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Decay estimates for quasilinear elliptic equations and a Brezis–Nirenberg result in $D^{1,p}(\mathbb{R}^N)$

Siegfried Carl[1] · Hossein Tehrani[2]

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Abstract

We prove decay estimates for solutions of quasilinear elliptic equations in the whole $\mathbb{R}^N$ of the form

$$u \in X : -\text{div}\, A(x, \nabla u) = a(x) f(x, u),$$

where $X = D^{1,p}(\mathbb{R}^N)$ is the Beppo-Levi space (also called homogeneous Sobolev space). Based on these decay estimates we are able to prove a Brezis–Nirenberg type result for the energy functional $\Phi : X \to \mathbb{R}$ related to the p-Laplacian equation in $\mathbb{R}^N$ in the form

$$u \in X : -\Delta_p u = a(x) g(u),$$

saying that for the subspace V of bounded continuous functions with weight $1 + |x|^{\frac{N-p}{p}}$, a local minimizer of Φ in the finer V topology is also a local minimizer in the X-topology. Global L^∞-estimates on the one hand and pointwise estimates for solutions of quasilinear elliptic equations in terms of nonlinear Wolff potentials on the other hand play a crucial role in the proofs.

Keywords Beppo-Levi space · Quasilinear elliptic equation · Pointwise estimate · Decay estimate · Nonlinear Wolff potential · Local minimizer

Hossein Tehrani contributed equally to this work.

✉ Siegfried Carl
siegfried.carl@mathematik.uni-halle.de

Hossein Tehrani
tehranih@unlv.nevada.edu

1 Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg, 06099 Halle, Germany

2 Department of Mathematical Sciences, University of Nevada Las Vegas, Box 454020, Las Vegas, USA

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1 Introduction and main results

In this paper we are going to prove decay estimates of solutions of the following quasilinear elliptic equation in the whole $\mathbb{R}^N$

$$u \in X : -\operatorname{div} A(x, \nabla u) = a(x)\, f(x, u), \tag{1.1}$$

where $X = D^{1,p}(\mathbb{R}^N)$ is the Beppo-Levi space (homogeneous Sobolev space) which is the completion of $C_c^\infty(\mathbb{R}^N)$ under the norm $\|u\|_X = (\int_{\mathbb{R}^N} |\nabla u|^p\, dx)^{1/p}$, and for which we have the continuous embedding $X \hookrightarrow L^{p^*}(\mathbb{R}^N)$, where $p^* = \frac{Np}{N-p}$ denotes the critical Sobolev exponent. Throughout this paper we assume $2 \le p < N$. The coefficient $a : \mathbb{R}^N \to \mathbb{R}$, the vector field $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and the nonlinear function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are supposed to satisfy the following conditions:

(A0) $a : \mathbb{R}^N \to \mathbb{R}$ is measurable and satisfies the following decay condition for some $\alpha,\ c_a > 0$

$$|a(x)| \le c_a \frac{1}{1 + |x|^{N+\alpha}}, \quad x \in \mathbb{R}^N. \tag{1.2}$$

$A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, i.e., $x \mapsto A(x, \xi)$ is measurable in $\mathbb{R}^N$ for all $\xi \in \mathbb{R}^N$, and $\xi \mapsto A(x, \xi)$ is continuous for a.a. $x \in \mathbb{R}^N$, and satisfies for a.a. $x \in \mathbb{R}^N$ and for all $\xi,\ \hat{\xi} \in \mathbb{R}^N$

(A1) $|A(x, \xi)| \le a_1 |\xi|^{p-1} + (a_2(x))^{p-1}$,
(A2) $(A(x, \xi) - A(x, \hat{\xi}))(\xi - \hat{\xi}) \ge \nu |\xi - \hat{\xi}|^p$,
(A3) $A(x, \xi)\, \xi \ge \nu |\xi|^p$,

where $a_1,\ \nu$ are positive constants, and the nonnegative function $a_2 : \mathbb{R}^N \to \mathbb{R}$ is measurable and for some $d > 0$ satisfies

$$a_2(x) \le d \frac{1}{1 + |x|^{\frac{N-1}{p-1}}}, \quad x \in \mathbb{R}^N, \tag{1.3}$$

which results in the following corollary.

Corollary 1.1 *$a_2 \in L^q(\mathbb{R}^N)$ for $p \le q \le \infty$, and thus $a_2^{p-1} \in L^{p'}(\mathbb{R}^N)$.*

Proof Clearly, $a_2 \in L^\infty(\mathbb{R}^N)$. Let $p \le q < \infty$.

$$\begin{aligned} \int_{\mathbb{R}^N} a_2^q\, dx &= \int_{B(0,1)} a_2^q\, dx + \int_{\mathbb{R}^N \setminus B(0,1)} a_2^q\, dx \\ &\le c|B(0,1)| + c \int_1^\infty \varrho^{-q\frac{N-1}{p-1} + N - 1} d\varrho < \infty, \end{aligned}$$

because $p < N$ and thus $N - q\frac{N-1}{p-1} < 0$ for all $q \in [p, \infty)$. □

The nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., $x \mapsto f(x, s)$ is measurable in $\mathbb{R}^N$ for all $s \in \mathbb{R}$, and $s \mapsto f(x, s)$ is continuous in $\mathbb{R}$ for a.a. $x \in \mathbb{R}^N$, and satisfies the growth condition

(F) $|f(x,s)| \le c_f\big(1 + |s|^{\gamma-1}\big)$, $\quad \forall s \in \mathbb{R}$, a.a $x \in \mathbb{R}^N$, and some $1 \le \gamma < p^*$.

We define a solution of (1.1) as follows.

Definition 1.2 The function $u \in X$ is a solution of (1.1) if

$$\int_{\mathbb{R}^N} A(x, \nabla u)\nabla\varphi \, dx = \int_{\mathbb{R}^N} a(x) f(x, u)\varphi \, dx, \quad \forall \varphi \in X.$$

We note that due to hypotheses (A1)–(A3) and Corollary 1.1, the operator $T : X \to X^*$ given by the left-hand side of (1.1) through

$$\langle Tu, \varphi\rangle = \int_{\mathbb{R}^N} A(x, \nabla u)\nabla\varphi \, dx$$

is easily seen to be bounded, continuous, strongly monotone and coercive, however T is neither homogeneous nor of variational type. Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual space X^*.

Before stating our results, first a few words on the notation. The standard norms of the Lebesgue spaces $L^r(\mathbb{R}^N)$ are denoted by $\|\cdot\|_r$. In addition, $\|\cdot\|$ is used to denote $\|m\| := \|m\|_1 + \|m\|_\infty$ for any function $m(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Also we use the convention of using C or a subscripted version of it to denote a constant whose exact value is immaterial and may change from line to line. In addition to indicate the dependence of such a constant on the data, we write $C = C(a, b, \cdot, \cdot, \cdot)$ with the understanding that this *dependence is increasing in any argument that is given as some integral norm of a data function*. Furthermore, except in cases where there is a need for emphasis, we generally do not explicitly indicate the dependence of these constants on data points that do not vary throughout the paper, such as p, N or structural constants in (A1)–(A3).

We begin by recalling the following lemma.

Lemma 1.3 [4, Lemma 6.6] *If $a : \mathbb{R}^N \to \mathbb{R}$ satisfies* (A0), *then a has the following properties*:

(a1) $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,
(a2) *There exists $\sigma > \frac{N}{p}$ and $D > 0$ such that*

$$|x|^{\frac{N}{\sigma'}} \|a\|_{L^\sigma(\mathbb{R}^N \setminus B(0,|x|))} \le D, \quad \forall x \in \mathbb{R}^N,$$

where $\frac{1}{\sigma'} + \frac{1}{\sigma} = 1$ and $B(0, |x|))$ is the open ball with radius $|x|$.

Our first main result is then the following decay theorem.

Theorem 1.4 *Assume hypotheses* (A0), (A1)–(A3) *and* (F). *Then any solution u of Eq.* (1.1) *satisfies $u \in X \cap C^1(\mathbb{R}^N)$ and the following decay estimate holds*:

$$|u(x_0)| \le \frac{C_1}{1+|x_0|^{\frac{N-p}{p}}} + \frac{C_2}{1+|x_0|^{\frac{N-p}{p-1}}}, \quad \forall x_0 \in \mathbb{R}^N, \tag{1.4}$$

where $C_1 = C_1(N, p, c_f, \|a\|, \|u\|_X)$, and $C_2 = C_2(N, p, c_f, \|a\|, \|u\|_X, \sigma, d, D)$, with σ, D as in (a2), *d as in* (1.3).

Remark 1.5 Regarding Theorem 1.4 a few remarks are in order.

