{S-1}$-norm and we obtain

$$(\mathrm{I} - \widetilde{V}^\varepsilon)^{-1} = \sum_{k=0}^\infty (\widetilde{V}^\varepsilon)^k = \mathrm{I} + \widetilde{W}.$$

Since all entries of $\widetilde{V}^\varepsilon$ are non-negative by definition, $\widetilde{W}$ also has only non-negative entries and preserves (pointwise) inequalities. Moreover, since $\widetilde{V}^\varepsilon$ is convolution-dominated, so is $\widetilde{W}$, and by (3.32) there exists an envelope $W \in \ell^1(\mathbb{Z}^d)$ such that

$$\widetilde{W}_{jk} \leq W(k-j), \quad j,k \in \mathbb{Z}^d.$$

Now, (3.29) yields

$$a = (\mathrm{I} + \widetilde{W})(\mathrm{I} - \widetilde{V}^\varepsilon)a \leq (\mathrm{I} + \widetilde{W})a_A, \tag{3.33}$$

or entrywise

$$\|\varphi^\varepsilon_k c\|^{p_0}_p \leq \|\varphi^\varepsilon_k Ac\|^{p_0}_p + \sum_{j\in\mathbb{Z}^d} W(k-j)\|\varphi^\varepsilon_j Ac\|^{p_0}_p. \tag{3.34}$$

Since $A$ is assumed to be invertible as a map from $\ell^p(\Pi)$ to $\ell^p(\Lambda)$, there exist $b_\lambda \in \ell^p(\Pi)$ such that $Ab_\lambda = \delta_\lambda$, whence the matrix $B$ with entries $b_{\rho,\lambda} = (b_\lambda)_\rho$ is the inverse of $A$. Using $b_\lambda$ in (3.34), we obtain

$$\|\varphi_k^\varepsilon b_\lambda\|_p^{p_0} \leq \|\varphi_k^\varepsilon \delta_\lambda\|_p^{p_0} + \sum_{j\in\mathbb{Z}^d} W(k-j)\|\varphi_j^\varepsilon \delta_\lambda\|_p^{p_0}. \tag{3.35}$$

Note that

$$\|\varphi_k^\varepsilon \delta_\lambda\|_p^{p_0} = \Big(\sum_{\rho\in\Lambda} \varphi(\varepsilon\rho - k)^p \delta_\lambda(\rho)\Big)^{p_0/p} = \varphi(\varepsilon\lambda - k)^{p_0}.$$

Let $\lambda \in \Lambda$ and $\rho \in \Pi$. For the off-diagonal decay, it suffices to consider only indices satisfying $\varepsilon|\lambda-\rho| > 4$. Choose $k_\rho \in \mathbb{Z}^d$ such that

$$|\varepsilon\rho - k_\rho| < 2 \quad \text{and} \quad \varphi(\varepsilon\rho - k_\rho)^{p_0} \geq c$$

for some constant $c$ (in fact, by (A.2) in the proof of Lemma 3.7, we have $c = \eta^{-1}$). Then

$$|\varepsilon\lambda - k_\rho| \geq \varepsilon|\lambda-\rho| - |\varepsilon\rho - k_\rho| > 2,$$

and consequently $\varphi(\varepsilon\lambda - k_\rho) = 0$ in (3.35) since $\operatorname{supp}(\varphi) \subseteq B_2(0)$. Now, (3.35) simplifies to

$$c|(b_\lambda)_\rho|^{p_0} \leq \|\varphi_{k_\rho}^\varepsilon b_\lambda\|_p^{p_0} \leq \sum_{j\in\mathbb{Z}^d} W(k_\rho - j)\varphi(\varepsilon\lambda - j)^{p_0} \leq \sum_{j:|j-\varepsilon\lambda|<2} W(k_\rho - j). \tag{3.36}$$

This inequality suggests the following envelope for $B = A^{-1}$. Let

$$H(x) = \sum_{l\in\mathbb{Z}^d:|l-\varepsilon x|<4} W(l). \tag{3.37}$$

To obtain a continuous envelope, we use a cut-off function $\psi \in C_c^\infty(\mathbb{R}^d)$ satisfying $0 \leq \psi \leq 1$ and $\psi(x) = 1$ for $|x| \leq 4$ and set

$$\tilde{H}(x) = \sum_{l\in\mathbb{Z}^d} W(l)\psi(\varepsilon x - l).$$

Since $\psi$ has compact support and $W(C_b, \ell^1)$ is translation invariant, we have

$$\|\tilde{H}\|_{W(C_b,\ell^1)} \leq \sum_{l\in\mathbb{Z}^d} |W(l)| \sup_{l\in\mathbb{Z}^d} \|\psi(\varepsilon\,\cdot\, - l)\|_{W(C_b,\ell^1)} \lesssim \|W\|_{\ell^1},$$

and therefore $\tilde{H} \in W(C_b, \ell^1)$.

Furthermore, since

$$|k_\rho - j - \varepsilon(\lambda - \rho)| \leq |k_\rho - \varepsilon\rho| + |\varepsilon\lambda - j| < 4,$$

equation (3.37) says that

$$|b_{\rho,\lambda}|^{p_0} \leq H(\rho - \lambda) \leq \tilde{H}(\rho - \lambda) \tag{3.38}$$

or

$$|b_{\rho,\lambda}| = |(b_\lambda)_\rho| \leq \tilde{H}(\rho - \lambda)^{1/p_0},$$

and $\tilde{H}^{1/p_0} \in W(C_b, L^{p_0})$, as claimed. □

This theorem enables us to extend the spectral invariance result of Corollary 3.14 to the case $p_0 \leq q \leq p$ as follows.

**Theorem 3.16.** *Let $p_0 \in (0,1]$, let $\Lambda, \Pi \subseteq \mathbb{R}^d$ be relatively separated and let $A \in C^{p_0}(\Lambda, \Pi)$ be invertible from $\ell^p(\Pi)$ to $\ell^p(\Lambda)$ for some $p \in [p_0, \infty]$. Then $A$ is invertible from $\ell^q(\Pi)$ to $\ell^q(\Lambda)$ for all $q \in [p_0, \infty)$.*

*Proof.* According to Corollary 3.14 (ii), $A$ is invertible on $\ell^q$ for $q \geq p$ and $A$ is one-to-one on all $\ell^q$. So, it remains to prove that $A$ is surjective for $p_0 \leq q < p$. We assume that $A$ is invertible on $\ell^p$ with inverse $A^{-1}$. Hence for $u \in \ell^q(\Lambda) \subseteq \ell^p(\Lambda)$, there is some $c \in \ell^p(\Pi)$ with $c = A^{-1}u$. By Theorem 3.15, $A^{-1}$ is bounded on $\ell^q$. Consequently, since $u \in \ell^q$, we have $c = A^{-1}u \in \ell^q$. Thus $A$ is onto $\ell^q(\Lambda)$. □

The following consequence explains why the statement of Theorem 3.16 is referred to as the spectral invariance property.

