Conditional Erlangen Program

Dissertation

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

von M.Sc. Xiangying Chen

geb. am 17.02.1995 in Shandong, China

genehmigt durch die Fakultät für Mathematik
der Otto-von-Guericke-Universität Magdeburg

Gutachter: Prof. Dr. Thomas Kahle
Prof. Dr. Alexandru Constantinescu

eingereicht am: 30.09.2024

Verteidigung am: 03.12.2024

Zusammenfassung

Die vorliegende Dissertation beschäftigt sich mit den kombinatorischen Strukturen, die bei der Axiomatisierung von Begriffen der Unabhängigkeit, vor allem der bedingten Unabhängigkeit unter den Zufallsvariablen vorkommen.

Einer der wichtigsten historischen Meilensteine in der Mathematik ist das Erlanger Programm von Felix Klein. Darin entwickelte er die Auffassung einer systematischen Klassifikation von Geometrien durch die Gruppe ihrer Symmetrien. Jede Geometrie ist durch eine Gruppe von Transformationen charakterisiert, und eine Geometrie ist die Invariantentheorie jener Transformationsgruppe. Joseph Kung folgte seiner Idee und initiierte die systematische Entwicklung von matroidartigen Strukturen und nannte sie das Kombinatorische Erlanger Programm. Mit einem anderen Ansatz führte Gelfand and Serganova die Coxetermatroide als eine Verallgemeinerung von Matroiden auf alle Coxetertypen ein.

Dem Erlanger Programm und dem Kombinatorischen Erlanger Programm nachzufolgen, ist das Ziel der Dissertation, die klassische Frage "Was ist mit den anderen Coxetertypen?" für bedingte Unabhängigkeit zu beantworten. Wir nennen die Klassifikation und die Axiomatisierung von CI-Strukturen in allen Coxetertypen das "Bedingte Erlanger Programm".

Kapitel 2 führt die Theorie der Strukturen der bedingten Unabhängigkeit (CI-Strukturen) von Typ A ein. Ein Überblick über Semigraphoide und Semimatroide und ihre Geometrie befindet sich in § 2.1. In § 2.2, nach einem Überblick über Gaußsche Zufallsvektoren, Gaussoide und ihre Darstellbarkeit über Körpern und geordneten Körpern, führen wir die Darstellungen von Gaussoiden über Schiefkörpern mit Antiautomorphismen ein. Wir diskutieren auch den Zusammenhang von Gaussoiden mit der Orthogonalität und führen eine verbandstheoretische Darstellung eines Gaussoids ein. In § 2.3 ist ein Überblick über aufsteigende Semigraphoide bzw. Gaussoide. Die sind die CI-Strukturen, die die verschiedenen Zusammenhangsbegriffe abstrahieren. In § 2.4 werden neue Axiomatisierungen von Matroiden als CI-strukturen und von orientierten Matroiden als orientierte CI-strukturen gegeben, welche eine starke Verbindung von der Matroidtheorie zur Theorie der bedingten Unabhängigkeit bieten.

Wir initiieren das Bedingte Erlanger Programm bei Einführung von Φ-Semigraphoiden und Φ-Semimatroiden für jedes Wurzelsystem Φ in § 3.1, und beschreiben sie explizit für die klassischen Typen B, C und D in § 3.2. Als Anwendung von dem Bedingten Erlanger Programm auf das Kombinatorische Erlanger Programm werden die Axiomatisierungen von Deltamatroiden als CI-Strukturen von Typ C und von orthogonalen Deltamatroiden als CI-Strukturen von Typ D in § 3.3 gegeben. In § 3.4 wird die Geometrie der verallgemeinerten Permutaeder von Typ B oder C untersucht. Wir beschreiben jeden verallgemeinerten Permutaeder von Typ B, C oder D explizit als eine vorgezeichnete Minkowski-Summe von Basispolytopen symplektischer Matroide von Rang eins. Anders gesagt, eine Basis von dem durch die bisubmodularen Funktionen aufgespannten linearen Raum wird gefunden und die entsprechende Basiswechselmatrix wird explizit gegeben. Danach wird den Zusammenhang in Typ B oder C diskutiert. Zusätzlich werden eine explizite Volumenformel für einen beliebigen verallgemeinerten Permutaeder von Typ B, C oder D und elementare Beweise für die Formeln für die gemischten Volumina von Standardsimplizen und Unabhängigkeitsmengenpolytopen symplektischer Matroide des Ranges eins hergeleitet. Wir beweisen die verschiedenen Heiratssätze in der Transversaltheorie mithilfe der elementaren Eigenschaften von gemischten Volumina.

Der Maximum-Likelihood-Grad (ML-Grad) eines durch einen generischen linearen Raum dargestellten linearen Konzentrationsmodells und der algebraische Grad der semidefiniten Optimierung sind grundlegende Maße der Komplexität von einem statistischen Modell bzw. einem semidefiniten Programm. Sie lassen sich in die abzählend geometrische Sprache umformulieren und auf die Typen A und D verallgemeinern. Alle von denen sind Polynomfunktionen. Die Beweise lassen sich auf die Polynomialität der Lascouxpolynome reduzieren. In Kapitel 4 werden explizite Formeln für den Grad und die führenden Koeffizienten der Lascoux(quasi-)polynome von Typen C, A und D gegeben. Als Anwendungen werden den Grad des Polynoms, des algebraischen Grades $\delta(m, n, n-s)$, und den führenden Koeffizienten für $s = 1$ in Typen C, A und D explizit hergeleitet. Dies ist gemeinsame Arbeit mit Alessio Borzì, Harshit J. Motwani, Lorenzo Venturello und Martin Vodička.

Summary

This thesis is concerned with combinatorial structures that arise in the axiomatization of notions of independence, in particular conditional independence among random variables.

One of the most important historic milestones in mathematics is Felix Klein's Erlangen Program. He initiated the classification of and the systematic study on geometries through the groups of their symmetries and suggested that each geometry can be characterized by a group of transformations and a geometry is the theory of invariants under this group of transformations. Following this idea, the systematic development of matroid-like structures is initiated by Kung under the name of the Combinatorial Erlanger Programm. From another approach Gelfand and Serganova introduced the Coxeter matroids as a generalization of matroids to all Coxeter types.

Following the Erlangen Program and the Combinatorial Erlangen Program, we here aim to answer the classic question "What about other Coxeter types" for conditional independence, and call the classification and axiomatization of conditional independence structures in all Coxeter types the "Conditional Erlangen Program".

Chapter 2 provides an introduction to the theory of conditional independence structures of type A. Section 2.1 reviews the background on semigraphoids and semimatroids and their geometry. In Section 2.2, after reviewing the background on Gaussians, gaussoids and the representability over fields and ordered fields, we introduce the gaussoid representations over skew fields with antiautomorphisms. We also discuss the relation of gaussoids to the orthogonality and introduce a lattice-theoretic representation of a gaussoid. Section 2.3 reviews the ascending semigraphoids and gaussoids, which are CI-structures abstracting various notions of connectedness. Section 2.4 gives new axiomatizations of matroids as CI-structures and of oriented matroids as oriented conditional independence structures, which provide a strong connection of matroid theory to the theory of conditional independence.

We initiate the Conditional Erlangen Program by introducing the Φ-semigraphoids and Φ-semimatroids for any root system Φ in Section 3.1, and describe them explicitly for the classical types B, C and D in Section 3.2. As an application of the Conditional Erlangen Program to the Combinatorial Erlangen Program, the axiomatizations of delta-matroids as CI-structures of type C and of orthogonal delta-matroids as CI-structures of type D are given in Section 3.3. Section 3.4 studies the geometry of the

generalized permutohedra of type B or C. We write every generalized permutohedron of type B, C or D explicitly as a signed Minkowski sum of rank 1 symplectic matroid basis polytopes. In other words, we found a basis of the linear space spanned by the bisubmodular functions and describe the exchange matrix explicitly. Then we discuss the connectedness of type B or C, give an explicit volume formula for any generalized permutohedron of type B, C or D and elementary proofs for the formulas for the mixed volumes of standard simplices and rank 1 symplectic matroid independent set polytopes, and prove the various marriage theorems in transversal theory using elementary properties of mixed volumes.

The maximum likelihood degree (ML-degree) of a linear concentration model represented by a generic linear space, the algebraic degree of semidefinite programming (SDP-degree) are fundamental measures for the computational complexity of the statistical model and the SDP, respectively. They can be expressed in the language of enumerative geometry and generalized to types A and D. All of them are polynomial functions. The proofs boils down to show the polynomiality of Lascoux polynomials. In Chapter 4 we give explicit formulas for the degrees and the leading coefficients of the Lascoux (quasi-)polynomials of types C, A and D. As an application, we give the degree of the polynomial, the algebraic degree $\delta(m, n, n - s)$, and the leading coefficient for $s = 1$ explicitly in types C, A and D. This is joint work with Alessio Borzì, Harshit J. Motwani, Lorenzo Venturello and Martin Vodička.

Contents

Chapter 1

Introduction

1.1 The Erlangen Program

This thesis is concerned with combinatorial structures that arise in the axiomatization of notions of independence, in particular conditional independence among random variables. These axiomatizations have a formally similar history to a development in geometry initiated by Felix Klein in the late 19th century. He initiated the classification of and the systematic study on geometries through the groups of their symmetries and suggested that each geometry can be characterized by a group of transformations and a geometry is the theory of invariants under this group of transformations. Well known as the "Erlanger Programm" [Kle93], it became one of the most important historic milestones in mathematics. We refer to [Haw84] for the history of the Erlangen Program. Here we quote [Kle25, § 3.I.1] written by Felix Klein himself: "Es sei irgendeine beliebige Gruppe räumlicher Transformationen gegeben, welche die Hauptgruppe als Teil umfaßt; dann gibt die Invariantentheorie dieser Gruppe eine bestimmte Art von Geometrie, und man kann so jede mögliche Geometrie erhalten. Als Charakteristikum jeder Geometrie wird ihre Gruppe stets in den Vordergrund der Betrachtung gestellt."

Following the Erlangen Program, the homogeneous spaces are studied. A homogeneous space is a space X with a transitive group action by G. Informally speaking, it is a space that remains invariant under a group of transformations G and its points are all "connected" by the transformations of G. By the orbit-stabilizer theorem, the space X can be identified with the quotient of G by a stabilizer subgroup, and the stabilizer subgroups of any point of X are conjugated, thus isomorphic, to each other. If G is a Lie group, according to Cartan's closed subgroup theorem, a closed subgroup of G admits a unique smooth structure which makes it an embedded Lie subgroup. Smooth homogeneous spaces with a smooth transitive action by a Lie group G are identified with the quotients of G by its closed subgroups. Compact Lie groups admits a nice classification. It can be reduced to the classification of complex semisimple Lie algebras, which are classified completely into several types via Dynkin diagrams. We refer to [BTD85].

Many mathematical objects have an underlying geometry, which can be associated to one of the Dynkin diagrams. We can look for analogous objects whose geometries are associated to the Dynkin diagrams of other types. This usually leads to new mathematical objects which are natural and interesting. Moreover, since the Dynkin diagrams are classified completely, we can find all plausible analogous objects and complete the list of generalizations. Therefore, given a geometric or combinatorial object, a nature question will almost always arise: What about other Coxeter types?

1.2 The Combinatorial Erlangen Program

"Anyone who has worked with matroids has come away with the conviction that matroids are one of the richest and most useful mathematical ideas of our day. It is as if one were to condense all trends of present day mathematics onto a single structure, a feat that any would *a priori* deem impossible, were it not for the fact that matroids do exist." [Rot08]

Matroids are a discrete structure with hundreds of equivalent axiom systems. The axiom systems arise from various areas of mathematics, they are elegant and define different natural-looking objects that are indeed equivalent, but the equivalence can be far from obvious. Every axiomatization of matroid provides a new insight into the relations between the areas of mathematics. Rota wrote in the foreword of [Whi86] that "New axiomatizations are still appearing. Matroid theory is unique in mathematics in the number and variety of its equivalent axiom systems; this accounts in part for the versatility and applicability of the subject."

Graphic matroids are an important class of matroids. Tutte famously pointed out that "If a theorem about graphs can be expressed in terms of edges and circuits alone it probably exemplifies a more general theorem about matroids" [Tut79]. From a graph G we can define the cycle matroid $M(G)$ of G. A matroid is graphic if it is the cycle matroid of a graph. Matroid operations are compatible with graph operations. In particular, the dual of a graphic matroid $M(G)$ is graphic iff the graph G is planar, in this case, $M(G)^*$ is the cycle matroid of the dual G^* of a planar embedding of G [Oxl06, § 5.2]. Many graphs are not planar, they can only be embedded on orientable or nonorientable surfaces of higher (Euler) genera. Matroid duality is not compatible with the duality of embedded graphs [EMM13]. The correct generalization of embedded graphs is delta-matroids. For delta-matroids and embedded graphs we refer to [Mof19; CMNR19].

Another important class of matroids the class of matroids representable over a field $\mathbb{K}$. Such a matroid encodes the vanishing of maximal minors of a matrix $A \in \mathbb{K}^{r\times n}$. In other words, the row space of A is a point in the Grassmannian $\mathrm{Gr}(r, n)$, and the matroid encodes which Plücker coordinates of this point are zero. A delta-matroid, however, encodes the vanishing of principal minors of a symmetric matrix, that is, the vanishing of the generalized Plücker coordinates of a point in a Lagrangian Grassmannian. The same can be done for any compact homogeneous space G/P where G is a complex semisimple Lie group and P a standard parabolic subgroup of

G. The vanishing of generalized Plücker coordinates forms a matroid-like discrete structure [GS87].

In Kung's dissertation [Kun78a], he observed that the basis exchange axiom of matroid theory are translations of the fundamental theorems of projective invariant theory into a coordinate-free language and matroid theory can be regarded as the study of those geometric properties of finite sets of vectors which can be stated in purely set-theoretic terms. Replacing the general linear groups by other classical groups can yield combinatorial structures analogous to a matroid. Following Klein's Erlanger Programm, he call the systematic development of these structures the *combinatorial Erlanger Programm*, or *Bowdoin Program* since it was suggested by Rota in his 1971 Bowdoin lectures [Kun78a; Kun78b; Kun97].

Later in [GS87], Gelfand and Serganova introduced the Coxeter matroids from a geometric approach. Let G be a semisimple algebraic group over $\mathbb{C}$, $P \subseteq G$ a parabolic subgroup and $H \subseteq P$ a maximal torus. The image of the torus orbit closure $\overline{H \cdot x}$ of a point $x \in G/P$ under the moment map is a convex polytope whose edges are parallel to the roots of the Lie algebra of G. In particular, if $G = \mathrm{SL}_n(\mathbb{C})$ is the special linear group and P is a maximal parabolic subgroup of G, then G/P is the Grassmannian and the convex polytope one get from $x \in G/P$ is the basis polytope of the matroid of the subspace x. In this way Coxeter matroids are defined for any Weyl group and classified combinatorially. It leads to a surprisingly intuitive and simple cryptomorphic definition of Coxeter matroids: Coxeter matroids are kaleidoscopes which generate only finitely many mirror images. We refer to [BGW03] for Coxeter matroids.

Stochastic independence and linear independence are two kinds of independence notions an undergraduate student encounters. However, stochastic independence behaves quite differently from linear independence. The linear dependence is defined by equations, while stochastic **in**dependence is defined by equations. Intuitively, linear dependence and stochastic **in**dependence are rare. There are many connections between these two kinds of independence. For instance, the linear independence of some vectors can always be interpreted as the conditional independence of some discrete random vector, see § 2.4.4. A matroid is a discrete structure which abstracts and generalizes linear independence. Likewise, conditional independence can also be encoded in a discrete structure. The structures which are abstraction of conditional independence can be defined axiomatically and enjoy many parallels to matroids. In parallel to the Combinatorial Erlangen Program, we here aim to answer the classic questions "What about other Coxeter types" for conditional independence. Following the Erlangen Program and the Combinatorial Erlangen Program, the classification and axiomatization of conditional independence structures in all Coxeter types are called the "Conditional Erlangen Program".

1.3 The Conditional Erlangen Program

The conditional independence of an random vector ξ is encoded in a discrete structure consisting of triples $(ij|K)$ such that the i-th and j-th random variables are independent under the condition $\{\xi_k : k \in K\}$.

We consider a conditional independence structure (CI-structure) on a ground set E as a set of triples $(ij|K)$, called CI-statements, where $K \subseteq E$ and $i \neq j \in E \backslash K$. A semigraphoid is a set of such triples $(ij|K)$ satisfying the condition (SG), which is satisfied by the conditional independence of any random vector. In [MPSSW09] a geometric characterization of semigraphoids is given as follows. A CI-statement $(ij|K) \in \mathcal{A}_n$ is associated to a set of walls in the permutohedral fan $\Sigma_{A_{n-1}}$. Semigraphoids are exactly the subsets $\mathcal{G}$ of $\mathcal{A}_n$ such that the removal of all walls corresponding to the elements of $\mathcal{G}$ from the permutohedral fan $\Sigma_{A_{n-1}}$ results in a coarser fan. The background on semigraphoids and its geometry will be reviewed in § 2.1.1 and § 2.1.2.

An important subclass of semigraphoids is semimatroids. A CI-structure is a semimatroid if there exists a submodular function such that iK and jK form a modular pair iff $(ij|K)$ is in the CI-structure. A prototype of the submodular function is the entropy function. Given a discrete random vector, the set function which maps every subset of the set of indices to the Shannon entropy of the corresponding subvector is submodular, and modularity occurs exactly at the independent pairs. As their support functions, submodular functions that take value 0 on the empty set correspond bijectively to generalized permutohedra, and semimatroids encode the combinatorial information of generalized permutohedra. Semimatroids and generalized permutohedra will be discussed in § 2.1.3.

Introduced in [LM07] and studied in [BDKS19; Boe22b], gaussoids are an abstraction of conditional independence relations among normally distributed random vectors. In § 2.2.1 we review the background on Gaussians, gaussoids and the representability over fields and ordered fields. Using quasideterminants, we extend the notion of representability to any skew field $\mathbb{K}$ and any antiautomorphism of $\mathbb{K}$ in § 2.2.2. We discuss the relation of gaussoids to the orthogonality and introduce a lattice-theoretic representation of a gaussoid in § 2.2.3. The results in these two subsections are new.

Although not so many as matroids, topological spaces also possess many cryptomorphic axiomatizations. The attempts to axiomatize the topological spaces by the connected sets lead to various generalizations of topology. In § 2.3.1 we review three generalizations of finite topological spaces from the point of view of connectedness. The connectedness is encoded in a CI-structure consisting of the CI-statements $(ij|K)$ such that i and j are separated by K. The CI-structures coming from each notion of the connectedness can be classified in terms of inference rules. The class of CI-structures encoding the connectedness in graphs is well-studied as graphical models and has a wide application in statistics. We survey the various classifications of this class in § 2.3.2.

Matroids are also semimatroids because the rank function of a matroid is submodular. In other words, matroid polytopes are generalized permutohedra. Section 2.4 is based on the paper [Che24]. In § 2.4.2 it is shown that loopless matroids are cryptomorphic to semigraphoids $\mathcal{G}$ satisfying $(ij|K) \in \mathcal{G} \Rightarrow (i\ell|jKL) \notin \mathcal{G}$. A similar axiomatization of oriented matroids as oriented conditional independence structures is given in § 2.4.3. In § 2.4.4 we survey the results on the probabilistic representability of matroids, and show that no interesting matroid is a gaussoid.

In [MPSSW09], a semigraphoid is referred to as a *convex rank test*. It is motivated by the rank tests in nonparametric statistics, which is to test the (partial) ranking from the data. For example, n pizzas are scored by customers and should be (partially) ranked from the scores. The data are the vectors in $\mathbb{R}^n$ which are the evaluations of the n pizzas. The statistics are the partial rankings of the n pizzas. The rank tests can be generalized to type B or C if we want to test the partial ranking of n pizzas and a copy of these n pizzas which are exactly so bad as how good the original pizzas are, and to type D if additionally nobody wants to distinguish the most similar pair of a pizza and its copy.

The convex rank tests of these types, are the coarsenings of the corresponding Coxeter complexes. A criterion for which walls whose removal from the Coxeter complex results in a fan is given in [Rea12]. The semimatroids are the combinatorial types of deformations of the permutohedron of this type. We review some preliminaries on root systems, Weyl groups and the Dynkin classification in § 3.1.1, and on Coxeter complexes and Coxeter permutohedra in § 3.1.2. Then we introduce in § 3.1.3 the Φ-CI-statements, the Φ-semigraphoids and the Φ-semigraphoids for any root system Φ according to the results in [Rea12; ACEP20]. In Section 3.2 we describe these CI-structures in the classical types.

In Section 3.3 we present an application of the Conditional Erlangen Program on the Combinatorial Erlangen Program. After reviewing some background on Coxeter matroids in § 3.3.1, we show that loopless delta-matroids are cryptomorphic to C_n-semigraphoids satisfying (ΔMCI) in § 3.3.2. A similar result for type D is given as a direct corollary.

In Section 3.4 we study the geometry of generalized permutohedra of types B and C. In the type A case, every generalized permutohedron is a signed Minkowski sum of standard simplices. We write every generalized permutohedron of type B, C or D explicitly as a signed Minkowski sum of rank 1 symplectic matroid basis polytopes. In § 3.4.2 we discuss the applications to the definition of connectedness of types B and C. In § 3.4.3 we deduce an explicit volume formula for any generalized permutohedron of type B, C or D. In § 3.4.4 we point out the relation between mixed volumes the various marriage theorems in transversal theory.

1.4 Overview of main results

In matroid theory one considers representations of matroids over various algebraic structures, including skew fields. We apply this idea to gaussoid theory and show that it make sense to seek representations over skew fields with antiautomorphisms. Theorem 2.2.20 says that a Hermitian matrix over a skew field satisfies the gaussoid axioms if all its principal submatrices are invertible. The notions of gaussoid representations are extended to modular lattices in Theorem 2.2.32.

We give a new axiomatization of matroids in Theorem 2.4.1. We embed the class of matroids into the class of CI-structures. It does not only provide a strong connection of matroid theory to the theory of conditional independence, but also extends the points of view on matroids from the basis exchanges or the normal fan of matroid polytopes. A similar axiomatization of oriented matroids as oriented conditional independence structures is given in Theorem 2.4.6.

Theorem 2.4.1. Loopless matroids are cryptomorphic to semigraphoids satisfying $(ij|K) \in \mathcal{G} \Rightarrow (i\ell|jKL) \notin \mathcal{G}$.

Theorem 2.4.6. Loopless oriented matroids are cryptomorphic to oriented CI-structures satisfying (OCI1)–(OCI5).

We initiate the Conditional Erlangen Program by introducing the Φ-semigraphoids and Φ-semimatroids for any root system Φ, and describe them explicitly for types B, C and D in Definition 3.2.2 and Definition 3.2.4. As an application of the Conditional Erlangen Program to the Combinatorial Erlangen Program, we found an axiomatization of delta-matroids as CI-structures of type C in Theorem 3.3.6. An axiomatization of orthogonal delta-matroids as CI-structures of type D is given in Corollary 3.3.7.

Theorem 3.3.6. Loopless delta-matroids are cryptomorphic to C_n-semigraphoids satisfying $i \not\perp\!\!\!\perp j|K \Rightarrow \bar{i} \perp\!\!\!\perp j|K \wedge i \perp\!\!\!\perp \ell|jKL \wedge \bar{i} \perp\!\!\!\perp \ell|jKL \wedge i \perp\!\!\!\perp \bar{i}|jKL'$.

Corollary 3.3.7. Loopless orthogonal delta-matroids are cryptomorphic to D_n-semigraphoids satisfying $i \not\perp\!\!\!\perp j|K \Rightarrow \bar{i} \perp\!\!\!\perp j|K \wedge i \perp\!\!\!\perp \ell|jKL \wedge \bar{i} \perp\!\!\!\perp \ell|jKL$.

We write every generalized permutohedron of type B, C or D explicitly as a signed Minkowski sum of rank 1 symplectic matroid basis polytopes. In other words, we found a basis of the linear space spanned by the bisubmodular functions which consists of the support functions of rank 1 symplectic matroid basis polytopes, and give the exchange matrices between two bases explicitly. This gives a full answer to [ACEP20, Question 9.3] for types B and C. Three bases are found earlier in the literature, but no exchange matrix between bases is known. As a result, we have an explicit volume formula for any generalized permutohedron of type B, C or D.

Theorem 3.4.2. Every generalized C_n-permutohedron

$$\Pi_{C_n}(h) := \{x \in \mathbb{R}^n : \langle x, \mathbf{e}_T \rangle \leq h(T) \ \forall T \sqsubseteq [\pm n]\} = \sum_{\emptyset \neq S_1 S_2 \subseteq [n]} y_{S_1 \overline{S_1} S_2} \Delta_{S_1 \overline{S_1} S_2}$$

with support function h can be written uniquely as a signed Minkowski sum of the

$3^n - 1$ rank 1 symplectic matroid basis polytopes $\Delta_{S_1\overline{S_1}S_2}$, $\emptyset \neq S_1S_2 \subseteq [n]$, where

$$y_{S_1\overline{S_1}S_2} = (-1)^{|S_2|} \sum_{J \subseteq S_1S_2} (-1)^{|J|} \left(\frac{1}{2} h\left(([n]\backslash J)\overline{J}\right) - h\left(([n]\backslash S_1S_2)\overline{J}\right) \right) - \\ - (-1)^{|S_2|} \sum_{\substack{I \subseteq S_1 \\ J \subsetneq S_2}} (-1)^{|IJ|} \left(h\left(([n]\backslash S_1J)\overline{IJ}\right) - h\left(([n]\backslash S_1S_2)\overline{IJ}\right)\right).$$

In § 3.4.3 we prove the formulas for the mixed volumes of standard simplices and of rank 1 symplectic matroid independent set polytopes using only elementary properties of mixed volumes. All previous proofs in the literature are using the BKK Theorem or (tropical) intersection theory. In § 3.4.4 we point out the relation between mixed volumes the various marriage theorems in transversal theory. The connection between mixed volumes and combinatorial marriage theorems was not pointed out before in the literature.

The last chapter is based on [BCMVV23], where all authors contributed equally significantly. The maximum likelihood degree (ML-degree) of a linear concentration model represented by a generic linear space, the algebraic degree of semidefinite programming (SDP-degree) are fundamental measures for the computational complexity of the statistical model and the SDP, respectively. They can be expressed in the language of enumerative geometry. In [MMMSV23], the analogues of the ML-degree and the SDP-degree in other classical types are introduced, and all of them are proven to be polynomial functions. In order to prove the results, the Lascoux (quasi-)polynomials of types C, A and D are introduced and their (quasi-)polynomiality are proven.

In the last chapter we give explicit formulas for the degrees and the leading coefficients of the Lascoux (quasi-)polynomials of types C, A and D. The formula for type C in Theorem 4.3.2 was already given in [MMMSV23], we provide a more direct proof. The formulas for types A and D in Theorem 4.4.2 and Theorem 4.5.1 are not known before. It follows that the Lascoux (quasi-)polynomials, the ML-degrees and SDP-degrees in type C, A and D are (quasi-)polynomials. As an application, we give the degree of the polynomial $\delta(m, n, n-s)$ and the leading coefficient for $s = 1$ explicitly in types C, A and D.

In order to prove the formulas for the leading coefficients, we found the following mysterious formulas for sums and products as rational functions.

Corollary 4.2.2 (Sum Lemma)**.**

$$x_1 + \cdots + x_r = \sum_{l=1}^{r} x_l \prod_{j \neq l} \frac{(x_j - x_l + 1)(x_j + x_l)}{(x_j - x_l)(x_j + x_l - 1)}.$$

Corollary 4.2.4 (Product Lemma)**.**

$$x_1 \cdots x_r = \prod_{j=1}^{r} (x_j + 2) - 2 \sum_{l=1}^{r} \prod_{l \neq j=1}^{r} \frac{(x_j + 2)(x_j - x_l - 1)(x_j + x_l + 2)}{(x_j - x_l)(x_j + x_l + 3)}.$$

Chapter 2

Conditional independence, matroids and generalized permutohedra

2.1 The geometry of semigraphoids and semimatroids

2.1.1 Conditional independence and semigraphoids

For a finite ground set E we denote by

$$\mathcal{A}_E := \{(ij|K) : K \subseteq E, i \neq j \in E \backslash K\}$$

the set of *conditional independent statements* (*CI-statements*) $(ij|K)$ on E. A *conditional independence structure* (*CI-structure*) $\mathcal{G} \subseteq \mathcal{A}_E$ on E is a subset of $\mathcal{A}_E$. We usually identify the CI-structures up to isomorphy, that is, relabeling the ground set elements, therefore we often use $[n] = \{1, \dots, n\}$ as the ground set and write $\mathcal{A}_n := \mathcal{A}_{[n]}$ for convenience.

Notational convention: In this thesis we use the "Matúš notation". Subsets (possibly empty) are denoted by upper case letters, singletons are denoted by lower case letters, and concatenation of letters means disjoint union of sets. In particular, i and j are exchangeable in the notation $(ij|K)$ of a CI-statement. Sets are assumed to be disjoint whenever they are written in concatenation in a condition. If the CI-structure $\mathcal{G}$ is clear in the context, we write $i \perp\!\!\!\perp j|K$ iff $(ij|K) \in \mathcal{G}$ and $i \not\!\perp\!\!\!\perp j|K$ iff $(ij|K) \notin \mathcal{G}$. If a condition appears as an axiom, it should be valid for all subsets and singletons of the ground set such that the expression makes sense. Isomorphic objects are sometimes identified.

Let $\xi = (\xi_1, \dots, \xi_n)$ be an n-dimensional random vector. We denote "random variables ξ_i and ξ_j are conditionally independent given $\{\xi_k : k \in K\}$" by $i \perp\!\!\!\perp j|K$. Then the CI-structure $[[\xi]] := \{(ij|K) \in \mathcal{A}_n : i \perp\!\!\!\perp j|K\}$ satisfies the *semigraphoid axiom* [Daw79]

(SG) $i \perp\!\!\!\perp j|K \wedge i \perp\!\!\!\perp \ell|jK \Rightarrow i \perp\!\!\!\perp \ell|K \wedge i \perp\!\!\!\perp j|\ell K$.

We refer to [Stu06, Appendix A.7] for a σ-theoretic proof. A CI-structure is a *semigraphoid* if it satisfies the axiom (SG). As in the case of matroids and linear independence, there exist semigraphoids which do not come from the conditional independence among the components of any random vector. The CI-structures coming from random vectors are not finitely axiomatizable [Stu92], but the semigraphoid axioms describe an upper approximation to this set of CI-structures [Stu94]. We refer to [Stu06] for more details about conditional independence.

Example 2.1.1. Let ξ be an n-dimensional discrete random vector, that is, ξ only takes a finite number of values. In the following Theorem 2.1.2 a condition always satisfied by $[[\xi]]$ which cannot be deduced from (SG) is given. This condition will be used in Subsection 2.3.1. More inference rules satisfied by any discrete random vector can be found in [Š07, Lemma 23].

Theorem 2.1.2 ([Stu89])**.** *If ξ is a discrete random vector, then the semigraphoid $[[\xi]]$ satisfies*

$$\{(ij|L),(k\ell|iL),(k\ell|jL),(ij|k\ell L)\}\subseteq[[\xi]] \\ \Rightarrow \{(k\ell|L),(ij|kL),(ij|\ell L),(k\ell|ijL)\}\subseteq[[\xi]]. \quad (2.1)$$

Example 2.1.3. If the random vector ξ has a positive density function, then $[[\xi]]$ satisfies the *intersection axiom* [PP87]

(Int) $i \perp\!\!\!\perp j|\ell K \wedge i \perp\!\!\!\perp \ell|jK \Rightarrow i \perp\!\!\!\perp j|K \wedge i \perp\!\!\!\perp \ell|K$.

A CI-structure is a *graphoid* if it satisfies (SG) and (Int). In [Fin11] a characterization of the conditions on the density function which guarantee (Int) to hold is given for discrete random variables and in [Pet15] for continuous ones.

Example 2.1.4. Let $\mathcal{C} \subseteq 2^{[n]}$ be a set family on $[n]$. Then the CI-structure

$$[[\mathcal{C}]] = \{(ij|K) \in \mathcal{A}_n : \nexists C \in \mathcal{C} \text{ such that } ij \subseteq C \subseteq [n]\backslash K\}.$$

is a semigraphoid. Moreover, it satisfies (2.1) and the *ascension axiom* [Mat92]

(Asc) $(ij|L) \in \mathcal{G} \Rightarrow (ij|kL) \in \mathcal{G}$.

Any semigraphoid satisfying (2.1) and (Asc) is of the form $[[\mathcal{C}]]$ for some set system $\mathcal{C}$. In particular, if $\mathcal{C}$ is the set of connected subsets of $[n]$ on which a topology is equipped, or the set of vertex subsets of a graph on $[n]$ that induce connected subgraphs, then $[[\mathcal{C}]]$ is the set of CI-statements $(ij|K)$ that "i and j are separated by K". CI-structures of these kinds will be discussed in Section 2.3.

Example 2.1.5. Let M be a matroid on $[n]$ and $r_M \colon 2^{[n]} \to \mathbb{N}$ be the rank function of M. Then

$$[[M]] := \{(ij|K) \in \mathcal{A}_n : r_M(iK) + r_M(jK) = r_M(ijK) + r_M(K)\}$$

is a semigraphoid. An cryptomorphic axiomatization of matroids as certain semigraphoids is given in § 2.4.2. Moreover, one can get a semigraphoid from a polymatroid or a submodular function, see § 2.1.3.

Remark 2.1.6. A semigraphoid is originally defined as a set of triples $A \perp\!\!\!\perp B | C$ of pairwise disjoint subsets $A, B, C \subseteq E$ which satisfies certain axioms [Daw79; Pea88]. It is determined by the subset of it consisting of all triples with $|A| = |B| = 1$ and therefore is equivalent to the definition used here [Mat92, Proposition 1].

In [Mat93], the notions of matroid operations are adapted to CI-structures. Let $\mathcal{G} \subseteq \mathcal{A}_n$ be a CI-structure. The *deletion* and *contraction* of $\mathcal{G}$ by $A \subseteq [n]$ are

$$\mathcal{G} \backslash A := \{(ij|K) \in \mathcal{G} : ijK \subseteq [n] \backslash A\},$$
$$\mathcal{G}/A := \{(ij|K) \in \mathcal{A}_{[n]\backslash A} : (ij|KA) \in \mathcal{G}\},$$

respectively. They reflect marginalization and conditioning in probability theory. The *restriction* $\mathcal{G}|A$ of $\mathcal{G}$ to A is $\mathcal{G}\backslash([n]\backslash A)$. The *dual* of $\mathcal{G}$ is $\mathcal{G}^* := \{(ij|[n]\backslash ijK) : (ij|K) \in \mathcal{G}\}$. The usual rules for matroid operations also work for CI-structure operations, for example, deletion and contraction commute and are dual operations of each other. A CI-structure $\mathcal{G}'$ is a *minor* of a CI-structure $\mathcal{G}$ if $\mathcal{G}'$ can be obtained from $\mathcal{G}$ by applying any sequence of deletion and contraction. The *direct sum* of two CI-structures $\mathcal{G}_1 \subseteq \mathcal{A}_{E_1}$ and $\mathcal{G}_2 \subseteq \mathcal{A}_{E_2}$ is the CI-structure

$$\mathcal{G}_1 \oplus \mathcal{G}_2 = \{(ij|K) \in \mathcal{A}_{E_1E_2} : i \in E_1, j \in E_2 \text{ or } ij \subseteq E_1, (ij|K \cap E_1) \in \mathcal{G}_1 \\ \text{ or } ij \subseteq E_2, (ij|K \cap E_2) \in \mathcal{G}_2\}$$

on E_1E_2.

2.1.2 Semigraphoids and fans

A geometric characterization of semigraphoids is given in [MPSSW09]. Consider the hyperplanes $\{x_i = x_j\}$, $1 \leq i < j \leq n$ in $\mathbb{R}^n$. These hyperplanes define a complete fan $\Sigma_{A_{n-1}}$ in $\mathbb{R}^n$ whose *chambers* (maximal cones) are $\{x \in \mathbb{R}^n : x_{\delta(1)} \geq \cdots \geq x_{\delta(n)}\}$, $\delta \in \mathfrak{S}_n$. We denote by the *descent vector* $(\delta(1)| \cdots |\delta(n))$ the permutation $\delta \in \mathfrak{S}_n$ corresponding to this cone. As $\Sigma_{A_{n-1}}$ has the lineality space $\mathbb{R}\mathbf{1} = \{(a, \ldots, a) : a \in \mathbb{R}\}$, we usually consider $\Sigma_{A_{n-1}}$ as a fan in the quotient space $\mathbb{R}^n/\mathbb{R}\mathbf{1}$.

The fan $\Sigma_{A_{n-1}}$ is called the *permutohedral fan* because it is the normal fan of the *permutohedron*

$$\begin{aligned}\Pi_{n-1} &= \operatorname{conv}\left\{(\delta^{-1}(n), \ldots, \delta^{-1}(1)) : \delta \in \mathfrak{S}_n\right\} \\ &= \left\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \frac{n(n+1)}{2}, \forall S \subseteq E : \sum_{i \in S} x_i \geq \frac{|S|(|S|+1)}{2}\right\} \\ &= \frac{n}{2}\mathbf{1} + \sum_{1 \leq i < j \leq n} \left[-\frac{\mathbf{e}_j - \mathbf{e}_i}{2}, \frac{\mathbf{e}_j - \mathbf{e}_i}{2}\right],\end{aligned}$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector in $\mathbb{R}^n$.

A *wall* (1-codimensional cone) of $\Sigma_{A_{n-1}}$ is of the form $\{x : x_{\delta(1)} \geq \cdots \geq x_{\delta(i)} = x_{\delta(i+1)} \geq \cdots \geq x_{\delta(n)}\}$, which is the intersection of two maximal cones corresponding to the permutations $(\delta(1)|\cdots|\delta(i)|\delta(i{+}1)|\cdots|\delta(n))$ and $(\delta(1)|\cdots|\delta(i{+}1)|\delta(i)|\cdots|\delta(n))$. We associate this wall to the CI-statement $\delta(i)\perp\!\!\!\perp\delta(i+1)|\delta(1)\cdots\delta(i-1)$ in $\mathcal{A}_n$.

Every CI-statement $(ij|K) \in \mathcal{A}_n$ corresponds to $|K|!(n-|K|-2)!$ walls of the form

$$\{x : x_{k_1} \geq \cdots \geq x_{k_s} \geq x_i = x_j \geq x_{\ell_1} \geq \cdots \geq x_{\ell_{n-s-2}}\} \tag{2.2}$$

for $k_1 \cdots k_s = K$ and $\ell_1 \cdots \ell_{n-s-2} = [n]\backslash ijK$ in the permutohedral fan $\Sigma_{A_{n-1}}$, or equivalently, $|K|!(n-|K|-2)!$ edges of the permutohedron Π_{n-1}.

The *ridges* (2-codimensional cones) of $\Sigma_{A_{n-1}}$ are either of the form

$$\{x : x_{\delta(1)} \geq \cdots \geq x_{\delta(i)} = x_{\delta(i+1)} = x_{\delta(i+2)} \geq \cdots \geq x_{\delta(n)}\}$$

for some $\delta \in \mathfrak{S}_n$ and $1 \leq i \leq n-2$, or

$$\{x : x_{\delta(1)} \geq \cdots \geq x_{\delta(i)} = x_{\delta(i+1)} \geq \cdots \geq x_{\delta(j)} = x_{\delta(j+1)} \geq \cdots \geq x_{\delta(n)}\}$$

for some $\delta \in \mathfrak{S}_n$ and $1 \leq i \leq j-2 \leq n-3$.

For $i = 1, \ldots, n-1$ let s_i be the reflection across the hyperplane $\{x_i = x_{i+1}\}$. They generate the symmetric group $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1}\rangle$ which acts transitively on the vertices of Π_{n-1}. It is easy to see that the vertices of a 2-face (or the chambers containing a ridge) is a coset $\delta \cdot \langle s_i, s_j\rangle$. Moreover, this 2-face is a square if $j > i+1$ and a hexagon if $j = i+1$. Figure 2.1 shows the permutations and CI-statements in a general 2-face of the permutohedron Π_{n-1}.

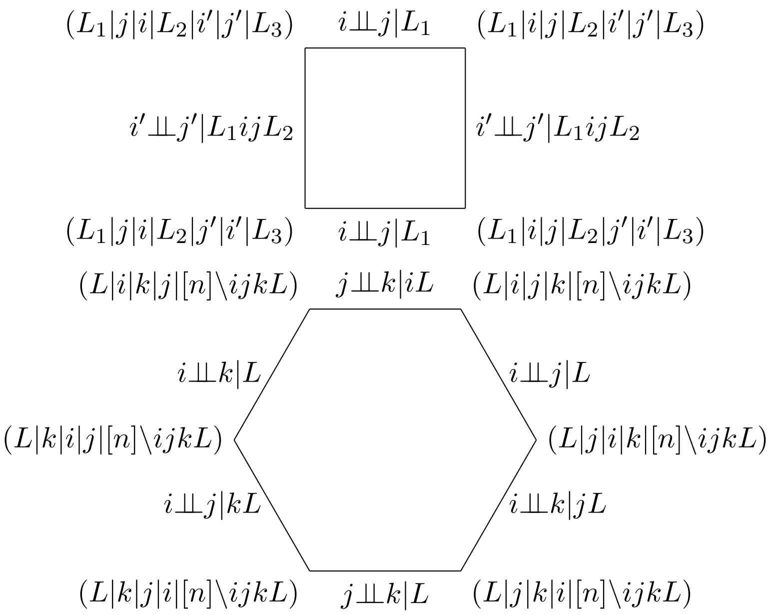


Figure 2.1: 2-faces of Π_{n-1}

We are ready to translate the semigraphoid axiom into geometric language.

Proposition 2.1.7 ([MPSSW09, Observation 11]). *A set E of edges of the permutohedron Π_{n-1} is a semigraphoid iff it satisfies*

(4gon) *if an edge of a square is in E, then the opposite edge is also in E.*

(6gon) *if two adjacent edges of a hexagon are in E, then the two opposite edges are also in E.*

Notice that (4gon) reflects the fact that $(ij|K)$ and $(ji|K)$ are not to distinguish, and (6gon) is merely a reformulation of (SG). Consider the normal fan of a square or a hexagon. It is easy to see that removing a set of walls results in a coarser fan iff the set of edges corresponding to the removed walls satisfies (4gon) respective (6gon). In fact, this is valid for permutohedra in any dimension. That is, semigraphoids are exactly the subsets $\mathcal{G}$ of $\mathcal{A}_n$ such that the removal of all walls corresponding to the elements of $\mathcal{G}$ from the permutohedral fan $\Sigma_{A_{n-1}}$ results in a coarser fan [MPSSW09].

Theorem 2.1.8 ([MPSSW09, Theorem 9]). *A set of walls in the permutohedral fan $\Sigma_{A_{n-1}}$ is corresponding to a semigraphoid iff removing them from $\Sigma_{A_{n-1}}$ results in a fan.*

A fan Σ' is a *coarsening* of a fan Σ if every cone in Σ is a subset of a cone Σ'. If Σ and Σ' are complete and have the same support, then Σ' can be obtained from Σ by removing certain walls.

So we identify a coarsening of the fan $\Sigma_{A_{n-1}}$ with the semigraphoid on $[n]$ consisting of the CI-statements corresponding to the missed walls.

2.1.3 Semimatroids and generalized permutohedra

A set function $\omega\colon 2^{[n]} \to \mathbb{R}$ is *submodular* if for all $A, B \subseteq [n]$,

$$\omega(A) + \omega(B) \geq \omega(A \cap B) + \omega(A \cup B).$$

Definition 2.1.1. The *semimatroid* of a submodular function $\omega\colon 2^{[n]} \to \mathbb{R}$ is the CI-structure

$$[[\omega]] := \{(ij|K) \in \mathcal{A}_n : \omega(iK) + \omega(jK) = \omega(ijK) + \omega(K)\}. \tag{2.3}$$

A CI-structure $\mathcal{G}$ on $[n]$ is a *semimatroid* if there is a submodular function $\omega\colon 2^{[n]} \to \mathbb{R}$ such that $\mathcal{G} = [[\omega]]$.

Submodular functions are important in the modeling of conditional independence [Stu06, Chap. 5]. If ξ is an n-dimensional discrete random vector, that is, a random vector which takes a finite number of values, and $h_\xi\colon 2^{[n]} \to \mathbb{R}$ maps every subset $S \subseteq [n]$ to the Shannon entropy of the subvector ξ_S, then h is submodular and $[[\xi]] = [[h_\xi]]$ [Fuj78]. The same also holds for the class of measures with finite multiinformation [Stu06, Corollary 2.2], which is a quite wide class containing the class of discrete random vectors and the class of regular normally distributed random vectors.

A *polymatroid* is a monotonic submodular function $r\colon 2^{[n]} \to \mathbb{R}$ with $r(\emptyset) = 0$. If $\omega\colon 2^{[n]} \to \mathbb{R}$ is a submodular function, then the function $\omega'\colon 2^{[n]} \to \mathbb{R}$ defined by $\omega'(A) = \omega(A) - \omega(\emptyset) + m|A|$ is a polymatroid for large enough $m \in \mathbb{R}$, and it satisfies $[[\omega']] = [[\omega]]$. Therefore every semimatroid is the semimatroid of some polymatroid.

Let $P \subseteq \mathbb{R}^n$ be a polytope. The normal fan of P is denoted by Σ_P. The *support function* $h_P\colon \mathbb{R}^n \to \mathbb{R}$ of P is defined by

$$h_P(u) := \max_{v \in P} \langle u, v \rangle. \tag{2.4}$$

And we can recover the polytope P from h_P by

$$P = \{ v \in \mathbb{R}^n : \langle u, v \rangle \leq h_P(u) \text{ for all } u \in \mathbb{R}^n \}. \tag{2.5}$$

The support function h_P is a continuous real-valued function on $|\Sigma_P| = \mathbb{R}^n$ which is linear on each cone. We denote the space of such functions on the support $|\Sigma|$ of a fan Σ by $\mathrm{PL}(\Sigma)$. Every function in $\mathrm{PL}(\Sigma)$ is uniquely determined by its restrictions to the rays $\Sigma(1)$ of Σ. For each ray $\rho \in \Sigma(1)$ of a rational fan Σ, let u_ρ be the first lattice point on the ray ρ. Let $\mathcal{R} = \{u_\rho : \rho \in \Sigma(1)\}$, we identify a function $h \in \mathrm{PL}(\Sigma)$ with the function $h\colon \mathcal{R} \to \mathbb{R}$ on the rays determining it. Moreover, $\mathrm{PL}(\Sigma) \cong \mathbb{R}^{\mathcal{R}}$ if Σ is simplicial.

