Dominance of Positivity of the Green's Function associated to a Perturbed Polyharmonic Dirichlet Boundary Value Problem by Pointwise Estimates

Dissertation

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

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geb. am 16.03.1985 in Halle/Saale

genehmigt durch die Fakultät für Mathematik
der Otto-von-Guericke-Universität Magdeburg

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eingereicht am: 25.11.2014

Verteidigung am: 25.02.2015

Abstract

In this work we study the behaviour of the Green function for a linear higher-order elliptic problem. More precisely, we consider the Dirichlet boundary value problem in a bounded $C^{2m,\gamma}$-smooth domain in $\mathbb{R}^n$, $n \geq 2$, for a linear symmetric coercive differential operator of order $2m$, $m \geq 2$. The polyharmonic operator $(-\Delta)^m$ builds the main part of the differential operator and the sufficiently smooth coefficient functions of the lower-order partial derivatives are assumed to be uniformly bounded. The Green function associated to this problem may be sign changing, even if the right-hand side or source term in the differential equation is positive. The focus of this thesis is to show that the negative part of the Green function is small compared with the dominant positive part.

For second-order differential equations this positivity preserving property can be shown by the maximum principle. By positivity preserving property we mean that a positive right-hand side gives rise to a positive solution. Certainly, the maximum principle is in general no longer true for higher-order equations. As a goal of this work, we prove two-sided estimates for the Green function, from which the dominance of the positive part can be seen.

The estimates from above for the Green function are due to Krasovskiĭ [43, 44], cf.[16], where higher smoothness of the boundary of the domain is assumed. In this work we present a proof for the estimates from above in $C^{2m,\gamma}$-smooth domains, similar to that in [23, Sec. 4.4] for the biharmonic Green function.

To prove the estimates from below, we use a blow-up argument developed in [27] and [29] in the case of the biharmonic problem ($m = 2$). We extend this procedure to higher order differential operators and generalise it, in the sense that we allow lower-order perturbations of the differential operator.

Since the coefficient functions of the lower-order perturbations are assumed to be uniformly bounded, they vanish during the blow-up. Therefore, the behaviour of the Green function is determined by the main part of the operator, that is $(-\Delta)^m$. Subsequently, the estimates from below can be shown with the help of Boggio's explicit formula of the polyharmonic Green function in the ball from [9, p. 126].

Zusammenfassung

In dieser Arbeit studieren wir das Verhalten der Greenschen Funktion eines linearen elliptischen Problems höherer Ordnung. Genauer gesagt betrachten wir das Dirichletsche Randwertproblem in einem beschränkten $C^{2m,\gamma}$-glatten Gebiet in $\mathbb{R}^n$, $n \geq 2$, für einen linearen symmetrischen koerzitiven Differentialoperator der Ordnung $2m$, $m \geq 2$. Der polyharmonische Operator $(-\Delta)^m$ bildet den Hauptteil des Differentialoperators und die hinreichend glatten Koeffizientenfunktionen der partiellen Ableitungen niederer Ordnung werden als gleichmäßig beschränkt vorausgesetzt. Selbst wenn die rechte Seite bzw. der Quellterm in der Differentialgleichung positiv ist, kann die zugehörige Greensche Funktion ihr Vorzeichen wechseln. Der Fokus der Arbeit richtet sich darauf zu zeigen, dass der negative Teil der Greenschen Funktion klein im Vergleich zum dominanten positiven Teil ist.

Für Differentialgleichungen zweiter Ordnung kann diese positivitätserhaltende Eigenschaft mit Hilfe des Maximumprinzips gezeigt werden. Für uns bedeutet die positivitätserhaltende Eigenschaft, dass eine positive rechte Seite zu einer positiven Lösung führt. Allerdings gilt, im Allgemeinen, das Maximumprinzip nicht mehr für Gleichungen höherer Ordnung. Als Ziel dieser Arbeit beweisen wir zweiseitige Abschätzungen der Greenschen Funktion, die die Dominanz des positiven Teils zeigen.

Die Abschätzungen von oben für die Greensche Funktion gehen zurück auf Krasovskiĭ [43, 44], vgl. [16], wobei hier höhere Glattheit des Randes des Gebiets vorausgesetzt wurde. In dieser Arbeit präsentieren wir einen Beweis für die Abschätzungen von oben in $C^{2m,\gamma}$-glatten Gebieten, ähnlich dem in [23, Sec. 4.4] für die biharmonische Greensche Funktion.

Um die Abschätzungen von unten zu beweisen, benutzen wir ein in [27] und [29] für das biharmonische Problem ($m = 2$) entwickelte Blow-up-Argument. Wir erweitern dieses Vorgehen auf Differentialoperatoren höherer Ordnung und verallgemeinern es in dem Sinne, dass wir Störungen niederer Ordnung für den Differentialoperator zulassen.

Da die Koeffizientenfunktionen der Störungen niederer Ordnung als gleichmäßig beschränkt vorausgesetzt wurden, verschwinden sie während des Blow-ups. Daher bestimmt der Hauptteil des Operators, das heißt $(-\Delta)^m$, das Verhalten der Greenschen Funktion. Mit Hilfe Boggios expliziter Formel für die polyharmonische Greensche Funktion in der Kugel aus [9, S. 126] können dann die Abschätzungen von unten gezeigt werden.

Contents

Introduction

In the year 1905 the Italian mathematician Tommaso Boggio answered the following question in the affirmative [9]:

> If a clamped plate of circular shape will be pushed upwards, will the clamped plate bend upwards everywhere, too? (Q)

More precisely, he found an explicit formula for the Green function $G_{(-\Delta)^m,B_1}$ of the following Dirichlet boundary value problem:

$$\begin{cases} (-\Delta)^m u = f & \text{in } B_1, \\ \dfrac{\partial^j}{\partial \nu^j} u = 0 & \text{on } \partial B_1,\ j = 0, \dots, m-1, \end{cases} \tag{0.1}$$

where f denotes a suitable datum, $B_1 = B_1(0) \subset \mathbb{R}^n$ the unit ball, ν the exterior unit normal at the boundary ∂B_1 and m, n are integers greater or equal than two. If the dimension n and the power of the operator m are equal two the Dirichlet problem (0.1) can be seen as a model for a clamped plate of circular shape, where f denotes the pushing force and the function $u : \overline{B}_1 \to \mathbb{R}$ describes the deflection of the plate, see Gazzola et al. [23, p. 5 ff.] and the references therein. To give a better understanding, why an explicit formula of the Green function helps to answer the question (Q) let us look at the following formula for the solution of (0.1):

$$u(x) = \int_{B_1} G_{(-\Delta)^m,B_1}(x,y) f(y) dy,$$

where $0 \not\equiv f \geq 0$. Therefore, $G_{(-\Delta)^m,B_1} \geq 0$ will imply a positive answer of question (Q). This implication holds true for the ball since the formula given by Boggio reads for example if $n = m = 2$ as

$$G_{(-\Delta)^2,B_1}(x,y) = \frac{1}{8\pi} |x-y|^2 \int_1^{\left| |x|y - \frac{x}{|x|} \right| / |x-y|} \frac{(v^2-1)}{v} dv,$$

which is strictly positive.

Going a step further it is quite natural to ask if this positivity preserving property, i.e.

$$0 \not\equiv f \geq 0 \Rightarrow u \geq 0,$$

is true even in more general domains Ω, where u is a solution of

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \dfrac{\partial^j}{\partial \nu^j} u = 0 & \text{on } \partial\Omega,\ j = 0, \dots, m-1. \end{cases} \tag{0.2}$$

It goes back to the so called Boggio-Hadamard conjecture [8, 9, 36, 37], where Boggio and Hadamard conjectured, that the Green's function $G_{(-\Delta)^2,\Omega}$ for the clamped plate boundary value problem, i.e. $m = 2$, in convex two-dimensional domains is positive. This is in general not the case, since many counterexamples exist.

Garabedian showed sign-changing of the Green's function for an ellipse [21], [22, p. 275], which is a counterexample in the class of smooth convex domains. For a long rectangle the conjecture is disproven by an example of Duffin [19, 20]. In domains with corners sign-changing of $G_{(-\Delta)^2,\Omega}$ is proven by Coffmann and Duffin [12]. Moreover, the conjecture for $G_{(-\Delta)^2,\Omega}$ is not true in higher dimensions [48].

For $m \geq 2$ and $n \geq 2$, Kozlov et al. [42] constructed a strictly convex smooth domain with large curvature of the boundary where $G_{(-\Delta)^m,\Omega}$ changes its sign. More precisely, they have smoothed the boundary of a strictly convex domain with a corner, the angle of which is small, on which they have proved that $G_{(-\Delta)^m,\Omega}$ changes its sign.

Even if the right-hand side of (0.2) is a constant, e.g. $f \equiv 1$, Grunau and Sweers constructed in [33, 34] domains such that the solution of (0.2) for $m = 2$ changes sign. For a deeper discussion of the Boggio-Hadamard conjecture, see [23, p. 9 ff] and the references therein.

Within the precision of measurement, an engineer would expect that the positivity preserving property for the clamped plate holds in smooth domains or, if not, the negative deflection of the plate is very small. If we consider the corresponding second-order problem, i.e. $m = 1$, for the deflection of a membrane, i.e. a soap film, this expectation coincides with mathematical observations because the positivity preserving property follows directly from the maximum principle. However, this powerful tool is no longer available for fourth- or higher-order problems, which can be seen by the biharmonic functions $x \mapsto \pm|x|^2$.

As mentioned before, in general one cannot expect positivity of the polyharmonic Green function $G_{(-\Delta)^m,\Omega}$ in domains other than the ball. Therefore much work has been done to find families of domains, where positivity is still true. This was done for two-dimensional domains which are close to the ball in the C^{2m}-sense by Grunau and Sweers [30] and relaxed to $C^{m,\gamma}$-closeness by Sassone [53]. For non-convex two-dimensional domains an example, precisely the Limaçon de Pascal, was given by Dall'Acqua and Sweers [17] in the biharmonic case. An extension to higher dimensional domains C^4-close to the ball were proven by Grunau and Robert [27] for the biharmonic Green function. The authors mentioned that their result should hold true for higher-order problems in dimensions $n \geq 2m - 1$.

In order to identify regions of positivity Grunau and Sweers [32] generalised the result and the methods of Nehari [49] to find a constant $\delta_{m,n}$ independent of the domain Ω, such that for all $x, y \in \Omega$ with

$$|x-y| < \delta_{m,n} \max\{d(x), d(y)\},$$

where here and in the following $d(x) := \operatorname{dist}(x, \partial\Omega) = \inf_{x^* \in \partial\Omega} |x - x^*|$, it yields

$$G_{(-\Delta)^m,\Omega}(x,y) > 0.$$

This was proven for $n > 2m$ and extended by Köckritz [41] to the case $n = 2m$. Therefore, a uniform bound of the negative part of the Green function is obtained, as long as x and y stay uniformly away from the boundary $\partial\Omega$. In the biharmonic case and if the pole of the Green function approaches the boundary, a minimal distance $\delta = \delta(\Omega)$ can be found such that for any $x, y \in \Omega$ with $x \neq y$:

$$|x-y| < \delta \quad \text{implies that} \quad G_{(-\Delta)^2,\Omega}(x,y) > 0. \tag{LP}$$

This local positivity result goes back to Grunau and Robert [27] for $n \geq 3$ and to Dall'Acqua et al. [15] for $n = 2$, see also [23, Theorem 6.24].

After these results we want to go a step further and look for bounds of the negative part of the polyharmonic Green function. For a generalisation of the situation we will look for the Green function G of the following Dirichlet boundary value problem with lower-order terms

$$\begin{cases} (-\Delta)^m u(x) + \sum\limits_{\ell=0}^{m-1} \sum\limits_{|\alpha|=|\beta|=\ell} D^\beta \left(a^\ell_{\alpha,\beta}(x) D^\alpha u(x) \right) = f(x) & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \dfrac{\partial^j}{\partial \nu^j} u(x) = 0 & \text{for } x \in \partial\Omega,\ j = 0, \dots, m-1, \end{cases} \tag{0.3}$$

where the datum f is in a suitable function space, and the coefficient functions $a^\ell_{\alpha,\beta}$ are assumed to be sufficiently smooth, i.e. $a^\ell_{\alpha,\beta} \in C^{m-1,\gamma}(\overline{\Omega})$ and symmetric, i.e. $a^\ell_{\alpha,\beta} = a^\ell_{\beta,\alpha}$. In addition we assume uniform boundedness, i.e. we find a $K > 0$, such that for all ℓ we have that

$$\left\| a^\ell_{\alpha,\beta} \right\|_{C^{m-1,\gamma}(\overline{\Omega})} \leq K.$$

Moreover, we assume that the bilinear form associated to the differential operator in (0.3) is coercive on $W^{m,2}_0(\Omega)$.

Let us now state the main result of this work.

Theorem 0.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2m,\gamma}$-smooth domain, $m \geq 2$. Let G denote the Green function in Ω for* (0.3)*. Then there exist constants $c_1 \geq 0$, $c_2 > 0$ and $c_3 > 0$, depending on the domain Ω, m and K, such that we have the following Green function estimate:*

$$c_2^{-1} H_\Omega(x,y) \leq G(x,y) + c_1 \mathbb{1}_{\{|x-y| \geq c_3\}}(x,y) d(x)^m d(y)^m \leq c_2 H_\Omega(x,y)$$

for all $x, y \in \Omega$, where

$$H_\Omega(x,y) := \begin{cases} |x-y|^{2m-n} \min\left\{1, \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n > 2m, \\ \log\left(1 + \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \dfrac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} & \text{if } n < 2m, \end{cases}$$

and

$$\mathbb{1}_{\{|x-y| \geq c_3\}}(x,y) := \begin{cases} 1 & \text{if } |x-y| \geq c_3, \\ 0 & \text{if } |x-y| < c_3, \end{cases}$$

is the indicator function.

Since a blow-up argument will be used to prove Theorem 0.1 uniform boundedness of the coefficients will be crucial. Note that the Green function for $(-\Delta)^m + a$ is always sign-changing whenever a is large enough [13], cf. [23, Corollary 5.5].

We give a short overview of what is done for characterisations of Green's functions like in Theorem 0.1.

As from Boggio's formula the polyharmonic Green's function $G_{(-\Delta)^m, B_1}$ in a ball is explicitly known, Grunau and Sweers gave the following optimal two-sided estimates

$$c_2^{-1} H_{B_1}(x,y) \leq G_{(-\Delta)^m, B_1}(x,y) \leq c_2 H_{B_1}(x,y), \tag{0.4}$$

see [31, Proposition 2.3]. In general domains, the absolute value of the polyharmonic Green function $G_{(-\Delta)^m,\Omega}$ of a Dirichlet boundary value problem could be estimated from above by the function H_Ω, as shown by Dall'Acqua and Sweers [16]. The authors stated, that their estimates hold for general uniformly elliptic differential operators of order $2m$. As a starting point, they used estimates without boundary terms of Green's functions for general higher order elliptic operators, which are due to Krasovskiĭ [43, 44].

A goal of further research was the improvement of the estimates from below, since the estimates from above from [16] seem to be optimal for $G_{(-\Delta)^m,\Omega}$, see (0.4). As a first step in this direction in the biharmonic case, the following bound from below

$$G_{(-\Delta)^2,\Omega}(x,y) \geq -c(\Omega) d(x)^2 d(y)^2$$

was proven by Grunau and Robert [27] for $n \geq 3$ and by Dall'Acqua et al. [15] for $n = 2$. Recently, Grunau et al. extended this result in [29] and proved Theorem 0.1 in the biharmonic case. The proof uses a blow-up procedure developed in [27]. Their ideas serve as the foundation of this work.

Therefore, our result Theorem 0.1 continues this process and gives a characterisation for the Green's function G of (0.3) by two-sided estimates, where the estimates from below are the main result.

Moreover, with the help of Theorem 0.1 a local positivity result as (LP) directly follows, i.e. for any $x, y \in \Omega$ with $x \neq y$:

$$|x-y| < c_3 \quad \text{implies that} \quad G(x,y) \geq c_2^{-1} H_\Omega(x,y) > 0.$$

For second-order differential operators, i.e. $m = 1$, two-sided estimates as (0.4) for the Green's function for general, sufficiently smooth domains are known. In dimensions higher than two, Grüter and Widman [56, 35] found estimates from above for the Green's function by H_Ω. Two-sided estimates for the Green's function of the Laplace operator were proven by Zhao [57, 58], see also [11]. For more general second-order differential operators, estimates like (0.4) in general, sufficiently smooth domains are due to Ancona [5], Hueber and Sieveking [39], and Cranston et al. [14], see also [55].

For estimates for the polyharmonic Green function in non-smooth domains we refer to Mayboroda and Maz'ya [46].

Now, let us briefly describe the idea of the proof for the estimates from below, since the estimates from above can be proven as in [23, Chapter 4], see also [16]. The main work has to be done for points near the boundary $\partial\Omega$. Here we will use a proof by contradiction and a blow-up or rescaling argument, such that the rescaled Green functions will converge locally uniformly to the polyharmonic Green function $G_{(-\Delta)^m,\mathcal{H}}$ of the half space $\mathcal{H} := \{x \in \mathbb{R}^n : x_1 < 0\}$, while the domain of the functions will converge locally uniformly to $\mathcal{H}$ itself. Using known estimates from below for the polyharmonic Green function $G_{(-\Delta)^m,\mathcal{H}}$, which goes back to Boggio's formula previously mentioned, leads to the desired contradiction.

We will briefly outline the parts of this work.

The first chapter is dedicated to some preliminary facts about the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^n$ and some useful estimates for the fundamental solution itself. Some facts about polyharmonic functions, which can be seen as the regular part of the polyharmonic Green function $G_{(-\Delta)^m,\Omega}$ will be given. Moreover, we will state the polyharmonic Green function for the half space and since we will use a rescaling argument, convergence of domains will be explained.

The second part is devoted to the perturbed Dirichlet boundary value problem. We will construct the Green function for the perturbed operator and find estimates for the Green function and it's derivatives proceeding as in [23, Chapter 4]. At the end, we will show the uniform convergence of the Green function to $G_{(-\Delta)^m,\mathcal{H}}$.

Since the coefficients of the perturbed operator are uniformly bounded, the main part $(-\Delta)^m$ of the differential operator plays an important role for the behaviour of the Green function. Therefore, we will prove in chapter 3 our main theorem in the polyharmonic case. For large dimensions a Nehari-type result together with the blow-up argument will be used. For the small dimensions we will first prove the uniqueness of $G_{(-\Delta)^m,\mathcal{H}}$ under some growth conditions at infinity. Then, a blow-up argument leads to the desired result.

Finally, in chapter 4 we will prove the main theorem for the perturbed operator. Since in many cases this can be done like the polyharmonic case, we will describe the changes.

Notation: Here we follow mostly the notations as in the book of Gazzola et al. [23]. In particular Ω denotes a domain, an open and connected subset of $\mathbb{R}^n$, $n \geq 2$. The positive constants C, c may change from term to term and depend on the parameters given in brackets. For further notations look at the list of symbols.

Acknowledgements: First of all I would like to thank Prof. H.-Ch. Grunau. His guidance and ideas gave me the opportunity to write this thesis under his supervision. Being his student since my first semester, he introduced and educated me in doing analysis.

I am grateful to my colleagues and friends Dr. Carlos Cueto Camejo, Robin Gröpler and Dr. Arthur Schlichting for many constructive conversations and their support beyond that. Thanks Robin, for your patience and the long time we spent together in learning mathematics.

A special thank goes out to Iris. Your love and affection is a great encouragement for me. Thank you for being there whenever I needed your support. I am grateful for all the beautiful time I could share with you beside this work.

I would like to thank my parents, my siblings Claudia and Maximilian and my whole family. You are always behind me. Your support and happiness in your midst means a lot to me.

Thanks to my flatmates Claudia and Klaus for being a part of my life and for the cheerful evenings in our kitchen.

1 Preliminaries and Auxiliary Results

In order to understand this work we need good knowledge about the polyharmonic fundamental solution and Green's Function. Therefore, we will state some basic facts in the following two sections. The last section of this chapter is devoted to a precise explanation to our understanding of the convergence of domains. This will be needed for a blow-up argument in the next chapters.

1.1 A Polyharmonic Fundamental Solution

We recall a fundamental solution for the polyharmonic operator $(-\Delta)^m$ on $\mathbb{R}^n$, cf. [40, p. 43 f.]:

$$F_{m,n}(x) = \begin{cases} \dfrac{2\Gamma(n/2-m)}{ne_n 4^m \Gamma(n/2)(m-1)!}|x|^{2m-n} & \text{if } n > 2m \text{ or } n \text{ is odd,} \\ \dfrac{(-1)^{m-n/2}}{ne_n 4^{m-1}\Gamma(n/2)(m-n/2)!(m-1)!}|x|^{2m-n}(-\log|x|) & \text{if } n \leq 2m \text{ is even,} \end{cases} \tag{1.1}$$

where $e_n := \int_{B_1(0)} dx$, such that

$$(-\Delta)^m F_{m,n} = \delta_0$$

in the distributional sense. If we assume

$$\lim_{|x|\to\infty} F_{m,n}(x) = 0, \tag{1.2}$$

the fundamental solution is unique for $n > 2m$. For $n \leq 2m$ condition (1.2) is not satisfied by a fundamental solution and it seems, that there exists no natural condition to achieve uniqueness in this dimensions, cf. [23, p. 50 f.].

From [6, Proposition 3.3] we get some basic estimates for the derivatives of the fundamental solution:

$$|D^\alpha F_{m,n}(x)| \leq C(m,n,|\alpha|)|x|^{2m-n-|\alpha|} \cdot \begin{cases} 1 & \text{if } n > 2m \text{ or } n \text{ is odd,} \\ 1 + |\log|x|| & \text{if } n \leq 2m \text{ is even.} \end{cases} \tag{1.3}$$

Remark 1.1. Observe that for all $\gamma' \in (0,1)$ it holds

$$1 + |\log|x|| \le \frac{1}{\gamma'}|x|^{-\gamma'} + |x|. \tag{1.4}$$

Moreover, if Ω is a bounded domain, there exists a constant $C = C(\Omega) > 0$ such that for all $x, y \in \Omega$ with $x \neq y$ we can show that

$$1 + |\log|x-y|| \le C \log\left(1 + \frac{1}{|x-y|}\right). \tag{1.5}$$

We give a short proof of (1.5). If $|x-y| \le 1$, we use Bernoulli's inequality and $e = \exp(1)$ to see

$$1 + |\log|x-y|| = \log\left(\frac{e}{|x-y|}\right) \le \log\left(1 + \frac{e}{|x-y|}\right) \le e\log\left(1 + \frac{1}{|x-y|}\right).$$

If $|x-y| > 1$, we have

$$\begin{aligned} 1 + |\log|x-y|| &\le 1 + \log(\operatorname{diam}(\Omega)) \le C(\Omega)\log\left(1 + \frac{1}{\operatorname{diam}(\Omega)}\right) \\ &\le C(\Omega)\log\left(1 + \frac{1}{|x-y|}\right), \end{aligned}$$

where $\operatorname{diam}(\Omega) := \sup\{|x-y| : x, y \in \Omega\}$.

From Lemma A.1 in the Appendix we derive a stronger estimate if n is even and $|\alpha| > 2m - n$:

$$|D^\alpha F_{m,n}(x)| = \left|D^\alpha\left(c_{m,n}|x|^{2m-n}\log|x|\right)\right| \le C(m,n,|\alpha|)|x|^{2m-n-|\alpha|}. \tag{1.6}$$

1.2 A Polyharmonic Green Function and Green's Second Identity

We cite some facts about the polyharmonic Green function from [23, Section 2.6]. Here let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and f be in a suitable function space. Then we have the following definition, cf. [23, Definition 2.26].

Definition 1.2. A Green function for the Dirichlet problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ D^\alpha u = 0 & \text{on } \partial\Omega,\ |\alpha| \le m-1, \end{cases} \tag{1.7}$$

is a function $(x,y) \mapsto G_{(-\Delta)^m,\Omega}(x,y) : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \cup \{\infty\}$ satisfying:

1. $x \mapsto G_{(-\Delta)^m,\Omega}(x,y) - F_{m,n}(x-y) \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ for all $y \in \Omega$ if defined suitably for $x = y$;

2. $(-\Delta_x)^m \left(G_{(-\Delta)^m,\Omega}(x,y) - F_{m,n}(x-y)\right) = 0$ for all $(x,y) \in \Omega^2$ if defined suitably for $x = y$;

3. $D_x^\alpha G_{(-\Delta)^m,\Omega}(x,y) = 0$ for all $(x,y) \in \partial\Omega \times \Omega$ and $|\alpha| \le m-1$.

The Green function for the Dirichlet problem with $\Omega = B_1(0) = B_1 \subset \mathbb{R}^n$ was explicitly calculated by Boggio [9, p. 126], see also [23, Lemma 2.27], and reads as follows:

$$G_{(-\Delta)^m,B_1}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (v^2-1)^{m-1}v^{1-n}dv, \tag{1.8}$$

where

$$k_{m,n} := \frac{1}{ne_n 4^{m-1}((m-1)!)^2}.$$

Applying the Cayley transform to (1.8) for $x, y \in \mathcal{H} := \{x \in \mathbb{R}^n : x_1 < 0\}$ we have that

$$G_{(-\Delta)^m,\mathcal{H}}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{|x^*-y|/|x-y|} (v^2-1)^{m-1}v^{1-n}dv, \tag{1.9}$$

where $x^* = (-x_1, x_2, \ldots, x_n)$, see [23, Remark 2.28].

Like in the case $m = 1$ we have Green's second identity.

Proposition 1.3 (Green's second identity)**.** *Let Ω be a domain for which the divergence theorem holds and let $u, v \in C^{2m}(\overline{\Omega})$. Then it holds*

$$\sum_{\ell=0}^{m-1} \int_{\partial\Omega} \frac{\partial}{\partial\nu}(-\Delta)^\ell u(-\Delta)^{m-1-\ell}v - \frac{\partial}{\partial\nu}(-\Delta)^{m-1-\ell}v(-\Delta)^\ell u \, d\sigma + \int_\Omega v(-\Delta)^m u - u(-\Delta)^m v \, dx = 0, \tag{1.10}$$

where ν denotes the exterior unit normal at the boundary $\partial\Omega$.

Proof. We insert $(-\Delta)^\ell u$ and $(-\Delta)^{m-1-\ell}v$ for $\ell = 0, \ldots, m-1$ in the usual second Green's identity. Summing up shows the result. □

As an easy consequence of Proposition 1.3 we have the following corollary.

Corollary 1.4. *Let Ω a domain for which the divergence theorem holds and $u, v \in C^{2m}(\overline{\Omega})$. Let $\frac{\partial^j}{\partial\nu^j}v = 0$ on $\partial\Omega$ for $j = 0, \ldots, m-1$. Then the following holds for $k \in \mathbb{N}_0$.*

1. *If* $m = 2k$*:*

$$\sum_{\ell=1}^{k} \int_{\partial\Omega} \frac{\partial}{\partial\nu}(-\Delta)^{\ell-1}u(-\Delta)^{m-\ell}v - \frac{\partial}{\partial\nu}(-\Delta)^{m-\ell}v(-\Delta)^{\ell-1}u \, d\sigma + \int_{\Omega} v(-\Delta)^m u - u(-\Delta)^m v \, dx = 0. \quad (1.11)$$

2. *If* $m = 2k+1$*:*

$$\sum_{\ell=1}^{k} \int_{\partial\Omega} \frac{\partial}{\partial\nu}(-\Delta)^{\ell-1}u(-\Delta)^{m-\ell}v - \frac{\partial}{\partial\nu}(-\Delta)^{m-\ell}v(-\Delta)^{\ell-1}u \, d\sigma - \int_{\partial\Omega} \frac{\partial}{\partial\nu}(-\Delta)^k v(-\Delta)^k u \, d\sigma + \int_{\Omega} v(-\Delta)^m u - u(-\Delta)^m v \, dx = 0. \quad (1.12)$$

Let us make the following definitions to abbreviate the boundary integrals.

1. If $m = 2k$:

$$I_{\partial\Omega}(u,v) := \sum_{\ell=1}^{k} \int_{\partial\Omega} \frac{\partial}{\partial\nu}(-\Delta)^{\ell-1}u(-\Delta)^{m-\ell}v - \frac{\partial}{\partial\nu}(-\Delta)^{m-\ell}v(-\Delta)^{\ell-1}u \, d\sigma. \quad (1.13)$$

2. If $m = 2k+1$:

$$I_{\partial\Omega}(u,v) := \sum_{\ell=1}^{k} \int_{\partial\Omega} \frac{\partial}{\partial\nu}(-\Delta)^{\ell-1}u(-\Delta)^{m-\ell}v - \frac{\partial}{\partial\nu}(-\Delta)^{m-\ell}v(-\Delta)^{\ell-1}u \, d\sigma - \int_{\partial\Omega} \frac{\partial}{\partial\nu}(-\Delta)^k v(-\Delta)^k u \, d\sigma. \quad (1.14)$$

The polyharmonic Green function is given by $G_{(-\Delta)^m,\Omega}(x,y) = F_{m,n}(x-y) + u(x,y)$, where $u(x,.) = u_x(.)$ is a solution of the Dirichlet problem stated in the following Lemma 1.5. Since we want to show estimates for the Green function in the next chapter, a good knowledge of the behaviour of the regular part u_x of the Green function is needed. Therefore we need the following Lemma.

Lemma 1.5. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded* $C^{2m,\gamma}$*-smooth domain,* $x \in \Omega$ *and* u_x *a solution of the following Dirichlet problem:*

$$\begin{cases} (-\Delta_y)^m u_x = 0 & \text{in } \Omega, \\ \dfrac{\partial^j}{\partial\nu_y^j} u_x(y) = -\dfrac{\partial^j}{\partial\nu_y^j}\Gamma_0(x,y) & \text{for } y \in \partial\Omega \text{ and } j = 0,\dots,m-1, \end{cases} \quad (1.15)$$

where $\Gamma_0(x,y) := F_{m,n}(x-y)$*. Then we have the following estimate:*

$$\|u_x\|_{C^{m-1,\gamma}(\overline{\Omega})} \le C(\Omega,m,n) \cdot \begin{cases} d(x)^{m-n+1-\gamma} & \text{if } n > m, \\ 1 & \text{if } n \le m, \end{cases} \quad (1.16)$$

where $d(x) := \operatorname{dist}(x,\partial\Omega) = \inf_{x^*\in\partial\Omega} |x-x^*|$*.*

Proof. Since $\partial\Omega$ is $C^{2m,\gamma}$-smooth, there exists for small enough $\varepsilon > 0$ a finite number of points $y_i \in \partial\Omega, i = 1, \ldots, N$, such that $\partial\Omega \subset \bigcup_{i=1}^{N} B_\varepsilon(y_i)$ and $\partial\Omega \cap B_{2\varepsilon}(y_i)$ is the graph of a $C^{2m,\gamma}$ function. Let $x \in \Omega$ be fixed, so $d(x) > 0$. Let us first collect some facts for different locations of $y', y'' \in \partial\Omega$ and $y_i \in \partial\Omega$ fixed.

Case $y' \in \partial\Omega \cap B_\varepsilon(y_i)$ and $|y' - y''| < \varepsilon$. With the mean value theorem we see

$$\begin{aligned}\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} &\le C \frac{\left\|\nabla_y D_y^\alpha \Gamma_0(x,\ .\ )\right\|_{C^0(\partial\Omega \cap B_{2\varepsilon}(y_i))} L(y',y'')}{|y'-y''|^\gamma} \\ &\le C \left\|\nabla_y D_y^\alpha \Gamma_0(x,\ .\ )\right\|_{C^0(\partial\Omega \cap B_{2\varepsilon}(y_i))} |y'-y''|^{1-\gamma},\end{aligned}$$

where $L(y', y'')$ denotes the length of a path from y' to y'' in $\partial\Omega$. If $\varepsilon \le d(x)$ or $\varepsilon > d(x) \ge |y' - y''|$ we get

$$\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} \le C \left\|\nabla_y D_y^\alpha \Gamma_0(x,\ .\ )\right\|_{C^0(\partial\Omega \cap B_{2\varepsilon}(y_i))} d(x)^{1-\gamma}. \tag{1.17}$$

If $\varepsilon > |y' - y''| > d(x)$ we see

$$\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} \le 2 \left\|D_y^\alpha \Gamma_0(x,\ .\ )\right\|_{C^0(\partial\Omega \cap B_{2\varepsilon}(y_i))} d(x)^{-\gamma}. \tag{1.18}$$

Case $y' \in \partial\Omega \cap B_\varepsilon(y_i)$ and $|y' - y''| \ge \varepsilon$. For $\varepsilon \le d(x)$ and $|y' - y''| > d(x)$ we get

$$\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} \le 2 \left\|D_y^\alpha \Gamma_0(x,\ .\ )\right\|_{C^0(\partial\Omega)} d(x)^{-\gamma}, \tag{1.19}$$

and for $\varepsilon \le |y' - y''| \le d(x)$ we see

$$\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} \le 2 \left(\frac{\operatorname{diam}(\Omega)}{\varepsilon}\right)^\gamma \left\|D_y^\alpha \Gamma_0(x,\ .\ )\right\|_{C^0(\partial\Omega)} d(x)^{-\gamma}. \tag{1.20}$$

If $\varepsilon > d(x)$ we have

$$\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} \le 2 \left\|D_y^\alpha \Gamma_0(x,\ .\ )\right\|_{C^0(\partial\Omega)} d(x)^{-\gamma}. \tag{1.21}$$

Now, we are ready to show for any $n \ge 2$ that

$$\|\Gamma_0(x,\ .\ )\|_{C^{m-1,\gamma}(\partial\Omega)} \le C \cdot \begin{cases} d(x)^{m-n+1-\gamma} & \text{if } n > m, \\ 1 & \text{if } n \le m. \end{cases} \tag{1.22}$$

Note that $|x - y| \ge d(x)$ for all $y \in \partial\Omega$.

