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RESEARCH ARTICLE

Vector Optimization with Variable Domination Structure: A Unifying Approach

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Abstract
We revisit two types of constrained vector optimization problems driven by set-valued maps, where the domination structure is defined by a cone-valued map. Within the framework of variable domination structures, we demonstrate that the approaches used in the literature cover each other. This observation enables us to design unified methods for deriving necessary optimality conditions in both cases. Our results rely on key concepts such as the Extremal Principle and the inherent incompatibility between openness and efficiency, encompassing several well-known assertions in this area of research.

Keywords Variable domination structure · Unifying approach · Extremal principle · Openness

Mathematics Subject Classification 49J53 · 49K27 · 46G05

1 Introduction

The research area of Pareto optimization with set-valued maps acting between normed vector spaces extends the classical Pareto optimization framework involving single-valued maps. Over the past 30 years, this field has undergone significant development. Recently, researchers have explored a further generalization of the classical case, where the order is

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defined by a fixed convex cone, to a more dynamic scenario in which the order is defined by a set-valued map. In this context, two main approaches have been considered: one in which the domination map acts between the same spaces as the objective set-valued map (see [11]), and another where the set-valued domination map has the same input and output space, coinciding with the range of the objective map (see [13]). Nevertheless, it can be said that the second approach (which was developed first, chronologically) has garnered more attention within the research community (see, e.g., [4, 17], and the references therein). Moreover, several applications of this type of variable domination structure have been described (see [15]).

The primary goal of this work is to demonstrate that, from a theoretical perspective, each approach encompasses the other. However, when addressing optimality conditions, there are two standard methodologies to consider. On one hand, the Extremal Principle can be employed (see [33]); on the other hand, an alternative principle – referred to here as the incompatibility principle – can be used. This principle highlights the opposition between the openness of a map at a point and the efficiency of that point (see, e.g., [9]). While other specialized tools may be applicable in certain cases, the two principles discussed here are highly general and can be applied to a broad range of optimization problems. This versatility suggests the possibility of embedding the first type of variable domination into the second, and vice versa. Once these embeddings are achieved, these naturally lead to the development of a unified framework for investigation.

The paper is organized as follows. After a preliminaries section introducing the main concepts and tools of investigation, Sect. 3 describes how both types of variable domination structures can be viewed in a unified way. Specifically, we show that, through meaningful modifications of the involved set-valued maps, each of the approaches discussed earlier can be interpreted as a particular case of the other. Section 4 focuses on deriving necessary optimality conditions using the Extremal Principle. The reduction method introduced in the previous section leads to similar proofs for both types of domination cone maps. Section 5 presents openness results tailored to sums of set-valued maps from multiple perspectives, with the ultimate goal of applying the incompatibility principle to derive optimality conditions. Finally, Sect. 6 integrates all these tools, offering alternative proofs and assumptions for the main results of the paper. Additionally, several strategies are outlined, and the derived assertions are compared with corresponding results in the existing literature. A short final section presents some further possible directions of research.

2 Preliminaries

For a nontrivial Banach space X over the real field $\mathbb{R}$, we denote by $B(x,\varepsilon)$ and $\overline{B}(x,\varepsilon)$ the open ball and the closed ball, respectively, with center $x \in X$ and radius $\varepsilon > 0$; B_X, $\overline{B}_X$, S_X are the open unit ball, the closed unit ball, and the unit sphere of X, respectively. The notation X^* stands for the topological dual space of X. If $\Omega \subseteq X$, we denote the indicator function of Ω by δ_Ω (that is, $\delta_\Omega(x) := 0$ if $x \in \Omega$ and $\delta_\Omega(x) := \infty$, if $x \notin \Omega$). As usual, $\operatorname{cl}\Omega$, $\operatorname{int}\Omega$, $\operatorname{bd}\Omega$ denote the topological closure, the topological interior, and the topological boundary of Ω, respectively. A set Ω is called closed around a point $\overline{x} \in \Omega$ if there exists a closed neighborhood U of $\overline{x}$ such that $\Omega \cap U$ is closed. If $\Omega \subseteq X$ is a cone, we denote its positive polar by

$$\Omega^+ := \left\{x^* \in X^* \mid x^*(x) \geq 0, \forall x \in \Omega\right\}.$$

Some tools of variational analysis and generalized differentiation, mainly taken from [33], are presented next. Although the mathematical objects are constructed on Banach spaces, the full calculus holds on a special class of spaces, i.e., the Asplund spaces: a Banach space X is Asplund if, and only if, every continuous convex function on any nonempty open convex subset U of X is Fréchet differentiable on a dense G_δ-subset of U. One of the most useful properties of the Asplund spaces, exploited also in this work, is that every bounded sequence from the topological dual admits a weak* convergent subsequence.

