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Multi-Valued Parabolic Variational Inequalities and Related Variational-Hemivariational Inequalities

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Abstract

In this paper we study multi-valued parabolic variational inequalities involving quasilinear parabolic operators, and multi-valued integral terms over the underlying parabolic cylinder as well as over parts of the lateral parabolic boundary, where the multi-valued functions involved are assumed to be upper semicontinuous only. Note, since lower semicontinuous multi-valued functions allow always for a Carathéodory selection, this case can be considered as the trivial case, and therefore will be omitted. Our main goal is threefold: First, we provide an analytical frame work and an existence theory for the problems under consideration. Unlike in recent publications on multi-valued parabolic variational inequalities, the closed convex set K representing the constraints is not required to possess a nonempty interior. Second, we prove enclosure and comparison results based on a recently developed sub-supersolution method due to the authors. Third, we consider classes of relevant generalized parabolic variational-hemivariational inequalities that will be shown to be special cases of the multi-valued parabolic variational inequalities under consideration.

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Key words. Parabolic variational inequality, variational-hemivariational inequality, upper semicontinuous multi-valued operator, pseudomonotone multi-valued operator, comparison principle, sub-supersolution.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $Q = \Omega \times (0,\tau)$ a space-time cylindrical domain, and $\Gamma = \partial\Omega \times (0,\tau)$ its lateral boundary with $\tau > 0$. We assume that $\partial\Omega$ admits the decomposition

$$\partial\Omega = \overline{\Sigma_D} \cup \overline{\Sigma_N},$$

with pairwise disjoint, relatively open subsets Σ_D, Σ_N, where Σ_N is supposed to have positive surface measure, i.e., $\mathrm{meas}(\Sigma_N) \neq 0$. The corresponding parts of the lateral boundary are denoted by

$$\Gamma_D = \overline{\Sigma_D} \times (0,\tau), \quad \Gamma_N = \overline{\Sigma_N} \times (0,\tau),$$

such that $\Gamma = \Gamma_D \cup \Gamma_N$.

Let $W^{1,p}(\Omega)$ be the usual Sobolev space with its dual space $(W^{1,p}(\Omega))^*$, and denote by p' the Hölder conjugate satisfying $1/p + 1/p' = 1$. By V_0 we denote the closed subspace of $W^{1,p}(\Omega)$ given by

$$V_0 = \{u \in W^{1,p}(\Omega) : \gamma_{\partial\Omega} u|_{\Sigma_D} = 0\}, \tag{1.1}$$

where $\gamma_{\partial\Omega} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ stands for the trace operator which is known to be linear and compact. For the sake of simplicity we assume throughout this paper $2 \leq p < \infty$. Then $W^{1,p}(\Omega) \subset L^2(\Omega) \subset (W^{1,p}(\Omega))^*$ forms an evolution triple with all the imbeddings being dense and compact, cf. [27], and the same holds true for $V_0 \subset L^2(\Omega) \subset V_0^*$. Let $X := L^p(0,\tau; W^{1,p}(\Omega))$, and denote by $X_0 \subset X$ the subspace of X defined by

$$X_0 = \{u \in X : \gamma u|_{\Gamma_D} = 0\} = L^p(0,\tau; V_0),$$

where $\gamma : X \to L^p(\Gamma)$ stands for the trace operator, which is linear and compact. In what follows we are going to use the following notation

$$\gamma_N u := \gamma u|_{\Gamma_N}.$$

In this paper we are going to study the following multi-valued parabolic variational inequality: Find $u \in W_0 \cap K$, $\eta \in L^{p'}(Q)$ and $\zeta \in L^{p'}(\Gamma_N)$ such that

$$u(\cdot,0) = 0 \ \text{ in } \Omega, \ \eta \in f(\cdot,\cdot,u), \ \zeta \in f_N(\cdot,\cdot,\gamma_N u), \ \text{and} \tag{1.2}$$

$$\langle u_t + Au, v - u\rangle + \int_Q \eta\,(v-u)\,dxdt + \int_{\Gamma_N} \zeta\,(\gamma_N v - \gamma_N u)\,d\Gamma \geq 0, \tag{1.3}$$

for all $v \in K$, where K is a closed, convex subset of X_0, $W_0 = \{u \in X_0 : u_t \in X_0^*\}$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X_0^* and X_0. The operator $A : X_0 \to X_0^*$ is assumed to be a second order quasilinear differential operator of Leray-Lions type of the form

$$Au(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x,t,\nabla u(x,t)),$$

and $f : Q\times\mathbb{R} \to 2^{\mathbb{R}}$ with $(x,t,s) \mapsto f(x,t,s) \in 2^{\mathbb{R}}$ and $f_N : \Gamma_N\times\mathbb{R} \to 2^{\mathbb{R}}$ with $(x,t,s) \mapsto f_N(x,t,s) \in 2^{\mathbb{R}}$ are supposed to be upper semicontinuous multi-valued functions with respect to s that will be specified later.

Problem (1.2)–(1.3) was studied in [8] in the special case where $f : Q \times \mathbb{R} \to \mathbb{R}$ is a (single-valued) Carathéodory function and $\mathrm{meas}(\Gamma_N) = 0$, i.e. with homogeneous Dirichlet boundary conditions on Γ. An extension of the latter to the multi-valued case with $f(x,t,s) = \partial j(x,t,s)$, where

$s \mapsto \partial j(x,t,s)$ denotes Clarke's generalized gradient of some locally Lipschitz function $s \mapsto j(x,t,s)$ defined by

$$\partial j(x,t,s) := \{\zeta \in \mathbb{R} : j^o(x,t,s;r) \geq \zeta r, \ \forall\, r \in \mathbb{R}\}$$

(cf., e.g., [15, Chap. 2]), leads to the following associated multi-valued parabolic variational inequality: Find $u \in W_0 \cap K$ and $\eta \in L^{p'}(Q)$ such that

$$u(\cdot,0) = 0 \ \text{ in } \Omega, \ \ \eta(x,t) \in \partial j(x,t,u(x,t)), \tag{1.4}$$

$$\langle u_t + Au, v - u\rangle + \int_Q \eta\,(v-u)\,dxdt \geq 0, \ \ \forall\, v \in K. \tag{1.5}$$

Problem (1.4)–(1.5) has been considered in [13], where among others it was shown that (1.4)–(1.5) is equivalent to the following parabolic variational-hemivariational inequality: Find $u \in W_0 \cap K$ with $u(\cdot,0) = 0$ in Ω such that

$$\langle u_t + Au, v - u\rangle + \int_Q j^o(x,t,u;v-u)\,dxdt \geq 0, \ \ \forall\, v \in K, \tag{1.6}$$

where for a.e. $(x,t) \in Q$, $(s,r) \mapsto j^o(x,t,s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $s \mapsto j(x,t,s)$ at s in the direction r which is defined by

$$j^o(x,t,s;r) = \limsup_{y \to s,\ \varepsilon \downarrow 0} \frac{j(x,t,y+\varepsilon\, r) - j(x,t,y)}{\varepsilon},$$

(cf., e.g., [15, Chap. 2]). It is well known that Clarke's generalized gradient $s \mapsto \partial j(x,t,s)$ of a locally Lipschitz function $s \mapsto j(x,t,s)$ is an upper semicontinuous multi-valued function with closed and convex values. However, the reverse is not true, i.e., there are upper semicontinuous multi-valued functions that cannot be represented as a Clarke's gradient. For illustration let us consider the following simple but relevant example, which has been used in [2] to model certain friction laws.

Example 1.1. Define $f : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$f(s) = g(s) + h(s) \tag{1.7}$$

where $s \mapsto h(s)$ is the multi-valued function given by

$$h(s) = \begin{cases} -1 & \text{if} \quad s < 0, \\ [-1,1] & \text{if} \quad s = 0, \\ 1 & \text{if} \quad s > 0, \end{cases}$$

and $g : \mathbb{R} \to \mathbb{R}$ is the following (single-valued) discontinuous function:

$$g(s) = \begin{cases} \frac{1}{2} & \text{if} \quad s < 0, \\ 0 & \text{if} \quad s = 0, \\ -\frac{1}{2} & \text{if} \quad s > 0. \end{cases}$$

The multi-valued function f defined in (1.7) is apparently upper semicontinuous. However, it is easily seen that it cannot be represented by Clarke's generalized gradient of some locally Lipschitz function.

While for variational-hemivariational inequalities a rather complete mathematical theory based on Clarke's generalized gradient has been developed in recent years, a need in applications requires to consider more general parabolic variational-hemivariational inequalities of the following form: Find $u \in W_0 \cap K$ with $u(\cdot, 0) = 0$ in Ω such that

$$\begin{cases} \langle u_t + Au, v - u\rangle + \displaystyle\int_Q j^o(x, t, u, u; v - u)\, dxdt \\ + \displaystyle\int_{\Gamma_N} j_N^o(x, t, \gamma_N u, \gamma_N u; \gamma_N v - \gamma_N u)\, d\Gamma \geq 0, \ \forall\, v \in K, \end{cases} \tag{1.8}$$

where j, j_N given by

$$j : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ \text{ with } (x, t, r, s) \mapsto j(x, t, r, s),$$
$$j_N : \Gamma_N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ \text{ with } (x, t, r, s) \mapsto j_N(x, t, r, s),$$

are supposed to be locally Lipschitz functions with respect to s, and $j^o(x, t, r, s; \varrho)$ and $j_N^o(x, t, r, s; \varrho)$ denote Clarke's generalized directional derivatives at s in the direction ϱ for fixed (x, t, r). In particular, the following special case of (1.8) will be considered:

$$\begin{cases} \langle u_t + Au, v - u\rangle + \displaystyle\int_Q h(x, t, u)\hat{j}^o(x, t, u; v - u)\, dxdt \\ + \displaystyle\int_{\Gamma_N} h_N(x, t, u)\hat{j}_N^o(x, t, \gamma_N u; \gamma_N v - \gamma_N u)\, d\Gamma \geq 0, \ \forall\, v \in K, \end{cases} \tag{1.9}$$

where j and j_N of (1.8) have the special form:

$$j(x, t, r, s) = h(x, t, r)\hat{j}(x, t, s), \quad j_N(x, t, r, s) = h_N(x, t, r)\hat{j}_N(x, t, s), \tag{1.10}$$

where h, $\hat{j} : Q \times \mathbb{R} \to \mathbb{R}$ and h_N, $\hat{j}_N : \Gamma_N \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and $\hat{j} : Q \times \mathbb{R} \to \mathbb{R}$ and $\hat{j}_N : \Gamma_N \times \mathbb{R} \to \mathbb{R}$ are, in addition, locally Lipschitz with respect to s. In problem (1.8) (as well as in its special case (1.9)) the functions $s \mapsto j(\cdot, \cdot, s, s)$ and $s \mapsto j_N(\cdot, \cdot, s, s)$ may be not locally Lipschitz but only partially locally Lipschitz. This enlarges the class of variational-hemivariational inequalities considerably.

Our main goal is threefold: First, we provide an analytical frame work and an existence theory for the multi-valued parabolic variational inequality (1.2)–(1.3) with multi-valued upper semicontinuous functions in Q and on parts of the lateral parabolic boundary Γ_N. Here we remark that the closed convex set K representing the constraints is not required to possess a nonempty interior. Second, we prove enclosure and comparison results based on a recently developed sub-supersolution method due to the authors. The sub-supersolution method, which is of interest in its own right, will allow us to relax certain coercivity conditions required in the general existence theory. Moreover, this method will serve us as a tool to show that classes of parabolic variational-hemivariational inequalities of the form (1.8) or (1.9) are equivalent to a subclass of the multi-valued parabolic variational inequality (1.2)–(1.3). It should be noted that the treatment of parabolic variational inequalities with general upper semicontinuous multi-valued functions, that is one of the main goal of this manuscript, is not at all a straightforward matter, as new tools have to be developed. Moreover, such an extension is desirable not only for disciplinary reasons, but because it also meets the needs in applications. It is needless to say that (1.2)–(1.3) covers a wide range of parabolic problems when specifying K and/or f and f_N such as the special cases mentioned above as well as parabolic initial-boundary value problem in the case when $K = X_0$, and $f : Q\times\mathbb{R} \to \mathbb{R}$ and $f_N : \Gamma_N\times\mathbb{R} \to \mathbb{R}$ are (single-valued) Carathéodory functions.

This paper is organized as follows: After providing important preliminary results about the pseudomonotonicity (w.r.t. $D(L)$) of certain multi-valued Nemytskij operators related to f and f_N, which are of interest in its own right, we present a general existence result under coercivity assumptions, where some relative growth condition of A and f, f_N for u with large norm is imposed. In this case the existence of solutions of (1.2)–(1.3) follows from penalty arguments and the solvability of equations with multi-valued pseudomonotone operators. Then we establish a sub-supersolution method that will allow us to prove existence, comparison and enclosure results without imposing coercivity conditions as before. The concepts of sub- and supersolutions and the arguments in our case here are combinations of those for parabolic variational inequalities in [8] and those for multi-valued elliptic and parabolic variational inequalities in [11], [21] and [12, Chap.3]. Finally, we show that problems (1.8) and (1.9) are special cases of (1.2)–(1.3) only. Several notions here were originally presented in [8, 10, 11]; their detailed inclusion in this paper is only for the sake of completeness.

2 Notations, hypotheses, and a preliminary result

In this section, we introduce some notations and definitions, as well as a key preliminary result, which assures that the multi-valued Nemytskij operators generated by the multi-valued functions $f : Q \times \mathbb{R} \to 2^{\mathbb{R}}$ and $f_N : \Gamma_N \times \mathbb{R} \to 2^{\mathbb{R}}$ are multi-valued pseudomonotone operators with respect to the graph norm topology of the time-derivative operator $L := \partial/\partial t : D(L) \to X_0^*$ which will be specified later.

