Analytical and Numerical Aspects of Non-Smooth Mechanics in the Context of Measure Differential Inclusions

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Institut für Mathematik

vorgelegt von
Frau Manuela Paschkowski

Gutachter: Prof. Dr. Martin Arnold
Prof. Dr. Timo Reis
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Danke für das Vertrauen, die Geduld, die fachliche und mentale Unterstützung.

Contents

List of Abbreviations and Symbols

Abbreviations

CP	Complementarity problem
DAE	Differential-algebraic equation
DI	Differential inclusion
LIMDI	Linear implicite measure differential inclusion
MDI	Measure differential inclusion
NCI	Normal cone inclusion
ODE	Ordinary differential equation

Symbols

1^n	$(1 \dots 1)^\top \in \mathbb{R}^n$
$AC(I, E)$	Space of all absolutely continuous functions mapping from I to E
$A_{j,:}$	j-th row of a matrix A
$A_{:,j}$	j-th column of a martix A
$B(z, r)$	Ball with centre z and radius r
$BV(I, E)$	Space of all functions of bounded variation mapping from I to E
$C(I, E)$	Space of all continuous functions mapping from I to E
χ_A	Characteristic function of set A
δ_s	Dirac measure to point s
$\mathrm{d}x$	Differential measure of function x
I^n	n-dimensional unity matrix of $\mathbb{R}^{n\times n}$
e_k^n	k-th unity vector of dimension n
$\mathcal{L}^1(I, E)$	Lebesgue integrable functions mapping from I to E
$N_C(x)$	Normal cone to set C in x
$\partial^C f$	Clarke differential of function f
$P(X)$	Power set of set X
ψ_A	Indicator function of set A
$\mathbb{R}_+$	Set $\{x \in \mathbb{R} \mid x \geq 0\}$ of non-negative real numbers
$SBV(I, E)$	Space of all special functions of bounded variation mapping from I to E
$SBV^+(I, E)$	Right-continuous functions of $SBV(I, E)$
$T^*(g, I)$	Set of all points of I where g is discontinuous
$x^\top, A^\top$	Transposed vector of $x \in \mathbb{R}^n$, transposed matrix of $A \in \mathbb{R}^{m\times n}$
$x^+(t)$	Right-side limit of function x in t
$x^-(t)$	Left-side limit of function x in t
$x_{a:b}$	The a-th to b-th components of $x \in \mathbb{R}^n$
$\dot{x}(t)$	Time derivative of x in t
0^n	$(0 \dots 0)^\top \in \mathbb{R}^n$

1. Introduction

In classical mechanics [8, 54], the motion of a system of rigid bodies is described by a function

$$q : I \subset \mathbb{R} \to \mathbb{R}^n, \ n \in \mathbb{N},$$

representing the generalised coordinates of the system at time $t \in I := [0, T], T > 0$. If q is twice continuously differentiable with respect to t - which is a common assumption - then the first derivative $v := \dot{q} : I \to \mathbb{R}^n$ characterises the velocity and $a := \dot{v} : I \to \mathbb{R}^n$ the acceleration of the mechanical system. Already in the early stages of mechanical engineering, it was recognised that mechanical systems including contact and impact forces as well as friction forces could not be depicted by continuously differentiable trajectories. The benchmark problem of the bouncing ball will easily underline this claim.

1.1. Motivating Benchmark Problem

Example 1.1 (Bouncing ball [2, 39, 59, 80, 88]) From an initial point $q(0) := q_0 \in \mathbb{R}_+$, a ball with mass $m \in \mathbb{R}_+$ and initial velocity $v(0) := v_0 \in \mathbb{R}$ is dropped to a ground obstacle (see Figure 1.1). Using variational principles [54], the motion in free flight can be described without consideration of air friction as a differential equation with respect to the position q and the velocity v

$$\dot{q}(t) = v(t), \qquad m\dot{v}(t) = -mg + \lambda(t), \qquad t \in I. \tag{1.1}$$

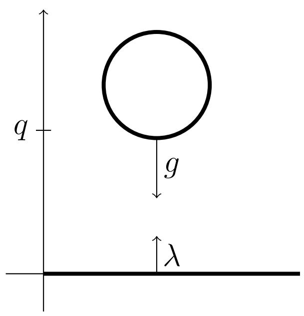


Figure 1.1.: Bouncing ball.

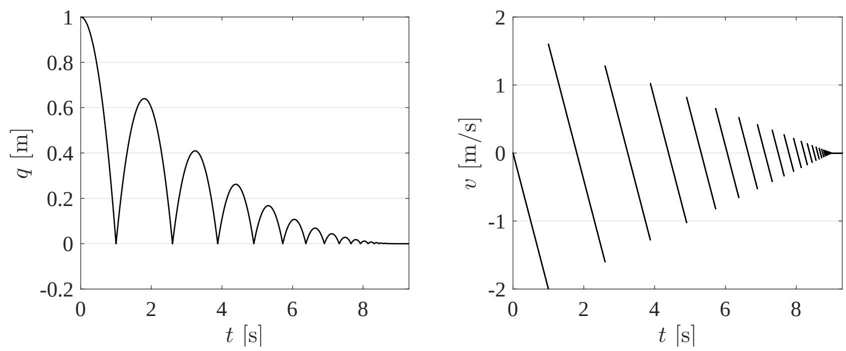


Figure 1.2.: Bouncing ball: Exemplary trajectories.

Here g is the gravitational acceleration and $\lambda(t) \geq 0$ is the contact force acting in case of hiding the obstacle. Over the whole time interval I, the ball can not penetrate the ground such that the inequality $q(t) \geq 0$ has to be satisfied additional to (1.1). If this constraint is active, i.e. $q(t) = 0$, the force $\lambda(t) \geq 0$ in (1.1) avoids penetration. If the constraint is not active this contact force is zero. The summarised problem has the form

$$0 \le q(t) \perp \lambda(t) \ge 0, \tag{1.2}$$

where the mathematical formulation $0 \le a \perp b \ge 0$ is equivalent to

$$a, b \in \mathbb{R}^n, \quad a^\top \cdot b = 0, \qquad a_i, b_i \ge 0, i = 1, \dots, n.$$

and is called complementarity problem (CP). Such problems, their properties und numerical treatment are analysed in Appendix B.

An active constraint symbolises a contact of ball and ground which are of special interests when discussing the dynamics of mechanical systems. In the moment the constraint becomes active at time $\tau \in I$, the left-side limit of the velocity satisfies

$$v^-(\tau) = \dot{q}^-(\tau) := \lim_{s \uparrow \tau} \dot{q}(s) \le 0,$$

because the ball falls down to the ground. In order to sustain the principle of non-penetrability of the ball and the ground, the right-side limit of the velocity must simultaneously fulfil the condition

$$v^+(\tau) = \dot{q}^+(\tau) := \lim_{s \downarrow \tau} \dot{q}(s) \ge 0.$$

Hence, the velocity function jumps maybe from a non-positive value $v^-(\tau)$ to a non-negative value $v^+(\tau)$ in τ. This is caused by the impact force which works in the moment where the ball and the ground get in contact. For the sake of uniqueness, the system consisting of equations (1.1) and (1.2) has to be completed by an impact law to describe the relationship between v^- and v^+ in such impact points. In this thesis the simple impact law of Newton

$$q(\tau) = 0 \quad \Rightarrow \quad v^+(\tau) = -\epsilon v^-(\tau), \qquad \epsilon \in [0, 1], \tag{1.3}$$

is used. The resolution number ϵ describes the elasticity of the impact. Alternatives are the impact law of Poisson or energy based models [88].

In Figure 1.2 the solution trajectories of (1.1)-(1.3) with parameters

$$q_0 = 1 \text{ m}, \quad v_0 = 0 \; \frac{\text{m}}{\text{s}}, \quad m = 1 \text{ kg}, \quad \epsilon = 0.8, \quad g = 2 \; \frac{\text{m}}{\text{s}^2}$$

are plotted. It can be observed that the position q is not longer differentiable in the impact points and so does not define a classical derivative v. In the same time points v is not continuous anymore.

1.2. Literature Survey and Outline

The motion of mechanical systems with impacts is a complex physical topic which will be analysed from a mathematical point of view in this thesis. Developments in mathematical and physical research are often closely related. Physical problems can be formulated and investigated in mathematical language. Theoretical results of mathematics in turn influence physical experiments, views and findings. Non-smooth analysis and mechanics with inequality constraints are one exemplary pair for this constant exchange of physics and mathematics.

In the 18th century, physicists tried to describe the motion of systems consisting of several rigid bodies mathematically and found a suitable way in Hamilton's principle, for example [54]. As a result, differential equations were obtained that describe the dynamics of mechanical systems through one single function. One advantage of this principle was that constraining forces can be easily included in the approach. Due to its simplicity and versatility, scientists in various application areas tried to use this modelling method. The problem was that frictional, contact and impact forces can only be described with the help of inequalities. At this time, it was not known if the combined formulation of differential systems with inequalities has solutions and of which nature the solutions could be. Therefore, it was not possible to assess whether the modelling approach correctly captured the behaviour of mechanical systems with contact and impact forces.

In the year 1798, Fourier published a paper in which he investigated contact problems that are very close to the understanding of non-smooth mechanics of today [73]. He combined Hamilton's principle with inequalities representing contact condition and dealt with the solvability of the analytical problems for the first time. However, since other areas of research were the focus of interest in the field of mechanics at that time, this was not yet the major breakthrough for multibody systems with contact forces. For a long time, there were no more theoretical investigations in this field. Mainly the much acclaimed mathematical publications by Moreau from the 1970s and 1980s [64, 65] awakened not only the interest in non-smooth analytical problems, but also again in unilaterally constrained mechanical systems. He established modelling with variational inequalities and introduced for the first time the comprehensive formulation of the equations of motion of mechanical systems as measure differential inclusions (MDI). Together with Jean and Panagiotopuolos, Moreau [50, 51, 66, 68] formulated important theoretical results using convex and non-smooth analysis. These mathematical research areas developed rapidly at that time, which was largely driven by optimisation. Additionally, the three pioneers developed first powerful numerical methods for MDIs. The simulation results matched very good to the experimental achievements for mechanical systems. Further theoretical treatises like solvability, boundedness or uniqueness on the complex field of MDI can nevertheless only be found sporadically, e.g. by Monteiro-Marques and Stewart [63, 87]. Much literature on non-smooth mechanics still refers to these basic explanations. This thesis contains a lot of new or advanced theoretical results.

In the field of engineering and numerical mathematics, greater interest in non-smooth mechanical systems and faster developments of theoretical investigations could be observed. The most common application areas for mechanical systems with inequalities are vehicle construction, biotechnology, electrical engineering, robotics or mechanical engineering [41, 84, 88] and all these areas of application were in the strong focus of science at the time of Moreau's, Jean's and Panagiotopuolos' publications and after that. Glocker and Pfeiffer [41, 73] reformulated the equations of motion as a MDI on velocity and acceleration levels with the help of linear complementarity problems. This facilitated the access to simulation and development of numerical methods immensely. A multitude of application problems were mathematically captured by these formulations [35, 39, 73]. Until today, new methods for MDIs are constantly being developed and analysed [1, 2, 24, 53, 70, 71, 80, 87, 91].

To get an overview on mechanical systems with inequality constraints, the modelling process of the motion of mechanical systems with impacts and contacts is described in Chapter 2. Two aspects of modelling and applications are particularly focused on:

- **A1** Modelling without a minimum number of coordinates or minimum necessary constraints. This aspects results from simplifications of the modelling process to save time and computational effort. Redundant constraints or dependent variables are the consequences. This efficiency approach is increasingly observed in practice [14, 17, 47, 49, 92].
- **A2** The occurrence of infinitely many impacts in a finite time interval. The resulting accumulation point of velocity jumps - as already observed in Example 1.1 - poses a complex analytical challenge and is known in the literature as the Zeno phenomenon [59, 70, 82].

For both aspects, academic examples are given at the end of the second chapter with the two-masses oscillator [17] and the slider-crank mechanism [82]. They will be examined in all further chapters of the thesis.

As Example 1.1 motivates, the equations of motion of such mechanical systems with impacts are differential systems complemented by complementarity problems [41, 88]. The example trajectories provide a hint that the solution space of such mathematical problems does not correspond to the set of arbitrarily often continuously differentiable functions, also called smooth functions. An introduction to the basics of non-smooth analysis is the beginning of the the second chapter. The absolutely continuous functions and the functions of bounded variation are introduced. These sets of functions form physically plausible spaces from which the position and velocity functions can originate. It is possible to define generalised derivative concepts both for absolutely continuous functions and for functions with bounded variation. These are needed to be able to relate position and velocity function and to obtain a general concept of acceleration.

The two mentioned function spaces are the solution spaces of certain non-smooth analytic problems such as MDIs. These are also introduced in more detail in Chapter 2, since the equations of motion of the non-smooth mechanical systems used in this thesis can be classified in this class of problems [65]. In addition, Chapter 2 presents known analytical and numerical results on MDIs which are developed further in the latter chapters

One open problem in previous investigations is that there is no generally valid existence result for applications that satisfy both **A1** and **A2**. In order to guarantee the correctness of the modelling approaches, the existence proof of solutions and the proof of their boundedness are carried out under weak assumptions in Chapter 3. It is worth mentioning, that there are already existence results in the literature on which the mentioned results base. When considering applications that satisfy **A1** but not **A2**, one can decompose the relevant time interval along the finitely many impact times. In the open intervals in between, one obtains differential-algebraic equations (DAE) or differential complementarity problems with redundant conditions or dependent variables. Both for the first [6] and for the second problem class [17] existence results have already been proved. Among others in [63, 87], the equations of motion of applications satisfying **A2** but not **A1** were considered in reverse. Solvability results could be derived there as well. In Chapter 3, two new analytical results on the existence of solutions and their boundedness are presented, which are also applicable to problems satisfying both **A1** and **A2**. So far, no similar result could be found in the literature. The examples from Chapter 2 are taken up again and it is proven that they fulfil the assumptions of the theoretical investigations.

Chapter 4 then focusses on further analytical properties of non-smooth mechanical systems. In particular, the focus is on the mathematical pendulum with an obstacle [18]. First, two new theorems regarding the existence of periodic solutions are presented. In the second part, stability properties for equilibria and periodic solutions are discussed. This is done by generalising the theory of invariant limit cycles for ordinary differential equations (ODEs) to MDIs and using known stability results of [59] for non-smooth systems.

Periodic solutions are also important for the final Chapter 5. Often it is not possible to obtain the exact solution of the equations of motion due to the complexity of the model. For this purpose, in Chapter 5 we present numerical algorithms for non-smooth problems that compute approximate solutions [2, 91]. Since our focus is on problems with the Zeno phenomenon, so-called time-stepping methods [1, 80] must be used for the numerical solution, as it will be explained. One problem of this class of methods is that the classical comparison tool, the convergence order, does not work for non-smooth problems [80]. In order to nevertheless be able to assess the approximation quality of time-stepping methods for non-smooth mechanical problems, we explain another comparison tool with the orbital convergence in Chapter 5. However, this is only applicable to problems with invariant limit cycles. This criterion is applied numerically to all applications of the chapters before. Finally, we prove that the orbital convergence is a suitable tool for certain time-stepping methods.

At the end of this thesis, the analytical and numerical results will be summarised once again and classified in terms of their relevance for practice. In the appendix, the basics of measure theory and convex analysis are summarised, which are relevant for our investigations but are not found in all mathematical studies. In addition, two further application examples that fulfil aspects **A1** and **A2** are described in the appendix. However, they are not examined further in the thesis.

2. Preliminaries

In this chapter, the theoretical and practical setting of the present thesis will be introduced. The exemplary trajectories of the bouncing ball in Example 1.1 motivate a study of non-smooth dynamical systems. Such problems are often of set-valued structure and their solution spaces include non-differentiable or even discontinuous functions. In Section 2.1., possible solution spaces and suitable generalised derivative concepts are introduced. Subsequently, a little survey to non-smooth dynamical systems is given. In Section 2.2., the equations of motion for mechanical systems with contact and impact forces are derived. A special issue of this thesis will be theoretical investigations on applications with singular mass matrices and redundant constraints. Because of that some examples of such kind complete this chapter.

2.1. Non-Smooth Analysis

In all following considerations, $I \subset \mathbb{R}$ ist a real time interval $[0, T]$ with $T > 0$.

Definition 2.1 (Smooth function) A function

$$f : I \to \mathbb{R}^n, \qquad t \mapsto f(t)$$

is called *smooth* on I if it is continuously differentiable up to any order for all t in the interior of I. If f is at least for one t in the interior of I not continuously differentiable we call f *non-smooth*.

Smooth functions play an important role in many analytical studies, e.g., in the theory of ordinary differential equations (ODEs)

$$f(t, q, \dot{q}) = 0$$

They describe the devolopment of a function $q : I \to \mathbb{R}^n$ and its first derivative $v := \dot{q}$ using a smooth function f. Classical and simple mechanical problems like an oscillator or a pendulum can be captured as an ODE since their motion is only influenced by smooth forces like gravitational, weight and spring forces. Contact and impact forces can not be described by continuously differentiable functions as mentioned before. Such forces are non-smooth.

Nevertheless, the motion of a mechanical system, which is influencend by non-smooth forces, can also be described by a position function q. But this function is in general not differentiable, such that the velocity and the acceleration can not be a classical first and second derivative. In this section suitable function spaces for the generalised coordinate and velocity function of non-smooth mechanical systems are introduced with the absolutely continuous functions and the functions of bounded variation. They form the base to understand the behaviour of solutions of the mechanical models and to investigate in further analytical studies like existence, boundedness or stability questions. For a more detailed introduction to the topic of non-smooth analysis, we refer to [9, 26, 56, 63, 65].

2.1.1. Generalised Continuity and Derivative Concepts

The following definition is a generalistion of continuity and permits Lebesgue integrability. It is a suitable function space for the position function q and preserves the possibility of relating velocity and q via the fundamental theorem of calculus.

Definition 2.2 (Absolutely continuous function) A function $q : I \to \mathbb{R}^n$ is called *absolutely continuous* if for all $\epsilon > 0$ a $\delta > 0$ exists such that

$$\sum_{i=1}^{m} \|q(t_i) - q(s_i)\| < \epsilon$$

for all disjoint partitions $(s_i, t_i) \subset I, i = 1, \ldots, m$, with $\sum_{i=1}^{m}(t_i - s_i) < \delta$. Notation: $q \in AC(I, \mathbb{R}^n)$.

Example 2.3 (Lipschitz continuous function) A subclass of $AC(I, \mathbb{R}^n)$ are the Lipschitz continuous functions [59] which play also an important role in the theory of ODEs. A function $q : I \to \mathbb{R}^n$ is *Lipschitz continuous* with the Lipschitz constant $L > 0$ if

$$\|q(s) - q(t)\| \leq L|s - t|, \qquad \forall\ s, t \in I.$$

A function $q \in AC(I, \mathbb{R}^n)$ is continuous, but not mandatory differentiable. But the set of all time points in which the first derivative does not exist is a Lebesgue null-set (see Appendix A, Definition A.4. and Example A.8). A Lebesgue null-set has Lebesgue measure zero. For example, a countable set is a Lebesgue null-set. This property of q is the prerequisite for the plausibility of the following theorem.

Lemma 2.4 (Fundamental theorem of calculus) Let $q \in AC(I, \mathbb{R}^n)$. Referring to [30, Theorem 4.14, Corollary 4.6], the function q is almost everywhere differentiable. Hence, $w \in \mathcal{L}^1(I, \mathbb{R}^n)$ exists pointwise with

$$q(t) = q(s) + \int_{[s,t]} w(\tau)\ \mathrm{d}\tau, \qquad \forall\ s, t \in I, s \leq t.$$

The function w is said to be the weak derivative of q and can be denoted by $\dot{q}$.

Proposition 2.5 (Banach space $AC(I, \mathbb{R}^n)$) The space $AC(I, \mathbb{R}^n)$ is a Banach space with the weak norm

$$\|q\|_{1,\infty} := \max\left(\|q\|_\infty, \|\dot{q}\|_\infty\right),$$

where $\|q\|_\infty := \sup_{t \in I} \|q(t)\|$ is the supremum norm and $\|.\|$ an arbitrary norm on $\mathbb{R}^n$ [38].

Example 2.6 (Absolute value function $|t|$) The function $h : I \to \mathbb{R}, t \mapsto |t|$, is an absolutely continuous function. It is in all $t \in I$ continuously differentiable except the origin and for all $a \in \mathbb{R}$ the function

$$\dot{h}_a(t) := \begin{cases} -1, & t < 0, \\ a, & t = 0, \\ 1, & t > 0 \end{cases}$$

defines a weak derivative in the sense of Lemma 2.4. The left- and right-side limit of these functions in $t = 0$

$$\dot{h}_a^-(0) := \lim_{s \uparrow 0} \dot{h}_a(s) = -1, \qquad \dot{h}_a^+(0) := \lim_{s \downarrow 0} \dot{h}_a(s) = 1$$

exist and do not match for all a. A classical and application related approach of generalised derivative concepts is to define the line segment $[-1, 1]$ between both limits as the set of possible values for $\dot{h}(0)$. The set-valued function

$$\partial h(t) := \begin{cases} -1, & t < 0, \\ [-1, 1], & t = 0, \\ 1, & t > 0 \end{cases} \tag{2.1}$$

is a generalised derivative of h and is noted by $\mathrm{Sign}(t)$. All pointwise elements $w(t) \in \mathrm{Sign}(t), t \in \mathbb{R}$, are weak derivatives of $|t|$ in the sense of Lemma 2.4.

Definition 2.7 (Clarke differential, [22]) Let $\mathrm{co}(C)$ be the convex hull of a set C and $f : I \to \mathbb{R}^n$ an absolutely continuous function. The set

$$\partial^C f(t) := \overline{\mathrm{co}} \left\{ \lim_{t_i \to t} \dot{f}(t_i) \mid (t_i)_{i \in \mathbb{N}} \subset I : (t_i \to t) \wedge (\forall\, i \in \mathbb{N} : \dot{f}(t_i) \text{ exists}) \right\} \tag{2.2}$$

is called the *Clarke differential*. If f is continuous and convex, the Clarke differential (2.2) is equal to the subdifferential [30].

Example 2.8 (Indicator function) For a set $D \subset \mathbb{R}^n$ the function

$$\psi_D(x) := \begin{cases} 0, & x \in D \\ \infty, & x \notin D \end{cases}$$

is called the indicator function. It holds $\partial^C \psi_D = N_D(x)$ [59] where

$$N_D(x) := \{ y \in D : y^\top (x^* - x) \leq 0, \forall\, x^* \in D \}$$

is the normal cone of D.

To study the generalised derivative concept (2.2) of Clarke, some properties of one-valued functions should be generalised to set-valued functions. In the following, F maps from $\mathbb{R}^n$ to the power set $P(\mathbb{R}^n)$. Henceforth, the image contains sets, not vectors.

Definition 2.9 (Monotone set-valued function) The function $F : \mathbb{R}^n \to P(\mathbb{R}^n)$ is said to be monotone if for all $x_1, x_2 \in \mathbb{R}^n$ and $y_1 \in F(x_1), y_2 \in F(x_2)$ the following relationship is satisfied

$$(y_1 - y_2)^\top (x_1 - x_2) \geq 0.$$

Example 2.10 (The Sign- and Upr-function) The Clarke differential $\mathrm{Sign}(t)$ of $|t|$ and the unilateral primitive function

$$\mathrm{Upr}(t) = \begin{cases} \emptyset, & t < 0, \\ (-\infty, 0], & t = 0, \\ 0, & t > 0 \end{cases} \tag{2.3}$$

are monotone set-valued functions. Both functions are used to formulate physical laws like frictional or impulsive ones as we will see in the following section.

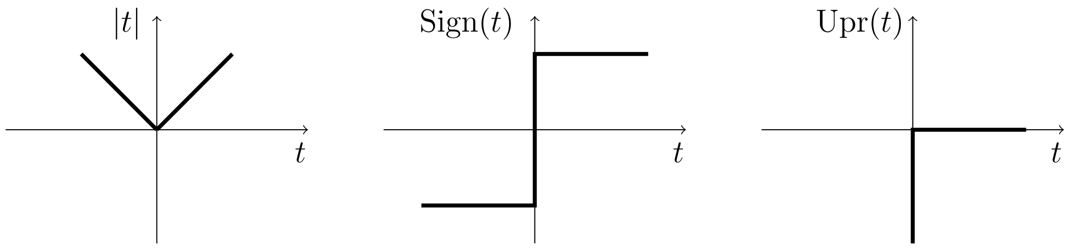


Figure 2.1.: Absolutely continuous and monotone functions.

The following definition is a generalisation of piecewise continuity for one-valued functions to set-valued ones.

Definition 2.11 (Upper semi-continuous set-valued function) Let $\emptyset \neq D \subset X$ and X a Banach space. A function $F : D \to P(X)$ is *upper semi-continuous* in $x \in D$ if for all $\epsilon > 0$ there exists a $\delta > 0$ with

$$y \in D \cap B(x, \delta) \quad \Rightarrow \quad F(y) \subset F(x) + B(0, \epsilon).$$

The set

$$B(z, r) := \{y \in D : \|y - z\| < r\}$$

is the open ball with the centre z and radius r. If F is upper semi-continuous in all $x \in D$, it is upper semi-continuous on D.

Theorem 2.12 Let $\emptyset \neq D \subset X$ and X be a Banach space. Referring to [26, Proposition 1.2], it is satisfied:

i) If $F : D \to P(X)$ is upper semi-continuous, then the set

$$graph(F) := \{(x, y) \in D \times X \mid y \in F(x)\}$$

is closed.

ii) If $graph(F) \subset D \times X$ is compact, then $F : D \to P(X)$ is upper semi-continuous.

The function q representing the generalised coordinates of a mechanical system with inequality constraints can be interpreted as an absolutely continuous function. The velocity should be a weak derivative of q which is often introduced by set-valued force laws using monotone upper semi-continuous functions. In the following, a suitable function space for the generalised velocity, the weak derivative of q, is presented.

Definition 2.13 (Variation of a function) Let $v : I \to \mathbb{R}^n$. The non-negative number

$$\mathrm{var}(v, I) := \sup \sum_{i=1}^{p} \|v(t_i) - v(t_{i-1})\|$$

is said to be the *variation* of v on I, where the supremum is taken over all increasing monotone sequences $(t_i)_{i=0,\ldots,p} \subset I$, and $p \in \mathbb{N}$.

Example 2.14 With $\dot{h}_a, a \in \mathbb{R}$, defined in Example 2.6, it is

$$\text{var}(|t|, [-2,2]) = 4 \qquad \text{var}(\dot{h}_a(t), [-2,2]) = |-1-a| + |1-a|.$$

The variation is the sum of all jump heights and the differences of functional values from one extreme value to another.

Definition 2.15 (Function of bounded variation) The function $v : I \to \mathbb{R}^n$ is of bounded variation on I (Notation $v \in BV(I, \mathbb{R}^n)$), if

$$\text{var}(v, I) < \infty.$$

Example 2.16 The functions $g_i : I \to \mathbb{R}, I = [-1,1], i \in \{1,2\}$, with

$$g_1(t) := \begin{cases} 0, & t = 0 \\ \sin\left(\frac{1}{t}\right), & t \neq 0 \end{cases} \qquad g_2(t) := \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \notin \mathbb{Q} \end{cases}$$

are not of bounded variation on I since they have infinitely many montonicity changes from -1 to 1 or infinitely many jumps of constant height.

Proposition 2.17 (Properties of $v \in BV(I, \mathbb{R}^n)$)

(i) The function v defines a right- and left-side limit in every point $t \in I$ [65, Proposition 4.2]

$$v^+(t) := \lim_{s \downarrow t} v(s), \qquad v^-(t) := \lim_{s \uparrow t} v(s).$$

(ii) The function v has countable discontinuity points in a closed time interval $[a,b] \subset I$ [65, Corollary 4.4]. More precisely, it has a finite number of discontinuities or accumulation points $t_1^*, \ldots, t_p^*, p \in \mathbb{N}$, of discontinuities with

$$\lim_{t \to t_d^*} \left(v^+(t) - v^-(t)\right) = 0, \qquad d = 1, \ldots, p.$$

This later case is called **Zeno phenomenon** and is of special interest in this thesis.

(iii) Following [65, p. 16] it is satisfied $\forall\ \lambda_1, \lambda_2 \in \mathbb{R}, v_1, v_2 \in\ BV(I, \mathbb{R}^n)$

$$\text{var}(\lambda_1 v_1 + \lambda_2 v_2) \leq |\lambda_1| \text{var}(v_1) + |\lambda_2| \text{var}(v_2).$$

(iv) All elements of $AC(I, \mathbb{R}^n)$ are also elements of $BV(I, \mathbb{R}^n)$ [30, Conclusion 4.12].

(v) Referring to [38], $BV(I, \mathbb{R}^n)$ is a Banach space with $\|v\|_{BV} := \|v(0)\| + \text{var}(v, I)$ for every closed real time interval I.

Definition 2.18 (Critical set) Let $v \in BV(I, \mathbb{R}^n)$. The set

$$T^*(v, I) := \left\{t \in I \mid v^+(t) \neq v^-(t)\right\}$$

of all discontinuity points of v is called *critical set* of v on I.

Lemma 2.19 Let $v \in BV(I, \mathbb{R}^n)$. Then the function $q : I \to \mathbb{R}^n$ defined by

$$q(t) := q(0) + \int_{[0,t]} v(\tau)\ \mathrm{d}\tau, \quad t \in I,$$

with $q(0) \in \mathbb{R}^n$ is an absolutely continuous function [30, Theorem 4.14].

Remark 2.20 (The differential measure, [65]) Like absolutely continuous functions, functions of bounded variation have a generalised derivative, the so called differential measure. In the following, classical elements and notations of measure and integration theory (see Appendix A) are utilised to define this tool which was established by Moreau [65]. Let $v \in BV(I, \mathbb{R}^n)$, $\phi \in C(I, \mathbb{R})$ be a continuous function on I, the values $0 = t_0 < t_1 < \ldots < t_p = T$ a sequence of time points in I and $\theta_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, p$, intermediate values. Referring to Moreau [65, Proposition 6.1], the sum

$$H((t_i)_{i=0,\ldots,p}, (\theta_i)_{i=1,\ldots,p}, \phi, v) := \sum_{i=1}^{p} \phi(\theta_i)(v(t_i) - v(t_{i-1}))$$

converges for $p \to \infty$ to a limit being independent of (t_i) and (θ_i). This limit

$$\int_I \phi\ \mathrm{d}v := \lim_{p\to\infty} H((t_i)_{i=0,\ldots,p}, (\theta_i)_{i=1,\ldots,p}, \phi, v) \tag{2.4}$$

corresponds to the Lebesgue-Stieltjes integral of ϕ with respect to $\mathrm{d}v$. The function $\mathrm{d}v : \phi \mapsto \int \phi\ \mathrm{d}v$ is said to be the *differential measure* of v and describes the change of v on I weighted by a function ϕ.

Definition 2.21 (Step function) A function $v_S : I \to \mathbb{R}^n$ is called a *step function* if there a monotone sequence $0 = t_0 \le t_1 \le \ldots \le t_p \le T$ on I and $c_1, \ldots, c_p, c_{p+1} \in \mathbb{R}^n$ with

(i) $I = (\bigcup_{i=1}^{p} [t_{i-1}, t_i]) \cup [t_p, T]$,

(ii) $t_1, \ldots, t_p \in T^*(v_S, I)$

(ii) $v_S = \left(\sum_{i=1}^{p} c_i \chi_{[t_{i-1},t_i)}\right) + c_{p+1}\chi_{[t_p,T]}$.

The function χ_A is the characteristic function of a set $A \subset I$

$$\chi_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin I \backslash A. \end{cases}$$

and could also be defined using the Dirac measure of x

$$\delta_x(A) := \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

through $\chi_A(x) = \delta_x(A)$.

Proposition 2.22 (Differential measure of a step function) Let $v_S \in BV(I, \mathbb{R}^n)$ be a step function. Following [65, Proposition 6.5] and using Definition 2.21, one has

$$\int_I \phi\ \mathrm{d}v_S = \sum_{i=1}^{p} \phi(t_i)(v_S^+(t_i) - v_S^-(t_i))$$

where $t_1, \dots t_p \in T^*(v_S, I)$. The differential measure $\mathrm{d}v_S$ is equivalent to

$$(v_S^+ - v_S^-) \sum_{t \in T^*(v_S, I)} \mathrm{d}\delta_t$$

in the sense of

$$\int_A \mathrm{d}v_S = \int_A (v_S^+ - v_S^-) \sum_{t \in T^*(v_S,I)} \mathrm{d}\delta_t = \sum_{t \in T^*(v_S,I)} \int_A (v_S^+ - v_S^-) \mathrm{d}\delta_t, \qquad A \subseteq I.$$

Following Example A.5, for a Dirac measure δ_s depending on $t \in \mathbb{R}$ it holds

$$\int_A \phi \, \mathrm{d}\delta_s = \begin{cases} \phi(s), & s \in A \\ 0 & s \notin A \end{cases}$$

Theorem 2.23 (Lebesgue decomposition, see Theorem A.37) For $v \in BV(I, \mathbb{R}^n)$ there exists a decomposition $v = v_{\mathrm{abs}} + v_S + v_{\mathrm{sing}}$ with the Lebesgue decomposition of the differential measure

$$\mathrm{d}v = \mathrm{d}v_{\mathrm{abs}} + \mathrm{d}v_S + \mathrm{d}v_{\mathrm{sing}}.$$

The function v_{abs} is absolutely continuous with a weak derivative $\dot{v}_{\mathrm{abs}}$ such that $\mathrm{d}v_{\mathrm{abs}} = \dot{v}_{\mathrm{abs}} \, \mathrm{d}t$ meaning

$$\int_{[c,d]} \mathrm{d}v_{\mathrm{abs}} = v_{\mathrm{abs}}(d) - v_{\mathrm{abs}}(c), [c, d] \subset I.$$

The step function v_S is discountiuous in the same time points like v and is almost everywhere constant. Following Proposition 2.22, this differential measure fulfils

$$\int_{[c,d]} \mathrm{d}v_S = \sum_{t \in T^*(v,[c,d])} v^+(t) - v^-(t).$$

The function v_{sing} is singular and its differential measure $\mathrm{d}v_{\mathrm{sing}}$ is so singular to the Lebesgue measure (see Appendix A Definition A.33). This means, that v_{sing} is not zero only on a Lebesgue null set. Since we consider differential systems which are formulated with Lebesgue integrals, the parts belonging to v_{sing} vanish.

As we will see in the next section, the velocity v of a mechanical system can be interpreted as a function of bounded variation. The absolutely continuous part v_{abs} describes the behaviour in time intervals without impacts und v_S the dynamics in impact points. The singular part has no practical relevance in our abstract model. Therefore, in this thesis only functions of bounded variation without a singular part are considered as it is often done in literature considering non-smooth mechanical systems [41, 59, 80].

Definition 2.24 (Special functions of bounded variation) The set

$$SBV(I, \mathbb{R}^n) := \{v \in BV(I, \mathbb{R}^n) \mid v_{\mathrm{sing}} \equiv 0\}$$

of all functions of bounded variation without a singular part is the space of all special functions of bounded variation. The set

$$SBV^+(I, \mathbb{R}^n) := \left\{v \in SBV(I, \mathbb{R}^n) \mid v = v^+\right\}$$

is the space of all right-continuous functions in $SBV(I,\mathbb{R}^n)$.

Theorem 2.25 Following [3, Corollary 4.3], $SBV(I,\mathbb{R}^n)$ and $SBV^+(I,\mathbb{R}^n)$ are closed subspaces of $BV(I,\mathbb{R}^n)$.

Theorem 2.26 (Variation and Differential measure, [63]) Let $v \in SBV^+(I,\mathbb{R}^n)$. Then

$$\operatorname{var}(v,I) = \int_I |dv| = \int_I |\dot{v}_{\text{abs}} dt + v_S \sum_{s\in T^*(v,I)} d\delta_s| = \int_I \|\dot{v}_{\text{abs}}\|\, dt + \sum_{s\in T^*(v,I)} \left\|v^+(s) - v^-(s)\right\|$$

where $|dv|$ ist the variation of the Lebesgue-Stieltjes measure of v (see Appendix A, Definition A.31).

2.1.2. Set-Valued Differential Systems

The function classes of the previous section are the solution spaces of certain analytical systems that are special dynamical systems. These problems are used to describe real world processes whose state evolves with time. Usually, this development is described by an ODE

$$\dot{x}(t) = f(t, x(t)), \quad t \in I. \tag{2.5}$$

If the right-hand side f is continuous, a solution of system (2.5) exists and it is unique if f is Lipschitz continuous. However for systems (2.5) with only piecewise continuous functions f, a generalised solution concept should be introduced.

Example 2.27 (ODE with discontinuous right-hand side) A classical example for differential systems with non-smooth nature are differential equations with a discontinuous right-hand side f, for example

$$\dot{x}(t) = f(x(t)) = \begin{cases} 1, & x(t) < 0, \\ -1, & x(t) \geq 0, \end{cases} \qquad t \in I,$$

with a piecewise constant right-hand side (compare Example 2.15 in [2]). Every solution ends up in zero and can not leave this value. There is no solution in the classical sense.

Remark 2.28 (Solution concept of Filippov) In [34] a generalised solution concept for differential systems is developed by replacing (2.5) by

$$\dot{x}(t) \in F(x(t)) := \bigcap_{\epsilon>0} \bigcap_{\lambda_n(N)=0} \overline{\text{co}}\, f((x(t) + B(0,\epsilon))\backslash N)$$

where λ_n is the Lebesgue measure of dimension n. This set-valued problem with respect to a weak derivative of x is a differential inclusion. In Example 2.27, one obtains the problem

$$\dot{x}(t) \in -\text{Sign}(x(t))$$

that has an absolutely continuous solution. If $x(0) \neq 0$, the function x has a kink in the time point in which it gets zero. General set-valued differential problems with absolutely continuous solutions are called differential inclusions.

Definition 2.29 (Differential inclusion) Let $F : \mathbb{R}^n \to P(\mathbb{R}^n)$ be a set-valued function. The problem to find an absolutely continuous function $x : I \to \mathbb{R}^n$ with

$$\dot{x}(t) \in F(x(t)), \qquad t \in I, \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{2.6}$$

is called a *differential inclusion* where $\dot{x} : I \to \mathbb{R}^n$ is a weak derivative of x.

Theorem 2.30 (Smirnov [86]) The differential inclusion (2.6) has an absolutely continuous solution $x(t)$, if the set-valued function F is Lipschitz continuous, i.e.

$$\exists\, L \geq 0 : F(x_1) \subset F(x_2) + L\,\|x_1 - x_2\|\, B(0,1), \qquad x_1, x_2 \in \mathbb{R}^n,$$

and if the set $F(x)$ is closed and convex for all $x \in \mathbb{R}^n$. The set $\alpha \cdot B, B \subset \mathbb{R}^n, \alpha \in \mathbb{R}$, is equivalent to $\{\alpha \cdot b \mid b \in B\}$.

Theorem 2.31 (Deimling [26]) The differential inclusion (2.6) has an absolutely continuous solution $x(t)$, if the set-valued function F is upper semi-continuous and linearly bounded, i.e.

$$\exists\, c > 0 : \|F(x)\| \leq c(1 + \|x\|), \qquad x \in \mathbb{R}^n,$$

and if the set $F(x)$ is closed and convex for all $x \in \mathbb{R}^n$.

Remark 2.32 (Survey to non-smooth analytical problems) Other methods to model non-smooth dynamical systems are, inter alia, Moreau's sweeping process [63, 64], evolution inequalities [60] or differential variational inequalities [69]. The solution of all these problems is an absolutely continuous function. As it was described by Example 1.1, solutions of non-smooth mechanical systems are not only non-differentiable, they are even discontinuous. Therefore, Moreau [65] introduced the measure differential inclusions.

Definition 2.33 (Measure differential inclusion) Let F be a set- and measure-valued function on $I \subset \mathbb{R}$. That means

$$F : BV(I, \mathbb{R}^n) \to P(\mathcal{S}_I), \qquad \mathcal{S}_I := \{\nu : P(I) \to \mathbb{R}^n \mid \ \nu \text{ is a signed measure on } I\}$$

A signed measure (see Appendix A Definition A.26) is a function that has all properties of a measure but could also be negative. The problem to find a function $x : I \to \mathbb{R}^n$ of bounded variation such that

$$\mathrm{d}x \in F(x), \qquad x(0) = x_0, \tag{2.7}$$

is said to be a *measure differential inclusion* (MDI). That means there is a signed measure $\nu : P(I) \to \mathbb{R}^n$ with $\nu \in F(x)$ and $\nu([0,t]) = \int_{[0,t]} \mathrm{d}x$ for all $t \in I$.

Example 2.34 (Reformulation: The bouncing ball) The equations of motion of the bouncing ball in Example 1.1 can be rewritten as

$$m\dot{v}(t) \in -mg - N_{\mathbb{R}_+}(q(t)), \qquad (t \in I \text{ s.t. } q(t) \neq 0), \tag{2.8}$$
$$v^+(t) = -\epsilon v^-(t), \qquad (t \in I \text{ s.t. } q(t) = 0). \tag{2.9}$$

In Appendix B, it is proven that $0 \leq \ \lambda \perp q \geq 0$ is equivalent to $-\lambda \in N_{\mathbb{R}_+}(q)$. Referring to Example 1.1, a separated analytical study of the differential inclusion and the discrete problem in the impact points is not possible since an accumulation point of discontinuities can be observed. The equivalent formulation as a measure differential inclusion is

$$m\ \mathrm{d}v \in -mg - N_{\mathbb{R}_+}(q(t))\ \mathrm{d}t - N_{\mathbb{R}_+}(v^+(t) + \epsilon v^-(t))\ \mathrm{d}\eta, \qquad \eta = \sum_{t \in T^*(v,I)} \delta_t$$

where $\mathrm{d}t$ is the one dimensional Lebesgue measure. The transformation of the impact law is explained in the following section.

Remark 2.35 (Decomposition of MDIs) In non-smooth mechanics, there are two sets of time points in I. As it is described in the next section, the first set contains all time points where the motion could be described by a differential inclusion (DI). The second one is a sequence of time points in which the velocity jumps. Remembering Example 1.1, the ball is either in free flight or an impact happens. Often numerical and analytical studies [17, 61, 69] based on the separated analytical consideration of DIs and the discrete problems in the impact points. For example, for both systems existence arguments are studied and are coupled by the implicit function theorem. However, if there is an accumulation point of discontinuities this procedure would fail. In contrast the summarised formulation as an MDI (2.7) is covering such complex behaviour.

2.2. Non-Smooth Mechanics

In the following, a system of a finite number of bodies connected by joints or other mechanical elements like springs or dampers is considered. Special interest is aimed to the simulation of its dynamics in a certain time interval I. The degree of freedom of the motion is reduced by the connecting elements and interaction forces like contacts and impacts affecting the multibody system. The equations of motion must be formulated as a differential system with inequality constraints. Those problems are investigated in the field of non-smooth analysis. Detailed introductions to the topic of mechanical systems can be found in [43, 84, 88].

The generalised coordinates, which describe uniquely the position of the mechanical system, at time $t \in I$ including angles or centres of masses are presented by the function $q : I \to \mathbb{R}^n$ and the generalised velocity by a weak derivative $v(t) = \dot{q}(t)$, $v : I \to \mathbb{R}^n$. In the following, it will be described that the dynamics of a mechanical system with contacts and impacts can be characterised through a measure differential inclusion.

2.2.1. Equations of Motion

There are two important aspects to set up the equations of motion for non-smooth mechanical systems. At first, the so called Lagrange equations for the unconstrained dynamics of the system must be derived, e.g., with Hamilton's principle. They form a differential system with respect to q and v. Furthermore, different types of constraints can influence the dynamics of the system. The physical laws of contact and impact forces can be formulated as normal cone inclusions [41]. They form in connection with the differential system a measure differential inclusion. Analytical results for non-smooth mechanical systems can be deduced from the results for measure differential inclusions which are derived for example in Chapter 3. For a more detailed introduction we refer to [41].

Hamilton's Principle

Remark 2.36 (Variational calculus) Let $S : u \mapsto S(u)$ be a functional that maps a function u to a real number $S(u) \in \mathbb{R}$. A typical task in calculus of variations is to determine a function u for which $S(u)$ is minimal or maximal. Those variational problems could be equivalently rewritten as partial differential equations. This problem class plays an important role in theoretical physics. It can even be said, especially elegant methods to formulate physical laws base on the application of variational principles. In [11, 76], further explanations of variational calculus and theoretical physics are discussed.

Definition 2.37 (Lagrangian) The Lagrange formalism assumes that the dynamics of a system can be described by using only one function called the Lagrangian. Let $T(q, v)$, $T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the kinetic energy of the system and $U(q)$, $U : \mathbb{R}^n \to \mathbb{R}$ the potential energy. The Lagrangian of the system is

$$\mathscr{L}(q, v) := T(q, v) - U(q). \tag{2.10}$$

Definition 2.38 (Action) The action of a mechanical system is defined as

$$A(t) := \int_{[0,t]} \mathscr{L}(q(s), v(s)) \; \mathrm{d}s. \tag{2.11}$$

Theorem 2.39 (Hamilton's principle of least action) The principle of least action of Hamilton states that a multibody system moves with minimal action. In a minimum of A, the first variation, a generalised directional derivative, vanishes. This means

$$\delta A(t) = \int_{[0,t]} \delta\mathscr{L}(q(s), v(s)) \; \mathrm{d}s = \int_{[0,t]} \delta T(q(s), v(s)) - \delta U(q(s)) \; \mathrm{d}s = 0.$$

Conclusion 2.40 (Euler-Lagrange equations) From the variational principle in Theorem 2.39, the so called Lagrange equations of the first kind or Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}(q, v)}{\partial v}\right) - \frac{\partial \mathscr{L}(q, v)}{\partial q} = 0 \tag{2.12}$$

follows [84]. Using the energy formula

$$T(q, v) := \frac{1}{2} v^\top M(q) v$$

with the symmetric positive semi-definite mass matrix $M : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, the equations (2.12) can be transformed to

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} &= f(q, v) \end{aligned} \tag{2.13}$$

with

$$f(q, v) := -\left(\frac{\partial}{\partial v}\frac{\partial T(q, v)}{\partial q}\right) \cdot v - \left(\frac{\partial U(q)}{\partial q} - \frac{\partial T(q, v)}{\partial q}\right)^\top .$$

The term $f^{nc}(q, v) := f(q, v) + \left(\frac{\partial U(q)}{\partial q} - \frac{\partial T(q,v)}{\partial q}\right)^\top$ will play a special role in Chapter 4 where periodic solutions are considered. It includes all non-conservative and gyroscopic forces.