**Corollary 3.17.** *Let $0 < p_0 \leq 1$ be arbitrary. If $A \in \mathcal{C}^{p_0}(\Lambda)$ and $A$ is invertible on $\ell^p(\Lambda)$ for some $p \in [p_0, \infty]$, then $A^{-1} \in \mathcal{C}^{p_0}(\Lambda)$ and*

$$Sp_{\mathcal{B}(\ell^q)}(A) = Sp_{\mathcal{C}^{p_0}}(A) \quad \text{for all } q \in [p_0, \infty),$$

*where $Sp_{\mathcal{A}}(A)$ denotes the spectrum of $A$ in the algebra $\mathcal{A}$.*

In the literature, many variations of this spectral invariance result exist. For an overview of those variations we refer to [21]. Here we just want to mention the following ones: In case $p_0 = 1$ and $p = q = 2$ the previous theorem also holds in the weighted case. It was proved by Baskakov, e.g., in [3, 4] and by Sjöstrand [36] in the unweighted case.

# 4 Spectral invariance of pseudodifferential operators

The aim of this section is to transfer the results on the spectral invariance of matrices to pseudodifferential operators on unweighted modulation spaces. In the proofs, we mainly follow [20, 24] and replace an arbitrary spectrally invariant Banach algebra of matrices by the quasi-Banach algebra $\mathcal{C}^{p_0}$.

First, we list all needed definitions and properties to reach that aim. We start with recalling the modulation spaces.

## 4.1 Modulation spaces $M^{p,q}$

For the definition of the modulation spaces, we need the short-time Fourier transform, which we recall now.

Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ be fixed. For every $f \in \mathcal{S}'(\mathbb{R}^d)$, the *short-time Fourier transform* $V_g f$ is the function on $\mathbb{R}^{2d}$ defined by the formula

$$V_g f(x, \xi) := \langle f, g(\cdot - x) e^{2\pi i \xi \cdot \cdot} \rangle. \tag{4.1}$$

Here $\langle \cdot, \cdot \rangle$ is the unique extension of the $L^2$ scalar product on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$. We observe that if $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then $V_g f$ is given by (1.1). If $g$ and $f$ are both defined on $\mathbb{R}^{2d}$, then $V_g f$ is a function on $\mathbb{R}^{4d}$.

We recall that, for $0 < p, q \leq \infty$ and fixed $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that (1.2) is finite. If $1 \leq p, q \leq \infty$, then $M^{p,q}(\mathbb{R}^d)$ is a Banach space with norm $\|\cdot\|_{M^{p,q}}$ (see [14]). Otherwise, $M^{p,q}(\mathbb{R}^d)$ is a quasi-Banach space with quasi-norm $\|\cdot\|_{M^{p,q}}$. We write $M^p = M^{p,p}$.

It is well known that the definition of $M^{p,q}(\mathbb{R}^d)$ is independent of the choice of the window function $g \in \mathcal{S}(\mathbb{R}^d)$ (see [19]). For $p < 1$ or $q < 1$, the proof of this fact can be found in [18]. Additionally, the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)$ in the case $p, q < \infty$; cf. [18, Remark 14].

Due to [31, Theorem 3.6] and [18, Theorem 3.4], the following continuous embeddings of modulation spaces hold.

**Proposition 4.1.** *Let $0 < p_1 \leq p_2 \leq \infty$ and $0 < q_1 \leq q_2 \leq \infty$. Then*

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow M^{p_1,q_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

## 4.2 Gabor frames

For the definition of Gabor frames, it is convenient to use time-frequency shifts $\pi(z)f$ of $f \in \mathcal{S}'(\mathbb{R}^d)$ given by

$$(\pi(z)f)(t) := e^{2\pi i \xi \cdot t} f(t - x), \quad z = (x, \xi) \in \mathbb{R}^{2d},\ t \in \mathbb{R}^d.$$

The *Gabor system* with respect to the *(Gabor) atom* $g \in M^1(\mathbb{R}^d) \setminus \{0\}$ and lattice $\Lambda \subseteq \mathbb{R}^{2d}$ is given by

$$\mathcal{G}(g, \Lambda) = \{\pi(\lambda) g : \lambda \in \Lambda\}.$$

Then the *analysis operator* and *synthesis operator*

$$C_g : M^\infty(\mathbb{R}^d) \to \ell^\infty(\Lambda) \quad \text{and} \quad D_g : \ell^\infty(\Lambda) \to M^\infty(\mathbb{R}^d),$$

respectively, with respect to $g$ and $\Lambda$, are given by

$$C_g f = (\langle f, \pi(\lambda)g\rangle)_{\lambda\in\Lambda} \quad \text{and} \quad D_g c = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g,$$

when $f \in M^\infty(\mathbb{R}^d)$ and $c = (c_\lambda)_{\lambda\in\Lambda} \in \ell^\infty(\Lambda)$. Here the series converges in $\mathcal{S}'(\mathbb{R}^d)$.

The (Gabor) frame operator

$$S = S_{g_1,g_2,\Lambda} : M^\infty(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d), \quad g_1, g_2 \in M^1(\mathbb{R}^d),$$

is defined by

$$S_{g_1,g_2,\Lambda} f := \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g_1\rangle \pi(\lambda) g_2. \tag{4.2}$$

If $g_1 = g_2 = g$, we write $S_{g,\Lambda}$ instead of $S_{g_1,g_2,\Lambda}$. It follows that $C_g, D_g$ and $S_{g,\Lambda}$ are well-defined and continuous (see, e.g., [19, Chapters 11–14]). Let $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. If $S_{g,\Lambda}$ is bounded and invertible on $L^2(\mathbb{R}^d)$, then we call $\mathcal{G}(g, \Lambda)$ *Gabor frame*. For rectangular lattices $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$, the existence of Gabor frames is well understood; see [12, 19, 30, 44].

For every Gabor frame $\mathcal{G}(g, \Lambda)$ over a lattice, there exists a dual window $\gamma = S^{-1}g \in L^2(\mathbb{R}^d)$ so that every $f$ can be expanded into a Gabor expansion

$$f = D_g C_\gamma f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda) g, \tag{4.3}$$

$$f = D_\gamma C_g f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda) \gamma. \tag{4.4}$$

For $g \in \mathcal{S}(\mathbb{R}^d)$, a fundamental result of Janssen [30] asserts that also $\gamma \in \mathcal{S}(\mathbb{R}^d)$. Then the expansion formulas (4.3) and (4.4) hold for every $f \in \mathcal{S}'(\mathbb{R}^d)$ with weak-$*$-convergence.

A Gabor frame $\mathcal{G}(g, \Lambda)$ is called tight if $S_{g,\Lambda} = C\,\mathrm{Id}$ for some $C > 0$. In this case, $\gamma = S_{g,\Lambda}^{-1} g = C^{-1} g$, and the Gabor expansion

$$f = C^{-1} \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda) g$$

looks like an orthonormal expansion.