A polytope Q is a *deformation* of polytope P if the normal fan Σ_Q is a coarsening of the normal fan Σ_P.

Proposition 2.1.9 ([CLS11, § 6.1])**.** *Let $P \subseteq \mathbb{R}^n$ be a polytope. A function $h \in \mathrm{PL}(\Sigma_P)$ is convex iff it is the support function of a deformation Q of P. The correspondence is bijective by* (2.4) *and* (2.5)*.*

The set of support functions of deformations of a polytope P form a full-dimensional cone in $\mathrm{PL}(\Sigma)$.

A *generalized permutohedron* is a deformation of a permutohedron. Generalized permutohedra can be characterized in several ways [PRW08, Theorem 15.3].

Theorem 2.1.10 ([Pos09; PRW08])**.** *A polytope $P \subseteq \mathbb{R}^n$ is a generalized permutohedron iff one of the following equivalent conditions is satisfied:*

(1) *there exists a submodular function $\omega\colon 2^{[n]} \to \mathbb{R}$ with $\omega(\emptyset) = 0$ such that*

$$P = \left\{ x \in \mathbb{R}^n : \sum_{i \in I} x_i \leq \omega(I) \ \forall \emptyset \neq I \subseteq [n], \sum_{i \in [n]} x_i = \omega([n]) \right\}; \tag{2.6}$$

(2) *the normal fan Σ_P of P coarsens $\Sigma_{A_{n-1}}$;*

(3) *P is a Minkowski summand of $\lambda \Pi_{n-1}$ for some $\lambda > 0$;*

(4) *all edge directions are of the form $\mathbf{e}_i - \mathbf{e}_j$, $i \neq j$.*

Submodular functions $h\colon 2^{[n]} \to \mathbb{R}$ with $h(\emptyset) = 0$ correspond bijectively to generalized permutohedra as their support functions. Since we consider the fan $\Sigma_{A_{n-1}}$ as a complete fan in $\mathbb{R}/\mathbb{R}\mathbf{1}$ whose rays are $\operatorname{conv}(u)$, $u \in \mathcal{R}_{A_{n-1}} := \{\overline{\mathbf{e}}_S : \emptyset \neq S \subsetneq [n]\}$, where $\mathbf{e}_S := \sum_{i\in S} \mathbf{e}_i$ and $\overline{\mathbf{e}}_S$ is the image of $\mathbf{e}_S$ under the projection $\mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{R}\mathbf{1}$, we assume that generalized permutohedra lie in the dual space $\mathbf{1}^\perp = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$ of $\mathbb{R}/\mathbb{R}\mathbf{1}$. By Theorem 2.1.10, these polytopes correspond bijectively to the submodular functions $h\colon 2^{[n]} \to \mathbb{R}$ with $h(\emptyset) = h([n]) = 0$, which are precisely the convex functions in $\mathbb{R}^{\mathcal{R}_{A_{n-1}}} \cong \mathrm{PL}(\Sigma_{A_{n-1}})$ by identifying $S \mapsto h(S)$ with $\overline{\mathbf{e}}_S \mapsto h(S)$. The cone of such functions is the *submodular cone*

$$\mathrm{SF}_{A_{n-1}} := \left\{h\colon 2^{[n]} \to \mathbb{R} \text{ submodular} : h(\emptyset) = h([n]) = 0\right\} \subseteq \mathbb{R}^{\mathcal{R}_{A_{n-1}}}.$$

We conclude that a semimatroid is always a semigraphoid. The *semimatroid* of a generalized permutohedron P is $[[h_P]]$. Semimatroids encode the combinatorial data of generalized permutohedra: Two generalized permutohedra define the same semimatroid iff their face lattices are same. The lattice of semimatroids ordered by inclusion is isomorphic to the dual of the face lattice of the submodular cone [SK16]. In particular, the atoms of the lattice of all semimatroids on $[n]$, which are the singletons $\{(ij|K)\}$ for all $(ij|K) \in \mathcal{A}_n$, are corresponding bijectively to the facets of the submodular cone $\mathrm{SF}_{A_{n-1}}$. We will extend these concepts to other root systems in § 3.1.3.

Theorem 2.1.11 ([MPSSW09, Theorem 17][MUWY18, Appendix B])**.** *A semigraphoid $\mathcal{G}$ is a semimatroid iff the corresponding coarsening of the permutohedral fan is polytopal. In particular, it is the normal fan of the generalized permutohedron* (2.6) *defined by any submodular function whose semimatroid is $\mathcal{G}$.*

Support functions and semimatroids behave well with respect to Minkowski arithmetic. The *common refinement* $\Sigma_1 \wedge \Sigma_2$ of fans Σ_1 and Σ_2 in the same space is $\Sigma_1 \wedge \Sigma_2 := \{\sigma \cap \sigma' : \sigma \in \Sigma_1, \sigma' \in \Sigma_2\}$.

Proposition 2.1.12. *For any two generalized permutohedra P_1 and P_2 we have $h_{P_1+P_2} = h_{P_1} + h_{P_2}$, $\Sigma_{P_1+P_2} = \Sigma_{P_1} \wedge \Sigma_{P_2}$ and $[[h_{P_1+P_2}]] = [[h_{P_1}]] \cap [[h_{P_2}]]$.*

A polytope $P \subseteq \mathbb{R}^n$ is a *Minkowski summand* of another polytope $Q \subseteq \mathbb{R}^n$ if there is a polytope $R \subseteq \mathbb{R}^n$ such that $P + R = Q$, in this case, R is the *Minkowski difference* of Q and P, denoted by $R = Q - P$. A *signed Minkowski sum* of polytopes of polytopes $P_1, \ldots, P_m$ is a formal sum $\sum_{i\in[m]} y_i P_i$ with real coefficients $y_1, \ldots, y_m$. The signed Minkowski sum $R = \sum_{i\in[m]} y_i P_i$ defines a polytope if $P = \sum_{i\in[m]:y_i<0} (-y_i) P_i$ is a Minkowski summand of $Q = \sum_{i\in[m]:y_i\geq 0} y_i P_i$, in this case R defines the Minkowski difference $Q - P$.

The support functions of the $2^n - 1$ standard simplices $\Delta_I := \operatorname{conv}(\mathbf{e}_i : i \in I) \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq [n]$ form a basis of $\mathbb{R}^{\mathcal{R}_{A_{n-1}}}$.

Proposition 2.1.13 ([ABD10, Proposition 2.3])**.** *Every generalized permutohedron $P \subseteq \mathbb{R}^n$ can be written uniquely as a signed Minkowski sum of simplices as $P = \sum_{\emptyset\neq I\subseteq[n]} y_I \Delta_I$ where $y_I = \sum_{J\subseteq I} (-1)^{|I|-|J|} h_P(J)$ for $\emptyset \neq I \subseteq [n]$.*

Theorem 2.1.14 ([JR22, Theorem 2.4])**.** *The signed Minkowski sum $\sum_{I\subseteq[n]} y_I \Delta_I$ defines a generalized permutohedron in $\mathbb{R}^n$ iff for all $E \subseteq T \subseteq [n]$ with $|E| = 2$, $\sum_{E\subseteq I\subseteq T} y_I \geq 0$.*

Example 2.1.15. Let M be a matroid on $[n]$ with rank function r. Since r is submodular, the matroid defines a semimatroid $[[r]] \subseteq \mathcal{A}_n$. Moreover,

$$(ij|K) \notin [[r]] \quad \Leftrightarrow \quad r(K) + 1 = r(iK) = r(jK) = r(ijK).$$

The corresponding coarsening of $\Sigma_{A_{n-1}}$ is the outer normal fan of the matroid polytope $\mathcal{P}(M)$.

2.2 Gaussoids and their realizations

2.2.1 Gaussians and Gaussoids

For a matrix $M \in \mathbb{K}^{m\times n}$ and $A \subseteq [m]$, $B \subseteq [n]$, we denote by $M_{A,B}$ the submatrix of M with rows in A and columns in B, and write $M_A := M_{A,A}$. We also denote by x_B the subvector of $x \in \mathbb{K}^n$ in $\mathbb{K}^B$.

Let $\xi = \xi_{[n]}$ be an n-dimensional *(regular) normally distributed* random vector (or *Gaussian*), that is, ξ has the density, with respect to the Lebesgue measure on $\mathbb{R}^n$, of the form

$$f_\xi(x) = \frac{\exp\left(-\frac{1}{2}(x-\mu)^\top \Sigma^{-1}(x-\mu)\right)}{\sqrt{(2\pi)^n \det(\Sigma)}},$$

where $\mu \in \mathbb{R}^n$ is the *mean vector* and the positive definite symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$ is the *covariance matrix* of ξ.

For $A \subseteq [n]$, the *marginal* distribution of ξ_A is normal with mean μ_A and covariance matrix Σ_A, and the *conditional* distribution of $\xi_{[n]\setminus A}$ on $\xi_A = x_A \in \mathbb{R}^A$ is also normal with mean $\mu_{[n]\setminus A} + \Sigma_{[n]\setminus A,A}\Sigma_A^{-1}(x_A - \mu_A)$ and covariance matrix $\Sigma_{/A} := \Sigma_{[n]\setminus A} - \Sigma_{[n]\setminus A,A}\Sigma_A^{-1}\Sigma_{A,[n]\setminus A}$, which is the *Schur complement* of Σ_A in Σ.

Because the components of a normally distributed random vector are independent iff they are uncorrelated, the CI-statement $(ij|K) \in \mathcal{A}_n$ is in $[[\xi]]$ iff

$$0 = \left((\Sigma_{ijK})_{/K}\right)_{i,j} = \Sigma_{i,j} - \Sigma_{i,K}\Sigma_K^{-1}\Sigma_{K,j} = (\Sigma_{ij|K})_{/K},$$

where $\Sigma_{ij|K} := \Sigma_{iK,jK}$ is the almost-principal submatrix of Σ. As $\det(\Sigma_{ij|K}) = \det((\Sigma_{ij|K})_{/K})\det(\Sigma_K)$ and $\det(\Sigma_K) > 0$, we have $(ij|K) \in [[\xi]]$ iff $\det(\Sigma_{ij|K}) = 0$. Therefore the CI-structure $[[\xi]]$ is determined by the covariance matrix Σ of ξ, and is denoted by

$$[[\Sigma]] := [[\xi]] = \{(ij|K) \in \mathcal{A}_n : \det(\Sigma_{ij|K}) = 0\}. \tag{2.7}$$

The following quadratic trinomial relations among principal and almost-principal minors of a symmetric matrix Σ can be deduced easily by expanding the determinant of the Schur complements $(\Sigma_{ijK})_{/K}$ and $(\Sigma_{ij|kL})_{/L}$, respectively.

Lemma 2.2.1 (Dodgson[1] Condensation [Dod67]).

$$\det(\Sigma_{ij|K})^2 = \det(\Sigma_{iK})\det(\Sigma_{jK}) - \det(\Sigma_{ijK})\det(\Sigma_K). \tag{2.8}$$

Lemma 2.2.2 ([Mat05, Lemma 1]).

$$\det(\Sigma_{kL})\det(\Sigma_{ij|L}) = \det(\Sigma_L)\det(\Sigma_{ij|kL}) + \det(\Sigma_{ik|L})\det(\Sigma_{jk|L}). \tag{2.9}$$

Note that we use the convention that for any almost-principal submatrix $\Sigma_{ij|K}$ we sort the rows and columns such that the row indexed by i and the column indexed by j come first and are followed by those indexed by K, that is, the i-row and the j-column are paired. Establishing the convention has the advantage of simplifying the sign in [Mat05, Lemma 1], and making the sign of $\Sigma_{ij|K}$ consistent with its statistical interpretation as the sign of conditional correlation.

Gaussoids are introduced in [LM07] as an abstraction of the conditional independence among regular normally distributed random variables.

Definition 2.2.1. A CI-structure $\mathcal{G} \subseteq \mathcal{A}_n$ is a *gaussoid* on $[n]$ if it satisfies the following axioms:

(SG) $\{(ij|L), (ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ik|L), (ij|kL)\} \subseteq \mathcal{G}$,

(Int) $\{(ij|kL), (ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ij|L), (ik|L)\} \subseteq \mathcal{G}$,

(Comp) $\{(ij|L), (ik|L)\} \subseteq \mathcal{G} \Rightarrow \{(ij|kL), (ik|jL)\} \subseteq \mathcal{G}$,

(WT) $\{(ij|L), (ij|kL)\} \subseteq \mathcal{G} \Rightarrow (ik|L)$ or $(jk|L) \in \mathcal{G}$.

The axioms (Int), (Comp) and (WT) are known as the *intersection axiom*, the *composition axiom* and the *weak transitivity axiom*, respectively.

A symmetric matrix Σ over a field $\mathbb{K}$ is *principally regular* if all principal minors of Σ do not vanish. We can also define $[[\Sigma]]$ for any symmetric $\Sigma \in \mathbb{K}^{n\times n}$ by (2.7). From (2.9) we can deduce that the vanishing of almost-principal minors of principally regular Σ satisfies the gaussoid axioms.

Corollary 2.2.3 ([Mat05, Corollary 1], [Boe22b, Proposition 3.8]). *For any principally regular symmetric matrix Σ, $[[\Sigma]]$ is a gaussoid. In particular, $[[\xi]]$ is a gaussoid for any regular normally distributed random vector ξ.*

In this case, we call $[[\Sigma]]$ and $[[\xi]]$ the *gaussoid* of Σ and of ξ, respectively.

Being a gaussoid is closed under the restriction, contraction, dual and direct sum operations [LM07], [Boe22a, Remark 5.2]. These operations are well compatible with the corresponding operations in probability theory and of matrices. For a Gaussian ξ on $[n]$ and $A \subseteq [n]$, the restriction $[[\xi]]|A$ is the gaussoid of the marginalization of ξ to A, and the contraction $[[\xi]]/A$ is the gaussoid of the conditioning of ξ to A. The direct sum is the gaussoid of the joint distribution of two Gaussians that are

[1] a.k.a. Lewis Carroll

independent to each other. The duality is the Fourier transformation: The Fourier transform of a Gaussian density function with covariance matrix Σ is also a Gaussian density function whose covariance matrix is Σ^{-1}, and $[[\Sigma^{-1}]] = [[\Sigma]]^*$.

If the symmetric matrices $\Sigma \in \mathbb{K}^{n\times n}$ and $\Sigma' \in \mathbb{K}^{m\times m}$ are principally regular and $A \subseteq [n]$, then $[[\Sigma]]|A = [[\Sigma_A]]$, $[[\Sigma]]/A = [[\Sigma_{/A}]]$, $[[\Sigma]]^* = [[\Sigma^{-1}]]$ and $[[\Sigma]] \oplus [[\Sigma']] = [[\Sigma \oplus \Sigma']]$ [Boe22b, § 3.3].

In parallel to the problems of matroid representability, various gaussoid representations are studied.

Definition 2.2.2. A gaussoid $\mathcal{G}$ on $[n]$ is

- *representable* over field $\mathbb{K}$ if $\mathcal{G} = [[\Sigma]]$ for some principally regular symmetric matrix $\Sigma \in \mathbb{K}^{n\times n}$;
- *positively representable* over ordered field $\mathbb{K}$ if $\mathcal{G} = [[\Sigma]]$ for some positive definite symmetric matrix $\Sigma \in \mathbb{K}^{n\times n}$.
- *Gaussian representable* if it is positively representable over $\mathbb{R}$, that is, $\mathcal{G} = [[\xi]]$ for some Gaussian ξ.

In the following subsections wider notions of representability are discussed.

Example 2.2.4 ([BDKS19, Example 13][Boe22b, Example 3.34])**.** The gaussoid

$$\mathcal{V} := \{(12|), (13|4), (14|5), (23|5), (35|1), (45|2), (15|23), (25|34), (34|12), (24|135)\}$$

on $[5]$ is not representable over any field. It is not a semimatroid [Boe22a]. $\mathcal{V}' := \mathcal{V}\backslash(25|34)$ is also a gaussoid which is not representable over any field. It is not a semimatroid as well.[2]

Example 2.2.5. In [HMSSW08], a semigraphoid on $[5]$ with 44 CI-statements is given, which is shown to be a gaussoid [MUWY18] and a maximal not-full-semigraphoid, but not a semimatroid.

Similarly to the semigraphoid case, the CI-structures coming from regular normally distributed random vectors are not finitely axiomatizable [Sul09], but they are axiomatically approximated by gaussoids [Boe22a].

A gaussoid may not be semimatroid, but Gaussian representable gaussoids are semimatroids. For a Gaussian ξ with covariance matrix Σ, by (2.8), the set function $A \mapsto \log\det(\Sigma_A)$ is submodular and satisfies $\log\det(\Sigma_\emptyset) = 0$ and the equality holds exactly at the triples $(ij|K) \in [[\Sigma]]$. Remark that scaling this set function by a negative constant then adding it with a modular function yields the multiinformation of ξ [Stu06, Corollary 2.6]. More generally, positively representable gaussoids are semimatroids [Boe22b, Theorem 6.25].

Because all gaussoids are semigraphoids, by Theorem 2.1.8, we can reformulate the gaussoid axioms (SG), (Int), (Comp) and (WT) in the same way as in Proposition 2.1.7.

[2] Tobias Boege, private communication.

Proposition 2.2.6. *A set E of edges of the permutohedron Π_{n-1} is a gaussoid iff it satisfies*

(4gon) *if an edge of a square is in E, then the opposite edge is also in E.*

(6gon*) *if two edges of a hexagon are in E, then either all edges or all but a pair of opposite edges of that hexagon are also in E.*

Remark 2.2.7. All Gaussians considered here are assumed to be regular. It is also possible to consider *singular Gaussians* by allowing covariance matrices to be positive semidefinite. The conditioning of a singular Gaussian can be defined via the disintergration theorem [CP97] and we get the CI-structure of a singular Gaussian ξ whose positive semidefinite covariance matrix is Σ is

$$[[\xi]] = \{(ij|K) \in \mathcal{A}_n : \Sigma_{i,j} - \Sigma_{i,K}\Sigma_K^{\dagger}\Sigma_{K,j} = 0\},$$

where $\Sigma_K^{\dagger}$ is the Moore-Penrose inverse of Σ_K. For example, the singular Gaussian with the covariance matrix

$$\begin{pmatrix} 5 & 0 & -3 & 4 \\ 0 & 5 & -4 & 3 \\ -3 & -4 & 5 & 0 \\ 4 & -3 & 0 & 5 \end{pmatrix}$$

has the CI-structure $\{(12|), (34|), (12|34), (13|24), (14|23), (23|14), (24|13), (34|12)\}$, which satisfies (SG), (Comp) and (WT) but not (Int). In fact, the CI-structure of any singular Gaussian satisfies (SG), (Comp) and (WT). See [Š07, Lemma 15] for a collection of inference rules satisfied by a singular or regular Gaussian and compare to the regular Gaussian inference rules in [Š07, Lemma 16].

Remark 2.2.8. The conditional uncorrelation of a *complex Gaussian* [And+95], or the conditional independence of a *circular* complex Gaussian [Pic96], corresponds exactly to the vanishing of the almost-principal minors of its covariance matrix, which is a positive definite Hermitian matrix. It also forms a semimatroid due to (2.8). The *quaternionic Gaussians* and their applications to signal processing are also studied [VC10; TM11; SW14]. In the next subsection we discuss the representation of a gaussoid as a principally invertible ι-Hermitian matrix over a skew field $\mathbb{K}$ where ι is an involutive antiautomorphism of $\mathbb{K}$.

2.2.2 Gaussoids over skew fields and orthogonality

In this subsection we investigate the representations of gaussoids over skew fields using the notion of quasideterminants introduced in [GR91]. Let $\mathcal{X} = \{x_{ij} : i, j \in [n]\}$ be a set of n^2 elements indexed by $[n] \times [n]$, and $F(\mathcal{X})$ be the free skew field generated by $\mathcal{X}$. Roughly speaking, it is the skew field of noncommutative rational functions with variables in $\mathcal{X}$. Let $X = (x_{ij})$ be the $n \times n$-matrix over $F(\mathcal{X})$.

Proposition 2.2.9 ([GGRW05, Proposition 1.2.1])**.** *The matrix X is invertible over $F(\mathcal{X})$.*

Proof. We proceed by induction on n. For $n = 1$, $X = (x_{11})$ and $X^{-1} = (x_{11}^{-1})$. For $n \geq 2$, write

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

where $X_{11}, X_{12}, X_{21}, X_{22}$ are blocks of size $(n-1) \times (n-1), (n-1) \times 1, 1 \times (n-1)$ and 1×1, respectively. Then

$$X^{-1} = \begin{pmatrix} (X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1} & -X_{11}^{-1}X_{12}(X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1} \\ -X_{22}^{-1}X_{21}(X_{11} - X_{12}X_{22}^{-1}X_{21})^{-1} & (X_{22} - X_{21}X_{11}^{-1}X_{12})^{-1} \end{pmatrix}.$$

□

Therefore it is possible to define the quasideterminants of a generic square matrix as an element in $F(\mathcal{X})$.

Definition 2.2.3. For $i, j \in [n]$, the (i, j)-th *quasideterminant* $|X|_{ij}$ of X is the element of $F(\mathcal{X})$ defined by $|X|_{ij} = \left((X^{-1})_{ji}\right)^{-1} \in F(\mathcal{X})$.

Example 2.2.10. For $n = 2$, the quasideterminants of $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ are

$$|X|_{11} = x_{11} - x_{12}x_{22}^{-1}x_{21}, \qquad |X|_{12} = x_{12} - x_{11}x_{21}^{-1}x_{22},$$
$$|X|_{21} = x_{21} - x_{22}x_{12}^{-1}x_{11}, \qquad |X|_{22} = x_{22} - x_{21}x_{11}^{-1}x_{12}.$$

The quasideterminants are invariant under permutations of the rows and columns (together with (i, j)).[3]

Proposition 2.2.11 ([GR91, § 1.1])**.** *For $(\sigma, \tau) \in \mathfrak{S}_n \times \mathfrak{S}_n$, we have*

$$(\sigma, \tau)|X|_{\sigma(i)\tau(j)} = |X|_{ij}.$$

Now we can define the quasideterminants of a square matrix over a ring with a unit when the corresponding generic quasideterminants can be evaluated in this ring.

Definition 2.2.4. Let $A = (a_{ij})$ be an $n \times n$-matrix over a ring R with a unit. Let $\alpha_A \colon \mathcal{X} \to R$ be the map determined by $\alpha_A(x_{ij}) = a_{ij}$. If $|X|_{ij} \in F(\mathcal{X})$ can be evaluated at α_A, then we say that the (i, j)-th *quasideterminant* exists and is equal to $\alpha_A(|X|_{ij})$.

The quasideterminant behaves similarly to the Schur complement (hence the conditional covariance of a Gaussian) rather than the usual determinant.

Proposition 2.2.12 ([GGRW05, Proposition 1.2.6])**.** *Let $n \geq 1$ and $i, j \in [n]$. Assume that $A_{[n]\setminus i,[n]\setminus j}$ is invertible over R. Then*

$$|A|_{ij} = a_{ij} - A_{i,[n]\setminus j}\left(A_{[n]\setminus i,[n]\setminus j}\right)^{-1} A_{[n]\setminus i,j}.$$

[3] Proposition 1.2.4 in [GGRW05] was wrong. Here we cite [GR91].

The quasideterminant $|A|_{ij}$ can be equivalently defined as above if the submatrix $A_{[n]\setminus i,[n]\setminus j}$ is invertible.

If the elements a_{ij} of the matrix A commute, then

$$|A|_{ij} = (-1)^{i+j}\frac{\det A}{\det A_{[n]\setminus i,[n]\setminus j}} \tag{2.10}$$

is the Schur complement of $A_{[n]\setminus i,[n]\setminus j}$ after moving the i-th row and the j-th column to the very first places.

The quasideterminants over skew fields are better behaved, almost every property of determinants in linear algebra over fields has a quasideterminantal counterpart. We state the following relations between the vanishing of quasideterminants and the invertibility of a matrix.

Proposition 2.2.13 ([GGRW05, Proposition 1.4.6]). *Let A be a matrix over a skew field. If $|A|_{ij}$ is defined, then the following are equivalent:*

(1) $|A|_{ij} = 0$,

(2) *the i-th row of the matrix A is a left linear combination of the other rows of A,*

(3) *the j-th column of the matrix A is a right linear combination of the other columns of A.*

Theorem 2.2.14 ([GR91, Theorem 1.6]). *Let A be a matrix over a skew field.*

(1) *The inverse matrix $B = A^{-1}$ exists iff*

  (a) *for any $i, j \in [n]$, if the quasideterminant $|A|_{ij}$ is defined, then $|A|_{ij} \neq 0$;*

  (b) *for each row index p there exists q such that the quasideterminant $|A|_{pq}$ is defined;*

  (c) *for each column index q there exists p such that the quasideterminant $|A|_{pq}$ is defined.*

(2) *If the inverse matrix $B = (b_{ij})$ is defined, then for $i, j \in [n]$, the element b_{ji} is equal to $|A|_{ij}^{-1}$ if the quasideterminant $|A|_{ij}$ is defined and to zero in the opposite case.*

We refer to [GR91; GR92; GGRW05] for more nice properties enjoyed by quasideterminants such as row and column operations, homological relations, decompositions, multiplicativity, Cramer's rule, Cayley-Hamilton, Plücker relations and so on. The Sylvester's Identity is important for deducing the relations among principal and almost-principal quasiminors.

Theorem 2.2.15 (Noncommutative Sylvester's Identity [GR92, Theorem 1.2]). *Let $A = (a_{ij})$ be an $n \times n$-matrix over a ring R. Fix $i, j \in [n]$. Let $P = (p_1, \ldots, p_k) \subseteq [n]$, $Q = (q_1, \cdots, q_k) \subseteq [n]$. Define the matrix $B = (b_{kl})_{k\in[n]\setminus P, l\in[n]\setminus Q}$ where $b_{kl} =$*

$|A_{Pk,Ql}|_{kl}$. *(The elements b_{kl} do not depend on the ordering of the sets Pk and Ql.) Then for any $k \notin P$ and $l \notin Q$,*

$$|A|_{kl} = |B|_{kl}.$$

The following noncommutative analogues of (2.8) and (2.9) are immediate consequences of the theorem above by expanding 2×2-quasideterminants.

Corollary 2.2.16 (Noncommutative Dodgson condensation)**.**

$$|A_{ijK}|_{ij} = |A_{iK,jK}|_{ij} - |A_{iK}|_{ii} \cdot |A_{jK,iK}|_{ji}^{-1} \cdot |A_{jK}|_{jj}.$$

Corollary 2.2.17 (Noncommutative Matúš Lemma)**.** *For $L \subseteq [n]$, $k \in [n] \backslash L$ and $i, j \in [n] \backslash kL$,*

$$|A_{ikL,jkL}|_{ij} = |A_{iL,jL}|_{ij} - |A_{iL,kL}|_{ik} \cdot |A_{kL}|_{kk}^{-1} \cdot |A_{kL,jL}|_{kj}.$$

The identities (2.8) and (2.9) in the commutative case can be recovered from these two corollaries by (2.10). The sign $(-1)^{[i,j;k]}$ in [Mat05, Lemma 1] is not needed in the noncommutative case because of Proposition 2.2.11.

Let R be a ring with a unit. A map $\iota\colon R \to R$ is an *antiautomorphism* in R if ι is bijective and for any $x, y \in R$ we have $\iota(x+y) = \iota(x) + \iota(y)$, $\iota(xy) = \iota(y)\iota(x)$ and $\iota(1) = 1$. An antiautomorphism ι in R is *involutive* if $\iota(\iota(x)) = x$ for any $x \in R$. Examples for involutive antiautomorphisms are the matrix transposition, the complex conjugation and the quaternion conjugation.

Let ι be an antiautomorphism in a ring R with a unit and V a left module over R. A *ι-sesquilinear form* on V is a map $f\colon V \times V \to R$ such that for any $x, y, z \in V$ and $t \in R$,

- $f(x+y, z) = f(x, z) + f(y, z), \quad f(x, y+z) = f(x, y) + f(x, z),$
- $f(tx, y) = tf(x, y), \quad f(x, ty) = f(x, y)\iota(t).$

A ι-sesquilinear form f on V is *reflexive* if $f(x, y) = 0$ implies $f(y, x) = 0$ for any $x, y \in V$, and is *Hermitian* if $f(x, y) = \iota(f(y, x))$ for any $x, y \in V$.

The same can be applied for a right R-module after a suitable reordering of expressions. It is clear that if a ι-sesquilinear form f is Hermitian, then necessarily f is reflexive and the antiautomorphism ι is involutive.

Now assume that $\mathbb{K}$ is a skew field, in this case every left (or right) $\mathbb{K}$-module is free. We will use the word $\mathbb{K}$-*vector spaces* instead of $\mathbb{K}$-modules, which is commonly used in geometry, in order to emphasize that the motivation is geometric representations. Let V be a finite dimensional left $\mathbb{K}$-vector space and $b_1, \ldots, b_n$ be a basis of V. A ι-sesquilinear form f on V is determined by $a_{ij} = f(b_i, b_j)$, $i, j \in [n]$. For any $x = \sum x_i b_i$ and $y = \sum y_i b_i$,

$$f(x, y) = f\left(\sum_i x_i b_i, \sum_j y_j b_j\right) = \sum_{i,j} x_i a_{ij} \iota(y_j) = x^\top A \iota(y),$$

where $A = (a_{ij})$ is an $n \times n$-matrix over $\mathbb{K}$. Conversely, any $n \times n$-matrix A over $\mathbb{K}$ determines a ι-sesquilinear form f by $f(x, y) = x^\top A\iota(y)$.

Let $A = (a_{ij})$ be an $n \times n$-matrix over a skew field $\mathbb{K}$ and ι be an involutive antiautomorphism in $\mathbb{K}$. We denote $\iota(A) := (\iota(a_{ij}))$ and

$$A^* := \iota(A^\top) = (\iota(A))^\top.$$

The map $^*\colon A \mapsto A^*$ is also an involutive antiautomorphism in the ring of $n \times n$-matrices over $\mathbb{K}$. The ι-sesquilinear form defined by A is Hermitian iff $A = A^*$.

Proposition 2.2.18 ([HJL12, pp.456-457])**.** *A ι-sesquilinear form f is reflexive iff there exists $\epsilon \in \mathbb{K}\backslash\{0\}$ such that for any $x, y \in V$:*

$$f(x, y) = \epsilon \cdot \iota(f(y, x)). \tag{2.11}$$

A ι-sesquilinear form f satisfying (2.11) is called (ι, ϵ)-*Hermitian.* The corresponding matrix A satisfies $A = \epsilon \cdot \iota(A^\top)$. It follows that if f is a nondegenerate (ι, ϵ)-Hermitian form, then $\iota(\epsilon) = \epsilon^{-1}$ and $\iota(\iota(a)) = \epsilon^{-1}a\epsilon$ for all $a \in \mathbb{K}$. Remark that this contains the cases of symmetric ($\iota = \mathrm{id}, \epsilon = 1$), skew-symmetric ($\iota = \mathrm{id}, \epsilon = -1$), ($\iota$-)Hermitian ($\epsilon = 1$), ($\iota$-)skew-Hermitian ($\epsilon = -1$) matrices.

Lemma 2.2.19. *Let A be an $n \times n$-matrix over some ring R and $i, j \in [n]$. If $A_{[n]\backslash i,[n]\backslash j}$ is invertible over R, then $|\iota(A^\top)|_{ij} = \iota(|A|_{ji})$.*

Proof. $|\iota(A^\top)|_{ij} = \iota\left(a_{ji} - A_{[n]\backslash i,j}\left(A_{[n]\backslash j,[n]\backslash i}\right)^{-1} A_{i,[n]\backslash j}\right) = \iota(|A|_{ji}).$ □

For an $n \times n$-matrix A over a ring R, we define $[[A]]$ to be the set

$$[[A]] := \{(ij|K) \in \mathcal{A}_n : |A_{iK,jK}|_{ij} = 0\}.$$

Theorem 2.2.20. *Let Σ be an $n \times n$-matrix over a skew field $\mathbb{K}$ such that*

- *$\Sigma = \epsilon \cdot \iota(\Sigma^\top)$ for some antiautomorphism ι in $\mathbb{K}$ and $\epsilon \in \mathbb{K}\backslash\{0\}$, and*
- *all principal submatrices of Σ are invertible over $\mathbb{K}$.*

Then $[[\Sigma]]$ is a gaussoid.

Proof. By Proposition 2.2.12 and Theorem 2.2.14, the principal quasiminors $|\Sigma_{kL}|_{kk}$ are defined and nonzero.

Observe that $\Sigma_{kL,jL} = \epsilon \cdot \iota((\Sigma_{jL,kL})^\top)$. By Lemma 2.2.19,[4]

$$|\Sigma_{kL,jL}|_{kj} = |\epsilon \cdot \iota((\Sigma_{jL,kL})^\top)|_{ij} = \epsilon|\iota(\Sigma_{jL,kL})^\top|_{kj} = \epsilon|\Sigma_{jL,kL}|_{jk},$$

[4]Note that to the contrary to the usual determinant of an $n \times n$-matrix over some field which satisfies $\det(\epsilon A) = \epsilon^n \det(A)$, the quasideterminant satisfies $|\epsilon A|_{ij} = \epsilon|A|_{ij}$, see [GGRW05, §1.3].

hence $|\Sigma_{kL,jL}|_{kj} = 0$ iff $|\Sigma_{jL,kL}|_{jk} = 0$. It follows from Corollary 2.2.17 that

$$|\Sigma_{ikL,jkL}|_{ij} = |\Sigma_{iL,jL}|_{ij} - |\Sigma_{iL,kL}|_{ik} \cdot |\Sigma_{kL}|_{kk}^{-1} \cdot |\Sigma_{kL,jL}|_{kj}. \tag{2.12}$$

The gaussoid condition (WT) is satisfied immediately.

To prove (SG), one switch j and k in (2.12)

$$|\Sigma_{ijL,kjL}|_{ik} = |\Sigma_{iL,kL}|_{ik} - |\Sigma_{iL,jL}|_{ij} \cdot |\Sigma_{jL}|_{jj}^{-1} \cdot |\Sigma_{jL,kL}|_{jk} \tag{2.13}$$

and combine this equality with (2.12). The condition (Comp) also follows directly from (2.12) and (2.13).

When $|\Sigma_{ikL,jkL}|_{ij} = |\Sigma_{ijL,kjL}|_{ik} = 0$, the left hand-sides of (2.12) and (2.13) vanish. Substitute $|\Sigma_{iL,jL}|_{ij} = |\Sigma_{iL,kL}|_{ik} \cdot |\Sigma_{kL}|_{kk}^{-1} \cdot |\Sigma_{kL,jL}|_{kj}$ into (2.13), we have

$$0 = |\Sigma_{iL,kL}|_{ik} - |\Sigma_{iL,kL}|_{ik} \cdot |\Sigma_{kL}|_{kk}^{-1} \cdot |\Sigma_{kL,jL}|_{kj} \cdot |\Sigma_{jL}|_{jj}^{-1} \cdot |\Sigma_{jL,kL}|_{jk}.$$

Suppose that $|\Sigma_{iL,kL}|_{ik} \neq 0$. Then we get

$$|\Sigma_{kL}|_{kk} - |\Sigma_{kL,jL}|_{kj} \cdot |\Sigma_{jL}|_{jj}^{-1} \cdot |\Sigma_{jL,kL}|_{jk} = 0.$$

By Corollary 2.2.17, the left-hand side of the above equation is exactly $|\Sigma_{kjL}|_{kk}$, which cannot vanish by assumption, a contradiction. Hence, $|\Sigma_{iL,jL}|_{ij} = 0$ and (Int) follow. □

Notice that if $\Sigma \in \mathbb{C}^{n\times n}$ is a principally regular complex skew-Hermitian matrix, then $i\Sigma$ is a Hermitian matrix and $[[i\Sigma]] = [[\Sigma]]$. In general, if f is a (ι, ϵ)-Hermitian form and $\lambda \in \mathbb{K}\backslash\{0\}$, then λf is (η, δ)-Hermitian where $\delta = \lambda\iota(\lambda)^{-1}\epsilon$ and $\delta(x) = \lambda\iota(x)\lambda^{-1}$ for every $x \in \mathbb{K}$. Moreover, f and λf define the same polarity on V. Every nondegenerate reflexive sesquilinear form is either skew-symmetric or a nonzero scalar multiple of a non-alternating Hermitian form [BC13, Corollary 7.3.16].

Remark 2.2.21. The dualities of a Desarguesian projective geometry are represented by the nondegenerate ι-sesquilinear forms, while the polarities are represented by the nondegenerate reflexive ι-sesquilinear forms [Bae52, § IV]. The reflexivity is equivalent to the symmetry of orthogonality relation ($x \perp y \Leftrightarrow y \perp x$). We refer to [Bae52; Jac53] and [BC13, § 7] for linear algebra over skew fields and geometry.

Every polarity of a Desarguesian projective geometry is coordinatizable over a skew field $\mathbb{K}$ with an involutive antiautomorphism $\alpha\colon \mathbb{K} \to \mathbb{K}$. The matrix Σ in Theorem 2.2.20 represents exactly a polarity in a Desarguesian projective geometry. Conversely, any polarity in a Desarguesian projective geometry can be represented by such a matrix Σ if a basis is given.

Geometrically, principal invertibility means that the intersection of any coordinate subspace with its orthogonal complement is trivial. In other words, the coordinate subspaces are non-isotropic [Jac53, § V.11]. Let Σ be the matrix of a ι-sesquilinear form f on the left vector space $\mathbb{K}^n$ with respect to the basis $b_1, \dots, b_n$ of $\mathbb{K}^n$.

Lemma 2.2.22. *For $K \subseteq [n]$, Σ_K is invertible iff the coordinate subspace $U_K :=$ $\operatorname{span}(b_k : k \in K)$ satisfies $U_K \cap U_K^\perp = \{\mathbf{0}\}$.*

Proof. By Theorem 2.2.14, if $\Sigma_K = (f(b_i, b_j))_{i,j\in K}$ is invertible, then for every $p \in K$ there exists some $q \in K$ such that $|\Sigma_K|_{pq}$ is defined and $|\Sigma_K|_{pq} \neq 0$. Suppose that $\mathbf{0} \neq x \in U_K \cap U_K^\perp$, that is, $x = \sum_{i\in K} \alpha_i b_i$ for some $\alpha_i \in \mathbb{K}$, $i \in K$ and $f(x, b_j) = f(\sum_{i\in K} \alpha_i b_i, b_j) = \sum_{i\in K} \alpha_i f(b_i, b_j) = 0$ for all $j \in K$. Hence $\sum_{i\in K} \alpha_i \Sigma_{i,K} = \mathbf{0}$. As $x \neq \mathbf{0}$, let $p \in K$ be such that $\alpha_p \neq 0$. Then the p-th row $\Sigma_{p,K} = -\alpha_p^{-1} \sum_{i\in K\setminus p} \Sigma_{i,K}$ is a left linear combination of the other rows of Σ_K. By Proposition 2.2.13, $|\Sigma_K|_{pj} = 0$ whenever it is defined, a contradiction.

If $U_K \cap U_K^\perp = \{\mathbf{0}\}$, that is,

$$\left(f\left(\sum_{i\in K} \alpha_i b_i, b_j\right)\right)_{j\in K} = \sum_{i\in K} \alpha_i \left(f(b_i, b_j)\right)_{j\in K} = \sum_{i\in K} \alpha_i \Sigma_{i,K} = \mathbf{0}$$

implies $\alpha_i = 0$ for all $i \in K$. The rows of Σ_K are linearly independent and span the left vector space $\mathbb{K}^K$. The standard basis vectors $\mathbf{e}_k$, $k \in K$, are linear combinations of the rows of Σ_K whose coefficients form the inverse matrix of Σ_K. □

Remark that $U_K \cap U_K^\perp = \{\mathbf{0}\}$ is equivalent to $U_K \oplus U_K^\perp = \mathbb{K}^n$ [Jac53, § V.11].

Let $\mathbb{K}$ be a skew field and $\iota\colon \mathbb{K} \to \mathbb{K}$ an involutive antiautomorphism. A gaussoid $\mathcal{G}$ on $[n]$ is *representable* over $(\mathbb{K}, \iota)$ if $\mathcal{G} = [[\Sigma]]$ for some principally invertible ι-Hermitian matrix $\Sigma \in \mathbb{K}^{n\times n}$.

Because antiautomorphisms of skew fields can be complicated, not much about the $(\mathbb{K}, \iota)$-representability of gaussoids is known. However, by modifying the calculation in [Boe22b, Example 3.34] slightly, we can see that quaternionic Hermitian matrices are not strong enough to represent the gaussoids $\mathcal{V}$ and $\mathcal{V}'$ in Example 2.2.4.

Proposition 2.2.23. *The gaussoids $\mathcal{V}$ and $\mathcal{V}'$ are not representable over $(\mathbb{H}, \bar{\cdot})$, where $\mathbb{H}$ is the skew field of quaternions and $\bar{\cdot}$ is the quaternion conjugation.*

The proof that $\mathcal{V}$ and $\mathcal{V}'$ are not representable over $(\mathbb{H}, \bar{\cdot})$ does not work for general skew fields, because it relies on the property that the fixed points of quaternion conjugation are the real numbers, which lie in the center of $\mathbb{H}$.

Problem 2.2.24. Is there a gaussoid that is not representable over any skew field and any involutive antiautomorphism?

Conjecture 2.2.25. *The gaussoids $\mathcal{V}$ and $\mathcal{V}'$ are not representable over any skew field and any involutive antiautomorphism.*

Another approach to distinguish the representability of gaussoids is using the von Staudt construction in [Boe22b, § 5.4]. It should work over any skew field and any involutive antiautomorphism because a Desarguesian projective geometry with a polarity will be enough for the construction.

Problem 2.2.26. Construct the non-Pappus configuration using the von Staudt construction in [Boe22b, § 5.4].

Problem 2.2.27. In [Bue93] there is an elegant equivalent axiomatization of projective spaces with polarities. Can we force the coordinate skew field and the antiautomorphism by a substructure of the space?

The difference between [AAV24a] and [AAV24b] provides a hint on the difference between the representability over $(\mathbb{C}, \mathrm{id})$ and $(\mathbb{C}, \bar{\cdot})$.

Problem 2.2.28. Is there a gaussoid which is representable over one of $(\mathbb{C}, \mathrm{id})$ and $(\mathbb{C}, \bar{\cdot})$, but not over the other?

2.2.3 Gaussoids from modular lattices

In the last subsection we have seen that a gaussoid encodes certain information about orthogonality in a Desarguesian projective geometry with a polarity. The modeling of orthogonality as conditional independence dates back to [Daw01]. In that paper Dawid introduced a general structure, called *separoid*, which is a set of triples of elements in a joint semilattice $\mathcal{L}$ satisfying certain axioms. If the joint semilattice $\mathcal{L}$ is the Boolean lattice $(2^E, \subseteq)$, then a separoid is equivalent to a *general semigraphoid* consisting of triples of the kind $A \perp\!\!\!\perp B | C$ where $A, B, C \subseteq E$ are not necessarily disjoint. General semigraphoids are more general than semigraphoids because they also involve the functional dependence, see [MDLW18, § 1.5]. If $\mathcal{L}$ is modular, a separoid is equivalent to a *orthogonoid* [Daw01, Theorem 3.1, 3.2], which models the relative orthogonality of subspaces in an inner product space.

Let $X, Y, Z \in \mathcal{L}$ be subspaces in an inner product space whose lattice of flats is $\mathcal{L}$. The subspace X is *orthogonal independent* of Y given Z if the image of X under the orthogonal projection onto $Y \vee Z$ lies in Z. The set of all such $(X \perp\!\!\!\perp Y | Z)$ is a separoid and $(X \vee Z, Y \vee Z)$ is an orthogonoid [Daw01, § 4.3]. Theorem 2.2.20 implies that if $(a_1, \ldots, a_n)$ is a basis of the space $\mathbb{K}^n$ and the condition

$$\mathrm{span}_{\mathbb{K}}(a_k : k \in K) \oplus \mathrm{span}_{\mathbb{K}}(a_k : k \in K)^{\perp} = \mathbb{K}^n$$

for all $K \subseteq [n]$ in Lemma 2.2.22 is fulfilled, then the CI-structure

$$\{(ij|K) \in \mathcal{A}_n : \mathrm{span}_{\mathbb{K}}(a_i) \text{ and } \mathrm{span}_{\mathbb{K}}(a_j) \text{ are orthogonal independent given } \mathrm{span}_{\mathbb{K}}(a_k : k \in K)\}$$

is a gaussoid. We illustrate this with a Gaussian.

Every positive definite real symmetric $n \times n$-matrix Σ can be written as the Gram matrix of n linearly independent vectors $v_1, \ldots, v_n \in \mathbb{R}^n$, namely,

$$\Sigma = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_n \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \langle v_n, v_2 \rangle & \cdots & \langle v_n, v_n \rangle \end{pmatrix}.$$

For $S \subseteq [n]$ we denote the subspace $\mathrm{span}_{\mathbb{R}}(v_s : s \in K) \subseteq \mathbb{R}^n$ by U_S.

Lemma 2.2.29. *We have $(ij|K) \in [[\Sigma]]$ iff the hyperplanes U_{iK} and U_{jK} of U_{ijK} are orthogonal, that is,*

$$U_{iK}^{\perp} \cap U_{ijK} \subseteq U_{jK}.$$

Proof. Let

$$0 \neq w = \alpha_i v_i + \alpha_j v_j + \sum_{k \in K} \alpha_k v_k \in U_{iK}^{\perp} \cap U_{ijK}.$$

We write $K = \{k_1, \ldots, k_s\}$. Then

$$0 = \begin{pmatrix} \langle v_i, w \rangle \\ \langle v_{k_1}, w \rangle \\ \vdots \\ \langle v_{k_s}, w \rangle \end{pmatrix} = \alpha_i \begin{pmatrix} \langle v_i, v_i \rangle \\ \langle v_{k_1}, v_i \rangle \\ \vdots \\ \langle v_{k_s}, v_i \rangle \end{pmatrix} + \Sigma_{ij|K} \begin{pmatrix} \alpha_j \\ \alpha_{k_1} \\ \vdots \\ \alpha_{k_s} \end{pmatrix}.$$

If $(ij|K) \in [[\Sigma]]$ and $\alpha_i \neq 0$, then $s = \mathrm{rk}(\Sigma_{ij|K}) < \mathrm{rk}(\Sigma_{iK,ijK}) = s+1$, the linear equation system in $\alpha_j, \alpha_{k_1}, \ldots, \alpha_{k_s}$ is not solvable. If $(ij|K) \notin [[\Sigma]]$ and $\alpha_i = 0$, then $\Sigma_{ij|K}$ is invertible and all α's have to be 0. Therefore, $(ij|K) \in [[\Sigma]]$ iff $\alpha_i = 0$ iff $U_{iK}^{\perp} \cap U_{ijK} \subseteq U_{jK}$. □

In this setting, the four gaussoid axioms can be understood intuitively by looking at three fingers representing v_i, v_j and v_k and a little bit of imagination on how the orthogonalities imply each other.

