Reflection length in infinite non-affine Coxeter groups

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Abstract

This thesis is devoted to the study of the reflection length function on infinite non-affine Coxeter groups. Infinite non-affine Coxeter groups are Coxeter groups that do not split into a direct product of spherical and Euclidean reflection groups. Reflection length is well understood for direct products of spherical and Euclidean reflection groups and there exist formulas in this case (see [Car72; Bre+19]). Whilst the reflection length is bounded on this type of Coxeter group, the reflection length function is unbounded on infinite non-affine Coxeter groups shown by Kamil Duszenko (see [Dus12]). Apart from this, very little is known about the reflection length in infinite non-affine Coxeter groups. In this work, we investigate the asymptotic behaviour of the reflection length function on infinite non-affine Coxeter groups to identify repetitive patterns and to prove global geometric results. For this, we use the rich duality of combinatorics and geometry inherent to Coxeter groups.

The geometric and combinatorial foundations of Coxeter groups are discussed in the second and third chapters.

In Chapter 4, we present a new proof of the unboundedness of the reflection length function on infinite non-affine Coxeter groups. This proof is based on the Brooks construction for acylindrically hyperbolic groups.

As a first main result, we prove a formula for the reflection length of powers of Coxeter elements in a universal Coxeter group of arbitrary rank in Chapter 5. The formula allows us to deduce results about the minimal word length for a given reflection length and vice versa in universal Coxeter groups. These results are proved combinatorially with the properties of the reflection length function and a result by Matthew J. Dyer in [Dye01].

The sixth chapter deals with infinite non-affine Coxeter groups that are discrete groups generated by finitely many hyperplane reflections in the n-dimensional hyperbolic space. The action of such a Coxeter group induces a tessellation of the hyperbolic space. For a fixed fundamental domain, there exists a bijection between the tiles and the group elements. The second main result describes points in the visual boundary of the n-dimensional hyperbolic space for which every neighbourhood contains tiles of every reflection length. For the proof of this result, we show that two disjoint hyperplanes in the n-dimensional hyperbolic space without common boundary points have a unique common perpendicular. This generalises a result of David Hilbert in [Hil13].

The last main result shows that the reflection lengths of the sequence of growing powers of a Coxeter element tend to infinity for Coxeter groups with sufficiently large braid relations. To obtain this result, we compare the reflection length function of an arbitrary Coxeter group and the reflection length function of the universal Coxeter group of the same rank in the last chapter. By applying a solution to the word problem for Coxeter groups, we derive a lower bound for the reflection length in an arbitrary Coxeter group. For Coxeter groups corresponding to a Coxeter matrix with equal non-diagonal entries, sharp upper bounds for the reflection length of the powers of Coxeter elements are established.

Zusammenfassung

Diese Arbeit befasst sich mit dem Studium der Spiegelungslänge in unendlichen nicht-affinen Coxetergruppen. Unendliche nicht-affine Coxetergruppen sind Coxetergruppen, die nicht das direkte Produkt von sphärischen und euklidischen Spiegelungsgruppen sind. Die Spiegelungslänge ist in direkten Produkten von sphärischen und euklidischen Spiegelungsgruppen gut verstanden und es existieren Formeln in diesem Fall (siehe [Car72; Bre+19]). Während sie beschränkt auf diesem Typ Coxetergruppe ist, ist die Spiegelungelänge unbeschränkt auf unendlichen nicht-affinen Coxetergruppen gemäß eines Resultats von Kamil Duszenko (siehe [Dus12]). Darüber hinaus ist wenig bekannt über die Spiegelungslänge in unendlichen nicht-affinen Coxetergruppen. Ziel dieser Arbeit ist es Methoden für die Erforschung des asymptotischen Verhaltens der Spiegelungslänge in unendlichen nicht-affinen Coxetergruppen zu entwickeln, repetitive Muster aufzudecken und globale geometrische Resultate über die Spiegelugnslänge zu beweisen. Dafür nutzen wir die vielfältige Dualität von Kombinatorik und Geometrie der Coxetergruppen.
Die geometrischen und kombinatorischen Grundlagen für Coxetergruppen werden im zweiten und dritten Kapitel zusammengefasst.
In Kapitel 4 wird ein neuer Beweis der Unbeschränktheit der Spiegelungslänge in unendlichen nicht-affinen Coxetergruppen vorgestellt, der auf der Brooks Konstruktion für azylindrische hyperbolische Gruppen beruht.
Als erstes Hauptresultat, beweisen wir eine Formel für die Spiegelungslänge der Potenzen der Coxeterelemente in universellen Coxetergruppen beliebigen Ranges in Kapitel 5. Resultate über die minimale Wortlänge für eine gegebene Spiegelungslänge und umgekehrt leiten wir daraus ab. Diese Ergebnisse werden kombinatorisch mit den Eigenschaften der Spiegelungslängenfunktion und einem Theorem von Matthew J. Dyer in [Dye01] bewiesen.
Das sechste Kapitel befasst sich mit unendlichen nicht-affinen Coxetergruppen, die diskrete, von endlich vielen Hyperebenenspiegelungen im n-dimensionalen hyperbolischen Raum erzeugte Gruppen sind. Die Wirkung einer solchen Coxetergruppe induziert eine Parkettierung des hyperbolischen Raums. Es existiert eine Bijektion zwischen den Kacheln und den Gruppenelementen für einen festen Fundamentalbereich. Das zweite Hauptresultat ist die Identifikation spezieller Punkte auf dem visuellen Rand des n-dimensionalen hyperbolischen Raums, für die alle ihre Umgebungen Kacheln beliebiger Spiegelungslänge beinhalten. Für den Beweis dieses Resultats zeigen wir, dass zwei disjunkte Hyperebenen im n-dimensionalen hyperbolischen Raum ohne gemeinsame Randpunkte eine eindeutige gemeinsame Senkrechte haben. Dies verallgemeinert ein Resultat von David Hilbert in [Hil13].
Das letzte Hauptresultat ist, dass die Spiegelungslängen der Folge von wachsenden Potenzen eines Coxeterelements gegen unendlich strebt in Coxetergruppen mit genügend großen Zopfrelationen. Um dieses Resultat zu beweisen, wird im letzten Kapitel die Abbildung der Spiegelungslänge einer beliebigen Coxetergruppe mit der Abbildung der Spiegelungslänge der universellen Coxetergruppe gleichen Ranges verglichen. Mit einer Lösung des Wortproblems für Coxetergruppen leiten wir eine untere Schranke für die Spiegelungslänge in beliebigen Coxetergruppen ab. Für Coxetergruppen mit dem gleichen Eintrag überall abseits der Diagonalen in der Coxetermatrix werden scharfe obere Schranken für die Spiegelungslänge der Potenzen der Coxeterlemente bewiesen.

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1. Introduction

1.1 Coxeter groups

Coxeter groups are named after Harold Scott MacDonald Coxeter, who investigated finitely generated groups with an abstract presentation of the form

$$\langle s_1, \ldots, s_n \mid s_i^2 = (s_i s_j)^{m_{ij}} = \mathbb{1}\,,\ \forall i, j \in \{1, \ldots, n\},\ m_{ij} \in \mathbb{N}_{\geq 2} \cup \{\infty\}\rangle$$

in the article [Cox34] from the year 1934. The article was motivated by his study of the symmetry groups of regular convex polytopes, which are generated by reflections. This translates to the condition $s_i^2 = \mathbb{1}$ in the abstract presentation above and justifies why he called these *discrete groups generated by reflections*. In [Cox34] and a subsequent work [Cox35], he classified all finite groups that admit a presentation of this specific form and also classified all discrete groups of isometries of the n-dimensional Euclidean space generated by reflections. On account of Coxeter's preliminary work on these groups, groups that admit a presentation as above were named *Coxeter groups* by Jacques Tits, who also made great contributions and further formalisations in [Bou02] (see [Vin72; Hum90; Ell+03]).

Coxeter groups appear in many contexts in mathematics such as Lie theory, the theory of buildings, algebra, combinatorics and topology. They have a versatile nature in the sense that it is often possible to describe properties of Coxeter groups and related objects geometrically as well as combinatorially. Both of these approaches date back to the beginnings of the study of Coxeter groups. Important instances of Coxeter groups are *geometric reflection groups*. These include finite groups generated by reflections like the groups of symmetries of regular convex polytopes. More generally, discrete groups generated by reflections across the sides of a polytope with additional local finiteness properties in the n-dimensional sphere $\mathbb{S}^n$, the n-dimensional Euclidean space $\mathbb{E}^n$ or the n-dimensional hyperbolic space $\mathbb{H}^n$ are geometric reflection groups. Not all Coxeter groups are geometric reflection groups (see [FT05]). Nevertheless, there are different constructions of geometric objects for every Coxeter group W like the geometric representation or the Coxeter complex, on which W acts as a discrete group with reflection-like generators. Figure 1.1 shows a white polytope in $\mathbb{H}^2$ in the Poincaré disc model on the left. The reflections across the hyperplanes containing the sides of the polytope generate the projective general linear group over the integers $\mathrm{PGL}_2(\mathbb{Z})$. The intersecting angles of the lines containing the sides of the polytope determine the relations between the generators of the group. On the right, the Coxeter complex of $\mathrm{PGL}_2(\mathbb{Z})$ is displayed embedded in the Poincaré disc model for $\mathbb{H}^2$.

Finite and Euclidean geometric reflection groups are classified by Coxeter as mentioned above. The classification is a list of finite and Euclidean reflection groups that do not split into a direct product of geometric reflection groups. Finite and Euclidean reflection groups are well understood due to their geometric nature and classification. Coxeter groups can be

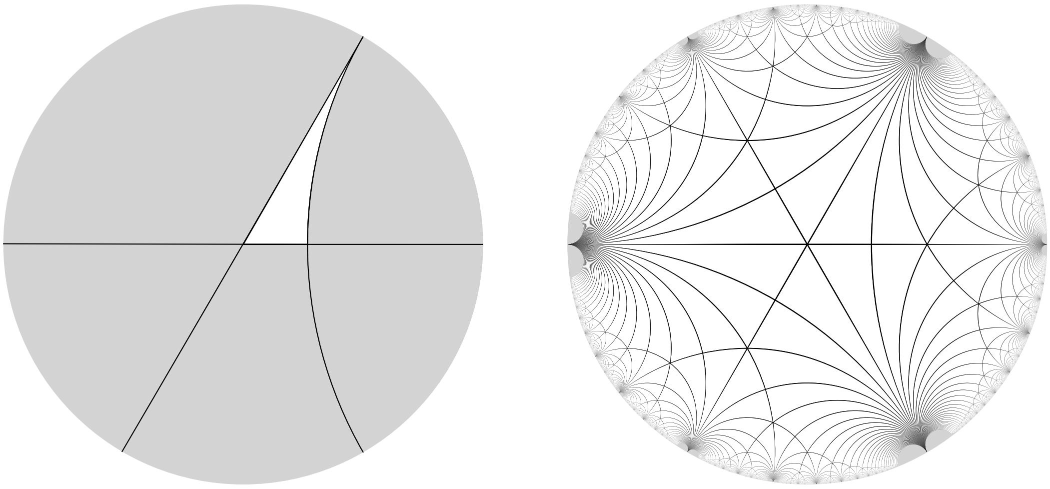


Figure 1.1: Generating polytope marked white and Coxeter complex embedded in $\mathbb{H}^2$ in the Poincaré disc model of hyperbolic reflection group $\mathrm{PGL}_2(\mathbb{Z})$.

divided generally into direct products of finite reflection groups, direct products of Euclidean reflection groups and finite reflection groups, and all Coxeter groups that are not of these two types. The latter are called *infinite non-affine* Coxeter groups. One example of an infinite non-affine Coxeter group is $\mathrm{PGL}_2(\mathbb{Z})$. Infinite non-affine Coxeter groups are much less understood than the direct products of finite and Euclidean reflection groups. They are either hyperbolic reflection groups or they are not geometric reflection groups.

However, all Coxeter groups are generated by involutions, reflection-like elements, and reflections play a central role in geometric reflection groups. The natural question arises of which other elements of a Coxeter group act like reflections on the associated geometric realisations, besides the standard generators. The answer is that an element acts as a reflection if and only if it is conjugated to a standard generator. Hence, elements that are conjugated to a standard generator are called *reflections*. Reflections are involutions and the set of reflections is a generating set of the Coxeter group.

The study of Coxeter groups together with the set of reflections as a generating set goes back to David Bessis (see [Bes03]). This approach has recently yielded significant outcomes in the research of Artin groups like the solution of the word problem and the $K(\pi, 1)$ conjecture for affine Artin groups (see [PS21; MS17]). For a summary of this so-called dual approach to the $K(\pi, 1)$ conjecture, we refer the reader to [Pao21]. Additionally, this dual approach is also connected to the Chavalley-Bruhat order and Kazhdan-Lusztig polynomials (see [Dye01; Dye91; KL79]).

1.2 Reflection length

Every generating set of a group has an associated length function. The *reflection length* is the function

$$l_R : W \to \mathbb{N}_0$$

that maps every element of a Coxeter group W to the minimal number of reflections sufficient to factor this element. The identity has reflection length 0 and reflections have reflection length 1.

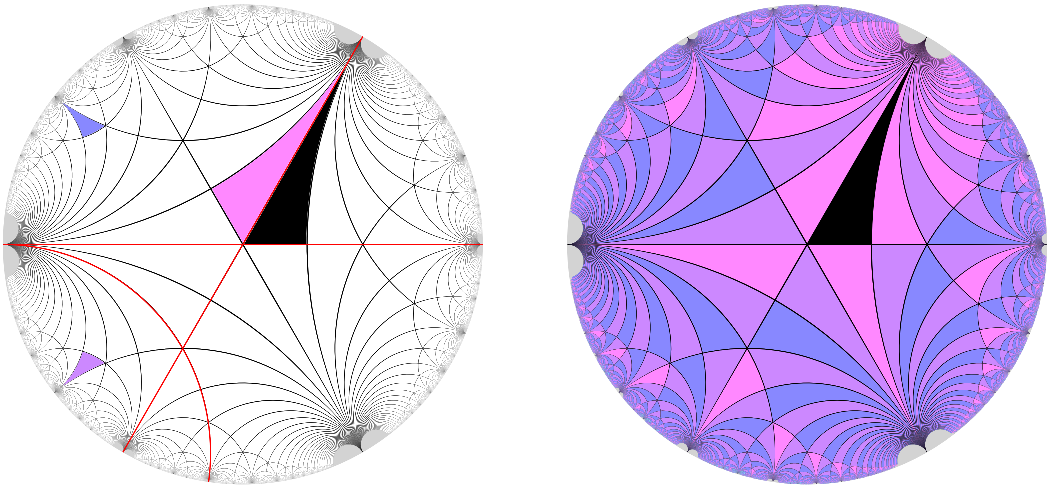


Figure 1.2: Geometric interpretation of reflection length in the Coxeter complex of $\mathrm{PGL}_2(\mathbb{Z})$ and full reflection length pattern in the Poincaré disc model.

In Figure 1.2, we fix a top-dimensional simplex ▲ in the Coxeter complex of the Coxeter group $\mathrm{PGL}_2(\mathbb{Z})$ and label it with the identity. Consider the map $w \mapsto w(\blacktriangle)$ for w in $\mathrm{PGL}_2(\mathbb{Z})$. It induces a bijection between the top-dimensional simplices of the Coxeter complex and the elements in W. The reflection length of $w \in W$ is equal to the minimal number of reflections across hyperplanes needed to reflect $w(\blacktriangle)$ to ▲. To reflect the blue simplex ▲ to ▲, reflections across the three red hyperplanes are needed. On the right side in Figure 1.2, a full pattern of coloured simplices according to their reflection length is shown. For an explanation of the colouring see Paragraph 1.4.1.

The reflection length function of finite and Euclidean reflection groups is well understood. In fact, for these groups, formulas for the reflection length exist. In the finite case, Roger William Carter proved in [Car72] that the reflection length of an element in a finite Coxeter group is equal to the dimension of a subspace associated to the action of the element on the geometric realisation mentioned above. Joel Brewster Lewis, Jon McCammond, T. Kyle Petersen and Petra Schwer established a formula of a similar form that applies to all affine Coxeter groups (see [Bre+19]). These are direct products of Euclidean reflection groups. It is based on the fact that affine Coxeter groups split into a semidirect product of the normal subgroup of translations and a finite Coxeter group. Before, McCammond and Petersen showed in [MP11] that the reflection length is bounded on these Coxeter groups.

Contrarily, the reflection length is an unbounded function on infinite non-affine Coxeter groups. This result is due to Kamil Duszenko and its proof is non-constructive (see [Dus12]). Very little is known beyond this result about the behaviour of the reflection length function on infinite non-affine Coxeter groups. An infinite non-affine geometric reflection group is a hyperbolic reflection group. Thus, infinite non-affine geometric reflection groups are in many instances of a higher geometric complexity than finite and Euclidean reflection groups. The only way known to compute reflection length in this case, by the time of writing, is by exhaustive search. Therefore, the computation of the reflection length of elements with a large word length is resource-costly.

Duszenko's theorem immediately raises questions. Which elements of an infinite non-affine Coxeter group have a large reflection length? What do sequences of elements with growing reflection length look like? What form does a formula for the reflection length have in these groups? How do the word length and the reflection length relate to one another? What is the minimal word length for a fixed reflection length and vice versa? This thesis is dedicated to the investigation of the reflection length in infinite non-affine Coxeter groups and the initial steps towards the answers to these questions.

1.3 Results and Methods

We summarize the main results presented in this thesis. Some of them appear in [Lot24a] and [Lot24b]. These articles are the bases of Chapter 6 and Chapter 7.

We give an alternative proof for Duszenko's Theorem stated as Theorem 4.1.3 in Section 4.2. The outline of this proof was communicated to the author by Andreas Thom. It is based on the fact that irreducible infinite non-affine Coxeter groups are acylindrically hyperbolic (see Theorem 4.2.9). This result is known to experts. However, we could not find a published proof of this fact. This is why, we include a proof. The outline of which was communicated to the author by Anthony Genevois. The second ingredient of the alternative proof of Duszenko's Theorem is a generalisation of the Brooks construction to acylindrically hyperbolic groups (see [BBF19]).

In connection to the conjecture that the reflection length is unbounded on infinite non-affine Coxeter groups in [MP11], McCammond and Petersen mention that they found a formula for the reflection length of the n-th power of a Coxeter element in a universal Coxeter group with three generators without giving a proof. A *Coxeter element* is a product of all standard generators in a Coxeter group. The *universal Coxeter group* of rank n is the Coxeter group with n generators $s_1, \ldots, s_n$ and no relations ($m_{ij} = \infty$) except $s_i^2 = \mathbb{1}$. In Chapter 5, we establish a formula for the powers of the Coxeter elements in a universal Coxeter group with arbitrary rank.

Theorem 1 (Formula for powers of Coxeter elements)**.** *In a universal Coxeter group W_n of rank $n \geq 2$, the following formula holds*

$$l_{R_n}((s_1 \cdots s_n)^\lambda s_1 \cdots s_i) = \lambda \cdot (n-2) + i,$$

for $\lambda \in \mathbb{N}_0$ and $1 \leq i \leq n$.

The property of the Coxeter group being universal plays a major role in the proof of this theorem. Using this theorem, we deduce formulas for the functions that describe the minimal word length for a fixed reflection length and the minimal reflection length for a fixed word length in universal Coxeter groups. These results are obtained by repeated use of the properties of the reflection length function and a technical result by Matthew J. Dyer in [Dye01].

From Theorem 1, we know that the reflection lengths of the sequence of powers of a Coxeter element in a universal Coxeter group are unbounded. Moreover, there are formulas to compute the reflection length for the elements in these specific sequences. On the other hand, the sequences of powers of Coxeter elements do not have unbounded reflection length in all infinite non-affine Coxeter groups. Counterexamples are infinite non-affine Coxeter groups of rank 3 with two commuting standard generators. So for all infinite non-affine Coxeter groups that are not universal, the result above does not give more than a hint of how a sequence of elements with unbounded reflection length might look like.

The next result is about infinite non-affine hyperbolic reflection groups. Considering only hyperbolic reflection groups allows geometric tools to be used to investigate the reflection length at a global level in the hyperbolic space $\mathbb{H}^n$. More precisely, it is investigated where top-dimensional simplices are located in the Coxeter complex that correspond to elements with large reflection lengths. We find points on the visual boundary of the compactification of $\mathbb{H}^n$ for which every neighbourhood contains simplices of every reflection length.

Theorem 2 (Boundary points close to arbitrary reflection length)**.** *Let (W, S) be a hyperbolic reflection group with fundamental domain P in $\mathbb{H}^n$. Let R be the set of reflections in W. Let U be a neighbourhood in $\overline{\mathbb{H}}^n$ of a point ξ in $\partial\mathbb{H}^n$. Suppose ξ satisfies one of the following conditions:*

(i) ξ is a common point of two parallel hyperplanes $H_r, H_{r'}$ with $r, r' \in R$.

(ii) ξ is an endpoint of the common perpendicular of two ultra-parallel hyperplanes $H_r, H_{r'}$ with $r, r' \in R$.

For every $k \in \mathbb{N}$, there exists $w \in W$ with $l_R(w) = k$ such that the domain wP is contained in U.

In contrast to the combinatorial methods used to obtain Theorem 1, this theorem together with its proof has a strong geometric flavour. One main ingredient of its proof is that two disjoint hyperplanes in the n-dimensional hyperbolic space without common boundary points have a unique common perpendicular. If two hyperplanes do not have common boundary points they are called *ultra-parallel*. We obtain the following result about hyperbolic geometry, which may be of interest regardless of our application.

Theorem 3 (Ultra-parallel Theorem for subspaces)**.** *Every pair of ultra-parallel geodesic subspaces in $\mathbb{H}^n$ has a common perpendicular. A pair of distinct hyperplanes in $\mathbb{H}^n$ is ultra-parallel if and only if it has a common unique perpendicular. Every hyperplane intersecting both hyperplanes at a right angle contains this perpendicular.*

This generalises the result of David Hilbert in [Hil13] that any two ultra-parallel geodesic lines in $\mathbb{H}^2$ have a common perpendicular.

Theorem 2 does not delimit regions close to the boundary $\partial\mathbb{H}^n$ where large reflection length occurs. It describes neighbourhoods where large reflection length surely occurs. This is without the description of the form of elements with large reflection lengths. The next results are a step towards answering the question about the form of elements with high reflection length. We establish a connection between the reflection length function l_R on an arbitrary Coxeter group and the reflection length function l_{R_n} on a universal Coxeter group of the same rank through the solution of the word problem for Coxeter groups.

Theorem 4 (Lower bound for rank n)**.** *Let w be an element in a Coxeter system (W, S) of rank n represented by an S-reduced word $\mathbf{s} = u_1 \cdot \cdots \cdot u_p$. Further, let $\tilde{\mathbf{s}}$ be a word obtained from $\mathbf{s}$ by omitting all letters in a deletion set $D(\mathbf{s})$. Let m be the minimal number of braid-moves necessary to transform $\tilde{\mathbf{s}}$ into the identity. The reflection length $l_R(w)$ in W is bounded from below:*

$$l_{R_n}(\omega_n(\mathbf{s})) - 2m \leq l_R(w).$$

For the powers of Coxeter elements in the universal Coxeter group, there exists a formula for the reflection length (see Theorem 1). Therefore, the powers of Coxeter elements are the first candidates to find unbounded reflection length in other Coxeter groups as well. Using this lower bound and counting special subwords in the powers of a Coxeter element, the next result follows.

Theorem 5 (Power sequences with unbounded reflection length)**.** *Let (W, S) be a Coxeter system of rank n and let $M = (m_{ij})_{i,j \in I}$ denote its Coxeter matrix. Further, let w be a Coxeter element in W. Then,*

(i) if $n = 3$ and $\min\{m_{ij} \mid i \neq j,\ i, j \in I\} \geq 5$, or

(ii) if $n \geq 4$ and $\min\{m_{ij} \mid i \neq j,\ i, j \in I\} \geq 3$,

we have

$$\lim_{\lambda \to \infty} l_R(w^\lambda) = \infty.$$

So in Coxeter groups with sufficiently large braid relations, the sequence of powers of a Coxeter element has unbounded reflection length.

A Coxeter group with n generators corresponding to a Coxeter matrix with non-diagonal entries all equal to k is called *single braided* and denoted with W_k^n. For single braided Coxeter groups, sharp upper bounds for the reflection length of the powers of Coxeter elements can be established using Theorem 4 and combinatorially counting specific subwords.

Theorem 6 (Upper bound for single braided power sequences in rank 3)**.** *Let (W_k^3, S) be a single braided Coxeter system with $k \geq 3$. The reflection length of elements of the form $(s_1 s_2 s_3)^\lambda s_1 \cdots s_r$ in (W_k^3, S) with $1 \leq r \leq 3$ and $\lambda \in \mathbb{N}_0$ is bounded from above by*

$$l_R(\omega((s_1 s_2 s_3)^\lambda s_1 \cdots s_r)) \leq \lambda + r - 2 \cdot \left\lfloor \frac{\lambda + \mathbb{1}_{r \geq 2}}{k} \right\rfloor.$$

Theorem 7 (Upper bound for single braided power sequences in higher rank)**.** *In a single braided Coxeter system (W_k^n, S) with $n \geq 4$, the reflection length of the element represented by the word $\mathbf{s} = (s_1 s_2 \cdots s_n)^\lambda s_1 \cdots s_r$ with $1 \leq r \leq n$ and $\lambda \in \mathbb{N}_0$ is bounded from above by*

$$l_R(\omega(\mathbf{s})) \leq \lambda(n-2) + r - 2 \cdot \mathbb{1}_{(\lambda + \mathbb{1}_{r \geq 2}) \geq k} \cdot \left(1 + \left\lfloor \frac{\lambda - k + \mathbb{1}_{r \geq 2}}{k - 1} \right\rfloor\right).$$

We conjecture that the bounds given in Theorem 6 and Theorem 7 are equal to the reflection length.

1.4 Structure

This thesis is organised as follows. Chapter 2 contains a summary of the foundations of Coxeter groups and geometric reflection groups with examples. The focus is on the duality between abstract group theory and the geometric objects related to Coxeter groups. In Chapter 3, we state the main definitions and theorems related to word length and reflection length in Coxeter groups. In particular, the word problem for Coxeter groups and the properties of the reflection length are explained. Furthermore, the aforementioned formulas for the reflection length in finite and affine Coxeter groups are discussed. Chapter 4 deals with the theorem of Duszenko and its proof. Our alternative proof is presented, too. As an ingredient, it is proved that irreducible infinite non-affine Coxeter groups are acylindrically hyperbolic.

The remaining chapters contain our results. The fifth chapter is devoted to the study of the reflection length function of universal Coxeter groups. Theorem 1 is proved. Further, we deduce results regarding minimal word length for a fixed reflection length and vice versa. Theorem 2 and Theorem 3 are stated and proved in Chapter 6 among other geometric flavoured results about infinite non-affine hyperbolic reflection groups. In Chapter 7, we study the relation between the reflection length function of an arbitrary Coxeter group and the reflection length function of the universal Coxeter group of the same rank. The proofs of Theorem 4 as well as of Theorem 5 are to be found in this chapter. Furthermore, this chapter contains the proofs of Theorem 6 and Theorem 7. At the end of Chapter 7, we state Conjecture 7.3.1 about the general relation between the reflection length function of an arbitrary Coxeter group and the reflection length function of the universal Coxeter group of the same rank. Finally, we give an outlook for further research and follow-up questions. In Appendix A, we present the SageMath code that we use to compute the reflection length in some examples throughout this work.

1.4.1 Colouring of the reflection length

The reflection length is encoded by the same colour scheme in all images in this work. The colours are totally ordered in ascending order of wavelength in the light spectrum. A small reflection length corresponds to a colour with a small wavelength and a large reflection length corresponds to a colour with a high wavelength. For example, reflection length 1 is represented by the colour and reflection length 3 is represented by the colour . One exception is reflection length 0, which is represented by the colour black.

2. Coxeter groups and reflection groups

This chapter contains some foundations of the theory of Coxeter groups and their geometry. For a more profound treatment of this topic, we refer the reader to [AB08], [Hum90], [Tho18] and [BB05]. The majority of this chapter is compiled from these sources. First, the definition and some examples of Coxeter groups are given. After discussing geometric reflection groups and their classification, we introduce geometric constructions for abstract Coxeter groups. The last section deals with roots and their relation to affine Coxeter groups.

2.1 Abstract Coxeter groups

Definition 2.1.1. A group W with a finite set of generators $S = \{s_1, \dots s_n\}$ that admits a presentation of the form

$$W = \langle S \mid s_i^2 = (s_i s_j)^{m_{ij}} = \mathbb{1} \; \forall i, j \in \{1, \dots, n\}, \; m_{ij} \in \mathbb{N}_{\geq 2} \cup \{\infty\} \rangle,$$

is called *Coxeter group*. The pair (W, S) is called *Coxeter system*. The cardinality $|S|$ is called the *rank* of the Coxeter group or Coxeter system, respectively. The relations $(s_i s_j)^{m_{ij}} = \mathbb{1}$ with $m_{ij} \in \mathbb{N}_{\geq 2}$ are called *braid relations*. The elements in W that can be expressed by words, in which each generator appears exactly once, are named *Coxeter elements* (e.g. $s_1 \cdots s_n$).

Definition 2.1.2. Let (W, S) be a Coxeter system. For a subset $S' \subseteq S$, the subgroup W' generated by S' is called *standard parabolic subgroup* and again a Coxeter group. A conjugate of a standard parabolic subgroup is called *parabolic subgroup*.

As a convention, we omit the ∞-relations, when writing out a presentation explicitly. Every element w in a Coxeter system (W, S) can be written as a product $w = u_1 \cdots u_p$ of generators in S with $u_i \in S$ for all $1 \leq i \leq p$.

Example 2.1.3. The simplest example of a Coxeter group is a group generated by a single involution. It is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and can be interpreted geometrically as a group generated by a single reflection across a point on the real line $\mathbb{R}$.

Example 2.1.4. (Symmetric groups) The symmetry group of the $(n-1)$-dimensional simplex is the group $\mathrm{Sym}(n)$. It is generated by $n-1$ reflections and admits the following presentation

$$\mathrm{Sym}(n) = \langle \{s_1, \dots, s_{n-1}\} \mid s_i^2 = (s_j s_{j+1})^3 = \mathbb{1} \text{ for } 1 \leq i \leq n-1, 1 \leq j \leq n-2 \rangle.$$

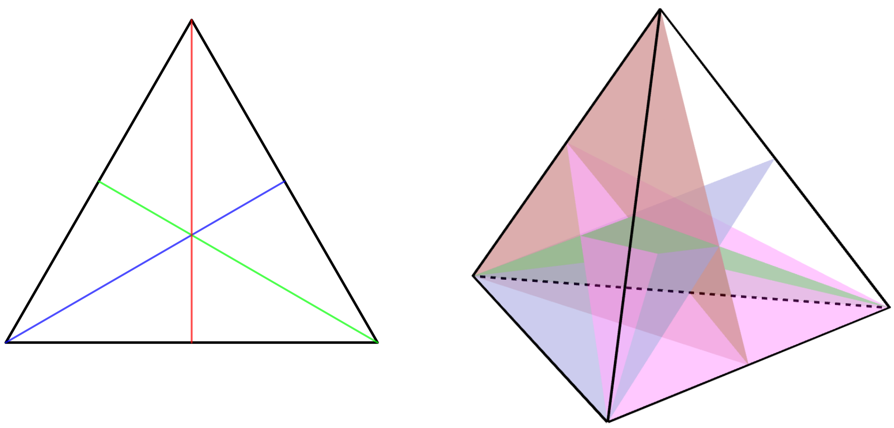


Figure 2.1: A 2-dimensional simplex with all and a 3-dimensional simplex with some coloured reflection axes.

Hence, symmetric groups of all finite ranks are Coxeter groups. Figure 2.1 displays a 2-dimensional simplex with all reflection axes and a 3-dimensional simplex with some reflection axes. The two reflections across the red and blue axes on the left generate $\mathrm{Sym}(3)$ and the three reflections across the green, blue and purple axes on the right generate $\mathrm{Sym}(4)$.

Coxeter groups can be of finite and infinite cardinality as the next example shows.

Example 2.1.5. (Dihedral groups) Define the *dihedral group* D_m to be the group with presentation

$$D_m := \langle s_1, s_2 \mid s_i^2 = (s_1 s_2)^m = \mathbb{1} \text{ for } i = 1, 2\rangle,$$

for an integer $m \geq 2$. The dihedral group D_m has order $2m$. Geometrically, the dihedral groups can be understood as the isometry groups of regular m-gons in the Euclidean plane $\mathbb{E}^2$ (regular means that the sides are isometric). So in fact, with our last example, we already introduced the group D_3, which is isomorphic to $\mathrm{Sym}(3)$. The two generators correspond to two orthogonal reflections across lines l_1 and l_2 through the centre of the m-gon such that the intersecting angle between the lines is $\frac{\pi}{m}$. This is shown in Figure 2.2 for $m = 3$. Further, D_m acts on the unit sphere $\mathbb{S}^1$. The intersections of $\mathbb{S}^1$ with the images of the lines reflected across each other divide the sphere into $2m$ isometric segments as shown in the figure. We obtain a $2m$-gon by connecting the adjacent intersections, which are points, with lines in $\mathbb{E}^2$. This illustrates why D_m is a subgroup of D_{2m}. In the figure, it can be seen that D_6 contains D_3 as a subgroup. The product of the two reflections $s_1 s_2$ is a rotation of the angle $\frac{2\pi}{m}$ about the origin. The relation $(s_1 s_2)^m = \mathbb{1}$ follows thereby. In total, the group D_m consists of m orthogonal reflections and m rotations of $\frac{2k\pi}{m}$ with $0 \leq k < m$ about the origin.

The only infinite Coxeter group of rank 2 is the *infinite dihedral group* D_∞. It is generated by two involutions, whose product has infinite order:

$$D_\infty := \langle s_0, s_1 \mid s_i^2 = \mathbb{1} \text{ for } i = 1, 2\rangle.$$

It is isomorphic to a subgroup W of isometries of the real Euclidean line $\mathbb{E}^1$ generated by the reflections $s_0 : x \mapsto -x$ and $s_1 : x \mapsto 2 - x$ across the points 0 and 1. The element $s_0 s_1$ is a translation by 2 to the left and therefore $\langle s_0 s_1\rangle \cong \mathbb{Z}$ (see Figure 2.3). The group

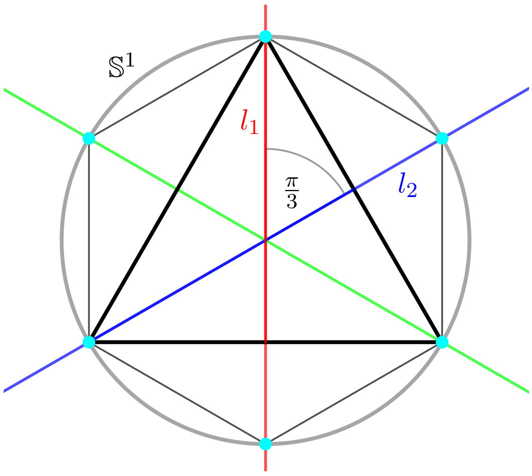


Figure 2.2: Reflections of the dihedral group D_3 with generators l_1 and l_2 and the action on $\mathbb{S}^1$.

W contains the reflection across every integer in $\mathbb{E}^1$. For $z \in \mathbb{Z}$, the reflection across z is a conjugate of s_0 if z is even and otherwise a conjugate of s_1. Hence, the points in the orbits of the points 1 and 0 under the action of W result in a tessellation of $\mathbb{E}^1$ into unit intervals (see Figure 2.3).

For further examples of finite Coxeter groups and their combinatoric interpretation, we refer the reader to [BB05].

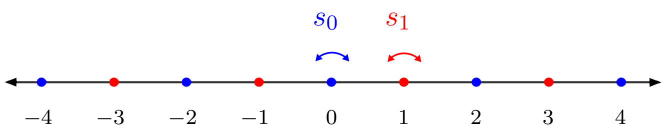


Figure 2.3: Action of the infinite dihedral group D_∞ on $\mathbb{E}^1$, orbit of 0 marked blue and orbit of 1 marked red.

It can be difficult to read off properties of the group from an abstract presentation like in the definition of a Coxeter group above. Different presentations can lead to isomorphic groups. For a Coxeter system (W, S), let $\omega(s)$ be the element in W represented by $s \in S$. A classical and important statement about the presentation of a Coxeter group is the following.

Lemma 2.1.6 ([BB05, Proposition 1.1.1])**.** *Let (W, S) be the Coxeter system and $s_i, s_j \in S$ be two distinct standard generators. Then, the following hold:*

(i) $\omega(s_i) \neq \omega(s_j)$ in W.

(ii) The order of $\omega(s_i s_j)$ in W is m_{ij}.

Coxeter systems are generally encoded in the literature by finite loop-free graphs and an edge labelling that is either a natural number greater than 2 or infinity.

Definition 2.1.7. Let (W, S) be a Coxeter system of rank n. The *Coxeter graph* (or *Coxeter diagram*) Γ has the vertex set S and the edge set $E := \{\{s_i, s_j\} \mid s_i, s_j \in S, i \neq j, m_{ij} \geq 3\}$. The edge labelling function $m : E \to \mathbb{N}_{\geq 3} \cup \{\infty\}$ is given by $m(\{s_i, s_j\}) = m_{ij}$. The convention when drawing these graphs is to leave out 3 as a label. Note that vertices corresponding to generators that commute are not connected by an edge. The *Coxeter matrix* of (W, S) is the matrix $(m_{ij})_{i,j\in[n]}$ with $m_{i,i} := 1$ for all indices $i \in [n] = \{1, \dots, n\}$.

Remark 2.1.8. The information encoded in the Coxeter matrix or Coxeter graph is equivalent to the Coxeter presentation from Definition 2.1.1. Lemma 2.1.6 implies that every Coxeter graph and every Coxeter matrix determines a unique Coxeter group up to isomorphism (see [BB05, Theorem 1.1.2]).

Example 2.1.9. The Coxeter graphs of the dihedral groups D_m and Sym(4) are the following.

The instances of Coxeter groups given above include finite groups of any rank and an infinite group D_∞ of rank 2. The property of having solely ∞-relations can be transferred to higher-rank Coxeter groups.

Example 2.1.10. (Universal Coxeter groups) The *universal Coxeter group* W_n of rank n is defined by the complete graph K_n with n vertices and the constant labelling $m_{ij} = \infty$ for all $i \neq j \in \{1, \dots, n\}$. It admits the presentation

$$W_n = \langle \{s_1, \dots, s_n\} \mid s_i^2 = \mathbb{1} \rangle.$$

Every element is represented by a unique reduced word of letters in S (see [BB05, p. 4]). In rank 2, W_2 is the infinite dihedral group D_∞.

Definition 2.1.11. A Coxeter group W is called *irreducible* (*reducible*) if the corresponding Coxeter graph is connected (is not connected).

Lemma 2.1.12 (see [Hum90, p. 129])**.** *Let (W, S) be a Coxeter system. Let $\Gamma_1, \dots, \Gamma_r$ be the connected components of the Coxeter graph with corresponding subsets $S_1, \dots, S_r \subseteq S$. Then W is the direct product of the standard parabolic subgroups $W_{S_1}, \dots, W_{S_r}$ and each Coxeter system (W_i, S_i) is irreducible.*

2.2 Geometric reflection groups

As mentioned above, the definition of a Coxeter group evolved from studying discrete groups of isometries of geometric objects generated by reflections. This section deals with the geometric approach to Coxeter groups. Geometric reflection groups are defined and partially classified in this section. Basic concepts of Riemannian geometry are assumed. The first two chapters in [BH99] and Chapter 6 in [Dav08] are the basis of this section.

