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Achievements in arithmetic and measurement units predict fraction understanding in an additive and linear manner

Markus Wolfgang Hermann Spitzer[a,*,1], Miguel Ruiz-Garcia[b], Younes Strittmatter[c], Eileen Richter[a], Raphael Gutsfeld[a], Korbinian Moeller[d,e]

[a] *Department of Psychology, Martin-Luther University Halle, Halle, Germany*
[b] *Departamento de Estructura de la Materia, Física Térmica y Electrónica, Universidad Complutense Madrid, Madrid 28040, Spain*
[c] *Department of Psychology, Princeton University, NJ, United States*
[d] *Centre for Mathematical Cognition, Loughborough University, UK*
[e] *LEAD Graduate School, University of Tuebingen, Germany*

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ABSTRACT

Learning fractions is one of the most difficult but nevertheless critical mathematical topics in school as understanding fractions significantly predicts later mathematics achievement and vocational prospects. Importantly, mastery of basic mathematical topics (e.g., arithmetic skills) was repeatedly observed to serve as a stepping stone for learning fractions. However, it has not yet been investigated in detail whether achievements on such basic mathematical topics predict fraction understanding uniquely and linearly or whether there are also multiplicative and non-linear dependencies. Such multiplicative and/or non-linear dependencies would suggest that closing knowledge gaps on key topics is of paramount importance, as knowledge gaps on these topics could have negative consequences for the understanding of fractions. Therefore, we predicted students' fraction understanding by their performance on four prior topics (i.e., *Geometry*, *Basic Arithmetic*, *Measurement Units*, and *Advanced Arithmetic*) and compared the fits of different regression models (including topics as main effects only vs. also including interaction and quadratic terms). Our analyses considered three cohorts of students (approximate age range: 12–13 years) attending different school tracks that vary in difficulty (i.e., 6468 students of academic track schools; as well as 4598 students, and 1743 students of two vocational track schools) who used an intelligent tutor system. Results were similar across all three cohorts substantiating the robustness of our results: students' fraction understanding was linearly predicted by achievements in basic mathematical skills (i.e., arithmetic and measurement units). We found no substantial support favoring more complex models across all three cohorts. As such, the results suggested that achievements in arithmetic and measurement units serve as unique and linear stepping stones for later fraction understanding. These findings suggest that those students with knowledge gaps in arithmetic and measurement units should be encouraged to revise these topics before moving on to more advanced topics—such as fractions—as these more advanced topics build on them.

* Corresponding author.
E-mail address: markus.spitzer@psych.uni-halle.de (M.W.H. Spitzer).

[1] The data that support the findings of this study are available from the corresponding author

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1. Introduction

It is well known that many students face considerable difficulties when learning fractions (Jordan et al., 2013; Lortie-Forgues et al., 2015; Siegler & Lortie-Forgues, 2014; Siegler & Braithwaite, 2017a; National Mathematics Advisory Panel, 2008; Siegler et al., 2012; Siegler & Braithwaite, 2013; Wortha et al., 2023; Braithwaite et al., 2017; Braithwaite & Siegler, 2024; Braithwaite et al., 2019). At the same time, however, understanding fractions seems critical for the acquisition of more advanced mathematical skills, as it was repeatedly shown to predict later mathematical achievements in general and algebra in particular (Bailey et al., 2012; Bailey et al., 2014; Siegler & Pyke, 2013; Mou et al., 2016; Park & Esposito, 2024; Torbeyns et al., 2015). This seems specifically relevant as better overall mathematics achievement was also observed to significantly predict later job prospects, well-being, as well as socioeconomic status (Murnane et al., 1995; National Mathematics Advisory Panel, 2008; Ritchie & Bates, 2013; Rivera-Batiz, 1992; Parsons & Bynner, 2007; Gerardi et al., 2013; Reyna et al., 2009). However, despite this long-term relevance of fraction understanding, research on identifying predictors of fraction understanding is still in progress.

Importantly, mastery of basic mathematical concepts (e.g., achievements in whole number magnitude, arithmetic, and measurement units) was repeatedly observed to significantly predict fraction understanding (Jordan et al., 2013; Bailey et al., 2014; Hansen et al., 2015; Vukovic et al., 2014; Siegler & Lortie-Forgues, 2014; Spitzer & Moeller, 2022; Siegler & Pyke, 2013; Siegler & Braithwaite, 2017a; Xu et al., 2024b; Xu et al., 2024a). However, one commonality of almost all previous studies is that linear regression models were applied. In addition, previous studies only evaluated main effects when predicting fraction understanding based on the achievements on specific basic mathematical topics (Jordan et al., 2013; Hansen et al., 2015; Vukovic et al., 2014; Bailey et al., 2014; Spitzer & Moeller, 2022). This allowed the evaluation of the unique (additive) and linear contribution of multiple predictor variables on fraction understanding. However, it remains an open question whether fraction understanding is indeed only predicted by basic mathematical topics in an additive (i.e., main effects only) manner, or whether i) there are also multiplicative effects between different basic mathematical skills (i.e., interaction effects) and ii) this prediction is of a linear or non-linear nature.

More insights into the specific dependencies between specific basic mathematical topics and their effects on fraction understanding are important to improve both our theoretical understanding of numerical development and to provide evidence-based guidance for practitioners and software developers on how to best facilitate fraction learning. For example, consider the case that a specific basic mathematical topic (e.g., whole number arithmetic) can be identified to multiply the effect of another basic mathematical topic (e.g., measurement units) on fraction understanding (reflected by a significant interaction effect between arithmetic and measurement units in predicting fraction understanding). In this case, focusing on these specific topics would be of utmost relevance to create a solid foundation for learning more complex topics, such as fractions (also see Figure 1A for an illustration of this point). In addition, when, for instance, non-linear relationships between basic mathematical topics and fraction understanding would be observed, this might hint towards fostering students in optimal zones for learning where little improvements on a basic mathematical topic leads to a considerably higher probability of understanding fractions (see Figure 1B). In this study, we therefore sought to investigate whether performance on basic mathematical topics predict fraction understanding in an additive and linear manner or rather multiplicatively and/or non-linearly.

