Commentary

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On I. Meghea and C. S. Stamin review article "Remarks on some variants of minimal point theorem and Ekeland variational principle with applications," Demonstratio Mathematica 2022; 55: 354–379

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Abstract: Being informed that one of our articles is cited in the paper mentioned in the title, we downloaded it, and we were surprised to see that, practically, all the results from our paper were reproduced in Section 3 of Meghea and Stamin's article. Having in view the title of the article, one is tempted to think that the remarks mentioned in the paper are original and there are examples given as to where and how (at least) some of the reviewed results are effectively applied. Unfortunately, a closer look shows that most of those remarks in Section 3 are, in fact, extracted from our article, and it is not shown how a specific result is used in a certain application. So, our aim in the present note is to discuss the content of Section 3 of Meghea and Stamin's paper, emphasizing their Remark 8, in which it is asserted that the proof of Lemma 7 in our article is "full of errors."

Keywords: minimal point, Ekeland variational principle

MSC 2020: 49J27, 49J40

1 Introduction

In the introduction of the paper [1] by Meghea and Stamin, it is said that "In this paper, the authors start from some series of results from [16] and [17] for the first part, [18] and [19] for the second part aiming to make remarks on statements of minimal points in relation to vector variants of Ekeland variational principle and similar assertions in uniform spaces, to compare them and to discuss on their links, implications, and applications in models evolved from real phenomena."[1]

We consider that the declared aims correspond to those of a review article.

Of course, even a review article has to bring something new, and the authors consider that this goal is accomplished, as seen in the conclusion "The novelty of this work consists in juxtaposing these results, to make observations, remarks, and comments, and finally to highlight appropriate applications."

1 The references [16]–[19] mentioned in the quoted text are our references [2]–[5], respectively.

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Unfortunately, in our opinion, there is a big gap between the declared goals and the achieved ones, as well as in what concerns the novelty of the Meghea and Stamin's article, at least in what concerns the Section 3 "Some variants of minimal point theorem," which is deeply connected to our paper [2], that is, the reference [16] from Meghea and Stamin article [1]. Furthermore, Meghea and Stamin give unproven statements concerning the results in [2]. The remarks given by Meghea and Stamin are saying that assertions are "full of errors" and then repeating the assertions as well as the proofs without any new ideas in their paper is not acceptable.

2 Discussion about Section 3 of [1]

Having in view the declared aims and novelty of [1] (recalled above), we have to understand that those statements called "Remark" (see also the title of [1]), and, maybe, those statements which are not attributed to some references, are original.

Related to the originality of the remarks in [1], observe the following:

- [1, Remark 2] is a reformulation of the following observation from [2, p. 912]: "Theorem 1 does not ensure, effectively, a minimal point. In the next section we shall derive an authentic minimal point theorem";
- After [1, Remark 2] the authors are saying: "As consequences of Theorem 1, we will see in the following two vector variants of Ekeland principle"; compare with the text "We want to apply the preceding results to obtain two vectorial EVP" from [2, p. 911], immediately after the end of the proof of [2, Th. 1].
- [1, Remark 4] is contained in the paragraph beginning with "Traditionally, in the statements of the EVP appears an $\varepsilon > 0$ and an estimate of $d(\overline{x}, x_0)$" and ending with "since $(d(\overline{x}, x_0) - \lambda)k^0 + k \in K \setminus \{0\}$" on [2, p. 912];
- [1, Remark 5] is contained in the first paragraph of [2, p. 912];
- [1, Remark 6] is contained in the paragraph immediately after the proof of [2, Cor. 3];
- [1, Attention]: This remark is essentially contained in the following paragraph from [2, p. 912]: "Note that taking $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ from Corollary 3 one obtains the EVP for functions that are not necessarily lower semicontinuous. For example the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \exp(-|x|)$ for $x \neq 0$ and $f(0) = 2$, satisfies the hypothesis of Corollary 3";
- [1, Remark 7]: The authors are right. In fact on [2, p. 914] it is said: "Of course, Theorem 1 is an immediate consequence of Corollary 5," and, before the statement of [2, Cor. 5], it is said "An immediate consequence of the preceding theorem is the following weaker result." Therefore, in [2] it is mentioned that [2, Th. 1] (i.e., [1, Theorem 1]) follows from [2, Th. 4] (i.e., [1, Theorem 4]). Even more, we mentioned the following in the paragraph after [2, Ex. 1]: "We preferred to formulate and give a direct proof of Theorem 1 because it has the advantage of not containing any reference to an element $x^* \in K^+$ and the proof is interesting, in our opinion, for itself."
- [1, Remark 9]: This is contained in the text of the proof of [2, Lem. 7], more precisely, "(even φ is continuous if $\operatorname{int} K \neq \emptyset$, because it is bounded above by 1 on the neighborhood of 0, namely, $k^0 - K$)."

