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$D^{1,p}(\mathbb{R}^N)$ versus $C_b\left(\mathbb{R}^N, 1+|x|^{\frac{N-p}{p-1}\alpha}\right)$ local minimizers

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ABSTRACT

Let $X = D^{1,p}(\mathbb{R}^N)$ be the Beppo-Levi space (homogeneous Sobolev space) with $2 \le p < N$, and for $\frac{p-1}{p} < \alpha \le 1$ let $V_\alpha = X \cap C_b\left(\mathbb{R}^N, 1+|x|^{\frac{N-p}{p-1}\alpha}\right)$ be the subspace of bounded continuous functions with weight $1+|x|^{\frac{N-p}{p-1}\alpha}$. In this paper we prove a Brezis-Nirenberg type result for the energy functional $\Phi : X \to \mathbb{R}$ related to the quasilinear elliptic equation in $\mathbb{R}^N$ of the form

$$u \in X : -\Delta_p u = a(x)g(u) \quad \text{in } \mathbb{R}^N,$$

which states that a local minimizer of Φ in the V_α-topology must be a local minimizer in the "bigger" X-topology.

Global L^∞-estimates for solutions of general quasilinear elliptic equations of divergence type in $\mathbb{R}^N$ on the one hand, and decay estimates for solutions of p-Laplace equations via nonlinear Wolff potentials as well as comparison theorems for p-Laplacian type operators on the other hand play an important role in the proofs.

1. Introduction and main results

Let $X = D^{1,p}(\mathbb{R}^N)$ be the Beppo-Levi space (homogeneous Sobolev space) which is the completion of $C_c^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\right)^{1/p},$$

and for which we have the continuous embedding $X \hookrightarrow L^{p^*}(\mathbb{R}^N)$, where $p^* = \frac{Np}{N-p}$ denotes the critical Sobolev exponent. Consider the following quasilinear elliptic equation in $\mathbb{R}^N$

$$u \in X : -\Delta_p u = a(x)g(u), \tag{1.1}$$

where throughout we assume $2 \le p < N$ and that the coefficient a and the nonlinearity g satisfy the assumptions:

(A0) $a : \mathbb{R}^N \to \mathbb{R}_+$ is measurable and satisfies the following decay condition for some $\beta, c_a > 0$

$$0 \le a(x) \le c_a w(x), \quad \text{where} \quad w(x) = \frac{1}{1+|x|^{N+\beta}}, \quad x \in \mathbb{R}^N. \tag{1.2}$$

(G) $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies for some positive constant c_g the conditions

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(i) $0 < g(s)$, for $s \in (0, \infty)$;
(ii) $|g(s)| \le c_g\left(1 + |s|^{\gamma-1}\right)$, $\forall\ s \in \mathbb{R}$, where $1 \le \gamma < p^*$.

With the following lemma we are able to characterize solutions of (1.1) as critical points of the energy functional Φ given by

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx - \int_{\mathbb{R}^N} a(x)G(u)\,dx, \quad \text{with } G(s) = \int_0^s g(t)\,dt. \tag{1.3}$$

Lemma 1.1 *([1, Lemma 6.1]). Let $L^q(\mathbb{R}^N, w)$ be the weighted Lebesgue space with weight w given in (1.2). Then the embedding $X \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ is compact for $1 < q < p^*$.*

Taking into account the weak lower semicontinuity of the norm and the compact embedding due to Lemma 1.1 we have the following result.

Lemma 1.2. *Let g satisfy (G)(ii), and let a fulfill (A0). Then $\Phi : X \to \mathbb{R}$ is a well defined C^1-functional, which is weakly lower semicontinuous. Moreover, critical points of Φ are solutions of (1.1).*

Next for $\frac{p-1}{p} < \alpha \le 1$, let $V_\alpha = X \cap C_b\left(\mathbb{R}^N, 1 + |x|^{\frac{N-p}{p-1}\alpha}\right)$ be the subspace of bounded continuous functions with weight $1 + |x|^{\frac{N-p}{p-1}\alpha}$ defined by

$$V_\alpha := \left\{ v \in X : v \in C(\mathbb{R}^N) \text{ with } \sup_{x\in\mathbb{R}^N} \left(1 + |x|^{\frac{N-p}{p-1}\alpha}\right)|v(x)| < \infty \right\},$$

which is a closed subspace of X with norm

$$\|v\|_{V_\alpha} := \|v\|_X + \sup_{x\in\mathbb{R}^N} \left(1 + |x|^{\frac{N-p}{p-1}\alpha}\right)|v(x)|,\ v \in V_\alpha\,.$$

Our main result is the following X versus V_α local minimizer theorem:

Theorem 1.3. *Let g satisfy (G), and let $a : \mathbb{R}^N \to \mathbb{R}$ fulfill (A0). In addition assume*

($\tilde{G}$) $\liminf_{s\to 0^+} \frac{g(s)}{s^q} > 0$ *for some* $0 \le q \le (p-1)(1-\alpha)$.

Suppose $u_0 \in X$ is a nonnegative solution of Eq. (1.1) and a local minimizer in the V_α-topology of the functional $\Phi : X \to \mathbb{R}$, that is, there exists $\varepsilon_0 > 0$ such that

$$\Phi(u_0) \le \Phi(u_0 + h), \quad \forall\ h \in V_\alpha : \|h\|_{V_\alpha} < \varepsilon.$$

Then u_0 is a local minimizer of Φ with respect to the X-topology, that is, there is $\varepsilon_1 > 0$ such that

$$\Phi(u_0) \le \Phi(u_0 + h), \quad \forall\ h \in X : \|h\|_X < \varepsilon_1.$$

Theorem 1.3 is in the spirit of and extends the classical result due to Brezis and Nirenberg for a semilinear elliptic equation on bounded domains (see [2]) in two directions. First, unlike in [2] the leading operator is the p-Laplacian, and more importantly, second, the unboundedness of the domain. While extensions of the Brezis-Nirenberg result on bounded domains with leading p-Laplacian type variational operators have been obtained by several authors (see [3–8]), the literature about extensions to unbounded domains, in particular to the whole $\mathbb{R}^N$, is much less developed. Extensions to $\mathbb{R}^N$ with the Laplacian or the fractional Laplacian as leading operators within the Beppo-Levi space $D^{1,2}(\mathbb{R}^N)$ or fractional Beppo-Levi space $D^{s,2}(\mathbb{R}^N)$, respectively, can be found in [9–11]. An extension of the Brezis-Nirenberg result to the (unbounded) exterior domain $\mathbb{R}^N \setminus \overline{B(0,1)}$ was obtained in [12] for the N-Laplacian equation in the Beppo-Levi space $D_0^{1,N}(\mathbb{R}^N \setminus \overline{B(0,1)})$, which is based on Kelvin transform. The latter, however, only works for p-Laplacian equations with $p = 2$ or $p = N$.