Firstly, we recall the concepts of normals to sets. For a nonempty subset Ω of the Asplund space X and $x \in \Omega$, the Fréchet normal cone to Ω at x is

$$\widehat{N}(\Omega, x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{x^*(u-x)}{\|u-x\|} \leq 0 \right\}, \tag{2.1}$$

where $u \xrightarrow{\Omega} x$ means that $u \to x$ and $u \in \Omega$. If $x \notin \Omega$, we let $\widehat{N}(\Omega, x) := \emptyset$. If Ω is closed around $\overline{x}$, the limiting (Mordukhovich) normal cone is given by

$$N(\Omega, \overline{x}) := \left\{ x^* \in X^* \mid \exists x_n \xrightarrow{\Omega} \overline{x}, x_n^* \xrightarrow{*} x^*, x_n^* \in \widehat{N}(\Omega, x_n), \forall n \in \mathbb{N} \right\}, \tag{2.2}$$

where $\xrightarrow{*}$ means the convergence in the weak* topology.

Next, we present the associated coderivative constructions for set-valued maps. Let $F : X \rightrightarrows Y$ be a set-valued map with the domain and the graph defined by

$$\operatorname{Dom} F := \{x \in X \mid F(x) \neq \emptyset\} \quad \text{and} \quad \operatorname{Gr} F := \{(x, y) \mid y \in F(x)\},$$

and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then the Fréchet coderivative at $(\overline{x}, \overline{y})$ is the set-valued map $\widehat{D}^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}, \tag{2.3}$$

while the Mordukhovich coderivative of F at $(\overline{x}, \overline{y})$ is the set-valued map $D^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}. \tag{2.4}$$

Moreover, we will work also with the mixed coderivative $D_M^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$, given by

$$\begin{aligned} D_M^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid \exists\, (x_n, y_n) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y}), x_n^* \xrightarrow{*} x^*, y_n^* \to y^* \\ \text{s.t. } x_n^* \in \widehat{D}^* F(x_n, y_n)\left(y_n^*\right), \forall n \in \mathbb{N}\}. \end{aligned} \tag{2.5}$$

If $\Omega \subseteq X$, we denote its associated indicator set-valued map from X to the implied output space by Δ_Ω ($\Delta_\Omega(x) := \{0\}$ if $x \in \Omega$ and $\Delta_\Omega(x) := \emptyset$ if $x \notin \Omega$).

For a function $f : X \to \mathbb{R} \cup \{+\infty\}$ finite at $\overline{x} \in X$ and lower semicontinuous around $\overline{x}$, the Fréchet subdifferential of f at $\overline{x}$ is the set

$$\widehat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where $\operatorname{epi} f$ denotes the epigraph of f, while the limiting (Mordukhovich) subdifferential of f at $\overline{x}$ is given by

$$\partial f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\}.$$

It is well-known that if f is a convex function, then $\widehat{\partial} f(\overline{x})$ and $\partial f(\overline{x})$ coincide with the Fenchel subdifferential. However, in general, $\widehat{\partial} f(\overline{x}) \subseteq \partial f(\overline{x})$, and the following generalized Fermat rule holds: if $\overline{x} \in X$ is a local minimum point for $f : X \to \mathbb{R} \cup \{+\infty\}$, then $0 \in \widehat{\partial} f(\overline{x})$. Some other properties of these subdifferentials will be reminded and used when needed.

Let us recall also some generalized compactness notions. For a set $\Omega \subseteq X$ closed around $\overline{x} \in \Omega$, one says that Ω is sequentially normally compact (SNC, for short) at $\overline{x}$ if

$$\left[x_n \overset{\Omega}{\to} \overline{x}, x_n^* \overset{*}{\to} 0, x_n^* \in \widehat{N}(\Omega, x_n)\right] \Rightarrow x_n^* \to 0.$$

Remark that in the case where $\Omega := C$ is a proper closed convex cone, the SNC property at 0 is equivalent to

$$\left[(x_n^*) \subseteq C^+, x_n^* \overset{*}{\to} 0\right] \Rightarrow x_n^* \to 0.$$

In particular, if $\operatorname{int} C \neq \emptyset$, then C is SNC at 0.

Let X, Y be normed vector spaces and $\Gamma : X \rightrightarrows Y$ be a set-valued map closed around $(\overline{x}, \overline{y}) \in \operatorname{Gr}\Gamma$ (that is, $\operatorname{Gr}\Gamma$ is closed around $(\overline{x}, \overline{y})$). One says that Γ is SNC at $(\overline{x}, \overline{y})$ if $\operatorname{Gr}\Gamma$ is SNC at $(\overline{x}, \overline{y})$, i.e.,

$$\left[(x_n, y_n) \overset{\operatorname{Gr}\Gamma}{\to} (\overline{x}, \overline{y}), (x_n^*, y_n^*) \overset{*}{\to} (0, 0), (x_n^*, y_n^*) \in \widehat{N}(\operatorname{Gr}\Gamma, (x_n, y_n))\right] \Rightarrow (x_n^*, y_n^*) \to (0, 0).$$

Moreover, following [33, pages 76, 266], one says that $\Gamma : X \rightrightarrows Y$ is partially sequentially normally compact (PSNC, for short) at $(\overline{x}, \overline{y})$ if

$$\left[(x_n, y_n) \overset{\operatorname{Gr}\Gamma}{\to} (\overline{x}, \overline{y}), x_n^* \overset{*}{\to} 0, y_n^* \to 0, (x_n^*, y_n^*) \in \widehat{N}(\operatorname{Gr}\Gamma, (x_n, y_n))\right] \Rightarrow x_n^* \to 0.$$