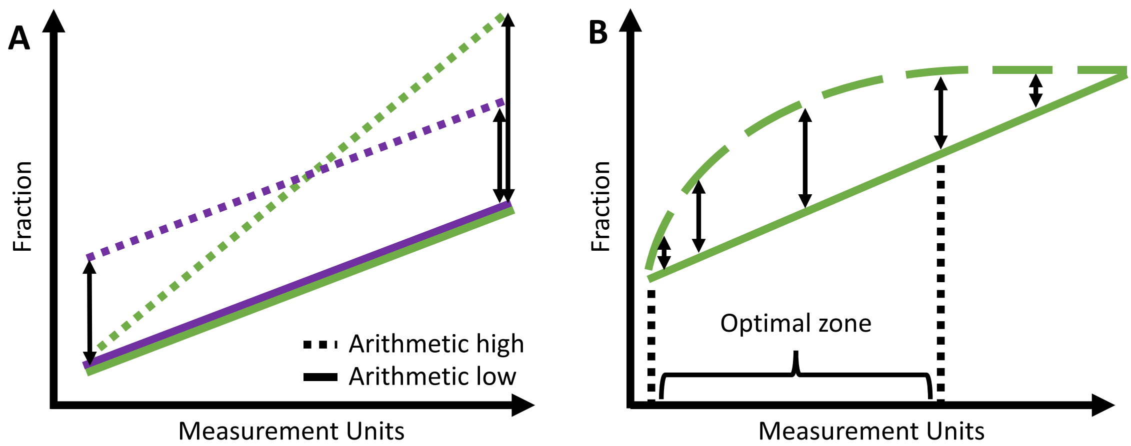

Fig. 1. Theoretical predictions of linear main effects vs. interaction effects (A) and linear vs. non-linear effects (B). A: A hypothetical main effect is depicted by parallel regression lines indicating that better measurement unit skills are associated with better fraction understanding consistently across differing levels of achievements in arithmetic (dashed vs. solid line). A hypothetical interaction effect is reflected by regression lines indicating that better arithmetic skills lead to a steeper increase of fraction understanding based on measurement units for those with better arithmetic skills (dashed line) compared to those performing poorer on arithmetic (solid line). Note that the interaction effect suggests that the better the mastery of arithmetic, the larger the performance boost of measurement units on fraction understanding. B: A hypothetical higher-order non-linear relationship of measurement unit skills on fraction understanding is depicted by the dashed line while a linear effect of measurement unit skills on fraction understanding is illustrated by the solid line. Solid arrows show the additional benefit students would have from improving their measurement unit skills for learning fractions in case the relationship is non-linear. Note that these additional benefits would be especially high within an optimal zone of learning where increases in measurement unit skills would particularly boost fraction learning.

In the following, we briefly introduce previous findings on the prediction of fraction understanding based on achievements in basic mathematical topics. We then elaborate the integrated theory of numerical development and how it explains why achievements in basic mathematical topics predict fraction understanding. Another section briefly elaborates on the importance of finding out whether basic mathematical skills predict fraction understanding in an additive or multiplicative and/or linear or non-linear manner. Finally, we describe the present study.

1.1. Achievements in basic mathematical topics predict fraction understanding

Previous findings on predicting fraction understanding based on achievements in basic mathematical topics mostly stem from longitudinal studies employing face-to-face testing and applying linear regression analysis considering main effects (Jordan et al., 2013; Bailey et al., 2014; Hansen et al., 2015; Vukovic et al., 2014). For instance, Jordan et al. (2013) observed that students' achievements in whole number arithmetic, as well as number line estimation assessed at the beginning of third grade, uniquely and significantly predicted fraction understanding at the end of fourth grade while controlling for variables such as attention, literacy, and working memory. These results are in line with those by Hansen et al. (2015), who also found that achievements in whole number arithmetic, number line estimation, and attention in fifth grade robustly and significantly predicted fraction understanding in sixth grade (Hansen et al., 2015). Two further longitudinal studies substantiated the main effects of achievements of whole number arithmetic on fraction understanding (Vukovic et al., 2014; Bailey et al., 2014). Together, these studies corroborate the idea of a hierarchical development of mathematical topics with more advanced aspects, such as fractions understanding, building on earlier mathematical topics (also see Xu et al. (2024a)).

Furthermore, a recent study found similar results when analyzing data from an intelligent tutoring system (ITS) for learning mathematics (Spitzer & Moeller, 2022). In their study, Spitzer & Moeller (2022) analyzed data from more than 5000 students who completed over 1 million problem sets within the *bettermarks* ITS. They observed that arithmetic performance significantly predicted fraction understanding. Moreover, Spitzer & Moeller (2022) extended previous findings reporting that beyond arithmetic, also performance in geometry and measurement units significantly predicted fraction understanding. A theoretical account for why such predictions should be observed is proposed by the integrated theory of numerical development.

1.2. The integrated theory of numerical development

The integrated theory of numerical development provides a comprehensive framework for understanding how children's development of numerical and mathematical skills progresses from whole number knowledge to basic arithmetic skills to even more advanced mathematical concepts, such as fractions (Siegler et al., 2011; Siegler & Lortie-Forgues, 2014; Siegler, 2016; Siegler & Braithwaite, 2017b). It suggests a hierarchical development of mathematical topics with the acquisition of numerical magnitude understanding as the common core of numerical development from whole to rational number understanding. Numerical magnitude understanding refers to the ability to conceptualize and compare the magnitudes of numbers on a mental number line. The theory further posits that numerical development involves a gradual expansion in ranges (e.g., up to 10, up to 1000), but also types of numbers (i.e., positive numbers, natural numbers, rational numbers) whose magnitudes can be accurately represented (Siegler & Braithwaite, 2017b). Accordingly, a proficient understanding of whole numbers (including basic arithmetic operations on them) is essential for developing an understanding of more complex number types such as rational numbers (e.g., fractions; Siegler (2016); Siegler & Braithwaite (2017b)).

Acquiring whole number magnitude understanding and arithmetic skills is therefore vital for understanding fractions later on, as it allows students to grasp how fractions represent parts of a whole and how these parts compare to one another. As such, the integrative theory of numerical development suggests that students who excel in whole number understanding and arithmetic should develop a better representation of (whole) number magnitude, which, in turn, should facilitate their understanding of fractions. The longitudinal findings reported above from studies using face-to-face testing and online learning substantiated that mastery of basic arithmetic skills uniquely contributes to the prediction of later fraction understanding and thus corroborates with the predictions of the integrated theory of numerical development. However, the question remains whether this development is indeed linear and additive in nature.

1.3. Linear and additive vs. non-linear and interactive development of mathematical skills

The specific predictive patterns of basic mathematical topics for fraction understanding are still elusive. For instance, so far, it remained unclear whether mastery of basic mathematical topics found to predict fraction understanding in isolation, also influences other later topics known to also predict fraction understanding. This, for example, would be reflected in significant interactions between predictors when applying linear regressions (see Figure 1A). For instance, one might expect that a better understanding of basic arithmetic operations not only influences later fraction understanding directly. Instead, basic arithmetic may also scale the effect of measurement units (which is typically taught after basic arithmetic in the curriculum) on later fraction understanding because most of the problems that include measurement units will also involve basic arithmetic operations and thus basic arithmetic skills are needed to work with measurement units successfully (e.g., "Ben buys 1,5 kg of apples, Liz 750 g plums, and Steve 2 kg of peaches. What is the overall weight of the fruits they bought?"). In other words, it is unclear whether mastery of (or knowledge gaps when put negatively) specific basic mathematical topics (e.g., arithmetic) may influence the effect of other basic mathematical topics (e.g., measurement units) on fraction understanding.

If so, this would have significant implications for educators and software developers. It would suggest to prioritize mastery of specific fundamental topics and addressing knowledge gaps in those topics, which amplify the effect of other fundamental topics on understanding fractions. Gaps in understanding these topics would not only directly hinder learning about fractions but also enhance the influence of other topics on fraction understanding, worsening the negative outcomes (see Figure 1A).