The duality of q-matroids also corresponds to the orthogonality. The dual q-matroid is well-defined up to isomorphy, that is, the isomorphy type of the dual q-matroid do not depend on which inner product is chosen [JP18, Theorem 42, Proposition 43]. However, the relation of a q-matroid to its dual depends on the inner product. Such information is partially encoded by a gaussoid.

We show a lattice-theoretic version of Theorem 2.2.20, which generalizes Theorem 2.2.20 slightly because of the following classifications of atomic modular lattices.

A lattice is *atomic* if every element is a join of atoms. It is *ranked* if for any $x \in \mathcal{L}$, all maximal chains from $\hat{0}$ to x have the same length $\mathrm{rk}(x)$, called the *rank* of x; and *modular* if $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for all $x, y, z \in \mathcal{L}$.

Theorem 2.2.30 ([BC95, Theorem 2.5 and 2.6])**.** *The following are equivalent for any lattice $\mathcal{L}$:*

1. *$\mathcal{L}$ is atomic and modular;*
2. *$\mathcal{L}$ is atomic and ranked, and $\mathrm{rk}(x \vee y) + \mathrm{rk}(x \wedge y) = \mathrm{rk}(x) + \mathrm{rk}(y)$ for all $x, y \in \mathcal{L}$;*
3. *$\mathcal{L}$ is the lattice of flats of a direct sum of finitely many finite-dimensional projective spaces;*
4. *$\mathcal{L}$ is the lattice of flats of a geometry which satisfies the following axioms:*
   - *each line contains at least two points;*

- *two points lie on a unique line;*
- *(Veblen's Axiom) if a line meets two sides of a triangle, not at a vertex, then it meets the third side also;*
- *any chain of subspaces has finite length.*

We remark the following classifications for finite atomic modular lattices.

Theorem 2.2.31 ([Oxl06, Proposition 6.9.1, Exercise 6.9.11])**.** *The following are equivalent for any finite lattice $\mathcal{L}$:*

(1) *$\mathcal{L}$ is atomic and modular, that is, $\mathcal{L}$ is the lattice of flats of a modular matroid;*

(2) *$\mathcal{L}$ is the lattice of flats of a direct sum of finitely many finite projective spaces;*

(3) *$\mathcal{L}$ is the lattice of flats of a matroid which have the same number of hyperplanes and points.*

Theorem 2.2.32. *Let $\mathcal{L}$ be an atomic modular lattice of finite rank n, and $\perp\colon \mathcal{L} \to \mathcal{L}^\vee$, $a \mapsto a^\perp$ be an order isomorphism from $\mathcal{L}$ to its opposite lattice. Let $a_1, \ldots, a_n$ be atoms of $\mathcal{L}$ such that $a_1 \vee \cdots \vee a_n = \hat{1}$ and $a_K \wedge a_K^\perp = \hat{0}$ for every $K \subseteq [n]$, where $a_K := \bigvee_{k\in K} a_k$. Then*

$$\mathcal{G}(\mathcal{L}, \perp, \{a_1, \ldots, a_n\}) := \Big\{(ij|K) \in \mathcal{A}_n : (a_i \vee a_K) \wedge (a_j \vee a_K)^\perp \neq \hat{0}\Big\}$$

is a gaussoid.

Proof. Note that for $(ij|K) \in \mathcal{A}_n$, we have $(ij|K) \in \mathcal{G} := \mathcal{G}(\mathcal{L}, \perp, \{a_1, \ldots, a_n\})$ iff

$$(a_i \vee a_K) \wedge (a_j \vee a_K)^\perp = (a_i \vee a_K) \wedge a_K^\perp \wedge a_j^\perp \neq \hat{0},$$

that means,

$$\operatorname{rk}((a_i \vee a_K) \wedge a_K^\perp \wedge a_j^\perp) = \operatorname{rk}((a_i \vee a_K) \wedge a_K^\perp) = 1,$$

which is equivalent to $(a_i \vee a_K) \wedge a_K^\perp \leq a_j^\perp$.

First we show that $\mathcal{G}$ satisfies (Comp). Assume that $(ij|L), (ik|L) \in \mathcal{G}$. Then we have $(a_i \vee a_L) \wedge a_L^\perp \leq a_k^\perp$ and $(a_i \vee a_L) \wedge a_L^\perp \leq a_j^\perp$. Then

$$(a_i \vee a_L) \wedge a_L^\perp = (a_i \vee a_L) \wedge a_L^\perp \wedge a_k^\perp \leq a_j^\perp.$$

Because $\operatorname{rk}((a_i \vee a_L) \wedge a_L^\perp \wedge a_k^\perp) = \operatorname{rk}((a_i \vee a_L \vee a_k) \wedge a_L^\perp \wedge a_k^\perp) = 1$ and $(a_i \vee a_L) \wedge a_L^\perp \wedge a_k^\perp \leq (a_i \vee a_L \vee a_k) \wedge a_L^\perp \wedge a_k^\perp$, we have

$$(a_i \vee a_L \vee a_k) \wedge a_L^\perp \wedge a_k^\perp = (a_i \vee a_L) \wedge a_L^\perp \wedge a_k^\perp \leq a_j^\perp.$$

Therefore, we have $(ij|kL) \in \mathcal{G}$, and also $(ik|jL) \in \mathcal{G}$ by symmetry.

Now assume that $(ij|L), (ik|jL) \in \mathcal{G}$. Then

$$(a_i \vee a_L) \wedge a_L^\perp = (a_i \vee a_j \vee a_L) \wedge (a_i \vee a_L) \wedge a_L^\perp \wedge a_L^\perp \leq (a_i \vee a_j \vee a_L) \wedge a_j \wedge a_L^\perp \leq a_k^\perp.$$

Therefore, $(ik|L) \in \mathcal{G}$. And by (Comp) we have $(ij|kL) \in \mathcal{G}$.

To show that $\mathcal{G}$ satisfies (Int) we assume now $(ij|kL), (ik|jL) \in \mathcal{G}$. Then $(a_j \vee a_k \vee a_L) \wedge a_j^\perp \wedge a_L^\perp \leq a_i^\perp$ and $(a_j \vee a_k \vee a_L) \wedge a_k^\perp \wedge a_L^\perp \leq a_i^\perp$. Therefore,

$$\begin{aligned} a_i^\perp &\geq ((a_j \vee a_k \vee a_L) \wedge a_j^\perp \wedge a_L^\perp) \vee ((a_j \vee a_k \vee a_L) \wedge a_k^\perp \wedge a_L^\perp) \\ &= (((a_j \vee a_k \vee a_L) \wedge a_j^\perp \wedge a_L^\perp) \vee a_k^\perp) \wedge (a_j \vee a_k \vee a_L) \wedge a_L^\perp \\ &= (a_j \vee a_k \vee a_L) \wedge a_L^\perp, \end{aligned}$$

where the first equality follows from the modularity of $\mathcal{L}$ and the second equality is valid because

$$\begin{aligned} &\mathrm{rk}(((a_j \vee a_k \vee a_L) \wedge a_j^\perp \wedge a_L^\perp) \vee a_k^\perp) \\ &= \mathrm{rk}((a_j \vee a_k \vee a_L) \wedge a_j^\perp \wedge a_L^\perp) + \mathrm{rk}(a_k^\perp) - \mathrm{rk}((a_j \vee a_k \vee a_L) \wedge a_j^\perp \wedge a_L^\perp \wedge a_k^\perp) \\ &= 1 + (n-1) - 0 = n. \end{aligned}$$

Thus $(a_j \vee a_L) \wedge a_L^\perp \leq a_i$ and $(a_k \vee a_L) \wedge a_L^\perp \leq a_i$, that is, $(ij|L), (ik|L) \in \mathcal{G}$.

At last, assume that $(ij|L), (ij|kL) \in \mathcal{G}$ and $(ik|L) \notin \mathcal{G}$. That is, $(a_i \vee a_L) \wedge a_L^\perp \leq a_j^\perp$ and $(a_i \vee a_k \vee a_L) \wedge a_k^\perp \wedge a_L^\perp \leq a_j^\perp$, and

$$\hat{0} = (a_i \vee a_L) \wedge a_L^\perp \wedge a_k^\perp = ((a_i \vee a_L) \wedge a_L^\perp) \wedge ((a_i \vee a_k \vee a_L) \wedge a_k^\perp \wedge a_L^\perp)$$

is the intersection. By modularity, the union $((a_i \vee a_L) \wedge a_L^\perp) \vee ((a_i \vee a_k \vee a_L) \wedge a_k^\perp \wedge a_L^\perp)$ has rank 2 and equals to $(((a_i \vee a_L) \wedge a_L^\perp) \vee a_k^\perp) \wedge ((a_i \vee a_k \vee a_L) \wedge a_L^\perp)$. Moreover, it is less or equal to $(a_i \vee a_k \vee a_L) \wedge a_L^\perp$, which also has rank 2. Therefore,

$$(a_i \vee a_k \vee a_L) \wedge a_L^\perp = (((a_i \vee a_L) \wedge a_L^\perp) \vee a_k^\perp) \wedge ((a_i \vee a_k \vee a_L) \wedge a_L^\perp) \leq a_j^\perp,$$

so we have $(a_k \vee a_L) \wedge a_L^\perp \leq a_j^\perp$, that is, $(jk|L) \in \mathcal{G}$. □

2.3 Separation and connectedness

2.3.1 Connectedness and ascending semigraphoids

Now we consider the CI-structures $\mathcal{G} \subseteq \mathcal{A}_n$ which model relations of the form

$$\mathcal{G} = \{(ij|K) \in \mathcal{A}_n : i \text{ and } j \text{ are separated by } K\}.$$

Relations of this form satisfy the *ascension axiom*

(Asc) $(ij|L) \in \mathcal{G} \Rightarrow (ij|kL) \in \mathcal{G}$.

Given a set system $\mathcal{C} \subseteq 2^{[n]}$, we define the CI-structure $[[\mathcal{C}]]$ by

$$[[\mathcal{C}]] := \{(ij|K) \in \mathcal{A}_n : \nexists C \in \mathcal{C} \text{ such that } ij \subseteq C \subseteq [n]\backslash K\}.$$

The CI-structure $[[\mathcal{C}]]$ is a semimatroid as follows [MPSSW09, Proposition 23]. For $C \subseteq [n]$, the support function of the standard simplex $\Delta_C := \mathrm{conv}(\mathbf{e}_i : i \in C)$ is

$$h_{\Delta_C}(S) = \begin{cases} 1 & \text{if } C \cap S \neq \emptyset, \\ 0 & \text{if } C \cap S = \emptyset. \end{cases}$$

Therefore the CI-statement $(ij|K) \in \mathcal{A}_n$ is not in the semimatroid $[[h_{\Delta_C}]]$ iff $ij \subseteq C \subseteq [n]\backslash K$. Moreover, by Proposition 2.1.12, for a set system $\mathcal{C} \subseteq 2^{[n]}$ we have

$$[[\mathcal{C}]] = \bigcap_{C \in \mathcal{C}} [[h_{\Delta_C}]] = [[\sum_{C \in \mathcal{C}} h_{\Delta_C}]] = [[h_{\Delta_\mathcal{C}}]],$$

where $\Delta_\mathcal{C} := \sum_{C \in \mathcal{C}} \Delta_C$ is a Minkowski sum of simplices (*MSS*). We conclude that a CI-structure is of the form $[[\mathcal{C}]]$ for some $\mathcal{C} \subseteq 2^{[n]}$ iff it is the semimatroid of a MSS. In this case it is $[[\Delta_\mathcal{C}]]$.

Remark 2.3.1. In this section we can consider equivalently the *descending semigraphoids* by taking dual, that is, the semigraphoids $\mathcal{G}$ satisfying the *descension axiom*

(Desc) $(ij|kL) \in \mathcal{G} \Rightarrow (ij|L) \in \mathcal{G}$

which models the dual structure

$$\mathcal{G} = \{(ij|K) \in \mathcal{A}_n : i \text{ and } j \text{ are separated by } [n]\backslash K\}.$$

The following properties are desired by the set $\mathcal{C} \subseteq 2^X$ of "connected" subsets of X.

(c0) $C \in \mathcal{C}$ for all $C \subseteq X$ with $|C| \leq 1$.

(c1$^-$) If for any $ij \subseteq C$ there exists some $C' \in \mathcal{C}$ such that $ij \subseteq C' \subseteq C$, then $C \in \mathcal{C}$.

(c1) $\{C_i\}_{i \in I} \subseteq \mathcal{C}, \bigcap_{i \in I} C_i \neq \emptyset \Rightarrow \bigcup_{i \in I} C_i \in \mathcal{C}$.

(c2) For any nonempty $A, B \in \mathcal{C}$ with $A \cup B \in \mathcal{C}$ there exists $x \in A \cup B$ such that $x \cup A \in \mathcal{C}$ and $x \cup B \in \mathcal{C}$.

(c3) If $A, B, C_i \in \mathcal{C}$ $(i \in I)$ disjoint and $A \cup B \cup \bigcup_{i \in I} C_i \in \mathcal{C}$, then there is a $J \subseteq I$ such that $A \cup \bigcup_{j \in J} C_j \in \mathcal{C}$ and $B \cup \bigcup_{i \in I \backslash J} C_i \in \mathcal{C}$.

Definition 2.3.1. Let X be a set and $\mathcal{C} \subseteq 2^X$. The pair $(X, \mathcal{C})$ is a

- *weak c-space* if it satisfies (c0) and (c1$^-$);
- *c-space* if it satisfies (c0) and (c1);
- *c^2-space* if it satisfies (c0), (c1), (c2) and (c3).

In these cases, the set system $\mathcal{C}$ is a *weak c-*, a *c-* and a *c^2-structure* on X, respectively.

Note that if a CI-structure $\mathcal{G}$ satisfies (Asc), then for $\mathcal{G}$ the semigraphoid axiom (SG) is equivalent to

$\{(ij|L),(ik|jL)\} \subseteq \mathcal{G} \Rightarrow (ik|L) \in \mathcal{G}$,

and the weak transitivity axiom is equivalent to the *transitivity axiom*

(T) $(ij|L) \in \mathcal{G} \Rightarrow (ik|L)$ or $(jk|L) \in \mathcal{G}$.

Moreover, the condition

(Stud) $\{(ij|kL),(ij|\ell L),(k\ell|L)\} \subseteq \mathcal{G} \Rightarrow (ij|L) \in \mathcal{G}$

is equivalent to (2.1) in Theorem 2.1.2, which is satisfied by the CI-structure of any discrete random vector [Stu89] and any semimatroid [Stu21, Corollary 6]. Conversely, any ascending semigraphoid satisfying (Stud) is the CI-structure of some discrete random vector [Mat92, Theorem 3].

Example 2.3.2. The ascending semigraphoid

$$\{(12|3),(12|4),(12|34),(34|),(34|1),(34|2),(34|12)\}$$

satisfies (Int), (Comp) but not (WT) and (Stud). It is not a semimatroid. Suppose this ascending semigraphoid is $[[\mathcal{C}]]$ for some $\mathcal{C} \subseteq 2^{[4]}$. Because $\{(12|3),(12|4)\} \in [[\mathcal{C}]]$ but $(12|) \notin [[\mathcal{C}]]$, there is some $C \in \mathcal{C}$ containing 12 and every $C \in \mathcal{C}$ containing 12 must contain 3 and 4. Therefore $1234 \in \mathcal{C}$, contradicting $(34|) \in [[\mathcal{C}]]$.

Theorem 2.3.3 ([Mat92, Theorem 1])**.** *The map $\mathcal{C} \mapsto [[\mathcal{C}]]$ is a bijection from the set of all weak c-structures $\mathcal{C}$ on $[n]$ to the set of all ascending semigraphoids satisfying (Stud).*

In this case, the weak c-structure can be recovered from the semigraphoid $\mathcal{G}$ by

$$\mathcal{C}_{\mathcal{G}} := \{C \subseteq [n] : C \cap (ijK) \neq ij \quad \forall (ij|K) \in \mathcal{G}\}.$$

Theorem 2.3.4.

- *A set system $\mathcal{C} \subseteq 2^{[n]}$ is a weak c-structure iff $\mathcal{C} = \mathcal{C}_{\mathcal{G}}$ for some CI-structure $\mathcal{G}$ on $[n]$.*
- *The following are equivalent for a CI-structure $\mathcal{G}$ on $[n]$:*
  - *$\mathcal{G}$ is an ascending semigraphoid satisfying (Stud);*
  - *$\mathcal{G} = [[\mathcal{C}]]$ for some set system $\mathcal{C} \subseteq 2^{[n]}$;*
  - *$\mathcal{G} = [[\mathcal{C}]]$ for some weak c-structure $\mathcal{C} \subseteq 2^{[n]}$;*
  - *$\mathcal{G}$ is the semimatroid of a MSS. That is, as a set of walls in $\Sigma_{A_{n-1}}$, its removal from $\Sigma_{A_{n-1}}$ results in the normal fan of a MSS.*

Proof. The statements can be found in the proof of [Mat92, Theorem 1] which gives the Galois connection $\mathcal{G} \mapsto \mathcal{C}_{\mathcal{G}}$ and $\mathcal{C} \mapsto [[\mathcal{C}]]$ between all CI-structures on $[n]$ and all set systems on $[n]$, and also follow from the discussion that $[[\mathcal{C}]] = [[\Delta_{\mathcal{C}}]]$ for all $\mathcal{C} \subseteq 2^{[n]}$. □

It is easy to check that (c1) implies (c1$^-$), and ascending semigraphoids satisfying (WT) also satisfy (Stud).

Theorem 2.3.5 ([Mat92, Proposition 4]). *Let $\mathcal{G}$ be an ascending semigraphoid on $[n]$ which satisfies (Stud). Then $\mathcal{G}$ satisfies (WT) iff $\mathcal{C}(\mathcal{G})$ is a c-structure.*

A c-structure $\mathcal{C}$ on X is *2-generated* if there is a subset $\mathcal{B} \subseteq \{C \in \mathcal{C} : |C| \leq 2\}$ such that $\mathcal{C}$ is the smallest c-structure containing $\mathcal{B}$.

Theorem 2.3.6 ([RMN13, Theorem 3.8 and Remark 3.9]). *The following are equivalent for a c-structure $\mathcal{C}$ on $[n]$:*

(1) *$\mathcal{C}$ is a c^2-structure;*

(2) *$\mathcal{C}$ is a 2-generated c-structure;*

(3) *There is a graph G on $[n]$ such that $\mathcal{C} = \{C \subseteq [n] : G[C]$ is connected$\}$.*

We focus on the CI-structures $[[\mathcal{C}]]$ of c^2-structures $\mathcal{C}$ in the next subsection.

Remark 2.3.7. The attempts to axiomatize the connectedness were motivated by topology but lead to various generalizations of topology. Every finite topological c-space is a c^2-space. But cofinite spaces violate (c2). There are non-topologizable finite c^2-spaces, for example, there is no topology on $[n]$ whose connected sets are the sets inducing connected subgraphs in the cycle graph on $[n]$ for any odd $n \geq 5$. There are other equivalent classifications for the connectedness using closure, separation, opening, etc. We refer to [RMN13; MB+06; SS15].

2.3.2 Graphical models and graph associahedra

Let $G = ([n], E)$ be a (simple) graph. Define

$$[[G]] := \{(ij|K) \in \mathcal{A}_n : K \text{ separates } i \text{ and } j\},$$

where K separates i and j means that i and j are in different connected components in the induced subgraph $G[[n]\backslash K]$. In other words, $[[G]] = [[\mathcal{C}(G)]]$ where $\mathcal{C}(G) \subseteq 2^{[n]}$ is the set of all vertex subsets of G that induce connected subgraphs in G.

Theorem 2.3.8 ([Mat97, pp.108]). *A CI-structure $\mathcal{G}$ satisfies (Asc), (Int) and (T) (or equivalently, (WT)) iff $\mathcal{G} = [[G]]$ for some graph G. Equivalently, $\mathcal{G}$ is an ascending gaussoid.*

A gaussoid $\mathcal{G}$ is *graphic* if $\mathcal{G} = [[G]]$ for some graph G, and *cographic* if $\mathcal{G} = [[G]]^*$ for some graph G.

A random vector ξ is *Markovian* with respect to the graph G if $[[G]] \subseteq [[\xi]]$, and *perfectly Markovian* if $[[G]] = [[\xi]]$.

Remark 2.3.9. There are notions pairwise, local and global Markovian properties, arranged in order of increasing strength. There are discrete random variables that distinguish the strength of three properties. However, if a random vector ξ satisfies (Int), in particular, if ξ has a positive density, then the three properties are equivalent. Collections of Markovian distributions with respect to graphs are *graphical models*, which are intensively researched and widely used in statistics. We refer to [Lau96; Sul18] for graphical models.

There are explicit construction for representations of graphic and cographic gaussoids. Given a graph $G = ([n], E)$, define the matrix $A \in \mathbb{K}^{n\times n}$ by

$$A_{i,j} = \begin{cases} 1 & i = j, \\ \epsilon_{ij} & ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

For generic $(\epsilon_{ij}) \in \mathbb{K}^E$, the matrix A satisfies $[[A]] = [[G]]^*$ and hence $[[A^{-1}]] = [[G]]$. In particular, if $\mathbb{K} = \mathbb{R}$ and ϵ_{ij} are close enough to 0, the matrix A is diagonally dominant and therefore positive definite. The Gaussian whose covariance matrix is A is perfectly Markovian with respect to G.

Theorem 2.3.10 ([LM07], [Boe22b, Theorem 4.6])**.** *For any graph G, $[[G]]$ and $[[G]]^*$ are gaussoids which are representable over all infinite fields and positively representable over all ordered fields. In particular, they are Gaussian representable.*

Definition 2.3.2. An *oriented gaussoid* on $[n]$ is a map $\sigma : \mathcal{A}_n \to \{-1, 0, 1\}$ which is compatible to (2.9), that is, σ satisfies

(OGß) $\{\sigma(ik|L)\sigma(jk|L), \sigma(ij|kL), -\sigma(ij|L)\} \in \{\{0\}, \{-1, 1\}, \{-1, 0, 1\}\}$.

An oriented gaussoid σ is *positively oriented* if $\sigma(\mathcal{A}_n) \subseteq \{0, 1\}$ and *negatively oriented* if $\sigma(\mathcal{A}_n) \subseteq \{0, -1\}$.

If σ is an oriented gaussoid, then $\underline{\sigma} := \sigma^{-1}(0)$ is a gaussoid, called the *underlying gaussoid* of σ. A gaussoid is (*positively/negatively*) *orientable* if it is the underlying gaussoid of a (positively/negatively) oriented gaussoid.

Inspired by the theory of oriented matroids and positroids, oriented gaussoids are introduced in [BDKS19, § 5]. Indeed there are many analogies: Every positively orientable matroid is a positroid, that is, representable over $\mathbb{R}$ [ARW17]. Every positively orientable gaussoid is representable over $\mathbb{R}$, moreover, the positively orientable gaussoids on $[n]$ are exactly the graphic gaussoids [BDKS19, Theorem 5.6]. A matroid is positively orientable iff its basis polytope is an alcoved polytope [LP24, Theorem 2.1]. A gaussoid is positively orientable iff it is the semimatroid of a MSS. This kind of MSSs are exactly the graph associahedra $\Delta_{\mathcal{C}(G)}$ of graphs G [CD06].

Following [Mat97], gaussoids and subclasses of gaussoids are classified by minors [BK20]. A CI-structure is a gaussoid iff all of its 3-minors are gaussoids, which have exactly the following five isomorphy types [BK20, Lemma 3.3]

$$\mathsf{E} = \emptyset,\ \mathsf{L} = \{(12|)\},\ \mathsf{U} = \{(12|3)\},\ \mathsf{B} = \{(13|), (23|), (13|2), (23|1)\} \text{ and } \mathsf{F} = \mathcal{A}_3.$$

Graphic gaussoids are exactly the gaussoids without minor isomorphic to L [BK20, Lemma 4.1].

We summarize the criteria in the following theorem.

Theorem 2.3.11. *Let $\mathcal{G} \subseteq \mathcal{A}_n$ be a CI-structure. The following are equivalent:*

(1) *$\mathcal{G} = [[G]]$ for some graph G, i.e. $\mathcal{G}$ is a graphic gaussoid;*

(2) $\mathcal{G}$ *satisfies* (Int), (WT) *and* (Asc)*;*

(3) $\mathcal{G}$ *is a gaussoid satisfying* (Asc)*;*

(4) $\mathcal{G}$ *is a gaussoid without minor* L*, that is, the 3-minors of* $\mathcal{G}$ *are* EUBF*;*

(5) $\mathcal{G}$ *is a positively orientable gaussoid;*

(6) $\mathcal{G}$ *is the semimatroid of a graph associahedron;*

(7) $\mathcal{G}$ *is a gaussoid and the semimatroid of a MSS.*

Proof. The equivalences are stated in [LM07, Remark 2], [BK20, §4.3], [BDKS19, Theorem 5.6], [MPSSW09, Theorem 25], and follow from the fact that the semimatroid of any MSS is an ascending semigraphoid [MUWY18, Lemma 3.7].[5] □

By duality we get the following classifications of cographic gaussoids.

Corollary 2.3.12. *Let* $\mathcal{G} \subseteq \mathcal{A}_n$ *be a CI-structure. The following are equivalent:*

(1) $\mathcal{G} = [[G]]^*$ *for some graph* G*, i.e.* $\mathcal{G}$ *is a cographic gaussoid;*

(2) $\mathcal{G}$ *satisfies* (Comp), (WT) *and* (Desc)*;*

(3) $\mathcal{G}$ *is a gaussoid satisfying* (Desc)*;*

(4) $\mathcal{G}$ *is a gaussoid without minor* U*, that is, the 3-minors of* $\mathcal{G}$ *are* ELBF*;*

(5) $\mathcal{G}$ *is a negatively orientable gaussoid;*

(6) $\mathcal{G}$ *is the semimatroid of a graph associahedron;*

(7) $\mathcal{G}$ *is a gaussoid and the semimatroid of the minus of a MSS.*

Remark 2.3.13. There are other popular notions of separation. For example, similarly to graphical models defined on graphs, we can define probabilistic models on directed acyclic graphs (DAG), known as *Bayesian networks* [Pea85]. Let $\vec{G}$ be a DAG on $[n]$ and $(ij|K) \in \mathcal{A}_n$. A *Bayes ball path* from i to j given K in $\vec{G}$ is a walk from i to j in the underlying graph of $\vec{G}$ such that

- if $a \to b \to c$ or $a \leftarrow b \to c$ or $a \leftarrow b \leftarrow c$ is on the path, then $b \notin K$;
- if $a \to b \leftarrow c$ (possibly $a = c$) is on the path, then $b \in K$.

Let $[[\vec{G}]]$ be the set of CI-statements $(ij|K) \in \mathcal{A}_n$ such that there is no Bayes-ball path from i to j given K in $\vec{G}$, in this case, we call i and j are *d-separated* by K. Then $[[\vec{G}]]$ is a Gaussian representable gaussoid [MUWY18, § 4]. Moreover, it is the semimatroid of a Minkowski sum of matroid polytopes (*MSMP*), which is constructed in [MUWY18, § 5].

[5] The Venn diagram [MUWY18, Figure 7] should collapse because the gaussoids that are also MSSs are exactly the undirected graphical models.

2.4 Matroids as CI-structures

2.4.1 Matroids and oriented matroids

A submodular function $r\colon 2^{[n]} \to \mathbb{Z}$ is the *rank function* of a *matroid* M on $[n]$ if it is integer-valued, subcardinal $r(A) \leq |A|$ $\forall A \subseteq [n]$ and monotonic $r(A) \leq r(B)$ $\forall A \subseteq B \subseteq [n]$. A set $S \subseteq [n]$ is *independent* in M if $r(S) = |S|$ and otherwise *dependent*. Minimal dependent sets are *circuits* and maximal independent sets are *bases* of M. The *basis polytope* $\mathcal{P}_M \subseteq [0,1]^n$ of the matroid M is the convex hull of the indicator vectors $\mathbf{e}_B = \sum_{i \in B} \mathbf{e}_i \in \mathbb{R}^n$ of all bases B of M. It is the generalized permutohedron whose support function is r [ABD10].

One of the most fascinating facts about matroids is that they can be defined in hundreds of different equivalent ways. The axiom systems arise from various areas of mathematics, they are elegant and define different natural-looking objects that are indeed equivalent, but the equivalence can be far from obvious. This equivalence of axiomatically defined objects is called "cryptomorphism". For instance, rank functions of matroids are cryptomorphic to polytopes whose vertex coordinates are 0 or 1 and edges are parallel to $\mathbf{e}_i - \mathbf{e}_j$, by taking their basis polytopes. A precise definition of cryptomorphism can be found in [Whi86, § 2.2]. Classic axiom systems of matroids and the cryptomorphisms among them can be found in the Appendix in [Whi86]. Recently, matroids have been proven to be cryptomorphic to tropical varieties of degree 1 [Fin13], supports of multiaffine Lorentzian polynomials [BH20], and moreover, a simplicial complex is the independence complex of a matroid

- iff all symbolic powers of its Stanley-Reisner ideal are Cohen-Macaulay [Var11; MT11];
- iff the m-th symbolic power of its Stanley-Reisner ideal is Cohen-Macaulay (or equivalently Buchsbaum or quasi-Buchsbaum or satisfies the Serre condition (S_2)) for some $m \geq 3$. [TT12];
- iff its combinatorial atlas is hyperbolic [CP22];
- iff the dimensions of the components of its first cotangent cohomology module attain certain upper bounds [BC23].

In this section, we show that matroids are cryptomorphic to semigraphoids satisfying a single additional axiom (MCI) and oriented matroids are cryptomorphic to oriented CI-structures satisfying the axioms (OCI1)–(OCI5). We assume that the reader is familiar with the terminology for and basic facts about matroids and oriented matroids, which can be found in [Oxl06; Whi86; BLVSWZ99].

2.4.2 Matroids as CI-structures

A matroid M on $[n]$ with rank function r defines a CI-structure $[[M]] := [[r]] = \{(ij|K) \in \mathcal{A}_n : r(iK) + r(jK) = r(ijK) + r(K)\}$. In other words, $[[M]]$ is the semimatroid of the submodular function r.

If M is loopless, one can recover its rank function from its CI-structure $[[M]]$ recursively by $r(\emptyset) = 0$, $r(i) = 1$ for all $i \in [n]$ and

$$r(ijK) = \begin{cases} r(iK) + r(jK) - r(K) & i \perp\!\!\!\perp j | K \\ r(iK) + r(jK) - r(K) - 1 & i \not\perp\!\!\!\perp j | K. \end{cases} \tag{2.14}$$

Alternatively, we can recover the independent sets $\mathcal{I}(M)$ of the matroid M by

$$\mathcal{I}(M) = \{S \subseteq [n] : \mathcal{A}_S \subseteq [[M]]\}, \tag{2.15}$$

where $\mathcal{A}_S := \{(ij|K) : K \subseteq S, i \neq j \in S \backslash K\}$.

Replacing a loop $i \in [n]$ by a coloop and replacing a coloop i by a loop correspond to the translation of matroid polytope by $-\mathbf{e}_i$ and $\mathbf{e}_i$, respectively. They do not affect the normal fan and thus the semimatroid stays unchanged. Here we consider loopless matroids as the representatives in every class of matroids on $[n]$ whose basis polytopes have the same normal fan.

In [Mat93], Matúš connected conditional independence to matroid theory by embedding matroids into CI-structures and investigating their probabilistic representability, and introducing matroid-theoretic tools to the research of conditional independence. He proved that a CI-structure can be defined by a matroid iff the sets obtained by (2.15) are the independent sets of a matroid whose CI-structure is again the original one. Here we characterize these CI-structures by axioms in the style of inference rules.

Theorem 2.4.1. *A CI-structure $\mathcal{G} \subseteq \mathcal{A}_n$ is defined by a matroid M iff it satisfies*

(MCI) $i \not\perp\!\!\!\perp j | K \Rightarrow i \perp\!\!\!\perp \ell | jKL,$

(SG) $i \perp\!\!\!\perp j | K \wedge i \perp\!\!\!\perp \ell | jK \Rightarrow i \perp\!\!\!\perp \ell | K \wedge i \perp\!\!\!\perp j | \ell K.$

Moreover, the correspondence between loopless matroids and CI-structures satisfying (MCI) and (SG) is one-to-one.

Proof. Let M be a matroid on $[n]$ with rank function r. The rank function axioms of matroids imply that $i \not\perp\!\!\!\perp j | K$ in $[[M]]$ iff

$$r(iK) = r(jK) = r(ijK) = r(K) + 1.$$

From the second equality and submodularity $r(ijK) + r(jKL) \geq r(ijKL) + r(jK)$ and monotonicity $r(ijKL) \geq r(jKL)$ of r, we have $r(ijKL) = r(jKL)$. This contradicts $i \not\perp\!\!\!\perp \ell | jKL$. Thus (MCI) is proven. The condition (SG) is the semigraphoid axiom. It is always satisfied by semimatroids.

Now let $\mathcal{G} \subseteq \mathcal{A}_n$ be a CI-structure satisfying (MCI) and (SG). We need to show that a function $r \colon 2^{[n]} \to \mathbb{N}$ is uniquely defined by the recursion (2.14) and it is the rank function of a matroid.

To show the well-definedness of $r(S)$ we apply induction on the cardinality c of S. For $c \in \{0, 1\}$ the function value $r(S)$ is given by the initial condition. For $c = 2$,

$r(ij) = 1$ if $i \not\perp\!\!\!\perp j|$ and 2 otherwise. Assume that $c \geq 3$ and $r(S)$ is uniquely defined for any S with $|S| < c$. What is left to check is that we get the same value $r(ijkL)$ for $|ijkL| = c$ by applying (2.14) to different conditional (in)dependence statements, namely,

$$\begin{aligned} &\begin{cases} r(ikL) + r(jkL) - r(kL) & i \perp\!\!\!\perp j|kL \\ r(ikL) + r(jkL) - r(kL) - 1 & i \not\perp\!\!\!\perp j|kL \end{cases} \\ = &\begin{cases} r(ijL) + r(jkL) - r(jL) & i \perp\!\!\!\perp k|jL \\ r(ijL) + r(jkL) - r(jL) - 1 & i \not\perp\!\!\!\perp k|jL. \end{cases} \end{aligned}$$

1. $i \perp\!\!\!\perp j|kL$ and $i \not\perp\!\!\!\perp k|jL$: From (SG) we have $i \not\perp\!\!\!\perp k|L$. By $i \not\perp\!\!\!\perp k|jL$ and (MCI) we have $i \perp\!\!\!\perp j|L$. Therefore

$$\begin{aligned} r(ikL) + r(jkL) - r(kL) &= r(iL) - r(L) - 1 + r(jkL) \\ &= r(ijL) + r(jkL) - r(jL) - 1. \end{aligned}$$

   The case $i \perp\!\!\!\perp k|jL$ and $i \not\perp\!\!\!\perp j|kL$ follows by symmetry.

2. $i \not\perp\!\!\!\perp j|kL$ and $i \not\perp\!\!\!\perp k|jL$: By (MCI) we have $i \perp\!\!\!\perp j|L$ and $i \perp\!\!\!\perp k|L$.

3. $i \perp\!\!\!\perp j|kL$ and $i \perp\!\!\!\perp k|jL$: If $i \perp\!\!\!\perp j|L$, by (SG) we have $i \perp\!\!\!\perp k|L$. If $i \not\perp\!\!\!\perp j|L$, we have $i \not\perp\!\!\!\perp k|L$ instead by (SG). In both cases the equality follows.

To show that r is the rank function of a matroid, what is left to show is $r(i_1 \cdots i_s) - r(i_2 \cdots i_s) \in \{0, 1\}$. Applying (2.14) $s - 2$ times, we have

$$r(i_1 \cdots i_s) - r(i_2 \cdots i_s) = r(i_1) - r(\emptyset) - a = 1 - a,$$

where

$$a = |\{(i_1 i_s | i_2 \cdots i_{s-1}), (i_1 i_{s-1} | i_2 \cdots i_{s-2}), \ldots, (i_1 i_2 |)\} \backslash \mathcal{G}| \, .$$

The condition (MCI) implies that at most one of

$$(i_1 i_s | i_2 \cdots i_{s-1}), (i_1 i_{s-1} | i_2 \cdots i_{s-2}), \ldots, (i_1 i_3 | i_2), (i_1 i_2 |)$$

can be a conditional dependence statement, i.e. not in $\mathcal{G}$, therefore a is either 0 or 1. □

We remark that sometimes it may be more convenient to write the condition (MCI) in the following form

$$i \perp\!\!\!\perp j|K \vee i \perp\!\!\!\perp \ell|jKL \quad \forall i\ell jKL \subseteq [n].$$

Since semigraphoids satisfying (MCI) are cryptomorphic to loopless matroids, we call such CI-structures *matroids* as well, and we can read off the CI-structure from other formulations of the matroid. The following lemma will be used repeatedly.

Lemma 2.4.2. *For any $K \subseteq [n]$ and $i \neq j \in [n] \backslash K$, $i \not\perp\!\!\!\perp j|K$ iff there is a circuit C such that $ij \subseteq C \subseteq ijK$ and every circuit in ijK contains either both i, j or none of them.*

Proof. Suppose that there is no circuit in ijK containing i, j, then i and j are coloops in ijK. Then $r(ijK) = r(iK) + 1$, so we have $i \perp\!\!\!\perp j | K$. Suppose now that there is a circuit C such that $i \in C \subseteq iK$, then $r(C) = r(C \backslash i)$. Together with $r(iK) = r(K)+1$, which is implied from $i \not\perp\!\!\!\perp j | K$, it contradicts the submodularity

$$r(C) + r(K) \geq r(iK) + r(C \backslash i).$$

Now assume that $i \perp\!\!\!\perp j | K$ and there is a circuit ijK' such that $K' \subseteq K$. We want to show that there exists a circuit containing exactly one of i, j. From $i \perp\!\!\!\perp j | K$ we have

$$r(iK) + r(jK) = r(ijK) + r(K),$$

and as ijK' is a circuit, $r(ijK') = r(jK')$. By submodularity

$$r(jK) + r(ijK') \geq r(ijK) + r(jK')$$

and monotonicity, we have $r(jK) = r(ijK)$ and therefore $r(iK) = r(K)$. So there is a circuit C' such that $i \in C' \subseteq iK$. □

We summarize the following characterizations for the conditional dependence $(ij|K) \notin [[M]]$. They are merely straightforward reformulations of previous statements.

Proposition 2.4.3. *Let M be a matroid on $[n]$ and $[[M]] \subseteq \mathcal{A}_n$ be the CI-structure associated to M. The following are equivalent for any $K \subseteq [n]$ and $i \neq j \in [n] \backslash K$:*

(1) $i \not\perp\!\!\!\perp j | K$.

(2) $r(iK) = r(jK) = r(ijK) = r(K) + 1$.

(3) *There exists a circuit C of M such that $ij \subseteq C \subseteq ijK$ and every circuit in ijK contains either both i, j or none of them.*

(4) *The set ij is a cocircuit (or equivalently, K is a hyperplane) in the restriction of M to ijK.*

(5) *For any basis B of K, iB and jB are bases of ijK.*

(6) *There exists a basis B of K such that iB and jB are bases of ijK.*

(7) *Any wall of $\Sigma_{A_{n-1}}$ corresponding to $(ij|K)$ by (2.2) is in some wall of the normal fan $\mathcal{N}(\mathcal{P}_M)$ of the basis polytope $\mathcal{P}_M$ of M.*

The operations deletion, contraction, dual and direct sum on CI-structures are compatible with the operations on matroids, namely, $[[M \backslash A]] = [[M]] \backslash A$, $[[M/A]] = [[M]]/A$ and $[[M^*]] = [[M]]^*$ for any matroid M on $[n]$ and any $A \subseteq [n]$, and $[[M_1 \oplus M_2]] = [[M_1]] \oplus [[M_2]]$ for any matroids M_1, M_2 on disjoint ground sets [Mat93]. In particular, being a matroid is a minor-closed property of CI-structures. In [Mat97], classes of CI-structures are investigated via forbidden minors. The CI-structures that are matroids, although having a short axiomatization by (MCI) and (SG), cannot be characterized by a finite set of forbidden minors.

Theorem 2.4.4. *The class of CI-structures that are matroids cannot be characterized by a finite set of forbidden minors.*

Proof. Let $\mathcal{G}_m := \mathcal{A}_m \backslash \{(12|), (ij|K) : ijK = [m]\}$ for $m \geq 4$. In other words, $\mathcal{G}_m$ is the semimatroid defined by the sum of rank functions of matroids $U_{1,2} \oplus U_{m-2,m-2}$ and $U_{m-1,m}$, where $U_{r,m}$ is the rank-r uniform matroid on $[m]$. Then $\mathcal{G}_m$ is not a matroid because (MCI) is violated by $(12|), (13|245 \cdots m) \notin \mathcal{G}_m$. But any proper minor of $\mathcal{G}_m$ is a matroid as $\mathcal{G}_m \backslash 1 \cong \mathcal{G}_m \backslash 2 \cong [[U_{m-1,m-1}]], \mathcal{G}_m \backslash i \cong [[U_{1,2} \oplus U_{m-3,m-3}]]$ for any $i \in [m] \backslash \{1, 2\}$ and $\mathcal{G}/j \cong [[U_{m-2,m-1}]]$ for any $j \in [m]$. □

2.4.3 Oriented matroids as oriented CI-structures

An *oriented CI-structure* on $[n]$ is a map $\sigma \colon \mathcal{A}_n \to \{-1, 0, 1\}$. In [BDKS19], oriented gaussoids are introduced for modeling the signs of the partial correlations among regular normally distributed random vectors. General oriented CI-structures are introduced in [Boe22b].

For notions regarding oriented matroids we use the notations in [BLVSWZ99]. A *signed subset* of $[n]$ is a map $X \colon [n] \to \{-1, 0, 1\}$. Write $X^+ := X^{-1}(1)$, $X^- := X^{-1}(-1)$ and $\underline{X} := X^+X^-$. A collection $\mathcal{C}$ of signed subsets of $[n]$ is the set of *signed circuits* of an *oriented matroid* on $[n]$ if it satisfies

(OC0) $\emptyset \notin \mathcal{C}$,

(OC1) $\mathcal{C} = -\mathcal{C}$,

(OC2) for all $X, Y \in \mathcal{C}$ with $\underline{X} \subseteq \underline{Y}$, either $X = Y$ or $X = -Y$,

(OC3) for all $X, Y \in \mathcal{C}$, $X \neq -Y$ and $e \in X^+ \cap Y^-$ there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \backslash e$ and $Z^- \subseteq (X^- \cup Y^-) \backslash e$.

If (OC0), (OC1) and (OC2) are satisfied, then (OC3) is equivalent to the following condition known as the *strong elimination axiom* [BLVSWZ99, Theorem 3.2.5]:

(OC3') For all $X, Y \in \mathcal{C}$, $e \in X^+ \cap Y^-$ and $f \in (X^+ \backslash Y^-) \cup (X^- \backslash Y^+)$, there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X^+ \cup Y^+) \backslash e$, $Z^- \subseteq (X^- \cup Y^-) \backslash e$ and $f \in Z$.

If $\mathcal{C}$ is a *circuit signature* of a matroid M, that is, $\mathcal{C}$ consists of two opposite signed sets X and $-X$ supported by C for each circuit C of M, then $\mathcal{C}$ clearly satisfies (OC0)–(OC2). In this case, we only need to check the X, Y in (OC3) such that $\underline{X}$ and $\underline{Y}$ are a *modular pair* in M, that is, $r(X) + r(Y) = r(X \cup Y) + r(X \cap Y)$ [BLVSWZ99, Theorem 3.6.1]. In this case, Z is unique [BLVSWZ99, Exercise 3.7].

Let $\mathcal{M}$ be an oriented matroid on $[n]$ and $\mathcal{C}$ be the set of signed circuits of $\mathcal{M}$. We associate an oriented CI-structure $\sigma_{\mathcal{M}} \colon \mathcal{A}_n \to \{-1, 0, 1\}$ to $\mathcal{M}$ by assigning $\sigma_{\mathcal{M}}(ij|K) = 0$ whenever $(ij|K) \in [[\underline{\mathcal{M}}]]$, and otherwise $\sigma_{\mathcal{M}}(ij|K) = X(i)X(j)$ for any $X \in \mathcal{C}$ such that $ij \subseteq \underline{X} \subseteq ijK$. The following lemma ensures the well-definedness of $\sigma_{\mathcal{M}}$.

Lemma 2.4.5. *If* $(ij|K) \notin [[\underline{\mathcal{M}}]]$*, then* $X(i)X(j) = Y(i)Y(j)$ *for any* $X, Y \in \mathcal{C}$ *such that* $ij \subseteq \underline{X}, \underline{Y} \subseteq ijK$*.*

Proof. Suppose that $X(i)X(j) = -Y(i)Y(j)$ for some $X, Y \in \mathcal{C}$ with $ij \subseteq \underline{X}, \underline{Y} \subseteq ijK$. Without loss of generality we can assume that $X(i) = X(j) = Y(i) = 1$ and $Y(j) = -1$ by changing the signs of X and/or Y by (OC1), and/or exchanging X and Y, if necessary. That is, $j \in X^+ \cap Y^-$ and $i \in X^+ \backslash Y^-$. The strong elimination axiom (OC3') implies that there is a $Z \in \mathcal{C}$ such that $i \in \underline{Z} \subseteq iK$ and $j \notin \underline{Z}$, which contradicts Lemma 2.4.2. □

We prove that loopless oriented matroids can be axiomatized in terms of oriented conditional independence.