Further to the notations already given in Section 1 we introduce the space W defined by

$$W = \{u \in X : u_t \in X^*\},$$

where $X^* = L^{p'}(0, \tau; (W^{1,p}(\Omega))^*)$ is the dual of X, and the derivative $u_t := \partial u/\partial t$ is understood in the sense of vector-valued distributions. The space W endowed with the graph norm of the operator $\partial/\partial t$

$$\|u\|_W = \|u\|_X + \|u_t\|_{X^*}$$

is a Banach space which is separable and reflexive due to the separability and reflexivity of X and X^*, where $\|\cdot\|_X$ and $\|\cdot\|_{X_0}$ are the usual norms defined on X and X_0 (and similarly on X^* and X_0^*) :

$$\|u\|_X = \left(\int_0^\tau \|u(t)\|_{W^{1,p}(\Omega)}^p \, dt\right)^{1/p}, \ \|u\|_{X_0} = \left(\int_0^\tau \|u(t)\|_{V_0}^p \, dt\right)^{1/p}.$$

It is well known that the embedding $W \hookrightarrow C([0, \tau], L^2(\Omega))$ is continuous, and by Aubin's lemma the embedding $W \hookrightarrow\hookrightarrow L^p(Q)$ is compact due to the compact embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$. Similar properties hold true for the subspace W_0, i.e.,

$$W_0 = \{u \in X_0 : u_t \in X_0^*\},$$

introduced in Section 1. The notation $\langle \cdot, \cdot \rangle$ stands for any of the dual pairings between X and X^*, X_0 and X_0^*, $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$, and V_0 and V_0^*, such as for example, if $h \in X^*$ and $u \in X$, then

$$\langle h, u \rangle = \int_0^\tau \langle h(t), u(t) \rangle \, dt.$$

Here we should remark that $u \mapsto \left(\int_\Omega |\nabla u|^p \, dx\right)^{1/p}$, in general, does not define a norm in V_0, since Σ_D may be empty. Likewise $u \mapsto \left(\int_Q |\nabla u|^p \, dxdt\right)^{1/p}$ may be not a norm in $X_0 = L^p(0, \tau; V_0)$.

In what follows we denote by $L := \partial/\partial t$ when its domain of definition, $D(L)$, is given by

$$D(L) = \left\{u \in X_0 : u_t \in X_0^* \text{ and } u(\cdot, 0) = 0\right\}. \tag{2.1}$$

It is known that the linear operator $L : D(L) \subset X_0 \to X_0^*$ is closed, densely defined and maximal monotone, e.g., cf. [27, Chap. 32].

We assume the following Leray–Lions conditions on the coefficient functions a_i, $i = 1, \dots, N$, of the operator A.

(A1) $a_i : Q \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, i.e., $a_i(\cdot, \cdot, \xi) : Q \to \mathbb{R}$ are measurable for all $\xi \in \mathbb{R}^N$, and $a_i(x, t, \cdot) : \mathbb{R}^N \to \mathbb{R}$ are continuous for a.e. $(x, t) \in Q$. In addition, the following growth condition holds:
$$|a_i(x, t, \xi)| \le c_1|\xi|^{p-1} + c_2(x, t)$$
for a.e. $(x, t) \in Q$ and for all $\xi \in \mathbb{R}^N$, for some constant $c_1 > 0$ and some function $c_2 \in L_+^{p'}(Q)$.

(A2) For a.e. $(x, t) \in Q$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \ne \xi'$ the following monotonicity in ξ holds:
$$\sum_{i=1}^N (a_i(x, t, \xi) - a_i(x, t, \xi'))(\xi_i - \xi_i') > 0.$$

(A3) There is some constant $c_3 > 0$ such that for a.e. $(x, t) \in Q$ and for all $\xi \in \mathbb{R}^N$ the inequality
$$\sum_{i=1}^N a_i(x, t, \xi)\xi_i \ge c_3|\xi|^p - c_4(x, t)$$
is satisfied for some function $c_4 \in L^1(Q)$.

By (A1) the operator A defined by
$$\langle Au, \varphi\rangle := \int_Q \sum_{i=1}^N a_i(x, t, \nabla u)\frac{\partial \varphi}{\partial x_i}\, dx\, dt, \quad \forall \varphi \in X_0$$
is continuous and bounded from X (resp. X_0) into $X^* \subset X_0^*$. We denote by $L_+^p(Q)$ the positive cone of nonnegative elements of $L^p(Q)$. The natural partial ordering in $L^p(Q)$ is defined by $u \le v$ if and only if $v - u \in L_+^p(Q)$. If $\underline{u}, \overline{u} \in L^p(Q)$ with $\underline{u} \le \overline{u}$, we denote by
$$[\underline{u}, \overline{u}] = \{u \in L^p(Q) : \ \underline{u} \le u \le \overline{u}\}$$
the ordered interval formed by $\underline{u}$ and $\overline{u}$. The positive cone $L_+^p(Q)$ induces a corresponding partial ordering also in its subspaces. For functions w, z and sets W and Z of functions we use the notations: $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z : w \in W, z \in Z\}$, $W \vee Z = \{w \vee z : w \in W, z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. In particular, we denote $w^+ = w \vee 0$. For any normed vector space V we denote by $\mathcal{K}(V) \subset 2^V$ the following family of subsets of V
$$\mathcal{K}(V) = \{A \subset V : A \ne \emptyset, A \text{ is closed and convex}\}.$$

Let us recall the notion of (Vietoris) upper semicontinuous multi-valued functions. We refer to [1] (Chapter 1), [17] (Chapter 8), or [18] (Chapter 1) for more details on different types of continuities of multi-valued functions with respect to some usual topologies, such as the Vietoris, Hausdorff, Mosco, and Attouch–Wets topologies, on power sets of topological vector spaces.

Definition 2.1 *Let V, Y be Banach spaces, and $T : V \to 2^Y$ be a multi-valued function. T is called* **upper semicontinuous** *at $x_0 (\in V)$ if for every open subset $O \subset Y$ with $T(x_0) \subset O$, there exists a neighborhood $U(x_0)$ such that $T(U(x_0)) \subset O$. If T is upper semicontinuous at every $x_0 \in V$, we call T upper semicontinuous in V.*

Next we introduce the multi-valued Nemytskij operators F and F_N associated with the multi-valued functions $f : Q \times \mathbb{R} \to 2^{\mathbb{R}}$ and $f_N : \Gamma_N \times \mathbb{R} \to 2^{\mathbb{R}}$, respectively, by

$$\begin{aligned} F(u) &= \{\eta : Q \to \mathbb{R} : \eta \text{ is measurable in } Q \text{ and } \eta \in f(\cdot, \cdot, u)\}, \\ F_N(\gamma_N u) &= \{\zeta : \Gamma_N \to \mathbb{R} : \zeta \text{ is measurable on } \Gamma_N \text{ and } \zeta \in f_N(\cdot, \cdot, \gamma_N u)\} \end{aligned} \tag{2.2}$$

where $\eta(x,t) \in f(x,t,u(x,t))$ for a.e. $(x,t) \in Q$, and $\zeta(x,t) \in f_N(x,t,\gamma_N u(x,t))$ for a.e. $(x,t) \in \Gamma_N$. We impose the following hypotheses on f and f_N:

(F1) $f : Q \times \mathbb{R} \to \mathcal{K}(\mathbb{R}) \subset 2^{\mathbb{R}}$ and $f_N : \Gamma_N \times \mathbb{R} \to \mathcal{K}(\mathbb{R}) \subset 2^{\mathbb{R}}$ are graph measurable on $Q \times \mathbb{R}$ and $\Gamma_N \times \mathbb{R}$, respectively, that is,

$$\begin{aligned} &\mathrm{Gr}(f) := \{(x,t,u,\eta) \in Q \times \mathbb{R} \times \mathbb{R} : \eta \in f(x,t,u)\} \text{ and },\\ &\mathrm{Gr}(f_N) := \{(x,t,u,\zeta) \in \Gamma_N \times \mathbb{R} \times \mathbb{R} : \zeta \in f_N(x,t,\gamma_N u)\} \end{aligned}$$

belong to $[\mathcal{L}(Q) \times \mathcal{B}(\mathbb{R})] \times \mathcal{B}(\mathbb{R})$ and $[\mathcal{L}(\Gamma_N) \times \mathcal{B}(\mathbb{R})] \times \mathcal{B}(\mathbb{R})$, respectively, where $\mathcal{L}(Q)$ and $\mathcal{L}(\Gamma_N)$ are the families of Lebesgue measurable subsets of Q and Γ_N, respectively, and $\mathcal{B}(\mathbb{R})$ is the σ-algebra of Borel sets in $\mathbb{R}$.

(F2) For a.e. $(x,t) \in Q$, the function $f(x,t,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous, and for a.e. $(x,t) \in \Gamma_N$, $f_N(x,t,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous as well in the sense of Definition 2.1.

(F3) f satisfies the growth condition

$$\sup\{|\eta| : \eta \in f(x,t,s)\} \le \alpha(x,t) + \beta|s|^{p-1} \tag{2.3}$$

for a.e. $(x,t) \in Q$, $\forall\ s \in \mathbb{R}$, where $\alpha \in L^{p'}(Q)$, and $\beta \ge 0$. Similarly, f_N satisfies the growth condition

$$\sup\{|\zeta| : \zeta \in f_N(x,t,s)\} \le \alpha_N(x,t) + \beta_N|s|^{p-1} \tag{2.4}$$

for a.e. $(x,t) \in \Gamma_N$, $\forall\ s \in \mathbb{R}$, where $\alpha_N \in L^{p'}(\Gamma_N)$, and $\beta_N \ge 0$.

It follows from (F1) and (F2) that the functions $(x,t) \mapsto f(x,t,u(x,t))$ and $(x,t) \mapsto f_N(x,t,v(x,t))$ are also measurable functions from Q to $2^{\mathbb{R}}$ and Γ_N to $2^{\mathbb{R}}$, for any measurable functions $u : Q \to \mathbb{R}$ and $v : \Gamma_N \to \mathbb{R}$, respectively. Further, by (F3), the multi-valued Nemytskij operators $F : L^p(Q) \to 2^{L^{p'}(Q)}$ and $F_N : L^p(\Gamma_N) \to 2^{L^{p'}(\Gamma_N)}$ are well defined.

Let $i : X_0 \hookrightarrow L^p(Q)$ be the (continuous) embedding of X_0 into $L^p(Q)$, and let $i^* : L^{p'}(Q) \hookrightarrow X_0^*$ be its adjoint. Denoting by $\gamma_N^* : L^{p'}(\Gamma_N) \to X_0^*$ the adjoint operator of the trace operator $\gamma_N : X_0 \to L^p(\Gamma_N)$, then the composed multi-valued operator

$$\mathcal{F} = i^* \circ F \circ i : X_0 \to 2^{X_0^*} \quad \text{and} \quad \mathcal{F}_N = \gamma_N^* \circ F_N \circ \gamma_N : X_0 \to 2^{X_0^*}$$

will be shown to possess a certain pseudomonotonicity property which is important for our main existence result to be proved in the next section, and which is of interest in its own right. To this end let us first provide the following definition of a multi-valued pseudomonotone operator with respect to the graph norm topology of the domain $D(L)$ (w.r.t. $D(L)$ for short) of some linear, closed, densely defined and maximal monotone operator $L : D(L) \subset Y \to Y^*$. We refer to [7] for the original concept of multi-valued pseudomonotone operators and to [26] (see also [16]) for that of multi-valued pseudomonotone operators w.r.t. $D(L)$. As is seen in the proof of Proposition 2.3, the operator $\mathcal{F}$ introduced above is pseudomonotone w.r.t. $D(L)$ with $L = \partial/\partial t$ but not necessarily pseudomonotone in the regular sense. This holds likewise for $\mathcal{F}_N$.

Definition 2.2 *Let Y be a reflexive Banach space, and let $L : D(L) \subset Y \to Y^*$ be a linear, closed, densely defined and maximal monotone operator. The operator $\mathcal{T} : Y \to 2^{Y^*}$ is called* **pseudomonotone w.r.t.** *$D(L)$ if the following conditions are satisfied:*

(i) The set $\mathcal{T}(u)$ is nonempty, bounded, closed and convex for all $u \in Y$.

(ii) $\mathcal{T}$ is upper semicontinuous from each finite dimensional subspace of Y to Y^* equipped with the weak topology.

(iii) If $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in Y, $Lu_n \rightharpoonup Lu$ in Y^*, $u_n^* \in \mathcal{T}(u_n)$ with $u_n^* \rightharpoonup u^*$ in Y^* and $\limsup\langle u_n^*, u_n - u\rangle \le 0$, then $u^* \in \mathcal{T}(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

We have the following result about the pseudomonotonicity of $\mathcal{F} = i^* \circ F \circ i$, which is an appropriate adaptation of Lemma 3.6 in [21] to the evolutionary case.

Proposition 2.3 *Under conditions (F1)–(F3), the mapping $\mathcal{F} = i^* \circ F \circ i : X_0 \to 2^{X_0^*}$ is pseudomonotone with respect to $D(L)$, where $L = \partial/\partial t$ and $D(L)$ is given by (2.1).*

Proof. The proof of this proposition is based on the ideas and arguments in the proof of Lemma 3.6, [21], and is divided into several steps.

Step 1:
First, we prove that for any $u \in L^p(Q)$, $F(u)$ is a nonempty, bounded, closed, and convex subset of $L^{p'}(Q)$, and thus in particular, $F(u) \in \mathcal{K}(L^{p'}(Q))$. Moreover, $F : L^p(Q) \to 2^{L^{p'}(Q)}$ is shown to be a bounded mapping. The convexity of $F(u)$ follows from the fact that $f(x, t, u)$ is a closed interval in $\mathbb{R}$. Let $\eta \in F(u)$. As a consequence of (2.3),

$$|\eta(x, t)| \le \alpha(x, t) + \beta|u(x, t)|^{p-1}, \quad \text{a.e. } (x, t) \in Q. \tag{2.5}$$

Since $|u|^{p-1} \in L^{p'}(Q)$, we have the boundedness of $F(u)$ in $L^{p'}(Q)$. To prove that $F(u)$ is closed in $L^{p'}(Q)$, let $\{\eta_n\}$ be a sequence in $F(u)$ such that $\eta_n \to \eta$ in $L^{p'}(Q)$. By passing to a subsequence, we can assume without loss of generality that $\eta_n(x, t) \to \eta(x, t)$ for a.e. $(x, t) \in Q$. Since $\eta_n(x, t) \in f(x, t, u(x, t))$ for a.e. $(x, t) \in Q$, all $n \in \mathbb{N}$, and $f(x, t, u(x, t))$ is closed in $\mathbb{R}$, we have $\eta(x, t) \in f(x, t, u(x, t))$. Since this holds for a.e. $(x, t) \in Q$, we have $\eta \in F(u)$, which proves the closedness of $F(u)$ in $L^{p'}(Q)$. Inequality (2.5) also proves that if Z is a bounded set in $L^p(Q)$ then $F(Z)$ is a bounded set in $L^{p'}(Q)$, that is, F is a bounded mapping from $L^p(Q)$ to $2^{L^{p'}(Q)}$.