We introduce the three complete, simply connected, Riemannian n-manifolds of constant sectional curvature $-1, 0$ and 1. These three manifolds admit totally geodesic submanifolds of codimension 1. The only three irreducible symmetric spaces that admit such submanifolds are $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$. This follows from the classification of symmetric spaces by Élie Cartan (see [Car26]). This fact is important because every totally geodesic submanifold of codimension 1 can be associated with a geometric reflection. These submanifolds are called *hyperplanes.* For each hyperplane H in a metric space $\mathbb{X}^n$, the complement $\mathbb{X}^n \setminus H$ has two open connected components. The associated (closed) *half-spaces* H^+ and H^- are the unions of the connected components with H. The reflection across H maps one half-space to the other.

Since the Riemannian structure is not needed to define reflection groups, we define the manifolds as metric spaces and keep the Riemannian structure in the background.

Definition 2.2.1. Let $(\mathbb{X}, d)$ be a metric space and $I \subseteq \mathbb{R}$ be an interval.

1. A *geodesic line* in $\mathbb{X}$ is an isometry $\lambda : I \to \mathbb{X}$ with $I = (-\infty, \infty)$.
2. A *geodesic ray* in $\mathbb{X}$ is an isometry $\lambda : I \to \mathbb{X}$ with $I = [a, \infty)$.
3. A *geodesic segment* in $\mathbb{X}$ is an isometry $\lambda : I \to \mathbb{X}$ with $I = [a, b]$.

We write $\lambda \subseteq \mathbb{X}$ instead of $\lambda(I) \subseteq \mathbb{X}$.

The metric space $(\mathbb{X}, d)$ is called *geodesic space* if for every pair of points, x, y in $\mathbb{X}$ there exists a geodesic segment $\lambda : [a, b] \to \mathbb{X}$ with $\lambda(a) = x$ and $\lambda(b) = y$. If for every pair of points the images of all geodesic segments is equal, we call $\mathbb{X}$ a *uniquely geodesic space.*

The Euclidean n-space. Let d_2 be the metric corresponding to the Euclidean scalar product $\langle X, Y\rangle_2 = \sum_{i=1}^n x_i y_i$, where $X = (x_1, \dots, x_n), Y = (y_1, \dots, y_n) \in \mathbb{R}^n$. We denote the vector space $\mathbb{R}^n$ equipped with the metric d_2 with $\mathbb{E}^n$. It is

$$d_2(X, Y) = \left(\sum_{i=1}^n (x_i - y_i)\right)^{\frac{1}{2}}.$$

The subset of the form $(A, B) := \{A + t(B - A) \mid t \in \mathbb{R}\}$ for $A, B \in \mathbb{E}^n$ is the geodesic line containing the points A and B and $\mathbb{E}^n$ is a uniquely geodesic space. Choose $0 \leq t \leq 1$ to obtain the geodesic segment $[A, B]$. Let $[C, A]$ and $[C, B]$ be two geodesic segments with $A, B, C \in \mathbb{E}^n$ and abbreviate the distances $d := d_2(B, C), e := d_2(A, C)$ and $f := d_2(A, B)$. According to the *law of cosines*, the angle γ between these segments at the point C is

$$\gamma = \cos^{-1} \frac{d^2 + e^2 - f^2}{2de}.$$

A hyperplane in $\mathbb{E}^n$ is an affine subspace of codimension 1. A half-space in $\mathbb{E}^n$ is an affine half-space. Every hyperplane H induces an isometry of $\mathbb{E}^n$. Namely, the reflection r_H across H. Fix a point P in H and let u be an orthogonal unit vector (with respect to the scalar product). Then, the reflection r_H can be defined as

$$r_H(A) := A - 2\langle A - P, u\rangle_2 u$$

for all $A \in \mathbb{E}^n$. The hyperplane H is the set of fixed points of r_H.

The n-sphere. The n-dimensional sphere is denoted by $\mathbb{S}^n$ and it is embedded in $\mathbb{E}^{n+1}$ in the following way: $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \langle x, x\rangle_2 = 1\}$, where $\langle \cdot, \cdot \rangle_2$ is the Euclidean scalar product from the last paragraph. For two points $A, B \in \mathbb{S}^n$, the metric d on $\mathbb{S}^n$ is defined as $d(A, B) = \cos^{-1}\langle A, B\rangle_2$. The metric space $(\mathbb{S}^n, d)$ is a geodesic space. The intersection of $\mathbb{S}^n$ with a 2-dimensional vector subspace of $\mathbb{E}^{n+1}$ is called *great circle*. The great circles in $\mathbb{S}^n$ are the geodesic lines. The shortest path between two distinct points $A, B \in \mathbb{S}^n$ (or paths if A and B are antipodal) is contained in the great circle corresponding to the subspace spanned by A and B. Let $[C, A]$ and $[C, B]$ be two geodesic segments in $\mathbb{S}^n$ with $A, B, C \in \mathbb{S}^n$ and abbreviate the distances $g := d(B, C)^2, e := d(A, C)^2$ and $f := d(A, B)^2$. According to the *spherical law of cosines*, the angle γ between these segments at the point C is

$$\gamma = \cos^{-1} \frac{\cos f - \cos g \cos e}{\sin g \sin e}.$$

The intersection of $\mathbb{S}^n$ with an n-dimensional vector subspace of $\mathbb{E}^{n+1}$ is a hyperplane in $\mathbb{S}^n$. Similarly, a half-space in $\mathbb{S}^n$ is the intersection of a linear half-space in $\mathbb{E}^{n+1}$ with $\mathbb{S}^n$. A hyperplane $H \subseteq \mathbb{S}^n$ is isometric to $\mathbb{S}^{n-1}$ with the induced metric for $\mathbb{S}^n$. The corresponding reflection r_H is defined as the isometric reflection across the hyperplane in $\mathbb{E}^{n+1}$ spanned by H restricted to $\mathbb{S}^n$. With the definition of a hyperplane reflection in $\mathbb{E}^{n+1}$ and the properties of symmetrical bilinear forms, the well-definedness of this restriction follows.

2.2.1 The hyperbolic n-space

We introduce two model spaces for the hyperbolic n-space, both of which we use in the further course of this work. First, the *hyperboloid model* which is embedded in $\mathbb{R}^{n+1}$ equipped with a symmetric bilinear form in a similar way as the sphere. The strength of this model is that it can be fully exhausted with linear algebra. On the other hand, the *Poincaré ball model* is useful for visualizing the hyperbolic n-space and using tools from geometry.

The hyperboloid model. Let $\mathbb{E}^{n,1}$ be the real vector space $\mathbb{R}^{n+1}$ equipped with the symmetric bilinear form $\langle \cdot, \cdot \rangle_{-1}$ of type $(n, 1)$ (n positive and 1 negative eigenvalues). We define

$$\langle X, Y \rangle_{-1} := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1},$$

where $X = (x_1, \ldots, x_{n+1}), Y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$. The n-dimensional *hyperboloid model* is the upper sheet of a hyperboloid and defined as follows:

$$\mathbb{H}^n := \{v = (v_1, \ldots, v_{n+1}) \in \mathbb{E}^{n,1} \mid \langle v, v\rangle_{-1} = -1, v_{n+1} > 0\}.$$

The bilinear form induces a metric $d(x, y) = \cosh^{-1}(-\langle x|y\rangle_{-1})$ for $x, y \in \mathbb{H}^n$. We always consider $\mathbb{H}^n$ as a metric space. Non-empty intersections of 2-dimensional subspaces of $\mathbb{E}^{n,1}$ with $\mathbb{H}^n$ are the geodesic lines in $\mathbb{H}^n$. For two geodesic lines γ_1 and γ_2 intersecting in a point $p \in \mathbb{H}^n$, there exist unit vectors $u_i \in \mathbb{E}^{n,1}$ with $\langle u_i|p\rangle_{-1} = 0$ and such that u_i is contained in the 2-dimensional subspace of $\mathbb{E}^{n,1}$ according to γ_i. The hyperbolic angle between γ_1 and γ_2 is the unique number $\alpha \in [0, \pi]$ with $\alpha = \cos^{-1}(\langle u_1|u_2\rangle_{-1})$.

Hyperplanes are defined as non-empty intersections of n-dimensional subspaces of $\mathbb{E}^{n,1}$ with $\mathbb{H}^n$. They are isometric to $\mathbb{H}^{n-1}$ equipped with the induced metric from $\mathbb{H}^n$. For a hyperplane H, we denote the corresponding subspace in $\mathbb{E}^{n,1}$ with V_H. We have two (closed) half-spaces $V_H^+, V_H^- \subseteq \mathbb{E}^{n,1}$ with $V_H^+ \cup V_H^- = \mathbb{E}^{n,1}$ and $H^\pm = V_H^\pm \cap \mathbb{H}^n$ with $H_i^+ \cup H_i^- = \mathbb{H}^n$. Every hyperplane H induces an isometry on $\mathbb{H}^n$, the reflection r_H across H:

$$r_H : \mathbb{H}^n \to \mathbb{H}^n; \qquad x \mapsto x - 2\langle u_H | x \rangle_{-1} u_H, \tag{2.2.1}$$

where u_H is the unique (modulo sign) unit vector orthogonal to V_H in $\mathbb{E}^{n,1}$ with respect to the bilinear from $\langle \cdot | \cdot \rangle_{-1}$. The fixed point set of r_H is exactly H and one half-space is mapped to the other one. The points at infinity in this model correspond to the set of 1-dimensional subspaces of points $x \in \mathbb{E}^{n,1}$ with $\langle x | x \rangle_{-1} = 0$.

The Poincaré ball model. In the *Poincaré ball model*, the points of the hyperbolic space are represented by points in the open unit ball D^n in the Euclidean n-space $\mathbb{E}^n$. There exists a homeomorphism $D^n \to \mathbb{H}^n$ by which the metric on $\mathbb{H}^n$ can be pulled back to D^n. Geodesic lines are the intersections of D^n with lines and circles in $\mathbb{E}^n$ that meet the boundary $\partial D^n = \mathbb{S}^{n-1}$ in a right angle. The angle between two segments issuing from a point is equal to the Euclidean angle between the segments. Let $\widehat{\mathbb{E}}^n := \mathbb{E}^n \cup \{\infty\}$ denote the one-point compactification of $\mathbb{E}^n$. The hyperplanes in this model are the intersections of D^n with $(n-1)$-dimensional spheres in $\widehat{\mathbb{E}}^n$ that intersect $\mathbb{S}^{n-1}$ orthogonally. If a hyperplane H is the intersection of an $(n-1)$-dimensional sphere containing ∞ with D^n, the reflection on H is the reflection on H as an $(n-1)$-dimensional subspace in $\mathbb{E}^n$ restricted to D^n. Otherwise, the hyperplane H is represented by a sphere S with radius r and centre c. In this case, the reflection across H is i_S, the inversion on S in the one-point compactification $\widehat{\mathbb{E}}^n$ restricted to D^n:

$$i_S : \widehat{\mathbb{E}}^n \to \widehat{\mathbb{E}}^n; \qquad x \mapsto \frac{r^2}{||x-c||^2} \cdot (x-c) + c. \tag{2.2.2}$$

The points at infinity in this model are the points in $\mathbb{S}^{n-1}$.

Remark 2.2.2. The points at infinity in both models are also called *ideal points.*

2.2.2 The action of the group of isometries $\mathrm{Iso}(\mathbb{X}^n)$

The goal is to connect the abstract definition of Coxeter groups to geometry. Let $\mathbb{X}^n$ denote either $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{X}^n$. Before restricting our considerations to discrete groups of isometries of $\mathbb{X}^n$ generated by reflections across hyperplanes, we state the following important results about the general group of isometries of $\mathbb{X}^n$.

Theorem 2.2.3 (see [BH99, Proposition 2.18])**.** *Let γ be an isometry of $\mathbb{X}^n$.*

(i) If γ is not the identity, then the set of points that it fixes is contained in a hyperplane.

(ii) If γ acts as the identity on a hyperplane H, then γ is either the identity or the reflection r_H through H.

(iii) γ can be written as the composition of $n+1$ or fewer reflections through hyperplanes in $\mathbb{X}^n$.

Let $O(n)$ denote the group of orthogonal matrices contained in the general linear group $GL_n(\mathbb{R})$. These are the real $n \times n$-matrices A that satisfy $A^T A = \mathbb{1}$, where A^T is the transpose of A. Moreover, we define the group $O(n,1)$ to be the subgroup of $GL_{n+1}(\mathbb{R})$ consisting of the matrices that leave invariant the bilinear form $\langle \cdot | \cdot \rangle_{-1}$. Thus, the elements of $O(n,1)$ preserve the hyperboloid $\{x \in \mathbb{E}^{n,1} \mid \langle x|x\rangle_{-1} = -1\}$ but possibly exchange the sheets. Equivalently, a matrix $A \in GL_{n+1}(\mathbb{R})$ is contained in $O(n,1)$ if and only if $A^T JA = J$, where J is diagonal matrix with diagonal $(1,\dots,1,-1)$. Let $O(n,1)_+ \subseteq O(n,1)$ be the index-two subgroup of elements that preserve the upper sheet $\mathbb{H}^n$ (see [BH99, pp. 29-30]).

Theorem 2.2.4 (see [BH99, Proposition 2.24]). *The groups of isometries of $\mathbb{X}^n$ are the following:*

(i) $\mathrm{Iso}(\mathbb{E}^n) \cong \mathbb{R}^n \rtimes O(n)$,

(ii) $\mathrm{Iso}(\mathbb{S}^n) \cong O(n+1)$,

(iii) $\mathrm{Iso}(\mathbb{H}^n) \cong O(n,1)_+$.

In all three cases, the stabilizer of a point is isomorphic to $O(n)$.

Here, $\rtimes$ denotes the semi-direct product.

Definition 2.2.5. Suppose a group G acts on a topological space Y. A closed subset $A \subseteq Y$ is a *strict fundamental domain* for the G-action if each G-orbit intersects A in exactly one point. The action of G is called *cocompact* if the quotient space ${}^{Y}\!/_{G}$ is compact.

Definition 2.2.6. Let G be a discrete group. A G-action on a Hausdorff space Y is *proper* (or *properly discontinuous*) if the following three conditions are satisfied:

(i) Y/G is Hausdorff.

(ii) For each $y \in Y$, the stabilizer $G_y = \{g \in G \mid gy = y\}$ is finite.

(iii) Each $y \in Y$ has a G_y-stable neighbourhood U_y such that $gU_y \cap U_y = \varnothing$ for all $g \in G - G_y$.

A simple example is the following.

Example 2.2.7. Consider the real Euclidean line $\mathbb{E}^1$ and the reflections s_0 and s_1 across the two points 0 and 1 as illustrated in Example 2.1.5. The group generated by $\{s_1, s_2\}$ is the infinite dihedral group D_∞. Every closed interval $[n, n+1]$ with $n \in \mathbb{Z}$ is a strict fundamental domain for the action of D_∞ on $\mathbb{E}^1$.

2.2.3 Convex polyhedra and geometric reflection groups

An important connection between Coxeter groups and convex polyhedra is established in this section. A convex polyhedron with simple combinatorial properties yields a Coxeter group that is a discrete subgroup generated by reflections of the group of isometries of the underlying space. The polyhedron is a strict fundamental domain of the action of the Coxeter group on the underlying space.

Definition 2.2.8.
Let $\mathbb{X}^n$ be one of the three spaces $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$.

1. An *n-dimensional convex polyhedron* P in $\mathbb{X}^n$ is an intersection of finitely many closed half-spaces in $\mathbb{X}^n$ having a non-empty interior.

2. An *n-dimensional convex polytope* P' in $\mathbb{X}^n$ is a convex and compact intersection of finitely many closed half-spaces in $\mathbb{X}^n$ with non-empty interior.

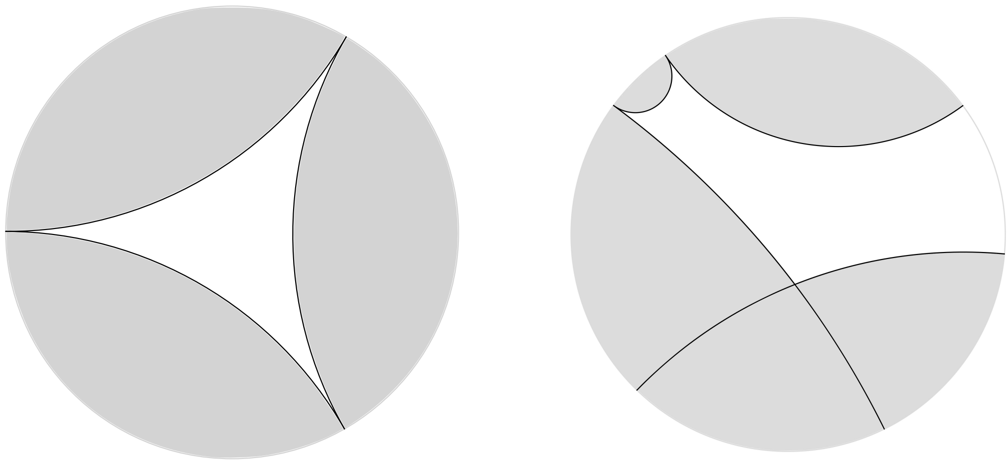


Figure 2.4: Two convex polyhedra in $\mathbb{H}^2$ in the Poincaré disc model marked white, with finite volume on the left and infinite volume on the right.

Remark 2.2.9. We always assume in the following text without further mention that no half-space in the definition above contains the intersection of all other half-spaces. With this assumption, the half-spaces are uniquely determined by the polyhedron (see [VS88, p. 104]).

Remark 2.2.10. We follow the denotation in [VS88] and use the word *polyhedron* instead of *polytope* to distinguish between a finite intersection of half-spaces and the convex hull of finitely many points. This is not the same in our setting. By the Weyl-Minkowski Theorem (see [Wey35]), a bounded convex polyhedron in the Euclidean space is the convex hull of finitely many points and vice versa (see [VS88, p.104]). The distinction between polyhedron and polytope is particularly important in $\mathbb{H}^n$ for us. Here, there exist unbounded polyhedra of finite volume and unbounded polyhedra with infinite volume (see Figure 2.4).

The following definition is a restriction on the combinatorial structure of a polyhedron in $\mathbb{X}^n$.

Definition 2.2.11. Let $P = \cap_{i \in I} H_i^{\varepsilon_i}$ be a convex polyhedron in $\mathbb{X}^n$ with $\varepsilon_i \in \{+, -\}$.

1. An $(n-k)$-*dimensional face* of P is an intersection contained in P of k hyperplanes in $\{H_i \mid i \in I\}$ with $1 \leq k \leq n$. A 0-dimensional face is called *vertex*.
2. The convex polyhedron P is called *simple in its* $(n-k)$-*dimensional face* F if F is contained in exactly k (the least possible number) $(n-1)$-dimensional faces. This implies that it is simple in any face containing F. The convex polyhedron P is called *simple* if it is simple in each of its faces.

Remark 2.2.12. A bounded polyhedron is simple if it is simple in its vertices. This is why, the simplicity for polytopes can also be defined in the following way: An n-dimensional polytope P' is called simple if for every vertex $v \in P'$ the intersection of a sufficiently small sphere centred at v with P' (this is commonly referred to as the *link* of v) is an $(n-1)$-dimensional simplex (see for example [Dav08, Definition 6.3.8.]).

Definition 2.2.13. Let H_1 and H_2 be two intersecting hyperplanes in $\mathbb{X}^n$, which bound half-spaces $H_1^{\varepsilon_1}$ and $H_2^{\varepsilon_2}$ with $H_1^{\varepsilon_1} \cap H_2^{\varepsilon_2} \neq \varnothing$. Further, let u_i be the inward-pointing unit vector in the orthogonal complement of H_i at a point $x \in H_1 \cap H_2$. The *dihedral angle* along $H_1 \cap H_2$ is defined as $\measuredangle(H_i, H_j) := \pi - \cos^{-1}\langle u_1, u_2\rangle$, where $\langle \cdot, \cdot \rangle$ is the bilinear form of the underlying space $\mathbb{E}^n$ or $\mathbb{E}^{n,1}$ for $\mathbb{X}^n$.

Definition 2.2.14. A convex polyhedron (polytope)

$$P = \bigcap_{i \in I} H_i^{\varepsilon_i}$$

with $\varepsilon_i \in \{+, -\}$ is called a *Coxeter polyhedron* (*Coxeter polytope*) if the dihedral angle $\measuredangle(H_i, H_j)$ is a submultiple of π for all intersecting hyperplanes H_i, H_j with $i, j \in I,\ i \neq j$. We call a hyperplane H_i *wall* if the closure $\overline{P}$ and H_i intersect.

Remark 2.2.15. We may assume that all the hyperplanes H_i are walls (see Remark 2.2.9). The hyperplanes of non-adjacent faces of a Coxeter polyhedron do not intersect. Every Coxeter polyhedron is simple (see [VS88, Theorem 1.8.]).

The following theorem establishes an important connection between geometry and Coxeter groups.

Theorem 2.2.16 (see [VS88, pp. 199-200])**.** *Let $P = \bigcap_{i \in I} H_i^{\varepsilon_i}$ be a Coxeter polyhedron in $\mathbb{X}^n$ with $n \geq 2$ and $W(P)$ the group generated by the isometric reflections $\{s_i \mid i \in I\}$ across the walls of P. Under these assumptions, the following holds:*

(i) $W(P)$ is a discrete subgroup of $\mathrm{Iso}(\mathbb{X}^n)$ *generated by hyperplane reflections.*

(ii) $W(P)$ acts properly on $\mathbb{X}^n$.

(iii) P is a strict fundamental domain for the $W(P)$-action on $\mathbb{X}^n$.

(iv) $W(P)$ is a Coxeter group with defining relations $s_i^2 = \mathbb{1}$ for all $i \in I$ and $(s_j s_k)^{m_{jk}} = \mathbb{1}$ for intersecting H_j and H_k with dihedral angle $\measuredangle(H_j, H_k) = \frac{\pi}{m_{jk}}$.

(v) The stabilizer $W(P)_x$ of any point $x \in P$ (including points at infinity if $\mathbb{X}^n = \mathbb{H}^n$) is generated by reflections across the walls of P containing x.

Corollary 2.2.17. *The fundamental domains wP with $w \in W(P)$ cover the space $\mathbb{X}^n$ and there is a bijection between $W(P)$ and the set of fundamental domains wP. Namely,*

$$w \mapsto wP \quad \text{with } w \in W(P).$$

Remark 2.2.18. A few remarks on the theorem above:

1. The dimension 1 is explicitly excluded in the theorem because the 1-dimensional sphere $\mathbb{S}^1$ allows hyperplanes (here points) that do not intersect while the product of corresponding reflections $s_1 s_2$ has finite order. This concerns precisely the finite dihedral groups D_m (see Example 2.1.5). The group action of D_m on $\mathbb{S}^1$ has a closed interval of length $\frac{\pi}{m}$ as a strict fundamental domain and is generated by the two reflections across the points that bound this interval. The action induces a tessellation of $\mathbb{S}^1$ by copies of this polytope as to be seen in Figure 2.2 (see [Tho18, p. 9]).

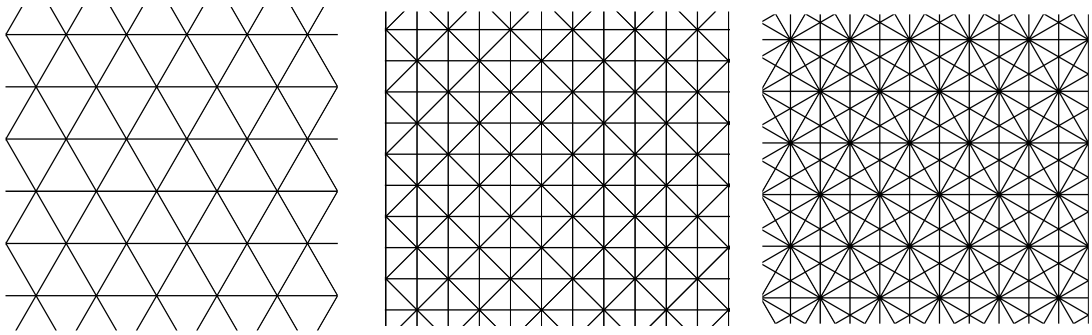


Figure 2.5: Tessellations of $\mathbb{E}^2$ induced by the three Euclidean triangle groups.

2. The infinite dihedral group D_∞ can be obtained analogously to the theorem by taking an interval P as the polyhedron in $\mathbb{E}^1$ or $\mathbb{H}^1$.

3. As seen in Theorem 2.2.4, the isometry group $\mathrm{Iso}(\mathbb{X}^n)$ is a Lie group. Thereby, the theorem gives examples of Coxeter groups that are discrete subgroups of Lie groups. The isometry groups $\mathrm{Iso}(\mathbb{S}^n) \cong O(n+1)$ are compact. Thus, any discrete subgroup of $\mathrm{Iso}(\mathbb{S}^n)$ is finite and so are the groups $W(P)$ in the theorem for $\mathbb{X}^n = \mathbb{S}^n$ (see [Tho18, p. 9]).

Definition 2.2.19. A *geometric reflection group* is a discrete subgroup of $\mathrm{Iso}(\mathbb{X}^n)$ generated by finitely many reflections across the $(n-1)$-dimensional faces of a Coxeter polyhedron. The reflection group is called *spherical*, *Euclidean* (also *affine*) or *hyperbolic* as $\mathbb{X}^n$ is equal to $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{X}^n$, respectively.

Remark 2.2.20. Geometric reflection groups are Coxeter groups according to Theorem 2.2.16.

Example 2.2.21 (Dihedral groups)**.** All finite and infinite dihedral groups (Example 2.1.5) are geometric reflection groups as to be seen from the Figures 2.2 and 2.3.

Example 2.2.22 (Triangle groups)**.** Let (p, q, r) be a triplet of integers with $2 \leq p \leq q \leq r$. There exists a triangle P in $\mathbb{X}^2$ with vertex angles $\frac{\pi}{p}, \frac{\pi}{q}$ and $\frac{\pi}{r}$. The angle sum $z = \frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r}$ determines if $\mathbb{X}^2$ is $\mathbb{S}^2$ $(z > \pi)$, $\mathbb{E}^2$ $(z = \pi)$ or $\mathbb{H}^2$ $(z < \pi)$. The spherical triples are $(2, 2, r)$ as well as the three triplets $(2, 3, 3), (2, 3, 4)$ and $(2, 3, 5)$ that correspond to irreducible Coxeter groups. The latter three are the symmetry groups of the Platonic solids. The first of the three triplets stands for the symmetry group of the tetrahedron, $(2, 3, 4)$ abbreviates the symmetry group of the cube and octahedron and the last triplet is the symmetry group of the dodecahedron and icosahedron.

There are three Euclidean triplets and infinitely many hyperbolic ones (see [Tho18, Example 1.12]). The three Euclidean triplets are $(3, 3, 3), (2, 4, 4)$ and $(2, 3, 6)$, which are all irreducible (see [Tho18, pp.10-11] and [Dav08, p. 86]).

Figure 2.5 shows the tessellations induced by the three Euclidean triangle groups. The tessellations of the spherical group $(2, 3, 5)$ and the hyperbolic group $(3, 4, 4)$ in the Poincaré disc model are depicted in Figure 2.6.

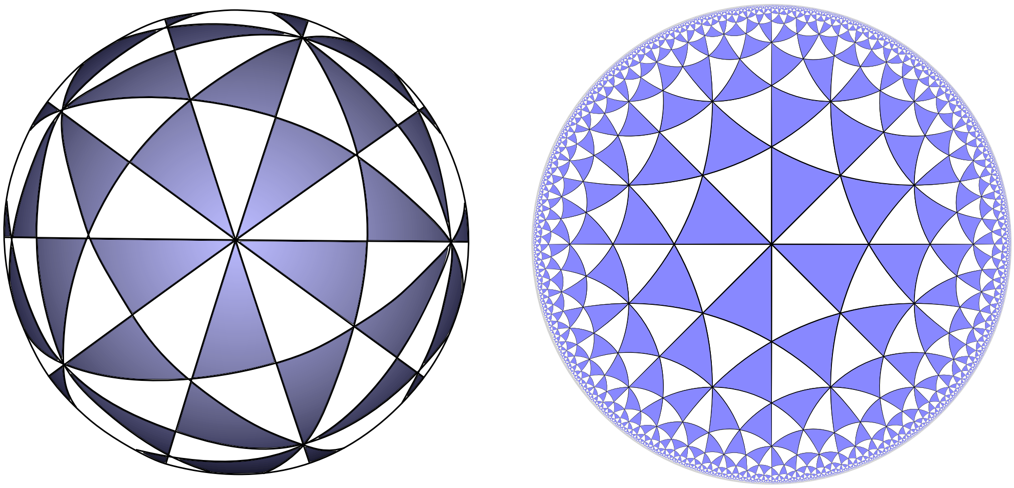


Figure 2.6: Tessellation of $\mathbb{S}^2$ induced by the triangle group $(2, 3, 5)$ on the left and tessellation of $\mathbb{H}^2$ in the Poincaré disc model induced by the triangle group $(3, 4, 4)$ on the right.

Example 2.2.23 (Polygon groups)**.** A spherical polygon P in $\mathbb{S}^2$ with m vertices and angles α_i at the vertices that satisfies $\alpha_i \leq \frac{\pi}{2}$ and $\sum \alpha_i > (m-2)\pi$ has maximally 3 vertices. Thus, all reflection groups acting on a sphere $\mathbb{S}^2$ with a polygon as a fundamental domain are triangle groups. The exterior angle sum of an Euclidean polygon is 2π. Hence, the angles α_i of an m-gon P in $\mathbb{E}^2$ satisfy $\sum \alpha_i = (m-2)\pi$. The interior angle sum of a Coxeter polytope is smaller than the exterior angle sum. This implies $m \leq 4$. So, apart from the Euclidean triangle groups, the only other possibility for P is a rectangle. For every m-tupel of angles $(\frac{\pi}{m_1}, \ldots, \frac{\pi}{m_m})$ satisfying $\sum_{i=1}^{m} \frac{1}{m_i} < m-2$, there exists a convex m-gon in $\mathbb{H}^2$ with this interior angles and a corresponding reflection group. There exists a right-angled hyperbolic m-gon P in $\mathbb{H}^2$ for every $m > 4$. Two reflections corresponding to adjacent sides of P commute while the product of two reflections of non-adjacent sides of P has infinite order. Such two reflections generates an infinite dihedral group.

In total, any m-tupel of angles yields a geometric reflection group acting on $\mathbb{X}^2$ with an m-gon with these angles as a fundamental domain. With finitely many exceptions it is $\mathbb{X}^2 = \mathbb{H}^2$ (see [Dav08, pp. 86-87]).

Example 2.2.24 (Polyhedra groups)**.** Let $P = \cap_{i \in I} H_i^{\epsilon_i}$ be a polyhedron in $\mathbb{H}^n$ such that any pair of hyperplanes in the set of walls $\{H_i \mid i \in I\}$ has an empty intersection. The corresponding reflection group $W(P)$ is the universal Coxeter group of rank $|I|$ from Example 2.1.10.

Remark 2.2.25. The last Example illustrates in particular that different Coxeter polyhedra can yield the same abstract Coxeter group. However, different Coxeter polyhedra in $\mathbb{X}^n$ imply different group actions on $\mathbb{X}^n$.

2.2.4 Classification of geometric reflection groups

Spherical and Euclidean reflection groups are completely classified and therefore well understood. This allows us to distinguish between Coxeter groups that behave like finite and affine reflection groups and the ones that do not. The former groups split into direct products of spherical and Euclidean reflection groups.

On the other hand, the classification of general hyperbolic reflection groups is not complete. A reason for this is the existence of non-trivial unbounded polyhedra in $\mathbb{H}^n$ because parallelism is not a transitive relation for hyperplanes in $\mathbb{H}^n$. Hyperbolic reflection groups and polyhedra are not completely classified yet.

Geometric reflection groups are Coxeter groups and as such they have a Coxeter graph. For the classification of irreducible finite and irreducible Euclidean reflection groups, the Coxeter graph is commonly used (see Definition 2.1.7).

2.2.4.1 Classification of spherical and Euclidean reflection groups

Coxeter classified spherical and Euclidean reflection groups in [Cox34] and [Cox35]. The listing of the Coxeter graphs corresponding to the irreducible spherical and Euclidean reflection groups is to be found in Table 2.1. There are two columns of Coxeter graphs. The left column shows the list of Coxeter graphs associated with the irreducible finite reflection groups and the right column shows the list of graphs associated with the irreducible affine reflection groups. The different types of spherical and Euclidean graphs are denoted by letter abbreviations.

The first line of the left column contains the graphs of type $\mathbf{A}_n$. These graphs correspond to the symmetric groups of any finite rank. Graphs of the dihedral groups are $\mathbf{A}_2, \mathbf{B}_2, \mathbf{G}_2$ and $\mathbf{I}_2(m)$. In the right column, the graph of the infinite dihedral group D_∞ can be seen in the first line. The three Euclidean triangle groups from Example 2.2.22 are represented by the graphs of types $\widetilde{\mathbf{A}}_2, \widetilde{\mathbf{C}}_2$ and $\widetilde{\mathbf{G}}_2$.

Definition 2.2.26. We classify Coxeter groups in the following way:

1. A Coxeter group is called *spherical* or *finite* if it is the direct product of spherical reflection groups.
2. A Coxeter group is called *affine* if it is the direct product of Euclidean reflection groups.
3. An *infinite non-affine* Coxeter group is a Coxeter group that does not split into a direct product of spherical and Euclidean reflection groups.

Remark 2.2.27. Every finite Coxeter group is the product of finite reflection groups (see [Hum90, p. 133]).

Definition 2.2.28. An infinite non-affine Coxeter system (W, S) is called *minimal* if every proper standard parabolic subgroup of W is a direct product of spherical and Euclidean reflection groups.

Remark 2.2.29. Every infinite non-affine Coxeter group has a minimal infinite non-affine standard parabolic subgroup; it is any standard parabolic subgroup minimal with respect to inclusion among the infinite non-affine ones.

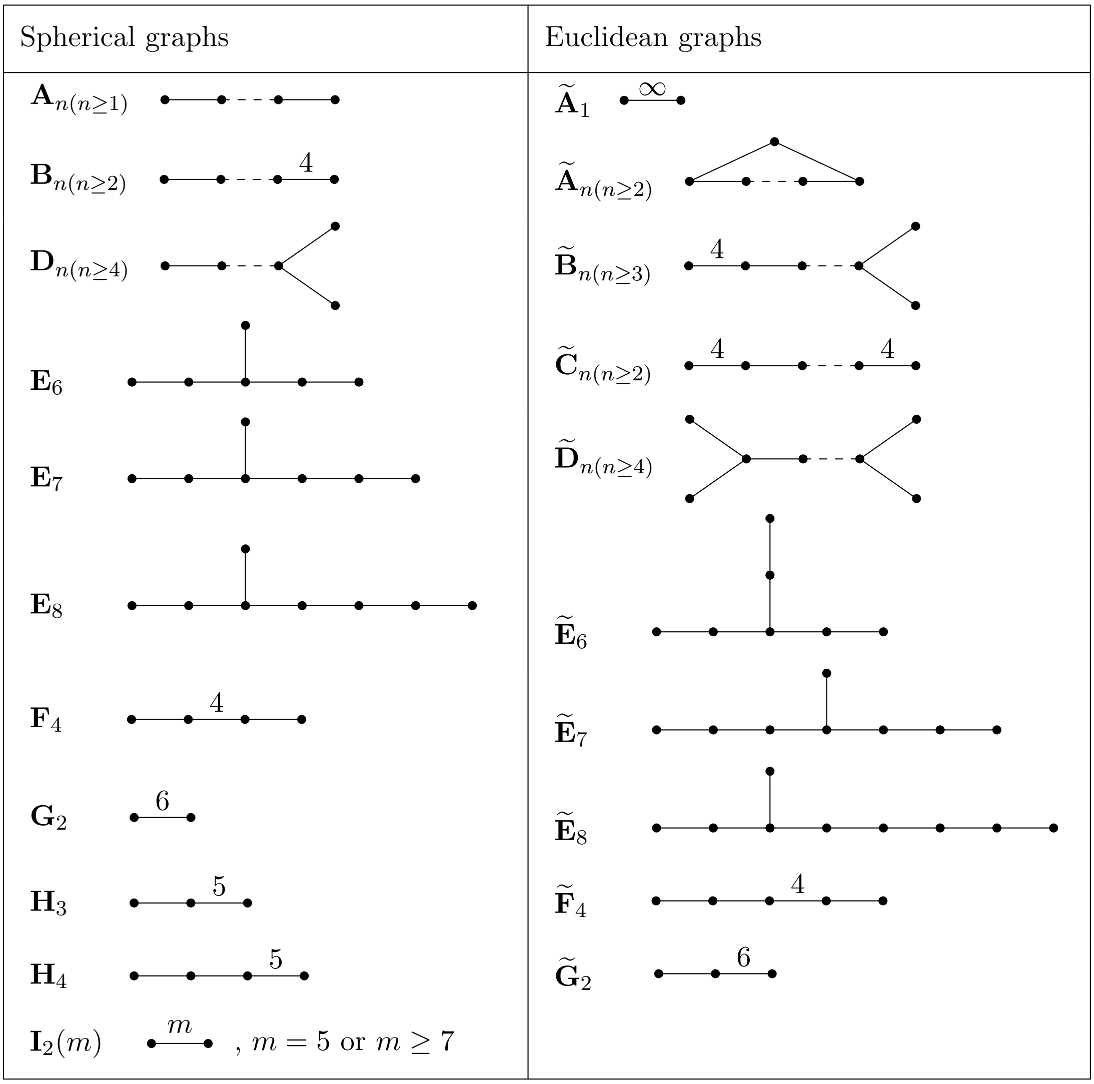


Table 2.1: Coxeter graphs of irreducible spherical and Euclidean reflection groups.

2.2.4.2 Classification of hyperbolic reflection groups.

Different aspects of the classification of hyperbolic reflection groups are discussed briefly in this paragraph.

Simplices as fundamental domains. We call a geometric reflection group $W(P)$ *simplicial reflection group* if the polyhedron from Theorem 2.2.16 is an n-dimensional simplex Δ^n. The next theorem is an implication of Lannér's Theorem in [Lan50] and combines a list of results of Chapter V in [Bou02]. It states that minimal infinite non-affine Coxeter groups can be represented as simplicial hyperbolic reflection groups.

Theorem 2.2.30 (see [Dus12, Proposition 2.1])**.** *Every minimal infinite non-affine Coxeter system (W, S) can be faithfully represented as a discrete reflection group acting on the hyperbolic space $\mathbb{H}^n$, where $n = |S| - 1$. The elements of S act as reflections with respect to codimension-1 faces of a simplex Δ (some of the vertices of Δ might be ideal).*

Example 2.2.31. Every infinite non-affine Coxeter group with exactly three generators that does not split into a direct product of Coxeter groups is isomorphic to a hyperbolic reflection group with a simplex as a fundamental polygon (with possibly ideal vertices).

Remark 2.2.32. Minimal infinite non-affine Coxeter groups are sometimes called *hyperbolic Coxeter groups* in standard literature about Coxeter groups like in [Bou02] and [Hum90]. However, a hyperbolic reflection group does not have to be a hyperbolic Coxeter group. Hyperbolic Coxeter groups are exactly the hyperbolic reflection groups that have a simplex as a fundamental domain (see [Hum90, pp. 140-141]).

The minimal infinite non-affine Coxeter groups are completely classified due to the work of Folke Lannér in [Lan50], Jean-Louis Koszul in [Kos67], Nicolas Bourbaki in [Bou02] and Michel Chein in [Che69] (see [Hum90, p. 141] and [VS88, p. 204]). They only exist in ranks from three up to ten and for each rank higher than three there exist only finitely many. These simplicial hyperbolic reflection groups can be divided into the ones with compact fundamental domains and the ones with non-compact fundamental domains. The former ones exist only in ranks $3, 4$ and 5 (see [Hum90, pp. 141-144]). A list of the compact and non-compact simplicial hyperbolic reflection groups is to be found in Section 6.9 in [Hum90].