Remark 2.41 (Alternative variational principles) The Euler-Lagrange equations (2.13) can also be derived with other variational principles like the ones of d'Alembert or Jourdain [11, 19, 41].

Inequality Constraints

Definition 2.42 (Inequality constraints) Similar to Example 1.1, the motion of multibody systems is often influenced by physical phenomena that can be mathematically described by inequalities. Such constraints are said to be unilateral. We consider constraints on the dynamics of the systems that can be represented by $m \in \mathbb{N}$ additional inequalities

$$g(q(t)) \geq 0, t \in I, \tag{2.14}$$

with a two times differentiable constraint function $g : \mathbb{R}^n \to \mathbb{R}^m$ and its Jacobian

$$G(q) = \frac{\partial g(q)}{\partial q} \in \mathbb{R}^{m \times n}.$$

The matrix G is called the constraint matrix. The relation $g \geq 0, g \in \mathbb{R}^m$, is equivalent to $g_i \geq 0, i = 1, 2, \ldots m$.

Remark 2.43 (Differential-algebraic equation) If we would combine (2.13) with an equality system $g(q(t)) = 0$ instead of (2.14), the total formulation of the equations of motion is a differential-algebraic equation (DAE). Such a problem combines a differential equality system with an algebraic one. The analytical investigation and numerical solution of DAEs is a well-studied mathematical field of research. Overviews about this topic can be found in [6, 43].

Conclusion 2.44 (Optimisation problem with inequality constraints) If the motion of a multibody system is restricted by additional inequality constraints, the equations of motion are equivalent to a non-linear optimization problem with inequality constraints

$$\min_{q \in \mathbb{R}^n} A(q) \qquad \text{s.t.} \qquad g(q) \geq 0 \tag{2.15}$$

where A is the action in (2.11).

Theorem 2.45 (Optimality conditions - Karush-Kuhn-Tucker conditions) We consider a generalised non-linear optimization problem (NOP)

$$\min_{x \in \mathbb{R}^p} h(x) \qquad \text{s.t.} \qquad g(x) \geq 0$$

with continuously differentiable functions $h : \mathbb{R}^p \to \mathbb{R}^n, g : \mathbb{R}^p \to \mathbb{R}^m$. Referring to [15], for a solution $x^* \in \mathbb{R}^p$ of (NOP), there must exist so called Lagrange multipliers $\lambda \in \mathbb{R}^m$ with

$$\begin{aligned} \frac{\partial h}{\partial x}(x^*) - \frac{\partial g}{\partial x}(x^*)\lambda &= 0, & (2.16)\\ \lambda \geq 0, \quad g(x^*) \geq 0, \quad \lambda^\top g(x^*) &= 0. & (2.17) \end{aligned}$$

Remark 2.46 (Complementarity problem) The problem (2.17) is a non-linear complementarity problem. Following Appendix B, it can be equivalently written as a normal cone inclusion

$$-\lambda \in N_{\mathbb{R}^m_+}(g(x^*))$$

where $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m \mid x \geq 0\}$. We prefer the latter formulation to underline that this is a set-valued problem.

Conclusion 2.47 (Equations of motion with inequality constraints) If one applies Theorem 2.45 to (2.15) the formulation of the equations of motion including unilateral constraints

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} &= f(q,v) + G(q)^\top \lambda, \\ -\lambda &\in N_{\mathbb{R}^m_+}(g(q)) \end{aligned} \tag{2.18}$$

results. The Lagrange multipliers $\lambda(t) \in \mathbb{R}^m$ represent the normal force $G(q)^\top \lambda(t)$ which prevent the violation of the constraints.

Remark 2.48 (Index formulations for DAEs, [6, 43]) For mechanical systems with equality constraints the motion can be described by a DAE of the form

$$M(q)\dot{v} = f(q,v) + G(q)^\top \lambda, \qquad g(q) = 0. \tag{2.19}$$

The so called index of a DAE plays important role for numerical and analytical investigations. The index is the number of times, the system (2.19) has to be differentiated with respect to t for a unique specification of $\dot{\lambda}$. For equation (2.19) this should be done three times. First the equality constraint is differentiated two times to get

$$0 = \frac{\partial^2 g}{\partial t^2}(q) = G(q)\dot{v} + \gamma(q,v), \qquad \gamma(q,v) := \frac{\partial^2 g}{\partial q^2}(v,v)$$

and combined with (2.19) to

$$\begin{pmatrix} M(q) & G(q)^\top \\ G(q) & 0 \end{pmatrix} \begin{pmatrix} \dot{v} \\ \lambda \end{pmatrix} = \begin{pmatrix} f(q,v) \\ -\gamma(q,v) \end{pmatrix}. \tag{2.20}$$

Following [6, Lemma 1], this coefficient matrix is regular if the conditions

(H1) M is symmetric and positive semi-definite,

(H2) M is positive definite at the null space of G

are satisfied. Then, formulation (2.20) has a unique solution for $\dot{v}$ and λ. By differentiating one more time we get an explicit description of $\dot{\lambda}$. If the initial value $q(0) := q_0$ satisfies the condition $g(q_0) = 0$ the DAE (2.19) is equivalent to the index two formulation with constraints on velocity level

$$M(q)\dot{v} = f(q,v) + G(q)^\top \lambda, \qquad G(q)v = 0. \tag{2.21}$$

If the initial values q_0 and $v(0) = v_0$ fulfil in addition the hidden constraint $G(q_0)v_0 = 0$ it is equivalent to the index one formulation with constraints on acceleration level

$$M(q)\dot{v} = f(q,v) + G(q)^\top \lambda, \qquad G(q)\dot{v} + \gamma(q,v) = 0. \tag{2.22}$$

Definition 2.49 (Active sets) In order to characterise the index term for inequality constraints, two important sets must be defined. The active set on position level is

$$J^1(q) := \{i \in \{1,\dots,m\} \mid g_i(q) = 0\}. \tag{2.23}$$

The set (2.23) contains all indices of constraint components that are active. An inequality $g_i(q) \geq 0$ is said to be active if $g_i(q) = 0$. The active set on velocity level is

$$J^2(q,v) := \left\{i \in J^1(q) \mid h_i(q,v) = 0\right\}. \tag{2.24}$$

where $h(q,v) := G(q)v$.

Definition 2.50 (Row reduction) Let $G \in \mathbb{R}^{m\times n}$ be a matrix and J be a subset of $\{1,2,\dots,m\}$. The matrix G_J is the matrix containing only the rows of G which number is an element of J.

Example 2.51 (Row reduction) Let G be the 5×5 unity matrix and $J = \{1,4,5\}$. Then we get

$$G_J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Conclusion 2.52 (Index formulations for DIs, [88]) Inspired by the index term for DAEs, it is defined that the differential complementarity problem

$$\begin{aligned} M(q)\dot{v} &= f(q,v) + G(q)^\top \lambda, \\ 0 \leq \lambda &\perp g(q) \geq 0, \\ \lambda &: I \to \mathbb{R}^m, \end{aligned}$$

has index three. The analytically equivalent formulation on velocity level

$$\begin{aligned} M(q)\dot{v} &= f(q,v) + G_{J^1(q)}(q)^\top \lambda, \\ 0 \leq \lambda &\perp G_{J^1(q)}(q)v \geq 0, \\ \lambda &: I \to \mathbb{R}^{m_1}, \quad m_1 = |J^1(q)|, \end{aligned}$$

has index two and the equivalent formulation on acceleration level

$$\begin{aligned} M(q)\dot{v} &= f(q,v) + G_{J^2(q,v)}(q)^\top \lambda, \\ 0 \leq \lambda &\perp G_{J^2(q,v)}(q)\dot{v} + \gamma_{J^2(q,v)}(q,v) \geq 0, \\ \lambda &: I \to \mathbb{R}^{m_2}, \quad m_2 = |J^2(q,v)| \end{aligned}$$

index one.

Example 2.53 (Contact forces) Typical inequality constraints (2.14) are given by distance conditions, i.e. the requirement that rigid bodies can not penetrate each other. They are based on purely geometric considerations. In Example 1.1 the distance of ball and ground obstacle must be non-negative over the whole time interval. Introduced in Conclusion 2.47, the Lagrange multiplier represents the normal contact force $G(q)^\top \lambda$ which guarantees the non-penetration.

Impact law

Definition 2.54 (Impact point) A time point τ is said to be an impact point if a non-active constraint gets active, i.e.

$$\exists\, \delta > 0,\ i \in J^1(q(\tau))\ \forall\, t \in [\tau - \delta, \tau) \subset I : i \notin J^1(q(t)).$$

Remark 2.55 (Velocity jump in impact points) Let $\tau \in I$ be an impact point. That means there is an index $j \in \{1, \dots, m\}$ such that the relations

$$g_j(t) = \begin{cases} > 0, & t \in [\tau - \delta_1, \tau), \\ = 0, & t = \tau, \\ \geq 0, & t \in (\tau, \tau + \delta_2] \end{cases}$$

are satisfied with sufficiently small $\delta_1, \delta_2 > 0$. The left-side limit of the derivative with respect to time calculated to

$$0 \geq \lim_{t \uparrow \tau} \frac{\mathrm{d}g_j}{\mathrm{d}t}(q(t)) = \lim_{t \uparrow \tau} G_j(q(t))v(t) =: G_j(q(\tau))v^-(\tau)$$

and the right-side limit to

$$0 \leq \lim_{t \downarrow \tau} \frac{\mathrm{d}g_j}{\mathrm{d}t}(q(t)) = \lim_{t \downarrow \tau} G_j(q(t))v(t) =: G_j(q(\tau))v^+(\tau)$$

where we use that g_j is twice times continuously differentiable. Therefore, it is possible that the pre-impact velocity v^- and the post-impact velocity v^+ have not the same value and a velocity jump can be observed. The change in the active set symbolises e.g., two bodies which get in contact. Physically, an impact happens.

A simple result is that impact points can be elements of the critical set $T^*(v, I)$ defined in Definition 2.18. Henceforth, impact points, which are also discontinuouty points of v, are a countable sequence $\tau_1, \tau_2, \dots$ in I.

Remark 2.56 (Impact law, [40]) In order to describe impacts, a physical law needs to be introduced. It relates v^- to v^+ in impact points $\tau \in T^*(v, I)$. In this thesis, the impact law of Newton

$$G_{J^1(q)}(q(\tau))v^+(\tau) = -\epsilon G_{J^1(q)}(q(\tau))v^-(\tau) \tag{2.25}$$

is utilised to relate the pre-impact velocity $v^-(\tau)$ and the post-impact velocity $v^+(\tau)$ in an impact point $\tau \in T^*(v, I)$. The so called resolution number $\epsilon \in [0, 1]$ is here constant. It characterises the elasticity of the impact. The condition $\epsilon = 0$ means that the impact is totally inelastic and $\epsilon = 1$ means that the impact is totally elastic. A normal impact force $\Lambda > 0$ accompanies the impact. Following [40, Section 2 (c)], there are reasons such that the contact does not participate in the impact. A very common situation is that multi-contact collisions happens. In this case, $G_{J^1(q)}(q(\tau))v^+(\tau)$ has to be greater or equal than $-\epsilon G(q)v^-(\tau)$ and the non-negative force Λ is orthogonal to the sum of both terms. These two cases can be combined to

$$0 \leq \Lambda \perp G_{J^1(q)}(q)(v^+ + \epsilon v^-) \geq 0$$

or equivalently

$$-\Lambda \in N_{\mathbb{R}^m_+}(G_{J^1(q)}(q)(v^+ + \epsilon v^-)).$$

Remark 2.57 (Consistent initial values) We assume that the velocity function v is right-continuous, i.e. $v(t) = v^+(t), t \in I$. Especially, in $t = 0$ it follows $v(0) = v^+(0) = v_0$. A solution of the equations of motion exists at $t = 0$ only, if $g_i(q_0) > 0$ or $g_i(q_0) = 0, G_{i,:}(q_0)v_0 \geq 0$ for all $i \in \{1, \ldots, m\}$. Therefore, we must assume that the initial point $\tau = 0$ is not an impact point. If $g_i(q_0) = 0$, $G_{i,:}(q_0)v_0 < 0$ is given for any $i = 1, \ldots, m$, we recalculated the initial velocity to $v_0 := v^+(0)$ by using the impact law of Newton.

Conclusion 2.58 (Equations of motion in form of an MDI) Summarising all steps, the equations of motion can be formulated as a measure differential inclusion

$$\begin{aligned} \dot{q} &= v, \\ M(q)\mathrm{d}v &= f(t, q, v)\mathrm{d}t + G(q)^\top \mathrm{d}P. \end{aligned} \tag{2.26}$$

The differential measures $\mathrm{d}v$ and $\mathrm{d}P$ are decomposed by the impact and the smooth parts

$$\mathrm{d}v = \dot{v}(t)\ \mathrm{d}t + \sum_{\tau_i \in T^*(v,I)} (v^+ - v^-)\ \mathrm{d}\delta_{\tau_i}$$

and

$$\mathrm{d}P = \lambda(t)\ \mathrm{d}t + \sum_{\tau_i \in T^*(v,I)} \Lambda\ \mathrm{d}\delta_{\tau_i}.$$

The smooth part λ of the normal force P is determined by the normal cone inclusion

$$-\lambda \in N_{\mathbb{R}^m_+}(g(q))$$

and the impact part Λ in impact points by

$$\tau \in T^*(v, I): \quad -\Lambda(\tau) \in N_{\mathbb{R}^{m_1}_+}(G_{J^1(q(\tau))}(q(\tau))(v^+(\tau) + \epsilon v^-(\tau)))$$

with $\epsilon \in [0, 1]$. We can ignore impact points which are not elements of $T^*(v, I)$. For impact points $\tau \notin T^*(v, I)$ it holds $G_{J^1(q(\tau))}(q)v = 0$ and the constraints with index in $J^1(q(\tau))$ can be treated as continuous constraints described in (2.14). The total formulation of the equations of motion for non-smooth mechanical systems is an MDI

$$
\begin{aligned}
\dot{q} &= v, \\
M(q)\mathrm{d}v &= f(t,q,v)\ \mathrm{d}t + G(q)^\top \mathrm{d}P, \\
-\mathrm{d}P &\in N_{\mathbb{R}^m_+}(g(q))\ \mathrm{d}t + N_{\mathbb{R}^{m_1}_+}(G_{J^1(q)}(q)(v^+ + \epsilon v^-))\ \mathrm{d}\eta, \\
\eta &= \textstyle\sum_{\tau \in T^*(v,I)} \delta_\tau.
\end{aligned} \tag{2.27}
$$

with consistent initial values.

Remark 2.59 (Function spaces) According to the description in the previous section, the generalised velocity can be interpreted as a function of bounded variation in $SBV^+(I, \mathbb{R}^n)$. The equations of motion must be rewritten in terms of the right-side limit $v^+(t) := \dot{q}^+(t)$. Referring to Lemma 2.19, the function

$$q(t) = q(0) + \int_{[0,t]} v(\tau)\mathrm{d}\tau$$

is absolutely continuous. The acceleration as the time derivative of $v \in SBV^+(I, \mathbb{R}^n)$ is not defined and must be replaced by the generalised derivative concept of differential measures $\mathrm{d}v$. That contains the differentiable part in the smooth phases and the discontinuous nature when impacts happen. There are also physical forces of fractial type like air turbulences. They are described by singular functions. We do not involve such forces in our modell and consider only contact and impact forces.

2.2.2. Applications

In literature about mechanical systems, it is very common to assume that

(i) the constraint equations are independent and therefore the Jacobian matrix $G(q)$ has full rank,

(ii) the mass matrix $M(q)$ is symmetric positive definite and the modelling process uses a minimal number of coordinates to describe the full system uniquely.

For example [61, 63, 69, 80, 87] deal with the mentioned requirements. However, in many situations the equations of motion are set up with rank-deficient mass or constraint matrices. Singular mass matrices occur in applications that utilise Euler parameters or natural coordinates to define the position of a rigid body [2, 16, 17, 47, 48]. Sometimes inertia is captured by zero if it is very small or if zero-mass points are used [12]. In such situations or by model reduction [92] singular mass matrices appear, too. Redundant constraints are derived from over-parametrized systems [55], simplifications of modelling and implementation processes [36] or singular positions.

Of course, the mentioned problems can also be analysed with a set of independent generalised coordinates by elimination of the redundant ones. In practice, the identification and elimination process is often expensive or even impossible, such that the formulation with the redundant coordinates or constraints is utilised. Therefore, generalised analytical studies including rank-deficient mass and constraint matrices are needed. In the literature, they are only occasionally available [6, 7, 17, 49, 92].

Example 2.60 (Two-masses-spring system with periodic forcing) The modelling of a two-masses-spring system is taken from [92] and is expanded by a unilateral constraint in

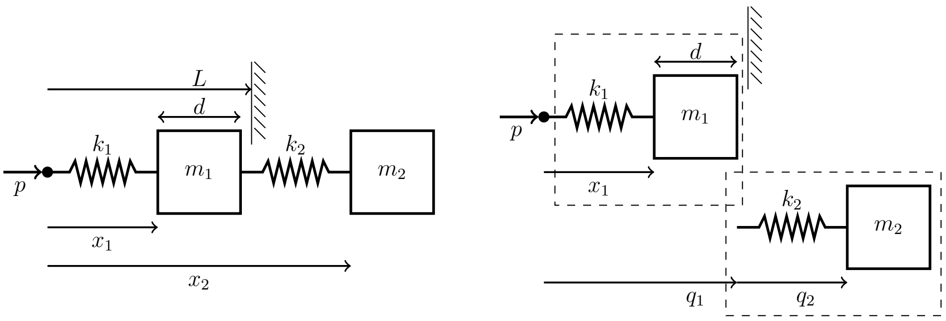


Figure 2.2.: Two-masses-spring system: Construction.

[17]. It consists of two masses $m_1 > 0$ and $m_2 > 0$ connected by springs with non-stretched lengths l_1 and l_2 and stiffnesses $k_1 > 0, k_2 > 0$. According to Figure 2.2 the position is presented by $q = (q_0, q_1, \bar{q}_2)^\top$ with $q_0 = x_1 - l_1$, $\bar{q}_2 = q_2 - l_2$. Referring to [92] the system is understood as two subsystems which are interconnected through an equality $f(q) = 0$ and as in [17] the motion of m_1 is further limited by a rigid obstacle at position L. In addition to the mentioned forces the first spring is influenced by an ω-periodic forcing $p(t)$. The resulting equations of motion have the form

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{pmatrix} \dot{v} = - \begin{pmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_2 \end{pmatrix} q + \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} + \begin{pmatrix} p(q) \\ 0 \\ 0 \end{pmatrix},$$

$$f(q) = q_1 - q_0 - l_1 - d = 0,$$
$$0 \le \lambda_2(t) \perp h_1(q) = L - q_0 - l_1 - d \ge 0,$$
$$0 \le \lambda_3(t) \perp h_2(q) = \bar{q}_2 \ge 0,$$

The mass matrix is obviously singular. These equations of motion are solved with the explicit Moreau-Jean scheme (see Chapter 5). With the parameter configuration

$$l_1 = l_2 = 1 \text{ m}, \quad d = 1 \text{ m}, \quad L = 2 \text{ m}, \quad m_1 = m_2 = 2 \text{ kg},$$
$$k_1 = 10 \,\frac{\text{kg}}{\text{s}^2}, \quad k_2 = 1 \,\frac{\text{kg}}{\text{s}^2}, \quad \epsilon = 0.9, \quad p(q) = \cos(\pi q/8),$$

in Figure 2.3 the periodic limit cycle of mass m_1 is plotted. It deals with accumulation points of velocity jumps

Example 2.61 (Slider-crank mechanism) A planar impacting slider-crank mechanism [35, 82, 83] is considered. Its motion is described by the angles $q = (\theta_1, \theta_2, \theta_3)^\top, \theta_i \in [0, 2\pi)$ and the angular velocity $v = (\omega_1, \omega_2, \omega_3)^\top$ (see Figure 2.4). Parameters belonging to the crank are indexed with 1: mass m_1, gravity J_1 and length l_1. Analogous characteristics of the connecting rod are m_2, J_2 and l_2. The slider has mass m_3, gravity J_3, length $2a$ and height $2b$. It is located between a ground and ceiling barrier which have a distance of d. The clearance of the slider between the obstacles is $2c = d - 2b$. The gap functions representing the geometrical constraints must be defined as

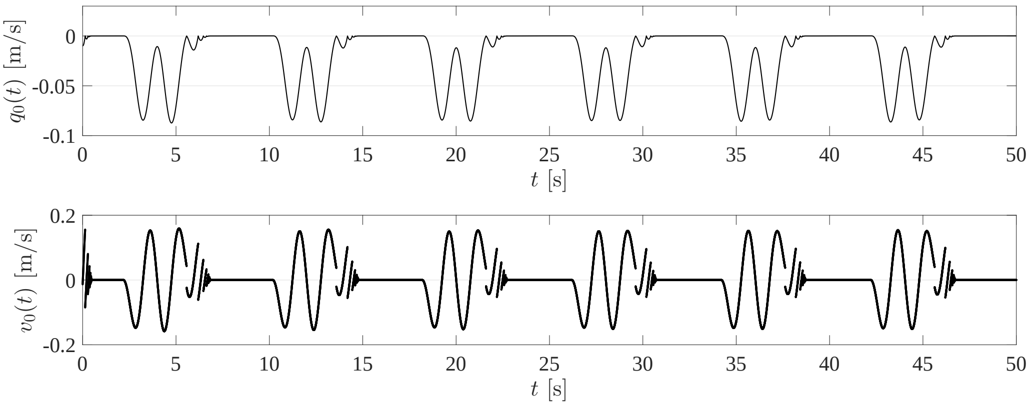


Figure 2.3.: Two-masses-spring system: Position q_0 and velocity v_0 of the first mass m_1.

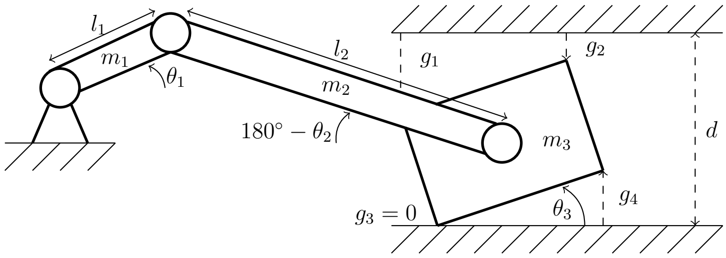


Figure 2.4.: Slider-crank mechanism: Construction.

$$
\begin{aligned}
g_1(q) &:= \frac{d}{2} - l_1 \sin\theta_1 - l_2 \sin\theta_2 + a \sin\theta_3 - b \cos\theta_3, \\
g_2(q) &:= \frac{d}{2} - l_1 \sin\theta_1 - l_2 \sin\theta_2 - a \sin\theta_3 - b \cos\theta_3, \\
g_3(q) &:= \frac{d}{2} + l_1 \sin\theta_1 + l_2 \sin\theta_2 - a \sin\theta_3 - b \cos\theta_3, \\
g_4(q) &:= \frac{d}{2} + l_1 \sin\theta_1 + l_2 \sin\theta_2 + a \sin\theta_3 - b \cos\theta_3,
\end{aligned}
$$

such that the constraint matrix satisfies

$$
G(q) = \begin{pmatrix}
-l_1 \cos\theta_1 & -l_2 \cos\theta_2 & a\cos\theta_3 + b\sin\theta_3 \\
-l_1 \cos\theta_1 & -l_2 \cos\theta_2 & -a\cos\theta_3 + b\sin\theta_3 \\
l_1 \cos\theta_1 & l_2 \cos\theta_2 & -a\cos\theta_3 + b\sin\theta_3 \\
l_1 \cos\theta_1 & l_2 \cos\theta_2 & a\cos\theta_3 + b\sin\theta_3
\end{pmatrix}.
$$

This matrix has not full rank which identify redundant constraints. The position of the rectangular slider is uniquely determined by the position of three corners and therefore the additional fourth gap function does not give new information. Assuming gravitation in negative y-direction the equations of motion fit exactly the form (2.27) with

$$M(q) = \begin{pmatrix} J_1 + l_1^2(m_1/4 + m_2 + m_3) & l_1 l_2 \cos(\theta_1 - \theta_2)(m_2/2 + m_3) & 0 \\ l_1 l_2 \cos(\theta_1 - \theta_2)(m_2/2 + m_3) & J_2 + l_2^2(m_2/4 + m_3) & 0 \\ 0 & 0 & J_3 \end{pmatrix},$$

$$f(q,v) = \begin{pmatrix} -l_1 l_2 \sin(\theta_1 - \theta_2)(m_2/2 + m_3)\omega_2^2 - \gamma l_1 \cos\theta_1(m_1/2 + m_2 + m_3) \\ l_1 l_2 \sin(\theta_1 - \theta_2)(m_2/2 + m_3)\omega_1^2 - \gamma l_2 \cos\theta_2(m_2/2 + m_3) \\ 0 \end{pmatrix}.$$

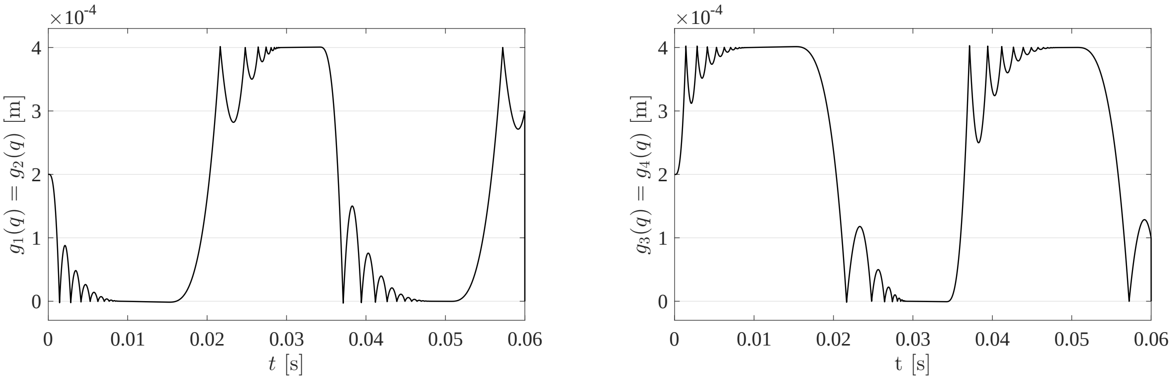


Figure 2.5.: Slider-crank mechanism: Gap functions.

The equations of motion are again solved with the explicit Moreau-Jean scheme of Chapter 5 and the configuration of parameters

Geometrys:	$l_1 = 0.153$ m,	$l_2 = 0.306$ m,	$a = 0.05$ m,
	$b = 0.05$ m,	$c = 0.0002$ m,	
Inertia properties:	$m_1 = 0.038$ kg,	$m_2 = 0.038$ kg,	$m_3 = 0.076$ kg,
	$J_1 = 7.4\ 10^{-5}\ \text{kg}\cdot\text{m}^2$,	$J_2 = 5.9\ 10^{-4}\ \text{kg}\cdot\text{m}^2$,	
	$J_3 = 2.7\ 10^{-6}\ \text{kg}\cdot\text{m}^2$,		
Force elements:	$\gamma = 9.81\ \text{m/s}^2$,	$\epsilon = 0.7$,	
Initial conditions:	$q_0 = (0,0,0)^\top$ m,	$v_0 = (150,-75,0)^\top\ \dfrac{\text{m}}{\text{s}}$	

as in [35]. As in Example 1.1, an accumulation point of impact points can be observed in numerical experiments, see Figure 2.5. Because the initial velocity is zero in the third component, the slider does not rotate. The rank of the constraint matrix even reduces to two and the gap functions g_1 and g_2 and the functions g_3 and g_4 are identical.

Further examples with rank-deficient mass or constraint matrices are listed in the Appendix C.

3. Solvability of Linear Implicit Measure Differential Inclusions

If real world processes are captured in mathematical models, the first question should be, whether the model has a solution and is so well-defined. In this chapter, the MDI (2.27) representing the equations of motion of mechanical systems including impacts is studied. There are already some detailed studies of the solvability of this problem. But they have to be generalised to match to our applications of interest.

Theorem 3.1 (Monteiro-Marques [63]) Let

(i) $m = 1$,

(ii) g and f be Lipschitz continuous,

(iii) $M(q) \equiv M \in \mathbb{R}^{n\times n}$ be regular.

Then, the MDI (2.27) has at least one solution $q \in AC(I, \mathbb{R}^n)$, $v \in BV(I, \mathbb{R}^n)$.

This theorem covers problems with a constant regular mass matrix and only a scalar inequality constraint.

Theorem 3.2 (Stewart [87]) MDI (2.27) has at least one solution $q \in AC(I, \mathbb{R}^n)$, $v \in BV(I, \mathbb{R}^n)$ if $g : \mathbb{R}^n \to \mathbb{R}^m$, $f : I \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous and

(i) $M(q), q \in \mathbb{R}^n$, is smooth, Lipschitz continuous, bounded and positive definite,

(ii) $G(q), q \in \mathbb{R}^n$, is smooth, Lipschitz continuous, bounded and has full rank.

Theorem 3.2 generalises Theorem 3.1 to time varying mass matrices and multi-dimensional constraints since $m \geq 1$. In the previous chapter, the description of mechanical systems with singular mass matrices and redundant constraints is considered. The existence results above do not cover the mathematical formulation of those problems. The possible singularity of the mass matrix concludes that instead of explicit problems (2.7) linear implicit measure differential inclusions are studied in this thesis. Due to the modeling approach, we are only interested in solutions $v \in SBV^+(I, \mathbb{R}^n)$.

3.1. Abstract Setting

Problem 3.3 Let $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be continuous and $F : BV(I, \mathbb{R}^n) \to P(\mathcal{S}_I)$ a set- and measure-valued function with

$$F(x) = F_1(x(t))\mathrm{d}t + F_2(x^-)\mathrm{d}\eta(x)$$
$$\eta(x) = \sum_{s\in T^*(x,I)} \delta_s$$

for $x \in BV(I,\mathbb{R}^n)$ and $F_i : \mathbb{R}^n \to P(\mathbb{R}^n), i = 1,2$. Here, d$t$ is the n-dimensional Lebesgue measure and η a sum of Dirac measures. We search for a solution $x \in SBV^+(I,\mathbb{R}^n)$ of the linear implicit measure differential inclusion (LIMDI)

$$A(x)\mathrm{d}x \in F(x), \qquad x(0) = x_0, \tag{3.1}$$

The inclusion (3.1) can be understood in the sense of Lebesgue-Stieltjes integrals

$$\begin{aligned}\int_{[0,t]} A(x)\mathrm{d}x &= \int_{[0,t]} f_1(x(s))\mathrm{d}s + \int_{[0,t]} f_2(x)\mathrm{d}\eta(x)\\ &= \int_{[0,t]} f_1(x(s))\mathrm{d}s + \sum_{s\in T^*(x,[0,t])} f_2(x(s)), \qquad \forall\, t \in I.\end{aligned}$$

The inclusions $f_1(x(t)) \in F_1(x(t)), t \in I\backslash T^*(x,I), f_2(x(s)) \in F_2(x^-(s)), s \in T^*(x,I)$, are defined pointwise.

To show the solvability of the dynamical problem (3.1), it is important to know that the discrete problem in a fixed time point has a solution. We call this property *consistency.*

Definition 3.4 (Consistency) Problem (3.1) is called consistent, if there exists a constant $K > 0$ such that for all $\bar{x} \in \mathbb{R}^n$ with $F_i(\bar{x}) \neq \emptyset, i = 1$ or 2, the following requirements are true

(C1) $\exists\, m_i \in \mathbb{R}^n :\; A(\bar{x})m_i \in F_i(\bar{x})$

(C2) $\exists\, h_0 > 0\; \forall\, h \in (0,h_0] : F_1(\bar{x} + hm_i) \neq \emptyset$

(C3) $\|m_i\| \leq K \cdot (1 + \|\bar{x}\|)$

We summarise all m fulfilling (C1) in the admissable sets $Z_i(\bar{x}), i = 1,2$. These sets have to be convex (see Theorem 2.31.).

Properties (C1)-(C2) are equivalent to the solvability of the discrete problem in an open neighbourhood. Referring to Theorem 2.31, the linear boundedness (C3) is already a sufficient criterion for solvability of differential inclusions.

We will now prove the solvability of LIMDI (3.1) with a set-valued fixed point theorem. This strategy is also used in the solvability proof for MDIs in [69]. After that we define A and F in (3.1) for Problem (2.27) which represent the equations of motion of non-smooth mechanical systems.

Theorem 3.5 (Fixed point theorem for set-valued functions, Himmelberg [46]) Let $D \neq \emptyset$ be a convex subset of a separated locally convex space X and $\Gamma : D \to P(D)\backslash\emptyset$ be an upper semi-continuous function with closed and convex values. If $\Gamma(D)$ is contained in some compact subset $C \subset D$ then Γ has a fixed point, i.e. $\exists\, x^* \in D$ with $x^* \in \Gamma(x^*)$.

A second important abstract argument of [63] is often used in the proof of solvability. It is a compactness result for functions of bounded variation.

Theorem 3.6 Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of functions in $BV(I,\mathbb{R}^n)$ with

$$\begin{aligned}\|x_k(t)\| &\leq L, \qquad t \in I,\\ \mathrm{var}(x_k, I) &\leq M\end{aligned}$$

for all $k \in \mathbb{N}$ and some constants $L, M > 0$. Then, there exists a subsequence (x_{k_l}) of (x_k) which converges pointwise to a function $x \in BV(I, \mathbb{R}^n)$ with

$$\begin{aligned} \|x(t)\| &\leq L, \qquad t \in I \\ \operatorname{var}(x, I) &\leq M. \end{aligned}$$

If all x_{k_l} are right-continuous, the differential measures $\mathrm{d}x_{k_l}$ converge weakly* to $\mathrm{d}x$, i.e. for all continuous functions $\phi : I \to \mathbb{R}$ and all $[c, d] \subset I$ the relationship

$$\lim_{l\to\infty} \left\| \int_{[c,d]} \phi \, \mathrm{d}x_{k_l} - \int_{[c,d]} \phi \, \mathrm{d}x \right\| = 0$$

is true.

Remark 3.7 (Convergence in $SBV^+(I, \mathbb{R}^n)$) A function sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in SBV^+(I, \mathbb{R}^n), \forall\, k \in \mathbb{N}$, converges pointwise to a limit $x : I \to \mathbb{R}^n$ if

$$\lim_{k\to\infty} x_k(t) - x(t) = 0, \qquad \forall\, t \in I.$$

It converges to $x \in SBV^+(I, \mathbb{R}^n)$ if

$$0 = \lim_{k\to\infty} \|x_k - x\|_{BV} = \lim_{k\to\infty} \|x_k(0) - x(0)\| + \operatorname{var}(x_k - x, I).$$

This means $\lim_{k\to\infty} (x_k(0) - x(0)) = 0$ is true and following Theorem 2.26 the equations

$$\begin{aligned} \lim_{k\to\infty} \operatorname{var}(x_k - x, I) = \lim_{k\to\infty} \int_I |d(x_k - x)| = \lim_{k\to\infty} \int_I \|\dot{x}_k(t) - \dot{x}(t)\| \, \mathrm{d}t + \\ + \sum_{s \in T^*(x_k, I) \cup T^*(x, I)} \left\| x_k^+(s) - x_k^-(s) - x^+(s) + x^-(s) \right\|. \end{aligned}$$

hold. On one hand, $\|\dot{x}_k(t) - \dot{x}(t)\|$ should converge for almost all $t \in I$ uniform to zero. On the other hand, it holds

$$\lim_{k\to\infty} \left\| x_k^+(s) - x_k^-(s) - x^+(s) + x^-(s) \right\| = 0, \qquad \forall s \in T^*(x_k, I) \cup T^*(x, I).$$

This could be only true if there is an $N \in \mathbb{N}$ such that

$$\forall\, l \geq N : T^*(x, I) \subset T^*(x_l, I)$$

Otherwise there exists an $s \in T^*(x, I)$ and a subsequence (x_{k_l}) of (x_k) with $s \in T^*(x, I)$ ans $s \notin T^*(x_{k_l}, I), \forall\, l \in \mathbb{N}$. The conclusion

$$\lim_{l\to\infty} \left\| x_{k_l}^+(s) - x_{k_l}^-(s) - x^+(s) + x^-(s) \right\| = \lim_{l\to\infty} \left\| x^+(s) - x^-(s) \right\| \neq 0$$

contradicts $\lim_{k\to\infty} \operatorname{var}(x_k - x, I) = 0$. Equality $T^*(x, I) = T^*(x_l, I), \forall\, l \geq N$, has not to be fulfilled. If there is an $s \notin T^*(x, I)$ with $s \in T^*(x_l, I)$, the Zeno phenomenon

$$\lim_{l\to\infty} \left\| x_l^+(s) - x_l^-(s) \right\| = 0$$

could be observed in s for x_l and continuous behaviour of x in s. Convergence implies as always pointwise convergence and the limit of a convergent sequence is also in $SBV^+(I, \mathbb{R}^n)$ since this space is closed (see Theorem 2.25).

Now the first new result of this thesis follows.

Theorem 3.8 (Solvability of LIMDI (3.1)) Let $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a continuous function with $A(x) = A(x_{\text{abs}})$ for all $x \in BV(I, \mathbb{R}^n)$ and $F_i : \mathbb{R}^n \to P(\mathbb{R}^n)\backslash\emptyset, i = 1, 2$, are upper semi-continuous with closed values. Here x_{abs} is the absolutely continuous part of the Lebesgue decomposition of a function $x \in BV(I, \mathbb{R}^n)$. Then all consistent LIMDI (3.1) with an initial value $x(0) = x_0$ satisfying $F_1(x_0) \neq \emptyset$ have a local solution $x \in SBV^+(I, \mathbb{R}^n)$ on I.

Proof: We define

$$\begin{aligned} C(t) &:= (1 + \|x_0\|)e^{Kt} - 1, \qquad t \in I, \\ C_0 &:= (1 + \|x_0\|)(e^{KT} - 1), \end{aligned}$$

where K is the positiv constant from Definition 3.4. The origin of the constant C_0 and $C(t)$ arises in the course of the proof. Since Theorem 3.5 shall be used to prove this statement, it is defined

$$\begin{aligned} X &:= BV(I, \mathbb{R}^n), \\ D &:= \left\{x \in SBV^+(I, \mathbb{R}^n) \mid \|x(t)\|_2 \leq C(t), t \in I,\ \text{var}(x, I) \leq C_0, x(0) = x_0\right\}, \\ \Gamma(x) &:= \{y \in D \mid A(x)\text{d}y \in F(x)\}. \end{aligned}$$

According to [38], the function space X is a Banach space with norm $\|\cdot\|_{BV}$ and therefore even a separated locally convex space.

With $x_1, x_2 \in D, \alpha \in [0, 1]$, the linear combination $x_3 := \alpha x_1 + (1 - \alpha)x_2$ is defined. Referring to Theorem 2.25, it can be concluded $x_3 \in SBV^+(I, \mathbb{R}^n)$. Following Proposition 2.17 (iii),

$$\begin{aligned} \text{var}(x_3, I) &\leq \alpha\text{var}(x_1, I) + (1 - \alpha)\text{var}(x_2, I) \leq C_0, \\ \|x_3(t)\|_2 &\leq \alpha \|x_1(t)\|_2 + (1 - \alpha) \|x_2(t)\| \leq C(t), \\ x_3(0) &= \alpha x_1(0) + (1 - \alpha)x_2(0) = \alpha x_0 + (1 - \alpha)x_0 = x_0 \end{aligned}$$

are satisfied and so $x_3 \in D$. The set D is convex and trivially non-empty ($D \ni x(t) \equiv x_0$). It remains to prove all properties of the multi-function Γ.

(i) $\Gamma(x)$ is convex for all $x \in D$: Let $y_1, y_2 \in \Gamma(x)$. Since D is convex, the linear combination $y_3 := \alpha y_1 + (1 - \alpha)y_2 \in SBV^+(I, \mathbb{R}^n), \alpha \in [0, 1]$, defines also an element of D. Furthermore, for all $t \in I\backslash T^*(x, I)$ it is

$$A(x(t))\dot{y}_3(t) \in F_1(x(t))$$

since $Z_1(x(t))$ is convex and for $s \in T^*(x, I)$

$$A(x(s))(y_3^+(s) - y_3^-(s)) \in F_2(x^-(s))$$

since $Z_2(x^-(s))$ is convex. Here Z_1, Z_2 are the admissable sets of the discrete inclusions which are defined in Definition 3.4. It follows $A(x)\text{d}y_3 \in F(x)$. Therefore, Γ has convex values.

(ii) $\Gamma(x)$ is closed for all $x \in D$: Let $x \in D$ be arbitrary and $(y_k) \subset \Gamma(x)$ with

$$\lim_{k\to\infty} \|y_k - y\|_{BV} = 0.$$

We have also pointwise convergence in the sense of $\lim_{k\to\infty} y_k(t) = y(t), t \in I$, with

$$y(0) = \lim_{k\to\infty} y_k(0) = x_0.$$

Due to Theorem 3.6 a sequence of functions y_k which is uniformly bounded in norm $\|y_k(t)\| \le C(t) \le C(T)$ and variation $\mathrm{var}(y_k, I) \le C_0$ converges to a limit $y \in BV(I, \mathbb{R}^n)$ with the same properties $\|y(t)\| \le C(T)$, $\mathrm{var}(y_k, I) \le C_0$. Since $\|y_k(t)\| \le C(t), t \in I$, the same inequality holds for the limit

$$\|y(t)\| = \left\| \lim_{k\to\infty} y_k(t) \right\| \le C(t).$$

Using Theorem 2.25, the space $SBV^+(I, \mathbb{R}^n)$ is a closed subset of $BV(I, \mathbb{R}^n)$ and so it can be concluded from $y_k \in SBV^+(I, \mathbb{R}^n)$ the inclusion $y \in SBV^+(I, \mathbb{R}^n)$ and finally $y \in D$.

Furthermore, Theorem 3.6 states that $\mathrm{d}y_k \to \mathrm{d}y$ in the sense of

$$\lim_{k\to\infty} \int_{[0,t]} \phi \; \mathrm{d}y_k = \int_{[0,t]} \phi \; \mathrm{d}y$$

for all continuous ϕ. The function $A(x)$ is continuous since A is continuous and depend only on the continuous part of x by assumption. Due to consistency, there exist sequences $(f_1^k)_{k\in\mathbb{N}}$ and $(f_2^k)_{k\in\mathbb{N}}$ with

$$\begin{aligned} f_1^k(x(t)) &\in F_1(x(t)), \qquad & t \in I \backslash T^*(x, I), \\ f_2^k(x(t)) &\in F_2(x^-(t)), \qquad & t \in T^*(x, I) \end{aligned}$$

for all $k \in \mathbb{N}$ fulfilling

$$\int_{[0,t]} A(x)\mathrm{d}y = \lim_{k\to\infty} \int_{[0,t]} A(x)\mathrm{d}y_k = \lim_{k\to\infty} \int_{[0,t]} f_1^k(x(s))\mathrm{d}s + \sum_{s\in T^*(x,[0,t])} f_2^k(x(s))$$

Since $A(x_{\mathrm{abs}})$ is continuous on $[0, t]$, its norm is bounded by a constant $\beta_A > 0$ and it holds

$$\left\| f_1^k(x(s)) \right\| = \|A(x_{abs}(s))\dot{y}_k(s)\| \le \beta_A \|\dot{y}_k(s)\| \le \beta_A \cdot K(1+\|x(s)\|) \le \beta_A \cdot K(1+C(T))$$

Using that the set $F_1(x(s))$ is closed, there exists a subsequence $(f_1^{k_l}(x(s)) \subset$

$f_1^k(x(s))$ which converges pointwise to $f_1(x(s)) \in F_1(x(s))$.

$$\int_{[0,t]} A(x)\mathrm{d}y = \lim_{l\to\infty}\left(\int_{[0,t]} f_1^{k_l}(x(s))\mathrm{d}s + \sum_{s\in T^*(x,[0,t])} f_2^{k_l}(x(s))\right).$$

If a function sequence $f_1^{k_l}$ converges pointwise to a function f_1 and there exists an upper bound $w \in \mathcal{L}^1([0,t],\mathbb{R}^n)$ for $\left\|f_1^{k_l}(x(s))\right\| \leq \|w(x(s))\|$, we can change the order of limit and Lebesgue integral [30]. Here, the conditions are fulfilled with $w(s) = \beta_A \cdot K(1 + C(T))$. The integral can be transformed to

$$\int_{[0,t]} A(x(s))\mathrm{d}y = \int_{[0,t]} f_1(x(s))\mathrm{d}s + \lim_{l\to\infty} \sum_{s\in T^*(x,[0,t])} f_2^{k_l}(x(s))$$

with $f_1(x(s)) \in F_1(x(s))$. Since $f_2^l(x(s))$ is analoguesly bounded and a sequence of the closed set $F_2(x^-(s))$, we get with the same argument a limit $f_2(x(s)) \in F_2(x(s))$ in every $s \in I$ of a subsequence with

$$\int_{[0,t]} A(x(s))\mathrm{d}y = \int_{[0,t]} f_1(x(s))\mathrm{d}s + \sum_{s\in T^*(x,[0,t])} f_2(x(s)).$$

This is the integral sense of $A(x)\mathrm{d}y \in F(x)$. So $y \in \Gamma(x)$ holds and it can be concluded that $\Gamma(x)$ is closed.

(iii) $\exists\, C \subset D$ with $\Gamma(x) \subset C, \forall\, x \in D$, and C compact: Let

$$C = \left\{y \in SBV^+(I,\mathbb{R}^n) : \|y\|_{BV} \leq \|x_0\|_2 + C_0, y(0) = x_0\right\}.$$

In metric spaces a set is compact if it is sequentially compact. Let $(y_k) \subset C$ be a sequence satisfying

$$\mathrm{var}(y_k, I) \leq C_0, \qquad \|y_k(t)\|_2 \leq \|x_0\|_2 + \mathrm{var}(y_k, I) \leq \|x_0\|_2 + C_0.$$

As in proof step (ii), from Theorem 3.6 it follows that there is a subsequence y_{k_l} with $\|y_{k_l} - y\|_{BV} \to 0, l \to \infty$, and $y \in C$. Therefore, the set C is compact.