Theorem 2.4.6. *An oriented CI-structure* $\sigma\colon \mathcal{A}_n \to \{-1, 0, 1\}$ *is the associated oriented CI-structure* $\sigma_{\mathcal{M}}$ *of an oriented matroid* $\mathcal{M}$ *iff it satisfies*

(OCI1) $\sigma(ij|K) \neq 0 \Rightarrow \sigma(i\ell|jKL) = 0,$

(OCI2) $\sigma(ij|K) = \sigma(i\ell|jK) = 0 \Rightarrow \sigma(ij|\ell K) = \sigma(i\ell|K) = 0,$

(OCI3) $\sigma(ij|K) \neq 0 \Rightarrow \sigma(ij|L) \in \{0, \sigma(ij|K)\}$ *for any* $L \subseteq K$ *or* $L \supseteq K,$

(OCI4) $\sigma(i\ell|K)\sigma(ij|K)\sigma(j\ell|K) \leq 0,$

(OCI5) $\sigma(i\ell|jK)\sigma(ij|\ell K)\sigma(j\ell|iK) \geq 0.$

In this case, the loopless oriented matroid $\mathcal{M}$ *can be determined uniquely from* σ *by first recovering the underlying matroid* $\underline{\mathcal{M}}$*, and then assigning to each circuit* C *of* $\underline{\mathcal{M}}$ *two opposite signed circuits* $\{X, -X\}$*, where*

$$X(c) := \begin{cases} 1 & c = c_0 \text{ or } c \in C,\ \sigma(cc_0|C\backslash cc_0) = 1 \\ -1 & c \in C,\ \sigma(cc_0|C\backslash cc_0) = -1 \\ 0 & c \notin C, \end{cases} \tag{2.16}$$

and $c_0 \in C$ *can be chosen arbitrarily in each circuit* C*. The pair of signed circuits defined by* (2.16) *is independent of the choice of* c_0*.*

Proof. Let $\mathcal{M}$ be an oriented matroid. By the construction of $\sigma_{\mathcal{M}}$, $\sigma^{-1}(0)$ is the CI-structure associated to the underlying matroid $\underline{\mathcal{M}}$, so (OCI1) and (OCI2) are guaranteed by Theorem 2.4.1. The condition (OCI3) follows by definition and Lemma 2.4.5.

If $\sigma(i\ell|K)\sigma(ij|K)\sigma(j\ell|K) \neq 0$, then by Lemma 2.4.2, there are signed circuits X, Y of $\mathcal{M}$ such that $ij \subseteq \underline{X}$, $\ell \notin \underline{X}$, $i\ell \subseteq \underline{Y}$ and $j \notin \underline{Y}$. By (OC1) we can assume $i \in X^+ \cap Y^-$. Then by (OC3) there is a signed circuit Z of $\mathcal{M}$ such that $Z^+ \subseteq (X^+ \cup Y^+)\backslash e$ and $Z^- \subseteq (X^- \cup Y^-)\backslash e$. It follows from $\sigma(j\ell|K) \neq 0$ that $j\ell \subseteq \underline{Z}$ and

$$\begin{aligned} \sigma(j\ell|K) = Z(j)Z(\ell) = X(j)Y(\ell) &= X(i)\sigma(ij|K)Y(i)\sigma(i\ell|K) \\ &= -\sigma(ij|K)\sigma(i\ell|K), \end{aligned}$$

so (OCI4) is proven.

If $\sigma(i\ell|jK)\sigma(ij|\ell K)\sigma(j\ell|iK) \neq 0$, then by Lemma 2.4.2, for any signed circuit X of $\mathcal{M}$ such that $\underline{X} \subseteq ij\ell K$, either $ij\ell \subseteq \underline{X}$ or $\{i, j, l\} \cap \underline{X} = \emptyset$, and there exists a signed circuit X such that $ij\ell \subseteq \underline{X} \subseteq ij\ell K$. Let X be such a signed circuit. Then we conclude (OCI5) as

$$\sigma(i\ell|jK)\sigma(ij|\ell K)\sigma(j\ell|iK) = X(i)^2 X(j)^2 X(\ell)^2 = 1.$$

Now let $\sigma\colon \mathcal{A}_n \to \{-1, 0, 1\}$ be such that (OCI1)–(OCI5) are satisfied, and let M be the loopless matroid obtained from the CI-structure $\sigma^{-1}(0) \subseteq \mathcal{A}_n$ by Theorem 2.4.1. Let $\mathcal{C}$ be the circuit signature of M given by (2.16). It follows from (OCI5) that the element c_0 in (2.16) can be chosen arbitrarily in each C.

Let $X, Y \in \mathcal{C}$ be such that $\underline{X}$ and $\underline{Y}$ are a modular pair in M. Let $e \in X^+ \cap Y^-$. By the construction (2.16) we have $X(x) = \sigma(xe|\underline{X}\backslash xe)$ for all $x \in \underline{X}\backslash e$ and $Y(y) = -\sigma(ye|\underline{Y}\backslash ye)$ for all $y \in \underline{Y}\backslash e$.

Let C be the unique circuit of M such that $C \subseteq (\underline{X} \cup \underline{Y})\backslash e$. Explicitly, $C = [n]\backslash \operatorname{cl}^*(([n]\backslash(\underline{X} \cup \underline{Y}))e)$, where cl^* is the closure operator of the dual matroid of M.

Let $f \in \underline{X}\backslash\underline{Y}$ and $g \in \underline{Y}\backslash\underline{X}$. The existence of f and g are ensured by the incomparability of circuits of a matroid. Because C is the unique circuit of M in $(\underline{X} \cup \underline{Y})\backslash e$, by the strong circuit elimination axiom of matroids, we have $f, g \in C$. Let Z be the signed subset of $[n]$ supported by C with $Z(f) = X(f) = \sigma(ef|\underline{X}\backslash ef)$ which is defined by (2.16), that is,

$$Z(z) = \sigma(zf|C\backslash zf)\sigma(ef|\underline{X}\backslash ef).$$

We show that Z is the desired circuit in the condition (OC3).

Case 1: If $z \in C \cap \underline{X}\backslash\underline{Y}$, then $\sigma(zf|(\underline{X} \cup \underline{Y})\backslash zf) \neq 0$ because

$$r(\underline{X} \cup \underline{Y}) = r((\underline{X} \cup \underline{Y})\backslash z) = r((\underline{X} \cup \underline{Y})\backslash f)$$

and

$$\begin{aligned} r(\underline{X} \cup \underline{Y}) - 1 &\leq r((\underline{X} \cup \underline{Y})\backslash zf) \\ &\leq r(\underline{X}\backslash zf) + r(\underline{Y}) - r(\underline{X} \cap \underline{Y}) \\ &= r(\underline{X}) - 1 + r(\underline{Y}) - r(\underline{X} \cap \underline{Y}) = r(\underline{X} \cup \underline{Y}) - 1. \end{aligned}$$

It follows from $\sigma(zf|C\backslash zf), \sigma(zf|\underline{X}\backslash zf), \sigma(zf|(\underline{X} \cup \underline{Y})\backslash zf) \neq 0$ and (OCI3) and (OCI5) that

$$\begin{aligned} Z(z) &= \sigma(zf|C\backslash zf)\sigma(ef|\underline{X}\backslash ef) \\ &= \sigma(zf|(\underline{X} \cup \underline{Y})\backslash zf)\sigma(ef|\underline{X}\backslash ef) \\ &= \sigma(zf|\underline{X}\backslash zf)\sigma(ef|\underline{X}\backslash ef) = \sigma(ez|\underline{X}\backslash ez) = X(z). \end{aligned}$$

Case 2: If $z \in C \cap \underline{Y}\backslash\underline{X}$, we need to show $Z(g) = Y(g)$, and the rest of this case is the same as Case 1. It follows from (OCI3) and (OCI4) that

$$\begin{aligned} Z(g) = Z(f)\sigma(fg|C\backslash fg) &= \sigma(ef|\underline{X}\backslash ef)\sigma(fg|C\backslash fg) \\ &= \sigma(ef|\underline{X}\backslash ef)\sigma(fg|(\underline{X} \cup \underline{Y})\backslash efg) \\ &= -\sigma(ef|\underline{X}\backslash ef)\sigma(ef|(\underline{X} \cup \underline{Y})\backslash efg)\sigma(eg|(\underline{X} \cup \underline{Y})\backslash efg) \\ &= -\sigma(ef|\underline{X}\backslash ef)\sigma(ef|\underline{X}\backslash ef)\sigma(eg|\underline{Y}\backslash eg) = -\sigma(eg|\underline{Y}\backslash eg) = Y(g), \end{aligned}$$

where all σ-values are nonzero because of Lemma 2.4.2.

Case 3: If $z \in C \cap \underline{X} \cap \underline{Y}$, suppose on the contrary that $X(z) = Y(z) = -Z(z)$. We have

$$\begin{aligned}\sigma(ez|\underline{X}\backslash ez) = X(z) = -Z(z) &= -\sigma(zf|C\backslash zf)\sigma(ef|\underline{X}\backslash ef)\\ &= -\sigma(zf|C\backslash zf)\sigma(zf|\underline{X}\backslash zf)\sigma(ez|\underline{X}\backslash ez),\end{aligned}$$

thus $\sigma(zf|C\backslash zf)\sigma(zf|\underline{X}\backslash zf) = -1$. Similarly, $\sigma(zg|C\backslash zg)\sigma(zg|\underline{Y}\backslash zg) = -1$ because

$$\begin{aligned}\sigma(ez|\underline{Y}\backslash ez) &= -Y(z) = Z(z) = Z(g)\sigma(zg|C\backslash zg) = Y(g)\sigma(zg|C\backslash zg)\\ &= -\sigma(eg|\underline{Y}\backslash eg)\sigma(zg|C\backslash zg) = -\sigma(ez|\underline{Y}\backslash ez)\sigma(zg|\underline{Y}\backslash zg)\sigma(zg|C\backslash zg).\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma(fg|C\backslash fg) &= \sigma(zg|C\backslash zg)\sigma(zf|C\backslash zf) = \sigma(zg|\underline{Y}\backslash zg)\sigma(zf|\underline{X}\backslash zf)\\ &= \sigma(eg|\underline{Y}\backslash eg)\sigma(ez|\underline{Y}\backslash ez)\sigma(ef|\underline{X}\backslash ef)\sigma(ez|\underline{X}\backslash ez)\\ &= \sigma(eg|\underline{Y}\backslash eg)(-Y(z))\sigma(ef|\underline{X}\backslash ef)X(z)\\ &= -\sigma(eg|\underline{Y}\backslash eg)\sigma(ef|\underline{X}\backslash ef) = -Z(f)Z(g),\end{aligned}$$

which contradicts (2.16).

We have shown that an oriented CI-structure σ satisfying (OCI1)–(OCI5) defines an oriented matroid $\mathcal{M}$. The fact $\sigma_{\mathcal{M}} = \sigma$ follows from (OCI3) and Lemma 2.4.2, namely, if $\sigma(ij|K) \neq 0$, then for any signed circuit X of $\mathcal{M}$ with $ij \subseteq \underline{X} \subseteq ijK$, we have $\sigma_{\mathcal{M}}(ij|K) = X(i)X(j) = \sigma(ij|\underline{X}\backslash ij) = \sigma(ij|K)$. □

Oriented matroids are known to be cryptomorphic to chirotopes, which abstract the possible signs of Plücker coordinates of points in the Grassmannian. By the following proposition, $\sigma_{\mathcal{M}}$ can be obtained directly from the chirotope of $\mathcal{M}$.

Proposition 2.4.7. *Let $\chi\colon \binom{[n]}{r} \to \{-1,0,1\}$ be the chirotope of an oriented matroid $\mathcal{M}$ on $[n]$. The oriented CI-structure $\sigma_{\mathcal{M}}\colon \mathcal{A}_n \to \{-1,0,1\}$ associated to $\mathcal{M}$ can be obtained from the chirotope χ by*

$$\sigma_{\mathcal{M}}(ij|K) = -\chi(i, b_1, \ldots, b_s, a_1, \ldots, a_{r-s-1})\chi(j, b_1, \ldots, b_s, a_1, \ldots, a_{r-s-1}),$$

where $b_1 \cdots b_s$ is a basis of K such that $ib_1 \cdots b_s$ and $jb_1 \cdots b_s$ are bases of ijK, and $a_1 \cdots a_{r-s-1}$ is a subset of $[n]\backslash ijK$ such that $\chi(i, b_1, \ldots, b_s, a_1, \ldots, a_{r-s-1}) \neq 0$; and $\sigma_{\mathcal{M}}(ij|K) = 0$ if no such $b_1 \cdots b_s$ exists.

Proof. Let $\sigma_\chi(ij|K)$ be the oriented CI-structure defined in the proposition. By Proposition 2.4.3, we have $\sigma_\chi(ij|K) = 0$ iff $(ij|K) \in [[\underline{\mathcal{M}}]]$. Assume that $(ij|K) \notin [[\underline{\mathcal{M}}]]$. The chirotope χ is a basis orientation of the oriented matroid $\mathcal{M}$ [BLVSWZ99, Theorem 3.5.5]. By the pivoting property [BLVSWZ99, Definition 3.5.1], we have

$$-\chi(i, b_1, \ldots, b_s, a_1, \ldots, a_{r-s-1})\chi(j, b_1, \ldots, b_s, a_1, \ldots, a_{r-s-1}) = X(i)X(j),$$

where X is one of the two opposite signed circuits of $\mathcal{M}$ such that $\underline{X}$ is contained in the set $ijb_1 \cdots b_s a_1 \cdots a_{r-s-1}$. Moreover, the circuit $\underline{X}$ of $\underline{\mathcal{M}}$ must satisfy $ij \subseteq \underline{X} \subseteq ijb_1 \cdots b_s$ because $ib_1 \cdots b_s$ and $jb_1 \cdots b_s$ are bases of ijK. Therefore, by definition, $\sigma_{\mathcal{M}}(ij|K) = X(i)X(j) = \sigma_\chi(ij|K)$. Note that it is independent of the choice of $b_1, \ldots, b_s, a_1, \ldots, a_{r-s-1}$. □

Problem 2.4.8. Find an axiomatization of valuated matroids as valuated CI-structures.

2.4.4 Remarks on representations of matroids

In [Mat93], Matúš introduced representations of matroids in probability theory and information theory which lead to new insight into representability of matroids. A matroid M is *probabilistically representable* if its CI-structure coincides with the conditional independence among a discrete random vector ξ, that is, $[[M]] = [[\xi]] = [[h_\xi]]$. As the rank function of any connected matroid spans an extreme ray in the cone of submodular functions, probabilistic representability is equivalent to entropicness in the case of connected matroids, where a matroid is *entropic* if its rank function is a positive multiple of the entropy function of some discrete random vector. Linear and multilinear matroids are entropic [Mat97, Lemma 10], however, the direct sum of Fano matroid and non-Fano matroid is algebraic but not entropic [Mat18]. The rank function of any algebraic matroid is the pointwise limit of a sequence of entropy functions [Mat23]. Such matroids are called *almost entropic* [Mat06]. The dual of an almost entropic matroid is not necessarily almost entropic [Kac18], while a long standing open problem in matroid theory is whether the dual of any algebraic matroid is algebraic.

A Gaussian conditional independence, however, can never represent an interesting matroid. In [MUWY18, §7] it was computationally verified that for $3 \leq n \leq 8$ no CI-structure corresponding to a connected matroid on $[n]$ is a gaussoid. We confirm this computation by showing that a loopless matroid is a gaussoid iff it is a direct sum of copies of uniform matroids $U_{1,1}$ and $U_{1,2}$.

Proposition 2.4.9. *Let M be a loopless matroid. Then $[[M]]$ is a gaussoid iff $M \cong U_{1,1}^{\oplus m_1} \oplus U_{1,2}^{\oplus m_2}$.*

Proof. The necessity follows from the observation that $[[U_{1,1}]] = [[(1)]]$ and $[[U_{1,2}]] = [[\begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}]]$ are gaussoids and $[[\Sigma_1]] \oplus [[\Sigma_2]] = [[\Sigma_1 \oplus \Sigma_2]]$ for any positive definite matrices Σ_1, Σ_2 [Boe22b, Lemma 3.14].

Now assume that M is a loopless matroid on the ground set E such that $[[M]]$ is a gaussoid. We deduce the following three inference rules for any $ij\ell K \subseteq E$ from (MCI), (SG), (Int) and (Comp):

Claim 1. $i \not\perp\!\!\!\perp j|K \Rightarrow i \not\perp\!\!\!\perp j|\ell K$. It follows by (MCI) from $i \not\perp\!\!\!\perp j|K$ that $i \perp\!\!\!\perp \ell|jK$ and $j \perp\!\!\!\perp \ell|iK$. And by (Int) we get $i \perp\!\!\!\perp \ell|K$. But by (SG) we have $i \not\perp\!\!\!\perp \ell|K$ or $i \not\perp\!\!\!\perp j|\ell K$ from $i \not\perp\!\!\!\perp j|K$. Thus we have $i \not\perp\!\!\!\perp j|\ell K$.

Claim 2. $i \not\perp\!\!\!\perp j|\ell K \Rightarrow i \not\perp\!\!\!\perp j|K$. It follows by (MCI) from $i \not\perp\!\!\!\perp j|\ell K$ that $i \perp\!\!\!\perp \ell|K$ and $j \perp\!\!\!\perp \ell|K$. And by (Comp) we get $i \perp\!\!\!\perp \ell|jK$. But by (SG) we have $i \not\perp\!\!\!\perp \ell|jK$ or $i \not\perp\!\!\!\perp j|K$ from $i \not\perp\!\!\!\perp j|\ell K$. Thus we have $i \not\perp\!\!\!\perp j|K$.

Claim 3. $i \not\perp\!\!\!\perp j| \Rightarrow i \perp\!\!\!\perp \ell|$. By applying (MCI) we get $i \perp\!\!\!\perp \ell|j$ and $j \perp\!\!\!\perp \ell|i$ from $i \not\perp\!\!\!\perp j|$. Then $i \perp\!\!\!\perp \ell|$ follows from (Int).

By Claim 3, the ground set E can be partitioned into 2-element sets $\{a_\iota, b_\iota\}$, $\iota \in [m_2]$ and singletons $\{c_\iota\}$, $\iota \in [m_1]$ such that $a_\iota \not\perp\!\!\!\perp b_\iota|$ for any $\iota \in [m_2]$ and $i \perp\!\!\!\perp j|$ if $i, j \in E$ are in different blocks. By Claim 1 and Claim 2, for any two subsets $K, K' \subseteq E \setminus ij$ we have $i \perp\!\!\!\perp j|K$ iff $i \perp\!\!\!\perp j|K'$. Therefore, the loopless matroid M is isomorphic to the direct sum of m_2 rank one uniform matroids on $\{a_\iota, b_\iota\}$, $\iota \in [m_2]$ and m_1 rank one uniform matroids on $\{c_\iota\}$, $\iota \in [m_1]$. □

Chapter 3

Conditional Erlangen Program and Combinatorial Erlangen Program

3.1 The Coxeter conditional independence

We have reviewed several classes of conditional independence structures and their geometry in the last chapter. In this chapter we aim to answer the classic question: What about other Coxeter types?

3.1.1 Root systems and Dynkin classification

Let $V \cong \mathbb{R}^d$ be a finite dimensional real vector space with a positive definite inner product $\langle \cdot, \cdot \rangle$. We identify V by the inner product with its dual V^*. Every vector $v \in V$ defines a linear automorphism s_v on V by reflecting across the hyperplane orthogonal to v, that is,

$$s_v(x) := x - \frac{2\langle x, v\rangle}{\langle v, v\rangle} v.$$

The reflection s_v fixes all points of the hyperplane $H_v := v^\perp = \{u \in V : \langle u, v\rangle = 0\}$, which is called the *mirror* of s_v.

A *root system* $\Phi \subseteq V$ is a finite set of vectors, called *roots*, which satisfies

(R0) $\operatorname{span}_{\mathbb{R}}(\Phi) = V$,

(R1) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for any $\alpha \in \Phi$,

(R2) $s_\alpha(\Phi) = \Phi$ for any $\alpha \in \Phi$.

A root system Φ is *crystallographic* if it satisfies additionally

(R3) $\frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}$ for any $\alpha, \beta \in \Phi$.

The *Weyl group* $W_\Phi := \langle s_\alpha : \alpha \in \Phi \rangle$ is the group generated by the reflections. It is a finite subgroup of $\mathrm{GL}(V)$.

A root system Φ gives rise to a hyperplane arrangement $\mathcal{H}_\Phi := \{H_\alpha : \alpha \in \Phi\}$, called the *Coxeter arrangement*. The *Coxeter complex* Σ_Φ is its associated simplicial fan, that is, Σ_Φ is the complete simplicial fan in V whose chambers are the Euclidean closures of the connected components of the complement of the Coxeter arrangement. Fix a chamber D, called the *fundamental domain*, of Σ_Φ. The *simple roots* $\Delta = \{\alpha_1, \dots, \alpha_d\} \subseteq \Phi$ are the roots in Φ which are the inner normals of the walls in D. The simple roots form a basis for V, we call $d = \dim V$ the *rank* of the root system Φ. The *positive roots* $\Phi_+ \subseteq \Phi$ are the roots which are nonnegative combinations of simple roots Δ. Remark that $\Phi = \Phi_+ \sqcup (-\Phi_+)$.

The *coroot* $\alpha^\vee$ of a root $\alpha \in \Phi$ is

$$\alpha^\vee := \frac{2}{\langle \alpha, \alpha \rangle} \alpha \in V^* = V.$$

The *fundamental weights* $(\lambda_1, \dots, \lambda_d)$ is the basis of V dual to the simple coroots $(\alpha_1^\vee, \dots, \alpha_d^\vee)$, that is, $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$. Also write $\lambda_{\alpha_i} := \lambda_i$ for the fundamental weight corresponding to the simple root α_i.

Let Φ be a root system and $\Delta = \{\alpha_1, \dots, \alpha_d\} \subseteq \Phi$ a set of simple roots. The *Cartan matrix* is the $d \times d$-matrix A whose entries are

$$A_{i,j} := 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = \langle \alpha_i^\vee, \alpha_j \rangle$$

for $i, j \in [d]$. There exist positive integers $m_{ij} = m_{ji}$ such that

$$A_{i,j} A_{j,i} = 4\cos^2(\pi/m_{ij}).$$

The entries m_{ij}, $i, j \in [d]$, form the *Coxeter matrix* of Φ. In this case, m_{ij} is the order of $s_{\alpha_i} s_{\alpha_j}$ in the Weyl group W_Φ, and the Weyl group is given by the following generators and relations

$$W_\Phi = \langle s_{\alpha_1}, \dots, s_{\alpha_d} \,|\, (s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = 1 \rangle.$$

The *Dynkin diagram* of Φ is the graph which has vertices $[d]$ and an edge labeled m_{ij} between i and j whenever $m_{ij} > 2$. Labels equal to 3 occur frequently and are therefore omitted when drawing pictures.

The *direct sum* of two root systems Φ_1 and Φ_2, spanning V_1 and V_2, respectively, is the root system

$$\Phi_1 \oplus \Phi_2 := \{(\alpha, 0) \in V_1 \oplus V_2 : \alpha \in \Phi_1\} \cup \{(0, \beta) \in V_1 \oplus V_2 : \beta \in \Phi_2\}$$

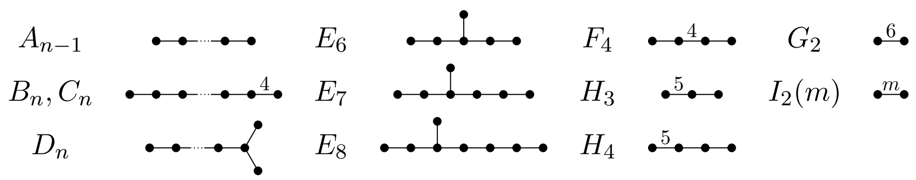
which spans $V_1 \oplus V_2$. A root system is *irreducible* if it is not a non-trivial direct sum of root systems.

The Cartan matrix A of any root system is positive definite. It can be decomposed as $A = DS$ where $D = \operatorname{diag}\big((\langle \alpha_i, \alpha_i \rangle^{-1})_i\big)$ is a positive definite diagonal matrix and

$S = (2\langle \alpha_i, \alpha_j \rangle)_{ij}$ is a positive definite symmetric matrix. A complete classification of root systems can be done by classifying the Dynkin diagrams obtained from positive definite matrices.

Theorem 3.1.1 ([Hum90, § 2])**.** *The irreducible root systems can be completely classified into four infinite families A_n, B_n, C_n, D_n, the exceptional types E_6, E_7, E_8, F_4, G_2, H_3, H_4 and $I_2(m)$ for $m \geq 3$.*

The ranks of the root systems in Theorem 3.1.1 are labeled by their subscripts. Root systems A_n, B_n, C_n, D_n are *classical* as they correspond to the Lie algebras associated to the complex classical groups [Wey46], the others are *exceptional*. The following are their Dynkin diagrams.



A *Coxeter group* is a group given by generators and relations of the form

$$\langle s_1, \dots, s_d \mid (s_i s_j)^m_{ij} = 1 \rangle$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$. Weyl groups are finite Coxeter groups, and conversely, every finite Coxeter group is isomorphic to a Weyl group [Hum90, § 6.4][BGW03, § 5.10].

A *finite reflection group* is a finite subgroup of $\mathrm{GL}(\mathbb{R}^d)$ generated by reflections. The Weyl group of a root system is a finite reflection group. Conversely, for any element s of a finite reflection group, take two unit vectors $\pm\alpha$ perpendicular to the mirror of s. Then the collection of such vectors is a root system [BGW03, § 5.4.2]. If Φ is a root system, then $\{\alpha/||\alpha|| : \alpha \in \Phi\}$ is a root system with the same Weyl group. In this dissertation the lengths of roots do not matter.

Remark 3.1.2. The notion of "root system" commonly used in Lie theory means crystallographic root system, that is, the irreducible root systems $H_3, H_4, I_2(m)$ are not considered. We do not require root systems to be crystallographic, therefore in our context Weyl groups, finite reflection groups and finite Coxeter groups are same [Hum90, § 2.9].

3.1.2 Coxeter permutohedra and Coxeter complexes

The Φ-*permutohedron* Π_Φ is the Minkowski sum of the positive roots of Φ, that is, the zonotope

$$\Pi_\Phi := \sum_{\alpha \in \Phi_+} [-\alpha/2, \alpha/2] = \mathrm{conv}\{w \cdot \rho : w \in W\}$$

of the Coxeter arrangement $\mathcal{H}_\Phi$, where $\rho := \frac{1}{2}(\sum_{\alpha \in \Phi_+} \alpha) = \lambda_1 + \cdots + \lambda_d$ is the sum of the fundamental weights. The Coxeter complex Σ_Φ is the normal fan of the Φ-permutohedron.

The Weyl group $W = W_\Phi = \langle s_\alpha : \alpha \in \Delta\rangle$ is generated by simple reflections. It acts *regularly* on the set $\Sigma_\Phi(d)$ of the chambers of Σ_Φ, that is, for any $\sigma, \sigma' \in \Sigma_\Phi(d)$ there is a unique $w \in W$ such that $w \cdot \sigma = \sigma'$. The *parabolic subgroups* of W_Φ are the subgroups

$$W_I := \langle s_\alpha : \alpha \in I\rangle \subseteq W$$

for all $I \subseteq \Delta$. Given $\Delta = \{\alpha_1, \dots, \alpha_d\}$ we also write $W_J := W_{\{\alpha_j : j \in J\}}$ for $J \subseteq [d]$. The *parabolic cosets* are the cosets of the parabolic subgroups.

The faces of the fundamental domain D are

$$\begin{aligned} C_I &:= \{x \in D : \langle x, \alpha\rangle = 0 \ \text{ for all } \ \alpha \in I,\ \langle x, \alpha\rangle \geq 0 \ \text{ for all } \ \alpha \in \Delta\backslash I\} \\ &= D \cap \bigcap_{\alpha \in I} H_\alpha = \operatorname{cone}(\lambda_\alpha : \alpha \notin I) \end{aligned}$$

for all $I \subseteq \Delta$.

Theorem 3.1.3 ([Hum90, § 1.12, § 1.15])**.** *The stabilizer subgroup of the face C_I is precisely the parabolic subgroup W_I. More generally, if V' is any subset of V then the subgroup of W fixing V' pointwise is generated by those reflections s_α whose mirrors H_α contain V'.*

Proposition 3.1.4 ([Hum90, § 1.15][ACEP20, Proposition 3.11])**.** *The faces of the Coxeter complex are in bijection with the parabolic cosets of the Weyl group W. Explicitly, The faces C_I, $I \subseteq \Delta$, of the fundamental domain D are in bijection with the the parabolic subgroups W_I, and each W-conjugate vC_I is labeled with the coset vW_I for $v \in W$.*

In particular:

- The chambers of Σ_Φ are $\{w \cdot D : w \in W\}$. The vertices of Π_Φ are $\{w \cdot \rho : w \in W\}$.
- The walls of Σ_Φ are labeled by the pairs $\{w, ws_i\} = wW_{\{i\}}$ for $w \in W$ and $i \in [d]$. The wall labeled by $wW_{\{i\}}$ is the intersection of the chambers $w \cdot D$ and $ws_i \cdot D$. This correspondence is bijective as $wW_{\{i\}} = ws_iW_{\{i\}}$.
- The ridges of Σ_Φ are labeled by $wW_{\{i,j\}}$ for $w \in W$, $s_i, s_j \in S$, $i < j$. As $W_{\{i\}} \subseteq W_{\{i,j\}}$ is a subgroup of order 2, by Lagrange's theorem, every ridge of Σ_Φ is the intersection of an even number of chambers. Thus every 2-face of Π_Φ is a $2k$-gon for some integer $k \geq 2$.
- The d rays of the fundamental domain D are $\operatorname{cone}(\lambda_1), \dots, \operatorname{cone}(\lambda_d)$. The rays of Σ_Φ are $\operatorname{cone}(u)$, $u \in \mathcal{R}_\Phi$, where $\mathcal{R}_\Phi := W \cdot \{\lambda_1, \dots, \lambda_d\}$ is the set of *fundamental weight conjugates* of Φ.

3.1.3 Coxeter semigraphoids and Coxeter semimatroids

Following § 2.1.2, we generalize the definition of semigraphoids to any root system as follows.

Definition 3.1.1. Let Φ be a root system. A Φ-*semigraphoid* is a fan which is a coarsening of the Coxeter complex Σ_Φ.

As before, we can identify a coarsening of Σ_Φ with the set of removed walls, and equivalently, the set of edges of Π_Φ corresponding to the removed walls. Theorem 2.1.8 is generalized to a general class of polyhedral complexes in [Rea12] and in particular, to Coxeter complexes.

Theorem 3.1.5 ([Rea12, Corollary 1.3])**.** *Let Z be a zonotope and let Σ_Z be the normal fan of Z. Then a set E of edges of Z is corresponding to the set of walls of Σ_Z whose removal results in a coarser fan if and only if for every $2k$-gonal 2-face F of Z, whenever E contains any $k-1$ consecutive edges of F, then E also contains the opposite $k-1$ consecutive edges of F.*

Therefore we have the following characterization for Φ-semigraphoids.

Definition 3.1.2. A set of edges E of Π_Φ is a Φ-*semigraphoid* if it satisfies

(ΦSG) For every $2k$-gonal 2-face F of Π_Φ, whenever E contains any $k-1$ consecutive edges of F, then E also contains the opposite $k-1$ consecutive edges of F.

Now we extend Theorem 2.1.11 to any root system Φ.

Definition 3.1.3. A Φ-semigraphoid, regarded as a fan, is a Φ-*semimatroid* if it is polytopal. That is, the normal fan of a polytope Q which coarsens Σ_Φ. Such a polytope Q is a deformation of Π_Φ, called a *generalized* Φ-*permutohedron*.

Let Φ be a root system of rank d and $\mathcal{R} = \mathcal{R}_\Phi := W_\Phi \cdot \{\lambda_1, \dots, \lambda_d\}$ be the set of the fundamental weight conjugates of Φ. As a piecewise linear function $h \in \mathrm{PL}(\Sigma_\Phi)$ is determined by its restriction to the rays and each ray contains a conjugate to a fundamental weight, we identify $\mathrm{PL}(\Sigma_\Phi)$ with $\mathbb{R}^{\mathcal{R}}$.

A function $h\colon \mathcal{R} \to \mathbb{R}$ is Φ-*submodular* if it is convex when regarded as a piecewise linear function in $\mathrm{PL}(\Sigma_\Phi)$. The correspondence between Φ-submodular functions and generalized Φ-permutohedra is bijective by Proposition 2.1.9. Two classifications for Φ-submodular functions are as follows.

Theorem 3.1.6 ([ACEP20, Theorem 5.2])**.** *A function $h\colon \mathcal{R} \to \mathbb{R}$ is Φ-submodular iff the following two equivalent sets of inequalities hold:*

- (local Φ-submodularity) *For every $w \in W_\Phi$ and every simple reflection s_i,*

$$h(w\lambda_i) + h(ws_i\lambda_i) \geq \sum_{j \in N(i)} -2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} h(w\lambda_j), \tag{3.1}$$

  where $N(i)$ is the set of neighbors of i, including i, in the Dynkin diagram, and α_i and λ_i are the simple root and the fundamental weight corresponding to s_i, respectively.

- (global Φ-submodularity) *For any two fundamental weight conjugates $\lambda, \lambda' \in \mathcal{R}$,*

$$h(\lambda) + h(\lambda') \geq h(\lambda + \lambda').$$

where h is regarded a piecewise linear function in $\mathrm{PL}(\Sigma_\Phi)$*.*

The Φ-*submodular cone* $\mathrm{SF}_\Phi \subseteq \mathbb{R}^{\mathcal{R}}$ is the cone of all Φ-submodular functions. The cone SF_Φ is therefore the parameter space of all generalized Φ-permutohedra. The facets of SF_Φ are described and enumerated in [ACEP20, § 7].

Theorem 3.1.7 ([ACEP20, Theorem 7.1 and 7.2])**.** *Each inequality* (3.1) *associated with a pair* (w, s_i)*,* $w \in W = W_\Phi$ *and* $i \in [d]$*, gives a facet of the* Φ*-submodular cone* SF_Φ*. Two pairs* (w, s_i) *and* $(w', s_{i'})$ *define the same facet iff* $i = i'$ *and* $w^{-1}w' \in W_{[d]\setminus N(i)}$*. In particular, the number of facets of* SF_Φ *is*

$$\sum_{i=1}^{d} \frac{|W|}{|W_{[d]\setminus N(i)}|}.$$

Semimatroids encode the combinatorial information of generalized permutohedra (or submodular functions), Φ-semimatroids encode analogously the combinatorics of generalized Φ-permutohedra. The lattice of Φ-semimatroids, ordered by inclusion, is isomorphic to the dual of the face lattice of SF_Φ. Every face of SF_Φ is an intersection of the facets of SF_Φ. Thus every Φ-semimatroid is a union of the atoms of the lattice of Φ-semimatroids. The Φ-CI-statements are defined to be corresponding to the atoms.

Definition 3.1.4. Let $\sim$ be the equivalence relation on the edges $\{wW_{\{i\}} : w \in W, i \in [d]\}$ of Π_Φ, defined by

$$(w, s_i) \sim (w', s_{i'}) :\Leftrightarrow i = i' \text{ and } w^{-1}w' \in W_{[d]\setminus N(i)}.$$

The set of Φ-*conditional independence statements* (Φ-*CI-statements*) is

$$\mathcal{A}_\Phi := \{wW_{\{i\}} : w \in W, i \in [d]\}/\sim .$$

For each $i \in [d]$, the parabolic subgroup $W_{[d]\setminus N(i)}$ acts on the set of edges $\{wW_{\{i\}} : w \in W\}$ faithfully. The Φ-CI-statements are exactly the orbits

$$\mathcal{A}_\Phi = \{W_{[d]\setminus N(i)} \cdot wW_{\{i\}} : w \in W, i \in [d]\}.$$

The atoms of the lattice of Φ-semimatroids, which correspond bijectively to the facets of SF_Φ, are precisely the singletons of each Φ-CI-statement.

Definition 3.1.5. A Φ-*CI-structure* is a subset $\mathcal{G}$ of $\mathcal{A}_\Phi$. A Φ-CI-structure $\mathcal{G} \subseteq \mathcal{A}_\Phi$ is a *semigraphoid* (*semimatroid*) if $\bigcup \mathcal{G}$ is a semigraphoid (semimatroid) as a set of edges of Π_Φ.

A Φ-semimatroid is therefore a Φ-CI-structure for which there is a Φ-submodular function h such that the inequalities in (3.1) attend the equality exactly at its elements.

Problem 3.1.8. One can mimic Proposition 2.2.6 and define a Φ-*gaussoid* to be a subset $\mathcal{G}$ of $\mathcal{A}_\Phi$ such that the corresponding edges $E = \bigcup \mathcal{G}$ of Π_Φ satisfies

(ΦGß) For every $2k$-gonal 2-face F of Π_Φ, whenever E contains any $k-1$ edges of F, then E contains either all edges or all but two opposite edges of F.

Does this definition agree with the vanishing of variables in some polynomial equation system, just like gaussoids are defined by the quadratic trinomials (2.8) and (2.9)? In other words, is there a polynomial system which tropical prevariety with respect to the trivial valuation is the set of all Φ-gaussoids?

Problem 3.1.9. Develop a theory of CI-structures over infinite Coxeter groups.

Problem 3.1.10. Find interpretations of Φ-CI-structures in probability theory and statistics.

3.2 The CI-structures of classical types

In this section we describe the semigraphoids and the semimatroids in the classical types explicitly.

3.2.1 Type A

The root system A_{n-1} is

$$A_{n-1} = \{\pm(\mathbf{e}_i - \mathbf{e}_j) : 1 \le i < j \le n\}.$$

The roots in A_{n-1} span the hyperplane $\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$ in $\mathbb{R}^n$. We choose the simple roots to be

$$\Delta_{A_{n-1}} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n\}.$$

The corresponding fundamental weights are $\{\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_1 + \overline{\mathbf{e}}_2, \ldots, \overline{\mathbf{e}}_1 + \overline{\mathbf{e}}_2 + \cdots + \overline{\mathbf{e}}_{n-1}\} \subseteq \mathbb{R}^n/\mathbb{R}\mathbf{1}$, where $\overline{\mathbf{e}}_i$ is the image of $\mathbf{e}_i$ under the projection $\mathbb{R}^n \twoheadrightarrow \mathbb{R}^n/\mathbb{R}\mathbf{1}$.

The Weyl group of A_{n-1} is $W = W_{A_{n-1}} = \langle s_1, \ldots, s_{n-1}\rangle \subseteq \mathrm{GL}(\mathbb{R}^n)$. It is generated by the simple reflections $s_1, \ldots, s_{n-1}$, where $s_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is to exchange the i-th coordinate and the $(i+1)$-th coordinate. The Weyl group $W_{A_{n-1}}$ is isomorphic to the symmetric group $\mathfrak{S}_n$.

The fundamental weights conjugates are $\mathcal{R}_{A_{n-1}} = \{\overline{\mathbf{e}}_S : \emptyset \subsetneq S \subsetneq [n]\}$ where $\mathbf{e}_S := \sum_{i\in S} \mathbf{e}_i$.

The objects of type A are the objects in the classic setting. The parabolic subgroup $W_{[n]\setminus\{|K|,|K|+1,|K|+2\}}$ acts faithfully on the set of cosets $\{wW_{\{|K|+1\}} : w \in W\}$. The set of $|K|!(n-|K|-2)!$ walls (2.2) corresponding to a CI-statement $(ij|K) \in \mathcal{A}_n$ is exactly an orbit in $\{wW_{\{|K|+1\}} : w \in W\}$ under the action of $W_{[n]\setminus\{|K|,|K|+1,|K|+2\}}$.

3.2.2 Type B or C

The root systems B_n and C_n are

$$B_n = \{\pm\mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le n\} \cup \{\pm\mathbf{e}_i : 1 \le i \le n\} \subseteq \mathbb{R}^n,$$
$$C_n = \{\pm\mathbf{e}_i \pm \mathbf{e}_j : 1 \le i < j \le n\} \cup \{\pm 2\mathbf{e}_i : 1 \le i \le n\} \subseteq \mathbb{R}^n.$$

They only differ in the length of some roots, therefore the Coxeter complexes Σ_{B_n} and Σ_{C_n} are same. They make no difference in this dissertation. We focus on type C.

We choose the simple roots of C_n to be

$$\Delta_{C_n} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n, 2\mathbf{e}_n\},$$

and the corresponding fundamental weights are $\{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \dots, \mathbf{e}_1 + \cdots + \mathbf{e}_n\}$.

The Weyl group of C_n is $W = W_{C_n} = \langle s_1, \dots, s_n \rangle \subseteq \mathrm{GL}(\mathbb{R}^n)$. It is generated by the simple reflections $s_1, \dots, s_{n-1}$, where $s_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is to exchange the i-th and the $(i+1)$-th coordinates for $i = 1, \dots, n-1$, and s_n, which is to change the sign of the n-th coordinate. The Weyl group W_{C_n} is the hyperoctahedral group $\mathfrak{S}_2 \wr \mathfrak{S}_n$, which is isomorphic to the group of signed permutations. As a matrix group, it is the group of $n \times n$ permutation matrices with signed entries.

Let $[\pm n] := \{1, \dots, n, -1, \dots, -n\}$. We write $S \sqsubseteq [\pm n]$ if $S \subseteq [\pm n]$ and $(j \in S \Rightarrow -j \notin S)$. For $i \in [\pm n]$ and $K \subseteq [\pm n]$ we also denote $-i$ by $\bar{i}$ and $-K = \{-i : i \in K\}$ by $\overline{K}$ for convenience.

The fundamental weight conjugates are

$$\mathcal{R}_{C_n} = \{\mathbf{e}_S : \emptyset \neq S \sqsubseteq [\pm n]\},$$

where $\mathbf{e}_{\bar{i}} := -\mathbf{e}_i$. We identify every function $h \in \mathbb{R}^{\mathcal{R}_{C_n}}$ with the function $\{S \sqsubseteq [\pm n]\} \to \mathbb{R}$ satisfying $\emptyset \mapsto 0$ and $S \mapsto h(\mathbf{e}_S)$. The functions in $\mathbb{R}^{\mathcal{R}_{C_n}}$ are identified with the functions in $\{h \colon \{S \sqsubseteq [\pm n]\} \to \mathbb{R} : h(\emptyset) = 0\}$.

The C_n-permutohedron Π_{C_n} is the *omnitruncated n-cube*.

- The vertices of Π_{C_n} are $w \cdot \rho = (w(n), \dots, w(1))^\top$, $w \in W_{C_n}$, where $\rho = \lambda_1 + \cdots + \lambda_n = (n, \dots, 1)^\top$. The vertex $w \cdot \rho$ can also be denoted by its *descent vector* $(\delta(1)|\cdots|\delta(n))$, where $\delta = w^{-1} \in W_{C_n}$. It corresponds to the chamber $\{x \in \mathbb{R}^n : x_{\delta(1)} \geq \cdots \geq x_{\delta(n)} \geq 0\}$, where $x_{\bar{i}} := -x_i$ for $i \in [n]$.
- There are two kinds of edges in Π_{C_n}. An edge with vertices $(\delta(1)|\cdots|\delta(i)|\delta(i+1)|\cdots|\delta(n))$ and $(\delta(1)|\cdots|\delta(i+1)|\delta(i)|\cdots|\delta(n))$ is associated to the C_n-CI-statement $\delta(i) \perp\!\!\!\perp \delta(i+1) | \delta(1) \cdots \delta(i-1) \in \mathcal{C}_n$. It corresponds to the wall $\{x \in \mathbb{R}^n : x_{\delta(1)} \geq \cdots \geq x_{\delta(i)} = x_{\delta(i+1)} \geq \cdots \geq x_{\delta(n)} \geq 0\}$ of Σ_{C_n}.

  An edge of the other kind corresponds to a coset of the parabolic subgroup $\langle s_n \rangle$. It has the vertices $(\delta(1)|\cdots|\delta(n-1)|\delta(n))$ and $(\delta(1)|\cdots|\delta(n-1)|\overline{\delta(n)})$ and corresponds to the wall $\{x \in \mathbb{R}^n : x_{\delta(1)} \geq \cdots \geq x_{\delta(n)} = 0\}$ of Σ_{C_n}. It is associated to the C_n-CI-statement $\delta(n) \perp\!\!\!\perp \overline{\delta(n)} | \delta(1) \cdots \delta(n-1) \in \mathcal{C}_n$.
- A 2-face of Π_{C_n} is either a square, or a hexagon, or an octagon. The 2-faces are labeled by cosets of parabolic subgroups $W_{\{i,j\}}$, $j > i+1$, $W_{\{i,i+1\}}$, $i < n-1$ and $W_{\{n-1,n\}}$, respectively.

Figure 3.1 shows a part of the C_3-permutohedron Π_{C_3}, also known as the *truncated cuboctahedron*. The vertices are labeled by their coordinates and descent vectors, the edges are labeled by the C_n-CI-statements and the 2-faces are labeled by the parabolic subgroups.

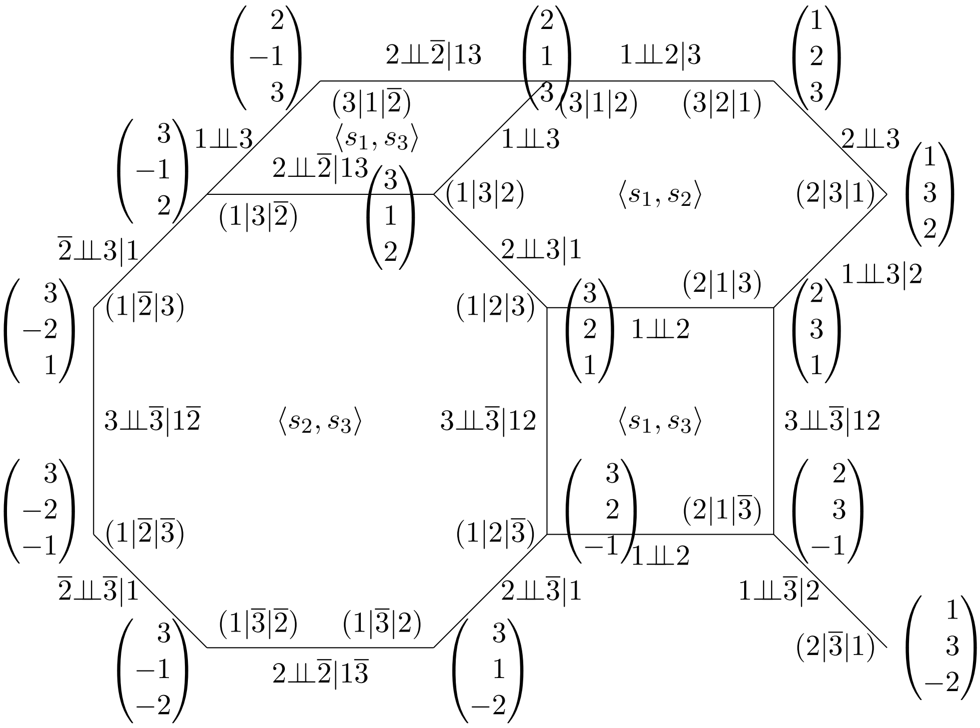


Figure 3.1: a piece of Π_{C_3}

For $f\colon \mathcal{R}_{C_n} \to \mathbb{R}$, write $f(S) = f(\mathbf{e}_S)$ for any $S \sqsubseteq [\pm n]$ and $f(\emptyset) = 0$. The local C_n-submodularity inequalities are

1. $f(aS) + f(bS) \geq f(S) + f(abS)$ for $S \sqsubseteq [\pm n]$, $ab \sqsubseteq [\pm n] \backslash S\overline{S}$.
2. $f(aS) + f(\overline{a}S) \geq 2f(S)$ for $S \sqsubseteq [\pm n]$, $|S| = n - 1$, $a \in [\pm n] \backslash S\overline{S}$.