Coxeter polyhedra as fundamental domains. There is not much known in general about hyperbolic reflection groups with a general Coxeter polyhedron as a fundamental domain. Compact as well as finite volume fundamental domains exist only until certain dimensions. Compact hyperbolic Coxeter polytopes do not exist in dimensions higher than 29 by a result of Èrnest Borisovich Vinberg in [Vin81]. Finite volume hyperbolic Coxeter polyhedra do not exist in dimensions higher than 995 (see [Kho86; Pro87]).

Remark 2.2.33. Since geometric reflection groups are Coxeter groups, the question of whether all Coxeter groups are geometric reflection groups is immediate. This is proven to be wrong. There exist infinite non-affine Coxeter groups that cannot be embedded as discrete subgroups generated by hyperplane reflections in $\mathrm{Iso}(\mathbb{H}^n)$ for any $n \in \mathbb{N}$ (see [FT05]). To our knowledge, there are no general criteria for Coxeter groups that specify when the group is isomorphic to a hyperbolic reflection group.

2.3 Geometric constructions for Coxeter groups

Not every Coxeter group is a geometric reflection group as described in the last remark. Nevertheless, there exist constructions of geometric objects for every Coxeter group such that the Coxeter group acts as a group generated by reflections on the according geometric objects. In this context, the term *reflection* is weakened such that the generators might not correspond to orthogonal reflections in $\mathbb{R}^n$.

We discuss two constructions: the *geometric representation* by Tits and the *basic construction*. The latter leads to the Cayley graph, the Coxeter complex and the Davis complex of a Coxeter system.

2.3.1 The geometric representation

The following theorem is due to Tits (see [Bou02, pp. 89-90]). It is based on a construction for a Coxeter system (W, S) that yields a faithful representation as a group generated by linear transformations. Each of these linear transformations fixes a hyperplane pointwise in a vector space V over $\mathbb{R}$ and sends an associated non-zero vector to its negative.

Theorem 2.3.1 (see [Tho18, p. 35]). *Let (W, S) be a Coxeter system with Coxeter matrix $(m_{ij})_{i,j\in[n]}$ and $S = \{s_1, \ldots, s_n\}$. There exists a faithful representation*

$$\rho : W \to GL_n(\mathbb{R}),$$

such that

(i) for all $s_i \in S$, the image $\rho(s_i) = \sigma_i$ is a linear involution with a hyperplane as a fixed point set;

(ii) for all distinct $s_i, s_j \in S$, the product $\sigma_i\sigma_j$ has order m_{ij}.

This representation is called the *geometric representation* of W. Other names for it in the literature are *standard representation* or *Tits representation*.

Let (W, S) be a Coxeter system. The construction of this representation starts with a vector space V over $\mathbb{R}$ with a bijection between its basis $\{e_1, \ldots e_n\}$ and the finite generating set S. Therefore, we have $GL(V) \cong GL_n(\mathbb{R})$. To have the angle between e_i and e_j compatible to m_{ij}, define a symmetric bilinear form B in the following way:

$$B(e_i, e_j) = \begin{cases} -\cos\frac{\pi}{m_{ij}} & \text{if } m_{ij} \in \mathbb{N} \\ -1 & \text{if } m_{ij} = \infty. \end{cases}$$

It is $B(e_i, e_i) = 1$ and $B(e_i, e_j) \leq 0$ if $i \neq j$. The image $\sigma_i : V \to V$ of the generator s_i from the theorem is defined as the linear map

$$\sigma_i(v) = v - 2B(e_i, v)e_i.$$

The fixed point set of σ_i is the subspace $H_i = \{v \in V \mid B(e_i, v) = 0\}$ orthogonal to e_i with codimension 1 (a hyperplane). Moreover, σ_i is an involution, reverses e_i and preserves the bilinear form B.

Remark 2.3.2. For an irreducible Coxeter system (W, S) of rank n, we can read off from the bilinear form B if W is a spherical or Euclidean reflection group. B is positive definite if and only if W is finite. An element s_i of S corresponds to a reflection across a codimension-1 face F_i of a simplex in $\mathbb{S}^{n-1}$ such that two different faces F_i and F_j intersect in a dihedral angle $\frac{\pi}{m_{ij}}$.

The bilinear form B is positive semidefinite of corank 1 if and only if W is an Euclidean reflection group. An element s_i of S corresponds to a reflection across a codimension-1 face F_i of either

1. an interval in $\mathbb{E}^1$ if $n = 2$ (for $W = D_\infty$), or
2. a simplex in $\mathbb{E}^{n-1}$ if $n \geq 3$ such that two different faces F_i and F_j intersect in a dihedral angle $\frac{\pi}{m_{ij}}$

(see [Dav08, Theorem 6.8.12]).

The geometric representation has some crucial implications, of which the most important ones for us are the following.

Corollary 2.3.3. *A Coxeter group of rank n is isomorphic to a discrete subgroup of $GL_n(\mathbb{R})$.*

The next implication of the geometric representation states the existence of a finite index subgroup of a group-theoretic property. In general, a group G is called *virtually* **P** for a property **P** if there exists a finite index subgroup $H \subseteq G$ satisfying **P**.

Definition 2.3.4. A group G is called *torsion-free* if the identity element is the only element in G with finite order.

Definition 2.3.5. A group is *residually finite* if, for each $g \in G \setminus \{e\}$, there exists a finite group H_g and a homomorphism $\varphi : G \to H_g$ such that $\varphi(g) \neq \mathbb{1}$.

The next corollary is a direct implication of Selberg's Lemma in [Sel60] and Malcev's Theorem in [Mal40]. These are both results about groups that have a faithful representation onto $GL_n(\mathbb{R})$ for $n \in \mathbb{N}$. Such groups are called *linear groups*.

Corollary 2.3.6. *Coxeter groups are virtually torsion-free and residually finite.*

2.3.2 The basic construction

The basic construction is a method to recover an action of a group G on a space X from the quotient X/G and the structure of its stabilizers $\{G_x \subseteq G \mid x \in X\}$ in X/G. It leads to many examples and is very useful in geometric group theory. The basic construction is reproduced here partially as it is described in [BH99]. However, for Coxeter groups, it already goes back to Tit's works [Tit75] and [Tit86]. It is classical for geometric reflection groups and dates back even further for this type of group (see [Kos65; Vin72]).

For Coxeter groups, modern literature like [Dav08] and [Tho18] introduce the basic construction to construct the *Cayley graph*, *Coxeter complex* and the *Davis-Moussong complex*. In this text, the basic construction is restricted to the action of Coxeter groups on topological spaces. Thus, most of the technical details in the general basic construction are left out.

Let (W, S) be a Coxeter system. For a topological space X, we construct a geometric object $\mathcal{U}(W, X)$, on which W acts as a group generated by reflections. The first step is the next definition.

Definition 2.3.7. A *mirror structure* on a connected Hausdorff space X is a family $(X_s)_{s\in S}$ of non-empty, closed subspaces X_s. For each $x \in X$ define $S(x)$ to be the following subset of S:

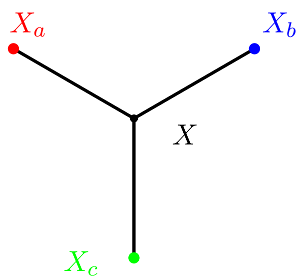
$$S(x) := \{s \in S \mid x \in X_s\}.$$

We assume that there exists an $\tilde{x} \in X$ such that there exists no $s \in S$ with $\tilde{x} \in X_s$. So the set $S(\tilde{x})$ is empty.

Example 2.3.8. The next two examples are important for the construction of concrete geometric objects from the basic construction.

1. Let X be a star graph with valence $|S|$ and leaves $\{v_s \in X \mid s \in S\}$. Define the mirror structure as $X_s := v_s$. The case of three leaves is displayed in Figure 2.7 on the left.

2. Let X be the n-dimensional simplex with $|S| = n + 1$. The codimension-1 faces $\{F_s \subset X \mid s \in S\}$ are a mirror structure ($X_s = F_s$). This situation is displayed in Figure 2.7 for $|S| = 3$. More generally, the codimension-1 faces of a simple convex polytope P are a mirror structure on P.



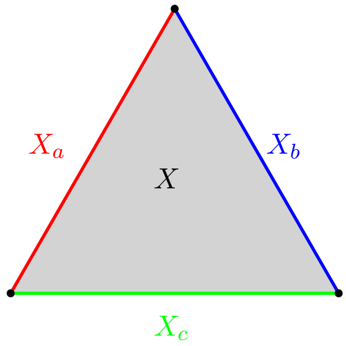


Figure 2.7: Mirror structures for $S = \{a, b, c\}$ of a star graph on the left and a 2-dimensional simplex on the right.

The construction of $\mathcal{U}(W, X)$ is as follows: After defining what the mirrors in X are, the reflection on a mirror is imitated by glueing copies of X along mirrors together in a way that respects the braid-relations between the corresponding generators in S. For every group element $w \in W$, there exists a copy of X. This is exactly the image of the copy of X associated with the identity under the action of w.

Write $W_{S'}$ for the standard parabolic subgroup generated by $S' \subseteq S$ with $W_\varnothing = \{\mathbb{1}\}$ (see Definition 2.1.1). Define the equivalence relation $\sim$ on $W \times X$ as

$$(w, x) \sim (w', x') :\Leftrightarrow x = x' \text{ and } w^{-1}w' \in W_{S(x)}.$$

Definition 2.3.9 (Basic construction)**.** Equip the product $W \times X$ with the product topology, where W has the discrete topology. We write $\mathcal{U}(W, X)$ for the quotient space

$$\mathcal{U}(W, X) := {(W \times X)}/{\sim}.$$

The equivalence class of an element (w, x) is denoted with $[w, x]$.

Since W is equipped with the discrete topology and X is a Hausdorff space, the equivalence classes are closed. Left multiplication with an element w is a well-defined automorphism on $\mathcal{U}(W, X)$. For every $w \in W$, the map $i : X \to \mathcal{U}(W, X)$ with $x \mapsto [w, x]$ is an embedding. The image of $\{w\} \times X$ in $\mathcal{U}(W, X)$ under the projection $p : W \times X \to \mathcal{U}(W, X)$ is called *chamber* and denoted with wX. The chamber $\mathbb{1}X$ is called *fundamental chamber*. There is a bijection between W and the set of chambers. The orbit space ${\mathcal{U}(W, X)}/{W}$ is homeomorphic to X and $\mathbb{1}X$ is a strict fundamental domain for the W-action on $\mathcal{U}(W, X)$ (see [Dav08, p. 65]).

The next theorem illustrates, how the whole space $\mathbb{X}^n$ can be recovered topologically from a geometric reflection group W on $\mathbb{X}^n$ with a fundamental polytope P.

Theorem 2.3.10 (see [Dav08, Theorem 6.4.3.])**.** *Let $P = \bigcap_{i \in I} H_i^{\varepsilon_i}$ be a Coxeter polytope in $\mathbb{X}^n$ with $n \geq 2$ and $W(P)$ the group generated by the isometric reflections $\{s_i \mid i \in I\}$ across the walls F_i of P. With the mirror structure $(F_i)_{i \in I}$ on P, the group action of $W(P)$ on $\mathbb{X}^n$ induces a homeomorphism*

$$\mathcal{U}(W(P), P) \to \mathbb{X}^n \text{ with } [w, p] \mapsto w(p).$$

Remark 2.3.11. Note that the assumptions of the theorem above are a special case of the assumptions in Theorem 2.2.16.

For a more detailed discussion of properties of the basic construction we refer the reader to Section 4.3 in [Tho18].

The following examples contain the geometric constructions associated with a Coxeter system most popular in literature besides the geometric representation: the Cayley graph, the Coxeter complex and the Davis complex.

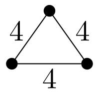
Example 2.3.12 (Cayley graph)**.**

1. Let (W, S) be a Coxeter system. Define X to be the topological cone

$$X := S \times [0, 1]/\{(s, 0) \mid s \in S\}$$

   over S. The mirror structure on X is defined as $((s, 1))_{s \in S}$. With the basic construction, a space $\mathcal{U}(W, X)$ homeomorphic to the Cayley graph of (W, S) is obtained.

2. Let (W, S) be the triangle group with Coxeter graph



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   The standard parabolic subgroup $W_{S(x)}$ for $x \in X$ is non-empty if and only if $x = (s, 1)$ for some $s \in S$. In this case, $W_{S(x)}$ is $\{\mathbb{1}, s\}$ and otherwise $W_{S(x)}$ is trivial. The definition of the equivalence relation implies

$$(w, (s, 1)) \sim (w', (s, 1)) \Longleftrightarrow w^{-1}w' \in \{\mathbb{1}, s\} \Longleftrightarrow w = w' \text{ or } w' = ws.$$

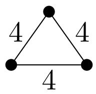
   So if $x = (s, r)$ for $s \in S$ and $r \in [0, 1)$, then $[w, x] = \{(w, x)\}$. In the other case where $x = (s, 1)$ for $s \in S$, we have $[w, x] = \{(w, s), (ws, s)\}$. Hence, the space $\mathcal{U}(W, X)$ is obtained by gluing together the chambers wX and wsX along the mirror $(s, 1)$ for all $w \in W$.

Example 2.3.13 (Coxeter complex)**.**

1. Let $(W, S = \{s_1, \dots, s_n\})$ be a Coxeter system of rank n. Denote the $(n-1)$-dimensional simplex with Δ^{n-1}. Since Δ^{n-1} has n codimension-1 faces $\{F_1, \dots, F_n\}$, a mirror structure $(\Delta^{n-1}_s)_{s \in S}$ on Δ^{n-1} can be defined by $\Delta^{n-1}_{s_i} := F_i$. The basic construction yields to a simplicial complex $\mathcal{U}(W, \Delta^{n-1})$, which is called the *Coxeter complex* of (W, S).

   If W is finite, $\mathcal{U}(W, \Delta^{n-1})$ is homeomorphic to an $(n-1)$-dimensional sphere. If W is infinite, the Coxeter complex of (W, S) is contractible (see [Dav08, p. 68] and [Ser71, pp. 340-342]).

2. For the triangle group with Coxeter graph



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the simplex Δ^{n-1} is a 2-dimensional simplex. The standard parabolic subgroup $W_{S(x)}$ for $x \in \Delta^{n-1}$ is non-empty if and only if $x \in F_i$ for some $i \in \{1, \dots, n\}$. If the point $x = F_i \cap F_j \in \Delta^2$ is the intersection of two distinct faces F_i and F_j, then the standard parabolic subgroup $W_{S(x)} = \langle s_i, s_j \rangle \cong D_4$ is a dihedral group and the point x in the following eight different chambers $w\Delta^{n-1}$, $ws_i\Delta^{n-1}$, $ws_is_j\Delta^{n-1}$, $ws_is_js_i\Delta^{n-1}$, $ws_is_js_is_j\Delta^{n-1} = ws_js_is_js_i\Delta^{n-1}$, $ws_js_is_j\Delta^{n-1}$, $ws_js_i\Delta^{n-1}$ and $ws_j\Delta^{n-1}$ is identified to one point in $\mathcal{U}(W, S)$ for every $w \in W$.

Since W is a minimal infinite non-affine Coxeter group, it can be faithfully represented as a discrete reflection group acting on $\mathbb{H}^2$, where the elements of S act as reflections across the codimension-1 faces of a 2-dimensional simplex (see Theorem 2.2.30). So the Coxeter complex $\mathcal{U}(W, \Delta^2)$ can be pictured as the tessellation of $\mathbb{H}^2$ generated by the hyperplane reflections corresponding to the generators in S. Figure 2.8 shows the tessellation in the Poincaré disc model associated with W. The different colours of codimension-1 faces of the 2-simplices indicate different corresponding generators in the mirror structure.

Remark 2.3.14. A priori, there is no metric defined on the Coxeter complex. For irreducible spherical and affine Coxeter groups, the Coxeter complex is metrisable in a way that allows to identify it with tessellated sphere or tessellated Euclidean space, respectively (see [Tho18, p. 52]). From Theorem 2.2.30, we know that minimal infinite non-affine Coxeter groups are simplicial hyperbolic reflection groups. This allows the conclusion that a minimal infinite non-affine Coxeter group of rank n with a compact simplex as a fundamental domain has a Coxeter complex that, metrised properly, is isometric to $\mathbb{H}^{n-1}$. More generally, for a geometric reflection group W with a simple convex polytope P as a fundamental domain in $\mathbb{X}^n$, the space $\mathcal{U}(W, P)$ is homeomorphic to $\mathbb{X}^n$ (see Theorem 2.3.10). Thus, every simplicial Coxeter group with a compact fundamental domain has a Coxeter complex homeomorphic to the underlying metric space $\mathbb{X}^n$.

Remark 2.3.15. As illustrated by Figure 2.8, the Cayley graph $\mathcal{U}(W, X)$ and the Coxeter complex $\mathcal{U}(W, \Delta^{n-1})$ are always dual to each other for a Coxeter system (W, S). This is due to their construction. It is important to fix a Coxeter group W together with a generating set S since both constructions depend not only on the group but also on the generating set.

Example 2.3.16 (Davis complex)**.**

1. Let (W, S) be a Coxeter system. Define the partially ordered set

$$\mathcal{S} := \{T \subseteq S \mid T \neq \varnothing,\ W_T \text{ is finite}\}$$

and let $L(W, S)$ denote the corresponding simplicial complex. $L(W, S)$ is called the *nerve* of the Coxeter system (W, S). The vertices v_s of $L(W, S)$ are in bijection with the generators in S. The cone K of the barycentric subdivision L_b of $L(W, S)$ is a connected Hausdorff space. We equip it with a mirror structure $(K_s)_{s \in S}$ such that K_s is the union of the closed simplices in L_b with vertex v_s. For a subset $\varnothing \neq T \subseteq S$ the intersection $\cap_{t \in T} K_t$ is non-empty if and only if W_T is finite. This implies that $W_{S(x)}$ is finite for all $x \in K$. The basic construction yields the *Davis complex* $\mathcal{U}(W, K)$.

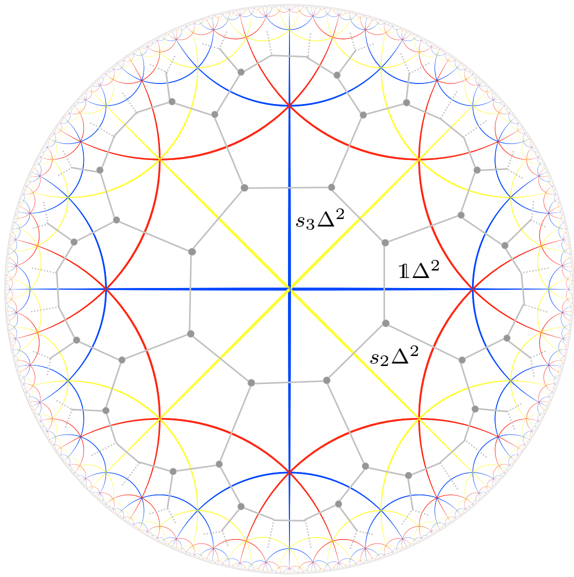

Figure 2.8: Cayley graph in grey and Coxeter complex with white simplices and coloured codimension-1 faces of the triangle group $(4, 4, 4)$ embedded in $\mathbb{H}^2$ in the Poincaré disc model.

2. For the triangle group $(4, 4, 4)$ from above, the nerve $L := L(W, S)$ is the hollow triangle with vertices $\{v_{s_1}, v_{s_2}, v_{s_3}\}$. The cone K over the barycentric subdivision of L is the barycentric subdivision of a 2-dimensional simplex. The standard parabolic subgroup $W_{S(x)}$ for $x \in K$ is non-empty if and only if $x \in K_s$ for some $s \in S$. The intersection of two different mirrors $K_{s_i} \cap K_{s_j}$ contains exactly one point x_{ij}. The point x_{ij} is the point that subdivides the 1-dimensional simplex corresponding to $\{s_i, s_j\} \subseteq S$ in the barycentric subdivision of L. Just analogously to the Coxeter complex for this Coxeter system, the point x_{ij} in the following eight chambers wK, ws_iK, ws_is_jK, $ws_is_js_iK$, $ws_is_js_is_jK = ws_js_is_js_iK$, $ws_js_is_jK$, ws_js_iK and ws_jK is identified to one point in the Davis complex $\mathcal{U}(W, K)$ for every $w \in W$. This is why in this case, the Davis complex is the barycentric subdivision of the Coxeter complex. The coloured and grey edges together with the white simplices in Figure 2.8 are the Davis complex of the triangle group $(4, 4, 4)$.

Remark 2.3.17. Comparing the construction of the Cayley graph and the Davis complex with the basic construction, we see that the Cayley graph is included in the Davis complex of the same Coxeter system as the 1-skeleton of the Davis complex.

Remark 2.3.18. The Coxeter complex as well as the Davis complex can also be defined as the geometric realisations of abstract simplicial complexes. The simplicial complex underlying the Coxeter complex of a Coxeter group W, for example, is the poset of cosets of the form $w\langle A\rangle$ with reverse inclusion ordering, where $w \in W$ and $A \subsetneq S$. For more details, we refer the reader to Chapter 7 in [Dav08].

The Davis complex is also called *Davis-Moussong complex* since Gabor Moussong proved that it can be equipped with a piecewise Euclidean metric so that it has non-positive curvature.

Theorem 2.3.19 (see [Mou88, Theorem A]). *Let (W, S) be an arbitrary Coxeter system. The Davis complex Σ of (W, S) equipped with its canonic piecewise Euclidean metric is* CAT(0)*.*

Remark 2.3.20. This especially means that a Coxeter group acts by isometries properly discontinuously and cocompactly on a CAT(0) space, its Davis complex.

Moussong's theorem has further consequences. One of these is the following result, which is used later.

Theorem 2.3.21 (see [Dav08, Theorem 12.3.5.]). *Let W be a virtually abelian irreducible Coxeter group. Then W is either finite or a cocompact Euclidean reflection group.*

2.4 Roots

In a geometric reflection group, each reflection corresponds to a hyperplane, across which it reflects all points of the underlying space. This is inherent from Definition 2.2.19. A hyperplane is an affine codimension-1 subspace and uniquely determined by its 1-dimensional orthogonal complement. It is sometimes more convenient to build the connection between group theory and geometry or linear algebra via directional vectors contained in this complement rather than working with the corresponding hyperplanes.

The affine 1-dimensional orthogonal complements get permuted by the according Coxeter group W just as the hyperplanes corresponding to the reflections in W do. We associate to every such complement two directional vectors α and $-\alpha$.

Historically, roots and root systems emerged from Lie Theory. There is a connection between certain finite Coxeter groups and semisimple Lie algebras. The content of this section is taken from [Hum90], [Deo82] and [Deo86].

Definition 2.4.1. Let (W, S) be an arbitrary Coxeter group and V be a vector space over $\mathbb{R}$ with basis $\mathcal{B} := \{\alpha_s \mid s \in S\}$. Consider the W action $w(v) = \rho(w)(v)$ from the geometric representation (see Subsection 2.3.1). The *root system* Φ_W of W is the orbit of $\mathcal{B}$ under the W-action: $\Phi_W := \{w(\alpha_s) \mid s \in S, w \in W\}$.

Remark 2.4.2. The root system exclusively contains unit vectors since W preserves the bilinear form B on V. Furthermore, the root system is symmetric, $\Phi_W = -\Phi_W$ because $s(\alpha_s) = -\alpha_s$ for all $s \in S$. A root $\alpha \in \Phi_W$ can be factored uniquely into a linear combination $\alpha = \sum_{s \in S} r_s \alpha_s$ with $r_s \in \mathbb{R}$, where all scalars r_s have the same sign (see [Deo82, Proposition 2.1]).

Every group that has a representation over $\mathbb{R}$ and a corresponding root system satisfying this condition and two additional technical conditions is a Coxeter group. This is another characterisation of Coxeter groups (see [Deo86]).

Definition 2.4.3. A root α is called *positive* (*negative*) if $r_s \geq 0$ ($r_s \leq 0$) for all $s \in S$ in the unique linear combination $\alpha = \sum_{s \in S} r_s \alpha_s$. We denote the set of positive roots with Φ_W^+ and the set of negative roots with Φ_W^-.

Lemma 2.4.4 (see [Hum90, Section 5.4]). *For a Coxeter group W, the union of positive and negative roots is the whole root system: $\Phi_W^+ \cup \Phi_W^- = \Phi_W$.*

Definition 2.4.5. Let V be an Euclidean vector space with a root system Φ. A subset $U \subseteq V$ is called *root space* if it is the span of the roots it contains: $U = \langle U \cap \Phi \rangle$. Since Φ is a finite set, there are only finitely many root spaces. The set of all root spaces in V is called *root space arrangement* and denoted with $\mathrm{Arr}(\Phi) = \{U \subset V \mid U = \langle U \cap \Phi \rangle\}$.

Definition 2.4.6. The *root dimension* $\dim_\Phi(A)$ of a subset $A \subseteq V$ is the minimal dimension of a root space in $\mathrm{Arr}(\Phi)$ that contains A. It is defined for every $A \subseteq V$ because V is a root space itself.

2.4.1 Affine reflection groups and root systems of finite Coxeter groups

All affine Coxeter groups have a certain structure that relates them strongly to a corresponding finite Coxeter group and its root system. An affine Coxeter group W acts on an Euclidean vector space V such that a finite reflection subgroup $W_0 \subseteq W$ is isomorphic to the spherical subgroup generated by all reflections in W that fix the origin in V.

Definition 2.4.7. For $\lambda \in V$, the isomorphism $t_\lambda(x) = x + \lambda$ for all $x \in V$ is called *translation.*

The following theorem is a composition of several results (see [Hum90, Chapter 4], [Bou02, §2, Chapter VI] and [IM65, pp. 7-10]).

Theorem 2.4.8. *Let (W, S) be an affine Coxeter system. The group T of translations in W is a normal subgroup. The quotient $W_0 := {}^{W}\!/_{T}$ is a finite Coxeter group. W is isomorphic to the semidirect product $T \rtimes W_0$.*

Remark 2.4.9. The resulting finite Coxeter system (W_0, S_0) always has only entries of the form $m_{ij} = 2, 3, 4$ and 6 besides the diagonal $(i \neq j)$ in its Coxeter matrix. This type of finite Coxeter group is called *crystallographic* or *Weyl group* (see [Hum90, Chapters 2.8 and 2.9]).

The root system Φ_{W_0} of the finite Coxeter system (W_0, S_0) is contained in a Euclidean vector space V equipped with a symmetric bilinear form B (see Paragraph 2.3.1). The vector space V has dimension $|S_0|$. For each root $\alpha \in \Phi_{W_0}$ and $k \in \mathbb{Z}$, define an affine hyperplane

$$H_{\alpha,k} := \{a \in V \mid B(a, \alpha) = k\}.$$

Let $s_{\alpha,k}$ be the reflection across $H_{\alpha,k}$ defined as

$$s_{\alpha,k}(a) = a - (B(a, \alpha) - k)\frac{2\alpha}{B(\alpha, \alpha)}.$$

Then, W is the group generated by all reflections $s_{\alpha,k}$ for $\alpha \in \Phi_{W_0}$ and $k \in \mathbb{Z}$. This induces a W-action on the space V.

Remark 2.4.10. Starting with a crystallographic Coxeter group, it is possible with the steps above to construct a group generated by reflections across affine hyperplanes in a Euclidean space. The resulting group is always an affine Coxeter group (see [Bou02, Proposition 9, §2, Chapter VI]).

Remark 2.4.11. Note that the construction above is distinct from the geometric representation for affine W and the corresponding root system.

3. Word length and reflection length

This chapter covers the fundamental combinatorics of the word length and the reflection length in Coxeter groups. Before defining the reflection length function on a Coxeter group, some combinatorial theory of Coxeter groups is necessary. We introduce the word length in Coxeter groups and discuss the word problem. In Section 3.3, reflections and the combinatorial theory of the reflection length function on Coxeter groups are examined. Lastly, formulas for the reflection length in finite and affine Coxeter groups are presented.

Some passages of this chapter are taken with minor deviations from the author's works [Lot24b] and [Lot24a]. A general reference for the topics discussed in this chapter is [BB05].

3.1 Words and group elements

Let (W, S) be a Coxeter system for this section. It is sometimes important to distinguish between words over the alphabet S and the elements they represent in the group W.

Definition 3.1.1. The *free monoid* $(S^*, \cdot)$ is the set of finite words consisting of letters in S together with concatenation of words as an operation on S^*. A word $\mathbf{s} = u_1 \cdots u_p$ in S^* is abbreviated with a bold variable. Let $\omega : S^* \to W$ be the canonical surjection that maps a word to the corresponding group element. We write $\omega(\mathbf{s})$ for the element in W that is represented by the word $\mathbf{s}$. In fact, for the equivalence relation $\sim$ generated by insertion and deletion of words $(s_i s_j)^{m_{ij}}$ and $s_i s_i$, the quotient $S^*/\sim$ is a group isomorphic to the Coxeter group W (see [BB05, p. 3]).

The first questions that may arise from the distinction between words and elements are if $s_i \neq s_j$ in S implies $\omega(s_i) \neq \omega(s_j)$ in W and if the order of $\omega(s_i s_j)$ in W is m_{ij} for $i \neq j$. To both questions, the answer is positive (see Lemma 2.1.6).

In general, for every group G with generating set Y, there exists an associated statistic $l_Y : G \mapsto \mathbb{N}_0$. The map l_Y counts for every $g \in G$ the minimal number of elements in $Y \cup Y^{-1}$ that suffice to factor g.

Definition 3.1.2. For a group G with generating set Y, set $\bar{Y} := Y \cup Y^{-1}$. The according *length function* l_Y is defined as

$$l_Y : G \to \mathbb{N}_0\,; \qquad g \mapsto \min\{n \in \mathbb{N}_0 \mid g \in \bar{Y}^n\}$$

with $\bar{Y}^n = \{y_1 \cdots y_n \in G \mid y_i \in \bar{Y}\}$. The identity element $\mathbb{1} \in G$ has length 0.

The next two sections are about the two most important generating sets for Coxeter groups and the corresponding length functions.

3.2 Word length

The standard generating set S in a Coxeter system (W, S) induces a statistic l_S on W according to Definition 3.1.2.

Definition 3.2.1. In a Coxeter system (W, S), every standard generator $s \in S$ is an involution. The length function $l_S : W \to \mathbb{N}_0$ corresponding to S is called the *word length*. Minimal standard generator factorisations of an element $w \in W$ with respect to the word length are called *S-reduced*.

Remark 3.2.2. Geometrically, the word length $l_S(w)$ of an element w in a Coxeter system (W, S) is the length of the shortest path between the vertices corresponding to $\mathbb{1} \in W$ and $w \in W$ in the Cayley graph. Since the Cayley graph and the Coxeter complex of the same Coxeter system are dual, the word length is the minimal number of simplices between the simplex corresponding to $\mathbb{1} \in W$ and the simplex corresponding to $w \in W$ in the Coxeter complex of (W, S) (see Examples 2.3.12, 2.3.13 and Remark 2.3.14).

The next example illustrates that the word length depends strongly on the braid relations in W.

Example 3.2.3. Fix a finite set of generators $S = \{s_1, \ldots, s_n\}$ and consider the word $\mathbf{s} = s_1s_2s_3s_2s_3 \in S^*$. In the following enumeration of Coxeter groups W_i with generating set S and canonic projection $\omega_i : S^* \to W_i$, let l_S^i be the word length in W_i.

1. Let (W_1, S) be the universal Coxeter system with no braid relations. Since there are no consecutive subwords s_is_i in $\mathbf{s}$, the factorisation $\mathbf{s} = s_1s_2s_3s_2s_3$ is S-reduced in W_1 and the word length is $l_S^1(\omega_1(\mathbf{s})) = 5$.
2. For the Coxeter system (W_2, S) that contains only the braid relation $(s_2s_3)^2 = \mathbb{1}$ in its Coxeter presentation, the factorisation $\mathbf{s} = s_1s_2s_3s_2s_3$ is not S-reduced. We have $\omega_2(s_2s_3s_2s_3) = \mathbb{1}$. A single generator is different from the identity. Hence, an S-reduced factorisation is $\omega_2(\mathbf{s}) = s_1$ and the word length is $l_S^2(\omega_2(\mathbf{s})) = 1$.

In this context, the following theorem is crucial for the later consideration of the powers of Coxeter elements.

Theorem 3.2.4 (see [Spe09, Theorem 1])**.** *Let W be an infinite, irreducible Coxeter group and let $s_1, \ldots, s_n$ be any ordering of generators in S. Then the word $(s_1 \cdots s_n)^\lambda$ is S-reduced for any $\lambda \in \mathbb{N}$.*

Two important conditions connected to the word length lead to a characterisation of Coxeter groups. Let W be a group generated by a set S of distinct involutions in W.

Notation 3.2.5. A hat over a letter in a word means omitting this letter:

$$u_1 \cdots \hat{u}_i \cdots u_n = u_1 \cdots u_{i-1}u_{i+1} \cdots u_n.$$

Deletion Condition 3.2.6. The tuple (W, S) satisfies the *Deletion Condition* if the following holds: If $l_S(\omega(u_1 \cdots u_k)) < k$ for a word $u_1 \cdots u_k \in S^*$, then there exist indices $i < j$ such that $\omega(u_1 \cdots u_k) = \omega(u_1 \cdots \hat{u}_i \cdots \hat{u}_j \cdots u_k)$.

Exchange Condition 3.2.7. The tuple (W, S) satisfies the *Exchange Condition* if the following holds: For $w \in W$ and $s \in S$ and every S-reduced factorisation $\omega(u_1 \cdots u_p) = w$ of w, either $l_S(sw) = p + 1$ or there exists an index i such that $w = \omega(su_1 \cdots \hat{u}_i \cdots u_p)$.

The next theorem is a characterisation of Coxeter groups and illustrates how the word length is connected to the presentation of a Coxeter system.

Theorem 3.2.8 (see [Hum90, Sections 1.7 and 4.6])**.** *Let W be a group generated by a set S of distinct involutions in W. The following statements are equivalent:*

(i) The tuple (W, S) is a Coxeter system.

(ii) The tuple (W, S) satisfies the Deletion Condition.

(iii) The tuple (W, S) satisfies the Exchange Condition.

The Deletion Condition implies the next corollary.

Corollary 3.2.9 (see [BB05, p.17])**.** *Let (W, S) be a Coxeter system.*

(i) For a factorisation $w = \omega(u_1 \cdots u_p)$ with $u_i \in S$ of an element $w \in W$, an S-reduced expression for w is obtained from $u_1 \cdots u_p$ by omitting an even number of letters in the word $u_1 \cdots u_p$.

(ii) For two S-reduced expressions $w = \omega(u_1 \cdots u_p) = \omega(u'_1 \cdots u'_p)$, the set of $s \in S$ appearing in the first expression is equal to the set of letters that are appearing in the second expression

(iii) No generator $s \in S$ can be expressed just with the other generators in $S \setminus \{s\}$. This means that S is a minimal generating set for W.

Before treating the computability of the word length, there is a more fundamental classical problem of whether the identity can always be distinguished from other elements in a group.

3.2.1 The word problem

For groups that are given by a finite generating set and relations, like Coxeter groups are, Max Dehn introduced three *fundamental problems.* The first of these problems is the *word problem.* The word problem is the question of whether there exists an algorithm that decides for every element $\mathbf{s} \in S^*$ if $\omega(\mathbf{s}) = \mathbb{1}$ in W (see [Deh12]). This is equivalent to the existence of an algorithm that decides if two words in S^* represent the same group element. If there exists an algorithm for a group G then G has a *solvable word problem.*

It is well known that the word problem in Coxeter groups is solvable. In this section, we state a specific theorem by Tits in [Tit69] leading to a solution to the word problem. This solution also provides a way to compute the word length of every element in a Coxeter group. Before stating the result, some definitions are necessary.

Definition 3.2.10. A *subword* of a word $\mathbf{s} = u_1 \cdots u_p$ over the alphabet S is a product $u_{i_1} \cdots u_{i_q}$ with $1 \leq i_1 < \cdots < i_q \leq p$. A subword is *consecutive* if the subsequence $i_1, \ldots, i_q$ is consecutive.

Definition 3.2.11. We distinguish two types of relations in the definition of a Coxeter group. Substituting the consecutive subword $s_i s_i$ with the empty word e in a word is called a *nil-move.* For $m_{ij} \in \mathbb{N}$, let $\mathbf{b}_{ij}$ be the word of length m_{ij} consisting only of alternating letters s_i and s_j starting with s_i. Substituting a consecutive subword $\mathbf{b}_{ij}$ with the subword $\mathbf{b}_{ji}$ in a word in S^* is called a *braid-move.*

Remark 3.2.12. Let $\mathbf{a}$ and $\mathbf{b}$ be two elements in S^* and $s_i, s_j \in S$. From the relations of the type $s_i^2 = \mathbb{1}$, the following nil-move can be derived

$$\omega(\mathbf{a} \cdot s_i s_i \cdot \mathbf{b}) = \omega(\mathbf{a} \cdot \mathbf{b}).$$

Equivalent to a braid relation $(s_i s_j)^{m_{ij}} = \mathbb{1}$ is the braid-move

$$\omega(\mathbf{a} \cdot \mathbf{b}_{ij} \cdot \mathbf{b}) = \omega(\mathbf{a} \cdot \mathbf{b}_{ji} \cdot \mathbf{b}).$$

Braid-moves do not change the length of a word. Braid-moves on subwords $\mathbf{b}_{ij}$ of even S-length do not change the number of letters of a certain type in a word. Braid-moves on subwords of odd S-length change the number of letters of a certain type by ± 1.

Theorem 3.2.13 (see [Tit69, Theorem 3])**.** *Let (W, S) be a Coxeter system and w be an element in W.*

(i) For every word $\mathbf{s} \in S^*$ with $\omega(\mathbf{s}) = w$, there exists a finite sequence of nil-moves and braid-moves that transforms $\mathbf{s}$ into an S-reduced expression for w.

(ii) For every pair of S-reduced expressions for w, there exists a finite sequence of braid-moves that transforms one of the S-reduced expressions into the other.

The second statement is due to Hideya Matsumoto (see [Mat64]). It is also called the *Matsumoto property* of a Coxeter group.

Corollary 3.2.14. *The numbers of letters in two words $\mathbf{s}$ and $\mathbf{s}'$ representing the same element $\omega(\mathbf{s}) = \omega(\mathbf{s}') = w \in W$ have the same parity.*

Remark 3.2.15. For a fixed word $\mathbf{s} \in S$, it is possible to enumerate algorithmically all words obtainable from $\mathbf{s}$ via nil-moves and braid-moves. Thus, it is decidable if $\mathbf{s}$ is S-reduced and Theorem 3.2.13 is a solution to the word problem in Coxeter groups (see [AB08, p. 86]). So the theorem above also yields an algorithm to estimate the word length of any element in a Coxeter group.

3.3 Reflection length

One of the simplest types of isometries of a metric space are reflections across hyperplanes. A classical result is that the group of isometries $\mathrm{Iso}(\mathbb{X}^n)$ is generated by reflections across hyperplanes and every isometry can be factored into maximally $n+1$ reflections (see Theorem 2.2.3). Geometric reflection groups are finitely generated discrete subgroups of $\mathrm{Iso}(\mathbb{X}^n)$. This motivates the question of which elements in geometric reflection groups are reflections. More generally with regards to Section 2.3, the question is which elements of a Coxeter group act as reflections on the geometric representation and the basic construction. Once these questions are answered, we study the length function of the set of reflections in this section.

3.3.1 Reflections in Coxeter groups

Reflections are involutions and orientation reversing. Therefore, a reflection in a Coxeter group should be expressed by an odd number of generators, which act as reflections.

From the geometric representation of a Coxeter system (W, S), we deduce the definition of the set of reflections in W: By the definition of the geometric representation (see Paragraph 2.3.1), every generator $s_i \in S$ corresponds to a reflection σ_i across a hyperplane $H_i = \{v \in V \mid B(e_i, v) = 0\}$. Let $R \subset W$ be the set of elements in W that act as reflections across hyperplanes on V. This means that $\rho(r)$ is a reflection in V for all $r \in R$.