(iv) Γ is upper semi-continuous: Following Theorem 2.12 it must be proven that $graph(\Gamma)$ is compact. A sequence $((x_k, y_k))_{k\in\mathbb{N}}$ in $graph(\Gamma)$ is considered. That means

$$(x_k, y_k) \in D \times D \qquad \text{and} \qquad A(x_k)dy_k \in F(x_k).$$

Referring to Theorem 3.6, there is a subsequence $((x_{k_l}, y_{k_l}))_{l\in\mathbb{N}}$ that converges to $(x, y) \in D\times D$. Following Remark 3.7, there is again a subsequence $((x_{k_m}, y_{k_m}))_{m\in\mathbb{N}}$ of $((x_{k_l}, y_{k_l}))$ with $T^*(x, I) \subset T^*(x_m, I)$. So for all $s \in T^*(x, I)$, we have

$$\lim_{m\to\infty} A(x_{k_m}(s))(y_{k_m}^+(s) - y_{k_m}^-(s)) = A(x(s))(y^+(s) - y^-(s)).$$

It is assumed that $A(x_{k_m})(y_{k_m}^+(s) - y_{k_m}^-(s)) \in F_2(x_{k_m}^-(s))$ and that F_2 is upper semi-continuous. Theorem 2.12 states that $graph(F_2)$ must then be closed. So one can conclude that the limit $(x(s), A(x(s))(y^+(s)-y^-(s)))$ of $(x_{k_m}(s), A(x_{k_m}(s))(y^+_{k_m}(s)-y^-_{k_m}(s)))$ is also an element of $graph(F_2)$, i.e.

$$A(x(s))(y^+(s)-y^-(s)) \in F_2(x^-(s)), \qquad \forall\ s \in T^*(x, I).$$

For almost all $t \in I$, the inclusion $A(x_{k_m}(t))\dot{y}_{k_m}(t) \in F_1(x_{k_m}(t))$ and the relation $\lim_{m\to\infty} A(x_{k_m}(t))\dot{y}(t) = A(x(t))\dot{y}(t)$ is true. Since F_1 is also upper semi-continuous, it can be concluded analogously $A(x(t))\dot{y}(t) \in F_1(x(t))$ for almost all $t \in I$. In summary, there is always a subsequence $((x_{k_m}, y_{k_m}))_{m\in\mathbb{N}}$ of an arbitrary sequence $((x_k, y_k))$ of $graph(\Gamma)$ that converges to a limit $(x, y) \in graph(\Gamma)$ since

$$(x, y) \in D \times D \qquad \text{and} \qquad A(x)\mathrm{d}y = A(x)\dot{y}\mathrm{d}t + A(x)(y^+ - y^-)\mathrm{d}\eta(x) \in F(x).$$

So $graph(\Gamma)$ is compact and Γ is upper semi-continuous.

(v) $\Gamma(x)$ is non-empty, $\forall\ x \in D$:

Let $h = T/n$ with $n \in \mathbb{N}$. We define a vector sequence in $\mathbb{R}^n$

$$\begin{aligned} y_{n,0} &= x_0, \\ y_{n,i+1} &= y_{n,i} + hm_{n,i}, \qquad A(x(ih))m_{n,i} \in F(x(ih)) \end{aligned}$$

and a sequence of functions

$$y_n(t) := y_{n,i}, \qquad t \in [ih, (i+1)h]\,,$$

with $i \in \{0, \dots, n-1\}$. The vector sequence fulfills pointwise

$$\|y_{n,i}\|_2 \le \|x_0\|_2 + ihK(1 + C(ih)). \tag{3.2}$$

Equation (3.2) will be proven by induction. The inequality $\|y_{n,0}\|_2 \le \|x_0\|_2$ is trivially true for $i = 0$. If (3.2) is satisfied for an arbitrary $i = k \in \mathbb{N}$ it can be concluded

$$\begin{aligned} \|y_{n,k+1}\| &\le \|y_{n,k}\|_2 + h\,\|m_{n,k}\|_2 \\ &\le \|x_0\|_2 + khK(1 + C(kh)) + hK(1 + \|x(kh)\|) \\ &\le \|x_0\|_2 + (k+1)hK(1 + C(kh)) \end{aligned}$$

since $x \in D$. Furthermore, we have for $i = 1, \dots, n-1$

$$\|y_{n,i} - y_{n,i-1}\|_2 = h\,\|m_{n,i}\| \le hK(1 + \|x(ih)\|_2) \le hK(1 + C(ih)). \tag{3.3}$$

Using equations (3.2)-(3.3), the function sequence $(y_n) \subset SBV^+(I, \mathbb{R}^n)$ satisfies

$$\|y_n(t)\|_2 \le \|x_0\|_2 + T_0K(1 + C(T_0)),$$

and in addition

$$\begin{aligned}\operatorname{var}(y_n, I) &= \sum_{j=1}^{n-1} \|y_{n,i} - y_{n,i-1}\|_2 \\ &\leq (n-1)hK(1 + C(T)) \\ &\leq T_0 K(1 + C(T)).\end{aligned}$$

Following [63, p. 11] and [3, Theorem 4.8], there must be a subsequence y_{n_k} converging in the sense of

$$\exists\, \check{y} \in SBV^+(I_0, \mathbb{R}^n) : \|y_{n_k} - \check{y}\| \to 0$$

with

$$\|\check{y}(t)\| \leq \|x_0\|_2 + TK(1 + C(T)), \forall\, t \in I_0.$$

We will prove that $y(t) := \check{y}^+(t) \in SBV^+(I, \mathbb{R}^n)$ is an element of $\Gamma(x)$.

Because of the consistency of the LIMDI, for $0 < h \leq h_0$ it holds

$$y_{n,i} = y_{n,i-1} + hm_{n,i} \in Z_1(x(ih))$$

with $y_{n,0} = x_0 \in Z_1(x_0)$. Since convergence implies pointwise convergence and $Z_1(x(ih))$ is closed, we have

$$y(t) = \lim_{n\to\infty} y_{n,\lceil \frac{nt}{T} \rceil} \in Z_1(x(t)).$$

In the following, we use that $T^*(x, I)$ is a set of Lebesgue measure zero. For all $t \in I$, it is satisfied

$$\begin{aligned}\int_0^t A(x(s))\mathrm{d}y &= \lim_{n\to\infty} \int_0^t A(x)\mathrm{d}y_n = \lim_{n\to\infty} \sum_{i=1}^{n-1} A(x(ih))(y_{n,i} - y_{n,i-1}) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n-1} A(x(ih))hm_{n,i-1} \\ &= \lim_{n\to\infty} \sum_{i=1,\ldots,n-1, ih\in I\setminus T^*(x)} f_1(x(ih))h + \sum_{i=1,\ldots,n-1, ih\in T^*(x)} f_2(x(ih))h \\ &= \lim_{n\to\infty} \sum_{i=1,\ldots,n-1, ih\in I\setminus T^*(x)} \int_{(i-1)h}^{ih} f_1(x(ih))\mathrm{d}s + \\ &+ \sum_{i=1,\ldots,n-1, ih\in T^*(x)} \int_{(i-1)h}^{ih} f_2(x(ih))\mathrm{d}s \\ &= \int_0^t f_1(x(s))\mathrm{d}s + \sum_{s\in T^*(x)} f_2(x(s)) \in F(x)\end{aligned}$$

Since $A(x)\mathrm{d}y \in F(x)$ follows, it holds

$$\begin{aligned}\mathrm{var}(y,I) &= \int_0^{T_0} |\mathrm{d}y| \leq \int_0^T K(1+\|x(s)\|)\mathrm{d}s \\ &\leq \int_0^T K(1+C(s))\mathrm{d}s = \int_0^T K(1+\|x_0\|_2)e^{L_M s}\mathrm{d}s \\ &= (1+\|x_0\|_2)(\mathrm{e}^{KT}-1) = C_0.\end{aligned}$$

In the same way we get

$$\|y(t)\|_2 \leq \|x_0\|_2 + \int_0^t |\mathrm{d}y| \leq \|x_0\|_2 + (1+\|x_0\|_2)(\mathrm{e}^{Kt}-1) = C(t)$$

i.e., $y \in \Gamma(x)$. □

After the solvability, we also give a new boundedness theorem for LIMDI.

Theorem 3.9 (Boundedness of Solutions) On the interval $[0,T] = I$, the solution $x \in SBV^+(I,\mathbb{R}^n)$ of the LIMDI (3.1) fulfills the inequalities

$$\begin{aligned}\|x(t)\|_2 &\leq (\|x_0\|_2+1)\mathrm{e}^{Kt}-1, \quad t \in I,\\ \mathrm{var}(x,I) &\leq (\|x_0\|_2+1)(\mathrm{e}^{KT}-1).\end{aligned}$$

Proof: Using the consistency, we get

$$\begin{aligned}\|x(t)\|_2 &\leq \|x_0\|_2 + \textstyle\int_0^t |\mathrm{d}x|\\ &\leq \|x_0\|_2 + \textstyle\int_0^t K(1+\|x(s)\|_2)\mathrm{d}s,\end{aligned}$$

where $|x|$ is the modulus function to x (see Corollary A.21). Let $w,z \in \mathbb{R}^n$ be. Following the Cauchy-Schwarz inequality $\langle w,z\rangle_2 \leq \|w\|_2\|z\|_2$ for the standard scalar product and Euclidean norm, it is satisfied

$$\begin{aligned}(\|w\|_2-\|z\|_2)^2 &= \|w\|_2^2 - 2\|w\|_2\|z\|_2 + \|z\|_2^2\\ &\leq \|w\|_2^2 - 2\langle w,z\rangle_2 + \|z\|_2^2\\ &= \|w-z\|_2^2\end{aligned}$$

and so $|\|w\|_2 - \|z\|_2| \leq \|w-z\|_2$. If x is a function of bounded variation on J than $\|x\|_2 : t \mapsto \|x(t)\|_2$ is also a function of bounded variation since

$$\begin{aligned}\mathrm{var}(\|x\|_2, J) &= \sup_{(t_i)\subset J} \textstyle\sum \|\|x(t_i)\|_2 - \|x(t_{i-1})\|_2\|_2\\ &\leq \sup_{(t_i)\subset J} \textstyle\sum \|x(t_i)-x(t_{i-1})\|_2\\ &= \mathrm{var}(x,J) < \infty.\end{aligned}$$

In [75, Theorem 3.1.] a Theorem of Gronwall type for functions of bounded variation was proven: If $y(t)$ is a non-negative function of bounded variation on J, $C \geq 0$ and $K(t)$ a non-negative Lebesgue-integrable function on J with

$$y(t) \leq C + \int_0^t K(s)y(s)\mathrm{d}s, \qquad t \in J,$$

then

$$y(t) \leq C(1 + \int_0^t K(s)\exp\left(\int_s^t K(\eta)\mathrm{d}\eta\right)\mathrm{d}s), \qquad t \in J.$$

With $y(t) = 1 + \|x(t)\|_2 \in BV(I, \mathbb{R}^n), C = \|x_0\| + 1, K(t) \equiv K$, the inequality for $\|x(t)\|$ holds since

$$\begin{aligned} 1 + \|x(t)\|_2 &\leq (\|x_0\| + 1)(1 + \textstyle\int_0^t K\exp\left(\int_s^t K\mathrm{d}s\right)\mathrm{d}s) \\ &= (\|x_0\| + 1)\mathrm{e}^{Kt} \end{aligned}$$

For the variation of the solution x on I, it is satisfied

$$\begin{aligned} \mathrm{var}(x, I) &= \textstyle\int_0^T |\mathrm{d}x| \\ &\leq \textstyle\int_0^T K(1 + \|x(s)\|_2)\mathrm{d}s \\ &\leq \textstyle\int_0^T K\mathrm{e}^{L_M s}(\|x_0\|_2 + 1)\mathrm{d}s \\ &= (\|x_0\|_2 + 1)(\mathrm{e}^{KT} - 1). \end{aligned}$$

□

The question of uniqueness of solutions of LIMDI is not discussed here. The existing literature [10, 78, 79] about such problems points towards a verdict of non-uniqueness in general. In [78, Example 3b] a one-dimensional example is constructed with a non-analytic force $f(t, q, v)$. The resulting MDI has on one hand the constant solution $x(t) \equiv x_0$ and on the other hand a second solution with an infinite number of discontinuity points in the neighbourhood of x_0. So non-uniqueness is already given in case of explicit measure differential inclusion. In [79] for the analytic case the following uniqueness result in the one-dimensional case was proven with rather technical methods.

Theorem 3.10 (Schatzman [79]) The problem

$$\dot{q} = v, \mathrm{d}v \in f(t, q, v) + N_{\mathbb{R}_+}(q)\mathrm{d}t + N_{\mathbb{R}_+}(v^+ + \epsilon v^-)\mathrm{d}\eta,$$

with $q(0) = q_0, v(0) = v_0$ has at most one solution on $I \subset \mathbb{R}$ if $f : I \times \mathbb{R}^n \times \mathbb{R}^n$ is an analytic function and the initial values fulfill $q_0 \geq 0$ and $q_0 = 0 \Rightarrow v_0 \geq 0$.

The bouncing ball in Example 1.1 defines a gravitational force $f(t, q, v) = -mg$. Since a constant function is analytic, there is for all consistent initial values a unique solution.

3.2. Setting in Non-Smooth Mechanics

The aim of this section is to prove the existence of solutions q and v for the equations of motion (2.27) in case of singular mass matrices and redundant constraints. This is equivalent to show that the modelling strategies of Chapter 2 are appropriate. In [17, 49], it is similarly proven that there exist solutions for mechanical systems with equality constraints or inequality constraints without impacts and singular mass matrices and redundant constraints. For a more general and detailed overview of overdetermined and underdetermined DAEs and differential variational inequalities we refer to [7, 14, 36, 37, 55].

Theorem 3.11 (Garcia de Jalón, Gutierrez-Lopez [49]) The equations of motion of mechanical systems with equality constraints are a DAE given in (2.20). Let $M(q) \in \mathbb{R}^{n\times n}$ be a symmetric positive semi-definite matrix for all $q \in \mathbb{R}^n$ and let the block matrix

$$\left[M(q)\ G(q)^\top\right]$$

have full rank n. Then, problem (2.20) has a unique solution.

Theorem 3.12 (Brogliato, Goeleven [17]) Let $M(q) \in \mathbb{R}^{n\times n}$ be symmetric positive semi-definite and the assumption

$$\operatorname{Im}(G_{J^1(q)}(q)) \cap \left\{z \in \mathbb{R}^m : z + \gamma_{J^1(q)}(q, \dot{q}) \in \mathbb{R}^m_+\right\} \neq \emptyset \tag{3.4}$$

is true. If one of the following statements is satisfied

(a) $\{x \in \mathbb{R}^n : G_{J^1(q)}(q)x \geq 0\} \cap \ \ker(M(q)) = \{0\}$,

(b) $\{x \in \mathbb{R}^n : G_{J^1(q)}(q)x \geq 0\} \cap \ \ker(M(q)) \neq \{0\}$ and $f(q,v)^\top z < 0,\ \forall\ z \in \{x \in \mathbb{R}^n : G_{J^1(q)}(q)x \geq 0\} \cap \ \ker(M(q))\backslash\{0\}$,

then the mixed complementarity problem

$$\begin{aligned} &M(q)x = f(q,v) + G_{J^1(q)}(q)^\top \lambda \\ &0 \leq \lambda \perp G_{J^1(q)}(q)x + \gamma_{J^1(q)}(q,v) \geq 0 \end{aligned}$$

has at least one solution x and λ. The vector x is unique if

(i) $z \in \ \ker(M(q)) \Rightarrow z \in \ \ker(G_{J^1(q)}(q))$,

(ii) $z \in \ \ker(M(q))\backslash\{0\} \Rightarrow f(q,v)^\top z \neq 0$.

The underlying works [17, 49] of Theorems 3.11 and 3.12 conclude from these discrete arguments the solvability of the dynamical system with the implicit function theorem and certain smoothness assumptions.

Remark 3.13 (Physical interpretation) Let $M(q) \in \mathbb{R}^{n\times n}$ be the symmetric positive semi-definite mass matrix describing the kinetic energy $0.5v^\top M(q)v$ and $G_{J^1(q)}(q) \in \mathbb{R}^{m_1\times n}$ the Jacobian matrix of the constraint function $g : \mathbb{R}^n \to \mathbb{R}^m$. The kernel of $G_{J^1(q)}$ spans the tangent space of the constraint manifold $\{z \mid g(z) = 0\}$ at point q. The condition

$$\operatorname{rank}\left(\left[M(q)\ G_{J^1(q)}(q)^\top\right]\right) = n$$

is important to get the existence of solutions in the DAE case. It can be interpreted as the fact that any possible movement must be associated with non-zero kinetic energy [49]. If $G_{J^1(q)}(q)v = 0$ it can immediately be concluded $v \notin \ker(M(q))$ and so $T(q,v) = v^\top M(q)v \neq 0$.

The equivalent conditions for the existence of solutions for contact systems are (a)-(b) in Theorem 3.12. The velocity is admissible if $G_{J^1(q)}(q)v \geq 0$. Either condition (a) is satisfied, than $T(q,v) \neq 0$ follows, or (b) is satisfied. Assume $v \neq 0$ with $M(q)v = 0, G_{J^1(q)}(q)v \geq 0$ is admissible and the MLCP in Theorem 3.12 has a solution (x, λ) that satisfy the condition $\lambda^\top G_{J^1(q)}(q)v = 0$. The contradiction

$$0 = v^\top M(q)x = v^\top f(q,v) + v^\top G^\top_{J^1(q)}(q)\lambda = v^\top f(q,v) > 0$$

follows. Just as in the case of equality constraints, an admissible velocity must cause kinetic energy.

Like Theorems 3.11 and 3.12 show, equations of motion with singular mass matrices and redundant constraints can be well-defined in case of equality constraints and inequality constraints without impacts. A generalisation to impact systems seems to be logically. We call mechanical systems with the impact law also systems with impulsive forces.

Remark 3.14 (Problem parameters) The problem (2.27) is a LIMDI for $x = (q^\top, v^\top)^\top$ with parameters

$$A(q,v) = \begin{pmatrix} I & 0 \\ 0 & M(q) \end{pmatrix},$$

$$F(q,v) = F_1(q,v)\mathrm{d}t + F_2(q,v^-)\mathrm{d}\eta(v)$$

$$= \begin{pmatrix} v \\ f(q,v) + G^\top(q) N_{\mathbb{R}^m_+}(g(q)) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ G^\top_{J^1}(q) N_{\mathbb{R}^{m_1}_+}(G_{J^1}(q)(v^+ + \epsilon v^-)) \end{pmatrix} \mathrm{d}\eta(v).$$

Remark 3.15 (Discrete system) For $q, v \in \mathbb{R}^n$ the problem $A(q,v)x \in F(q,v)$ is equivalent to the problem to find

(i) vectors $\dot v \in \mathbb{R}^n, \lambda \in \mathbb{R}^{m_2}, x = (\dot v, \lambda)$ with

$$\begin{aligned} M(q)\dot v &= f(q,v) + G_{J^2(q,v)}(q)^\top \lambda \\ 0 \le \lambda &\perp G_{J^2(q,v)}\dot v + \gamma_{J^2(q,v)}(q,v) \ge 0 \end{aligned}$$

if for all $i = 1, \ldots, m, g_i(q) > 0$ or $g_i(q) = 0, G_i(q)v > 0$ or $g_i(q) = 0, G_i(q)v = 0, g_i(q - hv) = 0$, for all $h \in (0, h_0)$ and h_0 small enough.

These conditions can be associated with inactive constraints, constraints which get inactive or active constraints which stay active.

(ii) vectors $v^+ \in \mathbb{R}^n, \Lambda \in \mathbb{R}^{m_1}, x = (v^+, \Lambda)$ with

$$\begin{aligned} M(q)(v^+ - v^-) &= G_{J^1(q)}(q)^\top \Lambda \\ 0 \le \Lambda &\perp G_{J^1(q)}(v^+ + \epsilon v^-) \ge 0 \end{aligned}$$

if there is an index $i = 1, \ldots, m$, with $g_i(q) = 0, G_i(q)v^- < 0$ or $g_i(q) = 0, G_i(q)v^- = 0, g_i(q - hv^-) > 0$, for all $h \in (0, h_0)$ and h_0 small enough.

These conditions can be associated with constraints which get active and therefore an impact happens.

We consider the discrete problem to find $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^{m_i}$ with

$$\begin{aligned} M(q)x &= \tilde f(q,v) + G_{J^i(q,v)}(q)^\top \lambda \\ 0 \le \lambda &\perp G_{J^i(q,v)}x + \tilde\gamma(q,v) \ge 0 \end{aligned} \tag{3.5}$$

and

$$\tilde{f}(q,v) = \begin{cases} f(q,v), & \text{in case (i)}, \\ M(q)v^-, & \text{in case (ii)} \end{cases}$$

$$J^i(q,v) = \begin{cases} J^2(q,v), & \text{in case (i)}, \\ J^1(q), & \text{in case (ii)} \end{cases}$$

$$\tilde{\gamma}(q,v) = \begin{cases} \gamma_{J^2(q,v)}(q,v), & \text{in case (i)}, \\ \epsilon G_{J^1(q)}v^-, & \text{in case (ii)} \end{cases}$$

Referring to Theorem B.14, the problem in Equation (3.5) has a solution if $M(q)$ is positive semi-definite and the problem is feasible. Feasibility is equivalent to the aspect that there exist $x \in \mathbb{R}^n, \lambda \in \mathbb{R}^{m_i}$ with

$$\lambda \geq 0, \quad M(q)x + G_{J^i(q,v)}(q)\lambda = \tilde{f}(q,v), \quad G_{J^i(q,v)}(q)x + \tilde{\gamma}(q,v) \geq 0. \tag{3.6}$$

Theorem 3.16 (Feasibility) Let $M(q)$ be positive semi-definite for all $q \in \mathbb{R}^n$. The inequality system (3.7) has for all $q, v \in \mathbb{R}^n$ a solution if

$$\left.\begin{array}{rcl} y & \in & \ker(M(q)) \\ z & \in & \ker(G_{J^i(q,v)}(q)^\top) \\ G_{J^i(q,v)}(q)y & = & u \\ u, z & \geq & 0 \end{array}\right\} \Rightarrow -\tilde{f}(q,v)^\top y + \tilde{\gamma}(q,v)^\top z \geq 0. \tag{3.7}$$

Proof: Following the Lemma of Farkas [33], a system $Ax + b \geq 0$ has a solution if

$$y \geq 0, A^\top y = 0 \Rightarrow y^\top b \geq 0$$

is true. System (3.6) is equivalent to $Ax + b \geq 0$ with

$$A = \begin{pmatrix} 0 & I \\ M & -G \\ -M & G \\ G & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ -\tilde{f} \\ \tilde{f} \\ \tilde{\gamma} \end{pmatrix}.$$

The arguments and indices are omitted for better readability. The Farkas condition for feasibility of system (3.6) is

$$\left.\begin{array}{rcl} y_i & \geq & 0, i = 1,2,3,4, \\ y_1 - G(y_2 - y_3) & = & 0 \\ M(y_2 - y_3) + G^\top y_4 & = & 0 \end{array}\right\} \Rightarrow \tilde{f}^\top(y_3 - y_2) + \tilde{\gamma}^\top y_4 \geq 0.$$

First we define $y := y_2 - y_3 \in \mathbb{R}^n$ and transform

$$\left.\begin{array}{rcl} y_i & \geq & 0, i = 1,4, \\ Gy & = & y_1 \\ My + G^\top y_4 & = & 0 \end{array}\right\} \Rightarrow -\tilde{f}^\top y + \tilde{\gamma}^\top y_4 \geq 0.$$

If the second equation is multiplied with $y^\top$ from the left and the first equation is put in, we get $y^\top My + y^\top G^\top y_4 = y^\top My + y_1^\top y_4 = 0$. Since $y_1, y_4 \geq 0$, this equation can only be satisfied if $y_1^\top y_4 = 0$ and $y^\top My = 0$. Since M is positive semi-definite, the relation

$y \in \ker(M)$ results. From $My + G^\top y_4 = 0$, the property $y_4 \in \ker(G^\top)$ and the claim follow. □

Remark 3.17 (Comparison to [17]) In [17] only case (i) without impacts is considered. Assumption (3.4) is equivalent to

$$\exists\, x \in \mathbb{R}^n : G_{J^2(q,v)}(q)x + \gamma_{J^2(q,v)}(q,v) \geq 0.$$

Following the lemma of Farkas that can be transformed to

$$y \geq 0, G_{J^2(q,v)}(q)y = 0 \Rightarrow y^\top \gamma_{J^2(q,v)}(q,v) \geq 0. \tag{3.8}$$

Cases (a) and (b) in Theorem 3.12 are subcases of

$$G_{J^2(q,v)}(q)x \geq 0, M(q)x = 0 \Rightarrow f(q,v)^\top x \leq 0. \tag{3.9}$$

The requirements of Theorem 3.16 are trivially true if (3.8)-(3.9) are satisfied. So Theorem 3.16 is a generalisation of the results in [17]. If we consider the impact case the feasibility condition is equivalent to

$$\left.\begin{array}{rcl} y & \in & \ker(M(q)) \\ z & \in & \ker(G_{J^1(q)}(q)^\top) \\ G_{J^1(q)}(q)y & = & u \\ u, z & \geq & 0 \end{array}\right\} \Rightarrow -(M(q)v^-)^\top y + (\epsilon G_{J^1(q)}(q)v^-)^\top z \geq 0.$$

Since $y \in \ker(M(q)), z \in \ker(G_{J^1(q)}(q)^\top)$, the inequality reduces to $0 \geq 0$ which is true.

Problem 3.18 (Mechanical LIMDI) We consider an MDI of the form (2.27) with the following properties

(P1) f, g, G, M, γ are Lipschitz continuous on I with Lipschitz constants $L_f, L_g, L_G, L_M, L_\gamma$. This implies that there are for example ρ_f, ρ_γ with $\|f(x)\| \leq \rho_f(1 + \|x\|), \|\gamma(x)\| \leq \rho_\gamma(1 + \|x\|)$.

(P2) $M(q)$ is symmetric and positive semi-definite for all $q \in \mathbb{R}^n$.

(P3) M, G are uniformly bounded by $b_M, b_G > 0$. That means $\forall\, q \in \mathbb{R}^n : \|M(q)\|_M \leq b_M, \|G(q)\|_M \leq b_G$ with a matrix norm $\|.\|_M$.

(P4) g is two times continuously differentiable.

(P5) Condition (3.7) is satisfied and $\left[M(q)\ G^\top_{J^3(q,v,a)}(q)\right]$ has for all $q, v, a \in \mathbb{R}^n$ full rank with
$$J^3(q,v,a) := \left\{i \in J^2(q,v) \mid \left[G_{J^2(q,v)}a + \gamma_{J^2(q,v)}(q,v)\right]_i = 0\right\}.$$

Example 3.19 Condition (P5) is important to guarantee the existence of solutions. Consider the simple problem

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \lambda,$$
$$q_1 + 2q_2 = 0.$$

Regardless of the initial value, the differential system has no solution.

Theorem 3.20 (Existence result) The problem (2.27) with (P1)-(P5) and a consistent initial value x_0 has at least one solution.

Proof: Following Theorem 3.8, we must show that the parameters in Remark 3.14 fulfill the properties

(i) A is continuous with $A(x) = A(x_{\text{abs}})$ for all x fulfilling Equation (3.1)
Following (P1) A is continuous. If $x(t) = (q(t)^\top, v(t)^\top), t \in I$, fulfills Equation (3.1) it is a function of bounded variation. More precisely, $q, v : I \to \mathbb{R}^n$ are functions of bounded variation. The first n rows of $A(x)\mathrm{d}x \in F(x)$ are equivalent to $\mathrm{d}q = v\mathrm{d}t$ or

$$\int_{[0,t]} \mathrm{d}q = \int_{[0,t]} v(t)\mathrm{d}t \quad \Leftrightarrow q(t) - q(0) = \int_{[0,t]} v(t)\mathrm{d}t.$$

Lemma 2.19 states that q is absolutely continuous and therefore also continuous. We get the claim $A(x) = A(x_{\text{abs}})$.

(ii) (2.27) is consistent.
To repeat Definition 3.4, an MDI is consistent if the discrete problem in an admissible state $x \in \mathbb{R}^{2n}$ with $F_1(x) \neq \emptyset$ or $F_2(x) \neq \emptyset$

(C1) has a solution

(C2) stays admissible

(C3) has linear bounded growth

and the solution spaces $Z_1(x), Z_2(x)$ are convex. Let $x = (q^\top, v^\top)^\top \in \mathbb{R}^{2n}$ be an admissible vector with

a) $F_1(x) \neq \emptyset$ if for all $i = 1, \dots, m$ either $g_i(q) > 0$ or $g_i(q) = 0, G_i(q)v \geq 0$ or

b) $F_2(x) \neq \emptyset$ if there is an $i \in \{1, \dots m\}$ with $g_i(q) = 0$ and $G_i(q)v < 0$

Referring to Theorems B.14 and 3.16, the MLCP $A(x)m \in F_i(x), i = 1, 2$, has a solution m and (C1) is satisfied.

Following Definition 3.4, the solution $m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ with $m_1, m_2 \in \mathbb{R}^n$ stays admissible if there is an $h_0 > 0$ such that $F_1(x + hm) \neq \emptyset, \forall\, h \in (0, h_0]$. Function F_1 is defined through

$$F_1(q, v) = f(q, v) + N_{R_+^m}(g(q)).$$

The inclusion $0 \in N_{R_+^m}(g(q))$ is true if $g(q) \geq 0$. It follows $f(q, v) \in F_1(q, v)$ and F_1, q, v is not empty. Now the inequality $g(q + hm_1) \geq 0$ is proven for one $h > 0$ to follow (C2).

a) In case (i) of Remark 3.15, m as a solution of $A(x)m \in F_1(x)$ fulfills $m_1 = v$. From the Taylor expansion of g follows

$$g_i(q + hm_1) = g_i(q) + G_i(q)v \cdot h + \mathcal{O}(h^2) \tag{3.10}$$

If either $g_i(q) > 0$ or $g_i(q) = 0, G_i(q)v > 0$ you can choose $h_i > 0$ small enough such that for all $h \in (0, h_i]$ from equation (3.10) also $g(q + hm_1) > 0$ follows. If

$g_i(q) = 0, G_i(q)v = 0$ and $g(q - hv) = 0$, for all $h \in (0, h_0)$ with a small enough h_0, it holds

$$\exists h_i > 0 \; \forall \; h \in (0, h_i) : g(q + hv) = 0.$$

Otherwise the left-side limit $\lim_{h \uparrow 0} \frac{\partial^2 g}{\partial q^2}(q + hv) = 0$ and the right-side limit $\lim_{h \downarrow 0} \frac{\partial^2 g}{\partial q^2}(q + hv) \neq 0$. That contradicts the requirement that g is two times continuously differentiable.

b) In case (ii) of Remark 3.15, m is a solution of $A(x)m \in F_2(x)$ and $m_1 = 0$. The simple relation $g(q + hm_1) = g(q) = 0$ follows. This represents the case of an impact. The position does not change but a postimpact velocity $v^+ = m_2$ with

$$G(q + hm_1)m_2 = G(q)m_2 \geq -\epsilon G(q)v \geq 0$$

is calculated. The constraints stays active or get inactive and we get a solution as in case (i).

Let $h_0 := \max(h_1, \dots, h_m)$ be. For all $h \in (0, h_0)$, the inequality $g(q + hm_1) \geq 0$ stays true and so we get an admissible solution.

It remains to show the linearly bounded growth of the solution set. All solutions $m \in \mathbb{R}^{2n}$ can be split in $(m_1^\top, m_2^\top)^\top, m_1, m_2 \in \mathbb{R}^n$, with $\|m_1\|_2 = \|v\|_2 \leq 1(1+\|v\|_2)$. The second part $m_2 \in \mathbb{R}^n$ fulfills

$$\begin{aligned} M(q)m_2 &= \tilde{f}(q, v) + G^\top_{J^3(q,v,m_2)}(q)\lambda, \\ G_{J^3(g,v,m_2)}(q)m_2 + \tilde{\gamma}_{J^3(q,v,m_2)}(q, v) &= 0. \end{aligned}$$

If $G_{J^3(q,v,m_2)}(q) \in \mathbb{R}^{|J^3| \times n}$ has rank $r \leq |J^3|$, there is an orthogonal matrix $Q \in \mathbb{R}^{|J_3| \times |J_3|}$ such that

$$Q^\top G_{J^3}(q) = \begin{pmatrix} G^1_{J^3} & G^2_{J^3} \\ 0 & G^3_{J^3} \end{pmatrix}$$

where $G^1_{J^3} \in \mathbb{R}^{r \times r}$ has full rank. We define

$$S := \begin{pmatrix} (G^1_{J^3})^{-1} \\ 0 \end{pmatrix} \in \mathbb{R}^{|J^3| \times r}, \qquad b := Q^\top \tilde{\gamma}_{J^3}$$

and $R \in \mathbb{R}^{n \times n - r}$ as a matrix which columns are a basis of the null space of G_{J^3} and so it is satisfied

$$G_{J^3} R = 0.$$

Referring to [49] and using (P5), we can transform the constraints $G_{J^3(g,v,m_2)}(q)m_2 + \tilde{\gamma}_{J^3(q,v,m_2)}(q, v) = 0$ to

$$\begin{aligned} m_2 = S(q)b(q, v) + R(q, v)(R^\top(q, v)M(q)R(q, v))^{-1}(R^\top(q, v)\tilde{f}(q, v) \\ -R^\top(q, v)M(q)S(q, v)b(q, v). \end{aligned}$$

Since G, f and γ are Lipschitz continuous, the same is true for S, R, b. For a Lipschitz continuous function $k : (q, v) \mapsto k(q, v)$ with Lipschitz constant C_k, it can be proven that there is a constant $\beta_k(C_k) > 0$ with $\|k(q,v)\| \leq \beta_k(1+\|(q,v)\|)$. Following this, there must be a constant β_m depending on $L_M, L_G, L_g, L_\gamma, L_f$ with

$$\|m_2\| \leq \beta_m(1+\|(q,v)\|).$$

So we get that m is linear bounded and (C3) fulfilled. The set $Z_i(x) = \{m \mid A(x)m \in F_i(x)\}, i = 1,2,$ is for a fixed $x \in \mathbb{R}^n$ the solution space of a mixed linear complementarity problem. Following [31, Theorem 2.5.17], this solution set is convex, if

$$(x^1 - x^2)^T M(\bar{q})(x^1 - x^2) = 0$$

for all solutions x^1 and x^2 of (3.5). If x^1, x^2 are solutions of this mixed complementarity problem, there exists λ^1, λ^2 such that

$$\begin{aligned}
(x^1 - x^2)^\top M(q)(x^1 - x^2) &= (x^1 - x^2)^T (G^\top(q)\lambda^1 + f(q,v) - G^\top(q)\lambda^2 - f(q,v)) \\
&= (x^1 - x^2)^\top G^\top(q)(\lambda^1 - \lambda^2) \\
&= (G(q)x^1 + \gamma(q,v) - G(q)x^2 - \gamma(q,v))^\top(\lambda^1 - \lambda^2) \\
&= -(G(q)x^1 + \gamma(q,v))^\top\lambda^2 - (G(q)x^2 + \gamma(q,v))^\top\lambda^1
\end{aligned}$$

Since $\lambda^i \geq 0, G(q)x^i + \gamma(q,v) \geq 0$, the last term is non-positive and so $(x^1 - x^2)^\top M(q)(x^1 - x^2) \leq 0$ is satisfied. Using that M positive semi-definite, we get $(x^1 - x^2)^\top M(q)(x^1 - x^2)$ should be zero and so $Z_i(x)$ convex. The LIMDI fulfills Definition 3.4 and is so consistent.

(iii) <u>F_1, F_2 are upper semi-continuous with non-empty and closed values</u>
Since $0 \in N_{\mathbb{R}^m_+}(\phi(x))$ for all functions ϕ

$$f(q,v) = f(x) \in F_1(x), \qquad 0 \in F_2(x), \forall\, x \in \mathbb{R}^{2n}$$

and so F_1, F_2 have non-empty values.

Following (P1), both F_1, F_2 are continuous transformations of a normal cone inclusion. Therefore, it is enough to show that

$$N_C(x) = \left\{x^* \in C : (y-x)^\top x^* \leq 0, \forall\, y \in C\right\}$$

is closed for all closed cones C as $\mathbb{R}^m_+, m \in \mathbb{N}$, and vectors $x \in C$. Let $(u_k) \subset N_C(x)$ be a sequence with $\lim_{k\to\infty} u_k \to u$. As $(u_k) \subset C$ and C closed, u is also an element of C. Furthermore, the inequality

$$(y-x)^\top u = \lim_{k\to\infty}(y-x)^\top u_k \leq 0$$

holds where $y \in C$ is arbitrary. The inclusion $u \in N_C(x)$ results and as well $N_C(x)$ as $F_1(x), F_2(x)$ have closed values.

In [25, Corollar 7.2.3], it is proven that for every closed cone like $\mathbb{R}_+^m, m \in \mathbb{N}$, the function $x \mapsto N_C(g(x))$ is upper semi-continuous. Since $F_i(x) = \phi_i \circ N_{C_i}(x), i = 1, 2$, are function compositions of a continuous function ϕ_i and an upper semi-continuous (usc) function, from [26] we can conclude that F_1, F_2 are usc.

□

Example 3.21 (Two-masses-spring system) In Example 2.60, a two-masses-spring system is considered. There is one equality constraint and one inequality constraint. Since equality constraints are always active, the combined matrix $\left[M(q)\ G_{J^3(q)}(q)^\top\right]$ has always full rank since it has the form

$$\begin{pmatrix} m_1 & 0 & 0 & 1 \\ 0 & m_2 & m_2 & -1 \\ 0 & m_2 & m_2 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} m_1 & 0 & 0 & 1 & -1 \\ 0 & m_2 & m_2 & -1 & 1 \\ 0 & m_2 & m_2 & 0 & -1 \end{pmatrix}.$$

The assumptions $\gamma(q, v) \equiv 0$ and $y \in \ker(M(q)), G_{J^1}(q)y \geq 0$ imply $y = 0$ and so the condition (3.7) is fullfilled. Therefore, (P5) is satisfied. The other smoothness conditions (P1)-(P4) are trivially fulfilled, since all functions are linear or constant ones. For all consistent initial values, the problem has at least one solution.

Example 3.22 (Slider-crank mechanism) The determinant of the mass matrix $M(q)$ of the slider-crank mechanism in Example 2.61 fulfils

$$\begin{aligned}\det(M(q)) \geq & J_3 \left(\left(J_1 + l_1^2 \left(\frac{m_1}{4} + m_2 + m_3 \right) \right) J_2 + l_2^2 \left(J_1 + l_1^2 \frac{m_1}{4} \right) \left(\frac{m_2}{4} + m_3 \right) \right. \\ & \left. + l_1^2 l_2^2 \left(\frac{m_2}{4} + m_3 \right)^2 \left(1 - \cos^2(\theta_1 - \theta_2) \right) \right).\end{aligned}$$

Since all parameters J_i, m_i, l_i are positive, the mass matrix has rank three. The combined matrix $\left[M(q)\ G_{J^3}(q)^\top\right]$ has also rank three. The smoothness requirements (P1)-(P4) are also fulfilled. Therefore, the equations of motion have for all consistent initial values at least one solution.

Remark 3.23 (Non-uniqueness of Lagrange-multipliers) However, q and v as the solution of (2.27) exist and are provided as unique, the same holds for the contact and impact forces

$$G_{J_1(q)}(q)^\top \mathrm{d}P = M(q)\mathrm{d}v - f(q, v)\mathrm{d}t.$$

But if $G_{J_1(q)}(q)^\top$ has not full rank because of redundant constraints, the term $\mathrm{d}P$ is not necessarily unique. Following $\mathrm{d}P = \lambda \mathrm{d}t + \Lambda \mathrm{d}\eta$, there is maybe a whole set of Lagrange multipliers which fulfil together with q and v the equations of motion. Let $\mathrm{d}P = \lambda \mathrm{d}t + \Lambda \mathrm{d}\eta$ be one solution. If

$$L(t) := \ker\, G_{J_1(q)}(q(t))^\top \cap \mathbb{R}_+^m \neq \{0\}$$

holds, then every force $\mathrm{d}P^* := \lambda^* \mathrm{d}t + \Lambda^* \mathrm{d}\eta$ with $\lambda^*(t) = \lambda(t) + l(t), l(t) \in L(t), t \notin T^*(v, I)$, and $\Lambda^*(t) = \Lambda(t) + l(t), l(t) \in L(t), t \in T^*(v, I)$, is another solution.

Remark 3.24 (Numerical treatment) In Chapter 5 the discretisation of the equations of motion (2.27) is explained. The values $(\dot{v}, \lambda)$ representing the acceleration and Lagrange multipliers must be evaluated in every timestep using a discrete (MLCP)

$$M\dot{v} + G^\top \lambda = b, \quad 0 \leq \lambda \perp G\dot{v} + c \geq 0.$$

Referring to the appendix, this problem is equivalent to a least squares problem

$$\|Ax - b\| \to \min .$$

The matrix A has the same rank deficit as G. In case of redundant constraints, an underestimated problem must be solved. One method to solve the problem is the QR-decomposition $A = QRP^\top$. So the least square problem is equivalent to

$$\left\| \begin{pmatrix} R_1 & R_2 \\ 0 & 0 \end{pmatrix} P^\top x - Q^\top b \right\| \to \min$$

with a permutation matrix $P \in \mathbb{R}^{2n\times 2n}$, $R_2 \in \mathbb{R}^{r\times 2n-r}$, a regular upper triangular matrix $R_1 \in \mathbb{R}^{r\times r}$ and an orthogonal matrix $Q \in \mathbb{R}^{2n\times 2n}$ where r is the rank of A. If all matrices and vectors are split after r components the residuum is equivalent to

$$\left\|R_1(P^\top x)_1 + R_2(P^\top x)_2 - (Q^\top b)_1\right\| + \left\|(Q^\top b)_2\right\| \to \min .$$

As R_1 is regular, $(P^\top x)_1$ calculates to $R_1^{-1}((Q^\top b)_1 - R_2(P^\top x)_2)$ where the components $(P^\top x)_2$ are arbitrarily chosen. So you get the minimum of the function. In numerical experiments as in Example 3.25 it can be observed as we expect that the components belonging to $(P^\top x)_1$ represent the acceleration and some Lagrange multipliers and $(P^\top x)_2$ the remaining Lagrange multipliers. If we would solve the least square problem in MATLAB with the backslash operator, it would use $(P^\top x)_2 = 0$. We call this the basic solution. In addition there are other common methods to choose $(P^\top x)_2$

(i) the minimum norm solution $(P^\top x)_2 = \arg\min_y \left\|R_1^{-1}((Q^\top b)_1 - R_2 y)\right\|$.

(ii) the unit solution where all components of $(P^\top x)_2$ are one.

(iii) or a random solution where all compionents are a normally distributed pseudorandom number with expectation between zero and one.

Example 3.25 (Slider-crank mechanism) The slider-crank mechanism of Example 2.61 is solved with the half-explicit timestepping scheme on velocity level of [82]. The upcoming least squares problems are solved with the QR-decomposition using the four different strategies which are described in Remark 3.24. In Figure 3.1 the Lagrange multiplier λ_1 is plotted for all strategies. We see a completely different behaviour for almost all strategies especially if you take a look at the magnitudes. Only the basic solution and the minimum norm solution strategy match.
In Figure 3.2, we also see the corresponding position functions for the crank q_1, rod q_2 and the slider q_3. They match in amount of machine accuracy. This numerical test underlines the uniqueness of the position and velocity function in contrast to the non-uniqueness of the Lagrange multipliers.

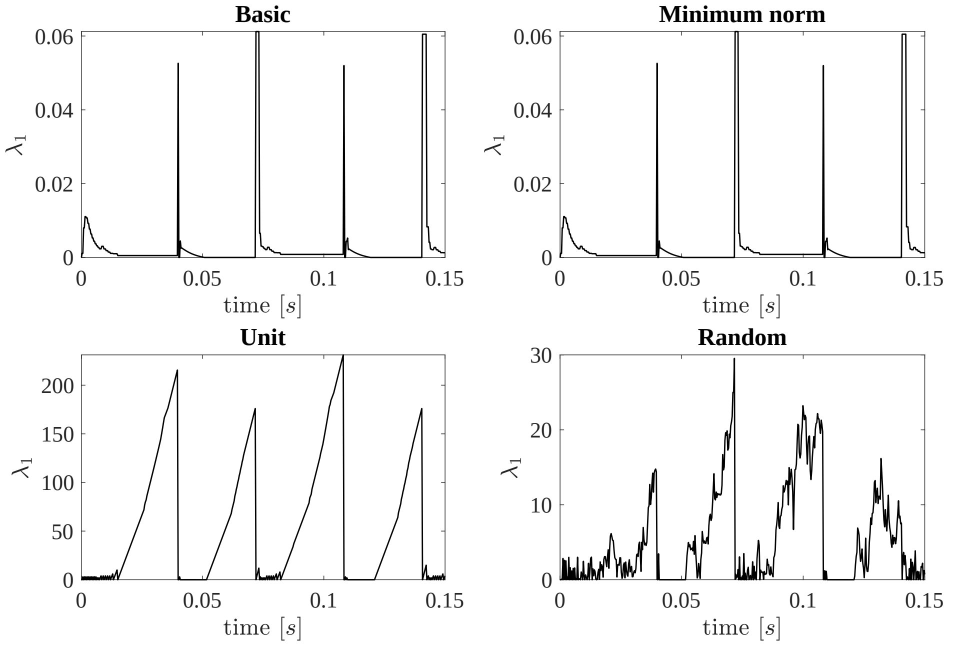


Figure 3.1.: Slider-crank mechanism: λ_1 for different strategies

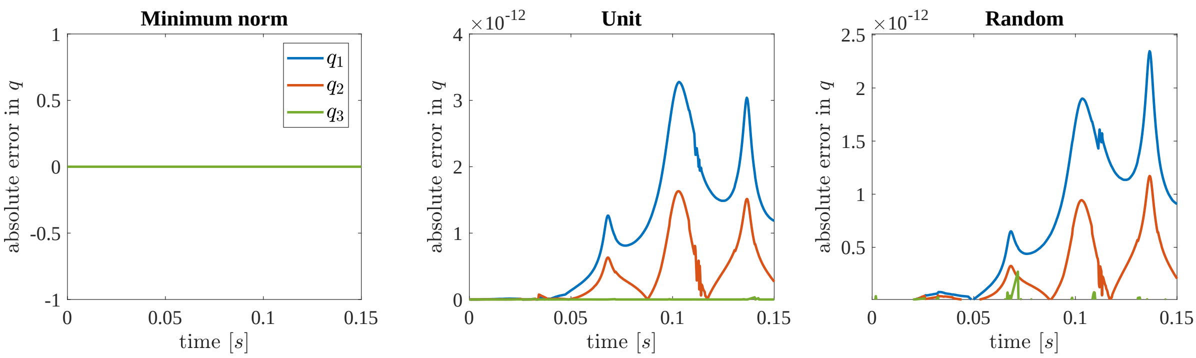


Figure 3.2.: Slider-crank mechanism: Difference between numerical solutions of q for different strategies

4. Qualitative Behaviour of Solutions of Measure Differential Inclusions

Besides the solvability, the qualitative theory of dynamical systems is of particular interest in this thesis. This includes the existence of equilibrium point or periodic solutions. In [20], the occurence of those special phenomena is analysed for a non-smooth mechanical system with high applicational relevance. In this mentioned paper, a single block on a free rocking base is considered. The block is set into a swaying motion by a non-smooth impulse of the platform. The focus of the numerical and experimental investigations is whether the motion changes into a periodic oscillation, into the resting state of the initial situation or an overturn. This example is a simple model for a building during an earthquake. The simulation results provide statements about which strengths and frequencies of earthquakes lead to possible collapses and which are relatively harmless for cities.