For $u \in X_0$, from the boundedness of i^* and the above arguments we see that $\mathcal{F}(u)$ is a nonempty, convex and bounded subset of X_0^*. Moreover, since $\|i^*\eta\|_{X_0^*} \le C\|\eta\|_{L^{p'}(Q)}, \forall\, \eta \in L^{p'}(Q)$ for some constant C, it follows from the boundedness of F that $\mathcal{F}$ is also a bounded mapping. Next, we prove that $\mathcal{F}(u)$ is closed in X_0^*. In fact, assume $\{\eta_n\} \subset \mathcal{F}(u)$, $\eta_n = i^*\tilde{\eta}_n$ with $\tilde{\eta}_n \in F(iu) = F(u)$, $\forall n \in \mathbb{N}$, and

$$\eta_n \to \eta \text{ in } X_0^*. \tag{2.6}$$

Because $\{\tilde{\eta}_n :\in \mathbb{N}\} \subset F(u)$, $\{\tilde{\eta}_n\}$ is a bounded sequence in $L^{p'}(Q)$. By passing to a subsequence if necessary we can assume without loss of generality that

$$\tilde{\eta}_n \rightharpoonup \tilde{\eta}_0 \text{ in } L^{p'}(Q). \tag{2.7}$$

Since $F(u)$ is weakly closed in $L^{p'}(Q)$, $\tilde{\eta}_0 \in F(u)$ and thus $i^*\tilde{\eta}_0 \in i^*F(u) = \mathcal{F}(u)$. On the other hand, since i^* is continuous from $L^{p'}(Q)$ to X_0^* both with weak topologies, we have from (2.7) that

$$\eta_n = i^*\tilde{\eta}_n \rightharpoonup i^*\tilde{\eta}_0 \text{ in } X_0^*,$$

which combined with (2.6) yields $\eta = i^*\tilde{\eta}_0 \in \mathcal{F}(u)$. Hence, $\mathcal{F}(u)$ is closed in X_0^*.

Step 2:
Let V be a finite dimensional subspace of X_0. We prove in this step that the restriction $\mathcal{F}|_V$ of $\mathcal{F}$ on V is upper semicontinuous from V into $2^{X_0^*}$ with respect to the weak topology of X_0^*.

In fact, assume $u_0 \in V$. To prove the upper semicontinuity of $\mathcal{F}|_V$ at u_0, we assume by contradiction that there is a weakly open neighborhood U of $\mathcal{F}(u_0)$ in X_0^* and a sequence $(u_n) \subset V$ such that $u_n \to u_0$ in V and there exists a sequence $(\eta_n) \subset X_0^*$ such that $\eta_n \in \mathcal{F}(u_n) \setminus U$, $\forall n \in \mathbb{N}$. We see that $\tilde{U} = (i^*)^{-1}(U)$ is a weakly open neighborhood of $F(u_0)$ in $L^{p'}(Q)$. Moreover, since $\eta_n \in i^*F(u_n)$, there exists $\tilde{\eta}_n \in F(u_n)$ such that

$$\eta_n = i^*\tilde{\eta}_n. \tag{2.8}$$

We have $\tilde{\eta}_n \notin \tilde{U}$ for all $n \in \mathbb{N}$. As (u_n) is a bounded sequence in $L^p(Q)$, it follows from Step 1 that $(\tilde{\eta}_n)$ is a bounded sequence in $L^{p'}(Q)$. Also, as above by passing to a subsequence we can assume that

$$\tilde{\eta}_n \rightharpoonup \tilde{\eta}_0 \text{ in } L^{p'}(Q). \tag{2.9}$$

Since $u_n \to u_0$ in $L^p(Q)$, we have $h^*(F(u_n), F(u_0)) \to 0$ (see [18, Theorem 7.26]), where

$$h^*(A, B) = h^*_{L^{p'}(Q)}(A, B) = \sup_{u\in A}\left(\inf_{v\in B} \|u - v\|_{L^{p'}(Q)}\right)$$

is part of the Hausdorff distance between subsets A, B of $L^{p'}(Q)$, which is defined as $h(A, B) = \max\{h^*(A, B), h^*(B, A)\}$. (Note that this property of F is referred in [18] as its Hausdorff upper semicontinuity (h-upper semicontinuity). According to Definition 2.60, [18], a multi-valued function F is h-upper semicontinuous at u_0 if the function $u \mapsto h^*(F(u), F(u_0))$ is continuous at u_0. On the other hand, F is h-lower semicontinuous at u_0 if the function $u \mapsto h^*(F(u_0), F(u))$ is continuous at u_0. It follows immediately from these definitions that F is continuous at u_0 with respect to the topology generated by the Hausdorff distance on $\mathcal{K}(L^{p'}(Q))$ if and only if F is both h-upper semicontinuous and h-lower semicontinuous at u_0.)

Since

$$h^*(F(u_n), F(u_0)) \geq \text{dist}_{L^{p'}(Q)}(\tilde{\eta}_n, F(u_0)) = \inf\{\|\tilde{\eta}_n - v\|_{L^{p'}(Q)} : v \in F(u_0)\},$$

there is a sequence $(\bar{\eta}_n) \subset F(u_0)$ such that $\|\tilde{\eta}_n - \bar{\eta}_n\|_{L^{p'}(Q)} \to 0$. From (2.9), we have $\bar{\eta}_n \rightharpoonup \tilde{\eta}_0$ in $L^{p'}(Q)$. Since $F(u_0)$ is weakly closed in $L^{p'}(Q)$, we get $\tilde{\eta}_0 \in F(u_0)$ and thus $\tilde{\eta}_0 \in \tilde{U}$. Again from (2.9) we have $\tilde{\eta}_n \in \tilde{U}$ for all n sufficiently large, contradicting (2.8) and the assumption on η_n, and therefore proving the upper semicontinuity of $\mathcal{F}|_V$.

Step 3:
First, let us prove that $\mathcal{F}$ is sequentially weakly closed from X_0 with respect to the $D(L)$-graph topology into $2^{X_0^*} \setminus \{\emptyset\}$ with respect to the weak topology of X_0^*, that is, if (u_n) and (η_n) are sequences in $D(L)$ and X_0^* respectively such that

$$u_n \rightharpoonup u \text{ in } X_0, u_{nt} \rightharpoonup u_t \text{ in } X_0^*, \tag{2.10}$$

$$\eta_n \rightharpoonup \eta \text{ in } X_0^*, \tag{2.11}$$

and

$$\eta_n \in \mathcal{F}(u_n), \ \forall n \in \mathbb{N}, \tag{2.12}$$

then,

$$\eta \in \mathcal{F}(u). \tag{2.13}$$

In fact, assume (2.10)–(2.12). We have $i(u_n) = u_n$ and $i^*(\eta_n) = \eta_n|_{X_0^*}$. From (2.12), for each $n \in \mathbb{N}$, there exists $\tilde{\eta}_n \in F(i(u_n)) = F(u_n)$ such that $\eta_n = i^*(\tilde{\eta}_n) = \tilde{\eta}_n|_{X_0^*}$. From (2.10) and Aubin' lemma (cf. [22]), we have

$$u_n = i(u_n) \to i(u) = u \text{ in } L^p(Q). \tag{2.14}$$

Again, from the h-upper semicontinuity of F from $L^p(Q)$ to $2^{L^{p'}(Q)}$ (see [18, Theorem 7.26]), we have

$$h^*(F(u_n), F(u)) \to 0. \tag{2.15}$$

Since $\tilde{\eta}_n \in F(u_n)$,

$$\inf_{v \in F(u)} \|\tilde{\eta}_n - v\|_{L^{p'}(Q)} \le h^*(F(u_n), F(u)).$$

Hence, $\inf_{v\in F(u)} \|\tilde{\eta}_n - v\|_{L^{p'}(Q)} \to 0$ as $n \to \infty$, and there exists a sequence $(\eta_n^*) \subset F(u)$ such that

$$\lim_{n\to\infty} \|\tilde{\eta}_n - \eta_n^*\|_{L^{p'}(Q)} = 0. \tag{2.16}$$

Since $(\eta_n^*) \subset F(u)$ and $F(u)$ is a bounded subset of $L^{p'}(Q)$, by passing to a subsequence if necessary, we can assume that

$$\eta_n^* \rightharpoonup \eta_0 \text{ in } L^{p'}(Q) \tag{2.17}$$

for some $\eta_0 \in L^{p'}(Q)$. As $F(u)$ is weakly closed in $L^{p'}(Q)$, $\eta_0 \in F(u)$. Hence, (2.16) and (2.17) imply that

$$\tilde{\eta}_n \rightharpoonup \eta_0 \text{ in } L^{p'}(Q). \tag{2.18}$$

Since i^* is continuous in the weak topologies of both $L^{p'}(Q)$ and X_0^*, it follows from (2.18) that

$$\eta_n = i^*(\tilde{\eta}_n) = \tilde{\eta}_n|_{X_0^*} \rightharpoonup i^*(\eta_0) = \eta_0|_{X_0^*} \tag{2.19}$$

weakly in X_0^*. From (2.11) and (2.19), we have $\eta = i^*(\eta_0) \in i^*F(u)$, since $\eta_n \rightharpoonup \eta$ and $\eta_n \rightharpoonup i^*(\eta_0)$ both in the sense of distribution. The inclusion (2.13) is thus verified, which completes our proof of the weakly closed property of $\mathcal{F}$.

Next, we prove that if $(u_n) \subset D(L)$, $(\eta_n) \subset X_0^*$ are sequences satisfying (2.10)–(2.12) then

$$\langle \eta_n, u_n\rangle_{X_0^*,X_0} \to \langle \eta, u\rangle_{X_0^*,X_0}. \tag{2.20}$$

In fact, let $(\tilde{\eta}_n)$ and η_0 be as above. We have

$$\begin{aligned} \langle \eta_n, u_n\rangle_{X_0^*,X_0} &= \langle \tilde{\eta}_n|_{X_0^*}, u_n\rangle_{X_0^*,X_0} \\ &= \langle i^*(\tilde{\eta}_n), u_n\rangle_{X_0^*,X_0} \\ &= \langle \tilde{\eta}_n, i(u_n)\rangle_{L^{p'}(Q),L^p(Q)} = \langle \tilde{\eta}_n, u_n\rangle_{L^{p'}(Q),L^p(Q)}. \end{aligned} \tag{2.21}$$

From (2.14) and (2.18), we have

$$\begin{aligned} \langle \tilde{\eta}_n, u_n\rangle_{L^{p'}(Q),L^p(Q)} &\to \langle \eta_0, u\rangle_{L^{p'}(Q),L^p(Q)} = \langle \eta_0, i(u)\rangle_{L^{p'}(Q),L^p(Q)} \\ &= \langle i^*(\eta_0), u\rangle_{X_0^*,X_0} \\ &= \langle \eta, u\rangle_{X_0^*,X_0}. \end{aligned}$$

This limit, together with (2.21), proves (2.20).

The weakly closed property of $\mathcal{F}$ and (2.20) show that $\mathcal{F}$ is pseudomonotone from X_0 to $2^{X_0^*}$ with respect to $D(L)$. □

Proposition 2.4 *Under conditions (F1)–(F3), the mapping* $\mathcal{F}_N = \gamma_N^* \circ F_N \circ \gamma_N : X_0 \to 2^{X_0^*}$ *is pseudomonotone with respect to* $D(L)$.

Proof. Since the trace operator $\gamma : X \to L^p(\Gamma)$ is linear and continuous, as well as $\gamma : W \to L^p(\Gamma)$ is linear and compact (see, e.g., [10]), the proof can be done in just the same way as the proof of Proposition 2.3. □

3 Existence results

In this section we prove an existence results about problem (1.2)–(1.3), which can be equivalently rewritten in the form: Find $u \in D(L) \cap K$, $\eta \in L^{p'}(Q)$ and a $\zeta \in L^{p'}(\Gamma_N)$ such that

$$\begin{cases} \eta \in F(u), \ \zeta \in F_N(\gamma_N u) \\ \langle u_t + Au, v - u\rangle + \displaystyle\int_Q \eta\,(v-u)\,dxdt + \int_{\Gamma_N} \zeta\,(\gamma_N v - \gamma_N u)\,d\Gamma \geq 0, \end{cases} \tag{3.1}$$

for all $v \in K$, which in turn can be rewritten in the form: Find $u \in D(L) \cap K$, $\eta \in L^{p'}(Q)$ and a $\zeta \in L^{p'}(\Gamma_N)$ such that

$$\begin{cases} \eta \in F(u), \ \zeta \in F_N(\gamma_N u) \\ \langle Lu + Au + i^*\eta + \gamma^*\zeta, v - u\rangle \geq 0, \ \ \forall\, v \in K. \end{cases} \tag{3.2}$$

The proof of the existence of solutions of (3.1) is based on a penalty approach. For this purpose, let us first recall the following general definition of a penalty operator associated with a convex set C in a reflexive Banach space Y.

Definition 3.1 *Let $C \neq \emptyset$ be a closed and convex subset of a reflexive Banach space Y. A bounded, hemicontinuous and monotone operator $P : Y \to Y^*$ is called a penalty operator associated with $C \subset Y$ if*

$$P(u) = 0 \iff u \in C.$$

In what follows, we assume that there exists a penalty operator $P : X_0 \to X_0^*$ associated with the given convex set $K \subset X_0$ satisfying the following properties:

(P) For each $u \in D(L)$, there exists $w = w(u) \in X_0$, with $w \neq 0$ if $P(u) \neq 0$, such that

$$\begin{array}{ll} \text{(i)} & \langle u_t + Au, w\rangle \geq 0, \ \text{ and} \\ \text{(ii)} & \langle Pu, w\rangle \geq D\|Pu\|_{X_0^*}(\|w\|_{L^p(Q)} + \|\gamma_N w\|_{L^p(\Gamma_N)}), \end{array} \tag{3.3}$$

for some constant $D > 0$ independent of u and w.

Application: Penalty operator of an obstacle

We consider an obstacle problem, where the convex, closed set K is given by

$$K = \{u \in X_0 : u \leq \psi \ \text{ a.e. in } \ Q\},$$

with any obstacle function ψ specified as follows:

(i) $\psi \in W$ and $\psi(\cdot, 0) \geq 0$ on Ω, $\gamma\psi|_{\Gamma_D} \geq 0$, and

(ii) $\psi_t + A\psi \geq 0$ in X_0^*, i.e., $\langle \psi_t + A\psi, v\rangle \geq 0$, $\forall v \in X_0 \cap L^p_+(Q)$.