Let $s_i \in S$ be a generator and $r \in R$ be a reflection distinct from s_i such that $\rho(r)$ is a reflection across a hyperplane H_r with a unit vector α_r generating the orthogonal complement of H_r with respect to the bilinear form B. Hence, the hyperplane H_r can be written as $H_r = \{v \in V \mid B(\alpha_r, v) = 0\}$. The image of H_r under σ_i is

$$\begin{aligned}\sigma_i(H_r) &= \{\sigma_i(v) \mid v \in V \text{ and } B(\alpha_r, v) = 0\} \\ &= \{v \in V \mid B(\alpha_r, \sigma_i(v)) = 0\} \\ &= \{v \in V \mid B(\sigma_i(\alpha_r), v) = 0\}.\end{aligned}$$

The second equation is true since σ_i is an involution by definition. The image $\rho(W)$ preserves the bilinear form B and this implies the third equation. In conclusion, the vector $\sigma_i(\alpha_r)$ is orthogonal to $\sigma_i(H_r)$. The reflection across $\sigma_i(H_r)$ is equal to $\sigma_i \circ \rho(r) \circ \sigma_i$. This is the motivation for the following definition.

Definition 3.3.1. For a Coxeter system (W, S), a conjugate r of a generator in S is called *reflection*. The set of reflections in W is

$$R := \{wsw^{-1} \in W \mid w \in W,\ s \in S\}.$$

Remark 3.3.2. Note that if the Coxeter group is infinite, the set of reflections R is infinite, too. In literature, the pair (W, R) is sometimes called *dual Coxeter system.* This originates from [Bes03], where Garside monoids associated with certain Artin groups are investigated.

Since the standard generators in S are all involutions, every reflection is an involution and $S \subseteq R$. So R is a generating set for W, too.

Roots and reflections. From the discussion above, it follows that a reflection $r = wsw^{-1}$ in W acts as a reflection $\rho(r)$ across a hyperplane H_r on V. If α_s is the root of s, then $\alpha_r := w(\alpha_s)$ is the root corresponding to r and a unit vector in the orthogonal complement of H_r. On the other hand, every root $\alpha \in \Phi_W$ corresponds to a reflection r_α in W. For $\alpha = w'(\alpha_{s'})$, we have $r_\alpha = w's'w'^{-1}$. The set of all reflections R can also be written as $R = \{r_\alpha \in W \mid \alpha \in \Phi_W\}$. The roots α and $-\alpha$ yield the same reflection $r_\alpha = r_{-\alpha}$ (see [Hum90, Section 5.79]).

3.3.2 The reflection length function

As a generating set, the set of reflections R in a Coxeter system (W, S) induces the second example of a length function l_R on W (see Definition 3.1.2). This manuscript is about reflection length in infinite non-affine Coxeter groups. Before we restrict ourselves to infinite non-affine Coxeter groups, we state some basic results about reflection length in general Coxeter groups. This section also contains proofs to be self-contained.

Definition 3.3.3. The length function $l_R : W \to \mathbb{N}_0$ corresponding to the generating set R of reflections is called the *reflection length*. Minimal reflection factorisations of an element $w \in W$ are called *R-reduced*. Sometimes the reflection length is also called *absolute length*.

Remark 3.3.4. Whether a reflection factorisation is R-reduced, depends strongly on the relations in the Coxeter group. We have $l_R(w) \leq l_S(w)$ for all $w \in W$ in an arbitrary Coxeter system (W, S) because $S \subseteq R$.

Remark 3.3.5. The reflection length $l_R(w)$ is the minimal number of hyperplane reflections sufficient to reflect the simplex corresponding to $w \in W$ onto the simplex corresponding to $\mathbb{1} \in W$ in the Coxeter complex of (W, S).

Example 3.3.6. Consider the Coxeter groups W_1 and W_2 over three generators a, b, c defined by the following two graphs from left to right:

The element represented by the word $\mathbf{w} = abcabc$ in W_1 has reflection length 4. An R-reduced factorisation is $abcabc = aba \cdot aca \cdot b \cdot c$. In contrast, the element represented by $\mathbf{w}$ in W_2 has reflection length 2. This is the minimal possible reflection length because of parity reasons. An R-reduced factorisation is $abcabc = aba \cdot cbc$. The elements in both groups represented by $\mathbf{w}$ have word length 6.

In the example, we claim without proof for the moment that the factorisations are R-reduced. This is also the case for the next geometric example.

Example 3.3.7. Let W be the Coxeter group defined by the Coxeter graph

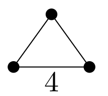

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Figure 3.1 shows the Coxeter complex of W with some coloured top-dimensional simplices and numbered black hyperplanes. The reflection length of the element w corresponding to the 2-dimensional simplex ▲ is 4. According to Remark 3.3.5, the minimal number of hyperplane reflections to reflect ▲ back to the identity simplex ▲ is 4. One way to reflect ▲ back is the sequence of reflections across the black hyperplanes indicated by the numbering. Let $r_i \in W$ be the reflection across the hyperplane H_i. The reflection r_1 reflects ▲ to ▲, r_2 reflects ▲ to ▲, r_3 reflects ▲ to ▲ and r_4 reflects ▲ to ▲. A minimal reflection factorisation for w is $w = r_1 r_2 r_3 r_4$.

The exchange condition in Paragraph 3.2.7 can be strengthened to the next theorem.

Theorem 3.3.8 (Strong Exchange Condition, see [BB05, Theorem 1.4.3])**.** *Let w be an element in a Coxeter system (W, S) with a factorisation $w = \omega(u_1 \cdots u_p)$ (not necessarily S-reduced) with $u_i \in S$ for $1 \leq i \leq p$. For every reflection $r \in R$ with $l_S(rw) < l_S(w)$, there exists a letter $u_j \in \{u_1, \ldots, u_p\}$ such that $rw = \omega(u_1 \cdots \hat{u}_j \cdots u_p)$.*

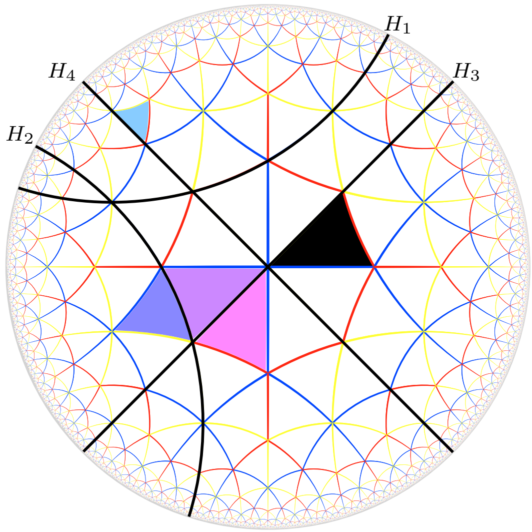


Figure 3.1: Geometric reflection factorisation of the simplex ▲ in the Coxeter complex of the triangle group $(3,3,4)$ embedded in $\mathbb{H}^2$ in the Poincaré disc model.

The following corollary illustrates the relation between a factorisation of r and an S-reduced factorisation of w.

Corollary 3.3.9 (see [BB05, Corollary 1.4.4]). *If $w = \omega(u_1 \cdots u_m)$ is S-reduced and $r \in R$, then the following are equivalent:*

(i) $l_S(rw) < l_S(w)$,

(ii) $rw = \omega(u_1 \cdots \hat{u}_i \cdots u_m)$ *for some* $i \in \{1, \ldots, m\}$,

(iii) $r = \omega(u_1 u_2 \cdots u_i \cdots u_2 u_1)$ *for some* $i \in \{1, \ldots, m\}$.

The index i in (ii) and (iii) is uniquely determined.

Lemma 3.3.10 (see [Bre+19, Remark 1.3]). *In a Coxeter system (W, S), the reflection length function $l_R : W \to \mathbb{N}_0$ is constant on conjugacy classes.*

Proof. Let w be an element in W with R-reduced factorisation $w = r_1 \cdots r_q$. For an element $w' = vwv^{-1}$ in the same conjugacy class as w with $v \in W$, we have

$$w' = vr_1 \cdots r_q v^{-1} = vr_1 v^{-1} \cdots vr_q v^{-1}$$

with $vr_i v^{-1} \in R$. So the right term is a reflection factorisation for w'. This implies the inequality $l_R(w') \leq l_R(w)$. The same argument with reversed roles of w and w' yields the equality $l_R(w) = l_R(w')$. □

The lemma below is elementary for the basic properties of the reflection length function.

Lemma 3.3.11 (see [BB05, Lemma 1.4.1]). *For a Coxeter system (W, S), the map $s \mapsto 1$ for all $s \in S$ extends to a group homomorphism $\tau : W \to \mathbb{Z}/2\mathbb{Z}$. For all reflections $r \in R$, it is $\tau(r) = 1$.*

Proof. Let $\mathbf{s}$ and $\mathbf{s}'$ be two words representing the same element $\omega(\mathbf{s}) = \omega(\mathbf{s}') = w \in W$. The numbers of letters in $\mathbf{s}$ and $\mathbf{s}'$ have the same parity (see Corollary 3.2.14). Thus, the rule $s \mapsto 1$ for all $s \in S$ directly implies $\tau(w) = l_S(w) \mod 2$ for all $w \in W$. The map $\tau : W \to {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$ is well-defined because all the relations in the definition of a Coxeter group have even lengths. Further, τ is a group homomorphism: The numbers $l_S(\omega(\mathbf{uv}))$ and $l_S(\omega(\mathbf{u})) + l_S(\omega(\mathbf{v}))$ have the same parity for all words $\mathbf{u}, \mathbf{v} \in S^*$ since the group elements $\omega(\mathbf{uv})$ and $\omega(\mathbf{u}) \cdot \omega(\mathbf{v})$ are equal. Reflections have odd S-length. This is why, $\tau(r) = 1$ for all $r \in R$. □

Corollary 3.3.12. *For an element w in a Coxeter system (W, S), its word length $l_S(w)$ and its reflection length $l_R(w)$ have the same parity: $l_S(w) \equiv l_R(w) \mod 2$.*

Some more basic properties of the reflection length function can be deduced from the previous lemma. Analogous results hold for the word length (see [BB05, Proposition 1.4.2]).

Lemma 3.3.13 (see [Bre+19, Remark 1.3])**.** *Let (W, S) be a Coxeter system, $u, v \in W$ and $r \in R$.*

(i) $l_R(uv) = l_R(vu)$.

(ii) $|l_R(u) - l_R(v)| \leq l_R(uv) \leq l_R(u) + l_R(v)$.

(iii) $l_R(uv) \equiv l_R(u) + l_R(v) \mod 2$.

(iv) $l_R(ru) = l_R(ur) = l_R(u) \pm 1$.

Proof. The first assertion is a direct consequence of Lemma 3.3.10 and the fact that the standard generators in S are involutions.

Let $r_1 \cdots r_p$ and $t_1 \cdots t_q$ be R-reduced reflection factorisations for u and v, respectively. Then, $r_1 \cdots r_p \cdot t_1 \cdots t_q$ is a reflection factorisation for uv. Consequently, we obtain the inequality $l_R(uv) \leq l_R(u) + l_R(v)$. On the other hand, if the inverse of one R-reduced factorisation appears in the other R-reduced factorisation at the beginning of v or the end of u as a consecutive subword, we have $|l_R(u) - l_R(v)| = l_R(uv)$. This proves the second assertion.

With Lemma 3.3.11, we have $\tau(w) \equiv l_R(w) \mod 2$ in ${}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$ for all $w \in W$. The third assertion follows with the group homomorphism property of τ. We have $\tau(uv) = \tau(u) + \tau(v)$ (see Lemma 3.3.11).

From (i), we know $l_R(ru) = l_R(ur)$. Substituting v with r in (ii) and (iii) implies the fourth assertion. □

The following lemma allows a crucial restriction for the investigation of reflection length in Coxeter groups. The reflection length function is additive on direct products. Thus, for the investigation of the reflection length in Coxeter groups, we only need to consider irreducible Coxeter groups.

Lemma 3.3.14 (see [MP11, Proposition 1.2])**.** *When (W, S) is a reducible Coxeter system, S has a non-trivial partition $S = S_1 \sqcup S_2$ such that every element in S_1 commutes with every element in S_2. In this context, $W = W_{S_1} \times W_{S_1}$ and the reflections R in W can be partitioned as well $R = R_1 \sqcup R_2$, where R_i is the set of reflections in W_{S_i}. An element $w \in W$ can be written as $w = w_1 w_2$ with $w_i \in W_{S_i}$. For the reflection length, we have $l_R(w) = l_{R_1}(w_1) + l_{R_2}(w_2)$.*

Proof. Let w be an arbitrary element in W with $w = r_1 \cdots r_m$ as an R-reduced reflection factorisation. Every reflection that is conjugated to an element in S_i is contained in R_i, since reflections can be represented as palindromes of odd length by definition and all generators in S_1 commute with all generators in S_2. This is the reason why, we can rewrite $r_1 \cdots r_m$ as $r_{i_1} \cdots r_{i_p} r_{i_{p+1}} \cdots r_{i_m}$ such that $r_{i_1}, \ldots, r_{i_p} \in R_1$ and $r_{i_{p+1}}, \ldots, r_{i_m} \in R_2$. This is done by solely commuting reflections in R_1 with reflections in R_2 and not changing any reflection. Hence, we have $w = w_1 w_2$ where $w_1 = r_{i_1} \cdots r_{i_p} \in W_{S_1}$ and $w_2 = r_{i_{p+1}} \cdots r_{i_m} \in W_{S_2}$. These reflection factorisations are R-reduced because $w = r_1 \cdots r_m$ is R-reduced. Since $W = W_{S_1} \times W_{S_1}$, it follows $l_R(w) = l_{R_1}(w_1) + l_{R_2}(w_2)$. □

Sometimes it is useful to rewrite reflection factorisations.

Lemma 3.3.15 (see [MP11, Lemma 3.5])**.** *Let $w = r_1 \cdots r_m$ be a reflection factorisation of an element w in a Coxeter system (W, S). For any selection $1 \leq i_1 < \cdots < i_k \leq m$ of indices, there is a reflection factorisation of w of length m whose first k reflections are $r_{i_1} \cdots r_{i_k}$. There is another reflection factorisation of w of length m, where the reflections corresponding to the selected indices are the last k reflections.*

Proof. The set of reflections R in W is closed under conjugation by definition. Reflections are involutions. For $r, r' \in R$, the elements $t = rr'r$ and $t' = r'rr'$ are also in R and we have $rr' = tr = r't'$. Hence, in any reflection factorisation a consecutive pair of reflections rr' may be replaced with the pairs tr or $r't'$. This rewriting rule does not change the length of the factorisation. Iterating this procedure suffices to move successively any subword of a reflection factorisation into a certain position (see [MP11, p. 716]). □

3.3.3 Automorphisms and reflection length

This paragraph discusses briefly how reflections and reflection length behave under group automorphisms. We restrict ourselves to automorphisms as the current state of research on isomorphisms between Coxeter groups is still in the early stages.

Lemma 3.3.16. *Let $\varphi : W \to W$ be an automorphism of a Coxeter group W. Further, let R be the set of reflections. If $\varphi(R) \subseteq R$, the automorphism φ preserves reflection length:*

$$l_R(w) = l_R(\varphi(w)) \qquad \textit{for all}\, w \in W.$$

Proof. With the conditions as above, it follows that $\varphi(R) = R$ and φ^{-1} also preserves reflections (see [Fra01, Lemma 1.33]). Let w be an element in W expressed by an R-reduced factorisation $w = r_1 \cdots r_k$ with $r_i \in R$ for $1 \leq i \leq k$. By applying the automorphism φ, we obtain a reflection factorisation for the image $\varphi(w) = \varphi(r_i) \cdots \varphi(r_k)$. Thus, we have $l_R(w) \geq l_R(\varphi(w))$. Consider the inverse φ^{-1} and an R-reduced factorisation for $\varphi(w)$ to arrive at $l_R(w) \leq l_R(\sigma(w))$ with the same argument. In total, it is $l_R(w) = l_R(\varphi(w))$. □

This result has direct consequences for the group of automorphisms of universal Coxeter groups. Recall that the universal Coxeter group W_n of rank n has no relations between all distinct generators.

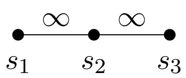
Corollary 3.3.17. *Let w be an element in the universal Coxeter group W_n with $n \geq 1$ and let σ be an element in the group of automorphisms $\mathrm{Aut}(W_n)$. Then, $l_R(w) = l_R(\sigma(w))$.*

Proof. From the homomorphism property of σ, it follows $\sigma(R) \subseteq R$ because in W_n an element has order 2 if and only if it is contained in R. This is true since there are no braid relations in W_n. $\square$

Remark 3.3.18. William N. Franzsen investigated the group of automorphisms of Coxeter groups in [Fra01]. An important property in his work is that automorphisms preserve reflections. Together with the lemma above, Franzsen's results imply that the group of automorphisms of a Coxeter group W preserves reflection length if all standard parabolic subgroups of W are finite or if W is minimal non-affine (see [Fra01, Lemma 4.4 and §4.3]).

A necessary condition for a homomorphism $\varphi : W \to W'$ between Coxeter groups to preserve reflection length is to be injective. Otherwise, the reflection length is not preserved for the non-trivial elements in the kernel of φ. Not all Coxeter groups have a group of automorphisms that preserves reflection length as the following example shows. Injectivity is especially not a sufficient condition for a homomorphism to preserve reflection length.

Example 3.3.19. Let W be the Coxeter group of rank 3 defined by the Coxeter diagram:



Define the automorphism $\alpha : W \to W$ via $\alpha(s_1) = s_1, \alpha(s_2) = s_2$ and $\alpha(s_3) = s_1 s_3$. Note that the homomorphism α is an endomorphism, since $\alpha(s_1)\alpha(s_3) = s_3$. Further, it has order 2. Thus, it is an automorphism (see [Fra01, Lemma 5.13]).

3.3.4 Computing reflection length

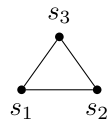
Every factorisation into generators of an element is a reflection factorisation for this element. Thus, finding a reflection factorisation is generally easy. However, it can be difficult to prove the minimality of a reflection factorisation. Since the word problem for Coxeter groups is solvable, it is possible to decide whether a reflection factorisation represents a certain element. On the other hand, the set of reflections is infinite in an infinite Coxeter group and until now there is no dual version known for Theorem 3.2.13 (see [Bes03; Bau+14; Weg20; WY23] for a dual version of the Matsumoto property).

The theorem below by Matthew Dyer provides an effective way to compute the reflection length of an element from an S-reduced factorisation. Together with its proof, it is closely related to the Strong Exchange Condition. The method for computing the reflection length deduced from this theorem is the only one known, at the time of writing, to compute the reflection length in an arbitrary Coxeter group of an arbitrary element. So it is essential for the study of the reflection length in Coxeter groups and we use it for many proofs in this text.

Theorem 3.3.20 (see [Dye01, Theorem 1.1.]). *Let $\mathbf{s} = u_1 \cdots u_p$ be an S-reduced expression in a Coxeter system (W, S). Then, $l_R(\omega(\mathbf{s}))$ is the minimum of the natural numbers q for which there exist $1 \leq i_1 < \cdots < i_q \leq p$ such that $\mathbb{1} = \omega(u_1 \cdots \hat{u}_{i_1} \cdots \hat{u}_{i_q} \cdots u_p)$.*

The theorem shows that the reflection length can be understood as a measure of how many standard generators an element differs from the identity.

Example 3.3.21. Consider the Coxeter group W of type $\tilde{\mathbf{A}}_2$ corresponding to the graph



.

Let $\mathbf{w} := s_1 s_2 s_3 \in S^*$. The reflection length of $\omega(\mathbf{w}^4 s_1 s_2)$ is 2. This is to be seen by omitting the following letters:

$$\begin{aligned} s_1 s_2 s_3 s_1 \hat{s}_2 s_3 s_1 s_2 s_3 \hat{s}_1 s_2 s_3 s_1 s_2 &= s_1 s_2 \cdot s_3 s_1 s_3 s_1 \cdot s_2 s_3 s_2 s_3 \cdot s_1 s_2 \\ &= s_1 s_2 \cdot s_1 s_3 \cdot s_3 s_2 \cdot s_1 s_2 = \mathbb{1}. \end{aligned}$$

Hence, by Theorem 3.3.20 the reflection length is at most 2. Since the word length is even, this implies $l_R(w^4 s_1 s_2) = 2$.

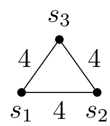
The proof in [Dye01] for Theorem 3.3.20 does not use the fact that the word $\mathbf{s}$ is S-reduced. So we state a slightly more general result.

Corollary 3.3.22. *Let (W, S) be a Coxeter system and let $\mathbf{s} = u_1 \cdots u_p \in S^*$ be a word (not necessarily S-reduced). Then $l_R(\omega(\mathbf{s}))$ is the minimum of the natural numbers q for which there exist $1 \leq i_1 < \cdots < i_q \leq p$ such that $\mathbb{1} = \omega(u_1 \cdots \hat{u}_{i_1} \cdots \hat{u}_{i_q} \cdots u_p)$.*

Corollary 3.3.23 (see [Dye01, Corollary 1.4]). *Let (W, S) be a Coxeter system and let $W' \subseteq W$ be a parabolic subgroup. Let $R' = W' \cap R$ be the set of reflections in W'. For the reflection lengths, we have $l_R(w') = l_{R'}(w')$ for all $w' \in W'$.*

Definition 3.3.24. In a Coxeter system (W, S), let $\mathbf{w} = u_1 \cdots u_p \in S^*$ be a word representing an element $w = \omega(\mathbf{w}) \in W$. We call a minimal set of indices like in Theorem 3.3.20 *deletion set* for $\mathbf{w}$. For the cardinality of a deletion set $D(\mathbf{w})$, we have $|D(\mathbf{w})| = l_R(w)$.

Example 3.3.25. Consider the Coxeter group W defined by the Coxeter graph



.

Define $\mathbf{w} := s_1 s_2 s_3 \in S^*$. The reflection length of $\omega(\mathbf{w}^5 s_1 s_2)$ is 5 and there are multiple deletion sets.

1. $\omega(s_1 s_2 \hat{s}_3 s_1 s_2 \hat{s}_3 s_1 s_2 s_3 s_1 \hat{s}_2 s_3 s_1 \hat{s}_2 s_3 s_1 s_2) = \omega(s_2 s_1 \cdot s_1 s_3 s_2)$ has reflection length 1.
2. $\omega(s_1 \hat{s}_2 s_3 s_1 s_2 \hat{s}_3 s_1 s_2 \hat{s}_3 s_1 s_2 \hat{s}_3 s_1 s_2 s_3 s_1 \hat{s}_2) = \mathbb{1}$.

Lemma 3.3.26 (see [Lot24a, Lemma 1.24.]). *Let w be an element of a Coxeter system (W, S) represented by a word $\mathbf{s} = u_1 \cdots u_p \in S^*$ and let $D(\mathbf{s}) = \{i_1, \ldots, i_q\}$ be a deletion set. For every proper subset $N =\subsetneq D(\mathbf{s})$, let $w^{\setminus N}$ be the element represented by the word that we obtain from $\mathbf{s}$ by removing all letters with indices in N. With $w' := w^{\setminus N}$, we have*

$$l_R(w') = q - |N| \quad \textit{and} \quad l_R(w^{\setminus N \cup \{i_j\}}) = l_R(w') - 1 \quad \textit{for all}\, i_j \in \{i_1, \ldots, i_q\} \setminus N.$$

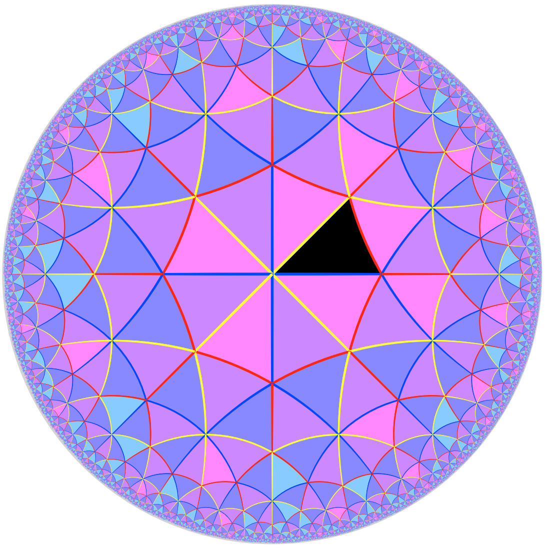


Figure 3.2: Coxeter complex of the triangle group $(3,3,4)$ with coloured simplices according to reflection length embedded in $\mathbb{H}^2$ in the Poincaré disc model.

Proof. Define the reflection $r_i := u_1 \cdots u_{i-1} u_i u_{i-1} \cdots u_1$. A deletion set $D(\mathbf{s})$ of indices $1 \leq i_1 < \cdots < i_q \leq n$ corresponds to the following minimal reflection factorisation of w:

$$w = r_{i_q} \cdots r_{i_1}.$$

A proper subset $N \subsetneq D(\mathbf{s})$ is a totally ordered set $i_{n_1} < \cdots < i_{n_m}$ with $m < q$. To remove all the letters corresponding to the indices $i_{n_j} \in N$ from the word $\mathbf{s}$, we multiply from the left with $r_{i_{n_1}} \cdots r_{i_{n_m}}$. This results in

$$\omega(u_1 \cdots \hat{u}_{i_{n_j}} \cdots u_p) = r_{i_{n_1}} \cdots r_{i_{n_m}} \cdot w = r_{i_{n_1}} \cdots r_{i_{n_m}} \cdot r_{i_q} \cdots r_{i_1}.$$

For every $r_{i_{n_j}}$ exists an r_{i_k} such that $r_{i_{n_j}} = r_{i_k}$ because $N \subsetneq D(\mathbf{s})$. Thus, we have

$$w^{\setminus N} = r_{i_{n_1}} \cdots r_{i_{n_m}} \cdot r_{i_q} \cdots r_{i_1} = r_{i_{n_1}} \cdots r_{i_{n_m}} \cdot r_{i_q} \cdots r_{i_{n_m}} \cdots r_{i_{n_1}} \cdots r_{i_1}.$$

In the reflection factorisation $r_{i_{n_1}} \cdots r_{i_{n_n}} \cdot r_{i_p} \cdots r_{i_{n_n}} \cdots r_{i_{n_1}} \cdots r_{i_1}$, all reflections between two equal reflections $r_{i_{n_j}}$ can be written as reflections conjugated with $r_{i_{n_j}}$. The reflection length only changes by ± 1 when multiplying with a reflection (see Lemma 3.3.13). We conclude $l_R(w^{\setminus N}) = l_R(w) - |N|$. With the same arguments we obtain $l_R(w^{\setminus N \cup \{i_j\}}) = l_R(w^{\setminus N}) - 1$ for every $i_j \in \{i_1, \ldots, i_q\} \setminus N$. □

Example 3.3.27. Based on Theorem 3.3.20, we implement an algorithm in Appendix A to compute the reflection length of arbitrary elements in arbitrary Coxeter groups. With this algorithm it is possible to colour top-dimensional simplices in the Coxeter complex according to the reflection length of the corresponding element. An example is displayed in Figure 3.2. For an explanation of the colour scheme see Paragraph 1.4.1.

3.3.5 Reflection length in reflection subgroups

Subgroups of a Coxeter group generated by a finite subset of the set of reflections have a reflection length function, too. We show via an example that the reflection length functions of the Coxeter group and of the subgroup are not equal in general.

The basic object considered in this paragraph is the following.

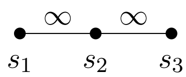
Definition 3.3.28. A *reflection subgroup* of a Coxeter group W is a subgroup $W' \subseteq W$ generated by a finite subset $T \subseteq R$ of the set of reflections in W.

Reflection subgroups of Coxeter groups are again Coxeter groups.

Theorem 3.3.29 (see [Deo89; Dye90])**.** *Let (W, S) be a Coxeter system and R be its set of reflections. A subgroup W_T generated by a finite subset $T \subseteq R$ is a Coxeter group with respect to a canonical set of generators.*

Hence, every reflection subgroup of a Coxeter group admits a reflection length function. The following example shows that the reflection length function of a Coxeter group and a proper reflection subgroup are not equal on the reflection subgroup in general.

Example 3.3.30. Consider the Coxeter system (W, S) with Coxeter diagram



and the reflection subgroup W' generated by s_1, s_2 and $s_3 s_2 s_3$. It is to be seen with Dyer's Theorem 3.3.20 that the element $w = s_1 s_2 s_3 s_2 s_3 s_1 s_2 s_3 s_2 s_3$ has reflection length 2 in W. The element w has an even word length and we have

$$s_1 s_2 \hat{s}_3 s_2 s_3 s_1 s_2 \hat{s}_3 s_2 s_3 = \mathbb{1}.$$

However, the reflection length of $s_1 s_2 s_3 s_2 s_3 s_1 s_2 s_3 s_2 s_3$ is not 2 in the reflection subgroup W'. The Coxeter generating set of W' is $S' = \{a = s_1, b = s_2, c = s_3 s_2 s_3\}$ (this follows with the results in [Dye90]). The order of all products of two generators is ∞. This is why, W' is a universal Coxeter group of rank 3. The word length of $(abc)^2$ is even. Omitting all possible pairs of generators in $(abc)^2$ does not yield the identity. The reflection length of $(abc)^2$ is 4. This follows from Theorem 1.

3.4 Reflection length in finite and affine Coxeter groups

The reflection length function on finite and affine Coxeter systems is well understood. The reflection length is a bounded function on both types of these groups. Moreover, there exist formulas that connect the reflection length with other geometric statistics. According to Lemma 3.3.14, the reflection length function is additive on direct products of Coxeter groups. This is why, it suffices to understand the reflection length function on irreducible Coxeter systems, which are the spherical and Euclidean reflection groups in the classification discussed in Section 2.2.4.1. We follow the publications [Car72] by Carter as well as [Bre+19] by Lewis, McCammond, Petersen and Schwer, in which the main results regarding the reflection length in finite and affine Coxeter groups are established.

Each element w_0 in a spherical Coxeter group is an orthogonal transformation (see Theorem 2.2.4). Two subspaces can be associated with an orthogonal transformation of the Euclidean space $\mathbb{E}^m$: The set of vectors $v \in \mathbb{E}^m$ that are fixed by w_0 and the set of vectors $\mu \in \mathbb{E}^m$ representing the different movements of w_0, These concepts extend to affine Coxeter groups.

Definition 3.4.1. Let w be an isometry of a real vector space V.

1. The *fixed space* $\mathrm{Fix}(w)$ of w is defined as the kernel $\mathrm{Fix}(w) := \ker(w - \mathbb{1})$. It consists of all points $x \in V$ that fulfil $w(x) = x$. If $\mathrm{Fix}(w)$ is not empty, it is an affine subspace of V.

2. The *motion* of an element $x \in V$ under w is $\lambda \in V$ with $w(x) = x + \lambda$. The *move-set* $\mathrm{Mov}(w)$ of w is defined as the image $\mathrm{Mov}(w) := \mathrm{im}(w - \mathbb{1})$. Equivalently, it is the set of all $\lambda \in V$ such that there exists $x \in V$ with $w(x) = x + \lambda$.

Example 3.4.2. For a reflection $r \in R$ in a geometric reflection group W across a hyperplane $H_r \subseteq \mathbb{E}^n$, the fixed space is $\mathrm{Fix}(r) = H_r$. The motion of any $x \in \mathbb{E}^n$ under r is λ orthogonal to H_r, which is a scalar multiple of the root corresponding to r. The move-set $\mathrm{Mov}(r)$ is the line passing through the origin with direction λ.

Remark 3.4.3. If the Coxeter group W is finite, each element w is an orthogonal transformation and the fixed space and the move-set are orthogonal complements. This does not hold in general (see [Bre+19, Remark 1.8]).

3.4.1 Reflection length in finite Coxeter groups

The first general result regarding reflection length is about finite Coxeter groups and due to Carter. We remind the reader that the root α_r corresponding to a reflection $r = wsw^{-1} \in R$ in a Coxeter group W is $w(\alpha_s)$ (see Section 3.3.1).

Theorem 3.4.4 (see [Car72, pp.3-4])**.** *Let w be an element in a finite reflection group W. Its reflection length $l_R(w)$ is equal to the codimension of the fixed space* $\mathrm{Fix}(w)$*. A reflection factorisation $w = r_1 \cdots r_k$ is R-reduced if and only if the roots $\{\alpha_{r_1}, \ldots, \alpha_{r_k}\}$ corresponding to the reflections are linearly independent.*

Remark 3.4.5. The finite case harbours a particularly strong duality between the word length and the reflection length. There are also just finitely many reflections in these groups. The maximum of the word length l_S is exactly $|R|$ and the maximum of the reflection length l_R is exactly $|S|$ (see [Bes03, Section 1]).

Corollary 3.4.6 (see [MP11, Corollary 2.6])**.** *Let W be finite Coxeter group W of rank n whose geometric representation acts on $\mathbb{E}^n$ by orthogonal transformations. All $w \in W$ have reflection length $l_R(w) \leq n$ and for every element $w_n \in W$ that only fixes the origin in $\mathbb{R}^n$, we have $l_R(w_0) = n$. Precisely, every Coxeter element w_1 in W has the maximal reflection length $l_R(w_1) = n$.*

3.4.2 Reflection length in affine Coxeter groups

This section summarizes two results from [Bre+19]. One is a formula for the reflection length in affine Coxeter groups. The other one is about a factorisation for every element that behaves well with the reflection length function. To describe the reflection length function on affine Coxeter groups, it is necessary to understand the relation between affine Coxeter groups and the associated finite Coxeter group illustrated in Section 2.4.

Let (W, S) be an affine Coxeter system of rank n. W is isomorphic to the semidirect product $T \rtimes W_0$, where T is the normal subgroup of translations in W and $W_0 := {}^{W}/_{T}$ is a finite Coxeter group (see Theorem 2.4.8).

Definition 3.4.7. An element $w' \in W$ is called *elliptic* if $\mathrm{Fix}(w') \neq \varnothing$. The *elliptic part* w_e of an arbitrary element $w \in W$ is its image $p(w) = w_e$ in W_0 under the projection $p : W \to W_0$. The elliptic elements in W are exactly the elements of finite order.

Definition 3.4.8. For an affine Coxeter group W and an element $w \in W$, a factorisation $w = t_\lambda \cdot e$, where t_λ is a translation and e is an elliptic element, is called *translation-elliptic factorisation* of w. The translation t_λ is called *translation part* and e is called *elliptic part.*

Choose an inclusion map $i : W_0 \to W$ that is a section of the projection

$$p : W \to W_0 = {}^{W}/_{T}.$$

Theorem 2.4.8 says that there exists a unique isomorphism between W and $T \rtimes W_0$. The kernel $\ker(p)$ is exactly the set of translations in W. The origin $0 \in V$ is the unique point that is fixed by $i(W_0)$. Every element $w \in W$ has a unique factorisation $w = t_\lambda u$, where $t_\lambda \in T$ is a translation and u is an elliptic element in $i(W_0)$. The image $i(w_e)$ of the elliptic part w_e of w is exactly u. Not all translation-elliptic factorisations come from an identification of W with $T \rtimes W_0$ (see Remark 3.4.13).

Now, we define all statistics necessary to state the formula for reflection length in affine Coxeter groups.

Definition 3.4.9. When w is an element in a spherical or affine Coxeter group, its move-set is contained in an Euclidean vector space V that also contains the corresponding root system Φ. The *dimension* $\dim(w)$ of such an element is defined to be the root dimension (Definition 2.4.6) of its move-set. In symbols, $\dim(W) := \dim_\Phi(\mathrm{Mov}(w))$. Let W be an affine Coxeter group acting on an Euclidean space E and let $p : W \to W_0$ be its projection map. For each element $w \in W$, we can compute the dimension of w and the dimension of its elliptic part $w_e = p(w) \in W_0$. We call $e = e(w) = \dim(w_e)$ the *elliptic dimension* of w. Instead of focusing on the dimension of w itself, we focus on the number $d = d(w) = \dim(w) - \dim(w_e)$, which we call the *differential dimension* of w.

Remark 3.4.10. Note that $dim(w) = d + e$. Both statistics $d(w)$ and $e(w)$ have a geometric meaning. An element is a translation if and only if its elliptic dimension is 0 and analogously an element is elliptic if and only if its differential dimension is 0. Both of these statistics are computable (see [Bre+19, Remarks 1.36 and 1.37]).

Theorem 3.4.11 (see [Bre+19, Theorem A])**.** *Let W be an Euclidean reflection group and let $p : W \to W_0$ be the projection onto its associated spherical Coxeter group. For any element $w \in W$, its reflection length is*

$$l_R(w) = 2 \cdot \dim(w) - \dim(p(w)) = 2d + e,$$

where $e = \dim(p(w))$ and $d = \dim(w) - \dim(p(w))$.

This formula for the reflection length is accompanied by a well-chosen translation-elliptic factorisation of each element.

Theorem 3.4.12 (see [Bre+19, Theorem B])**.** *Let W be an affine Coxeter group. For every element $w \in W$, there exists a translation-elliptic factorisation $w = t_\lambda u$ such that $l_R(t_\lambda) = 2d(w)$ and $l_R(u) = e(w)$. In particular, $l_R(w) = l_R(t_\lambda) + l_R(u)$ holds for this factorisation of w.*

Remark 3.4.13. The translation-elliptic factorisation from the theorem above does not come from an isomorphism between W and $T \rtimes W_0$. It is proven that it is not always possible to choose $u = p(w)$ in the theorem above, where $p : W \to W_0$ is the projection map (see [Bre+19, Example 2.4]).

4. Unbounded reflection length in infinite non-affine Coxeter groups

This chapter deals with two proofs for the unboundedness of reflection length function in infinite non-affine Coxeter groups. The infinite non-affine case includes all scenarios that are not covered in the sections before. Jon McCammond and T. Kyle Petersen conjecture in [MP11] that the reflection length function is unbounded on infinite non-affine Coxeter groups. The first section briefly discusses the proof of Kamil Duszenko in [Dus12] for the unboundedness. A new proof for the unboundedness is presented in the second section. Here, we also give a proof that irreducible infinite non-affine Coxeter groups are acylindrically hyperbolic.

4.1 Unbounded reflection length

We reproduce Duszenko's result Theorem 4.1.3 in this section and give an alternative proof in the next section. Both proofs rely on the existence of an unbounded *homogeneous quasi-morphisms* on minimal infinite non-affine Coxeter groups.

Definition 4.1.1. A *quasi-morphism* of a group G is a map $f : G \to \mathbb{R}$ such that there exists a constant $D(f) \in \mathbb{R}$, called the *defect* of f, and

$$|f(xy) - g(x) - f(y)| \leq D(f)$$

holds for all $x, y \in G$. A quasi-morphism $f : G \to \mathbb{R}$ is called *homogeneous* if $f(g^n) = nf(g)$ for all $g \in G$ and $n \in \mathbb{N}$. A quasi-morphism $f : G \to \mathbb{R}$ is called *anti-symmetric* if $f(g^{-1}) = -f(g)$ for all $g \in G$.

Remark 4.1.2. Homogeneous quasi-morphisms are constant on conjugacy classes (see [Kot04]).

Duszenko states the following theorem.

Theorem 4.1.3 (see [Dus12, Theorem 1.1])**.** *For any infinite non-affine Coxeter group W, the reflection length is an unbounded function on W.*

Every infinite non-affine Coxeter group has a minimal infinite non-affine Coxeter group as a standard parabolic subgroup (see Remark 2.2.29). With Lemma 3.3.23, it is sufficient to prove Theorem 4.1.3 for minimal infinite non-affine Coxeter groups.

4.1.1 Proof of the unboundedness

The following two definitions are needed to understand the main ingredients of the proof of Theorem 4.1.3 in [Dus12].

Definition 4.1.4. Let $\delta > 0$. A geodesic triangle in a metric space is said to be *δ-slim* if each of its sides is contained in the δ-neighbourhood of the union of the two other sides. A geodesic space X is said to be *δ-hyperbolic* if every triangle in X is δ-slim.

Definition 4.1.5. A finitely generated group is *hyperbolic* (in the sense of Gromov) if its Cayley graph is a δ-hyperbolic metric space for some $\delta > 0$. A hyperbolic group is called *elementary* if it is finite or virtually infinite cyclic.