Tight Gabor frames with the constant $C = 1$ can be constructed as follows. Let $\mathcal{G}(g, \Lambda)$ be a Gabor frame with frame operator $S_{g,\Lambda}$. Due to [20, Lemma 5.16], $S_{g,\Lambda}^{-1/2}\mathcal{G}(g, \Lambda)$ is a tight Gabor frame with constant $C = 1$. By applying this procedure to a Gaussian window, one sees that there exist tight Gabor frames $\mathcal{G}(g, \Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$.

For future references, we remark that if $p \in (0, \infty]$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, then

$$f \in M^p(\mathbb{R}^d) \iff C_g f \in \ell^p(\Lambda) \iff C_\gamma f \in \ell^p(\Lambda), \tag{4.5}$$

with norm equivalence $\|f\|_{M^p} \asymp \|C_g f\|_{\ell^p} \asymp \|C_\gamma f\|_{\ell^p}$; see, e.g., [15].

## 4.3 Pseudodifferential operators

For a real-valued $d \times d$-matrix $A \in \mathbb{R}^{d\times d}$ and a symbol $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$, the pseudodifferential operator $Op_A(\mathfrak{a})$ is defined by

$$Op_A(\mathfrak{a})u(x) = \iint \mathfrak{a}(x - A(x - y), \xi) f(y) e^{2\pi i (x-y)\cdot\xi}\, dy\, d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

where the integrals should be interpreted in distribution sense, if necessary. If $A = 0$, then $Op_A(\mathfrak{a})$ agrees with the Kohn–Nirenberg or normal representation $a(x, D)$. If instead $A = \frac{1}{2}I$, where $I$ is the $d \times d$ identity matrix, then $Op_A(\mathfrak{a})$ is the Weyl quantization $\mathfrak{a}^w$ of $\mathfrak{a}$.

By [28, 43], for each symbol $\mathfrak{a}_1 \in \mathcal{S}'(\mathbb{R}^{2d})$ and each $A_1, A_2 \in \mathbb{R}^{d\times d}$, there is a unique $\mathfrak{a}_2 \in \mathcal{S}'(\mathbb{R}^{2d})$ such that $Op_{A_1}(\mathfrak{a}_1) = Op_{A_2}(\mathfrak{a}_2)$ and such that

$$Op_{A_1}(\mathfrak{a}_1) = Op_{A_2}(\mathfrak{a}_2) \quad \text{if and only if} \quad \mathfrak{a}_2(x,\xi) = e^{2\pi it((A_1-A_2)D_\xi)\cdot D_x}\mathfrak{a}_1(x,\xi). \tag{4.6}$$

We refer to [43] for the proof of the following result.

**Proposition 4.2.** *Let $A, A_1, A_2 \in \mathbb{R}^{d\times d}$ and $p, q \in (0,\infty]$. Then the following assertions hold:*
(i) *$e^{2\pi it(AD_\xi)\cdot D_x}$ is an isomorphism on $M^{p,q}(\mathbb{R}^{2d})$.*
(ii) *$Op_{A_1}(M^{p,q}(\mathbb{R}^{2d})) = Op_{A_2}(M^{p,q}(\mathbb{R}^{2d}))$.*

We can write a Weyl operator by means of the Wigner distribution $W$ of $f, g \in L^2(\mathbb{R}^d)$, which is defined by

$$W(f,g)(x,\xi) := \int_{\mathbb{R}^d} f\Big(x+\frac{t}{2}\Big)\overline{g\Big(x-\frac{t}{2}\Big)}e^{2\pi i\xi\cdot t}\,dt, \quad x,\xi \in \mathbb{R}^d. \tag{4.7}$$

Denote the inversion $\breve{g}$ of $g \in \mathcal{S}'(\mathbb{R}^d)$ by $\breve{g}(x) := g(-x)$ for all $x \in \mathbb{R}^d$. Then

$$W(f,g)(x,\xi) = 2^d e^{4\pi ix\cdot\xi}V_{\breve{g}}f(2x,2\xi),$$

and the Wigner distribution is just a slight modification of the short-time Fourier transform. Since the short-time Fourier transform satisfies $V_\cdot : \mathcal{S}(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d})$, we immediately get $W(f,g) \in \mathcal{S}(\mathbb{R}^{2d})$ for all $f, g \in \mathcal{S}(\mathbb{R}^d)$.

By means of the Wigner distribution the Weyl operator of a symbol $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$ is given by the formula

$$\langle \mathfrak{a}^w f, g\rangle = \langle \mathfrak{a}, W(g,f)\rangle \quad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^d).$$

Pseudodifferential operators of Weyl form are continuous maps from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$; see [28, 41]. Moreover, they are continuous as maps between certain modulation spaces; cf. [19, 22].

**Proposition 4.3.** *Let $0 < p_0 \le 1$, $p, q \in [p_0,\infty]$ and $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$. Then $\mathfrak{a}^w$ is bounded on $M^{p,q}(\mathbb{R}^d)$.*

As proved in [42, Theorem 3.1], this theorem also holds for more general weighted modulation spaces.

**Remark 4.4.** Due to [41], it follows that $M^{p,q}(\mathbb{R}^{2d})$ is invariant under actions with chirps $e^{i(AD_\xi)\cdot D_x}$ with $A \in \mathbb{R}^{d\times d}$ for all $p, q \in (0,\infty]$. Hence all results concerning Weyl operators of this paper also hold for operators of the form $Op_A(a)$.

Next, we show that Gabor frame operators with windows in $M^{p_0}$ are pseudodifferential operators with symbols in $M^{\infty,p_0}$.

**Proposition 4.5.** *Let $p_0 \in (0,1]$, let $g_1, g_2 \in M^{p_0}(\mathbb{R}^d)$, let $A \in M(d\times d;\mathbb{R})$ be a $d\times d$-matrix, let $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice, and let $\mathfrak{a} \in \mathscr{S}'(\mathbb{R}^{2d})$ be such that $S_{g_1,g_2,\Lambda} = Op_A(\mathfrak{a})$. Then $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$.*

*Proof.* In view of Remark 4.4, we may assume that $A = 0$. The Weyl symbol of $S_{g_1,g_2,\Lambda}$ with $g_1 = g_2$ was calculated in [2, p. 12], and by similar arguments it follows that the Kohn–Nirenberg symbol is

$$\mathfrak{a}(x,\xi) = \sum_{(l,\lambda)\in\Lambda} g_1(x-l)\overline{\widehat{g}_2(\xi-\lambda)}e^{2\pi i(x-l)\cdot(\lambda-\xi)}. \tag{4.8}$$

In order to estimate the $M^{\infty,p_0}$-norm of $\mathfrak{a}$, we choose the window $\Psi \in \mathcal{S}(\mathbb{R}^{2d})\setminus\{0\}$ as

$$\Psi(x,\xi) := \psi(x)\overline{\widehat{\psi}(\xi)}e^{-2\pi ix\cdot\xi}.$$

By straightforward applications of the Fourier inversion formula, it follows that (see, e.g., [26])

$$(V_\Psi\mathfrak{a})(z,w) = \sum_{l,\lambda\in\Lambda} e^{-2\pi i(y\cdot\xi+l\cdot\eta)}(V_\psi g_1)(x-l,\xi+\eta-\lambda)\cdot\overline{(V_\psi g_2)(x+y-l,\xi-\lambda)}, \tag{4.9}$$

with $z = (x,\xi) \in \mathbb{R}^d\times\mathbb{R}^d$ and $w = (\eta,y) \in \mathbb{R}^d\times\mathbb{R}^d$.