The penalty operator $P : X_0 \to X_0^*$ can be chosen as

$$\langle P(u), v\rangle = \int_Q [(u-\psi)^+]^{p-1}\, v\,dxdt + \int_{\Gamma_N} [(\gamma_N u - \gamma_N \psi)^+]^{p-1}\,\gamma_N v\,d\Gamma, \tag{3.4}$$

for all $u, v \in X_0$. Indeed, P is bounded, continuous and monotone. Let us check that

$$P(u) = 0 \iff u \in K.$$

If $P(u) = 0$, then $(u - \psi)^+ = 0$ a.e. in Q, i.e.,

$$u \le \psi \text{ a.e. in } Q,$$

that is $u \in K$. Conversely, assume that $u \in K$. Then, for a.a. $t \in (0, \tau)$, we have $u(\cdot, t) \le \psi(\cdot, t)$ a.e. in Ω, which implies that

$$\gamma_{\partial\Omega} u(\cdot, t) \le \gamma_{\partial\Omega} \psi(\cdot, t) \text{ a.e. on } \partial\Omega$$

($\gamma_{\partial\Omega}$ is the trace operator on $\partial\Omega$). This means that $\gamma u \le \gamma\psi$ a.e. on Γ, and thus, in particular, $\gamma_N u \le \gamma_N \psi$ showing that $P(u) = 0$. Now we need to check that P satisfies properties (3.3) of (P). For each $u \in D(L)$ we choose $w = (u - \psi)^+$. Then, $w \in X_0$ and $w \ne 0$ whenever $P(u) \ne 0$. We justify that (3.3)(i) is satisfied. According to assumption (i) for ψ, $(u - \psi)^+(\cdot, 0) = 0$, we have

$$\langle u_t - \psi_t, (u - \psi)^+ \rangle = \frac{1}{2} \|(u - \psi)^+(\cdot, \tau)\|^2_{L^2(\Omega)} \ge 0.$$

Combining the last inequality with $\langle Au - A\psi, (u - \psi)^+ \rangle \ge 0$, we arrive at

$$\langle u_t + Au, (u - \psi)^+ \rangle \ge \langle \psi_t + A\psi, (u - \psi)^+ \rangle \ge 0,$$

because $(u - \psi)^+ \in X_0 \cap L^p_+(Q)$. So we have checked (3.3)(i) of (P). To verify (3.3)(ii), we note that

$$\langle P(u), w \rangle = \|(u - \psi)^+\|^p_{L^p(Q)} + \|(\gamma_N u - \gamma_N \psi)^+\|^p_{L^p(\Gamma_N)}, \tag{3.5}$$

which yields by applying Hölder's inequality: There exists some constant $c > 0$ such that

$$\begin{aligned} |\langle P(u), v \rangle| &\le \|(u - \psi)^+\|^{p-1}_{L^p(Q)} \|v\|_{L^p(Q)} + \|(\gamma_N u - \gamma_N \psi)^+\|^{p-1}_{L^p(\Gamma_N)} \|v\|_{L^p(\Gamma_N)} \\ &\le c(\|(u - \psi)^+\|^{p-1}_{L^p(Q)} + \|(\gamma_N u - \gamma_N \psi)^+\|^{p-1}_{L^p(\Gamma_N)}) \|v\|_{X_0}, \end{aligned}$$

for all $v \in X_0$. Hence,

$$\|P(u)\|_{X_0^*} \le c(\|(u - \psi)^+\|^{p-1}_{L^p(Q)} + \|(\gamma_N u - \gamma_N \psi)^+\|^{p-1}_{L^p(\Gamma_N)}), \ \forall u \in X_0,$$

which, by taking into account (3.5), finally implies (P)(ii). Note also, for our example of K, the following lattice conditions:

$$K \wedge K \subset K, \quad K \vee K \subset K \tag{3.6}$$

are satisfied.

Our main result in this section is the following existence result of solutions of (3.1) (resp.(3.2). For its formulation and proof we are going to use the notation:

$$\eta^* = i^* \eta \in \mathcal{F}(u) \text{ iff } \eta \in F(u), \quad \zeta^* = \gamma_N^* \zeta \in \mathcal{F}_N(u) \text{ iff } \zeta \in F_N(\gamma_N u).$$

Theorem 3.2 *Assume (A1)–(A3) and that f, f_N satisfy hypotheses (F1)–(F3). Suppose $D(L) \cap K \ne \emptyset$ and that u_0 is an element of $D(L) \cap K$. Then, under the coercivity condition*

$$\lim_{\|u\|_{X_0} \to \infty} \left[\inf_{\substack{\eta^* \in \mathcal{F}(u) \\ \zeta^* \in \mathcal{F}_N(u)}} \frac{\langle Au + \eta^* + \zeta^*, u - u_0 \rangle}{\|u\|_{X_0}} \right] = \infty, \tag{3.7}$$

the multi-valued parabolic variational inequality (3.1) has solutions.

Proof. For $\varepsilon > 0$, let us consider the following penalized equation: Find $u \in D(L)$, $\eta^* \in \mathcal{F}(u)$, and $\zeta^* \in \mathcal{F}_N(u)$ such that

$$\langle u_t, v\rangle + \langle Au + \eta^* + \zeta^*, v\rangle + \frac{1}{\varepsilon}\langle Pu, v\rangle = 0, \ \forall v \in X_0, \tag{3.8}$$

where P is a penalty operator (associated to K) that satisfies (3.3). From Proposition 2.3, $\mathcal{F}$ and $\mathcal{F}_N$ are pseudomonotone with respect to $D(L)$. Since A and $\varepsilon^{-1}P$ are monotone and hemicontinuous, they are pseudomonotone and thus pseudomonotone with respect to $D(L)$ (cf. e.g. Proposition 27.6, [27]). As a consequence, the operator $A + \mathcal{F} + \mathcal{F}_N + \varepsilon^{-1}P$ is pseudomonotone with respect to $D(L)$. Moreover, it is bounded since $A, P, \mathcal{F}$ and $\mathcal{F}_N$ are bounded mappings. From the coercivity condition (3.7) and the monotonicity of $\varepsilon^{-1}P$, it is easy to see that $A + \mathcal{F} + \mathcal{F}_N + \varepsilon^{-1}P$ is coercive on X_0 in the following sense:

$$\lim_{\|u\|_{X_0}\to\infty}\left[\inf_{\substack{\eta^* \in \mathcal{F}(u)\\ \zeta^* \in \mathcal{F}_N(u)}} \frac{\langle (A + \varepsilon^{-1}P)(u) + \eta^* + \zeta^*, u - u_0\rangle}{\|u\|_{X_0}}\right] = \infty. \tag{3.9}$$

According to the surjectivity result of [16, Theorem 1.3.73, p. 62], (3.8) has solutions for each $\varepsilon > 0$. Let $u_\varepsilon, \eta^*_\varepsilon, \zeta^*_\varepsilon$ satisfy (3.8), and let $(u_\varepsilon), (\eta^*_\varepsilon), (\zeta^*_\varepsilon)$ stand for sequences where $\varepsilon > 0$ is the sequence parameter tending to zero. We show that the family $\{u_\varepsilon : \varepsilon > 0, \text{ small}\}$ is bounded with respect to the graph norm of $D(L)$. In fact, let u_0 be a (fixed) element of $D(L) \cap K$. Putting $v = u_\varepsilon - u_0$ into (3.8) (with u_ε) and noting the monotonicity of L and that $Pu_0 = 0$, one gets

$$\begin{aligned}\langle -u_{0t}, u_\varepsilon - u_0\rangle &= \langle u_{\varepsilon t} - u_{0t}, u_\varepsilon - u_0\rangle + \langle Au_\varepsilon + \eta^*_\varepsilon + \zeta^*_\varepsilon, u_\varepsilon - u_0\rangle\\ &\quad + \frac{1}{\varepsilon}\langle Pu_\varepsilon - Pu_0, u_\varepsilon - u_0\rangle\\ &\geq \langle Au_\varepsilon + \eta_\varepsilon + \zeta^*_\varepsilon, u_\varepsilon - u_0\rangle.\end{aligned}$$

Thus,

$$\frac{\langle Au_\varepsilon + \eta^*_\varepsilon + \zeta^*_\varepsilon, u_\varepsilon - u_0\rangle}{\|u_\varepsilon - u_0\|_{X_0}} \leq \|u_{0t}\|_{X_0^*},$$

for all $\varepsilon > 0$. From the last inequality together with (3.7), we imply that $\|u_\varepsilon\|_{X_0}$ is bounded. As a consequence, we see that (Au_ε) is a bounded sequence in X_0^*. Moreover, from the growth conditions (2.3) and (2.4), we see that (η_ε) and (ζ_ε) are bounded sequences in $L^{p'}(Q)$ and $L^{p'}(\Gamma_N)$, respectively. (Note: $\eta^*_\varepsilon = i^*\eta_\varepsilon$ and $\zeta^*_\varepsilon = \gamma^*_N\zeta_\varepsilon$.) Next, let us check that the sequence $(\varepsilon^{-1}Pu_\varepsilon)$ is also bounded in X_0^*. To see this, for each ε, we choose $w = w_\varepsilon$ to be an element satisfying (3.3) with $u = u_\varepsilon$. From (3.8), we have

$$\langle u_{\varepsilon t}, w_\varepsilon\rangle + \langle Au_\varepsilon + \eta^*_\varepsilon + \zeta^*_\varepsilon, w_\varepsilon\rangle + \frac{1}{\varepsilon}\langle Pu_\varepsilon, w_\varepsilon\rangle = 0.$$

From (3.3)(i), we see that $\langle u_{\varepsilon t}, w_\varepsilon\rangle + \langle Au_\varepsilon, w_\varepsilon\rangle \geq 0$. Therefore,

$$\frac{1}{\varepsilon}\langle Pu_\varepsilon, w_\varepsilon\rangle \leq \langle -\eta^*_\varepsilon - \zeta^*_\varepsilon, w_\varepsilon\rangle. \tag{3.10}$$

Since $(\|\eta_\varepsilon\|_{L^{p'}(Q)})$ and $(\|\zeta_\varepsilon\|_{L^{p'}(\Gamma_N)})$ are bounded we get: there exists a constant $c > 0$ such that

$$|\langle \eta^*_\varepsilon, w_\varepsilon\rangle| \leq \int_Q |\eta_\varepsilon|\,|w_\varepsilon|\,dxdt \leq c\|w_\varepsilon\|_{L^p(Q)}, \ \forall \varepsilon,$$

and

$$|\langle \zeta_\varepsilon^*, w_\varepsilon \rangle| \le \int_{\Gamma_N} |\zeta_\varepsilon| \, |\gamma_N w_\varepsilon| \, d\Gamma \le c \|\gamma_N w_\varepsilon\|_{L^p(\Gamma_N)}, \ \forall \varepsilon,$$

which results in

$$|\langle \eta_\varepsilon^* + \zeta_\varepsilon^*, w_\varepsilon \rangle| \le c(\|w_\varepsilon\|_{L^p(Q)} + \|\gamma_N w_\varepsilon\|_{L^p(\Gamma_N)}), \ \forall \varepsilon.$$

This last inequality and (3.3)(ii) imply that

$$\frac{1}{\varepsilon} \|P u_\varepsilon\|_{X_0^*} \le \frac{c}{D}, \ \forall \varepsilon.$$

On the other hand, since

$$u_{\varepsilon t} = -(A + \varepsilon^{-1} P)(u_\varepsilon) - \eta_\varepsilon^* - \zeta_\varepsilon^*$$

in X_0^*, the above estimate implies that $(u_{\varepsilon t})$ is also bounded in X_0^*. Thus, we have shown that (u_ε) is bounded with respect the graph norm of $D(L)$. Hence, there exist $u \in X_0$ (with $u_t \in X_0^*$) and a subsequence of (u_ε), still denoted by (u_ε), such that

$$u_\varepsilon \rightharpoonup u \text{ in } X_0, \ u_{\varepsilon t} \rightharpoonup u_t \text{ in } X_0^* \ (\varepsilon \to 0^+). \tag{3.11}$$

Since $D(L)$ is closed in W_0 and convex, it is weakly closed in W_0, and thus $u \in D(L)$. Now, let us prove that u is a solution of the variational inequality (3.1). First, note that $Pu = 0$. In fact, we have $P u_\varepsilon \to 0$ in X_0^*. It follows from the monotonicity of P that

$$\langle Pv, v - u \rangle \ge 0, \ \forall v \in X_0.$$

As in the proof of Minty's lemma (cf. [19]), one obtains from this inequality that

$$\langle Pu, v \rangle \ge 0, \ \forall v \in X_0.$$

Hence, $Pu = 0$ in X_0^*, that is, $u \in K$. On the other hand, (3.11) and Aubin's lemma as well as the compactness of the trace operator $\gamma : W \to L^p(\Gamma)$ (resp. $\gamma_N : W_0 \to L^p(\Gamma_N)$) imply that

$$u_\varepsilon \to u \ \text{ in } L^p(Q) \text{ and } \gamma_N u_\varepsilon \to \gamma_N u \ \text{ in } L^p(\Gamma_N)). \tag{3.12}$$

As a consequence, we obtain

$$\langle \eta_\varepsilon^* + \zeta_\varepsilon^*, u_\varepsilon - u \rangle = \int_Q \eta_\varepsilon (u_\varepsilon - u) \, dxdt + \int_{\Gamma_N} \zeta_\varepsilon (\gamma_N u_\varepsilon - \gamma_N u) \, d\Gamma \to 0, \tag{3.13}$$

as $\varepsilon \to 0^+$. For $w \in K$, letting $v = w - u_\varepsilon$ in (3.8) (with $u = u_\varepsilon$), one gets

$$\langle u_{\varepsilon t}, w - u_\varepsilon \rangle + \langle A u_\varepsilon + \eta_\varepsilon^* + \zeta_\varepsilon^*, w - u_\varepsilon \rangle = \frac{1}{\varepsilon} \langle -P u_\varepsilon, w - u_\varepsilon \rangle \ge 0. \tag{3.14}$$

By choosing $w = u$ in (3.14), we have

$$\begin{aligned} \langle A u_\varepsilon, u - u_\varepsilon \rangle &\ge -\langle \eta_\varepsilon^* + \zeta_\varepsilon^*, u - u_\varepsilon \rangle - \langle u_t, u - u_\varepsilon \rangle + \langle u_t - u_{\varepsilon t}, u - u_\varepsilon \rangle \\ &\ge -\langle \eta_\varepsilon^* + \zeta_\varepsilon^*, u - u_\varepsilon \rangle - \langle u_t, u - u_\varepsilon \rangle. \end{aligned}$$