Only recently in [13] the authors proved a "X versus $X \cap V_\alpha$ local minimizers" result for $\alpha = \frac{p-1}{p}$ supposing only the general growth restriction (G)(ii). Assuming additional conditions (G)(i) and ($\tilde{G}$), Theorem 1.3 provides "X versus $X \cap V_\alpha$ local minimizers" results for α in the range $\frac{p-1}{p} < \alpha \le 1$, which in a way may be considered as an "interpolation" between the authors' result in [13] under the general growth on g and that given by Theorem 1.3 under additional restrictions on g. Unlike in [13], here the additional assumptions imposed on g enable us to use a different approach to deal with the "better" weights $1 + |x|^{\frac{N-p}{p-1}\alpha}$.

Global L^∞-estimates for solutions of general quasilinear elliptic equations of divergence type in $\mathbb{R}^N$ on the one hand, and decay estimates for solutions of p-Laplace equations via nonlinear Wolff potentials as well as comparison theorems for p-Laplacian type operators on the other hand play an important role in the proofs.

The outline of this paper is as follows: In Section 2 we provide preliminary results which will be used in Section 3 to prove Theorem 1.3. In Section 4 we demonstrate the applicability of our main result to prove the existence of solutions within an interval of sub- and supersolutions that are in fact local minimizer of the associated energy functional Φ.

2. Preliminaries

Before we present our result, first a few words on the notation. For an open set $\Omega \subset \mathbb{R}^N$, the standard norms of the Lebesgue spaces $L^r(\Omega)$ are denoted by $\|\cdot\|_{r,\Omega}$, or whenever it is convenient and not confusing, by $\|\cdot\|_r$. The weighted Lebesgue space $L^r(\mathbb{R}^N, w)$ with weight function w given by (1.2) is defined by

$$L^r(\mathbb{R}^N, w) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} w|u|^r\,dx < \infty \right\},$$

which is separable and reflexive for $1 < r < \infty$ under the norm

$$\|u\|_{r,w} = \left(\int_{\mathbb{R}^N} w|u|^r\,dx \right)^{\frac{1}{r}}.$$

One readily verifies that the weight function w belongs to $L^q(\mathbb{R}^N)$ for all q with $1 \le q \le \infty$. Thus $a \in L^q(\mathbb{R}^N)$ for all $q \in [1, \infty]$. We use $\|m\|$ defined by $\|m\| = \|m\|_1 + \|m\|_\infty$ for any function $m \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Finally we use C, to denote a constant whose exact value is immaterial and may change from line to line. To indicate the dependence of the constant on the data, we write $C = C(a, b, \cdot, \cdot, \cdot)$ with the understanding that this dependence is increasing in its variables.

We begin by recalling the following lemma.

Lemma 2.1 (*[1, Lemma 6.6]*). *If $a : \mathbb{R}^N \to \mathbb{R}$ satisfies (A0), then a has the following properties:*

(a1) $a \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$,

(a2) *There exists $\sigma > \frac{N}{p}$ and $D > 0$ such that*

$$|x|^{\frac{N}{\sigma'}} \|a\|_{L^\sigma(\mathbb{R}^N \setminus B(0,|x|))} \le D, \quad \forall x \in \mathbb{R}^N,$$

where $\frac{1}{\sigma'} + \frac{1}{\sigma} = 1$ and $B(0, |x|)$ is the open ball with radius $|x|$.

Lemma 2.2. *Let (A0) and (G) be satisfied. If $u \in X$ is a nonnegative solution of Eq. (1.1), then $u \in X \cap C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$, $\lambda \in (0, 1)$, and the following decay estimate holds:*

$$0 \le u(x) \le \frac{C}{1 + |x|^{\frac{N-p}{p-1}}}, \quad \forall x \in \mathbb{R}^N \tag{2.1}$$

where $C = C(N, p, \sigma, c_g, \|a\|, \|u\|_X, D)$ with σ and D as in $(a2)$ above and c_g given in $(G)(ii)$.

Furthermore, if u is not identically zero, then there exists a positive constant C', depending on u, such that:

$$\frac{C'}{1 + |x|^{\frac{N-p}{p-1}}} \le u(x), \quad \forall x \in \mathbb{R}^N \tag{2.2}$$

Proof. From [14, Corollary 3.1] we obtain $u \in L^\infty(\mathbb{R}^N)$ satisfying the estimate

$$\|u\|_\infty \le \tilde{C}(N, p, c_g, \|a\|, \|u\|_{p^*}) \max\{\|u\|_{p^*}, \|u\|_{p^*}^{\theta_0}\}.$$