In what concerns [1, Remark 8] (this is really an original formulation by Meghea and Stamin), their statement is as follows:

"Remark 8. *In* [16], *the proof of the Lemma* 7, here Lemma 1, *is full of errors*. For this reason, the authors presented here the recovered proof."

We only succeeded to detect 2 "errors"; they are: "−" instead of "+" (at the end of the sixth line of [2, p. 916]) and "a neighborhood of 0" instead of "the neighborhood of 0" (see the quoted text mentioned in the above comment on [1, Remark 9]). Neither of the two "errors" affects the result.

- [1, Remark 11]: This remark is provided after the proof of [2, Th. 8].
- [1, Remark 12]: This observation is done in the following paragraph from [2, p. 919]: "The above corollary shows that Theorem 4.1 of Tammer [14] is valid without the boundedness from below of $f(X)$ and even for cones K with empty interior."[2]

2 Paper [14] from the quoted text is our reference [6].

About the statements which are not attributed to some references, observe the following:

- [1, Proposition 4] is a reformulation of the next remark from [2, p. 910]: "Note that if A satisfies (H2) and K has closed lower sections with respect to $\mathbb{R}_+k^0$, i.e. $K \cap (y - \mathbb{R}_+k^0)$ is closed for every $y \in K$, then (H1) is also satisfied";
- [1, Proposition 5]: This result is contained in the first paragraph of the proof of [2, Lem. 7];
- [1, Proposition 6]: This assertion is mentioned on [2, p. 920], immediately after (H6).

  Also, notice that for Theorems 1, 5, and 7 from [1], one mentions Theorems 1, 8, and 10 from [2] (respectively); for Theorems 2, 3, 6, and 8 one mentions [2], but not Corollaries 2, 3, 9, and 12 from [2] (with which they are equivalent, respectively); Theorem 4 is attributed to [7], even if [1, Theorem 4] is nothing else than [2, Th. 4].

Related to [1, Remark 8], quoted above, we asked the authors of [1] (on August 29, 2022) to provide the complete list of errors they detected in the proof of [2, Lem. 7]. We did not receive directly such a list. However (on January 6, 2023), we received, from the Managing Editor of Demonstratio Mathematica, the following **response of one of the authors of** [1] concerning [1, Remark 8]:

"*Extended remarks – detailed for the Remark 8* (related to the proof of Lemma 7, page 916 from [16])

In [16], φ, K and Λ_y are L, C, and M_y in my paper.

Lines 1–7 of the proof are dedicated to proving a property which is given as the Proposition 5 from my paper. On the line 4, the last sign is "−", not "+", being, probably, a typing error.

Lines 8–10: From the properties displayed on the line 9, it results, with some additional arguments that φ is l.s.c. when $\varphi(y) < +\infty$, but the case $\varphi(y) = +\infty$ is not treated.

Lines 10–12: It is not proven nor said that $T^{-1}(K) \neq \emptyset$ and from there will result that φ is proper.

Lines 13–14: True assertions – not justified. It is possible to be proven in other places (works), but there is no citation. The corresponding proofs are not immediately obtained.

Line 18 contains two no justified assertions. Similar considerations as above.

Line 20 contains one no justified assertion. Similar considerations as above.

Line 21: The first equality is not obvious. It has a substantial proof.