As a consequence, one gets

$$\liminf_{\varepsilon \to 0^+} \langle A u_\varepsilon, u - u_\varepsilon \rangle \ge 0.$$

Note that A is of class (S_+) with respect to $D(L)$ (cf. e.g. [4, 5] or [10], we recall that A is said to be of class (S_+) with respect to $D(L)$ if for any sequences $\{u_n\} \subset D(L)$, the conditions $u_n \rightharpoonup u$ in X_0, $Lu_n \rightharpoonup Lu$ in X_0^* and $\liminf_{n\to\infty}\langle Au_n, u - u_n\rangle \geq 0$ imply that $u_n \to u$ in X_0). Therefore, we deduce from (3.11) and the above limit that

$$u_\varepsilon \to u \text{ in } X_0. \tag{3.15}$$

On the other hand, since (η_ε^*) as well as (ζ_ε^*) are bounded in X_0^*, by passing to a subsequence still denoted by (η_ε^*) and (ζ_ε^*), respectively, for simplicity of notation, we have

$$\eta_\varepsilon^* \rightharpoonup \eta^*, \quad \zeta_\varepsilon^* \rightharpoonup \zeta^* \text{ in } X_0^*. \tag{3.16}$$

From (3.11) and the property of the mappings $\mathcal{F}$ and $\mathcal{F}_N$ to be sequentially weakly-closed with respect to $D(L)$ (as for the proof, see Step 3 of Proposition 2.3), we have

$$\eta^* \in \mathcal{F}(u) \text{ and } \zeta^* \in \mathcal{F}_N(u). \tag{3.17}$$

Letting $\varepsilon \to 0$ in (3.14) and taking (3.11), (3.15), and (3.16) into account, we obtain

$$\langle u_t, w - u\rangle + \langle Au + \eta^* + \zeta^*, w - u\rangle \geq 0.$$

This holds for all $w \in K$ which together with (3.17) proves that u is in fact a solution of (3.1). □

In the case where Σ_D has positive surface measure, let us consider some simple conditions that ensure the coercivity condition (3.7) is satisfied and thus the existence of solutions of the inequality (3.1). In this case, we can take as norm on X_0 the usual norm defined only by the gradient of functions in X_0:

$$\|u\|_{X_0} = \left(\int_Q |\nabla u|^p dxdt\right)^{1/p} \quad (u \in X_0).$$

Let μ and μ_N be the best constants corresponding to the continuous mappings $X_0 \to L^p(Q)$, $u \mapsto u$, and $X_0 \to L^p(\Gamma_N)$, $u \mapsto \gamma_N u$, that is,

$$\|u\|_{L^p(Q)} \leq \mu\|u\|_{X_0}, \ \|\gamma_N u\|_{L^p(\Gamma_N)} \leq \mu_N\|u\|_{X_0}, \ \forall u \in X_0.$$

Some sufficient conditions for (3.7) are given in the following result.

Corollary 3.3 *Assume* $\text{meas}(\Sigma_D) \neq 0$.

(a) If

$$c_3 > \beta\mu^p + \beta_N\mu_N^p, \tag{3.18}$$

then the coercivity condition (3.7) holds.

(b) In particular, if (2.3) and (2.4) hold with $\sigma \in [0, p-1)$ *instead of* $p - 1$*, that is,*

$$\sup\{|\eta| : \eta \in f(x,t,s)\} \leq \alpha(x,t) + \beta|s|^\sigma \tag{3.19}$$

for a.e. $(x,t) \in Q$, $\forall\, s \in \mathbb{R}$*, and*

$$\sup\{|\zeta| : \zeta \in f_N(x,t,s)\} \leq \alpha_N(x,t) + \beta_N|s|^\sigma \tag{3.20}$$

for a.e. $(x,t) \in \Gamma_N$, $\forall\, s \in \mathbb{R}$*, where* $\alpha \in L^{p'}(Q)$*,* $\alpha_N \in L^{p'}(\Gamma_N)$*,* $\beta, \beta_N \geq 0$*, then the coercivity condition (3.7) holds.*

Proof. (a) Let u_0 be any (fixed) element of $D(L) \cap K$. It follows from (A3) and (A1) that

$$\langle Au, u\rangle \geq c_3\|u\|_{X_0}^p - \|c_4\|_{L^1(Q)}, \tag{3.21}$$

and

$$|\langle Au, u_0\rangle| \leq Nc_1\|u\|_{X_0}^{p-1}\|u_0\|_{X_0} + N\|c_2\|_{L^{p'}(Q)}\|u_0\|_{X_0}, \forall u \in X_0. \tag{3.22}$$

On the other hand, for any $u \in X_0$, any $\eta^* \in \mathcal{F}(u)$ and $\zeta^* \in \mathcal{F}_N(u)$, (F3) implies that

$$\begin{aligned} |\langle \eta^*, u - u_0\rangle| \quad \leq \quad & \mu\|\alpha\|_{L^{p'}(Q)}(\|u\|_{X_0} + \|u_0\|_{X_0}) + \beta\mu^p\|u\|_{X_0}^p \\ & +\beta\mu^p\|u\|_{X_0}^{p-1}\|u_0\|_{X_0}, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} |\langle \zeta^*, u - u_0\rangle| \quad \leq \quad & \mu_N\|\alpha_N\|_{L^{p'}(\Gamma_N)}(\|u\|_{X_0} + \|u_0\|_{X_0}) + \beta_N\mu_N^p\|u\|_{X_0}^p \\ & +\beta_N\mu_N^p\|u\|_{X_0}^{p-1}\|u_0\|_{X_0}. \end{aligned} \tag{3.24}$$

Combining (3.21)–(3.24) yields

$$\begin{aligned} \langle Au + \eta^* + \zeta^*, u - u_0\rangle \quad & \geq (c_3 - \beta\mu^p - \beta_N\mu_n^p)\|u\|_{X_0}^p \\ & -(Nc_1 + \beta\mu^p + \beta_N\mu_N^p)\|u\|_{X_0}^{p-1}\|u_0\|_{X_0} \\ & -(\mu\|\alpha\|_{L^{p'}(Q)} + \mu_N\|\alpha_N\|_{L^{p'}(\Gamma_N)})(\|u\|_{X_0} + \|u_0\|_{X_0}) \\ & -\|c_4\|_{L^1(Q)} - N\|c_2\|_{L^{p'}(Q)}\|u_0\|_{X_0}. \end{aligned}$$

It is clear from this estimate that (3.18) implies (3.7).

(b) In view of Young's inequality (with ϵ), we see that conditions (3.19) and (3.20) imply conditions (2.3) and (2.4) with arbitrarily small choices of β and β_N in (2.3) and (2.4), which in their turns, implies (3.18). □

Remark 3.4 Theorem 3.2, in particular, allows to treat rather general parabolic obstacle problems, as for those a penalty operator satisfying the required property (P) can explicitly be constructed, see (3.4). As for parabolic obstacle problems with (single-valued) Carathéodory nonlinearities, see e.g. [6, 14, 20].

4 Enclosure and comparison results in the noncoercive case

Note that when the growth conditions (2.3), (2.4), or the coercivity condition (3.7) is not fulfilled then problem (3.1) (resp. (3.2)) may not have solutions. However, without these conditions, we can still have the existence and other properties of solutions of (3.1) provided that sub- and supersolutions of (3.1), defined in a certain appropriate sense, exist. In this section we establish a sub-supersolution method for (3.1), which will allow us to derive existence, enclosure and comparison results for (3.1) (which is equivalent to (1.2)–(1.3)).

Let us first introduce our basic notions of sub-supersolution for the multi-valued parabolic variational inequality (3.1).

Definition 4.1 *A function $\underline{u} \in W$ is called a* **subsolution** *of (3.1) if there is an $\underline{\eta} \in L^{p'}(Q)$ and a $\underline{\zeta} \in L^{p'}(\Gamma_N)$ such that the following holds:*

$$
\begin{aligned}
&(i)\ \underline{u} \vee K \subset K, \quad \underline{u}(\cdot, 0) \le 0 \ \text{ in } \Omega,\\
&(ii)\ \underline{\eta} \in F(\underline{u}),\ \underline{\zeta} \in F_N(\gamma_N \underline{u})\\
&(iii)\ \langle \underline{u}_t + A\underline{u}, v - \underline{u} \rangle + \int_Q \underline{\eta}\,(v - \underline{u})\,dxdt + \int_{\Gamma_N} \underline{\zeta}\,(\gamma_N v - \gamma_N \underline{u})\,d\Gamma \ge 0,\\
&\quad \text{for all } v \in \underline{u} \wedge K.
\end{aligned}
$$

We have a similar definition for supersolutions of (3.1).

Definition 4.2 *A function $\overline{u} \in W$ is called a* **supersolution** *of (3.1) if there is an $\overline{\eta} \in L^{p'}(Q)$ and a $\overline{\zeta} \in L^{p'}(\Gamma_N)$ such that the following holds:*

$$
\begin{aligned}
&(i)\ \overline{u} \wedge K \subset K, \quad \overline{u}(\cdot, 0) \ge 0 \ \text{ in } \Omega,\\
&(ii)\ \overline{\eta} \in F(\overline{u}),\ \overline{\zeta} \in F_N(\gamma_N \overline{u})\\
&(iii)\ \langle \overline{u}_t + A\overline{u}, v - \overline{u} \rangle + \int_Q \overline{\eta}\,(v - \overline{u})\,dxdt + \int_{\Gamma_N} \overline{\zeta}\,(\gamma_N v - \gamma_N \overline{u})\,d\Gamma \ge 0,\\
&\quad \text{for all } v \in \overline{u} \vee K.
\end{aligned}
$$

Remark 4.3 Note that the notions for sub- and supersolution defined in Definition 4.1 and Definition 4.2 have a symmetric structure, i.e., one obtains the definition for the supersolution $\overline{u}$ from the definition of the subsolution by replacing $\underline{u}, \underline{\eta}, \underline{\zeta}$ in Definition 4.1 by $\overline{u}, \overline{\eta}, \overline{\zeta}$, and interchanging $\vee$ by $\wedge$, and "$\le$" in (i) by "$\ge$".

Throughout this section instead of the growth condition (F3) of the preceding section we assume the following local growth assumption with respect to the ordered interval of sub-supersolutions.

(F4) Assume that there exists a pair of sub-supersolutions $\underline{u}$ and $\overline{u}$ of (3.1) such that $\underline{u} \le \overline{u}$. For f and f_N we require the following growth between $\underline{u}$ and $\overline{u}$:

$$
|\eta| \le c_5(x,t), \quad \forall\, \eta \in f(x,t,s), \tag{4.1}
$$

for some $c_5 \in L^{p'}(Q)$, for a.e. $(x,t) \in Q$, and all $s \in [\underline{u}(x,t), \overline{u}(x,t)]$, as well as

$$
|\zeta| \le c_6(x,t), \quad \forall\, \zeta \in f_N(x,t,s), \tag{4.2}
$$

for some $c_6 \in L^{p'}(\Gamma_N)$, for a.e. $(x,t) \in \Gamma_N$, and all $s \in [\gamma_N \underline{u}(x,t), \gamma_N \overline{u}(x,t)]$,

We are now ready to state and prove our main existence and comparison result.

Theorem 4.4 *Assume (A1)–(A3) and that (3.1) has an ordered pair of sub- and supersolutions $\underline{u}$ and $\overline{u}$, and that (F1)–(F2), and (F4) are satisfied. Suppose furthermore that $D(L) \cap K \neq \emptyset$. Then, (3.1) has a solution u such that $\underline{u} \le u \le \overline{u}$ a.e. in Q.*

Proof. We define the following cut-off function $b : Q \times \mathbb{R} \to \mathbb{R}$:

$$
b(x,t,s) = \begin{cases} [s - \overline{u}(x,t)]^{p-1} & \text{if } \ s > \overline{u}(x,t) \\ 0 & \text{if } \ \underline{u}(x,t) \le s \le \overline{u}(x,t) \\ -[\underline{u}(x,t) - s]^{p-1} & \text{if } \ s < \underline{u}(x,t), \end{cases}
$$

for $(x,t,s) \in Q \times \mathbb{R}$. It is easy to check that b is a Carathéodory function satisfying the following growth condition

$$|b(x,t,s)| \le c_7(x,t) + c_8|s|^{p-1}, \text{ for a.e. } (x,t) \in Q, \text{ all } s \in \mathbb{R}, \tag{4.3}$$

with $c_7 \in L^{p'}(Q)$, $c_8 > 0$. Hence, the Nemytskij operator $B : u \mapsto b(\cdot,\cdot,u)$ is a continuous and bounded mapping from $L^p(Q)$ to $L^{p'}(Q)$ and the composed operator $\mathcal{B} = i^* \circ B \circ i : X_0 \to X_0^*$ given by

$$\langle \mathcal{B}u, v\rangle = \int_Q b(\cdot,\cdot,u)\, v\, dxdt, \ \forall u, v \in X_0 \tag{4.4}$$

is (single-valued) pseudomonotone w.r.t. $D(L)$ due to the compact imbedding $W \hookrightarrow\hookrightarrow L^p(Q)$). Moreover, there are $c_9, c_{10} > 0$ such that

$$\int_Q b(\cdot,\cdot,u)u\, dxdt \ge c_9\|u\|^p_{L^p(Q)} - c_{10}, \ \forall u \in L^p(Q). \tag{4.5}$$

Let $\underline{\eta}, \underline{\zeta}$ and $\overline{\eta}, \overline{\zeta}$ correspond to $\underline{u}$ and $\overline{u}$ as in definitions 4.1 and 4.2. We define multi-valued truncation functions $f_0 : Q \times \mathbb{R} \to 2^{\mathbb{R}}$ and $f_{N0} : \Gamma_N \times \mathbb{R} \to 2^{\mathbb{R}}$ related to the multi-valued function f and f_N, respectively, by

$$f_0(x,t,s) = \begin{cases} \{\underline{\eta}\} & \text{if } \ s < \underline{u}(x,t) \\ f(x,t,s) & \text{if } \ \underline{u}(x,t) \le s \le \overline{u}(x,t) \\ \{\overline{\eta}\} & \text{if } \ s > \overline{u}(x,t), \end{cases} \tag{4.6}$$

for $(x,t,s) \in Q \times \mathbb{R}$, and

$$f_{N0}(x,t,s) = \begin{cases} \{\underline{\zeta}\} & \text{if } \ s < \gamma_N\underline{u}(x,t) \\ f_N(x,t,s) & \text{if } \ \gamma_N\underline{u}(x,t) \le s \le \gamma_N\overline{u}(x,t) \\ \{\overline{\zeta}\} & \text{if } \ s > \gamma_N\overline{u}(x,t), \end{cases} \tag{4.7}$$

for $(x,t,s) \in \Gamma_N \times \mathbb{R}$.