The proof of Theorem 4.1.3. The proof of Theorem 4.1.3 relies on a result that every minimal infinite non-affine Coxeter group admits a surjection onto a non-elementary δ-hyperbolic group (see [Dus12, Theorem 1.2]). This answers the general question of whether every infinite non-affine Coxeter group admits a surjection onto a non-elementary hyperbolic group by Tadeusz Januszkiewicz in [Bri+10] in the special case of minimal infinite non-affine Coxeter groups. The basis for the proof of this result is that Coxeter groups are virtually torsion-free (see Corollary 2.3.6) and that minimal infinite non-affine Coxeter groups are all hyperbolic reflection groups with a possibly unbounded simplex as a fundamental domain (see Theorem 2.2.30).

Duszenko constructs a simply connected negatively curved space for every minimal infinite non-affine Coxeter group on which a quotient of the Coxeter group acts cocompactly and properly discontinuously. It follows that every minimal infinite non-affine Coxeter group admits a surjection onto a non-elementary δ-hyperbolic group. Every non-elementary δ-hyperbolic group G has an unbounded homogeneous quasi-morphism $\varphi : G \to \mathbb{R}$. Every bi-invariant word metric on such a group is unbounded (see [Dus12, Proposition 3.2.] and [GK11, Lemma 3.7]). The unboundedness of the reflection length on infinite non-affine Coxeter groups follows with these results. Here, *bi-invariance* means invariant under conjugation like the reflection length is.

4.2 Alternative proof of the unboundedness

In this section, we discuss an alternative proof of the unboundedness of the reflection length function on infinite non-affine Coxeter groups. It is based on the fact that irreducible infinite non-affine Coxeter groups are acylindrically hyperbolic (see Theorem 4.2.9). The outline of this alternative proof was communicated to the author by Andreas Thom. For the proof that infinite non-affine Coxeter groups are acylindrically hyperbolic, we follow a sketch of a proof communicated to the author by Anthony Genevois.

Definition 4.2.1 (see [Bow08, p. 284])**.** An isometric action of a group G on a metric space X is *acylindrical* if, for every $\varepsilon > 0$, there exists $R, N > 0$ such that for every two points $x, y \in X$ with $d(x, y) \leq R$, there are at most N elements $g \in G$ satisfying

$$d(x, gx) \leq \varepsilon \quad \text{and} \quad d(y, gy) \leq \varepsilon.$$

Definition 4.2.2. Two geodesic rays $\gamma, \gamma' : [0, \infty) \to \mathbb{X}$ in a metric space $\mathbb{X}$ are called *asymptotic* if $\sup\{d(\gamma(x), \gamma'(x)) \mid x \in [0, \infty)\} < \infty$. This is an equivalence relation on the set of geodesic rays. The *visual boundary* $\partial\mathbb{X}$ is the set of equivalence classes of geodesic rays. We call the elements *ideal points* and denote them with $\gamma(\infty)$. The union of the space and its visual boundary is denoted $\overline{\mathbb{X}} := \mathbb{X} \cup \partial\mathbb{X}$.

Remark 4.2.3. Two geodesic rays are asymptotic if and only if their Hausdorff distance is finite. For $\mathbb{H}^n$, the visual boundary is the same as the Gromov boundary. For further details, see Section 1.8 in [Gro87].

The following definition is part of the standard classification of groups acting on hyperbolic spaces from Gromov (see [Gro87, Section 8.2]).

Definition 4.2.4. Let G be a group acting isometrically on a metric δ-hyperbolic space X. The *limit set* $\Lambda(G)$ of G is the set of accumulation points in ∂X of an orbit $G(s)$:

$$\Lambda(G) := \overline{G(s)} \cap \partial X.$$

The group action of G is *non-elementary* if $|\Lambda(G)| = \infty$.

Definition 4.2.5 (see [Osi16])**.** A group G is called *acylindrically hyperbolic* if it has a non-elementary acylindrical action on a δ-hyperbolic space.

Remark 4.2.6. Every acylindrically hyperbolic group G is *SQ-universal* (see [DGO16, Theorem 8.1]). This means that every countable group can be embedded into a quotient of G.

Definition 4.2.7. A geodesic line $\lambda(-\infty, \infty) \subseteq X$ in a CAT(0) space X has *rank-one* if it is not the boundary of a flat half-plane in X. An isometry γ of a CAT(0) space has *rank-one* if it has no fixed point and an invariant geodesic line that has rank-one.

The next result is due to Alessandro Sisto in [Sis18] but was reformulated to the following theorem by Denis Osin.

Theorem 4.2.8 (see [Osi16, Section 8, (d)])**.** *Let G be a group acting properly on a proper* CAT(0) *space. If G contains a rank-one element, G is either virtually cyclic or acylindrically hyperbolic.*

Let Σ be the Davis complex of an arbitrary irreducible infinite non-affine Coxeter system (W, S). From Remark 2.3.20, we know that the elements of W act cocompactly and properly discontinuous as isometries on Σ. Further, Σ is CAT(0). Every Coxeter element acts as a rank-one isometry on Σ (this follows from [CF10, Proposition 4.5]). Hence, every irreducible infinite non-affine Coxeter group is either virtually cyclic or acylindrically hyperbolic. A cyclic group is abelian. Virtually abelian irreducible Coxeter groups are characterized by Theorem 2.3.21. By the definition of infinite non-affine Coxeter groups, we obtain the theorem:

Theorem 4.2.9. *Irreducible infinite non-affine Coxeter groups are acylindrically hyperbolic.*

In [BBF19], the *Brooks construction* is generalised to acylindrically hyperbolic groups. It defines a family of anti-symmetric quasi-morphisms on the free group F_2 of rank two. These quasi-morphisms are not a bounded distance from a homomorphism. This implies that these quasi-morphisms are unbounded considering the unbounded distance from the trivial homomorphism. The extension of the Brooks construction to acylindrically hyperbolic groups is the next theorem.

Theorem 4.2.10 (see [BBF19, Theorem 2.2]). *Let G be an acylindrically hyperbolic group. Then, there exists a free subgroup $F_2 \subseteq G$ such that for every Brooks quasi-morphism L_2 there is a quasi-morphism $L : G \to \mathbb{R}$ such that $L_{|F_2} = L_2$. Further, there exists an element w in the subgroup F_2 and a quasi-homomorphism $L' : G \to \mathbb{R}$ with $L'(w^n) \geq n$ for all $n \in \mathbb{N}$.*

Corollary 4.2.11. *For every acylindrically hyperbolic group G, there exists an unbounded homogeneous quasi-morphism $\varphi : G \to \mathbb{R}$.*

Proof. We homogenize the quasi-morphism L' from the theorem by defining

$$\varphi(x) := \lim_{n\to\infty} \frac{L'(x^n)}{n}.$$

The resulting quasi-morphism φ is anti-symmetric and homogeneous. It is not trivial and unbounded since it has a bounded distance from L' (see [Cal09, Lemma 2.21]). □

We are ready to state the alternative proof.

Proof of Theorem 4.1.3. The finite union of the conjugacy classes of generators in S in a Coxeter system (W, S) is the set of reflections $\bigcup_{s\in S}[s] = R$. Every homogeneous quasi-morphism $\varphi : W \to \mathbb{R}$ is constant on conjugacy classes (see Remark 4.1.2). Since S is finite, there exists a constant C such that $\varphi(r) < C$ for all $r \in R$. Hence, for all $w \in W$ we have

$$\varphi(w) < l_R(w) \cdot (D(\varphi) + C) - D(\varphi)$$

by applying the quasi-morphism property. This implies that if an unbounded homogeneous quasi-morphism exists, the reflection length function l_R is unbounded on W, too. So it remains to show that on every minimal infinite non-affine Coxeter group there exists an unbounded homogeneous quasi-morphism.

According to Theorem 4.2.9, minimal infinite non-affine Coxeter group are acylindrically hyperbolic. Every acylindrically hyperbolic group admits an unbounded homogeneous quasi-morphism $\varphi : G \to \mathbb{R}$ (see Corollary 4.2.11). With the first paragraph, the alternative proof is complete. □

Many questions arise directly from the general assertion of Theorem 4.1.3 with non-constructive proofs. Which ones are the elements with large reflection lengths in infinite non-affine Coxeter groups? Is there a formula for the reflection length in infinite non-affine Coxeter groups? The remaining part of this work is dedicated to the investigation of the reflection length in infinite non-affine Coxeter groups and partial answers to these questions.

5. Reflection length in universal Coxeter groups

The braid relations are decisive for the factorisation of an element into reflections. This can be seen by looking at the solution to the word problem for Coxeter groups. The simplest starting point for the investigation of the reflection length in Coxeter groups is therefore to consider universal Coxeter groups. These have a minimal number of relations among all Coxeter groups. In [MP11], McCammond and Petersen mention that the n-th power of a Coxeter element in the universal Coxeter group of rank 3 has reflection length $n+2$ without providing a proof. We generalise this result to arbitrary rank universal Coxeter groups. Results about the relation between word length and reflection length as well as an upper bound for the reflection length in Coxeter groups in general follow in Section 5.2 and Section 5.3.

Some of the results presented in this chapter also appear in the article [DP21] by Brian Drake and Evan Peters. The results were obtained independently by the author. For a detailed breakdown of this, see Authorship Comment 5.3.4.

5.1 Formula for powers of Coxeter elements

Recall that the universal Coxeter group W_n of rank n is generated by n involutions in $S = \{s_1, \dots, s_n\}$ and there is no braid relation between two distinct generators (see Example 2.1.10). Every element in W_n is represented by a unique reduced word in S^*. This is a direct consequence of the solution to the word problem in Coxeter groups (see Theorem 3.2.13). In this section, a formula for the reflection length of the elements of the form

$$(s_1 \cdots s_n)^\lambda s_1 \cdots s_i$$

with $1 \leq i \leq n$ and $\lambda \in \mathbb{N}_0$ is proved. The formula is a simple combination of the variables n, λ and i.

Since the S-reduced words representing an element are unique for every element in a universal Coxeter group, it is practical to count generators in words and relate these numbers to the reflection length. Further, $n = 3$ is the smallest rank such that W_n is an infinite non-affine Coxeter group. The reflection length function of the universal Coxeter group W_n is abbreviated with l_{R_n}.

Lemma 5.1.1. *For $w \in W_3$ with $l_{R_n}(w) = m \geq 3$, an expression $\mathbf{e} \in S^*$ representing w contains all three different types of generators and these three types each occur in $\mathbf{e}$ at least $m-2$ times.*

Proof. We prove the lemma by induction. For $m = 3$ the assumption is true. Consider an element $w \in W_3$ with $l_{R_n}(w) = m + 1$. We assume $m \geq 3$. So the S-reduced expression for w contains all three different types of generators s_1, s_2 and s_3. Otherwise, the reflection length of w would be maximally 2. If we omit an arbitrary letter in an expression of w, we obtain a word with reflection length m or $m + 2$ (see Lemma 3.3.13). With Theorem 3.3.20 and the induction hypothesis, this word has to contain at least $m - 2$ generators of every kind. Since the omitted generator is arbitrary, we obtain that there are at least $m + 1 - 2$ generators of every kind in an expression for w. □

Remark 5.1.2. This lower bound for the number of generators in the reduced expression is sharp according to the next theorem.

Theorem 1 (Formula for powers of Coxeter elements)**.** *In a universal Coxeter group W_n of rank $n \geq 2$, the following formula holds*

$$l_{R_n}((s_1 \cdots s_n)^\lambda s_1 \cdots s_i) = \lambda \cdot (n-2) + i,$$

for $\lambda \in \mathbb{N}_0$ and $1 \leq i \leq n$.

Proof. Dyer's Theorem 3.3.20 gives us the correct upper bound for every λ and n. The word $(s_1 \cdots s_n)^m$ is reduced. Removing all generators except s_1 and s_2 yields to the word $(s_1 s_2)^{\lambda+1}$ or $(s_1 s_2)^\lambda s_1$ since we are assuming $1 \leq i \leq n$. In this procedure, in total $\lambda \cdot (n-2)$ generators are removed if $i \leq 2$. Otherwise, in total $\lambda \cdot (n-2) + i - 2$ generators are removed. According to Theorem 3.3.20, we have

$$l_{R_n}((s_1 \cdots s_n)^\lambda s_1 \cdots s_i) \leq \lambda \cdot (n-2) + i.$$

For an arbitrary n, we prove the formula by induction over λ. It is $l_{R_n}(s_1 \cdots s_i) = i$ for $i \leq n$ with Theorem 3.3.20 because each generator appears once and no nil-moves are possible. Hence, we assume the formula

$$l_{R_n}((s_1 \cdots s_n)^\lambda s_1 \cdots s_i) = \lambda \cdot (n-2) + i$$

holds for all natural numbers smaller or equal to λ.

Firstly, we show for $\lambda + 1$ by induction over i:

$$l_{R_n}((s_1 \cdots s_n)^{\lambda+1} \cdot s_1 \cdots s_i) = l_{R_n}((s_1 \cdots s_n)^{\lambda+1}) + i - 2 \quad \text{for } 1 \leq i \leq n.$$

According to Lemma 3.3.13, we have $l_{R_n}((s_1 \cdots s_n)^{\lambda+1} \cdot s_1) = l_{R_n}((s_1 \cdots s_n)^{\lambda+1}) \pm 1$. Removing all generators except s_1 and s_2 means removing $(\lambda+1) \cdot (n-2)$ generators and leaves the element $(s_1 s_2)^{\lambda+1} \cdot s_1$, which is a reflection and therefore has reflection length 1. The induction hypothesis gives us

$$l_{R_n}((s_1 \cdots s_n)^{\lambda+1}) = \lambda \cdot (n-2) + n = (\lambda+1) \cdot (n-2) + 2.$$

Again, with Theorem 3.3.20, we obtain $l_{R_n}((s_1 \cdots s_n)^{\lambda+1} \cdot s_1) = l_{R_n}((s_1 \cdots s_n)^{\lambda+1}) - 1$.

For $i \mapsto (i+1)$, we know from Lemma 3.3.13 that multiplying with a reflection increases or decreases the reflection length by one. We assume the latter now, which means

$$l_{R_n}((s_1 \cdots s_n)^{\lambda+1} s_1 \cdots s_{i+1}) = l_{R_n}((s_1 \cdots s_n)^{\lambda+1} s_1 \cdots s_i) - 1 \tag{5.1.1}$$

what is equivalent to

$$l_{R_n}((s_1\cdots s_n)^{\lambda+1}s_1\cdots s_{i+1}\cdot s_{i+1}) = l_{R_n}((s_1\cdots s_n)^{\lambda+1}s_1\cdots s_{i+1})+1.$$

Accordingly, there exists an R-reduced expression $r_1\cdots r_p\cdot s_{i+1} = (s_1\cdots s_n)^{\lambda+1}s_1\cdots s_i$ with $r_i\in R$. By the word property (see Theorem 3.2.13), equivalent words can be transformed into each other by nil-moves and braid-moves. Since there are no braid relations in universal Coxeter groups, we obtain $r_1\cdots r_p s_{i+1}$ by inserting subwords of the form ss into the reduced word $(s_1\cdots s_n)^{\lambda+1}\ s_1\cdots s_i$ (see Theorem 3.2.4). We may assume that all r_i are S-reduced for themselves. This means that the first and the last letter of an inserted subword ss have to belong to different reflections in the factorisation $r_1\cdots r_p\cdot s_{i+1}$.

What follows is that one inverse nil-move has to be inserting $s_{i+1}s_{i+1}$ at the end, where the second s_{i+1} is the last reflection in $r_1\cdots r_p s_{i+1}$. The first s_{i+1} is not a reflection in the R-reduced expression. This would contradict the fact that $r_1\cdots r_p s_{i+1}$ is R-reduced. Further, there exists no S-reduced word $s_{i+1}ws_{i+1}$ with $w\in W_n$ for r_1, since this would mean an insertion of $s_{i+1}s_{i+1}$ at the beginning of $(s_1\cdots s_n)^{\lambda+1}s_1\cdots s_i$ and therefore also a contradiction to $r_1\cdots r_p\cdot s_{i+1}$ being R-reduced. Hence, the last j reflections $r_j, r_{j+1},\ldots,r_p$ in the R-reduced expression are conjugated with s_{i+1} where $1<j\le p$. It follows

$$(s_1\cdots s_n)^a s_1\cdots s_i = r_1\cdots r_{j-1} \quad \text{and} \quad s_{i+1}\cdots s_n(s_1\cdots s_n)^b s_1\cdots s_{i+1} = r_j\cdots r_p$$

with $1\le a,b\in\mathbb{N}$ and $b=\lambda+1-a-1$. In total, we obtain

$$\begin{aligned}
&l_{R_n}((s_1\cdots s_n)^{\lambda+1}s_1\cdots s_i) = l_{R_n}((s_1\cdots s_n)^{\lambda+1}s_1\cdots s_{i+1}s_{i+1})\\
&= l_{R_n}((s_1\cdots s_n)^{\lambda+1}s_1\cdots s_{i+1})+1\\
&= l_{R_n}((s_1\cdots s_n)^a s_1\cdots s_i) + l_{R_n}(s_{i+1}\cdots s_n(s_1\cdots s_n)^b s_1\cdots s_{i+1})+1.
\end{aligned}$$

From the induction hypotheses, we know that $l_{R_n}((s_1\cdots s_n)^a s_1\cdots s_i) = a\cdot(n-2)+i$ since $a<\lambda+1$. Before we can apply the induction assumption to the other term, we have to modify it slightly. Since reflection length is invariant under conjugation (see Lemma 3.3.10), we have

$$\begin{aligned}
l_{R_n}(s_{i+1}\cdots s_n(s_1\cdots s_n)^b s_1\cdots s_{i+1}) &= l_{R_n}(s_{i+2}\cdots s_n(s_1\cdots s_n)^b s_1\cdots s_i)\\
&= l_{R_n}(s_{i+1}s_i\cdots s_1(s_1\cdots s_n)^{b+1}s_1\cdots s_i).
\end{aligned}$$

The second equality is obtained by inserting the word $s_{i+1}s_i\cdots s_1 s_1\cdots s_{i+1}$. It is $b\le\lambda$ and the induction assumption gives us

$$l_{R_n}(s_i\cdots s_1(s_1\cdots s_n)^{b+1}s_1\cdots s_i) = (b+1)\cdot(n-2)+2.$$

Lemma 3.3.13 yields to $l_{R_n}(s_{i+1}s_i\cdots s_1(s_1\cdots s_n)^{b+1}s_1\cdots s_i) = l_{R_n}((s_1\cdots s_n)^{b+1})\pm 1$. Composing everything, we get

$$\begin{aligned}
&l_{R_n}((s_1\cdots s_n)^{\lambda+1}s_1\cdots s_i) = l_{R_n}(r_1\cdots r_p s_{i+1})\\
&= l_{R_n}((s_1\cdots s_n)^a s_1\cdots s_i) + l_{R_n}(s_{i+1}\cdots s_n(s_1\cdots s_n)^b s_1\cdots s_{i+1})+1\\
&= a\cdot(n-2)+i+(b+1)\cdot(n-2)+2\pm 1+1\\
&= (\lambda+1)\cdot(n-2)+2+i+1\pm 1 > l_{R_n}((s_1\cdots s_n)^{\lambda+1})+i-2.
\end{aligned}$$

This contradicts the induction assumption for i. Consequently, our assumption in Equation 5.1.1 is wrong and the following equation is proven

$$l_{R_n}((s_1 \cdots s_n)^{\lambda+1} s_1 \cdots s_{i+1}) = l_{R_n}((s_1 \cdots s_n)^{\lambda+1} s_1 \cdots s_i) + 1.$$

The induction hypothesis implies for $\lambda + 1$ that

$$l_{R_n}((s_1 \cdots s_n)^{\lambda+1} \cdot s_1 \cdots s_i) = l_{R_n}((s_1 \cdots s_n)^{\lambda+1}) + i - 2 \qquad \text{for } 1 \leq i \leq n.$$

□

Corollary 5.1.3. *In a universal Coxeter group W_n of rank $n > 1$, the following formula for the reflection length holds*

$$l_{R_n}((s_1 \cdots s_n)^{\lambda}) = \lambda \cdot (n-2) + 2$$

for natural numbers $\lambda \in \mathbb{N}$.

Remark 5.1.4. All parabolic subgroups of a universal Coxeter group are universal Coxeter groups. Additionally, permuting the generators does not change the formula, because this induces automorphisms of the universal Coxeter group and the group of automorphisms of W_n preserves reflection length for all $n \geq 1$ (see Corollary 3.3.17). For the same reason, the formula holds for all elements in the same conjugacy class as the powers of a Coxeter element. Exploiting automorphisms together with the properties of the reflection length function possibly leads to formulas for further elements.

Remark 5.1.5. The formula in Theorem 1 is easily computable independent of the input size. Contrary, the formula proved in [Bre+19] reduces the computation of the reflection length partially to the computation of the nullity of a vector. This can be reduced to an NP-complete problem (see [Bre+19, Appendix A]).

Remark 5.1.6. In an arbitrary infinite non-affine Coxeter group, Theorem 1 does not hold because of the braid-relations. Moreover, the sequence of powers of a Coxeter element possibly has a bounded reflection length, even though the reflection length function is generally unbounded by Theorem 4.1.3! To see this, it is sufficient to look at a Coxeter system of rank 3, in which two generators commute like the following lemma shows.

Lemma 5.1.7. *Let $(W, S = \{s_1, s_2, s_3\})$ be a Coxeter system with two distinct commuting generators. For $\lambda \in \mathbb{N}$, the powers of the Coxeter element $(s_1 s_2 s_3)^{\lambda} \in W$ have the following reflection length:*

$$l_R((s_1 s_2 s_3)^{\lambda}) = \begin{cases} 2 & \text{for even } \lambda \\ 3 \text{ or } 1 & \text{for odd } \lambda. \end{cases}$$

Proof. Since two generators commute, there are distinct $1 \leq i, j \leq 3$ with $m_{ij} = 2$. Reflection length is invariant under conjugation (see Lemma 3.3.10). Exchanging the two generators that commute in $(s_1 s_2 s_3)^{\lambda}$ does not change the element, since they are adjacent and commute. This is why, we may assume that $m_{13} = 2$ without loss of generality.

Take the following subdivision

$$(s_1 s_2 s_3)^{\lambda} = (s_1 s_2 s_3)^{m} \cdot (s_1 s_2 s_3)^{\lambda - m},$$

where $m = \frac{\lambda}{2} - 1$ if λ is even and $m = \frac{\lambda-1}{2}$ if λ is odd. Next, we apply the braid-move $s_3 s_1 \mapsto s_1 s_3$ to all consecutive subwords $s_3 s_1$ in the word $(s_1 s_2 s_3)^{\lambda - m}$. Braid-moves do not change the element represented by the word. When λ is even, we obtain the reflection factorisation

$$(s_1s_2s_3)^m \cdot s_1s_2s_1 \cdot (s_3s_2s_1)^{\lambda-m-2} \cdot s_3s_2s_3. \tag{5.1.2}$$

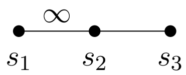
In this case, it is $\lambda - m - 2 = m$. When λ is odd, we obtain the factorisation

$$(s_1s_2s_3)^{m-1} \cdot s_1s_2s_3s_1s_2s_1 \cdot (s_3s_2s_1)^{\lambda-m-2} \cdot s_3s_2s_3. \tag{5.1.3}$$

In the latter case, it is $\lambda - m - 2 = m - 1$.

From both factorisations, we can read off an upper bound for the reflection length of $(s_1s_2s_3)^\lambda$. The first factorisation (5.1.2) is a reflection factorisation and the reflection length has the same parity as the word length (see Corollary 3.3.12). The word does not represent the identity, because it is reduced (see Theorem 3.2.4). Hence, it is $l_R((s_1s_2s_3)^\lambda) = 2$ for even λ. The second factorisation (5.1.3) is not a reflection factorisation. Theorem 3.3.20 gives us $l_R((s_1s_2s_3)^\lambda) \leq 3$ by removing the two letters in the middle of the word $s_1s_2s_3s_1s_2s_1$. $\square$

Example 5.1.8. The projective linear group $\mathrm{PGL}_2(\mathbb{Z})$ over the integers is a Coxeter group defined by the Coxeter graph



.

Consider the element represented by the periodic word $(s_1s_2s_3)^\lambda$ of all generators. We have $l_R((s_1s_2s_3)^\lambda)$ is equal to 2 if λ is even, and smaller or equal to 3 if λ is odd.

5.2 Minimal word length and minimal reflection length

In this section, the function $m_S^n : \mathbb{N} \to \mathbb{N}$ with

$$m_S^n(m) := \min\{l_S(w) \mid l_{R_n}(w) = m, w \in W_n\}$$

is investigated. We know that the reflection length increases or decreases by 1 when an element is multiplied by a generator (see Lemma 3.3.13). Hence, m_S^n grows strictly monotonous. The formula from Theorem 1 connects the word length and reflection length in the following way

$$\begin{aligned} l_{R_n}((s_1 \cdots s_n)^\lambda s_1 \cdots s_i) &= \lambda \cdot (n-2) + i = \lambda n + i - 2\lambda \\ &= l_S((s_1 \cdots s_n)^\lambda s_1 \cdots s_i) - 2\lambda. \end{aligned} \tag{5.2.1}$$

The symmetric group $\mathrm{Sym}(n)$ acts on W_n by permuting the generators in expressions for the elements in W_n. The action is well defined since for all distinct $i, j \in \{1, \ldots, |S|\}$ we have $m_{ij} = \infty$. The action preserves the reflection length. This follows from Corollary 3.3.17. The following lemma states that elements of the form $(s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ have minimal word length with regard to their reflection length.

Lemma 5.2.1. *Let W_n be the universal Coxeter group of rank n and $m \in \mathbb{N}$. There exists an element $w \in W_n$ of the form $w = (s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ with $2 < i \leq n$ such that $l_{R_n}(w) = m$ and*

$$m_S^n(m) = l_S(w) = m + 2\lambda = \lambda n + i.$$

Proof. For reflection length $m \leq n$, words of the form $s_1 \cdots s_m$ are the words of minimal S-length with reflection length m. Words of this form and other words of the same word length in a common orbit of the action of $\mathrm{Sym}(n)$ are the only ones with this property (this follows from Theorem 3.3.20).

For a fixed reflection length m', assume that there exists an element $\tilde{w}$ that has smaller S-length than the word of the form $w = (s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ with $2 < i \leq n$ and reflection length m'. According to Corollary 3.3.12, S-length and reflection length have the same parity. So $l_S(\tilde{w}) \leq l_S(w) - 2$. With Theorem 3.3.20, we find an upper bound for the reflection length of $\tilde{w}$. This upper bound cannot be below m' according to our assumption. In the S-reduced expression for $\tilde{w}$ remove all generators except the two most occurring ones. This way, we remove maximally $(n-2)$ different types of generators. Let $n_{\tilde{w}}(s_i)$ be the number of occurrences of the generator s_i in the reduced expression of $\tilde{w}$. Without loss of generality, we may assume that s_1 and s_2 are the generators that occur the most in the reduced expression for $\tilde{w}$. Thus, we have

$$\sum_{i=3}^{n} n(s_i) \geq m' - 2.$$

So the number of all letters in the reduced expression of $\tilde{w}$ that are not the two most occurring generators is at least the same as in the reduced expression for w. For w, this number is $m' - 2$ since $2 < i \leq n$. The generator s_3 occurs as many times as the two most occurring generators s_1 and s_2 in the reduced expression for w. It is $(\lambda + 1)$ times. The occurrence of generators is as equally divided as possible among the generators in the reduced expression of w. Hence, there exists a generator s_i with $2 < i \leq n$ that occurs at least $(\lambda+1)$ times in the reduced expression for $\tilde{w}$, too. This is a contradiction to $l_S(\tilde{w}) \leq l_S(w) - 2$. Consequently, there exists no word with reflection length m' that has a smaller word length than a word of the form $(s_1 \cdots s_n)^\lambda s_1 \cdots s_i$. ∎

Remark 5.2.2. In the situation of the lemma, let w' be an element with $l_{R_n}(w') = m$ and $m_S^n(m) = l_S(w')$. The action of $\mathrm{Sym}(n)$ on W_n and conjugation do not change the reflection length. Nevertheless, in general there does not always exist $\sigma \in \mathrm{Sym}(n)$ and $y \in W_n$ such that $y\sigma(w')y^{-1} = w$. For $n = 5$, we have for example

$$l_{R_n}(s_1 \cdots s_5 s_1 s_2 s_4) = 6 = l_{R_n}(s_1 \cdots s_5 s_1 s_2 s_3).$$

This is true since there is just one letter of type s_5 occurring in the S-reduced factorisation $s_1 \cdots s_5 s_1 s_2 s_4$. So applying Theorem 3.3.20 leads necessarily to removing this letter. It is

$$l_{R_n}(s_1 \cdots \hat{s}_5 s_1 s_2 s_4) = l_{R_n}(s_4 s_1 \cdots s_3 s_4 s_1 s_2)$$

since conjugacy preserves reflection length. Theorem 1 is applicable and we obtain

$$l_{R_n}(s_1 \cdots s_5 s_1 s_2 s_4) = 1 + l_{R_n}(s_4 s_1 \cdots s_3 s_4 s_1 s_2) = 1 + 5.$$

This is also the reflection length of $s_1 \cdots s_5 s_1 s_2 s_3$ (see Theorem 1). Note that $s_1 \cdots s_5 s_1 s_2 s_3$ and $s_1 \cdots s_5 s_1 s_2 s_4$ are not conjugated to each other in W_5.

One consequence of the lemma is an explicit description of the function m_S^n by using the Equation 5.2.1 obtained from Theorem 1 and solving it for l_S. The reflection length grows strictly monotonously if we extend words of the form $(s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ generator by generator and leave out $i = 1, 2$. Thus, the set $\{j \in \mathbb{N} \mid 2 < j \leq n, (n-2) \text{ divides } (m-j)\}$ is always a singleton.

Corollary 5.2.3. *Define $\kappa(n,m)$ to be the unique number $j \in \mathbb{N}$ with $2 < j \leq n$ such that $(n-2)$ divides $(m-j)$. For the function $m_S^n : \mathbb{N} \to \mathbb{N}$, we have*

$$m_S^n(m) = m + 2 \cdot \frac{m - \kappa(n,m)}{n-2}.$$

The reverse question of what the maximum reflection length is for a fixed word length is also relevant to us. This is especially important with regard to deriving an upper bound for the reflection length in general Coxeter groups from the formula for reflection length in universal Coxeter groups. Define the map

$$m_R^n : \mathbb{N} \to \mathbb{N} \qquad k \mapsto \max\{l_{R_n}(w) \mid w \in W_n, l_S(w) = k\}.$$

Lemma 5.2.1 implies an explicit description for the statistic m_R^n, too.

Lemma 5.2.4. *Define $\iota(n,k)$ to be the unique number $j \in \mathbb{N}$ with $1 \leq j \leq n$ such that n divides $(k-j)$. For the function $m_R^n : \mathbb{N} \to \mathbb{N}$ with $n \geq 3$, we have*

$$m_R^n(k) = k - 2 \cdot \frac{k - \iota(n,k)}{n}.$$

Proof. Let $\tilde{w} \in W_n$ be an element with $m_R^n(l_S(\tilde{w})) = l_{R_n}(\tilde{w})$. According to Lemma 5.2.1, there exists an element $(s_1 \cdots s_n)^\lambda s_1 \cdots s_i \in W_n$ with $l_{R_n}(\tilde{w}) = l_{R_n}((s_1 \cdots s_n)^\lambda s_1 \cdots s_i)$ and $l_S(\tilde{w}) \geq l_S((s_1 \cdots s_n)^\lambda s_1 \cdots s_i)$ for $\lambda \in \mathbb{N}_0$ and $2 < i \leq n$.

If $l_S(\tilde{w}) = l_S((s_1 \cdots s_n)^\lambda s_1 \cdots s_i$, the explicit description can be obtained from solving Equation 5.2.1 for l_{R_n}. In case $l_S(\tilde{w}) > l_S((s_1 \cdots s_n)^\lambda s_1 \cdots s_i)$, both word lengths have the same parity (see Theorem 3.2.13). Extend the factorisation $(s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ generator by generator to a word of the same form with word length equal to $l_S(\tilde{w})$. We assume $m_R^n(l_S(\tilde{w})) = l_{R_n}(\tilde{w})$. The only two scenarios, where the extension has a reflection length equal to $l_{R_n}(\tilde{w})$ and not larger, are $i = n-1$ and $i = n$. This holds, because the reflection length of the elements of the form $(s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ grows strictly monotonously in dependency on the word length leaving out $i = 1, 2$ (see Theorem 1). Consequently, we have $l_S(\tilde{w}) = l_S((s_1 \cdots s_n)^{\lambda+1} s_1)$ or $l_S(\tilde{w}) = l_S((s_1 \cdots s_n)^{\lambda+1} s_1 s_2)$. So in this case, there also exists an element $w \in W_n$ of the form $(s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ with $m_R^n(l_S(w)) = l_{R_n}(w)$ and the assertion follows by solving Equation 5.2.1 for l_{R_n}. □

The proof contains the result analogous to Lemma 5.2.1 for m_R^n.

Corollary 5.2.5. *Let W_n be the universal Coxeter group of rank n and $k \in \mathbb{N}$. There exists an element $w \in W_n$ of the form $w = (s_1 \cdots s_n)^\lambda s_1 \cdots s_i$ with $1 \leq i \leq n$ such that $l_s(w) = k$ and*

$$m_R^n(k) = l_{R_n}(w) = \lambda \cdot (n-2) + i.$$

5.3 Upper bound for reflection length in general infinite non-affine Coxeter groups

In general, braid relations shrink the reflection length of an element. By comparing the reflection lengths of the elements represented by the same word in an arbitrary and a universal Coxeter group of the same rank, we obtain upper bounds for the reflection length function on the arbitrary Coxeter group.

In finite and affine Coxeter groups, upper bounds for the reflection length function are well known and constants (see Remark 3.4.5 and [MP11]). On the contrary, an upper bound for infinite non-affine Coxeter group cannot be constant because of Theorem 4.1.3.

Notation 5.3.1. Given a generating set S with n elements, we write ω_n for the canonic surjection $S^* \to W_n$.

Take an S-reduced word $\mathbf{s}$ in an arbitrary Coxeter group W of rank n corresponding to the element $\omega(\mathbf{s})$. Every reflection factorisation of $\omega_n(\mathbf{s})$ in W_n is also a reflection factorisation of the element $\omega(\mathbf{s})$ in W.

Lemma 5.3.2. *Let (W, S) be an arbitrary Coxeter system of rank n and let R be the set of reflections in W. For every element $v \in W$ represented by an S-reduced word $\mathbf{s} \in S^*$, the reflection length is bounded by*

$$l_R(v) \leq l_{R_n}(\omega_n(\mathbf{s})).$$

Proof. Let $\omega_n(\mathbf{r}_1 \cdots \mathbf{r}_l) = \omega_n(\mathbf{s})$ be a R_n-reduced reflection factorisation in W_n. The absence of braid relations in universal Coxeter groups implies that the word $\mathbf{r}_1 \cdots \mathbf{r}_l$ can be transformed to $\mathbf{s}$ with a sequence of nil-moves and that the words $\mathbf{r}_i \in S^*$ are palindromes (see Theorem 3.2.13). So we have $\omega(\mathbf{r}_i) \in R$ for all $\mathbf{r}_i$ and $v = \omega(\mathbf{r}_1) \cdots \omega(\mathbf{r}_l)$. □

So Lemma 5.2.4 implies a sharp upper bound for the reflection length function on arbitrary infinite non-affine Coxeter groups. For a Coxeter system (W, S) and $w \in W$, let $W(w)$ be the smallest standard parabolic subgroup containing w. With this notation, we obtain a function of the rank and the word length that is an upper bound for the reflection length. Recall that $\iota(n, k)$ is the unique number $j \in \mathbb{N}$ with $1 \leq j \leq n$ such that n divides $(k - j)$.

Lemma 5.3.3. *Let (W, S) be an arbitrary Coxeter system and $w \in W$. Let n be the rank of $W(w)$. The reflection length of w is bounded from above by*

$$l_R(w) \leq m_R^n(l_S(w)) = l_S(w) - 2 \cdot \frac{l_S(w) - \iota(n, l_S(w))}{n}.$$

Proof. The lemma follows directly from Lemma 5.2.4 applied to the standard parabolic subgroup $W(w)$. Take a reduced word $\mathbf{s}$ representing w in W. Every reflection factorisation of the element $\omega_n(\mathbf{s})$ in W_n is a reflection factorisation for w since all relations from W_n also appear in $W(w)$. The reflection length function on W restricted to $W(w)$ is equal to the reflection length function on $W(w)$ (see Corollary 3.3.23). □

This upper bound is sharp. This is to be seen with Corollary 5.2.5.

Authorship Comment 5.3.4. The results in this chapter were obtained by the author independently and autonomously. A part of them also got published afterwards by Brian Drake and Evan Peters in [DP21]. There exists no entry on the distribution service `arxiv.org` for the article [DP21]. The author came across this publication before he was able to publish the results himself. The following findings also appear in [DP21]: Theorem 1 as Lemma 7, Corollary 5.1.3 as Lemma 6 and Lemma 5.3.3 as Theorem 1. The proofs of Theorem 1 and of Lemma 5.3.3 presented here are different from the ones of Drake and Peters. First, they prove Lemma 5.3.3 with a generalization of the pigeon-hole principle. Afterwards, they prove Theorem 1 with the help of this upper bound. Whereas, we prove the formula first to deduce the upper bound.

6. Reflection length at infinity in hyperbolic reflection groups

We exclusively consider infinite non-affine hyperbolic reflection groups in this chapter. We mean infinite non-affine hyperbolic reflection group when we write hyperbolic reflection group. The action of such a rank $n+1$ Coxeter group induces a tessellation of $\mathbb{H}^n$. After fixing a fundamental domain, there exists a bijection between the tiles and the group elements (see Theorem 2.2.16). We describe certain points in the visual boundary of the n-dimensional hyperbolic space for which every neighbourhood contains tiles of every reflection length (see Theorem 2). Additionally, we show that two disjoint hyperplanes in the n-dimensional hyperbolic space without common boundary points have a unique common perpendicular (see Theorem 3). In the first section, we discuss a compactification of $\mathbb{H}^n$ and ultra-parallel geodesic subspaces. Theorem 3 is proved in Section 6.2 and Theorem 2 is proved in Section 6.3. The next section contains results about groups generated by two parallel hyperplanes in $\mathbb{H}^n$. The last section discusses Theorem 2 in the case where the fundamental domain of the action of a hyperbolic reflection group is a polytope. Most of this chapter appeared as the article [Lot24b] by the author.

Let (W, S) be a hyperbolic reflection group with fundamental polyhedron P in $\mathbb{H}^n$, walls $\{H_1, \ldots, H_m\}$ and generating set $S = \{s_1, \ldots, s_m\}$. For $w \in W$ and $1 \leq i \leq m$, the reflection $ws_iw^{-1} \in R$ acts on $\mathbb{H}^n$ as the hyperplane reflection across wH_i. For each polyhedron vP with $v \in W$, the minimal number of hyperplane reflections across hyperplanes in $\{wH_i \subseteq \mathbb{H}^n \mid w \in W, 1 \leq i \leq m\}$ that suffices to reflect vP onto P is exactly the reflection length of v. This follows from our studies in Chapter 2 and Chapter 3. For a reflection $r \in R$, we denote the corresponding hyperplane in $\mathbb{H}^n$ with H_r.

6.1 Geodesic subspaces and their boundary

To investigate the behaviour of the reflection length far away from the fundamental polyhedron, we introduce a compactification of the hyperbolic space $\mathbb{H}^n$ and a topology on the latter.

We remind the reader that the *visual boundary* $\partial\mathbb{X}$ of a CAT(0) metric space $\mathbb{X}$ is the set of equivalence classes of geodesic rays. Fix an origin x_0 in $\mathbb{X}$. Two geodesic rays in $\mathbb{X}$ $\gamma_1, \gamma_2 : [0, \infty) \to \mathbb{X}$ originating from x_0 are said to be equivalent if there exists a constant $C > 0$ such that $d(\gamma_1(a), \gamma_2(a)) < C$ for all $a \in [0, \infty)$. The resulting boundary $\partial\mathbb{X}$ is independent of the choice of an origin. See Definition 4.2.2 for further details.