Now, let $\Lambda^2 = \Lambda \times \Lambda$, and $\Lambda^4 = \Lambda^2 \times \Lambda^2$ and choose $\varepsilon = \frac{1}{N}$ with integer $N \geq 1$ large enough such that

$$\{\psi(x-m)e^{2\pi i x\cdot\mu}\}_{m,\mu\in\varepsilon\Lambda} \quad \text{and} \quad \{\Psi(z-w_1)e^{2\pi i z\cdot w_2}\}_{w_1,w_2\in\varepsilon\Lambda^2}$$

are Gabor frames of $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^{2d})$, respectively. Then

$$\|g_j\|_{M^{p_0}} \asymp \|V_\psi g_j\|_{\ell^{p_0}(\varepsilon\Lambda^2)} \quad \text{and} \quad \|\mathfrak{a}\|_{M^{\infty,p_0}} \asymp \|V_\Psi \mathfrak{a}\|_{\ell^{\infty,p_0}(\varepsilon\Lambda^4)},$$

in view of a result in [18].

Let $F_j = |V_\psi g_j|$. If $w_1 = (m,\mu) \in \varepsilon\Lambda^2$ and $w_2 = (\nu, n) \in \varepsilon\Lambda^2$, then (4.9) yields

$$\begin{aligned}|(V_\Psi \mathfrak{a})(w_1,w_2)|^{p_0} &\leq \sum_{l,\lambda\in\Lambda} F_1(m-l,\mu+\nu-\lambda)^{p_0} F_2(m+n-l,\mu-\lambda)^{p_0}\\ &= (|\check{F}_1|^{p_0} * |F_2|^{p_0})(Tw_2), \quad Tw_2 = (n,-\nu).\end{aligned}$$

The right-hand side does not depend on $w_1$, and therefore

$$\begin{aligned}\|\mathfrak{a}\|_{M^{\infty,p_0}} &\asymp \|V_\Psi \mathfrak{a}\|_{\ell^{\infty,p_0}(\varepsilon\Lambda^4)}\\ &\lesssim \|\check{F}_1^{p_0} * F_2^{p_0}\|_{\ell^1(\varepsilon\Lambda^2)}^{1/p_0}\\ &\leq \big(\|F_1^{p_0}\|_{\ell^1(\varepsilon\Lambda^2)}\|F_2^{p_0}\|_{\ell^1(\varepsilon\Lambda^2)}\big)^{1/p_0}\\ &= \|F_1\|_{\ell^{p_0}(\varepsilon\Lambda^2)}\|F_2\|_{\ell^{p_0}(\varepsilon\Lambda^2)}\\ &\asymp \|g_1\|_{M^{p_0}}\|g_2\|_{M^{p_0}}.\end{aligned}$$

Thus $\mathfrak{a} \in M^{\infty,p_0}$. □

## 4.4 Almost diagonalization of pseudodifferential operators

In this section, we list several characterizations of symbols in $M^{\infty,p_0}(\mathbb{R}^{2d})$, $p_0 \in (0,\infty]$. The following characterization of a symbol class by means of the almost diagonalization of the associated pseudodifferential operator was found in [20, Theorem 3.2] for $p_0 = 1$, and subsequently generalized to the full range of $p_0$ in [5, Theorem 3.2] (with almost the same proof). This characterization helps to deduce spectral properties of pseudodifferential operators from spectral properties of infinite matrices. We only formulate the unweighted case of [5, Theorem 3.2], which is sufficient for this paper.

**Theorem 4.6** (Almost diagonalization). *We fix a non-zero $g \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ and a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ such that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then, for any $p_0 \in (0,\infty]$, the following properties are equivalent:*
(i) $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$.
(ii) $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a function $H \in L^{p_0}(\mathbb{R}^{2d})$, in fact $H \in W(C_b, L^{p_0})$, such that*

$$|\langle \mathfrak{a}^w \pi(z)g, \pi(w)g\rangle| \leq H(w-z) \quad \textit{for all } w,z\in\mathbb{R}^{2d}. \tag{4.10}$$

(iii) $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$ *and there exists a sequence $h \in \ell^{p_0}(\Lambda)$ such that*

$$|\langle \mathfrak{a}^w \pi(\rho)g, \pi(\lambda)g\rangle| \leq h(\lambda-\rho) \quad \textit{for all } \lambda,\rho\in\Lambda. \tag{4.11}$$

As an immediate application, we obtain the following result.

**Corollary 4.7.** *Under the hypotheses of Theorem 4.6, we assume that $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is continuous and satisfies*

$$|\langle T\pi(\rho)g, \pi(\lambda)g\rangle| \leq h(\lambda-\rho) \quad \textit{for all } \lambda,\rho\in\mathbb{Z}^{2d},$$

*for some $h \in \ell^{p_0}(\mathbb{Z}^{2d})$. Then $T = \mathfrak{a}^w$ for some $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$.*

*Proof.* Because of the continuity of $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, the Schwartz kernel theorem asserts the existence of a symbol $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$ with $T = \mathfrak{a}^w$. An application of Theorem 4.6 yields the claim. □

## 4.5 Matrix formulation

We fix a lattice $\Lambda$ and a window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ such that $\mathcal{G}(g, \Lambda)$ is a frame with dual window $\gamma$ and associated Gabor expansion (4.3) and (4.4).