Our investigation on the qualitive behaviour of non-smooth dynamical problems starts with a much simpler model. In this chapter, the solutions of an one-dimensional example, the example of the impact oscillator, are considered. It is shown whether they have periodic behaviour or equilibria. What are its asymptotes for long running times or $t \to \infty$? How does the qualitative behaviour of the solutions change when parameters or initial values are modified? For multi-dimensional applications, the investigation of such questions is much more complex. In general, one cannot assume the existence of equilibria or periodic solutions, however Examples 2.60 and 2.61 are higher-dimensional applications and numerical tests suggest that they have periodic solutions.

For all investigations in this chapter, it is important to assume that a LIMDI (3.1) with a fixed consistent initial value $x_0 \in Z_1$ has always a solution $x(t, x_0)$.

Example 4.1 (Forced impact oscillator) The motion of a mass with position q and velocity v attached to a linear spring is observed. It is influenced by a force p and an obstacle at position σ (see Figure 4.1). A typical question is whether there are periodic solutions of the equations of motion or whether all solutions to different initial values have no special behaviour.

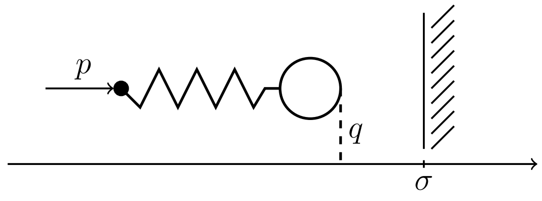


Figure 4.1.: Impact oscillator with forcing p: modell

The equations of motion can be formulated in the form

$$\begin{aligned} \dot{v}(t) + q(t) &= p(t, q(t)), \quad (q(t) < \sigma), \\ v^{+}(t) &= -\epsilon \, v^{-}(t), \quad (q(t) = \sigma). \end{aligned} \tag{4.1}$$

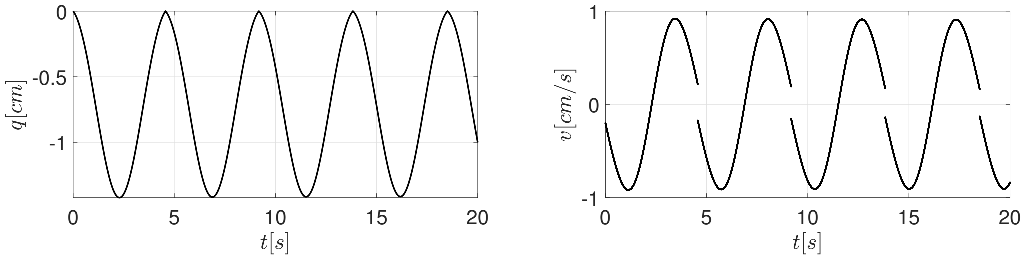


Figure 4.2.: Periodic solution of the impact oscillator with $\sigma = 0$, $\epsilon = 0.8$, $p(q) = -\cos(q)$

In Figures 4.2 and 4.3, numerically determined periodic solutions of (4.1) with different problem parameters are depicted. In a closed time period, q fulfills the requirements of an absolutely continuous function and v that of a function of bounded variation.

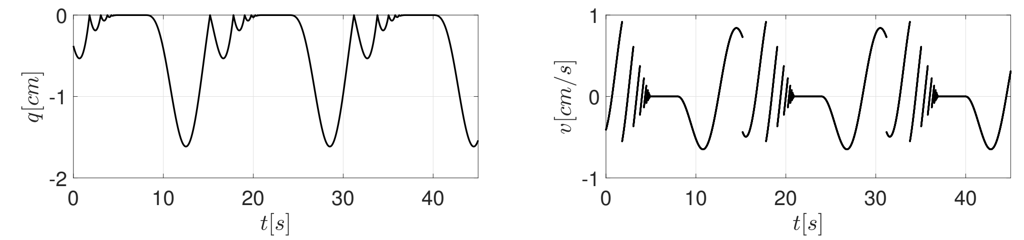


Figure 4.3.: Periodic solution of the impact oscillator with $\sigma = 0$, $\epsilon = 0.6$, $p(t) = \sin(\pi t/8)$

Example 4.2 (Higher dimensional dynamical systems) For planar autonomous smooth systems we know from the Poincaré-Bendixson theorem that limit cycles without equilibria are always an orbit of a periodic solution. The qualitive behaviour of solutions for dynamical systems with a dimension greater than two is not generally understood even in the smooth case. One observes not only equilibria and periodic solutions, but also other interesting attractors, which represent the final states of dynamical processes. So-called strange attractors record the chaotic behaviour of dynamical systems which nevertheless obey regularities. There is no general concept to prove the existence of periodic solutions or strange attractors. It often has to be examined individually for each example or is not possible analytically at all. Using numerical simulations of the non-smooth mechanical systems of higher dimension from Chapter 2, one can assume that periodic behaviour or attractors also exist for our problems of interest which have higher dimension.

In Figure 4.4 you can see an solution of the two-masses-spring system of Example 2.60 with time-independent force $p(q) = (\cos(4q_2), 0, 0)^\top$ which seems to be periodic. In Figure 4.5 (a) the angular velocity of the slider-crank mechanism from Example 2.61 with $v_3(0) = 0$ cm/s is shown in the phase space. This characterises that the slider moves up and down by the crank mechanism, but does not rotate. A periodic solution is observed. If the slider rotates with $v_3(0) = 0.1$ cm/s, the impacts are clearly visible. In Figure 4.5

(b) and (c), the orbits of the angular velocity after 0.5 seconds and 10 seconds are shown. They seem to move chaotically on a two-dimensional surface. This could be understood as chaotic behaviour in a certain bounded set, as in the case of a strange attractor.

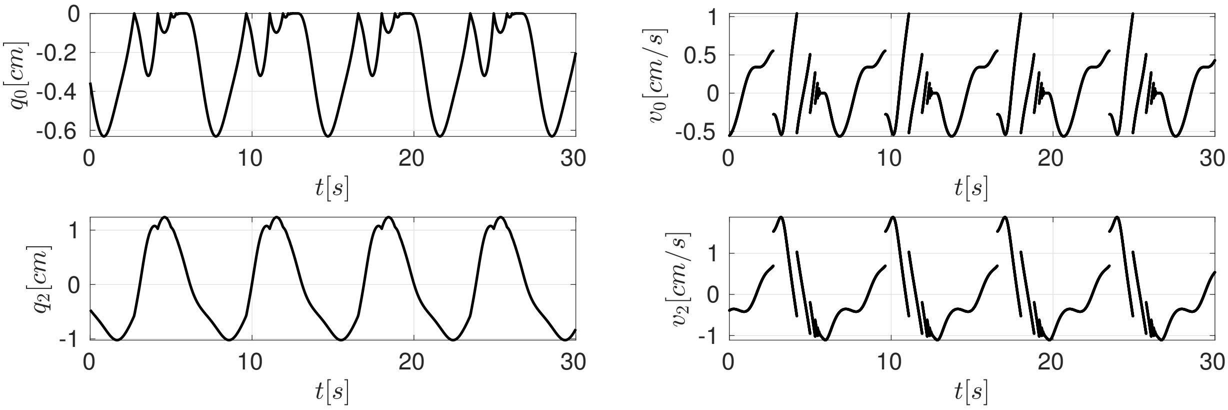
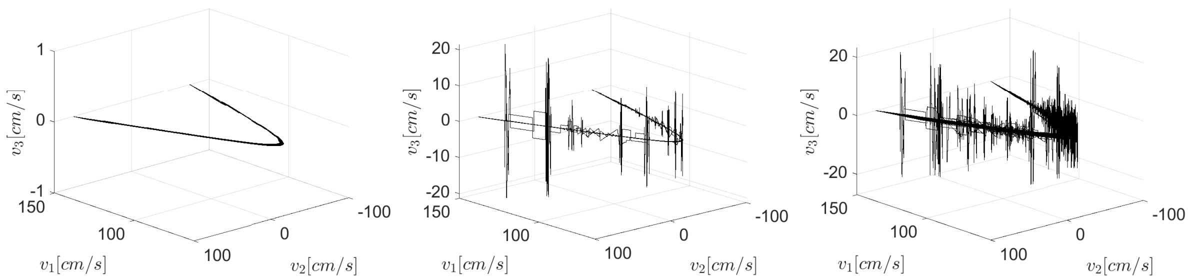


Figure 4.4.: Solution of the two-masses-spring system with $\epsilon = 0.5$, $p(q) = (\cos(4q_2), 0, 0)^\top$



(a) $v_3(0) = 0, T = 10s$ (b) $v_3(0) = 0.1, T = 0.5s$ (c) $v_3(0) = 0.1, T = 10s$

Figure 4.5.: Velocity orbits of the slider-crank mechanism

4.1. Equilibria and Periodic Solutions

In this section, we consider the qualitative behaviour of solutions $x := (q, v)$ of the measure differential inclusion

$$\begin{aligned} \mathrm{d}x \in F(t,x) &:= Ax + F_1(t,x)\mathrm{d}t + F_2(x)\mathrm{d}\eta, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ F_1(t,x) &:= \begin{pmatrix} 0 \\ p(t,x_1) - N_{\mathbb{R}_+}(\sigma - x_1) \end{pmatrix}, \quad F_2(x) := \begin{pmatrix} 0 \\ N_{\mathbb{R}_+}(x_2^+ + \epsilon x_2^-) \end{pmatrix}, \end{aligned} \tag{4.2}$$

which is a reformulation of (4.1) as an MDI. The admissible set for the position and the velocity is partitioned in

$$Z = Z_1 \cup Z_2 := \left\{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 < \sigma\right\} \cup \left\{(x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 = \sigma, x_2 \leq 0\right\}.$$

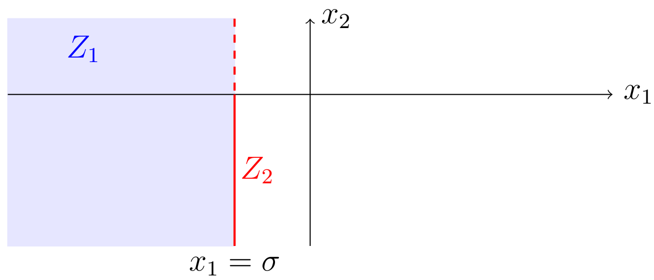


Figure 4.6.: Impact oscillator: admissible set

Definition 4.3 a) A point $x^* \in \mathbb{R}^n$ is an *equilibrium* of (4.2) if $F(t, x^*) \equiv 0$. All equilibria are summarised in the set Σ.

b) A function $x : \mathbb{R}_+ \to \mathbb{R}^n$ is *ω-periodic* if $x(t+\omega) = x(t)$ for all $t \geq 0$ and there is no value $0 < \theta < \omega$ with the same property $x(t+\theta) = x(t)$. A trivial periodic function is a constant one. Therefore, trivial periodic solutions of (4.2) are equilibria.

c) For a function $x : \mathbb{R}_+ \to \mathbb{R}^n$, the set $\gamma(x, I) := \{x(t) \mid t \in I\}$ is the *orbit* of x on $I \subseteq \mathbb{R}_+$. It is abbreviated $\gamma(x) := \gamma(x, \mathbb{R}_+)$.

d) The set $\Phi(x) := \{y \in \mathbb{R}^n \mid \exists\, (t_k)_{k\in\mathbb{N}} \text{ with } (\forall\, k \in \mathbb{N} : t_k \in \mathbb{R}_+) \wedge (t_k \to \infty) \wedge (x(t_k) \to y)\}$ is the (positive) **limit set** of x.

Obviously $\overline{\gamma(x)} = \gamma(x) \cup \Phi(x)$ holds and $\Phi(x)$ is closed. For periodic solutions, it follows

$$\gamma(x) \subset \Phi(x) \Rightarrow \overline{\gamma(x)} = \Phi(x).$$

In the smooth case, even $\gamma(x) = \Phi(x)$ is true. But the discontinuous character of the solutions prevent this. If $t \in T^*(x, \mathbb{R}_+)$, there is a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to t, (k \to \infty)$, with $x^+(t_k) \to x^-(t)$ and $x^-(t) \notin \gamma(x)$ since x is rightcontinuous.

Using the Theorem of Bolzano-Weierstraß, we get that $\Phi(x)$ is non-empty if $\underline{\lim}_{t\to\infty}|x(t)| < \infty$. For periodic solution, both sets $\gamma(x)$ and $\Phi(x)$ are bounded.

Definition 4.4 (Positively invariant) A set $P \subset Z$ is *positively invariant* with respect to LIMDI (3.1) if any solutions $x(t, x_0)$ starting with $x_0 \in P$ remains in this set

$$x(t, x_0) \in P, \;\forall\, t \geq 0.$$

Example 4.5 Let $x(t, x_0)$ be a ω-periodic solution of an autonomuos LIMDI (3.1) to an initial value $x_0 \in Z$. The sets $\gamma(x)$ and $\Phi(x) \cap Z$ are positively invariant.

Conclusion 4.6 The point $(q^*, 0) \in \mathbb{R}^2$ with $q^* \leq \sigma$ is an equilibrium of (4.1) if and only if one of the conditions

(i) $p(t, q^*) - q^* = 0$ for all $t \in I$

(ii) $q^* = \sigma$ and $p(t, \sigma) - \sigma > 0$ for all $t \in I$

are satisfied. The second case characterises that the oscillator is continuously pressed against the obstacle.

Before the equations of motion of the impact oscillator are analysed, the existence of periodic solutions for oscillators without obstacles is summarised.

Remark 4.7 (Periodic solutions of ordinary differential equations and differential inclusions, [56, 74]) The generalised solution of the ordinary differential equation

$$\ddot{q} + q = c \sin\left(\frac{2\pi}{\omega} t + \varphi\right), c, \varphi \in \mathbb{R}, \qquad \text{is}$$

$$q(t) = a\sin(t) + b\cos(t) + \frac{c\sin(\frac{2\pi}{\omega}t + \varphi)}{1 - \left(\frac{2\pi}{\omega}\right)^2}, a, b \in \mathbb{R}.$$

This describes the motion of an oscillator with an ω-periodic force $p(t) = \sin(\frac{2\pi}{\omega}t)$. With $a = b = 0$, you get an ω-periodic solution. Fourier proves that every ω-periodic continuous function $p(t)$ has a Fourier series

$$p(t) = \sum_{n=0}^{\infty} \left(a_n \sin\left(n\frac{2\pi}{\omega}t\right) + b_n \cos\left(n\frac{2\pi}{\omega}t\right)\right).$$

So the generalised solution of $\ddot{q} + q = p(t)$ has the form $q(t) = a\sin(t) + b\cos(t) + \sum_{n=0}^{\infty}(a_n \sin(n\frac{2\pi}{\omega}t) + b_n\cos(n\frac{2\pi}{\omega}t))/(1 - (2n\pi/\omega)^2)$. If $\omega \in \mathbb{Z}$ and

$$\exists\, N \in \mathbb{N}\; \forall\, n \geq N : a_n = 0,$$

a periodic solution with $a = 0 = b$ can be defined. For all these results it is important that p is a non-resonant force, i.e., that $\omega \neq 2\pi k, k \in \mathbb{Z}$.

If $\omega = 2\pi k, k \in \mathbb{N}$, the solutions of the ordinary differential equation generally explode. For example, the system $\ddot{q} + q = \cos(t)$ has the generalised solution

$$q(t) = a\sin(t) + b\cos(t) + \sin(t)t/2$$

which explode for $t \to \infty$.

If the force p depends only on q and is differentiable, the equation of motion $\ddot{q} + q = p(q)$ is an autonomous differential equation. It can be transformed into a planar autonomous system

$$\begin{aligned}\dot{q} &= v,\\ \dot{v} &= -q + p(q).\end{aligned}$$

Limit sets of two-dimensional autonomous systems are very well analysed. Poincaré-Bendixson's theorem states that the limit set of the system is already an orbit of a periodic solution if it does not contain any equilibria. That means, if one knows that the solution remains in a bounded set of $\mathbb{R}^2$ that does not contain equilibria, then there is a periodic solution.

We will also consider this three cases of resonant, non-resonant and autonomous force p for the impact oscillator. In [56], Kunze considered a pendulum in a straight tube with continuous ω-periodic force p depending on time t and with dry friction. Its equations of motion can represented by the differential inclusion

$$\ddot{q} + q \in p(t) + \mathrm{Sign}(\dot{q}).$$

In [56, Theorem 3.2.1], it is proven that for $\omega \neq 2\pi k, k \in \mathbb{N}$, an ω-periodic solution of this differential inclusion exists. This can also be an equilibrium. If $\omega = 2\pi k$ there exists an ω-periodic solution of this differential inclusion if one of the condtions

(i)
$$\left| \int_0^{2\pi} p(t) e^{\mathrm{i}t} \mathrm{d}t \right| < 4,$$

where i is the imaginary unit,

(ii)
$$\left| \int_0^{2\pi} p(t) e^{\mathrm{i}t} \mathrm{d}t \right| = 4, \qquad \int_0^{\pi} p(t + \tau - \pi/2) \cos(t) \mathrm{d}t = 0,$$

where τ is the argument of $\int_0^{2\pi} p(t) e^{\mathrm{i}t} \mathrm{d}t$

is true. As far as we know, the equations of motion of an oscillator with dry friction and an autonomous force $p(q)$ has not yet been considered in the literature.

In the following the existence of periodic solutions of (4.1) with different periodic forces p is examined. A distinction is made between

(P1) The ω-periodic function p depends only on time t and $\omega \neq 2\pi k, k \in \mathbb{Z}$ (*non-resonant force*).

(P2) The ω-periodic function p depends only on time t and $\omega = 2\pi k, k \in \mathbb{Z}$ (*resonant force*).

(P3) The ω-periodic function p depends only on position q. That means an autonomous dynamical system is considered.

Theorem 4.8 (Periodic solutions in case (P1)) Let $T > \omega > 0$ and $p(t) \in C([0,T], \mathbb{R})$ be ω-periodic. The measure differential inclusion (4.2) has an ω-periodic solution in $SBV^+([0,T], \mathbb{R})$, if $\omega \neq 2\pi k, k \in \mathbb{N}$.

Proof: In [56], the existence of periodic solutions of a similar problem is analysed. They form the equations of motion of an oscillator with Coloumb friction, but without impulsive forces. The concept of proof can be set up in an analogous way, but has to take account special requirements for functions of bounded variation. For this function class, the classical Fréchet derivative and the Riemann integral are in general not defined and must be replaced by the differential measure and the Riemann-Stieltjes integral. Since the solution of the smooth homogeneous problem is known, the proof is based on the idea of a non-smooth version of the method of variation of parameters. We define

$$g(t,s) := U(t)(I - U(\omega))^{-1} U(s)^{-1}, t, s \in [0,T], \text{ with } U(t) := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

as the fundamental matrix of the ordinary differential equation $\dot{y} = Ay$. The function g is well-defined since $\omega \neq 2\pi k$. With the constant $C := \sqrt{2}/\sin^2(\omega/2) > 0$, the function g fulfills

$$\|g(t_1, s) - g(t_2, s)\| \leq C\,|t_1 - t_2| \text{ und } \|g(t_1, s)\| \leq C, t_1, t_2 \in [0, T]\,.$$

That are the Lipschitz and boundedness criteria for g. Furthermore, we need the matrix function

$$\Gamma(t,s) := \begin{cases} g(t,s), & 0 \leq s \leq t \leq \omega, \\ g(t+\omega, s), & 0 \leq t < s \leq \omega \end{cases}.$$

and the function set

$$D := \left\{ y \in SBV^+([0,\omega]\,, \mathbb{R}^2) : \|y\|_{BV} \leq W \right\} \tag{4.3}$$

where $W := \omega P(CP + \omega CP + 1) > 0$, C is the upper bound of $\|g(t,s)\|_2$ and P is the upper bound of $\|p(t)\|_2$ on I. On the subspace D of the Banach space $SBV^+([0,\omega]\,, \mathbb{R}^2)$ we consider the set-valued function $G : D \to 2^D$ with

$$G(y) := \Big\{ z \in D : z(t) = \int_0^\omega \Gamma(t,s)\mathrm{d}f \text{ with } \mathrm{d}f = f_1(s)\mathrm{d}s + f_2 \mathrm{d}\eta(y),$$
$$f_1(t) \in F_1(t, y(t)), f_2(t) \in F_2(y^-(t)), \forall\, t \in [0,\omega] \Big\}.$$

Similar to the existence proof, we can use the Fixed Point Theorem 3.5 to show that the set-valued function G has a fixed point on D. The function $y \equiv 0$ is an element of D and therefore, D is non-empty. As to the proof of Theorem 3.8 in Chapter 3, D is convex and a subset of the Banach space $SBV^+([0,\omega]\,, \mathbb{R}^2)$. Only the properties of G from Theorem 3.8 must be proven. The function G has to be upper semi-continuous with non-empty closed and convex values and its image must be contained in a compact subset of D.

Let $z_1, z_2 \in G(y), y \in D$. That means there are $f_1^1, f_1^2, f_2^1, f_2^2$ with

$$f_1^1(s), f_1^2(s) \in F_1(s, y(s)), f_2^1(s), f_2^2(s) \in F_2(y^-(s)), \forall\, s \in [0,\omega]$$

pointwise. With the differential measures $\mathrm{d}f^1 := f_1^1(s)\mathrm{d}s + f_2^1\mathrm{d}\eta$ and $\mathrm{d}f^2 := f_1^2(s)\mathrm{d}s + f_2^2\mathrm{d}\eta$, the elements z_1, z_2 are pointwise defined as

$$z_1(t) = \int_0^\omega \Gamma(t,s)\mathrm{d}f^1, \qquad z_2(t) = \int_0^\omega \Gamma(t,s)\mathrm{d}f^2.$$

Since the convexity of a normal cone [25, Theorem 3.58] implies the convexity of $F_1(s, y(s))$ and $F_2(y^-(s))$ the linear combination

$$z_3(t) := \alpha z_1(t) + (1-\alpha) z_2(t), \qquad t \in [0,\omega]\,, \alpha \in [0,1],$$

is also an element of $G(y)$ with

$$z_3(t) = \int_0^\omega \Gamma(t,s)\mathrm{d}f^3, \qquad \mathrm{d}f^3 = f_1^3(s)\mathrm{d}s + f_2^3\;\mathrm{d}\eta(y)$$

using $f_1^3 = \alpha f_1^1 + (1-\alpha) f_1^2, f_2^3 = \alpha f_2^1 + (1-\alpha) f_2^2$. We can summarise that for all $y \in D$ the function value $G(y)$ is a convex set.

Since $\{0\} \times \{p(s)\} \in F_1(s, y(s)), 0 \in F_2(y^-(s))$ for all $s \in I$ the function $z \in AC([0,\omega], \mathbb{R}^2) \subset SBV^+([0,\omega], \mathbb{R}^2)$ defined by

$$z(t) = \int_0^\omega \Gamma(t,s) p(s) \mathrm{d}s$$

is an element of $G(y)$ if $\|z\|_{BV} \leq W$. Using

$$\frac{\mathrm{d}g}{\mathrm{d}t}(t,s) = Ag(t,s), \qquad g(t,t) - g(t+\omega, t) = I^2 \in \mathbb{R}^{2\times 2}, \quad \forall\, t, s \in [0,\omega],$$

this can be concluded from

$$\begin{aligned}
\|z\|_{BV} &= \|z(0)\|_2 + \operatorname{var}(z, [0,\omega]) \\
&= \left\| \int_0^\omega \Gamma(0,s) p(s) \mathrm{d}s \right\|_2 + \int_0^\omega \|\dot{z}(s)\| \,\mathrm{d}s \\
&\leq \omega C P + \int_0^\omega \left\| A \int_0^\omega \Gamma(s,r) p(r) \mathrm{d}r + p(s) \right\| \mathrm{d}s \\
&\leq \omega C P + \omega^2 C P + \omega P = W.
\end{aligned}$$

The definition of the set-valued function G implies $G(D) \subset D$ which is compact according to Theorem 3.6. Using the same strategy as in Theorem 3.8, we know that $G(y)$ is closed for every function $y \in D$.

It still remains to show that G is usc. Let $((y_k, z_k))_{k\in\mathbb{N}}$ be a sequence of $graph(G)$. That means

$$y_k, z_k \in D, \qquad z_k(t) = \int_{[0,\omega]} \Gamma(t,s) \mathrm{d}f^k$$

$$\mathrm{d}f^k = f_1^k(s)\mathrm{d}s + f_2^k \mathrm{d}\eta(y_k), \qquad f_1^k(s) \in F_1(y_k(s)), f_2^k(s) \in F_2(y_k^-(s)).$$

Since $\|y_k\|_{BV} = \|y_k(0)\| + \operatorname{var}(y_k, I) \leq W$, it follows

$$\begin{aligned}
\operatorname{var}(y_k, I) &\leq W \\
\|y_k(t)\| &= \left\| y_k(0) + \int_{[0,t]} \mathrm{d}y \right\| \\
&\leq \|y_k(0)\| + \int_I |\mathrm{d}y| \\
&= \|y_k(0)\| + \operatorname{var}(y_k, I) \leq W
\end{aligned}$$

The same holds for $z_k \in D$. Using Theorem 3.6, a subsequence $((y_{k_l}, z_{k_l}))_{l\in\mathbb{N}}$ exists with $y_{k_l} \to y \in D, z_{k_l} \to z \in D, l \to \infty$, in the BV-norm. Furthermore, it can be split

$$\begin{aligned}
\dot{z}_{k_l}(t) &= \Gamma(t,t) f_1^{k_l}(t), \forall\, t \in [0,\omega] \setminus T^*(y_{k_l}, [0,\omega]), \\
z_{k_l}^+(s) - z_{k_l}^-(s) &= (g(t,t) - g(t+\omega,t)) f_2^{k_l}(s) = f_2^{k_l}(s), \forall\, s \in T^*(y_{k_l}, [0,\omega]).
\end{aligned}$$

Since $z_{k_l}(t), \dot{z}_{k_l}(t)$ and $\Gamma(t,t)$ are bounded on $[0,\omega]$, the same is true for $f_1^{k_l}(t)$ and $f_2^{k_l}(t)$.

Again a subsequence $(f_1^{k_{l_p}}(s), f_2^{k_{l_p}}(s))_{p\in\mathbb{N}}$ exists with components which converge pointwise to limits $f_1(s)$ and $f_2(s)$ for $s \in [0,\omega]$. The set-valued functions F_1 and F_2 are continuous transformations of a normal cone inclusion. Following [25, Corollary 7.2.3], the solution operator of normal cone inclusions is upper semi-continuous. Because Theorem 2.12 states that $graph(F_1), graph(F_2)$ are closed, the inclusions

$$f_1(s) \in F_1(y(s)), \qquad f_2(s) \in F_2(y^-(s)), \forall\ s \in [0,\omega]$$

result. Summarising, we have

$$\begin{aligned} z(t) &= \lim_{p\to\infty} z_{k_{l_p}}(t) = \lim_{p\to\infty} \int_{[0,\omega]} \Gamma(t,s)\mathrm{d}f^{k_{l_p}} \\ &= \lim_{p\to\infty} \int_{[0,\omega]} \Gamma(t,s) f_1^{k_{l_p}}(s)\ \mathrm{d}s + \sum_{s\in T^*(y_p,[0,\omega])} f_2^{k_{l_p}}(s) \end{aligned}$$

We can change limit and Lebesgue integral and sum sign (see [30]) because f_1, f_2 converge pointwise and are bounded.

$$\begin{aligned} z(t) &= \int_{[0,\omega]} \Gamma(t,s) \lim_{p\to\infty} f_1^{k_{l_p}}(s)\ \mathrm{d}s + \lim_{p\to\infty} \sum_{s\in T^*(y_{k_{l_p}},[0,\omega])} f_2^{k_{l_p}}(s) \\ &= \int_{[0,\omega]} \Gamma(t,s) f_1(s)\ \mathrm{d}s + \sum_{s\in T^*(y,[0,\omega])} f_2(s) \end{aligned}$$

with $f_1(s) \in F_1(y(s)), f_2(s) \in F_2(y^-(s)), s \in [0,\omega]$. We get the inclusion $(y,z) \in graph(G)$ for the limit and G is so usc.

The existing fixed point $y \in G(y)$ is an ω-periodic solution since

$$y(0) = \int_0^\omega \Gamma(0,s)\mathrm{d}f = \int_0^\omega g(\omega,s)\mathrm{d}f = \int_0^\omega \Gamma(\omega,s)\mathrm{d}f = y(\omega)$$

by definition of Γ. For further considerations, we split

$$y(t) = \int_0^\omega \Gamma(t,s)\mathrm{d}f = \int_0^t g(t,s)\mathrm{d}f + \int_t^\omega g(t+\omega,s)\mathrm{d}f.$$

Therefore, y satisfies

$$\begin{aligned} y^+(t) - y^-(t) &= \int_{\{t\}} g(t,s) f_2(s)\mathrm{d}\eta - \int_{\{t\}} g(t+\omega,s) f_2(s)\mathrm{d}\eta \\ &= \begin{cases} (g(t,t) - g(t+\omega,t)) f_2(t), & t \in T^*(y,[0,\omega]) \\ 0, & t \notin T^*(y,[0,\omega]). \end{cases} \end{aligned}$$

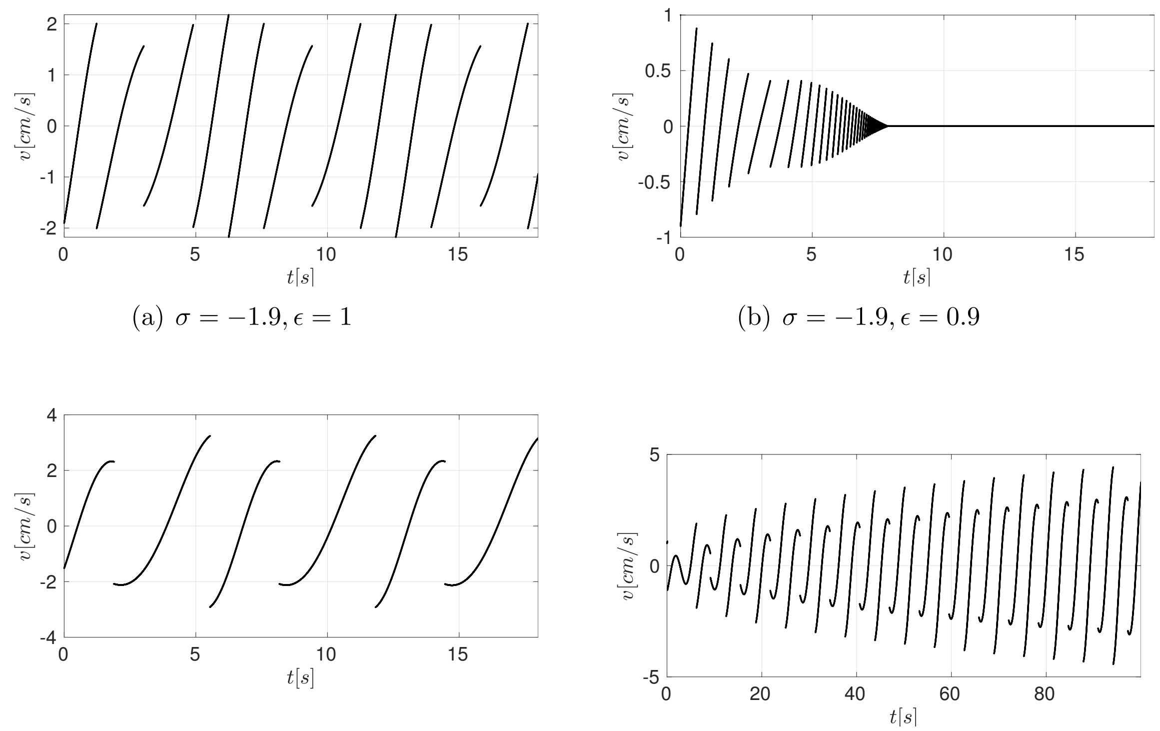
$$
\begin{aligned}
\dot{y}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t g(t,s)\mathrm{d}f + \int_t^\omega g(t+\omega,s)\mathrm{d}f\right) \\
&= \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} g(t,s)\mathrm{d}f + \int_t^\omega \frac{\mathrm{d}}{\mathrm{d}t} g(t+\omega,s)\mathrm{d}f + (g(t,t) - g(t+\omega,t))f_1(t) \\
&= \int_0^\omega A\Gamma(t,s)\mathrm{d}f + f_1(t) \\
&= Ay(t) + f_1(t)
\end{aligned}
$$

With these preliminary considerations and $f_1(s) \in F_1(y(s)), f_2(s) \in F_2(y^-(s))$, y fulfills the measure differential inclusion $\mathrm{d}y \in Ay + F_1(y)\mathrm{d}s + F_2(y^-)\mathrm{d}\eta$ in the sense

$$
\begin{aligned}
\int_0^t \mathrm{d}y &= \int_0^t \dot{y}(s)\mathrm{d}s + \sum_{s\in T^*(y,[0,t])} \big(y^+(s) - y^-(s)\big) \\
&= \int_0^t Ay(s) + f_1(s)\mathrm{d}s + \sum_{s\in T^*(y,[0,t])} f_2(s) \\
&= \int_0^t Ay(s) + f_1(s)\mathrm{d}s + \int_0^t f_2(s)\mathrm{d}\eta \\
\Rightarrow \qquad & \mathrm{d}y \in Ay + F_1(y)\mathrm{d}s + F_2(y^-)\mathrm{d}\eta.
\end{aligned}
$$

□

The ω-periodic solution in case (P1) can of course be trivial in the sense that it is an equilibrium.



(a) $\sigma = -1.9, \epsilon = 1$ (b) $\sigma = -1.9, \epsilon = 0.9$

(c) $\sigma = 0, \epsilon = 0.9$ (d) $\sigma = 0, \epsilon = 1$

Figure 4.7.: Solutions of the impact oscillator with resonant force $p(t) = \cos(t)$

Remark 4.9 (Periodic solutions in case (P2)) As already mentioned, the solutions of the smooth oscillator generally grows unboundelly if p has the period 2π. Since in the non-smooth case, the mass meets the obstacle at some point, the solutions for $p(t) = \cos(t)$ have a completely different structure. As shown in Figure 4.7, they can still explode, but also converge towards an equilibrium or a periodic limit cycle. This is strongly related to the choice of σ and ϵ. A precise analysis should not be carried out here, since autonomous differential systems are of particular interest for the numerical investigations in the next chapter and so we concentrate on case (P3).

Next, a criterion for the existence of periodic solutions of (4.1) is given if the force p depends only on the position q. In this case (P3), the system is autonomous. The fundamental idea is based on the transfer of the transversal theory and the Poincaré-Bendixson theorem from smooth systems to a non-smooth example. It is supposed that the system (4.1) has a unique solution for all initial values $(q_0, v_0) \in Z$ and that p is continuous.

Definition 4.10 (Transversal) A set $S = \{\tau u + (1-\tau)w \mid \tau \in [0,1]\}$ with $u, w \in Z$ is a *transversal* to (4.1) if

$$h(q,v) := \begin{pmatrix} v \\ p(q) - q \end{pmatrix} \quad \text{and} \quad u - w$$

are linearly independent for all $(q, v) \in S$. Since Z is convex, the line S is a subset of Z.

Remark 4.11 If $x = (q, v) \in Z$ is not an equilibrium of (4.1) then there exists a transversal with $x \in S$.

(i) Let $x = (q, v) \in Z_1$, i.e. $q < \sigma$. Then, a ball $B(x, \delta)$ exists where $\bar{q} < \sigma$ for all $(\bar{q}, \bar{v}) \in B(x, \delta)$. This case is similar to smooth systems and a transversal exists because of [74, Bem. 9.1.2.].

(ii) If $x = (q, v) \in Z_2$ with $q = \sigma, v < 0$ we can define $u = (\sigma, v/2)^\top, w = (\sigma, 3v/2)$. That means, S is a segment of Z_2. In all points $(\bar{q}, \bar{v}) \in S$, it is $h(\bar{q}, \bar{v}) = (\beta \cdot v, p(\sigma) - \sigma)^\top, \beta \in [0.5, 1.5]$. It is linearly independent of the direction vector $u - w = (0, -v)^\top$ of S.

(iii) Let $x = (q, v) = (\sigma, 0) \in Z_2$. Since x is no equilibrium, Conclusion 4.6 states that $p(\sigma) - \sigma < 0$. There has to be an $\delta > 0$ such that $p(\sigma - s) - (\sigma - s) < 0$ for all $s \in [0, \epsilon]$ because p is continuous. The set S with $u = (\sigma, 0), w = (\sigma - \delta, 0)$ is a transversal through x as $h(\bar{q}, \bar{v}) = (0, p(\sigma - s) - s), s \in [0, \delta]$ is always linearly independent of the direction vector $u - w = (\delta, 0)$ of S.

Theorem 4.12 Let $x = (q, v) \in Z$ be not an equilibrium and S a transversal of (4.1) with $x \in S$. Then, there is for all $\epsilon > 0$ a ball $B(x, \delta)$ such that for all solutions $x(t, x_1)$ with initial value $x_1 \in B(x, \delta) \cap Z$ a time point $t \in [-\epsilon, \epsilon]$ exists with $x(t, x_1) \in S$.

Proof:

(i) Let $x \in Z_1$. This is similar to the smooth case. The existence can be concluded with [74, Prop. 9.1.3].

(ii) Let $x \in Z_2, v < 0$. Assume the statement is false. Then, there exists an $\epsilon_0 > 0$ and a sequence $Z \ni x_k = (q_k, v_k) \to x = (\sigma, v)$ of initial values such that

$$x(t, x_k) \notin S, \ \forall\, t \in [-\epsilon_0, \epsilon_0],$$

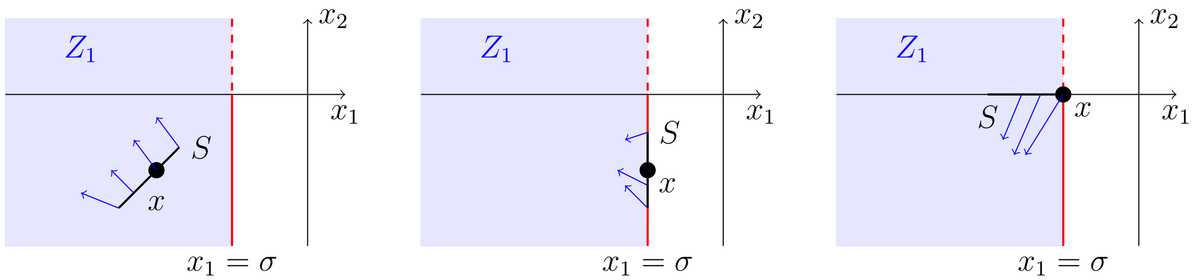


Figure 4.8.: Impact oscillator: Tranversals for $x \in Z$ in situations (i)-(iii) from Remark 4.11

for an transversal $S = \{\sigma\} \times [3v/2, v/2]$. If $v_k \to v < 0, q_k \to \sigma$ the inequalities $3v/2 < v_k < v/2$ and $q_k \geq \sigma + v\epsilon_0/4$ are true for $k \geq N$ with one $N \in \mathbb{N}$. As long as $q(s) \neq \sigma$, the functions v and q are continuous and so satisfy this equation in a sufficiently short time interval $[-t_0, t_0]$. So for $\max(-\epsilon_0, -t_0) \leq t \leq 0$ it holds

$$q(t, q_k) = q_k - \int_t^0 v(s)\mathrm{d}s \geq q_k + \frac{v \cdot t}{2} \geq \sigma + \frac{v\epsilon_0}{4} - \frac{v\epsilon_0}{2} \geq \sigma - \frac{v\epsilon_0}{2} \geq \sigma.$$

There is a timepoint $\max(-\epsilon_0, -t_0) \leq \tau \leq 0$ where $q(\tau) = \sigma$ and the corresponding velocity $v(\tau)$ is in $[3v/2, v/2]$. So $x(\tau, x_k) \in S$.

(iii) Let $x = (\sigma, 0) \in Z_2$. Since x is not an equilibrium, it follows from Conclusion 4.6 that $p(\sigma) - \sigma = \theta < 0$. Assume the statement is false. Then, there exists a $\epsilon_0 > 0$ and a sequence $Z \ni x_k = (q_k, v_k) \to x = (\sigma, 0)$ of initial values such that

$$v(t, v_k) \neq 0, \quad \forall\, t \in [-\epsilon_0, \epsilon_0]\,.$$

That means in particular $v_k \neq 0$ for all $k \in \mathbb{N}$.

a) We consider the subsequence (q_{k_n}, v_{k_n}) with $v_{k_n} > 0$ for all $n \in \mathbb{N}$. For all $t \in [-\epsilon_0, \epsilon_0]$ the value $v(t, v_{k_n})$ stays positive. It could change the sign if $v(t, v_{k_n})$ gets zero - which we assume that it does not happend - or if an impact could be observed. For all $t < 0$ it holds

$$q(t, q_{k_n}) = q_k - \int_t^0 v(s, v_{k_n})\mathrm{d}s \leq q_{k_n} + \frac{v_{k_n} \cdot t}{2} < \sigma.$$

Therefore, no impact happens for $t < 0$ and v and q are continuous. Since p is assumed to be continuous there is an index $N \in \mathbb{N}$ and a constant $\delta > 0$ with

$$\theta - \delta \leq p(q_{k_n}(t, q_{k_n})) - q_{k_n}(t, q_{k_n}) \leq \theta + \delta < 0, \quad \forall\, n \geq N.$$

It follows

$$v(t, v_{k_n}) = v_{k_n} + \int_t^0 p(q(s, q_{k_n})) - q(s, q_{k_n})\mathrm{d}s \leq v_{k_n} + \int_t^0 (\theta + \delta)\mathrm{d}s = v_{k_n} - (\theta + \delta)t.$$

Since $-\epsilon_0 \leq t < 0$ and $v_{k_n} \to 0$ this term is getting negativ. In particular, there is a timepoint when the velocity becomes zero. This is a contradiction.

b) If we consider the subsequence with $v_{k_n} < 0$, the argumentation in a) is true for $t > 0$.

□

Remark 4.13 The basic idea of the following theorem is based on Jordan's curve theorem [44]. This states that every double point free closed curve $\Gamma \subset \mathbb{R}^2$ divides the space into an interior Γ_i and an exterior Γ_e, which are open subsets of $\mathbb{R}^2$ with $\Gamma \cup \Gamma_i \cup \Gamma_e = \mathbb{R}^2$. The three sets are pairwise disjoint. A solution curve of (4.1) can switch between the interior and the exterior only if it crosses Γ or jumps at a time $t \in T^*(x, \mathbb{R}_+)$ from $x^-(t) \in \Gamma_i$ into the other set $x^+(t) \in \Gamma \cup \Gamma_e$. Of course, the jump also works the other way around.

Theorem 4.14 Let x be a bounded and non periodic solution of (4.1) and S a transversal. Furthermore, there are three values $x(t_i) \in S, i = 1, 2, 3$, with $t_1 < t_2 < t_3$. That means, there are $\tau_i, i = 1, 2, 3$, with $x(t_i) = \tau_i u + (1 - \tau_i)w$. Furthermore, it is $\gamma(x, [t_1, t_3]) \cap S = \{t_1, t_2, t_3\}$. Then $\tau_1 < \tau_2 < \tau_3$ or $\tau_1 > \tau_2 > \tau_3$.

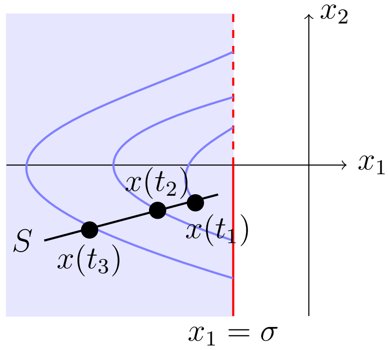


Figure 4.9.: Impact oscillator: sequence of transversal crossings in case (i)

Proof: Because it was assumed that the solutions of (4.1) are unique and non-periodic, all $\tau_i, i = 1, 2, 3$, must be different.

Suppose that $\tau_1 < \tau_2$ holds for $t_1 < t_2$. The intersection $\gamma(x, [t_1, t_2]) \cap T^*(x, \mathbb{R}_+)$ can be at most one-element. We justify this statement for the three transversal types from Theorem 4.12.

(i) Let $s_1, s_2 \in (t_1, t_2) \cap T^*(x, \mathbb{R}_+)$ with $s_1 < s_2$ be the first two discontinuouty timepoints. Then, $\gamma(x, [t_1, s_1))$ and $\gamma(x, (s_2, s_2 + \delta]), \delta > 0$, are connected curves which are disjoint with the closed curve $\Omega := \gamma(x, [s_1, s_2)) \cup L$ where

$$L = \{\tau x^-(s_2) + (1 - \tau)x^+(s_1) \mid \tau \in [0, 1]\},$$

because x is not periodic. There are two possible situations

a) The curve $\gamma(x, [t_1, s_1))$ is in the interior Ω_i. Then $0 < x^-(s_1) < x^-(s_2)$. With the impact law and $0 \leq \epsilon \leq 1$ we get $x^+(s_2) < x^+(s_1) < 0$. So the curve stays in Ω_e for $t \geq s_2$ since $x(t_2) \in \Omega_e$. So the curve $\gamma(x, (s_2, t_2)) \cup S$ must cross Ω. This is a contradiction. There is no intersection with S in (t_1, t_2) and there can not be an intersection with $\omega(x, (t_1, t_2))$ since x is non-periodic.

b) The curve $\gamma(x,[t_1,s_1))$ is in the exterior Ω_e. Then $x^-(s_1) > x^-(s_2) > 0$. With the impact law and $0 \le \epsilon \le 1$ we get $x^+(s_2) > x^+(s_1) < 0$. So the curve stays in Ω_i for $t \ge s_2$ since $x(t_2) \in \Omega_1$. So the curve $\gamma(x,(s_2,t_2)) \cup S$ must cross Ω. This is a contradiction. There is no intersection with S in (t_1,t_2) and there can not be an intersection with $\omega(x,(t_1,t_2))$ since x is non-periodic.