Example 6.1.1. In the Poincaré ball model, the visual boundary $\partial\mathbb{H}^n$ is exactly the unit sphere $\mathbb{S}^{n-1}$. In the hyperboloid model, if two geodesic rays are asymptotic, then the intersection of the corresponding subspaces in $\mathbb{E}^{n,1}$ is a line contained in the boundary of the light cone $\{v = (v_1, \dots, v_{n+1}) \in \mathbb{E}^{n,1} \mid \langle v|v\rangle_{-1} = 0, v_{n+1} > 0\}$ (see [BH99, pp. 262-263]).

We equip $\overline{\mathbb{X}} = \mathbb{X} \cup \partial\mathbb{X}$ with the cone topology.

Definition 6.1.2. Fix a point x_0 in a CAT(0) metric space $(\mathbb{X}, d)$. Let c be a geodesic ray with $c(0) = x_0$ and let $r, \epsilon > 0$ be real numbers. Let p_r be the projection of $\overline{\mathbb{X}}$ onto the closed ball $\overline{B}(c(0), r)$: For $x \notin B(c(0), r)$, the projection $p_r(x)$ is the point in the segment $[x_0, x]$, geodesic ray respectively, with distance r from x_0. We define

$$U(c, r, \epsilon) := \{x \in \overline{\mathbb{X}} \mid d(x, c(0)) > r,\ d(p_r(x), c(r)) < \epsilon\}.$$

The sets $U(c, r, \epsilon)$ form a neighbourhood basis for the ideal point $c(\infty) \in \partial\mathbb{X}$ and the set of all open balls $B(x, r)$ together with all sets of the form $U(c, r, \epsilon)$, where c is a geodesic ray with $c(0) = x_0$, is a basis of the *cone topology* on $\overline{\mathbb{X}}$. The cone topology is independent of the choice of the point x_0.

Now, we turn to geodesic subspaces of $\mathbb{H}^n$ of arbitrary dimension in the hyperboloid model. Geodesic lines in $\mathbb{H}^n$ are exactly the non-empty intersections of 2-dimensional vector subspaces of $\mathbb{E}^{n,1}$ with $\mathbb{H}^n$. The m-dimensional geodesic subspaces of $\mathbb{H}^n$ are the non-empty intersections of $(m+1)$-dimensional vector subspaces of $\mathbb{E}^{n,1}$ with $\mathbb{H}^n$ $(m \leq n)$. These subspaces are isometric to $\mathbb{H}^m$. For a subspace $H \subseteq \mathbb{H}^n$, we denote the corresponding subspace in $\mathbb{E}^{n,1}$ with V_H.

Definition 6.1.3. Two disjoint arbitrary dimensional geodesic subspaces H_1 and H_2 of $\mathbb{H}^n$ are called *ultra-parallel* if there exist no geodesic rays $\gamma_1 \subseteq H_1$ and $\gamma_2 \subseteq H_2$ that are asymptotic. In other words, the subspaces do not have a common point in $\partial\mathbb{H}^n$.

Definition 6.1.4. Two intersecting geodesic subspaces $H_1, H_2 \subseteq \mathbb{H}^n$ are *intersecting at a right angle* if their intersection is non-empty in $\mathbb{H}^n$, and there exist non-trivial vectors $u_1, \dots, u_m$ with $m > 0$ in the orthogonal complement $V_{H_1}^{\perp} \subseteq \mathbb{E}^{n,1}$ such that

$$\langle (V_{H_1} \cap V_{H_2}) \cup \{u_1, \dots, u_m\}\rangle = V_{H_2}.$$

A *perpendicular* of a geodesic subspace $H \subseteq \mathbb{H}^n$ is a geodesic line that intersects H at a right angle as a subspace.

Remark 6.1.5. The definition above is symmetric. By definition, the subspace V_{H_1} is contained in $\langle\{u_1, \dots, u_m\}\rangle^{\perp}$ and we have $V_{H_2}^{\perp} = \langle\{u_1, \dots, u_m\}\rangle^{\perp} \cap (V_{H_1} \cap V_{H_2})^{\perp}$. So we can select vectors $\{v_1, \dots, v_n\} \subseteq V_{H_2}^{\perp}$ such that

$$\langle (V_{H_1} \cap V_{H_2}) \cup \{v_1, \dots, v_n\}\rangle = V_{H_1}.$$

Example 6.1.6. Let H_1 be a hyperplane and let H_2 be a geodesic line with $H_1 \cap H_2 \neq \varnothing$ and $H_2 \nsubseteq H_1$. The subspaces $V_{H_1} \cap V_{H_2}$ and $V_{H_1}^{\perp}$ are 1-dimensional. In this case, H_2 is a perpendicular of H_1 if and only if $V_{H_1}^{\perp} \subseteq V_{H_2}$. Two hyperplanes intersect at a right angle if and only if the corresponding orthogonal unit vectors are orthogonal to each other.

Lemma 6.1.7. *Let $H_1, H_2 \subseteq \mathbb{H}^n$ be two geodesic subspaces intersecting at a right angle. For a subspace $A \subseteq H_1$, A and H_2 are intersecting at a right angle if*

$$\varnothing \neq H_2 \cap A \neq A$$

Proof. By definition, there exist non-trivial vectors $u_1, \dots, u_m$ with $m > 0$ in the orthogonal complement $V_{H_2}^{\perp} \subseteq \mathbb{E}^{n,1}$ such that $\langle (V_{H_1} \cap V_{H_2}) \cup \{u_1, \dots, u_m\} \rangle = V_{H_1}$. For the intersection of the corresponding subspaces, it is $V_A \cap V_{H_2} \subseteq V_{H_1} \cap V_{H_2}$ and we obtain

$$\langle (V_A \cap V_{H_2}) \cup (V_A \cap \{u_1, \dots, u_m\}) \rangle = V_A$$

by intersecting with V_A. It is $|V_A \cap \{u_1, \dots, u_m\}| > 0$ because $H_2 \cap A \neq A$. This completes the proof. ☐

6.2 Ultra-parallel theorem for subspaces in $\mathbb{H}^n$

David Hilbert proved the following theorem based on his system of axioms for $\mathbb{H}^2$.

Theorem 6.2.1 (see [Hil13, p. 149])**.** *Any two ultra-parallel geodesic lines in $\mathbb{H}^2$ have a common perpendicular.*

We extend this theorem to ultra-parallel geodesic subspaces in $\mathbb{H}^n$ and prove the equivalence of the existence of a common perpendicular and ultra-parallelism in case both subspaces are hyperplanes. Our proof is different from Hilbert's proof and includes his theorem. Therefore, we need the following geometric lemma, which follows from basic Euclidean geometry.

Lemma 6.2.2. *Let $\mathbb{S}^n$ be the unit sphere embedded in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ and let S_1, S_2 be two spheres of dimension n or lower intersecting $\mathbb{S}^n$ orthogonally.*

(i) If S_1 and S_2 are not intersecting in the closed unit ball $\overline{D}^{n+1}$, the line (C_1, C_2) through the centres C_1 and C_2 of S_1 and S_2 intersects $\mathbb{S}^n$ exactly two times.

(ii) If S_1 and S_2 are both of dimension n and not intersecting in the open unit ball D^{n+1}, $S_1 \cap S_2$ is a point in $\mathbb{S}^n$ or empty.

Proof. S_1 and S_2 intersect $\mathbb{S}^n$ orthogonally. This implies that C_i is the intersection of all tangent spaces of points in $\mathbb{S}^n \cap S_i$ for $i = 1, 2$. On the other hand, S_i is the unique sphere intersecting $\mathbb{S}^n$ orthogonally with centre C_i. Thus, if there is just one point on a line through C_i also contained in $\mathbb{S}^n$, this point is in S_i.

The line (C_1, C_2) intersects $\mathbb{S}^n$ maximally two times. Assume that (C_1, C_2) intersects $\mathbb{S}^n$ exactly once in a point P. According to the first paragraph of this proof, we have $P \in S_1 \cap S_2$ and $P \in \overline{D}^{n+1}$. Assume that (C_1, C_2) does not intersect $\mathbb{S}^n$. In this case, there exists a tangent space T_1 of $\mathbb{S}^n$ containing C_1 separating $\mathbb{S}^n$ and C_2 and vice versa. Accordingly, $T_1 \cap T_2$ is non-empty and so is $S_1 \cap S_2 \cap \overline{D}^{n+1}$. This proves the first assertion.

Assume that the intersection $S_1 \cap S_2 \cap \mathbb{S}^n$ is neither a point in $\mathbb{S}^n$ nor empty. So there are at least two points P_1, P_2 contained in $S_1 \cap S_2 \cap \mathbb{S}^n$. Thus, the intersection $S_1 \cap S_2$ is either S_1 for $S_1 = S_2$ or an $(n-1)$-dimensional sphere for $S_1 \neq S_2$. If $n = 1$, it follows directly $S_1 = S_2$ and $S_1 \cap S_2 \cap D^{n+1} \neq \varnothing$, because S_i intersects $\mathbb{S}^n$ orthogonally. For $n \geq 2$, an $(n-1)$-dimensional sphere intersecting $\mathbb{S}^n$ in two non-antipodal points also intersects D^{n+1} non-trivially. So $S_1 \cap S_2 \cap D^{n+1}$ is non-empty and the second assertion is proven. ☐

In the Poincaré ball model for $\mathbb{H}^n$, the hyperplanes are represented by $(n-1)$-dimensional spheres that intersect $\mathbb{S}^{n-1}$ orthogonally. This is why the following corollary is immanent.

Corollary 6.2.3. *Let H_1 and H_2 be two parallel hyperplanes in $\mathbb{H}^n$. The intersection of ∂H_1 and ∂H_2 is empty or a point in $\partial \mathbb{H}^n$.*

Theorem 3 (Ultra-parallel Theorem for subspaces)**.** *Every pair of ultra-parallel geodesic subspaces in $\mathbb{H}^n$ has a common perpendicular. A pair of distinct hyperplanes in $\mathbb{H}^n$ is ultra-parallel if and only if it has a common unique perpendicular. Every hyperplane intersecting both hyperplanes at a right angle contains this perpendicular.*

Proof. The proof of the existence of a unique common perpendicular consists of two parts. The existence part is shown constructively in the Poincaré ball model and the uniqueness part follows in the upper hyperboloid model.

Let H_a and H_b be two ultra-parallel geodesic subspaces in $\mathbb{H}^n$. In the Poincaré ball model, these are represented by spheres S_a and S_b that intersect $\mathbb{S}^{n-1}$ orthogonally. Since they are ultra-parallel, S_a and S_b do not intersect in the closed unit ball $\overline{D}^n$. Without loss of generality, we can assume that there exists a hyperplane S through the centre M of $\mathbb{S}^{n-1}$ such that we have $S_a \subseteq \mathring{S}^+$ and $S_b \subseteq \mathring{S}^-$ for the open half-spaces. If necessary, apply a translation on the Poincaré ball model. Translations are isometries and preserve Euclidean angles (see [BH99, Sections 6.5 and 6.11]). We proceed with the following construction.

Let M_a, M_a be the centres of the spheres S_a and S_b. Since we assumed that S_a and S_b can be separated by a hyperplane through M, neither M_a nor M_b is ∞. The line (M_a, M_b) in $\mathbb{E}^n$ intersects $\mathbb{S}^{n-1}$ in two points I_a and I_b, because S_a and S_b are ultra-parallel (see Lemma 6.2.2). In addition, it intersects S_a and S_b orthogonally since the line contains the centres of these spheres. If $M \in (M_a, M_b)$, the common perpendicular in $\overline{\mathbb{H}}^n$ is represented by the segment $[I_a, I_b]$.

In case $M \notin (M_a, M_b)$, the triangle $\Delta M I_a I_b$ is an isosceles triangle and the segments $[M, I_a]$ and $[M, I_b]$ are contained in a unique plane $E \subseteq \mathbb{E}^n$. Let X be the middle point of the segment $[I_a, I_b]$. Since the angles $\measuredangle I_b I_a M$ and $\measuredangle M I_b I_a$ are equal, the unique tangent lines to $\mathbb{S}^{n-1}$ in E at I_a and I_b, as well as the line (M, X), intersect all in a point Q. We can construct a circle C_Q with centre Q through I_a and I_b, because the triangles $\Delta I_a X Q$ and $\Delta I_b X Q$ are congruent. The circle C_Q intersects $\mathbb{S}^{n-1}$ orthogonally since its centre is on a tangent line. Thus, it represents a geodesic line in the Poincaré ball model. It remains to show that C_Q intersects S_a and S_b orthogonally. The following equations are implied by the Pythagorean theorem, where r_M, r_a, r_b and r_Q are the radii of the corresponding spheres in $\mathbb{E}^n$:

$$\begin{aligned}
r_M^2 &= d(I_a, X)^2 + d(M, X)^2 && \text{(i)}\\
r_Q^2 &= d(I_a, X)^2 + d(Q, X)^2 && \text{(ii)}\\
d(M, M_a)^2 &= d(M_a, X)^2 + d(M, X)^2 && \text{(iii)}\\
d(M_a, Q)^2 &= d(Q, X)^2 + d(M_a, X)^2 && \text{(iv)}\\
d(M, M_a)^2 &= r_M^2 + r_a^2. && \text{(v)}
\end{aligned}$$

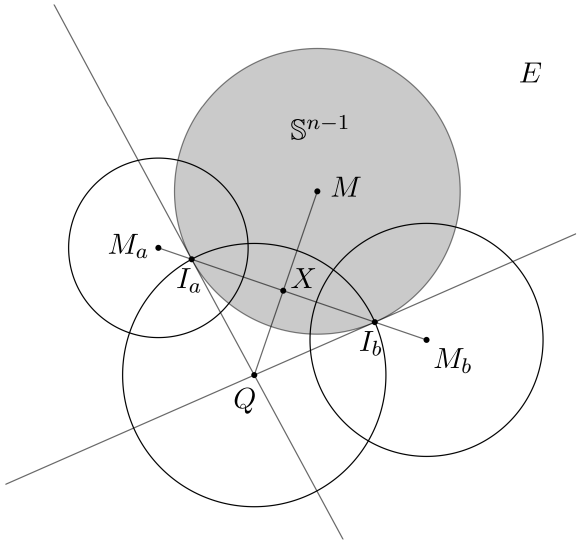


Figure 6.1: Cross section by E

By substituting, we get:

$$\begin{aligned} r_a^2 + r_Q^2 &\overset{\text{(v)}}{=} d(M, M_a)^2 - r_M^2 + r_Q^2 \overset{\text{(iii)}}{=} d(M_a, X)^2 + d(X, M)^2 - r_M^2 + r_Q^2 \\ &\overset{\text{(ii)}}{=} d(M_a, X)^2 + d(M, X)^2 - r_M^2 + d(I_a, X)^2 + d(Q, X)^2 \\ &\overset{\text{(iv)}}{=} d(M_a, Q)^2 + d(M, X)^2 - r_M^2 + d(I_a, X)^2 \\ &\overset{\text{(i)}}{=} d(M_a, Q)^2. \end{aligned}$$

By the Pythagorean theorem, $\measuredangle I_a Q_a Q$ is a right angle, where Q_a is the intersection of C_Q and S_a in D^n. Analogously, we conclude that $\measuredangle I_b Q_b Q$ is a right angle. Hence, the circle C_Q represents a common perpendicular of S_a and S_b and the existence is proven.

To prove the uniqueness, we change to the hyperboloid model. We assume that H_a and H_b are distinct ultra-parallel hyperplanes. According to Definition 6.1.4, a hyperplane H and a geodesic line L intersect at a right angle in $\mathbb{H}^n$ if there exists $v \in V_L \cap V_H$ such that $\langle\{u_H, v\}\rangle = V_L$, where u_H is the unique (modulo sign) unit vector in $V_H^\perp$. From the first part of this proof, we know that H_a and H_b have a common perpendicular. Let u_{H_a} and u_{H_b} be the corresponding unit vectors. The only possibility for a 2-dimensional subspace that represents a common perpendicular is $\langle\{u_{H_a}, u_{H_b}\}\rangle$ since we assume the hyperplanes and their orthogonal complements to be different. It follows that the existing common perpendicular is $\langle\{u_{H_a}, u_{H_b}\}\rangle \cap \mathbb{H}^n$ and unique.

For the implication in the other direction, assume that H_a and H_b are distinct hyperplanes with a unique perpendicular p. Moreover, assume that the hyperplanes are not parallel and let N be a point in $H_a \cap H_b$. If p intersects H_a and H_b in a point in $H_a \cap H_b$, it follows $H_a = H_b$ directly from Definition 6.1.4, because hyperplanes are maximally dimensional subspaces. This is a contradiction to our assumption that H_a and H_b are distinct. If p intersects H_a and H_b in points X_a and X_b not in $H_a \cap H_b$, the lines (X_a, N) and (X_b, N) intersect with p at a right angle (see Lemma 6.1.7). Two lines which are perpendicular to the same line are ultra-parallel to one another (see [Cox42, p. 9.63.]). So this case leads to a contradiction, too. Thus, we may assume that H_a and H_b are parallel or ultra-parallel.

Additionally, the hyperplanes H_a and H_b cannot be parallel. The argument above for two lines which are perpendicular to the same line also is applicable to the lines containing the intersection points X_a and X_b and a common boundary point. So distinct hyperplanes with a unique common perpendicular are ultra-parallel.

Let $\tilde{H}$ be a hyperplane in $\mathbb{H}^n$ intersecting H_a and H_b at a right angle. hence, we have $\langle\{u_{H_i}\} \cup (V_{H_i} \cap V_{\tilde{H}})\rangle = V_{\tilde{H}}$ for $i = a, b$. This implies $u_{H_i} \in V_{\tilde{H}}$ for $i = a, b$. It follows that every hyperplane orthogonal to two ultra-parallel hyperplanes contains their unique common perpendicular. □

Remark 6.2.4. Once the existence is proven, the uniqueness for hyperplanes also follows with the proof of Theorem 3.2.7 and Exercise 3.1.9. in [Rat19]. The next corollary follows from these results, too.

Corollary 6.2.5. *Let H_a and H_b be a pair of ultra-parallel hyperplanes in $\mathbb{H}^n$. For the corresponding unit vectors u_{H_a} and u_{H_b} the following holds.*

(i) The vectors are not contained in the light cone: $\langle u_H, u_H\rangle_{-1} > 0$ for $H = H_a, H_b$.

(ii) The vectors span a hyperbolic geodesic line: $\langle\{u_{H_a}, u_{H_b}\}\rangle \cap \mathbb{H}^n \neq \varnothing$.

(iii) The vectors satisfy $||\langle u_{H_a}, u_{H_b}\rangle_{-1}|| > (\langle u_{H_a}, u_{H_a}\rangle \cdot \langle u_{H_b}, u_{H_b}\rangle_{-1})^{\frac{1}{2}}$, where $||\cdot||$ is the Euclidean norm.

The uniqueness of the common perpendicular gives us a notion of distance between hyperplanes in $\overline{\mathbb{H}}^n$.

Definition 6.2.6. Let H_a and H_b be two hyperplanes in $\mathbb{H}^n$. The distance $\bar{d}(H_a, H_b)$ between these hyperplanes is defined as follows:

$$\bar{d}(H_a, H_b) := \begin{cases} d(\rho(a), \rho(b)) & H_a, H_b \text{ ultra-parallel} \\ 0 & \text{else,} \end{cases}$$

where $d(\rho(a), \rho(b))$ is the distance between the intersections $\rho(a), \rho(b)$ of H_a and H_b with their unique common perpendicular ρ. It is the unique number $\eta(u_{H_a}, u_{H_b})$ such that $||\langle u_{H_a}, u_{H_b}\rangle_{-1}|| = (\langle u_{H_a}, u_{H_a}\rangle - -1 \cdot \langle u_{H_b}, u_{H_b}\rangle - -1)^{\frac{1}{2}} \cosh \eta(u_{H_a}, u_{H_b})$ (see [Rat19, Theorem 3.2.8]).

The rest of this section is devoted to the following lemma, which provides a group theoretic meaning for *ultra-parallel* in hyperbolic reflection groups with a polytope as a fundamental domain.

Lemma 6.2.7. *Let (W, S) be a hyperbolic reflection group in $\mathbb{H}^n$ with Coxeter polyhedron P. Let $\{H_1, \dots, H_m\}$ be the hyperplanes corresponding to elements in S and R be the set of reflections. Further, let H_{r_1}, H_{r_2} be two distinct hyperplanes with $r_i \in R$.*

(i) H_{r_1}, H_{r_2} intersect in $\mathbb{H}^n$ if and only if there exist $s_i, s_j \in S$ with $m_{ij} < \infty$ and $w \in W$ such that

$$r_1 r_2 = w(s_i s_j)^k w^{-1} \quad \textit{with } k \in \mathbb{Z} \setminus \{0\}.$$

(ii) If P is a convex polytope, H_{r_1}, H_{r_2} are not ultra-parallel if and only if there exist $s_i, s_j \in S$ and $w \in W$ such that

$$r_1 r_2 = w(s_i s_j)^k w^{-1} \quad \textit{with } k \in \mathbb{Z} \setminus \{0\}.$$

Proof. We prove (i). Assume that the hyperplanes H_{r_1} and H_{r_2} intersect in $\mathbb{H}^n$. Hence, there exists $w_1, w_2 \in W$ such that $H_{r_1} = w_1 H_i$ and $H_{r_2} = w_2 H_j$ for some $i, j \in \{1, \dots, m\}$. Further, we have $\tilde{w} = w_1^{-1} \cdot w_2 = (s_i s_j)^l$ with $l \in \mathbb{Z}$. Since the intersection $H_{r_1} \cap H_{r_2}$ is non-empty in $\mathbb{H}^n$, we have $m_{ij} < \infty$. It follows that $r_1 = w_1 s_i w_1^{-1}$ and $r_2 = w_1 \tilde{w} s_x \tilde{w}^{-1} w_1^{-1}$, where $s_x \in \{i, j\}$. Thus, we obtain

$$r_1 r_2 = w_1 s_i \tilde{w} s_x \tilde{w}^{-1} w_1^{-1} = w_1 (s_i s_j)^k w_1^{-1}$$

with $k \in \mathbb{Z} \setminus \{0\}$.

Now, let $r_1 r_2 = w(s_i s_j)^k w^{-1}$ for some $k \in \mathbb{Z} \setminus \{0\}$, $m_{ij} < \infty$ and $w \in W$. Without loss of generality, we may assume additionally $r_1 = s \in S$ (conjugating both sides). Multiplying with r_1 yields $r_2 = s \cdot w(s_i s_j)^k w^{-1}$. Both sides have reflection length 1. It also may be assumed that k is maximal in the sense that the last letters of a word representing w are not s_i or s_j. If $s \notin \{s_i, s_j\}$, we have $l_R(s \cdot w(s_i s_j)^k w^{-1}) = 3$ with Theorem 3.3.20. We also obtain $l_R(s \cdot w(s_i s_j)^k w^{-1}) = 3$ with the same criterion if we assume that s does not commute with w. Hence, we get $s \in \{s_i, s_j\}$ and s commutes with w. Let $v = s_i s_j \cdots$ be a dihedral element with S-length k. We have $r_1 = wsw^{-1}$ and $r_2 = s \cdot w(s_i s_j)^k w^{-1} = wvs_x v^{-1} w^{-1}$ with $s_x \in \{s_i, s_j\}$. The hyperplanes H_i and H_j are intersecting because $m_{ij} < \infty$. Thus, the hyperplanes H_s and $H_{vs_x v^{-1}}$ intersect and so do H_{r_1} and H_{r_2}. The proof of the second statement works analogously except that we ignore the condition $m_{ij} < \infty$. □

6.3 Arbitrary reflection length close to boundary points

After stating a series of lemmata, boundary points with neighbourhoods that contain copies of the fundamental domain of arbitrary large reflection length are described in Theorem 2, which is one of the main results of this chapter.

Lemma 6.3.1. *Let $\mathbb{S}^n$ be the unit sphere in the Euclidean space $\mathbb{E}^{n+1}$. For every $(n-1)$-dimensional sphere O in $\mathbb{S}^n$ that is not a great circle, there exists a unique n-dimensional sphere S_O in $\mathbb{E}^{n+1}$ with*

$$\mathbb{S}^n \cap S_O = O$$

and S_O intersects $\mathbb{S}^n$ at a right angle.

Proof. We give a sketch of a proof and leave the details to the reader. Given $O \subseteq \mathbb{S}^n$, we obtain S_O by the following construction: Take $n+1$ disjoint points $p_1, \dots, p_{n+1}$ in O and consider the tangent spaces $T_1, \dots, T_{n+1}$ at this points. These tangent spaces are distinct hyperplanes in $\mathbb{E}^{n+1}$ intersecting in a single point c. The point c is the centre of S_O and together with a point in O it determines S_O uniquely. □

We recall that D^n abbreviates the open unit ball in $\mathbb{E}^n$ and that $\widehat{\mathbb{E}}^n$ abbreviates the one-point compactification of $\mathbb{E}^n$ (see Chapter 2 for further details).

Corollary 6.3.2. *The inversion on a sphere in $\widehat{\mathbb{E}}^{n+1}$ intersecting $\mathbb{S}^n$ orthogonally fixes $\mathbb{S}^n$ and D^{n+1} as sets.*

Proof. The inversion on a sphere in $\widehat{\mathbb{E}}^n$ maps spheres to spheres and preserves the Euclidean angle between intersecting spheres (see [BH99, Proposition 6.5]). Together with Lemma 6.3.1, this implies the corollary. □

The principal significance of the following two lemmata is to draw conclusions from conditions (i) and (ii) in Theorem 2. These are important ingredients for the proof of Theorem 2.

Lemma 6.3.3. *Let H_1 and H_2 be two parallel hyperplanes in $\mathbb{H}^n$ with a common point $\xi \in \partial H_1 \cap \partial H_2 \subseteq \partial\mathbb{H}^n$. Let $S = \{s_1, s_2\} \subseteq \mathrm{Iso}(\mathbb{H}^n)$ be the corresponding reflections across H_1, H_2, respectively. For every neighbourhood U of ξ there exists a reflection r in the Coxeter group $\langle S\rangle \subseteq \mathrm{Iso}(\mathbb{H}^n)$ such that $H_r \subseteq U$.*

Proof. The proof is conducted in the Poincaré ball model. The hyperplanes H_i are represented by spheres S_i in the one-point compactification $\widehat{\mathbb{E}}^n$ that intersect the unit sphere $\mathbb{S}^{n-1}$ orthogonally. We can assume that these spheres have radii r_i, centre c_i and an antipode a_i to ξ (apply an inversion on the other sphere, in case one sphere contains ∞). Let $(\hat{S}_i)_{i\in\mathbb{N}}$ be the sequence of spheres defined by

$$\hat{S}_0 := S_2 \text{ and } \hat{S}_i = i_{\hat{S}_{i-1}}(S_1),$$

where $i_{\hat{S}_{i-1}}$ is the inversion on the sphere $\hat{S}_{i-1}$. According to Corollary 6.3.2, the unit sphere $\mathbb{S}^{n-1}$ is fixed as a set by all inversions $i_{\hat{S}_i}$. All spheres $(\hat{S}_i)_{i\in\mathbb{N}}$ contain ξ, because ξ is fixed by each $i_{\hat{S}_i}$. Let $\hat{c}_i$ be the centre and $\hat{r}_i$ be the radius of $\hat{S}_i$. The points $\xi, c_i, , a_i, \hat{c}_i$ are collinear in $\widehat{\mathbb{E}}^n$ for all $i \in \mathbb{N}$. This follows from the definition of the inversion on a sphere. We have $d_2(\xi, \hat{c}_i) = \hat{r}_i$. Given the centre $\hat{c}_{i-1}$ and the radius $\hat{r}_{i-1}$, the following equations hold according to Formula (2.2.2) for an inversion on a sphere:

$$\begin{aligned}
\hat{c}_i &= \frac{1}{2} \cdot (\xi - i_{\hat{S}_{i-1}}(a_1)) + i_{\hat{S}_{i-1}}(a_1),\\
\hat{r}_i &= \frac{1}{2} \cdot d_2(i_{\hat{S}_{i-1}}(a_1), \xi),\\
i_{\hat{S}_{i-1}}(a_1) &= \frac{{\hat{r}_{i-1}}^2}{||a_1 - \hat{c}_{i-1}||^2} \cdot (a_1 - \hat{c}_{i-1}) + \hat{c}_{i-1}.
\end{aligned}$$

All $i_{\hat{S}_i}(a_1)$ lay in the segment $[\xi, \hat{c}_{i-1}]$. Since the sequence $(d_2(i_{\hat{S}_i}(a_1), \xi))_{i\in\mathbb{N}}$ is monotonously decreasing and bounded by 0, it converges to 0, the only possible limit.

Hence, by reflecting S_1 across $\hat{S}_i$, we obtain spheres with arbitrary small diameter in $\mathbb{E}^n$ that contain ξ as $i \to \infty$. The spheres represent hyperplanes corresponding to reflections in $\{wsw^{-1} \mid s \in S, w \in \langle S\rangle\}$ and we find a sufficiently small sphere that represents a hyperplane H_r contained in U. □

Lemma 6.3.4. *Let H_1 and H_2 be two ultra-parallel hyperplanes in $\mathbb{H}^n$ and let the corresponding reflections be $S = \{s_1, s_2\} \subseteq \mathrm{Iso}(\mathbb{H}^n)$. Let γ be a geodesic ray contained in the unique common perpendicular $\bar{\gamma}$ of H_1 and H_2. For every neighbourhood U of $\gamma(\infty)$, there exists a reflection r in the Coxeter group $\langle S\rangle \subseteq \mathrm{Iso}(\mathbb{H}^n)$ such that $H_r \subseteq U$.*

Proof. Isometries preserve angles and intersection at a right angle of subspaces. Let R be the set of reflections in $\langle S\rangle$. From Formula (2.2.1) in Section 2.2.1 for a reflection on a hyperplane in the hyperboloid model, it is evident that $\bar{\gamma}$ gets fixed as a set by a reflection across $H_{\tilde{r}}$ for all $\tilde{r}$ in R. Thus, all hyperplanes in $\mathcal{H} = \{H_{\tilde{r}} \mid \tilde{r} \in R\}$ have $\bar{\gamma}$ as a common perpendicular.

Just as in the proof of Lemma 6.3.3, we continue in the Poincaré ball model and define a sequence of spheres. Let S_i be the sphere in $\widehat{\mathbb{E}}^n$ representing H_i. Without loss of generality, we can assume that S_i has radius r_i, centre c_i, intersection x_i with the segment $[c_1, c_2]$ in $\mathbb{E}^n$ and antipode a_i to x_i (apply an inversion on the other sphere, in case one sphere contains ∞). Let $(\hat{S}_i)_{i\in\mathbb{N}}$ be the sequence of spheres defined by

$$\hat{S}_0 := S_2 \text{ and } \hat{S}_i = i_{\hat{S}_{i-1}}(S_1),$$

where $i_{\hat{S}_{i-1}}$ is the inversion on the sphere $\hat{S}_{i-1}$. For all $\hat{S}_i$, a reflection $r_i \in R$ exists such that $\hat{S}_i$ represents H_{r_i}. Let $\hat{c}_i$ be the centre of $\hat{S}_i$, let $\hat{x}_i$ be the intersection of $\hat{S}_i$ with the segment $[c_1, c_2]$ in $\mathbb{E}^n$ and let $\hat{a}_i$ be the antipode of $\hat{x}_i$ in $\hat{S}_i$. From the proof of Theorem 3 and from the Formula (2.2.2) in Section 2.2.1, we deduce that all $\hat{c}_i, \hat{x}_i, \hat{a}_i$ and $\gamma(\infty)$ lay on the geodesic segment $[c_1, c_2]$. Given the points $\hat{c}_{i-1}, \hat{a}_{i-1}, \hat{x}_{i-1}$, Formula (2.2.2) implies:

$$\begin{aligned}
\hat{x}_i &= i_{\hat{S}_{i-1}}(x_1) = \frac{d_2(\hat{a}_{i-1}, \hat{x}_{i-1})^2}{4 \cdot ||x_1 - \hat{c}_{i-1}||^2} \cdot (x_1 - \hat{c}_{i-1}) + \hat{c}_{i-1} \quad \in (\hat{x}_{i-1}, \gamma(\infty)),\\
\hat{a}_i &= i_{\hat{S}_{i-1}}(a_1) = \frac{d_2(\hat{a}_{i-1}, \hat{x}_{i-1})^2}{4 \cdot ||a_1 - \hat{c}_{i-1}||^2} \cdot (a_1 - \hat{c}_{i-1}) + \hat{c}_{i-1} \quad \in (\gamma(\infty), \hat{c}_{i-1}),\\
\hat{c}_i &= \frac{1}{2}(\hat{x}_i - \hat{a}_i) + \hat{a}_i \quad \in (\gamma(\infty), \hat{a}_i).
\end{aligned}$$

We consider the sequence of distances $(d_2(\hat{c}_i, \hat{x}_i))_{i\in\mathbb{N}}$, where $\hat{c}_i$ is outside the closed unit ball $\overline{D}^n$ and $\hat{x}_i$ is always inside the open unit ball D^n since the inversion on a sphere intersecting $\mathbb{S}^{n-1}$ orthogonally is a bijection on $\widehat{\mathbb{E}}^n$ and fixes $\mathbb{S}^{n-1}$ as well as D^n as sets (Corollary 6.3.2). This sequence is monotonously decreasing and converges to 0 since the sequences of points $(\hat{c}_i)_{i\in\mathbb{N}}$ and $(\hat{x}_i)_{i\in\mathbb{N}}$ both converge to $\gamma(\infty)$. Hence, the sequence of spheres $(\hat{S}_i)_{i\in\mathbb{N}}$ contains elements with arbitrary small radius and centre arbitrarily close to $\gamma(\infty)$ in $\mathbb{E}^n$. This implies the existence of a hyperplane H_r in the sequence $(H_i)_{i\in\mathbb{N}}$ of hyperplanes corresponding to $(\hat{S}_i)_{i\in\mathbb{N}}$ such that $H_r \subseteq U$ with $r \in R$. □

Theorem 2 (Boundary points close to arbitrary reflection length)**.** *Let (W, S) be a hyperbolic reflection group with fundamental domain P in $\mathbb{H}^n$. Let R be the set of reflections in W. Let U be a neighbourhood in $\overline{\mathbb{H}}^n$ of a point ξ in $\partial\mathbb{H}^n$. Suppose ξ satisfies one of the following conditions:*

(i) ξ is a common point of two parallel hyperplanes $H_r, H_{r'}$ with $r, r' \in R$.

(ii) ξ is an endpoint of the common perpendicular of two ultra-parallel hyperplanes $H_r, H_{r'}$ with $r, r' \in R$.

For every $k \in \mathbb{N}$, there exists $w \in W$ with $l_R(w) = k$ such that the domain wP is contained in U.

Proof. For the point $\xi \in \partial\mathbb{H}^n$, let $U(c, d, \epsilon)$ be contained in the neighbourhood U, where c is a geodesic ray with $c(\infty) = \xi$. In both cases, there exists a hyperplane H_r with $r \in R$ completely contained in $U(c, d, \epsilon)$ by Lemma 6.3.3 and Lemma 6.3.4. So the image rP of P under r is contained in U. This proves the theorem for reflection length 1. In general, l_R is unbounded on W (see Theorem 4.1.3). Thus, there exists $w \in W$ with reflection length $l_R(w) = k + 1$. In case wP is in the half-space of H_r^- that is contained in U, the proof is complete. Otherwise, $wP \subseteq H_r^+$ and we reflect wP across H_r to obtain $wrP \subseteq U$. The element wr has reflection length $n + 1 \pm 1$ (see [Bre+19, Remark 1.3]). A sequence $(w_1P, \ldots, w_lP)$ of adjacent tiles in U between rP and wrP contains tiles w_iP of all reflection lengths $l_R(w_i)$ between 1 and $l_R(wr)$ (see [Bre+19, Remark 1.3]). □

6.4 Hyperplanes not generating a hyperbolic reflection group

In this section, we state three results that we need in the last section of this chapter but hold in a more general setting.

Lemma 6.4.1. *For $k \geq 3$, let $\{H_{r_1}, \dots, H_{r_k}\}$ be a set of pairwise parallel hyperplanes in $\mathbb{H}^n$. If all H_{r_i} have a common point ξ in $\partial\mathbb{H}^n$ and the group D generated by the reflections r_i across H_{r_i} is a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^n)$, then D is isomorphic as a group to the infinite dihedral group D_∞.*

Proof. The uniqueness of $\xi \in \partial\mathbb{H}^n$ as a common point of $\partial H_{r_1}, \dots, \partial H_{r_k}$ follows from Corollary 6.2.3. The hyperplane reflection r_i across H_{r_i} maps hyperplanes in $\{H_{r_1}, \dots, H_{r_k}\}$ to hyperplanes with ξ in their boundary because H_{r_i} and ∂H_{r_i} are fixed pointwise by r_i. Let R be the set of all reflections in the group $D = \langle r_i \mid i \in \{1, \dots, k\}\rangle$. The corresponding hyperplanes all contain ξ in their boundary.

We fix a hyperplane $H \in \mathcal{H} := \{H_r \mid r \in R\}$. Since W is discrete, we can choose $\varepsilon, \delta \in \{+, -\}$ such that the intersection of half-spaces $H^\varepsilon \cap H_1^\delta$ is neither empty nor a half-space and contains no other hyperplane in $\mathcal{H}$. We show that the reflections s_1 and s corresponding to H_1 and H generate W. Therefore, we assume that there exists an H_{r_i} such that r_i cannot be written as a product of s_1 and s. There exists an element $w \in \langle\{s, s_1\}\rangle$ such that

$$H_{r_i} \subseteq wH^\varepsilon \cap wH_1^\delta.$$

In words, the hyperplane H_{r_i} is contained in the intersection of two half-spaces of copies of H and H_1, because H_{r_i} contains ξ in its boundary, too. Thus, the hyperplane $w^{-1}H_{r_i} \in \mathcal{H}$ is contained in $H^\varepsilon \cap H_1^\delta$. This contradicts our assumption and we proved $\langle\{s, s_1\}\rangle = D$. Since H and H_1 do not intersect in $\mathbb{H}^n$, ss_1 has infinite order and D is isomorphic to D_∞. □

Lemma 6.4.2. *Let $\{H_{r_1}, \dots, H_{r_k}\}$ be a set of pairwise ultra-parallel hyperplanes in $\mathbb{H}^n$ with $k \geq 3$ and a common perpendicular ϱ. If the group D generated by the reflections r_i across H_{r_i} is a discrete subgroup of $\mathrm{Iso}(\mathbb{H}^n)$, then D is isomorphic as a group to D_∞.*

Proof. The hyperplane reflection r_i across H_{r_i} maps hyperplanes in $\{H_{r_1}, \dots, H_{r_k}\}$ to pairwise ultra-parallel hyperplanes with ϱ as their common perpendicular, because r_i fixes ϱ as a set. Let R be the set of all reflections in the group $D = \langle r_i \mid i \in \{1, \dots, k\}\rangle$. The hyperplanes in the set $\mathcal{H} := \{H_r \mid r \in R\}$ are all pairwise ultra-parallel and have ϱ as their common perpendicular. The rest of the proof is analogous to the one of Lemma 6.4.1. □

Lemma 6.4.3. *Let (W, S) be a hyperbolic reflection group with fundamental polyhedron P. Let R be the set of reflections. For every $r \in R$ and every $\epsilon > 0$, there exists $r_\epsilon \in R$ such that the hyperplanes H_{r_ϵ} and H_r are ultra-parallel and*

$$\bar{d}(H_r, H_{r_\epsilon}) > \epsilon.$$

Proof. It suffices to show that there exists $\tilde{r} \in R$ with $\bar{d}(H_r, H_{\tilde{r}}) > 0$ since we can reflect these hyperplanes several times on each other to obtain the H_{r_ϵ}. Hyperplane reflections fix perpendiculars as sets.