(i) Serrin in [21] studied the local behaviour of weak solutions of general quasilinear equations of the form

$$\operatorname{div} \mathcal{A}(x, u, \nabla u) = \mathcal{B}(x, u, \nabla u)$$

in a domain Ω of $\mathbb{R}^N$, in particular $\Omega = \mathbb{R}^N$. In [21, Theorem 1] a local L^∞-estimate for the solution u in a ball $B(x_0, R)$ is obtained, from which—when applied to Eq. (1.1)—we can derive an estimate of the form (for $|x_0|$ large)

$$\begin{aligned} |u(x_0)| \le \|u\|_{\infty, B(x_0,R)} &\le C\left(\frac{\|u\|_{p^*, B(x_0,2R)}}{|x_0|^{\frac{N-p}{p}}} + \frac{1}{|x_0|^{\frac{N-p}{p-1}}}\right) \\ &\le C\left(\frac{\|u\|_{p^*,\mathbb{R}^N}}{|x_0|^{\frac{N-p}{p}}} + \frac{1}{|x_0|^{\frac{N-p}{p-1}}}\right), \end{aligned}$$

where the constant $C = C(N, p, \varepsilon, |x_0|^\varepsilon(\|a\|, \|u\|_X))$ with $\varepsilon \in (0, 1]$ depends on x_0. Our Theorem 1.4 is obtained through a different approach and provides a global pointwise estimate meaning that the constants C_i do not depend on x_0.

(ii) For u fixed the first term in (1.4) dominates the second term for $|x_0|$ large, and one might ask why to keep the second term. The answer is given in Sect. 3, where the estimate (1.4) is applied to a sequence (h_n) with $\|h_n\|_X \to 0$, which yields (see formula (3.22))

$$|h_n(x)| \le C_1 \frac{C(\|h_n\|_X)}{1+|x|^{\frac{N-p}{p}}} + C_2 \frac{1}{1+|x|^{\frac{N-p}{p-1}}},$$

where $C(\|h_n\|_X) \to 0$ as $n \to \infty$, and the C_i, $i = 1, 2$ stay bounded.

In the special case where the function a_2 in hypothesis (A1) is zero, that is, where the operator $-\operatorname{div} A(x, \nabla u)$ behaves like the p-Laplacian, we are going to prove the following refined decay result.

Theorem 1.6 *Assume hypotheses* (A0), (A1)–(A3) *with $a_2(x) = 0$, and* (F). *Then any solution u of Eq.* (1.1) *satisfies $u \in X \cap C^1(\mathbb{R}^N)$ and the following decay estimate holds*:

$$|u(x_0)| \le C_2 \frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \quad \forall x_0 \in \mathbb{R}^N, \tag{1.5}$$

where $C_2 = C_2(N, p, \sigma, c_f, \|a\|, \|u\|_X, D)$.

Even though (1.1) is of nonvariational structure, the motivation for investigating the decay of solutions of the quasilinear elliptic equation (1.1) under the conditions (A1)–(A3) and (F) comes from the variational study of the following p-Laplacian equation in the whole $\mathbb{R}^N$

$$u \in X : -\Delta_p u = a(x)g(u), \tag{1.6}$$

where $a : \mathbb{R}^N \to \mathbb{R}$ is given by (A0) and the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

(G) $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the growth condition

$$|g(s)| \le c_g(1 + |s|^{\gamma-1}), \ \forall s \in \mathbb{R}, \ \text{and some } 1 \le \gamma < p^*.$$

With the following lemma we are able to characterize solutions of (1.6) as critical points of the energy functional Φ given by

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \, dx - \int_{\mathbb{R}^N} a(x)G(u)\, dx, \quad \text{with } G(s) = \int_0^s g(t)\, dt. \tag{1.7}$$

Lemma 1.7 [4, Lemma 6.1] *Let* $L^q(\mathbb{R}^N, w)$ *be the weighted Lebesgue space with weight*

$$w(x) = \frac{1}{1 + |x|^{N+\alpha}}, \quad x \in \mathbb{R}^N, \ \alpha > 0. \tag{1.8}$$

Then the embedding $X \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ *is compact for* $1 < q < p^*$.

Taking into account the weak lower semicontinuity of the norm, the compact embedding due to Lemma 1.7 yields the following result.

Lemma 1.8 *Let g satisfy* (G), *and let a fulfill* (A0). *Then* $\Phi : X \to \mathbb{R}$ *is a well defined* C^1*-functional, which is weakly lower semicontinuous. Moreover, critical points of* Φ *are solutions of* (1.6).

Let V be the subspace of bounded continuous functions with weight $1 + |x|^{\frac{N-p}{p}}$ defined by

$$V := \left\{ v \in X : v \in C(\mathbb{R}^N) \text{ with } \sup_{x\in\mathbb{R}^N} \left(1 + |x|^{\frac{N-p}{p}}\right)|v(x)| < \infty \right\},$$

which is a closed subspace of X with norm

$$\|v\|_V := \|v\|_X + \sup_{x\in\mathbb{R}^N} \left(1 + |x|^{\frac{N-p}{p}}\right)|v(x)|, \ v \in V.$$

Our second main result is the following X versus V local minimizer theorem

Theorem 1.9 *Let g satisfy* (G), *and let $a : \mathbb{R}^N \to \mathbb{R}$ fulfill* (A0). *Suppose $u_0 \in X$ is a solution of the Eq.* (1.6) *and a local minimizer in the V-topology of the functional $\Phi : X \to \mathbb{R}$, that is, there exists $\varepsilon > 0$ such that*

$$\Phi(u_0) \le \Phi(u_0 + h), \quad \forall\, h \in V : \|h\|_V < \varepsilon.$$

Then u_0 is a local minimizer of Φ with respect to the X-topology, that is, there is $\delta > 0$ such that

$$\Phi(u_0) \le \Phi(u_0 + h), \quad \forall\, h \in X : \|h\|_X < \delta.$$

As will be seen later the proof of Theorem 1.9 requires decay results for nonvariational type quasilinear elliptic equations of the form (1.1) and makes use of the decay result given by Theorem 1.4. The main ingredients for proving our decay results are on the one hand local pointwise estimates of solutions of (1.1) and their gradients due to [11, 17, 19, 20] in terms of nonlinear Wolff potentials, and on the other hand on a global L^∞-estimates obtained recently by the authors in [9].

The Brezis–Nirenberg result presented here extends the classical result due to Brezis and Nirenberg for a semilinear elliptic equation on bounded domains (see [3]) in two directions. First, unlike in [3] the leading operator is the p-Laplacian, and more importantly, second, the unboundedness of the domain. While extensions of the Brezis–Nirenberg result on bounded domains with leading p-Laplacian type variational operators have been obtained by several authors (see [2, 12–14, 16, 22]), the literature about extensions to unbounded domains, in particular to $\mathbb{R}^N$, is much less developed. Extensions to $\mathbb{R}^N$ with the Laplacian or the fractional Laplacian as leading operators within the Beppo-Levi space $D^{1,2}(\mathbb{R}^N)$ or fractional Beppo-Levi space $D^{s,2}(\mathbb{R}^N)$, respectively, can be found in [1, 7, 8]. An extension of the Brezis–Nirenberg result to the (unbounded) exterior domain $\mathbb{R}^N \setminus \overline{B(0,1)}$ was obtained in [5] for the N-Laplacian equation in the Beppo-Levi space $D_0^{1,N}(\mathbb{R}^N \setminus \overline{B(0,1)}$, which is based on Kelvin transform. The latter, however, only works for p-Laplacian equations with $p = 2$ or $p = N$.

In Sect. 2 we are going to prove Theorem 1.4 and Theorem 1.6, and in Sect. 3 we are going to provide the proof of Theorem 1.9.

2 Decay estimate

As a preliminary result let us recall a recent global L^∞-estimate obtained by the authors in [9] which when applied to Eq. (1.1) yields the following lemma.

Lemma 2.1 *Assume* (A0), (A1)–(A3) *and* (F). *If $u \in X$ is a solution of* (1.1) *then u is bounded and satisfies an L^∞-estimate of the form*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\phi(\|u\|_X), \tag{2.1}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a data independent function satisfying $\phi(s) \to 0$ as $s \to 0$ and $C = C(c_f, \|a\|, \|u\|_X)$.