Case $n > 2m$. The estimate (1.3) shows that

$$|D_y^\alpha \Gamma_0(x,y)| \le C d(x)^{m-n+1}$$

for all $y \in \partial\Omega$ if $|\alpha| \le m-1$ and

$$|\nabla_y D_y^\alpha \Gamma_0(x,y)| \le C d(x)^{m-n}$$

for all $y \in \partial\Omega$ if $|\alpha| = m-1$. Hence, with (1.17)–(1.21) we get (1.22).

Case $2m > n > m$ and n is odd. The same estimate as in the previous case, see (1.3), leads to

$$|D_y^\alpha \Gamma_0(x,y)| \le C \cdot \begin{cases} 1 & \text{if } 2m-n-|\alpha| \ge 0, \\ d(x)^{m-n+1} & \text{if } 2m-n-|\alpha| < 0, \end{cases} \qquad |\alpha| \le m-1;$$
$$|\nabla_y D_y^\alpha \Gamma_0(x,y)| \le C d(x)^{m-n}, \qquad |\alpha| = m-1.$$

Again, using (1.17)–(1.21) we obtain (1.22).

Case $2m \ge n > m+1$ and n is even. Let $|\alpha| \le m-2$. Then, the estimate (1.3) shows that

$$|D_y^\alpha \Gamma_0(x,y)| \le C \cdot \begin{cases} 1 & \text{if } 2m-n-|\alpha|-1 \ge 0, \\ d(x)^{m-n+1-\gamma} & \text{if } 2m-n-|\alpha|-1 < 0, \end{cases}$$

since $|\alpha| < m-2+\gamma$. Using (1.6) for $|\alpha| \ge m-1$ we can see $|D_y^\alpha \Gamma_0(x,y)| \le C d(x)^{2m-n-|\alpha|}$. Thus, using (1.17)–(1.21) we can show (1.22).

Case $n \le m$ and n is odd. For $|\alpha| \le m$ we see with (1.3) that $|D_y^\alpha \Gamma_0(x,y)| \le C$. Therefore, if $|\alpha| = m-1$, we obtain

$$\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} \le C \cdot \begin{cases} \left\| \nabla_y D_y^\alpha \Gamma_0(x,\,.\,) \right\|_{C^0(\partial\Omega \cap B_{2\varepsilon}(y_i))} \varepsilon^{1-\gamma} & \text{if } |y'-y''| \le \varepsilon, \\ \left\| D_y^\alpha \Gamma_0(x,\,.\,) \right\|_{C^0(\partial\Omega)} \varepsilon^{-\gamma} & \text{if } |y'-y''| > \varepsilon, \end{cases}$$
$$\le C.$$

Case $n = m+1$ and n is even. Let $|\alpha| \le m-2$. Then, (1.3) shows that

$$|D_y^\alpha \Gamma_0(x,y)| \le C|x-y|^{m-1-|\alpha|-1} \le C.$$

For $|\alpha| = m-1$ we use (1.3) and (1.4) with $\gamma' = \gamma$ to end up with

$$|D_y^\alpha \Gamma_0(x,y)| \le C d(x)^{-\gamma}.$$

Since $|\alpha| + 1 = m > 2m-n = m-1$, using (1.6), we get that

$$|\nabla_y D_y^\alpha \Gamma_0(x,y)| \le C d(x)^{-1}.$$

Similar as for (1.17)–(1.21) we have:

- Let $y' \in \partial\Omega \cap B_\varepsilon(y_i)$ and $|y'-y''| \ge \varepsilon$. Then, for $|\alpha| = m-1$, we get
$$\frac{|D_y^\alpha \Gamma_0(x,y') - D_y^\alpha \Gamma_0(x,y'')|}{|y'-y''|^\gamma} \le C \left\| D_y^\alpha \Gamma_0(x,\,.\,) \right\|_{C^0(\partial\Omega)} \le C d(x)^{-\gamma}.$$

- Let $|y' - y''| < \varepsilon$ and $|\alpha| = m - 1$. Then, by the mean value theorem, if $\varepsilon \le d(x)$ or $|y' - y''| \le d(x) < \varepsilon$, it holds

$$\frac{|D_y^\alpha \Gamma_0(x, y') - D_y^\alpha \Gamma_0(x, y'')|}{|y' - y''|^\gamma} \le C \left\| \nabla_y D_y^\alpha \Gamma_0(x, \ . \ ) \right\|_{C^0(\partial\Omega)} d(x)^{1-\gamma} \le C d(x)^{-\gamma}.$$

- Let $\varepsilon > |y' - y''| > d(x)$ and $|\alpha| = m - 1$. Using Lemma A.1 we get

$$\begin{aligned}\frac{|D_y^\alpha \Gamma_0(x, y') - D_y^\alpha \Gamma_0(x, y'')|}{|y' - y''|^\gamma} \le C &\frac{|\log|x - y'| - \log|x - y''||}{|y' - y''|^\gamma} \\ &+ \frac{\left|P_2^{m-1}(x - y')|x - y'|^{-m+1} - P_2^{m-1}(x - y'')|x - y''|^{-m+1}\right|}{|y' - y''|^\gamma},\end{aligned}$$

where P_2^{m-1} is a homogeneous polynomial of degree $m - 1$. Using the inequality

$$\left|\log|x - y'| - \log|x - y''|\right| \le \frac{1}{\gamma}|y' - y''|^\gamma \left(|x - y'|^{-\gamma} + |x - y''|^{-\gamma}\right),$$

cf. [45, p. 225], we can see

$$\begin{aligned}\frac{|D_y^\alpha \Gamma_0(x, y') - D_y^\alpha \Gamma_0(x, y'')|}{|y' - y''|^\gamma} &\le C|y' - y''|^{-\gamma}\Big(|y' - y''|^\gamma \left(|x - y'|^{-\gamma} + |x - y''|^{-\gamma}\right) + 1\Big) \\ &\le C\left(|x - y'|^{-\gamma} + |x - y''|^{-\gamma} + |y' - y''|^{-\gamma}\right) \\ &\le C d(x)^{-\gamma}.\end{aligned}$$

Case $n = m$ and n is even. The estimate (1.3) shows

$$|D_y^\alpha \Gamma_0(x, y)| \le C$$

for all $y \in \partial\Omega$ if $|\alpha| \le m - 1$. Moreover, for $|\alpha| = m - 1$ by using Lemma A.1 we see

$$D_y^\alpha \Gamma_0(x, y) = \left(P_1^{m-1}(x - y)\log|x - y| + P_2^{m-1}(x - y)\right)|x - y|^{-m+2},$$

where as before P_1^{m-1} and P_2^{m-1} are homogeneous polynomials of degree $m - 1$. Since $x \mapsto |x|\log|x|$ is Hölder continuous on compact sets for all $\gamma \in (0, 1)$, we get (1.22) in this case.

Case $n < m$ and n is even. The estimate (1.3) shows that

$$|D_y^\alpha \Gamma_0(x, y)| \le C$$

for all $y \in \partial\Omega$, $|\alpha| \le m$. Hence, (1.22) follows.

After collecting all the cases, we use for all $x \in \Omega$ and $y \in \partial\Omega$ that

$$\partial_{\nu_y} u_x(y) = -\partial_{\nu_y} \Gamma_0(x, y) = -\nabla_y \Gamma_0(x, y) \cdot \nu(y)$$

to estimate for all $j = 0, \dots, m-1$:

$$\|\partial_{\nu_y}^j u_x\|_{C^{m-1-j,\gamma}(\Omega)} \le C(\partial\Omega) \cdot \begin{cases} d(x)^{m-n+1-\gamma} & \text{if } n > m, \\ 1 & \text{if } n \le m. \end{cases} \tag{1.23}$$

Note that $\partial\Omega$ is $C^{2m,\gamma}$-smooth and $(-\Delta)^m$ is coercive, see [23, p. 40]. Then, a-priori estimates for boundary value problems from [3] or [23, Theorem 2.19] prove the claim. □

1.3 Convergence of Domains

Suppose that $n \ge 2$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain with $0 \in \partial\Omega$ such that the first unit vector e_1 is the exterior unit normal to $\partial\Omega$ at 0. Let $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subset \overline{\Omega}$ with $x_k \to 0, y_k \to 0$ for $k \to \infty$. Since $\partial\Omega$ is smooth enough, we can find for k large enough a uniquely determined $\tilde{x}_k \in \partial\Omega$ such that $d(x_k) = |x_k - \tilde{x}_k|$, cf. [24, Sect. 14.6]. We want to study the rescaled and translated domains

$$\Omega_k := \frac{1}{|x_k - y_k|} \left(-\tilde{x}_k + \Omega\right).$$

Let $\mathcal{H} := \{x \in \mathbb{R}^n : x_1 < 0\}$. Since Ω is a $C^{2m,\gamma}$-smooth bounded domain, we can find a local $C^{2m,\gamma}$-smooth coordinate chart $\Phi : V \to U$, where $U, V \subset \mathbb{R}^n$ are open neighbourhoods of $0 \in \partial\Omega$, such that

$$\Phi(V \cap \mathcal{H}) = U \cap \Omega, \quad \Phi(V \cap \partial\mathcal{H}) = U \cap \partial\Omega$$

and

$$\begin{aligned} \Phi(\xi) &= \xi + O(|\xi|^2), \\ D\Phi(\xi) &= \mathrm{I}^{n\times n} + O(|\xi|), \end{aligned}$$

for $\xi \in V$. For a fixed $k \in \mathbb{N}$, i.e. a fixed domain Ω_k, we define

$$V_k := \frac{1}{|x_k - y_k|} \left(-\Phi^{-1}(\tilde{x}_k) + V\right).$$

This allows us to define for all $\xi \in V_k$

$$\Phi_k(\xi) := \frac{1}{|x_k - y_k|} \left(-\tilde{x}_k + \Phi\left(\Phi^{-1}(\tilde{x}_k) + |x_k - y_k|\xi\right)\right)$$

as a local coordinate chart for Ω_k.

Let us take a fixed $r > 0$. Since $|x_k - y_k| \to 0$ and $\tilde{x}_k \to 0$, which implies that also $\Phi^{-1}(\tilde{x}_k) \to 0$, we can find a $k_0 \in \mathbb{N}$ such that $B_r(0) \cap \overline{\mathcal{H}} \subset V_k$ for all $k \ge k_0$. We prove, as in [23, p. 220], that $\Phi_k \to \mathrm{Id}$ in $C^{2m,\gamma}(B_r(0) \cap \overline{\mathcal{H}})$.

For all ξ in $B_r(0) \cap \overline{\mathcal{H}}$ we get with a Taylor expansion for $k \to \infty$

$$\Phi(\Phi^{-1}(\tilde{x}_k) + |x_k - y_k|\xi) = \Phi(\Phi^{-1}(\tilde{x}_k)) + |x_k - y_k| D\Phi(\Phi^{-1}(\tilde{x}_k))\xi + O(|x_k - y_k|^2 |\xi|^2).$$

Since $D\Phi$ is continuous and $\Phi^{-1}(\tilde{x}_k) \to 0$ we get

$$D\Phi(\Phi^{-1}(\tilde{x}_k))\xi = D\Phi(0)\xi + o(1)|\xi| = \xi + o(1)|\xi|,$$

where $o(1) \to 0$ as $k \to \infty$. Therefore, it follows

$$\Phi(\Phi^{-1}(\tilde{x}_k) + |x_k - y_k|\xi) = \tilde{x}_k + |x_k - y_k|\xi + o(1)|x_k - y_k||\xi| + O(|x_k - y_k|^2|\xi|^2).$$

This shows for $k \geq k_0$ large enough and for all ξ in $B_r(0) \cap \overline{\mathcal{H}}$:

$$|\Phi_k(\xi) - \xi| = \left| o(1)|\xi| + O(|x_k - y_k||\xi|^2) \right|,$$

so

$$\lim_{k\to\infty} \|\Phi_k - \mathrm{Id}\|_{C^0(B_r(0)\cap\overline{\mathcal{H}})} = 0.$$

Applying the chain rule we get

$$D^\alpha \Phi_k = |x_k - y_k|^{|\alpha|-1} D^\alpha \Phi \circ h_k,$$

for $|\alpha| = 1, 2, \ldots, 2m$, where $h_k(\xi) := \Phi^{-1}(\tilde{x}_k) + |x_k - y_k|\xi$.

Now, we can find a $M > 0$, such that $\|\Phi\|_{C^{2m,\gamma}(B_r(0)\cap\overline{\mathcal{H}})} \leq M$. Then, since $h_k \to 0$ uniformly and $D\Phi(0) = \mathrm{I}^{n\times n}$, we have that

$$\lim_{k\to\infty} \left\| D\Phi_k - \mathrm{I}^{n\times n} \right\|_{C^0(B_r(0)\cap\overline{\mathcal{H}})} = 0.$$

Moreover, for $|\alpha| = 2, 3, \ldots, 2m$ we find that:

$$\|D^\alpha \Phi_k\|_{C^0(B_r(0)\cap\overline{\mathcal{H}})} \leq M|x_k - y_k|^{|\alpha|-1} \to 0.$$

For $|\alpha| = 2m$ we get

$$\begin{aligned}
\frac{|D^\alpha \Phi_k(\xi) - D^\alpha \Phi_k(\xi')|}{|\xi - \xi'|^\gamma} &= |x_k - y_k|^{|\alpha|-1} \frac{|D^\alpha \Phi \circ h_k(\xi) - D^\alpha \Phi \circ h_k(\xi')|}{|\xi - \xi'|^\gamma} \\
&\leq M|x_k - y_k|^{|\alpha|-1} \frac{|h_k(\xi) - h_k(\xi')|^\gamma}{|\xi - \xi'|^\gamma} \\
&= M|x_k - y_k|^{|\alpha|-1+\gamma} \to 0,
\end{aligned}$$

where $\xi, \xi' \in B_r(0) \cap \overline{\mathcal{H}}$ with $\xi \neq \xi'$. Therefore, $\Phi_k \to \mathrm{Id}$ in $C^{2m,\gamma}(B_r(0) \cap \overline{\mathcal{H}})$, i.e. $\Omega_k \to \mathcal{H}$ locally uniformly.

2 A Perturbed Polyharmonic Operator

This chapter is devoted to the perturbed polyharmonic Dirichlet boundary value problem (0.3). In the first section, we state our assumptions which ensure existence, uniqueness and regularity for a solution of (0.3). We construct a Green function for (0.3) and show numerous estimates for it in the subsequent sections. In the last section we show the convergence of a rescaled Green function to the polyharmonic Green function $G_{(-\Delta)^m,\mathcal{H}}$, which is needed for the blow-up procedure.

2.1 Assumptions and Definitions

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2m,\gamma}$-smooth domain with exterior unit normal ν, $m \geq 2$ and $\gamma \in (0,1)$. We consider the following Dirichlet boundary value problem

$$\begin{cases} (-\Delta)^m u(x) + \sum_{\ell=0}^{m-1} \sum_{|\alpha|=|\beta|=\ell} D^\beta \left(a^\ell_{\alpha,\beta}(x) D^\alpha u(x) \right) = f(x) & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad \dfrac{\partial^j}{\partial \nu^j} u(x) = 0 & \text{for } x \in \partial\Omega,\ j = 0, \ldots, m-1. \end{cases} \tag{2.1}$$

In the following we use $\sum_*$ as an abbreviation of $\sum_{\ell=0}^{m-1} \sum_{|\alpha|=|\beta|=\ell}$.

To the Dirichlet boundary value problem we associate for all $u, v \in W_0^{m,2}$ the following bilinear form

$$B(u,v) := \langle u, v \rangle_{W_0^{m,2}} + \sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(x) D^\alpha u(x) D^\beta v(x) dx,$$

where

$$\langle u, v \rangle_{W_0^{m,2}} \mapsto \begin{cases} \displaystyle\int_\Omega \Delta^k u \Delta^k v \; dx & \text{if } m = 2k, \\ \displaystyle\int_\Omega \nabla(\Delta^k u) \cdot \nabla(\Delta^k v) \; dx & \text{if } m = 2k+1, \end{cases}$$

is a scalar product on $W_0^{m,2}$, which induces a norm equivalent to $\left(\sum_{k=0}^m \| D^k \,.\, \|^2_{L^2} \right)^{1/2}$, cf. [23, Theorem 2.2].

For the coefficient functions of the operator in (2.1) we make the following assumptions.

(A1) Symmetry: $a^\ell_{\alpha,\beta} = a^\ell_{\beta,\alpha}$, for all $\ell = 0, \dots, m-1$.

(A2) Regularity: $a^\ell_{\alpha,\beta} \in C^{m-1,\gamma}(\overline{\Omega})$, for all $\ell = 0, \dots, m-1$.

(A3) Boundedness: There is a $K > 0$, such that for all ℓ it holds $\left\|a^\ell_{\alpha,\beta}\right\|_{C^{m-1,\gamma}(\overline{\Omega})} \le K$.

For the bilinear form we assume coercivity:

(A4) There exists a $\lambda > 0$ such that for all $v \in W^{m,2}_0(\Omega)$ we have

$$B(v,v) \ge \lambda \|v\|^2_{W^{m,2}_0}. \tag{2.2}$$

Thanks to elliptic Schauder theory, see [23, Theorem 2.19], cf. [3], the Dirichlet boundary value problem (2.1) admits together with the assumptions (A2) and (A4) for $f \in C^{0,\gamma}(\overline{\Omega})$ a unique solution $u \in C^{2m,\gamma}(\overline{\Omega})$.

2.2 Construction of the Green Function

In this section, we want to construct a Green function for the Dirichlet boundary value problem (2.1):

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded $C^{2m,\gamma}$-smooth domain. Let the assumptions (A1)–(A4) of Section 2.1 hold. Then for every $x \in \Omega$, there exists a unique function $G_x \in L^1(\Omega) \cap C^{2m,\gamma}(\overline{\Omega} \setminus \{x\})$ with the following properties:*

1. *$\partial^j_\nu G_x|_{\partial\Omega} = 0$, $j = 0, \dots, m-1$;*
2. *for all $\varphi \in C^{2m}(\overline{\Omega})$ with $\partial^j_\nu \varphi|_{\partial\Omega} = 0$, $j = 0, \dots, m-1$, we have for all $x \in \Omega$ the representation formula*
$$\varphi(x) = \int_\Omega (-\Delta)^m \varphi(y) G_x(y)\, dy + \sum_* \int_\Omega D^\alpha \left(a^\ell_{\alpha,\beta}(y) D^\beta \varphi(y)\right) G_x(y)\, dy; \tag{2.3}$$
3. *$G_x(y) = G_y(x)$, $x \ne y$.*
4. *If $R > 0$ is such that $\Omega \subset B_R(0)$ then the following estimate holds:*
$$|G_x(y)| \le C \cdot \begin{cases} |x-y|^{2m-n} + \max\{d(x), d(y)\}^{2m-n} & \text{if } n > 2m, \\ \log\left(1 + |x-y|^{-1} + \max\{d(x), d(y)\}^{-1}\right) & \text{if } n = 2m, \\ 1 & \text{if } n < 2m, \end{cases} \tag{2.4}$$
where $C = C(m, n, R, K, \lambda, \partial\Omega)$. For $n \le 2m$ also the following gradient estimate holds:
$$\left|\nabla_{(x,y)} G_x(y)\right| \le C(m,n,R,K,\lambda,\partial\Omega) \cdot \begin{cases} |x-y|^{-1} + \max\{d(x), d(y)\}^{-1} & \text{if } n = 2m, \\ 1 & \text{if } n < 2m. \end{cases} \tag{2.5}$$

For the proof of Proposition 2.1 we need the following fundamental Lemma of Giraud [25, p. 150], cf. [7, Proposition 4.12].

Lemma 2.2 (Giraud)**.** *Let Ω be a bounded open set of $\mathbb{R}^n$ and let $\Gamma_1, \Gamma_2 \in C^0\big(\overline{\Omega}\times\overline{\Omega}\setminus\{(x,y) : x = y\}\big)$ with*

$$|\Gamma_1(x,y)| \le c|x-y|^{a-n}$$

and

$$|\Gamma_2(x,y)| \le c|x-y|^{b-n}$$

for $a, b \in (0,n)$. Then

$$\Gamma_3(x,y) := \int_\Omega \Gamma_1(x,z)\Gamma_2(z,y)\,dz$$

is continuous for $x \neq y$ and satisfies:

$$|\Gamma_3(x,y)| \le c\cdot \begin{cases} |x-y|^{a+b-n} & \text{if } a+b<n,\\ 1+|\log|x-y|| & \text{if } a+b=n,\\ 1 & \text{if } a+b>n; \end{cases}$$

in the last case Γ_3 is continuous on $\Omega \times \Omega$.

For the proof of Proposition 2.1 we adopt the the ideas from [27, Proposition 1], cf. [23, Lemma 4.18].

Proof of Proposition 2.1. <u>*Case $n > 2m$.*</u> For $x \in \Omega$ we define

$$G_x(y) := \Gamma_0(x,y) + \sum_{j=1}^{k} \Gamma_j(x,y) + u_x(y), \tag{2.6}$$

where

$$\Gamma_0(x,y) := F_{m,n}(x-y) = c_{m,n}|x-y|^{2m-n}$$

is a fundamental solution of the polyharmonic operator $(-\Delta)^m$ in $\mathbb{R}^n$, $\Gamma_0 \in C^\infty\big(\overline{\Omega}\times\overline{\Omega}\setminus\{(x,y) : x = y\}\big)$. The function u_x and $k \in \mathbb{N}$ are specified later. For $j \ge 0$ we define:

$$\Gamma_{j+1}(x,y) := -\sum_{s=0}^{m-1}\sum_{|\sigma|=|\mu|=s}\int_\Omega D_z^\mu\left(a^s_{\sigma,\mu}(z)D_z^\sigma\Gamma_j(x,z)\right)\Gamma_0(z,y)\,dz.$$

Due to the definition of Γ_j we have $y \mapsto \Gamma_j(x,y) \in C^{2m,\gamma}(\overline{\Omega}\setminus\{x\})$ for $j \ge 1$. Moreover, by induction and with the help of Giraud's lemma 2.2 and (1.3) we get for $|\alpha|+|\beta| \le 2m-2$ and $j \ge 1$:

$$\left|D_y^{\alpha+\beta}\Gamma_j(x,y)\right| \le \begin{cases} C_j|x-y|^{2m+2j-|\alpha|-|\beta|-n} & \text{if } 2m+2j-|\alpha|-|\beta|<n,\\ C_j\left(1+|\log|x-y||\right) & \text{if } 2m+2j-|\alpha|-|\beta|=n,\\ C_j & \text{if } 2m+2j-|\alpha|-|\beta|>n, \end{cases} \tag{2.7}$$

where $C_j = C_j(m,n,R,K)$. For all $|\alpha|+|\beta| \le 2m-2$ we have that

$$2m + 2(k-1) - |\alpha| - |\beta| \ge 2k.$$

Now, we fix $k > 0$ such that $2k > n$.

For $\varphi \in C^{2m}(\overline{\Omega})$ with $\partial_\nu^j \varphi|_{\partial\Omega} = 0$, $j = 0, \dots, m-1$ we define

$$b_x(\varphi, G) := \int_\Omega (-\Delta)^m \varphi(y) G_x(y)\, dy + \sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \varphi(y) D_y^\alpha G_x(y)\, dy. \qquad (2.8)$$

Inserting (2.6) in (2.8) and using Corollary 1.4 we see that:

$$\begin{aligned} b_x(\varphi, G) = {} & \varphi(x) + I_{\partial\Omega}\left(\Gamma_0(x,y), \varphi(y)\right) + \sum_{j=0}^{k-1} \int_\Omega (-\Delta)^m \varphi(y) \Gamma_{j+1}(x,y)\, dy \\ & + \int_\Omega (-\Delta)^m \varphi(y) u_x(y)\, dy + \sum_{j=0}^{k} \left(\sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \varphi(y) D_y^\alpha \Gamma_j(x,y)\, dy \right) \\ & + \sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \varphi(y) D_y^\alpha u_x(y)\, dy. \end{aligned}$$

Then, Fubini's theorem leads us to

$$\begin{aligned} b_x(\varphi, G) = {} & \varphi(x) + I_{\partial\Omega}\left(\Gamma_0(x,y), \varphi(y)\right) \\ & - \sum_{j=0}^{k-1} \left(\sum_\circ \int_\Omega D_z^\mu \left(a^s_{\sigma,\mu}(z) D_z^\sigma \Gamma_j(x,z) \right) \left(\int_\Omega (-\Delta)^m \varphi(y) \Gamma_0(z,y)\, dy \right) dz \right) \\ & + \int_\Omega (-\Delta)^m \varphi(y) u_x(y)\, dy + \sum_{j=0}^{k} \left(\sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \varphi(y) D_y^\alpha \Gamma_j(x,y)\, dy \right) \\ & + \sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \varphi(y) D_y^\alpha u_x(y)\, dy, \end{aligned}$$

where we use $\sum_\circ$ as an abbreviation of $\sum_{s=0}^{m-1} \sum_{|\sigma|=|\mu|=s}$. Now, it follows from Corollary 1.4 that

$$\begin{aligned} b_x(\varphi, G) = {} & \varphi(x) - \sum_{j=0}^{k-1} \left(\sum_\circ \int_\Omega D_z^\mu \left(a^s_{\sigma,\mu}(z) D_z^\sigma \Gamma_j(x,z) \right) \varphi(z)\, dz \right) \\ & + \int_\Omega (-\Delta)^m \varphi(y) u_x(y)\, dy + \sum_{j=0}^{k} \left(\sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \varphi(y) D_y^\alpha \Gamma_j(x,y)\, dy \right) \\ & + \sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \varphi(y) D_y^\alpha u_x(y)\, dy + I_{\partial\Omega}\left(\Gamma_0(x,y), \varphi(y)\right) \\ & - \sum_{j=0}^{k-1} \left(\sum_\circ \int_\Omega D_z^\mu \left(a^s_{\sigma,\mu}(z) D_z^\sigma \Gamma_j(x,z) \right) I_{\partial\Omega}\left(\Gamma_0(z,y), \varphi(y)\right) dz \right). \end{aligned}$$

Since φ has zero boundary conditions, integrating by parts shows:

$$
\begin{aligned}
b_x(\varphi, G) = \varphi(x) &- \sum_{j=0}^{k-1}\left(\sum_{\circ}(-1)^{|\mu|}\int_\Omega a^s_{\sigma,\mu}(z)D^\sigma_z\Gamma_j(x,z)D^\mu_z\varphi(z)\,dz\right)\\
&+\int_\Omega(-\Delta)^m\varphi(y)u_x(y)\,dy+\sum_{j=0}^{k}\left(\sum_{*}(-1)^{|\beta|}\int_\Omega a^\ell_{\alpha,\beta}(y)D^\beta_y\varphi(y)D^\alpha_y\Gamma_j(x,y)\,dy\right)\\
&+\sum_{*}(-1)^{|\beta|}\int_\Omega a^\ell_{\alpha,\beta}(y)D^\beta_y\varphi(y)D^\alpha_y u_x(y)\,dy + I_{\partial\Omega}\left(\Gamma_0(x,y),\varphi(y)\right)\\
&-\sum_{j=0}^{k-1}\left(\sum_{\circ}\int_\Omega D^\mu_z\left(a^s_{\sigma,\mu}(z)D^\sigma_z\Gamma_j(x,z)\right)I_{\partial\Omega}\left(\Gamma_0(z,y),\varphi(y)\right)\,dz\right).
\end{aligned}
$$

Using Corollary 1.4 once more, we finally find that

$$
\begin{aligned}
b_x(\varphi, G) = \varphi(x) &+ \sum_{*}\int_\Omega\varphi(y)D^\beta_y\left(a^\ell_{\alpha,\beta}(y)D^\alpha_y\Gamma_k(x,y)\right)\,dy+\int_\Omega\varphi(y)(-\Delta)^m u_x(y)\,dy\\
&+\sum_{*}\int_\Omega\varphi(y)D^\beta_y\left(a^\ell_{\alpha,\beta}(y)D^\alpha_y u_x(y)\right)\,dy + I_{\partial\Omega}\left(u_x(y),\varphi(y)\right)\\
&+\sum_{j=0}^{k} I_{\partial\Omega}\left(\Gamma_j(x,y),\varphi(y)\right). \qquad (2.9)
\end{aligned}
$$

Therefore, if u_x solves the Dirichlet boundary value problem

$$
\begin{cases}
(-\Delta)^m u_x(y)+\sum_{*}D^\beta_y\left(a^\ell_{\alpha,\beta}(y)D^\alpha_y u_x(y)\right) = -\sum_{*}D^\beta_y\left(a^\ell_{\alpha,\beta}(y)D^\alpha_y\Gamma_k(x,y)\right) & \text{in } \Omega,\\
\dfrac{\partial^i}{\partial\nu^i_y}u_x(y) = -\dfrac{\partial^i}{\partial\nu^i_y}\Gamma_0(x,y)-\sum_{j=1}^{k}\dfrac{\partial^i}{\partial\nu^i_y}\Gamma_j(x,y),\quad i=0,\dots,m-1, & \text{on } \partial\Omega,
\end{cases}
\qquad (2.10)
$$

G_x has zero boundary values, i.e. satisfies claim 1. From elliptic theory, see [23, Theorem 2.19], and the coercivity (2.2), we derive the existence of a unique solution $u_x \in C^{2m,\gamma}(\overline{\Omega})$ for (2.10). Then, after integrating by parts in (2.9), we have claim 2. Moreover, we have $G_x \in C^{2m,\gamma}(\overline{\Omega}\setminus\{x\})$.

Let us now justify the symmetry, i.e. $G_x(y) = G_y(x)$, $x \neq y$, from which we get claim 3. Let $\psi,\phi \in W^{2m,2}(\Omega)\cap W^{m,2}_0(\Omega)$. Using the zero boundary values of ψ and ϕ, Corollary 1.4, the symmetry $a^\ell_{\alpha,\beta} = a^\ell_{\beta,\alpha}$ and $|\alpha| = |\beta|$ we see by partial integration that

$$
\begin{aligned}
&\int_\Omega(-\Delta)^m\phi(y)\psi(y)\,dy+\sum_{*}\int_\Omega D^\beta\left(a^\ell_{\alpha,\beta}(y)D^\alpha\phi(y)\right)\psi(y)dy\\
&=\int_\Omega\phi(y)(-\Delta)^m\psi(y)\,dy+\sum_{*}(-1)^{|\beta|+|\alpha|}\int_\Omega\phi(y)D^\alpha\left(a^\ell_{\alpha,\beta}(y)D^\beta\psi(y)\right)dy\\
&=\int_\Omega\phi(y)(-\Delta)^m\psi(y)\,dy+\sum_{*}\int_\Omega\phi(y)D^\alpha\left(a^\ell_{\beta,\alpha}(y)D^\beta\psi(y)\right)dy. \qquad (2.11)
\end{aligned}
$$

Therefore, the linear differential operator L is self-adjoint on $W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega)$, where we define

$$L := (-\Delta)^m + \sum_* D^\beta \left(a^\ell_{\alpha,\beta} D^\alpha \right).$$

Let $f, g \in C_c^\infty(\Omega)$ and $\phi, \psi \in C^{2m}(\overline{\Omega})$ such that

$$\begin{cases} L\phi = f & \text{in } \Omega, \\ \dfrac{\partial^j}{\partial \nu^j}\phi = 0 & \text{on } \partial\Omega,\ j = 0, \dots, m-1 \end{cases} \quad \text{and} \quad \begin{cases} L\psi = g & \text{in } \Omega, \\ \dfrac{\partial^j}{\partial \nu^j}\psi = 0 & \text{on } \partial\Omega,\ j = 0, \dots, m-1. \end{cases}$$

Using the representation formula (2.3) we get that

$$\phi(x) = \int_\Omega G_x(y) f(y) dy \quad \text{and} \quad \psi(x) = \int_\Omega G_x(y) g(y) dy. \tag{2.12}$$

The symmetry of G follows by applying (2.11) to (2.12).

It is left to show the estimates, i.e. claim 4. With the help of local Schauder estimates, the uniform Hölder continuous right hand side of (2.10) and the coercivity of the differential operators in $x \in \Omega$, we see that

$$\|u_x\|_{C^{m-1,\gamma}(\overline{\Omega})} \le C(m, n, R, K, \lambda, \partial\Omega) d(x)^{m-n+1-\gamma}. \tag{2.13}$$

The estimate (2.13) can be proven like Lemma 1.5 since Γ_0 has the strongest singularity and we are in the case $n > 2m$.

Let $d(y) \le d(x)$ and $y' \in \partial\Omega$ such that $d(y) = |y - y'|$. We perform a Taylor expansion to obtain

$$\begin{aligned} u_x(y) \le |u_x(y)| &\le C \left(\sum_{|\alpha|=0}^{m-1} \|D_y^\alpha u_x\|_{C^0(\partial\Omega)} d(y)^{|\alpha|} + |D_y^{m-1} u_x(y^*) - D_y^{m-1} u_x(y')| d(y)^{m-1} \right) \\ &\le C \left(\sum_{|\alpha|=0}^{m-1} \|D_y^\alpha u_x\|_{C^0(\partial\Omega)} d(y)^{|\alpha|} + \|u_x\|_{C^{m-1,\gamma}(\overline{\Omega})} d(y)^{m-1+\gamma} \right), \end{aligned}$$

where y^* is on the line segment between y and y'. Then, by using (1.3), (2.7) and (2.13), we have

$$u_x(y) \le |u_x(y)| \le C(m, n, R, K, \lambda, \partial\Omega, C_0) d(x)^{2m-n}. \tag{2.14}$$

Since $G_x(y) = \Gamma_0(x, y) + \sum_{j=1}^k \Gamma_j(x, y) + u_x(y)$, this shows

$$|G_x(y)| \le C \left(|x - y|^{2m-n} + d(x)^{2m-n} \right). \tag{2.15}$$

Let $d(y) > d(x)$. We use the symmetry of the Green function to get

$$|G_x(y)| = |G_y(x)| \le C \left(|x - y|^{2m-n} + d(y)^{2m-n} \right). \tag{2.16}$$

Combining inequalities (2.15) and (2.16) we finally get the estimate and $G_x \in L^1(\Omega)$ for $n > 2m$.