Lines 22–23: The proof of the last assertion is, indeed, based on the second relation from the line 22, but it needs more arguments."[3]

Before discussing the above text, one must clarify the meaning of the word *error*; as per Oxford dictionary: https://www.oxfordlearnersdictionaries.com/definition/english/error

error means a mistake, especially one that causes problems or affects the result of something.

Related to the previous quoted text, let us first discuss the following texts referring to the proof of [2, Lem. 7]:

(a) "Lines 8–10: From the properties displayed on the line 9, it results, with some additional arguments that φ is l.s.c. when $\varphi(y) < +\infty$, but the case $\varphi(y) = +\infty$ is not treated";

(b) "Lines 10–12: It is not proven nor said that $T^{-1}(K) \neq \emptyset$ and from there will result that φ is proper."

Concerning "the case $\varphi(y) = +\infty$ is not treated" from (a), note that it is not necessary to consider it because for a function $f : (T, \tau) \to \overline{\mathbb{R}}$, f is l.s.c. iff $\{t \in T | f(t) \leq \lambda\}$ is closed for every $\lambda \in \mathbb{R}$, (T, τ) being a topological space (cf. https://en.wikipedia.org/wiki/Semi-continuity for a rapid reference). In what concerns "some additional arguments that φ is l.s.c. when $\varphi(y) < +\infty$," let us look into the "additional arguments that φ is l.s.c. when $\varphi(y) < +\infty$" provided in the proof of [1, Lemma 1]. So, the proof begins with "Let y be arbitrary from Y. Assume $L(y) < +\infty$. Then for any t from $\boldsymbol{R}$, we have $L^{-1}((-\infty, t]) = tk_0 - C$, as, for instance, $L(y) \leq t \overset{5}{\Rightarrow} y \in tk_0 - C$. Since $tk_0 - C$ is closed, one concludes L l.s.c. in y." One must justify the correctness of the last assertion based on what is previously known. Said differently, where (how) the fact that $L(y) < +\infty$ is used and $[L^{-1}((-\infty, t]) = tk_0 - C$ for every $t \in \mathbb{R}]$ for getting the conclusion "L l.s.c. in y"?

3 Paper [16] in the quoted text is our paper [2].

Concerning “It is not proven nor said that $T^{-1}(K) \neq \emptyset$” from (b), observe the trivial fact that $0 \in T^{-1}(K)$ because T is a linear operator and $0 \in K$. In what concerns “from there will result that φ is proper,” on line 8 of the proof of [1, Lemma 1] one asserts “As $T^{-1}(C) \neq \emptyset$ ($k_0 = T(0,1)$!), L is proper”; of course, in order to have a complete *justification*, one had to invoke [1, Proposition 5]!

The other remarks are complaints that certain assertions are not justified (sufficiently).

So, does that “typing error” justify the assertion “*In [16], the proof of the Lemma 7,* here Lemma 1, *is full of errors*” from [1, Remark 8]? Or, maybe, is this an error?

In the present context, note that the appreciation that a detail is trivial or not depends on the author as well as the reader. We, the authors of [2], consider that we gave sufficient details in the proofs of our results.

In the same context, a natural question is if there are errors in [1]; we detected three false assertions in the part in which we are mostly concerned. More precisely, one asserts:

(A) “Let E be a real vector space. (i) If E is a preordered, then the set $C = \{x \in E : x \geq 0\}$ is a convex cone with vertex”; see [1, Proposition 3].

(B) “$M^i \subset M$, M convex $\Rightarrow M \subset M^a$”; see [1, p. 361, line 15].

(C) “$\mathbf{R}k_0 - C$ is closed since $\mathbf{R}k_0$ has a finite dimension”; see [1, p. 362, line 5] (i.e., line 4 of the proof of [1, Lemma 1]). Recall that “Here the convex cone C with vertex is assumed closed. So $k_0 \in C \setminus (-C)$” (cf. [1, p. 361]).