Proof The global L^∞-estimate in [9], having the form stated in (2.1), is obtained for $u \in X$ solutions of

$$-\text{div}\, A(x, u, \nabla u) = B(x, u, \nabla u)$$

where the forms A and B satisfy some basic structural assumptions. These in case of A, is a more general version of $(A3)$ above. As for B, it is required that

$$|B(x, u, \nabla u)| \le b_1(x)|\nabla u|^{p-1} + b_2(x)|u|^{p-1} + b_3(x),$$

with

$$b_1 \in L^\infty(\mathbb{R}^N), \quad b_2 \in L^{\frac{q}{p}}(\mathbb{R}^N) \text{ for some } q > N, \quad \text{and} \quad b_3 \in L^{\frac{p^*}{p-1}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),$$

in which case $C = C(\|b_1\|_\infty, \|b_2\|_{\frac{q}{p}}, \|b_3\|_{\frac{p^*}{p-1}} + \|b_3\|_\infty)$. In our case $B(x, u, \nabla u) = a(x) f(x, u)$, with f satisfying (F). Without loss of generality, we may assume $p < \gamma$, and therefore

$$|a(x) f(x, u)| \le c_f\big(|a(x)||u(x)|^{\gamma - p}\big)|u|^{p-1} + c_f|a(x)|.$$

Hence taking $b_1(x) = 0$, $b_2(x) = c_f|a(x)||u(x)|^{\gamma-p}$, and $b_3(x) = c_f|a(x)|$, to apply the result in [9], we only need to verify that $b_2(x) = c_f|a(x)||u(x)|^{\gamma-p}$ is in $L^{\frac{q}{p}}(\mathbb{R}^N)$ for some $q > N$, as by $(A0)$

$$\|b_3\|_{\frac{p^*}{p-1}} + \|b_3\|_\infty \le c_f\|a\|.$$

On the other hand since $\frac{N}{p}(\gamma - p) < p^*$ (recall that $p < N$), we may choose $q > N$ such that $\frac{q}{p}(\gamma - p) < p^*$. Hence, taking into account the definition of $\|a\|$, and using Hölder inequality

$$\|au^{\gamma-p}\|_{\frac{q}{p}} \le \|a\|\|u\|_{p^*}^{(\gamma-p)} \le C\|a\|\|u\|_X^{(\gamma-p)}$$

from which one obtains (2.1) with $C = C(c_f, \|a\|, \|u\|_X)$. □

Remark 2.2 If the right-hand side function f of (1.1), i.e., $f = f(x, u)$ is independent of u, that is, $f(x, u) = f(x)$, then the constant C in the statement of Lemma 2.1 is independent of $\|u\|_X$, that is, $C = C(c_f, \|a\|)$.

An immediate consequence of Lemma 2.1 along with regularity results due to DiBenedetto (see [10]) is as follows.

Corollary 2.3 *Assume* (A0), (A1)–(A3) *and* (F). *If* $u \in X$ *is a solution of* (1.1), *then* u *is* $C^{1,\lambda}_{\text{loc}}(\mathbb{R}^N)$*-regular with* $\lambda \in (0, 1)$.

Let us assume hypotheses (A0), (A1)–(A3) and (F) throughout this section. From Lemma 2.1 it follows that the right-hand side of (1.1) is bounded by

$$|a(x) f(x, u(x)| \le C|a(x)|, \quad \text{with the positive constant } C = C(c_f, \|a\|, \|u\|_X). \tag{2.2}$$

Furthermore by the result of Lemma 1.3, the function $a(x)$ satisfies conditions (a1)–(a2). For the rest of this section, given $\sigma > \frac{N}{p}$ and $D > 0$, let us denote

$$\mathcal{A}_{\sigma,D} = \Big\{a : \mathbb{R}^N \to \mathbb{R} : \, a \text{ satisfies (a1)}-\text{(a2)}\Big\}.$$

For $\hat{a} \in \mathcal{A}_{\sigma,D}$ let us consider the equations

$$v \in X : -\text{div}\, A(x, \nabla v) = |\hat{a}(x)| \tag{2.3}$$

and

$$w \in X : -\text{div}\, A(x, \nabla w) = -|\hat{a}(x)|, \tag{2.4}$$

with A satisfying hypotheses (A1)–(A3).

Lemma 2.4 *The Eq.* (2.3) *has a unique positive solution* $v \in X \cap L^\infty \cap C^1(\mathbb{R}^N)$, *and* (2.4) *has a unique negative solution* $w \in X \cap L^\infty \cap C^1(\mathbb{R}^N)$.

Proof Since $\hat{a} \in \mathcal{A}_{\sigma,D}$, $\hat{a} \in L^r(\mathbb{R}^N)$ for all $r \in [1, \infty]$, and thus it belongs, in particular, to $L^{p^{*'}}(\mathbb{R}^N)$, which is continuously embedded into X^*. From hypotheses (A1)–(A3) it follows that $T = -\text{div}\, A(x, \cdot) : X \to X^*$ defines a bounded, continuous, strongly monotone (note $2 \le p < N$) and hence coercive operator. Thus $T : X \to X^*$ is bijective, which yields the existence of a unique solution v of (2.3). Again using Lemma 2.1, v is in L^∞ with a norm depending on $\|\hat{a}\|$ which is even $C^{1,\lambda}_{\text{loc}}(\mathbb{R}^N)$-regular. Next, we show that $v(x) \ge 0$. As a weak solution v satisfies

$$\int_{\mathbb{R}^N} A(x, \nabla v)\, \nabla\varphi\, dx = \int_{\mathbb{R}^N} |\hat{a}|\varphi\, dx.$$

Testing this relation with $\varphi = v^- = \max\{-v, 0\}$, we get by means of (A3)

$$0 \le \int_{\mathbb{R}^N} A(x, \nabla v)\, \nabla v^-\, dx \le -\nu \int_{\mathbb{R}^N} |\nabla v^-|^p\, dx \le 0,$$

which implies that $\|v^-\|_X = 0$ and thus $v^- = 0$, that is, $v(x) \ge 0$ for all $x \in \mathbb{R}^N$, and by Harnack's inequality it follows that $v(x) > 0$ for all $x \in \mathbb{R}^N$. The proof for the unique solution w follows by similar arguments and can be omitted. □

Lemma 2.5 *If* u *is a solution of* (1.1), *then* $w(x) \le u(x) \le v(x)$, *where* v *and* w *are unique positive and negative solutions of* (2.3) *and* (2.4), *respectively, with* $\hat{a}(x) = Ca(x)$, *and* C *as in* (2.2).

Proof From (2.2) it follows that

$$-\text{div}\, A(x, \nabla u) - (-\text{div}\, A(x, \nabla v)) \leq 0,$$

which by testing this inequality with $\varphi = (u - v)^+$ and applying (A2) yields

$$\nu \|\nabla (u - v)^+\|_p^p \leq \int_{\mathbb{R}^N} (A(x, \nabla u) - A(x, \nabla v) \nabla (u - v)^+ \, dx \leq 0,$$

and thus $\|(u - v)^+\|_X = 0$, that is $(u - v)^+ = 0$, and thus $u \leq v$.

Multiplying Eqs. (1.1) and (2.4) by (-1) and setting $\hat{u} = -u$ and $\hat{w} = -w$, respectively, we get

$$-\text{div}\,(-A(x, -\nabla \hat{u})) = -a(x) f(x, -\hat{u}) \quad \text{and} \quad -\text{div}\,(-A(x, -\nabla \hat{w})) = C|a(x)|.$$

Set $\hat{A}(x, \xi) = -A(x, -\xi)$, then $\hat{A} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the same hypotheses as A, that is, $\hat{A}$ satisfies (A1)–(A3). Taking (2.2) into account we thus obtain the inequality

$$\langle -\text{div}\, \hat{A}(x, \nabla \hat{u}) - (-\text{div}\, \hat{A}(x, \nabla \hat{w})), (\hat{u} - \hat{w})^+ \rangle \leq 0,$$

which by applying (A2) and following the arguments above yields $\hat{u} \leq \hat{w}$, that is, $w \leq u$ completing the proof. □

From Lemma 2.5 we infer that a decay estimate of a solution u of (1.1) is obtained via a decay estimate of the positive solution v of (2.3) and the positive solution $\hat{w} = -w$, where w is the unique solution of (2.4), with $\hat{a}(x) = Ca(x)$. Let us focus first on the decay estimate for the positive solution v of (2.3). For that we make use of pointwise estimates for nonnegative supersolutions of general quasilinear elliptic equations with nonnegative Radon measure right-hand side of the form

$$-\,\text{div} A(x, u, \nabla u) + B(x, u, \nabla u) = \mu, \tag{2.5}$$

where $\mu : \mathbb{R}^N \to [0, \infty)$ is a nonnegative Radon measure, see [20, Chapter 4.4]. When specifying to Eq. (2.3), we have $B(x, u, \nabla u) = 0$ and the nonnegative Radon measure μ is given by the function $x \mapsto |\hat{a}(x)|$ through

$$\mu(E) := \int_E |\hat{a}(x)| \, dx, \quad E \subset \mathbb{R}^N, \; E \text{ Lebesgue measurable.} \tag{2.6}$$

The point of departure for the decay estimate is the following pointwise estimate of nonnegative supersolutions of (2.5) in terms of the nonlinear Wolff potential $W_p^\mu(x_0, r_0)$, and the data of the operators A and B due to Maly–Ziemer. Applying [20, Theorem 4.36] to the nonnegative solution of (2.3) yields the following Wolff potential estimates. (Note: $\Omega = \mathbb{R}^N$.)