(ii) In this case $t_1, t_2 \in T^*(x,\mathbb{R}_+)$ and there is no $s \in T^*(x,(t_1,t_2))$. The curve $\gamma(x,[t_1,t_2)) \cup L$ with

$$L = \{\tau x^-(t_2) + (1-\tau)x^+(t_1) \mid \tau \in [0,1]\}$$

is again closed and has no doublepoint. Since the non-periodic solution $x^+(t_2) \in \Omega_e$. Then, x can not cross Ω in Z_1 and if it reaches Z_2 in a timepoint s, it is $x^-(s) > x^-(t_2)$. With the impact law we get $x^+(s) < x^+(t_2)$. So the solution curve stays alwas in the exterior. If $x^+(t_2) \in \Omega \cup \Omega_i$ we get that the curve stays in the interior because of the impact law.

(iii) Is similar to (i).

With the argumentation from (ii), in this case it follows immediately from $x^+(t_1) < x^+(t_2)$ also $x^+(t_2) < x^+(t_3)$ and thus the assumption. Analogously, of course, from $x^+(t_1) > x^+(t_2)$ also $x^+(t_2) > x^+(t_3)$ and thus the assumption. In the other two cases we define with

$$\Omega := \gamma(x,[t_1,t_2]) \cup S \cup \big\{\tau x^-(s) + (1-\tau)x^+(s) \mid \tau \in [0,1]\big\}$$

a closed doublepoint-free curve in $\mathbb{R}^2$. Here s is the only element in $T^*(x,[t_1,t_2])$. If it does not exists the third set is empty and Ω is nevertheless closed and doublepoint-free. With [74, Prop. 9.1.4] it follows $\tau_3 > \tau_2$ since S is a transversal. The argumentation in [74] is, that Ω is a closed double-point free curve and the vectorfield $h(q,v)$ points to Ω_e for all $(q,v) \in S$, since $\tau_1 < \tau_2$. Since $x(t_3) \in S$ is true with $x(t_3-\delta) \in \Omega_2, \delta > 0$, the condition $\tau_3 > \tau_2$ follows. □

Next a conclusion of Theorems 4.12 and 4.14 is formulated which can be shown as in the smooth case [74, Prop. 9.1.5,9.1.6.].

Conclusion 4.15 Let x be a bounded solution of (4.1), S a transversal and $y \in S \cap \Phi(x)$. Then, there is a sequence $t_k \to \infty$ with $x(t_k) \in S$ and $x(t_k) \to y$. Furthermore, there is at most one element in $S \cap \Phi(x)$. .

The following Theorem is again a conclusion of Theorem 4.12 and Conclusion 4.15 like for smooth systems. Hence, it is referred to [74, Lem. 9.2.1.]

Conclusion 4.16 Let $x = (q,v)$ be a bounded solution of (4.1) with $\gamma(x) \cap \Phi(x) \neq \emptyset$. Then x is periodic.

Theorem 4.17 (Periodic solutions in case (P3)) Let p be continuous, linearly bounded and $(q,v) \in AC(\mathbb{R}_+,\mathbb{R}) \times SBV^+(\mathbb{R}_+,\mathbb{R})$ be a bounded solution of (4.1) with $\Phi(q,v) \cap Z$ containing no equilibrium. There exists a periodic solution of (4.1).

Proof: If $x = (q,v)$ is periodic, the statement is of course true. Let us assume in the following that (q,v) is not periodic. Since (q,v) are bounded functions, as mentioned $\Phi(q,v) \cap Z$ is also bounded and non-empty. We want to prove that $\Phi(q,v) \cap Z$ is positive invariant. Let $y \in \Phi(x) \cap Z$. There is a sequence $(t_k)_{k\in\mathbb{N}}$ with $x(t_k) \to y, t_k \to \infty$. The functions $y_k(s) := x(t_k + s), s \geq 0$, are also solutions of (4.1) with initial value $x(t_k) \in Z$. With the same argument as in (ii) of the proof of Theorem 3.8 it follows that there is a subsequence $(y_{k_l})_{l\in\mathbb{N}}$ which converges to a bounded solution z of (4.1) with initial value $z(0) = y$. Since $t_m + s \to \infty$ with $x(t_m + s) \to z(s)$, we get $z(s) \in \Phi(x) \cap Z$ and so $\Phi(x) \cap Z$ is positive invariant.

Since $\gamma(z) \subset \Phi(x) \cap Z$, the function z is also bounded. So it exists $y \in \Phi(z) \cap Z \subset \Phi(x) \cap Z$. According to the prerequisite, this is not an equilibrium. Using Theorem 4.12, there is thus a transversal S through y.

Following Conclusion 4.15, the set $\Phi(x) \cap S$ is equivalent to $\{y\}$ since there is not a second element in the intersection. In addition the sequence created in Conclusion 4.15 shows that $\gamma(z) \cap S \neq \emptyset$. Since $\gamma(z) \subset \gamma(x)$, it follows $\gamma(x) \cap S = \{y\}$ and $y \in \gamma(z) \cap \Phi(z)$. From Theorem 4.16 we get that z is periodic. □

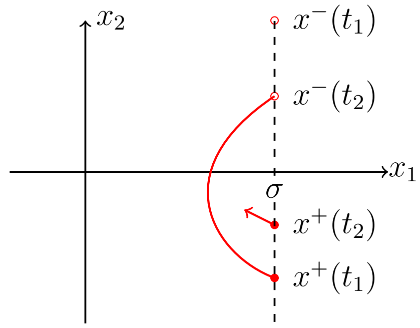
Theorem 4.18 (Number of velocity jumps during one period in case (P3)) Let x be an ω-periodic solution of (4.2) with an autononous ω-periodic force $p \in \mathbb{R} \to \mathbb{R}$. Then, x has at most one discontinuity point during the period $[0, \omega]$.

Proof: We assume there are two points of discontinuity. That means there are two different time points $t_1, t_2 \in [0, \omega], t_1 < t_2$, with

$$x_2^+(t_1) \neq x_2^-(t_1), \qquad x_2^+(t_2) \neq x_2^-(t_2)$$

and in the non-empty interval (t_1, t_2) the solution x is continuous. There are two possible cases

(i) The relationship $x_2^-(t_1) > x_2^-(t_2) > 0$ is satisfied. From Newtons impact law, it follows $0 > x_2^+(t_2) > x_2^+(t_1)$.



Since x is continuous in (t_1, t_2), the orbit of x on this interval forms a closed set and

$$\Gamma := \gamma(x, (t_1, t_2)) \cup \big(\{\sigma\} \times [x_2^+(t_1), x_2^-(t_2)]\big)$$

a closed, double-point free, continuous curve. According to Jordan's curve theorem [74], Γ decomposes the plane $\mathbb{R}^2$ into two disjoint open sets Γ_i, Γ_e with

$$\partial\Gamma_i = \Gamma = \partial\Gamma_e,$$

the exterior of Γ and the interior. By assumption, $x^-(t_1) \in \Gamma_e \cup \gamma(x, \mathbb{R})$ and $x^+(t_2) \in \overline{\Gamma_i}$.

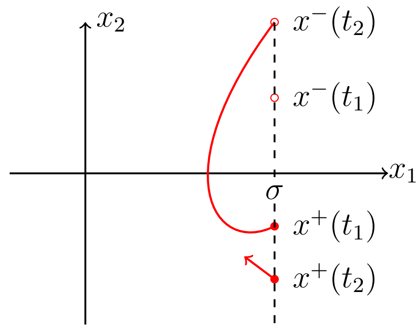
Let $t_3 > t_2$ be the next time point with $x^-(t_3) \neq x^+(t_3)$. Then the partial orbit $\gamma(x, (t_2, t_3)) \subseteq \{z \in \mathbb{R} \mid z \leq \sigma\} \times \mathbb{R}$

is again a connected curve, since it belongs to a continuous function. This either remains in the interior Γ_i or is part of the edge $\partial\Gamma_i = \Gamma$ at a time $t \leq t_3$. If $x(t) \in \gamma(x, (t_1, t_2))$, x remains on the edge and $x^-(t_2) = x^-(t_3)$. The second case is

$$x^-(t) = x^-(t_3) \in \left(\{\sigma\} \times [x_2^+(t_1), x_2^-(t_2)]\right).$$

Since $x^-(t_3) \leq x^-(t_2)$, according to Newton's impact law the inequality $x^+(t_3) \geq x^+(t_2)$ follows and thus $x^+(t_3) \in \overline{\Gamma_i}$. This reasoning can be continued over $(t_2, +\infty)$ and we get $\gamma(x, (t_2, +\infty)) \subseteq \overline{\Gamma_i}$. Thus there is no $t > t_2$ with $x^-(t) = x^-(t_1)$, since $x^-(t_1) \in \Gamma_e$. But this is a contradiction to the periodicity with $\omega > 0$.

(ii) The relationship $x_2^-(t_2) > x_2^-(t_1) > 0$ is satisfied. From Newtons impact law, it follows $0 > x_2^+(t_1) > x_2^+(t_2)$.



In this case one can argue analogously that

$$\gamma(x, (t_1, t_2)) \cup \left(\{\sigma\} \times [x_2^+(t_1), x_2^-(t_2)]\right)$$

forms a closed curve with $x^-(t_1)$ on the inside and $x^+(t_2)$ on the outside. However, the solution can never reach the inside again and thus a contradiction arises again.

□

In this section we have given criteria for the existence of equilibria and periodic solutions in case (P1) and (P3). In the next chapter we analyse the stability of this solutions. This is important if we try to study periodic solutions numerically and do not know how to choose the initial value to get periodic solutions.

4.2. Stability Issues of Equilibria

There are different types of stability of solutions. The interesting aspect for us is whether solutions remain close to equilibrium points or periodic solutions when you change the initial value. This can be investigated with the well-known method of Lyapunov, which is based on the functions of the same name. In addition, we will also investigate the stronger property of attractivity. This states that all solutions converge towards the equilibrium or the periodic solution if the initial value is close to the original one.

A detailed introduction to Lyapunov stability theory of MDIs is given in [59]. In this thesis, some issues are generalised to the implicit formulation of MDIs.

Definition 4.19 (Lyapunov stability of equilibria) Let $x^* \in Z$ be an equilibrium of a LIMDI (3.1). It is called *Lyapunov stable* if for all $\mu > 0$ there is a $\delta > 0$ such that for all initial values $x_0 \in Z_1$ with $\|x_0 - x^*\| \leq \delta$ each solution $x(t, x_0)$ of LIMDI (3.1) with $x(0) = x_0$ satisfies

$$\|x(t, x_0) - x^*\| \leq \mu, \quad \forall\, t \in I.$$

Definition 4.20 (Attractivity of equilibria) Let $x^* \in Z$ be an equilibrium of a LIMDI (3.1). It is called *Lyapunov attractive* if there is a $\delta > 0$ such that for all initial values $x_0 \in Z$ with $\|x_0 - x^*\| \leq \delta$ each solution $x(t, x_0)$ of LIMDI (3.1) with $x(0) = x_0$ satisfies

$$\lim_{t \to \infty} \|x(t, x_0) - x^*\| = 0.$$

For the next three theorems, we assume that $x^* = 0$ is an equilibrium of LIMDI (3.1) with initial value $x_0 = 0$ and non-empty closed convex admissible set Z. This holds if $0 \in Z$ and $0 \in F_1(0)$. It shell be emphasized that the next two theorems are strictly related to the results of [59]. The slightly changed formulation is important for the third theorem which generalises the results of [59] to implicite problems.

In this section, the expression $\mathrm{d}V \leq 0$ is used. This symbolises the condidtions

- $\dot{V}(q(t), v(t)) \leq 0, \forall\, t \notin T^*(V, I)$,
- $V^+(q(t), v(t)) - V^-(q(t), v(t)) \leq 0, \forall\, t \in T^*(V, I)$.

It can be summarised in the condition that V is monotonously decreasing along each solution trajectory.

Theorem 4.21 (Stability of equilibria) The equilibrium $x^* = 0$ is stable in the sense of Lyapunov if there exists an upper semi-continuous *Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ that is Lipschitz continuous on Z and a neighbourhood $U := B(x^*, h) \cap Z$ for an $h > 0$ with

(i) $V(x) \geq 0, \forall\, x \in U, V(x^*) = 0$

(ii) $\mathrm{d}(V \circ x) \leq 0, \forall\, x \in SBV^+(I, U)$

(iii) $\{x \in Z : V(x) = 0\} = \{x^*\}$.

Proof: For all $c > 0$ we can define a non-empty closed set $\Omega_c := \{x \in \mathbb{R}^n : V(x) \leq c\}$. Conditions (i)-(iii) state that $x^* = 0$ is the global minimum of V on the closed and convex set Z. From this fact it follows similar to [59, Prop. 6.2.] that there exists a $c^* > 0$ such that the intersection $\Omega_c \cap Z$ is closed and simply connected for all $c \leq c^*$. Let now

$$V_{\mu/2} := \sup_{x \in \overline{B}(0,\mu/2) \cap Z} V(x), \qquad V_{h/2} := \sup_{x \in \overline{B}(0,h/2) \cap Z} V(x)$$

with μ from the definition of stability be the suprema the upper semi-continuous function takes on compact sets. Since V is Lipschitz on Z the function $V \circ x$ is of bounded variation [59, Prop 6.3.]. Every function of bounded variation attains a finite supremum on compact sets.

Set $c := \min(V_{\mu/2}, V_{h/2}, c^*)$. It follows that $\Omega_c \cap Z$ is a subset of $\overline{B}(0, \mu/2)$ and $\overline{B}(0, h/2)$ and it holds

$$\Omega_c \cap Z \subset Z \cup B(0, \mu) \cup B(0, h).$$

According to [59, Prop 6.5.] you get from condition (ii) that $\Omega_c \subset Z$ is positive invariant for consistent LIMDI, i.e. for all $x_0 \in \Omega_c \cap Z$ it follows $x(t, x_0) \in \Omega_c \cap Z$ for each solution $x(t, x_0)$. Let now be $B(0, \delta)$ the largest ball which lies in $\Omega_c \cup Z_1^c$. The notation Z_1^c means the complement of the set Z_1. Its existence follows immeditally from the fact that the closed set $\Omega_c \cap Z$ is not empty. It holds $\sup_{x \in B(0,\delta) \cap Z} V(x) \leq c$ and so all solutions starting in $B(0, \delta)$ remain in $B(0, \mu)$. □

Theorem 4.22 (Attractivity of a equilibria) The equilibrium $x^* = 0$ is attractive in the sense of Lyapunov if there exists an upper semi-continuous *Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ that is Lipschitz continuous on Z and a neighbourhood $U := B(x^*, h) \cap Z$ for an $h > 0$ with

(i) $V(x) \geq 0, \forall\, x \in U, V(x^*) = 0$

(ii) $\mathrm{d}(V \circ x) \leq 0, \forall\, x \in SBV^+(I, U)$

(iii) $\{x \in Z : \mathrm{d}(V \circ x) = 0\} = \{x^*\}$.

Proof: According to (i)-(iii) there can not be a $y \in Z \backslash \{x^*\}$ with $V(y) = 0$ and so x^* is stable. Set again $c = \min(V_{h/2}, c^*)$ and $B(0, \delta) \subset B(0, h)$ as the largest ball in $\Omega_c \cap Z$. Because $\mathrm{d}(V \circ x) < 0$ for all $x \in B(0, h) \cap Z \backslash \{x^*\}$ it holds

$$\|x_0\| < \delta, x_0 \in Z \Rightarrow \lim_{t \to \infty} V(x(t, x_0)) =: a \leq 0.$$

Since $\mathrm{d}(V \circ x)$ is negative there exists a continuous strictly increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ and $\mathrm{d}V(x) \leq -\alpha(\|x\|)$. Using the same contradiction argument as in [59, Theorem 6.23] we can show that $a = 0$. Otherwise there should be a $d > 0$ such that $B(0, d) \subset \Omega_a \cap Z$ and $V(x(t, x_0))$ is outside $\Omega_a \cap Z$. From (ii) we get

$$V(x(t, x_0)) \leq V(x_0) + \int_{[0,t]} \mathrm{d}(V \circ x) \leq V(x_0) - \int_{[0,t]} \alpha(\|x(t, x_0)\|)\mathrm{d}t \leq V(x_0) - \alpha(d)(t - 0).$$

For $t \to \infty$ the value $V(x(t, x_0))$ is getting negative. From this contradiction to (i) you can follow $\lim_{t \to \infty} V(x(t, x_0)) = 0$ which implies attractivity. □

Remark 4.23 (Lyapunov function candidate) Referring to [16, 59], the sum of the total machanical energy and the indicator function of the admissible set Z

$$V(q, v) := T(q, v) + U(q) - U(0) + \psi_Z(q) = \frac{1}{2} v^\top M(q) v + U(q) - U(q^*) + \psi_Z(q) \quad (4.4)$$

is a suitable Lyapunov function with $q^* = 0$. The mass matrix $M(q)$ is again symmetric positive semi-definite and Lipschitz continuous. The potential energy $U : \mathbb{R}^n \to \mathbb{R}_+$ fulfills $U(q) > 0, \forall\, q \in Z \backslash \{0\}$ and $\min_{q \in \mathbb{R}^n} U(q) = U(q^*)$. In addition, the gradient of $U(q)$ does not vanish for all $q \in Z\backslash\{q^*\}$. For example, a potential energy function $q^\top K q$ with a positive definite $K \in \mathbb{R}^{n\times n}$ fulfills all requirements. Systems with fewer assumptions on the kinetic energy T and the potential energy U often have no equilibrium points

Theorem 4.24 (Stability result for equilibria of non-smooth mechanical systems) Let $M(q) \in \mathbb{R}^{n\times n}$ be symmetric positiv semi-definite for all $q \in \mathbb{R}^n$ and like g, G, f Lipschitz continuous. If in addition the property (P5) in Problem 3.18 and the conditions

(S1) $v^\top \cdot f^{nc}(q,v) \leq 0$, for all $q \in Z, v \in \{x \in \mathbb{R}^n \mid G_{J^1(q)}(q)x \geq 0\}$, with $v^\top f^{nc}(q,v) = 0 \Rightarrow v = 0$

(S2) $v^\top f(q,v) < 0$, for all $q \in Z, v \in \{x \in \mathbb{R}^n \mid G_{J^1(q)}(q)x \geq 0\} \cap \ker(M(q))$ except $(q,v) = (0,0)$,

(S3) $(q^*, v^*) = (0,0)$ is an equilibrium of LIMDI (3.1)

are satisfied, than (q^*, v^*) is stable. If furthermore $\epsilon < 1$ holds, the equilibrium is attractive.

Proof: The indicator function is upper semi-continuous for closed sets like Z such that V is also upper semi-continuous and Lipschitz continuous on Z. Since M is positive semi-definite, U positive with minimum q^* and ψ also, the Lyapunov candidate V in (4.4) is non negative with $V(q^*, v^*) = 0$. The differential measure can be transformed to

$$\begin{aligned}
\mathrm{d}V &= \frac{1}{2}(v^+ + v^-)^\top M(q)\mathrm{d}v + v^\top \left(\frac{\partial}{\partial q}(T(q,v) + U(q))\right)\mathrm{d}t \\
&= v^\top M(q)\dot{v} + v^\top \left(\frac{\partial}{\partial q}(T(q,v) + U(q))\right)\mathrm{d}t + \frac{1}{2}(v^+ + v^-)^\top M(q)(v^+ - v^-)\mathrm{d}\eta \\
&= v^\top (f(q,v) + G^\top_{J^1(q)}(q)\lambda) + v^\top \left(\frac{\partial}{\partial q}(T(q,v) + U(q))\right)\mathrm{d}t + \frac{1}{2}(v^+ + v^-)^\top G^\top_{J^1(q)}(q)\Lambda\mathrm{d}\eta \\
&= v^\top (f^{nc}(q,v) + G^\top_{J^1(q)}(q)\lambda)\mathrm{d}t + \frac{1}{2}(v^+ + v^-)^\top G^\top_{J^1(q)}(q)\Lambda\mathrm{d}\eta
\end{aligned}$$

Using (S1) and $-\lambda \in N_{\mathbb{R}^{m_1}_+}(G_{J^1(q)}(q)v) \Rightarrow v^\top G^\top_{J^1(q)}(q)\lambda = 0$ the weak derivative of the absolute continuous part is negative. That means $\dot{V}(q,v) \leq 0$. Furthermore, the discrete part in the impact points $s \in T^*(q, I)$ can be rewritten in form

$$\begin{aligned}
(v^+(s) + v^-(s))^\top G^\top_{J^1(q)}(q)\Lambda &= \frac{2}{1+\epsilon}(v^+ + \epsilon v^-)^\top G^\top_{J^1(q)}(q)\Lambda \\
&\quad - \left(\frac{2}{1+\epsilon} - 1\right)(v^+(s) - v^-(s))^\top G^\top_{J^1(q)}(q)\Lambda \\
&= \frac{2}{1+\epsilon}\cdot 0 - \left(\frac{2}{1+\epsilon} - 1\right)(v^+(s) - v^-(s))^\top M(q)(v^+ - v^-)
\end{aligned}$$

Since $\epsilon \in [0,1]$ and M is positive semi-definite, we get that the term is not positive. That means $V^+(q,v) - V^-(q,v) \leq 0$. Summarising, $\mathrm{d}V \leq 0$ can be concluded.

It remains to show (iii) in Theorem 4.21. We assume that there is an admissible (q,v) with $V(q,v) = 0$ and $(q,v) \neq (q^*, v^*)$. It follows $q = q^* = 0$ and $v \in \ker(M(q))$. The equation

$$0 = v^\top M(q)v = v^\top f(q,v) + v^\top G^\top_{J^1(q)}(q)\lambda = v^\top f(q,v)$$

contradicts (S2). So (iii) is also fullfilled and the equilibrium $(q^*, v^*) = (0, 0)$ is stable.

Let now $\mathrm{d}V = 0$. According to (S1) it follows $v(t) = 0, t \in I \backslash T^*(v, I)$, i.e., t is no impact point. In impact points $s \in T^*(v, I)$ it has to be $V^+(q, v) - V^-(q, v) = 0$ which conclude

$$\left(\frac{2}{1+\epsilon} - 1\right) v^+(s) - v^-(s) \in \ker M(q).$$

Since $\epsilon < 1$ this is equivalent to $v^+(s) - v^-(s) \in \ker M(q)$. Together with (S2) it can be concluded $v^+(s) - v^-(s) = 0$. So we have $v^+(s) = v^-(s) = v(s) = 0$ for all $s \in I$ and the differential measure reduces to

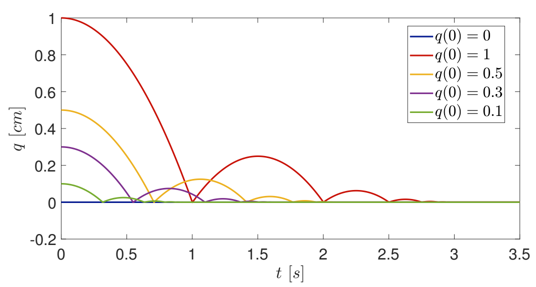
$$\mathrm{d}V = \frac{\partial}{\partial q} U(q) = 0.$$

The partial derivative of the potential energy is only zero if q^* is zero. Hence, the equilibrium is attractive. □

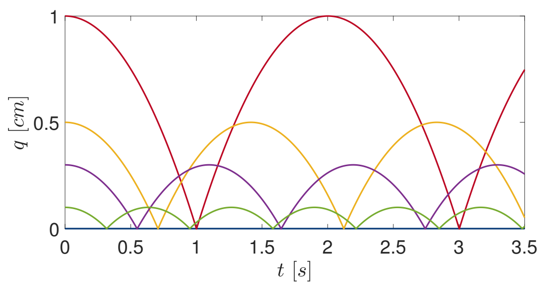
In the case $\epsilon = 1$, no generally valid result can be formulated.

Remark 4.25 (Stability and attractivity for other equilibria) The stability or attractivity of other equilibria $(q^*, v^*) \neq (0, 0)$ can be proven with the Lyapunov function $V((q, v) - (q^*, v^*))$. The fact that only the equilibrium $(0, 0)$ was considered in the previous theorems can only be justified by the more compact notation.

Example 4.26 (Bouncing ball) In Example 1.1, the motion of a bouncing ball with $M(q) = m > 0, f(q, v) = -mg, g(q) = q, \ \forall\, q \in \mathbb{R}^n$ is described. It follows $U(q) = mgq$ which is positive definite. Trivially M is symmetric and positive semi-definite and all functions are Lipschitz continuous. Since M has full rank, also (P5) is fulfilled. If the mass matrix is constant, we get $f(q, v) = -\frac{\partial}{\partial q} U(q)$ and so $f^{nc}(q, v) \equiv 0$. Therefore, (S1) is satisfied. The set $\ker(M(q)) \cap \{x \in \mathbb{R}^n \mid G_{J^1(q)}(q)v = v \geq 0\}$ contains only $v = 0$ since $M(q) = m > 0$. Hence, (S2) also follows. The trivial solution $(q(t), v(t)) \equiv (0, 0)$ is at all a solution of this benchmark problem with $T^*(v, I) = \emptyset, \lambda(t) = mg > 0$. This stable equilibrium is only attractive if $\epsilon < 1$. The following figure will underline this analytical result.



(a) $\epsilon = 0.5$

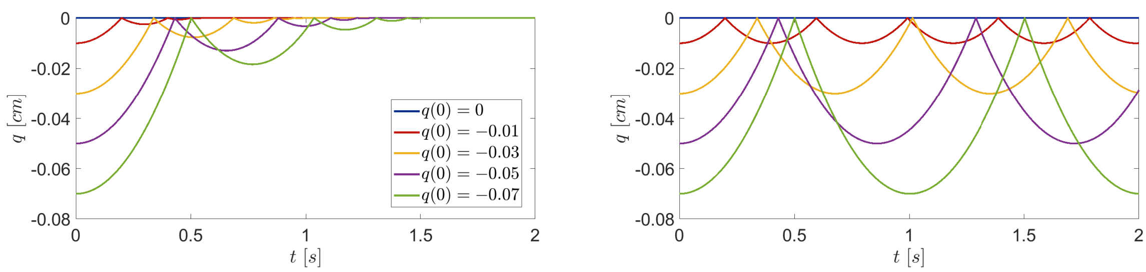


(b) $\epsilon = 1$

Figure 4.10.: Bouncing ball: Stability

Example 4.27 (Impact oscillator) For the impact oscillator we get from Equation (4.1) the functions $M(q) = 1$, $g(q) = \sigma - q$, $f(q, v) = -\frac{\partial}{\partial q} U(q) = p(t, q) - q$. Analogous to

Example 4.26, all requirements of Theorem 4.24 are fulfilled. One only has to presuppose that $(0, 0)$ is an equilibrium of the underlying MDI. Referring to Conclusion 4.6, it should be $p(t, 0) = 0$, for all $t \geq 0$ or $\sigma = 0, p(t, 0) \geq 0$ for all $t \geq 0$. Figure 4.11 underlines the expected attractivity for the equilibrium $(0, 0)$ when $\epsilon < 1$. For $\epsilon = 1$, the solution curve seems to be periodic.



(a) $\epsilon = 0.5$ (b) $\epsilon = 1$

Figure 4.11.: Oszillator: Stability with $p(q) = \frac{1}{2}\cos(q)$

Example 4.28 (Two-masses-spring system) We refer to Example 2.60 to describe the motion of a two-masses-spring system. Following Example 3.21, all smoothness assumptions in Theorem 4.24 are satisfied. Here the forces are given by

$$U(q) = q^\top \begin{pmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_2 \end{pmatrix} q, \quad f^{nc}(q, v) = (p(q), 0, 0)^\top, \quad f(q, v) = -\frac{\partial}{\partial q} U(q) + f^{nc}(q, v).$$

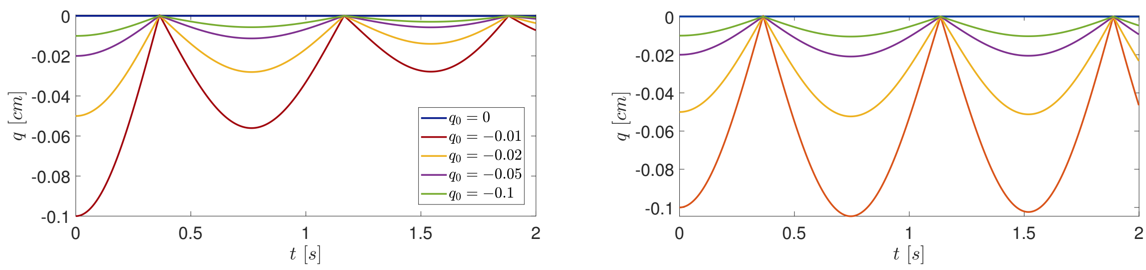
The state $(0, 0, 0)^\top$ is an equilibrium of this dynamical system if $p(0) \geq 0$. Referring to For $p(0) = 0$ we get an equilibrium in a resting position and for $p(0) > 0$ the first mass is pressed against the obstacle. The condition (S1) is also equivalent to these properties. If $v \in \ker(M(q))$ it follows $v_1 = 0$. Therefore,

$$v^\top f(q, v) = -k_2 v_3 q_3 \leq 0$$

with the positive stiffness parameter $k_2 > 0$ and the non-negative position $q_3 \geq 0$ of the seconde mass and the non-negative velocity $v_3 \geq 0$. Hence, (S2) is also fulfilled. If $(0, 0, 0)$ is an equilibrium of the two-masses-spring system, it is stable. As Figure 4.12 points out the numerical solutions converges to the equilibrium if $\epsilon < 1$. If $\epsilon = 1$ a periodic behaviour can be observed.

4.3. Stability Issues of Periodic Solutions

Now we want to investigate the stability of periodic solutions of autonomous systems. This means that F in LIMDI (3.1) cannot depend directly on the time variable t. Such systems also play a major role in the next chapter. As already stated in [74] for smooth systems, the previous notions of stability do not make sense for autonomous systems. It could be that a solution $x(t, x_0^1)$ to an initial value $x_0^1 \in Z$ differs only by a phase shift $\tau > 0$ from a second solution $x(t, x_0^2)$ to an initial value $x_0^2 \in Z$



(a) $\epsilon = 0.5$ (b) $\epsilon = 1$

Figure 4.12.: Two-masses-spring system: Stability with $p(q) = \sin(q_0)$

$$x(t+\tau, x_0^1) = x(t, x_0^2), t \geq 0.$$

The same qualitative behaviour is described by both solutions, but the definition of attractivity in Definition 4.20 is not fulfilled if it is not an equilibrium. We extend our understanding of stability by the following definitions.

Definition 4.29 (Orbital stability) Let $x^* : \mathbb{R}_+ \to Z$ be a ω-periodic solution of a LIMDI (3.1). It is called *orbital stable* if $\forall\ \mu > 0\ \exists\ \delta > 0$ such that for all initial values $x_0 \in Z$ with $\text{dist}(x_0, \gamma(x^*)) \leq \delta$ each solution $x(t, x_0)$ of LIMDI (3.1) with $x(0) = x_0$ satisfies

$$\text{dist}(x(t), \gamma(x^*)) \leq \mu, \quad \forall\ t \geq 0.$$

Definition 4.30 (Orbital attractivity) Let $x^* : \mathbb{R}_+ \to Z$ be a ω-periodic solution of a LIMDI (3.1). It is called *orbital attractive* if there exists a $\delta > 0$ such that for all initial values $x_0 \in Z$ with $\text{dist}(x_0, \gamma(x^*)) \leq \delta$ each solution $x(t, x_0)$ of LIMDI (3.1) with $x(0) = x_0$ satisfies

$$\lim_{t\to\infty} \text{dist}(x(t), \gamma(x^*)) = 0.$$

If no periodic solutions are considered, the limit set Φ must be used instead of γ. The condition $\lim_{t\to\infty} \text{dist}(x(t), A) = 0$ for a set $A \in \mathbb{R}^n$ is equivalent to

$$\exists\ a : I \to A : \lim_{t\to\infty} \|x(t) - a(t)\| = 0.$$

Theorem 4.31 (Orbital stability for periodic solutions) The ω-periodic solution $x^*(t, x_0) \in Z$ is orbital stable if there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}, V(x) := v(x) + \psi_Z(x)$ with v is continuous and positiv and a set $U(h) := \bigcup_{y\in\gamma(x^*)}(B(y,h) \cap Z)$ for an $h > 0$ with

(i) $V(x^*(t)) = 0, \forall\ t \geq 0$,

(ii) there is no $y \notin \gamma(x) : V(y) = 0$

(iii) $\mathrm{d}V(x) \leq 0, \forall\ x \in SBV^+(\mathbb{R}_+, U(h))$.

Proof: As in the proof of Theorem 4.21, we can construct a compact and positive invariant set C containing $\gamma(x^*)$. Let $\mu > 0$ be arbitrary and

$$V_{\mu/2} := \sup_{x\in U(\mu/2)} V(x), \qquad V_{h/2} := \sup_{x\in U(h/2)} V(x)$$

be suprema that upper semi-continuous functions takes on compact sets. Again there exists an $c^* > 0$ such that all sets $\Omega_c \cap Z$ with $\Omega_c := \{x \in \mathbb{R}^n \mid V(x) \leq c\} \neq \emptyset$ are closed and simply connected. Set $d := \min(V_{\mu/2}, V_{h/2}, c^*)$. According to condition (iii) the set $C := \Omega_d \cap Z$ is closed and positive invariant.

Let now be $U(\delta)$ be the largest neighbourhood which lies in $\Omega_c \cup Z_1^c$. Its existence follows from the fact that the closed set $\Omega_c \cap Z$ is not empty. It holds $\sup_{x \in U(\delta)} V(x) \leq c$ and so all solutions starting in $U(\delta)$ remains in $U(\mu)$. The periodic solution is orbital stable. □
The following theorem is stated in [59, Theorem 6.31].

Theorem 4.32 (LaSalle's Invariance principle) Let $C \subset Z$ be compact and positive invariant with respect to LIMDI (3.1) and $V(x) := v(x) + \psi_Z(x)$ a Lyapunov function candidate where v is continuous and bounded from below. Suppose furthermore $\mathrm{d}V(x) \leq 0$ for all solution curves of (3.1) starting in C and

$$B := \{x \mid x(t, x_0) \in C, \forall t \geq 0, A(x)\mathrm{d}x \in F(x)\}.$$

We define D as the largest positively invariant set in B. Then for every solution curve $x(t, x_0)$ of LIMDI (3.1) with $x_0 \in C$ the condition

$$\lim_{t \to \infty} \mathrm{dist}(x(t, x_0), D) = 0$$

is fulfilled.

Theorem 4.33 (Orbital attractivity for periodic solutions) The ω-periodic solution function $x^*(t, x_0) \in Z$ is orbital stable if there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}, V(x) := z(x) + \psi_Z(x)$ with z being continuous and positive and a set $U(h) := \bigcup_{y \in \gamma(x^*)} (B(y, h) \cap Z)$ for an $h > 0$ with

(i) $V(x^*(t)) = 0, \forall\, t \geq 0$,

(ii) $\mathrm{d}V(x) \leq 0, \forall\, x \in SBV^+(\mathbb{R}_+, U(h))$,

(iii) $dV(y) = 0 \Rightarrow y \in \gamma(x^*)$.

Proof: According to (i)-(iii) there can not be a $y \in Z \backslash \gamma(x^*)$ with $V(y) = 0$ and so x^* is stable. The set C constructed in the proof of Theorem 4.31 suits to Theorem 4.32. So condition

$$\lim_{t \to \infty} \mathrm{dist}(x(t, x_0), D) = 0$$

is fulfilled for the largest positive invariant set D belonging to B. Following (iii), B contains only elements of $\gamma(x^*)$ and $D = \gamma(x^*)$. The periodic solution is attractive. □

Remark 4.34 (Lyapunov function candidate) Let

$$(q^*, v^*) \in AC(\mathbb{R}_+, \mathbb{R}^n) \times SBV^+(\mathbb{R}_+, \mathbb{R}^n)$$

be a periodic solution of the equations of motion (2.27). Similar to Remark 4.23 a Lyapunov function candidate is

$$V(q, v) = T((q, v) - \varphi(q, v)) + U((q, v) - \varphi(q, v)) - U(0) + \psi_Z(q, v)$$

where $\varphi(q,v) = \arg(\min_{x\in\gamma(q^*,v^*)} \|(q,v)-x\|)$ is continuous. If all requirements of Theorem 4.24 are satisfied, this function fulfills the assumptions of Theorems 4.31 and 4.33. The periodic solution is orbital stable or even orbital attractive.

It should be mentioned that this proof strategy does not work for arbitrary solutions, since it is not certain whether $\gamma(x)$ and $\Phi(x)$ are bounded sets at all.

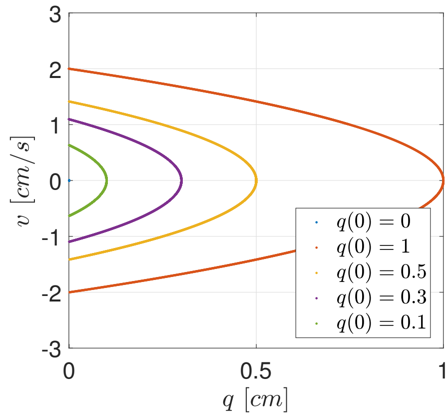
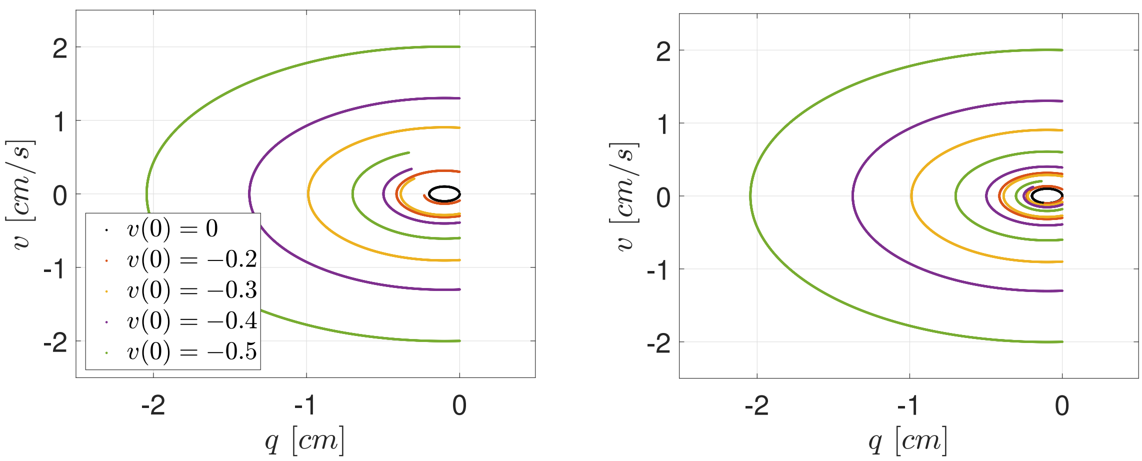


Figure 4.13.: Bouncing ball orbit: Stability with $\epsilon = 1$

Example 4.35 (Bouncing ball) The Bouncing ball has only periodic solutions if $\epsilon = 1$. The reverse is also true. If $\epsilon = 1$ holds, then the solution is periodic. These solutions are orbital stable, but never attractive.



(a) $T = 5\ s$ (b) $T = 10\ s$

Figure 4.14.: Impact oscillator: Orbits for different inital values after 5 and 10 seconds and $\epsilon = 0.7$

Example 4.36 (Impact oscillator and two-masses-spring system) An periodic solution $(q^*(t), v^*(t))$ of the impact oscillator (4.1) is orbitable stable if $p(q) \geq 0, \forall\ q \leq \sigma$, how it is justify in Example 4.11. For $\epsilon < 1$ it is orbital attractive.

The condition $p(q_1) \geq 0, \forall\ q \in Z$, states the results for the two-masses-spring system.

5. Orbital convergence

In this chapter, numerical methods for solving the equations of motion of non-smooth mechanics are discussed. We assume that the well-posedness assumptions of Chapter 3 hold to guarantee that an exact solution of the underlying measure differential inclusion exists and is unique in position, velocity and forces. The main aim of this chapter is to compare the approximation quality of the classical integration method of Moreau, Jean [51] with the approximation quality of the novel timestepping schemes constructed by Schindler, Acary [80]. The last methods base on the idea of discontinuous Galerkin methods and are developed to get higher order in phases without impacts. The classical convergence analysis of time-stepping methods for differential equation problems is based on strong smoothness assumptions. These are not satisfied for non-smooth mechanical problems. When measure differential inclusions are numerically solved with timestepping schemes, one difficulty, the peaking phenomen, arises. Nevertheless, it turns out in numerical tests that all mentioned methods converge and they can be compared with each other by their error bounds. We prove this analytically and transfer it to orbital convergence. This concept is well suited for autonomous problems with periodic solutions. A detailed overview on integration methods for non-smooth mechanics is given in [2].

5.1. Integration Methods for Non-Smooth Mechanics

There are two main groups of numerical methods to solve the equations of motion of non-smooth mechanics. The first one are the *event-driven* ones [2, 58, 62, 73]. They identify the next impact point exact and use up to this point a classical DAE-solver which can be of arbitrary order. In the impact point, the discrete impact problem is solved once. Then, the procedure is repeated over the entire time interval again and again. Those methods are very accurate when dealing with few critical points. If, however, the number increases or even the Zeno phenomenon arises, they get inconsistent [80]. For this situation, the second group of integration methods has been developed. The *timestepping schemes* define a sequence of time points

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$$

in $[0, T]$ and approximate the function values $q(t_i), v(t_i) \in \mathbb{R}^n, i = 0, 1, \ldots, N$, through numerical approximations $q_i, v_i \in \mathbb{R}^n, i = 0, 1, \ldots, N$. The partition of the time interval is never changed during the whole integration process and so it is independent of the location of impacts. The stepsize $h_i := t_i - t_{i-1}, i = 1, \ldots, N$, describes the distance between two consecutive time points. In this thesis, the stepsize is never adapted, such that $h_i \equiv h$. Timestepping schemes are robust and can have only order of convergence one which is explained in the second part of this section. For details about timestepping schemes we refer to [2, 91]. In the following, two very simple implementations of a mixed event-driven and a similar timestepping scheme applied to Example 1.1 will underline the advantages and the disadvantages of both approaches.

Algorithm 1 Naive mixed event-driven scheme for the bouncing ball

Require: time interval $[0, T]$, initial values $q_0 = q(0), v_0 = v(0)$, minimal number of intervals N, stepsize $h = T/N$, positive tolerances δ_1, δ_2

procedure EventDrivenBall$(T, N, q_0, v_0, \delta_1, \delta_2)$
  $i = 0$
  $t_0 = 0$
  $h_i = h$
  while $t_{i+1} < T$ **do**
    $v_{i+1} = v_i - h_i\ g$ ▷ Trapezoidal rule
    $q_{i+1} = q_i + h_i\ (v_i + v_{i+1})/2$
    if $q_{i+1} - r < -\delta_1$ and $v_{i+1} < 0$ **then**
      if $h_i < \delta_2$ **then** ▷ Impact identification with accuracy δ_2
        $v_{i+1} = -\epsilon v_i$
        $q_{i+1} = q_i + h_i\ (v_i + v_{i+1})/2$
        $h_{i+1} = h$
        $t_{i+1} = t_i + h_i$
        $i = i + 1$
      else $h_i = h_i/2$
      end if
    else $t_{i+1} = t_i + h_i; h_{i+1} = h; i = i + 1$
    end if
  end while
  return q, v
end procedure

Algorithm 2 Naive timestepping scheme for the bouncing ball

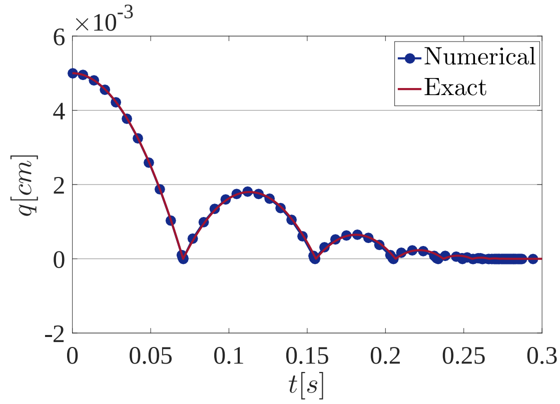
Require: time interval $[0, T]$, initial values $q_0 = q(0), v_0 = v(0)$, stepsize $h = T/N$, positive tolerance δ_1

procedure TimesteppingBall$(T, N, q_0, v_0, \delta_1)$
  $i = 0$
  $t_0 = 0$
  while $t_{i+1} < T$ **do**
    $v_{i+1} = v_i - h\ g$ ▷ Trapezoidal rule
    $q_{i+1} = q_i + h(v_i + v_{i+1})/2$
    if $q_i - r < -\delta_1$ and $v_{i+1} < 0$ **then** ▷ Impact identification
      $v_{i+1} = -\epsilon v_i$
    end if
    $t_{i+1} = t_i + h$
    $i = i + 1$
  end while
  return q, v
end procedure

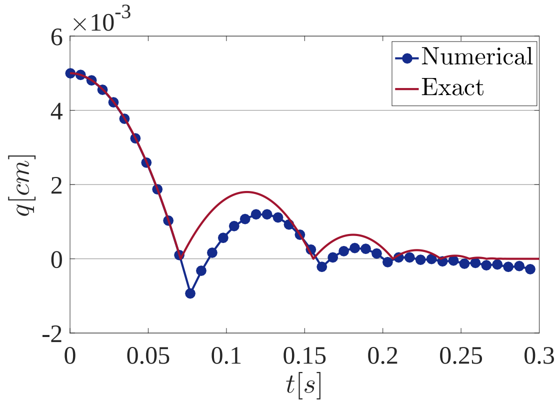
Example 5.1 (Bouncing ball) The generalised velocity $v(t) = \dot{q}(t)$ of a bouncing ball with centre of mass $q(t) \in \mathbb{R}, t \in [0, T]$, can be described through

$$\begin{aligned} \dot{v}(t) &= -g, && \text{if } q(t) - r > 0, \\ v^+(t) &= -\epsilon v^-(t), && \text{if } q(t) - r = 0, \end{aligned}$$

where $g > 0$ is the gravitational acceleration, $r > 0$ the radius of the ball and $\epsilon \in [0, 1]$ the impact number. If $\epsilon < 1$, the Zeno phenomenon arises and an event-driven scheme would fail. Therefore, only a mixed event-driven scheme in Algorithm 1 and a similar timestepping scheme in Algorithm 2 are used to solve these equations of motion. In smooth phases $(q(t) - r > 0)$, both methods apply the classical *trapezoidal rule* [90] to the differential system to get numerical approximations of order two for q and v in the time points (t_i). In $t_i \to t_{i+1}$, an impact is recognised if q_{i+1} will be smaller than r in a certain tolerance and $v_{i+1} < 0$ to exclude that the ball rests on the ground. Then, the timestepping scheme calculates $v_{i+1} = -\epsilon v_i$ directly and continues the numerical simulation with the next previously fixed integration step $t_{i+1} \to t_{i+2} = t_{i+1} + h$. In contrast the event-driven scheme locates the impact point $t^* \in [t_i, t_{i+1}]$ by reducing the stepsize to δ_2. If δ_2 is equivalent to the machine accuracy, Algorithm 1 would be a purely event-driven scheme. The order of convergence two of the trapezoidal rule can be transferred to the non-smooth phases, if $\delta_2 \in \mathcal{O}(h^2)$.