As in [21], we can check that f_0 and f_{N0} also satisfy (F1) and (F2). Moreover, as a consequence of (F4), f_0 satisfies (2.3), and f_{N0} satisfies (2.4) of (F3) with $\beta = 0$ and $\beta_N = 0$, and $\alpha = c_5 + |\underline{\eta}| + |\overline{\eta}| \in L^{p'}(Q)$ and $\alpha_N = c_6 + |\underline{\zeta}| + |\overline{\zeta}| \in L^{p'}(\Gamma_N)$, respectively. For $u : Q \to \mathbb{R}$ measurable, let

$$F_0(u) = \{\eta : Q \to \mathbb{R} : \eta \text{ is measurable on } Q \text{ and } \eta(x,t) \in f_0(x,t,u(x,t))\},$$

and for $v : \Gamma_N \to \mathbb{R}$ measurable, let

$$F_{N0}(v) = \{\zeta : \Gamma_N \to \mathbb{R} : \zeta \text{ is measurable on } \Gamma_N \text{ and } \zeta(x,t) \in f_0(x,t,v(x,t))\}.$$

Then, $F_0(u) \subset L^{p'}(Q)$ and $F_{N0}(v) \subset L^{p'}(\Gamma_N)$ for any measurable functions $u : Q \to \mathbb{R}$ and $v : \Gamma_N \to \mathbb{R}$, respectively. This allows us to define $F_0 : L^p(Q) \to 2^{L^{p'}(Q)}$, $u \mapsto F_0(u)$ and $\mathcal{F}_0 : X_0 \to 2^{X_0^*}$, where $\mathcal{F}_0 = i^* \circ F_0 \circ i$, as well as $F_{N0} : L^p(\Gamma_N) \to 2^{L^{p'}(\Gamma_N)}$, $v \mapsto F_{N0}(v)$ and $\mathcal{F}_{N0} : X_0 \to 2^{X_0^*}$, where $\mathcal{F}_{N0} = \gamma_N^* \circ F_{N0} \circ \gamma_N$. Then $\mathcal{F}_0$ and $\mathcal{F}_{N0}$ are pseudomonotone with respect to $D(L)$, according to Proposition 2.3 and Proposition 2.4, respectively. Let us consider the following auxiliary variational inequality: Find $u \in D(L) \cap K$, $\eta^* \in \mathcal{F}_0(u)$, and $\zeta^* \in \mathcal{F}_{N0}(u)$ such that

$$\langle Lu + Au + \mathcal{B}u + \eta^* + \zeta^*, v - u\rangle \ge 0. \ \forall v \in K, \tag{4.8}$$

By means of the existence result of Section 3 we are going to show first that problem (4.8) possesses solutions.

We already mentioned above that $\mathcal{B} : X_0 \to X_0^*$ is (single-valued) bounded, continuous, and pseudomonotone w.r.t. $D(L)$. Setting, $f_1 = b + f_0$, then in view of the properties of the truncation functions b, f_0, f_{N0} one readily verifies that f_1 and f_{N0} satisfy (F1)–(F3), and thus $\mathcal{F}_1 = \mathcal{B} + \mathcal{F}_0$ and $\mathcal{F}_{N0}$ are bounded and pseudomonotone with respect to $D(L)$ according to Proposition 2.3 and Proposition 2.4, respectively. In order to apply the existence result of Section 3, we need to check that the operator

$$A + \mathcal{B} + \mathcal{F}_0 + \mathcal{F}_{N0} : X_0 \to 2^{X_0^*}$$

is coercive on X_0 in the following sense:

$$\lim_{\|u\|_{X_0}\to\infty} \left[\inf_{\substack{\eta^* \in \mathcal{F}_0(u)\\ \zeta^* \in \mathcal{F}_{N0}(u)}} \frac{\langle Au + \mathcal{B}u + \eta^* + \zeta^*, u - \varphi\rangle}{\|u\|_{X_0}} \right] = \infty, \tag{4.9}$$

for any $\varphi \in X_0$, which then holds true also for $\varphi \in K$. In fact, from (A3), we have

$$\langle Au, u\rangle \geq c_3 \||\nabla u|\|^p_{L^p(Q)} - c_{11}, \ \forall u \in X_0, \tag{4.10}$$

with some constant $c_{11} > 0$. Now let $c > 0$ be a generic constant. For $\eta^* \in \mathcal{F}_0(u)$, $\eta^* = i^*\eta$ with $\eta \in F_0(u)$, we have

$$\begin{aligned} |\langle \eta^*, u\rangle| &= \left| \int_Q \eta\, u\, dxdt \right| \\ &\leq (\|c_5\|_{L^{p'}(Q)} + \|\underline{\eta}\|_{L^{p'}(Q)} + \|\overline{\eta}\|_{L^{p'}(Q)})\|u\|_{L^p(Q)} \\ &\leq c\,\|u\|_{L^p(Q)} \leq c\,\|u\|_{X_0}. \end{aligned} \tag{4.11}$$

For $\zeta^* \in \mathcal{F}_{N0}(u)$, $\zeta^* = \gamma_N^*\zeta$ with $\zeta \in F_{N0}(\gamma_N u)$, we have

$$\begin{aligned} |\langle \zeta^*, u\rangle| &= \left| \int_{\Gamma_N} \zeta\, \gamma_N u\, d\Gamma \right| \\ &\leq (\|c_6\|_{L^{p'}(\Gamma_N)} + \|\underline{\zeta}\|_{L^{p'}(\Gamma_N)} + \|\overline{\zeta}\|_{L^{p'}(\Gamma_N)})\|\gamma_N u\|_{L^p(\Gamma_N)} \\ &\leq c\,\|\gamma_N u\|_{L^p(\Gamma_N)} \leq c\,\|u\|_{X_0} \end{aligned} \tag{4.12}$$

Combining (4.5) with (4.10),(4.11) and (4.12), one gets for all $u \in X_0$

$$\begin{aligned} \langle (Au + \mathcal{B}u + \eta^* + \zeta^*, u\rangle &\geq c_3\|\nabla u\|^p_{L^p(Q)} + c_9\|u\|^p_{L^p(Q)} - c_{10} \\ &\quad -c\,\|u\|_{X_0}, \end{aligned} \tag{4.13}$$

and thus with $\tilde{c} = \min\{c_3, c_9\} > 0$

$$\langle (Au + \mathcal{B}u + \eta^* + \zeta^*, u\rangle \geq \tilde{c}\,\|u\|^p_{X_0} - c\,(\|u\|_{X_0} + 1). \tag{4.14}$$

For any $\varphi \in X_0$ fixed, it is inferred from (A1), (4.3), (4.2) and (4.1) that

$$|\langle Au + \mathcal{B}u + \eta^* + \zeta^*, \varphi\rangle| \leq c\,(\|u\|^{p-1}_{X_0} + 1), \ \forall u \in X_0. \tag{4.15}$$

From (4.14) and (4.15), we obtain (4.9). Let $u_0 \in D(L) \cap K$ be fixed. With the particular choice of $\varphi = u_0$, we see that all conditions of Theorem 3.2 are fulfilled with $\mathcal{F}_1 = \mathcal{B} + \mathcal{F}_0$ in place of $\mathcal{F}$ and $\mathcal{F}_{N0}$ in place of $\mathcal{F}_N$. According to Theorem 3.2, (4.8) has solutions.

Now, let us show that any solution u of (4.8) satisfies: $\underline{u} \le u \le \overline{u}$ a.e. in Q. We verify that $\underline{u} \le u$, the second inequality is proved in a similar way. Because $u \in K$, it follows that

$$u + (\underline{u} - u)^+ = \underline{u} \vee u \in K.$$

Letting $v = u + (\underline{u} - u)^+$ into (4.8), one gets

$$\langle u_t, (\underline{u} - u)^+\rangle + \langle Au + \mathcal{B}u + \eta^* + \zeta^*, (\underline{u} - u)^+\rangle \ge 0,$$

which is equivalent to

$$\begin{aligned} &\langle u_t, (\underline{u} - u)^+\rangle + \langle Au + \mathcal{B}u, (\underline{u} - u)^+\rangle \\ &+ \int_Q \eta\, (\underline{u} - u)^+ \, dxdt + \int_{\Gamma_N} \zeta\, (\gamma_N \underline{u} - \gamma_N u)^+ \, d\Gamma \ge 0, \end{aligned} \tag{4.16}$$

where $\eta^* = i^*\eta$ and $\zeta^* = \gamma_N^*\zeta$ with $\eta \in F_0(u)$ and $\zeta \in F_{N0}(\gamma_N u)$. On the other hand, since $\underline{u}$ is a subsolution, it follows with

$$v = \underline{u} - (\underline{u} - u)^+ = \underline{u} \wedge u \in \underline{u} \wedge K,$$

that

$$-\langle \underline{u}_t, (\underline{u} - u)^+\rangle - \langle A\underline{u}, (\underline{u} - u)^+ - \int_Q \underline{\eta}\, (\underline{u} - u)^+ \, dxdt - \int_{\Gamma_N} \underline{\zeta}\, (\gamma_N \underline{u} - \gamma_N u)^+ \, d\Gamma \ge 0. \tag{4.17}$$

Adding (4.16) and (4.17), we get

$$\begin{aligned} &\langle (u - \underline{u})_t, (\underline{u} - u)^+\rangle + \langle Au - A\underline{u} + \mathcal{B}u, (\underline{u} - u)^+\rangle \\ &+ \int_Q (\eta - \underline{\eta})\, (\underline{u} - u)^+ \, dxdt + \int_{\Gamma_N} (\zeta - \underline{\zeta})\, (\gamma_N \underline{u} - \gamma_N u)^+ \, d\Gamma \ge 0. \end{aligned} \tag{4.18}$$

We have $\underline{u} - u \in W$ and $(\underline{u} - u)^+(\cdot, 0) = 0$, and thus

$$\langle (\underline{u} - u)_t, (\underline{u} - u)^+\rangle = \frac{1}{2}\|(\underline{u} - u)^+(\cdot, \tau)\|^2_{L^2(\Omega)} \ge 0. \tag{4.19}$$

On the other hand, it is easy to check from (A2) that

$$\langle A\underline{u} - Au, (\underline{u} - u)^+\rangle \ge 0. \tag{4.20}$$

Moreover, because of (4.6), it follows

$$\int_Q (\eta - \underline{\eta})\, (\underline{u} - u)^+ \, dxdt = \int_{Q^+} (\eta - \underline{\eta})(\underline{u} - u) \, dxdt = 0, \tag{4.21}$$

where $Q^+ = \{(x,t) \in Q : \underline{u}(x,t) > u(x,t)\}$, and due to (4.6), we have

$$\eta(x,t) = \underline{\eta}(x,t) \text{ for a.e. } (x,t) \in Q^+.$$

Similarly, one has

$$\int_{\Gamma_N} (\zeta - \underline{\zeta})\, (\gamma_N \underline{u} - \gamma_N u)^+ \, d\Gamma = \int_{\Gamma_N^+} (\zeta - \underline{\zeta})\, (\gamma_N \underline{u} - \gamma_N u) \, d\Gamma = 0, \tag{4.22}$$

because due to (4.7), for $(x,t) \in \Gamma_N^+ = \{(x,t) \in \Gamma_N : \gamma_N \underline{u}(x,t) > \gamma_N u(x,t)\}$, it follows that

$$\zeta(x,t) = \underline{\zeta}(x,t) \text{ for a.e. } (x,t) \in \Gamma_N^+.$$

Combining (4.19)–(4.22) with (4.18), we obtain

$$0 \le \langle \mathcal{B}u, (\underline{u} - u)^+ \rangle = -\int_{Q^+} (\underline{u} - u)^p \, dxdt \le 0.$$

This proves that $\underline{u} - u = 0$ a.e. on Q^+ and thus $\underline{u} \le u$ a.e. on Q. A similar proof shows that $u \le \bar{u}$. From $\underline{u} \le u \le \bar{u}$, we have $\mathcal{B}u = 0$ as well as $\mathcal{F}_0 u \subset \mathcal{F} u$, and $\mathcal{F}_{N0} u \subset \mathcal{F}_N u$. Consequently, a solution u of (4.8) is also a solution of (3.1), which completes the proof. □

The construction of sub- and supersolutions of (3.1) depends on the specific properties of the data of the problem, i.e., on Ω, A, F, F_N and the convex set K. In the following application, we consider a noncoercive problem that can be treated by means of Theorem 4.4 via the construction of sub- and supersolutions for an obstacle problem with the Laplacian as principal operator.