For the manipulations to be meaningful, we assume of a symbol $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^d)$ that the Weyl operator $\mathfrak{a}^w$ is bounded on $M^p(\mathbb{R}^d)$ for some $p \in (0, \infty]$. Just keep in mind that, due to Proposition 4.1, $M^p(\mathbb{R}^d) \subseteq M^\infty(\mathbb{R}^d)$. On account of the Gabor expansion (4.3), we have $D_g C_\gamma f = f$ for all $f \in M^p(\mathbb{R}^d) \subseteq M^\infty(\mathbb{R}^d)$. Together with the continuity of $\mathfrak{a}^w$ on $M^p(\mathbb{R}^d)$, we obtain, for all $f \in M^p(\mathbb{R}^d)$,

$$C_g(\mathfrak{a}^w f)(\lambda) = \langle \mathfrak{a}^w f, \pi(\lambda)g\rangle = \sum_{\mu\in\Lambda} \langle f, \pi(\mu)\gamma\rangle \langle \mathfrak{a}^w \pi(\mu)g, \pi(\lambda)g\rangle \quad \text{for all } \lambda \in \Lambda. \tag{4.12}$$

We define the matrix $M(\mathfrak{a})$ of $\mathfrak{a}^w$ with respect to the frame $\mathcal{G}(g, \Lambda)$ by its entries

$$M(\mathfrak{a})_{\lambda,\mu} = \langle \mathfrak{a}^w \pi(\mu)g, \pi(\lambda)g\rangle \quad \text{for all } \lambda, \mu \in \Lambda.$$

We can then recast (4.12) as

$$C_g(\mathfrak{a}^w f) = M(\mathfrak{a})C_\gamma f. \tag{4.13}$$

Likewise, Theorem 4.6 now reads as follows.

**Corollary 4.8.** *Let $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$. Then $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$ if and only if $M(\mathfrak{a}) \in \mathcal{C}^{p_0}$.*

The matrix $P$ associated to the identity operator has the entries

$$P_{\lambda,\mu} = \langle \pi(\mu)g, \pi(\lambda)g\rangle \quad \text{for all } \lambda, \mu \in \Lambda. \tag{4.14}$$

To simplify the analysis of $P$, we assume from now on that $\mathcal{G}(g, \Lambda)$ is a tight frame with $S_{g,\Lambda} = \mathrm{I}$. As mentioned already, tight frames with a window in $\mathcal{S}(\mathbb{R}^d)$ always exist. The matrix $P$ has the following properties.

**Lemma 4.9.** *Assume that $\mathcal{G}(g, \Lambda)$ is a tight frame with $g \in \mathcal{S}(\mathbb{R}^d)$. Let $0 < p_0 \le p \le \infty$ and let $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$ be such that the associated pseudodifferential operator $\mathfrak{a}^w$ is bounded on $M^p$.*

(i) *$P$ is a projection from $\ell^p(\Lambda)$ to the range $C_g(M^p)$, i.e., $Pc = c \in \ell^p(\Lambda)$, if and only if there exists $f \in M^p$ such that $c_\lambda = \langle f, \pi(\lambda)g\rangle = (C_g f)_\lambda$ for $\lambda \in \Lambda$.*

(ii) *$PM(\mathfrak{a}) = M(\mathfrak{a})$ and $M(\mathfrak{a})P = M(\mathfrak{a})$.*

(iii) *$P \in \mathcal{C}^p(\Lambda)$.*

(iv) *For $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$, we have $M(\mathfrak{a}) + I - P \in \mathcal{C}^{p_0}(\Lambda)$.*

*Proof.* (i) For all $\lambda, \nu \in \Lambda$, the assumption that $S_{g,\Lambda} = \mathrm{Id}$ and (4.2) imply that

$$\begin{aligned}(P^2)_{\lambda,\nu} &= \Big\langle \sum_{\mu\in\Lambda} \langle \pi(\nu)g, \pi(\mu)g\rangle \pi(\mu)g, \pi(\lambda)g \Big\rangle \\ &= \langle S_{g,\Lambda}\pi(\nu)g, \pi(\lambda)g\rangle \\ &= \langle \pi(\nu)g, \pi(\lambda)g\rangle \\ &= P_{\lambda,\nu}.\end{aligned}$$

Consequently, $P^2 = P$ and $P$ is a projection. Next, let $c \in \ell^p(\Lambda)$ with $Pc = c$. With $f = \sum_{\mu\in\Lambda} c_\mu \pi(\mu)g$, we have, for $\lambda \in \Lambda$,

$$c_\lambda = (Pc)_\lambda = \sum_{\mu\in\Lambda} \langle \pi(\mu)g, \pi(\lambda)g\rangle c_\mu = \langle f, \pi(\lambda)g\rangle = (C_g f)_\lambda,$$

whence $c = C_g f$. Then the norm-equivalence (4.5) yields $f \in M^p(\mathbb{R}^d)$.

Conversely, assume that $c = C_g f$ for some $f \in M^p(\mathbb{R}^d)$. Then, due to (4.3), we get

$$(Pc)_\lambda = \Big\langle \sum_{\mu\in\Lambda} \langle f, \pi(\mu)g\rangle \pi(\mu)g, \pi(\lambda)g \Big\rangle = \langle S_{g,\Lambda} f, \pi(\lambda)g\rangle = \langle f, \pi(\lambda)g\rangle = c_\lambda, \quad \lambda \in \Lambda,$$

and (i) holds.

(ii) This is proved similarly by straightforward calculations using (4.2).

(iii) If $g \in \mathcal{S}(\mathbb{R}^d)$, then also $V_g g \in \mathcal{S}(\mathbb{R}^{2d})$, e.g., by [19, Theorem 11.2.5]. In particular, this implies that, for every $N \geq 0$,

$$|V_g g(z)| \lesssim (1 + |z|)^{-N},$$

and thus

$$|P_{\lambda,\mu}| = |\langle \pi(\mu)g, \pi(\lambda)g\rangle| = |V_g g(\lambda - \mu)| \lesssim (1 + |\lambda - \mu|)^{-N}.$$

By choosing $N$ large enough, we see that $H(z) = (1 + |z|)^{-N}$ is in $W(C_b, L^{p_0})$, and consequently $P \in \mathcal{C}^{p_0}(\Lambda)$.

(iv) Since all matrices $P$, I and $M(\mathfrak{a})$ are in $\mathcal{C}^{p_0}$, their sum $M(\mathfrak{a}) + \mathrm{Id} + P$ is also in $\mathcal{C}^{p_0}$. □

## 4.6 Spectral invariance

We have already seen before that it is possible to relate a pseudodifferential operator $\mathfrak{a}^w$ to an infinite matrix $M(\mathfrak{a})$. It turns out that there is a connection between the invertibility of $\mathfrak{a}^w$ and $M(\mathfrak{a})$.

**Lemma 4.10.** *Assume that $\mathcal{G}(g, \Lambda)$ is a tight frame with $g \in \mathcal{S}(\mathbb{R}^d)$. Let $0 < p \leq \infty$ and $\mathfrak{a} \in \mathcal{S}'(\mathbb{R}^{2d})$ be such that the associated Weyl operator $\mathfrak{a}^w$ is bounded on $M^p$. Then $\mathfrak{a}^w$ is invertible on $M^p$ if and only if the following assertions hold:*
(i) $\|M(\mathfrak{a})Pc\|_p \gtrsim \|Pc\|_p$ *for all* $c \in \ell^p(\Lambda)$.
(ii) *For every* $c_0 \in P\ell^p(\Lambda)$*, there is a* $c \in P\ell^p(\Lambda)$ *such that* $M(\mathfrak{a})Pc = Pc_0$.
*Here the projection $P$ is defined as in* (4.14).