Before providing a counterexample for assertion (A), let us see what is meant by “preorder” in [1]: “ *Explanation. Preorder relation means reflexivity + transitivity*” (cf. [1, p. 356, line 4]); “*Recapitulation.* Let A be a nonempty set and “≤” a binary relation in A. This is called *preorder relation* if it is *reflexive and transitive*. In this case, A becomes *preordered*” (cf. [1, p. 364, lines 2 and 3 from below]).

Example A. Take an arbitrary bijection $\psi : E := \mathbb{R} \to F := \mathbb{R}$, E, and F being endowed with the usual addition and multiplication (on $\mathbb{R}$), $\leq_E := \{(x,y) \in E \times E | y - x \in \mathbb{R}_+\}$ and $\leq_F := \{(u,v) \in F \times F | \psi^{-1}(v) - \psi^{-1}(u) \in \mathbb{R}_+\}$. (For example, take $\psi(x) := x$ for $x \in \mathbb{R} \setminus \{-1, 1\}$ and $\psi(x) := -x$ for $x \in \{-1, 1\}$.) It is obvious that $\leq_E$ and $\leq_F$ are preorder relations on E and F, respectively. However, $C := \{u \in F | u \geq_F 0\}$ is not, in general, a convex cone (in the previous example $C = [0, 1) \cup (1, \infty) \cup \{-1\}$, which is neither convex, nor cone).

Before providing a counterexample for assertion (B), let us see the definitions of the sets M^i and M^a in [1, p. 361, lines 5–13]: “Let E be a vector space on $\boldsymbol{K}$, $\boldsymbol{K} = \boldsymbol{R}$, or $\boldsymbol{K} = \boldsymbol{C}$, and x, y in E, $x \neq y$. The closed interval $[x,y]$ is $[x,y] = \{(1-\lambda)x + \lambda y : \lambda \in [0,1]\}$. If $\lambda \in [0,1)$, $\lambda \in (0,1]$, or $\lambda \in (0,1)$, one obtains $[x,y)$, $(x,y]$, and (x,y) (open interval), respectively. The straight line d which passes through x and y is $d = \{(1-\lambda)x + \lambda y : \lambda \in \boldsymbol{R}\}$. If $x_1 \in d$, we have $d = \{x_1 + \lambda(y - x) : \lambda \in \boldsymbol{R}\}$ ($x_1 \in d \Rightarrow x_1 = (1-\lambda_1)x + \lambda_1 y$, replace). Let M be a nonempty subset of E. x_0 from M is by definition *algebraic interior point*, if on each straight line that passes through x_0 there is an open interval included in M which contains x_0. The set of these points is denoted M^i or $\operatorname{aint} M$, the *algebraic interior* of M. y_0 from E is by definition *algebraic adherent point* to M if $\exists x$ in M so that $[x, y_0) \subset M$. The set of these points, the *algebraic closure* of M, is denoted M^a.”

Example B. Consider E, a real linear space with $\dim E \geq 1$ and $M \subset E$; by the very definition of M^i one has that $M^i \subset M$. Take now $\overline{x} \in E$ and $M := \{\overline{x}\}$; clearly, M is convex. Assume that there exists $y_0 \in M^a$. By the definition of M^a, there exists $x \in M$ such that $[x, y_0) \subset M$; hence, $x = \overline{x}$. By the definitions of the intervals (recalled above), $x \neq y_0$. Clearly, $(\frac{1}{2}\overline{x} + \frac{1}{2}y_0 =)$ $\frac{1}{2}x + \frac{1}{2}y_0 \in [x, y_0) \subset M$, and so $\frac{1}{2}\overline{x} + \frac{1}{2}y_0 = \overline{x}$, whence the contradiction $(x=)\overline{x} = y_0$. Hence, $M^a = \emptyset \not\supset M = \{\overline{x}\}$.

Example C. Consider $C := (\{(0,0)\} \times \mathbb{R}_+) \cup \{(x,y,z) \in \mathbb{R}^3 | x > 0, y^2 \leq 2xz\}$ and $k_0 := (0,0,1)$. C is a closed convex cone and $k_0 \in C \setminus (-C)$. It is clear that $\mathbb{R}k_0 - C \ni nk^0 - (\frac{1}{n}, -1, n) = (-\frac{1}{n}, 1, 0) \to (0,1,0) \notin \mathbb{R}k_0 - C$. Hence, $\mathbb{R}k_0 - C$ is not closed, and so assertion (C) is false.