Taking $X \hookrightarrow L^{p^*}(\mathbb{R}^N)$ into account we get

$$\|u\|_\infty \le C,$$

where $C = C(N, p, c_g, \|a\|, \|u\|_X)$ with $C(N, p, c_g, \|a\|, \|u\|_X) \to 0$ as $\|u\|_X \to 0$. Regularity results due to DiBenedetto (see [15]) yield $u \in X \cap C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, the right-hand side of (1.1) allows for the estimate

$$|a(x)g(u(x))| \le Ca(x) \tag{2.3}$$

where $C = C(N, p, c_a, c_g, \|a\|, \|u\|_X)$. Consider the equation

$$v \in X : -\Delta_p v = Ca(x). \tag{2.4}$$

Let us show that (2.4) has a unique positive solution $v \in X \cap C^{1,\lambda}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying

$$0 \le u(x) \le v(x).$$

Since $w \in L^r(\mathbb{R}^N)$ for all $r \in [1, \infty]$, it belongs, in particular, to $L^{p^{*\prime}}(\mathbb{R}^N)$, which is continuously embedded into X^*. It is well known that the operator $T = -\Delta_p$ defines a bounded, continuous, strongly monotone (note $2 \le p < N$) and coercive operator from X into its dual through

$$\langle Tv, \varphi \rangle = \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi\,dx, \quad \forall \varphi \in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. Thus $T : X \to X^*$ is bijective, which yields the existence of a unique solution v of (2.4), which is even $C^{1,\lambda}_{loc}(\mathbb{R}^N)$-regular due to [15]. Next, we show that $v(x) \geq 0$. As a weak solution v satisfies

$$\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\, \nabla\varphi\, dx = \int_{\mathbb{R}^N} Ca(x)\varphi\, dx.$$

Testing this relation with $\varphi = v^- = \max\{-v, 0\}$, we get

$$0 \leq \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v\, \nabla v^-\, dx = -\int_{\mathbb{R}^N} |\nabla v^-|^p\, dx \leq 0,$$

which implies that $\|v^-\|_X = 0$ and thus $v^- = 0$, that is, $v(x) \geq 0$ for all $x \in \mathbb{R}^N$, and by Harnack's inequality it follows that $v(x) > 0$ for all $x \in \mathbb{R}^N$.

From (1.1), (2.3), and (2.4) we get by comparison

$$\langle -\Delta_p u - (-\Delta_p v), \varphi\rangle \leq 0, \quad \forall \varphi \in X_+, \tag{2.5}$$

where $X_+ = \{\varphi \in X : \varphi \geq 0\}$. Taking in (2.5) the test function $\varphi = (u - v)^+$ we get

$$0 \geq \int_{\mathbb{R}^N} \Big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\Big)\nabla (u-v)^+ \geq c_p \|(u-v)^+\|^p_X,$$

and thus $(u - v)^+ = 0$, i.e., $u \leq v$. Finally, a pointwise estimate of v from above is provided by an estimate from above of the Wolff potential $W^\mu_{1,p}(x, \infty)$, which has been calculated in [16,17]. In particular, by [17, Lemma 2.1, Theorem 2.2] we obtain

$$0 \leq v(x) \leq \frac{C}{1 + |x|^{\frac{N-p}{p-1}}}, \quad \forall x \in \mathbb{R}^N$$

where $C = C(N, p, \sigma, c_g, \|a\|, \|u\|_X, D)$, which completes the proof of (2.1).

Finally to prove (2.2) we note that u solves $-\Delta_p u \geq 0$, which allows to make use of a Vázquez type maximum principle (see [1, Theorem 6.4] or [16, Theorem 3.1]) according to which there is some positive constant θ such that

$$u(x) \geq \frac{\theta}{|x|^{\frac{N-p}{p-1}}}, \quad \text{for } |x| \geq 1,$$

which implies

$$u(x) \geq \frac{\theta}{1 + |x|^{\frac{N-p}{p-1}}}, \quad \text{for } |x| \geq 1. \tag{2.6}$$

Also since $u \in C^1(\mathbb{R}^N)$, and by Harnack's inequality $u(x) > 0$ in $\mathbb{R}^N$, we have $\tilde\theta = \min_{x\in B(0,1)} u(x) > 0$, which yields

$$u(x) \geq \tilde\theta \geq \frac{\tilde\theta}{1 + |x|^{\frac{N-p}{p-1}}}, \quad \text{for } |x| \leq 1. \tag{2.7}$$

From (2.6) and (2.7) with $C' = \min\{\theta, \tilde\theta\}$ we get (2.2). □

For the rest of this section assume that $u_0 \in X$ is a fixed *positive* solution of (1.1). We define operators $\mathcal{A}_\mu$ as follows.

$$\mathcal{A}_\mu u = -\mathrm{div}\, A_\mu(x, \nabla u), \quad \mu \geq 0, \tag{2.8}$$

where the function $A_\mu : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$A_\mu(x,\xi) = \frac{1}{1+\mu}\Big[|\nabla u_0 + \xi|^{p-2}(\nabla u_0 + \xi) - |\nabla u_0|^{p-2}\nabla u_0 + \mu|\xi|^{p-2}\xi\Big], \quad \mu \geq 0. \tag{2.9}$$

Lemma 2.3. *$A_\mu : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, which satisfies the following properties uniformly for $\mu \geq 0$.*

(A1) $|A_\mu(x,\xi)| \leq 2^{p-1}|\xi|^{p-1} + 2^p|\nabla u_0(x)|^{p-1}$;
(A2) $(A_\mu(x,\xi) - A_\mu(x,\hat\xi))(\xi - \hat\xi) \geq 2^{2-p}|\xi - \hat\xi|^p$;
(A3) $A_\mu(x,\xi)\xi \geq 2^{2-p}|\xi|^p$.