However, it is still unclear whether the relationship between basic mathematical skills and fraction understanding is indeed linear in nature, as implicitly assumed by most previous studies when using linear regression models. Instead, the relationship between mastery of basic mathematical topics and fraction understanding may rather be of a non-linear nature, reflected in a better model fit of models considering a quadratic function (second polynomial) to the data instead of a linear function (first polynomial).[1] If so, this would indicate that revising basic mathematical topics for which there are severe knowledge gaps should have a stronger beneficial effect on fraction learning than revising basic mathematical topics with only moderate knowledge gaps. In particular, in case a non-linear relationship between basic mathematical topics and fraction understanding would be observed, this would suggest that children's development of fraction understanding may particularly benefit from revisiting basic mathematical concepts for which little improvements lead to larger improvements in fraction understanding (for a hypothetical example see Figure 1B). As such, a better understanding of the exact pattern of predictions would hold significant value for educators.

1.4. The present study

Accordingly, the present study evaluated three distinct data sets (see Methods for details) from the ITS for learning mathematics *bettermarks* to investigate whether i) mastery of basic mathematical topics additively predicted fraction understanding (reflected in a better model fit when including several basic mathematical skills as main effects only in a regression model), ii) basic mathematical skills predicted fraction understanding in a multiplicative manner (reflected in a better model fit when including several basic mathematical skills as main effects and interaction effects between these skills in a regression model), and iii) whether the prediction of fraction understanding based on basic mathematical skills is indeed of linear, or rather non-linear, nature. All three considered data sets had in common that students successively worked through the same four basic mathematical topics (i.e., geometry, basic arithmetic, measurement units, and advanced arithmetic) according to the curriculum before working through fraction problems.

In particular, we first ran a baseline model that predicted students' fraction understanding (i.e., their average accuracy on the fraction topic) based on their average achievements (i.e., their average accuracy) in each of the basic mathematical topics in an additive manner (i.e., only considering main effects in the regression model). In the next step, we then evaluated whether a more complex model with the same independent and dependent variables predicted fraction understanding better (with respect to BIC and R^2). To do so, we ran a backward regression analysis starting with all main effects and interaction effects (i.e., products between the four basic mathematical topics). Finally, we repeated the above steps but also fitted quadratic models. Again, we started with a full model including all interaction terms as well as quadratic terms and applied a backward regression analysis approach to evaluate whether one of these more complex quadratic models fitted the data better than the baseline model that only included main effects.

Our modeling approach allowed us to i) evaluate the potential contribution of interactions between several predictor variables (i.e., achievements in several mathematical topics) on the influence on fraction understanding and ii) investigate the nature of the prediction, this means, whether it reflects a linear and/or a quadratic relationship. As such, our model comparison approach might lead to a better understanding of whether specific combinations of basic mathematical topics predict fraction understanding significantly and whether the prediction of basic mathematical topics is of a linear or non-linear nature.

We expected to replicate previous findings on the relevance of basic mathematical skills for fraction understanding, with achievements in arithmetic, measurement units, and geometry significantly predicting fraction understanding (Spitzer & Moeller, 2022). However, due to the exploratory nature of this project, we did not have any a-priori hypotheses regarding whether the consideration of interaction terms would increase the model fits. Additionally, we had no hypotheses whether the relationship between basic mathematical topics would be linear or non-linear.

Given this exploratory approach, we evaluated a total of three different datasets to replicate the analyses on the first dataset twice. The rationale for including all three levels (i.e., a dataset considering students enrolled in different school tracks that vary in difficulty; a high-level, medium-level, and low-level track) was to determine whether we would observe robust and consistent results across the different datasets, in particular because our initial model fit analysis was exploratory without pre-established hypotheses.

2. Method

2.1. Participants

The study considered data from the curriculum-based ITS *bettermarks* for learning mathematics as used in the Netherlands Spitzer & Moeller (2022); Spitzer et al. (2024b). The data from students who use the ITS serves as a rich source to investigate students' longitudinal development of maths skills (e.g., see Spitzer & Moeller (2022); Stapel et al. (2016); Spitzer & Moeller (2024); Whalen et al. (2023); Spitzer et al. (2024a, 2024b); Spitzer (2022). Critically, the school system in the Netherlands differentiates between three tracks (academic track vs. two more vocational tracks) that vary in difficulty. Here, we refer to the three tracks as high-level, medium-level, and low-level. All considered students were within the first year of their secondary school and worked through problem sets

[1] Note that we only consider the comparison between a linear function and a higher-order non-linear quadratic function in this study. We did not consider a cubic function or any other higher-order non-linear function.

suited for the first year of their respective track. Thus, it is reasonable to assume that all considered students were between 12 and 13 years of age.

We only considered students who met the following inclusion criteria. First, we only included students who used the ITS in the Netherlands. Second, the present study considered students who used the ITS between January 1, 2016, and March 1, 2022. Third, we considered the data of all students from each of the three tracks resulting in a high-level, medium-level, and low-level dataset. The order of mathematical topics is prespecified by the ITS (it follows this order: Geometry, Basic Arithmetic, Measurement Units, Advanced Arithmetic, and Fractions) and reflects the curriculum in the Netherlands. However, teachers may assign problem sets in any other order. Thus, we only included students who completed the topics in the order as implemented in the ITS, which was (1) Geometry, (2) Basic Arithmetic, (3) Measurement Units, (4) Advanced Arithmetic, and (5) Fractions. This allowed us to evaluate data from the same mathematical topics across the three different tracks. Fourth, we included only students, that had worked through at least ten problem sets in each topic to get a reliable estimate of students' performance on a mathematical topic. Fifth, as students are able to repeat problem sets, we considered students' best result on each problem set as the estimate of students' performance. However, note that repetition rates were rather low (mean repetition rate = 1.6 %), and results were virtually identical when considering the results of students' first attempt instead of their best results.

After we applied these inclusion criteria, the three datasets comprised 12,573 students who computed 2,195,195 problem sets, resulting in a total of 18,049,172 mathematical problems. The first dataset comprised 6468 students of the academic track (labeled as high-level) who worked through 1,197,110 problem sets. The second dataset considered 4598 students of the second highest educational track (labeled as medium-level) who completed 814,774 problem sets. The third dataset included 1743 students of the third highest educational track (henceforth low-level) who worked through 183,311 problem sets.

The ITS logs several events when used by students and/or teachers and stores the data fully anonymized, without collecting personal data such as names, age, or gender from students or teachers. In particular, the ITS logs the accuracy of computed problem sets, the date when students computed a problem set, as well as the problem set ID to recover the content that was computed. All users agreed that their anonymized data was stored and analyzed when they signed up with the ITS. The data analysis was in accordance with the ethical standards of the national research committee and with the 1964 Helsinki declaration and its later amendments. The data can not be traced back to any individual student.