There are $2d(d-1)3^{d-2} + d2^{d-1}$ local C_n-submodularity inequalities. The equivalent global C_n-submodularity inequalities are

$$f(S) + f(T) \geq f(S \cap T) + f(S \sqcup T) \quad \text{ for } S, T \sqsubseteq [\pm n],$$

where $S \sqcup T := \{i \in S \cup T : -i \neq S \cup T\} \sqsubseteq [\pm n]$. A function $f\colon \{S \sqsubseteq [\pm n]\} \to \mathbb{R}$ with $f(\emptyset) = 0$ satisfying these inequalities is called *bisubmodular* [ACEP20, § 5.2 and Theorem 7.2].

By applying § 3.1.3 we define the CI-statements, semigraphoids and semimatroids of type B or C.

Definition 3.2.1. The set of C_n-*conditional independence statements* is

$$\begin{aligned}\mathcal{C}_n := \{&(ij|K) : K \sqsubseteq [\pm n], ij \sqsubseteq [\pm n] \backslash K\overline{K}\} \cup \\ \cup \{&(i\overline{i}|K) : K \sqsubseteq [\pm n], |K| = n-1, i \in [\pm n] \backslash K\overline{K}\}.\end{aligned}$$

Definition 3.2.2. A C_n-*semigraphoid* is a subset $\mathcal{G} \subseteq \mathcal{C}_n$ which satisfies (SG), and for every $L \sqsubseteq [\pm n]$, $|L| = n-2$, $ij \sqsubseteq [\pm n] \backslash L\overline{L}$,

(CSG1) $\{(ij|L), (j\bar{j}|iL), (i\bar{j}|L)\} \subseteq \mathcal{G} \Rightarrow \{(\overline{ij}|L), (j\bar{j}|\bar{i}L), (\bar{i}j|L)\} \subseteq \mathcal{G}$,

(CSG2) $\{(i\bar{i}|jL), (\bar{i}j|L), (j\bar{j}|\bar{i}L)\} \subseteq \mathcal{G} \Rightarrow \{(i\bar{i}|\bar{j}L), (i\bar{j}|L), (j\bar{j}|iL)\} \subseteq \mathcal{G}$.

We also write $i \perp\!\!\!\perp j|K$ iff $(ij|K) \in \mathcal{G}$.

A C_n-semigraphoid $\mathcal{G}$ is a C_n-*semimatroid* if there is a bisubmodular function $f \colon \{S \sqsubseteq [\pm n]\} \to \mathbb{R}$ such that the equality in the local C_n-submodularity inequalities is attended exactly at the triples $(ij|K) \in \mathcal{G}$.

3.2.3 Type D

The root system D_n is

$$D_n = \{\pm \mathbf{e}_i \pm \mathbf{e}_j : 1 \leq i < j \leq n\} \subseteq \mathbb{R}^n.$$

We choose the simple roots of D_n to be

$$\Delta_{D_n} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_{n-1} + \mathbf{e}_n\}.$$

The corresponding fundamental weights are

$$\{\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \dots, \mathbf{e}_1 + \dots + \mathbf{e}_{n-2}, (\mathbf{e}_1 + \dots + \mathbf{e}_{n-1} - \mathbf{e}_n)/2, (\mathbf{e}_1 + \dots + \mathbf{e}_{n-1} + \mathbf{e}_n)/2\}.$$

The Weyl group of D_n is $W = W_{D_n} = \langle s_1, \dots, s_n \rangle \subseteq \mathrm{GL}(\mathbb{R}^n)$. It is generated by the simple reflections $s_1, \dots, s_{n-1}$, where $s_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is to exchange the i-th and the $(i+1)$-th coordinates for $i = 1, \dots, n-1$, and s_n, which is to exchange the $(n-1)$-th and the n-th coordinates and then change the signs of the $(n-1)$-th and n-th coordinates. The Weyl group W_{D_n} is a subgroup of W_{C_n} consisting of the evenly signed permutations. As a matrix group, it is the group of the permutation matrices with signed entries and with an even number of (-1)s.

The 2-faces of Π_{D_n} are either a square or a hexagon. The hexagons are corresponding to the cosets of parabolic subgroups $\langle s_i, s_{i+1} \rangle \subseteq W_{D_n}$, $1 \leq i \leq n-2$, and $\langle s_{n-2}, s_n \rangle$.

The fundamental weight conjugates are

$$\mathcal{R}_{D_n} = \{\mathbf{e}_S : \emptyset \neq S \sqsubseteq [\pm n], |S| \leq n-2\} \cup \{\frac{1}{2}\mathbf{e}_S : S \sqsubseteq [\pm n], |S| = n\}.$$

For $f \colon \mathcal{R}_{D_n} \to \mathbb{R}$ we write $f(\emptyset) = 0$, $f(S) = f(\mathbf{e}_S)$ for any $S \sqsubseteq [\pm n]$ with $|S| \leq n-2$, and $g(S) = f(\frac{1}{2}\mathbf{e}_S)$ for any $S \sqsubseteq [\pm n]$ with $|S| = n$. There are $2d(d-1)3^{d-2} - d(d-1)2^{d-2}$ local D_n-submodularity inequalities:

$$f(aS) + f(bS) \geq f(S) + f(abS) \qquad S \sqsubseteq [\pm n], |S| \leq n-4, ab \sqsubseteq [\pm n] \backslash S\overline{S},$$
$$f(aS) + f(bS) \geq f(S) + g(abcS) + g(ab\bar{c}S) \qquad S \sqsubseteq [\pm n], |S| = n-3, abc \sqsubseteq [\pm n] \backslash S\overline{S},$$
$$g(abS) + g(\overline{ab}S) \geq f(S) \qquad S \sqsubseteq [\pm n], |S| = n-2, ab \sqsubseteq [\pm n] \backslash S\overline{S}.$$

A function $f\colon \mathcal{R}_{D_n} \to \mathbb{R}$ satisfying D_n-submodularity inequalities is called *disubmodular* [ACEP20, § 5.2 and Theorem 7.2]. Remark that every generalized D_n-permutohedron is a generalized C_n-permutohedron because Σ_{D_n} coarsens Σ_{C_n}.

Definition 3.2.3. Let

$$\tilde{\mathcal{D}}_n := \{(ij|K) : K \sqsubseteq [\pm n], ij \sqsubseteq [\pm n]\backslash K\overline{K}\} \subseteq \mathcal{C}_n.$$

The set of D_n*-CI-statements* is

$$\mathcal{D}_n := \tilde{\mathcal{D}}_n/\sim$$

where $\sim$ is the equivalence relation in $\mathcal{D}_n$ defined by

$$\sim := \{((ij|K), (\overline{ij}|K)) \in \tilde{\mathcal{D}}_n \times \tilde{\mathcal{D}}_n : |K| = n-2\}.$$

By abusing of notations, we write an element of $\tilde{\mathcal{D}}_n$ for its class in $\mathcal{D}_n$. In other words, we identify $(ij|K)$ with $(\overline{ij}|K)$ for $|K| = n-2$.

Remark 3.2.1. As before, we also denote by the descent vector $(\delta(1)|\cdots|\delta(n))$ the vertex $(\delta^{-1}(n)^\flat, \dots, \delta^{-1}(1)^\flat)$ of Π_{D_n}, where $k^\flat$ is defined to be $k-1$ if $k>0$, and $k+1$ if $k<0$. For $|K| = n-3$, the CI-statement $(ij|K)$ denotes the orbit of an edge $(K|i|j|\ell) - (K|j|i|\ell)$ (a coset of $\langle s_{n-2}\rangle$) under the action of $\mathfrak{S}_{n-3} = W_{[n-4]} = W_{[n]\backslash N(\{n-2\})}$. The last entry $\ell = \delta(n)$ for this vertex $\delta \in W_{\mathcal{D}_n}$ is uniquely determined because $W_{\mathcal{D}_n}$ is the group of evenly signed permutations.

For $|K| = n-2$, if the number of negation signs in $(ij|K)$ is even, then $(ij|K)$ denotes the orbit of an edge $(K|i|j) - (K|j|i)$ under the action of $\mathfrak{S}_{n-2} \times \langle s_n\rangle = W_{[n-3]\cup\{n\}}$. The reason for taking equivalence classes is that the simple reflection s_n is not adjacent to s_{n-1} in the Dynkin diagram. If the number of negation signs in $(ij|K)$ is odd, it denotes the orbit of an edge $(K|i|\bar{j}) - (K|j|\bar{i})$ under the action of $\mathfrak{S}_{n-2} \times \langle s_{n-1}\rangle$.

This notation can simplify the definition of a semigraphoid of type D. It is exactly same as the semigraphoid axiom of type A. One needs only to check the hexagons

$$(L|i|j|k) - (L|j|i|k) - (L|j|\bar{k}|\bar{i}) - (L|\bar{k}|j|\bar{i}) - (L|\bar{k}|i|\bar{j}) - (L|i|\bar{k}|\bar{j}) - (L|i|j|k)$$

corresponding to the cosets of the parabolic subgroup $\langle s_{n-2}, s_n\rangle$.

Figure 3.2 is a part of Π_{D_3}. Although Π_{D_3} is accidentally same as Π_{A_3}, its labeling reveals the structure of Π_{D_n} and D_n-CI-statements for larger n.

Definition 3.2.4. A D_n*-semigraphoid* is a subset $\mathcal{G} \subseteq \mathcal{D}_n$ satisfying

(DSG) $\{(ij|L), (ik|jL)\} \subseteq \mathcal{G} \Rightarrow \{(ik|L), (ij|kL)\} \subseteq \mathcal{G}$,

A D_n-semigraphoid $\mathcal{G}$ is a D_n*-semimatroid* if there is a disubmodular function $f\colon \mathcal{R}_{D_n} \to \mathbb{R}$ such that the equality in the local D_n-submodularity inequalities is attained exactly at the elements of $\mathcal{G}$.

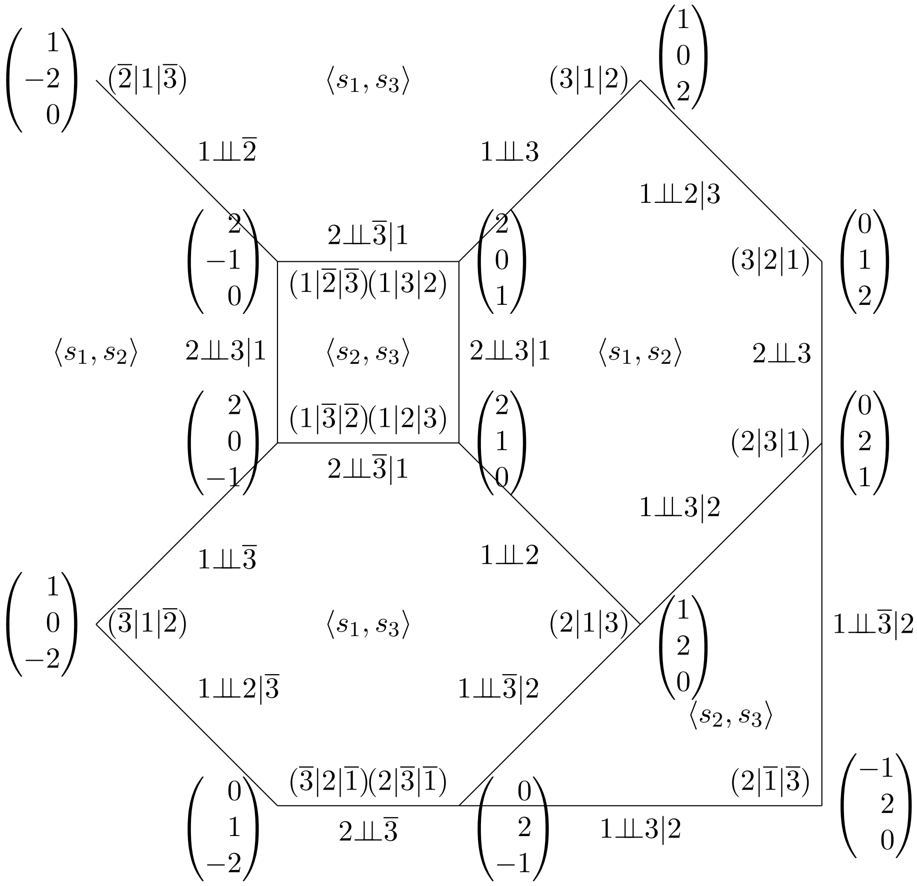


Figure 3.2: a piece of Π_{D_3}

The Coxeter complex Σ_{D_n} is a coarsening of Σ_{C_n} from which all walls whose normal vectors have form $\mathbf{e}_i$ are removed. Since being a coarsening is a transitive relation, any D_n-semigraphoid is also a C_n-semigraphoid. Conversely, if all C_n-CI-statements of the form $(i\bar{i}|K) \in \mathcal{C}_n$ are in a C_n-semigraphoid, then by (CSG2), $i \perp\!\!\!\perp j|K$ iff $\bar{i} \perp\!\!\!\perp \bar{j}|K$ for all $(ij|K) \in \mathcal{C}_n$ with $|K| = n-2$. What we got is exactly an D_n-semigraphoid. In other words, the D_n-semigraphoids are exactly the C_n-semigraphoids in which $i \perp\!\!\!\perp \bar{i}|K$ always hold for all $(i\bar{i}|K) \in \mathcal{C}_n$. Similarly, the D_n-semimatroids are exactly the C_n-semimatroids in which $i \perp\!\!\!\perp \bar{i}|K$ always hold for all $(i\bar{i}|K) \in \mathcal{C}_n$.

3.3 Combinatorial Erlangen Program

We have defined the Φ-semigraphoids and the Φ-semimatroids. As an application to the Combinatorial Erlangen Program, we deduce the analogues of Theorem 2.4.1 in types B, C and D, namely, an axiomatization of delta-matroids as C_n-CI-structures, and of orthogonal delta-matroids as D_n-CI-structures.

3.3.1 Coxeter matroids

In this subsection we explain that Coxeter matroids are kaleidoscopes which generate only finitely many mirror images. A polytope is a *Coxeter matroid polytope* if the group generated by the reflections in the mirrors of the symmetries of all edges is finite.

If $P \subseteq \mathbb{R}^d$ is a Coxeter matroid polytope, let $W = W(P)$ be the group in the definition, called the *exchange group* of P. Let $p \in \mathbb{R}^d$ be an arbitrary point. The barycenter $b = \frac{1}{|W|}\sum_{w \in W} w \cdot p$ of the orbit $W \cdot p$ is fixed by W because for any $w' \in W$,

$$w' \cdot b = w' \cdot \frac{1}{|W|}\sum_{w \in W} w \cdot p = \frac{1}{|W|}\sum_{w \in W} w'w \cdot p = b.$$

Moreover, the barycenter b lies on every mirror. Therefore, without loss of generality we can assume that the group W fixes the origin of $\mathbb{R}^d$ and hence is a linear group. Moreover it is a finite reflection group and hence a finite Coxeter group. All vertices of P belong to one W-orbit because the edge graph of P is connected.

Let Φ be the root system consisting of two unit vectors perpendicular to every mirror and Σ_Φ be the Coxeter complex of Φ. Choose a vertex δ of P and take a chamber D whose closure contains δ as the fundamental domain of Σ_Φ. Let $\Delta = \{\alpha_1, \ldots, \alpha_d\} \subseteq \Phi$ be the simple roots according to the fundamental domain D. By Theorem 3.1.3, the stabilizer subgroup of δ is a parabolic subgroup W_I of W, that is, it is generated by some simple reflections s_α, $\alpha \in I \subseteq \Delta$. Therefore, the set of vertices of P corresponds bijectively to a subset M of the quotient group W/W_I, and every edge of P is parallel to a root in Φ.

The converse also holds. Let W be a Weyl group and W_I be a parabolic subgroup of W. Fix a point δ in the relative interior of C_I. A subset $M \subseteq W/W_I$ is a *Coxeter matroid* if every edge of the polytope $Q := \mathrm{conv}\{w \cdot \delta : wW_I \in M\}$ is parallel to a root in Φ.

The Gelfand-Serganova Theorem states that the polytope Q is a Coxeter matroid polytope if M is a Coxeter matroid. By Theorem 3.1.3, the choice of the point $\delta \in C_I$ do not affect the normal fan Σ_Q of Q. We will choose δ as $\lambda_I := \sum_{\alpha \notin I} \lambda_\alpha$ and call

$$Q = Q(M) := \mathrm{conv}\{w \cdot \lambda_I : wW_I \in M\}$$

the *(canonical) Coxeter matroid polytope* or the *basis polytope* of M. The vertices of $Q(M)$ are the *bases* of M. We identify the coset $wW_I \in W/W_I$ with the point $w \cdot \lambda_I$.

Example 3.3.1. For $W = W_{A_{n-1}} \cong \mathfrak{S}_n$ and a maximal parabolic subgroup W_I where $I = \Delta_{A_{n-1}} \backslash \{\mathbf{e}_r - \mathbf{e}_{r+1}\}$, the Coxeter matroids are exactly the matroids of rank r on $[n]$. The basis polytope of such a Coxeter matroid is the convex hull of a subset of $W/W_I = \{\sigma \cdot (\bar{\mathbf{e}}_1 + \cdots + \bar{\mathbf{e}}_r) : \sigma \in \mathfrak{S}_n\}$ such that every edge is parallel to some A_{n-1} root $\mathbf{e}_i - \mathbf{e}_j$. This is exactly the basis polytope of a matroid [GGMS87].

Example 3.3.2. For $W = W_{C_n}$ and a maximal parabolic subgroup W_I with $I = \Delta_{C_n} \backslash \{\alpha_r\}$, where $\alpha_r = \mathbf{e}_r - \mathbf{e}_{r+1}$ for $r < n$ and $\alpha_n = 2\mathbf{e}_n$, we have

$$\begin{aligned} W/W_I &= \{\sigma \cdot (\mathbf{e}_1 + \cdots + \mathbf{e}_r) : \sigma \in W\} \\ &= \{\pm \mathbf{e}_{i_1} \pm \cdots \pm \mathbf{e}_{i_r} : 1 \leq i_1 < \cdots < i_r \leq n\} \\ &= \{x \in \{-1, 0, 1\}^n : ||x||_1 = r\}. \end{aligned}$$

The *basis polytope* $\mathcal{P}_M$ of a *symplectic matroid* M of rank r on $[n]$ is a polytope in $\mathbb{R}^n$ such that all vertices of $\mathcal{P}_M$ are in $\{-1, 0, 1\}^n$ and have 1-norm r, and all edges of $\mathcal{P}_M$ are parallel to the C_n roots. The *bases* of M are

$$\mathcal{B}(M) = \{S \sqsubseteq [\pm n] : \mathbf{e}_S \in \mathrm{Vert}(\mathcal{P}_M)\}.$$

An *independent set* of M is a subset of some basis of M. A *Lagrangian matroid* is a symplectic matroid of full rank.

Example 3.3.3. For $W = W_{D_n}$ and W_I a maximal parabolic subgroup, the Coxeter matroids are the *orthogonal matroids*. Because W_{D_n} is a subgroup of $W_{C_n} \cong W_{B_n}$ and the fundamental weight conjugates $\mathcal{R}_{D_n}$ is a subset of

$$\mathcal{R}_{B_n} = \{\mathbf{e}_S : \emptyset \neq S \sqsubseteq [\pm n], |S| \leq n-1\} \cup \{\frac{1}{2}\mathbf{e}_S : S \sqsubseteq [\pm n], |S| = n\},$$

every orthogonal matroid is a symplectic matroid. In particular, if $I = \Delta_{D_n} \backslash \{\alpha_{n-1}\}$, then the Coxeter matroids are the Lagrangian matroids of which every basis has an odd number of negated elements, and they are the Lagrangian matroids whose every basis has an even number of negated elements for $I = \Delta_{D_n} \backslash \{\alpha_n\}$. We refer to [BGW03, § 3 and § 4] for symplectic, orthogonal and Lagrangian matroids.

3.3.2 Delta-matroids as CI-structures of type C

General Coxeter matroids behave much worse than matroids. For instance, matroids have hundreds of equivalent axiom systems, but only about five axiom systems are known for symplectic matroids. Lagrangian matroids are the full rank Coxeter matroids for the root system B_n or C_n with respect to the parabolic subgroup generated by the first $n-1$ simple reflections. Lagrangian matroids are in several ways more closely related to matroids than general symplectic matroids. Many axiom systems of matroids have counterparts for Lagrangian matroids. Lagrangian matroids are introduced and studied from different perspectives in the literature under various names, we refer to the beginning of [BGW03, § 4.1] and the last remark in [Mof19, § 2.2] for a partial list of notions that are (almost) equivalent to Lagrangian matroids. In this subsection we use the equivalent notion of delta-matroids and give a new axiomatization as CI-structures of type B or C. As a direct corollary we get an axiomatization of orthogonal delta-matroids as CI-structures of type D.

A set family $\mathcal{B} \subseteq 2^{[n]}$ is the *bases* (or *feasible sets*) of a *delta-matroid* D on $[n]$ if it satisfies the following *symmetric exchange axiom*:

(ΔB) For all $B_1, B_2 \in \mathcal{B}$ and any $u \in B_1 \triangle B_2$, there exists $v \in B_1 \triangle B_2$ such that $B_1 \triangle \{u, v\} \in \mathcal{B}$.

Note that u, v may be same in the condition (ΔB). If u and v are not allowed to be same, then the delta-matroid is *orthogonal*, that is, it is of type D, see Example 3.3.3. The bases $\mathcal{B}$ of orthogonal delta-matroids can be classified by the following equivalent but seemingly stronger *strong exchange axiom* [BGW03, Theorem 4.2.4].

($\perp\Delta$B) For all $B_1, B_2 \in \mathcal{B}$ and any $u \in B_1 \triangle B_2$, there exists $v \in B_1 \triangle B_2$ with $v \neq u$ such that $B_1 \triangle \{u, v\}, B_2 \triangle \{u, v\} \in \mathcal{B}$.

We write $\mathcal{B}(D)$ for the set of bases of a delta-matroid D. Delta-matroids are exactly the Lagrangian matroids if we exclude the negative elements in each basis. The excluded negative elements are clear due to the full symplectic matroid rank. The *delta-matroid polytope* $\mathcal{P}_D$ of D is the convex hull of the indicator vectors of the bases of D. It is exactly the corresponding Lagrangian matroid polytope translated by $\mathbf{1} = (1, \ldots, 1)^\top$ and then dilated by the factor $\frac{1}{2}$. In particular, they share the same normal fan and the same semimatroid. The delta-matroid polytopes are exactly the polytopes whose vertices have coordinates either 0 or 1 and whose edges are parallel to $\mathbf{e}_i$, $\mathbf{e}_i + \mathbf{e}_j$ or $\mathbf{e}_i - \mathbf{e}_j$. A delta-matroid polytope is an orthogonal delta-matroid polytope iff no edge is parallel to any $\mathbf{e}_i$.

The *rank function* $r_D \colon \{S \sqsubseteq [\pm n]\} \to \mathbb{Z}$ of a delta-matroid D on $[n]$ is

$$r_D(S) = h_{\mathcal{P}_D}(\mathbf{e}_S) = \max_{B \in \mathcal{B}(D)} \left(|S \cap B| - |\overline{S} \cap B| \right),$$

where $h_{\mathcal{P}_D} \colon \mathcal{R}_{C_n} \to \mathbb{R}$ is the support function of the delta-matroid polytope $\mathcal{P}_D$. The set of bases $\mathcal{B}(D)$ of a delta-matroid D on $[n]$ can be recovered from the rank function r_D by

$$\mathcal{B}(D) = \left\{ S \subseteq [n] : r_D(S\overline{[n] \setminus S}) = |S| \right\}.$$

Delta-matroids can be defined equivalently using rank functions as follows.

Theorem 3.3.4 ([CK88, Theorem 3]). *A function $r \colon \{S \sqsubseteq [\pm n]\} \to \mathbb{Z}$ is the rank function of a delta-matroid on $[n]$ iff it satisfies*

(Δr1) *$r(\emptyset) = 0$ and $r(i) \leq 1$ for all $i \in [n]$,*

(Δr2) *r is* bimonotonic, *that is, $r(S\overline{T}) \leq r(S'\overline{T'})$ for all $S \subseteq S' \subseteq [n]$, $T' \subseteq T \subseteq [n]$ such that $S \cap T = S' \cap T' = \emptyset$,*

(Δr3) *r is bisubmodular.*

We define $[[D]]$ to be the C_n-semimatroid of the delta-matroid polytope $\mathcal{P}_D$. Explicitly,

$$\begin{aligned}[[D]] := \{(ij|K) \in \mathcal{C}_n : i \neq \bar{j}, r_D(iK) + r_D(jK) = r_D(ijK) + r_D(K)\} \cup \\ \cup \{(i\bar{i}|K) \in \mathcal{C}_n : r_D(iK) + r_D(\bar{i}K) = 2r_D(K)\}.\end{aligned}$$

An element $i \in [n]$ is a *loop* of D if i is in none of the bases of D, and a *coloop* if every basis of D contains i. If D is loopless, the rank function r_D can be recovered from the C_n-semimatroid $[[D]]$ by $r_D(\emptyset) = 0$, $r_D(i) = 1$ for all $i \in [n]$, and

$$r_D(\bar{i}) = \begin{cases} -1 & \text{if } (ij|K), (\bar{i}j|K), (i\bar{i}|L) \in [[D]] \quad \forall\, (ij|K), (\bar{i}j|K), (i\bar{i}|L) \in \mathcal{C}_n \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

for all $i \in [n]$, and recursively for $ijK \sqsubseteq [n]$,

$$r_D(ijK) = \begin{cases} r_D(iK) + r_D(jK) - r_D(K) & (ij|K) \in [[D]], \\ r_D(iK) + r_D(jK) - r_D(K) - 1 & (ij|K) \notin [[D]]. \end{cases} \tag{3.3}$$

The assumption of looplessness is not essential: changing a loop by a coloop or changing a coloop by a loop corresponds to a translation of the delta-matroid polytope. The normal fan of the delta-matroid polytope and the C_n-semimatroid stay unchanged. Note that for a loopless delta-matroid D and $i \in [n]$, $r_D(\bar{i}) = -1$ iff i is a coloop of D, iff $\mathcal{P}_D$ is in the hyperplane $\{x_i = 1\}$, iff $\mathrm{span}(\mathbf{e}_i)$ is in the lineality space of each cone of the normal fan of $\mathcal{P}_D$, iff any C_n-CI-statements of any of the forms $(ij|K)$, $(\bar{i}j|K)$ and $(i\bar{i}|L)$ is in $[[D]]$. This gives the condition for $r_D(\bar{i}) = -1$ in (3.2).

The following criteria for C_n-conditional dependence in a delta-matroid follows easily from the properties of delta-matroid rank functions.

Lemma 3.3.5. *Let D be a delta-matroid on $[n]$.*

(1) *If $(i\bar{i}|K) \in \mathcal{C}_n$ and $i > 0$, then*

$$(i\bar{i}|K) \notin [[D]] \quad \Leftrightarrow \quad r_D(\bar{i}K) = r_D(K) = r_D(iK) - 1.$$

(2) *If $(ij|K) \in \mathcal{C}_n$, $ijK \sqsubseteq [\pm n]$ and $i, j > 0$, then*

$$(ij|K) \notin [[D]] \quad \Leftrightarrow \quad r_D(iK) = r_D(jK) = r_D(ijK) = r_D(K) + 1.$$

(3) *If $(ij|K) \in \mathcal{C}_n$, $ijK \sqsubseteq [\pm n]$ and $i > 0$, $j < 0$ then*

$$(ij|K) \notin [[D]] \quad \Leftrightarrow \quad r_D(iK) - 1 = r_D(jK) = r_D(ijK) = r_D(K).$$

(4) *If $(ij|K) \in \mathcal{C}_n$, $ijK \sqsubseteq [\pm n]$ and $i, j < 0$, then*

$$(ij|K) \notin [[D]] \quad \Leftrightarrow \quad r_D(iK) = r_D(jK) = r_D(ijK) + 1 = r_D(K).$$

Now we give a cryptomorphic axiomatization for delta-matroids as C_n-semigraphoids.

Theorem 3.3.6. *Let $\mathcal{G} \subseteq \mathcal{C}_n$ be a C_n-semigraphoid. Then there exists a delta-matroid D on $[n]$ such that $\mathcal{G} = [[D]]$ iff $\mathcal{G}$ satisfies*

(ΔMCI) $(ij|K) \in \mathcal{C}_n \backslash \mathcal{G} \Rightarrow (\bar{\imath}j|K), (i\ell|jKL), (\bar{\imath}\ell|jKL), (i\bar{\imath}|jKL') \in \mathcal{G}$.

Moreover, the correspondence between the C_n-semigraphoids satisfying (ΔMCI) and loopless delta-matroids is one-to-one.

Proof. Let D be a delta-matroid on $[n]$. We show that $[[D]]$ satisfies (ΔMCI).

Assume that $(ij|K), (i\ell|jKL) \in \mathcal{C}_n \backslash [[D]]$ (possibly $\ell = \bar{\imath}$). By Lemma 3.3.5, if $i > 0$, then $r_D(jK) = r_D(ijK)$ and $r_D(ijKL) = r_D(jKL) + 1$. If $i < 0$, then $r_D(jK) = r_D(ijK) + 1$ and $r_D(ijKL) = r_D(jKL)$. In both cases we got $r_D(ijK) + r_D(jKL) < r_D(ijKL) + r_D(jK)$, which contradicts the bisubmodularity of r_D.

Now assume that $(ij|K), (\bar{\imath}\ell|jKL) \in \mathcal{C}_n \backslash [[D]]$. Again by Lemma 3.3.5, if $i > 0$, then $r_D(jK) = r_D(ijK)$ and $r_D(\ell jKL) = r_D(\bar{\imath}\ell jKL) + 1$. But by the bisubmodularity of r_D, we have

$$r_D(\ell jKL) + 1 \geq r_D(i\ell jKL) \geq 2r_D(\ell jKL) - r_D(\bar{\imath}\ell jKL) \geq r_D(\ell jKL) + 1,$$

thus $r_D(\ell jKL) + 1 = r_D(i\ell jKL)$. Together with $r_D(jK) = r_D(ijK)$ we have $r_D(\ell jKL) + r_D(ijK) < r_D(i\ell jKL) + r_D(jK)$, a contradiction to the bisubmodularity. If $i < 0$, we have $r_D(\bar{\imath}\ell jKL) = r_D(\ell jKL)$ and $r_D(jK) = r_D(ijK) + 1$. Analogously we get $r_D(i\ell jKL) = r_D(\ell jKL)$ and the contradiction.

Now let $(ij|K), (\bar{\imath}j|K) \in \mathcal{C}_n \backslash [[D]]$, $i > 0$. By Lemma 3.3.5 we have $r_D(jK) = r_D(ijK)$ and $r_D(jK) = r_D(\bar{\imath}jK) + 1$. Then $r_D(ijK) + r_D(\bar{\imath}jK) = 2r_D(jK) - 1$, a contradiction to the bisubmodularity.

Now let $\mathcal{G} \subseteq \mathcal{C}_n$ be a C_n-semigraphoid satisfying (ΔMCI). Let $r\colon \{S \sqsubseteq [\pm n]\} \to \mathbb{Z}$ be the function defined recursively by (3.2) and (3.3). We need to show that r is uniquely defined and is the rank function of a delta-matroid D, and $\mathcal{G} = [[D]]$.

To show the well-definedness of $r(S)$ we proceed by induction on the cardinality c of S. For the cases $c \in \{0, 1\}$ the function value $r(S)$ is given by the initial condition and (3.2). For $c = 2$, $r(ij) = r(i) + r(j)$ if $(ij|) \in \mathcal{G}$, and $r(ij) = r(i) + r(j) - 1$ if $(ij|) \notin \mathcal{G}$. Assume that $c \geq 3$ and $r(S)$ is uniquely defined for any S with $|S| < c$. What is left to check is that we got the same value $r(ijkL)$ for $|ijkL| = c$ by applying (3.3) to different C_n-conditional (in)dependence statements, namely,

$$\begin{aligned}
&\begin{cases} r(ikL) + r(jkL) - r(kL) & (ij|kL) \in \mathcal{G} \\ r(ikL) + r(jkL) - r(kL) - 1 & (ij|kL) \notin \mathcal{G} \end{cases} \\
= &\begin{cases} r(ijL) + r(jkL) - r(jL) & (ik|jL) \in \mathcal{G} \\ r(ijL) + r(jkL) - r(jL) - 1 & (ik|jL) \notin \mathcal{G}. \end{cases}
\end{aligned}$$

1. $(ij|kL) \in \mathcal{G}$ and $(ik|jL) \notin \mathcal{G}$: From (SG) we have $(ik|L) \notin \mathcal{G}$. By $(ik|jL) \notin \mathcal{G}$ and (ΔMCI) we have $(ij|L) \in \mathcal{G}$. Therefore

   $$r(ikL)+r(jkL)-r(kL) = r(iL)-r(L)-1+r(jkL) = r(ijL)+r(jkL)-r(jL)-1.$$

   For the case $(ij|kL) \notin \mathcal{G}$ and $(ik|jL) \in \mathcal{G}$ it follows by symmetry.

2. $(ij|kL), (ik|jL) \notin \mathcal{G}$: By ($\Delta$MCI) we have $(ij|L), (ik|L) \in \mathcal{G}$.

3. $(ij|kL), (ik|jL) \in \mathcal{G}$: If $(ij|L) \in \mathcal{G}$, by (SG) we have $(ik|L) \in \mathcal{G}$. If $(ij|L) \notin \mathcal{G}$, we have $(ik|L) \notin \mathcal{G}$ instead by (SG). In both cases the equality follows.

Now we show that r is the rank function of a delta-matroid. By Theorem 3.3.4, we need to show that r is bimonotonic and bisubmodular.

We define the function $\delta\colon \mathcal{C}_n \to \{0,1\}$ by $\delta(ij|K) = 0$ if $(ij|K) \in \mathcal{G}$ and $\delta(ij|K) = 1$ otherwise. Then for any $is_1 \cdots s_r \sqsubseteq [\pm n]$,

$$\begin{aligned} & r(is_1 \cdots s_r) - r(s_1 \cdots s_r) = r(is_2 \cdots s_r) - r(s_2 \cdots s_r) - \delta(is_1|s_2 \cdots s_r) \\ = {} & \cdots = r(i) - \delta(is_1|s_2 \cdots s_r) - \delta(is_2|s_3 \cdots s_r) - \cdots - \delta(is_r|). \end{aligned}$$

The condition (ΔMCI) implies that

$$\delta(is_1|s_2 \cdots s_r) + \delta(is_2|s_3 \cdots s_r) + \cdots + \delta(is_r|) \in \{0,1\}.$$

Therefore, $r(is_1 \cdots s_r) - r(s_1 \cdots s_r) \in \{0,1\}$ if $i > 0$, and $r(is_1 \cdots s_r) - r(s_1 \cdots s_r) \in \{0,-1\}$ if $i < 0$. So is the bimonotonicity proven. For the bisubmodularity it is left to show that

$$r(is_1 \cdots s_{n-1}) + r(\bar{i}s_1 \cdots s_{n-1}) \geq 2r(s_1 \cdots s_{n-1})$$

for any $is_1 \cdots s_{n-1}, \bar{i}s_1 \cdots s_{n-1} \sqsubseteq [\pm n]$. Say, $i > 0$, then

$$\begin{aligned} & r(is_1 \cdots s_{n-1}) + r(\bar{i}s_1 \cdots s_{n-1}) - 2r(s_1 \cdots s_{n-1}) \\ = {} & r(i) + r(\bar{i}) - \sum_{j=1}^{n-1} \delta(is_j|s_{j+1} \cdots s_{n-1}) - \sum_{j=1}^{n-1} \delta(\bar{i}s_j|s_{j+1} \cdots s_{n-1}). \end{aligned} \tag{3.4}$$

If $r(\bar{i}) = -1$, by definition (3.2), i and $\bar{i}$ are C_n-independent from anything under any condition. Therefore

$$r(is_1 \cdots s_{n-1}) + r(\bar{i}s_1 \cdots s_{n-1}) - 2r(s_1 \cdots s_{n-1}) = 1 - 1 = 0.$$

If $r(\bar{i}) = 0$, then (ΔMCI) implies that

$$|\{(is_j|s_{j+1} \cdots s_{n-1}), (\bar{i}s_j|s_{j+1} \cdots s_{n-1}) : j \in [n-1]\}| \leq 1,$$

therefore,

$$r(is_1 \cdots s_{n-1}) + r(\bar{i}s_1 \cdots s_{n-1}) - 2r(s_1 \cdots s_{n-1}) \in \{0,1\}.$$

We have shown that a C_n-semigraphoid $\mathcal{G}$ satisfying (ΔMCI) defines a function r by (3.2) and (3.3) which is the rank function of a delta-matroid D. Now we check

$[[D]] = \mathcal{G}$. It is clear by (3.3) that $(ij|K) \in \mathcal{G}$ iff $(ij|K) \in [[D]]$ for $(ij|K) \in \mathcal{C}_n$ with $i \neq \bar{j}$. What is left to show is $(i\bar{i}|S) \in \mathcal{G}$ iff $(i\bar{i}|S) \in [[D]]$. Say again, $S = s_1 \cdots s_{n-1}$. If $(i\bar{i}|S) \notin \mathcal{G}$, then by (3.2) and (ΔMCI), the right hand side of (3.4) equals $1+0-0=1$, therefore, $(i\bar{i}|S) \notin [[D]]$.

At last, suppose $(i\bar{i}|S) \in \mathcal{G}$ and $(i\bar{i}|S) \notin [[D]]$. By (3.4), necessarily we have $r(\bar{i}) = 0$ and $(ij|K), (\bar{i}j|K) \in \mathcal{G}$ for any $j \in S$ and $K \subseteq S\backslash j$. We will get a contradiction to $r(\bar{i}) = 0$ by showing that any C_n-CI-statement of any of the forms $(i\bar{i}|T), (ij|K), (\bar{i}j|K) \in \mathcal{C}_n$ is in $\mathcal{G}$. We proceed by induction on the number c of elements in T or jK, respectively, whose negations are in S. The case $c = 0$ is done. Assume that for all $(i\bar{i}|T), (ij|K), (\bar{i}j|K) \in \mathcal{C}_n$ with $|T \cap \overline{S}| = c$, $(i\bar{i}|T), (ij|K), (\bar{i}j|K) \in [[D]]$. Let $(i\bar{i}|T')$ be any element in $\mathcal{C}_n$ such that $|T' \cap \overline{S}| = c+1$ and let $\overline{t_0} \in T' \cap \overline{S}$. By the induction hypothesis, $(i\bar{i}|t_0T'\backslash\overline{t_0}), (it_0|T'\backslash\overline{t_0})$ and $(\bar{i}t_0|T'\backslash\overline{t_0})$ are in $\mathcal{G}$. Then $(i\bar{i}|T') \in \mathcal{G}$ by (CSG1). Now consider $(i\bar{j}|K'), (\overline{ij}|K') \in \mathcal{C}_n$ with $K' \cap \overline{S} = c$ and $j \in S$. Write $\ell_1 \cdots \ell_d = S\backslash(jK'\overline{jK'})$. By the induction hypothesis, $(ij|K'\ell_1 \cdots \ell_d)$, $(\bar{i}j|K'\ell_1 \cdots \ell_d)$ and $(i\bar{i}|jK'\ell_1 \cdots \ell_d)$ are elements of $\mathcal{G}$. Together with (CSG1) we have $(i\bar{j}|K'\ell_1 \cdots \ell_d), (\overline{ij}|K'\ell_1 \cdots \ell_d) \in \mathcal{G}$. By the induction hypothesis, we have

$$(i\ell_1|K'\ell_2 \cdots \ell_d), (i\ell_2|K'\ell_3 \cdots \ell_d), \ldots, (i\ell_d|K'), (\bar{i}\ell_1|K'\ell_2 \cdots \ell_d), \ldots, (\bar{i}\ell_d|K') \in \mathcal{G}.$$

Applying (SG) d times for each of $(i\bar{j}|K'\ell_1 \cdots \ell_d), (\overline{ij}|K'\ell_1 \cdots \ell_d) \in \mathcal{G}$ yields that $(i\bar{j}|K'), (\overline{ij}|K') \in \mathcal{G}$. Now consider $(ij|K'), (\bar{i}j|K') \in \mathcal{C}_n$ with $j \in S$ and $|K' \cap \overline{S}| = c+1$. Choose any $\bar{k} \in K' \cap \overline{S}$. It was shown that $(i\bar{k}|jK'\backslash\bar{k})$ and $(\bar{i}\bar{k}|jK'\backslash\bar{k})$ are in $\mathcal{G}$. By the induction hypothesis $(ij|K'\backslash\bar{k})$ and $(\bar{i}j|K'\backslash\bar{k})$ and (SG) we get $(ij|K'), (\bar{i}j|K') \in [[D]]$. This finishes the proof. □

Corollary 3.3.7. *Let $\mathcal{G} \subseteq \mathcal{D}_n$ be a D_n-semigraphoid. Then there exists an orthogonal delta-matroid D on $[n]$ such that $\mathcal{G} = [[D]]$ iff $\mathcal{G}$ satisfies*

(⊥ΔMCI) $(ij|K) \in \mathcal{D}_n\backslash\mathcal{G} \Rightarrow (\bar{i}j|K), (i\ell|jKL), (\bar{i}\ell|jKL) \in \mathcal{G}$.

Moreover, the correspondence between the D_n-semigraphoids satisfying (⊥ΔMCI) and loopless orthogonal delta-matroids is one-to-one.

Proof. A loopless delta-matroid on $[n]$ is orthogonal iff its delta-matroid polytope is a generalized D_n-permutohedron. By Theorem 3.3.6, loopless orthogonal delta-matroids on $[n]$ are the C_n-semigraphoids satisfying (ΔMCI) and containing all $(i\bar{i}|K) \in \mathcal{C}_n$. In other words, they are precisely the D_n-semigraphoids satisfying (⊥ΔMCI). □

3.4 The geometry of generalized C_n-permutohedra

In this section we study the geometry of generalized C_n-permutohedra. We write every generalized permutohedron of type B, C or D explicitly as a signed Minkowski sum of rank 1 symplectic matroid basis polytopes. We also discuss its application to the connectedness of type B or C, and give volume formulas using the basic properties of mixed volumes, and point out the connection of mixed volumes to the marriage theorems in transversal theory.

3.4.1 A symplectic-matroidal basis of generalized C_n-permutohedra

The standard simplices $\Delta_S = \mathrm{conv}(\mathbf{e}_i : i \in S)$, $\emptyset \neq S \subseteq [n]$ are the basis polytopes of rank 1 matroids on $[n]$. The support functions of the $2^n - 1$ standard simplices Δ_S, $\emptyset \neq S \subseteq [n]$ form a basis of $\mathbb{R}^{2^n-1}$, that means, every generalized permutohedron is a signed Minkowski sum of these simplices as in Proposition 2.1.13. But this is not the case for type B or C. The $3^n - 1$ nonempty faces of the crosspolytope $\mathrm{conv}(\mathbf{e}_i : i \in [\pm n])$ only span a subspace of dimension $\frac{1}{2}(3^n - (-1)^n)$ in $\mathbb{R}^{\mathcal{R}_{C_n}} \cong \mathbb{R}^{3^n-1}$ [Dok11, Corollary 5.3.10]. Moreover, every basis of $\mathbb{R}^{\mathcal{R}_{C_n}}$ contains at least 2^{n-1} full dimensional polytopes [Bas21, Proposition 6.8]. However, the support functions of the $4^n - 1$ rank 1 symplectic matroid basis polytopes $\Delta_S := \mathrm{conv}(\mathbf{e}_i : i \in S)$, $\emptyset \neq S \subseteq [\pm n]$ span the space $\mathbb{R}^{\mathcal{R}_{C_n}}$. In particular, we describe a basis consisting of $3^n - 1$ rank 1 symplectic matroid basis polytopes, which gives an answer to [ACEP20, Question 9.3].

The following three bases of $\mathbb{R}^{\mathcal{R}_{C_n}}$ are already found, namely, the support functions of

- type B shard polytopes [PPR23, Corollary 149],
- $\{\Delta_S, \Delta_S^0 : S \sqsubseteq [\pm n], \max S = \max S\overline{S}\}$ [Bas21, Theorem 6.9],
- $\{\Delta_S^0 : \emptyset \neq S \sqsubseteq [\pm n]\}$ [Bas21, Theorem 6.11][EFLS24, Theorem A(a)],

where $\Delta_S^0 := \mathrm{conv}(\mathbf{0}, \mathbf{e}_i : i \in S)$ for $\emptyset \neq S \subseteq [\pm n]$ are the *independent set polytopes* of rank 1 symplectic matroids. In Theorem 3.4.2 we give the exchange matrix between the basis in Theorem 3.4.1 and the standard basis of $\mathbb{R}^{\mathcal{R}_{C_n}}$ explicitly. No previous work gives explicitly the exchange matrices between two bases. Combining with [EFLS24, Theorem A(b)], we can get an explicit formula for the volume of an arbitrary generalized C_n-permutohedron, which answers [ACEP20, Question 9.3] for types B and C completely.