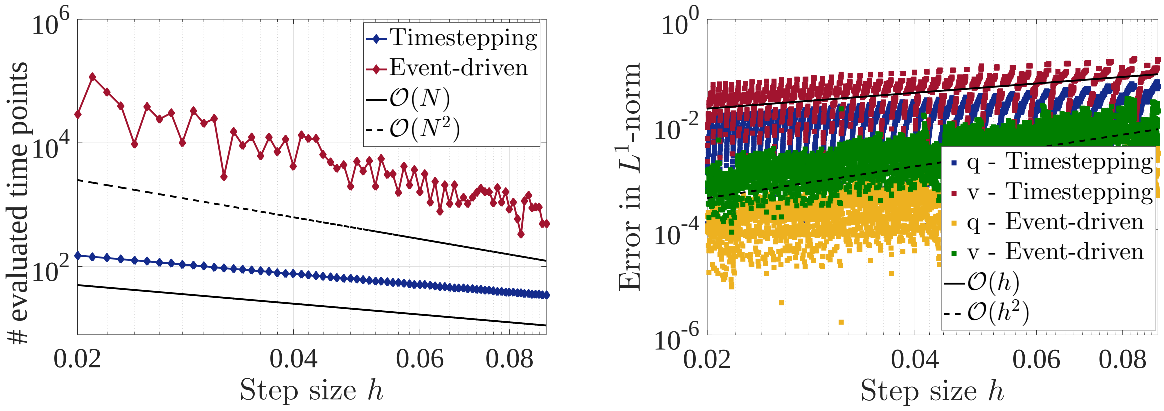
(a) Event-driven scheme in Algorithm 1



(b) Timestepping scheme in Algorithm 2

Figure 5.1.: Numerical solution of the bouncing ball with values $q_0 = 0.005$ cm, $v_0 = 0$ cm/s, $\epsilon = 0.6$, $h = 0.007$ s, $r = 0$ cm, $\delta_1 = 0, \delta_2 = 10^{-6}$ (V1).

Figure 5.1 underlines very well the benefits of the event-driven method with respect to the accuracy. If the numerical solution penetrated the constraint, the stepsize is reduced so that the impact time point is identified in the desired accuracy. The approximation of the exact solution is very accurate for this strategy. But especially for accumulation points of impacts, this requires the computation in a lot of additional time points.
In addition, Figure 5.2 emphasizes the differences between the two algorithms. The mixed method provides more accurate calculations with an error in accuracy $\mathcal{O}(h^2)$. The numerical solutions $q^h = (q_0, \ldots, q_N), v^h = (v_0, \ldots, v_N)$ have been compared with the exact solutions $q = (q(t_0), \ldots, q(t_N)), v = (v(t_0), \ldots, v(t_N))$ in L^1-norm. The timestepping method loses accuracy due to the inexact location of the impact, but instead of $\mathcal{O}(N^2)$ it only needs $\mathcal{O}(N)$ operations for the integration process. It is always recommendable to compare timestepping methods with smaller stepsizes with mixed ones with larger



(a) Number of evaluated time points

(b) L^1-norm

Figure 5.2.: Comparison of numerical parameters using Algorithm 1 and 2 with (V1).

stepsizes.

Including elements of event-driven schemes like in Algorithm 1 is only one strategy to increase the accuracy or order of timestepping schemes. More possibilities are described in the next section.

5.1.1. Timestepping Schemes

There are two maingroups of timestepping schemes: based on the ideas of Moreau, Jean [50, 51, 66] and based on the ideas of Paoli, Schatzman [70, 71]. The drawback of latter ones is that they formulate impact laws at position level and fulfil Newton's impact law only after several time steps. As we focus on applications that are significantly influenced by their behaviour during collisions, the focus is on the first group of methods.

Remark 5.2 (Timestepping schemes of Moreau-Jean type) The first timestepping schemes to solve the equations of motion of non-smooth mechanics are developed by *Moreau and Jean* [51]. To get numerical approximations $q_i \approx q(t_i), v_i \approx v(t_i)$, they use θ-schemes in smooth phases. It is not distinguished between contact and impact forces; both are always calculated with the impact law of Newton. If in $t_i \to t_{i+1}$ a contact or impact is recognised, v_i is taken as an approximation of the pre-impact velocity in the impact point. In this thesis only the explicit version in Algorithm 3 is considered. It uses the explicit Euler method in smooth phases to solve the underlying ODE. Since this method itself already has only *order of convergence one* for smooth systems, the timestepping method of Moreau-Jean type also has at most this order of convergence. This is proved among others in [29, 63].

Remark 5.3 (Higher-order timestepping schemes) Because the classical timestepping scheme in Algorithm 3 has only order of convergence one, in literature a lot of numerical possibilities are studied to increase the accuracy or the order of timestepping schemes based on Algorithm 3. There are different strategies

(i) **Augmented timestepping schemes**: this group of timestepping schemes are extended versions of methods of Moreau-Jean type (see Algorithm 3). If there is no contact or impact classical augmentation strategies like extrapolation or stepsize

Algorithm 3 Explicit Moreau-Jean timestepping scheme [51]

Require: time interval $[0, T]$, initial values $q_0 = q(0), v_0 = v(0)$, stepsize $h = T/N$
1: **procedure** MoreauJean$(T, N, q_0, v_0, \delta_1)$
2: $i = 0$
3: $t_0 = 0$
4: **while** $t_{i+1} < T$ **do**
5: $q_{i+1} = q_i + hv_i$
6: $M(q_i)(v_{i+1} - v_i) = h(f(q_i, v_i) + G(q_i)^T \Lambda)$ ▷ Solve lines 6 and 7 simultaneously
7: $\Lambda = \text{proj}_{\mathbb{R}_+^{m_1}}(\Lambda - G_{J^1(q_{i+1})}(q_i)(v_{i+1} + \epsilon v_i))$
8: $t = t + h$
9: $i = i + 1$
10: **end while**
11: **return** q, v
12: **end procedure**

adaptation are utilised to improve numerical approximations [43, 91]. Such strategies often deal with instabilities.

(ii) **Mixed timestepping** (cf. Algorithm 1): such timestepping schemes use stepsize adaptation in the non-smooth phases similar to event-driven schemes. They suffer often from lacking of appropriate strategies. Error estimations based on Richardson strategies as well as stepsize switching procedures $h_{\text{non-smooth}} = \mathcal{O}(h_{\text{smooth}})$ or retrospective bisections [1, 91] are not satisfactory understood and analysed.

Based on this unsatisfactory analysis and uncertain understanding of higher-order methods, in [80] a whole set of methods for the consistent numerical treatment of measure differential inclusions and their accuracy enhancement is developed.

Remark 5.4 (Timestepping schemes based on discontinuous Galerkin methods) To improve the behaviour of timestepping schemes in the smooth phases and to give a consistent treatment of contacts and impacts, Acary and Schindler [80] embedded these methods in the setting of *discontinuous Galerkin methods*. Their consideration is motivated by [57]. Discontinuous Galerkin methods are special mortar methods which use discontinuous ansatz and test functions in any situation not just if the situation demands it. The starting point is the weak formulation of the equations of motion

$$\begin{aligned} \langle \dot{q} \, \mathrm{d}t, \varphi_q \rangle &= \langle v \, \mathrm{d}t, \varphi_q \rangle \,, && \forall \, \varphi_q \in \mathcal{D}([0,T]), \\ \langle M(q)\mathrm{d}v, \varphi_v \rangle &= \left\langle f + G(q)^\top \lambda \, \mathrm{d}t + G(q)^\top \Lambda \, \mathrm{d}\eta, \varphi_v \right\rangle, && \forall \, \varphi_v \in \mathcal{D}([0,T]). \end{aligned}$$

The function space $\mathcal{D}([0,T])$ can be the space of all functions of bounded variation $BV([0,T], \mathbb{R}^n)$ and the product $\langle \cdot, \cdot \rangle$ is the primal-dual pairing. To get numerical approximations, $(r+1)$-dimensional subspaces of $\mathcal{D}([0,T])$ must be considered. The spaces $\Phi_q, \Phi_v \subset \mathcal{D}([0,T])$ are the spaces of test functions with bases $(\varphi_{q_k}), k = 0, 1, \ldots, r$, and (φ_{v_k}). In addition $\Psi_{\dot{q}}, \Psi_v$ with bases $(\psi_{\dot{q}_k}), (\psi_{v_k})$ are suitable $(r+1)$-dimensional spaces of ansatz functions for $\dot{q}$ and v. Then, the numerical solutions q^h, v^h can be presented by

$$q^h(t) = q_0 + \sum_{k=0}^{r} \int_{t_0}^{t} \psi_{\dot{q}_k}(s)\mathrm{d}s\ \dot{q}_k^h$$
$$v^h(t) = \sum_{k=0}^{r} \psi_{v_k}(t) v_k^h$$

with weights $(\dot{q}_k^h), (v_k^h)$. Inserting the ansatz and test functions in the weak formulation the discrete problem

$$\sum_{k=0}^{r} \langle \psi_{\dot{q}_k}, \varphi_{q_l} \rangle\, \dot{q}_k^h = \sum_{k=0}^{r} \langle \psi_{v_k}, \varphi_{q_l} \rangle\, v_k^h, \qquad \forall\ l = 0, 1, \cdots, r, \tag{5.1}$$

$$\sum_{k=0}^{r} M(q_k) \langle \mathrm{d}\psi_{v_k}, \varphi_{v_l} \rangle\, v_k^h = \left\langle f + G(q)^\top \lambda\ \mathrm{d}t + G(q)^\top \Lambda\ \mathrm{d}\eta, \varphi_{v_l} \right\rangle, \qquad \forall\ l = 0, 1, \cdots, r \tag{5.2}$$

must be solved. If the compact support of the test functions is chosen as $[t_i, t_{i+1}]$, every integration step $t_i \to t_{i+1}$ can be considered separately. Therefore, a one-step method is constructed. In [80] all ansatz and test function spaces are chosen to be the space of all piecewise polynomials of order r. An integration interval $I_i = [t_i, t_{i+1}]$ is divided by the Chebychev points (t_{i_l}) such that $\varphi_{q_l}, \varphi_{v_l}, \psi_{\dot{q}_l}, \psi_{v_l}$ can be chosen to be the *pruned Lagrange polynomials*

$$L_{i_l}(t) = \begin{cases} \prod_{j \neq l} \frac{t - t_{i_j}}{t_{i_l} - t_{i_j}}, & t \in I_i, \\ 0, & \text{elsewhere.} \end{cases}$$

It holds $L_{i_l}(t_{i_l}) = 1, L_{i_l}(t_{i_k}) = 0, k \neq l$, and that all these functions are continuous inside the interval (t_i, t_{i+1}). An important task is how to calculate the velocity jumps. In [80] only ansatz and test functions are considered that jump only one time per integration interval. The same holds then for $\dot{q}, v, \dot{v}$. It is distinguish between integration schemes with functions which evaluate the impact law at the left-side border or the right-side border of the integration interval. That means, if $L_{i_l}(t)$ is *left- or right-side continuous*. We only consider the last group to get the explicit formulation of this methods. Then, the numerical approximation is a right-side continuous function. In every integration step, the following stage values must be calculated

$$\begin{array}{llllllll} q_{i,0} & := q_i, & q_{i,1} & := q^h(t_{i_1}), & \dots & q_{i,r} & := q^h(t_{i_r}), & q_{i+1} := q_{i,r}, \\ \dot{q}_{i,0} & := \dot{q}_i^{h+}, & \dot{q}_{i,1} & := \dot{q}^h(t_{i_1}), & \dots & \dot{q}_{i,r} & := \dot{q}^{h-}(t_{i_r}), & \dot{q}_{i+1} := \dot{q}^{h+}(t_{i_r}), \\ v_{i,0} & := v_i^{h+}, & v_{i,1} & := v^h(t_{i_1}), & \dots & v_{i,r} & := v^{h-}(t_{i_r}), & v_{i+1} := v^{h+}(t_{i_r}), \\ \dot{v}_{i,0} & := \dot{v}_i^{h+}, & \dot{v}_{i,1} & := \dot{v}^h(t_{i_1}), & \dots & \dot{v}_{i,r} & := \dot{v}^{h-}(t_{i_r}), & \dot{v}_{i+1} := \dot{v}^{h+}(t_{i_r}). \end{array}$$

All products $\langle .,. \rangle$ are evaluated in [80] with the Clenshaw-Curtis quadrature. With the parameters

$$\beta_{i_k}(t_{i_l}) := \frac{1}{h} \int_{t_i}^{t_{i_l}} L_{i_k}(t)\mathrm{d}t, \qquad \beta_{i_k} := \beta_{i_k}(t_{i_r})$$

we get for $l = 0, 1, \ldots, r$ with relation (5.1) the following condition

$$q_{i,l} = q_{i,0} + \sum_{k=0}^{r} \int_{t_i}^{t_{i_l}} \psi_{\dot{q}_k}(t)\mathrm{d}t\ \dot{q}_k^h = q_{i,0} + \sum_{k=0}^{r} \int_{t_i}^{t_{i_l}} L_{i_k}(t)\mathrm{d}t\ \dot{q}_k^h$$
$$= q_{i,0} + h\sum_{k=0}^{r} \beta_{i_k}(t_{i_l})\dot{q}_k^h = q_{i,0} + h\sum_{k=0}^{r} \beta_{i_k}(t_{i_l}) v_{i,k}.$$

Because q and q^h are continuous, it holds $q_{i+1} = q_{i,r}$. Since $v^h, \dot{v}^h, \dot{q}^h$ are assumed to be continuous in $[t_i, t_{i+1})$, their values can be calculated in the same way. Referring to [57, 80], the derivative $\dot{v}^h$ of v^h with $\psi_{v_k} = L_{i_k}$ is equivalent to a polynomial of order $r-1$ of the form

$$\dot{v}^h(t) = \sum_{k=0}^{r-1} \tilde{L}_{i_k}(t)\dot{v}_{i,k}^h, \qquad \tilde{L}_{i_k}(t) := \prod_{j\neq k, j\neq r} \frac{(t - t_{i_j})}{(t_{i_k} - t_{i_j})}$$

Using also the Clenshaw-Curtis quadrature for (5.2) over $[t_i, t_{i_l}]$, it follows $M\dot{v}_{i-1,l} = f_{i_l} + G_{i_l}^\top \lambda_{i_l}$. Following this condition, you get for the velocity

$$M(v_{i,l} - v_{i,0}) = M\int_{t_i}^{t_{i_l}} \dot{v}^h(t)\mathrm{d}t = M\int_{t_i}^{t_{i_l}} \sum_{k=0}^{r-1} \tilde{L}_{i_k}(t)\dot{v}_{i,k}^h \mathrm{d}t$$
$$= \sum_{k=0}^{r-1} \int_{t_i}^{t_{i_l}} \tilde{L}_{i_k}(t)\mathrm{d}t\ M\dot{v}_{i,k}^h = h\sum_{k=0}^{r-1} \tilde{\beta}_{i_k}(t_{i_l})(f_{i_k} + G_{i_k}^\top \lambda_{i_k})$$

with $l = 0, 1, \ldots, r$

$$\tilde{\beta}_{i_k}(t^*) := \int_{t_i}^{t^*} \tilde{L}_{i_k}(t)\mathrm{d}t.$$

If we now include the impact at the end of the interval, the variation in the right-side border of the interval is described in [80] as

$$M(v_{i+1} - v_{i,r}) + M\dot{v}^h = h\beta_{i_r}(f_{i_r} + G_{i_r}^\top \lambda_{i_r}) + G_{i_r}^\top \Lambda_{i_r}.$$

The last stage value can be calculated by

$$M(v_{i+1} - v_{i,r}) = h\beta_{i_r}\left(f_{i_r} + G_{i_r}^\top \lambda_{i_r} - \sum_{k=0}^{r-1} \tilde{L}_{i_k}(t_{i+1})(f_{i_k} + G_{i_k}^\top \lambda_{i_k})\right) + G_{i_r}^\top \Lambda_{i_r}$$

Summarizing all calculations, you get Algorithm 4. The contact and impact forces are calculated with the contact or impact law based on the functional values in the Chebychev points.

Definition 5.5 (Matrix notation) In every integration step $t_i \to t_{i+1}$ numerical approximations in the Chebychev points $t_i = t_{i_0} < t_{i_1} < \ldots < t_{i_r} = t_{i+1}$ are calculated where $q_{i,l} \approx q(t_{i_l}), v_{i,l} \approx v^+(t_{i_l}), \lambda_{i,l} \approx \lambda(t_{i_l})$. With the vector-valued notation

$$\underline{q}_i := (q_{i,0}^\top\ q_{i,1}^\top\ \ldots\ q_{i,r}^\top)^\top, \qquad \underline{v}_i := (v_{i,0}^\top\ v_{i,1}^\top\ \ldots\ v_{i,r}^\top)^\top, \qquad \underline{\lambda}_i := (\lambda_{i,0}^\top\ \lambda_{i,1}^\top\ \ldots\ \lambda_{i,r}^\top)^\top,$$

Algorithm 4 can be written in a *compact matrix-version*, see Algorithm 5. The following

Algorithm 4 Forecasting timestepping scheme [80]

Require: time interval $[0, T]$, initial values $q_0 = q(0), v_0 = v(0)$, stepsize $h = T/N$

1: **procedure** TimesteppingDiscontinuousGalerkin$(T, N, q_0, v_0, \delta_1)$
2: $\quad i = 0$
3: $\quad t_0 = 0$
4: $\quad$ **while** $t_{i+1} < T$ **do**
5: $\qquad \triangleright$ Solve simultaneously lines 6,7,8 for $l = 0, 1, \dots, r$
6: $\qquad q_{i,l} = q_i + h \sum_{k=0}^{r} \beta_{i_k}(t_{i_l}) v_{i,k}$
7: $\qquad M(q_{i,l})(v_{i,l} - v_i) = h \sum_{k=0}^{r-1} \tilde{\beta}_{i_k}(t_{i_l})(f(q_{i,k}, v_{i,k}) + G^T(q_{i,k})\lambda_{i_k})$
8: $\qquad \lambda_{i_l} = \mathrm{proj}_{\mathbb{R}^m}(\lambda_{i_l} - g(q_{i,l}))$
9: $\qquad q_{i+1} = q_{i_r}$
10: $\qquad \triangleright$ Solve simultaneously lines 11,12,13,14
11: $\qquad M(q_{i+1})(v_{i+1} - v_{i,r}) = h\ \beta_{i_r}\Big((f(q_{i,r}, v_{i,r}) + G^T(q_{i,r})\lambda_{i_r})$
12: $\qquad\qquad - \sum_{k=0}^{r-1} \tilde{L}_{i_k}(t_{i+1})(f(q_{i_k}, v_{i_k}) + G^T(q_{i,k})\lambda_{i_k})\Big) + G^T(q_i)\Lambda_i$
13: $\qquad \lambda_{i_r} = \mathrm{proj}_{\mathbb{R}^m_+}(\lambda_{i_r} - g(q_{i,r}))$
14: $\qquad \Lambda_i = \mathrm{proj}_{\mathbb{R}^{m_1}_+}(\Lambda_i - G_{J^1(q_{i_r})}(q_{i_r})(v_{i+1} + \epsilon v_{i,r}))$
15: $\qquad t_{i+1} = t_i + h$
16: $\qquad i = i + 1$
17: $\quad$ **end while**
18: $\quad$ **return** q, v
19: **end procedure**

matrices are needed

$$
\begin{aligned}
B_r = (b_{l,k}) &:= (\beta_{i_{k-1}}(t_{i_{l-1}}))_{l=1,\dots,r+1,k=1,\dots,r+1}, & B_r^n &:= B_r \otimes I^n, \\
\tilde{B}_r = (\tilde{b}_{l,k}) &:= (\tilde{\beta}_{i_{k-1}}(t_{i_{l-1}}))_{l=1,\dots,r,k=1,\dots,r+1}, & \tilde{B}_r^n &:= \tilde{B}_r \otimes I^n, \\
\underline{M}_r &:= \mathrm{diag}(M(q_{i,0})\ M(q_{i,1})\ \dots\ M(q_{i,r})), & \tilde{L}_r &:= (\tilde{L}_{i_{k-1}}(t_{i+1}))_{k=1,\dots,r}, \\
\underline{G}_r &:= \mathrm{diag}(G(q_{i,0})\ G(q_{i,1})\ \dots\ G(q_{i,r})), & \underline{g}_r &:= (g^\top(q_{i,0})\ g^\top(q_{i,1})\ \dots\ g^\top(q_{i,r}))^\top, \\
\underline{f}_r &:= (f^\top(q_{i,0})\ f^\top(q_{i,1})\ \dots\ f^\top(q_{i,r}))^\top.
\end{aligned}
$$

Example 5.6 (Forecasting trapezoidal rule) For $r = 0$ no rule exists but a popular choice is $t_{i_0} = t_i$ such that you get the classical scheme of Moreau and Jean in Algorithm 3. For $r = 1$, Algorithm 4 reduces to the *trapezoidal rule* in Algorithm 6 which was derived in [80] and expanded and numerically studied in [82]. The coefficients $B_1, \tilde{B}_1$ and $\tilde{L}_1$ are

$$
B_1^\top = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.5 \end{pmatrix}, \qquad \tilde{B}_1^\top = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad \tilde{L}_1^\top = \begin{pmatrix} 1 \end{pmatrix}. \tag{5.3}
$$

Example 5.7 (Timestepping scheme of order three and four) Using all considerations in [80], we developed the schemes belonging to $r = 2$ and $r = 3$. They represent the methods constructed by [80] of order 3 and 4 in smooth phases. The coefficients of the method of order 3 are

Algorithm 5 Forecasting timestepping scheme [80] in matrix notation

Require: time interval $[0, T]$, initial values $q_0 = q(0), v_0 = v(0)$, stepsize $h = T/N$
1: **procedure** TimesteppingDiscontinuousGalerkinMatrix$(T, N, q_0, v_0, \delta_1)$
2: $i = 0$
3: $t_0 = 0$
4: **while** $t_{i+1} < T$ **do**
5: $\underline{q}_i = 1^r \otimes q_i + hB^n \underline{v}_i$ ▷ Solve simultaneously lines 5-10
6: $\underline{M}_r(\underline{v}_i - 1^r \otimes v_i) = h\tilde{B}^n_r(\underline{f}_{r-1} + \underline{G}^\top_{r-1}\underline{\lambda}_{i,1:r-1})$
7: $M(q_{i,r})(v_{i+1} - v_{i,r}) = h\beta_{i_r}(f(t_{i+1}, q_{i,r}) + G^\top(q_{i,r})\lambda_{i_r} - \tilde{L}(\underline{f}_{r-1} + \underline{G}^\top_{r-1}\underline{\lambda}_{i,1:r-1}))$
8: $+G^\top(q_{i,r})\Lambda_i$
9: $\underline{\lambda}_i = \text{proj}_{\mathbb{R}^{m*r}_+}(\underline{\lambda}_i - \underline{g})$
10: $\Lambda_i = \text{proj}_{\mathbb{R}^{m_1}_+}(\Lambda_i - G_{J^1(q_{i,r})}(q_{i,r})(v_{i+1} + \epsilon v_{i,r}))$
11: $q_{i+1} = q_{i,r}; v_{i+1} = v_{i+1}; t_{i+1} = t_i + h$
12: $i = i + 1$
13: **end while**
14: **return** q, v
15: **end procedure**

Algorithm 6 Forecasting trapezoidal rule [80]

Require: time interval $[0, T]$, initial values $q_0 = q(0), v_0 = v(0)$, stepsize $h = T/N$
procedure TrapezoidalRule$(T, N, q_0, v_0, \delta_1)$
$i = 0$
$t_0 = 0$
while $t_{i+1} < T$ **do**
$M(q_i)(v_{i,1} - v_i) = h(f(q_i, v_i) + G^T(q_i)\lambda_i)$
$\lambda_i = \text{proj}_{\mathbb{R}^m}(\lambda_i - g(q_i))$
$q_{i+1} = q_i + h(v_i + v_{i,1})/2$
$M(q_{i+1})(v_{i+1} - v_{i,1}) = h\left((f(q_i, v_i) + G^T(q_i)\lambda_i) + (f(q_{i+1}, v_{i,1}) + G^T(q_{i+1})\lambda_{i_1})\right)/2$
$+G^T(q_i)\Lambda_i$
$\lambda_{i_1} = \text{proj}_{\mathbb{R}^m}(\lambda_{i_1} - g(q_{i+1}))$
$\Lambda_i = \text{proj}_{\mathbb{R}^{m_1}}(\Lambda_i - G_{J^1(q_i)}(q_i)(v_{i+1} + \epsilon v_{i,1}))$
$t_{i+1} = t_i + h$
$i = i + 1$
end while
return q, v
end procedure

$$B_2^\top = \begin{pmatrix} 0 & 5/24 & 1/6 \\ 0 & 1/3 & 2/3 \\ 0 & -1/24 & 1/6 \end{pmatrix}, \qquad \tilde{B}_2^\top = \begin{pmatrix} 0 & 1/4 & 0 \\ 0 & 1/4 & 1 \end{pmatrix}, \qquad \tilde{L}_2^\top = \begin{pmatrix} -1 \\ 2 \end{pmatrix}. \tag{5.4}$$

In the following, this scheme is called Simpson method. The coefficients of the method of order 4 are

$$\begin{aligned} B_3^\top &= \begin{pmatrix} 0 & (8\sqrt{2}-5)/96 & -(8\sqrt{2}+5)/96 & -1/6 \\ 0 & (32-17\sqrt{2})/96 & (23\sqrt{2}+32)/96 & 2/3 \\ 0 & (32-23\sqrt{2})/96 & (17\sqrt{2}+32)/96 & 2/3 \\ 0 & (8\sqrt{2}-11)/96 & -(8\sqrt{2}+11)/96 & -1/6 \end{pmatrix}, \\ \tilde{B}_3^\top &= \begin{pmatrix} 0 & (\sqrt{2}-1)/6 & -(1+\sqrt{2})/6 & -1/3 \\ 0 & (8-3\sqrt{2})/48 & (24+17\sqrt{2})/48 & (4+\sqrt{2})/6 \\ 0 & (24-17\sqrt{2})/48 & (8+3\sqrt{2})/48 & (4-\sqrt{2})/6 \end{pmatrix}, \qquad \tilde{L}_3^\top = \begin{pmatrix} 1 \\ -\sqrt{2} \\ \sqrt{2} \end{pmatrix}. \end{aligned} \tag{5.5}$$

Remark 5.8 (Numerical treatment of non-linear terms) The numerical solution of contact and impact laws in the discretised form is a research topic of current interest. Detailed studies on the various aspects of the numerical treatment of non-smooth mechanics can be found in [2, 88, 91]. On one hand, there are different, *analytically equivalent formulation*s that result in different solution methods. In detail, the equivalent problems are discussed in the appendix. The force laws can be a *complementarity problem*, a *non-smooth equation* following the projected formulation or a *non-linear optimisation problem*. The oldest method to solve a complementarity problem is the direct method of Lemke [25, 67, 89]. It is similar to the simplex method for convex optimization problems. However, in comparison to other methods, it is considerably less robust and requires more computing time. For mechanical problems with redundant constraints or singular mass matrices, you can not use this algorithm. Further methods based on these analytic assumptions and solving either an equivalent optimisation problem or a root finding problem resulting from projection functions approaches are generalized Gauss-Jacobi or Gauss-Seidel projection methods [4], augmented Lagrangian methods [91], Krylov subspace methods [45] and inner point methods [53]. Due to the simple structure, the convergence properties described in the appendix, and the handling of redundant constraints and singular mass matrices, we selected *non-smooth Newton methods* to solve non-smooth equations.

On the other hand, the constraints can be formulated on *position, velocity or acceleration level*. A physically consistent treatment of the impact law on position level is not possible if the law of Newton is used. Considering only non-impulsive forces, it is possible to formulate the constraints on position level to avoid the *drift-off effect*. This effect describes the increasing violation of the constraints over time. It has already been described in [5] that for differential systems with constraints, this disadvantage increases as one proceeds in the formulation from position to velocity or from velocity to acceleration level. Another advantage of the formulation on velocity level is that friction laws can be easily included. Therefore, from our point of view, it is most appropriate to perceive the formulation of constraints at velocity level. Similar decisions are made in [80, 82].

Remark 5.9 (Non-smooth Newton method) To get a solution in one integration step

$t_i \to t_{i+1}$ of Algorithm 4, a *non-smooth Newton method* is used. Here similar to [83] we formulate the integration scheme for one non-smooth Newton step. We use the scheme in matrix notation to be able to formulate it in an elegant way. In each integration step, one must solve a system of non-linear equations of the form

$$0_{2(r+1)n+n+\bar{m}(r+1)+m_1} = \varphi(x_{i+1}) :=$$
$$\begin{pmatrix} \underline{q}_i - 1^n \otimes q_i - hB_r^n \underline{v}_i \\ \underline{M}_r(\underline{v}_i - 1^n \otimes v_i) - h\tilde{B}_r^n(\underline{f}_{r-1} + \underline{G}_{r-1}^\top \underline{\lambda}_{i,1:r-1}) \\ M_{i_r}(v_{i+1} - v_{i_r}) - h\beta_{i_r}(f_{i_r} + G_{i_r}^\top \lambda_{i,r} - \tilde{L}(\underline{f}_{r-1} + (\underline{G}^\top \underline{\lambda}_i)_{1:r-1})) - G_{i_r}^\top \Lambda_i \\ \underline{\lambda}_i - \mathrm{proj}_{\mathbb{R}_+^{\bar{m}*r}}(\underline{\lambda}_i - (\bar{G}\bar{v}_i)_{J^1}) \\ \Lambda_i - \mathrm{proj}_{\mathbb{R}_+^{m_1}}(\Lambda_i - G_{J^1(q_{i_r})}(q_{i_r})(v_{i+1} + \epsilon v_{i_r})) \end{pmatrix}$$

with respect to unknown variables

$$x_{i+1} = \begin{pmatrix} \underline{q}_i^\top & \underline{v}_i^\top & v_{i+1}^\top & \underline{\lambda}_i^\top & \Lambda_i^\top \end{pmatrix}^\top .$$

Using the non-smooth Newton method

$$\left\| \left(\frac{\partial^C \varphi(x)}{\partial x} \right) \Bigg|_{x = x_{i+1}^k} (x_{i+1}^{k+1} - x_{i+1}^k) - \varphi(x_{i+1}^k) \right\| \to \min$$

one can derive a Newton sequence that converges to the root of φ. We use the Clarke differential since proj is continuous but not differentiable in every point. In every step $k \to k+1$, a least square problem has to be solved. Since the Lagrange multipliers are not unique, this problem also has no unique solution if there are redundant constraints. As a starting point of the sequence we set

$$x_{i+1}^0 = ((1^n \otimes q_i)^\top, (1^n \otimes v_i)^\top, 0_{\bar{m}*r}^\top, 0_{m_1}^\top)^\top.$$

We consider here only constant mass matrices M and right hand sides f, which depend only on the position q. We define

$$\partial \underline{f}_r := \left(\partial f^\top(q)/\partial q|_{q=q_{i_0}} \; \partial f^\top(q)/\partial q|_{q=q_{i_1}} \; \ldots \; \partial f^\top(q)/\partial q|_{q=q_{i_r}} \right),$$
$$\partial \underline{G^\top \lambda}_r := \left(\partial G(q)^\top \lambda/\partial q|_{q=q_{i_0},\lambda=\lambda_{i_0}} \; \ldots \; \partial G(q)^\top \lambda/\partial q|_{q=q_{i_r},\lambda=\lambda_{i_r}} \right).$$

A Jacobian matrix $J\varphi(x) \in \frac{\partial^C \varphi(x)}{\partial x}$ calculates then to

$$J\varphi(x) = \begin{pmatrix} I^{(r+1)n} & -hB_r^n & 0 & 0 & 0 \\ jp_1 & \underline{M}_r & 0 & \left[-h\tilde{B}_r^n \underline{G}_{r-1}^\top \; 0\right] & 0 \\ jp_2 & [0 \ldots 0 \; -M(q_{i,r})] & M(q_{i,r}) & h\beta_{i_r}[\tilde{L} \; -I^n]\underline{G}^\top & -G^T(q_{i,r}) \\ jp_3 & 0 & 0 & jp_4 & 0 \\ jp_5 & jp_6 & jp_7 & 0 & jp_8 \end{pmatrix}$$

with

$$jp_1 = \left[-h\tilde{B}_r^n \left(\partial \underline{f} + \partial \underline{G^\top \lambda}\right)_{r-1} \;\; 0\right]$$

$$jp_2 = \left[-\tilde{L} \;\; - h\beta_{i_r} I_n\right] \left(\partial \underline{f}_r + \partial \underline{G^\top \lambda}_r\right) + \left[0 \;\; - \frac{\partial (G^T \Lambda)}{\partial q}|_{q=q_{i,r},\Lambda=\Lambda_i}\right]$$

$$(jp_3)_{j,:} = \begin{cases} (0^{(r+1)n})^\top, & \mathrm{proj}_{\mathbb{R}_+}(\underline{\lambda}_j - \left((\bar{G}\bar{v}_i)_{J^1}\right)_j) > 0 \\ \underline{G}_{i,:}, & \text{otherwise} \end{cases} \quad j = 1, \ldots, (r+1)m$$

$$(jp_4)_{j,j} = \begin{cases} 0, & \mathrm{proj}_{\mathbb{R}_+}(\underline{\lambda}_j - \left((\bar{G}\bar{v}_i)_{J^1}\right)_j) > 0 \\ 1, & \text{otherwise} \end{cases} \quad j = 1, \ldots, (r+1)m$$

$$p = \mathrm{proj}_{\mathbb{R}_+}\left(\Lambda_i - \left[G_{J^1(q_{i,r})}(q_{i,r})(v_{i+1} + \epsilon v_{i_r})\right]\right)$$

$$(jp_5)_{j,rn+1:(r+1)n} = \begin{cases} \left.\frac{\partial G_{J^1(q_{i,r})}(q)(v_{i+1}+\epsilon v_{i,r})}{\partial q}\right|_{q=q_{i,r}}, & p_j > 0 \\ (0^n)^\top, & \text{otherwise} \end{cases}$$

$$(jp_6)_{j,rn+1:(r+1)n} = \begin{cases} \epsilon\left[G_{J^1(q_{i,r})}(q_{i,r})\right]_{j,:}, & p_j > 0 \\ (0^n)^\top, & \text{otherwise} \end{cases}$$

$$(jp_7)_{j,:} = \begin{cases} \left[G_{J^1(q_{i,r})}(q_{i,r})\right]_{j,:}, & p_j > 0 \\ (0^n)^\top, & \text{otherwise} \end{cases}$$

$$(jp_8)_{j,j} = \begin{cases} 0, & p_j > 0 \\ 1, & \text{otherwise} \end{cases}$$

The terms jp_4 and jp_8 are diagonal matrices. In Theorem B.23 in the appendix, it is captured that the non-smooth Newton method converge for all elements of the Clarke differential $\frac{\partial^C \varphi(x)}{\partial x}$. In this thesis, the following element is choosen

$$\left.\frac{\partial}{\partial x}\mathrm{prox}_{\mathbb{R}_+}(f(x))\right|_{x_{i+1}^k} = \begin{cases} \frac{\partial}{\partial x} f(x), & f(x_{i+1}^k) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

5.1.2. Classical Comparison Tool: Local and Global error

We want to solve numerically an autonomous MDI on $I = [0, T]$

$$\begin{aligned} T^*(x, I) :=& \{t \in I \mid \exists\, i \in \{1, \ldots, m\} : (g_i(x(t)) = 0) \wedge \\ & (\exists\, \delta > 0\ \forall\, s \in [t - \delta, t) : g_i(x(s)) > 0)\}, \\ t \notin T^*(x, I) :\quad & \dot{x}(t) = \varphi_1(x), \\ t \in T^*(x, I) :\quad & x^+(t) - x^-(t) = G_{J^1(x^-(t))}(x^-(t)^\top \varphi_2(x^-(t)), \\ & G_{J^1(x^-(t))}(x^-(t))x^+(t) = -\epsilon G_{J^1(x^-(t))}(x^-(t))x^-(t), \\ x(0) =& x_0, \end{aligned} \tag{5.6}$$

and $\varphi_1 \in C^0(\mathbb{R}^{2n})$ representing the continuous but not necessarily differentiable forces and φ symbolising the discontinuous impact forces. We use this compact form of MDIs for reasons of readability.

If $T^*(x, I) = \emptyset$, system (5.6) would be an ODE. The methods which are discussed in Section 5.1.1 would solve an ODE with order $p \geq 1$ by construction (see Deuflhard,Bornemann [27]). It is not proven whether they solve differential inclusions with the same order. We reduce in the first step MDIs to an ODE with an impact conditions as it is formulated in (5.6). One advantage is that the methods of 5.1.1 would identify every impact point, if the step sizes are small enough, and would not identify an impact falsely. This can happened, if we would combine (5.6) with contact conditions like $g(x(\tau)) = 0$, $\frac{\partial}{\partial x}g(x(\tau)) = 0$. In this situtation, two bodies touch each other and no impact happens. A numerical scheme could falsely identify an impact.

The functions φ_1, φ_2 are Lipschitz continuous with Lipschitz constants L_1, L_2 and bounded with upper bounds β_1, β_2. We assume that the solution of (5.6) is uniquely determined. Furthermore, let $0, T \notin T^*(x)$ hold, i.e.

$$g(x_0) > 0, \qquad g(x(T)) > 0.$$

Thus there must exist a $\delta > 0$ with $t \notin T^*(x, I)$ for all $t \in (0, \delta) \cup (T - \delta, T)$. For proof of convergence for time-stepping methods in Theorem 5.25, we first have a look on solutions x of (5.6) with only finitely many points of discontinuity

$$0 < \tau_1 < \tau_2 < \ldots < \tau_m < T$$

with $\tau_i \in T^*(x, I)$. This means that in the first step of developing a comparison criterion for numerical methods we will not consider the Zeno phenomenon. The bouncing ball and the impact oscillator are problems of the form (5.6) and it is possible to find $T > 0$ such that $|T^*(x, I)| = m \in \mathbb{N}$ as explained in Chapter 4.

If the solution $x(t)$ of (5.6) fulfills the inequality $g(x(t)) > 0$ for all $t \in [0, T]$, the system (5.6) would be equivalent to an ODE. Hence, it is plausible to use timestepping methods which are originally devoloped for ODEs and then combined with the impact constraints. The timestepping schemes presented in Section 5.1.1. proceed in time steps $t_i \to t_{i+1} = t_i + h$ of equidistant stepsize

$$h = \frac{T}{N}, N \in \mathbb{N},$$

from an initial state x_0 in $t = 0$ to T. A sequence $(x_i)_{i=1}^N$ with $x_i \approx x(t_i)$ is computed. An abstract version of the methods from Section 5.1.1., which we examine here, is

$$
\begin{aligned}
&\tilde{x}_{i+1} = x_i + h\Phi(t_i, x_i; h, \varphi_1), \qquad n = 0, 1, \ldots, N-1\\
&\text{if } g(\hat{x}_{i+1}) \leq 0:\\
&\quad \hat{x}_{i+1} = x_i + h\Phi\left(t_i, x_i; \frac{r \cdot h}{r+1}, \varphi_1\right)\\
&\quad x_{i+1} = \hat{x}_{i+1} + h\Phi\left(t_i + \frac{r \cdot h}{r+1}, \hat{x}_{i+1}; \frac{h}{r+1}, \varphi_1\right) + G_{J^1(\hat{x}_{i+1})}(\hat{x}_{i+1})\varphi_2(\hat{x}_{i+1})\\
&\quad G_{J^1(\hat{x}_{i+1})}(\hat{x}_{i+1})x_{i+1} = -\epsilon G_{J^1(\hat{x}_{i+1})}(\hat{x}_{i+1})\hat{x}_{i+1}\\
&\text{else}\\
&\quad x_{i+1} = \tilde{x}_{i+1}
\end{aligned}
\tag{5.7}
$$

Here, Φ is the increment function of the numerical method for the ODE part of (5.6) of order $p \geq 1$ for smooth enough right hand sides which is also Lipschitz continuous with Lipschitz constant L_Φ and bounded by β_3. The parameter $r \in \mathbb{N}$ is the polonymial degree of the ansatz of test functions which are used to construct the timestepping methods.

If there is no impact in $[0,T]$, the numerical solution $(x_i)_{i=1}^N$ has a *global error*

$$\varepsilon := \max_{1\leq i\leq N} \|x_i - x(t_i)\| = \mathcal{O}(h^p).$$

We use here the supremum norm to underline that the error can be constant even if $h \to 0$. The classical convergence analysis of timestepping schemes supposes that the right-hand side is at least p times continuously differentiable on $[0,T]$ to get order p, see [27]. Since this assumption is violated for non-smooth problems, this error estimate can not be used. In the following section, we work out the error bounds observed in practice for systems (5.6) with the help of different examples (see also [80]). We prove this error bounds by using not the classical convergence. We will be explain later why we switch to orbital convergence.

As in [80], we take a look at the global error of numerical solutions of a ball in free fall, a ball in rest mode and a jumping ball. The respective equations of motion increase in complexity from an ordinary differential equation to a differential inclusion to a measure differential inclusion.

Problem 5.10 (Bouncing ball: free flight) First, we consider the equations of motion of a ball in free fall. An obstacle does not exist. The equations of motion result in the following *ODE*

$$q(0) = 1, \qquad v(0) = 0, \qquad \dot{q} = v, \qquad \dot{v} = -10t^4.$$

The analytical solution is given by

$$q(t) = 1 - \frac{1}{3}t^6, \qquad v(t) = -2t^5.$$

Algorithm 3 shows numerically convergence order one. The timestepping schemes based on parameters (5.3) and (5.4) have order two and three (cf. Figure 5.3). We does not use the realistic formulation $\dot{v}(t) = g$ where g is the acceleration of gravity because then no difference between the algorithms could be observed.

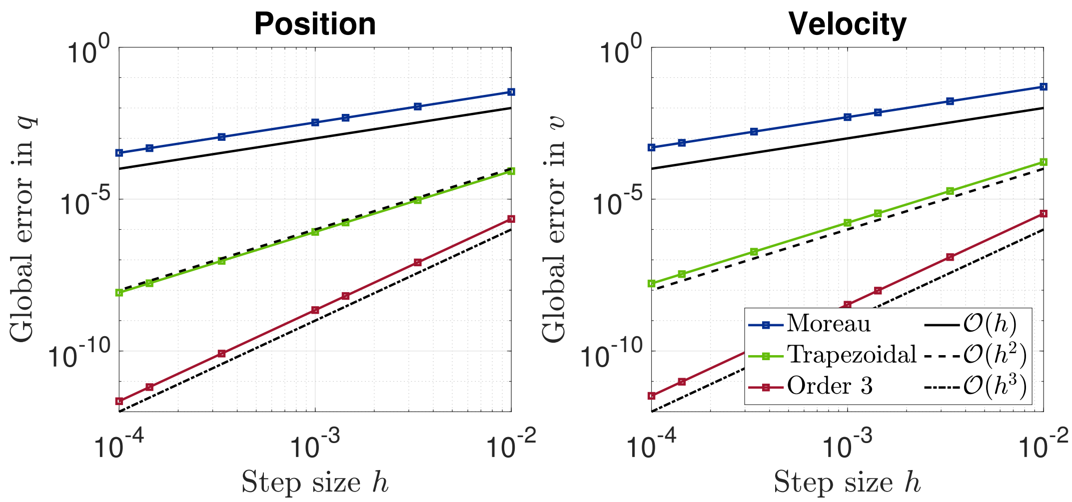


Figure 5.3.: Experimental convergence analysis for the bouncing ball: free flight.

Problem 5.11 (Bouncing ball: Rest phase) Now the dynamics of a ball is considered which starts in a resting position on a ground obstacle. It is influenced by the gravitational force. To guarantee the non-penetration of obstacle and ball, a contact force λ is added. The equations of motion form a *differential inclusion* of the following form

$$q(0) = 0, \qquad v(0) = 0, \qquad \dot{q} = v, \qquad \dot{v} = -10t^4 + \lambda(t), \qquad -\lambda(t) \in N_{\mathbb{R}^+}(q(t)).$$

The analytical solution is given by

$$q(t) = 0, \qquad v(t) = 0, \qquad \lambda(t) = 2t^5.$$

For this problem, Algorithm 3 has convergence order one, the numerical schemes based on parameters (5.3) order two, on parameters (5.4) order three (cf. Figure 5.4). The timestepping schemes have also for differential inclusions the expected convergence rate.