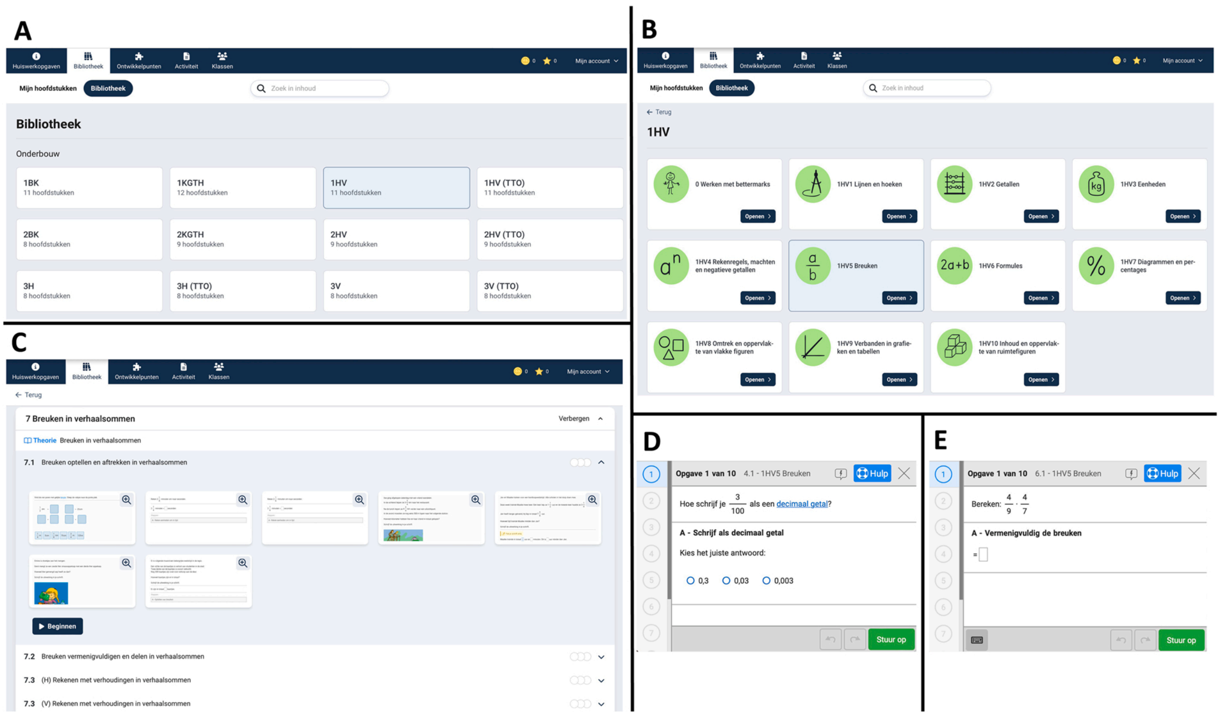

Fig. 2. The current bettermarks user interface. A: Users may first select one of the secondary tracks (here 1HV). B: This track covers different books. The first five books topics were considered in this study. For simplicity, we refer to "1TTO1 Lines and angles" as Geometry, "1TTO2 Numbers" as Basic Arithmetic, "1TTO3 Units of measurements" as Measurement Units, "1TTO4 Order of operations, powers, and negative numbers" as Advanced Arithmetic, and "1TTO5 Fractions" as Fractions. Book topics can be selected by pressing "Openen" (to open in Dutch). C: Users are provided with different chapters they may select. D: An exemplary problem within a problem set. E: Another exemplary problem.

2.2. Apparatus and material

The data used was leveraged from the ITS *bettermarks*. The ITS organizes the mathematical content in the following hierarchical structure from top to bottom (for the user interface, see Figure 2). At the highest level, there are over 100 distinctive "book topics", each of which covers a different mathematical topic (e.g., geometry, measurement units, arithmetic, or fractions). Each book topic typically comprises several chapters (second level), each of which comprises several problem sets (third level), which typically contain several (8–9 on average) individual mathematical problems (fourth level).

Table 1
Overview of books, chapters, and the number of unique problem sets within each chapter.

Book	Chapter name	Problem sets
Geometry	1 Lines, parallel and perpendicular	10
	2 Drawing line segments of a given length digitally and on paper	5
	3 Drawing and measuring angles	10
	4 Calculations on angles	10
	5 Circles, perpendiculars and bisectors	8
	6 Symmetry	8
Basic Arithmetic	1 Digits and numbers	7
	2 Compare numbers	6
	3 Calculating with large numbers	6
	4 Calculating with decimal numbers	11
	5 Rounding	12
	6 Estimating	6
Units	1 Measures of length	10
	2 Charts and scale	6
	3 Measures of mass	9
	4 Calculating with time	7
	5 All kinds of units of measurement	6
Advanced Arithmetic	1 Concepts of arithmetic	3
	2 Order of operations	7
	3 Real-world math	5
	4 Squares and roots	5
	5 Powers	7
	6 Negative numbers	5
	7 Adding and subtracting with negative numbers	8
	8 Calculating with negative numbers	11
	9 Powers with a minus sign	6
Fractions	1 Multiples and factors	10
	2 Prime numbers	8
	3 Set up and simplify fractions	15
	4 Fractions and decimal numbers	10
	5 Adding and subtracting fractions	8
	6 Multiplying and dividing fractions	10
	7 Fractions in context exercises	4
	8 Fractions and the rules of calculation	7
	9 Fractions and powers	4

2.3. Procedure

The Dutch educational system typically uses the ITS within the classroom context (Spitzer et al., 2024b; Spitzer & Moeller, 2022). Teachers assign problem sets to students, which students have to solve independently. When teachers assign problem sets, they receive feedback from the ITS on how well students solved the respective assignments. Students also receive feedback from the ITS while they work on the assignments. They may repeat problems, however, the parameters of the problems (e.g., the specific numerical values in the problem sets) change with each new attempt to prevent students from memorizing the result of their previous attempt (for more details on how the ITS is used, see Spitzer & Moeller (2022); Spitzer et al. (2023), Spitzer et al., 2024b).

Each of the three considered school tracks that is implemented within the ITS has separate books (referred to as mathematical topics here). When students work through problem sets within the ITS, the system tracks the specific problem sets of the specific book students worked on. This allowed us to evaluate accuracy for each school track separately in three distinct analyses reported below. Crucially, while the problem sets vary in difficulty between school tracks, all three tracks implemented the following five topics, which we considered in our analyses: Geometry, Basic Arithmetic, Measurement Units, Advanced Arithmetic, and Fractions (for more details on these topics, see Spitzer & Moeller (2022); also see Table 1 for a list of all chapters included within these books).

2.4. Data analysis

For each student, the average accuracy was calculated as the mean of the averages on the problem sets of the four book topics, respectively, (i.e., Geometry, Measurement Units, Basic Arithmetic, and Advanced Arithmetic) served as the independent variables and as a proxy for student's performance on each of the four book topics. The average accuracy of each student for the Fraction book topic served as the dependent variable and as a proxy for fraction understanding.

We used the R software for statistical modeling to conduct the analysis (R Core Team, 2023) and generated plots with the sjPlot package (Lüdecke, 2020). We also used the stepAIC function of the MASS package (Ripley et al., 2013) to conduct the backward regression procedure from which we obtained the Bayesian information criterion (BIC) and R^2 as model fit estimates (BIC in addition to R^2 as it penalizes model complexity). A smaller BIC generally indicates a better model fit (Burnham et al., 1998). When comparing two models, a difference in BIC of 10 or more indicates a meaningful difference in model fit (Burnham et al., 1998). We additionally considered R^2 as a measure as R^2 reflects the percent of variance explained by the model, and thus, we can directly interpret it.