<u>*Case* $n = 2m$.</u> Here we define

$$\Gamma_0(x,y) := F_{m,n}(x-y) = -c_{m,n} \log|x-y|.$$

As before, cf. (1.3), we derive some basic estimate for the derivatives of the fundamental solution if $|\alpha| \geq 1$:

$$|D_y^\alpha F_{m,n}(x-y)| \leq C(m,n,|\alpha|)|x-y|^{-|\alpha|}.$$

We define the iterated kernels Γ_j as above and for $|\alpha|+|\beta| \leq 2m-2$ and $j \geq 1$ we get

$$\left|D_y^{\alpha+\beta}\Gamma_j(x,y)\right| \leq \begin{cases} C_j|x-y|^{2j-|\alpha|-|\beta|} & \text{if } 2j-|\alpha|-|\beta| < 0, \\ C_j\left(1+|\log|x-y||\right) & \text{if } 2j-|\alpha|-|\beta| = 0, \\ C_j & \text{if } 2j-|\alpha|-|\beta| > 0, \end{cases} \tag{2.17}$$

where $C_j = C_j(m,n,R,K)$. For all $|\alpha|+|\beta| \leq 2m-2$ we have that

$$2(k-1)-|\alpha|-|\beta| \geq 2k-2m = 2(k-m).$$

We fix $k > 0$ such that $k > m$. Proceeding as above, we now see that

$$\|u_x\|_{C^{m-1,\gamma}(\overline{\Omega})} \leq C(m,n,R,K,\lambda,\partial\Omega)d(x)^{-m+1-\gamma}. \tag{2.18}$$

Performing a Taylor expansion we obtain

$$|\nabla_y u_x(y)| \leq C\left(\sum_{|\alpha|=0}^{m-2} \|D_y^\alpha \nabla_y u_x\|_{C^0(\partial\Omega)} d(y)^{|\alpha|} + \|u_x\|_{C^{m-1,\gamma}(\overline{\Omega})} d(y)^{m-2+\gamma}\right).$$

As the estimates of the fundamental solution show for $d(y) \leq d(x)$ that

$$|\nabla_y u_x(y)| \leq C\left(\partial\Omega\right) d(x)^{-1},$$

we get

$$|\nabla_y G_x(y)| \leq C(\partial\Omega)\left(|x-y|^{-1} + d(x)^{-1}\right). \tag{2.19}$$

For $d(x) \leq d(y)$ we get from the symmetry of the Green function

$$|\nabla_x G_x(y)| \leq C(\partial\Omega)\left(|x-y|^{-1} + d(y)^{-1}\right). \tag{2.20}$$

A similar estimate for $|\nabla_x G_x(y)|$ follows by differentiating (2.10) with respect to x as a parameter. Note that $y \mapsto D_y^{|\alpha|+|\beta|}\nabla_x \Gamma_k(x,y)$ is still Hölder continuous for all $|\alpha|+|\beta| \leq 2m-2$, since $k > m$. Proceeding as before leads to

$$|\nabla_x G_x(y)| \leq C(\partial\Omega)\left(|x-y|^{-1} + d(x)^{-1}\right), \tag{2.21}$$

if $d(y) \le d(x)$. As before, using the symmetry of the Green function for $d(x) \le d(y)$, we obtain

$$|\nabla_y G_x(y)| \le C(\partial\Omega)\left(|x-y|^{-1} + d(y)^{-1}\right). \tag{2.22}$$

This shows the estimate for the derivatives and integration finally proves the claim for G_x in the case $n = 2m$.

Case $m < n < 2m$. We define

$$\Gamma_0(x,y) := F_{m,n}(x-y) = \begin{cases} c_{m,n}|x-y|^{2m-n} & \text{if } n \text{ is odd,} \\ c_{m,n}|x-y|^{2m-n}\left(-\log|x-y|\right) & \text{if } n \text{ is even.} \end{cases}$$

Although the fundamental solution is bounded on bounded domains, its derivatives could become singular. For this reason, we have to work with iterated kernels to overcome this difficulty.

If n is odd we proceed as in the case $n > 2m$ and choose $k = k_{odd} > \frac{n}{2}$.

If n is even we obtain the estimate, cf. (1.3),

$$|D_y^\alpha F_{m,n}(|x-y|)| \le C(m,n,|\alpha|)|x-y|^{2m-n-|\alpha|-1},$$

from which we get

$$\left|D_y^{\alpha+\beta}\Gamma_j(x,y)\right| \le \begin{cases} C_j|x-y|^{2m+(j-1)-|\alpha|-|\beta|-n} & \text{if } 2m+(j-1)-|\alpha|-|\beta| < n, \\ C_j\left(1+|\log|x-y||\right) & \text{if } 2m+(j-1)-|\alpha|-|\beta| = n, \\ C_j & \text{if } 2m+(j-1)-|\alpha|-|\beta| > n, \end{cases} \tag{2.23}$$

where $C_j = C_j(m,n,R,K)$. For all $|\alpha| + |\beta| \le 2m-2$ we have that

$$2m + ((k_{even}-1)-1) - |\alpha| - |\beta| \ge k_{even}.$$

Here, we choose $k = k_{even} > n$.

We start by estimating $|\nabla_y G_x(y)|$. Using the estimates (1.3) for the fundamental solution we get with a Taylor expansion and (1.16):

$$|\nabla_y u_x(y)| \le C\left(\sum_{|\alpha|=0}^{m-2} \|D_y^\alpha \nabla_y u_x\|_{C^0(\partial\Omega)} d(y)^{|\alpha|} + \|u_x\|_{C^{m-1,\gamma}(\overline{\Omega})} d(y)^{m-2+\gamma}\right) \le C.$$

Note that for even n the case $n = 2m-1$ does not occur. Arguing as above we get

$$|\nabla_y G_x(y)| \le C.$$

Proceeding as in the case $n = 2m$ the claims for $m < n < 2m$ are proved.

Case $m \geq n \geq 2$. As before, we define

$$G_x(y) := \Gamma_0(x,y) + \sum_{j=1}^{k} \Gamma_j(x,y) + u_x(y), \tag{2.24}$$

where we set Γ_0 as the polyharmonic fundamental solution from (1.1). For n is odd we choose $k = k_{odd}$ as in the case $n > 2m$ and if n is even, k is chosen as in the previous case, i.e. $k = k_{even}$. Moreover, we have the estimate (2.7) if n is odd and (2.23) if n is even for the iterated kernels. Since $n \leq m$, using Lemma 1.5, we can see that $\|u_x\|_{C^{m-1,\gamma}(\overline{\Omega})} \leq C(m,n,R,K,\lambda,\partial\Omega)$. Combining these estimates, using (1.3) and the symmetry of G the claims

$$|G_x(y)| \leq C \qquad \text{and} \qquad |\nabla_{(x,y)} G_x(y)| \leq C$$

follow. This finishes the proof. □

From now on, we denote by G the Green function constructed in Proposition 2.1.

As in [27, Proposition 3], cf. [23, Proposition 4.17], the regularity of the Green function with respect to both variables follows:

Proposition 2.3. *Under the assumptions of Proposition 2.1 we have in addition that*

$$G \in C^{2m,\gamma}\left(\overline{\Omega} \times \overline{\Omega} \setminus \{(x,y) : x \neq y\}\right).$$

The proof can be done by a duality argument. As in [27, Proposition 3] we can use that the derivatives of $G(x,\ .\ ) = G_x(\ .\ )$ with respect to the x-variable satisfy (2.1) in the distributional sense. Note that the coefficient functions depend only on the y-variable. Since $G_x(\ .\ )$ has only derivatives up to the order $2m$ with respect to the y-variable, this restricts the order of derivatives with respect to the x-variable due to the symmetry of G.

2.3 Estimates for the Green Function

Let us now show some estimates for the Green function G itself and for their partial derivatives. Note that this was already done by Krasovskiĭ in [43, 44], see also [16, Theorem 3]. Since this was done in a very general context, higher regularity on the boundary was assumed. Since we only assume $\partial\Omega \in C^{2m,\gamma}$, we adopt the ideas from the proofs of [23, Theorem 4.20] and [23, Theorem 4.28].

The following theorem gives global estimates for the Green function G without boundary terms. For the corresponding result in the biharmonic setting see [23, Theorem 4.20].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2m,\gamma}$-smooth domain and let G be the Green function in Ω for the Dirichlet boundary value problem (2.1). Then there exists a constant $C = C(\Omega, K)$, such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq 2m$:*

- *If $|\alpha| + |\beta| + n > 2m$:*
$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C|x-y|^{2m-n-|\alpha|-|\beta|} \text{ for all } x, y \in \Omega.$$
- *If $|\alpha| + |\beta| + n = 2m$ and n is even:*
$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C \log\left(1 + |x-y|^{-1}\right) \text{ for all } x, y \in \Omega.$$
- *If $|\alpha| + |\beta| + n = 2m$ and n is odd, or if $|\alpha| + |\beta| + n < 2m$:*
$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C \text{ for all } x, y \in \Omega.$$

To prove Theorem 2.4, we proceed as in [23, Section 4.5.1]. For that reason, we divide the proof in several steps and show the following lemmas and propositions.

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2m$, be a bounded $C^{2m,\gamma}$-smooth domain. For any $q \in \left(\frac{n}{n-2m+1}, \frac{n}{n-2m}\right)$, there exists a constant $C(q, \Omega, K) > 0$ such that for all $x \in \Omega$ we have*

$$\|G(x,\ .\ )\|_{L^q(\Omega)} \leq C(q) d(x)^{2m-n+\frac{n}{q}}. \tag{2.25}$$

The proof is done as for [23, Lemma 4.21].

Proof. Let $\varphi \in C_c^\infty(\Omega)$ and $\Psi \in C^{2m,\gamma}(\overline{\Omega})$ such that

$$\begin{cases} (-\Delta)^m \Psi + \displaystyle\sum_{\ell=0}^{m-1} \sum_{|\alpha|=|\beta|=\ell} D^\beta \left(a_{\alpha,\beta}^\ell D^\alpha \Psi\right) = \varphi & \text{in } \Omega, \\ \dfrac{\partial^j}{\partial \nu^j} \Psi = 0 & \text{on } \partial\Omega \text{ and } j = 0, \dots, m-1. \end{cases}$$

From [3, Theorem 15.2], cf. [23, Theorem 2.20], and Sobolev's embeddings, cf. [1, Chapter V], for all $q' := \frac{q}{q-1} \in \left(\frac{n}{2m}, \frac{n}{2m-1}\right)$ and all $\mu := 2m - \frac{n}{q'} \in (0,1)$ we get

$$\|\Psi\|_{C^{0,\mu}(\overline{\Omega})} \leq C \|\Psi\|_{W^{2m,q'}(\Omega)} \leq C \|\varphi\|_{L^{q'}(\Omega)}.$$

Let $x \in \Omega$ and $x' \in \partial\Omega$. Then we obtain

$$|\Psi(x)| = |\Psi(x) - \Psi(x')| \leq \|\Psi\|_{C^{0,\mu}(\overline{\Omega})} |x - x'|^\mu \leq C \|\varphi\|_{L^{q'}(\Omega)} |x - x'|^\mu.$$

From Proposition 2.1 we have the representation formula $\Psi(x) = \int_\Omega \varphi(y) G(x,y)\, dy$ and we can see that

$$\left|\int_\Omega \varphi(y) G_x(y)\, dy\right| = |\Psi(x)| \leq C \|\varphi\|_{L^{q'}(\Omega)} \inf_{x' \in \partial\Omega} |x - x'|^\mu = C \|\varphi\|_{L^{q'}(\Omega)} d(x)^\mu.$$

Therefore, by duality, cf. [4, Folgerung 4.13], we have $y \mapsto G(x,y) \in L^q(\Omega)$ and

$$\|G(x,\ .\ )\|_{L^q(\Omega)} \leq C(q) d(x)^\mu.$$

□

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2m,\gamma}$-smooth domain. Then there exists a constant $C = C(\Omega, K) > 0$ such that for all $x, y \in \Omega$ with $x \neq y$ one has that*

$$|G(x,y)| \leq C \cdot \begin{cases} |x-y|^{2m-n} & \text{if } n > 2m, \\ \log\left(1 + |x-y|^{-1}\right) & \text{if } n = 2m, \\ 1 & \text{if } n < 2m. \end{cases} \tag{2.26}$$

For $n \leq 2m$ the following gradient estimates also hold:

$$\left|\nabla_{(x,y)} G(x,y)\right| \leq C \cdot \begin{cases} |x-y|^{-1} & \text{if } n = 2m, \\ 1 & \text{if } n < 2m. \end{cases} \tag{2.27}$$

The proof is done as for [23, Proposition 4.22].

Proof. Since we have Proposition 2.1, the case $n < 2m$ is proved.

Case $n > 2m$. We prove the claim by contradiction. Let $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subset \Omega$ such that $x_k \neq y_k$ for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} |x_k - y_k|^{n-2m} |G(x_k, y_k)| = \infty. \tag{2.28}$$

Using Proposition 2.1, we get

$$|x_k - y_k|^{n-2m} |G(x_k, y_k)| \leq C\left(1 + |x_k - y_k|^{n-2m} \max\left\{d(x_k), d(y_k)\right\}^{2m-n}\right).$$

Since Ω is bounded, we see, after passing to a further subsequence, that

$$x_\infty \in \partial\Omega \quad \text{and} \quad \lim_{k\to\infty} \frac{d(x_k)}{|x_k - y_k|} = 0. \tag{2.29}$$

Let us now show by contradiction that $\lim_{k\to\infty} |x_k - y_k| = 0$. After passing to a further subsequence, there exists a $\delta > 0$ such that for all k we have $x_k \in B_\delta(x_\infty)$ and $y_k \in \Omega \setminus \overline{B_{3\delta}(x_\infty)}$. Let $y \in \Omega \setminus \overline{B_{3\delta}(x_\infty)}$. Local elliptic estimates show that

$$\|G(x_k,\ .\ )\|_{W^{2m,p}(\Omega\cap B_\delta(y))} \leq C \|G(x_k,\ .\ )\|_{L^1(\Omega\cap B_{2\delta}(y))}.$$

Since $(\Omega \cap B_\delta(y)) \subset \left(\Omega \setminus \overline{B_{2\delta}(x_\infty)}\right)$, we use Sobolev's embedding theorem with $p \geq \frac{n}{2m}$ in $\Omega \setminus \overline{B_{2\delta}(x_\infty)}$ to see that

$$\|G(x_k,\ .\ )\|_{L^\infty(\Omega\cap B_\delta(y))} \leq C \|G(x_k,\ .\ )\|_{W^{2m,p}(\Omega\cap B_\delta(y))}.$$

Moreover, Lemma 2.5 shows that

$$\|G(x_k,\ .\ )\|_{L^q(\Omega)} \le C(q).$$

Combining these inequalities and using the Hölder inequality we obtain that

$$\|G(x_k,\ .\ )\|_{L^\infty\left(\Omega\setminus\overline{B_{3\delta}(x_\infty)}\right)} \le C(q,\delta),$$

which gives us in particular

$$|G(x_k,y_k)| \le C(q,\delta) \qquad \text{and} \qquad |x_k-y_k|^{n-2m}|G(x_k,y_k)| \le C(q,\delta).$$

This contradicts (2.28). Therefore, $|x_k-y_k|\to 0$ and we can use a fixed coordinate chart $\Phi : U \to \mathbb{R}^n$ for $\overline{\Omega}$ around x_∞ such that $\Phi(0)=x_\infty$,

$$\Phi(U\cap\{x_1<0\}) = \Phi(U)\cap\Omega \qquad \text{and} \qquad \Phi(U\cap\{x_1=0\}) = \Phi(U)\cap\partial\Omega.$$

From (2.29) we see

$$\lim_{k\to\infty} x'_k = 0 \quad \text{and} \quad \lim_{k\to\infty}\frac{x'_{k,1}}{|x'_k-y'_k|} = 0, \tag{2.30}$$

where $x_k=\Phi(x'_k)$ and $y_k=\Phi(y'_k)$. For $R>0$ and k large enough the function

$$\tilde{G}_k(z) := |x'_k-y'_k|^{n-2m} G\left(\Phi(x'_k), \Phi(x'_k+|x'_k-y'_k|(z-\rho_k e_1)\right)$$

is well defined on $B_R(0)\cap\{x_1<0\}$, where $\rho_k := \frac{x'_{k,1}}{|x'_k-y'_k|}$ and e_1 is the first unit vector. Since

$$\begin{cases} (-\Delta)^m G(x,\ .\ ) + \displaystyle\sum_{\ell=0}^{m-1}\sum_{|\alpha|=|\beta|=\ell} D^\beta\left(a^\ell_{\alpha,\beta}D^\alpha G(x,\ .\ )\right) = 0 & \text{in } \Omega\setminus\{x\}, \\ G(x,\ .\ ) = \partial_\nu G(x,\ .\ ) = \ldots = \partial_\nu^{(m-1)} G(x,\ .\ ) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.31}$$

we have that

$$\begin{cases} (-\Delta_{g_k})^m\tilde{G}_k + \displaystyle\sum_{0\le|\sigma|\le 2m-1} |x'_k-y'_k|^{2m-|\sigma|}\tilde{a}^k_\sigma D^\sigma\tilde{G}_k = 0 & \text{in } (B_R(0)\cap\{z_1<0\})\setminus\{\rho_k e_1\}, \\ \tilde{G}_k = \partial_1\tilde{G}_k = \ldots = \partial_1^{(m-1)}\tilde{G}_k = 0 & \text{on } \{z_1=0\}, \end{cases} \tag{2.32}$$

where $g_k(z) = \Phi^*(\mathscr{E})\left(\Phi(x'_k+|x'_k-y'_k|(z-\rho_k e_1)\right)$, $\mathscr{E}=\delta_{ij}$ the Euclidean metric and Δ_{g_k} denotes the Laplace-Beltrami operator with respect to this scaled and translated pull back of the Euclidean metric under Φ. Since Φ is a fixed diffeomorphism and $\left\|a^\ell_{\alpha,\beta}\right\|_{C^{m-1,\gamma}(\overline{\Omega})} \le K$, we have that the suitable coefficients $\tilde{a}^k_\sigma$ are Hölder continuous functions on $B_R(0)\cap\{z_1<0\}$ with uniformly bounded Hölder norm.

As above, we use Sobolev embeddings, elliptic estimates and Hölder's inequality to see for $z\in\left(B_{R/2}(0)\setminus B_{2\tau}(0)\right)\cap\{z_1\le 0\}$ that

$$|\tilde{G}_k(z)| \le C\|\tilde{G}_k\|_{L^q((B_R(0)\setminus B_\tau(0))\cap\{z_1<0\})},$$

where q is chosen as in Lemma 2.5, τ is suitably small and $C = C(R, q, \tau)$. From Lemma 2.5, since the Jacobian of Φ is bounded, we obtain

$$\begin{aligned}
&\int_{B_R(0)\cap\{z_1<0\}} |\tilde{G}_k(z)|^q dz \\
&= |x_k' - y_k'|^{q(n-2m)} \int_{B_R(0)\cap\{z_1<0\}} \left|G\left(\Phi(x_k'), \Phi(x_k' + |x_k' - y_k'|(z - \rho_k e_1))\right)\right|^q dz \\
&\leq C|x_k' - y_k'|^{q(n-2m)-n} \int_\Omega |G(x_k, y)|^q dy \\
&\leq C|x_k' - y_k'|^{q(n-2m)-n} d(x_k)^{(2m-n)q+n} \\
&= C\left(\frac{d(x_k)}{|x_k' - y_k'|}\right)^{(2m-n)q+n}.
\end{aligned}$$

Hence, (2.29) shows $\|\tilde{G}_k\|_{L^q((B_R(0)\setminus B_\tau(0))\cap\{z_1<0\})} \to 0$, so we have

$$\lim_{k\to\infty} \tilde{G}_k = 0 \quad \text{in } C^0\left(\left(B_{R/2}(0) \setminus B_{2\tau}(0)\right) \cap \{z_1 \leq 0\}\right). \tag{2.33}$$

Since $\left|\frac{y_k'-x_k'}{|y_k'-x_k'|}\right| = 1$ and $\lim_{k\to\infty} \rho_k = 0$, we get with (2.33) that

$$\begin{aligned}
0 &= \lim_{k\to\infty} \tilde{G}_k\left(\frac{y_k' - x_k'}{|y_k' - x_k'|} + \rho_k e_1\right) = \lim_{k\to\infty} |x_k' - y_k'|^{n-2m} G\left(\Phi(x_k'), \Phi(y_k')\right) \\
&= \lim_{k\to\infty} |x_k' - y_k'|^{n-2m} G(x_k, y_k).
\end{aligned}$$

Moreover, since Φ is a diffeomorphism, we get

$$0 = \lim_{k\to\infty} |x_k - y_k|^{n-2m} G(x_k, y_k),$$

which contradicts (2.28) and the claim is proved for $n > 2m$.

<u>*Case $n = 2m$.*</u> Since the Green's function is symmetric, we perform the same proof as in the case $n > 2m$ for $\nabla_y G$, cf. [23, proof of Prop. 4.22]. Integration proves the estimate for G. □

Proposition 2.7. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2m - 1$, be a bounded $C^{2m,\gamma}$-smooth domain. Then there exists a constant $C = C(\Omega, K) > 0$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $1 \leq |\alpha| + |\beta| \leq 2m$, and all $x, y \in \Omega$ with $x \neq y$ one has that*

$$\left|D_x^\alpha D_y^\beta G(x, y)\right| \leq C|x - y|^{2m-n-|\alpha|-|\beta|}.$$

The proof is done as for [23, Proposition 4.23], cf. [28, Theorem 2].

Proof. Let B_R and B_{2R} be two concentric balls with $B_R \subset B_{2R}$. For $|\alpha| \leq 2m$ we show the following estimate

$$\|D^\alpha u\|_{L^\infty(B_R\cap\Omega)} \leq \frac{C}{R^{|\alpha|}} \|u\|_{L^\infty(B_{2R}\cap\Omega)}, \tag{2.34}$$

where u is a solution of the homogeneous Dirichlet boundary value problem (2.1). We perform a scaling argument.

Let us define $v(x) := u(Rx)$. Then we have that $D^\alpha v(x) = (D^\alpha u)(Rx) \cdot R^{|\alpha|}$. Using local Schauder estimates (for fixed radii) we get for $x_0 \in \Omega$ and $|\alpha| \leq 2m$

$$\begin{aligned}\|D^\alpha u\|_{L^\infty(B_R(x_0)\cap\Omega)} \cdot R^{|\alpha|} &= \|D^\alpha v\|_{L^\infty\left(B_1\left(\frac{x_0}{R}\right)\cap\Omega\right)} \leq \|v\|_{C^{2m,\gamma}\left(B_1\left(\frac{x_0}{R}\right)\cap\overline{\Omega}\right)} \\ &\leq C\|v\|_{L^1\left(B_2\left(\frac{x_0}{R}\right)\cap\Omega\right)} \leq C \cdot \operatorname{vol}(B_2 \cap \Omega)\,\|v\|_{L^\infty\left(B_2\left(\frac{x_0}{R}\right)\cap\Omega\right)} \\ &\leq C\|u\|_{L^\infty(B_{2R}(x_0)\cap\Omega)}.\end{aligned}$$

Case $n > 2m$. Let $x \in \Omega,\ y \in \Omega \setminus \{x\}$ and $R = \frac{|x-y|}{4}$. For $|\alpha| = 0$ we find with the help of Proposition 2.6 and (2.34) in $B_R(y) \subset B_{2R}(y)$ that

$$\begin{aligned}\|D_y^\beta G(x,\ .\ )\|_{L^\infty(B_R(y)\cap\Omega)} &\leq \frac{C}{|x-y|^{|\beta|}}\|G(x,\ .\ )\|_{L^\infty(B_{2R}(y)\cap\Omega)} \\ &\leq \frac{C}{|x-y|^{|\beta|}}\left\||x-\ .\ |^{2m-n}\right\|_{L^\infty(B_{2R}(y)\cap\Omega)} \\ &\leq C|x-y|^{2m-n-|\beta|},\end{aligned}$$

where we used for $z \in B_{2R}(y)$ that

$$|x-z| \geq |x-y| - |y-z| \geq \frac{1}{2}|x-y|.$$

If $|\beta| = 0$ and $|\alpha| > 0$ since the Green function is symmetric, the analogue statement holds true. Moreover, since $y \mapsto D_x^\alpha G(x,y)$ solves the homogeneous Dirichlet boundary value problem, we can proceed as before for the mixed derivatives.

Case $n = 2m, n = 2m-1$. For $|\alpha| + |\beta| = 1$ the claim is already proven in Proposition 2.6. We follow the lines of the proof for the case $n > 2m$ starting with the first order derivative estimate from Proposition 2.6 to prove the higher derivative estimates. □

Lemma 2.8. *Let $\Omega \subset \mathbb{R}^n$, $2 \leq n < 2m-1$, be a bounded $C^{2m,\gamma}$-smooth domain and $\delta > 0$. Then there exists a constant $C = C(\delta, \Omega, K) > 0$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| + n \leq 2m$ and $x, y \in \Omega$ with $\max\{d(x), d(y)\} \geq \delta$ we have:*

- *If $|\alpha| + |\beta| + n < 2m$ and n is even, or if $|\alpha| + |\beta| + n \leq 2m$:*

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C.$$

- *If $|\alpha| + |\beta| + n = 2m$ and n is even:*

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C \log\left(1 + |x-y|^{-1}\right).$$

The proof is inspired by the proof of [23, Lemma 4.24].

Proof. In Proposition 2.1 we have constructed the Green function in the following way:

$$G(x,y) := \Gamma_0(x,y) + \sum_{j=1}^{k} \Gamma_j(x,y) + u(x,y).$$

For odd n and $|\alpha| = 0$, i.e. $|\beta| \le 2m-n$, using (1.3) and the estimates (2.17) and (2.23) we get

$$|D_y^\beta G(x,y)| \le C|x-y|^{2m-n-|\beta|} + \|u(x,\ .\ )\|_{C^{|\beta|}(\overline{\Omega})}\,.$$

If $d(x) > \delta$ from Schauder estimates or from Lemma 1.5 we get for all $x \in \Omega$ with $d(x) > \delta$ that

$$\|u(x,\ .\ )\|_{C^{|\beta|}(\overline{\Omega})} \le C.$$

From which $|D_y^\beta G(x,y)| \le C$ follows.

If n is even, proceeding in a similar way, we get the same estimate for $|\beta| < 2m-n$. For $|\beta| = 2m-n$ we see with (1.3) that

$$|D_y^\beta G(x,y)| \le C|x-y|^{-1} + C(\delta),$$

and integration shows the estimate for $|\alpha| = 0$.

If $|\alpha| > 0$, i.e. $|\beta| < 2m-n$, we start with the function $D_x^\alpha u(x,\ .\ )$ and follow a similar argumentation as before.

Since the Green function is symmetric, we can show the same result for $d(y) \ge \delta$. □

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^n$, $2 \le n < 2m-1$, be a bounded $C^{2m,\gamma}$-smooth domain. For $q \in \left(\frac{n}{n-m+1}, \frac{n}{n-m}\right)$ if $n > m$ or $q > n$ if $n \le m$, there exists a constant $C = C(q,\Omega,K) > 0$ such that*

$$\left\|\nabla_y^m G(x,\ .\ )\right\|_{L^q(\Omega)} \le C d(x)^{m-n+\frac{n}{q}}, \tag{2.35}$$

$$\left\|\nabla_x \nabla_y^{m-1} G(x,\ .\ )\right\|_{L^q(\Omega)} \le C d(x)^{m-n+\frac{n}{q}}, \tag{2.36}$$

where ∇^m means any m-th derivative.

The proof is done as for [23, Lemma 4.26].

Proof. We first prove (2.35). Let $\varphi \in W^{m,q'}(\Omega)$ be the solution of

$$\begin{cases} (-\Delta)^m \varphi + \displaystyle\sum_{\ell=0}^{m-1} \sum_{|\alpha|=|\beta|=\ell} D^\beta \left(a_{\alpha,\beta}^\ell D^\alpha \varphi\right) = \nabla^m \Psi & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \dfrac{\partial^j}{\partial \nu^j}\varphi = 0 & \text{on } \partial\Omega \text{ and } j = 0,\dots,m-1, \end{cases}$$

where $\Psi \in L^{q'}$, $q' = \frac{q}{q-1} \in \left(\frac{n}{m}, \frac{n}{m-1}\right)$ if $n > m$ and $q' = \frac{q}{q-1} \in \left(1, \frac{n}{n-1}\right)$ if $n \leq m$. Then, [3, Theorem 15.3'] shows

$$\|\varphi\|_{W^{m,q'}(\Omega)} \leq C \|\Psi\|_{L^{q'}(\Omega)} .$$

Let $n > m$. Since we have zero boundary values for φ and $m - \frac{n}{q'} \in (0, 1)$, using Sobolev's embedding theorem, we get for $d(x) = |x - \tilde{x}|$ with $\tilde{x} \in \partial\Omega$ that

$$|\varphi(x)| = |\varphi(x) - \varphi(\tilde{x})| \leq C \|\varphi\|_{W^{m,q'}(\Omega)} d(x)^{m-\frac{n}{q'}} . \tag{2.37}$$

For $n \leq m$ we have that $m - \frac{n}{q'} \in (m - n, m - n + 1)$ and Sobolev's embedding theorem shows $W^{m,q'}(\Omega) \subset C^{m-n,n/q}(\overline{\Omega})$. Using Taylor's formula, the zero boundary values for φ and its derivatives up to the order $m - 1$ we obtain that

$$|\varphi(x)| \leq C \|\varphi\|_{C^{m-n,n/q}(\overline{\Omega})} d(x)^{m-n+\frac{n}{q}} \leq C \|\varphi\|_{W^{m,q'}(\Omega)} d(x)^{m-\frac{n}{q'}} . \tag{2.38}$$

The estimates (2.37) and (2.38) lead us to

$$|\varphi(x)| \leq C \|\Psi\|_{L^{q'}(\Omega)} d(x)^{m-\frac{n}{q'}} .$$

From Proposition 2.1 we have the representation formula $\varphi(x) = \int_\Omega G(x, y)\nabla_y^m \Psi(y)\ dy$. After integration by parts we see together with the zero boundary values of φ that

$$\varphi(x) = (-1)^m \int_\Omega \nabla_y^m G(x, y)\Psi(y)\ dy.$$

Therefore, by duality, cf. [4, Folgerung 4.13], and $m - \frac{n}{q'} = m - n + \frac{n}{q}$:

$$\left\|\nabla_y^m G(x,\ .\ )\right\|_{L^q(\Omega)} \leq C d(x)^{m-n+\frac{n}{q}},$$

which is (2.35).

To prove (2.36) we consider the following Dirichlet boundary value problem

$$\begin{cases} (-\Delta)^m \varphi + \displaystyle\sum_{\ell=0}^{m-1} \sum_{|\alpha|=|\beta|=\ell} D^\beta \left(a_{\alpha,\beta}^\ell D^\alpha \varphi\right) = \nabla^{m-1}\Psi & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \dfrac{\partial^j}{\partial\nu^j}\varphi = 0 & \text{on } \partial\Omega \text{ and } j = 0, \dots, m-1. \end{cases}$$

Again, [3, Theorem 15.3'] shows

$$\|\varphi\|_{W^{m+1,q'}(\Omega)} \leq C \|\Psi\|_{L^{q'}(\Omega)} .$$

We use now the zero boundary values of $\nabla\varphi$ and its derivatives up to the order $m - 2$ to get with Sobolev's embedding theorem and Taylor's formula

$$|\nabla\varphi(x)| \leq C \|\varphi\|_{W^{m+1,q'}(\Omega)} d(x)^{m-\frac{n}{q'}} \leq C \|\Psi\|_{L^{q'}(\Omega)} d(x)^{m-\frac{n}{q'}} .$$

Then,

$$\nabla\varphi(x) = \int_\Omega \nabla_x G(x, y)\nabla_y^{m-1}\Psi(y)\ dy = (-1)^{m-1} \int_\Omega \nabla_x \nabla_y^{m-1} G(x, y)\Psi(y)\ dy.$$

By duality, we have (2.36). □

Proposition 2.10. *Let $\Omega \subset \mathbb{R}^n$, $2 \leq n < 2m-1$, be a bounded $C^{2m,\gamma}$-smooth domain. Then there exists a constant $C = C(\Omega, K) > 0$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| \leq 2m$, and all $x, y \in \Omega$ we have*

- *If $|\alpha| + |\beta| + n > 2m$:*

$$\left| D_x^\alpha D_y^\beta G(x,y) \right| \leq C|x-y|^{2m-n-|\alpha|-|\beta|}.$$

- *If $|\alpha| + |\beta| + n = 2m$ and n is odd:*

$$\left| D_x^\alpha D_y^\beta G(x,y) \right| \leq C.$$

The proof is done as for [23, Proposition 4.25].

Proof. We prove first the estimates for $|\alpha| + |\beta| + n > 2m$.