Theorem 2.6 [20, Theorem 4.36] *Let v be the nonnegative solution of* (2.3), *and $B(x_0, r_0)$ be the ball centered at x_0 with radius r_0. Then for any β with $p - 1 < \beta < (p-1)\frac{N}{N-(p-1)}$ the following estimate holds*

$$v(x_0) \leq C\left[\left(\frac{1}{r_0^N}\int_{B(x_0,r_0)\cap\{v>0\}} v^\beta dx\right)^{\frac{1}{\beta}} + W_p^\mu(x_0, r_0) + \int_0^{r_0} k(r)\frac{1}{r}dr\right], \quad (2.7)$$

with some constant $C = C(N, p, \beta)$ that can be verified to be independent of r_0, where $W_p^\mu(x_0, r_0)$ is the nonlinear Wolff potential defined by

$$W_p^\mu(x_0, r_0) = \int_0^{r_0}\left(\frac{\mu(B(x_0, r))}{r^{N-p}}\right)^{\frac{1}{p-1}}\frac{dr}{r}, \quad (2.8)$$

with $\mu(B(x_0, r)$ given by (2.6), *and the function $r \mapsto k(r)$ is given in terms of the function $a_2(x)$, which appears in condition* (A1) *as follows*:

$$k(r) = c\|a_2\|_{M(p,p);B(r)}, \quad (2.9)$$

where $\|\cdot\|_{M(p,p);B(r)}$ is the multiplier norm (*see* [20, p. 162]) *defined by*

$$\|w\|_{M(p,p);B(r)} = \sup\{\|w\varphi\|_{p,B(r)} : \varphi \in W_0^{1,p}(B(r)),\ \|\nabla\varphi\|_{p,B(r)} \leq 1\}.$$

The following corollary will be useful for estimating the third term in (2.7).

Corollary 2.7 *If $w \in L^q(B(r))$ with $q > N$, then*

$$\|w\|_{M(p,p);B(r)} \leq C(p, q, N)r^{1-\frac{N}{q}}\|w\|_{q,B(r)}.$$

Proof Let $\varphi \in W_0^{1,p}(B(r))$, and $w \in L^q(B(r))$. Then we have

$$\|w\varphi\|_{p,B(r)}^p = \int_{B(r)}|w|^p|\varphi|^p dx \leq \|w\|_{N,B(r)}^p\|\varphi\|_{p^*,B(r)}^p \leq C\|w\|_{N,B(r)}^p\|\nabla\varphi\|_{p,B(r)}^p$$

which, for $q > N$, yields (denoting $|B(x_0, r_0)| = \text{meas}(B(x_0, r_0))$

$$\|w\|_{M(p,p);B(r)} \leq C\|w\|_{N,B(r)} \leq C\|w\|_{q,B(r)}|B(r)|^{\frac{q-N}{Nq}}$$

and thus we finally obtain

$$\|w\|_{M(p,p);B(r)} \leq C(p, q, N)r^{\frac{q-N}{q}}\|w\|_{q,B(r)}.$$

□

We have the following decay theorem for the unique positive solution v of Eq. (2.3).

Theorem 2.8 *The unique positive solution v of Eq.* (2.3) *satisfies the following decay estimate*

$$v(x_0) \leq C_1 \frac{\|v\|_X}{1+|x_0|^{\frac{N-p}{p}}} + C_2 \frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \tag{2.10}$$

for all $x_0 \in \mathbb{R}^N$, with the constants $C_1 = C_1(N, p, \|\hat{a}\|)$, $C_2 = C_2(N, p, d, \sigma, D)$.

The proof of Theorem 2.8 is based on Theorem 2.6 in that the three terms are individually estimated in Lemmas 2.9–2.11 that follow next.

Lemma 2.9 *The first term on the right-hand side of* (2.7), *at $r_0 = \frac{|x_0|}{2}$ can be estimated as follows*:

$$\left(\frac{1}{r_0^N}\int_{B(x_0,r_0)\cap\{v>0\}} v^\beta dx\right)^{\frac{1}{\beta}} \leq C_1(N, p, \|\hat{a}\|)\frac{\|v\|_X}{1+|x_0|^{\frac{N-p}{p}}}.$$

Proof Since $\beta < p^*$ and $v \in X$ we have, using Hölder inequality, the following estimate

$$\left(\frac{1}{r_0^N}\int_{B(x_0,r_0)\cap\{v>0\}} v^\beta dx\right)^{\frac{1}{\beta}} \leq \left(\frac{1}{r_0^N}\int_{B(x_0,r_0)} v^{p^*} dx\right)^{\frac{1}{p^*}}$$
$$\leq C(p, N)\frac{1}{r_0^{\frac{N}{p^*}}}\|v\|_X.$$

Taking $r_0 = \frac{|x_0|}{2}$, and recalling that $\|v\|_\infty \leq C(\|\hat{a}\|)$ (see Remark 2.2), we get

$$\left(\frac{1}{(\frac{|x_0|}{2})^N}\int_{B(x_0,\frac{|x_0|}{2})\cap\{v>0\}} v^\beta dx\right)^{\frac{1}{\beta}} \leq C_1(N, p, \|\hat{a}\|)\frac{\|v\|_X}{1+|x_0|^{\frac{N-p}{p}}}, \tag{2.11}$$

completing the proof. □

Lemma 2.10 *For the second term on the right-hand side of* (2.7) *at $r_0 = \frac{|x_0|}{2}$, we have the estimate*

$$W_p^\mu\left(x_0, \frac{|x_0|}{2}\right) = \int_0^{\frac{|x_0|}{2}} \left(\frac{\mu(B(x_0,r))}{r^{N-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} \leq C_2(N, p, \sigma, D)\frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}.$$

Proof Consider

$$W_p^\mu\left(x_0, \frac{|x_0|}{2}\right) = \int_0^{\frac{|x_0|}{2}} \left(\frac{\mu(B(x_0,r))}{r^{N-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r},$$

where μ is given by (2.6). First, the fact that $\hat{a} \in \mathcal{A}_{\sigma,D}$ allows for the following estimate of $\mu(B(x_0,r))$: (note: $\sigma > \frac{N}{p}$)

$$\begin{aligned}\mu(B(x_0,r)) = \int_{B(x_0,r)} |\hat{a}(y)|\,dy = \int_{B(0,1)} |\hat{a}(x_0+rz)|r^N\,dz \\ \leq C(N,\sigma)\,r^N\Big(\int_{B(0,1)} |\hat{a}(x_0+rz)|^\sigma\,dz\Big)^{\frac{1}{\sigma}} \\ \leq C r^N r^{-\frac{N}{\sigma}}\Big(\int_{B(x_0,r)} |\hat{a}(y)|^\sigma\,dy\Big)^{\frac{1}{\sigma}} \\ \leq C\,r^{\frac{N}{\sigma'}}\|\hat{a}\|_{L^\sigma(B(x_0,r))}.\end{aligned}$$

Thus

$$\mu(B(x_0,r)) = \int_{B(x_0,r)} |\hat{a}(y)|\,dy \leq C\,r^{\frac{N}{\sigma'}}\|\hat{a}\|_{L^\sigma(B(x_0,r))} \text{ with } C = C(N,\sigma). \tag{2.12}$$

Using (2.12) we estimate the Wolff potential

$$W_p^\mu\Big(x_0,\frac{|x_0|}{2}\Big) = \int_0^{\frac{|x_0|}{2}} \Big(\int_{B(x_0,t)} |\hat{a}(y)|\,dy\Big)^{\frac{1}{p-1}} t^{\frac{1-N}{p-1}}\,dt,$$

by taking into account that by (a2) $\frac{N}{\sigma'}\frac{1}{p-1} + \frac{p-N}{p-1} > 0$.

$$\begin{aligned}W_p^\mu\Big(x_0,\frac{|x_0|}{2}\Big) &\leq \int_0^{\frac{|x_0|}{2}} \|\hat{a}\|_{L^\sigma(B(x_0,t))}^{\frac{1}{p-1}} t^{\frac{N}{\sigma'}\frac{1}{p-1}+\frac{1-N}{p-1}}\,dt \\ &\leq \|\hat{a}\|_{L^\sigma(\mathbb{R}^N\setminus B(0,\frac{|x_0|}{2}))}^{\frac{1}{p-1}} \int_0^{\frac{|x_0|}{2}} t^{\frac{N}{\sigma'}\frac{1}{p-1}+\frac{1-N}{p-1}}\,dt \\ &\leq C\|\hat{a}\|_{L^\sigma(\mathbb{R}^N\setminus B(0,\frac{|x_0|}{2}))}^{\frac{1}{p-1}} \Big(\frac{|x_0|}{2}\Big)^{\frac{N}{\sigma'}\frac{1}{p-1}+\frac{p-N}{p-1}} \\ &\leq C\Big[\Big(\frac{|x_0|}{2}\Big)^{\frac{N}{\sigma'}}\|\hat{a}\|_{L^\sigma(\mathbb{R}^N\setminus B(0,\frac{|x_0|}{2}))}\Big]^{\frac{1}{p-1}}\Big(\frac{|x_0|}{2}\Big)^{\frac{p-N}{p-1}},\end{aligned}$$

which, using (a2), yields for some positive constant $C_2 = C_2(N,p,\sigma,D) > 0$

$$W_p^\mu\Big(x_0,\frac{|x_0|}{2}\Big) \leq C_2\,|x_0|^{-\frac{N-p}{p-1}}, \quad |x_0| > 0,$$

and hence

$$W_p^{\mu}\Big(x_0, \frac{|x_0|}{2}\Big) \le C_2 \frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \quad \text{for } |x_0| \text{ large.} \tag{2.13}$$