Theorem 3.4.1. *The support functions of the $3^n - 1$ rank 1 symplectic matroid basis polytopes $\{\Delta_{S_1\overline{S_1}S_2} : \emptyset \neq S_1S_2 \subseteq [n]\}$ form a basis of $\mathbb{R}^{\mathcal{R}_{C_n}}$.*

Proof. We denote by $\mathfrak{M} := \{S_1\overline{S_1}S_2 : \emptyset \neq S_1S_2 \subseteq [n]\}$. For every $S_1\overline{S_1}S_2 \in \mathfrak{M}$, let $h_{S_1\overline{S_1}S_2} : \mathcal{R}_{C_n} \to \mathbb{R}$ be the support function of the polytope $\Delta_{S_1\overline{S_1}S_2}$. Explicitly,

$$h_{S_1\overline{S_1}S_2}(T) = \max_{i \in S_1\overline{S_1}S_2} \langle \mathbf{e}_i, \mathbf{e}_T \rangle = \begin{cases} 1 & \text{if } (S_1\overline{S_1}S_2) \cap T \neq \emptyset \\ -1 & \text{if } S_1\overline{S_1}S_2 \subseteq \overline{T} \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

for all $T \sqsubseteq [\pm n]$. We want to show that the $3^n - 1$ support functions $h_{S_1\overline{S_1}S_2}$, $S_1\overline{S_1}S_2 \in \mathfrak{M}$, are linearly independent in the (3^n-1)-dimensional vector space $\mathbb{R}^{\mathcal{R}_{C_n}}$. That is,

$$\sum_{S_1\overline{S_1}S_2 \in \mathfrak{M}} \alpha_{S_1\overline{S_1}S_2} h_{S_1\overline{S_1}S_2}(T) = 0 \;\; \forall T \sqsubseteq [\pm n] \;\Rightarrow\; \alpha_{S_1\overline{S_1}S_2} = 0 \;\; \forall S_1\overline{S_1}S_2 \in \mathfrak{M}.$$

1. By (3.5), for any $J \subseteq [n]$ we have

$$h_{S_1\overline{S_1}S_2}(([n]\backslash J)\overline{J}) = \begin{cases} -1 & S_1\overline{S_1}S_2 \subseteq \overline{[n]\backslash J}J \\ 0 & S_1\overline{S_1}S_2 \cap ([n]\backslash J)\overline{J} = \emptyset \text{ and } S_1\overline{S_1}S_2 \not\subseteq \overline{[n]\backslash J}J \\ 1 & \text{otherwise} \end{cases}$$
$$= \begin{cases} -1 & S_1 = \emptyset \text{ and } S_2 \subseteq J \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, for any $\emptyset \neq S \subseteq [n]$ we have

$$\sum_{J\subseteq S}(-1)^{|J|}h_{S_1\overline{S_1}S_2}\left(([n]\backslash J)\overline{J}\right) = \left(\sum_{J\subseteq S}(-1)^{|J|}\right) - 2\cdot\left(\sum_{\substack{S_2\subseteq J\subseteq S\\ S_1=\emptyset}}(-1)^{|J|}\right)$$
$$= \begin{cases} -2\cdot(-1)^{|S|} & S_1 = \emptyset \text{ and } S_2 = S \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$0 = \sum_{S_1\overline{S_1}S_2\in\mathfrak{M}} \alpha_{S_1\overline{S_1}S_2}\sum_{J\subseteq S}(-1)^{|J|}h_{S_1\overline{S_1}S_2}\left(([n]\backslash J)\overline{J}\right) = -2\cdot(-1)^{|S|}\alpha_S.$$

We have shown that $\alpha_S = 0$ for any $\emptyset \neq S \subseteq [n]$.

2. Now let $\emptyset \neq S \subseteq [n]$ and $J \subseteq S$. By (3.5) we have

$$h_{S_1\overline{S_1}S_2}\left(([n]\backslash S)\overline{J}\right) = \begin{cases} -1 & S_1\overline{S_1}S_2 \subseteq \overline{[n]\backslash S}J \\ 0 & S_1\overline{S_1}S_2 \cap \left(([n]\backslash S)\overline{J}\right) = \emptyset \text{ and } S_1\overline{S_1}S_2 \not\subseteq \overline{[n]\backslash S}J \\ 1 & \text{otherwise} \end{cases}$$
$$= \begin{cases} -1 & \text{if } S_1 = \emptyset \text{ and } S_2 \subseteq J \\ 0 & \text{else if } J \subseteq S\backslash S_1 \text{ and } S_1S_2 \subseteq S \\ 1 & \text{otherwise.} \end{cases}$$

For convenience we set $h_{S_1\overline{S_1}S_2}(\emptyset) := 0$. This convention satisfies (3.5) as well and does not affect the sum. So for every $\emptyset \neq S \subseteq [n]$,

$$\sum_{J\subseteq S}(-1)^{|J|}h_{S_1\overline{S_1}S_2}\left(([n]\backslash S)\overline{J}\right)$$
$$= \left(\sum_{J\subseteq S}(-1)^{|J|}\right) - \left(\sum_{\substack{J\subseteq S\backslash S_1\\ S_1S_2\subseteq S}}(-1)^{|J|}\right) - \left(\sum_{\substack{S_2\subseteq J\subseteq S\\ S_1=\emptyset}}(-1)^{|J|}\right)$$
$$= \begin{cases} -1 & \text{if } S_1 = S, S_2 = \emptyset \\ -(-1)^{|S|} & \text{if } S_1 = \emptyset, S_2 = S. \end{cases}$$

So,

$$0=\sum_{S_1\overline{S_1}S_2\in\mathfrak{M}}\alpha_{S_1\overline{S_1}S_2}\sum_{J\subseteq S}(-1)^{|J|}h_{S_1\overline{S_1}S_2}\left(([n]\backslash S)\overline{J}\right)=-\alpha_{S\overline{S}}-(-1)^{|S|}\alpha_S=-\alpha_{S\overline{S}}.$$

We have shown that $\alpha_{S\overline{S}}=0$ for any $\emptyset\neq S\subseteq[n]$.

3. Now let $\emptyset\neq S,T\subseteq[n]$ be nonempty subsets such that $S\cap T=\emptyset$. From (3.5) we get

$$\begin{aligned}&h_{S_1\overline{S_1}S_2}\left(([n]\backslash SJ)\,\overline{IJ}\right)-h_{S_1\overline{S_1}S_2}\left(([n]\backslash ST)\,\overline{IJ}\right)\\&=h_{S_1\overline{S_1}S_2}\left(([n]\backslash ST)\,\overline{IJ}(T\backslash J)\right)-h_{S_1\overline{S_1}S_2}\left(([n]\backslash ST)\,\overline{IJ}\right)\\&=\begin{cases}1 & (IJ)\cap S_1=\emptyset\ \wedge\ S_1S_2\subseteq ST\ \wedge\ (S_1S_2)\cap(T\backslash J)\neq\emptyset\\0 & \text{otherwise}\end{cases}\end{aligned}$$

for $I\subseteq S$ and $J\subsetneq T$. Therefore,

$$\begin{aligned}&\sum_{I,J:I\subseteq S,J\subsetneq T}(-1)^{|IJ|}\left(h_{S_1\overline{S_1}S_2}\left(([n]\backslash SJ)\,\overline{IJ}\right)-h_{S_1\overline{S_1}S_2}\left(([n]\backslash ST)\,\overline{IJ}\right)\right)\\&=\sum_{\substack{I,J:I\subseteq S,J\subsetneq T\\(IJ)\cap S_1=\emptyset\\S_1S_2\subseteq ST\\(S_1S_2)\cap(T\backslash J)\neq\emptyset}}(-1)^{|IJ|}=\sum_{\substack{J:J\subsetneq T\\J\cap S_1=\emptyset\\S_1S_2\subseteq ST\\(S_1S_2)\cap(T\backslash J)\neq\emptyset}}(-1)^{|J|}\sum_{\substack{I:I\subseteq S\\I\cap S_1=\emptyset}}(-1)^{|I|}=\sum_{\substack{J:J\subsetneq T\\J\cap S_1=\emptyset\\S_1S_2\subseteq ST\\(S_1S_2)\cap(T\backslash J)\neq\emptyset\\S\subseteq S_1}}(-1)^{|J|}.\end{aligned}$$

We decompose the sum into two parts:

- $T\cap S_1\neq\emptyset$. In this case, $T\backslash S_1\neq T$. So the sum above is

$$\sum_{\substack{J:J\subseteq T\backslash S_1\\S_1S_2\subseteq ST\\(S_1S_2)\cap(T\backslash J)\neq\emptyset\\S\subseteq S_1}}(-1)^{|J|},$$

  which equals to 1 iff $T\backslash S_1=\emptyset$, that is, $T\subseteq S_1$, so it follows from $ST\subseteq S_1\subseteq S_1S_2\subseteq ST$ that $S_1=ST,S_2=\emptyset$. In this case $(S_1S_2)\cap(T\backslash J)=T\backslash J\supseteq T\cap S_1\neq\emptyset$ is always satisfied. Otherwise the sum vanishes.

- $T\cap S_1=\emptyset$. In this case,

$$\begin{aligned}&\sum_{\substack{J:J\subsetneq T\\T\cap S_1=\emptyset\\S_1S_2\subseteq ST\\(S_1S_2)\cap(T\backslash J)\neq\emptyset\\S\subseteq S_1}}(-1)^{|J|}=\sum_{\substack{J:J\subsetneq T\\S_2\subseteq T\\S_2\cap(T\backslash J)\neq\emptyset\\S_1=S}}(-1)^{|J|}=\sum_{\substack{J:S_2\not\subseteq J\subsetneq T\\S_2\subseteq T\\S_1=S}}(-1)^{|J|}\\&=\sum_{\substack{J:J\subsetneq T\\S_2\subseteq T\\S_1=S}}(-1)^{|J|}-\sum_{\substack{J:S_2\subseteq J\subsetneq T\\S_2\subseteq T\\S_1=S}}(-1)^{|J|}=\begin{cases}-(-1)^{|T|} & \text{if } S_1=S,S_2=T\\0 & \text{otherwise.}\end{cases}\end{aligned}$$

So,

$$0 = \sum_{S_1\overline{S_1}S_2 \in \mathfrak{M}} \alpha_{S_1\overline{S_1}S_2} \sum_{I,J: I\subseteq S, J\subsetneq T} (-1)^{|IJ|} \left(h_{S_1\overline{S_1}S_2}\left(([n]\backslash SJ)\,\overline{IJ}\right) - h_{S_1\overline{S_1}S_2}\left(([n]\backslash ST)\,\overline{IJ}\right) \right)$$
$$= \alpha_{ST\overline{ST}} - (-1)^{|T|}\alpha_{S\overline{S}T} = \alpha_{S\overline{S}T}.$$

This completes the proof. □

Theorem 3.4.2. *Every generalized C_n-permutohedron*

$$\Pi_{C_n}(h) := \{x \in \mathbb{R}^n : \langle x, \mathbf{e}_T\rangle \leq h(T)\ \forall T \sqsubseteq [\pm n]\} = \sum_{\emptyset \neq S_1S_2 \subseteq [n]} y_{S_1\overline{S_1}S_2} \Delta_{S_1\overline{S_1}S_2}$$

with support function h can be written uniquely as a signed Minkowski sum of the $3^n - 1$ rank 1 symplectic matroid basis polytopes $\Delta_{S_1\overline{S_1}S_2}$, $\emptyset \neq S_1S_2 \subseteq [n]$, where

$$\begin{aligned} y_{S_1\overline{S_1}S_2} = (-1)^{|S_2|} \sum_{J\subseteq S_1S_2} (-1)^{|J|} \left(\frac{1}{2} h\left(([n]\backslash J)\overline{J}\right) - h\left(([n]\backslash S_1S_2)\overline{J}\right)\right) - \\ - (-1)^{|S_2|} \sum_{\substack{I\subseteq S_1 \\ J\subsetneq S_2}} (-1)^{|IJ|} \left(h\left(([n]\backslash S_1J)\overline{IJ}\right) - h\left(([n]\backslash S_1S_2)\overline{IJ}\right)\right). \end{aligned} \tag{3.6}$$

Proof. By Theorem 3.4.1, the map $\Psi\colon \mathbb{R}^{\mathfrak{M}} \to \mathbb{R}^{\mathcal{R}_{C_n}}$ is a linear isomorphism which maps the elements in the basis $(\mathbf{e}_{S_1\overline{S_1}S_2})_{S_1\overline{S_1}S_2\in\mathfrak{M}}$ of $\mathbb{R}^{\mathfrak{M}}$ to the elements $h_{S_1\overline{S_1}S_2}$, which form a basis of $\mathbb{R}^{\mathcal{R}_{C_n}}$. We show that the map $\Phi\colon \mathbb{R}^{\mathcal{R}_{C_n}} \to \mathbb{R}^{\mathfrak{M}}$ defined by (3.6) is the inverse of Ψ by showing that $\Phi(h_{S_1\overline{S_1}S_2}) = \mathbf{e}_{S_1\overline{S_1}S_2}$ for all $S_1\overline{S_1}S_2 \in \mathfrak{M}$.

We use the Iverson bracket $[\cdot]$ in this proof: For a statement P, let $[P] = 1$ if the statement P is true and $[P] = 0$ otherwise. The coefficient $y_{S_1'\overline{S_1'}S_2'}$ of $\Delta_{S_1'\overline{S_1'}S_2'}$ for $h_{S_1\overline{S_1}S_2}$ is

$$\begin{aligned} y_{S_1'\overline{S_1'}S_2'} = (-1)^{|S_2'|} \sum_{J\subseteq S_1'S_2'} (-1)^{|J|} \left(\frac{1}{2} h_{S_1\overline{S_1}S_2}\left(([n]\backslash J)\overline{J}\right) - h_{S_1\overline{S_1}S_2}\left(([n]\backslash S_1'S_2')\overline{J}\right)\right) - \\ -(-1)^{|S_2'|} \sum_{\substack{I\subseteq S_1' \\ J\subsetneq S_2'}} (-1)^{|IJ|} \left(h_{S_1\overline{S_1}S_2}\left(([n]\backslash S_1'J)\overline{IJ}\right) - h_{S_1\overline{S_1}S_2}\left(([n]\backslash S_1'S_2')\overline{IJ}\right)\right). \end{aligned} \tag{3.7}$$

If $S_1', S_2' \neq \emptyset$, by the identities in the proof of Theorem 3.4.1,

$$\begin{aligned} y_{S_1'\overline{S_1'}S_2'} &= (-1)^{|S_2'|} \cdot (-1) \cdot (-1)^{|S_1'S_2'|} [S_2 = S_1'S_2'][S_1 = \emptyset] + \\ &\quad + (-1)^{|S_2'|}[S_1 = S_1'S_2'][S_2 = \emptyset] + (-1)^{|S_2'|}(-1)^{|S_1'S_2'|}[S_1 = \emptyset][S_2 = S_1'S_2'] + \\ &\quad + (-1)\cdot(-1)^{|S_2'|}[S_1 = S_1'S_2'][S_2 = \emptyset] + (-1)^{|S_2'|}(-1)^{|S_2'|}[S_1 = S_1'][S_2 = S_2'] \\ &= [S_1 = S_1'][S_2 = S_2']. \end{aligned}$$

If $S_2' = \emptyset$, we have

$$\begin{aligned} y_{S_1'\overline{S_1'}S_2'} &= (-1)^{|S_2'|} \cdot (-1) \cdot (-1)^{|S_1'S_2'|}[S_2 = S_1'S_2'][S_1 = \emptyset] + \\ &\quad + (-1)^{|S_2'|}[S_1 = S_1'S_2'][S_2 = \emptyset] + (-1)^{|S_2'|}(-1)^{|S_1'S_2'|}[S_1 = \emptyset][S_2 = S_1'S_2'] \\ &= (-1) \cdot (-1)^{|S_1'|}[S_2 = S_1'][S_1 = \emptyset] + [S_1 = S_1'][S_2 = \emptyset] \\ &\quad + (-1)^{|S_1'|}[S_1 = \emptyset][S_2 = S_1'] \\ &= [S_1 = S_1'][S_2 = \emptyset]. \end{aligned}$$

If $S_1' = \emptyset$, we have

$$\begin{aligned} y_{S_1'\overline{S_1'}S_2'} &= (-1)^{|S_2'|} \sum_{J \subseteq S_2'} (-1)^{|J|} \left(\frac{1}{2} h_{S_1\overline{S_1}S_2}\left(([n]\backslash J)\overline{J}\right) - h_{S_1\overline{S_1}S_2}\left(([n]\backslash S_2')\overline{J}\right) \right) - \\ &\quad - (-1)^{|S_2'|} \sum_{J \subsetneq S_2'} (-1)^{|J|} \left(h_{S_1\overline{S_1}S_2}\left(([n]\backslash J)\overline{J}\right) - h_{S_1\overline{S_1}S_2}\left(([n]\backslash S_2')\overline{J}\right) \right) \\ &= -(-1)^{|S_2'|} \sum_{J \subseteq S_2'} (-1)^{|J|} \left(\frac{1}{2} h_{S_1\overline{S_1}S_2}\left(([n]\backslash J)\overline{J}\right) \right) \\ &= -(-1)^{|S_2'|} \cdot \frac{1}{2} \cdot (-2) \cdot (-1)^{|S_2'|}[S_1 = \emptyset][S_2 = S_2'] = [S_1 = \emptyset][S_2 = S_2']. \end{aligned}$$

In all cases,

$$y_{S_1'\overline{S_1'}S_2'} = [S_1 = S_1'][S_2 = S_2'] = \begin{cases} 1 & \text{if } S_1 = S_1' \text{ and } S_2 = S_2', \\ 0 & \text{otherwise.} \end{cases}$$

Therefore Φ is the inverse map of Ψ. □

Theorem 3.4.2 also works for generalized D_n-permutohedra because they form a subclass of generalized C_n-permutohedra. Note that the Minkowski additive inverse is not the negative dilation of a polytope. The negative dilation $-P$ of a polytope P is a signed Minkowski sum of the faces of P.

Proposition 3.4.3 ([She68][Sch13, Note 6.2.5])**.** *For $P \subseteq \mathbb{R}^n$ we have*

$$-h_{-P} = \sum_{F \in \mathcal{L}(P)\backslash\{\emptyset\}} (-1)^{\dim(F)} h_F,$$

or equivalently, when written as an equation of Minkowski sums,

$$-P + \sum_{\substack{F \in \mathcal{L}(P)\backslash\{\emptyset\} \\ 2 | \dim(F)}} F = \sum_{\substack{F \in \mathcal{L}(P)\backslash\{\emptyset\} \\ 2 \nmid \dim(F)}} F.$$

Corollary 3.4.4. *For $\emptyset \neq T \sqsubseteq [\pm n]$,*

$$\begin{aligned} h_{\Delta_T} &= h_{\Delta_T^0} + \sum_{F \subseteq T} (-1)^{|F|} h_{\Delta_{\overline{F}}^0}, \\ \Delta_T &= \Delta_T^0 + \sum_{F \subseteq T} (-1)^{|F|} \Delta_{\overline{F}}^0. \end{aligned} \tag{3.8}$$

Proof.

$$h_{\Delta_T^0} = -\sum_{\emptyset \neq F \subseteq T} (-1)^{|F|-1} h_{\Delta_{\overline{F}}} - \sum_{F \subseteq T} (-1)^{|F|} h_{\Delta_{\overline{F}}^0} = h_{\Delta_T} - \sum_{F \subseteq T} (-1)^{|F|} h_{\Delta_{\overline{F}}^0}.$$

□

3.4.2 The connectedness of type B or C

Recall that in Section 2.3, Theorem 2.3.4 states that the CI-structures which model separation are exactly the semimatroids of Minkowski sums of standard simplices. They are exactly the ascending semigraphoids satisfying (Stud). And Theorem 2.3.11 states that a CI-structure is a graphic gaussoid iff it is the semimatroid of a graph associahedron iff it is the semimatroid of a MSS that is also a gaussoid. It is natural to look for the notion of connectedness in other Coxeter types.

Problem 3.4.5. The counterparts of graph of types B, C and D are *signed graphs* [Zas81; Zas82]. Similarly to Theorem 2.3.11, we would like a classification of the separation in signed graphs as the C_n-semimatroids of a certain class of generalized C_n-permutohedron, or axiomatize it as inference rules on C_n-CI-statements. We also would like a classification of the C_n-semimatroids of the Minkowski sum of rank 1 symplectic matroid basis (or independent set) polytopes like Theorem 2.3.4. Some properties of such C_n-semimatroids are deduced as follows.

Lemma 3.4.6. *For $S \subseteq [\pm n]$, the C_n-semimatroid corresponding to Δ_S is*

$$\begin{aligned} &\left\{ i \not\perp\!\!\!\perp j | K : (ij|K) \in \mathcal{C}_n, i \neq \bar{j}, (ij \subseteq S \subseteq [\pm n] \backslash K \text{ or } \overline{ij} \subseteq S \subseteq \overline{ijK}) \right\} \cup \\ \cup &\left\{ i \not\perp\!\!\!\perp \bar{i} | K : (i\bar{i}|K) \in \mathcal{C}_n, i\bar{i} \subseteq S \subseteq \overline{i\bar{i}K} \right\}. \end{aligned}$$

Proof. The support function $h_S \colon \{T \sqsubseteq [\pm n]\} \to \mathbb{R}$ of Δ_S is given by

$$h_S(T) = \max_{i \in S} \langle \mathbf{e}_i, \mathbf{e}_T \rangle = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ -1 & \text{if } T \supseteq \overline{S} \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

Let $ijK \sqsubseteq [\pm n]$. If $ijK \not\supseteq \overline{S}$, $h_S(K), h_S(iK), h_S(jK)$ and $h_S(ijK)$ are all nonnegative and the condition

$$h_S(iK) + h_S(jK) - h_S(K) - h_S(ijK) > 0 \tag{3.10}$$

for $i \not\perp\!\!\!\perp j | K$ is valid iff $h_S(iK) = h_S(jK) = h_S(ijK) = 1$ and $h_S(K) = 0$. This happens iff $ij \subseteq S \subseteq [\pm n] \backslash K$. If, on the contrary, $ijK \supseteq \overline{S}$, then $h_S(ijK) = -1$ and (3.10) iff $h_S(iK) = h_S(jK) = h_S(K) = 0$, this happens iff $\overline{ij} \subseteq S \subseteq \overline{ijK}$.

Now let $(i\bar{i}|K) \in \mathcal{C}_n$. If $i \not\perp\!\!\!\perp \bar{i} | K$, that is,

$$h_S(iK) + h_S(\bar{i}K) > 2h_S(K), \tag{3.11}$$

then $h_S(K) = 0$ because otherwise $h_S(iK) = h_S(\bar{i}K) = h_S(K)$. That is, $S \subseteq \overline{i\bar{i}K}$ and $S \not\subseteq \overline{K}$. Moreover, both i and $\bar{i}$ are in S for $i \not\perp\!\!\!\perp \bar{i}|K$ because otherwise one term in the left-hand side of (3.11) would be -1 and (3.11) cannot be valid. Therefore $i\bar{i} \subseteq S \subseteq \overline{i\bar{i}K}$ and this implies (3.11) as well. □

For $\mathcal{S} \subseteq 2^{[\pm n]}$, we deduce a condition satisfied by the C_n-semimatroid $[[h_{\Delta_\mathcal{S}}]]$ of any Minkowski sum of rank 1 symplectic matroid basis polytopes (*MS1SMP*) $\Delta_\mathcal{S} := \sum_{S\in\mathcal{S}} \Delta_S$.

Proposition 3.4.7. *Let $\mathcal{G} \subseteq \mathcal{C}_n$ be a C_n-CI-structure. If $\mathcal{G} = [[h_{\Delta_\mathcal{S}}]]$ for some $\mathcal{S} \subseteq 2^{[\pm n]}$, then $\mathcal{G}$ satisfies*

(CItv) $\{(ij|K_1),(ij|K_2)\} \subseteq \mathcal{G} \Rightarrow (ij|K) \in \mathcal{G} \quad \forall K_1 \subseteq K \subseteq K_2.$

Proof. If $(ij|K) \notin \mathcal{G}$, $ijK \sqsubseteq [\pm n]$, by Lemma 3.4.6 and $[[h_{\Delta_\mathcal{S}}]] = \bigcap_{S\in\mathcal{S}} [[h_{\Delta_S}]]$, there exists some $S \in \mathcal{S}$ such that $ij \subseteq S \subseteq [\pm n]\backslash K$ or $ij \subseteq \overline{S} \subseteq ijK$. This implies that $ij \subseteq S \subseteq [\pm n]\backslash K_1$ for all $K_1 \subseteq K$ or $ij \subseteq \overline{S} \subseteq ijK_2$ for all $K_2 \supseteq K$. Hence $(ij|K_1) \notin \mathcal{G}$ for all $K_1 \subseteq K$ or $(ij|K_2) \notin \mathcal{G}$ for all $K_2 \supseteq K$. The property (CItv) of $\mathcal{G}$ is proven by contraposition. □

The C_n-semimatroid of a Minkowski sum of rank 1 symplectic matroid independent set polytopes (*MS1SMIP*) satisfies (Asc).

Lemma 3.4.8. *For $S \subseteq [\pm n]$, the C_n-semimatroid corresponding to Δ^0_S is*

$$\left\{ i \not\perp\!\!\!\perp j|K : (ij|K) \in \mathcal{C}_n, i \neq \bar{j}, ij \subseteq S \subseteq [\pm n]\backslash K \right\} \cup \\ \cup \left\{ i \not\perp\!\!\!\perp \bar{i}|K : (i\bar{i}|K) \in \mathcal{C}_n, S \subseteq \overline{i\bar{i}K}, i\bar{i} \cap S \neq \emptyset \right\}.$$

Proof. The support function $h_{\Delta^0_S} : \{T \sqsubseteq [\pm n]\} \to \mathbb{R}$ of Δ^0_S is given by

$$h_{\Delta^0_S}(T) = \max\{0, \langle \mathbf{e}_i, \mathbf{e}_T \rangle : i \in S\} = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

The lemma follows by modifying the proof of Lemma 3.4.6 such that the case $h_{\Delta^0_S}(ijK) = -1$ for (3.10) is not considered and (3.11) will be valid if one of the i and $\bar{i}$ is in S. □

Proposition 3.4.9. *Let $\mathcal{G} \subseteq \mathcal{C}_n$ be a C_n-CI-structure. If $\mathcal{G} = [[h_{\Delta^0_\mathcal{S}}]]$ for some $\mathcal{S} \subseteq 2^{[\pm n]}$, where $\Delta^0_\mathcal{S} := \sum_{S\in\mathcal{S}} \Delta^0_S$, then $\mathcal{G}$ satisfies (Asc).*

Proof. The property (Asc) follows directly from

$$[[h_{\Delta^0_\mathcal{S}}]] = \bigcap_{S\in\mathcal{S}} [[h_{\Delta^0_S}]] = \left\{ (ij|K) \in \mathcal{C}_n : i \neq \bar{j}, \nexists S \in \mathcal{S} : ij \subseteq S \subseteq [\pm n]\backslash K \right\} \cap \\ \cap \left\{ (i\bar{i}|K) \in \mathcal{C}_n : \nexists S \in \mathcal{S} : S \subseteq \overline{i\bar{i}K}, i\bar{i} \cap S \neq \emptyset \right\}.$$

□

3.4.3 The volumes of generalized C_n-permutohedra

In this subsection we give a formula for the volume of an arbitrary generalized B_n-permutohedron in term of its support function.

Let $P_1, \dots, P_m \subseteq \mathbb{R}^n$ be polytopes and $x_1, \dots, x_m$ be nonnegative real numbers. The volume

$$\mathrm{Vol}_n(x_1 P_1 + \cdots + x_m P_m) = \sum_{i_1,\dots,i_n=1}^{m} V(P_{i_1}, \dots, P_{i_n}) x_{i_1} \cdots x_{i_n} \tag{3.13}$$

of the Minkowski sum of scaled polytopes is a polynomial function in $x_1, \dots, x_m$. The *mixed volume* $V(P_1, \dots, P_n)$ of polytopes $P_1, \dots, P_n \subseteq \mathbb{R}^n$ is $\frac{1}{n!}$ times the coefficient of $x_1 \cdots x_n$ in $\mathrm{Vol}_n(x_1 P_1 + \cdots + x_n P_n)$. By the inclusion-exclusion principle,

$$V(P_1, \dots, P_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathrm{Vol}_n(P_{i_1} + \cdots + P_{i_k}).$$

We refer to [Sch13, § 5 and 7] and [CLO05, § 7] for mixed volumes. Remark that there are different conventions in the literature concerning volumes and mixed volumes. In some literature there is an extra factor $\frac{1}{n!}$ in the definition of the mixed volume, and in some literature the n-dimensional volume is normalized. We review some properties of mixed volumes.

Theorem 3.4.10 ([CLO05, § 7.4][Sch13, § 5])**.** *The mixed volumes have the following properties.*

(1) *The mixed volume is symmetric and Minkowski linear in each variable.*

(2) *The mixed volume is nonnegative, monotonic and invariant under a volume-preserving affine map.*

(3) *The mixed volume is valuative, that is, if we fix $\ell \in [n]$ and the polytopes $P_{\ell+1}, \dots, P_n \subseteq \mathbb{R}^n$. Let $f(Q) := V(Q, \dots, Q, P_{\ell+1}, \dots, P_n)$, then*

$$f(P) + f(P') = f(P \cap P') + f(P \cup P')$$

for any polytopes $P, P' \subseteq \mathbb{R}^n$ such that $P \cup P'$ is a polytope.

(4) $V(P, \dots, P) = \mathrm{Vol}_n(P)$.

(5) *The mixed volume satisfies*

$$V(P_1, \dots, P_n) = \frac{1}{n} \sum_{u} h_{P_1}(u) V^{(u)}(F(P_2, u), \dots, F(P_n, u)), \tag{3.14}$$

where the sum extends over the unit normal vectors of the facets of $P_2 + \cdots + P_n$. Here $F(P_i, u)$ denotes the face of P_i maximizing the linear functional $u^\top$, and $V^{(u)}(P'_1, \dots, P'_{n-1})$ denotes the $(n-1)$-dimensional mixed volume of the translations of $P'_1, \dots, P'_{n-1}$ in the hyperplane $u^\perp$ orthogonal to u.

(6) *If* $P_1, \dots, P_n$ *are lattice polytopes in* $\mathbb{R}^n$*, then* $n!V(P_1, \dots, P_n)$ *is an integer.*

The equation (3.13) also works for signed Minkowski sums.

Proposition 3.4.11 ([ABD10, Proposition 3.2])**.** *If* $x_1P_1 + \cdots + x_mP_m$ *is a signed Minkowski sum which defines a polytope in* $\mathbb{R}^n$*, then*

$$\mathrm{Vol}_n(x_1P_1 + \cdots + x_mP_m) = \sum_{i_1,\dots,i_n=1}^{m} V(P_{i_1}, \dots, P_{i_n})x_{i_1} \cdots x_{i_n}. \tag{3.15}$$

In [Sch13, Theorem 5.1.8] criteria for the positivity of the mixed volume of convex bodies are given. Modifying the proof of [Sch13, Theorem 5.1.8] slightly yields the following slight strengthening for polytopes.

Theorem 3.4.12. *For polytopes* $P_1, \dots, P_n \subseteq \mathbb{R}^n$*, the following are equivalent:*

(1) $V(P_1, \dots, P_n) > 0$*;*

(2) *there are edges* $E_1 \subseteq P_1, \dots, E_n \subseteq P_n$ *with linearly independent directions;*

(3) $\dim(P_{i_1} + \cdots + P_{i_k}) \geq k$ *for each choice of indices* $1 \leq i_1 < \cdots < i_k \leq n$ *and for all* $k \in [n]$*.*

Proof. We prove the (1)⇒(2) by induction on n, all other implications follow from Theorem 5.1.8 in [Sch13]. The case $n = 1$ being trivial, suppose that $n > 1$ and that the assertion is true in smaller dimensions. We can assume that $\mathbf{0} \in \mathrm{Vert}(P_1)$ by translating the polytope P_1. By (3.14),

$$0 < V(P_1, \dots, P_n) = \frac{1}{n} \sum_{u_i} h_{P_1}(u_i) V^{(u_i)}(F(P_2, u_i), \dots, F(P_n, u_i)),$$

where the sum extends over finitely many unit vectors u_i.

As $V^{(u_i)}(F(P_2, u_i), \dots, F(P_n, u_i)) \geq 0$ for any u_i, there is some u_j such that $h_{P_1}(u_j) > 0$ and $V^{(u_j)}(F(P_2, u_j), \dots, F(P_n, u_j)) > 0$. By the induction hypothesis, there are edges $E_2 \subseteq F(P_2, u_j), \dots, E_n \subseteq F(P_n, u_j)$ with linearly independent directions. Since $h_{P_1}(u_j) > 0$, there is an edge E_1 of the polytope P_1 containing the vertex $\mathbf{0}$ that is not orthogonal to u_j. We have the desired edges $E_1, \dots, E_n$ of $P_1, \dots, P_n$, respectively. □

We prove the following explicit formulas for the mixed volumes of standard simplices, the independent set polytopes of rank 1 matroids and of rank 1 symplectic matroids. These formulas are already proven, but the proofs were either using the BKK Theorem or (tropical) intersection theory. Here we give proofs using only basic properties of mixed volumes. The conditions for these mixed volume to be nonzero can be reformulated by the various marriage theorems. In § 3.4.4 we reprove these marriage theorems using the basic properties of mixed volumes.

Theorem 3.4.13 ([CDEHL24, Lemma 5.14])**.** *For* $S_1, \dots, S_n \subseteq [n]$*,*

$$V(\Delta^0_{S_1}, \dots, \Delta^0_{S_n}) = \begin{cases} \frac{1}{n!} & \text{if } \left|\bigcup_{i\in I} S_i\right| \geq |I| \text{ for all } I \subseteq [n], \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 3.4.10, $n!V(\Delta^0_{S_1},\dots,\Delta^0_{S_n})$ is a nonnegative integer not larger than $n!V(\Delta^0_{[n]},\dots,\Delta^0_{[n]}) = n!\mathrm{Vol}_n(\Delta^0_{[n]}) = 1$. By Theorem 3.4.12 (or [Sch13, Theorem 5.1.8]), it is positive iff $\dim \sum_{i\in I}\Delta^0_{S_i} \geq |I|$ for all $I \subseteq [n]$, which is equivalent to $\left|\bigcup_{i\in I} S_i\right| \geq |I|$ for all $I\subseteq [n]$. □

Using the same proof we also deduce the mixed volumes of standard simplices

$$\Delta_{S_1},\dots,\Delta_{S_{n-1}} \subseteq \mathbf{1}^\perp = \{x\in\mathbb{R}^n : \mathbf{1}^\top x = 0\}.$$

The $(n-1)$-dimensional volumes in $\mathbf{1}^\perp \subseteq \mathbb{R}^n$ are normalized with respect to the lattice generated by the simple roots $\Delta_{A_{n-1}}$, so the standard simplex $\Delta_{[n]}$ has volume $\frac{1}{(n-1)!}$.

Theorem 3.4.14 ([Pos09, § 9])**.** *For $S_1,\dots,S_{n-1}\subseteq [n]$,*

$$V(\Delta_{S_1},\dots,\Delta_{S_{n-1}}) = \begin{cases} \frac{1}{(n-1)!} & \text{if } \left|\bigcup_{i\in I} S_i\right| \geq |I|+1 \text{ for all } \emptyset\neq I\subseteq[n],\\ 0 & \text{otherwise.}\end{cases}$$

A formula for the mixed volumes of Δ^0_T, $T\sqsubseteq[n]$, is given in [EFLS24, Theorem A(b)]. The key point is that these polytopes are "saturated towards the origin", so are their Minkowski sums as well. Therefore they can be decomposed orthantwisely.

Theorem 3.4.15 ([EFLS24, Theorem A(b)])**.** *For $S_1,\dots,S_n\subseteq[\pm n]$,*

$$V(\Delta^0_{S_1},\dots,\Delta^0_{S_n}) = \frac{1}{n!}\left|\left\{O\sqsubseteq[\pm n] : |O| = n, \left|\bigcup_{i\in I}(S_i\cap O)\right| \geq |I|\ \forall I\subseteq[n]\right\}\right|. \quad (3.16)$$

Proof. Let $\mathbb{R}^n_O := \{x\in\mathbb{R}^n : \langle x,\mathbf{e}_i\rangle \geq 0\ \forall i\in O\}$ be the orthant whose coordinate signs are given by $O\sqsubseteq[n]$, $|O| = n$. Then

$$\left(\sum_{i\in J}\Delta^0_{S_i}\right)\cap\mathbb{R}^n_O = \sum_{i\in J}\left(\Delta^0_{S_i}\cap\mathbb{R}^n_O\right) = \sum_{i\in J}\Delta^0_{S_i\cap O}.$$

We show the inclusion $(\subseteq)$ of the first equality, the remaining part is clear. Let $x = \sum_{i\in J} x^{(i)}\in\mathbb{R}^n_O$ where $x^{(i)}\in\Delta^0_{S_i}$ for all $i\in J$. We can move the points $x^{(i)}$ to $\mathbb{R}^n_O$ without altering the sum x in the following way. For every $k\in[n]$, let σ be the sign of the element in $(k\bar{k})\cap O$. If there is some $i\in J$ such that $x^{(i)}_k$ has the sign $-\sigma$, then make the k-th coordinate of every point to be 0 whenever it has the sign $-\sigma$, and multiply every k-th coordinate with sign σ by the same factor so that they sum up to x_k. The new points are in the corresponding $\Delta^0_{S_i}\cap\mathbb{R}^n_O$.

Now we decompose the Minkowski sums of symplectic matroid independent set polytopes orthantwisely.

$$\begin{aligned}
V(\Delta^0_{S_1},\dots,\Delta^0_{S_n}) &= \frac{1}{n!}\sum_{J\subseteq[n]}(-1)^{n-|J|}\mathrm{Vol}_n\left(\sum_{i\in J}\Delta^0_{S_i}\right)\\
&= \frac{1}{n!}\sum_{J\subseteq[n]}(-1)^{n-|J|}\sum_{\substack{O\sqsubseteq[\pm n]\\|O|=n}}\mathrm{Vol}_n\left(\left(\sum_{i\in J}\Delta^0_{S_i}\right)\cap\mathbb{R}^n_O\right)\\
&= \sum_{\substack{O\sqsubseteq[\pm n]\\|O|=n}}\frac{1}{n!}\sum_{J\subseteq[n]}(-1)^{n-|J|}\mathrm{Vol}_n\left(\sum_{i\in J}\Delta^0_{S_i\cap O}\right)\\
&= \sum_{\substack{O\sqsubseteq[\pm n]\\|O|=n}} V(\Delta^0_{S_1\cap O},\dots,\Delta^0_{S_n\cap O}).
\end{aligned}$$

The equation (3.16) follows from Theorem 3.4.13 and the fact that the mixed volume is invariant under a volume-preserving map. □

Thus we have an explicit formula for the volume of any generalized C_n-permutohedron given its support function by combining (3.7), (3.8), (3.15) and (3.16).

Problem 3.4.16. Simplify the volume formula, as there should be massive cancellation in the formula.

3.4.4 Mixed volumes and marriage theorems

In this subsection we prove the classic marriage theorems in transversal theory using basic properties of mixed volumes. The connection between mixed volumes and combinatorial marriage theorems was not pointed out before in the literature.

A *transversal* of $S_1,\dots,S_m\subseteq[n]$ is a subset $I\subseteq[n]$ such that there is a bijection $\phi\colon I\to[m]$ such that $i\in S_{\phi(i)}$ for all $i\in I$.

Applying Theorem 3.4.12 on $\Delta^0_{S_1},\dots,\Delta^0_{S_m},\Delta^0_{[n]},\dots,\Delta^0_{[n]}\subseteq\mathbb{R}^n$ yields Hall's marriage theorem.

Theorem 3.4.17 (Hall's marriage theorem [Hal35])**.** *The sets $S_1,\dots,S_m\subseteq[n]$ has a transversal iff*

$$\left|\bigcup_{i\in I}S_i\right|\geq|I| \tag{3.17}$$

for all $I\subseteq[m]$.

Proof. Let $S_{m+1}=\dots=S_n:=[n]$. By Theorem 3.4.12, $V(\Delta^0_{S_1},\dots,\Delta^0_{S_n})>0$ iff

$$\dim\sum_{i\in I}\Delta^0_{S_i}\geq|I|\quad\text{ for all } I\subseteq[n].$$

Since $\dim \Delta^0_{S_i} = n$ for $i > m$, this is equivalent to

$$\dim \sum_{i\in I} \Delta^0_{S_i} \geq |I| \quad \text{for all } I \subseteq [m],$$

which is equivalent to (3.17).

If $S_1, \dots, S_m$ has a transversal $\{a_1, \dots, a_m\}$, where $a_1 \in S_1, \dots, a_m \in S_m$, then the n edges $\mathrm{conv}(\mathbf{0}, \mathbf{e}_{a_1}) \subseteq \Delta^0_{S_1}, \dots, \mathrm{conv}(\mathbf{0}, \mathbf{e}_{a_m}) \subseteq \Delta^0_{S_m}$ and $\mathrm{conv}(\mathbf{0}, \mathbf{e}_j) \subseteq \Delta^0_{[n]}$ for all $j \in [n]\backslash\{a_1, \dots, a_m\}$ are linearly independent. Conversely, assume that there are linearly independent edges $\mathrm{conv}(v_1, v_1') \subseteq \Delta^0_{S_1}, \dots, \mathrm{conv}(v_n, v_n') \subseteq \Delta^0_{S_n}$ where $v_1, v_1', \dots, v_n, v_n' \in \{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_n\}$. In other words, $T = \{v_1v_1', \dots, v_nv_n'\}$ is a spanning tree in the complete graph on the vertex set $\{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_n\}$. For each $i \in [n]$, let $a_i \in S_i$ be such that $\mathbf{e}_{a_i}$ is the farther vertex of the edge v_iv_i' from $\mathbf{0}$ in the tree T. Then we get a transversal $\{a_1, \dots, a_m\} \subseteq [n]$ such that $a_1 \in S_1, \dots a_m \in S_m$. □

A *partial transversal* of $S_1, \dots, S_m \subseteq [n]$ is a subset $I \subseteq [n]$ such that there is a subset $S \subseteq [m]$ and a bijection $\phi\colon I \to S$ such that $i \in S_{\phi(i)}$ for all $i \in I$.

Let $K := \{n+1, \dots, n+k\}$. Applying Theorem 3.4.12 on

$$\Delta^0_{S_1K}, \dots, \Delta^0_{S_mK}, \Delta^0_{[n+k]}, \dots, \Delta^0_{[n+k]} \subseteq \mathbb{R}^{n+k}$$

yields Ore's theorem.

Theorem 3.4.18 (Ore's theorem [Ore55]). *The sets $S_1, \dots, S_m \subseteq [n]$ has a partial transversal of size $m-k$ iff $\left|\bigcup_{i\in I} S_i\right| \geq |I| - k$ for all $I \subseteq [m]$.*

Applying Theorem 3.4.12 on $\Delta_{S_1}, \dots, \Delta_{S_{n-1}} \subseteq \mathbf{1}^\perp \subseteq \mathbb{R}^n$ yields the dragon marriage theorem.

Theorem 3.4.19 (Dragon marriage theorem [Pos09, Proposition 5.4]). *The following are equivalent for $S_1, \dots, S_{n-1} \subseteq [n]$:*

(1) *For any $\emptyset \neq I \subseteq [n-1]$, $\left|\bigcup_{i\in I} S_i\right| \geq |I| + 1$.*

(2) *For any $j \in [n]$, $[n]\backslash j$ is a transversal of $S_1, \dots, S_{n-1}$.*

(3) *There are 2-element subsets $a_1b_1 \subseteq S_1, \dots, a_{n-1}b_{n-1} \subseteq S_{n-1}$ that are the edges of a spanning tree in the complete graph on $[n]$.*

These three theorems can be generalized to the criteria for existence of a transversal which is independent in a given matroid, known as Rado's theorem [Rad42], Perfect's theorem [Mir71, Theorem 6.2.2] and dragon Hall-Rado theorem [BES23, Proposition 5.2.3], respectively. They also follow from Theorem 3.4.12 for matroids that are representable over $\mathbb{R}$. We sketch Rado's theorem as follows as an example.

Let M be a rank r matroid on $[n]$. Rado's theorem states that the sets $S_1, \dots, S_m \subseteq [n]$ has a transversal $I \in \mathcal{I}(M)$ iff $\mathrm{rk}\left(\bigcup_{i\in I} S_i\right) \geq |I|$ for all $I \subseteq [m]$. If M is representable over $\mathbb{R}$, let $v_1, \dots, v_n \in \mathbb{R}^r$ be a representation of M. By scaling and negating when necessary, we can assume that for each non-loop $i \in [n]$, the vector v_i is a unit vector, and the first nonzero coordinate of v_i is positive. Then any of $\{\mathbf{0}, v_1, \ldots, v_n\}$ is a vertex of $\mathrm{conv}(\mathbf{0}, v_1, \ldots, v_n)$. By replacing the $\mathbf{e}_i$ in the proof of Theorem 3.4.17 by v_i, we show that the existence of an independent transversal of $S_1, \ldots, S_m$ is equivalent to the positivity of $V\left(\mathrm{conv}(\mathbf{0}, v_i : i \in S_1), \ldots, \mathrm{conv}(\mathbf{0}, v_i : i \in S_n)\right)$, where $S_{m+1} = \cdots = S_n = [n]$.

Chapter 4

ML-degrees and algebraic degrees of semidefinite programming in classical types

4.1 Introduction

In statistics, a multivariate Gaussian distribution is an important family of parametric statistical models, whose parameters are given by a mean vector $\mu \in \mathbb{R}^n$ and covariance matrix Σ which is positive definite. The inverse Σ^{-1} is called the *concentration matrix*. The problems studied in this chapter are motivated by *linear concentration models*, introduced by Anderson [And70]. In these models, the concentration matrix Σ^{-1} is assumed to be in a d-dimensional linear subspace $\mathcal{L}$ of symmetric matrices, in particular Σ should belong to the set $\mathcal{L}^{-1}$ of the inverse matrices of $\mathcal{L}$.

An important invariant that measures the complexity of a linear concentration model is the *maximum likelihood degree* (ML-degree), which is the number of critical points of the rational score equations coming from generic data points. If the linear space $\mathcal{L}$ is generic, the ML-degree is the degree of the Zariski closure of $\mathcal{L}^{-1}$ (see [SU10, Theorem 1] or [MMW21, Corollary 2.6]). In this case, the ML-degree depends just on the size n of the symmetric matrices and the dimension d of $\mathcal{L}$, and it will be denoted by $\phi(n, d)$.