We conducted the data analysis in three steps and for each of the three datasets separately. In Step 1, we computed a baseline model where fraction understanding was predicted by the four other mathematical topics using linear regression considering main effects only with no interactions. We conducted this analysis to obtain the BIC and R^2 of the baseline model.

In Step 2, we expanded the linear regression model from Step 1 to consider all possible interactions between the four predictor variables. We then ran a backward regression analysis applying the stepAIC function to obtain the BIC of the backward regression procedure from the model that now considered main effects and all interaction terms between the independent variables. We compared the BIC and R^2 of the model of this backward regression procedure against the baseline model to examine which of the two models had a lower BIC and explained more variance.

In a final Step 3, we further expanded our model comparison approach by also considering quadratic terms in the regression model. We followed the same procedure as in Step 2 but started with a full model with all interactions and quadratic terms for each independent variable. We then ran the backward regression procedure to identify the best-fitting model of this backward regression procedure against the baseline model. Based on these three analysis steps, we sought to identify whether the consideration of additional interaction and/or additional quadratic terms led to a better model fit to the data as indicated by a smaller BIC and a higher R^2.

We identified the best-fitting model based on the following criteria. First, the best-fitting model was selected if the BIC was the lowest or at least 10 points lower than the BIC of the baseline model. However, it may be that a model fitted the data better according to BIC but without explaining substantially more variance than the baseline model. Thus, we also evaluated the R^2 and only selected a best-fitting model when it also explained more than 1 % variance more than the baseline model. Importantly, these two model selection criteria were set prior to any data analysis.

Table 2
Model comparison results.

Regression type	Formula	High-level dataset BIC	R^2	Formula	Medium-level dataset BIC	R^2	Formula	Low-level dataset BIC	R^2
BM	**G+BA+U+AA**	**−14265.04**	**.43**	**G+BA+U+AA**	**−10925.35**	**.40**	**G+BA+U+AA**	**−4206.28**	**.36**
BR	G*BA*AA+BA*U*AA	−14252.65	.43	BA+U+AA	−10933.64	.40	BA+U*AA	−4222.42	.36
QBR	poly(G,2)+poly(U,2)+ poly(AA,2)	−14252.6	.43	poly(BA,2)+poly (U,2)+poly(AA,2)	−10934.35	.40	poly(BA,2)+poly (U,2)+poly(AA,2)	− 4201.10	.36

Note. The model formula of the baseline model and the best-fitting models of both backward regression procedures are listed in the Formula column in Wilkinson notation. The Bayesian information criterion (BIC) of the winning model is indicated in bold font. BM: Baseline model; BR: Backward regression; QBR: Quadratic backward regression; G: Geometry; BA: Basic Arithmetic; U: Units; AA: Advanced Arithmetic.

3. Results

Table 1 depicts all included books, chapters, and the number of unique problem sets per chapter. All considered variables showed normal distribution (for the distribution of each variable see S1 in the Online supplementary material). The average performance for all three data sets is listed in Table 4. We first describe the results of the model comparison for each dataset below and also provide all regression models for each dataset in Table 2. We then focus on the results of the best-fitting model for each of the three datasets and report these results in Table 3. Table 4.

3.1. Model comparison results

The baseline regression model, which only included main effects, had the lowest BIC for the dataset including academic track students as compared to models including interaction terms and/or quadratic trends (delta BIC 0 > 10 for both comparisons) but did not explain a substantially smaller share of the variance (i.e., R^2 =.43 for all three models). For students of the medium track, the BIC of the baseline model did not differ in a meaningful way (all delta BIC < 10) compared to the models considering interactions and quadratic terms (see Table 2). Additionally, all models explained a comparable share of the variance (i.e., R^2 =.40 for all three models). For the data of students from the lowest track schools, the BIC of the backward regression model with quadratic terms was lower than the BIC of the baseline model (delta BIC = 5) and the model including interaction terms (delta BIC = 21). However, this model did not explain substantially more variance than the other two models (R^2 =.36 for all three models; also see Table 2).

In sum, across all three datasets, the two regression models resulting from the backward regression procedure did not fit the data better (with a delta BIC of at least 10) and at the same time did also not explain substantially more variance (a R^2 difference of at least 1 % variance explained) than the baseline model. Thus, applying Occam's razor, we selected the baseline model that only included main effects but no interaction terms and no quadratic terms as the best-fitting model for all three datasets. Below, we report the results of each winning model.

3.2. Regression results for each dataset

The results of each linear regression model are listed in Table 3. In short, units and advanced arithmetic showed the strongest prediction of fraction understanding across all three datasets. Interestingly, we observed that the variance explained as well as the effect of the predictors (indicated by beta) scaled with the datasets: for the dataset comprising academic track students, the regression model explained the largest share of variance also reflected in relatively higher beta values. In contrast, models for lower-track students explained less variance with relatively lower beta values. Moreover, we also observed a positive trend for the beta weights for advanced arithmetic across school track levels, with increasing beta weights for higher track levels. Interestingly, we also observed a negative trend for the beta weights for basic arithmetic and geometry across school track levels, with relatively lower beta weights for higher school track levels.

3.2.1. High-level dataset

The results of the linear regression model for the high-level dataset were in line with the results reported by Spitzer & Moeller (2022). Geometry, Measurement Units, and Advanced Arithmetic significantly predicted fraction understanding, with better accuracy in any of these mathematical topics predicting better fraction understanding (see Table 3). Basic Arithmetic did not significantly predict fraction understanding, as also reported by Spitzer & Moeller (2022). However, when Advanced Arithmetic was not included in the regression model, Basic Arithmetic significantly predicted fraction understanding (as also found by Spitzer & Moeller (2022)), indicating that in the full regression model, the effect of Basic Arithmetic on fraction understanding was presumably suppressed (see Table S1 for results on the reduced regression model).

3.2.2. Medium-level dataset

The results of this linear regression model suggested that Basic Arithmetic, Measurement Units, and Advanced Arithmetic significantly predicted fraction understanding. Again, higher accuracy in each of these mathematical topics significantly predicted better fraction understanding (see Table 3). However, Geometry did not significantly predict fraction understanding. As in the high-level analysis, we examined whether the effect of Geometry was suppressed by another mathematical topic and thus ran an additional

Table 3
Baseline Model Results.

	High-Level			Medium-Level			Low-Level		
	beta	t-Value	p-Value	beta	t-Value	p-Value	beta	t-Value	p-Value
Intercept	.08	5.41	***<.001***	.19	12.89	***<.001***	.27	11.96	***<.001***
Geometry	.10	6.00	***<.001***	−.01	−.38	.701	< . 01	.10	.919
Basic Arithmetic	.02	.85	.392	.11	5.19	***<.001***	.12	3.92	***<.001***
Units	.20	11.21	***<.001***	.25	14.96	***<.001***	.27	10.66	***<.001***
Advanced Arithmetic	.58	35.68	***<.001***	.46	29.54	***<.001***	.34	13.83	***<.001***
R^2 / R^2 adjusted			.434 /.433			.405 /.405			.361 /.359
$n_{students}$			6468			4598			1743

Table 4
Mean accuracy, standard deviation (SD), and the average number of worked-through problem sets for each topic and each data set.