Proof. As $2 \leq p < N$ we use the inequality

$$\Big(|\xi|^{p-2}\xi - |\hat\xi|^{p-2}\hat\xi\Big)(\xi - \hat\xi) \geq 2^{2-p}|\xi - \hat\xi|^p, \quad \forall \xi, \hat\xi \in \mathbb{R}^N \tag{2.10}$$

in the following estimates.

$$\begin{aligned}
|A_\mu(x,\xi)| &\leq \frac{1}{1+\mu}\Big[|\nabla u_0 + \xi|^{p-1} + |\nabla u_0|^{p-1} + \mu|\xi|^{p-1}\Big] \\
&\leq \frac{1}{1+\mu}\Big[2^{p-1}\Big(|\nabla u_0|^{p-1} + |\xi|^{p-1}\Big) + |\nabla u_0|^{p-1} + \mu|\xi|^{p-1}\Big] \\
&\leq \frac{1}{1+\mu}\Big[\big(2^{p-1} + \mu\big)|\xi|^{p-1} + \big(2^{p-1} + 1\big)|\nabla u_0|^{p-1}\Big]
\end{aligned}$$

$$\leq 2^{p-1}|\xi|^{p-1} + 2^p|\nabla u_0(x)|^{p-1},$$

which is (A1).

$$(A_\mu(x,\xi) - A_\mu(x,\hat{\xi}))(\xi - \hat{\xi}) \geq \frac{1}{1+\mu}\left[2^{2-p}|\xi-\hat{\xi}|^p + \mu 2^{2-p}|\xi-\hat{\xi}|^p\right] \geq 2^{2-p}|\xi-\hat{\xi}|^p,$$

which is (A2), and finally,

$$A_\mu(x,\xi)\xi \geq \frac{1}{1+\mu}\left[2^{2-p}|\xi|^p + \mu|\xi|^p\right] \geq \frac{2^{2-p}+\mu}{1+\mu}|\xi|^p \geq 2^{2-p}|\xi|^p,$$

which is (A3). □

From Lemma 2.3 we immediately get the following result.

Lemma 2.4. *The operator $\mathcal{A}_\mu : X \to X^*$ is bounded, continuous, strongly monotone, and thus coercive.*

Next for given $\sigma > \frac{N}{p}$ and $D > 0$, let us denote

$$C_{\sigma,D} = \left\{a : \mathbb{R}^N \to \mathbb{R} : a \text{ satisfies } (a1)-(a2) \text{ of Lemma 2.1}\right\}.$$

Given $\hat{a} \in C_{\sigma,D}$ let us consider the equations

$$v \in X : \mathcal{A}_\mu v = \hat{a}(x) \tag{2.11}$$

and

$$w \in X : \mathcal{A}_\mu w = -\hat{a}(x). \tag{2.12}$$

Lemma 2.5. *The Eq. (2.11) has a unique positive solution $v \in X \cap L^\infty \cap C^1(\mathbb{R}^N)$, and (2.12) has a unique negative solution $w \in X \cap L^\infty \cap C^1(\mathbb{R}^N)$.*

Proof. The right-hand side $\hat{a}$ of (2.11) belongs to $L^r(\mathbb{R}^N)$ for all $r \in [1,\infty]$, and thus, in particular, $\hat{a} \in L^{p^{*\prime}}(\mathbb{R}^N) \hookrightarrow X^*$. From Lemma 2.4 it follows that $\mathcal{A}_\mu : X \to X^*$ is bijective, which yields the unique solvability. Since $\hat{a}(x) \geq 0$, the unique solution v must be nonnegative. By regularity results due to DiBenedetto we get $v \in X \cap C^{1+\lambda}_{\text{loc}}(\mathbb{R}^N)$, and thus, in particular, $v \in X \cap C^1(\mathbb{R}^N)$. Moreover, by Harnack's inequality we obtain $v(x) > 0$ for all $x \in \mathbb{R}^N$.

Multiplying (2.12) by -1, and setting $\hat{w} = -w$, (2.12) becomes

$$\hat{w} \in X : -\text{div}\,(-A_\mu(x, -\nabla\hat{w})) = \hat{a}(x). \tag{2.13}$$

Set $\hat{A}_\mu(x,\xi) = -A_\mu(x,-\xi)$, then one readily observes that $\hat{A}_\mu : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies (A1)–(A3), and thus $\hat{w} \in X \cap C^1(\mathbb{R}^N)$ is the unique positive solution of (2.13), which implies that $w = -\hat{w}$ is the unique negative solution of (2.12). Finally the boundedness of v and w follows from [14, Corollary 3.1], which completes the proof. □

Next we consider the equation

$$u \in X : \mathcal{A}_\mu u = a(x)f(x,u), \tag{2.14}$$

where

(F) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carthéodory function satisfying, for some positive constant c_f, the conditions

(i) $|f(x,s)| \leq c_f\left(1+|s|^{\gamma-1}\right)$, $\forall s \in \mathbb{R}$, a.e $x \in \mathbb{R}^N$, and $1 \leq \gamma < p^*$;
(ii) $\lim_{s\to 0}|f(x,s)| = 0$ uniformly in $x \in \mathbb{R}^N$.

Lemma 2.6. *Assume (A0), and (F). If $u \in X$ is a solution of (2.14) then u is bounded and satisfies an L^∞-estimate of the form*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leq C\phi(\|u\|_X), \tag{2.15}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a data independent function satisfying $\phi(s) \to 0$ as $s \to 0$ and $C = C(c_f, \|a\|, \|u\|_X)$.

Proof. As for the proof we refer to [14, Section 3]. □

Taking $(F)(ii)$ into account we clearly get

$$|a(x)f(x,u(x))| \leq Ca(x) \tag{2.16}$$

where $C = C(\|u\|_X) \to 0$ as $\|u\|_X \to 0$. We can now state the following crucial result:

Proposition 2.7. *Assume (A0), (F), (G) and ($\tilde{G}$). If u is a solution of (2.14) then $w(x) \le u(x) \le v(x)$, where v and w are unique positive and negative solutions of (2.11) and (2.12), respectively, with $\hat{a}(x) = Ca(x)$, and $C = C(\|u\|_X)$ as in (2.16) above. Moreover, the following estimate holds true*

$$|u(x)| \le \frac{\hat{C}}{1 + |x|^{\frac{N-p}{p-1}\alpha}}, \quad \forall x \in \mathbb{R}^N, \ \hat{C} = \hat{C}(\|u\|_X), \tag{2.17}$$

where $\hat{C}(\|u\|_X) \to 0$ as $\|u\|_X \to 0$.