In [MMW21] a new connection of the ML-degree with enumerative geometry was found. This allowed new techniques and tools to study the ML-degree. For instance, $\phi(n, d)$ can be defined in pure enumerative terms, as being the number of nondegenerate quadrics in n variables, passing through $\binom{n+1}{2} - d$ general points and tangent to $d - 1$ general hyperplanes. Such problems can be solved by performing computations in the cohomology ring of the variety of *complete quadrics*. In light of this connection, later in [MMMSV23] the following polynomiality result, previously conjectured by Sturmfels and Uhler [SU10, p. 611] (see also [MMW21, Conjecture 2.8]) was settled:

Theorem 4.1.1 ([MMMSV23, Theorem 1.3])**.** *For any $d > 0$ fixed, the function $n \mapsto \phi(n, d)$ is polynomial.*

The proof of the previous theorem boils down to show the polynomiality of certain functions [MMMSV23, Theorem 4.3], called *Lascoux polynomials*, after Alain Lascoux [LLT89]. There are several equivalent ways to define Lascoux polynomials. For instance, in Section 4.3 we will give a definition in terms of Schur polynomials. Here we describe Lascoux polynomials in a more elementary manner. First, consider the infinite Pascal triangle matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & \dots \\ 1 & 3 & 3 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $E_{ij} = \binom{i}{j}$. For every pair of finite subsets $I, J \subseteq \mathbb{N}$, let $E_{I,J}$ be the submatrix of E with rows indexed by I and columns indexed by J. The *Lascoux coefficient* ψ_I of a finite subset $I \subseteq \mathbb{N}$ of cardinality r, is defined by

$$\psi_I = \sum_{J \subseteq \mathbb{N},\, |J|=r} \det(E_{I,J}).$$

Observe that the sum above has only finitely many non-zero terms. For every nonnegative integer $n \geq 0$, let $[n] := \{0, 1, \dots, n-1\}$ in this chapter. The Lascoux polynomial of a finite subset $I \subseteq \mathbb{N}$, is the function

$$\mathrm{LP}_I(n) = \begin{cases} \psi_{[n] \setminus I} & \text{if } I \subseteq [n], \\ 0 & \text{otherwise.} \end{cases}$$

Two proofs of the polynomiality of LP_I were provided in [MMMSV23]. The first uses two recursive formulas of the Lascoux polynomials. In the second, the authors dive in to the properties of the minors of the Pascal triangle matrix. Although the techniques used in the second proof are longer and more technical, they allow to find the degree and leading coefficient of the Lascoux polynomials:

Theorem 4.1.2 ([MMMSV23, Theorem 4.12])**.** (Type C, Theorem 4.3.2) Let $I = \{i_1 < \cdots < i_r\} \subseteq \mathbb{N}$. The polynomial LP_I has degree $\sum_k i_k + |I|$. Its leading coefficient is equal to

$$\frac{\prod_{j>k}(i_j - i_k)}{(i_1+1)! \cdots (i_r+1)! \prod_{j>k}(i_j + i_k + 2)}.$$

Our first main contribution is to provide a more direct proof of the previous theorem, starting from the recurrence relations of the Lascoux polynomials.

Further, all the results mentioned above have natural analogues if we replace the space of symmetric matrices ("type C", Section 4.3) with the space of general matrices ("type A", see Section 4.4) or with the space of skew-symmetric matrices ("type D", Section 4.5). This point of view was already taken in [MMMSV23]. We continue in this direction, finding formulas for the degree and leading coefficient of Lascoux polynomials for types A and D, which were not previously known.

Theorem 4.1.3 (Type A, Theorem 4.4.2)**.** *For sets $I = \{i_1, ..., i_r\}$, $J = \{j_1, ..., j_r\}$, the degree of the Lascoux polynomials $\mathrm{LP}^A_{I,J}(n)$ of type A is given by the following expression on I, J:*

$$\deg(\mathrm{LP}^A_{I,J}(n)) = |I| + \sum I + \sum J$$

and the leading coefficient of $\mathrm{LP}^A_{I,J}$ is

$$\frac{\prod_{k>l}(i_k - i_l)\prod_{k>l}(j_k - j_l)}{\prod_{k,l=1}^r(i_k + j_l + 1)\prod_{k=1}^r(i_k)!\prod_{k=1}^r(j_k)!}.$$

Theorem 4.1.4 (Type D, Theorem 4.5.1)**.** *Let $I = \{i_1 < \cdots < i_r\} \subset \mathbb{N}$ be a set of nonnegative integers. Then*

- *If $i_1 > 0$, then the Lascoux polynomials $\mathrm{LP}^D_I(2n)$ and $\mathrm{LP}^D_I(2n+1)$ of type D are polynomials in n of degree $\sum I$ and leading coefficient equal to*

$$\frac{2^{\sum I - |I|}\prod_{k>l}(i_k - i_l)}{\prod_{k>l}(i_k + i_l)\prod_k(i_k)!}.$$

- *If $i_1 = 0$, then $\mathrm{LP}^D_I(n) = \mathrm{LP}^D_{I\setminus\{0\}}(n)$ if $n - |I|$ is even, and $\mathrm{LP}^D_I(n) = 0$ if $n - |I|$ is odd.*

Lascoux coefficients also appear in the context of *semidefinite programming* (SDP), a subject in optimization theory that concerns the problem of optimizing a linear function over the cone of positive semidefinite matrices. An important invariant that addresses the complexity of these problems is the *algebraic degree of semidefinite programming*. For more information about the algebraic degree of SDP, we refer to [NRS10]. Following [MMMSV23, Definition 1.4], here we provide the following definition of the algebraic degree of SDP in the language of algebraic geometry. Let $\mathcal{L} \subseteq S^2\mathbb{C}^n$ be a general linear space of symmetric matrices of affine dimension $m+1$, and let $SD^{r,n}_m \subseteq \mathbb{P}(\mathcal{L})$ denote the projectivization of the cone of matrices of rank at most r in $\mathcal{L}$. The algebraic degree of SDP, denoted $\delta(m, n, r)$, is the degree of the projective dual of $SD^{r,n}_m$ if this dual is a hypersurface, and zero otherwise.

In [BR09] the authors found a formula that expresses $\delta(m, n, r)$ in terms of Lascoux coefficients. In addition, in [MMMSV23] the authors proved that the function $n \mapsto \delta(m, n, n-s)$ for fixed $m, s > 0$ is a polynomial, and provided another formula for $\delta(m, n, n-s)$ previously conjectured in [NRS10, Conjecture 21]. Similarly for Lascoux polynomials, the results in [MMMSV23] were also proved for the functions δ_A and δ_D of type A and D (see Section 4.6 for the related definitions). As an application of our previous results for the Lascoux polynomials, we find the degree of the polynomials $\delta(m, n, n-s)$, and their leading coefficient for $s = 1$, in type C, A and D.

Theorem 4.1.5. *Let $s > 0$.*

- (Type C, Theorem 4.6.1) *The polynomial $\delta(m, n, n-s)$ has degree m, for every $m \geq \binom{s+1}{2}$. Moreover*
$$LC(\delta(m, n, n-1)) = \frac{2^{m-1}}{m!},$$
for every $m > 0$.

- (Type A, Theorem 4.6.2) *The polynomial $\delta_A(m, n, n-s)$ has degree m, for every $m \geq s^2$. Moreover,*
$$LC(\delta_A(m, n, n-1)) = \frac{1}{m!}\binom{2(m-1)}{m-1},$$
for every $m > 0$.

- (Type D, Theorem 4.6.3) *The polynomial $\delta_D(m, n, n-s)$ has degree m, for every $m \geq \binom{2s}{2}$. Moreover,*
$$LC(\delta_D(m, n, n-1)) = \frac{2^{m-2}}{m!}\left(\frac{1}{m}\binom{2(m-1)}{m-1} + 1\right),$$
for every $m > 0$.

The ML-degree $\phi(n, d)$ is regarded as a type C object. Its analogues $\phi_A(n, d)$ in type A and $\phi_D(n, d)$ in type D are the degrees of the Zariski closures of $\mathcal{L}^{-1}$ where $\mathcal{L}$ is a generic d-dimensional linear subspace of the spaces of $n \times n$ general matrices and $2n \times 2n$ skew-symmetric matrices, respectively. They can be written in terms of the respective analogues of the algebraic degree of SDP as follows [MMMSV23, (3.3), (6.1) and (7.1)]
$$\phi(n, d) = \frac{1}{n} \sum_{1 \leq \binom{s+1}{2} \leq d} s\delta(d, n, n-s),$$
$$\phi_A(n, d) = \frac{1}{n} \sum_{s=1}^{n-1} r\delta_A(d, n, n-s),$$
$$\phi_D(n, d) = \frac{1}{n} \sum_{\binom{s}{2} \leq d} r\delta_D(d, n, n-s),$$
and their polynomiality follows.

Corollary 4.1.6 ([MMMSV23, Theorem 4.1, 6.13 and 7.12])**.** *For any fixed $d > 0$, the functions $\phi(n, d)$, $\phi_A(n, d)$ and $\phi_D(n, d)$ are polynomials for $n > 0$.*

This chapter is organized as follows. In Section 4.2 we prove some technical lemmas that will be used throughout this chapter, in Section 4.3, 4.4 and 4.5 we find the degree and the leading coefficient of the Lascoux polynomials for types C, A and D respectively. Finally, in Section 4.6 we find the algebraic degrees $\delta(m, n, n-1)$ for types C, A and D.

Remark 4.1.7. We would like to point out that the terminology "Lascoux polynomials" appears in the literature in more than one context not necessarily related to our setting. Our choice is motivated by the definitions in [MMMSV23].

4.2 Four identities

In this chapter we will need the following four identities of rational functions. All the identities are thought to be in $k(x_1, \dots, x_r, y_1, \dots, y_r)$, where k is a field of characteristic zero. We start with a "Double Sum Lemma", expressing the sum of two sets of r variables as a certain sum of rational functions.

Lemma 4.2.1 (Double Sum Lemma)**.** *The identity*

$$\begin{aligned}\sum_{i=1}^{r} x_i + \sum_{j=1}^{r} y_j + r = &\sum_{t=1}^{r} x_t \prod_{k\neq t} \frac{x_k - x_t + 1}{x_k - x_t} \prod_{l=1}^{r} \frac{x_t + y_l + 1}{x_t + y_l} + \\ &+ \sum_{m=1}^{r} y_m \prod_{k\neq m} \frac{y_k - y_m + 1}{y_k - y_m} \prod_{l=1}^{r} \frac{x_l + y_m + 1}{x_l + y_m}\end{aligned} \tag{4.1}$$

holds for every $r \geq 1$.

Proof. We write the right-hand side of (4.1) with a common denominator

$$\frac{\prod_{k>l}(y_k - y_l) A(x_1, \dots, x_r, y_1, \dots, y_r) + \prod_{k>l}(x_k - x_l) B(x_1, \dots, x_r, y_1, \dots, y_r)}{\prod_{k>l}(x_k - x_l)(y_k - y_l) \prod_{k,l=1}^{r}(x_k + y_l)}, \tag{4.2}$$

with

$$\begin{aligned}&A(x_1, \dots, x_r, y_1, \dots, y_r) \\ &= \sum_{t=1}^{r} (-1)^{t-1} x_t \prod_{\substack{k>l \\ k,l\neq t}} (x_k - x_l) \prod_{k\neq t} (x_k - x_t + 1) \prod_{l=1}^{r} (x_t + y_l + 1) \prod_{\substack{k,l=1 \\ k\neq t}}^{r} (x_k + y_l)\end{aligned}$$

and

$$\begin{aligned}&B(x_1, \dots, x_r, y_1, \dots, y_r) \\ &= \sum_{m=1}^{r} (-1)^{m-1} y_m \prod_{\substack{k>l \\ k,l\neq m}} (y_k - y_l) \prod_{k\neq m} (y_k - y_m + 1) \prod_{l=1}^{r} (x_l + y_m + 1) \prod_{\substack{k,l=1 \\ l\neq m}}^{r} (x_k + y_l).\end{aligned}$$

Claim 1: If we swap the roles of x_a and x_b, for some $1 \leq a < b \leq r$, then

$$A(x_1, \dots, x_b, \dots, x_a, \dots, x_r, y_1, \dots, y_r) = -A(x_1, \dots, x_r, y_1, \dots, y_r).$$

We analyze each summand in $A(x_1, \dots, x_r, y_1, \dots, y_r)$ separately. If $t \notin \{a, b\}$, then the only factor in the t-th summand which is affected by the swap is $\prod_{\substack{k>l \\ k,l\neq t}} (x_k - x_l)$. More precisely, the linear forms $(x_k - x_a)$, with $a < k \leq b$ and the linear forms

$(x_b - x_l)$, with $a < l < b$ change sign. As there are $2(b-a)-1$ such factors, there is a change of sign in $\prod_{\substack{k>l\\k,l\neq t}}(x_k - x_l)$. If $t = a$, then the only changes of sign are given by the linear forms $(x_b - x_k)$ with $a < k < b$, as each becomes $-(x_k - x_a)$. This accounts for a factor of $(-1)^{b-a-1}$. Together with $(-1)^{t-1} = (-1)^{a-1}$ we obtain $-(-1)^{b-1}$. Hence the a-th summand of $A(x_1, \dots, x_b, \dots, x_a, \dots, x_r, y_1, \dots, y_r)$ is equal to the b-th summand of $A(x_1, \dots, x_r, y_1, \dots, y_r)$, with the sign changed. The case $t = b$ is analogous. We then have that

$$A(x_1, \dots, x_r, y_1, \dots, y_r) = \prod_{k>l}(x_k - x_l)A'(x_1, \dots, x_r, y_1, \dots, y_r),$$

for some polynomial $A'(x_1, \dots, x_r, y_1, \dots, y_r)$ which is invariant under the transposition of any two x-variables. In the same way we can show that

$$B(x_1, \dots, x_r, y_1, \dots, y_r) = \prod_{k>l}(y_k - y_l)B'(x_1, \dots, x_r, y_1, \dots, y_r),$$

with $B'(x_1, \dots, x_r, y_1, \dots, y_r)$ invariant under the transposition of any two y-variables.

Claim 2: The evaluation of the numerator of (4.2) in $x_a = -y_b$ is equal to 0, for every $1 \leq a, b \leq r$. Let us fix a and b. Observe that $(x_a + y_b)$ is a factor in all summands of $A(x_1, \dots, x_r, y_1, \dots, y_r)$ with $t \neq a$, and it is a factor in all summands of $B(x_1, \dots, x_r, y_1, \dots, y_r)$ with $m \neq b$. We then have that, with $x_a = -y_b$, the numerator of (4.2) equals:

$$\prod_{k>l}(y_k - y_l)\Bigg((-1)^a \mathbf{y_b} \prod_{\substack{k>l\\k,l\neq a}}(x_k - x_l) \prod_{k\neq a}(\mathbf{x_k} + \mathbf{y_b} + \mathbf{1}) \prod_{\mathbf{l}\neq\mathbf{b}}(\mathbf{y_l} - \mathbf{y_b} + \mathbf{1}) \prod_{\substack{k,l=1\\k\neq a\\l\neq b}}^{\mathbf{r}}(\mathbf{x_k} + \mathbf{y_l}) \prod_{\substack{k=1\\k\neq a}}^{r}(x_k - x_a)\Bigg) +$$
$$+\prod_{k>l}(x_k - x_l)\Bigg((-1)^{b-1}\mathbf{y_b} \prod_{\substack{k>l\\k,l\neq b}}(y_k - y_l) \prod_{k\neq b}(\mathbf{y_k} - \mathbf{y_b} + \mathbf{1}) \prod_{\mathbf{l}\neq\mathbf{a}}(\mathbf{x_l} + \mathbf{y_b} + \mathbf{1}) \prod_{\substack{k,l=1\\l\neq b\\k\neq a}}^{\mathbf{r}}(\mathbf{x_k} + \mathbf{y_l}) \prod_{\substack{l=1\\l\neq b}}^{r}(y_l - y_b)\Bigg).$$

Here we have highlighted in bold the factors which are common to the two summands. To conclude the proof of claim 2 we observe that

$$\prod_{k>l}(x_k - x_l) = (-1)^{a-1} \prod_{\substack{k>l\\k,l\neq a}}(x_k - x_l) \prod_{\substack{k=1\\k\neq a}}^{r}(x_k - x_a)$$

and

$$\prod_{k>l}(y_k - y_l) = (-1)^{b-1} \prod_{\substack{k>l \\ k,l\neq b}} (y_k - y_l) \prod_{\substack{l=1 \\ l\neq b}}^{r}(y_l - y_b).$$

This implies that the two summands above contain precisely the same factors in absolute value. As the first is multiplied by $(-1)^{2a-1} = -1$ and the second is multiplied by $(-1)^{2b-2} = 1$, those cancel out.

We conclude that the numerator of (4.2) equals

$$\prod_{k>l}(x_k - x_l) \prod_{k>l}(y_k - y_l) \prod_{k,l=1}^{r} (x_k + y_l) Q(x_1, \dots, x_r, y_1, \dots, y_r),$$

for some polynomial $Q(x_1, \dots, x_r, y_1, \dots, y_r)$ invariant under the transposition of any two x-variables and any two y-variables. Counting the factors in (4.2) shows that the degree of the numerator is at most $2r^2 - r + 1$. Since the degree of the expression above equals $2r^2 - r + \deg(Q(x_1, \dots, x_r, y_1, \dots, y_r))$, this implies that the degree of $Q(x_1, \dots, x_r, y_1, \dots, y_r)$ is at most 1. The vector space of polynomials of degree at most 1 with this symmetry is 3-dimensional, and therefore we can write

$$Q(x_1, \dots, x_r, y_1, \dots, y_r) = \lambda \sum_{i=1}^{r} x_i + \mu \sum_{j=1}^{r} y_j + \nu,$$

for some $\lambda, \mu, \nu \in \mathbb{R}$. In order to prove the lemma we must show that Q is equal to the LHS of (4.1). In other words, we show that $\lambda = \mu = 1$. If we order the variables as $y_r > \cdots > y_1 > x_r > \cdots > x_1$, we obtain that the leading term of the numerator of (4.2) is $y_1^r y_2^{r+1} \cdots y_r^{2r-1} x_2 x_3^2 \cdots x_{r-1}^{r-2} x_r^r$, while the leading term of the denominator equals to $y_1^r y_2^{r+1} \cdots y_r^{2r-1} x_2 x_3^2 \cdots x_r^{r-1}$. The ratio of their coefficients, which is clearly equal to 1, is the coefficient of x_r in $Q(x_1, \dots, x_r, y_1, \dots, y_r)$, namely λ. If we order the variables as $x_r > \cdots > x_1 > y_r > \cdots > y_1$ we obtain in the same way that $\mu = 1$.

Finally, to conclude that $\nu = r$ we substitute $y_k = -x_k - 1$ for every $1 \leq k \leq r$ in the RHS of (4.1). It is immediate to see that it vanishes, as both summands have $(x_t + y_t + 1)$ as a factor, for some t. We deduce that $\sum_{i=1}^{r} x_i + \sum_{j=1}^{r}(-x_j - 1) + \nu = -r + \nu = 0$, and hence $\nu = r$. □

As a corollary, we obtain the following "Sum Lemma".

Corollary 4.2.2 (Sum Lemma). *For all positive integers r the following identity holds:*

$$x_1 + \cdots + x_r = \sum_{l=1}^{r} x_l \prod_{j\neq l} \frac{(x_j - x_l + 1)(x_j + x_l)}{(x_j - x_l)(x_j + x_l - 1)}. \tag{4.3}$$

Proof. On substituting $y_l = x_l - 1$ for all l in (4.1) we get

$$\begin{aligned}\sum_{i=1}^{r} 2x_i &= \sum_{t=1}^{r} x_t \prod_{k\neq t} \frac{x_k - x_t + 1}{x_k - x_t} \prod_{l=1}^{r} \frac{x_t + x_l}{x_t + x_l - 1} + \\ &+ \sum_{m=1}^{r} (x_m - 1) \prod_{k\neq m} \frac{x_k - x_m + 1}{x_k - x_m} \prod_{l=1}^{r} \frac{x_l + x_m}{x_l + x_m - 1} \\ &= \sum_{t=1}^{r} \frac{2x_t^2}{2x_t - 1} \prod_{k\neq t} \frac{(x_k - x_t + 1)(x_t + x_k)}{(x_k - x_t)(x_t + x_k - 1)} + \\ &+ \sum_{m=1}^{r} \frac{2x_m(x_m - 1)}{2x_m - 1} \prod_{k\neq m} \frac{(x_k - x_m + 1)(x_k + x_m)}{(x_k - x_m)(x_k + x_m - 1)} \\ &= \sum_{t=1}^{r} 2x_t \prod_{k\neq t} \frac{(x_k - x_t + 1)(x_t + x_k)}{(x_k - x_t)(x_t + x_k - 1)}.\end{aligned}$$

On cancelling 2 from both sides we get the desired identity. □

Next, we prove a "Double Product Lemma", involving the product of two sets of variables.

Lemma 4.2.3 (Double Product Lemma)**.** *The identity*

$$\begin{aligned}&\frac{\prod_{k=1}^{r} x_k \prod_{k=1}^{r} y_k}{\prod_{k=1}^{r}(x_k + 1) \prod_{k=1}^{r}(y_k + 1)} \\ &= 1 - \sum_{l=1}^{r} \frac{1}{x_l + 1} \prod_{k=1}^{r} \frac{x_l + y_k + 1}{x_l + y_k + 2} \prod_{l\neq k=1}^{r} \frac{x_k - x_l - 1}{x_k - x_l} - \\ &\quad - \sum_{l=1}^{r} \frac{1}{y_l + 1} \prod_{k=1}^{r} \frac{y_l + x_k + 1}{y_l + x_k + 2} \prod_{l\neq k=1}^{r} \frac{y_k - y_l - 1}{y_k - y_l}\end{aligned} \tag{4.4}$$

holds for every $r \geq 1$.

Proof. First, we multiply both sides of (4.4) by $\prod_{k=1}^{r}(x_k + 1) \prod_{k=1}^{r}(y_k + 1)$. Thus, we are proving the following identity:

$$\begin{aligned}\prod_{k=1}^{r} x_k \prod_{k=1}^{r} y_k &= \prod_{k=1}^{r}(x_k + 1)(y_k + 1) - \\ &- \sum_{l=1}^{r} \prod_{l\neq k=1}^{r} (x_k + 1) \prod_{k=1}^{r}(y_k + 1) \prod_{k=1}^{r} \frac{x_l + y_k + 1}{x_l + y_k + 2} \prod_{l\neq k=1}^{r} \frac{x_k - x_l - 1}{x_k - x_l} - \\ &- \sum_{l=1}^{r} \prod_{l\neq k=1}^{r} (y_k + 1) \prod_{k=1}^{r}(x_k + 1) \prod_{k=1}^{r} \frac{y_l + x_k + 1}{y_l + x_k + 2} \prod_{l\neq k=1}^{r} \frac{y_k - y_l - 1}{y_k - y_l}.\end{aligned}$$

We can put everything on the right-hand side to the common denominator to obtain

$$\text{RHS} = \frac{A(x_1, \ldots, x_r, y_1, \ldots, y_r)}{\prod_{1\leq k<l\leq r}(x_k - x_l)(y_k - y_l) \prod_{1\leq k,l\leq r}(y_l + x_k + 2)},$$

where $A(x_1,\dots,x_r,y_1,\dots,y_r)=A(\overline{x},\overline{y})$ is a polynomial of degree at most $2r^2+r$.

The next step is to see what happens when we exchange the values of x_i and x_j. Clearly, the right-hand side does not change its value. However, the denominator in the equation above changes sign, thus also the polynomial $A(\overline{x},\overline{y})$ must change sign. That means that $A(\overline{x},\overline{y})$ is divisible by $\prod_{1\le k<l\le r}(x_k-x_l)$ and after dividing we obtain a symmetric polynomial in $x_1,\dots,x_r$.

Analogously, the same holds when we exchange y_i and y_j, and we can write

$$A(\overline{x},\overline{y})=\prod_{1\le k<l\le r}(x_k-x_l)(y_k-y_l)\cdot B(\overline{x},\overline{y}),$$

where $B(\overline{x},\overline{y})$ is a polynomial of degree at most r^2+2r, symmetric in both $x_1,\dots x_r$ and $y_1,\dots,y_r$.

Now we multiply the RHS by (x_1+y_1+2) and plug in $x_1+y_1+2=0$. Clearly all summands except those corresponding to $l=1$ vanish. Moreover for the two summands left we have

$$\begin{aligned}&-(x_1+y_1+1)\prod_{k=2}^{r}(x_k+1)\prod_{k=1}^{r}(y_k+1)\prod_{k=2}^{r}\frac{(x_1+y_k+1)(x_k-x_1-1)}{(x_1+y_k+2)(x_k-x_1)}-\\&-(x_1+y_1+1)\prod_{k=2}^{r}(y_k+1)\prod_{k=1}^{r}(x_k+1)\prod_{k=2}^{r}\frac{(y_1+x_k+1)(y_k-y_1-1)}{(y_1+x_k+2)(y_k-y_1)}.\end{aligned}$$

After substituting $y_1=-2-x_1$, we obtain

$$\begin{aligned}&-(-1)(-x_1-1)\prod_{k=2}^{r}(y_k+1)(x_k+1)\prod_{k=2}^{r}\frac{(x_1+y_k+1)(x_k-x_1-1)}{(x_1+y_k+2)(x_k-x_1)}-\\&-(-1)(x_1+1)\prod_{k=2}^{r}(x_k+1)(y_k+1)\prod_{k=2}^{r}\frac{(x_k-x_1-1)(y_k+x_1+1)}{(x_k-x_1)(y_k+x_1+2)}=0.\end{aligned}$$

This implies that the polynomial $B(\overline{x},\overline{y})$ must be divisible by (x_1+y_1+2). From symmetry it is also divisible by (x_i+y_j+2) for any i,j.

Hence,

$$B(\overline{x},\overline{y})=\prod_{1\le k,l\le r}(x_k+y_l+2)\cdot C(\overline{x},\overline{y}),$$

where $C(\overline{x},\overline{y})$ is a polynomial of degree at most $2r$, symmetric in $\overline{x}$ and $\overline{y}$. We then have that RHS $=C(\overline{x},\overline{y})$.

Next we plug in $x_1=-1$. Again, every term, except the one from the first sum for

$l = 1$ is 0. Thus, we get

$$\begin{aligned} & C(-1, x_2, \ldots, x_r, y_1, \ldots, y_r) \\ = & -\prod_{k=2}^{r}(x_k+1)\prod_{k=1}^{r}(y_k+1)\prod_{k=1}^{r}\frac{x_1+y_k+1}{x_1+y_k+2}\prod_{k=2}^{r}\frac{x_k-x_1-1}{x_k-x_1} \\ = & -\prod_{k=2}^{r}(x_k+1)\prod_{k=1}^{r}(y_k+1)\prod_{k=1}^{r}\frac{y_k}{y_k+1}\prod_{k=2}^{r}\frac{x_k}{x_k+1} \\ = & -\prod_{k=2}^{r}(x_k)\prod_{k=1}^{r}(y_k) = \prod_{k=1}^{r}(x_k y_k). \end{aligned}$$

Therefore, for $x_1 = -1$, the desired equality holds. Analogously, the same is true for any $x_i = -1$ or $y_i = -1$. This means that $C(\overline{x}, \overline{y}) - \prod_{k=1}^{r}(x_k y_k)$ is a polynomial divisible by $\prod_{k=1}^{r}(x_k+1)(y_k+1)$.

Furthermore, the only term of degree $2r$ in RHS is the first term $\prod_{k=1}^{r}(x_k+1)(y_k+1)$, as all other summands have degree at most $2r-1$. In particular, the degree $2r$ part of $C(\overline{x}, \overline{y})$ is equal to $\prod_{k=1}^{r}(x_k y_k)$, and the difference $C(\overline{x}, \overline{y}) - \prod_{k=1}^{r}(x_k y_k)$ is of degree at most $2r-1$. Since it is divisible by a degree $2r$ polynomial, it must be 0, which proves the lemma. □

We conclude this section with a "Product Lemma" derived from Lemma 4.2.3, in a similar way as Corollary 4.2.2 was obtained from Lemma 4.2.1.

Corollary 4.2.4 (Product Lemma). *For every positive integer r the following identity holds:*

$$x_1 \cdots x_r = \prod_{j=1}^{r}(x_j+2) - 2\sum_{l=1}^{r}\prod_{l\neq j=1}^{r}\frac{(x_j+2)(x_j-x_l-1)(x_j+x_l+2)}{(x_j-x_l)(x_j+x_l+3)}.$$

Proof. For every $1 \le i \le r$, we specialize the identity in Lemma 4.2.3 with $y_i = x_i + 1$. The left-hand side of the identity is then equal to $\frac{\prod_{k=1}^{r} x_k}{\prod_{k=1}^{r}(x_k+2)}$. For the right-hand side, we obtain

$$\begin{aligned} & 1 - \sum_{l=1}^{r}\frac{1}{x_l+1}\prod_{k=1}^{r}\frac{x_l+x_k+2}{x_l+x_k+3}\prod_{l\neq k=1}^{r}\frac{x_k-x_l-1}{x_k-x_l} - \\ & - \sum_{l=1}^{r}\frac{1}{x_l+2}\prod_{k=1}^{r}\frac{x_l+x_k+2}{x_l+x_k+3}\prod_{l\neq k=1}^{r}\frac{x_k-x_l-1}{x_k-x_l} \\ = & 1 - \sum_{l=1}^{r}\left(\frac{1}{x_l+1}\cdot\frac{2x_l+2}{2x_l+3} + \frac{1}{x_l+2}\cdot\frac{2x_l+2}{2x_l+3}\right)\prod_{l\neq k=1}^{r}\frac{(x_k-x_l-1)(x_l+x_k+2)}{(x_k-x_l)(x_l+x_k+3)} \\ = & 1 - \sum_{l=1}^{r}\frac{2}{x_l+2}\prod_{l\neq k=1}^{r}\frac{(x_k-x_l-1)(x_l+x_k+2)}{(x_k-x_l)(x_l+x_k+3)}. \end{aligned}$$

Multiplying both sides by $\prod_{j=1}(x_j+2)$ yields the desired identity. □

4.3 Type C

The Lascoux polynomials play an essential role in proving the polynomiality of the ML-degrees of linear concentration models. In this section we study the leading coefficient of these polynomials. Following [MMMSV23], we start by setting the notation and recalling the definition of Schur polynomials, Lascoux coefficients and Lascoux polynomials.

A *partition* λ is a nonincreasing sequence $(\lambda_1, \dots, \lambda_r)$ of nonnegative integers λ_i. The *length* of the partition is the length of the sequence, the *weight* is $\sum \lambda = \sum_{i=1}^r \lambda_i$. For a set $I = \{i_1, \dots, i_r\}$ of nonnegative integers with $i_1 < i_2 < \cdots < i_r$, we denote with $|I|$ its cardinality and with $\sum I = \sum_{j=1}^r i_j$. We associate to I the corresponding partition

$$\lambda(I) = (i_r - (r-1), i_{r-1} - (r-2), \dots, i_2 - 1, i_1).$$

For a partition λ of length k its associated *Schur polynomial* s_λ is defined as follows:

$$s_\lambda(x_1, \dots, x_k) = \frac{\det(x_j^{\lambda_i + k - i})_{ij}}{\det(x_j^{k-i})_{ij}}.$$

Note that the denominator of s_λ is the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$. The degree of s_λ is equal to the weight $\sum \lambda$ of the partition. As an example, the elementary symmetric polynomial in k variables of degree r is the Schur polynomial with partition $\lambda = (\underbrace{1, \dots, 1}_{r}, 0, \dots, 0)$ of length k:

$$s_\lambda(x_1, \dots, x_k) = \sum_{i_1 < \cdots < i_r} x_{i_1} \dots x_{i_r}.$$

Throughout the chapter, the leading coefficient of a polynomial p will be denoted by $LC(p)$.

Definition 4.3.1. The *Lascoux coefficients* are the numbers ψ_I such that the following identity holds:

$$s_{(d,0,\dots,0)}(\{x_i + x_j : 1 \le i \le j \le k\}) = \sum_{\substack{|I|=k \\ \sum \lambda(I) = d}} \psi_I s_{\lambda(I)}(x_1, \dots, x_k).$$

The *Lascoux polynomial* is the following function:

$$\mathrm{LP}_I(n) = \begin{cases} \psi_{[n]\setminus I} & \text{if } I \subseteq [n], \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.3.1. Notice that the previous definition differs from the one given in the Introduction. In fact, there are many equivalent ways of defining the Lascoux coefficients. While the one given in the Introduction may be easier to state, the definition used here has the advantage of being naturally extended also for types

A and D. However, in this article we will just make use of the recurrence relations from [MMMSV23], without worrying too much about which definition we use. There is also a geometrical way of defining Lascoux coefficients. More precisely, they are the Segre classes of the second symmetric power of the universal bundle over the Grassmannian. For more definitions and formulas about Lascoux coefficients we refer the reader to [LLT89, Appendix].

The authors in [MMMSV23] give different proofs that the Lascoux polynomials are indeed polynomials. The first is the simplest one, and is based on the following recurrence relations:

We fix $I = \{i_1 < i_2 < \cdots < i_r\} \subset \mathbb{N}$.

1. If $i_1 = 0$, then

$$\mathrm{LP}_I(n) = (n - r + 1)\,\mathrm{LP}_{I\setminus\{0\}}(n) - 2 \sum_{\substack{l>1 \\ i_{l+1}>i_l+1}} \mathrm{LP}_{I\cup\{i_l+1\}\setminus\{0,i_l\}}(n). \tag{4.5}$$

2. If $i_1 > 0$, then

$$\mathrm{LP}_I(n) - \mathrm{LP}_I(n-1) = \sum_{\epsilon\in\{0,1\}^r\setminus 0} \mathrm{LP}_{I-\epsilon}(n-1), \tag{4.6}$$

   where $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$, $I - \epsilon := \{i_1 - \epsilon_1, \ldots, i_r - \epsilon_r\}$ and $\mathrm{LP}_{I-\epsilon} = 0$ if there is a repeated element in $I - \epsilon$.

The degree and leading coefficient of the Lascoux polynomials is also known:

Theorem 4.3.2 ([MMMSV23, Theorem 4.12]). $\deg \mathrm{LP}_I = |I| + \sum I$, *the leading coefficient is*

$$\frac{\prod_{j>k}(i_j - i_k)}{(i_1+1)!\cdots(i_r+1)!\prod_{j>k}(i_j+i_k+2)}.$$

However, the proof of this theorem in [MMMSV23] does not use the recurrence relations (4.5), (4.6), but a completely different approach, where the Lascoux coefficients ψ_I are expressed as a sum of minors of the Pascal triangle matrix, using the definition given in the Introduction. Here instead, we provide a direct proof of this theorem, by using just the recurrence relations (4.5), (4.6). Moreover, this method will be also useful in computing the degree and leading coefficient of Lascoux polynomials in types A and D, which is a new result in this article.

Proof of Theorem 4.3.2. We proceed analogously as in the first proof of polynomiality of Lascoux polynomials in [MMMSV23]. Thus, we proceed by induction, first on $|I|$, then on $\sum I$.

The base case is $I = \emptyset$, when $\mathrm{LP}_I = 1$ and the statement holds.

For the set I we define ℓp_I to be the coefficient of $n^{|I|+\sum I}$ in $\mathrm{LP}_I(n)$. Next, we fix I and assume that the statement is true for all I' with $|I'| < |I|$ or $|I'| = |I|$ and $\sum I' < \sum I$. we consider two cases.

Case 1: $i_1 = 0$. Then

$$\mathrm{LP}_I(n) = (n-r+1)\,\mathrm{LP}_{I\setminus\{0\}}(n) - 2 \sum_{\substack{l>1 \\ i_{l+1}>i_l+1}} \mathrm{LP}_{I\cup\{i_l+1\}\setminus\{0,i_l\}}(n).$$

By the induction hypothesis, all terms on the right-hand side are polynomials of degree $|I|+\sum I$. Also from the induction hypothesis we know their leading coefficients. Moreover, in the sum we can ignore the condition for $i_{l+1} > i_l$ simply by defining $\mathrm{LP}_{I'} := 0$, if I' has repeated elements. Note that the formula for the leading coefficient holds in this case, since it is 0. Thus, by comparing the coefficients of $n^{|I|+\sum I}$ on both sides we get:

$$\begin{aligned}
\ell p_I &= \ell p_{I\setminus\{0\}} - 2\sum_{l>1} \ell p_{I\cup\{i_l+1\}\setminus\{0,i_l\}} \\
&= \frac{\prod_{j>k>1}(i_j-i_k)}{(i_2+1)!\cdots(i_r+1)!\prod_{j>k>1}(i_j+i_k+2)} - \\
&\quad - 2\sum_{l=2}^{r} \frac{\prod_{j>k>1}(i_j-i_k)}{(i_2+1)!\cdots(i_r+1)!\prod_{j>k>1}(i_j+i_k+2)} \cdot \\
&\quad \cdot \frac{1}{i_l+2} \cdot \prod_{l\neq j=2}^{r} \frac{(i_j-i_l-1)(i_j+i_l+2)}{(i_j-i_l)(i_j+i_l+3)} \\
&= \frac{\prod_{j>k>1}(i_j-i_k)}{(i_2+1)!\cdots(i_r+1)!\prod_{j>k>1}(i_j+i_k+2)} \cdot \\
&\quad \cdot \left(1 - 2\sum_{l=2}^{r} \frac{1}{i_l+2} \prod_{l\neq j=2}^{r} \frac{(i_j-i_l-1)(i_j+i_l+2)}{(i_j-i_l)(i_j+i_l+3)}\right) \\
&= \frac{\prod_{j>k>1}(i_j-i_k)}{(i_2+1)!\cdots(i_r+1)!\prod_{j>k}(i_j+i_k+2)} \cdot \\
&\quad \cdot \left(\prod_{l}(i_l+2) - 2\sum_{l=2}^{r}\prod_{l\neq j=2}^{r} \frac{(i_l+2)(i_j-i_l-1)(i_j+i_l+2)}{(i_j-i_l)(i_j+i_l+3)}\right) \\
&= \frac{\prod_{j>k>1}(i_j-i_k)}{(i_2+1)!\cdots(i_r+1)!\prod_{j>k}(i_j+i_k+2)} \cdot i_2\cdots i_k \\
&= \frac{\prod_{j>k}(i_j-i_k)}{(i_2+1)!\cdots(i_r+1)!\prod_{j>k}(i_j+i_k+2)},
\end{aligned}$$

where we applied Corollary 4.2.4 for $i_2,\ldots,i_r$ with $r=k$, $x_1 = 0$ and $x_j = i_j$ for $2 \le j \le k$. This proves the theorem in this case.

Case 2: $i_1 > 0$. Then

$$\mathrm{LP}_I(n) - \mathrm{LP}_I(n-1) = \sum_{\epsilon\in\{0,1\}^r\setminus\mathbf{0}} \mathrm{LP}_{I-\epsilon}(n-1).$$

By the induction hypothesis, all terms on the right-hand side are polynomials of degree $|I|+\sum I-\sum_{i=1}^{r}\epsilon_i$ with positive leading coefficients. Thus, the right-hand side is a polynomial of degree $|I|+\sum I-1$, and to the coefficient of $n^{|I|+\sum I-1}$ contribute only terms for $\sum_{i=1}^{r}\epsilon_i=1$.

It follows that LP_I is a polynomial of degree $|I|+\sum I$ and the coefficient of $n^{|I|+\sum I-1}$ is $(|I|+\sum I)\ell p_I$. Using the induction hypothesis we can compare the leading coefficients of both sides and get:

$$\begin{aligned}
&(i_1+\cdots+i_r+r)\ell p_I\\
&=\sum_{l=1}^{r}\frac{\prod_{j>k}(i_j-i_k)}{(i_1+1)!\cdots(i_r+1)!\prod_{j>k}(i_j+i_k+2)}(i_l+1)\prod_{l\neq j=1}^{r}\frac{(i_j-i_l+1)(i_j+i_l+2)}{(i_j-i_l)(i_j+i_l+1)}\\
&=\frac{\prod_{j>k}(i_j-i_k)}{(i_1+1)!\cdots(i_r+1)!\prod_{j>k}(i_j+i_k+2)}\left(\sum_{l=1}^{r}(i_l+1)\prod_{l\neq j=1}^{r}\frac{(i_j-i_l+1)(i_j+i_l+2)}{(i_j-i_l)(i_j+i_l+1)}\right)\\
&=\frac{\prod_{j>k}(i_j-i_k)}{(i_1+1)!\cdots(i_r+1)!\prod_{j>k}(i_j+i_k+2)}\cdot(i_1+\cdots+i_r+r),
\end{aligned}$$

where we used Corollary 4.2.2 with $x_j=i_j+1$ for $1\leq j\leq r$. Then the statement follows by cancelling $(i_1+\cdots+i_r+r)$ from both sides. □

4.4 Type A

In [MMMSV23, Section 6], the authors have defined the Type A Lascoux functions. They have also proved that these functions are indeed polynomials in n. The aim of this section is to find a formula for the leading coefficients and the degrees of these polynomials using the recurrence relations given in [MMMSV23, Lemma 6.10, Theorem 6.11]. The proof is very similar to the one for type C given in Section 4.3.

Definition 4.4.1. For $X=(x_1,\dots,x_k)$ and $Y=(y_1,\dots,y_l)$ two sets of indeterminates, denote by $X+Y$ the set of indeterminates $\{x_i+y_j, 1\leq i\leq k, 1\leq j\leq l\}$. The *Lascoux coefficients of type* A are the numbers $d_{I,J}$ such that the following identity holds:

$$s_{(d,0,\dots,0)}(X+Y)=\sum_{\substack{|I|=k,|J|=l\\ \sum\lambda(I)+\sum\lambda(J)=d}} d_{I,J}s_{\lambda(I)}(X)s_{\lambda(J)}(Y).$$

The *Lascoux polynomials of type* A are given by

$$\mathrm{LP}^{A}_{I,J}(n)=\begin{cases} d_{[n]\setminus I,[n]\setminus J} & \text{if } I,J\subseteq[n],\\ 0 & \text{otherwise.}\end{cases}$$

The Lascoux polynomials of type A are indeed polynomial functions in n [MMMSV23, Theorem 6.11]. These polynomials satisfy the following two recurrence relations [MMMSV23, Lemma 6.10, Theorem 6.11]. Fix $I=\{i_1<i_2<\cdots<i_r\}\subset\mathbb{N}$ and $J=\{j_1<j_2<\cdots<j_r\}\subset\mathbb{N}$.

1. If $i_1 = 0$ and $j_1 = 0$, then

$$\begin{aligned} \mathrm{LP}^A_{I,J}(n) = (n-r+1)\,\mathrm{LP}_{I\setminus\{0\},J\setminus\{0\}}(n)- \\ - \sum_{\ell: i_{\ell+1} > i_\ell + 1} \mathrm{LP}^A_{I\setminus\{0,i_\ell\}\cup\{i_\ell+1\},J\setminus\{0\}}(n)- \\ - \sum_{\ell: j_{\ell+1} > j_\ell + 1} \mathrm{LP}^A_{I\setminus\{0\},J\setminus\{0,j_\ell\}\cup\{j_\ell+1\}}(n). \end{aligned} \tag{4.7}$$

2. Otherwise, if $i_1 > 0$ or $j_1 > 0$,

$$\mathrm{LP}^A_{I,J}(n) = \sum_{I',J'} \mathrm{LP}^A_{I',J'}(n-1) \tag{4.8}$$

   where the sum is over all pairs (I', J') of the form $(\{i_1-\epsilon_1,\dots,i_r-\epsilon_r\},\{j_1-\mu_1,\dots,j_r-\mu_r\})$, where $\epsilon_l, \mu_l \in \{0,1\}$.

Remark 4.4.1. The degree of the Lascoux polynomials of type A satisfies the following inequality:

$$\deg(\mathrm{LP}^A_{I,J}(n)) \le |I| + \sum I + \sum J.$$

Theorem 4.4.2. *For sets $I = \{i_1, ..., i_r\}$, $J = \{j_1, ..., j_r\}$, the degree of the Lascoux polynomials of type A is given by the following expression on I, J:*

$$\deg(\mathrm{LP}^A_{I,J}(n)) = |I| + \sum I + \sum J$$

and the leading coefficient of $\mathrm{LP}^A_{I,J}$ is

$$\frac{\prod_{k>l}(i_k - i_l)\prod_{k>l}(j_k - j_l)}{\prod_{k,l=1}^r (i_k + j_l + 1)\prod_{k=1}^r (i_k)!\prod_{k=1}^r (j_k)!}.$$

Proof. We will proceed by induction, first on $|I|$ and then on $\sum I + \sum J$. The proof is analogous to the proof of Theorem 4.3.2 in the Type C case. We denote $\ell_{I,J}$ to be the coefficient of $n^{|I|+\sum I + \sum J}$ in $\mathrm{LP}_{I,J}(n)$. As $\mathrm{LP}_{I,J}$ have two recurrence relations given in (4.7) and (4.8), we will get corresponding recurrence relations for $\ell_{I,J}$.