3 Other remarks

Now, we can compare (at least partially) the aims of the authors of [1], recalled in Introduction, with the achieved ones.

(1) Meghea and Stamin said that "the authors start from some series of results from [16] and [17] for the first part, [18] and [19] for the second part [...]."[4] In fact, excepting [2, Cor. 11], all the results from [2] are present in this review article (probably the same is true for those from [4] and [5], but one refers to [3] only for the preliminary part of Section 3.4 (without mentioning any result from [3])).

(2) Meghea and Stamin said "[...] aiming to make remarks on statements of minimal points in relation to vector variants of Ekeland variational principle and similar assertions in uniform spaces, to compare them and to discuss on their links [...]." In fact, we determined, after verification, that there are no "remarks on statements of minimal points in relation to vector variants of Ekeland variational principle" among the results from Section 3, other than those done in our article [2]; moreover, there is no comparison or discussion on the links of the results mentioned in Sections 3 and 4.

(3) In what concerns "[...] implications, and applications in models evolved from real phenomena," in the Section "5 Applications," there is no mention of any result from Sections 3 or 4 which is applied in a specific "model evolved from real phenomena" among those suggested by the following keywords: "equilibrium problem, location problem, multicriteria control problem, multicriteria fractional programming problem, stochastic efficiency."

(4) Note that in [1, Remark 3] it is said: "From Theorem 2 it can indeed obtain *a form of Ekeland principle*." In fact, from [1, Theorem 2] (i.e., [2, Th. 2]) one can obtain exactly the "Ekeland principle [1,20]"[5] from the beginning of Section "2.2 Preliminaries for Ekeland principle" in [1].

Indeed, applying [1, Theorem 2] (that is, [2, Th. 2]) for $x_0 \coloneqq u$ with $\varphi(u) \le \inf\varphi + \varepsilon, \lambda > 0$ and $k_0 \coloneqq \varepsilon\lambda^{-1}$, one obtains v_ε verifying (19) and (20) from [1]. Hence, $(\varphi(v_\varepsilon)\le)\, \varphi(v_\varepsilon) + \varepsilon\lambda^{-1}d(v_\varepsilon, u) \le \varphi(u) \le \varphi(v_\varepsilon) + \varepsilon$ by (19), and so $\varphi(v_\varepsilon) \le \varphi(u)$, $d(v_\varepsilon, u) \le \lambda$; by (20) one obtains $\varphi(x) + \varepsilon\lambda^{-1}d(v_\varepsilon, x) \le \varphi(v_\varepsilon) \Rightarrow x = v_\varepsilon$; hence $\varphi(v_\varepsilon) < \varphi(x) + \varepsilon\lambda^{-1}d(v_\varepsilon, x)$ for every $x \in X\setminus\{v_\varepsilon\}$, and so v_ε verifies (13)–(15) from [1, Ekeland principle [1,20], p. 357].

Therefore, from [1, Theorem 2] one obtains the version of EVP given in "Ekeland principle [1,20]," not only "a form of Ekeland principle."

(5) It is worth observing that except Sections "1 Introduction," "5 Applications," and "6 Conclusions" from [1], all the text (including [1, Ekeland principle [1,20], p. 357]) can be found, almost word by word, in Meghea's book [10].

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[1] I. Meghea and C. S. Stamin, *Remarks on some variants of minimal point theorem and Ekeland variational principle with applications*, Demonstr. Math. **55** (2022), 354–379.

4 See Note 1.

5 Note that the reference [1] in [1,20] seems to be a preprint of [8], while [20] is our reference [9]. Observe that the statement of [1, Ekeland principle [1,20], p. 357] is equivalent to [8, Th. 1.1], but there is no equivalent statement in [9]; in fact, [8] is not cited in [9].

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