Let us start with $|\alpha| = 0$ and $|\beta| = 2m-n+1$. We assume by contradiction that there exist $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}} \subset \Omega, x_k \neq y_k$, such that

$$\lim_{k\to\infty} |x_k - y_k||D_y^\beta G(x_k, y_k)| = \infty.$$

As in the proof of Proposition 2.6 we get $\lim_{k\to\infty} |x_k - y_k| = 0$. Since $\overline{\Omega}$ is compact, there exists a $x_\infty \in \overline{\Omega}$ such that, after choosing a suitable subsequence,

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x_\infty.$$

Thus, for k large enough, x_k, y_k are in a fixed neighbourhood of x_∞, where we can use local rescaled elliptic estimates which hold with uniform constants.

Case $d(x_k) < 2|x_k - y_k|$. Here, we have

$$\left(B_{4|x_k-y_k|}(x_k) \setminus B_{|x_k-y_k|/2}(x_k)\right) \cap \partial\Omega \neq \emptyset.$$

Since we have zero boundary values for $G(x_k, \ . \ )$, we can use local rescaled elliptic estimates, as in the proof of Proposition 2.7, and a localised Poincaré inequality, cf. [47, Theorem 3.2.1] and [10, 1.6 Bemerkung], to estimate

$$\begin{aligned}
|D_y^\beta G(x_k, y_k)| &\leq C|x_k - y_k|^{-|\beta|-\frac{n}{q}} \|G(x_k, \ . \ )\|_{L^q\left(B_{4|x_k-y_k|}(x_k)\setminus B_{|x_k-y_k|/2}(x_k)\right)} \\
&\leq C|x_k - y_k|^{-|\beta|+m-\frac{n}{q}} \left\|\nabla_y^m G(x_k, \ . \ )\right\|_{L^q\left(B_{4|x_k-y_k|}(x_k)\setminus B_{|x_k-y_k|/2}(x_k)\right)} \\
&\leq C|x_k - y_k|^{-|\beta|+m-\frac{n}{q}} d(x_k)^{m-n+\frac{n}{q}} \\
&\leq C|x_k - y_k|^{-1},
\end{aligned}$$

where we choose q as in Lemma 2.9. In this way, we have a contradiction.

Case $d(x_k) \geq 2|x_k - y_k|$. In this case we have

$$d(x_k) \leq |x_k - y_k| + d(y_k) \leq \frac{1}{2}d(x_k) + d(y_k),$$

from which it follows that $d(x_k) \leq 2d(y_k)$.

As in the proof of Proposition 2.1, cf. (2.21), we now look on y_k as a parameter and consider the boundary value problem for $D_y^\beta G(\ . \ , y_k)$. By using (1.3) and (1.6) the derivatives of the iterated kernels in (2.6) can be estimated by $C|x_k - y_k|^{-1}$. Note that $|\beta| = 2m - n + 1 > 2m - n$. Performing a Taylor expansion we have

$$|D_y^\beta u_{y_k}(x_k)| \leq C\left(\sum_{|\alpha|=0}^{m-1} \|D_x^\alpha D_y^\beta u_{y_k}\|_{C^0(\partial\Omega)} d(x_k)^{|\alpha|} + \|D_y^\beta u_{y_k}\|_{C^{m-1,\gamma}(\overline{\Omega})} d(x_k)^{m-1+\gamma}\right).$$

Similar as in Lemma 1.5, but looking at the polyharmonic boundary value problem for $\nabla_y^{2m-n+1} u_{y_k}(\ . \ )$, we can prove

$$\|D_y^\beta u_{y_k}\|_{C^{m-1,\gamma}(\overline{\Omega})} \leq C d(y_k)^{m-n+1-|\beta|-\gamma} = C d(y_k)^{-m-\gamma}.$$

Note that $|\beta| = 2m - n + 1 > m - n$, and therefore we have this estimate also for the case $n \leq m$.

Since $d(x_k) \leq 2d(y_k)$, the estimates (1.3) of the fundamental solution show again

$$|D_y^\beta u_{y_k}(x_k)| \leq C\left(d(x_k)^{-1} + \frac{d(x_k)^{m-1+\gamma}}{d(y_k)^{m+\gamma}}\right) \leq C d(x_k)^{-1}.$$

Then,

$$|D_y^\beta G(x_k, y_k)| \leq C\left(|x_k - y_k|^{-1} + d(x_k)^{-1}\right) \leq C|x_k - y_k|^{-1},$$

which is a contradiction.

Proceeding as before, we prove the claim for $|\alpha| = 1$ and $|\beta| = 2m - n$ in a similar way. For more details look at [23, Proposition 4.25]. To get the estimate for higher mixed derivatives we can proceed as in the proof of Proposition 2.7.

It is left to prove the estimate for $|\alpha| + |\beta| + n = 2m$ and odd n. Here we can proceed as for the case $|\alpha| + |\beta| + n > 2m$. Note that for odd n all derivatives of the iterated kernels in (2.6) are bounded. □

Proposition 2.11. *Let $\Omega \subset \mathbb{R}^n$, $2 \leq n < 2m-1$, be a bounded $C^{2m,\gamma}$-smooth domain. Then there exists a constant $C = C(\Omega, K) > 0$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| + |\beta| + n \leq 2m$, and all $x, y \in \Omega$ we have*

- *If $|\alpha| + |\beta| + n < 2m$:*

$$\left|D_x^\alpha D_y^\beta G(x, y)\right| \leq C.$$

- *If* $|\alpha| + |\beta| + n = 2m$ *and* n *is even:*

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C \log\left(1 + |x-y|^{-1}\right).$$

The proof is inspired by the proof of [23, Proposition 4.27].

Proof. Let $\delta > 0$ small enough and $y_0 \in \Omega$ with $d(y_0) > 2\delta$. Since the claim follows from Lemma 2.8 if $\max\{d(x), d(y)\} \geq \delta$, we consider $d(x) < \delta$ and $d(y) < \delta$. Moreover, let r_0 be small enough such that $\Omega \setminus B_{2r_0}(z)$ is connected for all $z \in \Omega$. Let $\omega \subset \Omega \setminus B_r(x)$ be a path from y_0 to y, where $r = \min\{r_0, |x-y|\}$. Then,

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq \left|D_x^\alpha D_y^\beta G(x,y_0)\right| + \int_\omega \left|D_x^\alpha \nabla_y D_y^\beta G(x,\omega(s))\right| d\omega(s).$$

In the following we use Lemma 2.8, Proposition 2.10, $|x - y_0| > \delta$ and $|x - \omega(s)| > r$ for all s to see that

- If $|\alpha| + |\beta| + n < 2m$ and n is odd: $\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C$.
- If $|\alpha| + |\beta| + n = 2m$ and n is even:

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C\left(1 + \int_r^{C(\Omega)} t^{-1}\, dt\right) \leq C \log\left(1 + |x-y|^{-1}\right).$$

- If $|\alpha| + |\beta| + n < 2m$ and n is even:

$$\left|D_x^\alpha D_y^\beta G(x,y)\right| \leq C\left(1 + \int_r^{C(\Omega)} |\log t|\, dt\right) \leq C.$$

Hence, the claims are proved. □

Proof of Theorem 2.4. For $2 \leq n < 2m - 1$ we use the estimates from Proposition 2.10 and Proposition 2.11. If $n \geq 2m - 1$ and $|\alpha| + |\beta| + n > 2m$ the result follows from Proposition 2.6 and Proposition 2.7. For even n, the case $|\alpha| + |\beta| + n = 2m$ and $n \geq 2m - 1$ occurs only if $n = 2m$ and is covered by Proposition 2.6. If $n \geq 2m - 1$ and $|\alpha| + |\beta| + n < 2m$ or $|\alpha| + |\beta| + n = 2m$ for odd n, only the case $n = 2m - 1$ is possible, and here we have Proposition 2.6. □

By the same integration process as in the proof of [16, Theorem 3], cf. [23, Theorem 4.28], the following corollary is given.

Corollary 2.12. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2m,\gamma}$-smooth domain. There exists a constant $C = C(\Omega, K) > 0$ such that for all $x, y \in \Omega$ with $x \neq y$ the following estimates hold:*

$$|G(x,y)| \leq C \cdot \begin{cases} |x-y|^{2m-n} \min\left\{1, \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n > 2m, \\ \log\left(1 + \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \dfrac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} & \text{if } n < 2m. \end{cases}$$

2.4 The Rescaled Green's Function

As explained in the introduction, the locally uniform convergence of a rescaled Green function to the polyharmonic Green function $G_{(-\Delta)^m,\mathcal{H}}$ of the half space $\mathcal{H} := \{x \in \mathbb{R}^n : x_1 < 0\}$ is needed for a blow-up argument, which we will use later in this work. The following proposition explains this convergence exemplary for the dimensions $n \geq 2m-1$. The ideas are taken from [27].

Proposition 2.13. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2m-1$, be a bounded $C^{2m,\gamma}$-smooth domain with $0 \in \partial\Omega$ such that the first unit vector e_1 is the exterior unit normal to $\partial\Omega$ at 0. Let $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \subset \overline{\Omega}$ with $x_k \to 0, y_k \to 0$ for $k \to \infty$. For k large enough let $\tilde{x}_k \in \partial\Omega$ be the uniquely determined point, such that $d(x_k) = |x_k - \tilde{x}_k|$. Let us define*

$$\Omega_k := \frac{1}{|x_k - y_k|}\left(-\tilde{x}_k + \Omega\right).$$

For $\xi, \eta \in \Omega_k$ we define

$$G_k(\xi,\eta) := |x_k - y_k|^{n-2m} G(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta).$$

Then it holds

$$G_k(\xi,\eta) \to G_{(-\Delta)^m,\mathcal{H}}(\xi,\eta) = k_{m,n}|\xi-\eta|^{2m-n} \int_1^{|\xi^*-\eta|/|\xi-\eta|} (v^2-1)^{m-1} v^{1-n}\, dv$$

in $C^{2m}_{loc}\left(\mathcal{H} \times \mathcal{H} \setminus \{(\xi,\xi) : \xi \in \mathcal{H}\}\right)$, where $\xi^* = (-\xi_1, \xi_2, \ldots, \xi_n)$ and

$$k_{m,n} = \frac{1}{n e_n 4^{m-1}((m-1)!)^2}.$$

As a first step to prove the proposition, uniform bounds of G_k are needed. Since we start in dimensions $n = 2m$ and $n = 2m-1$ with the derivatives of G_k, some integration along paths constructed in the following lemma has to be done. The following lemma is due to [16, Lemma 7], cf. [23, p. 136].

Lemma 2.14 ([16, Lemma 7]). *Let $\xi, \eta \in \Omega$. There exists a piecewise smooth curve $\omega_\xi : [0,1] \to \overline{\Omega}$ with $\omega_\xi(0) = \xi$, $\tilde{\xi} := \omega_\xi(1) \in \partial\Omega$ such that for every $t \in [0,1]$ we have that*

$$|\omega_\xi(t) - \eta| \geq \frac{1}{2}|\xi - \eta|.$$

Moreover, if we parametrise ω_ξ by arc length (again denoted by ω_ξ), $\omega_\xi : [0,\ell] \to \overline{\Omega}$, it yields for $s \in [0,\ell]$:

1. $\frac{2}{3}s \leq |\xi - \omega_\xi(s)| \leq s$;
2. $|\omega_\xi(s) - \eta| \geq \frac{1}{8}|\xi - \eta| + \frac{1}{8}|\omega_\xi(s) - \xi|$;
3. $\ell \leq \frac{1}{3}(1+\pi)d(\xi)$;

where ℓ is the length of ω_ξ.

Proof of Proposition 2.13. From Section 1.3 we have $\Omega_k \to \mathcal{H}$ locally uniformly. We divide the proof in several steps.

Step 1. With the estimates from Proposition 2.6 we observe uniformly in k, ξ and η:

- If $n > 2m$:

$$\begin{aligned}|G_k(\xi,\eta)| &= |x_k - y_k|^{n-2m}|G(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta)| \\ &\leq C|x_k - y_k|^{n-2m}\,|\tilde{x}_k + |x_k - y_k|\xi - (\tilde{x}_k + |x_k - y_k|\eta)|^{2m-n} \\ &= C|\xi - \eta|^{2m-n}.\end{aligned}$$

- If $n = 2m$:

$$\begin{aligned}|\nabla_{(\xi,\eta)}G_k(\xi,\eta)| &= |x_k - y_k||\nabla_{(x,y)}G(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta)| \\ &\leq C\,|x_k - y_k|\,|\tilde{x}_k + |x_k - y_k|\xi - (\tilde{x}_k + |x_k - y_k|\eta)|^{-1} \\ &= C|\xi - \eta|^{-1}.\end{aligned} \tag{2.39}$$

- If $n = 2m-1$:

$$|\nabla_{(\xi,\eta)}G_k(\xi,\eta)| \leq C. \tag{2.40}$$

For $n = 2m$ and $n = 2m-1$ some integration has to be done. We use the path ω_ξ from Lemma 2.14. From the properties of the path we can see for all s that

$$|\omega_\xi(s) - \eta| \geq \frac{1}{12}(|\xi - \eta| + s).$$

Let $s \mapsto \xi(s)$ parametrise ω_ξ as above by arc length. Since $\tilde{x}_k \in \partial\Omega$, i.e. $0 \in \partial\Omega_k$, we have $\ell \leq \frac{3}{2}d(\xi) \leq \frac{3}{2}|\xi|$. Then, for $\xi, \eta \in \Omega_k$ we have

$$G_k(\xi,\eta) = \underbrace{G_k(\tilde{\xi},\eta)}_{=0} + \int_{\omega_\xi} \nabla_\xi G_k(\xi(s),\eta)\cdot\tau(s)ds,$$

where $\tau(s)$ is the unit tangent vector.

Let $n = 2m-1$. By using (2.40) we get

$$|G_k(\xi,\eta)| \le \int_0^\ell |\nabla_\xi G_k(\xi(s),\eta)|\, ds \le C \int_0^{(3/2)|\xi|} 1\, ds \le C|\xi|.$$

Interchanging ξ and η shows that

$$|G_k(\xi,\eta)| \le C|\eta|.$$

Hence,

$$|G_k(\xi,\eta)| \le C \min\{|\xi|,|\eta|\}. \tag{2.41}$$

Let $n = 2m$. If $|\xi-\eta| \ge 1$ we obtain with the help of (2.39):

$$\begin{aligned} |G_k(\xi,\eta)| &\le C \int_0^\ell |\xi(s)-\eta|^{-1} ds \le C \int_0^{(3/2)|\xi|} (|\xi-\eta|+s)^{-1} ds \le C \int_0^{(3/2)|\xi|} (1+s)^{-1} ds \\ &\le C\ (1+\log(1+|\xi|)). \end{aligned} \tag{2.42}$$

Interchanging ξ, η shows that

$$|G_k(\xi,\eta)| \le C\ (1+\log(1+|\eta|)). \tag{2.43}$$

For $|\xi-\eta| < 1$ it follows that

$$\begin{aligned} |G_k(\xi,\eta)| &\le C \int_0^{(3/2)|\xi|} (|\xi-\eta|+s)^{-1} ds = C \log\left(\frac{3}{2}|\xi| + \underbrace{|\xi-\eta|}_{<1}\right) - C\ \underbrace{\log(|\xi-\eta|)}_{<0} \\ &\le C\ \left(\log(1+|\xi|) + \big|\log|\xi-\eta|\big|\right). \end{aligned} \tag{2.44}$$

Then, interchanging ξ and η we get

$$|G_k(\xi,\eta)| \le C\ \left(\log(1+|\eta|) + \big|\log|\xi-\eta|\big|\right). \tag{2.45}$$

Finally, we have by combining (2.42)–(2.45) that

$$|G_k(\xi,\eta)| \le C\ \left(1 + \big|\log|\xi-\eta|\big| + \log(1+|\xi|) + \log(1+|\eta|)\right). \tag{2.46}$$

Then we can conclude that

$$|G_k(\xi,\eta)| \le C \cdot \begin{cases} |\xi-\eta|^{2m-n} & \text{if } n > 2m, \\ 1 + \big|\log|\xi-\eta|\big| + \log(1+|\xi|) + \log(1+|\eta|)) & \text{if } n = 2m, \\ \min\{|\xi|,|\eta|\} & \text{if } n = 2m-1, \end{cases} \tag{2.47}$$

where C does not depend on k.

<u>*Step 2.*</u> We prove the following.

For each $\xi \in \mathcal{H}$ we find a function $\overline{G}(\xi, \,.\,) \in C^{2m}(\overline{\mathcal{H}} \setminus \{\xi\})$ such that $G_k(\xi, \,.\,) \to \overline{G}(\xi, \,.\,)$ in $C^{2m}_{loc}(\mathcal{H} \setminus \{\xi\})$.

We take an arbitrary $\eta_0 \in \mathcal{H} \setminus \{\xi\}$ and choose $r_1 > 0$ such that $\overline{B_{r_1}(\eta_0)} \subset \mathcal{H} \setminus \{\xi\}$. Since the rescaled domains Ω_k exhaust the whole $\mathcal{H}$ we find a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\overline{B_{r_1}(\eta_0)} \subset \Omega_k$. Then, with the estimates (2.47) we find a constant $C = C(r_1, \eta_0, \xi, n, m)$ such that for all $k \geq k_0$:

$$\|G_k(\xi, \,.\,)\|_{C^0(\overline{B_{r_1}(\eta_0)})} \leq C(r_1, \eta_0, \xi, n, m).$$

In order to compute the differential equation satisfied by G_k we define for $\ell \in \{0, \ldots, m-1\}$ and $\eta \in \Omega_k$:

$$a^{\ell,k}_{\alpha,\beta}(\eta) := a^{\ell}_{\alpha,\beta}(\tilde{x}_k + |x_k - y_k|\eta),$$

from which we see that $a^{\ell,k}_{\alpha,\beta} \in C^{m-1,\gamma}(\overline{\Omega}_k)$. Then, since $\xi \neq \eta$ implies $\tilde{x}_k + |x_k - y_k|\xi \neq \tilde{x}_k + |x_k - y_k|\eta$, we have for all $\eta \in \overline{B_{r_1}(\eta_0)}$:

$$\begin{aligned}
&(-\Delta_\eta)^m G_k(\xi,\eta) + \sum_* |x_k - y_k|^{2m-|\alpha|-|\beta|} D^\beta_\eta \left(a^{\ell,k}_{\alpha,\beta}(\eta) D^\alpha_\eta G_k(\xi,\eta) \right) \\
&= |x_k - y_k|^n \left(-\Delta_y\right)^m G \circ (\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta) \\
&+ \sum_* |x_k - y_k|^{n-|\beta|} D^\beta_\eta \left(a^{\ell}_{\alpha,\beta}(\tilde{x}_k + |x_k - y_k|\eta) \left(D^\alpha_y G \circ (\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta) \right) \right) \\
&= |x_k - y_k|^n \Big(\left(-\Delta_y\right)^m G \circ (\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta) \\
&\quad + \sum_* D^\beta_y \left(a^{\ell}_{\alpha,\beta} \circ (\tilde{x}_k + |x_k - y_k|\eta) \left(D^\alpha_y G \circ (\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta) \right) \right) \Big) \\
&= 0.
\end{aligned} \tag{2.48}$$

As a consequence, interior Schauder estimates, cf. [3] or [23, Theorem 2.19], show for a positive constant $C > 0$ that

$$\|G_k(\xi, \,.\,)\|_{C^{2m,\gamma}(\overline{B_{r_1/2}(\eta_0)})} \leq C \, \|G_k(\xi, \,.\,)\|_{C^0(\overline{B_{r_1}(\eta_0)})} . \tag{2.49}$$

Since $\overline{B_{r_1/2}(\eta_0)}$ is compact, we get for all $k \geq k_0$ that for all $\eta, \eta' \in \overline{B_{r_1/2}(\eta_0)}$ and for all $|\sigma| \leq 2m - 1$ it holds

$$\begin{aligned}
|D^\sigma_\eta G_k(\xi,\eta) - D^\sigma_\eta G_k(\xi,\eta')| &\leq \|G_k(\xi, \,.\,)\|_{C^{2m}(\overline{B_{r_1/2}(\eta_0)})} \, |\eta - \eta'| \\
&\leq C(r_1, \eta_0, \xi, n, m)|\eta - \eta'|.
\end{aligned} \tag{2.50}$$

Also for $|\sigma| = 2m$ the Schauder estimate (2.49) shows

$$\begin{aligned}
|D^\sigma_\eta G_k(\xi,\eta) - D^\sigma_\eta G_k(\xi,\eta')| &\leq C \, \|G_k(\xi, \,.\,)\|_{C^0(\overline{B_{r_1}(\eta_0)})} \, |\eta - \eta'|^\gamma \\
&\leq C(r_1, \eta_0, \xi, n, m)|\eta - \eta'|^\gamma.
\end{aligned} \tag{2.51}$$

As a consequence of (2.50) and (2.51) the sequence $(D_\eta^\sigma G_k(\xi,\ .\ ))_{k\in\mathbb{N}}$ is locally uniformly bounded in C^{2m} and locally equicontinuous for $|\sigma| \le 2m$. By the Arzela–Ascoli theorem, we can select a subsequence converging locally uniformly to a function $\overline{G}(\xi,\ .\ ) \in C^{2m}(\mathcal{H} \setminus \{\xi\})$, which satisfies (2.47). Moreover, since the coefficients $a^\ell_{\alpha,\beta}$ are uniformly bounded in $C^{m-1,\gamma}(\overline{\Omega})$ by K, we get

$$\|a^{\ell,k}_{\alpha,\beta}\|_{C^{m-1,\gamma}(\overline{\Omega}_k)} \le K.$$

Then, from (2.48), we get for $k \to \infty$ since $2m - |\alpha| - |\beta| > 0$:

$$(-\Delta_\eta)^m \overline{G}(\xi,\eta) = 0 \qquad \text{in } \mathcal{H} \setminus \{\xi\}.$$

Thus, in the limiting process, the lower order perturbations of the polyharmonic operator vanish.

Step 3. Let us now compute the differential equation satisfied by $\overline{G}$ near $\eta = \xi \in \mathcal{H}$.

Let $\Psi \in C_c^\infty(\mathcal{H})$ and $r_2 > 0$ such that $\operatorname{supp}\Psi \subset B_{r_2}(\xi) \subset \mathcal{H}$. For k large enough we have $B_{r_2}(\xi) \subset \Omega_k$ for all k. For $x \in \Omega$ and k large we define

$$\Psi_k(x) := \Psi\left(\frac{x - \tilde{x}_k}{|x_k - y_k|}\right),\ \ \Psi_k \in C_c^\infty(\Omega).$$

Using the representation formula (2.3), we have

$$\begin{aligned}
\Psi(\xi) &= \Psi_k(\tilde{x}_k + |x_k - y_k|\xi)\\
&= \int_\Omega (-\Delta_y)^m \Psi_k(y) G(\tilde{x}_k + |x_k - y_k|\xi, y)\, dy\\
&\quad + \sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D_y^\beta \Psi_k(y) D_y^\alpha G(\tilde{x}_k + |x_k - y_k|\xi, y)\, dy\\
&= \int_\Omega ((-\Delta_\eta)^m \Psi)\left(\frac{y - \tilde{x}_k}{|x_k - y_k|}\right) |x_k - y_k|^{-2m} G(\tilde{x}_k + |x_k - y_k|\xi, y)\, dy\\
&\quad + \sum_* (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) \left(D_\eta^\beta \Psi\right)\left(\frac{y - \tilde{x}_k}{|x_k - y_k|}\right) |x_k - y_k|^{-|\beta|} D_y^\alpha G(\tilde{x}_k + |x_k - y_k|\xi, y)\, dy.
\end{aligned}$$

Changing the variables shows that

$$\begin{aligned}
\Psi(\xi) &= \int_{\Omega_k} (-\Delta_\eta)^m \Psi\,(\eta)\, G(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta)\, |x_k - y_k|^{n-2m} d\eta\\
&+ \sum_* (-1)^{|\beta|} \int_{\Omega_k} a^{\ell,k}_{\alpha,\beta}(\eta) D_\eta^\beta \Psi\,(\eta)\, D_\eta^\alpha G(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta) |x_k - y_k|^{n-|\alpha|-|\beta|} d\eta.
\end{aligned}$$

Now, substitution and integration by parts gives

$$\begin{aligned}\Psi(\xi) &= \int_{B_{r_2}(\xi)} (-\Delta_\eta)^m \Psi(\eta)\, G_k(\xi,\eta)\, d\eta \\ &\quad + \sum_* (-1)^{|\beta|} \int_{B_{r_2}(\xi)} a^{\ell,k}_{\alpha,\beta}(\eta) D^\beta_\eta \Psi(\eta)\, D^\alpha_\eta G_k(\xi,\eta)\, |x_k - y_k|^{2m-|\alpha|-|\beta|} d\eta \\ &= \int_{B_{r_2}(\xi)} (-\Delta_\eta)^m \Psi(\eta)\, G_k(\xi,\eta)\, d\eta \\ &\quad + \sum_* \int_{B_{r_2}(\xi)} D^\alpha_\eta \left(a^{\ell,k}_{\alpha,\beta}(\eta) D^\beta_\eta \Psi(\eta)\right) G_k(\xi,\eta)\, |x_k - y_k|^{2m-|\alpha|-|\beta|} d\eta \end{aligned} \tag{2.52}$$

Since $G_k(\xi, \,.\,) \to \overline{G}(\xi, \,.\,)$ locally uniformly, moreover $G_k(\xi, \,.\,) \to \overline{G}(\xi, \,.\,)$ in L^1, and since the coefficients are uniformly bounded, we find for $k \to \infty$:

$$\Psi(\xi) = \int_{\mathcal{H}} (-\Delta_\eta)^m \Psi(\eta)\, \overline{G}(\xi,\eta)\, d\eta,$$

i.e.

$$(-\Delta_\eta)^m \overline{G}(\xi, \,.\,) = \delta_\xi \qquad \text{in } \mathcal{H}.$$

Remark. Note, since we will prove in step 5 below that any limit of any converging subsequence has to be the unique polyharmonic Green function in $\mathcal{H}$, we have convergence of the whole sequence to the limit function of step 2.

Step 4. Let us now prove which boundary conditions are attained by $\overline{G}$.

Let $\xi \in \mathcal{H}$ and $\eta_0 \in \overline{\mathcal{H}}$. Let us choose $\delta > 0$ such that $B_\delta(\eta_0) \cap \overline{\mathcal{H}} \subset \overline{\mathcal{H}} \setminus \{\xi\}$. Let $r_3 > 0$ such that $\left(B_\delta(\eta_0) \cap \overline{\mathcal{H}}\right) \subset \left(B_{r_3}(0) \cap \overline{\mathcal{H}}\right)$. We use the local parametrisation Φ_k of Ω_k from Section 1.3 to define a sequence of functions by

$$\tilde{G}_k(\xi, \,.\,) := G_k(\xi, \Phi_k(\,.\,)).$$

Since the domains Ω_k exhaust the whole $\mathcal{H}$ and $\Phi_k \to \mathrm{Id}$ uniformly we have for all k large enough

$$\xi \in \Omega_k \qquad \text{and} \qquad \|\Phi_k - \mathrm{Id}\,\|_{C^{2m,\gamma}(B_{r_3}(0)\cap\overline{\mathcal{H}})} \le \frac{\delta}{16}.$$

This shows

$$\Phi_k\left(B_{\delta/16}(\eta_0) \cap \overline{\mathcal{H}}\right) \subset \left(B_{\delta/8}(\eta_0) \cap \overline{\Omega}_k\right).$$

Next, by choosing $\eta' \in \mathcal{H}$ such that $\eta' \in B_{\delta/8}(\eta_0) \cap \Omega_k$ for all k large enough we can conclude that $B_{\delta/8}(\eta_0) \subset B_{\delta/4}(\eta')$. For all k we have that $\|\Phi_k\|_{C^{2m,\gamma}(B_{r_3}(0)\cap\overline{\mathcal{H}})}$ is uniformly bounded by a constant $C > 0$ and we find that

$$\begin{aligned}\|\tilde{G}_k(\xi,\ .\ )\|_{C^{2m,\gamma}(B_{\delta/16}(\eta_0)\cap\overline{\mathcal{H}})} &\leq C\|G_k(\xi,\Phi_k(\ .\ ))\|_{C^{2m,\gamma}(B_{\delta/16}(\eta_0)\cap\overline{\mathcal{H}})}\\ &\leq C\|G_k(\xi,\ .\ )\|_{C^{2m,\gamma}(B_{\delta/8}(\eta_0)\cap\overline{\Omega}_k)}\\ &\leq C\|G_k(\xi,\ .\ )\|_{C^{2m,\gamma}(B_{\delta/4}(\eta')\cap\overline{\Omega}_k)}\\ &\leq C\|G_k(\xi,\ .\ )\|_{L^1(B_{\delta/2}(\eta')\cap\Omega_k)}\\ &\leq C(\delta,\eta',\xi,n,m), \end{aligned}\tag{2.53}$$

where the last inequalities follow by local Schauder estimates and (2.47). Because of this uniform bound we can apply the Arzela-Ascoli theorem and conclude the existence of a subsequence converging locally uniformly to a function $\tilde{G} \in C^{2m}(\overline{\mathcal{H}} \setminus \{\xi\})$. In fact, the whole sequence converges, see the remark before this step.

Moreover, we have that

$$\tilde{G}(\xi,\eta) = \partial_{\eta_1}\tilde{G}(\xi,\eta) = \ldots = \partial_{\eta_1}^{(m-1)}\tilde{G}(\xi,\eta) = 0$$

for all $\xi \in \mathcal{H}$ and all $\eta \in \partial\mathcal{H}$.

With the same techniques, i.e. deriving uniform bounds as in (2.53), but now considering $G_k \circ (\Phi_k, \Phi_k)$ we can prove that $G_k \circ (\Phi_k, \Phi_k) \to \tilde{G}$ in $C^{2m}_{loc}\left(\overline{\mathcal{H}} \times \overline{\mathcal{H}} \setminus \{(\xi,\xi) : \xi \in \overline{\mathcal{H}}\}\right)$.

In this step it is left to prove $\overline{G}(\xi,\eta) = \tilde{G}(\xi,\eta)$ for all $\xi,\eta \in \mathcal{H}$.

Let $\xi_0,\eta_0 \in \mathcal{H}$ with $\xi_0 \neq \eta_0$ and let $0 < \delta < \frac{1}{2}\min\{|\xi_0 - \eta_0|, d(\xi_0), d(\eta_0)\}$. We define $\Omega' := B_\delta(\xi_0) \cup B_\delta(\eta_0)$. For k large enough we have that $\overline{B_\delta(\xi_0)}, \overline{B_\delta(\eta_0)} \subset \Omega_k$. Let $r_4 > 0$ such that $\overline{\Omega'} \subset B_{r_4}(0)$. Since $\Phi_k \to \mathrm{Id}$ uniformly in $\mathcal{H}$ we find for $\varepsilon > 0$ with $\varepsilon \leq \frac{\delta}{4}$ for k large enough

$$\|\Phi_k - \mathrm{Id}\,\|_{C^{2m,\gamma}(B_{r_4}(0)\cap\overline{\mathcal{H}})} \leq \varepsilon.$$

Then, for $\xi \in B_{\delta/4}(\xi_0)$ and $\eta \in B_{\delta/4}(\eta_0)$, we have

$$\begin{aligned}&|G_k(\xi,\eta) - \tilde{G}_k(\xi,\eta)|\\ &= |G_k(\xi,\eta) - G_k(\Phi_k(\xi),\Phi_k(\eta))|\\ &\leq |G_k(\xi,\eta) - G_k(\Phi_k(\xi),\eta)| + |G_k(\Phi_k(\xi),\eta) - G_k(\Phi_k(\xi),\Phi_k(\eta))|\\ &\leq C\,\|G_k(\ .\ ,\eta)\|_{C^1(\overline{B_{\delta/2}(\xi_0)})}\,|\xi - \Phi_k(\xi)| + C\,\|G_k(\Phi_k(\xi),\ .\ )\|_{C^1(\overline{B_{\delta/2}(\eta_0)})}\,|\eta - \Phi_k(\eta)|\\ &\leq C\left(\|G_k(\ .\ ,\eta)\|_{C^1(\overline{B_{\delta/2}(\xi_0)})} + \|G_k(\Phi_k(\xi),\ .\ )\|_{C^1(\overline{B_{\delta/2}(\eta_0)})}\right)\cdot\varepsilon. \end{aligned}$$

By using local Schauder estimates and the uniform estimates (2.47) it follows

$$|G_k(\xi,\eta) - \tilde{G}_k(\xi,\eta)| \leq C(\xi_0,\eta_0,\delta)\varepsilon$$

for all k large enough. And this shows that $\overline{G}(\xi,\eta) = \tilde{G}(\xi,\eta)$ for $\xi,\eta \in \mathcal{H}$.

From now on we denote by $\overline{G}$ the local uniform limit of the functions G_k extended up to the boundary, i.e. with the zero boundary values.

Step 5. To finish the proof we have to show that

$$\overline{G}(\xi,\eta) = G_{(-\Delta)^m,\mathcal{H}}(\xi,\eta)$$

for all $\xi,\eta \in \mathcal{H}$ with $\xi \neq \eta$.