First recursion:
From the first recursion (4.7) by comparing the coefficients of degree $|I| + \sum I + \sum J$ we get

$$\ell_{I,J} = \ell_{I_0,J_0} - \sum_{\substack{l>1 \\ i_{l+1} > i_l + 1}} \ell_{I_l,J_0} - \sum_{\substack{l>1 \\ j_{l+1} > j_l + 1}} \ell_{I_0,J_l}$$

where $I_0 = I\setminus\{0\}$, $J_0 = J\setminus\{0\}$, $I_l = I\cup\{i_l+1\}\setminus\{0,i_l\}$ and $J_l = J\cup\{j_l+1\}\setminus\{0,j_l\}$. Now write

$$\ell_{I_0,J_0} = \frac{\prod_{k>t>1}(i_k-i_t)\prod_{k>t>1}(j_k-j_t)}{\prod_{k,t=2}^{r}(i_k+j_t+1)\prod_{k=2}^{r}(i_k)!\prod_{k=2}^{r}(j_k)!}$$
$$\ell_{I_l,J_0} = \frac{\prod_{k>t>1}(i_k-i_t)\prod_{k>t>1}(j_k-j_t)}{\prod_{k,t=2}^{r}(i_k+j_t+1)\prod_{k=2}^{r}(i_k)!\prod_{k=2}^{r}(j_k)!}\cdot$$
$$\cdot\frac{1}{i_l+1}\prod_{k=2}^{r}\frac{i_l+j_k+1}{i_l+j_k+2}\prod_{l\neq k=2}^{r}\frac{i_k-i_l-1}{i_k-i_l}$$
$$\ell_{I_0,J_l} = \frac{\prod_{k>t>1}(i_k-i_t)\prod_{k>t>1}(j_k-j_t)}{\prod_{k,t=2}^{r}(i_k+j_t+1)\prod_{k=2}^{r}(i_k)!\prod_{k=2}^{r}(j_k)!}\cdot$$
$$\cdot\frac{1}{j_l+1}\prod_{k=2}^{r}\frac{j_l+i_k+1}{j_l+i_k+2}\prod_{l\neq k=2}^{r}\frac{j_k-j_l-1}{j_k-j_l}.$$

Note that

$$\ell_{I,J} = \frac{\prod_{k=1}^{r} i_k\prod_{k=1}^{r} j_k}{\prod_{k=1}^{r}(i_k+1)\prod_{k=1}^{r}(j_k+1)}\cdot\ell_{I_0,J_0}.$$

Now write

$$\ell_{I,J} = \ell_{I_0,J_0}\Bigg(1-\sum_{l>1}\frac{1}{i_l+1}\prod_{k=2}^{r}\frac{i_l+j_k+1}{i_l+j_k+2}\prod_{l\neq k=2}^{r}\frac{i_k-i_l-1}{i_k-i_l}-$$
$$-\sum_{l>1}\frac{1}{j_l+1}\prod_{k=2}^{r}\frac{j_l+i_k+1}{j_l+i_k+2}\prod_{l\neq k=2}^{r}\frac{j_k-j_l-1}{j_k-j_l}\Bigg)$$

and apply Lemma 4.2.3 with $x_k = i_k$ and $y_k = j_k$ for every $1\le k\le r$.

Second recursion:
From the second recursion (4.8) by comparing the coefficients of degree $|I|+\sum I+\sum J-1$ we get

$$\deg \mathrm{LP}_{I,J}\cdot\ell_{I,J} = \sum_{t=1}^{r}\ell_{I_t,J}+\sum_{t=1}^{r}\ell_{I,J_t}$$

where $I_t = \{i_1,\dots,i_t-1,\dots,i_r\}$ and $J_t = \{j_1,\dots,j_t-1,\dots,j_r\}$. Now write

$$\ell_{I_t,J} = \frac{\prod_{k>l}(i_k-i_l)\prod_{k>l}(j_k-j_l)}{\prod_{k,l=1}^{r}(i_k+j_l+1)\prod_{k=1}^{r}(i_k)!\prod_{k=1}^{r}(j_k)!}i_t\prod_{k\neq t}\frac{i_k-i_t+1}{i_k-i_t}\prod_{l=1}^{r}\frac{i_t+j_l+1}{i_t+j_l},$$

$$\ell_{I,J_t} = \frac{\prod_{k>l}(i_k-i_l)\prod_{k>l}(j_k-j_l)}{\prod_{k,l=1}^{r}(i_k+j_l+1)\prod_{k=1}^{r}(i_k)!\prod_{k=1}^{r}(j_k)!}j_t\prod_{k\neq t}\frac{j_k-j_t+1}{j_k-j_t}\prod_{l=1}^{r}\frac{i_l+j_t+1}{i_l+j_t}.$$

Therefore,

$$\deg(\mathrm{LP}^A_{I,J}(n)) = \sum_{t=1}^{r} i_t \prod_{k\neq t} \frac{i_k - i_t + 1}{i_k - i_t} \prod_{l=1}^{r} \frac{i_t + j_l + 1}{i_t + j_l} + \\ + \sum_{m=1}^{r} j_m \prod_{k\neq m} \frac{j_k - j_m + 1}{j_k - j_m} \prod_{l=1}^{r} \frac{i_l + j_m + 1}{i_l + j_m},$$

which is equal to $|I| + \sum I + \sum J$ by Lemma 4.2.1 with $x_k = i_k$ and $y_k = j_k$ for every $1 \leq k \leq r$. □

4.5 Type D

In this section, we turn our attention to the type D case, and proceed in a way analogous to the previous sections. The Lascoux functions for type D were first defined in [MMMSV23, Section 7]. Here we provide a formula for their degrees and leading coefficients.

Definition 4.5.1. The *Lascoux coefficients of type* D are the numbers α_I which verify the identity

$$s_{(d,0,\ldots,0)}(\{x_i + x_j : 1 \leq i < j \leq n\}) = \sum_{\substack{|I|=n \\ \sum \lambda(I)=d}} \alpha_I s_{\lambda(I)}(x_1, \ldots, x_n).$$

For any increasing sequence $I = \{i_1, \ldots, i_s\}$ of nonnegative integers the *Lascoux quasipolynomial of type* D is

$$\mathrm{LP}^D_I(n) = \begin{cases} \alpha_{[n]\setminus I} & I \subseteq [n], \\ 0 & \text{otherwise.} \end{cases}$$

In [MMMSV23, Theorem 7.10] it was proved that $\mathrm{LP}^D_I(n)$ is a quasipolynomial of period 2, in other words, $\mathrm{LP}^D_I(2n)$ and $\mathrm{LP}^D_I(2n-1)$ are polynomials in n. The proof of this result uses the following recursive relations. Fix $I = \{i_1 < i_2 < \cdots < i_r\} \subset \mathbb{N}$.

1. If $i_1 = 0$, then

$$\mathrm{LP}^D_I(n) = \begin{cases} \mathrm{LP}^D_{I\setminus\{0\}}(n) & \text{if } n - |I| \text{ is even}, \\ 0 & \text{if } n - |I| \text{ is odd}. \end{cases} \tag{4.9}$$

2. If $i_1 > 0$, then

$$\mathrm{LP}^D_I(n) - \mathrm{LP}^D_I(n-1) = \sum_{\epsilon \in \{0,1\}^n \setminus 0} \mathrm{LP}^D_{I-\epsilon}(n-1), \tag{4.10}$$

   where $\epsilon = (\epsilon_1, \ldots, \epsilon_r)$, $I - \epsilon := \{i_1 - \epsilon_1, \ldots, i_r - \epsilon_r\}$ and $\mathrm{LP}^D_{I-\epsilon} = 0$ if $|I - \epsilon| < r$. In the main result of this section we compute the degree and the leading coefficient of the quasipolynomials $\mathrm{LP}^D_I(n)$.

Theorem 4.5.1. *Let $I = \{i_1 < \cdots < i_r\} \subset \mathbb{N}$ be a set of nonnegative integers. Then*

- *If $i_1 > 0$, $\mathrm{LP}^D_I(2n)$ and $\mathrm{LP}^D_I(2n+1)$ are polynomials in n of degree $\sum I$ and leading coefficient equal to*

$$\frac{2^{\sum I - |I|}\prod_{k>l}(i_k - i_l)}{\prod_{k>l}(i_k + i_l)\prod_k (i_k)!}.$$

- *If $i_1 = 0$, then $\mathrm{LP}^D_I(n) = \mathrm{LP}^D_{I\setminus\{0\}}(n)$ if $n - |I|$ is even, and $\mathrm{LP}^D_I(n) = 0$ if $n - |I|$ is odd.*

Proof. We fix a set $I = \{i_1 < \cdots < i_r\} \subset \mathbb{N}$. For the case $i_1 = 0$, the statement follows from the induction hypothesis.

In the case $i_1 = 0$, the statement is nothing but the recurrence relation (4.9). Now we consider the case $i_1 > 0$ and proceed by induction.

Assume that the statement holds for all I' with $|I'| < |I|$ or $|I'| = |I|$ and $\sum I' < \sum I$. By applying (4.10) we obtain

$$\mathrm{LP}^D_I(2n) - \mathrm{LP}^D_I(2(n-1)) = \sum_{\epsilon\in\{0,1\}^n\setminus\mathbf{0}} \mathrm{LP}^D_{I-\epsilon}(2n-1) + \sum_{\epsilon\in\{0,1\}^n\setminus\mathbf{0}} \mathrm{LP}^D_{I-\epsilon}(2n-2). \quad (4.11)$$

On the right-hand side we get sum of polynomials of degree at most $\sum I - 1$. Moreover, the polynomials with this degree are only those with $\epsilon_1 + \cdots + \epsilon_r = 1$. Let ℓ^D_I be the coefficient of $n^{\sum I}$ in $\mathrm{LP}^D_I(2n)$, and let $\mathbf{e}_i \in \{0,1\}^n$ be the i-th vector of the canonical basis of $\mathbb{Z}^n$. Comparing the coefficients of $n^{\sum I - 1}$ of both sides of (4.11) and using the induction hypothesis on $\ell^D_{I-\mathbf{e}_j}$ we obtain

$$\begin{aligned}
\deg(\mathrm{LP}^D_I(2n))\ell^D_I &= 2\sum_{j=1}^{r} \ell^D_{I-\mathbf{e}_j} \\
= 2\sum_{j=1}^{r} &\left(\frac{\prod_{k>l}(i_k - i_l)2^{\sum I - |I| - 1}}{\prod_{k>l}(i_k + i_l)\prod_k (i_k)!} \cdot i_j \prod_{k\neq j} \frac{(i_k + i_j)(i_k - i_j + 1)}{(i_k - i_j)(i_k + i_j - 1)} \right) \\
= \frac{\prod_{k>l}(i_k - i_l)2^{\sum I - |I|}}{\prod_{k>l}(i_k + i_l)\prod_k (i_k)!} &\left(\sum_{j=1}^{r} i_j \prod_{k\neq j} \frac{(i_k + i_j)(i_k - i_j + 1)}{(i_k - i_j)(i_k + i_j - 1)} \right).
\end{aligned}$$

Notice that if $i_1 = 1$, one of $\mathrm{LP}_{\{0,i_2,\ldots,i_r\}}(2n-1)$ and $\mathrm{LP}_{\{0,i_2,\ldots,i_r\}}(2n-2)$ is zero and the other is equal to $\mathrm{LP}^D_{\{i_2,\ldots,i_r\}}(2n-1)$ (or $\mathrm{LP}^D_{\{i_2,\ldots,i_r\}}(2n-2)$) whose leading coefficient is also the same as the term included in the expression above. By Corollary 4.2.2, the last expression is equal to

$$\frac{\prod_{k>l}(i_k - i_l)2^{\sum I - |I|}}{\prod_{k>l}(i_k + i_l)\prod_k (i_k)!} \left(\sum_{j=1}^{r} i_j \right).$$

As this quantity is not zero for any set I, we have that $\ell_I^D \neq 0$, and hence that $\deg(\mathrm{LP}_I^D(2n)) = \sum I$. It follows that $n \mapsto \mathrm{LP}_I^D(2n)$ is a polynomial function in n with degree $\sum I$ and leading coefficient

$$\frac{\prod_{k>l}(i_k - i_l)2^{\sum I - |I|}}{\prod_{k>l}(i_k + i_l)\prod_k (i_k)!}.$$

The case of the polynomial function for $n \mapsto \mathrm{LP}_I^D(2n-1)$ is completely analog. This concludes the proof. □

4.6 The algebraic degrees $\delta(m,n,n-1)$, $\delta_A(m,n,n-1)$ and $\delta_D(m,n,n-1)$

One of the applications of the results in [MMMSV23] establishes polynomiality of a sequence of positive integers attached to *semidefinite programming*. This is the problem of optimizing a linear function over the cone of positive semidefinite matrices. In [NRS10] the authors study the complexity of computing an exact solution for this optimization problem, and they quantify this complexity via the degree of a projective variety. Similar degrees can be defined for optimization problems related to the space of general and skew-symmetric matrices. We recall that for a variety $X \subseteq \mathbb{P}^n$ the *projective dual* $X^* \subseteq (\mathbb{P}^n)^*$ is the closure of the set of hyperplanes tangent to X at a smooth point. In the next definition we follow the notation in [MMMSV23].

Definition 4.6.1 ([MMMSV23, Definition 1.4, 6.2 and 7.2])**.** We define the following three numbers:

- Let $SD_m^{r,n} \subseteq \mathbb{P}(S^2\mathbb{C}^n)$ be the intersection of the variety of $n \times n$ symmetric matrices of rank at most r with a general linear space of projective dimension m. We define $\delta(m,n,r)$ as the degree of $(SD_m^{r,n})^*$ if it is a hypersurface, and zero otherwise.
- Let $D_m^{r,n} \subseteq \mathbb{P}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the intersection of the variety of $n \times n$ matrices of rank at most r with a general linear space of projective dimension m. We define $\delta_A(m,n,r)$ as the degree of $(D_m^{r,n})^*$ if it is a hypersurface, and zero otherwise.
- Let $AD_m^{2r,2n} \subseteq \mathbb{P}(\bigwedge^2 \mathbb{C}^n)$ be the intersection of the variety of $2n \times 2n$ skew-symmetric matrices of rank at most $2r$ with a general linear space of projective dimension m. We define $\delta_D(m,n,r)$ as the degree of $(AD_m^{2r,2n})^*$ if it is a hypersurface, and zero otherwise.

Using Lascoux polynomials, in [MMMSV23] it is proved that the algebraic degrees $\delta(m,n,n-s)$, $\delta_A(m,n,n-s)$ and $\delta_D(m,n,n-s)$ are polynomials in n. We determine their degrees using our results on the leading coefficient of Lascoux polynomials. Moreover, we compute the leading coefficients of $\delta(m,n,n-s)$, $\delta_A(m,n,n-s)$ and $\delta_D(m,n,n-s)$ in the case when $s=1$ combining the formulas obtained in the previous sections together with the results in [MMMSV23]. These results should be regarded as asymptotic degrees.

Here and in the rest of the section we denote with $LC(f)$ the leading coefficient of a univariate polynomial f.

Theorem 4.6.1 (Type C)**.** *For every $s > 0$ and $m \geq \binom{s+1}{2}$, the polynomial $\delta(m, n, n-s)$ has degree m. Moreover*

$$LC(\delta(m, n, n-1)) = \frac{2^{m-1}}{m!},$$

for every $m > 0$.

Proof. By [BR09, Theorem 1.1] we have that

$$\delta(m, n, n-s) = \sum_{\substack{I \subseteq [n] \\ |I|=s \\ \sum I = m-s}} \psi_I \, \mathrm{LP}_I(n),$$

where ψ_I are the Lascoux coefficients as in [MMMSV23, Definition 2.5]. Observe that the last sum is not empty if and only if there exists $I \subseteq [n]$ with $\sum I \geq \binom{|I|}{2}$. This happens if and only if $m \geq \binom{s+1}{2}$. By Theorem 4.3.2 for every fixed m, s satisfying this inequality, $\delta(m, n, n-s)$ is a positive finite linear combination of polynomials of degree $|I| + \sum I = m$, which proves the first claim. For the second statement we have $\delta(m, n, n-1) = \psi_{\{m-1\}} \mathrm{LP}_{\{m-1\}}(n)$ and $LC(\delta(m, n, n-1)) = \psi_{\{m-1\}} LC(\mathrm{LP}_{\{m-1\}}(n))$. In [MMMSV23, Lemma 2.7] it is proved that $\psi_{\{m-1\}} = 2^{m-1}$, and by Theorem 4.3.2 we have that $LC(\mathrm{LP}_{\{m-1\}}(n)) = \frac{1}{m!}$. This concludes the proof. □

Hence, for large values of n we have that $\delta(m, n, n-1) \sim \frac{2^{m-1}}{m!} n^m$.

Theorem 4.6.2 (Type A)**.** *For every $s > 0$ and $m \geq s^2$ the polynomial $\delta_A(m, n, n-s)$ has degree m. Moreover,*

$$LC(\delta_A(m, n, n-1)) = \frac{1}{m!}\binom{2(m-1)}{m-1}.$$

Proof. By [MMMSV23, Theorem 6.8] we have that

$$\delta_A(m, n, n-s) = \sum_{\substack{I, J \subset [n] \\ |I|=|J|=s \\ \sum I + \sum J = m-s}} d_{I,J} \, \mathrm{LP}^A_{I,J}(n),$$

where $d_{I,J}$ are the type A Lascoux coefficients as defined in [MMMSV23, Definition 6.7]. The last sum is not empty if and only if the condition $\sum I + \sum J \geq \binom{|I|}{2} + \binom{|J|}{2}$ is satisfied by some $I, J \subseteq [n]$. This is equivalent to $m - s \geq 2\binom{s}{2}$, that is $m \geq s^2$. Hence by Theorem 4.4.2 for fixed m, s satisfying this inequality, $\delta_A(m, n, n-s)$ is a positive finite combination of polynomials of degree $|I| + \sum I + \sum J = m$. For the second statement we specialize to $r = n-1$ and obtain $\delta_A(m, n, n-1) =$

$\sum_{i=0}^{m-1} d_{\{i\},\{m-1-i\}} \operatorname{LP}^A_{\{i\},\{m-1-i\}}(n)$. As by Theorem 4.4.2 all the m polynomials on the right-hand side have the same degree we have that

$$LC(\delta_A(m,n,n-1)) = \sum_{i=0}^{m-1} d_{\{i\},\{m-1-i\}} LC(\operatorname{LP}^A_{\{i\},\{m-1-i\}}(n)).$$

By [MMMSV23, Proposition 6.9] we have that $d_{\{i\},\{m-1-i\}} = \binom{m-1}{i}$, and by Theorem 4.4.2 we have that $LC(\operatorname{LP}^A_{\{i\},\{m-1-i\}}(n)) = \frac{1}{m\cdot i!(m-1-i)!}$. Combining the two results we obtain

$$\begin{aligned} LC(\delta_A(m,n,n-1)) &= \sum_{i=0}^{m-1} \binom{m-1}{i} \frac{1}{m\cdot i!(m-1-i)!} \\ &= \frac{1}{m}\sum_{i=0}^{m-1} \binom{m-1}{i}^2 = \frac{1}{m!}\binom{2(m-1)}{m-1}. \end{aligned}$$

□

Theorem 4.6.2 implies that for large values of n, $\delta_A(m,n,n-1) \sim \frac{1}{m!}\binom{2(m-1)}{m-1} n^m$.

Finally, we present analog results for the type D case.

Theorem 4.6.3 (Type D)**.** *For every $s > 0$ and every $m \geq \binom{2s}{2}$, the polynomial $\delta_D(m,n,n-s)$ has degree m. Moreover,*

$$LC(\delta_D(m,n,n-1)) = \frac{2^{m-2}}{m!}\left(\frac{1}{m}\binom{2(m-1)}{m-1} + 1\right).$$

Proof. In [MMMSV23, Theorem 7.8] it is proved that

$$\delta_D(m,n,r) = \sum_{\substack{I\subset[2n] \\ |I|=2n-2r \\ \sum I = m}} \alpha_I \operatorname{LP}^D_I(2n).$$

The sum on the right-hand side is not empty if and only if $\sum I \geq \binom{|I|}{2}$, that is $m \geq \binom{2s}{2}$. When this inequality holds, by Theorem 4.5.1 we have that $\delta_D(m,n,n-s)$ is a positive finite combination of polynomials of degree $\sum I = m$. Moreover, $\delta_D(m,n,n-1) = \sum_{i=0}^{\lfloor \frac{m-1}{2}\rfloor} \alpha_{\{i,m-i\}} \operatorname{LP}^D_{\{i,m-i\}}(2n)$ and we obtain

$$LC(\delta_D(m,n,n-1)) = \sum_{i=0}^{\lfloor \frac{m-1}{2}\rfloor} \alpha_{\{i,m-i\}} LC(\operatorname{LP}^D_{\{i,m-i\}}(2n)).$$

By [LLT89, A.16.5] we have that $\alpha_{\{i,j\}} = \binom{i+j-1}{i} - \binom{i+j-1}{i-1}$, so in particular $\alpha_{i,m-i} = \binom{m-1}{i} - \binom{m-1}{i-1}$. Using Theorem 4.5.1 we conclude that

$$LC(\operatorname{LP}^D_{\{i,m-i\}}(2n)) = \begin{cases} \frac{m-2i}{4m\cdot i!(m-i)!} & i>0 \\ \frac{m-2i}{2m\cdot i!(m-i)!} & i=0 \end{cases}.$$

As a polynomial in n, the degree of $\mathrm{LP}^D_{\{i,m-i\}}(2n)$ is equal to m and its leading coefficient is then $2^m LC(\mathrm{LP}^D_{\{i,m-i\}}(2n))$. We obtain

$$\begin{aligned}
LC(\delta_D(m,n,n-1)) &= \frac{2^{m-1}}{m!} + \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} \left(\binom{m-1}{i} - \binom{m-1}{i-1} \right) \frac{2^{m-2}(m-2i)}{m \cdot i!(m-i)!} \\
&= \frac{2^{m-1}}{m!} + \frac{2^{m-2}}{m^2 \cdot m!} \sum_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} (m-2i)^2 \binom{m}{i}^2 \\
&= \frac{2^{m-1}}{m!} + \frac{2^{m-2}}{m^2 \cdot m!} \left(m \binom{2(m-1)}{m-1} - m^2 \right) \\
&= \frac{2^{m-2}}{m!} \left(\frac{1}{m} \binom{2(m-1)}{m-1} + 1 \right).
\end{aligned}$$

□

Hence, for large values of n we have that

$$\delta_D(m,n,n-1) \sim \frac{2^{m-2}}{m!} \left(\frac{1}{m} \binom{2(m-1)}{m-1} + 1 \right) n^m.$$

It is of course possible to follow the same idea to compute the leading coefficients of $\delta(m,n,n-s)$, $\delta_A(m,n,n-s)$ and $\delta_D(m,n,n-s)$ for higher values of s, even though the calculation becomes significantly more involved. We conclude this article with a natural question.

Problem 4.6.4. Find formulas in m and s for the leading coefficients of $\delta(m,n,n-s)$, $\delta_A(m,n,n-s)$ and $\delta_D(m,n,n-s)$.

Bibliography

[AAV24a] Abeer Al Ahmadieh and Cynthia Vinzant. “Characterizing principal minors of symmetric matrices via determinantal multiaffine polynomials”. In: *Journal of Algebra* 638 (2024), pp. 255–278 (cit. on p. 26).

[AAV24b] Abeer Al Ahmadieh and Cynthia Vinzant. “Determinantal representations and the image of the principal minor map”. In: *International Mathematics Research Notices* 2024.10 (2024), pp. 8930–8958 (cit. on p. 26).

[ABD10] Federico Ardila, Carolina Benedetti, and Jeffrey Doker. “Matroid polytopes and their volumes”. In: *Discrete & Computational Geometry* 43 (2010), pp. 841–854 (cit. on pp. 15, 35, 72).

[ACEP20] Federico Ardila, Federico Castillo, Christopher Eur, and Alexander Postnikov. “Coxeter submodular functions and deformations of Coxeter permutahedra”. In: *Advances in Mathematics* 365 (2020), p. 107039 (cit. on pp. 5, 6, 48–50, 53, 55, 64).

[And70] Theodore Wilbur Anderson. “Estimation of covariance matrices which are linear combinations or whose inverses are linear combinations of given matrices”. In: *Essays in Probability and Statistics*. Univ. of North Carolina Press, Chapel Hill, N.C., 1970, pp. 1–24 (cit. on p. 77).

[And+95] Heidi H Andersen et al. *The multivariate complex normal distribution*. Springer, 1995 (cit. on p. 19).

[ARW17] Federico Ardila, Felipe Rincón, and Lauren K Williams. “Positively oriented matroids are realizable”. In: *Journal of the European Mathematical Society* 19.3 (2017), pp. 815–833 (cit. on p. 33).

[Bae52] Reinhold Baer. *Linear algebra and projective geometry*. New York: Academic Press, 1952 (cit. on p. 24).

[Bas21] Jose Bastidas. “The polytope algebra of generalized permutahedra”. In: *Algebraic Combinatorics* 4.5 (2021), pp. 909–946 (cit. on p. 64).

[BC13] Francis Buekenhout and Arjeh M Cohen. *Diagram geometry: related to classical groups and buildings*. Vol. 57. Springer Science & Business Media, 2013 (cit. on p. 24).

[BC23] William Brehm and Alexandru Constantinescu. “The first cotangent cohomology module for matroids”. In: *Journal of Combinatorial Algebra* 7.3 (2023), pp. 327–345 (cit. on p. 35).

[BC95] Francis Buekenhout and Peter Cameron. “Projective and affine geometry over division rings”. In: *Handbook of incidence geometry*. Elsevier, 1995, pp. 27–62 (cit. on p. 27).

[BCMVV23] Alessio Borzì, Xiangying Chen, Harshit J Motwani, Lorenzo Venturello, and Martin Vodička. “The leading coefficient of Lascoux polynomials”. In: *Discrete Mathematics* 346.2 (2023), p. 113217 (cit. on p. 7).

[BDKS19] Tobias Boege, Alessio D’Alì, Thomas Kahle, and Bernd Sturmfels. “The geometry of gaussoids”. In: *Foundations of Computational Mathematics* 19 (2019), pp. 775–812 (cit. on pp. 4, 18, 33, 34, 39).

[BES23] Spencer Backman, Christopher Eur, and Connor Simpson. “Simplicial generation of Chow rings of matroids”. In: *Journal of the European Mathematical Society* 26.11 (2023), pp. 4491–4535 (cit. on p. 75).

[BGW03] Alexandre V Borovik, Israel Moiseevich Gelfand, and Neil White. *Coxeter matroids*. Birkhäuser Boston, 2003 (cit. on pp. 3, 47, 58, 59).

[BH20] Petter Brändén and June Huh. “Lorentzian polynomials”. In: *Annals of Mathematics* 192.3 (2020), pp. 821–891 (cit. on p. 35).

[BK20] Tobias Boege and Thomas Kahle. “Construction methods for gaussoids”. In: *Kybernetika* 56.6 (2020), pp. 1045–1062 (cit. on pp. 33, 34).

[BLVSWZ99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented Matroids*. 2nd ed. Encyclopedia of Mathematics and its Applications 46. Cambridge University Press, 1999 (cit. on pp. 35, 39, 42).

[Boe22a] Tobias Boege. “Gaussoids are two-antecedental approximations of Gaussian conditional independence structures”. In: *Annals of Mathematics and Artificial Intelligence* 90.6 (2022), pp. 645–673 (cit. on pp. 17, 18).

[Boe22b] Tobias Boege. “The Gaussian conditional independence inference problem”. PhD thesis. OvGU Magdeburg, 2022 (cit. on pp. 4, 17, 18, 25, 26, 33, 39, 43).

[BR09] Hans-Christian Graf von Bothmer and Kristian Ranestad. “A general formula for the algebraic degree in semidefinite programming”. In: *Bull. Lond. Math. Soc.* 41.2 (2009), pp. 193–197 (cit. on pp. 79, 96).

[BTD85] Theodor Bröcker and Tammo Tom Dieck. *Representations of compact Lie groups*. Vol. 98. Springer Science & Business Media, 1985 (cit. on p. 1).

[Bue93] Francis Buekenhout. “A theorem of Parmentier characterizing projective spaces by polarities”. In: *Finite Geometry and Combinatorics* (1993), pp. 69–72 (cit. on p. 26).

[CD06] Michael Carr and Satyan L Devadoss. “Coxeter complexes and graph-associahedra”. In: *Topology and its Applications* 153.12 (2006), pp. 2155–2168 (cit. on p. 33).

[CDEHL24] Emily Clader, Chiara Damiolini, Christopher Eur, Daoji Huang, and Shiyue Li. “Multimatroids and rational curves with cyclic action”. In:

International Mathematics Research Notices 2024.12 (2024), pp. 9743–9777 (cit. on p. 72).

[Che24] Xiangying Chen. “An axiomatization of matroids and oriented matroids as conditional independence models”. In: *SIAM Journal on Discrete Mathematics* 38.2 (2024), pp. 1526–1536 (cit. on p. 5).

[CK88] Ramaswamy Chandrasekaran and Santosh N Kabadi. “Pseudomatroids”. In: *Discrete Mathematics* 71.3 (1988), pp. 205–217 (cit. on p. 59).

[CLO05] David A Cox, John Little, and Donal O’Shea. *Using algebraic geometry.* Vol. 185. Springer Science & Business Media, 2005 (cit. on p. 71).

[CLS11] David A Cox, John B Little, and Henry K Schenck. *Toric varieties.* Vol. 124. American Mathematical Society, 2011 (cit. on p. 14).

[CMNR19] Carolyn Chun, Iain Moffatt, Steven D Noble, and Ralf Rueckriemen. “Matroids, delta-matroids and embedded graphs”. In: *Journal of Combinatorial Theory, Series A* 167 (2019), pp. 7–59 (cit. on p. 2).

[CP22] Swee Hong Chan and Igor Pak. “Introduction to the combinatorial atlas”. In: *Expositiones Mathematicae* 40.4 (2022), pp. 1014–1048 (cit. on p. 35).

[CP97] Joseph T Chang and David Pollard. “Conditioning as disintegration”. In: *Statistica Neerlandica* 51.3 (1997), pp. 287–317 (cit. on p. 19).

[Daw01] A Philip Dawid. “Separoids: A mathematical framework for conditional independence and irrelevance”. In: *Annals of Mathematics and Artificial Intelligence* 32 (2001), pp. 335–372 (cit. on p. 26).

[Daw79] A Philip Dawid. “Conditional independence in statistical theory”. In: *Journal of the Royal Statistical Society: Series B (Methodological)* 41.1 (1979), pp. 1–15 (cit. on pp. 9, 11).

[Dod67] Charles L Dodgson. “IV. Condensation of determinants, being a new and brief method for computing their arithmetical values”. In: *Proceedings of the Royal Society of London* 15 (1867), pp. 150–155 (cit. on p. 17).

[Dok11] Jeffrey Samuel Doker. *Geometry of generalized permutohedra.* University of California, Berkeley, 2011 (cit. on p. 64).

[EFLS24] Christopher Eur, Alex Fink, Matt Larson, and Hunter Spink. “Signed permutohedra, delta-matroids, and beyond”. In: *Proceedings of the London Mathematical Society* 128.3 (2024), e12592 (cit. on pp. 64, 73).

[EMM13] Joanna A Ellis-Monaghan and Iain Moffatt. *Graphs on surfaces: dualities, polynomials, and knots.* Vol. 84. Springer, 2013 (cit. on p. 2).

[Fin11] Alex Fink. “The binomial ideal of the intersection axiom for conditional probabilities”. In: *Journal of Algebraic Combinatorics* 33 (2011), pp. 455–463 (cit. on p. 10).

[Fin13] Alex Fink. “Tropical cycles and Chow polytopes”. In: *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry* 54 (2013), pp. 13–40 (cit. on p. 35).

[Fuj78] Satoru Fujishige. “Polymatroidal dependence structure of a set of random variables”. In: *Information and control* 39.1 (1978), pp. 55–72 (cit. on p. 13).

[GGMS87] Israel Moiseevich Gelfand, R Mark Goresky, Robert D MacPherson, and Vera V Serganova. “Combinatorial geometries, convex polyhedra, and Schubert cells”. In: *Advances in Mathematics* 63.3 (1987), pp. 301–316 (cit. on p. 58).

[GGRW05] Israel Moiseevich Gelfand, Sergei Gelfand, Vladimir Retakh, and Robert Lee Wilson. “Quasideterminants”. In: *Advances in Mathematics* 193.1 (2005), pp. 56–141 (cit. on pp. 19–21, 23).

[GR91] Israel Moiseevich Gelfand and Vladimir S Retakh. “Determinants of matrices over noncommutative rings”. In: *Functional Analysis and Its Applications* 25.2 (1991), pp. 91–102 (cit. on pp. 19–21).

[GR92] Israel Moiseevich Gelfand and Vladimir S Retakh. “A theory of noncommutative determinants and characteristic functions of graphs”. In: *Functional Analysis and Its Applications* 26.4 (1992), pp. 231–246 (cit. on p. 21).

[GS87] Israel Moiseevich Gelfand and Vera V Serganova. “Combinatorial geometries and torus strata on homogeneous compact manifolds”. In: *Russian Mathematical Surveys* 42.2 (1987), p. 133 (cit. on p. 3).

[Hal35] P Hall. “On Representatives of Subsets”. In: *Journal of the London Mathematical Society* 1.1 (1935), pp. 26–30 (cit. on p. 74).

[Haw84] Thomas Hawkins. “The Erlanger Programm of Felix Klein: Reflections on its place in the history of mathematics”. In: *Historia Mathematica* 11.4 (1984), pp. 442–470 (cit. on p. 1).

[HJL12] Marshall Hall Jr and Jacobus Hendricus van Lint. *Combinatorics: Proceedings of the NATO Advanced Study Institute held at Nijenrode Castle, Breukelen, The Netherlands 8–20 July 1974*. Vol. 16. Springer Science & Business Media, 2012 (cit. on p. 23).

[HMSSW08] Raymond Hemmecke, Jason Morton, Anne Shiu, Bernd Sturmfels, and Oliver Wienand. “Three counter-examples on semi-graphoids”. In: *Combinatorics, Probability and Computing* 17.2 (2008), pp. 239–257 (cit. on p. 18).

[Hum90] James E Humphreys. *Reflection groups and Coxeter groups*. 29. Cambridge university press, 1990 (cit. on pp. 47, 48).

[Jac53] Nathan Jacobson. *Lectures in Abstract Algebra: II. Linear Algebra*. Vol. 31. Springer Science & Business Media, 1953 (cit. on pp. 24, 25).

[JP18] Relinde Jurrius and Ruud Pellikaan. “Defining the q-Analogue of a Matroid”. In: *The Electronic Journal of Combinatorics* 25.3 (2018), pp. 3–2 (cit. on p. 27).

[JR22] Katharina Jochemko and Mohan Ravichandran. “Generalized permutahedra: Minkowski linear functionals and Ehrhart positivity”. In: *Mathematika* 68.1 (2022), pp. 217–236 (cit. on p. 16).

[Kac18] Tarik Kaced. "Information inequalities are not closed under polymatroid duality". In: *IEEE Transactions on Information Theory* 64.6 (2018), pp. 4379–4381 (cit. on p. 43).

[Kle25] Felix Klein. *Elementarmathematik vom höheren Standpunkte aus, II: Geometrie*. Vol. 15. Springer, 1925 (cit. on p. 1).

[Kle93] Felix Klein. "Vergleichende Betrachtungen über neuere geometrische Forschungen". In: *Mathematische Annalen* 43.1 (1893), pp. 63–100 (cit. on p. 1).

[Kun78a] Joseph Pee Sin Kung. "An Erlanger program for combinatorial geometries." PhD thesis. Massachusetts Institute of Technology, 1978 (cit. on p. 3).

[Kun78b] Joseph PS Kung. "Bimatroids and invariants". In: *Advances in Mathematics* 30.3 (1978), pp. 238–249 (cit. on p. 3).

[Kun97] Joseph PS Kung. "Pfaffian structures and critical problems in finite symplectic spaces". In: *Annals of Combinatorics* 1 (1997), pp. 159–172 (cit. on p. 3).

[Lau96] Steffen L Lauritzen. *Graphical models*. Vol. 17. Clarendon Press, 1996 (cit. on p. 32).

[LLT89] Dan Laksov, Alain Lascoux, and Anders Thorup. "On Giambelli's theorem on complete correlations". In: *Acta Math.* 162.3-4 (1989), pp. 143–199 (cit. on pp. 78, 88, 97).

[LM07] Radim Lněnička and František Matúš. "On Gaussian conditional independence structures". In: *Kybernetika* 43.3 (2007), pp. 327–342 (cit. on pp. 4, 17, 33, 34).

[LP24] Thomas Lam and Alexander Postnikov. "Polypositroids". In: *Forum of Mathematics, Sigma*. Vol. 12. Cambridge University Press. 2024, e42 (cit. on p. 33).

[Mat05] František Matúš. "Conditional independences in Gaussian vectors and rings of polynomials". In: *Conditionals, Information, and Inference: International Workshop, WCII 2002, Hagen, Germany, May 13-15, 2002, Revised Selected Papers*. Springer. 2005, pp. 152–161 (cit. on pp. 17, 22).

[Mat06] František Matúš. "Two constructions on limits of entropy functions". In: *IEEE Transactions on Information Theory* 53.1 (2006), pp. 320–330 (cit. on p. 43).

[Mat18] František Matúš. "Classes of matroids closed under minors and principal extensions". In: *Combinatorica* 38.4 (2018), pp. 935–954 (cit. on p. 43).

[Mat23] František Matúš. "Algebraic matroids are almost entropic". In: *Proceedings of the American Mathematical Society* (2023) (cit. on p. 43).

[Mat92] František Matúš. "Ascending and descending conditional independence relations". In: *Transactions of the 11th Prague Conference on Information Theory*. 1992, pp. 189–200 (cit. on pp. 10, 11, 31, 32).

[Mat93] František Matúš. "Probabilistic conditional independence structures and matroid theory: Background". In: *International Journal Of General System* 22.2 (1993), pp. 185–196 (cit. on pp. 11, 36, 38, 43).

[Mat97] František Matúš. "Conditional independence structures examined via minors". In: *Annals of Mathematics and Artificial Intelligence* 21 (1997), pp. 99–30 (cit. on pp. 32, 33, 38, 43).

[MB+06] Joseph Muscat, David Buhagiar, et al. "Connective spaces". In: *Mem. Fac. Sci. Eng. Shimane Univ. Series B: Math. Sci* 39 (2006), pp. 1–13 (cit. on p. 32).

[MDLW18] Marloes Maathuis, Mathias Drton, Steffen Lauritzen, and Martin Wainwright. *Handbook of graphical models*. CRC Press, 2018 (cit. on p. 26).

[Mir71] Leonid Mirsky. *Transversal theory: an account of some aspects of combinatorial mathematics*. Academic Press, 1971 (cit. on p. 75).

[MMMSV23] Laurent Manivel, Mateusz Michałek, Leonid Monin, Tim Seynnaeve, and Martin Vodička. "Complete quadrics: Schubert calculus for Gaussian models and semidefinite programming". In: *Journal of the European Mathematical Society* 26.8 (2023), pp. 3091–3135 (cit. on pp. 7, 77–81, 87, 88, 90, 93, 95–97).

[MMW21] Mateusz Michałek, Leonid Monin, and Jarosław A. Wiśniewski. "Maximum likelihood degree, complete quadrics, and $\mathbb{C}^*$-action". In: *SIAM J. Appl. Algebra Geom.* 5.1 (2021), pp. 60–85 (cit. on p. 77).

[Mof19] Iain Moffatt. "Delta-matroids for graph theorists". In: *Surveys in Combinatorics 2019* 456 (2019), p. 167 (cit. on pp. 2, 58).

[MPSSW09] Jason Morton, Lior Pachter, Anne Shiu, Bernd Sturmfels, and Oliver Wienand. "Convex rank tests and semigraphoids". In: *SIAM Journal on Discrete Mathematics* 23.3 (2009), pp. 1117–1134 (cit. on pp. 4, 5, 11, 13, 15, 30, 34).

[MT11] Nguyen Cong Minh and Ngo Viet Trung. "Cohen-Macaulayness of monomial ideals and symbolic powers of Stanley–Reisner ideals". In: *Advances in mathematics* 226.2 (2011), pp. 1285–1306 (cit. on p. 35).

[MUWY18] Fatemeh Mohammadi, Caroline Uhler, Charles Wang, and Josephine Yu. "Generalized permutohedra from probabilistic graphical models". In: *SIAM Journal on Discrete Mathematics* 32.1 (2018), pp. 64–93 (cit. on pp. 15, 18, 34, 43).

[NRS10] Jiawang Nie, Kristian Ranestad, and Bernd Sturmfels. "The algebraic degree of semidefinite programming". In: *Math. Program.* 122.2, Ser. A (2010), pp. 379–405 (cit. on pp. 79, 95).

[Ore55] Oystein Ore. "Graphs and matching theorems". In: *Duke Math. J.* 22.1 (1955), pp. 625–639 (cit. on p. 75).

[Oxl06] James G Oxley. *Matroid theory*. Vol. 3. Oxford University Press, USA, 2006 (cit. on pp. 2, 28, 35).

[Pea85] Judea Pearl. "Bayesian networks: A model of self-activated memory for evidential reasoning". In: *Proceedings of the 7th conference of the*

Cognitive Science Society, University of California, Irvine, CA, USA. 1985, pp. 15–17 (cit. on p. 34).

[Pea88] Judea Pearl. *Probabilistic reasoning in intelligent systems: networks of plausible inference*. Morgan Kaufmann, 1988 (cit. on p. 11).

[Pet15] Jonas Peters. "On the intersection property of conditional independence and its application to causal discovery". In: *Journal of Causal Inference* 3.1 (2015), pp. 97–108 (cit. on p. 10).

[Pic96] Bernard Picinbono. "Second-order complex random vectors and normal distributions". In: *IEEE Transactions on Signal Processing* 44.10 (1996), pp. 2637–2640 (cit. on p. 19).

[Pos09] Alexander Postnikov. "Permutohedra, associahedra, and beyond". In: *International Mathematics Research Notices* 2009.6 (2009), pp. 1026–1106 (cit. on pp. 14, 73, 75).

[PP87] Judea Pearl and Azaria Paz. "Graphoids: a graph-based logic for reasoning about relevance relations". In: *Advances in Artificial Intelligence* (1987), pp. 357–363 (cit. on p. 10).

[PPR23] Arnau Padrol, Vincent Pilaud, and Julian Ritter. "Shard polytopes". In: *International Mathematics Research Notices* 2023.9 (2023), pp. 7686–7796 (cit. on p. 64).

[PRW08] Alexander Postnikov, Victor Reiner, and Lauren Williams. "Faces of generalized permutohedra". In: *Documenta Mathematica* 13 (2008), pp. 207–273 (cit. on p. 14).

[Rad42] Richard Rado. "A theorem on independence relations". In: *The Quarterly Journal of Mathematics* 1 (1942), pp. 83–89 (cit. on p. 75).

[Rea12] Nathan Reading. "Coarsening polyhedral complexes". In: *Proceedings of the American Mathematical Society* 140.10 (2012), pp. 3593–3605 (cit. on pp. 5, 49).

[RMN13] KP Ratheesh and NM Madhavan Namboothiri. "On c-spaces and connective spaces". In: *South Asian Journal of Mathematics* 4.1 (2013), pp. 48–56 (cit. on p. 32).

[Rot08] Gian-Carlo Rota. *Indiscrete thoughts*. Springer Science & Business Media, 2008 (cit. on p. 2).

[Sch13] Rolf Schneider. *Convex bodies: the Brunn–Minkowski theory*. Vol. 151. Cambridge University Press, 2013 (cit. on pp. 68, 71–73).

[She68] GC Shephard. "Euler-type relations for convex polytopes". In: *Proceedings of the London Mathematical Society* 3.4 (1968), pp. 597–606 (cit. on p. 68).

[SK16] Milan Studený and Tomáš Kroupa. "Core-based criterion for extreme supermodular functions". In: *Discrete Applied Mathematics* 206 (2016), pp. 122–151 (cit. on p. 15).

[SS15] Bärbel MR Stadler and Peter F Stadler. "Connectivity spaces". In: *Mathematics in Computer Science* 9.4 (2015), pp. 409–436 (cit. on p. 32).

[Stu06] Milan Studený. *Probabilistic conditional independence structures*. Springer Science & Business Media, 2006 (cit. on pp. 10, 13, 18).

[Stu21] Milan Studený. "Conditional independence structures over four discrete random variables revisited: conditional Ingleton inequalities". In: *IEEE Transactions on Information Theory* 67.11 (2021), pp. 7030–7049 (cit. on p. 31).

[Stu89] Milan Studený. "Multiinformation and the problem of characterization of conditional independence relations". In: *Problems of Control and Information Theory* 18.1 (1989), pp. 3–16 (cit. on pp. 10, 31).

[Stu92] Milan Studený. "Conditional independence relations have no finite complete characterization". In: *Information Theory, Statistical Decision Functions and Random Processes. Transactions of the 11th Prague Conference vol. B.* 1992, pp. 377–396 (cit. on p. 10).

[Stu94] Milan Studený. "Semigraphoids are two-antecedental approximations of stochastic conditional independence models". In: *Uncertainty Proceedings 1994*. Elsevier, 1994, pp. 546–552 (cit. on p. 10).

[SU10] Bernd Sturmfels and Caroline Uhler. "Multivariate Gaussian, semidefinite matrix completion, and convex algebraic geometry". In: *Ann. Inst. Statist. Math.* 62.4 (2010), pp. 603–638 (cit. on p. 77).

[Sul09] Seth Sullivant. "Gaussian conditional independence relations have no finite complete characterization". In: *Journal of Pure and Applied Algebra* 213.8 (2009), pp. 1502–1506 (cit. on p. 18).

[Sul18] Seth Sullivant. *Algebraic statistics*. Vol. 194. American Mathematical Soc., 2018 (cit. on p. 32).

[SW14] Alba Sloin and Ami Wiesel. "Proper quaternion Gaussian graphical models". In: *IEEE Transactions on Signal Processing* 62.20 (2014), pp. 5487–5496 (cit. on p. 19).

[TM11] Clive Cheong Took and Danilo P Mandic. "Augmented second-order statistics of quaternion random signals". In: *Signal Processing* 91.2 (2011), pp. 214–224 (cit. on p. 19).

[TT12] Naoki Terai and Ngo Viet Trung. "Cohen-Macaulayness of large powers of Stanley–Reisner ideals". In: *Advances in Mathematics* 229.2 (2012), pp. 711–730 (cit. on p. 35).

[Tut79] William Thomas Tutte. "All the king's horses. A guide to reconstruction". In: *Graph theory and related topics* (1979), pp. 15–33 (cit. on p. 2).

[Var11] Matteo Varbaro. "Symbolic powers and matroids". In: *Proceedings of the American Mathematical Society* 139.7 (2011), pp. 2357–2366 (cit. on p. 35).

[VC10] NN Vakhania and GZ Chelidze. "Quaternion Gaussian random variables". In: *Theory of Probability & its Applications* 54.2 (2010), pp. 363–369 (cit. on p. 19).

[Š07] Petr Šimeček. "Nezávislostní modely". PhD thesis. Univerzita Karlova v Praze, 2007 (cit. on pp. 10, 19).

[Wey46] Hermann Weyl. *The classical groups: their invariants and representations*. 1. Princeton university press, 1946 (cit. on p. 47).

[Whi86] Neil White, ed. *Theory of Matroids*. Encyclopedia of Mathematics and its Applications 26. Cambridge University Press, 1986 (cit. on pp. 2, 35).

[Zas81] Thomas Zaslavsky. "The geometry of root systems and signed graphs". In: *The American Mathematical Monthly* 88.2 (1981), pp. 88–105 (cit. on p. 69).

[Zas82] Thomas Zaslavsky. "Signed graphs". In: *Discrete Applied Mathematics* 4.1 (1982), pp. 47–74 (cit. on p. 69).