	High-Level			Medium-Level			Low-Level		
	accuracy	SD	problem sets	accuracy	SD	problem sets	accuracy	SD	problem sets
Geometry	86.7 %	7.5 %	40.8	86.7 %	8 %	26.5	82.4 %	10.1 %	21.2
Basic Arithmetic	81.6 %	8.7 %	36.1	80.6 %	9 %	43.8	81.8 %	9.1 %	23.4
Units	87.5 %	6.6 %	30.9	86.3 %	7.1 %	37.4	88.4 %	7.3 %	18.6
Advanced Arithmetic	82.8 %	8.6 %	38.3	81.9 %	8.7 %	23.1	82.3 %	8.9 %	23.9
Fractions	82.2 %	10.5 %	38.9	85.9 %	9.5 %	29.1	87.3 %	8.9 %	17.9
$n_{students}$			6468			4598			1743

linear regression model without Advanced Arithmetic as a predictor, as Advanced Arithmetic is the last topic in order within the ITS. Geometry was a significant predictor of fraction understanding in this reduced regression model. In particular, better accuracy for Geometry was associated with better fraction understanding. This suggests that the effect of Geometry was presumably suppressed by Advanced Arithmetic (also see Table S1 for results on the reduced regression model).

3.2.3. Low-level dataset

The results of the baseline regression model replicated the results of the medium-level analysis. Basic Arithmetic, Measurement Units, and Advanced Arithmetic significantly predicted fraction understanding. Again, better accuracy in any one of these mathematical topics predicted better fraction understanding. Geometry did not significantly predict fraction understanding. However, in the case where Advanced Arithmetic was not included in the regression model, Geometry turned out to be a significant predictor of fraction understanding. This seems to indicate that in the full regression model, the effect of Geometry on fraction understanding was presumably suppressed (also see Table S1).

4. Discussion

In this study, we evaluated the specific nature of how fraction understanding is predicted by achievements in basic mathematical topics. In particular, we compared the fit of a model only including the main effects of the predictor variables Geometry, Basic Arithmetic, Measurement Units, and Advanced Arithmetic (baseline model) against two other models: one incorporating all main effects but also interaction terms between the predictor variables and another one which incorporated all main effects and interaction terms as well as quadratic terms. Our model selection procedure indicated that the baseline model, only including main effects, fitted the data better, or similarly well compared to the two more complex models. In other words, we found no systematic support for substantial influences of interaction terms or non-linear relationships between basic mathematical topics when predicting fraction understanding.

We observed that students' performance in measurement units and arithmetic significantly predicted their later fraction understanding in an additive and linear manner across all three school tracks. These results, stemming from over 12,000 students and over 18 million mathematical worked-through problems, substantiate the robustness of previous studies which reported that fraction understanding builds on students' achievements in mathematical topics taught earlier in the curriculum (Jordan et al., 2013; Hansen et al., 2015; Vukovic et al., 2014; Bailey et al., 2014; Spitzer & Moeller, 2022). These studies mostly reported linear relationships between basic mathematical achievements and fraction understanding, and it remained an open question whether these relationships are indeed additive and linear or whether multiplicative (linear or non-linear) effects between prior mathematical topics on fraction understanding exist. Our present results contribute to these previous studies, substantiating their findings that achievements in basic mathematical topics–such as measurement units and arithmetic skills–significantly predicted students' fraction understanding later in the curriculum in an additive and linear manner.

Our findings across the three data sets diverged with respect to the effect of Basic Arithmetic on fraction understanding. While the results for students of the lower- and medium-track schools indicated that achievements in Basic Arithmetic significantly predicted achievements in fraction understanding, the effect was not observed for students of the academic track schools. These inconsistent findings between datasets might be driven by several factors. First, and as suggested by additional data analyses (see Table S1), the effect of Basic Arithmetic on fraction understanding for the high-level data set may be suppressed by the predicting effect of advanced arithmetic. This seems to suggest a hierarchical development from Basic Arithmetic to Advanced Arithmetic and finally to Fractions. Similar hierarchical developments, suggesting that performance on more advanced mathematical topics build upon understanding of more basic mathematical topics, have recently also been reported by others (e.g., Xu et al. (2024a)), yet within slightly different domains (from whole numbers to fractions to word problems). Second, while the three different secondary school tracks covered the same book topics, the specific problem sets within books differed across tracks, with more complex problem sets for higher-level tracks (see Table 1). Thus, the inconsistent findings across the three school tracks could have also emerged from differing problem sets implemented within the ITS (also see the limitation section below).

Another result that needs further investigation is the specific contribution of geometry skills on fraction understanding. In particular, we only observed for students of academic track schools that geometry skills significantly and positively predicted fraction understanding. Future research is needed to better understand this finding. Moreover, future research may seek to investigate which specific aspects (e.g., which subtopics) of geometry specifically contribute to fraction understanding as also suggested by others (e.g., Geary (2011)).

Taken together, these results inform theories of numerical development, such as the integrated theory of numerical development (Siegler, 2016; Siegler et al., 2011; Siegler & Lortie-Forgues, 2014). In particular, our results substantiate that basic mathematical concepts (e.g., whole number arithmetic and converting measurement units) uniquely and linearly predict fraction understanding. This means that achievement on both basic mathematical topics seems to critically contribute to fraction understanding in an additive and linear way. The large data set and the largely consistent findings across three school tracks thus underscore the idea that students' basic magnitude understanding, including basic arithmetic (reflected in chapters on whole-number arithmetic and converting measurement units), seems essential for later fraction understanding. In sum, these results provide strong support for and further specify the idea that mathematical concepts hierarchically and linearly build upon each other (Siegler & Braithwaite, 2017b; Siegler, 2016; Siegler et al., 2011).

Furthermore, our results on the longitudinal contribution of arithmetic skills to fraction understanding provide further evidence of the similarity between the results stemming from face-to-face testings and data obtained from digital learning software when predicting fraction understanding. As digital learning software continuously log vast amounts of process data, the systematic consideration and evaluation of these data may well extend future research on the predictors of fraction understanding. Moreover, these data typically are longitudinal in nature (i.e., a student uses a digital learning software over a longer period of time) and are therefore highly informative on the development of mathematical skills (Spitzer & Moeller, 2022; Koedinger et al., 2023; Ritter et al., 2007; Rittle-Johnson & Koedinger, 2005,2009; Ten Braak & Storksen, 2021; Ten Braak et al., 2022; Storksen et al., 2023; Hilz et al., 2023b; Hilz et al., 2023a). In addition, these systems usually cover different mathematical topics or even entire curricula spanning several school grades (Spitzer, 2022; Meeter, 2021; Tomasik et al., 2020; Ritter et al., 2007; Koedinger et al., 1997). As such, future research may build upon our findings to investigate the longitudinal development of mathematical skills such as fraction understanding but also other more advanced later topics (e.g., algebra) based on their previous results using linear regression or more advanced analysis approaches such as psychological network analyses (Spitzer et al., 2024a; Spitzer et al., 2024b; van Hoogmoed et al., 2024).