□

Lemma 2.11 *For the third term on the right-hand side of (2.7) at* $r_0 = \frac{|x_0|}{2}$, *we have the estimate*

$$\int_0^{\frac{|x_0|}{2}} k(r)\frac{1}{r}\,dr \le C_3(N,p,d)\frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}.$$

Proof Let us consider the third term on the right-hand side of (2.7) at $r_0 = \frac{|x_0|}{2}$, that is,

$$\int_0^{\frac{|x_0|}{2}} k(r)\frac{1}{r}\,dr.$$

The function $r \mapsto k(r)$ is given by (2.9), which in view of Corollary 1.1 and Corollary 2.7 yields $a_2 \in L^q(\mathbb{R}^N)$ for $q > N$ and

$$k(r) \le c r^{1-\frac{N}{q}} \|a_2\|_{q,B(x_0,r)}. \tag{2.14}$$

Fixing $q > N$ and making use of the decay (1.3) of a_2 we get

$$\|a_2\|_{q,B(x_0,\frac{|x_0|}{2})} \le c\bigg(\int_0^{\frac{|x_0|}{2}} \bigg(\frac{1}{1+|x_0+\omega s|^{\frac{N-1}{p-1}}}\bigg)^q s^{N-1}ds\bigg)^{\frac{1}{q}}, \tag{2.15}$$

where $\omega \in S^{N-1}$. For $|x_0|$ sufficiently large (say $|x_0| > 4$) we obtain

$$|x_0+\omega s| \ge \frac{1}{2}|x_0| \ge 2,\ \forall \omega \in S^{N-1} \text{ and } 0 \le s \le \frac{|x_0|}{2},$$

and thus from (2.15) it follows

$$\|a_2\|_{q,B(x_0,\frac{|x_0|}{2})} \le c\bigg(\int_0^{\frac{|x_0|}{2}} \frac{1}{|x_0|^{q\frac{N-1}{p-1}}} s^{N-1}ds\bigg)^{\frac{1}{q}}. \tag{2.16}$$

Finally, applying (2.16) we have the following estimate

$$\begin{aligned}\int_0^{\frac{|x_0|}{2}} k(r)\frac{1}{r}\,dr &\le c\int_0^{\frac{|x_0|}{2}} r^{1-\frac{N}{q}}\|a_2\|_{q,B(x_0,\frac{|x_0|}{2})}\frac{1}{r}\,dr\\ &\le c|x_0|^{\frac{N}{q}-\frac{N-1}{p-1}}\int_0^{\frac{|x_0|}{2}} r^{-\frac{N}{q}}\,dr\\ &\le c|x_0|^{\frac{N}{q}-\frac{N-1}{p-1}}|x_0|^{1-\frac{N}{q}} = c|x_0|^{1-\frac{N-1}{p-1}} = c\frac{1}{|x_0|^{\frac{N-p}{p-1}}},\end{aligned}$$

which for $|x_0|$ large yields with some positive constant $C_3(N, p, d)$

$$\int_0^{\frac{|x_0|}{2}} k(r)\frac{1}{r}\,dr \le C_3(N,p,d)\frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}. \tag{2.17}$$

which completes the proof. □

Proof of Theorem 2.8 Taking into account that $v \in X \cap C^1(\mathbb{R}^N)$, from Lemmas 2.9–2.11 and the estimates (2.11), (2.13), and (2.17) it follows that (2.10) holds true for all $x_0 \in \mathbb{R}^N$, which completes the proof. □

The unique negative solution w of (2.4) implies that $\hat{w} = -w$ is the unique solution of

$$-\operatorname{div}(-A(x, -\nabla\hat{w})) = \hat{a}. \tag{2.18}$$

Since $\hat{A}(x,\xi) := -A(x,-\xi)$ fulfills the same conditions (A1)–(A3), we obtain as an immediate consequence from Theorem 2.8 the following result. □

Corollary 2.12 *The unique positive solution $\hat{w}$ of Eq. (2.18) satisfies the following decay estimate*

$$\hat{w}(x_0) \le C_1\frac{\|\hat{w}\|_X}{1+|x_0|^{\frac{N-p}{p}}} + C_2\frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \tag{2.19}$$

for all $x_0 \in \mathbb{R}^N$, where C_i, $i = 1, 2$ are of the same nature as in Theorem 2.8.

By means of Theorem 2.8 and Corollary 2.12 it is now easy to complete the proof of Theorem 1.4, that is one of our main results.

Proof of Theorem 1.4 If u is a solution of (1.1), then from Lemma 2.5 we have $w \le u \le v$ or equivalently $-\hat{w} \le u \le v$, where v and $\hat{w}$ are the unique solutions of the Eqs. (2.3) and (2.18), respectively, with $\hat{a} = Ca(x)$ and $C = C(c_f, \|a\|, \|u\|_X)$ given in Eq. (2.2). Also (2.18) is equivalent to

$$-\operatorname{div}\hat{A}(x,\nabla\hat{w}) = |\hat{a}|. \tag{2.20}$$

Testing Eq. (2.20) with $\varphi = \hat{w}$ and making use of (A2) we obtain

$$\|\hat{w}\|_X^p \leq \frac{c}{\nu} \|\hat{a}\|_{p^{*\prime}} \|\hat{w}\|_X,$$

which yields the following estimate

$$\|\hat{w}\|_X \leq \left(\frac{c}{\nu}\right)^{\frac{1}{p-1}} \|\hat{a}\|_{p^{*\prime}}^{\frac{1}{p-1}}.$$

In the same way from Eq. (2.3) we obtain

$$\|v\|_X \leq \left(\frac{c}{\nu}\right)^{\frac{1}{p-1}} \|\hat{a}\|_{p^{*\prime}}^{\frac{1}{p-1}}.$$

Finally note that $\hat{a} = Ca(x)$ with $C = C(c_f, \|a\|, \|u\|_X)$. Now by Lemma 1.3, $a \in \mathcal{A}_{\sigma,D}$ for some $\sigma > \frac{N}{p}$ and $D > 0$, and so $\hat{a} \in \mathcal{A}_{\sigma,CD}$. Therefore from Theorem 2.8 and Corollary 2.12 we have the following estimate

$$|u(x_0)| \leq \frac{C_1}{1 + |x_0|^{\frac{N-p}{p}}} + \frac{C_2}{1 + |x_0|^{\frac{N-p}{p-1}}}, \quad \forall x_0 \in \mathbb{R}^N$$

with $C_1 = C_1(N, p, c_f, \|a\|, \|u\|_X)$ and $C_2 = C_2(N, p, c_f, \|a\|, \|u\|_X, \sigma, d, D)$ which completes the proof. □

Special case: Proof of Theorem 1.6

For the special case we are dealing with here we only replace hypothesis (A1) by (A-1) with

(A-1) $|A(x, \xi)| \leq a_1 |\xi|^{p-1}$

while all other hypotheses, that is, (A0), (A2), (A3) and (F) remain the same. We follow the approach of the general case, where the decay of a solution u of (1.1) is based on the decay of the unique positive solution v and the unique negative solution w of Eqs. (2.3) and (2.4), respectively, with $\hat{a} = Ca(x)$, that is

$$v \in X : -\text{div}\, A(x, \nabla v) = C|a(x)|, \tag{2.21}$$
$$w \in X : -\text{div}\, A(x, \nabla w) = -C|a(x)|, \tag{2.22}$$

where $C = C(c_f, \|a\|, \|u\|_X)$. From Lemmas 2.4 and 2.5 we know that $v, w \in X \cap C^1(\mathbb{R}^N)$ and $w \leq u \leq v$. Again let us introduce the specific nonnegative Radon measure μ by means of the right-hand side $C|a(x)| \in L^r(\mathbb{R}^N)$ for $1 \leq r \leq \infty$ through

$$\mu(E) = \int_E C|a(x)|\, dx, \quad E \subset \mathbb{R}^N, \ E \text{ Lebesgue measurable.}$$

Then v (resp. $\hat{w} = -w$) is a special case of what is referred to as A-superharmonic function (note: $a_2(x) = 0$), which is a weak solution of

$$v \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N) : -\mathrm{div}\, A(x, \nabla v) = \mu, \tag{2.23}$$

see e.g. [15, 17, 18]. Thus we may apply pointwise estimates due to Kilpelainen–Maly in terms of the nonlinear Wolff potential $W^{\mu}_p(x_0, r_0)$ given in (2.8), which we recall here for convenience.