The proof of this proposition, and in particular estimate (2.17), is based on construction of a special super-subsolution pair of (2.11), respectively, (2.12), that depends on the positive solution u_0 of (1.1). This will be accomplished in the next few lemmas. But first recall that by Lemma 2.2, there exists $c_1, c_2 > 0$ such that

$$\frac{c_1}{1 + |x|^{\frac{N-p}{p-1}}} \le u_0(x) \le \frac{c_1}{1 + |x|^{\frac{N-p}{p-1}}}, \quad \forall x \in \mathbb{R}^N,$$

and also, in view of [17, Theorem 2.6], we have

$$|\nabla u_0(x)| \le \frac{c_3}{1 + |x|^{\frac{N-1}{p-1}}}, \quad \forall x \in \mathbb{R}^N,$$

for some constant $c_3 > 0$.

Lemma 2.8. *Let $\frac{p-1}{p} < \alpha \le 1$. Suppose $h \in C^1(0, \infty)$ is a positive function with $h'(s) \ge 0$ for $s \in (0, L]$, where $L = \|u_0\|_\infty$. Furthermore, assume*

$$\limsup_{s \to 0^+} h'(s) s^{1-\alpha} < \infty. \tag{2.18}$$

Let $\overline{v}(x) = h(u_0(x))$. Then

(i) $\overline{v}$ belongs to the space $D^{1,p}(\mathbb{R}^N)$.
(ii) If in addition we assume that $h \in C^2(0, \infty)$ and $h''(s) \le 0$, for $s \in (0, L]$, then

$$\mathcal{A}_\mu(\overline{v}) \ge \frac{1}{1+\mu}\Big[\big(1 + h'(u_0)\big)^{p-1} - 1 + \mu h'(u_0)^{p-1}\Big]\Big(-\Delta_p u_0\Big) \tag{2.19}$$

Proof. Note that $\nabla \overline{v} = h'(u_0)\nabla u_0$. Now using the above growth estimates on u_0 and $|\nabla u_0|$, (2.18) and the fact that $\frac{p-1}{p} < \alpha \le 1$, we have, for R sufficiently large

$$\begin{aligned}\int_{|x|>R} |h'(u_0)\nabla u_0|^p dx &\le C \int_{|x|>R} |u_0^{(\alpha-1)} \nabla u_0|^p dx \\ &\le C \int_R^\infty \big(r^{\frac{N-p}{p-1}(1-\alpha)} r^{-\frac{N-1}{p-1}}\big)^p r^{N-1} dr < \infty,\end{aligned}$$

from which (i) follows directly. Finally, a straight forward calculation implies:

$$\begin{aligned}\mathcal{A}_\mu(\overline{v}) = &\frac{1}{1+\mu}\Big[\big(1 + h'(u_0)\big)^{p-1} - 1 + \mu h'(u_0)^{p-1}\Big]\Big(-\Delta_p u_0\Big) \\ &- \frac{p-1}{1+\mu} h''(u_0)\Big[\big(1 + h'(u_0)\big)^{p-2} + \mu h'(u_0)^{p-2}\Big]|\nabla u_0|^p\end{aligned}$$

which, taking into account the sign of h'', yields (ii), that is (2.19). □

Using this result we can now prove:

Lemma 2.9. *Suppose g satisfies (G) and ($\tilde{G}$) for some $\frac{p-1}{p} < \alpha \le 1$. There exists $M > 0$ such that $\overline{v} = MC^{\frac{1}{p-1}} u_0^\alpha$ is a supersolution of (2.11) with $\hat{a}(x) = Ca(x)$ and C as in (2.16) above. Furthermore one can take*

$$M = M(\alpha, m) = \frac{1}{\alpha} m^{\frac{-1}{p-1}},$$

where

$$m = \inf\left\{ \frac{g(s)}{s^{(p-1)(1-\alpha)}} \ : \ 0 < s \le \|u_0\|_\infty \right\}.$$

Proof. As g satisfies (G)(i) and ($\tilde{G}$), we have $m > 0$. Next let $h(s) := MC^{\frac{1}{p-1}} s^\alpha$, which clearly satisfies the assumptions of the previous lemma, and in particular (2.18), and take $\overline{v} = h(u_0)$. By Lemma 2.8 and taking into account that u_0 is a positive solution of (1.1) we have

$$\mathcal{A}_\mu(\overline{v}) \geq \frac{1}{1+\mu}\Big[\big(1+h'(u_0)\big)^{p-1} - 1 + \mu h'(u_0)^{p-1}\Big]a(x)g(u_0)$$
$$\geq \big(h'(u_0)^{p-1}g(u_0)\big)a(x) = (\alpha M)^{p-1}\big[u_0^{(p-1)(\alpha-1)}g(u_0)\big]Ca(x).$$

Hence

$$\mathcal{A}_\mu(\overline{v}) \geq (\alpha M)^{p-1}mCa(x) = Ca(x) = \hat{a}(x),$$

which proves the result. □

In a similar manner one can prove

Lemma 2.10. *Suppose g satisfies (G) and $(\tilde{G})$ for some $\frac{p-1}{p} < \alpha \leq 1$. There exists $\tilde{M} = \tilde{M}(\alpha, m) > 0$ such that $\overline{w} = -\tilde{M}C^{\frac{1}{p-1}}u_0^\alpha$ is a subsolution of (2.12) with $\hat{a}(x) = Ca(x)$ and C as in (2.16) above, i.e.*

$$\mathcal{A}_\mu(\overline{w}) \leq -Ca(x) = -\hat{a}(x),$$

We are now ready to complete

Proof of Proposition 2.7. Using (2.16), the inequality $w(x) \leq u(x) \leq v(x)$ is shown by applying arguments as in the proof of Lemma 2.2. In order to prove the decay estimate (2.17), we are going to show that

$$|w(x)|,\ v(x) \leq \frac{\hat{C}}{1+|x|^{\frac{N-p}{p-1}\alpha}}, \quad \forall x \in \mathbb{R}^N,\ \hat{C} = \hat{C}(\|u\|_X). \tag{2.20}$$