Application: Multi-Valued Parabolic Obstacle Problem

Let us assume $\Gamma_D = \emptyset$, which implies $\Gamma = \Gamma_N$, $X_0 = X$, and let $\Omega = B(0,1)$ be the unit ball in $\mathbb{R}^N$. We consider the multi-valued parabolic obstacle problem: Find $u \in D(L) \cap K$, $\eta \in L^2(Q)$ and a $\zeta \in L^2(\Gamma_N)$ such that

$$\begin{cases} \eta \in F(u), \ \zeta \in F_N(\gamma_N u) \\ \langle u_t + Au, v - u \rangle + \int_Q \eta \, (v - u) \, dxdt + \int_{\Gamma_N} \zeta \, (\gamma_N v - \gamma_N u) \, d\Gamma \ge 0, \end{cases} \tag{4.23}$$

for all $v \in K$, where K is given by an obstacle

$$K = \{u \in X : u \le \psi \ \text{ a.e. on } \ Q\},$$

and $A = -\Delta$ is the Laplacian (i.e. $p = 2$ and (A1)–(A3) are apparently satisfied). We assume $\psi \in W \cap L^\infty(Q)$ to satisfy the conditions already formulated in Section 3. The multi-valued functions $f, f_N : \mathbb{R} \to \mathcal{K}(\mathbb{R}) \subset 2^{\mathbb{R}}$ are supposed to be upper semicontinuous, and for simplicity we assume these functions be bounded. More precisely, we assume the existence of positive constants d and d_N such that

$$d \le N : -d \le f(s) \le d, \ \ \forall \ s \in \mathbb{R}. \tag{4.24}$$

$$\begin{cases} d_N > 2 \ \text{ and } \ -d_N \le f_N(s) \le d_N, \ \ \forall \ s \in \mathbb{R}, \\ \exists \ s_0 > 0 : 2 \le f_N(s) \le d_N, \ \forall \ s \ge s_0, \\ -d_N \le f_N(s) \le -2, \ \forall \ s \le -s_0. \end{cases} \tag{4.25}$$

As $\|\nabla u\|_{L^2(Q)}$ is not an equivalent norm in $X_0 = X$, we readily observe that the coercivity condition (3.7) of Theorem 3.2 is not satisfied. Still we are able to construct an ordered pair $\underline{u}$, $\overline{u}$ of sub-supersolution of (4.23), which allows to apply Theorem 4.4. To this end let us introduce the following two (single-valued) parabolic initial boundary value problems:

$$u_t - \Delta u - d = 0, \ \ u(\cdot, 0) = 0 \text{ in } \ \Omega, \ \ \frac{\partial u}{\partial \nu} + \beta(u) = 0 \text{ on } \ \Gamma, \tag{4.26}$$

$$u_t - \Delta u + d = 0, \ \ u(\cdot, 0) = 0 \text{ in } \Omega, \ \ \frac{\partial u}{\partial \nu} + \beta(u) = 0 \text{ on } \Gamma, \tag{4.27}$$

where $\partial/\partial\nu$ is the outward normal derivative at Γ_N, and where $\beta : \mathbb{R} \to \mathbb{R}$ may be any smooth function satisfying, in addition, the following bounds for $s \geq s_0$ and $s \leq -s_0$ with s_0 as given in (4.25):

$$1 \leq \beta(s) \leq 2 \text{ for } s \geq s_0, \quad -2 \leq \beta(s) \leq -1 \text{ for } s \leq -s_0. \tag{4.28}$$

Then the following enclosure holds:

Corollary 4.5 *Assume assumptions (4.24) and (4.25) for the multi-valued functions $f, f_N : \mathbb{R} \to \mathcal{K}(\mathbb{R}) \subset 2^{\mathbb{R}}$. If $\overline{u}$ is a supersolution of the initial boundary value problem (4.26) satisfying $\overline{u}(x,t) \geq s_0$, then $\overline{u}$ is a supersolution of the obstacle problem (4.23). If $\underline{u}$ is a subsolution of the initial boundary value problem (4.27) satisfying $\underline{u}(x,t) \leq \min\{-s_0, -\|\psi\|_\infty\}$, then $\underline{u}$ is a subsolution of the obstacle problem (4.23), and $\underline{u} \leq \overline{u}$ holds true, which implies that (4.23) has solutions within the interval $[\underline{u}, \overline{u}]$.*

Proof. As a supersolution of (4.26) the function $\overline{u} \in W$ satisfies: $\overline{u}(\cdot, 0) \geq 0$, in Ω, and

$$\langle \overline{u}_t, \varphi\rangle + \int_Q (\nabla\overline{u}\nabla\varphi - d\varphi)\, dxdt + \int_\Gamma \beta(\gamma\overline{u})\gamma\varphi\, d\Gamma \geq 0, \ \ \forall\, \varphi \in X \cap L^2_+(Q). \tag{4.29}$$

We are going to verify that $\overline{u}$ is a supersolution in the sense of Definition 4.2 of the obstacle problem (4.23). First, for any $w \in K$ we have $\overline{u} \wedge w \in K$ which is (i) of Definition 4.2. Let $\overline{\eta}$ be any measurable selection of $f(\overline{u}(x,t))$, and let $\overline{\zeta}$ be any measurable selection of $f_N(\gamma\overline{u}(x,t))$. Then $-d \leq \overline{\eta}(x,t) \leq d$, $-d_N \leq \overline{\zeta}(x,t) \leq d_N$, and thus $\overline{\eta} \in F(\overline{u})$ and $\overline{\zeta} \in F_N(\gamma\overline{u})$ (which is (ii)). Moreover, since $\overline{u}(x,t) \geq s_0$ we have $\gamma\overline{u} \geq s_0$, and thus by (4.25) and (4.28) it follows $\overline{\zeta} \geq \beta(\gamma\overline{u})$. Apparently, $-d \leq \overline{\eta}$, which in view of (4.29) leads to

$$\langle \overline{u}_t - \Delta\overline{u} + i^*\overline{\eta} + \gamma^*\overline{\zeta}, \varphi\rangle \geq 0, \ \ \forall\, \varphi \in X \cap L^2_+(Q). \tag{4.30}$$

The last inequality, in particular holds for $\varphi = (w - \overline{u})^+$ for any $w \in K$, which implies that for $v = \overline{u} \vee w = \overline{u} + (w - \overline{u})^+$ $(w \in K)$, the following inequality is satisfied:

$$\langle \overline{u}_t - \Delta\overline{u} + i^*\overline{\eta} + \gamma^*\overline{\zeta}, v - \overline{u}\rangle \geq 0, \ \ \forall\, v \in \overline{u} \vee K,$$

which is (iii) of Definition 4.2.

Since the subsolution of (4.27) satisfies $\underline{u}(x,t) \leq \min\{-s_0, -\|\psi\|_\infty\}$, it readily follows $\underline{u} \vee w \in K$ for all $w \in K$, i.e., (i) of Definition 4.1. As a subsolution of (4.27) the function $\underline{u} \in W$ satisfies: $\underline{u}(\cdot, 0) \leq 0$, in Ω, and

$$\langle \underline{u}_t, \varphi\rangle + \int_Q (\nabla\underline{u}\nabla\varphi + d\,\varphi)\, dxdt + \int_\Gamma \beta(\gamma\underline{u})\gamma\varphi\, d\Gamma \leq 0, \ \ \forall\, \varphi \in X \cap L^2_+(Q). \tag{4.31}$$

Let $\underline{\eta}$ be any measurable selection of $f(\underline{u}(x,t))$, and let $\underline{\zeta}$ be any measurable selection of $f_N(\gamma\underline{u}(x,t))$ then $\underline{\eta} \in F(\underline{u})$ and $\underline{\zeta} \in F_N(\gamma\underline{u})$ (which is (ii)), and again we have $-d \leq \underline{\eta}(x,t) \leq d$. Since $\underline{u}(x,t) \leq \min\{-s_0, -\|\psi\|_\infty\}$, we have, in particular, that $\gamma\underline{u} \leq -s_0$, which, in view of (4.25) and (4.28), yields $\underline{\zeta} \leq \beta(\gamma\underline{u})$. Thus from (4.31) we get

$$\langle \underline{u}_t - \Delta\underline{u} + i^*\underline{\eta} + \gamma^*\underline{\zeta}, \varphi\rangle \leq 0, \ \ \forall\, \varphi \in X \cap L^2_+(Q). \tag{4.32}$$

Inequality (4.32), in particular, holds for $\varphi = (\underline{u} - w)^+$ for any $w \in K$, which implies that for $v = \underline{u} \wedge w = \underline{u} - (\underline{u} - w)^+$ $(w \in K)$, the following inequality is satisfied:

$$\langle \underline{u}_t - \Delta\underline{u} + i^*\underline{\eta} + \gamma^*\underline{\zeta}, -(v - \underline{u})\rangle \leq 0, \ \ \forall\, v \in \underline{u} \wedge K,$$

which yields

$$\langle \underline{u}_t - \Delta\underline{u} + i^*\underline{\eta} + \gamma^*\underline{\zeta}, v - \underline{u}\rangle \geq 0, \quad \forall\, v \in \underline{u} \wedge K. \tag{4.33}$$

But the last inequality (4.33) is property (iii) of Definition 4.1. Applying Theorem 4.4 completes the proof. □

Let us verify that sub-supersolutions as required in Corollary 4.5 do exist. As $\Omega = B(0,1)$ is the unit ball in $\mathbb{R}^N$ we claim that

$$\overline{u}(x,t) = -\frac{1}{2}\,|x|^2 + a, \quad \underline{u} = -\overline{u}, \quad a \geq \max\{\frac{1}{2} + s_0, \frac{1}{2} + \|\psi\|_\infty\} \tag{4.34}$$

satisfy the conditions of Corollary 4.5. To see that $\overline{u}$ is a supersolution of (4.26) we calculate:

$$\overline{u}_t - \Delta\overline{u} - d = N - d \geq 0, \text{ since } d \leq N.$$

$$\frac{\partial\overline{u}}{\partial\nu} + \beta(\overline{u})|_{\partial B(0,1)} = -1 + \beta(\gamma\overline{u}) \geq 0,$$

since $\overline{u} \geq s_0$ and thus $\gamma\overline{u} \geq s_0$, which implies $\beta(\gamma\overline{u}) \geq 1$. Similarly, one shows that $\underline{u} = -\overline{u}$ satisfies the condition of Corollary 4.5.

5 Generalized variational-hemivariational inequalities

As already mentioned in the introduction, a need in applications requires to consider more general parabolic variational-hemivariational inequalities of the following form: Find $u \in D(L) \cap K$ such that

$$\begin{cases} \langle u_t + Au, v - u\rangle + \displaystyle\int_Q j^o(x,t,u,u;v-u)\,dxdt \\ \quad + \displaystyle\int_{\Gamma_N} j_N^o(x,t,\gamma_N u,\gamma_N u;\gamma_N v - \gamma_N u)\,d\Gamma \geq 0, \ \forall\, v \in K, \end{cases} \tag{5.1}$$

where j, j_N given by

$$j : Q \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ \text{ with } (x,t,r,s) \mapsto j(x,t,r,s), \ (x,t) \in Q$$
$$j_N : \Gamma_N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ \text{ with } (x,t,r,s) \mapsto j_N(x,t,r,s), \ (x,t) \in \Gamma_N$$

are supposed to be locally Lipschitz functions with respect to s, and $j^o(x,t,r,s;\varrho)$ and $j_N^o(x,t,r,s;\varrho)$ denote Clarke's generalized directional derivative at s in the direction ϱ for fixed (x,t,r). In particular, the following special case of (5.1) will be considered:

$$\begin{cases} \langle u_t + Au, v - u\rangle + \displaystyle\int_Q h(x,t,u)\hat{j}^o(x,t,u;v-u)\,dxdt \\ \quad + \displaystyle\int_{\Gamma_N} h_N(x,t,\gamma_N u)\hat{j}_N^o(x,t,\gamma_N u;\gamma_N v - \gamma_N u)\,d\Gamma \geq 0, \ \forall\, v \in K, \end{cases} \tag{5.2}$$

where j and j_N of (5.1) now have the special form:

$$j(x,t,r,s) = h(x,t,r)\hat{j}(x,t,s), \qquad j_N(x,t,r,s) = h_N(x,t,r)\hat{j}_N(x,t,s), \tag{5.3}$$

We assume that $h, \hat{j} : Q \times \mathbb{R} \to \mathbb{R}$ and $h_N, \hat{j}_N : \Gamma_N \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, where $\hat{j} : Q \times \mathbb{R} \to \mathbb{R}$ and $\hat{j}_N : \Gamma_N \times \mathbb{R} \to \mathbb{R}$ are supposed, in addition, to be locally Lipschitz with respect to s. In problem (5.1) (as well as in its special case (5.2)) the functions $s \mapsto j(\cdot,\cdot,s,s)$ and $s \mapsto j_N(\cdot,\cdot,s,s)$ may be not locally Lipschitz but only partially locally Lipschitz. This enlarges the class of variational-hemivariational inequalities considerably. More precisely, we make the following assumptions on j and j_N:

(J1) $(x,t) \mapsto j(x,t,r,s)$ and $(x,t) \mapsto j_N(x,t,r,s)$ are measurable in Q and on Γ_N, respectively, for all $r, s \in \mathbb{R}$.

$r \mapsto j(x,t,r,s)$ and $r \mapsto j_N(x,t,r,s)$ are continuous for a.e. $(x,t) \in Q$ and $(x,t) \in \Gamma_N$, respectively, and for all $s \in \mathbb{R}$.

$s \mapsto j(x,t,r,s)$ and $s \mapsto j_N(x,t,r,s)$ are locally Lipschitz for a.e. $(x,t) \in Q$ and $(x,t) \in \Gamma_N$, respectively, and for all $r \in \mathbb{R}$.

(J2) Let $s \mapsto \partial j(x,t,r,s)$ and $s \mapsto \partial j_N(x,t,r,s)$ denote Clarke's generalized gradient of the functions j and j_N w.r.t s, respectively. Assume the following growth conditions: ∂j satisfies the growth condition

$$\sup\{|\eta| : \eta \in \partial j(x,t,s,s)\} \le \alpha(x,t) + \beta |s|^{p-1} \tag{5.4}$$

for a.e. $(x,t) \in Q$, $\forall\ s \in \mathbb{R}$, where $\alpha \in L^{p'}(Q)$, and $\beta \ge 0$. Similarly, ∂j_N satisfies the growth condition

$$\sup\{|\zeta| : \zeta \in \partial j_N(x,t,s,s)\} \le \alpha_N(x,t) + \beta_N |s|^{p-1} \tag{5.5}$$

for a.e. $(x,t) \in \Gamma_N$, $\forall\ s \in \mathbb{R}$, where $\alpha_N \in L^{p'}(\Gamma_N)$, and $\beta_N \ge 0$.

(J3) Let $s \mapsto j^o(x,t,r,s;\varrho)$ and $s \mapsto j^o_N(x,t,r,s;\varrho)$ denote Clarke's generalized directional derivative of the functions $s \mapsto j(x,t,r,s)$ and $s \mapsto j_N(x,t,r,s)$ at s, respectively, in the direction ϱ for fixed (x,t,r). Suppose that $s \mapsto j^o(x,t,s,s;\varrho)$ and $s \mapsto j^o_N(x,t,s,s;\varrho)$ are upper semicontinuous for a.e. $(x,t) \in Q$ and $(x,t) \in \Gamma_N$, respectively, and for all $\varrho \in \mathbb{R}$.

Define the multi-valued functions $f : Q \times \mathbb{R} \to 2^{\mathbb{R}}$ and $f_N : \Gamma_N \times \mathbb{R} \to 2^{\mathbb{R}}$ as follows:

$$f(x,t,s) = \partial j(x,t,s,s), \quad f_N(x,t,s) = \partial j_N(x,t,s,s). \tag{5.6}$$

For the so defined multi-valued functions the following lemma holds true.