4.1. Limitations and future research avenues

There are a number of aspects to be considered when interpreting the current findings. First, our study only considered students' achievements on four specific basic mathematical topics as predictors for fraction understanding. However, it may be that achievements in other topics, that were not included in this study, would considerably interact with other predictor variables, leading to better model fits compared to pure additive models. It may also be that the prediction of other topics is best described by non-linear functions. As such, it is important for future research to evaluate the nature of the prediction and thus the specific contribution of other (basic) mathematical topics to students' fraction understanding. Second, we only included quadratic terms as non-linear dependencies and more research is needed to rule other non-linear dependencies beyond quadratic terms such as third or fourth order terms. Third, the chapter topics and their respective problem sets for the same books slightly differed across school tracks. Thus, students from the three different tracks worked through slightly different problem sets which could have impacted the divergent findings for the predictive effect of Basic Arithmetic and Geometry on fraction understanding. Fourth, the ITS has different answer possibilities. Students may be provided with open-text answers or multiple-choice answers which have a 20 % guessing chance. As we computed the average accuracy for each student based on all problem sets for each mathematical topic, we did not account for guessing. Future research should consider to investigate differences in accuracy rate between open-text answers and multiple-choice answers within the considered ITS and may account for such potential differences. Fifth, our results suggest that understanding basic mathematical topics, such as whole number arithmetic and converting measurement units, serve as stepping stones for understanding fractions. However, our study evaluated the effects of compound scores on mathematical topics. However, mathematical topics are typically composed of several subtopics (see Table1). Thus, it remains to be investigated which specific subtopic students should revise if they struggle with a particular fraction subtopic. Future research may examine whether studying prior fraction subtopic is most beneficial when struggling on a specific fraction subtopic (if prior fraction subtopics exist), or whether students should rather focus on subtopics from other mathematical topics, such as whole number arithmetic. Finally, we considered students' average performance on different basic mathematical topics as an estimation of their performance on each of these topics. Further research is needed to evaluate the specific contribution of subtopics as well as the effect of different problem features (e.g., visualization problems, pure calculation problems, or word problems) and the specific effects of each of these features for each topic on students' fraction understanding.

5. Conclusion

In conclusion, we believe that our findings from three large cohorts, comprising more than 12,000 students who worked through more than 18 million problems, provide important evidence to the existing body of research that fraction understanding is predicted in an additive and linear manner by basic mathematical topics arithmetic and measurement units. This, in turn, indicates that fraction understanding builds upon several basic mathematical topics each of which serves as a unique stepping stone. Thus, our results suggest that students with knowledge gaps in arithmetic and measurement units are well advised to revise these topics as fraction understanding builds on these earlier topics. As our results revealed similar findings to those from face-to-face testings and were also robust across three cohorts, they also point researchers from the digital learning as well as educational community to the value of such large-scale naturalistic data from classroom settings collected through ITS. Future research may leverage such large-scale data sets obtained from digital learning software to further advance our understanding on the specific nature of students' numerical development.

CRediT authorship contribution statement

Eileen Richter: Writing – review & editing. **Younes Strittmatter:** Writing – review & editing, Conceptualization. **Miguel Ruiz-Garcia:** Writing – review & editing, Conceptualization. **Markus Wolfgang Hermann Spitzer:** Writing – original draft, Visualization, Methodology, Data curation, Conceptualization. **Korbinian Moeller:** Writing – review & editing, Conceptualization.

Data availability

Data will be made available on request.

Appendix A. Supporting information

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.cogdev.2024.101517.

References

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology, 113*(3), 447–455. https://doi.org/10.1016/j.jecp.2012.06.004

Bailey, D. H., Siegler, R. S., & Geary, D. C. (2014). Early predictors of middle school fraction knowledge. *Developmental Science, 17*(5), 775–785. https://doi.org/10.1111/desc.12155

Braithwaite, D. W., Leib, E. R., Siegler, R. S., & McMullen, J. (2019). Individual differences in fraction arithmetic learning. *Cognitive Psychology, 112*, 81–98.

Braithwaite, D. W., Pyke, A. A., & Siegler, R. S. (2017). A computational model of fraction arithmetic. *Psychological Review, 124*(5), 603.

Braithwaite, D. W., & Siegler, R. S. (2024). A unified model of arithmetic with whole numbers, fractions, and decimals. *Psychological Review, 13*(2), 431.

Burnham, K. P., Anderson, D. R., Burnham, K. P., & Anderson, D. R. (1998). *Practical use of the information-theoretic approach*. Springer.

Geary, D. C. (2011). Cognitive predictors of achievement growth in mathematics: a 5-year longitudinal study. *Developmental Psychology, 47*(6), 1539.

Gerardi, K., Goette, L., & Meier, S. (2013). Numerical ability predicts mortgage default. *Proceedings of the National Academy of Sciences, 110*(28), 11267–11271.

Hansen, N., Jordan, N. C., Fernandez, E., Siegler, R. S., Fuchs, L., Gersten, R., & Micklos, D. (2015). General and math-specific predictors of sixth-graders' knowledge of fractions. *Cognitive Development, 35*, 34–49. https://doi.org/10.1016/j.cogdev.2015.02.001

Hilz, A., Guill, K., Roloff, J., Aldrup, K., & Köller, O. (2023a). The relationship between individual characteristics and practice behaviour within an adaptive arithmetic learning program. *Journal of Computer Assisted Learning, 39*(3), 970–983.

Hilz, A., Guill, K., Roloff, J., Sommerhoff, D., & Aldrup, K. (2023b). How to continue? new approaches to investigating the effects of adaptive math learning programs on students' performance, self-concept, and anxiety. *Journal of Intelligence, 11*(6), 108.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology, 116*(1), 45–58. https://doi.org/10.1016/j.jecp.2013.02.001

Koedinger, K. R., Anderson, J. R., Hadley, W. H., & Mark, M. A. (1997). Intelligent tutoring goes to school in the big city. *International Journal of Artificial Intelligence in Education, 8*, 30–43.

Koedinger, K. R., Carvalho, P. F., Liu, R., & Mclaughlin, E. A. (2023). An astonishing regularity in student learning rate. *Proceedings of the National Academy of Sciences, 120*(13), Article e2221311120. https://doi.org/10.1073/pnas

Lortie-Forgues, H., Tian, J., & Siegler, R. S. (2015). Why is learning fraction and decimal arithmetic so difficult? *Developmental Review, 38*, 201–221. https://doi.org/10.1016/j.dr.2015.07.008

Lüdecke, D. (2020). sjPlot: Data Visualization for Statistics in Social Science. *sjPlot: Data Visualization for Statistics in Social Science*. https://doi.org/10.5281/zenodo.1308157

Meeter, M. (2021). Primary school mathematics during Covid-19: No evidence of learning gaps in adaptive practicing results. *Trends in Neuroscience and Education, 2* (October), Article 100163. https://doi.org/10.1016/j.tine.2021.100163

Mou, Y., Li, Y., Hoard, M. K., Nugent, L. D., Chu, F. W., Rouder, J. N., & Geary, D. C. (2016). Developmental foundations of children's fraction magnitude knowledge. *Cognitive Development, 39*, 141–153.

Murnane, R. J., Willett, J. B., & Levy, F. (1995). The growing importance of cognitive skills in wage determination. *Review of Economics and Statistics, 77*(2), 251–266.