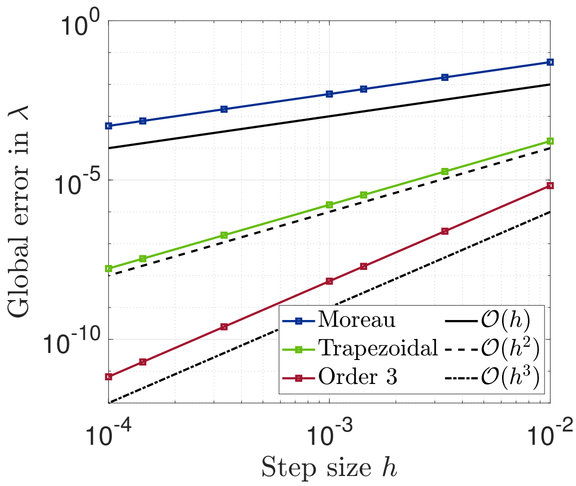


Figure 5.4.: Experimental convergence analysis for the bouncing ball: rest phase.

Problem 5.12 (Bouncing ball: Combined analysis [80]) Now the classical bouncing ball example with an obstacle is considered. The equations of motion

$$\begin{aligned} \dot{q} = v, \quad \dot{v} = -2, \\ q(t) = 0 \Rightarrow v^+(t) = v^-(t) + \max(0, -(1+\epsilon)v^-(t)), \\ q(0) = 1, v(0) = 0, \end{aligned}$$

are formulated state dependent. For $\epsilon = 0.5$, the analytical solution is given by

$$\begin{aligned} t \in [0,1): \quad & q(t) = 1 - t^2, \qquad v(t) = -2t, \\ t \in \left[3 - \frac{1}{2^{p-1}}, 3 - \frac{1}{2^p}\right), p \in \mathbb{N}: \quad & q(t) = -(t-3)^2 - \frac{3}{2^p}(t-1) + \frac{1}{2^{p-1}}\left(3 - \frac{1}{2^p}\right), \\ & v(t) = -2(t-3) - \frac{3}{2^p}. \end{aligned}$$

For this problem, for all tested numerical schemes the global error in q and v fulfills

$$\varepsilon_q = \mathcal{O}(h), \qquad \varepsilon_v = \mathcal{O}(1)$$

(cf. Figure 5.5). It does not matter how many computational effort is used to get better

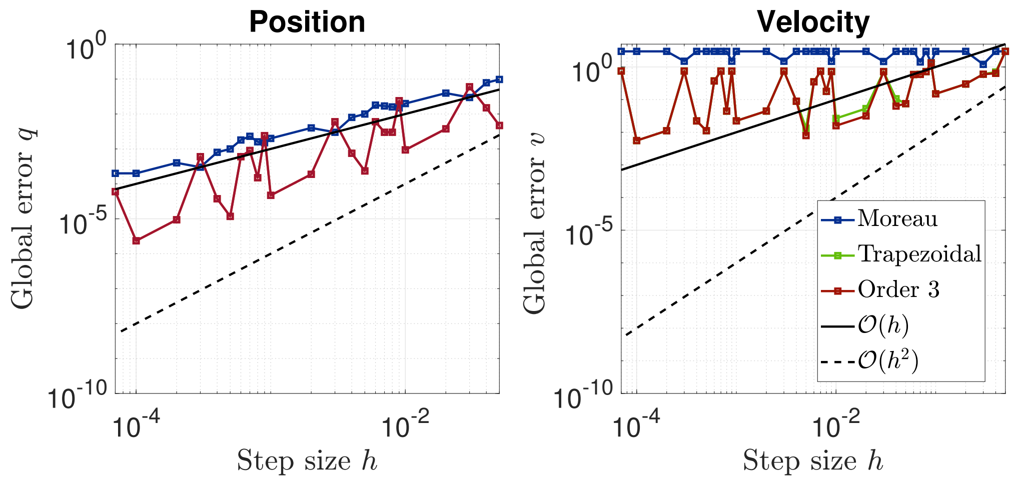


Figure 5.5.: Experimental convergence analysis for the bouncing ball: combined phase

approximations in smooth phases. For some schemes and stepsizes, the impact time points are hit better than with other combinations. However, this is always a random task and would require a prior knowledge of impact points. Therefore, there is at most only an upper bound for the error. You can't seem to get a better approximation with more effort.

Example 5.13 (Classical Order - Impact oscillator) As another application, the impact oscillator of Example 4.1 is considered. The reference solution is calculated with the Algorithm 4 and parameters (5.5) with high accuracy. It can again be observed

$$\varepsilon_q = \mathcal{O}(h), \qquad \varepsilon_v = \mathcal{O}(1)$$

for all timestepping schemes (cf. Figure 5.6).

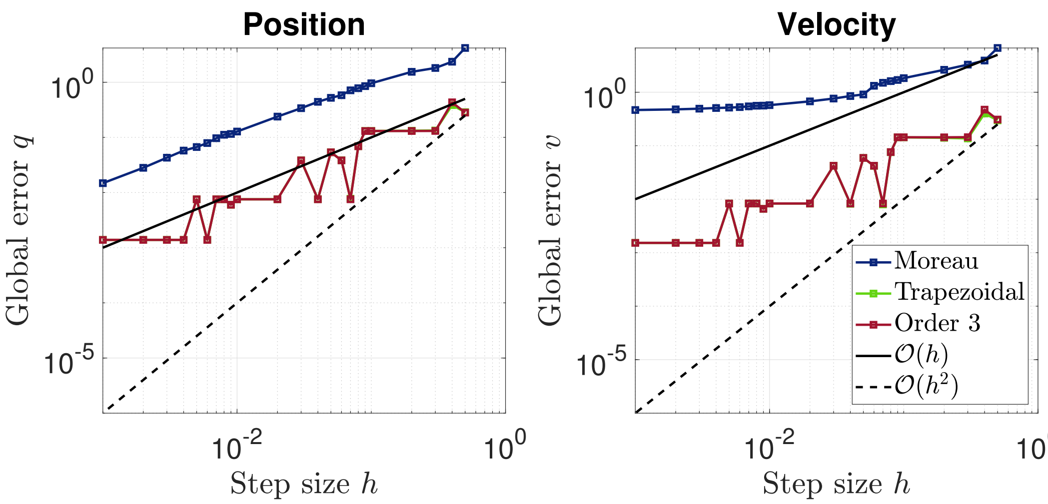


Figure 5.6.: Experimental convergence analysis for the impact oscillator.

Example 5.14 (Classical order - higher dimensional problem) In order to apply our considerations to higher dimensional problems with an impact condition, we consider the

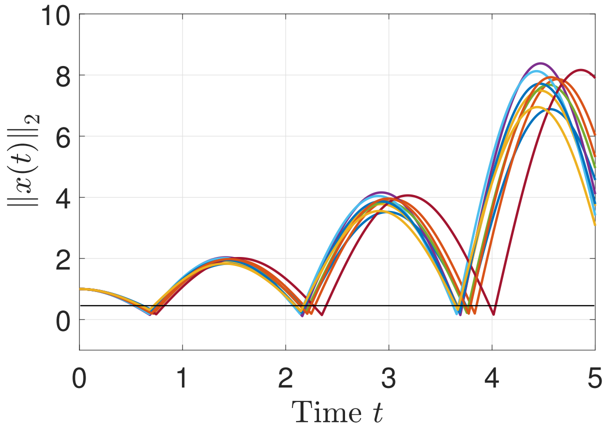


Figure 5.7.: 10 trajectories of Example 5.14 with $n = 5$

following abstract differential problem

$$\begin{cases} \ddot{x}(t) = -Ax(t), & \|x(t)\|_2^2 > 0.25 \\ \dot{x}^+(t) = -2\dot{x}^-(t), & \|x(t)\|_2^2 = 0.25 \end{cases}, \qquad x(0) = 1^n$$

with $n \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$. We select the entries of matrix A normally distributed between 0.5 and 1. A function $x \in C^0([0,T], \mathbb{R}^n)$ is the solution of this problem. The time derivative $\dot{x}$ is not continuous. We use Algorithm 4 and parameters (5.5) to solve the problem. Since $\|x(t)\|$ obviously has points where it is not differentiable, the time derivative must again have jumps.
As Figure 5.8 shows, the error ε of x is $\mathcal{O}(h)$. The equation $\varepsilon_{\dot{x}} = \mathcal{O}(1)$ follows.

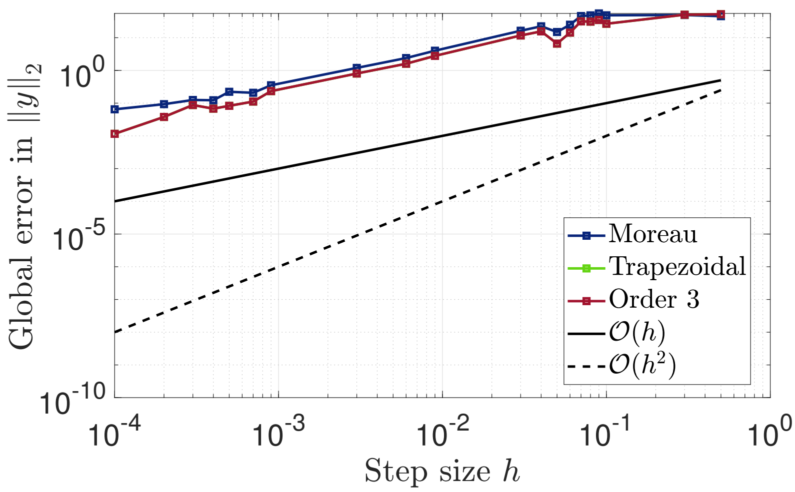


Figure 5.8.: Experimental convergence analysis for Example 5.14 and a fixed A.

Remark 5.15 (Peaking phenomenon) It will not make a difference which application example of non-smooth mechanics one considers, timestepping methods will always have the classical global error in $\mathcal{O}(1)$ for such non-smooth systems. An increased computational effort for a better approximation of the solution in the smooth phases is *not reflected* in the concept at all. The reason, the so-called *peaking phenomenon*, will be explained in this section. If one does not identify the critical points like it would be typically for event-driven schemes, it may happen that they are *not noticed in the right time step*. Figure 5.9 highlights visually the described phenomenon. If there is a jump in the solution as here in t^* and it is noticed regardless of the stepsize in the next time step, the error remains almost constant. In the iteration point next to t^*, the error is about as large as the jump. It may even happen that a jump is registered at the correct interval with stepsize h and with $h/2$ not. The convergence plots do not give such smooth curves as if smooth examples are used. In Figure 5.10 the absolute error in the velocity is plotted for different stepsizes. For the numerical solution the trapezoidal rule was used. In the neighbourhood of the critical time points, a peak is observed, which does not decrease even for smaller stepsizes. The reason is that the impact is not recognised in the correct time integration step.

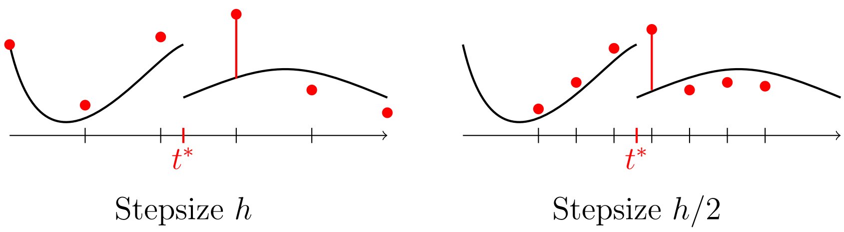


Figure 5.9.: Peaking phenomenon.

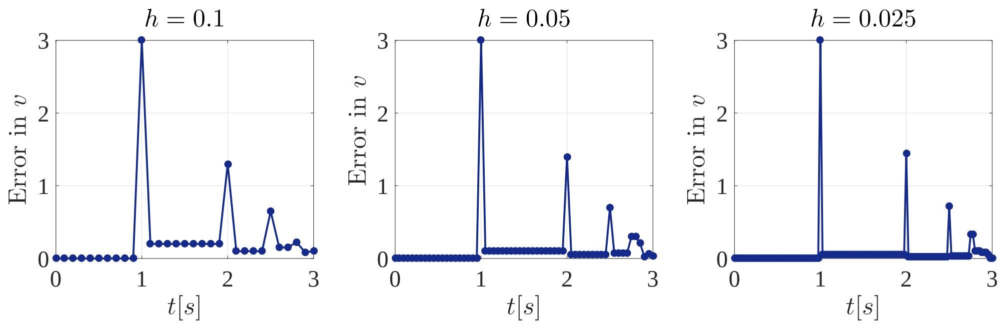


Figure 5.10.: Experimental convergence analysis for the bouncing ball: Peaks.

5.2. Orbital Convergence

In order to be able to compare different numerical methods in a better way, we use another tool to judge their approximation quality. In this section, the concept of orbital convergence will be introduced. There are applications with a higher relevance of a good numerical approximation of *limit sets* than of the approximation in *certain time points*. Noteworthy are invariant orbits such as of an equilibrium points or of a periodic solutions. In the following, we will consider the approximation quality of this whole solution set, so-called orbits. It is important to mention, however, that the concept is only plausible for applications with *invariant limit cycles*. Another important point, which will be discussed later, is that the system is autonomous. For chaotic solutions, the orbital convergence presented in the next section would not be a suitable comparising tool for numerical methods.

5.2.1. Concept

Definition 5.16 (Orbit, [74]) Let $x(t) \in SBV^+(I, \mathbb{R}^n)$ be a solution of (3.1) with initial value $x(t_0) = x_0$. Then we call the set $\gamma(x_0, I) = \{x(t), t \in I\}$ *orbit* in x_0 of I.

Definition 5.17 (Numerical orbit) Let $x_0, x_1, \ldots, x_N$ be numerical approximations of a timestepping scheme in time points $t_0, t_1, \ldots, t_N \in I$. Then the discrete set

$$\gamma^h(I) = \{x_i, i = 0, 1, \ldots, N\}$$

is called *numerical orbit* of I.

Definition 5.18 (One-sided Hausdorff distance, [85]) Let $\emptyset \neq A \subset \mathbb{R}^n$ and $\emptyset \neq B \subset \mathbb{R}^n$ two bounded sets. The *directed* or *one-sided Hausdorff distance* from A to B is the non-negative number

$$\rho(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|$$

where $\|.\|$ is a norm on $\mathbb{R}^n$.

Definition 5.19 (Pseudo-quasimetric) A function $d : X \times X \to \mathbb{R}_+$ on a set X is called a *pseudo-quasimetric* if $\forall\ x, y \in X$

(i) $d(x, y) \geq 0$ (positivity),

(ii) $x = y \Rightarrow d(x, y) = 0$ (indistancy),

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Theorem 5.20 The one-sided Hausdorff distance is a pseudo-quasimetric on the set of all bounded sets of $\mathbb{R}^n$ called X.

Proof: Let $A, B, C \subset \mathbb{R}^n$ and $\|.\|$ be a norm on $\mathbb{R}^n$. For all bounded sets, both the supremum and the infimum exist.

(i)
$$\begin{aligned} & & \|a - b\| \geq 0, \qquad & \forall\ a \in A, b \in B. \\ \Leftrightarrow & & \inf_{b \in B} \|a - b\| \geq 0, \qquad & \forall\ a \in A. \\ \Leftrightarrow & \quad \rho(A, B) = & \sup_{a \in A} \inf_{b \in B} \|a - b\| \geq 0. & \end{aligned}$$

(ii) Let $A = B$ and $a \in A$ fixed. Since $\|.\|$ is a norm and with $\|a - b\| \geq 0,\ \forall\ b \in A$, you get $\inf_{b \in A} \|a - b\| = \|a - a\| = 0,\ \forall\ a \in A$, and furthermore $\rho(A, A) = \sup_{a \in A} 0 = 0$.

(iii)

$$\begin{aligned}
& \|a-c\| \leq \|a-b\| + \|b-c\|, && \forall\, a \in A, b \in B, c \in C,\\
\Rightarrow\quad & \inf_{c\in C} \|a-c\| \leq \|a-b\| + \inf_{c\in C} \|b-c\|, && \forall\, a \in A, b \in B,\\
\Rightarrow\quad & \inf_{c\in C} \|a-c\| \leq \|a-b\| + \sup_{b\in B}\inf_{c\in C} \|b-c\|, && \forall\, a \in A, b \in B,\\
& \qquad = \|a-b\| + \rho(B,C), && \forall\, a \in A, b \in B,\\
\Rightarrow\quad & \inf_{c\in C} \|a-c\| \leq \inf_{b\in B} \|a-b\| + \rho(B,C), && \forall\, a \in A,\\
\Rightarrow\quad & \sup_{a\in A}\inf_{c\in C} \|a-c\| \leq \sup_{a\in A}\inf_{b\in B} \|a-b\| + \rho(B,C), &&\\
\Rightarrow\quad & \rho(A,C) \leq \rho(A,B) + \rho(B,C). &&
\end{aligned}$$

□

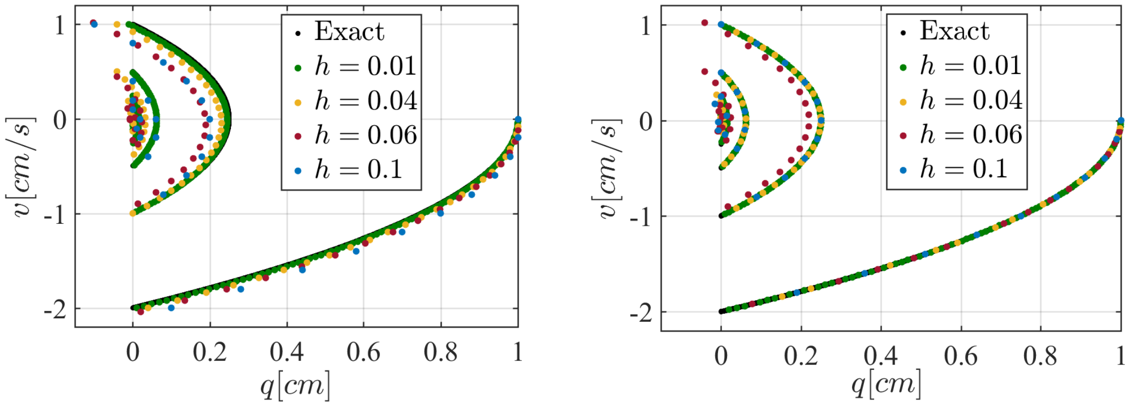
Definition 5.21 (Orbital convergence) Let $x(t) \in SBV^+(I, \mathbb{R}^n)$ be the solution of (3.1) to the initial value $x(t_0) = x_0$ with orbit $\gamma(x_0, I)$ and $x_0, x_1, \ldots, x_N$ numerical approximations in time points $t_0, t_1, \ldots, t_N$ summarised to the numerical orbit $\gamma^h(I)$. The numerical scheme *convergences orbitally* if

$$\rho\left(\gamma^h(I), \gamma(x_0, I)\right) = \mathcal{O}(h).$$

5.2.2. Numerical Studies

In the following section, three examples from the previous chapters with impact conditions are studied. The orbits are graphically illustrated for one dimensional problems. The respective subfigures refer to the different timestepping methods that have already been presented. Sometimes there is no visual difference between numerical and continuous orbit. Then the graphical representation was omitted. In each figure, the exact or numerical orbit of very high accuracy can be seen in black. In different colours four discrete numerical orbits are added which should approximate the black one. Next, the numerical orbits representing different stepsizes are compared with the continuous one using the one-sided Hausdorff distance. For the calculation of the distance we use the numerical orbit γ^{h^4} instead of the continuous one.

Example 5.22 (Bouncing ball - Orbits) First, the benchmark problem of the bouncing ball with the newly introduced concept will be examined. For this purpose, the equations of motion are solved numerically with the scheme of Moreau and the trapezoidal rule, each with four different stepsizes. In Figure 5.11, the corresponding four discrete numerical orbits are displayed together with the continuous orbit of the exact solution in black in a q-v-diagramm. The initial values are $q = 1, v = 0$. The ball drops on a smooth curve to $q = 0, v = -2$ and then jumps to $q = 0, v = 1$. After that, the ball lifts off with a positive post-impact velocity until it approaches the state $q = 0$ again along a smooth curve. It is observed that the trajectory winds closer and closer to the equilibrium point and accumulation point of impact times $(0, 0)$. In Figure 5.12 the Hausdorff distance between the continuous orbit in black and the discrete numerical orbit for different stepsizes is shown. We observe for all schemes that the Hausdorff distance is $\mathcal{O}(h)$. For some stepsizes, the discontinuity points are hit so well, that the Algorithm 5 for methods with order two or three solve the problem with machine accuracy. The present example is a differential problem with constant right-hand side.



(a) Moreau scheme (b) Trapezoidal rule

Figure 5.11.: Bouncing ball orbits, Problem 5.12: $q_0 = 1, v_0 = 0, \epsilon = 0.6$.

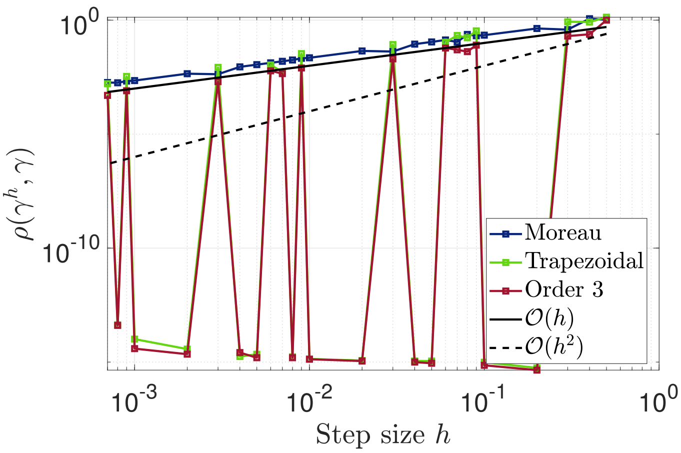
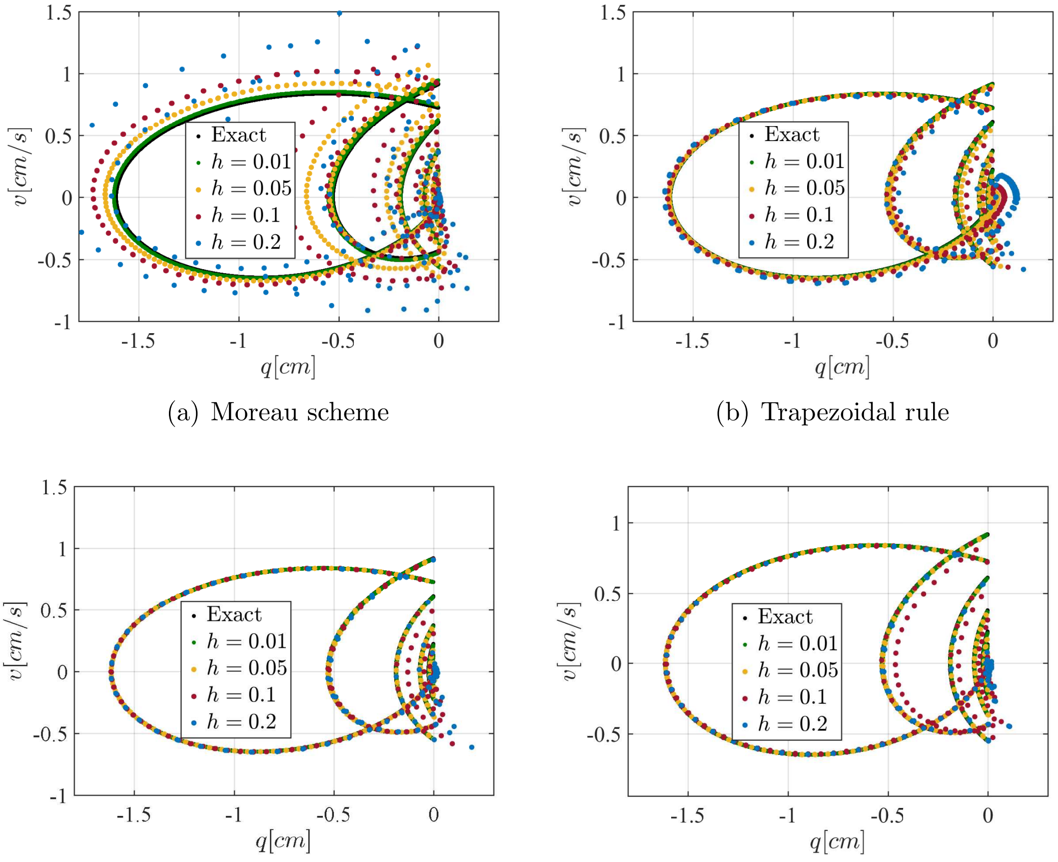


Figure 5.12.: Orbital convergence: Bouncing ball.

Example 5.23 (Impact oscillator - Orbits) Example 4.1 describes a one-dimensional impact oscillator. In Chapter 4, it has been proven that a periodic limit cycle exists for each parameter set. Moreover, in numerical experiments it has been shown that it is an asymptotically stable limit cycle. With the parameter set $\epsilon = 0.6, \sigma = 0$ and $p = \sin(q)$, the problem has been solved with the method of Moreau and the timestepping methods in Algorithm 5 of order 2, 3 and 4 for four different stepsizes.
The orbits are plotted in Figure 5.13. For $h \to 0$ one can observe a convergence against the limit cycle for all methods. In Figure 5.14, the Hausdorff distance between the exact orbit and the discrete numerical ones is plotted and the expected accurracy can be observed.

Remark 5.24 (Higher dimensional problem) The concept of orbital convergence is also applied to a higher dimensional problem. Of course, the orbits can not longer be visualised. Therefore, we do not show the orbits graphically. We use the problem in Example 5.14 with $n = 5$ and a fixed A.



(a) Moreau scheme (b) Trapezoidal rule

(c) Order 3 (d) Order 4

Figure 5.13.: Impact oscillator orbits.

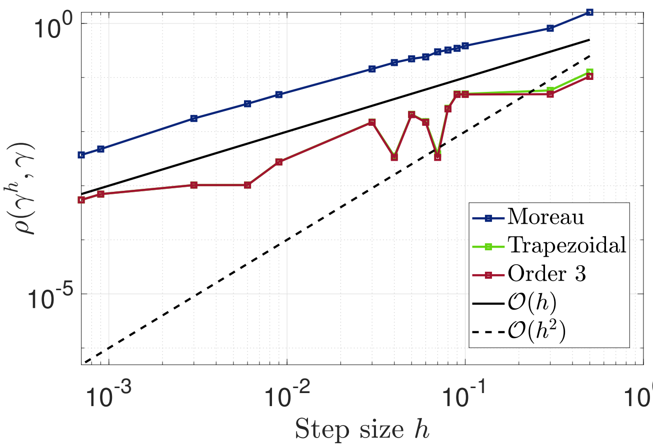


Figure 5.14.: Orbital convergence: Impact oscillator

5.2.3. Convergence Analysis

Since we consider first only problems with finitely many discontinuities $(\tau_j)_{j=1}^m$ in $[0, T]$ and a $\delta > 0$ exists with

$$((0, \delta) \cup (T - \delta, T)) \cap T^*(x) = \emptyset$$

one can find a stepsize $h_{\max}$ such that for all stepsizes $h \leq h_{\max}$ there is at most one discontinuity τ_j in each time step $t_i \to t_{i+1} = t_i + h$.

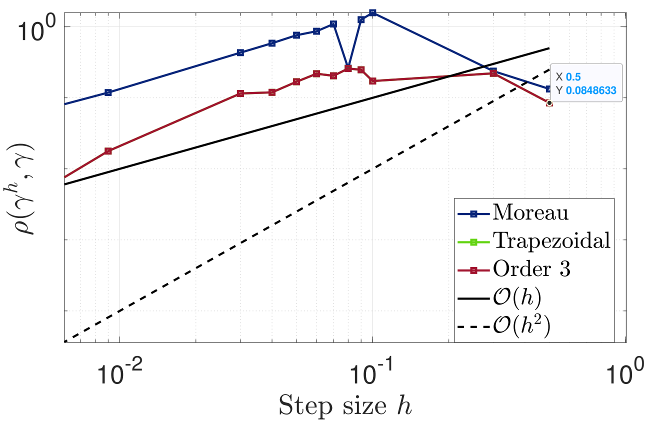


Figure 5.15.: Orbital convergence: Example 5.14

Theorem 5.25 Let $h_0 > 0$ and $\gamma^h([0,T])$ be the numerical orbit of (5.7) with a stepsize $h \in (0, h_0)$ applied to (5.6). Then

$$\rho(\gamma^h([0,T], \gamma(x_0, [0,T]))) = \mathcal{O}(h).$$

Proof: Since g is twice continuously differentiable, there is an $h_0 \leq \frac{h_{\max}}{3}$, so that in three time steps

$$t_{i-1} \to t_i \to t_{i+1} \to t_{i+2}$$

we have at most one discontinuity τ_j, detect no impact erroneously and capture all impacts. This is a consequence of the assumptions made in Remark **??**.

To derive the global error of (5.7), we first consider the local error in a time step $t_i \to t_{i+1} = t_i + h$ with an initial value $x_i = x(t_i)$. The following situations can occur:

(i) $T^*(x) \cap [t_i, t_{i+1}] = \emptyset, g(\hat{x}_{i+1}) > 0$: Both the analytical and numerical solutions detect no impact.

(ii) $T^*(x) \cap [t_i, t_{i+1}] = \{\tau_j\}, g(\hat{x}_{i+1}) \leq 0$: An impact is detected at the correct time step.

(iii) $T^*(x) \cap ([t_{i-1}, t_i] \cup [t_{i+1}, t_{i+2}]) = \{\tau_j\}, g(\hat{x}_{i+1}) \leq 0$: An impact is detected one time step too early or too late.

(iv) $T^*(x) \cap [t_i, t_{i+1}] = \{\tau_j\}, g(\hat{x}_{i+1}) \geq 0$: An impact is not detected.

We will find for each of the four situations a $t \in [t_{i-1}, t_{i+2}]$ with

$$\|x_{i+1} - x(t)\| = \mathcal{O}(h).$$

We start with situation (i). Then the problem (5.6) is an ODE in $[t_i, t_{i+1}]$ with continuous right-hand side solved by (5.7) with an increment function Φ of order p.

$$\begin{aligned}
\|x_{i+1} - x(t_{i+1})\| &= \|x_i + h\Phi(t_i, x_i; h, \varphi_1) - x(t_{i+1})\| \\
&= \|x(t_i) + h\Phi(t_i, x(t_i); h, \varphi_1) - x(t_{i+1})\| \\
&= \mathcal{O}(h^{p+1})
\end{aligned}$$

Let us look at situation (ii). Here the discontinuity $\tau_j \in [t_i, t_{i+1}]$ is detected at the right time step. Nevertheless, the numerical procedure does not solve the problem (5.6) in $[t_i, t_{i+1}]$, which has discontinuity in τ_j. It solves the MDI

$$\mathrm{d}\overline{x} = \varphi_1(\overline{x})\ \mathrm{d}t + \varphi_2(\overline{x}^-)\ \mathrm{d}\delta_{t_{i+1}}, \qquad \overline{x}(t_i) = x(t_i), \tag{5.8}$$

which jumps in t_{i+1}. Note that $\overline{x}$ coincides with x up to timepoint τ_j. The jump amount in t_{i+1} is computed numerically to arbitrary precision using, for example, non-smooth Newton methods for complementarity problems. If we start with the exact initial value $x_n = \overline{x}(t_i)$, the method (5.7) solves the problem (5.8) with accuracy $\mathcal{O}(h^p)$, because

$$\begin{aligned}
\|x_{i+1} - \overline{x}(t_{i+1})\| &= \|x_n + h\Phi(t_i, x_i; h, \varphi_1) + \varphi_2(\hat{x}_{i+1}) - \\
&\quad (x_i + \int_{[t_i,t_{i+1}]} \varphi_1(\overline{x}(t))\ \mathrm{d}t + \varphi_2(x^-(t_{i+1})))\Big\| \\
&\leq \left\| h\Phi(t_i, x_i; h, \varphi_1) - \int_{[t_i,t_{i+1}]} \varphi_1(\overline{x}(t))\ \mathrm{d}t \right\| + \left\|\varphi_2(\hat{x}_{i+1}) - \varphi_2(x^-(t_{i+1}))\right\| \\
&= \mathcal{O}(h^{p+1}) + \left\|\varphi_2(x^-(t_{i+1}) + \mathcal{O}(h^{p+1})) - \varphi_2(x^-(t_{i+1}))\right\| \\
&\leq \mathcal{O}(h^{p+1}) + L_2 \cdot \mathcal{O}(h^{p+1}) = \mathcal{O}(h^{p+1})
\end{aligned}$$

We define another auxiliary system on $[\tau_j, t_{i+1}]$ by applying to the analytic solution x for $t \in [\tau_j, t_{i+1}]$ a translation by the vector $\varrho := -\varphi_2(x^-(\tau_j))$. Let $\tilde{x}(t) := x(t) - m$ in $t \in [\tau_j, t_{i+1}]$ be the solution of the ODE

$$\begin{aligned}
\dot{\tilde{x}}(t) &= \dot{x}(t) = \varphi_1(x(t)) = \varphi_1(\tilde{x}(t) + \varrho) \\
\tilde{x}(\tau_j) &= x^+(\tau_j) + q = x^-(\tau_j)
\end{aligned} \tag{5.9}$$

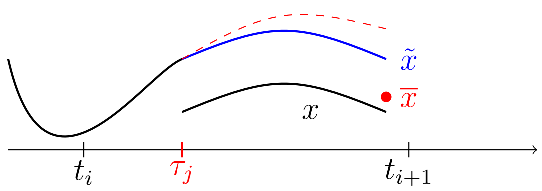


Figure 5.16.: Different solutions.

For this shifted problem (5.9) one can show with the Gronwall lemma

$$\begin{aligned}
\left\|\overline{x}^-(t_{i+1}) - \tilde{x}(t_{i+1})\right\| &= \left\| x(\tau_j) + \varrho + \int_{(\tau_j,t_{i+1}]} \varphi_1(\overline{x}(t))\ \mathrm{d}t - \varrho \right. \\
&\qquad \left. - \left(x(\tau_j) + \int_{[\tau_j,t_{i+1}]} \varphi_1(x(t))\ \mathrm{d}t \right) \right\| \\
&\leq C \cdot \|\varrho\| \left(e^{L_1(t_{i+1}-\tau_j)} - 1\right) \\
&\leq C \cdot \|\varrho\|\ L_1 e^{L_1 h} |t_{i+1} - \tau_j| = \mathcal{O}(h)
\end{aligned}$$

So with problems (5.8) and (5.9) we get, if $x_i = x(t_i)$,

$$\begin{aligned}
\|x(t_{i+1}) - x_{i+1}\| &\leq \|x_{i+1} - \overline{x}(t_{i+1})\| + \|\overline{x}(t_{i+1}) - x(t_{i+1})\| \\
&= \mathcal{O}(h^p) + \left\|\overline{x}^-(t_{i+1}) + \varphi_2(\overline{x}^-(t_{i+1})) - x(t_{i+1})\right\| \\
&\leq \mathcal{O}(h^p) + \left\|\overline{x}^-(t_{i+1}) - \tilde{x}(t_{i+1})\right\| + \left\|\tilde{x}^-(t_{i+1}) + \varphi_2(\overline{x}^-(t_{i+1})) - x(t_{i+1})\right\| \\
&= \mathcal{O}(h^p) + \mathcal{O}(h) + \left\|x(t_{i+1}) - q + \varphi_2(\overline{x}^-(t_{i+1})) - x(t_{i+1})\right\| \\
&= \mathcal{O}(h^p) + \mathcal{O}(h) + \left\|\varphi_2(\overline{x}^-(t_{i+1})) - \varphi_2(\overline{x}^-(\tau_j))\right\| \\
&= \mathcal{O}(h^p) + \mathcal{O}(h)
\end{aligned}$$

Let us now look at situation (iii). If $\tau_j \in T^*(x) \cap [t_{i-1}, t_i]$ and the jump is detected too late, $t_{i+1} - \tau_j \leq 2h$ holds and the proof from situation (ii) is also applicable here. Now let $\tau_j \in T^*(x) \cap [t_{i+1}, t_{i+2}]$ and the discontinuity is noticed in the previous time step. Then, if again $x_i = x(t_i) + \mathcal{O}(h)$ holds, it follows similarly

$$\begin{aligned}
\|x_{i+1} - x(\tau_j)\| &\leq \|x_{i+1} - \overline{x}(t_{i+1})\| + \|\overline{x}(t_{i+1}) - x(\tau_j)\| \\
&= \mathcal{O}(h) + \left\|\varphi_2(x^-(t_{i+1})) - \left(\int_{[t_{i+1},\tau_j]} \varphi_1(x(t))\,\mathrm{d}t + \varphi_2(x^-(\tau_j))\right)\right\| \\
&\leq L_2 \mathrm{e}^{L_2(\tau_j - t_{i+1})} + \beta_1 \cdot C \cdot (\tau_j - t_{i+1}) + \mathcal{O}(h) = \mathcal{O}(h)
\end{aligned}$$

Finally, we consider situation (iv). If a discontinuity point $\tau_j \in T^*(h) \cap [t_i, t_{i+1}]$ is not identified, it will be recognised at the next time step $t_{i+1} \to t_{i+2}$. For the numerical solution x_{i+1}, the following applies

$$\|x_{i+1} - x(t_i)\| = \|h\Phi(t_i, x(t_i), h, \varphi_1)\| = \mathcal{O}(h)$$

We have been able to infer in all four situations from the initial condition $x_i = x(t_i) + \mathcal{O}(h)$ at the time step $t_i \to t_{i+1} = t_i + h$ the conclusion

$$\exists\, t \in [t_i, t_{i+2}]: \quad x_{i+1} = x(t) + \mathcal{O}(h).$$

So the numerical solution remains in an h-tube around the analytic orbit. If we now proceed inductively and sum over all time steps, we get the numerical observed result

$$\rho(\gamma^h([0,T]), \gamma(x_0, [0,T])) = \mathcal{O}(h)$$

of orbital convergence. □

Next we consider the Zeno phenomen for $\epsilon \in (0,1)$. If $\epsilon = 0$ this is not possible.

Theorem 5.26 (Orbital convergence of systems with accumulation points of impacts) Let t^* be an accumulation point of critical points with $\epsilon \in (0,1)$. Therefore, there is a sequence $(t_k^*)_{k\in\mathbb{N}}$ with $t_k^* \to t^*$ and $v^+(t_k^*) - v^-(t_k^*) \to 0$ for $k \to \infty$. There exists an $h_{\max} > 0$ with $i \in \mathbb{N}$ such that $t_{i-1}^* < t^* - h_{\max} < t_i^*$ and

$$\sum_{j=i}^{\infty} \left\|v^+(t_j^*) - v^-(t_j^*)\right\| \leq C \cdot \left\|v^+(t_i^*) - v^-(t_i^*))\right\| + \mathcal{O}(h_{\max})$$

for some constant $C > 0$.

Proof: Let δ_i be the heigth of the i-th velocity jump, i.e.

$$\delta_i = \left\| v^+(t_i^*) - v^-(t_i^*) \right\| .$$

The first three jumps can be estimated

$$\begin{aligned}
\delta_0 &= \left\| v^+(t_0^*) - v^-(t_0^*) \right\| \leq \beta_G(1+\epsilon) \left\| v^+(t_0^*) \right\| , \\
\delta_1 &= \left\| v^+(t_1^*) - v^-(t_1^*) \right\| leq \beta_G(1+\epsilon) \left\| v^+(t_0^*) + \mathcal{O}(t_1^* - t_0^*) \right\| , \\
\delta_2 &= \left\| v^+(t_2^*) - v^-(t_2^*) \right\| leq \beta_G(1+\epsilon) \left\| \epsilon v^+(t_0^*) + \epsilon \mathcal{O}(t_1^* - t_0^*) + \mathcal{O}(t_1^* - t_2^*) \right\|
\end{aligned}$$

with β_G is an upper bound of $\|G\|$ and in general

$$\delta_i \leq \beta_G(1+\epsilon)\epsilon^{i-1} \left\| v^+(t_0^*) \right\| + \beta_G(1+\epsilon) \sum_{j=1}^{i} \epsilon^{j-1} \mathcal{O}(t_j^* - t_{j-1}^*)$$

If the sum over all velocity jumps is considered we get

$$\begin{aligned}
\sum_{i=0}^{\infty} \delta_i &\leq \beta_G(1+\epsilon) \sum_{i=0}^{\infty} \epsilon^{i-1} \left\| v^+(t_0^*) \right\| + \beta_G(1+\epsilon) \sum_{i=0}^{\infty} \sum_{j=1}^{i} \epsilon^{j-1} \mathcal{O}(t_j^* - t_{j-1}^*) \\
&= \beta_G(1+\epsilon) \left\| v^+(t_0^*) \right\| \cdot \frac{1}{\epsilon - \epsilon^2} + \beta_G(1+\epsilon) \sum_{i=0}^{\infty} \mathcal{O}(t_j^* - t_{j-1}^*) \cdot \frac{(1-\epsilon^{i-1})}{(1-\epsilon)} \\
&\leq \beta_G(1+\epsilon) \left\| v^+(t_0^*) \right\| \cdot \frac{1}{\epsilon - \epsilon^2} + \beta_G(1+\epsilon) \sum_{i=0}^{\infty} \mathcal{O}(h_{\max}) \cdot \frac{(1-\epsilon)^{i-1}}{(1-\epsilon)} \\
&= \frac{\beta_G(1+\epsilon)}{\epsilon - \epsilon^2} \left\| v^+(t_0^*) \right\| + \frac{\beta_G(1+\epsilon)}{(1-\epsilon)} \cdot \mathcal{O}(h_{\max}) \sum_{i=0}^{\infty} (1-\epsilon)^{i-1} \\
&= \frac{\beta_G(1+\epsilon)}{\epsilon - \epsilon^2} \left\| v^+(t_0^*) \right\| + \frac{\beta_G(1+\epsilon)}{\epsilon(1-\epsilon)} \cdot \mathcal{O}(h_{\max})
\end{aligned}$$

□

Remark 5.27 In the numerical tests we have deliberately considered only examples with invariant limit cycles, since for these the orbit plays a special role. As it has been proven, the numerical solution converges against the analytical orbit. For applications without invariant limit cycles, this is not a statement relevant for reality. Whether the error bound is dominated by a $\mathcal{O}(h)$ term or, as can be seen in the experiments, even of better accuracy, is a matter of chance. It depends on the location of the discontinuities relative to the location of the grid points. Other problem properties that are favourable for convergence, and possibly observed in the numerical experiments, are

- Attractiveness of the orbit
- Influence of the selection of initial values

These properties were not considered in more detail. No statement can be made how much they influence the error bounds.

6. Conclusion

In this thesis, various analytical and numerical aspects of non-smooth mechanical systems have been considered, re-derived and proved. The inclusion of contact and impact forces makes the discussion of the equations of motion more complex as for force-free systems or systems with smooth forces. New analytical problem classes and generalised solution spaces have to be added. However, due to similar structures, it is often possible to pick up and generalise many results and strategies for smooth systems to non-smooth ones.

In Chapter 3, two new theorems concerning the existence and boundedness are introduced and proved. These are immensely important to guarantee the correctness of current modelling strategies. With the quickly verifiable preconditions of the existence statements, one can now save time and effort without hesitation in order to work with a minimum number of coordinates or without redundant constraints. The criteria could be verified for a few academic examples. Their true manageability would only become apparent when they are applied to systems of very high dimension. These play a role in granular or building simulation, for example.

As the example of a house during an earthquake [20] shows, other properties of the solutions play an important role. For this purpose, the existence of equilibria and periodic solutions of MDIs of small dimension was analysed in Chapter four. The focus is on the impact oscillator. Again, two new theorems could be formulated and shown. They prove the existence of periodic solutions for different drive forms. In the first theorem about time-dependent systems, the idea of a fixed point theorem was used following [56]. In the cited work, frictional rather than impulsive forces were considered. In the second step, the drive depends only on the position. An autonomous system is considered. For this purpose, the transversal theory and the Poincaré-Bendixson theorem for smooth autonomous plane systems [74] were applied to the impact oscillator. Whether these strategies work for all plane systems is still unknown. In the next section, it is shown for equilibrium and periodic systems that they are stable under certain conditions. This means that initial value perturbations nevertheless cause similar behaviour and the system does not behave totally chaotically. The analytical results could again be supported with numerically generated experiments. In the future, it would be interesting to study the systems in terms of their dependence on initial values and parameters. After all, reality is always changing.

In Chapter 5, the numerical solution of MDIs is discussed. Since the difficult Zeno phenomenon is the focus of this thesis, time-stepping methods are concentrated on. These are difficult to compare with the usual numerical instruments for non-smooth problems. Therefore, a completely new instrument was developed using orbital convergence. The numerical experiments and the analytical proof showed the well-definedness of the concept. However, it can only be applied to autonomous problems with positively invariant limit cycles. The usefulness for other problems must be questioned.

A. Measure and Integration Theory

An interesting question is, whether it is possible to define functions μ called measures which map to a large class of sets $A \subset \mathbb{R}^n$ their volume or other important quantities. In $\mathbb{R}$ it could match the length of intervals, in $\mathbb{R}^2$ the classical surface of areas and in $\mathbb{R}^3$ the classical volume of bodies. With the rigorous theory of measure and integration theory this difficult topic could be handled for a large class of subsets $\mathbb{R}^n$. For more details we refer to the classical literature [23, 30, 77].

σ-algebras and measures

The most important role of σ-algebras is taken as domains of measures. Precursors of σ-algebras are rings and algebras. For a set $X \subset \mathbb{R}^n$ the set of all possible subsets including the empty set and X is called power set and is denoted by $P(X)$.

Definition A.1 (Ring, algebra, σ-algebra) A system $R \subset P(X)$ is called a ring over X if and only if it has the following three properties

(i) $\emptyset \in R$, (ii) $A, B \in R \Rightarrow A \cup B \in R$, (iii) $A, B \in R \Rightarrow A \backslash B \in R$.

A system $\mathcal{A} \subset P(X)$ is an algebra over X if and only if

(i) $X \in \mathcal{A}$, (ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$, (iii) $A \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}$.

A system $\mathcal{A} \subset P(X)$ is called σ-algebra over X if and only if

(i) $X \in \mathcal{A}$, $\emptyset \in \mathcal{A}$, (ii) $A_1, A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A}, A_1 \cap A_2 \in \mathcal{A}, A_1 \backslash A_2 \in \mathcal{A}$,
(iii) $A_1, A_2, ... \in \mathcal{A} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{A}, \cap_{i=1}^{\infty} A_i \in \mathcal{A}$.