Theorem 2.13 [17, Theorem 1.6] *Suppose that u is a nonnegative A-superharmonic function in $B(x_0, 3r)$. If $-\mathrm{div}\, A(x, \nabla u) = \mu$, then*

$$c_1\, W^{\mu}_p(x_0, r) \le u(x_0) \le c_2 \inf_{B(x_0,r)} u + c_3\, W^{\mu}_p(x_0, 2r), \tag{2.24}$$

where c_1, c_2 and c_3 are positive constants, depending only on N, p, and the structural constants a_1 and ν of (A-1) *and* (A3), *respectively.*

As a consequence of Theorem 2.13 we get the following pointwise estimate.

Corollary 2.14 [17, Corollary 4.13] *Let u be a nonnegative A-superharmonic function in $\mathbb{R}^N$ with $\inf_{\mathbb{R}^N} u = 0$. If $-\mathrm{div}\, A(x, \nabla u) = \mu$, then*

$$c_1\, W^{\mu}_p(x_0; \infty) \le u(x_0) \le c_2\, W^{\mu}_p(x_0; \infty), \tag{2.25}$$

where c_1, c_2 are positive constants, depending only on N, p, and the structural constants a_1 and ν of (A-1) *and* (A3), *respectively.*

We remark that in [17] an additional homogeneity condition on the vector function $A(x, \xi)$ was assumed, which, however, was found not needed for the validity of Theorem 2.13.

Since the unique positive solution v of (2.21) belongs to $X \cap C^1(\mathbb{R}^N)$, we have $\inf_{\mathbb{R}^N} v(x) = 0$, and thus we may apply Corollary 2.14, which yields

$$0 \le v(x_0) \le c\, W^{\mu}_p(x_0; \infty), \tag{2.26}$$

where c is a positive constant depending only on N, p, and the structural constants a_1 and ν of (A-1) and (A3), respectively. Thus it remains to estimate $W^{\mu}_p(x_0; \infty) = \lim_{r\to\infty} W^{\mu}_p(x_0, r)$ with $W^{\mu}_p(x_0, r)$ given by (2.8).

Let $x_0 \neq 0$ then

$$\begin{aligned}W_p^\mu(x_0;\infty) &= \int_0^\infty \left(\frac{\mu(B(x_0,t))}{t^{N-p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \int_0^\infty \left(\int_{B(x_0,t)} C|a(y)|\,dy\right)^{\frac{1}{p-1}} t^{\frac{1-N}{p-1}}\,dt \\ &= \int_0^{\frac{|x_0|}{2}} \left(\int_{B(x_0,t)} C|a(y)|\,dy\right)^{\frac{1}{p-1}} t^{\frac{1-N}{p-1}}\,dt \\ &\quad + \int_{\frac{|x_0|}{2}}^\infty \left(\int_{B(x_0,t)} C|a(y)|\,dy\right)^{\frac{1}{p-1}} t^{\frac{1-N}{p-1}}\,dt \\ &= I_1 + I_2.\end{aligned}$$

The first term I_1 has already been estimated in Lemma 2.10 of the proof of Theorem 2.8, which yields

$$I_1 = W_p^\mu\left(x_0, \frac{|x_0|}{2}\right) \leq C_2|x_0|^{-\frac{N-p}{p-1}}, \quad |x_0| > 0,$$

and hence

$$I_1 \leq C_2 \frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \quad \text{for } |x_0| \text{ large}, \tag{2.27}$$

where $C_2 = C_2(N, p, \sigma, CD)$.

Since a, in particular, belongs to $L^1(\mathbb{R}^N)$ we get for the second term I_2

$$I_2 \leq \left(C\|a\|_1\right)^{\frac{1}{p-1}} \int_{\frac{|x_0|}{2}}^\infty t^{\frac{1-N}{p-1}}\,dt = \left(C\|a\|_1\right)^{\frac{1}{p-1}} \frac{p-1}{N-p} \left(\frac{|x_0|}{2}\right)^{-\frac{N-p}{p-1}},$$

which results in

$$I_2 \leq C_2 \frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \quad \text{for } |x_0| \text{ large}. \tag{2.28}$$

As $v \in X \cap C^1(\mathbb{R}^N)$ from (2.27) and (2.28) we get

$$v(x_0) \leq C_2 \frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \quad \text{for all } x_0 \in \mathbb{R}^N. \tag{2.29}$$

Similarly, we get for $\hat{w} = -w$ an estimate of the form (2.29), which results in

$$|u(x_0)| \leq C_2 \frac{1}{1+|x_0|^{\frac{N-p}{p-1}}}, \quad \text{for all } x_0 \in \mathbb{R}^N,$$

where $C_2 = C_2(N, p, \sigma, CD)$ with $C = C(c_f, \|a\|, \|u\|_X)$ given in Eq. (2.2) which completes the proof of Theorem 1.6. ◻

3 $D^{1,p}(\mathbb{R}^N)$ versus V local minimizer

Before we prove Theorem 1.9, first we are providing some preliminary results of weak solutions of Eq. (1.6), which we recall here

$$u \in X : -\Delta_p u = a(x)g(u).$$

Let us assume throughout this section the hypotheses (A0) and (G). Since the vector function $A : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ related with the p-Laplacian $\Delta_p u = \operatorname{div} |\nabla u|^{p-2}\nabla u$ $(2 \le p < N)$ is $A(x, \xi) = |\xi|^{p-2}\xi$, one readily verifies that hypotheses (A-1), (A2) and (A3) are fulfilled. By (G) the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ satisfies also hypothesis (F), which by means of Theorem 1.6 yields the following result.

Corollary 3.1 *Any solution u of Eq.* (1.6) *satisfies $u \in X \cap C^1(\mathbb{R}^N)$ and the following decay estimate holds*:

$$|u(x_0)| \le C_2 \frac{1}{1 + |x_0|^{\frac{N-p}{p-1}}}, \quad \forall x_0 \in \mathbb{R}^N \tag{3.1}$$

where $C_2 = C_2(N, p, \sigma, c_g, \|a\|, \|u\|_X, D)$.

Next we are going to prove an estimate of the gradient $|\nabla u|$ of a solution of (1.6). In view of Corollary 3.1, the right-hand side of Eq. (1.6) can be estimated as

$$|a(x)g(u(x))| \le C|a(x)|, \tag{3.2}$$

where $C = C(c_g, \|a\|, \|u\|_X)$ is of the same quality as $C = C(c_f, \|a\|, \|u\|_X)$ in the preceding section. Thus Eq. (1.6) can be rewritten in the form

$$u \in X : -\Delta_p u = \mu(x), \quad \text{with } \mu(x) = a(x)g(u(x)). \tag{3.3}$$

Since the right-hand side of (3.3) is in $L^1(\mathbb{R}^N)$, we may apply the following gradient estimate which is based on Wolff potential estimates and which has been deduced from [11, Theorem 1.1] and [19, Theorem 2.3] for the case that the leading quasilinear elliptic operator is the p-Laplacian.

Lemma 3.2 *Let $2 \le p < N$, and let $u \in C^1(\Omega)$ be a weak solution of*

$$-\Delta_p u = \mu, \quad \textit{with } \mu \in L^1(\Omega).$$

Then there is a constant $c = c(p, N) > 0$ such that the pointwise estimate

$$|\nabla u(x_0)| \le c \frac{1}{|B(x_0, R)|} \int_{B(x_0,R)} |\nabla u|\, dx + c\, W^{\mu}_{\frac{1}{p},p}(x_0, 2R) \tag{3.4}$$

holds, whenever $B(x_0, 2R) \subset \Omega$, *where the Wolff potential* $W^{\mu}_{\frac{1}{p},p}(x_0, 2R)$ *is defined by*

$$W^{\mu}_{\frac{1}{p},p}(x_0, 2R) = \int_0^{2R} \left(\frac{|\mu|(B(x_0, t))}{t^{N-1}} \right)^{\frac{1}{p-1}} \frac{dt}{t}, \quad R > 0,\ x_0 \in \mathbb{R}^N. \tag{3.5}$$

Due to [11, Theorem 1.1] the estimate (3.4) holds without any restriction on R. Moreover, $|\mu|(B(x_0, t))$ with $\mu \in L^1(\Omega)$ stands for

$$|\mu|(B(x_0, t)) = \int_{B(x_0,t)} |\mu(y)|\, dy.$$

We are going to use Lemma 3.2 to estimate the gradient $|\nabla u(x)|$ with $x \in \Omega = \mathbb{R}^N$.