In order to show this we define $\Psi := \Psi_\xi := \overline{G}(\xi,\ .\ ) - G_{(-\Delta)^m,\mathcal{H}}(\xi,\ .\ )$. Since $\overline{G}(\xi,\ .\ )$ and $G_{(-\Delta)^m,\mathcal{H}}(\xi,\ .\ )$ satisfy the polyharmonic equation with the δ-distribution δ_ξ as right hand side and zero Dirichlet boundary conditions on $\{\eta_1 = 0\}$, we have that $\Psi \in C^\infty(\overline{\mathcal{H}})$ solves

$$\begin{cases} (-\Delta)^m \Psi = 0 & \text{in } \mathcal{H}, \\ \Psi = \partial_{\eta_1}\Psi = \ldots = \partial_{\eta_1}^{(m-1)}\Psi = 0 & \text{on } \{\eta_1 = 0\}. \end{cases} \tag{2.54}$$

Then it follows for all $\eta \in \mathcal{H}$ that

$$|\Psi(\eta)| \leq C \begin{cases} |\eta|^{2m-n} & \text{if } n > 2m, \\ 1 + |\log|\eta|| & \text{if } n = 2m, \\ 1 + |\eta| & \text{if } n = 2m-1, \end{cases} \tag{2.55}$$

and

$$|\nabla\Psi(\eta)| \leq C \begin{cases} |\eta|^{-1} & \text{if } n = 2m, \\ 1 & \text{if } n = 2m-1, \end{cases} \tag{2.56}$$

where $C = C(\xi)$. With the reflection principle for polyharmonic functions from [38] we have with $\overline{\eta} := (\eta_2, \ldots, \eta_n)$ that

$$\Psi^*(\eta) = \begin{cases} \Psi(\eta) & \text{if } \eta_1 \leq 0, \\ \displaystyle\sum_{j=0}^{m-1} \frac{\eta_1^{m+j}}{(j!)^2} \Delta^j \left(\frac{\Psi(-\eta_1, \overline{\eta})}{(-\eta_1)^{m-j}} \right) & \text{if } \eta_1 > 0. \end{cases}$$

Here, $\Psi^* \in C^{2m}(\mathbb{R}^n)$ is an entire polyharmonic function if Ψ/η_1^{m-1} assumes 0 on $\{\eta_1 = 0\}$. Since $\partial_{\eta_1}^{(m-1)}\Psi = 0$ on $\{\eta_1 = 0\}$ this condition is fulfilled.

Let us consider the case $n > 2m$. We prove the following.

For all $\eta \in \mathcal{H}$ and for all $j = 1, \ldots, 2m-2$ we have that

$$|\nabla^j \Psi(\eta)| \leq C|\eta|^{2m-n-j}, \tag{2.57}$$

where $C = C(\xi)$.

Assume by contradiction that there exists a sequence $(\eta_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{H}$ such that

$$|\nabla^j \Psi(\eta_\ell)||\eta_\ell|^{n-2m+j} \to \infty$$

for $\ell \to \infty$. Let $\eta_{\ell,1}$ be the first component of η_ℓ and

$$\tilde{\Psi}_\ell(\eta) := |\eta_\ell|^{n-2m}\Psi(\eta_\ell - \eta_{\ell,1}e_1 + |\eta_\ell|\eta).$$

Then $\tilde{\Psi}_\ell$ solves

$$\begin{cases} (-\Delta)^m\tilde{\Psi}_\ell = 0 & \text{in } \mathcal{H}, \\ \tilde{\Psi}_\ell = \partial_{\eta_1}\tilde{\Psi}_\ell = \ldots = \partial_{\eta_1}^{(m-1)}\tilde{\Psi}_\ell = 0 & \text{on } \{\eta_1 = 0\}. \end{cases}$$

With the assumption (2.57) we find that

$$\left|\nabla^j\tilde{\Psi}_\ell\left(\frac{\eta_{\ell,1}}{|\eta_\ell|}e_1\right)\right| = |\eta_\ell|^{n-2m+j}|\nabla^j\Psi(\eta_\ell)| \to \infty. \tag{2.58}$$

By using estimate (2.55) we obtain

$$|\tilde{\Psi}_\ell(\eta)| \le C|\eta_\ell|^{n-2m}\,|\eta_\ell - \eta_{\ell,1}e_1 + |\eta_\ell|\eta|^{2m-n} = C\left|\frac{\eta_\ell}{|\eta_\ell|} + \eta - \frac{\eta_{\ell,1}}{|\eta_\ell|}e_1\right|^{2m-n},$$

and we find that $\tilde{\Psi}_\ell$ is bounded in a neighbourhood of $\frac{\eta_{\ell,1}}{|\eta_\ell|}e_1$ in $\overline{\mathcal{H}}$. Therefore, with local Schauder estimates, cf. [3] or [23, Theorem 2.19], it follows that

$$\left|\nabla^j\tilde{\Psi}_\ell\left(\frac{\eta_{\ell,1}}{|\eta_\ell|}e_1\right)\right| \le C.$$

This contradicts (2.58) and completes the proof of (2.57).

Let us show that for all $\eta \in \mathbb{R}^n$ we have

$$|\Psi^*(\eta)| \le C|\eta|^{2m-n}. \tag{2.59}$$

If $\eta_1 \le 0$, (2.59) follows from (2.55). Let us take $\eta_1 > 0$. We use

$$\Delta^j u = \sum_{\ell_1+\ldots+\ell_n=j}\frac{j!}{\ell_1!\ldots\ell_n!}\frac{\partial^{2j}}{\partial\eta_1^{2\ell_1}\ldots\partial\eta_n^{2\ell_n}}u,$$

which can be seen by induction, cf. [23, p. 28], and the general Leibniz rule for functions,

$$(uv)^{(2\ell_1)} = \sum_{k=0}^{2\ell_1}\binom{2\ell_1}{k}u^{(k)}v^{(2\ell_1-k)}, \tag{2.60}$$

to obtain

$$\begin{aligned}\Psi^*(\eta) &= \sum_{j=0}^{m-1}\frac{\eta_1^{m+j}}{(j!)^2}\Delta^j\left(\frac{\Psi(-\eta_1,\overline{\eta})}{(-\eta_1)^{m-j}}\right)\\ &= \sum_{j=0}^{m-1}\frac{\eta_1^{m+j}}{(j!)^2}\sum_{\ell_1+\ldots+\ell_n=j}(-1)^{j-m}\frac{j!}{\ell_1!\ldots\ell_n!}\frac{\partial^{2j-2\ell_1}}{\partial\eta_2^{2\ell_2}\ldots\partial\eta_n^{2\ell_n}}\left(\frac{\partial^{2\ell_1}}{\partial\eta_1^{2\ell_1}}\left(\Psi\eta_1^{j-m}\right)\right)\\ &= \sum_{j=0}^{m-1}\sum_{\ell_1+\ldots+\ell_n=j}\sum_{k=0}^{2\ell_1}c_{j,\ell_1,k}(m)\frac{\partial^{2j-2\ell_1}}{\partial\eta_2^{2\ell_2}\ldots\partial\eta_n^{2\ell_n}}\frac{\partial^k\Psi}{\partial\eta_1^k}\eta_1^{2j-2\ell_1+k},\end{aligned} \tag{2.61}$$

where $|c_{j,\ell_1,k}(m)| \le C(m)$. Then

$$|\Psi^*(\eta)| \le C(m) \sum_{j=0}^{2m-2} |\nabla^j \Psi(\eta)||\eta|^j \tag{2.62}$$

and by using (2.57) the estimate (2.59) follows.

Moreover, (2.59) shows that Ψ^* is a bounded entire function. With Liouville's theorem for polyharmonic functions from [50, p. 19] and the boundary conditions for Ψ we have $\Psi^*(\eta) \equiv 0$ and the proposition for the case $n > 2m$ is proved.

It is left to show $\Psi^*(\eta) \equiv 0$ for the cases $n = 2m$ and $n = 2m-1$. We prove the following.

For all $\eta \in \mathcal{H}$ and for all $j = 0, \dots, 2m-2$ we have

$$|D^{2+j}\Psi(\eta)| \le C|\eta|^{2m-n-2-j}, \tag{2.63}$$

where $C = C(\xi)$.

The proof for (2.63) is similar to the case $n > 2m$. Assume that there exists a sequence $(\eta_\ell)_{\ell \in \mathbb{N}} \subset \mathcal{H}$ such that

$$|D^{2+j}\Psi(\eta_\ell)||\eta_\ell|^{2+j+n-2m} \to \infty \tag{2.64}$$

for $\ell \to \infty$. Let us define

$$\tilde{\Psi}_\ell(\eta) := |\eta_\ell|^{n-2m}\Psi(\eta_\ell - \eta_{\ell,1}e_1 + |\eta_\ell|\eta).$$

Then we have that $\tilde{\Psi}_\ell$ solves

$$\begin{cases} (-\Delta)^m \tilde{\Psi}_\ell = 0 & \text{in } \mathcal{H}, \\ \tilde{\Psi}_\ell = \partial_{\eta_1}\tilde{\Psi}_\ell = \ldots = \partial_{\eta_1}^{(m-1)}\tilde{\Psi}_\ell = 0 & \text{on } \{\eta_1 = 0\}. \end{cases}$$

With the assumption (2.64) we find that

$$\left| D^{2+j}\tilde{\Psi}_\ell\left(\frac{\eta_{\ell,1}}{|\eta_\ell|}e_1\right)\right| = |\eta_\ell|^{2+j+n-2m}|D^{2+j}\Psi(\eta_\ell)| \to \infty. \tag{2.65}$$

Using estimate (2.56) shows

$$|\nabla\tilde{\Psi}_\ell(\eta)| \le C|\eta_\ell|^{1+n-2m}\,|\eta_\ell - \eta_{\ell,1}e_1 + |\eta_\ell|\eta|^{2m-n-1} = C\left|\frac{\eta_\ell}{|\eta_\ell|} + \eta - \frac{\eta_{\ell,1}}{|\eta_\ell|}e_1\right|^{2m-n-1},$$

and we see that $\nabla\tilde{\Psi}_\ell$ is uniformly bounded outside $\frac{\eta_\ell}{|\eta_\ell|} - \frac{\eta_{\ell,1}}{|\eta_\ell|}e_1$. We can find a path from a neighbourhood from $\frac{\eta_{\ell,1}}{|\eta_\ell|}e_1$ to the boundary $\partial\mathcal{H}$ staying outside $\frac{\eta_\ell}{|\eta_\ell|} - \frac{\eta_{\ell,1}}{|\eta_\ell|}e_1$. Using the mean value theorem on this path and that $\tilde{\Psi}_\ell$ vanishes on $\partial\mathcal{H}$, we get that $\tilde{\Psi}$ is bounded in a neighbourhood of $\frac{\eta_{\ell,1}}{|\eta_\ell|}e_1$ in $\overline{\mathcal{H}}$. By local Schauder estimates we have a contradiction to (2.65) and the claim is proved.

Now, we prove for all $\eta \in \mathbb{R}^n$ that

$$|D^2\Psi^*(\eta)| \le C\,(1+|\eta|)^{2m-n-2}\,. \tag{2.66}$$

If $\eta_1 \le 0$, (2.66) follows from (2.63).

Let $\eta_1 > 0$. We show (2.66) for $\partial^2_{\eta_1\eta_1}\Psi^*$ since for any other partial derivative of second order we can proceed in the same way. From (2.61) we see that

$$\frac{\partial^2\Psi^*}{\partial\eta_1^2} = \sum_{j=0}^{m-1}\sum_{\ell_1+\ldots+\ell_n=j}\sum_{k=0}^{2\ell_1} c_{j,\ell_1,k}(m)\left(\frac{\partial^{2j-2\ell_1}}{\partial\eta_2^{2\ell_2}\ldots\partial\eta_n^{2\ell_n}}\frac{\partial^k\Psi}{\partial\eta_1^k}\frac{\partial^2}{\partial\eta_1^2}\left(\eta_1^{2j-2\ell_1+k}\right)\right.$$
$$\left.+2\frac{\partial^{2j-2\ell_1}}{\partial\eta_2^{2\ell_2}\ldots\partial\eta_n^{2\ell_n}}\frac{\partial^{k+1}\Psi}{\partial\eta_1^{k+1}}\frac{\partial}{\partial\eta_1}\left(\eta_1^{2j-2\ell_1+k}\right)+\frac{\partial^{2j-2\ell_1}}{\partial\eta_2^{2\ell_2}\ldots\partial\eta_n^{2\ell_n}}\frac{\partial^{k+2}\Psi}{\partial\eta_1^{k+2}}\eta_1^{2j-2\ell_1+k}\right).$$

In each term there are at least two partial derivatives of Ψ since

$$\frac{\partial^2}{\partial\eta_1^2}\left(\eta_1^{2j-2\ell_1+k}\right) \equiv 0 \quad \text{if} \quad 2j-2\ell_1+k<2$$

and

$$\frac{\partial}{\partial\eta_1}\left(\eta_1^{2j-2\ell_1+k}\right) \equiv 0 \quad \text{if} \quad 2j-2\ell_1+k<1.$$

Therefore, we find that

$$|D^2\Psi^*| \le C(m)\sum_{j=0}^{2m-2}|D^{2+j}\Psi||\eta|^j. \tag{2.67}$$

Applying (2.63) to (2.67) and using that $D^2\Psi$ is bounded near zero the claim is proved.

Since Ψ^* is an entire polyharmonic function, also $D\Psi^*$ and $D^2\Psi^*$ are. By using Liouville's theorem and (2.66) we have $D^2\Psi^*(\eta) \equiv 0$. With the boundary conditions for Ψ and $D\Psi$ we get $\Psi^*(\eta) \equiv 0$ and in this way we finally proved the claim of the proposition for $n = 2m$ and $n = 2m-1$. □

3 Pointwise Estimates for Polyharmonic Green Functions

In this chapter, which can be seen as a extension of [29], we focus on proving our main result Theorem 0.1 for the polyharmonic Green's function of the following Dirichlet boundary value problem

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega, \\ \dfrac{\partial^j}{\partial \nu^j} u = 0 & \text{on } \partial\Omega,\ j = 0, \dots, m-1, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^n$ is a $C^{2m,\gamma}$-smooth bounded domain with exterior unit normal ν, $n \geq 2$, $m \geq 2$ and $\gamma \in (0,1)$. For $f \in C^{0,\gamma}(\overline{\Omega})$ the unique solution $u \in C^{2m,\gamma}(\overline{\Omega})$ of (3.1) is given by

$$u(x) = \int_\Omega G_{(-\Delta)^m,\Omega}(x,y) f(y) dy,$$

where we define $G_{(-\Delta)^m,\Omega} : \overline{\Omega} \times \overline{\Omega} \setminus \{(x,x) : x \in \overline{\Omega}\} \to \mathbb{R}$ as the Green function of $(-\Delta)^m$ in the domain Ω with Dirichlet boundary conditions. First we recall the main result Theorem 0.1 in the polyharmonic setting which we will prove in this chapter.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded $C^{2m,\gamma}$-smooth domain, $m \geq 2$. Let $G_{(-\Delta)^m,\Omega}$ denote the polyharmonic Green function in Ω for* (3.1)*. Then there exist constants $c_1 \geq 0$, $c_2 > 0$ and $c_3 > 0$ depending on the domain Ω and m, such that we have the following Green function estimate:*

$$c_2^{-1} H_\Omega(x,y) \leq G_{(-\Delta)^m,\Omega}(x,y) + c_1 \mathbb{1}_{\{|x-y| \geq c_3\}}(x,y) d(x)^m d(y)^m \leq c_2 H_\Omega(x,y) \tag{3.2}$$

for all $x, y \in \Omega$, where

$$H_\Omega(x,y) := \begin{cases} |x-y|^{2m-n} \min\left\{1, \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n > 2m, \\ \log\left(1 + \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \dfrac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} & \text{if } n < 2m, \end{cases} \tag{3.3}$$

and

$$\mathbb{1}_{\{|x-y| \geq c_3\}}(x,y) := \begin{cases} 1 & \text{if } |x-y| \geq c_3, \\ 0 & \text{if } |x-y| < c_3, \end{cases}$$

is the indicator function.

After some auxiliary results we prove the estimates from below in Section 3.1.2 for $n \geq 2m-1$ and in Section 3.2.2 for $n < 2m-1$.

Remark 3.2. For the estimate from above it remains to show that on $\Omega \times \Omega$ we have

$$c_1 d(x)^m d(y)^m \leq c_2 H_\Omega(x,y) \tag{3.4}$$

since for $G_{(-\Delta)^m,\Omega}$ the estimate follows from Corollary 2.12.

Let us prove (3.4).

Case $n > 2m$. Let $d(x)d(y) < |x-y|^2$. Then, since $|x-y| \leq \operatorname{diam}(\Omega)$, we get

$$|x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} = \frac{d(x)^m d(y)^m}{|x-y|^n} \geq \frac{d(x)^m d(y)^m}{(\operatorname{diam}(\Omega))^n}.$$

For $d(x)d(y) \geq |x-y|^2$ we see that

$$\begin{aligned} |x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} &= |x-y|^{2m-n} \geq (\operatorname{diam}(\Omega))^{2m-n} \frac{d(x)^m d(y)^m}{(\operatorname{diam}(\Omega))^{2m}} \\ &\geq \frac{d(x)^m d(y)^m}{(\operatorname{diam}(\Omega))^n}. \end{aligned}$$

Case $n = 2m$. From [23, Lemma 4.5], which states

$$\log\left(1 + \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) \geq c \log\left(2 + \frac{d(y)}{|x-y|}\right) \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\},$$

we get with $d(y)/|x-y| \geq 0$

$$\log\left(1 + \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) \geq c \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}.$$

Examining the same cases as in $n > 2m$ and using $|x-y| \leq \operatorname{diam}(\Omega)$ we get the desired inequality.

Case $n < 2m$. Let $d(x)d(y) < |x-y|^2$. Then, since $|x-y| \leq \operatorname{diam}(\Omega)$, we get

$$d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} = \frac{d(x)^m d(y)^m}{|x-y|^n} \geq \frac{d(x)^m d(y)^m}{(\operatorname{diam}(\Omega))^n}.$$

For $d(x)d(y) \geq |x-y|^2$ we see with $d(x)d(y) \leq (\operatorname{diam}(\Omega))^2$ that

$$d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} = d(x)^{m-n/2} d(y)^{m-n/2} \geq \frac{d(x)^m d(y)^m}{(\operatorname{diam}(\Omega))^n},$$

which completes the proof.

Remark 3.3. With the help of Theorem 3.1 a uniform local positivity result, cf. (LP) in the introduction, follows. For any $x, y \in \Omega$ with $x \neq y$ we have that

$$|x-y| < c_3 \quad \text{implies} \quad G_{(-\Delta)^m,\Omega}(x,y) \geq c_2^{-1} H_\Omega(x,y) > 0,$$

where the constant c_3 is chosen as in Theorem 3.1.

3.1 Large Dimensions $n \geq 2m-1$

Here we want to prove Theorem 3.1 for $n \geq 2m-1$. We proceed in several steps. First, we prove estimates from below for the polyharmonic Green function $G_{(-\Delta)^m,\Omega}(x,y)$ if $x, y \in \Omega$ are closer to each other compared to their boundary distances. This is due to Proposition 3.4. For the opposite case we prove Proposition 3.5. After this is done, we are able to prove Theorem 3.1 in Section 3.1.2.

3.1.1 Some Auxiliary Results for $n \geq 2m-1$

We start with the following local estimate. Note that this result was already stated in [29, Proposition 3] for the biharmonic Green function.

Proposition 3.4. *Let $m \geq 2$, $n \geq 2m-1$. Then there exist constants $\delta_{m,n} > 0$ and $c_4 > 0$, which depend only on the dimension n and the order of the polyharmonic operator, such that the following holds true. Assume $\Omega \subset \mathbb{R}^n$ to be a $C^{2m,\gamma}$-smooth bounded domain and let $G_{(-\Delta)^m,\Omega}$ denote the Green function for the polyharmonic operator under Dirichlet boundary conditions. If*

$$|x-y| \leq \delta_{m,n} \max\{d(x), d(y)\},$$

then we have

$$G_{(-\Delta)^m,\Omega}(x,y) > \begin{cases} c_4 |x-y|^{2m-n} & \text{if } n > 2m, \\ c_4 \log\left(1 + \dfrac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ c_4 d(x)^{1/2} d(y)^{1/2} & \text{if } n = 2m-1. \end{cases} \tag{3.5}$$

Proof. The main part of the proof was done in [32] for $n > 2m$ and in [41] for $n = 2m$ developing ideas from Nehari [49] in dimension $n = 3$. Without loss of generality we are in the following situation for some $R > 1$

$$B_1 = B_1(0) \subset \Omega \subset B_R = B_R(0).$$

Note that the estimate (3.5) is invariant with respect to translation and scaling.

The Green function $G_{(-\Delta)^m,\Omega}$ can be decomposed into the fundamental solution of the polyharmonic operator $(-\Delta)^m$ in $\mathbb{R}^n$ and a polyharmonic function $H_{(-\Delta)^m,\Omega}(x,y) \in C^{2m,\gamma}(\overline{\Omega}^2)$ as

$$G_{(-\Delta)^m,\Omega}(x,y) = F_{m,n}(x-y) + H_{(-\Delta)^m,\Omega}(x,y)$$

with

$$F_{m,n}(x) = \begin{cases} c_{m,n}|x|^{2m-n} & \text{if } n > 2m \text{ or } n = 2m-1, \\ -2c_{m,n}\log|x| & \text{if } n = 2m, \end{cases} \tag{3.6}$$

and

$$c_{m,n} = \begin{cases} \dfrac{2\Gamma(n/2-m)}{ne_n 4^m \Gamma(n/2)(m-1)!} & \text{if } n > 2m \text{ or } n = 2m-1, \\ \dfrac{1}{8ne_n 4^{m-2}((m-1)!)^2} & \text{if } n = 2m, \end{cases}$$

cf. (1.1). Let us recall Boggio's formula for the Green function of the Dirichlet problem with $\Omega = B_1$

$$G_{(-\Delta)^m,B_1}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (v^2-1)^{m-1}v^{1-n}dv, \tag{3.7}$$

where $k_{m,n} = 1/\left(ne_n 4^{m-1}((m-1)!)^2\right)$, see (1.8).

For $n > 2m$ or $n = 2m$ we have for all $x, y \in B_1$ with $x \neq y$ that

$$\begin{aligned} &G_{(-\Delta)^m,\Omega}(x,y) \\ &\quad \geq \frac{1}{4}(H_{(-\Delta)^m,B_1}(x,x) - H_{(-\Delta)^m,B_R}(x,x) + H_{(-\Delta)^m,B_1}(y,y) - H_{(-\Delta)^m,B_R}(y,y)) \\ &\qquad + \frac{1}{2}(G_{(-\Delta)^m,B_1}(x,y) + G_{(-\Delta)^m,B_R}(x,y)), \end{aligned} \tag{3.8}$$

see [32, Lemma 5] for $n > 2m$ or [41, Satz 2] for $n = 2m$. Since the following identities hold

$$G_{(-\Delta)^m,B_R}(x,y) = R^{2m-n}G_{(-\Delta)^m,B_1}\left(\frac{1}{R}x, \frac{1}{R}y\right), \tag{3.9}$$

$$\begin{aligned} &H_{(-\Delta)^m,B_R}(x,x) \\ &\quad = k_{m,n} \cdot \begin{cases} -\dfrac{1}{n-2m}\left(R - \dfrac{|x|^2}{R}\right)^{2m-n} & \text{if } n > 2m \text{ or } n = 2m-1, \\ \log\left(R - \dfrac{|x|^2}{R}\right) - \displaystyle\sum_{j=1}^{m-1}\binom{m-1}{j}\frac{(-1)^j}{-2j} & \text{if } n = 2m, \end{cases} \end{aligned} \tag{3.10}$$

$$c_{m,n} = k_{m,n} \cdot \begin{cases} \dfrac{2^{m-1}(m-1)!}{\prod_{j=1}^{m}(n-2j)} & \text{if } n > 2m \text{ or } n = 2m-1, \\ \dfrac{1}{2} & \text{if } n = 2m, \end{cases}$$

we observe from (3.8) by letting $R \to \infty$ if $n > 2m$ that

$$G_{(-\Delta)^m,\Omega}(0,y) \geq \frac{c_{m,n}}{2}|y|^{2m-n} - \frac{k_{m,n}}{4(n-2m)}\left(1 + \left(1-|y|^2\right)^{2m-n}\right),$$

cf. [32, eq. (25)].

To prove the proposition, it is enough, by scaling and translation, to consider $x = 0$ and $y \in B_{\delta_{m,n}}(0)$, where we specify $\delta_{m,n} \in (0,1)$ below.

For $n > 2m$ we have

$$G_{(-\Delta)^m,\Omega}(0,y) \geq |y|^{2m-n} k_{m,n} \underbrace{\left(\frac{2^{m-2}(m-1)!}{\prod\limits_{j=1}^{m}(n-2j)} - \frac{1}{4(n-2m)} \left(|y|^{n-2m} + \left(\frac{|y|}{1-|y|^2} \right)^{n-2m} \right) \right)}_{=: f_{m,n}(|y|)}.$$

Since $f_{m,n}$ is positive near 0 and monotonically decreasing in $|y|$, we can find a $\delta_{m,n}$ such that $f_{m,n}$ is positive for all $|y| \in (0, \delta_{m,n})$. For a further discussion how to choose $\delta_{m,n}$ see [32, Theorem 3].

In the following, in addition to $x = 0$, $y \in B_{\delta_{m,n}}(0)$, we assume without loss of generality $d(0) = 1$. Then we have that

$$d(y) \leq |y| + d(0) < \delta_{m,n} + 1 < 2. \tag{3.11}$$

Let $n = 2m$. Using (3.7)-(3.10) we see that

$$\begin{aligned} G_{(-\Delta)^m,\Omega}(0,y) \geq{}& \frac{k_{m,2m}}{4} \left(-2\log R + \log\left(1-|y|^2\right) - \log\left(1 - \frac{|y|^2}{R^2}\right) \right) \\ &+ \frac{k_{m,2m}}{2} \left(\sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j \frac{|y|^{2j}}{-2j} + 2\sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j \frac{1}{2j} - \log|y| \right. \\ &\left. + \sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j \frac{|y|^{2j}}{-2jR^{2j}} + \log\left(\frac{R}{|y|}\right) \right), \end{aligned}$$

from which we get

$$\begin{aligned} G_{(-\Delta)^m,\Omega}(0,y) \geq{}& \frac{k_{m,2m}}{4} \left(\log\left(1-|y|^2\right) - \log\left(1 - \frac{|y|^2}{R^2}\right) + 2\sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j \frac{|y|^{2j}}{-2j} \right. \\ &\left. + 4\sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j \frac{1}{2j} - 4\log|y| + 2\sum_{j=1}^{m-1} \binom{m-1}{j} (-1)^j \frac{|y|^{2j}}{-2jR^{2j}} \right). \end{aligned}$$

For $R \to \infty$ it follows that

$$\begin{aligned} G_{(-\Delta)^m,\Omega}(0,y) \geq{}& \frac{k_{m,2m}}{4} \left(\log\left(1-|y|^2\right) - \log|y|^4 + \sum_{j=1}^{m-1} \binom{m-1}{j} \frac{(-1)^{j-1}}{j} |y|^{2j} \right. \\ &\left. - 2\sum_{j=1}^{m-1} \binom{m-1}{j} \frac{(-1)^{j-1}}{j} \right). \end{aligned}$$

We use the following combinatorial identity, cf. [26, No. 1.45] and Proposition A.2 in the appendix for a proof, to see that for $m-1 \geq 0$ and $|y| \leq 1$ it holds

$$\sum_{j=1}^{m-1} \binom{m-1}{j} \frac{(-1)^{j-1}}{j} |y|^{2j} = s(m) - \sum_{j=1}^{m-1} \frac{1}{j} \left(1-|y|^2\right)^j, \tag{3.12}$$

where $s(m) := \sum_{j=1}^{m-1} \frac{1}{j}$. Since

$$\sum_{j=1}^{m-1} \frac{1}{j} \left(1-|y|^2\right)^j \leq \sum_{j=1}^{\infty} \frac{1}{j} \left(1-|y|^2\right)^j = -\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left(|y|^2-1\right)^j = -\log|y|^2,$$

we have that

$$\frac{4}{k_{m,2m}} G_{(-\Delta)^m,\Omega}(0,y) \geq \log\left(\frac{1}{|y|^2}-1\right) - s(m). \tag{3.13}$$

For $|y|<1$ we get with (3.11) and $d(x)=d(0)=1$ that

$$\left(1+\frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right)^{\frac{1}{2m}} = \left(1+\frac{d(y)^m}{|y|^{2m}}\right)^{\frac{1}{2m}} < \frac{(1+2^m)^{1/2m}}{|y|}. \tag{3.14}$$

Now we choose $|y| \in (0,1)$ such that

$$0<|y|<-\frac{(1+2^m)^{1/2m}}{2}\exp\left(s(m)\right) + \sqrt{1+\frac{(1+2^m)^{1/m}}{4}\exp\left(2\,s(m)\right)} =: \delta_{m,2m},$$

which gives

$$\frac{(1+2^m)^{1/2m}}{|y|} < \exp(-s(m))\left(\frac{1}{|y|^2}-1\right). \tag{3.15}$$

Inequalities (3.13), (3.14) and (3.15) show for the chosen $|y|$ that

$$G_{(-\Delta)^m,\Omega}(0,y) \geq \frac{k_{m,2m}}{4}\frac{1}{2m}\log\left(1+\frac{d(y)^m}{|y|^{2m}}\right).$$

Let $n=2m-1$. Since the proof of (3.8) is independent of the dimension of the domain, we see that

$$\begin{aligned}G_{(-\Delta)^m,\Omega}(0,y) \geq{}& \frac{k_{m,2m-1}}{4}\left(2-2R-|y|^2+\frac{|y|^2}{R}\right)\\ &+\frac{k_{m,2m-1}}{2}\left(1+\sum_{j=1}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}|y|^{2j} - 2|y|\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}\right.\\ &\left.+R+\sum_{j=1}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}R^{1-2j}|y|^{2j}\right),\end{aligned}$$

from which we get

$$\begin{aligned}G_{(-\Delta)^m,\Omega}(0,y) \geq{}& \frac{k_{m,2m-1}}{4}\left(4-|y|^2+\frac{|y|^2}{R}+2\sum_{j=1}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}|y|^{2j}\right.\\ &\left.-4|y|\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}+2\sum_{j=1}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}R^{1-2j}|y|^{2j}\right).\end{aligned}$$

Taking $R \to \infty$ we obtain

$$G_{(-\Delta)^m,\Omega}(0,y)$$
$$\geq \frac{k_{m,2m-1}}{4}\left(4-|y|^2+2\sum_{j=1}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}|y|^{2j}-4|y|\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}\right).$$

With

$$\left|\sum_{j=1}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1}|y|^{2j}\right| \leq |y|^2\left(2^{m-1}-1\right)$$

and

$$\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{(-1)^{j-1}}{2j-1} = -\frac{\mathrm{B}\left(m,-\frac{1}{2}\right)}{2} = \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m-\frac{1}{2}\right)},$$

where $\mathrm{B}(\,.\,,\,.\,)$ denotes the Beta function, we get

$$G_{(-\Delta)^m,\Omega}(0,y) \geq \frac{k_{m,2m-1}}{4}\left(4+|y|^2\left(1-2^m\right)-4|y|\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m-\frac{1}{2}\right)}\right).$$

If we choose

$$\delta_{m,2m-1} := \frac{2}{1-2^m}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m-\frac{1}{2}\right)} + \sqrt{\left(\frac{2}{1-2^m}\frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m-\frac{1}{2}\right)}\right)^2 - \frac{4-\sqrt{2}}{1-2^m}},$$

we obtain with (3.11) for $|y| \in (0, \delta_{m,2m-1})$ that

$$\frac{4}{k_{m,2m-1}}G_{(-\Delta)^m,\Omega}(0,y) \geq \sqrt{2} > d(y)^{1/2}.$$

□

Due to Proposition 3.4 we may concentrate in the following on x, y such that

$$x, y \in \Omega,\ x \neq y,\ |x-y| > \delta_{m,n}\max\{d(x), d(y)\} \tag{3.16}$$

where the constant $\delta_{m,n}$ is chosen as in Proposition 3.4.

Lemma 3.5. *Suppose that $n \geq 2m-1$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then for each $x_0 \in \overline{\Omega}$ there exists a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ subject to condition* (3.16) *one has*

$$G_{(-\Delta)^m,\Omega}(x,y) \geq C|x-y|^{-n}d(x)^m d(y)^m. \tag{3.17}$$

The proof is inspired by the proof of [29, Lemma 4].

Proof. For $x_0 \in \Omega$ and $x, y \in B_{r_{x_0}}(x_0) \subset \Omega$ we can choose r_{x_0} small enough such that

$$|x-y| \le |x-x_0| + |y-x_0| < 2r_{x_0} \le \delta_{m,n} \max\{d(x), d(y)\}.$$

Then condition (3.16) is violated and (3.17) always holds.

From now on let $x_0 \in \partial\Omega$ and we assume by contradiction that there exist $x_k, y_k \in \Omega_{x_0,1/k} = \overline{\Omega} \cap B_{1/k}(x_0)$ subject to condition (3.16) such that

$$G_{(-\Delta)^m,\Omega}(x_k, y_k) < \frac{1}{k}|x_k - y_k|^{-n} d(x_k)^m d(y_k)^m. \tag{3.18}$$

For $x_k,\ y_k \in \Omega_{x_0,1/k}$ we find

$$|x_0 - x_k| < \frac{1}{k}, \quad |x_0 - y_k| < \frac{1}{k},$$
$$d(x_k) = \inf_{x^* \in \partial\Omega} |x_k - x^*| \le |x_k - x_0| + \underbrace{\inf_{x^* \in \partial\Omega} |x_0 - x^*|}_{=0} < \frac{1}{k}, \quad d(y_k) < \frac{1}{k},$$
$$|x_k - y_k| \le |x_k - x_0| + |y_k - x_0| \le \frac{2}{k},$$

which imply $x_k \to x_0$, $y_k \to x_0$, $d(x_k) \to 0$, $d(y_k) \to 0$ and $|x_k - y_k| \to 0$ when k tends to infinity.