To this end we note that by Lemma 2.9 the function $\overline{v} = MC^{\frac{1}{p-1}}u_0^\alpha$ is a supersolution of (2.11), with $C = C(\|u\|_X)$ as in (2.16) above. Thus by comparison arguments it follows $v(x) \leq MC^{\frac{1}{p-1}}u_0^\alpha(x)$. Taking Lemma 2.2 into account we obtain

$$v(x) \leq MC^{\frac{1}{p-1}}C_2(N, p, \sigma, c_g, \|a\|, \|u_0\|_X, D)\frac{1}{1+|x|^{\frac{N-p}{p-1}\alpha}}, \quad \forall x \in \mathbb{R}^N,$$

which proves half of the estimate (2.20) with

$$\hat{C}_1(\|u\|_X) := MC^{\frac{1}{p-1}}C_2(N, p, \sigma, c_g, \|a\|, \|u_0\|_X, D).$$

Similarly, using $\overline{w}$ the subsolution constructed in Lemma 2.10, one shows that $|w(x)| \leq \tilde{M}C^{\frac{1}{p-1}}u_0^\alpha(x)$ from which we conclude

$$|w(x)| \leq \hat{C}_2(\|u\|_X)\frac{1}{1+|x|^{\frac{N-p}{p-1}\alpha}}, \quad \forall x \in \mathbb{R}^N,$$

with $\hat{C}_2(\|u\|_X) = \tilde{M}C^{\frac{1}{p-1}}C_2(N, p, \sigma, c_g, \|a\|, \|u_0\|_X, D)$. This provides the other half of estimate (2.20).

Finally (2.20) follows with $\hat{C} = \max\{\hat{C}_1(\|u\|_X), \hat{C}_2(\|u\|_X)\}$, and the proof of proposition is complete. □

3. Proof of Theorem 1.3

Let u_0 be a nonnegative solution of (1.1) and a local minimizer of the functional

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N}|\nabla u|^p\,dx - \int_{\mathbb{R}^N}a(x)G(u)\,dx, \quad \text{with } G(s) = \int_0^s g(t)\,dt.$$

in the V_α-topology. Consider the functional $h \mapsto \Phi(u_0 + h)$, and let $h_n : \|h_n\|_X \leq \frac{1}{n}$ be such that

$$\Phi(u_0 + h_n) = \inf_{h\in B_n}\Phi(u_0 + h), \quad \text{where } B_n = \left\{h \in X : \|h\|_X \leq \frac{1}{n}\right\}.$$

The existence of a minimizer h_n is guaranteed, since $\Phi : X \to \mathbb{R}$ is C^1 and weakly lower semicontinuous and B_n is weakly compact in X. Set $u_n = u_0 + h_n$, that is,

$$\Phi(u_n) = \inf_{u\in B_n}\Phi(u), \quad \text{where } B_n = \left\{u \in X : \|u - u_0\|_X \leq \frac{1}{n}\right\}.$$

For $u_n \in B_n$ we have either $\|u_n - u_0\|_X < \frac{1}{n}$ or else $\|u_n - u_0\|_X = \frac{1}{n}$. In case $\|u_n - u_0\| < \frac{1}{n}$, u_n is a critical point of Φ, and thus u_n is a weak solution of (1.1), i.e., $-\Delta_p u_n = a(x)g(u_n)$, that is,

$$\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi - a(x)g(u_n)\varphi\Big)dx = 0, \quad \forall \varphi \in X.$$

In case $\|u_n - u_0\| = \frac{1}{n}$, there exists a Lagrange multiplier $\lambda_n \leq 0$ such that

$$\int_{\mathbb{R}^N}\Big(|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi - a(x)g(u_n)\varphi\Big)dx = \lambda_n\int_{\mathbb{R}^N}|\nabla(u_n - u_0)|^{p-2}\nabla(u_n - u_0)\varphi\,dx,$$

for all $\varphi \in X$, which (in a distributional sense) can be written as

$$-\Delta_p u_n - a(x)g(u_n) = -\lambda_n \Delta_p(u_n - u_0). \tag{3.1}$$

Taking into account that u_0 is a solution of (1.1) and using (3.1), we get

$$-(\Delta_p u_n - \Delta_p u_0) + \lambda_n \Delta_p(u_n - u_0) = a(x)(g(u_n) - g(u_0)). \tag{3.2}$$

Set $\mu_n = -\lambda_n \geq 0$. Thus $h_n = u_n - u_0$ satisfies the equation

$$-(\Delta_p(u_0 + h_n) - \Delta_p u_0) - \mu_n \Delta_p h_n = a(x)(g(u_0 + h_n) - g(u_0)). \tag{3.3}$$

Dividing (3.3) by $1 + \mu_n$ and taking into account (2.8), (2.9), (2.14) and (2.15) yields

$$\mathcal{A}_{\mu_n} h_n = a(x) f(x, h_n), \tag{3.4}$$

where we set

$$f(x, s) = \frac{1}{1 + \mu_n}(g(u_0(x) + s) - g(u_0(x))),$$

which clearly satisfies (F). Since $\|h_n\|_X \to 0$ as $n \to \infty$, by Lemma 2.6 it follows

$$\|h_n\|_\infty \leq C\phi\big(\|h_n\|_X\big) \to 0. \tag{3.5}$$

From (3.5), the fact that $u_0(x)$ is bounded and g is continuous, we get

$$a(x)|f(x, h_n)| \leq C(\|h_n\|_X)a(x), \tag{3.6}$$

where $C(\|h_n\|_X) \to 0$ as $n \to \infty$. Next recall that by Proposition 2.7 we have

$$w_n \leq h_n \leq v_n,$$

where v_n and w_n solve (2.11) and (2.12), respectively, with

$$\hat{a}(x) = C(\|h_n\|_X)a(x),$$

from which it follows

$$|h_n(x)| \leq \hat{C}\frac{1}{1 + |x|^{\frac{N-p}{p-1}\alpha}}, \quad \forall x \in \mathbb{R}^N, \ \hat{C} = \hat{C}(\|h_n\|_X), \tag{3.7}$$

where $\hat{C}(\|h_n\|_X) \to 0$ as $\|h_n\|_X \to 0$. In view of (3.7) and $\|h_n\|_X \to 0$ it follows that $\|h_n\|_{V_\alpha} \to 0$. Finally, since u_0 is a local minimizer of Φ in the V_α-topology we get with $h_n \to 0$ in V_α for n large

$$\Phi(u_0) \leq \Phi(u_0 + h_n) = \Phi(u_n) = \inf_{h \in B_n} \Phi(u_0 + h),$$

where

$$B_n = \Big\{u \in X : \|u - u_0\|_X \leq \frac{1}{n}\Big\},$$

which proves that u_0 must be a local minimizer of Φ in the X-topology completing the proof of Theorem 1.3. □

4. Existence of a positive local minimizer

In this section we present a useful application of Theorem 1.3. In fact we are going to prove the existence of a positive solution u_0 of (1.1) such that, in addition, u_0 is a local minimizer of the functional Φ in X.