*Proof.* Let $\mathfrak{a}^w$ be invertible on $M^p(\mathbb{R}^d)$. Using (4.13), we obtain

$$\|M(\mathfrak{a})C_g f\|_p = \|C_g(\mathfrak{a}^w f)\|_p \asymp \|\mathfrak{a}^w f\|_{M^p} \gtrsim \|f\|_{M^p} \asymp \|C_g f\|_p,$$

which is (i).

To prove (ii), let $c_0 \in P\ell^p(\Lambda)$ be arbitrary. Then there exists $h \in M^p(\mathbb{R}^d)$ with $C_g h = c_0 = Pc_0$ by Lemma 4.9. Since $\mathfrak{a}^w$ is bijective on $M^p(\mathbb{R}^d)$, there is a $f \in M^p(\mathbb{R}^d)$ such that

$$\mathfrak{a}^w f = h.$$

Then we obtain for $c = C_g f$, due to Lemma 4.9, that $Pc = c$ and that

$$M(\mathfrak{a})Pc = M(\mathfrak{a})c = M(\mathfrak{a})C_g f = C_g(\mathfrak{a}^w f) = C_g h = c_0 = Pc_0.$$

This implies (ii).

Conversely assume that (i) and (ii) hold. Using (4.13) (with $\gamma = g$) and (i), we obtain, for all $f \in M^p(\mathbb{R}^d)$,

$$\|\mathfrak{a}^w f\|_{M^p} \asymp \|C_g(\mathfrak{a}^w f)\|_p = \|M(\mathfrak{a})C_g f\|_p \gtrsim \|C_g f\|_p \asymp \|f\|_{M^p}.$$

Hence $\mathfrak{a}^w$ is one-to-one on $M^p(\mathbb{R}^d)$. To prove that $\mathfrak{a}^w$ is surjective, we choose an arbitrary $h \in M^p(\mathbb{R}^d)$ and let $c_0 := C_g h \in \ell^p(\Lambda)$. Then $c_0 = Pc_0 \in P\ell^p(\Lambda)$.

By assumption (ii), there is a $c \in P\ell^p(\Lambda)$ such that

$$M(\mathfrak{a})Pc = Pc_0 = c_0 = C_g h. \tag{4.15}$$

Since the image of $P$ is $C_g(M^p)$, there is a $f \in M^p(\mathbb{R}^d)$ with

$$Pc = C_g f.$$

A combination of (4.3), (4.13) and (4.15) yields

$$\mathfrak{a}^w f = D_g C_g(\mathfrak{a}^w f) = D_g M(\mathfrak{a})Pc = D_g C_g h = h.$$

This implies that $\mathfrak{a}^w$ maps onto $M^p(\mathbb{R}^d)$, and is hence invertible on $M^p(\mathbb{R}^d)$. □

In the previous lemma, we proved the equivalence of the invertibility of $\mathfrak{a}^w$ on $M^p(\mathbb{R}^d)$ and of $M(\mathfrak{a})$ on $P\ell^p(\Lambda)$. Since $\ker P \neq \{0\}$ and $M(\mathfrak{a}) = M(\mathfrak{a})P$, $M(\mathfrak{a})$ cannot be invertible on the whole space $\ell^p(\Lambda)$. In the literature, this problem is usually overcome by using the pseudo-inverse of $M(\mathfrak{a})$ and holomorphic functional calculus. Here we use a new trick, which may be of independent interest. Consider the matrix $A = M(\mathfrak{a}) + \mathrm{Id} - P$. We can then use the spectral invariance result for infinite convolution-dominated matrices of Theorem 3.16 to derive a spectral invariance result for pseudodifferential operators on modulation spaces.

**Theorem 4.11** (Spectral invariance on modulation spaces). *If $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$ for $p_0 \in (0,1]$ and $\mathfrak{a}^w$ is invertible on $M^p(\mathbb{R}^d)$ for some $p \in [p_0,\infty]$, then $\mathfrak{a}^w$ is also invertible on $M^q(\mathbb{R}^d)$ for all $q \in [p_0,\infty)$.*

*Proof.* Let $p \in [p_0,\infty]$ be the index for which $\mathfrak{a}^w$ is invertible on $M^p(\mathbb{R}^d)$ and let $A = M(\mathfrak{a}) + \mathrm{Id} - P$, where $P$ is the projection defined in (4.14). First, we check the assumptions of Theorem 3.16 and prove that

$$A = M(\mathfrak{a}) + I - P \quad \text{is invertible on} \quad \ell^p(\Lambda). \tag{4.16}$$

Assume that $Ac = 0$ for some $c \in \ell^p(\Lambda)$. Then by Lemma 4.9 (i) and (ii),

$$0 = Ac = (M(\mathfrak{a}) + I - P)(Pc + (I-P)c) = M(\mathfrak{a})Pc + (I-P)c.$$

If we apply $P$ (resp. $I - P$) to the previous equality and use Lemma 4.9 again, we obtain

$$M(\mathfrak{a})Pc = 0 \quad \text{and} \quad (I-P)c = 0.$$

Since $\mathfrak{a}^w$ is invertible on $M^p$, we obtain, by Lemma 4.10 and the previous estimate,

$$\|Pc\|_p \lesssim \|M(\mathfrak{a})Pc\|_p = 0,$$

and consequently $Pc = 0$. Hence $c = Pc + (I-P)c = 0$, which shows that $A$ is one-to-one.

To show the surjectivity of $A$, we let $c_0 \in \ell^p(\Lambda)$ be arbitrary. Since $\mathfrak{a}^w$ is invertible on $M^p$, Lemma 4.10 yields the existence of $c \in P\ell^p(\Lambda)$ with

$$M(\mathfrak{a})Pc = Pc_0. \tag{4.17}$$

Then $\widetilde{c} = Pc + (I-P)c_0 \in \ell^p(\Lambda)$ and, by Lemma 4.9 and (4.17),

$$A\widetilde{c} = M(\mathfrak{a})\widetilde{c} + (I-P)\widetilde{c} = M(\mathfrak{a})Pc + (I-P)c_0 = Pc_0 + (I-P)c_0 = c_0.$$

Thus $A$ is onto on $\ell^p(\Lambda)$, and therefore invertible on $\ell^p(\Lambda)$. Due to Lemma 4.9, we have $A \in \mathcal{C}^{p_0}(\Lambda)$. Since (4.16) also holds, we can apply Theorem 3.16 and get the invertibility of $A$ on $\ell^q(\Lambda)$ for all $q \in [p_0,\infty)$.