By rotation and translation we may assume that $x_0 = 0$ and that the first unit vector e_1 is the exterior unit normal to $\partial\Omega$ at x_0. Since $\partial\Omega$ is smooth enough, we can find for k large enough a uniquely determined $\tilde{x}_k \in \partial\Omega$, such that $d(x_k) = |x_k - \tilde{x}_k|$.

For $\xi, \eta \in \Omega_k := \frac{1}{|x_k - y_k|}(-\tilde{x}_k + \Omega)$ we define the rescaled polyharmonic Green function

$$G_k(\xi, \eta) := |x_k - y_k|^{n-2m} G_{(-\Delta)^m,\Omega}(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta)$$

as in Proposition 2.13. Then assumption (3.18) gives

$$G_k(\xi_k, \eta_k) = |x_k - y_k|^{n-2m} G_{(-\Delta)^m,\Omega}(x_k, y_k) < \frac{1}{k}|x_k - y_k|^{-2m} d(x_k)^m d(y_k)^m, \tag{3.19}$$

where

$$\xi_k = \frac{1}{|x_k - y_k|}(x_k - \tilde{x}_k) \qquad \text{and} \qquad \eta_k = \frac{1}{|x_k - y_k|}(y_k - \tilde{x}_k).$$

For ξ_k and η_k from condition (3.16) follows

$$|\xi_k| = \frac{d(x_k)}{|x_k - y_k|} \le \frac{1}{\delta_{m,n}}, \quad |\xi_k - \eta_k| = 1 \quad \text{and} \quad |\eta_k| \le 1 + \frac{1}{\delta_{m,n}}.$$

Therefore, after passing to a further subsequence we find $\xi_\infty, \eta_\infty \in \overline{\mathcal{H}}$ with $\xi_\infty = \lim_{k\to\infty} \xi_k$, $\eta_\infty = \lim_{k\to\infty} \eta_k$ and $|\xi_\infty - \eta_\infty| = 1$. To show a contradiction we prove the following claim:

Claim. There exists a constant $\sigma > 0$ such that for k large enough it yields

$$G_k(\xi_k, \eta_k) \geq \sigma d_k(\xi_k)^m d_k(\eta_k)^m = \sigma \left(\frac{d(x_k)}{|x_k - y_k|}\right)^m \left(\frac{d(y_k)}{|x_k - y_k|}\right)^m,$$

where $d_k := d(\,.\,, \partial\Omega_k)$.

We prove the claim in the following steps.

Step 1. From Section 1.3 we recall the local $C^{2m,\gamma}$-smooth coordinate charts $\xi' \to \Phi_k(\xi') = \xi$ for Ω_k with coordinates ξ' in bounded neighbourhoods of 0 in $\overline{\mathcal{H}}$. For $\tilde{G}_k := G_k \circ (\Phi_k, \Phi_k)$ in $\overline{\mathcal{H}} \times \overline{\mathcal{H}} \setminus \{\xi' = \eta'\}$ we see as in Proposition 2.13 that $\tilde{G}_k \to G_{(-\Delta)^m, \mathcal{H}}$ in $C^{2m}((\overline{\mathcal{H}} \cap B_{1/4}(\xi_\infty)) \times (\overline{\mathcal{H}} \cap B_{1/4}(\eta_\infty)))$. Since $\Phi_k \to \mathrm{Id}$, we have for k large enough that $\xi'_k = \Phi_k^{-1}(\xi_k) \in \overline{\mathcal{H}} \cap B_{1/4}(\xi_\infty)$ and $\eta'_k = \Phi_k^{-1}(\eta_k) \in \overline{\mathcal{H}} \cap B_{1/4}(\eta_\infty)$.

Let $\mu_k \in \partial\mathcal{H}$ such that

$$d_{\mathcal{H}}(\xi'_k) = \inf_{\mu \in \partial\mathcal{H}} |\xi'_k - \mu| = |\xi'_k - \mu_k|$$

and

$$\omega(t) := t\xi'_k + (1-t)\mu_k$$

for $t \in [0,1]$. Since $\Phi_k \to \mathrm{Id}$ uniformly in $\overline{\mathcal{H}} \cap B_{1/4}(\xi_\infty)$, there exists a positive constant $\sigma_1 > 0$ such that

$$\begin{aligned} d_k(\xi_k) &= \inf_{\xi_k^* \in \partial\Omega_k} |\xi_k - \xi_k^*| \leq |\xi_k - \underbrace{\Phi_k(\mu_k)}_{\in \partial\Omega_k}| \leq L(\Phi_k(\omega)) \leq \|\Phi_k\|_{C^1(\overline{\mathcal{H}} \cap B_{1/4}(\xi_\infty))} L(\omega) \\ &\leq \sigma_1 |\xi'_k - \mu_k| = \sigma_1 d_{\mathcal{H}}(\xi'_k), \end{aligned} \tag{3.20}$$

where L denotes the length of the path. Similarly we get

$$d_k(\eta_k) \leq \sigma_1 d_{\mathcal{H}}(\eta'_k). \tag{3.21}$$

Step 2. Now we use an estimate from below of Boggio's formula for the polyharmonic Green function: For $\xi', \eta' \in \overline{\mathcal{H}}$ there exists a constant $\sigma_2 > 0$ such that

$$G_{(-\Delta)^m, \mathcal{H}}(\xi', \eta') \geq \sigma_2 \begin{cases} |\xi' - \eta'|^{2m-n} \min\left\{1, \dfrac{d_{\mathcal{H}}(\xi')^m d_{\mathcal{H}}(\eta')^m}{|\xi' - \eta'|^{2m}}\right\} & \text{if } n > 2m, \\ \log\left(1 + \dfrac{d_{\mathcal{H}}(\xi')^m d_{\mathcal{H}}(\eta')^m}{|\xi' - \eta'|^{2m}}\right) & \text{if } n = 2m, \\ d_{\mathcal{H}}(\xi')^{\frac{1}{2}} d_{\mathcal{H}}(\eta')^{\frac{1}{2}} \min\left\{1, \dfrac{d_{\mathcal{H}}(\xi')^{m-\frac{1}{2}} d_{\mathcal{H}}(\eta')^{m-\frac{1}{2}}}{|\xi' - \eta'|^{2m-1}}\right\} & \text{if } n = 2m-1, \end{cases} \tag{3.22}$$

which can be proved like [31, Proposition 2.3].

Since $\xi'_k \in \overline{\mathcal{H}} \cap B_{1/4}(\xi_\infty)$ and $\eta'_k \in \overline{\mathcal{H}} \cap B_{1/4}(\eta_\infty)$ we get

$$|\xi'_k - \eta'_k| \leq |\xi_\infty - \eta_\infty| + |\xi_\infty - \xi'_k| + |\eta_\infty - \eta'_k| \leq 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2} \tag{3.23}$$

and

$$|\xi_k' - \eta_k'| \geq |\xi_\infty - \eta_\infty| - |\xi_\infty - \xi_k'| - |\eta_\infty - \eta_k'| \geq 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}. \tag{3.24}$$

Moreover, since $d_{\mathcal{H}}(\xi_k')$ and $d_{\mathcal{H}}(\eta_k')$ are uniformly bounded, we see for all k large enough that

$$\frac{d_{\mathcal{H}}(\xi_k')d_{\mathcal{H}}(\eta_k')}{|\xi_k' - \eta_k'|} \leq C(\xi_\infty, \eta_\infty). \tag{3.25}$$

Combining (3.22)-(3.25) and using $\log(1+z) \geq \frac{z}{1+z}$ for all $z > -1$ in the case $n = 2m$ we obtain for all k large enough

$$G_{(-\Delta)^m,\mathcal{H}}(\xi_k', \eta_k') \geq \sigma_3 d_{\mathcal{H}}(\xi_k')^m d_{\mathcal{H}}(\eta_k')^m, \tag{3.26}$$

where σ_3 is a positive constant.

Step 3. In what follows we prove the claim for the distinct possible locations of $\xi_\infty, \eta_\infty \in \overline{\mathcal{H}}$.

Case $\xi_\infty \in \mathcal{H},\ \eta_\infty \in \mathcal{H}$. Since ξ_∞ and η_∞ are interior points, we have that $d_{\mathcal{H}}(\xi_\infty), d_{\mathcal{H}}(\eta_\infty) > 0$. Let $\varepsilon < \frac{\sigma_3}{2}\left(\frac{d_{\mathcal{H}}(\xi_\infty)}{2}\right)^m\left(\frac{d_{\mathcal{H}}(\eta_\infty)}{2}\right)^m$. Since $\tilde{G}_k \to G_{(-\Delta)^m,\mathcal{H}}$ in $C^{2m}((\overline{\mathcal{H}} \cap B_{1/4}(\xi_\infty)) \times (\overline{\mathcal{H}} \cap B_{1/4}(\eta_\infty)))$, using (3.26) for all k large enough, we have

$$\begin{aligned} G_k(\xi_k, \eta_k) &= \tilde{G}_k(\xi_k', \eta_k') \\ &\geq G_{(-\Delta)^m,\mathcal{H}}(\xi_k', \eta_k') - \varepsilon \\ &\geq \sigma_3 d_{\mathcal{H}}(\xi_k')^m d_{\mathcal{H}}(\eta_k')^m - \frac{\sigma_3}{2}\left(\frac{d_{\mathcal{H}}(\xi_\infty)}{2}\right)^m\left(\frac{d_{\mathcal{H}}(\eta_\infty)}{2}\right)^m. \end{aligned}$$

From $\Phi_k \to \mathrm{Id}$ and $\xi_k \to \xi_\infty, \eta_k \to \eta_\infty$ we get $\xi_k' \to \xi_\infty, \eta_k' \to \eta_\infty$. Then, for k possibly larger, we have $\xi_k' \in B_{d_{\mathcal{H}}(\xi_\infty)/2}(\xi_\infty) \cap \overline{\mathcal{H}}$ and $\eta_k' \in B_{d_{\mathcal{H}}(\eta_\infty)/2}(\eta_\infty) \cap \overline{\mathcal{H}}$. It follows

$$d_{\mathcal{H}}(\xi_k') \geq \frac{1}{2}d_{\mathcal{H}}(\xi_\infty) \qquad \text{and} \qquad d_{\mathcal{H}}(\eta_k') \geq \frac{1}{2}d_{\mathcal{H}}(\eta_\infty). \tag{3.27}$$

Now, using (3.20) and (3.21) we obtain

$$G_k(\xi_k, \eta_k) \geq \frac{\sigma_3}{2} d_{\mathcal{H}}(\xi_k')^m d_{\mathcal{H}}(\eta_k')^m \geq \frac{\sigma_3}{2\sigma_1^{2m}} d_k(\xi_k)^m d_k(\eta_k)^m. \tag{3.28}$$

Case $\xi_\infty \in \mathcal{H},\ \eta_\infty \in \partial\mathcal{H}$. After possibly interchanging ξ_∞ and η_∞ this covers also the situation $\xi_\infty \in \partial\mathcal{H},\ \eta_\infty \in \mathcal{H}$.

We perform a Taylor expansion and see that for a suitable $\theta \in (0,1)$ due to the boundary conditions on $G_{(-\Delta)^m,\mathcal{H}}$ we have that

$$\begin{aligned} &G_{(-\Delta)^m,\mathcal{H}}(\xi_k', \eta_k') \\ &= G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \left(\eta_{k,1}', \overline{\eta_k'}\right)\right) \\ &= \sum_{j=0}^{m-1} \frac{1}{j!}\partial_{\eta_1}^{(j)} G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \left(0, \overline{\eta_k'}\right)\right)(\eta_{k,1}')^j + \frac{1}{m!}\partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \left(\theta\eta_{k,1}', \overline{\eta_k'}\right)\right)(\eta_{k,1}')^m \\ &= \frac{(-1)^m}{m!}\partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \left(\theta\eta_{k,1}', \overline{\eta_k'}\right)\right) d_{\mathcal{H}}(\eta_k')^m. \end{aligned}$$

From (3.26) it follows

$$\frac{(-1)^m}{m!}\partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \left(\theta\eta_{k,1}', \overline{\eta_k'}\right)\right) \geq \sigma_3 d_{\mathcal{H}}(\xi_k')^m. \tag{3.29}$$

Again with a Taylor expansion for a suitable $\tilde{\theta} \in (0,1)$ we have

$$\tilde{G}_k(\xi_k', \eta_k') = \frac{(-1)^m}{m!}\partial_{\eta_1}^{(m)} \tilde{G}_k\left(\xi_k', \left(\tilde{\theta}\eta_{k,1}', \overline{\eta_k'}\right)\right) d_{\mathcal{H}}(\eta_k')^m. \tag{3.30}$$

Since $\tilde{G}_k \to G_{(-\Delta)^m,\mathcal{H}}$ in C^{2m} locally uniformly, we have $\partial_{\eta_1}^{(m)}\tilde{G}_k \to \partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}$ locally uniformly. We choose $\varepsilon < \frac{\sigma_3}{2}\left(\frac{d_{\mathcal{H}}(\xi_\infty)}{2}\right)^m$ and using (3.29) for k large enough we get

$$\begin{aligned}
\frac{(-1)^m}{m!}\partial_{\eta_1}^{(m)} \tilde{G}_k\left(\xi_k', \left(\tilde{\theta}\eta_{k,1}', \overline{\eta_k'}\right)\right) &\geq \frac{(-1)^m}{m!}\partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \left(\theta\eta_{k,1}', \overline{\eta_k'}\right)\right) - \varepsilon \\
&\geq \sigma_3 d_{\mathcal{H}}(\xi_k')^m - \frac{\sigma_3}{2}\left(\frac{d_{\mathcal{H}}(\xi_\infty)}{2}\right)^m \\
&\geq \frac{\sigma_3}{2}\left(d_{\mathcal{H}}(\xi_k')\right)^m,
\end{aligned}$$

since $\xi_k' \in B_{d_{\mathcal{H}}(\xi_\infty)/2}(\xi_\infty) \cap \overline{\mathcal{H}}$ for k large enough. Then, using (3.30), (3.20) and (3.21) we obtain

$$G_k(\xi_k, \eta_k) = \tilde{G}_k(\xi_k', \eta_k') \geq \frac{\sigma_3}{2} d_{\mathcal{H}}(\xi_k')^m d_{\mathcal{H}}(\eta_k')^m \geq \frac{\sigma_3}{2\sigma_1^{2m}} d_k(\xi_k)^m d_k(\eta_k)^m.$$

Case $\xi_\infty \in \partial\mathcal{H},\ \eta_\infty \in \partial\mathcal{H}$. From above for a suitable $\theta_1 \in (0,1)$ we have

$$G_{(-\Delta)^m,\mathcal{H}}(\xi_k', \eta_k') = \frac{1}{m!}\partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \left(\theta_1\eta_{k,1}', \overline{\eta_k'}\right)\right)(\eta_{k,1}')^m.$$

Performing a Taylor expansion for $G_{(-\Delta)^m,\mathcal{H}}(\xi_k', \eta_k')$ with respect to ξ_1' for a suitable $\theta_2 \in (0,1)$ we obtain

$$G_{(-\Delta)^m,\mathcal{H}}(\xi_k', \eta_k') = \frac{1}{m!}\partial_{\xi_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\left(\theta_2\xi_{k,1}', \overline{\xi_k'}\right), \eta_k'\right)(\xi_{k,1}')^m,$$

where we used the boundary conditions of $G_{(-\Delta)^m,\mathcal{H}}$

$$\partial_{\xi_1}^{(j)} G_{(-\Delta)^m,\mathcal{H}}\left(\left(0, \overline{\xi_k'}\right), \eta\right) = 0$$

for all $\eta \in \mathcal{H}$ and $j \in \{0, \dots, m-1\}$. Then

$$\begin{aligned}
&\partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\xi_k', \,.\,\right)\big|_{\eta=\left(\theta_1\eta_{k,1}', \overline{\eta_k'}\right)} \\
&\qquad = \frac{1}{m!}\partial_{\eta_1}^{(m)}\partial_{\xi_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\left(\theta_2\xi_{k,1}', \overline{\xi_k'}\right), \left(\theta_1\eta_{k,1}', \overline{\eta_k'}\right)\right)(\xi_{k,1}')^m,
\end{aligned}$$

which shows

$$G_{(-\Delta)^m,\mathcal{H}}(\xi_k', \eta_k') = \frac{1}{(m!)^2}\partial_{\xi_1}^{(m)}\partial_{\eta_1}^{(m)} G_{(-\Delta)^m,\mathcal{H}}\left(\left(\theta_2\xi_{k,1}', \overline{\xi_k'}\right), \left(\theta_1\eta_{k,1}', \overline{\eta_k'}\right)\right) d_{\mathcal{H}}(\xi_k')^m d_{\mathcal{H}}(\eta_k')^m. \tag{3.31}$$

Now, using (3.26) we have

$$\frac{1}{(m!)^2}\partial_{\xi_1}^{(m)}\partial_{\eta_1}^{(m)}G_{(-\Delta)^m,\mathcal{H}}\left(\left(\theta_2\xi'_{k,1},\overline{\xi'_k}\right),\left(\theta_1\eta'_{k,1},\overline{\eta'_k}\right)\right)\geq\sigma_3.$$

In the same way for suitable $\tilde{\theta}_1,\tilde{\theta}_2\in(0,1)$ we have

$$\tilde{G}_k(\xi'_k,\eta'_k)=\frac{1}{(m!)^2}\partial_{\xi_1}^{(m)}\partial_{\eta_1}^{(m)}\tilde{G}_k\left(\left(\tilde{\theta}_2\xi'_{k,1},\overline{\xi'_k}\right),\left(\tilde{\theta}_1\eta'_{k,1},\overline{\eta'_k}\right)\right)d_{\mathcal{H}}(\xi'_k)^m d_{\mathcal{H}}(\eta'_k)^m.$$

With $\partial_{\xi_1}^{(m)}\partial_{\eta_1}^{(m)}\tilde{G}_k\to\partial_{\xi_1}^{(m)}\partial_{\eta_1}^{(m)}G_{(-\Delta)^m,\mathcal{H}}$ locally uniformly we choose $\varepsilon<\frac{\sigma_3}{2}$ and have for k large enough

$$\begin{aligned}&\frac{1}{(m!)^2}\partial_{\xi_1}^{(m)}\partial_{\eta_1}^{(m)}\tilde{G}_k\left(\left(\tilde{\theta}_2\xi'_{k,1},\overline{\xi'_k}\right),\left(\tilde{\theta}_1\eta'_{k,1},\overline{\eta'_k}\right)\right)\\ &\geq\frac{1}{(m!)^2}\partial_{\xi_1}^{(m)}\partial_{\eta_1}^{(m)}G_{(-\Delta)^m,\mathcal{H}}\left(\left(\theta_2\xi'_{k,1},\overline{\xi'_k}\right),\left(\theta_1\eta'_{k,1},\overline{\eta'_k}\right)\right)-\varepsilon\\ &\geq\frac{\sigma_3}{2}.\end{aligned}$$

Then, using (3.31), (3.20) and (3.21) we obtain

$$G_k(\xi_k,\eta_k)=\tilde{G}_k(\xi'_k,\eta'_k)\geq\frac{\sigma_3}{2}d_{\mathcal{H}}(\xi'_k)^m d_{\mathcal{H}}(\eta'_k)^m\geq\frac{\sigma_3}{2\sigma_1^{2m}}d_k(\xi_k)^m d_k(\eta_k)^m.$$

This proves the claim, i.e. we have a positive constant $\sigma>0$ such that for k large enough

$$\begin{aligned}G_k(\xi_k,\eta_k)\geq&\sigma d_k(\xi_k)^m d_k(\eta_k)^m=\sigma\left(\frac{d(x_k)}{|x_k-y_k|}\right)^m\left(\frac{d(y_k)}{|x_k-y_k|}\right)^m\\ =&\sigma|x_k-y_k|^{-2m}d(x_k)^m d(y_k)^m.\end{aligned}$$

This contradicts (3.19) and the proof of the lemma is complete. □

3.1.2 Proof of the Main Result for $n\geq 2m-1$

Now we are in the position to prove the bounds from below in Theorem 3.1 for $n\geq 2m-1$. This is done as for the biharmonic case in [29].

Proof of Theorem 3.1 for $n\geq 2m-1$. Let us first assume that for $x,y\in\Omega$ with $x\neq y$

$$|x-y|\leq\delta_{m,n}\max\{d(x),d(y)\}\tag{3.32}$$

holds, so we are in the situation of Proposition 3.4.

Without loss of generality let $d(x)=\max\{d(x),d(y)\}$ and $d(y)=\min\{d(x),d(y)\}$, then we get

$$\begin{aligned}(1-\delta_{m,n})|x-y|&\leq\delta_{m,n}(\max\{d(x),d(y)\}-|x-y|)=\delta_{m,n}(d(x)-|x-y|)\\ &\leq\delta_{m,n}(d(y)+|x-y|-|x-y|)=\delta_{m,n}\min\{d(x),d(y)\},\end{aligned}$$

and, by using (3.32), we see

$$\begin{aligned}(1-\delta_{m,n})|x-y|^2 &\leq \delta_{m,n}^2 \min\{d(x),d(y)\}\max\{d(x),d(y)\} \\ &= \delta_{m,n}^2 d(x)d(y).\end{aligned}$$

For $n > 2m$ using Proposition 3.4 and $d(x)d(y)/|x-y|^2 \geq (1-\delta_{m,n})/\delta_{m,n}^2$ we get

$$\begin{aligned}G_{(-\Delta)^m,\Omega}(x,y) &> c_4|x-y|^{2m-n} \\ &\geq c_4|x-y|^{2m-n}\min\left\{1,\frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}.\end{aligned}$$

For the case $n = 2m$ we use Proposition 3.4 to obtain

$$G_{(-\Delta)^m,\Omega}(x,y) > c_4 \log\left(1+\frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right).$$

If $n = 2m-1$ we use Proposition 3.4 to get

$$\begin{aligned}G_{(-\Delta)^m,\Omega}(x,y) &> c_4 d(x)^{1/2}d(y)^{1/2} \\ &\geq c_4 d(x)^{1/2}d(y)^{1/2}\min\left\{1,\frac{d(x)^{m-1/2}d(y)^{m-1/2}}{|x-y|^{2m-1}}\right\}.\end{aligned}$$

Now, it is left to prove the estimates when $x, y \in \Omega$ are subject to condition (3.16). First we show that there exist constants $c_5 > 0, r > 0$ such that $|x-y| < r$ yields

$$G_{(-\Delta)^m,\Omega}(x,y) \geq c_5|x-y|^{-n}d(x)^m d(y)^m.$$

We prove this by a compactness argument.

By Lemma 3.5 there exists for all $x_j \in \overline{\Omega}$ a constant $r_j := r_{x_j}$ such that for all $x, y \in \Omega_{x_j,r_j} := \overline{\Omega} \cap B_{r_j}(x_j)$ the following

$$G_{(-\Delta)^m,\Omega}(x,y) \geq C_{x_j}|x-y|^{-n}d(x)^m d(y)^m$$

holds. Since $\overline{\Omega}$ is compact, there exist $x_1, \dots, x_k \in \overline{\Omega}$ such that $\overline{\Omega} \subset \bigcup_{j=1}^k \Omega_{x_j,r_j/2}$. For $x, y \in \overline{\Omega}$ with $|x-y| < r$, where $r := \min\{\frac{r_1}{2}, \dots, \frac{r_k}{2}\}$, there exists a j such that $x \in \overline{\Omega} \cap B_{r_j/2}(x_j)$. Because of $|x-y| < r$ we get $y \in B_{r_j}(x_j)$ and the claim follows.

If $|x-y| \geq r$ we obtain from Corollary 2.12 and $\log(1+z) \leq z$ for all $z > -1$ that

$$G_{(-\Delta)^m,\Omega}(x,y) \geq -c_6|x-y|^{-n}d(x)^m d(y)^m$$

or rather

$$G_{(-\Delta)^m,\Omega}(x,y) + 2c_6|x-y|^{-n}d(x)^m d(y)^m \geq c_6|x-y|^{-n}d(x)^m d(y)^m,$$

for some constant $c_6 > 0$.

Since $|x-y|^{-n} \leq r^{-n}$ it follows for a positive constant c_7

$$G_{(-\Delta)^m,\Omega}(x,y) + c_7 d(x)^m d(y)^m \geq c_6|x-y|^{-n}d(x)^m d(y)^m,$$

which finishes the proof. □

3.2 Small Dimensions $n < 2m - 1$

In this section we prove Theorem 3.1 for $n < 2m-1$. As before, we proceed in several steps. Since we like to use a rescaling argument as in the proof of Proposition 3.5, uniqueness of the polyharmonic Green function in the half space $\mathcal{H}$ is needed. This is due to Lemma 3.6, where we assume a growth condition at infinity. After this is done, we prove local estimates from below for $G_{(-\Delta)^m,\Omega}$ in Corollary 3.9. Here, we will use a compactness argument to combine local estimates for $G_{(-\Delta)^m,\Omega}$ near the boundary of Ω, see Lemma 3.7, and in the interior of Ω, see Lemma 3.8. Section 3.2.2 is due to conclude the proof of Theorem 3.1 for $n < 2m - 1$.

In this section we use the ideas of [29, Sec. 4-5].

3.2.1 Some Auxiliary Results for $n < 2m - 1$

Lemma 3.6. *Let $n, m \geq 2$, $n < 2m - 1$ and let $\tilde{G} \in C^{2m}\left(\overline{\mathcal{H}} \times \overline{\mathcal{H}} \setminus \left\{(x,x) : x \in \overline{\mathcal{H}}\right\}\right)$ be a polyharmonic Green function with Dirichlet boundary condition, that is*

- *if $m = 2k$:* $\displaystyle\int_{\mathcal{H}} \tilde{G}(x,\,.\,)(-\Delta)^m \varphi \, dy$

$$= \varphi(x) + \sum_{\ell=1}^{k} \int_{\partial\mathcal{H}} \frac{\partial}{\partial\nu}(-\Delta)^{m-\ell}\tilde{G}(x,\,.\,)(-\Delta)^{\ell-1}\varphi - \frac{\partial}{\partial\nu}(-\Delta)^{\ell-1}\varphi(-\Delta)^{m-\ell}\tilde{G}(x,\,.\,)\, d\sigma,$$

- *if $m = 2k+1$:* $\displaystyle\int_{\mathcal{H}} \tilde{G}(x,\,.\,)(-\Delta)^m \varphi \, dy$

$$= \varphi(x) + \sum_{\ell=1}^{k} \int_{\partial\mathcal{H}} \frac{\partial}{\partial\nu}(-\Delta)^{m-\ell}\tilde{G}(x,\,.\,)(-\Delta)^{\ell-1}\varphi - \frac{\partial}{\partial\nu}(-\Delta)^{\ell-1}\varphi(-\Delta)^{m-\ell}\tilde{G}(x,\,.\,)\, d\sigma$$

$$+ \int_{\partial\mathcal{H}} \frac{\partial}{\partial\nu}(-\Delta)^{k}\tilde{G}(x,\,.\,)(-\Delta)^{k}\varphi \, d\sigma,$$

for all $\varphi \in C_c^{2m}\left(\overline{\mathcal{H}}\right)$ and $x \in \mathcal{H}$. Moreover, we assume that $\tilde{G}(x,y) = \tilde{G}(y,x)$ for all $x \neq y$ and that a growth condition at infinity holds

$$|\tilde{G}(x,y)| \leq C\left(1 + |x|^{2m-n} + |y|^{2m-n}\right)\left(1 + \log\left(1+|x|\right) + \log\left(1+|y|\right)\right). \tag{3.33}$$

Then $\tilde{G}$ is uniquely determined and given by Boggio's formula

$$\tilde{G}(x,y) = G_{(-\Delta)^m,\mathcal{H}}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{|x^*-y|/|x-y|} (v^2-1)^{m-1} v^{1-n} dv,$$

where $x^* = (-x_1, x_2, \ldots, x_n)$ and

$$k_{m,n} = \frac{1}{n e_n 4^{m-1}((m-1)!)^2}.$$

Proof. In what follows we fix an arbitrary $x \in \mathcal{H}$ and write $\tilde{G}(x,y) = G_{(-\Delta)^m,\mathcal{H}}(x,y) + H(x,y)$, where H is a regular function in $\overline{\mathcal{H}} \times \overline{\mathcal{H}}$ satisfying

$$\begin{cases} (-\Delta_y)^m H(x,\ .\ ) = 0 & \text{in } \mathcal{H}, \\ \dfrac{\partial^j}{\partial y_1^j} H(x,y) = 0 & \text{for } y_1 = 0 \text{ and } j = 0,\dots,m-1. \end{cases}$$

From [38] we have with $y^* := (-y_1, y_2, \dots, y_n)$ that

$$H^*(x,y) := \begin{cases} H(x,y) & \text{if } y_1 \le 0, \\ \displaystyle\sum_{j=0}^{m-1} \frac{y_1^{m+j}}{(j!)^2} \Delta_y^j \left(\frac{H(x,y^*)}{(-y_1)^{m-j}} \right) & \text{if } y_1 > 0, \end{cases}$$

satisfies $H^*(x,\ .\ ) \in C^{2m}(\mathbb{R}^n)$ and is polyharmonic on $\mathbb{R}^n$.

We prove the following growth behaviour for $G_{(-\Delta)^m,\mathcal{H}}$. For all $x, y \in \mathcal{H}$ we have that

$$|G_{(-\Delta)^m,\mathcal{H}}(x,y)| \le C \left(1 + |x|^{2m-n} + |y|^{2m-n}\right).$$

This is done for even $n < 2m-1$, since for odd $n < 2m-1$ the proof works in a similar way.

From Boggio's formula we get

$$G_{(-\Delta)^m,\mathcal{H}}(x,y) = k_{m,n}|x-y|^{2m-n} \sum_{j=0}^{m-1} \left(\binom{m-1}{j} (-1)^j \int_1^{|x^*-y|/|x-y|} v^{2m-n-1-2j}\,dv \right).$$

Then, taking the modulus, for a constant $c > 0$, we see that

$$\begin{aligned} |G_{(-\Delta)^m,\mathcal{H}}(x,y)| &\le c|x-y|^{2m-n} \sum_{j=0}^{m-1} \int_1^{|x^*-y|/|x-y|} v^{2m-n-1-2j}\,dv \\ &= c|x-y|^{2m-n} \left(\int_1^{|x^*-y|/|x-y|} v^{-1}\,dv + \sum_{j \in J} \int_1^{|x^*-y|/|x-y|} v^{2m-n-1-2j}\,dv \right), \end{aligned}$$

where $J := \{0, 1, \dots, m-1\} \setminus \{m - \frac{n}{2}\}$. Note that

$$2 \le n < 2m-1 \quad \Leftrightarrow \quad \frac{1}{2} < m - \frac{n}{2} \le m-1.$$

If n is odd we see $m - \frac{n}{2} \notin \mathbb{N}$ and the first integral does not occur in this case. Then

$$\begin{aligned} &|G_{(-\Delta)^m,\mathcal{H}}(x,y)| \\ &\le c|x-y|^{2m-n} \left(\log \frac{|x^*-y|}{|x-y|} + \sum_{j \in J} \frac{1}{2m-n-2j} \left(\left(\frac{|x^*-y|}{|x-y|} \right)^{2m-n-2j} - 1 \right) \right) \\ &\le c|x-y|^{2m-n} \left(\log \frac{|x^*-y|}{|x-y|} + \sum_{\substack{j \in J, \\ j < m - \frac{n}{2}}} \frac{1}{2m-n-2j} \left(\frac{|x^*-y|}{|x-y|} \right)^{2m-n-2j} - \sum_{\substack{j \in J, \\ j > m - \frac{n}{2}}} \frac{1}{2m-n-2j} \right). \end{aligned}$$

Since $\frac{|x^*-y|}{|x-y|} > 1$, it follows with $\log(z) < z$ for $z > 1$ that

$$\begin{aligned}|G_{(-\Delta)^m,\mathcal{H}}(x,y)| &\le c\left(|x-y|^{2m-n}\log\frac{|x^*-y|}{|x-y|} + |x^*-y|^{2m-n}\right)\\ &\le c\left(|x-y|^{2m-n-1}|x^*-y| + |x^*-y|^{2m-n}\right)\\ &\le c\,|x^*-y|^{2m-n}\\ &\le c\left(1+|x|^{2m-n}+|y|^{2m-n}\right).\end{aligned}$$

Therefore $H(x,y)$ satisfies (3.33). Hence,

$$|H(x,y)| \le C_x\left(1+|y|^{2m-n}\right)\log\left(1+|y|\right).$$

Now, from (2.62) it follows

$$|H^*(x,y)| \le C_x \sum_{j=0}^{2m-2} |\nabla_y^j H(x,y)||y|^j$$

and since $H(x,y)$ is polyharmonic and satisfies Dirichlet boundary conditions on $\partial\mathcal{H}$, we can use a scaling argument and local elliptic estimates to find for $R > 0$ that

$$\|D^\alpha H(x,\,.\,)\|_{L^\infty(B_R\cap\mathcal{H})} \le \frac{C}{R^{|\alpha|}}\,\|H(x,\,.\,)\|_{L^\infty(B_{2R}\cap\mathcal{H})},$$

cf. (2.34). Then we have for $j = 0,\ldots,2m-2$ that

$$|\nabla_y^j H(x,y)| \le \frac{C_x}{|y|^j}\left(1+|y|^{2m-n}\log\left(1+|y|\right)\right).$$

Noting that for $|y| \le 1$ the function $H^*(x,\,.\,) \in C^{2m}(\mathbb{R}^n)$ is bounded we get

$$|\nabla_y^{2m-1}H^*(x,y)| \le C_x\frac{(1+\log\left(1+|y|\right))}{1+|y|^{n-1}}. \tag{3.34}$$

Since $\nabla_y\Delta_y^{m-1}H^*(x,\,.\,)$ is harmonic we can apply the maximum principle to obtain

$$\left\|\nabla_y\Delta_y^{m-1}H^*(x,\,.\,)\right\|_{C^0(\overline{B_R(0)})} \le C_x\frac{(1+|\log|R||)}{1+|R|^{n-1}}.$$

Thus, taking $R\to\infty$ we have for a suitable function $a(\,.\,)$ that

$$\nabla_y\Delta_y^{m-1}H^*(x,\,.\,) \equiv 0 \qquad\text{and}\qquad \Delta_y^{m-1}H^*(x,\,.\,) = a(x).$$

Since

$$\Delta_y^{m-1}\left(\nabla_y^{2m-1}H^*(x,\,.\,)\right) = \nabla_y^{2m-1}\left(\Delta_y^{m-1}H^*(x,\,.\,)\right) = \nabla_y^{2m-1}a(x) \equiv 0,$$

any $\nabla_y^{2m-1}H^*(x,\,.\,)$ is polyharmonic and, since (3.34) shows that $\nabla_y^{2m-1}H^*(x,\,.\,)\to 0$ as $y\to\infty$, bounded. Furthermore, Liouville's theorem for polyharmonic functions [50, p. 19] shows that $\nabla_y^{2m-1}H^*(x,\,.\,) \equiv 0$.