We assume throughout this section hypotheses (A0) with $\beta \geq \frac{N}{p-1}$, (G), and $(\tilde{G})$ with $\alpha = 1$. In other words as far as g is concerned we assume

$(\hat{G})$ $g : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies for some positive constants δ, c_g the conditions

(i) $\delta \leq g(s)$, for $s \in [0, \infty)$;
(ii) $|g(s)| \leq c_g\big(1 + |s|^{\gamma-1}\big)$, $\forall\ s \in \mathbb{R}$, where $1 \leq \gamma < p^*$.

Lemma 4.1. *The function $\underline{u}(x) = 0$ is a subsolution of (1.1) and $\overline{u}$ given by*

$$\overline{u}(x) = \Big[1 + |x|^{\frac{p}{p-1}}\Big]^{\frac{p-N}{p}} \tag{4.1}$$

is a supersolution of (1.1) provided $2c_a c_g \leq \frac{N}{2^{N+1}}\left(\frac{N-p}{p-1}\right)^{p-1}$. Moreover, $\overline{u} \in V_1$.

Proof. Clearly, as $g(0) > 0$, $\underline{u}(x) = 0$ is a subsolution. Let us show that $\overline{u}$ is a supersolution. Elementary calculation yields

$$|\nabla\overline{u}(x)|^p = \left(\frac{N-p}{p-1}\right)^p \left(1 + |x|^{\frac{p}{p-1}}\right)^{-N} |x|^{\frac{p}{p-1}}.$$

By using spherical coordinates one obtains

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla\overline{u}(x)|^p\,dx &= \int_{B(0,1)} |\nabla\overline{u}(x)|^p\,dx + \int_{\mathbb{R}^N\setminus B(0,1)} |\nabla\overline{u}(x)|^p\,dx \\ &\le c_1 + c_2 \int_1^\infty \varrho^{(-N+1)\frac{p}{p-1}+N-1}\,d\varrho < \infty,\end{aligned}$$

since $(-N+1)\frac{p}{p-1} + N < 0$, which shows that $\overline{u} \in X$. Further, again by elementary calculation we obtain

$$-\Delta_p\overline{u}(x) = N\left(\frac{N-p}{p-1}\right)^{p-1} \frac{1}{\left(1 + |x|^{\frac{p}{p-1}}\right)^N} \left(1 + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}. \tag{4.2}$$

We estimate

$$\left(1 + |x|^{\frac{p}{p-1}}\right)^N \le 2^N\left(1 + |x|^{N\frac{p}{p-1}}\right) \le 2^{N+1}\left(1 + |x|^{N+\beta}\right), \quad \text{as } \beta \ge \frac{N}{p-1}. \tag{4.3}$$

From (4.2) and (4.3) we get

$$-\Delta_p\overline{u}(x) \ge \frac{N}{2^{N+1}}\left(\frac{N-p}{p-1}\right)^{p-1} \frac{1}{\left(1 + |x|^{N+\beta}\right)} \left(1 + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}. \tag{4.4}$$

As $0 < \overline{u}(x) \le 1$, we have $g(\overline{u}(x)) \le 2c_g$, and thus from (4.4) it follows that

$$-\Delta_p\overline{u}(x) \ge a(x)g(\overline{u}(x)),$$

that is, $\overline{u}$ is a supersolution provided that the following inequality

$$2c_a c_g \le \frac{N}{2^{N+1}}\left(\frac{N-p}{p-1}\right)^{p-1}$$

is fulfilled. Moreover, we have the estimate

$$\overline{u}(x) = \left[1 + |x|^{\frac{p}{p-1}}\right]^{\frac{p-N}{p}} \le \frac{2}{1 + |x|^{\frac{N-p}{p-1}}},$$

hence $\overline{u} \in V_1$, which completes the proof. □

We introduce the truncated function $\hat{g} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{g}(x,s) = \begin{cases} g(0) & \text{if} \quad s \le 0, \\ g(s) & \text{if} \quad 0 \le s \le \overline{u}(x), \\ g(\overline{u}(x)) & \text{if} \quad s \ge \overline{u}(x), \end{cases} \tag{4.5}$$

and define the following functional $\hat{\Phi}$

$$\hat{\Phi}(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx - \int_{\mathbb{R}^N} a(x)\hat{G}(u)\,dx \quad \text{with } \hat{G}(s) = \int_0^s \hat{g}(x,s)\,ds. \tag{4.6}$$

Lemma 4.2. *The functional $\hat{\Phi} : X \to \mathbb{R}$ is a well defined C^1-functional, which is bounded below, coercive, and weakly lower semicontinuous. Therefore, a global minimizer $\hat{u} \in X$ exists, which is a solution of (1.1) satisfying $0 < \hat{u}(x) \le \overline{u}(x)$. Moreover, $\hat{u} \in V_1 \cap C^1(\mathbb{R}^N)$.*