Often, σ-algebras are given by the specification of a generator. This procedure bases on a simple assumption: Every average of arbitrary many σ-algebras is again a σ-algebra. A deduction is that for every set $B \subset P(X)$ a with respect to set inclusion smallest σ-algebra over B exists (Notation $\mathcal{A}_B$). It is the average of all σ-algebras over B. This set is called the σ-algebra generated by B and B is the so called generator.

Example A.2 (Cuboids, figures) A subset $Q \subset X$ is called a half-open cuboid of $\mathbb{R}^n$ if it has the form

$$Q = [a_1, b_1) \times \ldots \times [a_n, b_n), a_i \leq b_i, i = 1, \ldots, n.$$

A set $A \subset \mathbb{R}^n$ is called figure if it is equivalent to the union of finite many pairwise disjoint half-open cuboids

$$A = \bigcup_{i=1}^{p} Q_i.$$

The set of all figures

$$\mathcal{R}_n := \{A \subset \mathbb{R}^n : A \text{ is a figure}\}$$

is a ring on $\mathbb{R}^n$, but no σ-algebra.

Example A.3 (Borel σ-algebra of $\mathbb{R}^n$) For the measure theory the most important σ-algebra is the σ-algebra generated by sets of all open subsets of $\mathbb{R}^n$. It is called the Borel σ-algebra of $\mathbb{R}^n$, noted by $\mathcal{B}(\mathbb{R}^n)$ and can be generated by the closed or compact subsets of $\mathbb{R}^n$, too.

Definition A.4 (Finitely additive measure, pre-measure, measure) Let R be a ring over X and $\mu : R \to [0, \infty]$. The function μ is called a finitely additive measure if

(i) $\mu(\emptyset) = 0$, (ii) $A, B \in R, A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$ (additivity).

Furthermore, μ is a pre-measure over an algebra $\mathcal{A}$ if

(i) $\mu(\emptyset) = 0$, (ii) $(A_n)_{n\in\mathbb{N}} \in \mathcal{A}, \cap_{n=1}^{\infty} A_n = \emptyset \Rightarrow \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ (σ-additivity).

If $\mu : \mathcal{A} \to [0, \infty]$ is a pre-measure over a σ-algebra $\mathcal{A}$ it is a measure. Then, the triple $(X, \mathcal{A}, \mu)$ is called a measure space and all elements of $\mathcal{A}$ are called measurable. If $A \in \mathcal{A}$ with $\mu(A) = 0$ then it is a μ-null-set.

Example A.5 (Dirac measure, characteristic function) Let $p \in X$. The Dirac measure δ_p indicates if p belonges to a set $A \subset X$, i.e.

$$\delta_p(A) := \begin{cases} 1, & p \in A, \\ 0, & p \notin A. \end{cases}$$

With the characteristic function

$$\chi_A(x) := \begin{cases} 1, & x \in A, \\ 0, & x \notin X \backslash A. \end{cases}$$

of a set A, the Dirac measure can be defined as $\delta_p(A) = \chi_A(p)$.

Example A.6 (Lebesgue pre-measure) Let $Q \subset \mathcal{R}_n$ be a half-open cuboid. The geometrical volume of Q is

$$v_n(Q) := \Pi_{i=1}^{n}(b_i - a_i).$$

The volume of a figure $A \in \mathcal{R}_n$ is defined as

$$\mu_n(A) := \begin{cases} \sum_{i=1}^{n} v_n(Q_i), & A \neq \emptyset, \\ 0, & A = \emptyset. \end{cases}$$

The function μ_n is a σ-finite pre-measure called the Lebesgue pre-measure. A function μ on $R \subset P(X)$ is called σ-finite if there are $(X_n)_{n\geq 0} \subset X$ with $X = \cup_{n=0}^{\infty} X_n$ and $\mu(X_n) < \infty, \forall\, n \geq 0$.

Remark A.7 (Construction of measure spaces according to Caratheodory) The specification of pre-measures is in contrast to the specification of measures relatively simple. Caratheodory construct a generalized method to continue pre-measures to measures.

A function $\mu^* : P(X) \to [0, \infty]$ is called an outer measure if

(i) $\mu^*(\emptyset) = 0$, (ii) $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$ (monotonicity), (iii) $\mu^*(\cap_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$ (σ-semi-additivity).

A set $A \subset X$ is called μ^*-measurable if

$$\exists\ E \subset X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C).$$

The set of all μ^*-measurable sets is noted by $\mathcal{A}_{\mu^*}$. Caratheodory has proved that $(X, \mathcal{A}_{\mu^*}, \mu^*|_{\mathcal{A}_{\mu^*}})$ is a measure space. For example, the function

$$\mu^*(E) := \inf \left(\sum_{n=1}^{\infty} \mu(A_n), E \subset \cup_{n=1}^{\infty} A_n, A_n \in R \right)$$

defines an outer measure when μ is a pre-measure over R [30, pp. 55-58].

Example A.8 (Lebesgue measure over $\mathbb{R}^n$) Let $\mathcal{R}_n$ be the ring of all figures of $\mathbb{R}^n$ and μ_n the Lebesgue pre-measure of Example A.6. The outer measure $\mu_n^* : P(\mathbb{R}^n) \to [0, \infty]$ generated by μ_n is called the outer Lebesgue measure. Following Remark A.7, the space $\mathcal{L}(\mathbb{R}^n) := \mathcal{A}_{\mu^*}$ is the σ-algebra of all Lebesgue measurable sets and $\lambda_n := \mu_n^*|_{\mathcal{L}(\mathbb{R}^n)}$ is the Lebesgue measure. Often the dimension n vanishes and the alternative notation $\mathrm{d}t := \lambda_n$ is used for the Lebesgue measure.

Example A.9 (Lebesgue-Stieltjes measure on $\mathbb{R}^n$) Let $f : \mathbb{R}^n \to \mathbb{R}$ be monotone increasing and right-side continuous and $A = \cup_{i=1}^{\infty} Q_i$ with half-open cuboids $Q_j = [a_1^j, b_1^j) \times \ldots \times [a_n^j, b_n^j)$ an element of $\mathcal{R}_n$. The function

$$\mu_f(A) := \begin{cases} \sum_{i=1}^{n} v(Q_i), & A \neq \emptyset, \\ 0, & A = \emptyset. \end{cases} \qquad \text{with } v(Q_j) := \Pi_{i=1}^{n}(f(b_i) - f(a_i))$$

is a σ-finite pre-measure on $\mathcal{R}_n$. The outer measure μ_f^* generated by μ_f is called outer Lebesgue-Stieltjes measure of f and $(\mathbb{R}^n, \mathcal{A}_{\nu_f^*}, \nu_f^*|_{\mathcal{A}_{\nu_f^*}})$ is a measure space where $\nu_f^*|_{\mathcal{A}_{\nu_f^*}}$ is the Lebesgue-Stieltjes measure of f.

Measurable Functions and the Lebesgue Integral

In the following, let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces.

Definition A.10 (Measurable function) A function $f : E \subset X \to Y$ is called $(\mathcal{A}, \mathcal{B})$-measurable if $f^{-1}(B) \in \mathcal{A}, \forall\ B \in \mathcal{B}$. The function $f : X \to Y$ is called ($\mathcal{A}$-)measurable if it is $(\mathcal{A}, \mathcal{B}(Y))$-measurable, where $\mathcal{B}(Y)$ are the Borel σ-algebra of Y. This is the σ-algebra generated by all open sets of Y.

Remark A.11 (Properties of measurable functions) Let $f : X \to Y, g : X \to Y, f_n : X \to Y, n \in \mathbb{N}$ measurable functions. The compositions $f + g$, $f \circ g$, $\min(f, g)$, $\max(f, g)$, $\inf(f_n)$, $\sup(f_n)$, $\limsup(f_n)$, $\liminf(f_n)$ are measurable.

Example A.12 (Lebesgue measurable functions) If f is $\mathcal{L}(\mathbb{R}^n)$-measurable it is called Lebesgue measurable. Every continuous function is Lebesgue measurable.

Theorem A.13 (Lusin) Let $E \subset \mathbb{R}^n$ a Lebesgue measurable set. Then $f : E \subset \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable if $\forall\ \epsilon > 0$ a closed $F_\epsilon \subset E$ exists with

(i) f is continuous on F_ϵ, (ii) $\lambda_n(E \backslash F_\epsilon) < \epsilon$.

Theorem A.14 (Fréchet) Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and $f : E \to \mathbb{R}$ Lebesgue measurable. Then, there is a sequence of continuous functions $f_n : E \to \mathbb{R}$ that converges λ_n-almost everywhere to f. If a statement is μ-almost everywhere true on X then there exists a μ-null set N such that is is satisfied for all $x \in X \backslash N$.

Corollary A.15 A step function is measurable.

Definition A.16 (Integral of non-negative step functions) Let $f : X \to \mathbb{R}$ a non-negative step function with $f = \sum_{i=1}^n c_i \chi_{A_i}, X = \cup_{i=1}^n A_i, \cap_{i=1}^n A_i = \emptyset$. The value

$$\int_X f \, \mathrm{d}\mu := \sum_{i=1}^n c_i \mu(A_i)$$

is called the integral of f over X with respect to the measure μ. If μ is the Lebesgue measure it is called the Lebesgue integral; if μ is the Lebesgue-Stieltjes measure of f it is the Lebesgue-Stieltjes integral.

Definition A.17 (Integral of non-negative functions) Let $f : X \to \mathbb{R}$ be a non-negative measurable function. The value

$$\int_X f \, \mathrm{d}\mu := \sup \left\{ \int_X \phi \, \mathrm{d}\mu : \phi \text{ is a non-negative step function with } \phi \leq f \right\}$$

is called the integral of f over X with respect to μ.

Theorem A.18 Let $f : X \to \mathbb{R}$ be a non-negative function. It exists a monotone increasing sequence of non-negative step functions $(f_n)_{n\in\mathbb{N}}$ with $f_n \leq f$ that converge pointwise to f (Notation $f_n \uparrow f$).

Definition A.19 (Beppo-Levi) Let $f : X \to \mathbb{R}$ be a non-negative function and $(f_n)_{n\in\mathbb{N}}$ a monotone increasing sequence of non-negative step functions with $f_n \uparrow f$ like in A.18. Then, the equivalence

$$\int_X f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu$$

is satisfied.

Remark A.20 (Properties of the integral) The integral is linear and preserves monotonicity of sets or functions.

Corollary A.21 (Modulus function) Let $f : X \to \mathbb{R}$ be measurable. Then, the functions

$$f^+ := \max(f, 0) \geq 0, \qquad f^- := \max(-f, 0) \geq 0$$

are measurable, too. It applies $f = f^+ - f^-$ and the modulus function $|f| := f^+ + f^-$ can be defined.

Definition A.22 (Integrable function) An arbitrary measurable function $f : X \to \mathbb{R}$ is called μ-integrable if

$$\int_X f^+ \, \mathrm{d}\mu < \infty, \qquad \int_X f^- \, \mathrm{d}\mu < \infty.$$

The value

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

is the integral of f over X with respect to μ. The space of all μ-integrable $\mathcal{A}$-measurable functions over X is noted by $\mathcal{L}(X, \mathcal{A}, \mu)$.

Remark A.23 (Properties of the integral) Let $f \in \mathcal{L}(X, \mathcal{A}, \mu)$. Then, f is μ-almost everywhere finite. If $f = h$ is satisfied μ-almost everywhere, the equivalence $\int_X f \, d\mu = \int_X h \, d\mu$ is true.

Remark A.24 (Comparison to Riemann integral) The integral A.22 introduced by Lebesgue has clear advantages in comparison to the older Riemann integral concept. If the proper Riemann integral exists both values match. The most significant advantage of the Lebesgue integral to Riemann ones is that much better convergence results exists.

Remark A.25 (L^p-spaces) Let

$$\mathcal{L}^p(X, \mathcal{A}, \mu) := \mathcal{L}^p := \left\{ f : X \to Y : f \text{ is } \mathcal{A}\text{-measurable and } \int_X |f|^p \, d\mu < \infty \right\}.$$

Using the inequalities of Hölder and Minkowski, it can be proved that

$$0 \leq \|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty$$

is a semi-norm on $\mathcal{L}^p$. If the set $\mathcal{N} := \{f : X \to Y : f \text{ is } \mathcal{A} - \text{measurable}, f = 0 \ \mu - \text{almost everywhere}\}$ is ignored, the function $\|\cdot\|_p$ is a norm on $L^p(X, \mathcal{A}, \mu) = L^p := \mathcal{L}^p \backslash \mathcal{N}$. For $p \geq 1$, $(L^p, \|\cdot\|_p)$ is a Banach space.

Absolute Continuity

Let $(X, \mathcal{A}, \mu)$ be a measure space.

Definition A.26 (Signed measure) A function $\nu : \mathcal{A} \to \mathbb{R}$ is called a signed measure if

(i) $\nu(\emptyset) = 0$, (ii) $\nu(\mathcal{A}) \subset (-\infty, \infty]$ or $\nu(\mathcal{A}) \subset [-\infty, \infty)$, (iii) $A = \cup_{i=1}^{\infty} A_i, \cap_{i=1}^{\infty} A_i = \emptyset \Rightarrow \nu(A) = \sum_{i=1}^{\infty} \nu(A_i)$.

Remark A.27 (Relation between measures and signed measures) Signed measures differ to measures through the non-fulfilment of the non-negativity.

Examples A.28 (i) Let $\rho, \mu : \mathcal{A} \to [0, \infty]$ be measures. The difference $\nu := \rho - \mu$ is a signed measure.

(ii) Let $\mu : \mathcal{A} \to [0, \infty]$ a measure and $f : X \to \mathbb{R}$ a μ-integrable function. Then, $\nu : \mathcal{A} \to \mathbb{R}$ with

$$\nu(A) := \int_A f(x) \, d\mu, A \in X$$

is a signed measure. It is called signed measure with density f with respect to μ and noted by $\nu = f \odot \mu$.

Definition A.29 A set $P \in \mathcal{A}$ is called ν-positive if $\nu(A) \geq 0, \forall\ A \subset P$. A set $N \in \mathcal{A}$ is called ν-negative if $\nu(A) \leq 0, \forall\ A \subset N$.

Theorem A.30 (Hahn decomposition theorem) Let $\nu : \mathcal{A} \to \mathbb{R}$ be a signed measure. Then a disjoint decomposition $X = P \cup N$ with a ν-positive set P and a ν-negative set N exists and is unique except for ν-null sets.

Definition A.31 Let $P \cup N = X$ be the decomposition of X in A.30. The function ν^+ defined by $\nu^+(A) := \nu(A \cap P), A \in X$ is the positive variation of ν, ν^- defined by $\nu^-(A) := \nu(A \cap N), A \in X$ the negative variation and $|\nu| := \nu^+ + \nu^-$ the variation. Clearly $\nu = \nu^+ - \nu^-$ is true.

Example A.32 Let f be measurable. Following A.21 there exists a decomposition $f = f^+ - f^-$ in non-negative functions f^+ and f^-. If ν is a signed function with density f with respect to μ ($\nu = f \odot \mu$) then $\nu^+ = f^+ \odot \mu$ and $\nu^- = f^- \odot \mu$.

Definition A.33 (Singular signed measures) Let $\nu, \mu : \mathcal{A} \to \mathbb{R}$ be signed measures. They are called singular if a decomposition $X = A \cup B$ with $A \cap B = \emptyset, A, B \in \mathcal{A}$, exists such that A is a ν-null set and B is a μ-null set. (Notation: $\nu \perp \mu$).

Example A.34 For every signed measure ν it follows $\nu^+ \perp \nu^-$.

Definition A.35 (Absolute continuity) Let $\nu, \mu : \mathcal{A} \to \mathbb{R}$ be signed measures. The function ν is called absolute continuous with respect to μ if $\mu(A) = 0, A \in X \to \nu(A) = 0$ (Notation $\nu << \mu$).

Theorem A.36 (Radon-Nikodým) Let μ be a σ-finite measure and $\nu << \mu$ a signed measure. Then, ν has a density f with respect to μ. This f is μ-almost everywhere uniquely defined.

Theorem A.37 (Lebesgue partition of the Lebesgue-Stieltjes measure) For all $f \in \mathrm{BV}(X, \mathbb{R})$, a partition

$$f = f_{abs} + f_S + f_{sing} \tag{A.1}$$

exists with the decomposition

$$\nu_f = \nu_{abs} + \nu_s + \nu_{sing}$$

of the signed Lebesgue-Stieltjes measure and the properties

i) f_{abs} is absolutely continuous with respect to the Lebesgue measure $\lambda_n = \mathrm{d}t$ with $\nu_{abs} << \lambda_n, \nu_{abs} = \dot{f}_{abs} \odot \lambda_n$,

ii) f_S is a step function, i.e. it is almost everywhere constant and its signed measure ν_S is a purely atomic measure

iii) f_{sing} is a singular function, i.e. it is continuous and almost everywhere constant and its measure ν_{sing} is orthogonal to λ_n.

If a vector-valued function $f : X \to \mathbb{R}^n$ is given, everything is considered componentwise.

B. Variational Inequalities and Complementarity Problems

Variational inequalities and the subclass of complementarity problems play an important role in non-smooth mechanics and other application fields and coherent mathematical topics like constrained optimisation problems, saddle problems, source problems, Nash equilibrium problems or option pricing. In this section, different equivalent formulations of variational inequalities are presented, properties are proved and a suitable numerical algorithm is discussed. For a detailed introduction on variational inequalities and complementarity problems we refer to [25, 31, 32, 52].

In this thesis, three analytically equivalent formulations of the impact law of Newton are considered

(i) as a complementarity problem, which has the clearest reference to reality,

(ii) as a normal cone inclusion, which is used for analytical considerations,

(iii) as a non-smooth equation, for which the most effective numerical solution strategy exists.

All of these are studied in the following section.

Definition B.1 (Variational inequality) Let $C \subset \mathbb{R}^n$ and $F : C \to \mathbb{R}^n$. The problem to find an element $x \in C$ such that

$$(y - x)^\top F(x) \geq 0, \qquad \forall\, y \in C \tag{B.1}$$

is called a variational inequality (Notation: $\mathrm{VI}(C, F)$). The set of all solutions x of (B.1) is denoted $\mathrm{SOL}(C, F)$.

Equivalent Formulations

Remark B.2 (Geometrical interpretation) The geometrical interpretation of Definition B.1 is, that $x \in C$ is a solution of $\mathrm{VI}(C, F)$ if and only if the angle between vector $F(x)$ and all vectors $y - x, y \in C$, is always non-obtuse. The formalisation can be done by using the definition of a normal cone.

Definition B.3 (Normal cone) Let $C \subset \mathbb{R}^n$. The set of all normal vectors in $x \in C$ is the normal cone

$$N_C(x) := \left\{ x^* \in \mathbb{R}^n : (y - x)^\top x^* \leq 0, \forall\, y \in C \right\}$$

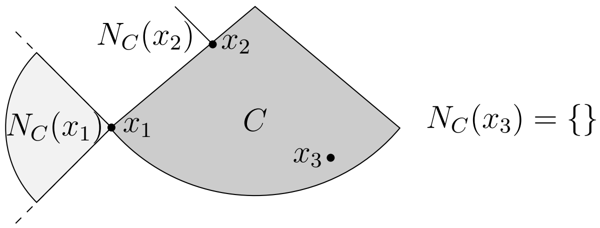


Figure B.1.: Normal Cone

Conclusion B.4 (Normal cone inclusion) All vectors that belongs to $N_C(x)$ are called normal vectors of C in x. By definition it is easy to prove that $x \in C$ solves VI(C, F) if and only if

$$-F(x) \in N_C(x)$$

is satisfied.

In the analytical part of this thesis, this formulation of the non-linear parts is used to underline the set-valued nature of the problems. Using projections, another important equivalent formulation can be derived for closed and convex C which is utilised for the numerical solution of such systems.

Definition B.5 (Euclidean projection) Let $C \subset \mathbb{R}^n$ be closed and convex. The Euclidean projection of $x \in \mathbb{R}^n$ on C is defined as

$$\text{proj}_C(x) := \arg\min_{z \in C} \|x - z\| .$$

The vector $\text{proj}_C(x)$ is the closest point of C to x. In [31, Theorem 1.5.5] it is proved that $\text{proj}_C(x)$ exists and is unique for closed and convex C.

Theorem B.6 Let $C \subset \mathbb{R}^n$ be closed and convex and $F : C \to \mathbb{R}^n$. Then, the statements

(i) x is a solution of VI(C, F),

(ii) $x = \text{proj}_C(x - F(x))$

are equivalent [31, Proposition 1.5.8].

Proof: First it is proven that x^* is the Euclidean projection of x on a convex C if and only if

$$\text{(iii)} \qquad (y - x^*)^\top (x^* - x) \geq 0, \qquad \forall\, y \in C.$$

Let x^* be $\text{proj}_C(x)$. It follows that x^* is the solution of

$$\min (y - x)^\top (y - x) \quad \text{subject to} \quad y \in C$$

or equivalent

$$\text{(iv)} \quad (y - x)^\top (y - x) \geq (x^* - x)^\top (x^* - x), \quad \forall\, y \in C.$$

Since C is convex, for all $\alpha \in [0, 1]$ an $y \in C$ the point $x_{\alpha,y} := x^* + \alpha(y - x^*)$ is in C, too. Following (iv) the function

$$\Psi(\alpha) = (x_{\alpha,y} - x)^\top (x_{\alpha,y} - x)$$

has its minimum independent of y in $\alpha = 0$ and so it holds

$$0 \leq \frac{\partial \Psi}{\partial \alpha}(0) = (y - x^*)^\top (x^* - x).$$

On the other side, (iii) can be transformed using the Cauchy-Schwarz inequality to

$$\begin{aligned}
& & (y - x^*)^\top (x^* - x) \geq 0 & \qquad \forall\, y \in C \\
&\Leftrightarrow & (y - x + x - x^*)^\top (x^* - x) \geq 0 & \qquad \forall\, y \in C \\
&\Leftrightarrow & (y - x)^\top (x^* - x) - \|x - x^*\|^2 \geq 0 & \qquad \forall\, y \in C \\
&\Leftrightarrow & (y - x)^\top (x^* - x) \geq \|x - x^*\|^2 & \qquad \forall\, y \in C \\
&\Leftrightarrow & \|y - x\| \, \|x^* - x\| \geq \|x - x^*\|^2 & \qquad \forall\, y \in C \\
&\Leftrightarrow & \|y - x\| \geq \|x - x^*\| & \qquad \forall\, y \in C \\
&\Leftrightarrow & x^* = \mathrm{proj}_C(x) & \qquad \forall\, y \in C.
\end{aligned}$$

Using this equivalence of $x^* = \mathrm{proj}(x)$ to (iii), we can rewrite (ii) to $(y - x)^\top F(x) \geq 0,\ \forall\, y \in C$, which is the definition of a solution of $\mathrm{VI}(C, F)$. □

In addition to [31], this projection consideration is picked up in [81] in the context of non-smooth mechanics. Since the projection function is non-differentiable at the boundery of C the problem to find an x fulfilling (ii) is a non-smooth equation. In [31, Proposition 1.1.3], furthermore the following equivalence is proved when C is a cone.

Theorem B.7 (Complementarity problem) Let $C \subset \mathbb{R}^n$ be a cone, i.e. $\alpha \geq 0, x \in C \Rightarrow \alpha x \in C$, and $F : C \to \mathbb{R}^n$. The vector $x \in C$ solves $\mathrm{VI}(C, F)$ if and only if x solves the following problem

$$x \in C, \quad F(x) \in C^*, \tag{B.2a}$$

$$x^\top F(x) = 0. \tag{B.2b}$$

The problem (B.2a)-(B.2b) is said to be a complementarity problem (Notation $\mathrm{CP}(C, F)$). The set C^* is the dual cone to a cone C that is defined by

$$C^* := \left\{ d \in \mathbb{R}^n : y^\top d \geq 0, y \in C \right\}.$$

An equivalent notation to (B.2a)-(B.2b) is

$$C \ni x \perp F(x) \in C^*$$

where $a \perp b$ means $a^\top b = 0$.

Proof: Let $x \in C$ be a solution of $\mathrm{VI}(C, F)$. From the definition of cones follows that $y_1 = 0 \in C$ and $y_2 = 2x \in C$. Condition (B.1) for this vectors is equivalent to

(i) $x^\top F(x) \leq 0$,

(ii) $x^\top F(x) \geq 0$.

Combining both, we deduce $x^\top F(x) = 0$. With this result, (B.1) transforms to

$$0 \leq (y - x)^\top F(x) = y^\top F(x), \forall\, y \in C,$$

i.e. $F(x) \in C^*$. Now let $x \in C$ be a solution of $\mathrm{CP}(K, F)$. That means for an arbitrary $y \in C$ the term $(y-x)^\top F(x)$ reduces to $y^\top F(x)$ which is non-negative because $F(x) \in C^*$. Therefore, x solves $\mathrm{VI}(C, F)$. □

Definition B.8 We call $x \in \mathbb{R}^n$ feasible to $\mathrm{CP}(C, F)$ if it satisfies (B.2a). All feasible vectors to $\mathrm{CP}(C, F)$ are summarized in $\mathrm{FEA}(C, F)$.

There are two special cases of variational inequalities (B.1) which are interesting in non-smooth mechanics.

Definition B.9 (Linear complementarity problem) The complementarity problem (B.2a)-(B.2b) with

$$C = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$$

and $F(x) = Mx + q, M \in \mathbb{R}^{n \times n}, q \in \mathbb{R}^n$, is called linear complementarity problem (Notation $\mathrm{LCP}(M, q)$) and denoted by

$$0 \leq x \perp Mx + q \geq 0. \tag{B.3}$$

Its solution set is $\mathrm{SOL}(M, q)$.

Definition B.10 (Mixed linear complementarity problem) Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times k}, D \in \mathbb{R}^{k \times n}, E \in \mathbb{R}^{k \times k}, a \in \mathbb{R}^m, b \in \mathbb{R}^k$. The problem to find $x \in \mathbb{R}^n, y \in \mathbb{R}^k$ such that

$$Ax + By + a = 0 \tag{B.4a}$$

$$0 \leq y \perp Dx + Ey + b \geq 0 \tag{B.4b}$$

is called mixed linear complementarity problem (Notation $\mathrm{MLCP}(A, B, D, E, a, b)$).

Conclusion B.11 If A is invertible than problem (B.4a)-(B.4b) is equivalent to $\mathrm{LCP}(E - DA^{-1}B, b - DA^{-1}a)$. Otherwise the following equivalence is useful.

Theorem B.12 The vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$ are the solution of $\mathrm{MLCP}(A, B, D, E, a, b)$ if and only if $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+k}$ is the solution of $\mathrm{VI}(C, F)$ with

$$C = \left\{ z \in \mathbb{R}^{n+k} : z = \begin{pmatrix} x \\ y \end{pmatrix}, x \in \mathbb{R}^n, y \in \mathbb{R}^k, y \geq 0 \right\},$$

$$F(z) = \begin{pmatrix} Ax + By + a \\ Dx + Ey + b \end{pmatrix}.$$

Proof: Let x, y be the solution of $\mathrm{MLCP}(A, B, D, E, a, b)$. Then $z = (x^\top \; y^\top)^\top$ is clearly an element of C. Let $\tilde{z} = (\tilde{x}^\top \; \tilde{y}^\top)^\top$ with $\tilde{x} \in \mathbb{R}^n, \tilde{y} \in \mathbb{R}^k, \tilde{y} \geq 0$ be an arbitrary element of C. It holds

$$\begin{aligned}(\tilde{z} - z)^\top F(z) &= (\tilde{x} - x)^\top (Ax + By + a) + (\tilde{y} - y)^\top (Dx + Ey + b) \\ &= (\tilde{x} - x)^\top 0 + \tilde{y}^\top (Dx + Ey + b) - y^\top (Dx + Ey + b) \\ &= \underbrace{\tilde{y}^\top}_{\geq 0} \underbrace{(Dx + Ey + b)}_{\geq 0} - 0 \geq 0.\end{aligned}$$

Therefore, z is a solution of $\mathrm{VI}(C, F)$. Let z be a solution of $\mathrm{VI}(C, F)$ and $x \in \mathbb{R}^n$ the first

n elements of z and $y \in \mathbb{R}^k$ the last k elements. Trivially it follows $y \geq 0$. Condition (B.1) is true for $\tilde{z}_1 = (x^\top\ 0)^\top \in C$ and $\tilde{z}_2 = (x^\top\ 2y^\top)^\top$. This means $y^\top(Dx + Ey + b) \leq 0$ and $y^\top(Dx + Ey + b) \geq 0$ are both satisfied such that $y^\top(Dx + Ey + b) = 0$ results. Because (B.1) is true for every $z_i = z + e_i, i = n+1, \dots, n+k$ the inequality $Dx + Ey + b \geq 0$ follows. The vector e_i is i-th unit vector. Similarly, from (B.1) for $z_i^1 = z + e_i$ and $z_i^2 = z - e_i$ follows $Ax + By + a = 0$. □

Lemma B.13 Let $x * y$ be the Hadamard product $(x_1y_1, \dots, x_ny_n)^\top \in \mathbb{R}^n$ for $x, y \in \mathbb{R}^n$. The following statements are equivalent for a symmetric matrix A

(i) A is positiv semi-definite

(ii) $x * (Ax) \leq 0 \Rightarrow x * (Ax) = 0$

Proof: Let A be positiv definite and $x*(Ax) \leq 0$. The inequality $x^\top Ax = \sum_{i=1}^n x_i(Ax)_i \leq 0$. Since A is positiv semi-definite we know $x^\top Ax = 0$. Because every summand $x_i(Ax)_i$ of the sum is non-positive, it is zero. You get the conclusion $x * (Ax) = 0$.

Let the conclusion $x * (Ax) \leq\Rightarrow x * (Ax) = 0$ be true. We assume that A is not positiv semi-definite. Then there is a negative eigenvalue $\lambda^i < 0$ with an eigenvector $x^i \neq 0$. The vector

$$x^i * (Ax^i) = \lambda^i \begin{pmatrix} (x_1^i)^2 \\ \vdots \\ (x_n^i)^2 \end{pmatrix}$$

has at least one component which is not zero since $x^i \neq 0$. This contradicts the assumption. □

Theorem B.14 (Solvability of (MLCP) [72]) Let $D = -B^\top, E = 0_{\mathbb{R}^{k\times k}}$. A feasible mixed complementarity problem (B.4a)-(B.4b) has at least one solution if A is symmetric and positiv semi-definite.

Proof: In [42] extended linear complementarity problems (XLCP) are analysed. Two vectors $y, z \in \mathbb{R}^m$ solve an (XLCP) if

$$0 \leq y \perp z \geq 0, \qquad My - Nz \in K$$

with $M, N \in \mathbb{R}^{n\times m}, \emptyset \neq K \subset \mathbb{R}^n$ is a polyhedron.An (MLCP) (B.4a)-(B.4b) is an (XLCP) with

$$M = \begin{pmatrix} B \\ E \end{pmatrix}, N = \begin{pmatrix} 0 \\ I \end{pmatrix}, K = \left\{ q \in \mathbb{R}^{n+m} | \exists\ x \in \mathbb{R}^n : \begin{pmatrix} -A \\ -D \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} = q \right\}.$$

Following [42], the (MLCP) with $A = A^\top, D = -B^\top, E = 0$ has one solution if

$$\left.\begin{array}{rcl} (B^\top u) * v & \leq & 0 \\ A^\top u - B^\top v & = & 0 \end{array}\right\} \Rightarrow (B^\top u) * v = 0.$$

If we use the equation and $(B^\top u) * v = u * (Bv)$ this property can be transformed to

$$u * (Au) \leq 0 \Rightarrow u * (Au) = 0.$$

With Lemma B.13 the claim follows. □

If we want to solve the variational inequality $VI(C, F)$ we could also use methods to solve inclusions, complementarity problems or non-smooth equations. A class of numerical methods for non-smooth equations are non-smooth Newton methods.

Non-smooth Newton Methods

Problem B.15 (Non-smooth equation) Let $G : C \subset \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function, but not necessarily differentiable. The problem to find an element $x \in C$ with

$$G(x) = 0$$

is said to be a non-smooth equation.

Remark B.16 (Smooth Newton-type methods) The most important methods to solve smooth non-linear equations with a differentiable function G are the Newton-type methods. The general idea of this schemes is to defining a sequence (x_k) that converges to the solution x. Therefore, G is replaced by an approximation depending on the current iterate which can be solved more easily to get the next iterate. If G is continuously differentiable this substitute term can be the linearisation

$$G(x_k) + \mathrm{J}G(x_k)(x - x_k)$$

where $\mathrm{J}G(x) \in \mathbb{R}^{m \times n}$ is the Jacobian of G in x. The zero of this function is the next iterate x_{k+1}. Using Taylor's expansion, it can be proofed that the sequence converges quadratically to the solution of Problem B.15 in a neighbourhood of x. Difficulties can arise in the determination of a suitable initial point x_0. If G is not differentiable it is not clear in general what assumptions on G and C are reasonable to define well-posed schemes. But for systems in non-smooth mechanics, generalised Newton-type methods are already established and studied. We utilise the rewritten formulation of the linearised problem

$$G(x_k) + \mathrm{J}G(x_k)d = 0, \qquad x_{k+1} = x_k + d.$$

During the numerical simulation, there could arise difficulties in the calculation of $\mathrm{J}G$ such that often approximations of the Jacobian are used. If $\mathcal{A}(x, d)$ is a family of approximations of $\mathrm{J}G(x)d$ all possible equations

$$G(x_k) + A(x_k, d) = 0, \qquad x_{k+1} = x_k + d$$

with $A(x, d) \in \mathcal{A}(x, d)$ define Newton-type sequences.

Remark B.17 (Clarke's calculus) Following Section 2.1.1, Lipschitz continuous functions are absolutely continuous function that define the generalised Clarke differential in Definition 2.7. For functions $G : \mathbb{R}^n \to \mathbb{R}^m$ this is defined as

$$\partial G(x) := \overline{\mathrm{co}} \left\{ \lim_{k \to \infty} \mathrm{J}G(x_k) : x_k \to x, \mathrm{J}G(x_k) \text{ exists} \right\}.$$

Theorem B.18 (Properties of the Clarke differential) Let $G : C \subset \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz continuous on a open set C. Then following [21], for all $x \in C$

(i) $\partial G(x) \neq \emptyset$,

(ii) $\partial G(x)$ is convex and compact,

(iii) $x \mapsto \partial G(x)$ is upper semi-continuous.

Example B.19 (Euclidean norm) Let $G(x) = \|x\|_2$. If $x \neq 0$ the gradient of calculates to $x/\|x\|_2$. In $x = 0$ the function is not differentiable but the Clarke differential exists with $\partial G^\top(0) = \partial B(0,1)$.

Following [32, Definition 7.2.2.], the following assumptions are needed to generalise Newton-type methods to non-smooth equations.

Definition B.20 (Newton approximation scheme) A function $G : C \subset \mathbb{R}^n \to \mathbb{R}^m$ has a Newton approximation scheme at point $x \in C$ if there is a neighbourhood $\bar{C} \subset C$ of x such that for all $\bar{x} \in \bar{C}$ a family $\mathcal{A}(x,d)$ of functions $\mathcal{A}(x,d) \ni A : \bar{C} \times \mathbb{R}^n \to \mathbb{R}^m$ exists with

(i) $A(x,0) = 0, A \in \mathcal{A}$,

(ii) $\lim_{\bar{x}\to x, A\in\mathcal{A}} \frac{\|G(x)-G(\bar{x})-A(\bar{x},x-\bar{x})\|}{\|\bar{x}-x\|} = 0$.

The family $\mathcal{A}$ is the Newton approximation scheme. It is called strong if (ii) is replaced by

(iii) $\lim_{\bar{x}\to x, A\in\mathcal{A}} \frac{\|G(x)-G(\bar{x})-A(\bar{x},x-\bar{x})\|}{\|\bar{x}-x\|^2} < \infty$.

The condition (ii) is a non-smooth correspondence to Taylor's expansion.

Theorem B.21 (Piecewise continuous functions) Let $G : C \subset \mathbb{R}^n \to \mathbb{R}^m, C$ open, be a piecewise continuous function, i.e.

$$G(\bar{x}) = G_l(\bar{x}), \quad x \in \mathcal{R}_l, \quad C = \cup_{l=1}^r \mathcal{R}_l$$

and G_l is continuously differentiable on $\mathcal{R}_l$. Furthermore, let $x \in C$ be a root of G sucht that all Jacobian matrices $JG_l(x)$ with $l \in \mathcal{P}(x) := \{i : G(x) = G_i(x) = 0\}$ are regular. Referring to [32, Theorem 7.2.15.], the set

$$\mathcal{A}(x,d) := \{JG_l(x)d, l \in \mathcal{P}(x)\}$$

is a Newton approximation scheme of G in x.

Example B.22 Let $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable in a neighbourhood of x with $f(x) = 0$ and $G : \mathbb{R} \to \mathbb{R}^+$ a composited function in the form

$$G(x) = \operatorname{proj}_{\mathbb{R}_+}(f(x)) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then, the set

$$\mathcal{A}(x,d) = \operatorname{co}\{0, f'(x)d\}$$

forms a Newton approximation scheme. The natural choice for implementations is to use

$$A(x,d) = \begin{cases} f'(x)d, & f(x) \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

If $G : \mathbb{R}^n \to \mathbb{R}^m_+$ with $G_i(x) = \operatorname{proj}_{\mathbb{R}_+}(f(x))$ this procedure is applied componentwise.

Algorithm 7 Non-smooth Newton method

Require: initial value x_0, TOL> 0, $\epsilon > 0$

procedure NonSmoothNewton(x_0)
    $k = 0$
    while $\|G(x_k)\| > \text{TOL}$ **do**
        Select $A(x_k, d) \in \mathcal{A}(x, d)$ and solve for $d \in B(0, \epsilon)$:
        $G(x_k) + A(x_k, d) = 0$
        $x_{k+1} = x_k + d$
        $k = k + 1$
    end while
    return x_k
end procedure

The difference of Algorithm 7 to a smooth Newton-type method is that d do not need to be unique.

Theorem B.23 (Local convergence result) Following [32, Theorem 7.2.5], let $G : C \subset \mathbb{R}^n \to \mathbb{R}^m$, C open, be locally Lipschitz in a neighbourhood of $x \in C$ with $G(x) = 0$ and has a Newton approximation scheme $\mathcal{A}(x, d)$ at x. Then, for all $\epsilon \in (0, \epsilon_{\mathcal{A}}]$ exists a ball $B(x, \delta)$ for which

(i) the sequence (x_k) defined by Algorithm 7 converges superlinear to x for $x_0 \in B(x, \delta)$.

(ii) If $\mathcal{A}(x, d)$ is strong, it converges quadratic.

C. Applications

Two further applications with a singular mass matrix or redundant constraints are enumerated.

Example C.1 (Shock absorber supported vehicle) Referring to [28], a model of a shock absorber supported vehicle (see Figure C.1) with zero-mass couplings is another application with a singular mass matrix and contact forces. A vehicle is travelling along a road of profile $h : I \to \mathbb{R}$ and consists of

(i) a tire with mass T, radius r and midpoint q_1 that is connected by a linear spring with $k_1 > 0$ and a damper with $c_1 > 0$ with

(ii) a vehicle with mass V and midpoint q_2,

(iii) a system of three spring-damping-systems with $k_i, c_i > 0, i = 2, 3, 4$, that symbol the spinal column of the driver and the head of the driver with mass D and midpoint q_7. The connection points between two springs or dampers are of zero-mass.

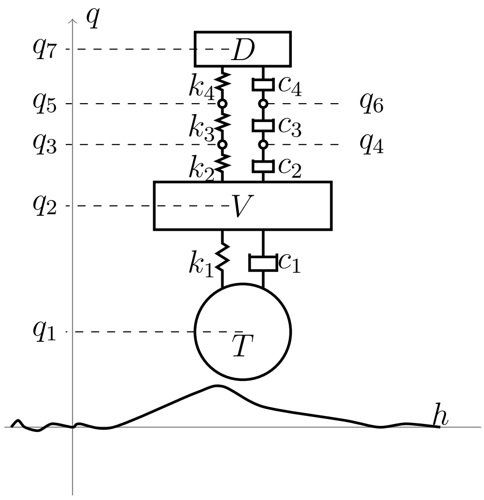


Figure C.1.: Shock absorber supported vehicle

The position coordinates are $q = (q_1, \ldots, q_7)^\top$. There is one geometrical constraint with

$$0 \le \lambda \perp q_1(t) - r - h(t) \ge 0.$$

This complementarity problem is combined with the differential system

$$\dot{q}(t) = v(t), \qquad M\mathrm{d}v(t) + Cv(t) + Kq(t) = f(t) + G(q)^\top \lambda$$

with $f = (-Tg\ \ -Vg\ 0\ 0\ 0\ 0\ \ -Dg)^\top, G(q) = (1\ 0\ 0\ 0\ 0\ 0\ 0)$ and

$$M = \begin{pmatrix} T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D \end{pmatrix}, C = \begin{pmatrix} c_1 & -c_1 & 0 & 0 & 0 & 0 & \\ -c_1 & c_1+c_2 & 0 & -c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_2 & 0 & c_2+c_3 & 0 & -c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_3 & 0 & c_3+c_4 & -c_4 \\ 0 & 0 & 0 & 0 & 0 & -c_4 & c_4 \end{pmatrix},$$

$$K = \begin{pmatrix} k_1 & -k_1 & 0 & 0 & 0 & 0 & \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & 0 & -k_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & k_3+k_4 & 0 & -k_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_4 & k_4 \end{pmatrix}$$

Example C.2 (Parallel five-bar linkage with side obstacle) The consideration of a planar parallel five-bar linkage (see Figure C.2) with one redundant constraint is taken from [13], studied also in [49] and expanded by an unilateral constraint.

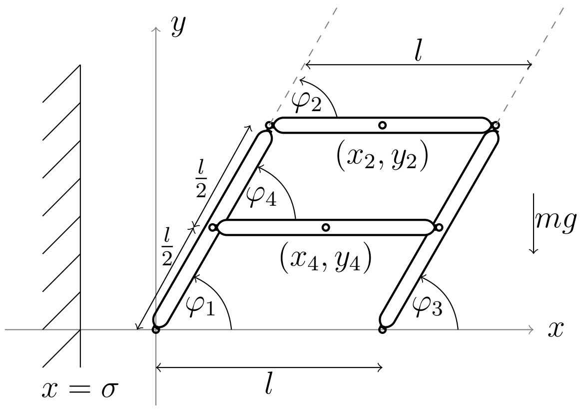


Figure C.2.: Parallel five bar linkage

The mixed Lagrangian and Cartesian coordinates are summarised to

$$q = (\varphi_1, x_2, y_2, \varphi_2, \varphi_3, x_4, y_4, \varphi_4)^\top.$$

The system is constrained by eight equalities

$$g_1(q) = x_2 - 0.5l\cos\varphi_2 - l\cos\varphi_1 = 0 \qquad g_5(q) = x_4 - 0.5l\cos\varphi_4 - 0.5l\cos\varphi_1 = 0$$
$$g_2(q) = y_2 - 0.5l\sin\varphi_2 - l\sin\varphi_1 = 0 \qquad g_6(q) = y_4 - 0.5l\cos\varphi_4 - 0.5l\sin\varphi_1 = 0$$
$$g_3(q) = x_2 + 0.5l\cos\varphi_2 - l\cos\varphi_3 = 0 \qquad g_7(q) = x_4 + 0.5l\cos\varphi_4 - 0.5l\cos\varphi_3 = 0$$
$$g_4(q) = y_2 + 0.5l\sin\varphi_2 - l\sin\varphi_3 = 0 \qquad g_8(q) = y_4 + 0.5l\sin\varphi_4 - 0.5l\sin\varphi_3 = 0$$

completed by two unilateral constraints

$$x_2 - \frac{l}{2} - \sigma \geq 0 \qquad x_4 - \frac{l}{2} - \sigma \geq 0.$$

The position of the system is uniquely determined by seven bilateral constraints such that one need to be redundant and the bilateral constraint matrix calculates to the rank-deficient matrix

$$G^b(q) = \begin{pmatrix} l\sin\varphi_1 & 1 & 0 & 0.5l\sin\varphi_2 & 0 & 0 & 0 & 0 \\ -l\cos\varphi_1 & 0 & 1 & -0.5l\cos\varphi_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.5l\sin\varphi_2 & l\sin\varphi_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5l\cos\varphi_2 & -l\cos\varphi_3 & 0 & 0 & 0 \\ 0.5l\sin\varphi_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0.5l\sin\varphi_4 \\ -0.5l\cos\varphi_1 & 0 & 0 & 0 & 0 & 0 & 1 & -0.5l\cos\varphi_4 \\ 0 & 0 & 0 & 0 & 0.5l\sin\varphi_3 & 1 & 0 & -0.5l\sin\varphi_4 \\ 0 & 0 & 0 & 0 & -0.5l\cos\varphi_3 & 0 & 1 & 0.5l\cos\varphi_4 \end{pmatrix}$$

and the unilateral constraint matrix to

$$G^u(q) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The equations of motion can be set up with

$$M = \operatorname{diag}\left(\frac{m_1 l^2}{3}, m_2, m_2, \frac{m_2 l^2}{12}, \frac{m_3 l^2}{3}, m_4, m_4, \frac{m_4 l^2}{12}\right),$$

$$f = \begin{pmatrix} -\frac{m_1 g l\cos\varphi_1}{2} & 0 & -m_2 g & 0 & -\frac{m_3 g l\cos\varphi_4}{2} & 0 & -m_4 g & 0 \end{pmatrix}^T.$$

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Selbstständigkeitsvereinbarung

Hiermit erkläre ich, dass ich die vorliegende Dissertation selbständig und ohne fremde Hilfe angefertigt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

Halle (Saale), den

Lebenslauf

	Manuela Paschkowski
Seit 04/19	Lehrkraft für besondere Aufgaben am Institut für Mathematik der Martin-Luther-Universität Halle-Wittenberg
02/15 – 03/19	Wissenschaftliche Mitarbeiterin am Institut für Mathematik der Martin-Luther-Universität Halle-Wittenberg, Fachbereich Numerische Mathematik
01/15	Master of Science in Wirtschaftsmathematik
11/12 – 01/15	Masterstudium der Wirtschaftsmathematik Martin-Luther-Universität Halle-Wittenberg
10/12	Bachelor of Science in Wirtschaftsmathematik Martin-Luther-Universität Halle-Wittenberg
10/08 – 10/12	Bachelorstudium der Wirtschaftsmathematik Martin-Luther-Universität Halle-Wittenberg
06/08–11/10	Werkstudentin Siemens AG, Leipzig, IT Solutions and Service
06/00-05/08	Georg-Cantor-Gymnasium Halle (Saale)

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