We perform a Taylor expansion in y and x using the boundary data to observe that

$$\begin{aligned} H^*(x,y) =& \frac{1}{m!}\frac{\partial^m H^*}{\partial y_1^m}(x,0)\cdot y_1^m + \frac{1}{(m+1)!}\sum_{j=1}^n \frac{\partial^{m+1}H^*}{\partial y_j \partial y_1^m}(x,0)\cdot y_j \cdot y_1^m + \dots \\ & \dots + \frac{1}{(2m-2)!}\sum_{j_1,\dots,j_{m-2}=1}^n \frac{\partial^{2m-2}H^*}{\partial y_{j_1}\dots\partial y_{j_{m-2}}\partial y_1^m}(x,0)\cdot y_{j_1}\cdot\dots\cdot y_{j_{m-2}}\cdot y_1^m \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} H^*(x,y) =& \frac{1}{m!}\frac{\partial^m H^*}{\partial x_1^m}(0,y)\cdot x_1^m + \frac{1}{(m+1)!}\sum_{k=1}^n \frac{\partial^{m+1}H^*}{\partial x_k \partial x_1^m}(0,y)\cdot x_k \cdot x_1^m + \dots \\ & \dots + \frac{1}{(2m-2)!}\sum_{k_1,\dots,k_{m-2}=1}^n \frac{\partial^{2m-2}H^*}{\partial x_{k_1}\dots\partial x_{k_{m-2}}\partial x_1^m}(0,y)\cdot x_{k_1}\cdot\dots\cdot x_{k_{m-2}}\cdot x_1^m. \end{aligned} \tag{3.36}$$

Then, by differentiating (3.36) with respect to y_1, we get with (3.35) for H

$$\begin{aligned} H(x,y) = y_1^m \cdot & \Bigg(\frac{\partial^m_{y_1^m}\partial^m_{x_1^m}H(0,0)}{(m!)^2}\cdot x_1^m + \sum_{k=1}^n \frac{\partial^m_{y_1^m}\partial^{m+1}_{x_k x_1^m}H(0,0)}{(m+1)!m!}\cdot x_k\cdot x_1^m + \dots \\ & \dots + \sum_{k_1,\dots,k_{m-2}=1}^n \frac{\partial^m_{y_1^m}\partial^{2m-2}_{x_{k_1}\dots x_{k_{m-2}}x_1^m}H(0,0)}{(2m-2)!m!}\cdot x_{k_1}\cdot\dots\cdot x_{k_{m-2}}\cdot x_1^m\Bigg) \\ + \sum_{j=1}^n y_j\cdot y_1^m & \Bigg(\frac{\partial^{m+1}_{y_j y_1^m}\partial^m_{x_1^m}H(0,0)}{(m+1)!m!}\cdot x_1^m + \sum_{k=1}^n \frac{\partial^{m+1}_{y_j y_1^m}\partial^{m+1}_{x_k x_1^m}H(0,0)}{((m+1)!)^2}\cdot x_k\cdot x_1^m + \dots \\ & \dots + \sum_{k_1,\dots,k_{m-2}=1}^n \frac{\partial^{m+1}_{y_j y_1^m}\partial^{2m-2}_{x_{k_1}\dots x_{k_{m-2}}x_1^m}H(0,0)}{(2m-2)!(m+1)!}\cdot x_{k_1}\cdot\dots\cdot x_{k_{m-2}}\cdot x_1^m\Bigg) \\ + \dots + \sum_{j_1,\dots,j_{m-2}=1}^n & y_{j_1}\cdot\dots\cdot y_{j_{m-2}}\cdot y_1^m\cdot\Bigg(\frac{\partial^{2m-2}_{y_{j_1}\dots y_{j_{m-2}}y_1^m}\partial^m_{x_1^m}H(0,0)}{(2m-2)!m!}\cdot x_1^m + \\ & + \sum_{k=1}^n \frac{\partial^{2m-2}_{y_{j_1}\dots y_{j_{m-2}}\partial y_1^m}\partial^{m+1}_{x_k x_1^m}H(0,0)}{(2m-2)!(m+1)!}\cdot x_k\cdot x_1^m + \dots \\ & \dots + \sum_{k_1,\dots,k_{m-2}=1}^n \frac{\partial^{2m-2}_{y_{j_1}\dots y_{j_{m-2}}y_1^m}\partial^{2m-2}_{x_{k_1}\dots x_{k_{m-2}}x_1^m}H(0,0)}{((2m-2)!)^2}\cdot x_{k_1}\cdot\dots\cdot x_{k_{m-2}}\cdot x_1^m\Bigg). \end{aligned} \tag{3.37}$$

Let us prove the following claim.

Claim. $H(x,y)\equiv 0$.

We assume by contradiction that there exists a $z^{(0)} = \big(x^{(0)}, y^{(0)}\big)\in\mathbb{R}^{2n}$ with $H\big(z^{(0)}\big)\neq 0$. For $v\in\mathbb{R}^{2n}$ we consider the one-dimensional polynomial $\mathbb{R}\ni\lambda\mapsto H\big(z^{(0)}+\lambda v\big)$, which is not the zero polynomial. The expansion (3.37) shows

$$\deg\left(H\left(z^{(0)}+\lambda v\right)\right)\in\{2m,\ldots,4m-4\},$$

where deg(.) gives the degree of a polynomial. Hence, for all $c>0$ exists a $\lambda_0>0$ such that for all $\lambda\geq\lambda_0$ we have that

$$\left|H\left(z^{(0)}+\lambda v\right)\right|\geq c\left|z^{(0)}+\lambda v\right|^{2m}.$$

This contradicts (3.33), which reads in this situation as

$$\left|H\left(z^{(0)}+\lambda v\right)\right|\leq C\left(1+\left|z^{(0)}+\lambda v\right|^{2m-n}\right)\left(1+\log\left(1+\left|z^{(0)}+\lambda v\right|\right)\right).$$

Note that $n\geq 2$.

This shows $H(x,y)\equiv 0$ and $\tilde{G}=G_{(-\Delta)^m,\mathcal{H}}$. □

Lemma 3.7 (Estimates near the boundary)**.** *Suppose that $n<2m-1$ and that $\Omega\subset\mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then for each $x_0\in\partial\Omega$ there exist a radius $r=r_{x_0}>0$ and a constant $C=C_{x_0}>0$ such that for all $x,y\in\Omega_{x_0,r}:=\overline{\Omega}\cap B_r(x_0)$ one has*

$$G_{(-\Delta)^m,\Omega}(x,y)\geq Cd(x)^{m-n/2}d(y)^{m-n/2}\min\left\{1,\frac{d(x)^{n/2}d(y)^{n/2}}{|x-y|^n}\right\}. \qquad (3.38)$$

Proof. We prove the lemma by assuming the contradiction: there exist $x_k,y_k\in\Omega_{x_0,1/k}=\overline{\Omega}\cap B_{1/k}(x_0)$ such that

$$G_{(-\Delta)^m,\Omega}(x_k,y_k)<\frac{1}{k}d(x_k)^{m-n/2}d(y_k)^{m-n/2}\min\left\{1,\frac{d(x_k)^{n/2}d(y_k)^{n/2}}{|x_k-y_k|^n}\right\}. \qquad (3.39)$$

Moreover, we have $x_k\to x_0, y_k\to x_0, d(x_k)\to 0, d(y_k)\to 0, |x_k-y_k|\to 0$.

By rotation and translation we may assume $x_0=0$ and that the first unit vector e_1 is the exterior unit normal to $\partial\Omega$ at x_0. We can pass to a subsequence to consider one of the following two cases.

Case $|x_k-y_k|<\frac{1}{2}\max\{d(x_k),d(y_k)\}$. We collect some facts in this case.

Since

$$d(x_k)\leq|x_k-y_k|+d(y_k)<\frac{1}{2}\max\{d(x_k),d(y_k)\}+d(y_k),$$

we get $d(x_k)<2d(y_k)$.

With $2d(y_k)\leq 2|x_k-y_k|+2d(x_k)<\max\{d(x_k),d(y_k)\}+2d(x_k)$, we see $2d(y_k)<4d(x_k)$.

Without loss of generality let $\max\{d(x_k), d(y_k)\} = d(x_k)$ and $\min\{d(x_k), d(y_k)\} = d(y_k)$. Hence,

$$\begin{aligned}\frac{1}{2}|x_k - y_k| &= |x_k - y_k| - \frac{1}{2}|x_k - y_k| \\ &< \frac{1}{2}\left(\max\{d(x_k), d(y_k)\} - |x_k - y_k|\right) = \frac{1}{2}(d(x_k) - |x_k - y_k|) \\ &\leq \frac{1}{2}(d(y_k) + |x_k - y_k| - |x_k - y_k|) = \frac{1}{2}\min\{d(x_k), d(y_k)\}.\end{aligned}$$

Together with our assumption we have

$$G_{(-\Delta)^m,\Omega}(x_k, y_k) < \frac{1}{k} d(x_k)^{m-n/2} d(y_k)^{m-n/2}. \tag{3.40}$$

Let $\tilde{x}_k \in \partial\Omega$ denote the closest boundary point to x_k and by

$$G_k(\xi, \eta) := d(x_k)^{n-2m} G_{(-\Delta)^m,\Omega}(\tilde{x}_k + d(x_k)\xi, \tilde{x}_k + d(x_k)\eta)$$

the scaled and translated polyharmonic Green function for

$$\xi, \eta \in \Omega_k := \frac{1}{d(x_k)}(-\tilde{x}_k + \Omega).$$

From Section 1.3 we see

$$\Omega_k \to \mathcal{H} := \{x : x_1 < 0\} \text{ locally uniformly for } k \to \infty.$$

For

$$\xi_k = \frac{1}{d(x_k)}(x_k - \tilde{x}_k), \quad \eta_k = \frac{1}{d(x_k)}(y_k - \tilde{x}_k),$$

we have

$$d_k(\xi_k) = 1 \quad \text{and} \quad d_k(\eta_k) = \frac{d(y_k)}{d(x_k)} < 2,$$

where $d_k := d(\,.\,, \partial\Omega_k)$. Thus, assumption (3.40) shows

$$\begin{aligned}G_k(\xi_k, \eta_k) = d(x_k)^{n-2m} G_{(-\Delta)^m,\Omega}(x_k, y_k) &< \frac{1}{k} d(x_k)^{n-2m} d(x_k)^{m-n/2} d(y_k)^{m-n/2} \\ = \frac{1}{k} d_k(\xi_k)^{m-n/2} d_k(\eta_k)^{m-n/2} &\leq \frac{1}{k} 2^{m-n/2}.\end{aligned} \tag{3.41}$$

Since $|\xi_k| = 1$ and

$$|\xi_k - \eta_k| = \frac{1}{d(x_k)}|y_k - x_k| < \frac{1}{d(x_k)} \min\{d(x_k), d(y_k)\} \leq 1,$$

which implies $|\eta_k| \leq 2$, the sequences $(\xi_k)_{k\in\mathbb{N}}, (\eta_k)_{k\in\mathbb{N}}$ are bounded. Together with $d_k(\xi_k) = 1$ and $d_k(\eta_k) = \frac{d(y_k)}{d(x_k)} > \frac{1}{2}$ we can choose subsequences, such that $\xi_k \to \xi_\infty \in \mathcal{H}$ and $\eta_k \to \eta_\infty \in \mathcal{H}$.

In the following we will show uniform estimates for G_k in k, which lead us to a convergence result like Proposition 2.13 since we have uniqueness due to Lemma 3.6.

From Theorem 2.4 we have uniformly in k

$$
\begin{aligned}
|\nabla^{2m-n+1}_{(\xi,\eta)} G_k(\xi,\eta)| &= d(x_k)|\nabla^{2m-n+1}_{(\xi,\eta)} G_{(-\Delta)^m,\Omega}(\tilde{x}_k + d(x_k)\xi, \tilde{x}_k + d(x_k)\eta)| \\
&\leq C d(x_k)|\tilde{x}_k + d(x_k)\xi - \tilde{x}_k - d(x_k)\eta)|^{-1} \\
&= C|\xi - \eta|^{-1}. \qquad (3.42)
\end{aligned}
$$

We use the boundary data to see that for all $\xi \in \partial\Omega_k$ and all $\eta \in \Omega_k$

$$\nabla^i_\xi \nabla^j_\eta G_k(\xi,\eta) = 0,$$

with $i + j \leq 2m - n$ and $i \leq m - 1$.

For all $\xi \in \Omega_k$ and all $\eta \in \partial\Omega_k$ it follows

$$\nabla^i_\xi \nabla^j_\eta G_k(\xi,\eta) = 0,$$

with $i + j \leq 2m - n$ and $j \leq m - 1$.

Then, if we use the path from Lemma 2.14 connecting $\xi \in \Omega_k$ with $\tilde{\xi} \in \partial\Omega_k$, or $\eta \in \Omega_k$ with $\tilde{\eta} \in \partial\Omega_k$, we get

$$D^\alpha_\xi D^\beta_\eta G_k(\xi,\eta) = \underbrace{D^\alpha_\xi D^\beta_\eta G_k(\tilde{\xi},\eta)}_{=0,\ \text{if } |\alpha| \leq m-1} + \int_{\omega_\xi} \nabla_\xi D^\alpha_\xi D^\beta_\eta G_k(\xi(s),\eta) \cdot \tau(s) ds, \qquad (3.43)$$

$$D^\alpha_\xi D^\beta_\eta G_k(\xi,\eta) = \underbrace{D^\alpha_\xi D^\beta_\eta G_k(\xi,\tilde{\eta})}_{=0,\ \text{if } |\beta| \leq m-1} + \int_{\omega_\eta} \nabla_\eta D^\alpha_\xi D^\beta_\eta G_k(\xi,\eta(s)) \cdot \tau(s) ds. \qquad (3.44)$$

Next, by integration, we estimate $D^\alpha_\xi D^\beta_\eta G_k(\xi,\eta)$ for all $0 \leq |\alpha| + |\beta| \leq 2m - n$.

For $n \geq m + 1$ we observe first that $|\alpha| \leq m - 1$ and $|\beta| \leq m - 1$.

If $2 \leq n < m+1$ we have either $|\alpha|, |\beta| \leq m-1$, $|\alpha| > m-1$ and $|\beta| \leq m-1$ or $|\beta| > m-1$ and $|\alpha| \leq m - 1$. Then, for all $0 \leq |\alpha| + |\beta| \leq 2m - n$, we can use (3.43) or (3.44) with zero boundary term.

Let $|\alpha| + |\beta| = 2m - n$. Proceeding as in step 1 of the proof of Proposition 2.13 for the case $n = 2m$, by using (3.42) and (3.43) or (3.44) we have

$$|D^\alpha_\xi D^\beta_\eta G_k(\xi,\eta)| \leq C\ \left(1 + |\log|\xi - \eta|| + \log(1 + |\xi|) + \log(1 + |\eta|)\right), \qquad (3.45)$$

uniformly in k.

Let $|\alpha| + |\beta| = 2m - n - 1$. Using (3.45) and the properties of the path connecting $\xi \in \Omega_k$ and $\tilde{\xi} \in \partial\Omega_k$ from Lemma 2.14 we get

$$\begin{aligned}
&|D_\xi^\alpha D_\eta^\beta G_k(\xi,\eta)| \\
&\le C \int_0^\ell 1 + |\log|\xi(s)-\eta|| + \log(1+|\xi(s)|) + \log(1+|\eta|)\, ds \\
&\le C \int_0^\ell |\log|\xi(s)-\eta||\, ds + C\int_0^{\frac{3}{2}|\xi|} 1 + \log(1+s+|\xi|) + \log(1+|\eta|)\, ds \\
&\le C \int_0^\ell |\log|\xi(s)-\eta||\, ds + C(1+|\xi|)(1+\log(1+|\xi|) + \log(1+|\eta|)).
\end{aligned}$$

It remains to estimate the integral. If $|\xi - \eta| > 2$, we have $|\xi(s) - \eta| \ge \frac{1}{2}|\xi - \eta| > 1$ for all $s \in [0, \ell]$ and therefore by integration

$$\begin{aligned}
\int_0^\ell |\log|\xi(s)-\eta||\, ds = \int_0^\ell \log|\xi(s)-\eta|\, ds &\le \int_0^{\frac{3}{2}|\xi|} \log(s+|\xi|+|\eta|)\, ds \\
&\le C \begin{cases} (1+|\xi|)\log(1+|\xi|) & \text{if } |\xi| \ge |\eta|, \\ (1+|\eta|)\log(1+|\eta|) & \text{if } |\xi| < |\eta|. \end{cases}
\end{aligned}$$

For $|\xi - \eta| \le 2$ one of the following situations for the path constructed as in Lemma 2.14 could happen

1. For all $s \in [0, \ell]$ we have $|\xi(s) - \eta| \ge 1$.
2. There exist $0 \le s_1 < s_2 \le \ell$, such that $|\xi(s) - \eta| < 1$ if $s \in [s_1, s_2]$ and $|\xi(s) - \eta| \ge 1$ if $s \notin [s_1, s_2]$.

For situation 1, we have

$$\begin{aligned}
\int_0^\ell |\log|\xi(s)-\eta||\, ds \le \int_0^\ell \log(|\xi(s)-\xi| + |\xi-\eta|)\, ds &\le \int_0^{\frac{3}{2}|\xi|} \log(s+2)\, ds \\
&\le C(1+|\xi|)(1+\log(1+|\xi|)).
\end{aligned}$$

For situation 2, we get

$$\int_0^\ell |\log|\xi(s)-\eta||\, ds = -\int_{s\in[s_1,s_2]} \log|\xi(s)-\eta|\, ds + \int_{s\notin[s_1,s_2]} \log|\xi(s)-\eta|\, ds.$$

Now, it is enough to consider the first integral since the second one is as in situation 1. Since $|\xi(s) - \eta| \ge \frac{1}{12}(|\xi - \eta| + s) \ge \frac{s}{12}$ it follows

$$-\int_{s\in[s_1,s_2]} \log|\xi(s)-\eta|\, ds \le \int_{s\in[s_1,s_2]} \log\left(\frac{12}{s}\right) ds \le C(1+|\xi|)(1+\log(1+|\xi|)),$$

and we finally have

$$|D_\xi^\alpha D_\eta^\beta G_k(\xi,\eta)| \le C\ (1+\log(1+|\xi|) + \log(1+|\eta|))(1+|\xi|+|\eta|)$$

uniformly in k.

Performing integration in the variable η and estimating in the same way, we see that for the full derivative it holds

$$|\nabla^{2m-n-1}_{(\xi,\eta)} G_k(\xi,\eta)| \le C \ (1+\log(1+|\xi|)+\log(1+|\eta|))(1+|\xi|+|\eta|)$$

uniformly in k. Proceeding as before we have

$$|\nabla^{2m-n-j}_{(\xi,\eta)} G_k(\xi,\eta)| \le C \ (1+\log(1+|\xi|)+\log(1+|\eta|))\left(1+|\xi|^j+|\eta|^j\right)$$

for $j \in \{1,2,\dots,2m-n\}$ and uniformly in k, which is for $j = 2m-n$ the desired estimate for G_k.

Note that we have local uniform bounds, even if $\xi = \eta$. Then, the convergence result from Proposition 2.13 works also in this case, which is important since $\xi_\infty = \eta_\infty$ is possible.

Thus, our convergence result together with (3.41) shows that $G_{(-\Delta)^m,\mathcal{H}}(\xi_\infty,\eta_\infty) \le 0$.

We conclude the proof by using Boggio's formula that shows $G_{(-\Delta)^m,\mathcal{H}}(\xi,\eta) > 0$ for all $\xi,\eta \in \mathcal{H}$, which gives us the desired contradiction.

If $\xi \ne \eta$, it follows directly from the formula since ξ,η are interior points.

For $\xi = \eta$, n is even and $J := \{0,1,\dots,m-1\} \setminus \{m-\frac{n}{2}\}$ we first see that

$$\begin{aligned}
G_{(-\Delta)^m,\mathcal{H}}(\xi,\eta) &= k_{m,n}|\xi-\eta|^{2m-n}\sum_{j=0}^{m-1}\left(\binom{m-1}{j}(-1)^j \int_1^{|\xi^*-\eta|/|\xi-\eta|} v^{2m-n-1-2j}\,dv\right)\\
&= k_{m,n}|\xi-\eta|^{2m-n}(-1)^{m-n/2}\binom{m-1}{m-\frac{n}{2}}\log\left(\frac{|\xi^*-\eta|}{|\xi-\eta|}\right)\\
&\quad + k_{m,n}|\xi-\eta|^{2m-n}\sum_{j\in J}\frac{(-1)^j}{2m-n-2j}\binom{m-1}{j}\left(\left(\frac{|\xi^*-\eta|}{|\xi-\eta|}\right)^{2m-n-2j}-1\right).
\end{aligned}$$

Since $m-\frac{n}{2} \ne 0$, we have $0 \in J$, and with $\xi \to \eta$ we get

$$G_{(-\Delta)^m,\mathcal{H}}(\eta,\eta) = \frac{k_{m,n}(2|\eta_1|)^{2m-n}}{2m-n} > 0,$$

which also holds for odd n.

Case $|x_k-y_k| \ge \frac{1}{2}\max\{d(x_k),d(y_k)\}$. This can be proved like Lemma 3.5 using the same estimates for G_k as in the previous case. □

Lemma 3.8 (Estimates in the interior)**.** *Suppose that* $n < 2m-1$ *and that* $\Omega \subset \mathbb{R}^n$ *is a bounded* $C^{2m,\gamma}$*-smooth domain. Then for each* $x_0 \in \Omega$ *there exist a radius* $r = r_{x_0} > 0$ *and a constant* $C = C_{x_0} > 0$ *such that for all* $x,y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ *one has*

$$G_{(-\Delta)^m,\Omega}(x,y) \ge C d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2}d(y)^{n/2}}{|x-y|^n}\right\}. \tag{3.46}$$

Proof. The proof is inspired by [49] and is based on results from [32] and [41].

Since x_0 is in the interior of Ω we can assume by scaling and translation without loss of generality that for some $R > 1$: $x_0 = 0$, $d(x_0) = d(0) = 1$ and

$$B_1 = B_1(0) \subset \Omega \subset B_R = B_R(0).$$

We recall Boggio's formula for the Green function for the Dirichlet problem with $\Omega = B_1$:

$$G_{(-\Delta)^m,B_1}(x,y) = k_{m,n}|x-y|^{2m-n} \int_1^{\left||x|y-\frac{x}{|x|}\right|/|x-y|} (v^2-1)^{m-1} v^{1-n}\, dv, \tag{3.47}$$

where $k_{m,n} = 1/\left(ne_n 4^{m-1}((m-1)!)^2\right)$. In what follows $n < 2m-1$ is even, since for odd n the proof can be done in the same way.

From the identity

$$G_{(-\Delta)^m,B_R}(x,y) = R^{2m-n} G_{(-\Delta)^m,B_1}\left(\frac{1}{R}x, \frac{1}{R}y\right)$$

we get

$$\begin{aligned}
&G_{(-\Delta)^m,B_R}(x,y)\\
&= k_{m,n}|x-y|^{2m-n} \int_1^{R\left|\frac{1}{R^2}|x|y-\frac{x}{|x|}\right|/|x-y|} (v^2-1)^{m-1} v^{1-n}\, dv\\
&= k_{m,n}|x-y|^{2m-n} \sum_{j=0}^{m-1} \left((-1)^j \binom{m-1}{j} \int_1^{R\left|\frac{1}{R^2}|x|y-\frac{x}{|x|}\right|/|x-y|} v^{2m-n-1-2j}\, dv \right).
\end{aligned}$$

Since n is even with $J := \{0,1,\dots,m-1\} \setminus \{m-\frac{n}{2}\}$ we obtain

$$\begin{aligned}
&G_{(-\Delta)^m,B_R}(x,y)\\
&= k_{m,n}|x-y|^{2m-n}(-1)^{m-n/2}\binom{m-1}{m-\frac{n}{2}} \log\left(\frac{R\left|\frac{1}{R^2}|x|y-\frac{x}{|x|}\right|}{|x-y|}\right)\\
&\quad + k_{m,n}|x-y|^{2m-n} \sum_{j\in J} \frac{(-1)^j}{2m-n-2j}\binom{m-1}{j}\left(\left(\frac{R\left|\frac{1}{R^2}|x|y-\frac{x}{|x|}\right|}{|x-y|}\right)^{2m-n-2j} - 1\right)\\
&= F_{m,n}(x-y) + H_{(-\Delta)^m,B_R}(x,y)
\end{aligned}$$

where

$$F_{m,n}(x) := c_{m,n}|x|^{2m-n}(-\log|x|) = k_{m,n}(-1)^{m-n/2}\binom{m-1}{m-\frac{n}{2}}|x|^{2m-n}(-\log|x|)$$

is a fundamental solution of the polyharmonic operator $(-\Delta)^m$ in $\mathbb{R}^n$ and

$$\begin{aligned}&H_{(-\Delta)^m,B_R}(x,y)\\&:= k_{m,n}(-1)^{m-n/2}\binom{m-1}{m-\frac{n}{2}}|x-y|^{2m-n}\log\left(R\left|\frac{1}{R^2}|x|y-\frac{x}{|x|}\right|\right)\\&+k_{m,n}\sum_{j\in J}\frac{(-1)^j}{2m-n-2j}\binom{m-1}{j}\left(\left(R\left|\frac{1}{R^2}|x|y-\frac{x}{|x|}\right|\right)^{2m-n-2j}|x-y|^{2j}-|x-y|^{2m-n}\right).\end{aligned}$$

From $m-\frac{n}{2}\neq 0$, we have $0\in J$, and we get

$$H_{(-\Delta)^m,B_R}(x,x)=\frac{k_{m,n}}{2m-n}\left(R-\frac{|x|^2}{R}\right)^{2m-n}.$$

Again, we use [32, Lemma 5], which states

$$\begin{aligned}&G_{(-\Delta)^m,\Omega}(x,y)\\&\quad\geq\frac{1}{4}\left(H_{(-\Delta)^m,B_1}(x,x)-H_{(-\Delta)^m,B_R}(x,x)+H_{(-\Delta)^m,B_1}(y,y)-H_{(-\Delta)^m,B_R}(y,y)\right)\\&\qquad\qquad+\frac{1}{2}\left(G_{(-\Delta)^m,B_1}(x,y)+G_{(-\Delta)^m,B_R}(x,y)\right)\end{aligned}$$

for all $x,y\in B_1$ with $x\neq y$. This is also valid for $n<2m-1$ and we see

$$\begin{aligned}&G_{(-\Delta)^m,\Omega}(0,y)\\&\quad\geq\frac{k_{m,n}}{4}\left(\frac{1}{2m-n}-\frac{R^{2m-n}}{2m-n}+\frac{\left(1-|y|^2\right)^{2m-n}}{2m-n}-\frac{1}{2m-n}\left(R-\frac{|y|^2}{R}\right)^{2m-n}\right)\\&\qquad+\frac{k_{m,n}}{2}\left(|y|^{2m-n}(-1)^{m-n/2}\binom{m-1}{m-\frac{n}{2}}\left(\log\left(\frac{1}{|y|}\right)+\log\left(\frac{R}{|y|}\right)\right)\right.\\&\qquad\qquad\left.+\sum_{j\in J}\frac{(-1)^j}{2m-n-2j}\binom{m-1}{j}\left(|y|^{2j}\left(1+R^{2m-n-2j}\right)-2|y|^{2m-n}\right)\right).\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}&G_{(-\Delta)^m,\Omega}(0,y)\\&\geq\frac{k_{m,n}}{4}\left(\frac{3}{2m-n}+\frac{R^{2m-n}}{2m-n}+\frac{\left(1-|y|^2\right)^{2m-n}}{2m-n}-\frac{1}{2m-n}\left(R-\frac{|y|^2}{R}\right)^{2m-n}\right.\\&\qquad+2(-1)^{m-n/2+1}\binom{m-1}{m-\frac{n}{2}}|y|^{2m-n}\left(\log|y|+\log\frac{|y|}{R}\right)-\frac{4|y|^{2m-n}}{2m-n}\\&\qquad\left.+2\sum_{\substack{j\in J,\\ j>0}}\frac{(-1)^j}{2m-n-2j}\binom{m-1}{j}\left(|y|^{2j}\left(1+R^{2m-n-2j}\right)-2|y|^{2m-n}\right)\right).\end{aligned}$$

Then, if $y\to 0$,

$$G_{(-\Delta)^m,\Omega}(0,0)\geq\frac{k_{m,n}}{2m-n}>0.$$

From which we get $G_{(-\Delta)^m,\Omega}(x_0, x_0) > 0$ for all $x_0 \in \Omega$. From the continuity of $G_{(-\Delta)^m,\Omega}$ we find $r, c > 0$ such that $B_r(x_0) \subset \Omega$ and $G_{(-\Delta)^m,\Omega}(x, y) > c$ for all $x, y \in B_r(x_0)$. Since Ω is bounded we get

$$\frac{c}{(\operatorname{diam}\Omega)^{2m-n}} d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} \le c < G_{(-\Delta)^m,\Omega}(x, y),$$

which proves the lemma. □

Corollary 3.9. *Suppose that $n < 2m - 1$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then there exist a radius $r > 0$ and a constant $C > 0$ such that for all $x_0 \in \overline{\Omega}$ and for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$G_{(-\Delta)^m,\Omega}(x, y) \ge C d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\}. \tag{3.48}$$

Proof. Combining Lemmas 3.7 and 3.8 we find for each $x_0 \in \overline{\Omega}$ a $r_0 = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r_0} := \overline{\Omega} \cap B_{r_0}(x_0)$ one has that

$$G_{(-\Delta)^m,\Omega}(x, y) \ge C d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\}.$$

We apply a compactness argument to $\overline{\Omega} = \bigcup_{x_0 \in \overline{\Omega}} \Omega_{x_0, r_{x_0}/2}$ to find $x_1, \ldots, x_k \in \overline{\Omega}$ such that $\overline{\Omega} \subset \bigcup_{j=1}^k \Omega_{x_j, r_j/2}$.

For $x, y \in \overline{\Omega}$ with $|x - y| < r$ where $r := \min\{\frac{r_1}{2}, \ldots, \frac{r_k}{2}\}$ exists a j such that $x \in \Omega_{x_j, r_j/2}$. Then $y \in \Omega_{x_j, r_j/2}$ and the claim follows. □

3.2.2 Proof of the Main Result for $n < 2m - 1$

Now we are able to prove the bounds from below in Theorem 3.1 for $n < 2m - 1$.

Proof of Theorem 3.1 for $n < 2m - 1$. We fix $r > 0$ as in Corollary 3.9. If $|x - y| < r$ there is nothing left to prove.

Let $|x - y| \ge r$. Then, with the help of Corollary 2.12 we have that there exist a constant $c_8 > 0$ such that

$$G_{(-\Delta)^m,\Omega}(x, y) \ge -c_8 |x - y|^{-n} d(x)^m d(y)^m.$$

Then,

$$\begin{aligned} &G_{(-\Delta)^m,\Omega}(x, y) + 2c_8 |x - y|^{-n} d(x)^m d(y)^m \\ &\qquad \ge c_8 d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\}. \end{aligned}$$

Taking $c_9 := 2c_8 r^{-n}$ it follows

$$G_{(-\Delta)^m,\Omega}(x,y) + c_9 d(x)^m d(y)^m \geq c_8 d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\}.$$

and the claim is proved. □

4 Pointwise Estimates for the Green Function of the Perturbed Problem

In this section, we will prove our main result Theorem 0.1 for the perturbed polyharmonic operator. As before, let G be the Green function in Ω from Proposition 2.1, i.e. G is the Green function for the following Dirichlet boundary value problem

$$\begin{cases} (-\Delta)^m u(x) + \sum_{\ell=0}^{m-1} \sum_{|\alpha|=|\beta|=\ell} D^\beta \left(a^\ell_{\alpha,\beta}(x) D^\alpha u(x) \right) = f(x) & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad \frac{\partial^j}{\partial \nu^j} u(x) = 0 & \text{for } x \in \partial\Omega,\ j = 0, \ldots, m-1. \end{cases} \tag{4.1}$$

Note that we assume boundedness for the coefficient functions: there is a $K > 0$, such that for all ℓ it holds $\left\| a^\ell_{\alpha,\beta} \right\|_{C^{m-1,\gamma}(\overline{\Omega})} \le K$, see (A3) in Section 2.1.