Proof. The proof for $\hat{\Phi}$ being C^1 and weakly lower semicontinuous is similar to Lemma 1.2. By definition $\hat{g}$ is uniformly bounded, that is, $|\hat{g}(x,s)| \le 2c_g$, since $0 < \overline{u} \le 1$. Thus

$$\left|\int_{\mathbb{R}^N} a(x)\hat{G}(u)\,dx\right| \le \int_{\mathbb{R}^N} a(x)2c_g|u(x)|\,dx \le 2c_g c_a\|u\|_{1,w} \le c\|w\|_{p^{*\prime}}\|u\|_X,$$

where c is some positive constant, and thus

$$\hat{\Phi}(u) \ge \frac{1}{p}\|u\|_X^p - c\|w\|_{p^{*\prime}}\|u\|_X,$$

which shows that $\hat{\Phi}$ is coercive and bounded below. Let $\hat{u}$ be a global minimizer of $\hat{\Phi}$, which is a critical point satisfying the equation

$$-\Delta_p\hat{u} = a(x)\hat{g}(x,\hat{u}(x)). \tag{4.7}$$

Since $\hat{g}(x, \hat{u}(x)) > 0$, by Harnack's inequality we obtain $\hat{u}(x) > 0$ for all $x \in \mathbb{R}^N$, and due to regularity results of [15], $\hat{u} \in X \cap C^1(\mathbb{R}^N)$. As the supersolution $\overline{u}$ satisfies the inequality

$$-\Delta_p \overline{u} \geq a(x) g(\overline{u}(x)),$$

we obtain by comparison with (4.7) the inequality

$$\int_{\mathbb{R}^N} \left(|\nabla \hat{u}|^{p-2}\nabla \hat{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\right)\nabla\varphi\, dx \leq \int_{\mathbb{R}^N} a(x)\big(\hat{g}(x,\hat{u}(x)) - g(\overline{u}(x))\big)\varphi\, dx,$$

for all $\varphi \in X$ with $\varphi \geq 0$. Testing the last inequality with $\varphi = (\hat{u} - \overline{u})^+$ yields

$$2^{2-p}\|(\hat{u} - \overline{u})^+\|_X^p \leq \int_{\{\hat{u}\geq \overline{u}\}} a(x)\big(\hat{g}(x,\hat{u}(x)) - g(\overline{u}(x))\big)(\hat{u}(x) - \overline{u}(x))\, dx = 0,$$

thus $(\hat{u} - \overline{u})^+ = 0$, that is, $\hat{u}(x) \leq \overline{u}(x)$. Since $\overline{u} \in V_1$, it follows that $\hat{u} \in V_1$ as well, which completes the proof. □

Lemma 4.3. *The global minimizer $\hat{u}$ of the functional $\hat{\Phi}$ satisfies the following inequality*

$$\hat{u}(x) \geq \frac{\varepsilon}{1 + |x|^{\frac{N-p}{p-1}}}, \tag{4.8}$$

for some positive constant ε.

Proof. The global minimizer $\hat{u}$ solves (4.7), and thus $-\Delta_p \hat{u} \geq 0$. The result now follows as in the proof of Lemma 2.2. □

Lemma 4.4. *The positive global minimizer $\hat{u}$ of the functional $\hat{\Phi}$ in the X-topology is a local minimizer of the original functional Φ with respect to the V_1-topology.*

Proof. We need to show that a ε-ball with center $\hat{u}$ in the V_1-topology belongs to the interval $[0, \overline{u}]$. In view of Lemma 4.3 it remains to show that

$$\hat{u}(x) + \frac{\varepsilon}{1 + |x|^{\frac{N-p}{p-1}}} \leq \overline{u}(x) \quad \text{for some } \varepsilon > 0. \tag{4.9}$$

The right-hand side of Eq. (4.7) is positive and bounded, that is,

$$a(x)\hat{g}(x, \hat{u}(x)) \leq 2c_a c_g w(x),$$

where c_a and c_g are the constants in (A0) and (G), respectively, and w given by (1.2). Consider the equation

$$v \in X : -\Delta_p v = 2c_a c_g w(x). \tag{4.10}$$

By arguments already used before, Eq. (4.10) has a unique positive solution $v \in X \cap C^1(\mathbb{R}^N)$. Based on Wolff potential estimates (see [18, Theorem 1.6, Corollary 4.13]), we get from [1, Theorem 6.5] the following estimate for v

$$v(x) \leq C\frac{1}{1 + |x|^{\frac{N-p}{p-1}}}, \quad x \in \mathbb{R}^N,$$

where $C = C(c_a, c_g)$ with $C(c_a, c_g) \to 0$ as $c_a c_g \to 0$. By comparison from Eqs. (4.7) and (4.10), we obtain

$$\hat{u}(x) \leq v(x) \leq C(c_a, c_g)\frac{1}{1 + |x|^{\frac{N-p}{p-1}}}. \tag{4.11}$$

On the other hand the function $\overline{u}$ given by (4.1) can be estimated below as follows

$$\overline{u}(x) = \left[1 + |x|^{\frac{p}{p-1}}\right]^{\frac{p-N}{p}} \geq \frac{1}{2^{\frac{N-p}{p}}}\frac{1}{1 + |x|^{\frac{N-p}{p-1}}}. \tag{4.12}$$

Thus, if $c_a c_g$ small such that $C(c_a, c_g) < \frac{1}{2^{\frac{N-p}{p}}}$, then from (4.11) and (4.12) it follows that there is a $\varepsilon > 0$ such that (4.9) is fulfilled, which completes the proof. □

Finally be means of the preceding Lemmata we are in the position to prove the following main result of this section.

Theorem 4.5. *Assume (A0) with $\beta \geq \frac{N}{p-1}$ and let g satisfies $(\hat{G})$. There exists $m > 0$ so that if $c_a c_g < m$ then Eq. (1.1) has a positive solution which is a local minimizer in the X-topology of the functional Φ given by (1.3).*

Proof. By Lemma 4.2 the global minimizer $\hat{u}$ of the functional $\hat{\Phi}$ is a solution of (1.1), and by Lemma 4.4, $\hat{u}$ is a local minimizer of the original functional Φ with respect to the V_1-topology. Thus, we may apply our main result Theorem 1.3, which completes the proof. □

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