The group W has universal Coxeter groups of arbitrary large rank as reflection subgroups (see [Edg13]). Let $W_3 = \langle r_1, r_2, r_3\rangle$ be a universal Coxeter group contained in W as a reflection subgroup with $\{r_1, r_2, r_3\} \subseteq R$ as a Coxeter generating set. H_{r_1}, H_{r_2} and H_{r_3} do not intersect pairwise in $\mathbb{H}^n$, since r_1, r_2, r_3 have pairwise infinite order. Suppose that all H_{r_i}

intersect with H_r in $\overline{\mathbb{H}}^n$. Otherwise, there would be a hyperplane with a distance greater than zero to H_r.

Assume that the hyperplanes H_{r_1} and H_r do not intersect in $\mathbb{H}^n$ and have a common point in $\partial\mathbb{H}^n$. H_r is contained in one half-space $H_{r_1}^{\varepsilon}$ with $\varepsilon \in \{+,-\}$. Each of H_{r_2} and H_{r_3} is also contained in a half-space. Since W_3 is not isomorphic to D_∞, the hyperplanes H_{r_1}, H_{r_2} and H_{r_3} have no common point in $\overline{\mathbb{H}}^n$ (see Lemma 6.4.1). This implies that there exists H_{r_j} with $j \in \{2,3\}$ such that H_{r_j} does not intersect with H_r in the point $H_r \cap H_{r_1} \subseteq \overline{\mathbb{H}}^n$ (see Lemma 6.2.3). Hence, H_{r_j} is contained in $H_{r_1}^{\varepsilon}$, too. The hyperplane's reflection $H_{r_1 r_j r_1}$ across H_{r_1} does not intersect with H_r in $\overline{\mathbb{H}}^n$.

Let us assume that H_r intersects all hyperplanes H_{r_i} in $\mathbb{H}^n$. Since $\langle\{r_1, r_2, r_3\}\rangle$ is isomorphic to a universal Coxeter group, the intersection of two distinct non-ultra-parallel hyperplanes H_{r_i} and H_{r_j}, $1 \leq i,j \leq 3$, is empty and $\partial H_{r_i} \cap \partial H_{r_j} = \{\delta\}$ (see Corollary 6.2.3). There exists a neighbourhood U of δ in $\overline{\mathbb{H}}^n$ such that $U \cap \overline{H}_r = \varnothing$ because H_r intersects all H_{r_i}. By Lemma 6.3.3, we obtain hyperplanes in arbitrary small neighbourhoods of $\delta \in \partial\mathbb{H}^n$ in $\overline{\mathbb{H}}^n$. This gives us $H_{\tilde{r}} \subseteq U$ with $\tilde{r} \in R$ and $H_{\tilde{r}}$ is ultra-parallel to H_r. Let the H_{r_i} be pairwise ultra-parallel. According to Lemma 6.4.2, we can assume that H_r is not orthogonal to H_{r_i}. This implies that H_r does not contain the unique common perpendicular γ of H_{r_i} and H_{r_j} (see Theorem 3). Hence, by Lemma 6.3.4 there exists a hyperplane $H_{\hat{r}}$ with $\hat{r} \in R$ in a neighbourhood V of an endpoint $\gamma(\infty)$ such that $V \cap H_r = \varnothing$. H_r and $H_{\hat{r}}$ are ultra-parallel. □

6.5 Polytopes as fundamental domains

Lemma 6.5.1. *Let (W,S) be a hyperbolic reflection group with a convex polytope P as a fundamental domain. Let R be the set of reflections in W. The set of ideal points of hyperplanes*

$$\mathcal{H}_\infty := \{\gamma(\infty) \mid \gamma \subseteq H_r \text{ geodesic ray, } r \in R\}$$

is dense in $\partial\mathbb{H}^n$.

Proof. Since W is discrete and the set $\mathcal{H}_\infty$ is a countable union of spheres homeomorphic to $\mathbb{S}^{n-2}$, we have $\mathcal{H}_\infty \subsetneq \partial\mathbb{H}^n$ (see Section 6.1). Let $\xi \in \partial\mathbb{H}^n \setminus \mathcal{H}_\infty$ and let $c : [0,\infty] \to \overline{\mathbb{H}}^n$ be a geodesic ray with $c(\infty) = \xi$. For $\epsilon, r > 0$ the sets $U(c,s,\epsilon)$ form a neighbourhood basis of ξ (see Definition 6.1.2). We fix arbitrary $\epsilon, r > 0$. In the Poincaré ball model, the set of the endpoints $C(\infty)$ of geodesic rays in

$$C := \{c^* \mid c^* \text{ geodesic ray, } c^*(0) = c(0),\ d(c^*(r), c(r)) = \epsilon\}$$

is a sphere in $\partial\mathbb{H}^n$. According to Lemma 6.3.1, there exists a hyperplane E, possibly not in $\{H_r \mid r \in R\}$, with $\partial E = C(\infty)$. Let E^+ be the half-space such that ξ is in the closure $\overline{E^+}$. To prove the lemma, it is sufficient to show that there exists a hyperplane $H_{r'}$ with $r' \in R$ that intersects the half-space E^+ non-empty. Then $U(c,s,\epsilon) \cap \partial H_r' \neq \varnothing$ because hyperplanes as well as $\mathbb{H}^n$ are uniquely geodesic subspaces (see [BH99, p. 21]). Since P is a strict fundamental domain, there exists $w \in W$ with $E^+ \cap wP \neq \varnothing$. This implies that there exists a hyperplane $H_{r'}$ with $r' \in R$ such that $E^+ \cap H_{r'} \neq \varnothing$ because P is a convex polytope. □

Notation 6.5.2. Let (W, S) be a hyperbolic reflection group in $\mathbb{H}^n$. Let R be the set of reflections. We write $\mathcal{I}_p(W)$ for the set of ideal points $\xi \in \partial\mathbb{H}^n$, such that there exists two parallel hyperplanes H_{r_1} and H_{r_2} corresponding to $r_1, r_2 \in R$ with $\xi \in \partial H_{r_1} \cap \partial H_{r_2}$. Furthermore, we write $\mathcal{P}_{up}(W)$ for the set of ideal points $\mu \in \partial\mathbb{H}^n$, such that μ is an endpoint of the common perpendicular of two ultra-parallel hyperplanes H_{r_3} and H_{r_4} corresponding to $r_3, r_4 \in R$.

Theorem 6.5.3. *Let (W, S) be a hyperbolic reflection group in $\mathbb{H}^n$ with a convex polytope P as a fundamental domain. The union $\mathcal{I}_p(W) \cup \mathcal{P}_{up}(W)$ is dense in the visual boundary $\partial\mathbb{H}^n$.*

Proof. Assume that there exists a point $\xi \in \partial\mathbb{H}^n$, real numbers $\epsilon, \delta > 0$ and a neighbourhood $U(c, \delta, \epsilon)$ of ξ such that $\mathcal{I}_p(W) \cap U(c, r, \epsilon) = \varnothing$. Let R be the set of reflections. We want to show $\mathcal{P}_{up}(W) \cap U(c, \delta, \epsilon) \neq \varnothing$ and begin by proving that there exists an endpoint $\varphi(\infty) \in U(c, \delta, \epsilon)$ of a geodesic ray φ contained in a 1-dimensional intersection of finitely many hyperplanes in $\mathcal{H} = \{H_r \mid r \in R\}$. Therefore, we successively follow the implications for each dimension lower hereafter:

Lemma 6.5.1 states that the endpoints $\mathcal{H}_\infty = \{\gamma(\infty) \mid \gamma \subseteq H_r \text{ geodesic ray}, r \in R\}$ are dense in $\partial\mathbb{H}^n$. Hence, there exists a hyperplane H_r with $r \in R$ and an ideal point ν such that $\nu \in \partial H_r \cap U(c, \delta, \epsilon)$. The hyperplane H_r is isometric to $\mathbb{H}^{n-1}$ (see Section 6.1) and the intersection $\partial H_r \cap U(c, \delta, \epsilon)$ is a neighbourhood of ν in $\overline{H_r}$. We consider hyperplanes in H_r that are intersections $H_r \cap H_t$ for $t \in R$. There exists $w \in W$ such that $P' = wP \cap H_r$ is a polytope in H_r and we can apply Lemma 6.5.1 again as well as analogous arguments in one dimension lower until dimension one.

Thus, there exists a geodesic line γ in $\mathbb{H}^n$ that is the intersection of finitely many hyperplanes in $\mathcal{H}$ and has an endpoint $\gamma(\infty)$ in $U(c, \delta, \epsilon)$. The geodesic ray γ contains dimension-1 faces of infinitely many uP with $u \in W$ since P is a polytope and we assume that $\mathcal{I}_p(W) \cap U(c, \delta, \epsilon)$ is empty. Considering the dimension-0 faces of the uP, this implies that γ is intersected punctually in intervals of finitely many different lengths by hyperplanes H_t with $t \in T \subseteq R$ with $|T| = \infty$.

In the Poincaré ball model, this setting translates to spheres S_t with $t \in T$ intersecting the circle C_γ corresponding to γ, where S_t and C_γ intersect $\mathbb{S}^{n-1}$ orthogonally for all $t \in T$. The spheres S_t intersect C_γ in a finite number of different angles because the only angles that can occur are inherited from P as it is a convex polytope and a strict fundamental domain. Thus, we can choose a sphere $S_{t'}$ with $t' \in T$ intersecting C_γ sufficiently close to $\gamma(\infty)$, such that the intersection $S_{t'} \cap D^n$ with the open unit ball D^n is contained in $U(c, \delta, \epsilon)$. One half-space $H_{t'}^\varepsilon$ of the hyperplane corresponding to $S_{t'}$ is contained in $U(c, \delta, \epsilon)$. For $H_{t'}$ exists a hyperplane $H_{\tilde{t}} \in \mathcal{H}$ that is ultra-parallel to $H_{t'}$ (see Lemma 6.4.3). The unique common perpendicular ϱ of H_t and $H_{t'}$, which exists by Theorem 3, has one endpoint $\varrho(\infty) \in \mathcal{P}_{up}(W)$ in the boundary $\partial H_{t'}^\varepsilon \subseteq U(c, \delta, \epsilon)$. It follows $\mathcal{P}_{up}(W) \cap U(c, \delta, \epsilon) \neq \varnothing$, which completes the proof. $\square$

Theorem 6.5.4. *Let (W, S) be a hyperbolic reflection group in $\mathbb{H}^n$ with a polytope P as a fundamental domain. Let R be the set of reflections in W. Let U be a neighbourhood in $\overline{\mathbb{H}}^n$ of a point ξ in $\partial\mathbb{H}^n$. For every $k \in \mathbb{N}$ there exists $w \in W$ with $l_R(w) = k$ such that the domain wP is contained in U.*

Proof. Theorem 6.5.3 states that there exists a point $\nu \in \partial\mathbb{H}^n$ in U such that ν is either an ideal point of two parallel hyperplanes corresponding to reflections in R or one endpoint of a common perpendicular of two ultra-parallel hyperplanes corresponding to reflections in R. So ν satisfies one of the conditions in Theorem 2 and in every neighbourhood of ν for every $k \in \mathbb{N}$ exists wP with $l_R(w) = k$. Since $\mathbb{H}^n$ is a metric space, there exists a neighbourhood U' of ν that is completely contained in U and the theorem is proven. $\square$

Remark 6.5.5. Theorem 6.5.4 is only true for reflection groups with a polytope as a fundamental domain. For reflection groups with polyhedra that are not polytopes as fundamental domains, the theorem is false as the following example shows.

Example 6.5.6. Consider the hyperbolic reflection group generated by three pairwise ultra-parallel hyperplanes in $\mathbb{H}^n$. This group is isomorphic to the universal Coxeter group W_n with three generators. Let w be in W_n and let R be the set of reflections in W_n. Since there exist neighbourhoods U_w of a point ξ_w in the boundary of every domain wP such that $U_w \cap wP = U_w$, the only reflection length arbitrarily close to $w\xi$ is the reflection length $l_R(w)$.

7. Powers of Coxeter elements with unbounded reflection length

For Coxeter groups with sufficiently large braid relations, we prove that the sequence of powers of a Coxeter element has unbounded reflection length. We establish a connection between the reflection length functions on arbitrary Coxeter groups and the reflection length function on the universal Coxeter groups of the same rank through the solution to the word problem for Coxeter groups. For Coxeter groups corresponding to a Coxeter matrix with the same entry everywhere except the diagonal, upper bounds for the reflection length of the powers of Coxeter elements are established. This chapter builds on Chapter 3 in particular. The first section contains the proofs of Theorem 4 and Theorem 5 including preparatory lemmata. In Section 7.2, we give sharp upper bounds in Theorem 6 and Theorem 7 for the reflection length of powers of Coxeter elements that have the same braid relation for all pairs of distinct generators. The last section contains a conjecture about the general relation between the reflection length function of an arbitrary Coxeter group and the universal Coxeter group of the same rank. We also prove the conjecture for reflections in the arbitrary Coxeter group. This chapter is based on the article with the same title [Lot24a] by the author. Most of the text is taken from [Lot24a] with minor modifications.

7.1 Comparing reflection length in arbitrary and universal Coxeter groups

We follow a new approach and compare the reflection length function of an arbitrary Coxeter group with the reflection length function of the universal Coxeter group of the same rank. Fix a word $\mathbf{s}$ over the alphabet S and compare the reflection length $l_R(\omega(\mathbf{s}))$ of the element $\omega(\mathbf{s})$ represented by $\mathbf{s}$ in the arbitrary Coxeter group and the reflection length $l_{R_n}(\omega_n(\mathbf{s}))$ of the element $\omega_n(\mathbf{s})$ represented by $\mathbf{s}$ in the universal Coxeter group. The braid relations are crucial for this. In comparison to the universal Coxeter group, the reflection length of an element possibly decreases because of the braid relations in an arbitrary Coxeter group.

Recall that a word $\mathbf{s}$ always can be transformed into an S-reduced word by a sequence of nil-moves and braid-moves in a Coxeter system (W, S) (see Theorem 3.2.13).

Definition 7.1.1. Let $(\alpha_i)_{i\leq a}$ be a finite sequence of nil-moves and braid-moves to transform a word $\mathbf{s}$ into an S-reduced word in a Coxeter system (W, S). We say that the sequence $(\alpha_i)_{i\leq a}$ is *braid-minimalistic* if the following to conditions are satisfied:

1. The move α_i is a braid-move if and only if $\alpha_{i-1}\circ\cdots\circ\alpha_1(\mathbf{s})$ has no consecutive subword of the form ss for $s \in S$.

2. If α_i is a braid-move on a consecutive subword $\mathbf{b}_{ij}$, there is no other braid-move α_j in $(\alpha_i)_{i\leq a}$ on the subword in the same position as $\mathbf{b}_{ij}$ in $\mathbf{s}$ involving the letters s_i, s_j.

This means especially that braid-moves just appear if there are no further nil-moves possible.

Example 7.1.2. Consider the word $\mathbf{s} := s_1s_2s_1s_2s_2s_3s_2s_3$ and the Coxeter group W_3^3 of type $\tilde{\mathbf{A}}_2$. The first of the following two different ways of reducing $\mathbf{s}$ in W_3^3 is braid-minimalistic.

$$s_1s_2s_1s_2s_2s_3s_2s_3 \mapsto s_1s_2s_1s_3s_2s_3 \mapsto s_2s_1s_2s_3s_2s_3 \mapsto s_2s_1s_3s_2 \tag{7.1.1}$$

$$s_1s_2s_1s_2s_2s_3s_2s_3 \mapsto s_1s_1s_2s_1s_3s_2s_3s_3 \mapsto s_2s_1s_3s_2 \tag{7.1.2}$$

The second one is not braid-minimalistic since the nil-move $s_2s_2 \mapsto e$ is not the first move.

Lemma 7.1.3. *Let $\mathbf{s} \in S^*$ be a word representing an element $\omega(\mathbf{s})$ in a Coxeter system (W, S). There exists a finite, braid minimalistic sequence $(\alpha)_{i\leq a}$ of nil-moves and braid-moves transforming $\mathbf{s}$ to an S-reduced expression for $\omega(\mathbf{s})$.*

Proof. From Theorem 3.2.13, we know that a sequence of nil-moves and braid-moves transforms $\mathbf{s}$ to a reduced expression. The braid-minimalistic sequence $(\alpha)_{i\leq a}$ is obtained as follows: Execute nil-moves on $\mathbf{s}$ and the resulting words until no more nil-moves are possible. Either the obtained word is S-reduced or there is a braid-move possible to obtain a word $\mathbf{s}'$. Again, we know from the solution to the word problem that $\mathbf{s}'$ is transformable to an S-reduced expression by nil-moves and braid-moves. So we repeat this procedure while keeping track of the executed braid-moves until we obtain a reduced word. This ensures that Property (1) from Definition 7.1.1 holds for the obtained sequence. Property (2) in the definition holds since we keep track of executed braid-moves and do not execute redundant braid-moves. □

Lemma 7.1.4. *Let (W, S) be a Coxeter system with presentation $\langle S \mid \mathcal{R}\rangle$. Let $\mathbf{s} \in S^*$ be a word with $\omega(\mathbf{s}) = \mathbb{1}$. If a braid-minimalistic sequence of nil-moves and braid-moves $(\alpha_i)_{i\leq a}$ transforming $\mathbf{s}$ into the empty word e contains a braid-move, $\mathbf{s}$ contains a subword of the form $(s_is_j)^{m_{ij}}$ with $(s_is_j)^{m_{ij}}$ in $\mathcal{R}$.*

Proof. We prove the assertion by induction over the number of braid-moves in $(\alpha_i)_{i\leq a}$. If $(\alpha_i)_{i\leq a}$ contains exactly one braid-move $\mathbf{b}_{12} \mapsto \mathbf{b}_{21}$, every letter in $\mathbf{b}_{21}$ is cancelled by a nil-move with a letter in $\mathbf{s}$ outside of $\mathbf{b}_{21}$. Hence, $\mathbf{s}$ either contains $(s_1s_2)^{m_{12}}$ or $(s_2s_1)^{m_{21}}$ as a subword. Additionally, the braid-move $\mathbf{b}_{12} \mapsto \mathbf{b}_{21}$ is executed on a subword of $(s_1s_2)^{m_{12}}$ or $(s_2s_1)^{m_{21}}$.

Assume that $(\alpha_i)_{i\leq a}$ contains $(n+1)$ many braid-moves and is braid-minimalistic. Let α_b be the first braid-move in the sequence. The sequence $(\alpha_i)_{b<i\leq a}$ transforms the word $\mathbf{s}' := \alpha_b \circ \cdots \circ \alpha_1(\mathbf{s})$ into the empty word and contains n braid-moves. According to the induction assumption, $\mathbf{s}'$ contains a subword $\mathbf{r}$ of the form $(s_is_j)^{m_{ij}}$ with $(s_is_j)^{m_{ij}}$ in $\mathcal{R}$. Braid-moves on subwords of even S-length do not change the number of letters of a certain type. Braid-moves on subwords of odd S-length change the number of letters of a certain

type by ± 1 (see Remark 3.2.12). In general, a word $\mathbf{b}_{ij}$ contains at least one s_i and one s_j. So in case α_b is not a braid-move solely on letters of $\mathbf{r}$, we can conclude directly that $\mathbf{s}$ contains a subword of the form $(s_i s_j)^{m_{ij}}$ with $(s_i s_j)^{m_{ij}}$ in $\mathcal{R}$.

Consider the case where α_b is a braid-move solely on letters of $\mathbf{r}$. Following the induction hypothesis, there exists another braid-move α_j in $(\alpha_i)_{i\leq a}$ on a subword of $\mathbf{r}$. Together with being braid-minimalistic, this implies that one of the braid-moves α_b and α_j acts on the first half of $(s_i s_j)^{m_{ij}}$ and the other one on the second half. There exists a letter s in $\mathbf{r}$ that is not touched by α_b and is adjacent to a letter touched by α_b in $\mathbf{r}$. In $\alpha_{b-1} \circ \cdots \circ \alpha_1(\mathbf{s})$ this letter s cannot be adjacent to a letter touched by α_b, because this contradicts being braid-minimalistic. Let $\bar{\mathbf{s}}$ be the consecutive subword of $\alpha_{b-1} \circ \cdots \circ \alpha_1(\mathbf{s})$ separating s and a letter that is touched by α_b.

If $\omega(\bar{\mathbf{s}}) = \mathbb{1}$ via a subsequence of $(\alpha_i)_{i\leq a}$, we can apply the induction hypothesis. Correspondingly, $\bar{\mathbf{s}}$ contains a subword of the form $(s_k s_l)^{m_{kl}}$ with $(s_k s_l)^{m_{kl}}$ in $\mathcal{R}$. Since $\bar{\mathbf{s}}$ is a subword of $\mathbf{s}$, the same holds for $\mathbf{s}$.

If there is no subsequence of $(\alpha_i)_{i\leq a}$ that transforms $\bar{\mathbf{s}}$ into e, there are no nil-moves within the letters of $\mathbf{r}$. Since $\omega(\mathbf{s}) = \mathbb{1}$, the letters in $\mathbf{r}$ have to be cancelled by nil-moves in pairs with letters outside of $\mathbf{r}$. The properties of possible braid-moves on these letters outside of $\mathbf{r}$ imply as before that $\mathbf{s}$ contains a subword of the form $(s_i s_j)^{m_{ij}}$ in this case, too. $\square$

We remind the reader of the following notation.

Notation 7.1.5. Given a generating set S with n elements, we write ω_n for the canonic surjection $S^* \to W_n$. The reflection length function of the universal Coxeter group W_n is abbreviated with l_{R_n}.

Lemma 7.1.6. *Let (W, S) be an arbitrary Coxeter system of rank n with a Coxeter presentation $\langle S \mid \mathcal{R}\rangle$ and relations $\mathcal{R}$. Further, let l_R be the reflection length function on W and $v \in W$ be an element represented by an S-reduced word* $\mathbf{s}$. *If v has no S-reduced expression that contains subwords of the form $(s_i s_j)^{m_{ij}}$ for all $(s_i s_j)^{m_{ij}}$ in $\mathcal{R}$, the reflection length of v is*

$$l_R(v) = l_{R_n}(\omega_n(\mathbf{s})).$$

Proof. Let $D(\mathbf{s})$ be a deletion set. Let $\mathbf{s}'$ be the word we obtain from $\mathbf{s}$ by deleting the letters in $D(\mathbf{s})$. It is $\omega(\mathbf{s}') = \mathbb{1}$ in W. According to Theorem 3.2.13, there exists a sequence of nil-moves and braid-moves that transforms $\mathbf{s}'$ into the empty word.
Under the assumptions, we show that there exists such a sequence without a braid-move. This implies the lemma because nil-moves are also allowed in W_n and for all $\mathbf{a} \in S^*$ we have $l_R(\omega(\mathbf{a})) \leq l_{R_n}(\omega_n(\mathbf{a}))$ (see Lemma 5.3.2).

Fix such a braid-minimalistic sequence $(\alpha_i)_{i\leq a}$ (see Lemma 7.1.3). We assume that it contains a braid-move. With Lemma 7.1.4, we conclude that $\mathbf{s}'$ contains a subword of the form $(s_i s_j)^{m_{ij}}$ with $(s_i s_j)^{m_{ij}}$ in $\mathcal{R}$. Hence, the same holds for $\mathbf{s}$ and we arrive at a contradiction. Correspondingly, the sequence $(\alpha_i)_{i\leq a}$ contains no braid-moves. This proves the assertion. $\square$

Definition 7.1.7. The *indicator function* $\mathbb{1}_A$ of a subset A of a set X is defined as

$$\mathbb{1}_A : X \to \{0,1\}; \quad x \mapsto \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}.$$

For convenience, we write for example $\mathbb{1}_{\{2,3,\dots\}}(a)$ as $\mathbb{1}_{a\geq 2}$ for $a \in \mathbb{Z}$ and do so analogously in similar cases.

Together with Theorem 1, Lemma 7.1.6 above implies that the reflection length of powers of a Coxeter element in arbitrary Coxeter systems behaves like in the universal case, if the power of the Coxeter element is small enough in relation to the braid relations of the Coxeter group.

Corollary 7.1.8. *Let (W,S) be an arbitrary Coxeter system of rank n with standard generators $S = \{s_1, \dots s_n\}$, reflection length function l_R and $\mathbf{w} = s_1 \cdots s_n$. The following equality holds*

$$l_R(\omega(\mathbf{w}^\lambda \cdot s_1 \cdots s_r)) = l_{R_n}(\omega_n(\mathbf{w}^\lambda \cdot s_1 \cdots s_r)) = \lambda(n-2) + r$$

if $\lambda + \mathbb{1}_{\{r\geq 2\}} < \min\{m_{ij} \mid s_i \neq s_j \in S\}$.

Lemma 7.1.6 shows that if an element has an S-reduced expression without subwords of the form $(s_i s_j)^{m_{ij}} \in \mathcal{R}$, the reflection length of the element represented by an S-reduced word in an arbitrary Coxeter group is equal to the reflection length of the element represented by the same word in the universal Coxeter group of the same rank. Now, we consider arbitrary words. So the corresponding elements in the Coxeter groups may have different reflection lengths.

Theorem 4 (Lower bound for rank n)**.** *Let w be an element in a Coxeter system (W,S) of rank n represented by an S-reduced word $\mathbf{s} = u_1 \cdot \dots \cdot u_p$. Further, let $\tilde{\mathbf{s}}$ be a word obtained from $\mathbf{s}$ by omitting all letters in a deletion set $D(\mathbf{s})$. Let m be the minimal number of braid-moves necessary to transform $\tilde{\mathbf{s}}$ into the identity. The reflection length $l_R(w)$ in W is bounded from below:*

$$l_{R_n}(\omega_n(\mathbf{s})) - 2m \leq l_R(w).$$

Proof. We have $w(\tilde{\mathbf{s}}) = \mathbb{1}$. Thus, a finite sequence $(\sigma_1, \dots, \sigma_t)$ of nil-moves and braid-moves transforms the word $\tilde{\mathbf{s}}$ into the empty word e (see Theorem 3.2.13). There exist no braid-moves in W_n. So if there exists a sequence of only nil-moves that transforms $\tilde{\mathbf{s}}$ into e, we have $w_n(\tilde{s}) = \mathbb{1}$ and therefore $l_R(x) = l_{R_n}(w(\mathbf{s}))$.

Assume that $(\sigma_1, \dots .\sigma_t)$ contains exactly m braid-moves. This translates into a finite sequence of words

$$(\mathbf{s}_1 = \mathbf{s}_1' \mathbf{b}_{i_1 j_1} \mathbf{s}_1'', \dots, \mathbf{s}_m = \mathbf{s}_m' \mathbf{b}_{i_m j_m} \mathbf{s}_m'', \mathbf{s}_{m+1}),$$

where $\mathbf{s}_1$ is obtained from $\tilde{\mathbf{s}}$ by finitely many nil-moves, $\mathbf{s}_{l+1}$ is obtained from $\mathbf{s}_l$ by applying the braid-braid move $\mathbf{b}_{i_l j_l} \mapsto \mathbf{b}_{j_l i_l}$ and finitely many nil-moves. The last entry $\mathbf{s}_{m+1}$ is transformable into the empty word only by nil-moves. For the last element of the sequence, we have $\omega(\mathbf{s}_{m+1})) = \omega_n(\mathbf{s}_{m+1}) = \mathbb{1}$.

On the level of group elements, we have $\omega(\tilde{\mathbf{s}}) = \omega(\mathbf{s}_i) = \mathbb{1}$ and in the universal Coxeter group W_n

$$\omega_n(\mathbf{s}_{l+1}) = \omega_n(\mathbf{s}_l) \cdot r_l r_l',$$

where r_l, r'_l are reflections in W_n. Executing a braid-move on a word is equivalent to removing the first (most left) letter and adding one letter on the right side of $\mathbf{b}_{ij}$:

$$s_i s_j \cdots s_i \mapsto \hat{s}_i s_j \cdots s_i \cdot s_j.$$

Removing as well as adding one letter translates to multiplying with a reflection from the left or right on the level of group elements. Applying Lemma 3.3.13 for every transition between elements in the sequence $\omega_n(\mathbf{s}_1), \ldots, \omega_n(\mathbf{s}_{m+1}))$, we obtain $l_{R_n}(\omega_n(\tilde{\mathbf{s}})) \leq 2m$. Together with Theorem 3.3.20, this implies

$$l_{R_n}(\omega_n(\mathbf{s})) \leq l_R(\omega(\mathbf{s})) + 2m,$$

which is equivalent to the assertion. □

Remark 7.1.9. Even though this lower bound is sharp in some cases, it is not in general. Take for example the Coxeter group W_3^3 of type $\tilde{\mathbf{A}}_2$ with generating set $S = \{s_1, s_2, s_3\}$. The element $v \in W_3^3$ represented by the word

$$\mathbf{r} := s_1 s_2 s_1 s_3 s_1 s_3 s_2 s_1 s_2 \in S^*$$

has reflection length 1, where $l_{R_n}(\omega_n(\mathbf{r})) = 3$. This can be seen by removing the letter in the middle (see Theorem 3.3.20). For W_3^3, we obtain the identity. For W_n, we do not. Thus, the reflection length $l_{R_n}(\omega_n(\mathbf{r}))$ has to be at least 3, because of parity reasons. The sequence of word transformations is

$$s_1 s_2 s_1 s_3 \hat{s}_1 s_3 s_2 s_1 s_2 \mapsto s_1 s_2 s_1 \cdot s_2 s_1 s_2 \mapsto s_2 s_1 s_2 \cdot s_2 s_1 s_2 \mapsto e.$$

The last $\mapsto$ includes 3 nil-moves. According to the theorem, we have

$$l_{R_n}(\omega_n(\mathbf{r})) - 2 \leq l_R(v)$$

and inserting the values leads to $1 \leq 1$. So in this case, the bound is sharp.

On the other hand, in Example 3.3.21 there are 3 braid-moves executed for the element $w^4 s_1 s_2 \in W_3^3$. We know from Theorem 1 that $l_{R_n}(\omega_n((s_1 s_2 s_3)^4 s_1 s_2)) = 6$. Consequently, the lower bound from the theorem is 0 but the reflection length of $w^4 s_1 s_2$ is 2.

Remark 7.1.10. The lower bound also depends on the deletion set $D(\mathbf{s})$. Consider the word $\mathbf{t} = s_3 s_1 s_2 s_1 s_3 s_2 s_1 s_2$ in a Coxeter group $W = \langle \{s_1, s_2, s_3\} \mid \mathcal{R} \rangle$, in which $(s_1 s_2)^3 \in \mathcal{R}$ and no braid relation exists with $m_{ij} = 2$. It is $l_R(\omega(\mathbf{t})) = l_{R_n}(\omega_n(\mathbf{t}))$. The reflection length is 2 because of parity reasons and the following deletion sets

$$\hat{s}_3 s_1 s_2 s_1 \hat{s}_3 s_2 s_1 s_2 \qquad \text{and} \qquad s_3 s_1 \hat{s}_2 s_1 s_3 s_2 \hat{s}_1 s_2.$$

Applying the theorem above with the left deletion set leads to a lower bound of 0 whereas the right side leads to a sharp lower bound of 2. In the case of the word $\mathbf{r}$ from the last remark, the lower bound cannot be sharpened by a different choice of a deletion set, because the only deletion set for a reduced word representing a reflection is the letter in the middle.

An immediate question is under which circumstances the lower bound is sharp. Follow-up questions are if the lower bound is sharp for special deletion sets and if the sharpness only holds for certain elements. These questions are not discussed further in this work.

The lower bound of Theorem 4 can be improved by the following lemmas. The lemmas are based on the observation that a finite sequence of braid-moves is in some cases equivalent to the concatenation with 2 reflections.

Lemma 7.1.11. *Let (W,S) be a Coxeter system. Further, let $\sigma = \sigma_n \circ \cdots \circ \sigma_1$ be a sequence of braid-moves on a word $\mathbf{s} \in S^*$ with $\sigma_i : \mathbf{b}^i_{x_i y_i} \mapsto \mathbf{b}^i_{y_i x_i}$ for $x_i, y_i \in S$. If the last letter of $\mathbf{b}^i_{y_i x_i}$ is the first letter of $\mathbf{b}^{i+1}_{x_{i+1} y_{i+1}}$ for all $i \in \{1, \ldots, n-1\}$, the execution of σ on the word $\mathbf{s}$ is equivalent to multiplying with two reflections on the group-element level in W_n:*

$$\omega(\sigma(\mathbf{s})) = \omega(\mathbf{s}), \;\; \omega_n(\sigma(\mathbf{s})) = \omega_n(\mathbf{s}) \cdot r_1 \cdot r_2 \quad \text{with} \quad r_i \in R_n.$$

Proof. Executing the braid move σ_i is the same as omitting the first letter (letter on the left side) of $\mathbf{b}^i_{x_i y_i}$ and extending $\mathbf{b}^i_{x_i y_i}$ with a letter $s \in \{x_i, y_i\}$ on the right side. If the corresponding m_{ij} is even, we have $\mathbf{b}_{ij} \mapsto s_i \cdot \mathbf{b}_{ij} \cdot s_i = \mathbf{b}_{ji}$. If the corresponding m_{ij} is odd, we have $\mathbf{b}_{ij} \mapsto s_i \cdot \mathbf{b}_{ij} \cdot s_j = \mathbf{b}_{ji}$. Since the last letter of $\mathbf{b}^i_{y_i x_i}$ is the first letter of $\mathbf{b}^{i+1}_{x_{i+1} y_{i+1}}$ for all $i \in \{1, \ldots, n-1\}$, the letter that gets inserted by applying σ_i is removed by applying σ_{i+1} for all $i \in \{1, \ldots, n-1\}$. Hence, executing σ in total corresponds to omitting the letter for σ_1 and inserting the letter for σ_n. On a group-element-level in W_n, this is equivalent to multiplying with two reflections that are represented by palindromes. □

Remark 7.1.12. The lemma is also true if the first letter of $\mathbf{b}^i_{y_i x_i}$ is the last letter of $\mathbf{b}^{i+1}_{x_{i+1} y_{i+1}}$ for all $i \in \{1, \ldots, n-1\}$. This condition reflects the analogous situation from the right to the left. Whereas, in the presuppositions of the lemma, a letter is "wandering" from left to right from braid-move to braid-move.

Another situation to be considered is that the first and the last letter of the first braid-move are involved in following braid-moves in both directions. In this case, the braid-moves in one direction do not influence the ones in the other direction and can be executed first. So the lemma can be applied in one direction. Afterwards, the braid-moves in the other direction are considered. Analogously to the lemma, we obtain that executing the braid-moves in both directions is equivalent to multiplying with four reflections on a group-element-level in W_n.

Lemma 7.1.13. *Let (W,S) be a Coxeter system. Further, define $\mathbf{B}^{-2}_{ij}$ to be the alternating word consisting of the two letters $s_i, s_j \in S$ with word length $2m_{ij} - 2$ starting with s_i. Let $s_1, s_2, s_3 \in S$ be distinct. The concatenation of two distinct braid-moves on the word $\mathbf{B}^{-2}_{12}\mathbf{B}^{-2}_{31}$:*

$$\mathbf{B}^{-2}_{12}\mathbf{B}^{-2}_{31} \mapsto s_2 s_3$$

is equivalent to multiplying with two reflections on the group-element-level in W_n.

Proof. Since both alternating words $\mathbf{B}^{-2}_{12}, \mathbf{B}^{-2}_{31}$ have even word length and consist exactly of two distinct letters, $\mathbf{B}^{-2}_{12}$ ends in s_2. Ignoring the first or the last letter of them yields palindromes of odd word length. Deleting these palindromes is equivalent on a group-element-level in W_n to multiplying with a reflection for each palindrome. The word that is left is $s_2 s_3$. □

Remark 7.1.14. With this lemma and Theorem 4 we can derive a sharp lower bound in the case of Example 3.3.21.

Notation 7.1.15. Let (W,S) be a Coxeter system. We write a word $\mathbf{s} := u_1 \cdots u_m \in S^*$ as $\mathbf{s} = (\mathbf{s}(1), \ldots, \mathbf{s}(m))$ with $u_i = \mathbf{s}(i)$ to keep track of the initial position of a letter while applying nil-moves and braid-moves. In the process of applying nil-moves and braid-moves, the i-th position can be filled with the according letter $\mathbf{s}(i) \in S$ or the empty word e. A subword $\mathbf{s}(i_1) \cdots \mathbf{s}(i_t)$ is called *consecutive* if the entries in $(\mathbf{s}(1), \ldots, \mathbf{s}(m))$ between the letters $\mathbf{s}(i_a)$ and $\mathbf{s}(i_{a+1})$ are all filled with e for all $a \in \{1, \ldots, n-1\}$. With this notation, nil-moves change two equal consecutive letters to the empty word. A braid-move on a consecutive subword $\tilde{\mathbf{s}}$ that consists of two alternating letters $s_i, s_j \in S$ with length m_{ij} permutes the two types of letters in $\tilde{\mathbf{s}}$.

Definition 7.1.16. Two braid-moves in a sequence of nil-moves and braid-moves on a word $\mathbf{s} = (\mathbf{s}(1), \dots, \mathbf{s}(m))$ *interfere* if there exists an $i \in \{1, \dots, m\}$ such that both braid-moves change the i-th entry in $(\mathbf{s}(1), \dots, \mathbf{s}(m))$. In this case, we say that the braid-moves are *interfering in the index* i.

Using this vocabulary and notation, Theorem 4 can be strengthened with the following lemma.

Lemma 7.1.17. *Let (W, S) be a Coxeter system with $m_{ij} > 2$ for all distinct $s_i, s_j \in S$. For every word $\mathbf{s} = (\mathbf{s}(1), \dots, \mathbf{s}(m))$ over the alphabet S^*, there exists a finite sequence of nil-moves and braid-moves $(\alpha)_{l \in L}$ transforming the word into an S-reduced word such that all pairs of braid-moves in $(\alpha)_{l \in L}$ interfere maximally in one index.*

Proof. Let $\mathbf{s} = (\mathbf{s}(1), \dots, \mathbf{s}(m))$ be a word over the alphabet S^*. From the solution of the word problem for Coxeter groups, we know that there exists a sequence $(a'_k)_{k \in K}$ that transforms $\mathbf{s} = (\mathbf{s}(1), \dots, \mathbf{s}(m))$ into an S-reduced word with some $\mathbf{s}(i) = e$ if $\mathbf{s}$ is not S-reduced (see Theorem 3.2.13). If $\mathbf{s}$ is already S-reduced, the sequence is empty. Since there is no braid-move necessary to reduce a word of word length 2, the statement of the lemma holds if only one or two indices are different from e. Define $c := |\{i \in \{1, \dots, m\} \mid \mathbf{s}(i) \neq e\}|$ to be the number of indices in $\mathbf{s}$ that are not filled with the empty word. From here, we prove the lemma by induction over c.

Let $\mathbf{s} = (\mathbf{s}(1), \dots, \mathbf{s}(m))$ be a word with $|\{i \in \{1, \dots, m\} \mid \mathbf{s}(i) \neq e\}| = c + 1$. If $\mathbf{s}$ is S-reduced, there exists the empty sequence as a sequence that does not contain pairs of braid-moves that interfere in more than one index. Otherwise, there exists a finite sequence $(a'_k)_{k \in K}$ of nil-moves and braid-moves that transforms $\mathbf{s}$ into an S-reduced word. Assume that $(a'_k)_{k \in K}$ contains two braid-moves a'_x and a'_y with $x < y$ such that a'_y interferes with a'_x in at least two indices. Further, we may assume that no nil-moves are appearing in the sequence $(a'_k)_{k \in K}$ before a_x because in this case a nil-move decreases $c+1$ by -2. Following the induction assumption $\mathbf{s}$ can be transformed into an S-reduced word by a sequence of nil-moves and braid-moves without braid-moves that interfere in more than one index.