Lemma 3.3 *There exists some positive constant* $C = C(c_g, \|a\|, \|u\|_X, \sigma, D)$ *such that the following estimate of* $|\nabla u|$ *holds*:

$$|\nabla u(x)| \leq C \frac{1}{1 + |x|^{\frac{N-1}{p-1}}}, \quad x \in \mathbb{R}^N. \tag{3.6}$$

Proof First, we are going to verify that the following is true:

$$|\nabla u(x)| \leq c W^{\mu}_{\frac{1}{p},p}(x, \infty) = \lim_{R\to\infty} W^{\mu}_{\frac{1}{p},p}(x, 2R). \tag{3.7}$$

We apply (3.4) of Lemma 3.2 and taking into account that the inequality holds without restriction on R we get

$$|\nabla u(x)| \leq c \lim_{R\to\infty} \frac{1}{|B(x, R)|} \int_{B(x,R)} |\nabla u|\, dx + c\, W^{\mu}_{\frac{1}{p},p}(x, \infty). \tag{3.8}$$

We have by Hölder inequality and $u \in X$

$$\begin{aligned} \frac{1}{|B(x, R)|} \int_{B(x,R)} |\nabla u|\, dx &\leq \frac{1}{|B(x, R)|} |B(x, R)|^{\frac{1}{p'}} \left(\int_{B(x,R)} |\nabla u|^p\, dx \right)^{\frac{1}{p}} \\ &\leq |B(x, R)|^{\frac{1}{p'}-1} \|u\|_X \to 0, \text{ as } R \to \infty, \end{aligned}$$

which yields (3.7). Hence, it remains to estimate $W^{\mu}_{\frac{1}{p},p}(x, \infty)$ with

$$|\mu| \leq C|a(x)| \leq C \frac{1}{1 + |x|^{N+\alpha}}.$$

We observe that with $\hat{\mu}(x) = C|a(x)|$ we get

$$W^{\mu}_{\frac{1}{p},p}(x, \infty) \leq W^{\hat{\mu}}_{\frac{1}{p},p}(x, \infty).$$

As for the estimate of $W^{\hat{\mu}}_{\frac{1}{p},p}(x,\infty)$ we refer to [6, Theorem 2.6] or [4, Theorem 6.7], which gives

$$W^{\hat{\mu}}_{\frac{1}{p},p}(x,\infty) \le C\frac{1}{1+|x|^{\frac{N-1}{p-1}}}, \quad \text{for all } |x| \ge 1,$$

which along with $u \in C^1(\mathbb{R}^N)$ proves (3.6) for all $x \in \mathbb{R}^N$. □

From Lemma 3.3 and Corollary 1.1 we immediately get

Corollary 3.4 *If u is a solution of the Eq. (1.6), then for its gradient it holds $|\nabla u| \in L^q(\mathbb{R}^N)$ for $p \le q \le \infty$.*

Proof of Theorem 1.9 Let u_0 be a solution of (1.6) and a local minimizer of the functional

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx - \int_{\mathbb{R}^N} a(x)G(u)\,dx, \quad \text{with } G(s) = \int_0^s g(t)\,dt,$$

in the V-topology. Consider the functional $h \mapsto \Phi(u_0+h)$, and let $h_n : \|h_n\|_X \le \frac{1}{n}$ be such that

$$\Phi(u_0+h_n) = \inf_{h\in B_n} \Phi(u_0+h), \quad \text{where } B_n = \left\{h \in X : \|h\|_X \le \frac{1}{n}\right\}.$$

The existence of a minimizer h_n is guaranteed, since $\Phi : X \to \mathbb{R}$ is C^1 and weakly lower semicontinuous and B_n is weakly compact in X. Set $u_n = u_0 + h_n$, that is,

$$\Phi(u_n) = \inf_{u\in B_n} \Phi(u), \quad \text{where } B_n = \left\{u \in X : \|u-u_0\|_X \le \frac{1}{n}\right\}.$$

For $u_n \in B_n$ we have either $\|u_n - u_0\|_X < \frac{1}{n}$ or else $\|u_n - u_0\|_X = \frac{1}{n}$. In case $\|u_n - u_0\| < \frac{1}{n}$, u_n is a critical point of Φ, and thus u_n is a weak solution of (1.6), i.e., $-\Delta_p u_n = a(x)g(u_n)$, that is,

$$\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi - a(x)g(u_n)\varphi\Big)dx = 0, \quad \forall \varphi \in X.$$

In case $\|u_n - u_0\| = \frac{1}{n}$, there exists a Lagrange multiplier $\lambda_n \le 0$ such that

$$\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p-2}\nabla u_n \nabla\varphi - a(x)g(u_n)\varphi\Big)dx = \lambda_n\int_{\mathbb{R}^N}|\nabla(u_n-u_0)|^{p-2}\nabla(u_n-u_0)\varphi\,dx,$$

for all $\varphi \in X$, which (in a distributional sense) can be written as

$$-\Delta_p u_n - a(x)g(u_n) = -\lambda_n\Delta_p(u_n-u_0). \tag{3.9}$$

Taking into account that u_0 is a solution of (1.6) and using (3.9), we get

$$-(\Delta_p u_n - \Delta_p u_0) + \lambda_n \Delta_p (u_n - u_0) = a(x)(g(u_n) - g(u_0)). \tag{3.10}$$

Thus $h_n = u_n - u_0$ satisfies the equation

$$-(\Delta_p (u_0 + h_n) - \Delta_p u_0) + \lambda_n \Delta_p h_n = a(x)(g(u_0 + h_n) - g(u_0)). \tag{3.11}$$

Let us introduce the operator $T_n : X \to X^*$, $h \mapsto T_n h$ given by

$$T_n h = -(\Delta_p (u_0 + h) - \Delta_p u_0) + \lambda_n \Delta_p h, \quad h \in X,$$

that is, (set $\mu_n = -\lambda_n \geq 0$)

$$\langle T_n h, \varphi \rangle = \int_{\mathbb{R}^N} \Big[\big(|\nabla(u_0 + h)|^{p-2} \nabla(u_0 + h) - |\nabla u_0|^{p-2} \nabla u_0 \big) \nabla\varphi + \mu_n |\nabla h|^{p-2} \nabla h \nabla\varphi \Big] \, dx. \tag{3.12}$$

Then the operator $T_n : X \to X^*$ is strongly monotone and coercive, which follows from

$$\begin{aligned} \langle T_n h, h \rangle &= \int_{\mathbb{R}^N} \Big[\big(|\nabla(u_0 + h)|^{p-2} \nabla(u_0 + h) - |\nabla u_0|^{p-2} \nabla u_0 \big) \nabla h + \mu_n |\nabla h|^p \Big] \, dx \\ &\geq (\nu + \mu_n) \int_{\mathbb{R}^N} |\nabla h|^p \, dx = (\nu + \mu_n) \|h\|_X^p, \end{aligned}$$

where $\nu = 2^{2-p}$, and similarly

$$\langle T_n h - T_n \hat{h}, h - \hat{h} \rangle \geq (\nu + \mu_n) \int_{\mathbb{R}^N} |\nabla(h - \hat{h})|^p \, dx = (\nu + \mu_n) \|h - \hat{h}\|_X^p.$$

Equation (3.11) is of divergence type and can be written in the form

$$T_n h_n := -\operatorname{div} A_n(x, \nabla h_n) = a(x)(g(u_0 + h_n) - g(u_0)), \tag{3.13}$$

where $A_n : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are given by

$$A_n(x, \xi) = |\nabla u_0 + \xi|^{p-2} (\nabla u_0 + \xi) - |\nabla u_0|^{p-2} \nabla u_0 + \mu_n |\xi|^{p-2} \xi, \quad \xi \in \mathbb{R}^N, \ \mu_n \geq 0. \tag{3.14}$$

Dividing both sides of Eq. (3.13) by $(\nu + \mu_n)$, the point of departure will be the following equivalent equation

$$\frac{1}{\nu + \mu_n} T_n h_n := -\operatorname{div} \left(\frac{1}{\nu + \mu_n} A_n(x, \nabla h_n) \right) = \frac{1}{\nu + \mu_n} a(x)(g(u_0 + h_n) - g(u_0)). \tag{3.15}$$