We recall our main result.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded $C^{2m,\gamma}$-smooth domain, $m \ge 2$. Then there exist constants $c_1 \ge 0$, $c_2 > 0$ and $c_3 > 0$, depending on the domain Ω, m and K, such that we have the following Green function estimate:*

$$c_2^{-1} H_\Omega(x,y) \le G(x,y) + c_1 \mathbb{1}_{\{|x-y| \ge c_3\}}(x,y) d(x)^m d(y)^m \le c_2 H_\Omega(x,y) \tag{4.2}$$

for all $x, y \in \Omega$, where

$$H_\Omega(x,y) := \begin{cases} |x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\} & \text{if } n > 2m, \\ \log\left(1 + \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right) & \text{if } n = 2m, \\ d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} & \text{if } n < 2m, \end{cases} \tag{4.3}$$

and

$$\mathbb{1}_{\{|x-y| \ge c_3\}}(x,y) := \begin{cases} 1 & \text{if } |x-y| \ge c_3, \\ 0 & \text{if } |x-y| < c_3, \end{cases}$$

is the indicator function.

Similar as for the polyharmonic case in the preceding Chapter 3, many of the following proofs are done by rescaling the Green function G as in Section 2.4. For example, if $n \geq 2m-1$ we define

$$G_k(\xi,\eta) := |x_k - y_k|^{n-2m} G(\tilde{x}_k + |x_k - y_k|\xi, \tilde{x}_k + |x_k - y_k|\eta),$$

for $\xi, \eta \in \Omega_k := \frac{1}{|x_k - y_k|}(-\tilde{x}_k + \Omega)$, see Section 1.3. Since the coefficient functions in (4.1) are uniformly bounded we have the convergence result from Proposition 2.13, i.e.

$$G_k(\xi,\eta) \to G_{(-\Delta)^m,\mathcal{H}}(\xi,\eta)$$

in $C^{2m}_{loc}(\mathcal{H} \times \mathcal{H} \setminus \{(\xi,\xi) : \xi \in \mathcal{H}\})$. This convergence was crucial to prove the estimates from below for the polyharmonic case. Thanks to the convergence result for the perturbed differential operator from Proposition 2.13, the following sections are devoted to demonstrate which changes and extensions compared to the polyharmonic case have to be made.

Remark 4.2. With the help of Theorem 4.1 a uniform local positivity result, cf. (LP) in the introduction, follows. For any $x, y \in \Omega$ with $x \neq y$ we have that

$$|x-y| < c_3 \quad \text{implies} \quad G(x,y) \geq c_2^{-1} H_\Omega(x,y) > 0,$$

where the constant c_3 is chosen as in Theorem 4.1.

4.1 Large Dimensions $n > 2m$

We prove Theorem 4.1 for $n > 2m$ after some auxiliary results.

Lemma 4.3 (Estimates in the interior)**.** *Suppose that* $n > 2m$ *and that* $\Omega \subset \mathbb{R}^n$ *is a bounded* $C^{2m,\gamma}$*-smooth domain. Then for each* $x_0 \in \Omega$ *there exist a radius* $r = r_{x_0} > 0$ *and a constant* $C = C_{x_0} > 0$ *such that for all* $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ *one has*

$$G(x,y) \geq C|x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}. \tag{4.4}$$

Proof. We prove the lemma by assuming the contradiction: there exist $x_k, y_k \in \Omega_{x_0,1/k} = \overline{\Omega} \cap B_{1/k}(x_0)$ such that

$$G(x_k,y_k) < \frac{1}{k}|x_k - y_k|^{2m-n} \min\left\{1, \frac{d(x_k)^m d(y_k)^m}{|x_k - y_k|^{2m}}\right\}. \tag{4.5}$$

We have $x_k \to x_0, y_k \to x_0$ and $|x_k - y_k| \to 0$. Moreover, since $x_0 \in \Omega$, i.e. $d(x_0) > c > 0$, we have for k large enough that $\Omega_{x_0,1/k} \subset \Omega$. Therefore, $d(x_k)$ and $d(y_k)$ are bounded from below and the case $|x_k - y_k| \geq \frac{1}{2}\max\{d(x_k), d(y_k)\}$ cannot occur.

Now, we can pass to a subsequence to consider the following:

$$|x_k - y_k| < \frac{1}{2}\max\{d(x_k), d(y_k)\}.$$

Then, as in the proof of Lemma 3.7, $\frac{d(x_k)^m d(y_k)^m}{|x_k-y_k|^{2m}}$ is uniformly bounded from below, and (4.5) becomes

$$G(x_k, y_k) < \frac{1}{k}|x_k - y_k|^{2m-n}. \tag{4.6}$$

Let $\varepsilon > 0$ such that $\overline{B_{2\varepsilon}(x_0)} \subset \Omega$. Then for k large enough we have that $|x_k - x_0|, |y_k - x_0| < \varepsilon$ and if $|\xi|, |\eta| < \frac{\varepsilon}{|x_k-y_k|}$ the following function is certainly defined

$$G_k(\xi, \eta) := |x_k - y_k|^{n-2m} G(x_k + |x_k - y_k|\xi, x_k + |x_k - y_k|\eta).$$

Using Theorem 2.4, we have uniform bounds for G_k, and since the sets $B_{\varepsilon/|x_k-y_k|}(0)$ exhaust the whole $\mathbb{R}^n$, we can prove exactly as in Proposition 2.13 that

$$G_k \to F_{m,n} \quad \text{in } C^{2m}_{loc}\left(\mathbb{R}^n \times \mathbb{R}^n \setminus \{(\xi, \xi) : \xi \in \mathbb{R}^n\}\right),$$

where $F_{m,n}(\xi) = c_{m,2m}|\xi|^{2m-n}$ is a fundamental solution for $(-\Delta)^m$ on $\mathbb{R}^n$.

Let $\varepsilon < \frac{c_{m,2m}}{2}$ and $\eta_k := \frac{y_k-x_k}{|x_k-y_k|}$. Then, for k large enough we have

$$G_k(0, \eta_k) \geq F_{m,n}(|\eta_k|) - \varepsilon = c_{m,2m}|\eta_k|^{2m-n} - \varepsilon > \frac{c_{m,2m}}{2} > 0.$$

On the other hand, (4.6) shows the contradicting statement

$$G_k(0, \eta_k) = |x_k - y_k|^{n-2m} G(x_k, y_k) < \frac{1}{k},$$

and we have finished the proof. □

Lemma 4.4 (Estimates near the boundary)**.** *Suppose that $n > 2m$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then for each $x_0 \in \partial\Omega$ there exist a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$G(x, y) \geq C|x - y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x - y|^{2m}}\right\}. \tag{4.7}$$

Proof. We prove the lemma by assuming the contradiction: there exist $x_k, y_k \in \Omega_{x_0,1/k} = \overline{\Omega} \cap B_{1/k}(x_0)$ such that

$$G(x_k, y_k) < \frac{1}{k}|x_k - y_k|^{2m-n} \min\left\{1, \frac{d(x_k)^m d(y_k)^m}{|x_k - y_k|^{2m}}\right\}. \tag{4.8}$$

We have $x_k \to x_0, y_k \to x_0, d(x_k) \to 0, d(y_k) \to 0$ and $|x_k - y_k| \to 0$.

By rotation and translation we may assume $x_0 = 0$ and that the first unit vector e_1 is the exterior unit normal to $\partial\Omega$ at x_0. Then, we can pass to a subsequence to consider one of the following two cases.

Case 1. *For all $\varepsilon \in \left(0, \frac{1}{2}\right)$ there exists a ℓ_0, such that for all $\ell \geq \ell_0$: $|x_\ell - y_\ell| \leq \varepsilon \max\{d(x_\ell), d(y_\ell)\}$.*

Moreover we have $|x_\ell - y_\ell| < \frac{1}{2} \max\{d(x_\ell), d(y_\ell)\}$ and as in the proof of Lemma 3.7, $\frac{d(x_\ell)^m d(y_\ell)^m}{|x_\ell - y_\ell|^{2m}}$ is uniformly bounded from below. Therefore, (4.8) becomes

$$G(x_\ell, y_\ell) < \frac{1}{\ell} |x_\ell - y_\ell|^{2m-n}.$$

In Proposition 2.1 we have constructed the Green function in the following way

$$G(x,y) = \Gamma_0(x,y) + \sum_{j=1}^{k} \Gamma_j(x,y) + u_x(y), \tag{4.9}$$

where

$$\Gamma_0(x,y) = c_{m,n} F_{m,n}(x-y) = c_{m,n} |x-y|^{2m-n}.$$

Using (2.14) and (2.7) we get

$$|u_x(y)| \leq c \max\{d(x), d(y)\}^{2m-n}$$

and

$$|\Gamma_j(x,y)| \leq \begin{cases} C_j |x-y|^{2m+2j-n} & \text{if } 2m+2j < n, \\ C_j \left(1 + |\log|x-y||\right) & \text{if } 2m+2j = n, \\ C_j & \text{if } 2m+2j > n. \end{cases} \tag{4.10}$$

Our assumption together with (4.9) and (4.10) shows for ε small enough and all $\ell \geq \ell_0$ large enough that

$$\begin{aligned} \frac{1}{\ell} &> |x_\ell - y_\ell|^{n-2m} G(x_\ell, y_\ell) \\ &\geq c_{m,n} - |x_\ell - y_\ell|^{n-2m} \sum_{j=1}^{k} |\Gamma_j(x,y)| - c \left(\frac{|x_\ell - y_\ell|}{\max\{d(x_\ell), d(y_\ell)\}} \right)^{n-2m} \\ &\geq \frac{c_{m,n}}{2}, \end{aligned}$$

where $c_{m,n} > 0$. And this is a contradiction. Thus, we have proved the claim in this case.

Case 2. *There exists an $\varepsilon \in \left(0, \frac{1}{2}\right)$, such that $|x_\ell - y_\ell| \geq \varepsilon \max\{d(x_\ell), d(y_\ell)\}$.* Here, (4.8) becomes

$$G(x_\ell, y_\ell) < \frac{1}{\ell} |x_\ell - y_\ell|^{-n} d(x_\ell)^m d(y_\ell)^m.$$

Now we can proceed exactly as in the proof of Lemma 3.5 to show the claim in this case.

Since we have proved the claim in both cases, the lemma is proved. □

Corollary 4.5. *Suppose that $n > 2m$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then there exist a radius $r > 0$ and a constant $C > 0$ such that for all $x_0 \in \overline{\Omega}$ and for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$G(x,y) \geq C |x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}. \tag{4.11}$$

Proof. Combining Lemmas 4.3 and 4.4 we find for every $x_0 \in \overline{\Omega}$ a $r_0 = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r_0} := \overline{\Omega} \cap B_{r_0}(x_0)$ one has

$$G(x,y) \geq C|x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}.$$

Applying a compactness argument to $\overline{\Omega} = \bigcup_{x_0 \in \overline{\Omega}} \Omega_{x_0, r_{x_0}/2}$ we find $x_1, \ldots, x_k \in \overline{\Omega}$ such that $\overline{\Omega} \subset \bigcup_{j=1}^{k} \Omega_{x_j, r_j/2}$. For $x, y \in \overline{\Omega}$ with $|x-y| < r$ where $r := \min\{\frac{r_1}{2}, \ldots, \frac{r_k}{2}\}$ exists a j such that $x \in \Omega_{x_j, r_j/2}$. Then $y \in \Omega_{x_j, r_j/2}$ and the claim follows. □

Proof of Theorem 4.1 for $n > 2m$. Only the bound from below has to be proven, since the bound from above follows from Corollary 2.12, see also Remark 3.2.

We fix $r > 0$ as in Corollary 4.5. If $|x-y| < r$ there is nothing left to prove.

Let $|x-y| \geq r$. With the help of Corollary 2.12 we have a constant $c_{10} > 0$ such that

$$G(x,y) \geq -c_{10}|x-y|^{-n} d(x)^m d(y)^m.$$

Then

$$G(x,y) + 2c_{10}|x-y|^{-n} d(x)^m d(y)^m \geq c_{10}|x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}.$$

With $c_{11} := 2c_{10} r^{-n}$ it follows

$$G(x,y) + c_{11} d(x)^m d(y)^m \geq c_{10}|x-y|^{2m-n} \min\left\{1, \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right\}.$$

and the theorem for $n > 2m$ is proved. □

4.2 Small Dimensions $n < 2m$

We prove Theorem 4.1 for $n < 2m$ after some auxiliary results.

Lemma 4.6 (Estimates in the interior)**.** *Suppose that $n < 2m$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then for each $x_0 \in \Omega$ there exist a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$G(x,y) \geq C d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\}. \tag{4.12}$$

Proof. From Proposition 2.1 we have that G is continuous on $\Omega \times \Omega$. Let us show $G(x_0, x_0) > 0$ for $x_0 \in \Omega$. Assuming this is true we can use

$$d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\} \leq \operatorname{diam}(\Omega)^{2m-n}$$

to show the lemma, since by the continuity of G there exist r_{x_0}, c_{x_0} such that $G(x,y) > c_{x_0}$ for all $x, y \in B_{r_{x_0}}(x_0) \subset \Omega$.

It is left to prove $G(x_0, x_0) > 0$. Using the representation formula (2.3) for $G(x_0, \,.\,)$ and the uniform coercivity, see (2.2), after integration by parts we get

$$G(x_0,x_0) = \langle G(x_0,\,.\,), G(x_0,\,.\,)\rangle_{W_0^{m,2}} + \sum_{*} (-1)^{|\beta|} \int_\Omega a^\ell_{\alpha,\beta}(y) D^\alpha_y G(x_0,y) D^\beta_y G(x_0,y)\,dy$$
$$\geq \lambda \|G\|^2_{W_0^{m,2}} > 0.$$

This proves the lemma. □

Lemma 4.7 (Estimates near the boundary)**.** *Suppose that $n < 2m$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then for each $x_0 \in \partial\Omega$ there exist a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$G(x,y) \geq C d(x)^{m-n/2} d(y)^{m-n/2} \min\left\{1, \frac{d(x)^{n/2} d(y)^{n/2}}{|x-y|^n}\right\}. \tag{4.13}$$

Proof. Since we have the estimates from Theorem 2.4 for G, the claim can be proven exactly as Lemma 3.7 for the polyharmonic case. □

Proof of Theorem 4.1 for $n < 2m$. Combining Lemmas 4.6 and 4.7 and applying a compactness argument as in the proof of Corollary 3.9, the claim can be proven like Theorem 3.1 for the polyharmonic case, see Section 3.2.2. □

4.3 Dimension $n = 2m$

We prove Theorem 4.1 for $n = 2m$ after some auxiliary results.

Lemma 4.8 (Estimates in the interior)**.** *Suppose that $n = 2m$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then for each $x_0 \in \Omega$ there exist a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$G(x,y) \geq C \log\left(1 + \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right). \tag{4.14}$$

Proof. We have constructed the Green function in Proposition 2.1 in the following way

$$G(x,y) = \Gamma_0(x,y) + \sum_{j=1}^{k} \Gamma_j(x,y) + u_x(y),$$

where

$$\Gamma_0(x,y) := c_{m,2m} F_{m,2m}(x-y) = c_{m,2m} \log\left(\frac{1}{|x-y|}\right).$$

Since $x_0 \in \Omega$, we have $d(x_0) > 0$ and furthermore $d(x) > \frac{d(x_0)}{2}, d(y) > \frac{d(x_0)}{2}$ for all $x, y \in B_{d(x_0)/2}(x_0)$. Then, as in the proof of Proposition 2.1, see also Lemma 1.5, we get from (2.14) that $|u_x(y)| \leq C(d(x_0))$ and $|u_y(x)| \leq C(d(x_0))$.

Moreover, since the iterated kernels are bounded, see (2.17), we get

$$C_{x_0} \geq |\Gamma_0(x,y) - G(x,y)| \geq \Gamma_0(x,y) - G(x,y),$$

for all $x, y \in B_{d(x_0)/2}(x_0)$, which shows

$$G(x,y) \geq c_{m,2m} \log\left(\frac{1}{|x-y|}\right) - C_{x_0} = c \log\left(\frac{1}{e^{C_{x_0}/c_{m,2m}}|x-y|}\right).$$

We also have

$$\left(1 + \frac{1}{|x-y|^{2m}}\right)^{1/4m} \leq \frac{\left((\operatorname{diam}(\Omega))^{2m} + 1\right)^{1/4m}}{|x-y|^{1/2}} \leq \frac{c(m,\Omega)}{|x-y|^{1/2}}.$$

Now, since

$$\frac{c(m,\Omega)}{|x-y|^{1/2}} \leq \frac{1}{e^{C_{x_0}/c_{m,2m}}|x-y|} \quad \text{if and only if} \quad |x-y| \leq \frac{1}{(e^{C_{x_0}/c_{m,2m}}c(m,\Omega))^2},$$

we define $r = r_{x_0} := \min\left\{\frac{d(x_0)}{2}, \frac{1}{\left(e^{C_{x_0}/c_{m,2m}}c(m,\Omega)\right)^2}\right\}$ and have for all $x, y \in B_r(x_0)$:

$$G(x,y) \geq C \log\left(1 + \frac{1}{|x-y|^{2m}}\right).$$

We conclude the proof by examining the cases $d(x)^m d(y)^m \leq 1$ and $d(x)^m d(y)^m > 1$. For the latter case we use $(\operatorname{diam}(\Omega))^{2m} \geq d(x)^m d(y)^m$ and Bernoulli's inequality. $\square$

Lemma 4.9 (Estimates near the boundary)**.** *Suppose that $n = 2m$ and that $\Omega \subset \mathbb{R}^n$ is a bounded $C^{2m,\gamma}$-smooth domain. Then for each $x_0 \in \partial\Omega$ there exist a radius $r = r_{x_0} > 0$ and a constant $C = C_{x_0} > 0$ such that for all $x, y \in \Omega_{x_0,r} := \overline{\Omega} \cap B_r(x_0)$ one has*

$$G(x,y) \geq C \log\left(1 + \frac{d(x)^m d(y)^m}{|x-y|^{2m}}\right). \tag{4.15}$$

Proof. We prove the claim by assuming the contradiction: there exist $x_k, y_k \in \Omega_{x_0,1/k} = \overline{\Omega} \cap B_{1/k}(x_0)$ such that

$$G(x_k,y_k) < \frac{1}{k} \log\left(1 + \frac{d(x_k)^m d(y_k)^m}{|x_k-y_k|^{2m}}\right). \tag{4.16}$$

We have $x_k \to x_0, y_k \to x_0, d(x_k) \to 0, d(y_k) \to 0$ and $|x_k - y_k| \to 0$.

By rotation and translation we may assume $x_0 = 0$ and that the first unit vector e_1 is the exterior unit normal to $\partial\Omega$ at x_0. Let us fix $\rho := \min\left\{\frac{1}{2}, \delta_{m,2m}\right\}$, where we choose $\delta_{m,2m}$ as in Proposition 3.4. Then, we can pass to a subsequence to consider one of the following two cases.

Case 1. For all $\varepsilon \in (0,\rho)$ there exists a ℓ_0, such that for all $\ell \geq \ell_0$: $|x_\ell - y_\ell| \leq \varepsilon \max\{d(x_\ell), d(y_\ell)\}$.

Since we want to use the result from Proposition 3.4, we construct the Green function G as in Proposition 2.1

$$G_x(y) = \Gamma_0(x,y) + \sum_{j=1}^{k} \Gamma_j(x,y) + u_x(y), \tag{4.17}$$

but starting now with $\Gamma_0 = G_{(-\Delta)^m,\Omega}$ as the polyharmonic Green function.

The proof, that $G_x(y)$ is indeed the Green function for (2.1), works in the same way as for Proposition 2.1, since we have similar estimates for $|G_{(-\Delta)^m,\Omega}|$ as for the modulus of the polyharmonic fundamental solution, see Theorem 2.4. The difference arises from the zero Dirichlet boundary conditions, which are satisfied by $G_{(-\Delta)^m,\Omega}$. Therefore u_x satisfies the boundary value problem (2.10) with zero Dirichlet boundary conditions. Now, elliptic estimates and Sobolev embeddings show for all $x, y \in \Omega$ that

$$|u_x(y)| \leq C(m, n, \operatorname{diam}(\Omega), K, \lambda, \partial\Omega).$$

Note that the iterated kernels are again bounded

$$|\Gamma_j| \leq C_j,$$

for $j = 1, \ldots, k$, see (2.17).

As in the proof of Theorem 3.1 for $n \geq 2m - 1$, see Section 3.1.2, we get that

$$\frac{d(x_\ell)^m d(y_\ell)^m}{|x_\ell - y_\ell|^{2m}} \geq \left(\frac{1-\varepsilon}{\varepsilon^2}\right)^m. \tag{4.18}$$

Then, using Proposition 3.4 and (4.16) we see

$$\begin{aligned}
\frac{1}{\ell} \log\left(1 + \frac{d(x_\ell)^m d(y_\ell)^m}{|x_\ell - y_\ell|^{2m}}\right) &> G(x_\ell, y_\ell) \\
&= G_{(-\Delta)^m,\Omega}(x_\ell, y_\ell) + \sum_{j=1}^{k} \Gamma_j(x_\ell, y_\ell) + u_{x_\ell}(y_\ell) \\
&> c_4 \log\left(1 + \frac{d(x_\ell)^m d(y_\ell)^m}{|x_\ell - y_\ell|^{2m}}\right) - \sum_{j=1}^{k} C_j - C.
\end{aligned}$$

Together with (4.18) we have for ε small enough and all $\ell \geq \ell_0$ large enough that

$$\frac{1}{\ell} > c_4 - \frac{\sum_{j=1}^{k} C_j + C}{\log\left(1 + \left(\frac{1-\varepsilon}{\varepsilon^2}\right)^m\right)} \geq \frac{c_4}{2},$$

which is a contradiction.

<u>*Case 2.*</u> *There exists an $\varepsilon \in (0, \rho)$, such that $|x_\ell - y_\ell| \geq \varepsilon \max\{d(x_\ell), d(y_\ell)\}$.* Again, we use the proof of Lemma 3.5 to show the claim.

□

Proof of Theorem 4.1 for $n = 2m$. Combining Lemmas 4.8 and 4.9 and applying a compactness argument as in Corollary 3.9, the claim can be proven like Theorem 3.1 for the polyharmonic case, see Section 3.2.2. □

A Appendix

Lemma A.1. *Let $x \in \mathbb{R}^n$, $r = |x|$ and $\alpha \in \mathbb{N}_0^n$. Then, for $k \in \mathbb{N}$ is even, it holds*

$$D^\alpha(r^k \log r) = \begin{cases} \left(P_1^{|\alpha|}(x)\log r + P_2^{|\alpha|}(x)\right) r^{k-2|\alpha|} & \text{if } |\alpha| < k/2, \\ P_1^{k-|\alpha|}(x)\log r + P_2^{|\alpha|}(x) r^{k-2|\alpha|} & \text{if } k/2 \le |\alpha| \le k, \\ P_3^{|\alpha|}(x) r^{k-2|\alpha|} & \text{if } |\alpha| > k, \end{cases} \tag{A.1}$$

where $P_i^\lambda(x_1, \dots, x_n)$, $i = 1, 2, 3$, are homogeneous polynomials of degree $\lambda \in \mathbb{N}_0$.

Proof. First, we prove the following claim by induction over $|\alpha| \in \mathbb{N}_0$:

$$D^\alpha(r^k) = \begin{cases} Q^{|\alpha|}(x) r^{k-2|\alpha|} & \text{if } |\alpha| < k/2, \\ Q^{k-|\alpha|}(x) & \text{if } k/2 \le |\alpha| \le k, \\ 0 & \text{if } |\alpha| > k, \end{cases} \tag{A.2}$$

where $Q^\lambda(x_1, \dots, x_n)$ is a suitable homogeneous polynomial of degree $\lambda \in \mathbb{N}_0$, which may change from step to step (even in the same line).

If $|\alpha| = 0$, the claim (A.2) holds true.

For $\alpha \in \mathbb{N}_0^n$, $1 \le |\alpha| \le \frac{k}{2} - 1$, and k is even we see that

$$\begin{aligned} \partial_{x_j}\left(D^\alpha(r^k)\right) &= \partial_{x_j}\left(Q^{|\alpha|}(x) r^{k-2|\alpha|}\right) = Q^{|\alpha|-1}(x) r^{k-2|\alpha|} + Q^{|\alpha|+1}(x) r^{k-2(|\alpha|+1)} \\ &= r^2 Q^{|\alpha|-1}(x) r^{k-2(|\alpha|+1)} + Q^{|\alpha|+1}(x) r^{k-2(|\alpha|+1)} \\ &= Q^{|\alpha|+1}(x) r^{k-2(|\alpha|+1)}. \end{aligned}$$

By induction, we get (A.2) for the case $|\alpha| < k/2$.

If $|\alpha| = \frac{k}{2} - 1$, we obtain from (A.2) that

$$D^\alpha(r^k) = r^2 Q^{k/2-1}(x) = Q^{k-(k/2-1)}(x) = Q^{k-|\alpha|}(x),$$

from where we get

$$\partial_{x_j}\left(D^\alpha(r^k)\right) = \partial_{x_j}\left(Q^{k-|\alpha|}(x)\right) = Q^{k-(|\alpha|+1)}(x).$$

The claim (A.2) follows by induction for the cases $k/2 \le |\alpha| \le k$ and $|\alpha| > k$.

Moreover, one can easily prove by induction for all $|\alpha| \geq 1$ that

$$D^\alpha(\log r) = r^{-2|\alpha|}\tilde{Q}^{|\alpha|}(x), \tag{A.3}$$

where $\tilde{Q}^\lambda(x)$ is a suitable homogeneous polynomial of degree λ.

The general Leibniz rule shows

$$D^\alpha(r^k \log r) = D^\alpha(r^k)\log r + \sum_{\beta<\alpha}\binom{\alpha}{\beta} D^\beta(r^k)\cdot D^{\alpha-\beta}(\log r). \tag{A.4}$$

Using (A.2) and (A.3) we obtain for $|\beta| < \frac{k}{2}$ that

$$\begin{aligned} D^\beta(r^k)\cdot D^{\alpha-\beta}(\log r) &= Q^{|\beta|}(x) r^{k-2|\beta|}\cdot r^{-2|\alpha|+2|\beta|}\tilde{Q}^{|\alpha|-|\beta|}(x) \\ &= \bar{Q}^{|\alpha|}(x) r^{k-2|\alpha|}, \end{aligned} \tag{A.5}$$

where $\bar{Q}^\lambda(x)$ is a suitable homogeneous polynomial of degree λ.

For $\frac{k}{2} \leq |\beta| \leq k$ we see for a suitable $s \in \mathbb{N}_0$ that $|\beta| = \frac{k}{2} + s$. This shows together with (A.2) and (A.3) that

$$\begin{aligned} D^\beta(r^k)\cdot D^{\alpha-\beta}(\log r) &= Q^{k-|\beta|}(x)\cdot r^{-2|\alpha|+2|\beta|}\tilde{Q}^{|\alpha|-|\beta|}(x) \\ &= Q^{k/2-s}(x)\cdot r^{2s} r^{k-2|\alpha|}\tilde{Q}^{|\alpha|-k/2-s}(x) \\ &= \bar{Q}^{|\alpha|}(x) r^{k-2|\alpha|}, \end{aligned} \tag{A.6}$$

where $\bar{Q}^\lambda(x)$ is a suitable homogeneous polynomial of degree λ.

Using (A.2) we see for $|\beta| > k$ that

$$D^\beta(r^k)\cdot D^{\alpha-\beta}(\log r) = 0. \tag{A.7}$$

Combining (A.5)–(A.7) and (A.2) with (A.4) the claim (A.1) follows. □

Proposition A.2. *Let $n \in \mathbb{N}$ and $\lambda \in [0,1]$. Then it holds:*

$$\sum_{j=1}^{n}\binom{n}{j}\frac{(-1)^{j-1}}{j}\lambda^{2j} = \sum_{j=1}^{n}\frac{1}{j}\left(1-\left(1-\lambda^2\right)^j\right).$$

Proof. If $\lambda = 0$ nothing is to show. Let $\lambda > 0$. We evaluate the integral

$$I_n(\lambda) := \int_0^\infty \left(1-\left(1-\lambda^2 e^{-t}\right)^n\right) dt$$

in two possible ways. First of all, using the substitution $x = 1-\lambda^2 e^{-t}$ we get

$$I_n(\lambda) = \int_{1-\lambda^2}^{1}\frac{1-x^n}{1-x}dx = \int_{1-\lambda^2}^{1}\left(1+x+x^2+\ldots+x^{n-1}\right)dx = \sum_{j=1}^{n}\frac{1}{j} - \sum_{j=1}^{n}\frac{1}{j}\left(1-\lambda^2\right)^j.$$

The other way around, by the binomial theorem we have

$$\left(1 - \lambda^2 e^{-t}\right)^n = 1 + \sum_{j=1}^{n} \binom{n}{j} (-1)^j \lambda^{2j} e^{-tj}.$$

Thus,

$$I_n(\lambda) = \sum_{j=1}^{n} \binom{n}{j} (-1)^{j-1} \lambda^{2j} \int_0^\infty e^{-tj} dt = \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^{j-1}}{j} \lambda^{2j}$$

and the proposition is proved. □

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List of Symbols

α, β	multiindices $\in \mathbb{N}_0^n$, $\lvert\alpha\rvert = \sum_{i=1}^n \alpha_i$.
$B_r(x_0)$	$= \{x \in \mathbb{R}^n : \lvert x - x_0\rvert < r\}$.
B_r	$= B_r(0)$.
C, c	positive constants in estimates, which may change from term to term and depend on the parameters given in brackets.
$\lVert u\rVert_{C(\overline{\Omega})}$	$= \sup_{x\in\Omega}\lvert u(x)\rvert$.
$\lVert u\rVert_{C^k(\overline{\Omega})}$	$= \sum_{\lvert\alpha\rvert\le k}\lVert D^\alpha u\rVert_{C(\overline{\Omega})}$.
$\lVert u\rVert_{C^{k,\gamma}(\overline{\Omega})}$	$= \sum_{\lvert\alpha\rvert\le k}\lVert D^\alpha u\rVert_{C(\overline{\Omega})} + \sup_{\lvert\alpha\rvert=k}\sup_{x\neq y}\left\{\frac{\lvert D^\alpha u(x)-D^\alpha u(y)\rvert}{\lvert x-y\rvert^\gamma}\right\}$.
$C_c^\infty(\Omega)$	$= \{u \in C^\infty(\Omega) : u$ has compact support in $\Omega\}$.
D^α	$= \prod_{i=1}^n \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}$.
$D^k u$	$= \{D^\alpha u : \lvert\alpha\rvert = k\}$.
$d(x)$	$= \operatorname{dist}(x, \partial\Omega) = \inf_{x^*\in\partial\Omega}\lvert x - x^*\rvert$, $x \in \Omega$.
$\operatorname{diam}(\Omega)$	$= \sup\{\lvert x - y\rvert : x, y \in \Omega\}$.
e_n	$= \int_{B_1(0)} dx$.
$G_{(-\Delta)^m,\Omega}$	Green's function for $(-\Delta)^m$ under Dirichlet boundary conditions in $\Omega \subset \mathbb{R}^n$.
G	Green's function for the perturbed polyharmonic operator under Dirichlet boundary conditions in $\Omega \subset \mathbb{R}^n$.
$\mathcal{H}$	$= \{x \in \mathbb{R}^n : x_1 < 0\}$, half space.
$I_{\partial\Omega}(\,.\,,\,.\,)$	boundary integral, see (1.13) and (1.14).
$\mathbb{1}_A(x)$	$= \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A; \end{cases}$ indicator function.
$k_{m,n}$	$= \frac{1}{ne_n 4^{m-1}((m-1)!)^2}$.
$\mathbb{N}_0$	$= \mathbb{N} \cup \{0\}$.
ν	exterior unit normal at $\partial\Omega$.
Ω	domain, an open and connected subset of $\mathbb{R}^n$.
$P^\lambda(x), Q^\lambda(x)$	homogeneous polynomials of degree $\lambda \in \mathbb{N}_0$, $P^\lambda(tx) = t^\lambda P^\lambda(x)$ for $t \in \mathbb{R}, x \in \mathbb{R}^n$.
$\operatorname{supp} u$	$= \overline{\{x \in \mathbb{R}^n : u(x) \neq 0\}}$.

$\mathrm{vol}(\Omega)$	$= \int_\Omega dx$.
$W^{m,p}(\Omega)$	Sobolev space of the m-times weakly differentiable functions in Ω with L^p-derivatives.
$\|u\|_{W^{m,p}}$	$= \left(\|u\|_{L^p}^p + \|D^m u\|_{L^p}^p\right)^{1/p}$.
$\|u\|_{W_0^{m,p}}$	$= \|D^m u\|_{L^p}$.
$\langle u, v\rangle_{W_0^{m,2}}$	$= \begin{cases} \int_\Omega \Delta^{m/2}u\Delta^{m/2}v\ dx & \text{if } m \text{ is even,} \\ \int_\Omega \nabla(\Delta^{(m-1)/2}u)\cdot\nabla(\Delta^{(m-1)/2}v)\ dx & \text{if } m \text{ is odd.} \end{cases}$
$W_0^{m,p}(\Omega)$	in bounded domains Ω, closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\,.\,\|_{W_0^{m,p}}$.

Index

Erklärung

Ich versichere hiermit, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; verwendete fremde und eigene Quellen sind als solche kenntlich gemacht.

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Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form als Dissertation eingereicht und ist als Ganzes auch noch nicht veröffentlicht.

Magdeburg, den 25.11.2014