Braid-moves are executed on consecutive subwords of $\mathbf{s} = (\mathbf{s}(1), \dots, \mathbf{s}(m))$. So a'_y interferes with a'_x in at least two adjacent indices that are both not e. This implies that a'_x and a'_y are braid-moves permuting the same two generators $s_i, s_j \in S$, because we have $m_{ij} > 2$ for all distinct $s_i, s_j \in S$. We can assume that a'_x and a'_y are interfering in less than m_{ij} indices because in this case a'_y would just reverse a'_x and both braid-moves can be omitted to obtain a reduced word. Thus, we additionally may assume that there exists a largest non-empty index i_e touched by a'_x and a'_y that is adjacent to a larger index i_f solely touched by a'_y (all other cases are symmetric). Since a'_y is executed after a'_x in $(a'_k)_{k \in K}$, it follows $\mathbf{s}(i_e) = \mathbf{s}(i_f) \in \{s_i, s_j\}$. This allows us to define a new sequence $(a_l)_{l \in L}$ of nil-moves and braid-moves. The fist move a_1 is a nil-move that substitutes both adjacent entries $\mathbf{s}(i_e)$ and $\mathbf{s}(i_f)$ with e. The resulting word $\mathbf{s}'$ has $c - 1$ non-empty indices. The rest of the sequence $(a_l)_{l \in L}$ is a sequence that reduces $\mathbf{s}'$ such that all pairs of braid-moves interfere maximally in one index, which exists by the induction assumption. This completes the proof. $\square$

By using the lemmas above, we can derive a lower bound for the reflection length function from Theorem 4 for some elements. This is without knowing a deletion set, which is stronger than knowing the reflection length. Consider a specific S-reduced word $\mathbf{s}$ for an element w

in a Coxeter group W and determine the maximal number of non-interfering braid-moves possible after omitting some letters in $\mathbf{s}$. The lower bounds obtained this way are not sharp in general. For Coxeter groups with sufficiently large braid relations, it implies that the powers of a Coxeter element have unbounded reflection length. This leads to the following Theorem.

Theorem 5 (Power sequences with unbounded reflection length)**.** *Let (W, S) be a Coxeter system of rank n and let $M = (m_{ij})_{i,j\in I}$ denote its Coxeter matrix. Further, let w be a Coxeter element in W. Then,*

(i) if $n = 3$ and $\min\{m_{ij} \mid i \neq j,\ i, j \in I\} \geq 5$, or

(ii) if $n \geq 4$ and $\min\{m_{ij} \mid i \neq j,\ i, j \in I\} \geq 3$,

we have

$$\lim_{\lambda\to\infty} l_R(w^\lambda) = \infty.$$

Proof. Assume $k = \min\{m_{ij} \mid i \neq j,\ i, j \in I\} > 2$ because Lemma 5.1.7 implies that this is a necessary condition for sequences of powers of Coxeter elements with unbounded reflection length. For $\lambda \in \mathbb{N}$, Theorem 3.2.13 implies that $\mathbf{s} = (s_1 \cdots s_n)^\lambda$ is the unique S-reduced word representing the element w^λ. Let $\mathbf{s}'$ be the word obtained from $(s_1 \cdots s_n)^\lambda$ by removing all letters in a deletion set $D(\mathbf{s})$. There exists a finite sequence $(\alpha_i)_{i\in I}$ of nil-moves and braid-moves on $\mathbf{s}'$ to transform $\mathbf{s}'$ into the empty word (see Theorem 3.2.13). According to Lemma 7.1.17, we may assume additionally that all pairs of braid-moves in $(\alpha_i)_{i\in I}$ interfere maximally in one index of $\mathbf{s} = (\mathbf{s}(1), \ldots, \mathbf{s}(\lambda \cdot n))$.

We do not know a concrete deletion set for $\mathbf{s}$ nor a sequence of nil-moves and braid-moves to reduce $\mathbf{s}'$. Hence, it is necessary to consider the lowest lower bound obtainable with Theorem 4 for the reflection length of w^λ. A sequence of interfering braid-moves is counted as maximally four braid-moves for the lower bound of Theorem 4 (see Lemma 7.1.11 and Remark 7.1.12). Therefore, we consider the maximal number of possible non-interfering braid-moves on $\mathbf{s}$. Let $\xi(\lambda)$ be the maximal number of non-interfering braid-moves possible on $(s_1 \cdots s_n)^\lambda$. Hence, a lower bound for the reflection length of w is

$$l_{R_n}(\omega_n((s_1 \cdots s_n)^\lambda)) - 2\xi(\lambda). \tag{7.1.3}$$

By counting letters, the minimal S-length of a minimal consecutive subword in $(s_1 \cdots s_n)^\lambda$ containing $\mathbf{b}_{ij}$ in dependency on m_{ij} is

$$\chi(m_{ij}) := \begin{cases} \frac{m_{ij}-1}{2} \cdot n + 1 & \text{for odd } m_{ij} \\ \frac{m_{ij}}{2} \cdot n - (n-2) & \text{for even } m_{ij} \end{cases}$$

for all distinct $s_i, s_j \in S$. The minimal word length of a minimal consecutive subword in $\mathbf{s}$ containing $\mathbf{b}_{ij}$ as a subword is obtained for $j = i+1$ (conjugacy yields other minimal cases).

The word length of w^λ is $\lambda \cdot n$. Assume $k = m_{ij}$ with $j = i + 1$. Thus, we have

$$\xi(\lambda) = \frac{\lambda \cdot n}{\chi(m_{ij})}.$$

Inserting this in Equation (7.1.3) together with Theorem 1 yields the following lower bound for the reflection length of w^λ:

$$\begin{aligned}(\lambda-1)\cdot(n-2)+n-2\cdot\frac{\lambda\cdot n}{\chi(m_{ij})} &\leq l_R(w^\lambda)\\ \Leftrightarrow\quad \lambda n\left(1-\frac{2}{\chi(m_{ij})}\right)-2\lambda+2 &\leq l_R(w^\lambda).\end{aligned}$$

The left term is an unbounded monotonous growing function in the variable λ if and only if

$$\left(1-\frac{2}{\chi(m_{ij})}\right)>\frac{2}{n}.$$

By inserting $\chi(m_{ij})$, we obtain that this is equivalent to the condition

$$\left(\frac{m_{ij}}{2}\cdot n-(n-2)\right)\cdot(n-2)-2n>0 \tag{7.1.4}$$

for even m_{ij}. For odd m_{ij}, it is equivalent to the condition

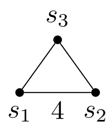
$$\left(\frac{m_{ij}-1}{2}\cdot n+1\right)\cdot(n-2)-2n>0. \tag{7.1.5}$$

This implies that the sequence $(l_R(w^\lambda))_{\lambda\in\mathbb{N}}$ diverges towards ∞ if the Inequality (7.1.4) is fulfilled for even k or if the Inequality (7.1.5) is fulfilled for odd k. Accordingly, the sequence of powers of a Coxeter element in a Coxeter group of rank 3 has unbounded reflection length if $k\geq 5$. This is to be seen by inserting the corresponding values in the inequality. For higher-rank Coxeter groups, the reflection length of the powers of a Coxeter element is unbounded for $k\geq 3$. This proves the lemma. □

Remark 7.1.18. The theorem above is independent of Duszenko's Theorem 4.1.3. So it can be seen as a constructive proof of the unboundedness of the reflection length function on Coxeter groups whose braid relations are large enough.

The theorem above does not cover all infinite non-affine Coxeter groups. We know that the reflection length is also an unbounded function on all infinite non-affine Coxeter groups. It is reasonable to conjecture that the same statement also applies to the exceptions with all $m_{ij}>2$. With Theorem 3.3.20, we implemented an algorithm to compute the reflection length (see Appendix A). This conjecture is supported by all our calculations of the reflection length for $n=3$ and small λ in the exceptional cases. If $m_{ij}=2$ appears as a minimal braid relation, the statement of Theorem 5 is false for rank-3 Coxeter groups and and presumably more complicated for higher ranks. This is illustrated by Lemma 5.1.7 and its proof.

Example 7.1.19. The Coxeter group W defined by the Coxeter graph



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is the infinite non-affine Coxeter group of rank 3 with the smallest braid relations and no commuting generators up to isomorphism. Table 7.1 shows the reflection length of $\omega(s_1s_2s_3)^\lambda$ as a function of λ. For $\lambda\leq 2$, the reflection length of $\omega(s_1s_2s_3)^\lambda$ behaves in W like $l_{R_n}(\omega_n(s_1s_2s_3)^\lambda)$ according to Lemma 7.1.6. For higher values of λ, it behaves differently. This is to be seen by comparing the values in Table 7.1 with the values of the formula in Theorem 1 for $\lambda\geq 3$.

λ	2	3	4	5	6	7	8	9	10	11	12	15
$l_R(\omega(s_1s_2s_3)^\lambda)$	4	3	4	5	4	5	4	5	6	5	6	7

Table 7.1: Reflection length of powers of a Coxeter element.

Example 7.1.20. Figure 7.1 shows four different sections of the coloured Coxeter complex (see Example 2.3.13) of the triangle group $(4,4,4)$ centred around the powers of a Coxeter element. The colour of a simplex corresponds to the reflection length of the associated group element. A small reflection length corresponds to a colour with a small wavelength and a large reflection length corresponds to a colour with a high wavelength (see Paragraph 1.4.1 for an explanation of the colour scheme). Black represents the reflection length 0. The chamber in the position of the black chamber in the upper left complex corresponds to e,$s_1s_2s_3$, $(s_1s_2s_3)^2$ and $(s_1s_2s_3)^3$ from left to right. It can be seen that the surrounding colours as well as the chamber itself change to larger wave length with increasing λ.

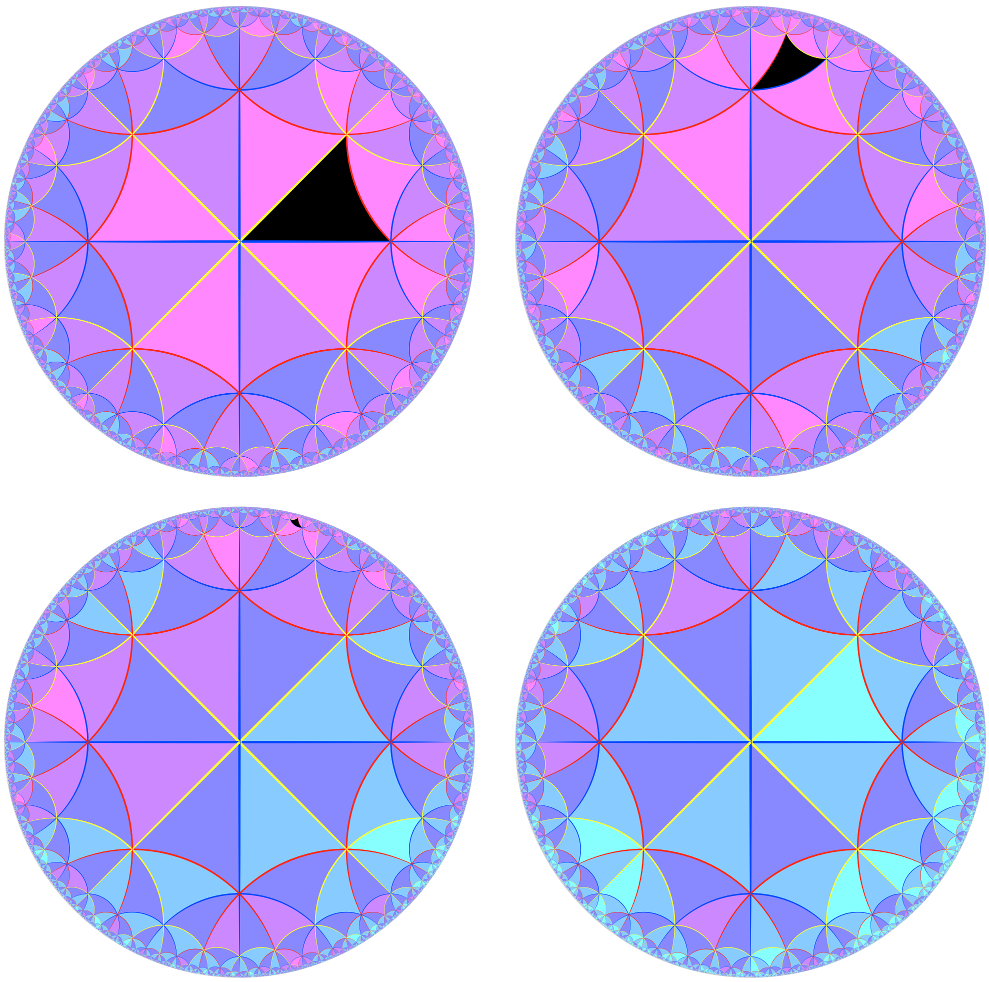


Figure 7.1: Different sections of the Coxeter complex of the triangle group $(4,4,4)$ centred around the powers of a Coxeter element embedded in $\mathbb{H}^2$ in the Poincaré disc model.

7.2 Upper bounds for reflection length in single braided Coxeter groups

In this section, we study upper bounds of the reflection length of powers of Coxeter elements in Coxeter groups corresponding to a Coxeter matrix with the same entry everywhere except the diagonal. From counting subwords of the form $(s_i s_j)^{m_{ij}}$ and with Theorem 4, we derive sharp upper bounds for the reflection length of the powers of Coxeter elements in these Coxeter groups. The proof of the upper bounds is inductive. We distinguish between rank-3 and higher-rank Coxeter groups because the reflection length of powers of elements with word length 3 behaves differently from the reflection length in higher-rank Coxeter groups.

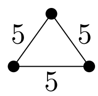
Lemma 7.1.6 is a first hint that not all braid-moves possible on a word are influencing the reflection length. Since braid-moves do not change the word length, every letter has to be deleted by a nil-move. For every consecutive subword of a power of a Coxeter element c^λ on which a braid-move was applied, letters to cancel this subword are to be found already in the right order in a reduced expression for c^λ. This is why, counting half of all possible braid-moves on a reduced expression of a power of a Coxeter element, is intuitive for estimating an upper bound for its reflection length. This section shows that this is indeed leading to a sharp upper bound for the reflection length.

Definition 7.2.1. We call a Coxeter group W *single braided* and denote it with W_k^n if it is defined by a complete graph over n vertices with a constant labelling function

$$m(\{u,v\}) = k \in \mathbb{N}_{\geq 2}.$$

If W is a single braided Coxeter group of rank n, every element σ in the symmetric group $S(n)$ defines a reflection length preserving automorphism of W by permuting generators.

Example 7.2.2. The single braided Coxeter group W_5^3 corresponds to the Coxeter graph:



This is neither a finite nor an Euclidean reflection group according to the classification of these groups.

Let (W_k^n, S) be a single braided Coxeter system of rank $n \geq 3$ with a constant edge labelling $m = k \in \mathbb{N}_{\geq 4}$. Theorem 5 covers all single braided infinite non-affine Coxeter groups with one exception, $W3_4$. This follows directly from the classification of Euclidean reflection groups.

Corollary 7.2.3. *The powers of every Coxeter element in a single braided infinite non-affine Coxeter system (W,S) have unbounded reflection length if (W,S) is not the single braided Coxeter system of rank 3 with $m_{ij} = 4$.*

Moreover, we can directly extract a lower bound for the reflection length of the powers of a Coxeter element from the proof of Theorem 5. Sharp upper bounds are given by the following results.

Theorem 6 (Upper bound for single braided power sequences in rank 3)**.** *Let (W_k^3, S) be a single braided Coxeter system with $k \geq 3$. The reflection length of elements of the form $(s_1s_2s_3)^\lambda s_1 \cdots s_r$ in (W_k^3, S) with $1 \leq r \leq 3$ and $\lambda \in \mathbb{N}_0$ is bounded from above by*

$$l_R(\omega((s_1s_2s_3)^\lambda s_1 \cdots s_r)) \leq \lambda + r - 2 \cdot \left\lfloor \frac{\lambda + \mathbb{1}_{r\geq 2}}{k} \right\rfloor .$$

Proof. We show the inequality by induction over λ. We assume k to be at least 3. Thus, for $\lambda \leq 1$ we have $\left\lfloor \frac{\lambda+\mathbb{1}_{r\geq 2}}{k} \right\rfloor = 0$ and $(s_1s_2s_3)^\lambda s_1 \cdots s_r$ has no subword of the form $(s_is_j)^3$. By Lemma 7.1.6 the reflection length $l_R(\omega((s_1s_2s_3)^\lambda s_1 \cdots s_r))$ is equal to the corresponding reflection length in the universal Coxeter group $l_{R_n}(\omega_n((s_1s_2s_3)^\lambda s_1 \cdots s_r))$ for $\lambda \leq 1$. Theorem 1 implies

$$l_R(\omega((s_1s_2s_3)^\lambda s_1 \cdots s_r)) = \lambda + r.$$

Hence, the statement of the lemma is true for $\lambda \leq 1$.

Now, consider $(s_1s_2s_3)^{\lambda+1}s_1$. Reflection length is invariant under conjugation and invariant under permutation of generators, since we only consider single braided Coxeter groups (see Lemma 3.3.10). Together with the induction hypothesis, this implies

$$l_R(\omega((s_1s_2s_3)^{\lambda+1}s_1)) = l_R(\omega((s_2s_3s_1)^\lambda s_2s_3)) = \lambda + 2 - 2 \cdot \left\lfloor \frac{\lambda+1}{k} \right\rfloor .$$

For the word $(s_1s_2s_3)^{\lambda+1}s_1s_2$, we obtain by conjugation, Lemma 3.3.13 and by the induction hypothesis

$$\begin{aligned}
l_R(\omega((s_1s_2s_3)^{\lambda+1}s_1s_2)) &= l_R(\omega(s_1s_2(s_1s_2s_3)^{k-2}s_1s_2(s_3s_1s_2)^{\lambda+1-(k-1)}s_3)) \\
&\leq l_R(\omega(s_1s_2(s_1s_2s_3)^{k-2}s_1s_2)) + l_R(\omega((s_3s_1s_2)^{\lambda+1-(k-1)}s_3)) \\
&= (k-2) + \lambda + 1 - (k-1) + 1 - 2\left\lfloor \frac{\lambda + 1 - (k-1)}{k} \right\rfloor \\
&= (\lambda+1) - 2\left\lfloor \frac{(\lambda+1)+1-k}{k} \right\rfloor \\
&= (\lambda+1) + 2 - 2\left\lfloor \frac{(\lambda+1)+1}{k} \right\rfloor .
\end{aligned}$$

So the lemma holds in this case.

For the reflection length of elements of the form $(s_1s_2s_3)^{\lambda+1}s_1s_2s_3$ we have

$$\begin{aligned}
l_R(\omega((s_1s_2s_3)^{\lambda+1}s_1s_2s_3)) &\leq l_R(\omega((s_1s_2s_3)^{\lambda+1}s_1s_2)) + 1 \\
&= (\lambda+1) + 2 - 2\left\lfloor \frac{(\lambda+1)+1}{k} \right\rfloor + 1
\end{aligned}$$

according to the induction hypothesis and Lemma 3.3.13. This completes the proof of the lemma. □

We state the following theorem analogously to Theorem 6 for single braided Coxeter groups of rank $n \geq 4$.

Theorem 7 (Upper bound for single braided power sequences in higher rank)**.** *In a single braided Coxeter system (W_k^n, S) with $n \geq 4$, the reflection length of the element represented by the word $\mathbf{s} = (s_1s_2 \cdots s_n)^\lambda s_1 \cdots s_r$ with $1 \leq r \leq n$ and $\lambda \in \mathbb{N}_0$ is bounded from above by*

$$l_R(\omega(\mathbf{s})) \leq \lambda(n-2) + r - 2 \cdot \mathbb{1}_{(\lambda+\mathbb{1}_{r\geq 2})\geq k} \cdot \left(1 + \left\lfloor \frac{\lambda - k + \mathbb{1}_{r\geq 2}}{k-1} \right\rfloor\right).$$

Proof. The assertion is proved by induction over λ. If $(\lambda + \mathbb{1}_{r\geq 2})$ is strictly smaller than k, the theorem is true because of Corollary 7.1.8 and Theorem 1. For the parameters we have $k \geq 3$ and $n \geq 4$.

Assume that the statement of the theorem is true for all $\lambda' < \lambda$ and all $1 \leq r \leq n$ with $\lambda, \lambda' \in \mathbb{N}$. For the reflection length of the element $w = \omega((s_1 s_2 \cdots s_n)^{\lambda} s_1)$ we have

$$l_R(w) = l_R(\omega((s_2 \cdots s_n s_1)^{\lambda-1} s_2 \cdots s_n))$$

because conjugacy in general and permuting the generators in a single braided Coxeter group preserves reflection length (see Lemma 3.3.10). The induction hypothesis gives us

$$\begin{aligned} l_R(w) &= (\lambda - 1)\cdot(n-2) + (n-1) - 2\cdot \mathbb{1}_{(\lambda-1+1)\geq k}\left(1 + \left\lfloor \frac{\lambda - 1 - k + 1}{k-1}\right\rfloor\right) \\ &= \lambda\cdot(n-2) + 1 - 2\cdot \mathbb{1}_{(\lambda+\mathbb{1}_{r\geq 2})\geq k}\left(1 + \left\lfloor \frac{\lambda - k}{k-1}\right\rfloor\right). \end{aligned}$$

Thus, the theorem is true for λ and $r = 1$.

To make a second induction argument according to r for a fixed λ, assume that the statement of the theorem is true for all $\lambda' < \lambda$ and all r' as well as for $\lambda' = \lambda$ and all r' with $1 < r' < r$. For the reflection length of the element $w = \omega((s_1 s_2 \cdots s_n)^{\lambda} s_1 \cdots s_r)$, we have

$$l_R(w) = l_R(\omega(s_1 \cdots s_r (s_1 \cdots s_n)^{k-2} s_1 s_2 (s_3 \cdots s_2)^{\lambda-(k-1)} s_3 \cdots s_n))$$

because conjugacy preserves reflection length. All exponents are non-negative since it is $(k-1) \leq \lambda$ (the other case is covered by the induction hypothesis). We obtain the identity element from the consecutive subword $s_1 \cdots s_r (s_1 \cdots s_n)^{k-2} s_1 s_2$ if we remove all letters distinct from s_1 and s_2 in it. This is true because $\omega((s_1 s_2)^k) = \mathbb{1}$ in W_k^n. Hence, we have the following inequality

$$l_R(w) \leq (k-2)\cdot(n-2) + (r-2) + l_R(\omega(s_3 \cdots s_2)^{\lambda-(k-1)} s_3 \cdots s_n)).$$

Permuting generators and the induction hypothesis imply

$$\begin{aligned} l_R(w) \leq\ & (k-2)\cdot(n-2) + (r-2) + (\lambda - (k-1))\cdot(n-2) + (n-2) \\ & - 2\cdot \mathbb{1}_{(\lambda-(k-1)+1)\geq k}\left(1 + \left\lfloor \frac{\lambda - (k-1) - k + \mathbb{1}_{r\geq 2}}{k-1}\right\rfloor\right) \\ =\ & \lambda\cdot(n-2) + r - 2 \\ & - 2\cdot \mathbb{1}_{(\lambda-(k-1)+1)\geq k}\left(1 + \left\lfloor \frac{\lambda - (k-1) - k + \mathbb{1}_{r\geq 2}}{k-1}\right\rfloor\right). \end{aligned}$$

There are two cases to be distinguished. The induction start is done for all words of the form $(s_1 s_2 \cdots s_n)^{\tilde{\lambda}} s_1 \cdots s_{\tilde{r}}$ with $(\tilde{\lambda} + \mathbb{1}_{\tilde{r}\geq 2}) < k$. So we assume $(\lambda + \mathbb{1}_{r\geq 2}) \geq k$.

In the case where $(\lambda - (k-1) + 1) < k$, it follows that $\lambda - (k-1) < (k-1)$. This implies $\left\lfloor \frac{\lambda-k+1}{k-1}\right\rfloor = 0$ and we have

$$\begin{aligned} l_R(w) \leq\ & \lambda\cdot(n-2) + r - 2 \\ &= \lambda(n-2) + r - 2\cdot \mathbb{1}_{(\lambda+\mathbb{1}_{r\geq 2})\geq k}\left(1 + \left\lfloor \frac{\lambda - k + \mathbb{1}_{r\geq 2}}{k-1}\right\rfloor\right). \end{aligned}$$

Otherwise, $(\lambda - (k-1) + 1) \geq k$ and it follows directly

$$\begin{aligned} l_R(w) \leq\ & \lambda \cdot (n-2) + r - 2 - 2 \cdot \left(1 + \left\lfloor \frac{\lambda - (k-1) - k + \mathbb{1}_{r\geq 2}}{k-1} \right\rfloor\right) \\ &= \lambda \cdot (n-2) + r - 2 \cdot \left(1 + \left\lfloor \frac{\lambda - k + \mathbb{1}_{r\geq 2}}{k-1} \right\rfloor\right) \\ &= \lambda(n-2) + r - 2 \cdot \mathbb{1}_{(\lambda + \mathbb{1}_{r\geq 2})\geq k} \left(1 + \left\lfloor \frac{\lambda - k + \mathbb{1}_{r\geq 2}}{k-1} \right\rfloor\right). \end{aligned}$$

In total, the inequality

$$l_R((s_1 s_2 \cdots s_n)^\lambda s_1 \cdots s_r) \leq \lambda(n-2) + r - 2 \cdot \mathbb{1}_{(\lambda + \mathbb{1}_{r\geq 2})\geq k} \left(1 + \left\lfloor \frac{\lambda - k + \mathbb{1}_{r\geq 2}}{k-1} \right\rfloor\right)$$

is proved by induction and the proof is complete. □

Remark 7.2.4. The question of how to connect the upper and the lower bound remains. The lower bound that we obtain from the proof of Theorem 5 for elements of the form $w = \omega((s_1 s_2 \cdots s_n)^\lambda)$ in a single braided Coxeter group W_k^n is

$$(\lambda - 1) \cdot (n-2) + n - 2 \cdot \frac{\lambda \cdot n}{\frac{k-1}{2} \cdot n + 1} \leq l_R(w)$$

for odd k. The negative term of his lower bound is roughly double the negative term in the upper bound from Theorem 7

$$l_R(w) \leq (\lambda - 1) \cdot (n-2) + n - 2 \cdot \mathbb{1}_{\lambda \geq k} \cdot \left(1 + \left\lfloor \frac{\lambda - k}{k-1} \right\rfloor\right)$$

(also true for Theorem 6). For the lower bound, the negative part of the term counts subwords of the form $\mathbf{b}_{ij}$ of word length k. Whereas for the upper bound, the negative part of the term counts subwords of the from $(s_i s_j)^k$ of word length $2k$.

Our computations of $l_R(\omega((s_1 s_2 \cdots s_n)^\lambda))$ for small λ in different single braided Coxeter groups show in all instances that the upper bounds established in this section are exactly the reflection length.

Based on this and Lemma 7.1.6, we conjecture the following:

Conjecture 7.2.5. The upper bounds from Theorem 6 and Theorem 7 are equal to the reflection length function itself.

7.3 The reflection length in arbitrary and universal Coxeter groups

The results obtained in this chapter are mostly based on the comparison between the reflection length of elements in an arbitrary and in the universal Coxeter group of the same rank. The statement of the following conjecture implies a complete understanding of the relationship between the reflection length function on these different Coxeter groups. We conjecture that for a fixed word $\mathbf{s}$ and an arbitrary Coxeter group W, there always exists a deletion set $D(\mathbf{s})$ such that $D(\mathbf{s})$ is a subset of a deletion set of $\mathbf{s}$ in the universal Coxeter group of the same rank. We prove our conjecture for the special case, where $\mathbf{s}$ represents a reflection in W

Conjecture 7.3.1. Let $W = \langle S \mid \mathcal{R} \rangle$ be a Coxeter group and $w \in W$ be an element. Further, let $u_1 \cdots u_p$ be an S-reduced expression for w in W with $u_i \in S$. There exists a letter s in $u_1 \cdots u_p$ such that omitting it results in

$$l_R(\omega(u_1 \cdots \hat{s} \cdots u_p)) = l_R(w) - 1 \text{ and}$$
$$l_{R_n}(\omega_n(u_1 \cdots \hat{s} \cdots u_p)) = l_{R_n}(\omega_n(u_1 \cdots u_p)) - 1.$$

A weaker version would be that every element has an S-reduced expression for which the statement of the conjecture is true. If the reflection length in W is 1, the conjecture is true. For the proof of this, we need the following definition.

Definition 7.3.2. We define pairs of reduced words $(\mathbf{s}_i, \mathbf{s}_{-i})$ with $i \in \mathbb{N}$ as words over an alphabet S with relation $s = s^{-1}$ for all $s \in S$ such that one of the following conditions hold

(i) $\mathbf{s}_{-i} = \mathbf{s}_i^{-1}$,

(ii) for two letters $s_1, s_2 \in S$ we have $\mathbf{s}_i \in \{s_1, s_2\}^*$, $l_S(\mathbf{s}_i) \geq 2$ and

$$\mathbf{s}_{-i} = \begin{cases} \tau_{1,2}(\mathbf{s}_i) & \text{for odd } l_S(\mathbf{s}_i) \\ \mathbf{s}_i & \text{for even } l_S(\mathbf{s}_i), \end{cases}$$

where $\tau_{1,2} : S^* \to S^*$ exchanges s_1 and s_2.

For $s \in S$, we define a *twisted palindrome* of odd word length to be a word

$$\mathbf{s}_1 \cdots \mathbf{s}_n \cdot s \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1},$$

where $(\mathbf{s}_i, \mathbf{s}_{-i})$ satisfies (i) or (ii) for all $1 \leq i \leq n$.

Remark 7.3.3. Twisted palindromes are special cases of twisted conjugates of the generators in S if (W, S) is a universal Coxeter system. Conditions (i) and (ii) are disjoint.

Lemma 7.3.4. *Let the word $\mathbf{s}_1 \cdots \mathbf{s}_n \cdot s \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1}$ be a twisted-palindrome. For the element $t = \omega_n(\mathbf{s}_1 \cdots \mathbf{s}_n \cdot s \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1})$ represented by this word, the following equation holds:*

$$l_{R_n}(\omega_n(\mathbf{s}_1 \cdots \mathbf{s}_n \cdot \hat{s} \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1})) = l_{R_n}(t) - 1,$$

where the hat over s means omitting s.

Proof. Since the element t has odd word length in W_n, we have $l_{R_n}(t) = 2k + 1$ for $k \in \mathbb{N}_0$. We prove the statement by induction over $k \in \mathbb{N}_0$. For $k = 0$, $l_{R_n}(t)$ is equal to 1 and we know that $\mathbf{s}_1 \cdots \mathbf{s}_n \cdot s \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1}$ is a palindrome since there are no braid relations in W_n. Thus, we have $\mathbf{s}_{-i} = \mathbf{s}_i^{-1}$ for all $i \in \{1, \ldots, n\}$ and therefore $l_{R_n}(\omega_n(\mathbf{s}_1 \cdots \mathbf{s}_n \cdot \hat{s} \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1})) = 0$.

In general, if we have $l_{R_n}(\mathbf{s}_i \mathbf{s} \mathbf{s}_{-i}) \geq l_{R_n}(\mathbf{s}) + 2$ for a quasi-palindrome $\mathbf{s}$ and a pair of words $(\mathbf{s}_i, \mathbf{s}_{-i})$ like in Definition 7.3.2, it follows that the inequality is an equality. The reflection length has to increase by an even number because of parity reasons. The reflection length increases maximally by two since a word of the form $\mathbf{s}_i \mathbf{s}_{-i}$ has maximal reflection length 2.

The pair satisfies condition (ii) from Definition 7.3.2 since the two conditions are disjoint and conjugacy preserves reflection length (Lemma 3.3.10). To neutralize this effect on the reflection length of adding a pair of words $(\mathbf{s}_i, \mathbf{s}_{-i})$ in the outlined way, we distinguish two cases. If the word length of $\mathbf{s}_i$ is odd and hence also the word length of $\mathbf{s}_{-i}$, we remove the middle of both words and they vanish completely by applying $s = s^{-1}$ for all $s \in S$. Otherwise, if the word length of both is an even number, we remove the first letter in both cases and obtain a word that is conjugated to $\mathbf{s}$. Conjugation preserves reflection length. We especially obtain a quasi-palindrome again.

Let $l_{R_n}(\mathbf{s}_1 \cdots \mathbf{s}_n \cdot s \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1}) = 2k + 1 + 2$. From above, we know that there exists a pair of words $(\mathbf{s}_i, \mathbf{s}_{-i})$ such that the reflection length is reduced by two after omitting one letter in each word as described above. Moreover, we obtain an odd quasi palindrome with reflection length $2k + 1$ and we can apply the induction assumption: By deleting the letter in the middle of the word we decrease the reflection length by 1 again. The deleted letters are elements in a deletion set of $\mathbf{s}_1 \cdots \mathbf{s}_n \cdot s \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1}$ in W_n. According to Lemma 3.3.26, we have $l_{R_n}(\mathbf{s}_1 \cdots \mathbf{s}_n \cdot \hat{s} \cdot \mathbf{s}_{-n} \cdots \mathbf{s}_{-1}) = 2k + 1$ and the induction is complete. $\square$

7.4 Outlook

We conclude this chapter with an outlook for further research and follow-up questions. With regard to Chapter 5, the question arises whether the reflection length of the powers of Coxeter elements can also be expressed by a simple formula in other cases. Conjecture 7.2.5 displays a possible answer to this question for single braided infinite non-affine Coxeter groups. Other subfamilies in which it may be possible to obtain results similar to the ones in Section 7.2 are infinite non-affine triangle groups and higher-rank right-angled infinite non-affine Coxeter groups with sufficiently many infinity-relations. The consideration of these subfamilies presupposes a criterion for the reflection length of powers of Coxeter elements being unbounded. We know from Lemma 5.1.7 that in rank-3 right angled Coxeter groups all powers of Coxeter elements have reflection length smaller than 4. Let (W, S) be an infinite non-affine Coxeter system.

Question 1. Is there a general criterion to detect whether the sequence of reflection lengths of powers of a Coxeter element in W tends to ∞?

This criterion could be for the Coxeter graph of (W, S) for example. In case the reflection length of the sequence of powers of all Coxeter elements is bounded, which are the elements with large reflection lengths?

Question 2. If the reflection length of powers of Coxeter elements is bounded, do sequences of powers of other elements with unbounded reflection length exist?

Given that the reflection length of powers of a Coxeter element in W tends to ∞, is an equivalent statement to Lemma 5.2.1 true in general? Recall the function $m_S : \mathbb{N} \to \mathbb{N}$ with

$$m_S(a) := \min\{l_S(w) \mid l_R(w) = a, w \in W\}.$$

Question 3. Let $v = s_1 \cdots s_n \in W$ be a Coxeter element. If $\lim_{\lambda \to \infty} l_R(v^\lambda) = \infty$, does there exist an element $u \in W$ of the form $u = v^\lambda \cdot s_1 \cdots s_i$ with $\lambda \in \mathbb{N}_0$ such that $l_R(u) = a$ and $m_S(a) = l_S(u) = \lambda n + i$ for every $a \in \mathbb{N}$?

The reverse question about a maximal reflection length for a given word length is also of interest. This opens up a whole type of questions. How many elements for each reflection length exist for a fixed word length? Questions like this could be answered if we knew a two-variable generator function for an infinite non-affine Coxeter group W. This generating function takes a tuple of natural numbers as an input and gives back the number of elements in W with word length and reflection length according to the tuple as an output.

Question 4. Let (W, S) be a infinite non-affine Coxeter system. What precise form does the two-variable generating function

$$\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}; \quad (n, m) \mapsto |\{w \in W \mid l_S(w) = n, l_R(w) = m\}|$$

have?

A generating function like this is also not known for universal Coxeter groups. By the time of writing, this question is open for all infinite non-affine Coxeter groups. A starting point to answer this problem is to determine the distribution of elements with reflection length 1.

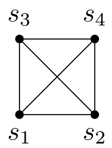
Tits gives a solution to the word problem for Coxeter groups in [Tit69]. This solution to the word problem is not limited to Coxeter groups with a finite generating set. So once we know the order of the product of every two reflections in a Coxeter group W, we can consider the Coxeter system (W, R) and apply the results of Tits. Then, every reflection factorisation of an element $w \in W$ can be transformed into an R-reduced reflection factorisation by a finite sequence of dual nil-moves and dual braid-moves. Moreover, two R-reduced reflection factorisations can be transformed into each other by applying a sequence of dual braid-moves.

Question 5. Given a Coxeter system (W, S), is there a way to determine the infinite Coxeter matrix of the dual Coxeter system (W, R)?

A. SageMath code to compute reflection length

The following SageMath code builds on the mathematics software system SageMath (see [The20]). We present a function `Rlength`, with which we can theoretically compute the reflection length of any element w in an arbitrary Coxeter group W. It is based on Dyer's Theorem 3.3.20. The algorithm for the computation of the reflection length works as follows. Take an S-reduced factorisation $\mathbf{s}$ of w. Try all possible combinations of omitting k letters in $\mathbf{s}$ and test if these lead to the identity. Here, $k \in \mathbb{N}$ is increased step by step.

The code. First, a Coxeter group object G is generated from a Coxeter matrix M. As an example, we take the Coxeter matrix corresponding to the Coxeter graph



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Coxeter groups are implemented in SageMath. The set S of standard generators is generated, too. We define an element $x = s_1 \cdots s_4 s_1 s_3$ for example.

```
import math
import numpy as np

M = CoxeterMatrix([[1,3,3,3],[3,1,3,3],[3,3,1,3], [3,3,3,1]])

#Generate Coxeter group object G from Coxeter matrix CM
G = CoxeterGroup(M)
S = G.simple_reflections()

x = S[1]*S[2]*S[3]*S[4]*S[1]*S[3]
```

The function `isRlength` takes an element w of the Coxeter group G and a natural number lr as an argument. It tests if $l_R(w) \leq lr$ and returns a boolean value. For each letter $u_i \in S$ in an S-reduced word $u_1 \cdots u_m \in S^*$ for w, it is checked if the inequality $l_R(\omega(u_1 \cdots u_{i-1} \hat{u}_i u_{i+1} \cdots u_m)) \leq lr - 1$ holds by calling the function `isRlength` recursively.

```
#Takes element w in G and tests if reflection length of w is equal
to lr.
def isRlength(w, lr):
    if(lr>1):
        GeneratorList=w.reduced_word()
        for i in range(len(GeneratorList)):
            l2 = copy(GeneratorList)
            l2.pop(i)
            cutword = product ([S[i] for i in l2])
            if(isRlength(cutword, lr-1)):
                return true
        return false
    elif(lr==1):
        return w.is_reflection()
    else:
        return "computation failed: lr<1"
```

To compute the reflection length, the parity of the word length $l_S(w)$ is determined first. Afterwards, the parameter lr is incremented from 1 or 2 in steps of two, depending on the parity. In each step, it is checked whether `isRlength(w,lr) == true`. The first value of lr for which `isRlength(w,lr)` equals `true` is returned. It is the reflection length of w.

```
#returns reflection length with Dyer-Theorem computation
def Rlength(w):
    genlist=w.reduced_word()
    if(genlist==[]):
        return 0
    else:
        if(len(genlist)%2==0):
            lr=2
        else:
            lr=1
        while((not isRlength(w, lr))):
            lr+=2
        return lr
```

Remark A.0.1. The code does an exhaustive search for a deletion set for $\mathbf{s}$ in W. To find a deletion set, the function `isRlength` is called

$$\frac{l_S(w)!}{l_R(w)!}$$

times with input $lr = l_R(w)$. This is why the computation of the reflection length of elements with a word length of 15 or higher is almost impracticable with the code (on an average computer).

Example A.0.2. To compute the reflection length of the element $s_1 \cdots s_4 s_1 s_3$ in the Coxeter system corresponding to the Coxeter graph displayed above, insert the following:

```
Rlength(x)
```

The output of this command is 4.

Remark A.0.3. Since the algorithm to compute the reflection length is a recursion, it is practical to store results obtained once in a database to accelerate further calculations that build on these results. The code has the potential to be optimized in terms of its runtime, as the same sets of indices are tested multiple times to find a deletion set. We know from Lemma 3.3.26 that the order of the omitted letters is irrelevant.

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