Next, we show that

$$M(\mathfrak{a}) \quad \text{is invertible on} \quad P\ell^q(\Lambda) \quad \text{for all } q \in [p_0,\infty). \tag{4.18}$$

Let $q \in [p_0,\infty)$ be arbitrary. Since $I - P \equiv 0$ on $P\ell^q(\Lambda)$, the injectivity of $A$ implies that $M(\mathfrak{a})$ is one-to-one on $P\ell^q(\Lambda)$. For an arbitrary $c_0 \in P\ell^q(\Lambda) \subseteq \ell^q(\Lambda)$, there is some $c \in \ell^q(\Lambda)$ with $Ac = c_0$, since $A$ is onto on $\ell^q(\Lambda)$. Since $(I-P)M(\mathfrak{a}) = 0$ by Lemma 4.9, we obtain, after applying $I - P$ to $Ac = c_0$, that

$$0 = (I-P)c_0 = (I-P)Ac = (I-P)c.$$

Then $c \in P\ell^q(\Lambda)$ and

$$M(\mathfrak{a})c = M(\mathfrak{a})c + (I-P)c = Ac = c_0.$$

Hence $M(\mathfrak{a})$ is onto $P\ell^q(\Lambda)$ and (4.18) holds. By Lemma 4.10, $\mathfrak{a}$ is invertible on $M^q$. □

Theorem 4.11 implies Theorem 1.2 of Section 1.

We now prove Theorem 1.1 and obtain more refined information about the inverse $(\mathfrak{a}^w)^{-1}$.

*Proof of Theorem 1.1.* By Theorem 4.11, we get the invertibility of $\mathfrak{a}^w$ on $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. By [20, Theorem 4.6] and the embedding $M^{\infty,p_0}(\mathbb{R}^{2d}) \subseteq M^{\infty,1}(\mathbb{R}^{2d})$, there is a symbol $\mathfrak{b} \in M^{\infty,1}(\mathbb{R}^{2d})$ with $\mathfrak{b}^w = (\mathfrak{a}^w)^{-1}$. We consider the associated matrices $M(\mathfrak{a})$ and $M(\mathfrak{b})$ with respect to a tight Gabor frame $\mathcal{G}(g,\Lambda)$ with $g \in \mathcal{S}(\mathbb{R}^d)$ and again denote by $P$ the projection with entries $P_{\lambda,\mu} = \langle \pi(\mu)g, \pi(\lambda)g\rangle$. On account of Lemma 4.9, we get for all $c \in \operatorname{ran} C_g$ the existence of an $f \in M^2(\mathbb{R}^d)$ with $c = C_g f = Pc$. Then, for all $c = C_g f = Pc \in \operatorname{ran} C_g$, using (4.13), we obtain

$$M(\mathfrak{b})M(\mathfrak{a})c = M(\mathfrak{b})M(\mathfrak{a})C_g f = M(\mathfrak{b})C_g(\mathfrak{a}^w f) = C_g(\mathfrak{b}^w \mathfrak{a}^w f) = C_g f = c.$$

If $Pc = 0$, then $M(\mathfrak{a})c = M(\mathfrak{a})Pc = 0$, and consequently on $\ell^2(\Lambda)$ we have

$$M(\mathfrak{b})M(\mathfrak{a}) = P.$$

It follows that

$$(M(\mathfrak{b}) + \mathrm{Id} - P)(M(\mathfrak{a}) + \mathrm{Id} - P) = (M(\mathfrak{b}) + \mathrm{Id} - P)A = \mathrm{Id}.$$

This means that $B = M(\mathfrak{b}) + \mathrm{Id} - P$ is the inverse of the invertible matrix $A$ (since the inverse is unique). Since $A \in \mathcal{C}^{p_0}$ by Lemma 4.9, Theorem 3.15 implies that also $B \in \mathcal{C}^{p_0}$.

Consequently, we have $M(\mathfrak{b}) \in \mathcal{C}^{p_0}$. Now, the characterization of Corollary 4.8 implies that $\mathfrak{b} \in M^{\infty,p_0}(\mathbb{R}^{2d})$, as claimed. □

By using Theorem 1.1, we can now deduce the invertibility of $\mathfrak{a}^w$ on more general modulation spaces, which generalizes Theorem 1.2.

**Theorem 4.12** (Spectral invariance on modulation spaces). *If $\mathfrak{a} \in M^{\infty,p_0}(\mathbb{R}^{2d})$ for $p_0 \in (0,1]$ and $\mathfrak{a}^w$ is invertible on $M^p(\mathbb{R}^d)$ for some $p \in [p_0,\infty]$, then $\mathfrak{a}^w$ is also invertible on $M^{r,q}(\mathbb{R}^d)$ for all $r, q \in [p_0,\infty)$.*

*Proof.* On account of Theorem 1.1, there is a $\mathfrak{b} \in M^{\infty,p_0}(\mathbb{R}^{2d})$ with $\mathfrak{b}^w = (\mathfrak{a}^w)^{-1}$ on $M^p(\mathbb{R}^d)$. By Proposition 4.3, $\mathfrak{b}^w$ is bounded on $M^{p,q}(\mathbb{R}^d)$. Since $\mathfrak{b}^w\mathfrak{a}^w = \mathfrak{a}^w\mathfrak{b}^w = I$ on $\mathcal{S}(\mathbb{R}^d) \subseteq M^p(\mathbb{R}^d)$, we obtain the invertibility of $\mathfrak{a}^w$ on $M^{p,q}(\mathbb{R}^d)$ by the density of $\mathcal{S}(\mathbb{R}^d)$ in $M^{p,q}(\mathbb{R}^d)$. □

This theorem is an extension of [20, Corollary 4.7] from the case $p_0 = 1$ to $p_0 < 1$. By using the arguments of [24], one can formulate the corollary for an even more general class of modulation spaces.

**Remark 4.13.** Let $A \in \mathbb{R}^{d\times d}$. Proposition 4.2 implies that the conclusions in Theorems 1.1, 1.2 and 4.12 remain true with $Op_A(\mathfrak{a})$ and $Op_A(\mathfrak{b})$ in place of $\mathfrak{a}^w$ and $\mathfrak{b}^w$, respectively, at each occurrence.

As an application of Theorem 4.12, we show the following property of the canonical dual window of an Gabor frame.

**Theorem 4.14.** *Let $0 < p \le 1$, let $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice and let $g \in M^p(\mathbb{R}^d)$ be such that $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$. Then the canonical dual window $\gamma$ satisfies*

$$\gamma = S_{g,\Lambda}^{-1} g \in M^p(\mathbb{R}^d). \tag{4.19}$$

*Proof.* We denote the Kohn–Nirenberg symbol of the frame operator $S_{g,\Lambda}$ by $\mathfrak{a}$. By Proposition 4.5, we have

$$\mathfrak{a} \in M^{\infty,p}(\mathbb{R}^{2d}).$$

Since Theorem 4.12 also holds for pseudodifferential operators in the Kohn–Nirenberg quantization, $S_{g,\Lambda}$ is invertible on $M^p(\mathbb{R}^d)$. Therefore, (4.19) holds. □

# A Proofs of some preparatory results

In this appendix, we prove some preparatory results from Sections 2 and 3.