National Mathematics Advisory Panel. (2008). Foundations for success: The final report of the National Mathematics Advisory Panel (Tech. Rep.).

Park, S., & Esposito, A. G. (2024). Rational number representation, math anxiety, and algebra performance in college students. *Cognitive Development, 69*, Article 101417.

Parsons, S. and Bynner, J. (2007). Research report over the lifecourse (Tech. Rep. No. November).

R Core Team. (2023). R: A Language and Environment for Statistical Computing. Vienna, Austria.⟨http://www.r-project.org/⟩.

Reyna, V. F., Nelson, W. L., Han, P. K., & Dieckmann, N. F. (2009). How numeracy influences risk comprehension and medical decision making. *Psychological Bulletin, 135*(6), 943.

Ripley, B., Venables, B., Bates, D. M., Hornik, K., Gebhardt, A., Firth, D., & Ripley, M. B. (2013). Package 'mass'. *Cran r, 538*, 113–120.

Ritchie, S. J., & Bates, T. C. (2013). Enduring links from childhood mathematics and reading achievement to adult socioeconomic status. *Psychological Science, 24*(7), 1301–1308. https://doi.org/10.1177/0956797612466268

Ritter, S., Anderson, J. R., Koedinger, K. R., & Corbett, A. (2007). Cognitive tutor: applied research in mathematics education. *Psychonomic Bulletin Review, 14*(2), 249–255.

Rittle-Johnson, B., & Koedinger, K. (2009). Iterating between lessons on concepts and procedures can improve mathematics knowledge. *British Journal of Educational Psychology, 79*(3), 483–500. https://doi.org/10.1348/000709908X398106

Rittle-Johnson, B., & Koedinger, K. R. (2005). Designing knowledge scaffolds to support mathematical problem solving. *Cognition and Instruction, 23*(3), 313–349. https://doi.org/10.1207/s1532690xci2303_1

Rivera-Batiz, F. L. (1992). Quantitative literacy and the likelihood of employment among young adults in the united states. *The Journal of Human Resources, 27*(2), 313–328.

Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science, 19*(3), 341–361.

Siegler, R. S., & Braithwaite, D. W. (2017a). Numerical development. *Annual Review of Psychology, 68*, 187–213. https://doi.org/10.1146/annurev-psych-010416-044101

Siegler, R. S., & Braithwaite, D. W. (2017b). Numerical development. *Annual Review of Psychology, 68*(1), 187–213.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., & Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691–697. https://doi.org/10.1177/0956797612440101

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences, 17*(1), 13–19. https://doi.org/10.1016/j.tics.2012.11.004

Siegler, R. S., & Lortie-Forgues, H. (2014). An integrative theory of numerical development. *Child Development Perspektives, 8*, 144–150.

Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology, 49*(10), 1994–2004. https://doi.org/10.1037/a0031200

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*(4), 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

Spitzer, M. W. H. (2022). Just do it! Study time increases mathematical achievement scores for grade 4-10 students in a large longitudinal cross-country study. *European Journal of Psychology of Education*. https://doi.org/10.1007/s10212-021-00546-0

Spitzer, M. W. H., Bardach, L., Strittmatter, Y., Meyer, J., & Moeller, K. (2024a). Evaluating the content structure of intelligent tutor systems—a psychological network analysis. *Computers and Education Open*, Article 100198.

Spitzer, M. W. H., & Moeller, K. (2022). Predicting fraction and algebra achievements online: A large-scale longitudinal study using data from an online learning environment. *Journal of Computer Assisted Learning, 38*(6), 1797–1806.

Spitzer, M. W. H., & Moeller, K. (2024). Performance increases in mathematics within an intelligent tutoring system during covid-19 related school closures: a large-scale longitudinal evaluation. *Computers and Education Open, 6*, Article 100162.

Spitzer, M. W. H., Moeller, K., & Musslick, S. (2023). Assignment strategies modulate students' academic performance in an online learning environment during the first and second COVID-19 related school closures. *PLoS ONE, 18*, 1–23. https://doi.org/10.1371/journal.pone.0284868

Spitzer, M. W. H., Ruiz-Garcia, M., & Moeller, K. (2024b). Basic mathematical skills and fraction understanding predict percentage understanding: Evidence from an intelligent tutoring system. *British Journal of Educational Technology*.

Stapel, M., Zheng, Z., & Pinkwart, N. (2016). An ensemble method to predict student performance in an online math learning environment. *International Educational Data Mining Society*.

Størksen, I., Rege, M., Solli, I. F., tenBraak, D., Lenes, R., & Geldhof, G. J. (2023). The playful learning curriculum: A randomized controlled trial. *Early Childhood Research Quarterly, 64*, 36–46.

TenBraak, D., Lenes, R., Purpura, D. J., Schmitt, S. A., & Størksen, I. (2022). Why do early mathematics skills predict later mathematics and reading achievement? the role of executive function. *Journal of Experimental Child Psychology, 214*, Article 105306.

Ten Braak, D., & Størksen, I. (2021). Psychometric properties of the ani banani math test. *European Journal of Developmental Psychology, 1*(4), 610–628.

Tomasik, M. J., Helbling, L. A., & Moser, U. (2020). Educational gains of in-person vs. distance learning in primary and secondary schools: A natural experiment during the COVID-19 pandemic school closures in Switzerland. *International Journal of Psychology, 56*(4), 566–576. https://doi.org/10.1002/ijop.12728

Torbeyns, J., Schneider, M., Xin, Z., & Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction, 37*, 5–13.

van Hoogmoed, A. H., Adriaanse, P., Vermeiden, M., & Weggemans, R. (2024). Combining cognitive and affective factors related to mathematical achievement in 4th graders: A psychological network analysis study. *Trends in Neuroscience and Education*, Article 100241.

Vukovic, R. K., Fuchs, L. S., Geary, D. C., Jordan, N. C., Gersten, R., & Siegler, R. S. (2014). Sources of individual differences in children's understanding of fractions. *Child Development, 85*(4), 1461–1476. https://doi.org/10.1111/cdev.12218

Whalen, K. A., Renkl, A., Eitel, A., & Glogger-Frey, I. (2023). Digital re-attributional feedback in high school mathematics education and its effect on motivation and achievement. *Journal of Computer Assisted Learning*.

Wortha, S. M., Klein, E., Lambert, K., Dackermann, T., & Moeller, K. (2023). The relevance of basic numerical skills for fraction processing: Evidence from cross-sectional data. *PLoS ONE, 18*, 1–15. https://doi.org/10.1371/journal.pone.0281241

Xu, C., Burr, S. D. L., Li, H., Liu, C., & Si, J. (2024a). From whole numbers to fractions to word problems: Hierarchical relations in mathematics knowledge for chinese grade 6 students. *Journal of Experimental Child Psychology, 242*, Article 105884.

Xu, C., Burr, S. D. L., McNally-Edwards, N., McShane, C., Wan, L., & Wylie, J. (2024b). A holistic investigation of fraction learning: examining the hierarchy of fraction skills, misconceptions, mathematics anxiety and response confidence. *Journal of Cognitive Psychology*, 1–19.