The vector functions $\frac{1}{\nu+\mu_n}A_n(x,\xi):\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ defining the operators $\frac{1}{\nu+\mu_n}T_n: X\to X^*$ have the following properties:

$$\begin{aligned}\frac{1}{\nu+\mu_n}|A_n(x,\xi)| &\le \frac{1}{\nu+\mu_n}\Big[c(p,N)(|\xi|^{p-1}+|\nabla u_0|^{p-1})+\mu_n|\xi|^{p-1}\Big]\\ &\le \frac{c(p,N)+\mu_n}{\nu+\mu_n}|\xi|^{p-1}+\frac{c(p,N)}{\nu+\mu_n}|\nabla u_0|^{p-1}\\ &\le a_1|\xi|^{p-1}+(a_2(x))^{p-1}, \end{aligned}\tag{3.16}$$

where the positive constant $a_1=\frac{c(p,N)}{\nu}+1$ and the nonnegative function $a_2(x)=\big(\frac{c(p,N)}{\nu}\big)^{\frac{1}{p-1}}|\nabla u_0|$ are independent of n, and

$$\frac{1}{\nu+\mu_n}\big(A_n(x,\xi)-A_n(x,\hat{\xi})\big)(\xi-\hat{\xi})\ge|\xi-\hat{\xi}|^p, \tag{3.17}$$

$$\frac{1}{\nu+\mu_n}A_n(x,\xi)\xi\ge|\xi|^p, \tag{3.18}$$

which shows that monotonicity and coercivity of $\frac{1}{\nu+\mu_n}T_n$ holds independently of n. In view of Lemma 3.3, the function $a_2(x)=\big(\frac{c(p,N)}{\nu}\big)^{\frac{1}{p-1}}|\nabla u_0|$ fulfills the condition (1.3). Thus the operators $\frac{1}{\nu+\mu_n}T_n$ satisfy all assumptions (A1)–(A3) independently of n. Denote $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ by

$$f(x,s)=\frac{1}{\nu+\mu_n}(g(u_0(x)+s)-g(u_0(x))). \tag{3.19}$$

Using hypotheses (G) and taking into account that $|u_0(x)|$ is bounded, we get the growth

$$|f(x,s)|\le c[1+|s|^{\gamma-1}],$$

where c is some positive constant independent of n, which is hypothesis (F). Since the crucial structure conditions (A1)–(A3) of the operators on the left-hand side and the condition (F) on the right-hand side of (3.15) are independent of n, we may apply the global L^∞-estimate (see Lemma 2.1), which gives (note: $\|h_n\|_X\le\frac{1}{n}\to 0$ as $n\to\infty$)

$$\|h_n\|_\infty\le C\phi\big(\|h_n\|_X\big)\to 0. \tag{3.20}$$

From (3.20), the fact that $u_0(x)$ is bounded and g is continuous, we get

$$|a(x)f(x,h_n)|=|a(x)(g(u_0+h_n)-g(u_0)))|\le C(\|h_n\|_X)|a(x)|, \tag{3.21}$$

where $C(\|h_n\|_X)\to 0$ as $n\to\infty$. Next recall that by Lemma 2.5 we have

$$w_n\le h_n\le v_n,$$

where v_n and w_n solve (2.3) and (2.4), respectively, with

$$A(x,\xi) = \frac{1}{\nu + \mu_n} A_n(x,\xi) \quad \text{and} \quad \hat{a}(x) = C(\|h_n\|_X)a(x),$$

from which it follows

$$\|v_n\|_X \leq C(\|h_n\|_X) \quad \text{and} \quad \|w_n\|_X \leq C(\|h_n\|_X).$$

Upon further application of Theorem 2.8 we get the following decay estimate:

$$|h_n(x)| \leq C_1 \frac{C(\|h_n\|_X)}{1+|x|^{\frac{N-p}{p}}} + \frac{C_2}{1+|x|^{\frac{N-p}{p-1}}}, \tag{3.22}$$

where $C(\|h_n\|_X) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, let $R \gg 1$ be sufficiently large such that

$$C_2 R^{\frac{p-N}{p(p-1)}} < \varepsilon. \tag{3.23}$$

Observe that

$$\sup_{\mathbb{R}^N} \left|\left(1+|x|^{\frac{N-p}{p}}\right)h_n(x)\right| \leq \sup_{|x|\leq R} \left|\left(1+|x|^{\frac{N-p}{p}}\right)h_n(x)\right| + 2 \sup_{|x|\geq R} |x|^{\frac{N-p}{p}} |h_n(x)|. \tag{3.24}$$

The first term on the right-hand side of (3.24) tends to zero as $n \to \infty$ due to (3.20). As for the second term on the right-hand side of (3.24) we get by using (3.22) and (3.23) the estimate

$$\begin{aligned} \sup_{|x|\geq R} |x|^{\frac{N-p}{p}} |h_n(x)| &\leq C_1 C(\|h_n\|_X) + C_2 R^{\frac{p-N}{p(p-1)}} \\ &\leq C_1 C(\|h_n\|_X) + \varepsilon, \end{aligned}$$

and thus in view of (3.20)

$$\sup_{|x|\geq R} |x|^{\frac{N-p}{p}} |h_n(x)| \leq \varepsilon, \quad \text{as } n \to \infty. \tag{3.25}$$

From (3.24) and (3.25) along with (3.20) and $\|h_n\|_X \to 0$ it follows that $\|h_n\|_V \to 0$. Finally, since u_0 is a local minimizer of Φ in the V-topology we get with $h_n \to 0$ in V for n large

$$\Phi(u_0) \leq \Phi(u_0 + h_n) = \Phi(u_n) = \inf_{h\in B_n} \Phi(u_0 + h), \text{ where } B_n = \left\{u \in X : \|u - u_0\|_X \leq \frac{1}{n}\right\},$$

which proves that u_0 must be a local minimizer of Φ in the X-topology completing the proof of Theorem 1.9. ◻

Remark 3.5 Consider the special case $p = 2$, that is, we are dealing with the semilinear equation

$$u \in X = D^{1,2}(\mathbb{R}^N) : -\Delta u = a(x)g(u), \tag{3.26}$$

where a fulfills (A0) and g satisfies (G). Let $\Phi : X \to \mathbb{R}$ be the associated energy functional given by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \int_{\mathbb{R}^N} a(x)G(u)\,dx, \quad \text{with } G(s) = \int_0^s g(t)\,dt.$$

Applying Theorem 1.6, then for any solution u of (3.26) we have $u \in X \cap C^1(\mathbb{R}^N)$ and the following decay

$$|u(x)| \le C_2 \frac{1}{1 + |x|^{N-2}}, \tag{3.27}$$

Let V_{N-2} be the subspace of bounded continuous functions with weight $1 + |x|^{N-2}$ defined by

$$V_{N-2} := \left\{ v \in X : v \in C(\mathbb{R}^N) \text{ with } \sup_{x\in\mathbb{R}^N} \left(1 + |x|^{N-2}\right)|v(x)| < \infty \right\},$$

which is a closed subspace of X with norm

$$\|v\|_{V_{N-2}} := \|v\|_X + \sup_{x\in\mathbb{R}^N} \left(1 + |x|^{N-2}\right)|v(x)|, \; v \in V_{N-2}.$$

Then the following X versus V_{N-2} local minimizer theorem holds:

Theorem 3.6 *Suppose $u_0 \in X$ is a solution of the Eq.* (3.26) *and a local minimizer in the V_{N-2}-topology of the functional $\Phi : X \to \mathbb{R}$, that means, there exists $\varepsilon > 0$ such that*

$$\Phi(u_0) \le \Phi(u_0 + h), \quad \forall\, h \in V_{N-2} : \|h\|_{V_{N-2}} < \varepsilon.$$

Then u_0 is a local minimizer of Φ with respect to the X-topology, that is, there is $\delta > 0$ such that

$$\Phi(u_0) \le \Phi(u_0 + h), \quad \forall\, h \in X : \|h\|_X < \delta.$$

We note that when applying our general result Theorem 1.9 to the special case considered here one would get only a X versus $V_{\frac{N-2}{2}}$ local minimizer theorem, where $V_{\frac{N-2}{2}}$ is the closed subspace of X defined by

$$V_{\frac{N-2}{2}} := \left\{ v \in X : v \in C(\mathbb{R}^N) \text{ with } \sup_{x\in\mathbb{R}^N} \left(1 + |x|^{\frac{N-2}{2}}\right)|v(x)| < \infty \right\},$$

which is V for $p = 2$. Clearly we have the continuous embedding $V_{N-2} \hookrightarrow V_{\frac{N-2}{2}}$. The reason for obtaining an X versus V_{N-2} local minimizer theorem within the finer topology of V_{N-2} is the linearity of the Laplacian, which by the Lagrangian multiplier method used in the proof of Theorem 1.9 yields again an equation with the Laplacian as the leading differential operator. In this case, by means of (3.27) we get the following estimate of the corresponding sequence (h_n):

$$|h_n(x_0)| \leq C_2(N, p, \sigma, D)\frac{C(\|h_n\|_X)}{1 + |x_0|^{N-2}},$$

which immediately implies the convergence of (h_n) in V_{N-2}, since $C(\|h_n\|_X) \to 0$ as $\|h_n\|_X \to 0$.

The situation for the p-Laplacian with $p > 2$ is, however, different, since in this case the Lagrangian method leads to an equation which does not preserve the structure of the p-Laplacian, and therefore the decay estimate for the solution of the Lagrangian equation cannot be obtained via the decay estimate for the p-Laplacian equation, and requires pointwise estimates for quasilinear elliptic operators of the more general structure given by the assumptions (A1)–(A3).

Although Theorem 3.6 has been proved in [7], here we provide a novel approach for its proof which is more efficient and which is based on two elements: the global L^∞-estimate $\|u\|_\infty \leq C\phi(\|u\|_X)$ and on the decay estimate (3.27) which immediately follows from Theorem 1.6.

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