Lemma 5.1 *Under the assumptions (J1)–(J3), the multi-valued functions* $f : Q \times \mathbb{R} \to 2^{\mathbb{R}}$ *and* $f_N : \Gamma_N \times \mathbb{R} \to 2^{\mathbb{R}}$ *defined by (5.6) satisfy hypotheses (F1)–(F3).*

Proof. From the definition of $\partial j(x,t,r,s)$ and the positive homogeneity of the mapping $\varrho \mapsto j^o(x,t,r,s;\varrho)$, we see that for almost all $(x,t) \in Q$, all $r, s \in \mathbb{R}$,

$$\partial j(x,t,r,s) = [-j^o(x,t,r,s;-1), j^o(x,t,r,s;1)].$$

Hence,

$$\begin{aligned} \mathrm{Gr}(f) &= \{(x,t,s,\eta) \in Q \times \mathbb{R} \times \mathbb{R} : \eta \in \partial j(x,t,s,s)\} \\ &= \{(x,t,s,\eta) \in Q \times \mathbb{R} \times \mathbb{R} : -j^o(x,t,s,s;-1) \le \eta \le j^o(x,t,s,s;1)\} \\ &= \{(x,t,s,\eta) \in Q \times \mathbb{R} \times \mathbb{R} : \eta \ge -j^o(x,t,s,s;-1)\} \\ &\quad \cap \{(x,t,s,\eta) \in Q \times \mathbb{R} \times \mathbb{R} : \eta \le j^o(x,t,s,s;1)\}. \end{aligned}$$

For each $\varrho \in \mathbb{R}$, it follows from (J1) that the function $(x,t,r,s) \mapsto j^o(x,t,r,s;\varrho)$ is measurable on $Q\times\mathbb{R}\times\mathbb{R}$ with respect to the measure $\mathcal{L}(Q)\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R})$, as "countable limit superior" of measurable functions there. Hence the functions $(x,t,s) \mapsto j^o(x,t,s,s;1)$ and $(x,t,s) \mapsto j^o(x,t,s,s;-1)$ are measurable on $Q \times \mathbb{R}$ with respect to the measure $\mathcal{L}(Q) \times \mathcal{B}(\mathbb{R})$. This implies that $\mathrm{Gr}(f)$ belongs to $[\mathcal{L}(Q) \times \mathcal{B}(\mathbb{R})] \times \mathcal{B}(\mathbb{R})$, i.e., f satisfies (F1).

To prove (F2), let $(x,t) \in Q$ be a point such that the functions $s \mapsto j^o(x,t,s,s;\pm 1)$ are upper semicontinuous on $\mathbb{R}$. Let $s_0 \in \mathbb{R}$ and U be an open neighborhood of $\partial j(x,t,s_0,s_0)$. There exists $\varepsilon > 0$ such that

$$(-j^o(x,t,s_0,s_0;-1) - \varepsilon, j^o(x,t,s_0,s_0;1) + \varepsilon) \subset U.$$

From the upper semicontinuity of the (single-valued) functions $s \mapsto j^o(x,t,s,s;\pm 1)$ at s_0, there exists an open neighborhood O of s_0 such that

$$\begin{cases} j^o(x,t,s,s;1) < j^o(x,t,s_0,s_0;1) + \varepsilon, \text{ and} \\ j^o(x,t,s,s;-1) < j^o(x,t,s_0,s_0;-1) + \varepsilon, \ \forall s \in O. \end{cases}$$

Hence, for all $s \in O$,

$$\begin{aligned} \partial j(x,t,s,s) &= [-j^o(x,t,s,s;-1), j^o(x,t,s,s;1)] \\ &\subset (-j^o(x,t,s_0,s_0;-1) - \varepsilon, j^o(x,t,s_0,s_0;1) + \varepsilon) \\ &\subset U. \end{aligned}$$

This shows the upper semicontinuity of f at s_0. Lastly, (F3) follows directly from (J2). The proof that f_N satisfies (F1)–(F3) is similar. □

Since by Lemma 5.1 the multi-valued functions f and f_N given by (5.6) satisfy hypotheses (F1)–(F3), we may consider the multi-valued parabolic variational inequality (1.2)–(1.3) or equivalently (3.1) with the special multi-valued functions (5.6), i.e, we consider the problem: Find $u \in D(L) \cap K$, $\eta \in L^{p'}(Q)$ and a $\zeta \in L^{p'}(\Gamma_N)$ such that

$$\begin{cases} \eta \in F(u), \ \zeta \in F_N(\gamma_N u) \\ \langle u_t + Au, v - u\rangle + \displaystyle\int_Q \eta\,(v-u)\,dxdt + \int_{\Gamma_N} \zeta\,(\gamma_N v - \gamma_N u)\,d\Gamma \ge 0, \end{cases} \tag{5.7}$$

for all $v \in K$, or equivalently: Find $u \in D(L) \cap K$, $\eta \in L^{p'}(Q)$ and a $\zeta \in L^{p'}(\Gamma_N)$ such that

$$\begin{cases} \eta \in F(u), \ \zeta \in F_N(\gamma_N u) \\ \langle Lu + Au + i^*\eta + \gamma^*\zeta, v - u\rangle \ge 0, \ \ \forall\, v \in K. \end{cases} \tag{5.8}$$

The main result of this section is to show that the parabolic variational-hemivariational inequality (5.1) and the related multi-valued variational inequality (5.7) are indeed equivalent provided K satisfies some lattice property. The following equivalence result holds:

Theorem 5.2 *Let (A1)–(A3), and (J1)–(J3) be satisfied and assume the following lattice condition for K to be fulfilled:*

$$K \vee K \subset K \ \textit{and} \ K \wedge K \subset K. \tag{5.9}$$

Then u is a solution of the parabolic variational-hemivariational inequality (5.1) if and only if u is a solution of the multi-valued variational inequality (5.7) with multi-functions f and f_N given by (5.6).

Proof. Let u be a solution of (5.7) (resp. (5.8)), which due to (5.6) means there is an $\eta \in L^{p'}(Q)$ and a $\zeta \in L^{p'}(\Gamma_N)$ such that

$$\eta(x,t) \in \partial j(x,t,u(x,t),u(x,t)), \ \zeta(x,t) \in \partial j_N(x,t,\gamma_N u(x,t),\gamma_N u(x,t))$$

and

$$\langle u_t + Au, v - u\rangle + \int_Q \eta\,(v-u)\,dxdt + \int_{\Gamma_N} \zeta\,(\gamma_N v - \gamma_N u)\,d\Gamma \ge 0, \tag{5.10}$$

for all $v \in K$. By the definition of ∂j and ∂j_N we get for any $v \in K$

$$\begin{aligned} &j^o(x,t,u,u;v-u) \ge \eta(x,t)\,(v-u), \text{ a.e. in } Q,\\ &j_N^o(x,t,\gamma_N u,\gamma_N u;\gamma_N v - \gamma_N u) \ge \zeta(x,t)\,(\gamma_N v - \gamma_N u), \text{ a.e. on } \Gamma_N. \end{aligned} \tag{5.11}$$

By (J1) and (J2) we can ensure that the left-hand sides of (5.11) belong to $L^1(Q)$ and $L^1(\Gamma_N)$, respectively, which in view of (5.10) implies (5.1).

Let us prove the reverse, and assume that u is a solution of (5.1). In order to show that u is a solution of the multi-valued variational inequality (5.7), we are going to show that u is both a subsolution and a supersolution for the multi-valued problem (5.7), which then by applying Theorem 4.4 completes the proof.

Since K has the lattice property (5.9), we can use in (5.1), in particular, $v \in u \wedge K$, i.e., $v = u \wedge \varphi = u - (u-\varphi)^+$ with $\varphi \in K$, which yields

$$\begin{cases} \langle u_t + Au, -(u-\varphi)^+\rangle + \displaystyle\int_Q j^o(x,t,u,u;-(u-\varphi)^+)\,dxdt \\ + \displaystyle\int_{\Gamma_N} j_N^o(x,t,\gamma_N u,\gamma_N u;-(\gamma_N u - \gamma_N\varphi)^+)\,d\Gamma \ge 0, \ \forall\, \varphi \in K. \end{cases}$$

Because $\varrho \mapsto j^o(\cdot,\cdot,r,s;\varrho)$ (resp. $\varrho \mapsto j_N^o(\cdot,\cdot,r,s;\varrho)$) is positively homogeneous, the last inequality is equivalent to

$$\begin{cases} \langle u_t + Au, -(u-\varphi)^+\rangle + \displaystyle\int_Q j^o(x,t,u,u;-1)(u-\varphi)^+\,dxdt \\ + \displaystyle\int_{\Gamma_N} j_N^o(x,t,\gamma_N u,\gamma_N u;-1)(\gamma_N u - \gamma_N\varphi)^+\,d\Gamma \ge 0, \ \forall\, \varphi \in K. \end{cases}$$

Using again for any $v \in u \wedge K$ its representation in the form $v = u - (u-\varphi)^+$ with $\varphi \in K$, the last inequality is equivalent to

$$\begin{cases} \langle u_t + Au, v-u\rangle + \displaystyle\int_Q -j^o(x,t,u,u;-1)(v-u)\,dxdt \\ + \displaystyle\int_{\Gamma_N} -j_N^o(x,t,\gamma_N u,\gamma_N u;-1)(\gamma_N v - \gamma_N u)\,d\Gamma \ge 0, \ \forall\, v \in u \wedge K. \end{cases} \tag{5.12}$$

By [15, Proposition 2.1.2] we have

$$\begin{aligned} &j^o(x,t,u(x,t),u(x,t));-1)\\ &= \max\{-\theta(x,t) : \theta(x,t) \in \partial j(x,t,u(x,t),u(x,t))\}\\ &= -\min\{\theta(x,t) : \theta(x,t) \in \partial j(x,t,u(x,t),u(x,t))\}\\ &=: -\underline{\eta}(x,t), \end{aligned} \tag{5.13}$$

where

$$\underline{\eta}(x,t) \in \partial j(x,t,u(x,t),u(x,t)). \tag{5.14}$$

Similarly, we get for j_N^o

$$\begin{aligned} & j_N^o(x,t,\gamma_N u(x,t),\gamma_N u(x,t)); -1) \\ & = \max\{-\zeta(x,t) : \zeta(x,t) \in \partial j_N(x,t,\gamma_N u(x,t),\gamma_N u(x,t))\} \\ & = -\min\{\zeta(x,t) : \zeta(x,t) \in \partial j_N(x,t,\gamma_N u(x,t),\gamma_N u(x,t))\} \\ & =: -\underline{\zeta}(x,t), \end{aligned} \tag{5.15}$$

and

$$\underline{\zeta}(x,t) \in \partial j_N(x,t,\gamma_N u(x,t),\gamma_N u(x,t)). \tag{5.16}$$

Since $(x,t) \mapsto j^o(x,t,u(x,t),u(x,t)); -1)$ as well as $(x,t) \mapsto j_N^o(x,t,\gamma_N u(x,t),\gamma_N u(x,t)); -1)$ are measurable functions, it follows that $(x,t) \mapsto \underline{\eta}(x,t)$ and $(x,t) \mapsto \underline{\zeta}(x,t)$ are measurable in Q and Γ_N, respectively, and in view of the growth conditions (J2) on the Clarke's gradients, we infer $\underline{\eta} \in L^{p'}(Q)$ and $\underline{\zeta} \in L^{p'}(\Gamma_N)$. Taking (5.13)–(5.16) into account, from (5.12) we get

$$\begin{cases} \langle u_t + Au, v - u\rangle + \displaystyle\int_Q \underline{\eta}(v-u)\,dxdt \\ + \displaystyle\int_{\Gamma_N} \underline{\zeta}(\gamma_N v - \gamma_N u)\,d\Gamma \ge 0, \ \forall\, v \in u \wedge K. \end{cases} \tag{5.17}$$

which together with (5.6) proves that u is a subsolution of (5.7). By similar arguments, one shows that u is a supersolution of (5.7) as well. By applying Theorem 4.4, there exists a solution $\hat{u}$ of (5.7) satisfying $u \le \hat{u} \le u$, i.e. $\hat{u} = u$ is a solution of (5.7), which completes the proof. □

Special Case

Let us consider the special case of (5.1) given by (5.2) with j and j_N of the special form (5.3), i.e.,

$$j(x,t,r,s) = h(x,t,r)\hat{j}(x,t,s), \quad j_N(x,t,r,s) = h_N(x,t,r)\hat{j}_N(x,t,s), \tag{5.18}$$

where $h, \hat{j} : Q \times \mathbb{R} \to \mathbb{R}$ and $h_N, \hat{j}_N : \Gamma_N \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. We suppose that $\hat{j} : Q\times\mathbb{R} \to \mathbb{R}$ and $\hat{j}_N : \Gamma_N\times\mathbb{R} \to \mathbb{R}$ are, in addition, locally Lipschitz with respect to s. In this case, in order for j and j_N to satisfy (J1)–(J3), only the following additional hypothesis for h, $\hat{j}$, h_N, $\hat{j}_N$ is required:

(HJ) $h(x,t,r) \ge 0$, $h_N(x,t,r) \ge 0$,

$$\sup\{|\eta| : \eta \in h(x,t,s)\partial\hat{j}(x,t,s)\} \le \alpha(x,t) + \beta|s|^{p-1} \tag{5.19}$$

for a.e. $(x,t) \in Q$, $\forall\, s \in \mathbb{R}$, where $\alpha \in L^{p'}(Q)$, and $\beta \ge 0$, and

$$\sup\{|\zeta| : \zeta \in h_N(x,t,s)\partial\hat{j}_N(x,t,s)\} \le \alpha_N(x,t) + \beta_N|s|^{p-1} \tag{5.20}$$

for a.e. $(x,t) \in \Gamma_N$, $\forall\, s \in \mathbb{R}$, where $\alpha_N \in L^{p'}(\Gamma_N)$, and $\beta_N \ge 0$.

We conclude this section with a few remarks.

Remark 5.3 (i) The results obtained in this paper may be extended and hold true if the operator A is replaced by a more general Leray-Lions operator of the form

$$A(x,t) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) + a_0(x,t,u,\nabla u).$$

(ii) Unlike in recent papers on multi-valued parabolic variational inequalities (e.g., [3]), the interior of the closed convex set K considered here may be empty, see for example the obstacle problem considered here.

(iii) Regarding the existence and enclosure result formulated in Theorem 4.4, more subtle consideration can be carried out to show that the solution set possesses a certain order structure. In particular, one can show that the solution set enclosed by sub-and supersolutions is a directed set, which then can be used to prove the existence of extremal solutions, see [11].

(iv) We remark that the lattice condition (5.9) for K, which is needed in the equivalence result Theorem 5.2 is satisfied, e.g., by obstacle problems and a number of further relevant constraints in applications.

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