*Proof of Lemma 2.3.* Let $\alpha > 0$ be chosen such that $[0,\alpha]^d \subseteq B_{1/2}(0)$ and $(B_{1/2}(\alpha k))_{k\in\mathbb{Z}^d}$ covers $\mathbb{R}^d$. Then

$$\|H\|_{\ell^{p_0}(\Lambda)}^{p_0} \le \sum_{k\in\mathbb{Z}^d} \sum_{\Lambda\cap B_{1/2}(\alpha k)} |H(\lambda)|^{p_0} \lesssim \sum_{k\in\mathbb{Z}^d} \operatorname{rel}(\Lambda) \int_{[0,\alpha]^d} \sup_{\lambda\in B_{1/2}(\alpha k)} |H(\lambda)|^{p_0}\, dy.$$

Since $B_{1/2}(\alpha k) \subseteq B_1(\alpha k + y)$ for each $y \in [0,\alpha]^d$, the integral just becomes larger if we take the supremum over all $\lambda \in B_1(\alpha k + y)$ instead of $\lambda \in B_{1/2}(\alpha k)$. A substitution then provides

$$\|H\|_{\ell^{p_0}(\Lambda)}^{p_0} \lesssim \sum_{k\in\mathbb{Z}^d} \mathrm{rel}(\Lambda) \int_{[0,\alpha]^d+\alpha k} \sup_{\lambda\in B_1(x)} |H(\lambda)|^{p_0}\, dx = \mathrm{rel}(\Lambda)\|H\|_{W(C_b,L^{p_0})}^{p_0}.$$

This concludes the proof. □

*Proof of Proposition 3.4.* For $\rho_0 \in \Pi$ and $\lambda_0 \in \Lambda$ fixed, let

$$s_{\Lambda,p_0}(\rho_0)^{p_0} := \sum_{\lambda\in\Lambda} H(\lambda-\rho_0)^{p_0} \quad \text{and} \quad s_{\Pi,p_0}(\lambda_0)^{p_0} := \sum_{\rho\in\Pi} H(\lambda_0-\rho)^{p_0}.$$

Then it follows by straightforward estimates that

$$s_{\Lambda,p_0}(\rho) \le C_\Lambda \|H\|_{W(C_b,L^{p_0})}, \quad s_{\Pi,p_0}(\lambda) \le C_\Pi \|H\|_{W(C_b,L^{p_0})}, \qquad \lambda\in\Lambda,\ \rho\in\Pi,$$

where $C_\Lambda = C_0\,\mathrm{rel}(\Lambda)^{1/p_0}$ for some constant $C_0 > 0$ which only depends on $d$.

Suppose $q \ge 1$. Then Hölder's inequality together with the fact that $s_{\Lambda,p_0}$ and $s_{\Pi,p_0}$ decrease with $p_0$ gives

$$\begin{aligned}
\|Ab\|_{\ell^q(\Lambda)}^q &\le \sum_{\lambda\in\Lambda}\Big(\sum_{\rho\in\Pi}(H(\lambda-\rho)^{1/q}|b_\rho|)H(\lambda-\rho)^{1/q'}\Big)^q \\
&\le \sum_{\lambda\in\Lambda}\Big(\sum_{\rho\in\Pi}H(\lambda-\rho)|b_\rho|^q\Big)\Big(\sum_{\rho\in\Pi}H(\lambda-\rho)\Big)^{q/q'} \\
&= \sum_{\lambda\in\Lambda}\Big(\sum_{\rho\in\Pi}H(\lambda-\rho)|b_\rho|^q\Big)s_{\Pi,1}(\lambda)^{q/q'} \\
&\le (C_\Pi\|H\|_{W(C_b,L^{p_0})})^{q/q'}\sum_{\rho\in\Pi}\Big(\sum_{\lambda\in\Lambda}H(\lambda-\rho)\Big)|b_\rho|^q \\
&= (C_\Pi\|H\|_{W(C_b,L^{p_0})})^{q/q'}\sum_{\rho\in\Pi}s_{\Lambda,1}(\rho)|b_\rho|^q \\
&\le C_\Pi^{q/q'}\|H\|_{W(C_b,L^{p_0})}^q C_\Lambda\|b\|_{\ell^q(\Pi)}^q,
\end{aligned}$$

giving the assertion when $q \ge 1$.

If instead $p_0 \le q \le 1$, then

$$\begin{aligned}
\|Ab\|_{\ell^q(\Lambda)}^q &\le \sum_{\lambda\in\Lambda}\Big(\sum_{\rho\in\Pi}H(\lambda-\rho)|b_\rho|\Big)^q \\
&\le \sum_{\rho\in\Pi}\Big(\sum_{\lambda\in\Lambda}H(\lambda-\rho)^q\Big)|b_\rho|^q \\
&\le \sum_{\rho\in\Pi}s_{\Lambda,q}(\rho)^q|b_\rho|^q \\
&\le C_\Lambda^q\|H\|_{W(C_b,L^{p_0})}^q\sum_{\rho\in\Pi}|b_\rho|^q \\
&= (C_\Lambda\|H\|_{W(C_b,L^{p_0})}\|b\|_{\ell^q(\Pi)})^q,
\end{aligned}$$

giving the result for $q \le 1$. □

*Proof of Lemma 3.7 for $q < 1$.* Since $\mathrm{supp}(\varphi) \subseteq B_2(0)$, we get

$$\eta := \sup_{\varepsilon>0}\sup_{x\in\mathbb{R}^d}\#\{k\in\mathbb{Z}^d : \varphi_k^\varepsilon(x)\neq 0\} = \sup_{\varepsilon>0}\sup_{x\in\mathbb{R}^d}\#\{k\in\mathbb{Z}^d\cap B_2(\varepsilon x)\} < \infty. \tag{A.1}$$

So, we obtain the following bound for all $x \in \mathbb{R}^d$:

$$1 \le \sum_{k\in\mathbb{Z}^d}\varphi_k^\varepsilon(x)^q = \sum_{k\in\mathbb{Z}^d:\varphi_k^\varepsilon(x)\neq 0}\varphi_k^\varepsilon(x)^q \le \eta\sup_{k\in\mathbb{Z}^d}\varphi_k^\varepsilon(x)^q \le \eta.$$

Therefore, for all $x \in \mathbb{R}^d$,

$$\frac{1}{\eta} \le \sup_{k\in\mathbb{Z}^d}\varphi_k^\varepsilon(x)^q \le \sum_{k\in\mathbb{Z}^d}\varphi_k^\varepsilon(x)^q \le \eta. \tag{A.2}$$

If $a \in \ell^q(\Pi)$, then

$$\frac{1}{\eta}\sum_{\rho\in\Pi}|a_\rho|^q \le \sum_{\rho\in\Pi}\sum_{k\in\mathbb{Z}^d}\varphi_k^\varepsilon(\rho)^q|a_\rho|^q \le \eta\sum_{\rho\in\Pi}|a_\rho|^q,$$

which implies the claim with constants independent of $\varepsilon$ and $q$. □

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