We next present the celebrated Extremal Principle. Let $\Omega_1, \ldots, \Omega_p$ be nonempty subsets of the Asplund space X with $p \geq 2$, and let $\overline{x}$ be a common point of them. We say that $\overline{x}$ is a local extremal point for the system $\{\Omega_1, \ldots, \Omega_p\}$ if there exist some sequences $(a_{in}) \subset X$, $a_{in} \to 0$ for any $i = 1, \ldots, p$, and a neighborhood U of $\overline{x}$ such that

$$\bigcap_{i=1}^{p} (\Omega_i - a_{in}) \cap U = \emptyset \text{ for } n \text{ sufficiently large.}$$

In this case $\{\Omega_1, \ldots, \Omega_p, \overline{x}\}$ is called an extremal system in X. In the proof of our main results, we will use the following (approximate) Extremal Principle (see Mordukhovich's monograph [33, Definition 2.5 (ii) and Theorem 2.20]).

Theorem 2.1 (The Extremal Principle) *Let $\{\Omega_1, \ldots, \Omega_p, \overline{x}\}$ be an extremal system in the Asplund space X. Then, for every $\varepsilon > 0$ there exist $x_i \in \Omega_i \cap \overline{B}(\overline{x}, \varepsilon)$ and $x_i^* \in X^*$ such that*

$$x_i^* \in \widehat{N}(\Omega_i, x_i) + \varepsilon \overline{B}_{X^*}, \quad i = 1, \ldots, p, \tag{2.6}$$

$$x_1^* + \cdots + x_p^* = 0, \quad \left\|x_1^*\right\| + \cdots + \left\|x_p^*\right\| = 1. \tag{2.7}$$

We end this section of preliminaries by briefly considering some topological and some regularity notions for set-valued maps.

We say that F is lower semicontinuous at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if for every sequence $(x_n) \to \overline{x}$, there exists a sequence $(y_n) \to \overline{y}$ with $(x_n, y_n) \in \operatorname{Gr} F$ for every n, and we say that F is lower semicontinuous at $\overline{x} \in X$ with $F(\overline{x}) \neq \emptyset$ if it is lower semicontinuous at every $(\overline{x}, y)$ with $y \in F(\overline{x})$.

A set-valued map $F : X \rightrightarrows Y$ is said to be open at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if the image through F of every neighborhood of $\overline{x}$ is a neighborhood of $\overline{y}$. Another, more useful openness property, is the openness at linear rate: one says that $F : X \rightrightarrows Y$ is open at linear rate $L > 0$ (or L-open) around $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if there exist $\varepsilon > 0$ and two neighborhoods U of $\overline{x}$, and V of $\overline{y}$ such that, for every $(x, y) \in \operatorname{Gr} F \cap (U \times V)$ and every $\rho \in (0, \varepsilon)$,

$$B(y, \rho L) \subseteq F(B(x, \rho)).$$

Moreover, F is said to be open at linear rate $L > 0$ or (L-open) at $(\overline{x}, \overline{y})$ if there exists a positive number $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$,

$$B(\overline{y}, \rho L) \subseteq F(B(\overline{x}, \rho)).$$

A well-known closely related property is the following one: The set-valued map F is said to have the Aubin property around $(\overline{x}, \overline{y})$ with constant $M > 0$ if there exist two neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that, for every $x, u \in U$,

$$F(x) \cap V \subseteq F(u) + M \|x - u\| \overline{B}_Y.$$

The infimum for all constants M satisfying the above inclusion is denoted by $\operatorname{lip} F(\overline{x}, \overline{y})$. It became classical that the latter property is equivalent to the openness at linear rate for F^{-1} around $(\overline{y}, \overline{x})$. If F has the Aubin property around $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, it easily follows that it is lower semicontinuous at $(\overline{x}, \overline{y})$. For more details and other topological interpretations, see, for instance, [1].

It is well-known that the mixed coderivative, combined with the PSNC property, are useful for characterizing the Aubin property of a set-valued map (see [33, Theorem 4.10]).

Lemma 2.2 (Mordukhovich Criterion) *Let X, Y be Asplund spaces, $G : X \rightrightarrows Y$ be a closed-graph set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} G$. Then the following properties are equivalent*:

(*a*) *G has the Aubin property around $(\overline{x}, \overline{y})$.*

(*b*) *G is PSNC at $(\overline{x}, \overline{y})$ and $D^*_M G(\overline{x}, \overline{y})(0) = \{0\}$.*

3 A Unifying Approach

Let us start with the following set-valued optimization problem:

$$(P) \qquad \text{minimize } F(x), \text{ subject to } x \in \Omega,$$

where $F : X \rightrightarrows Y$ is a set-valued map, and $\Omega \subseteq X$ is a closed set. Consider a proper closed and convex cone $C \subseteq Y$, which induces a reflexive preorder on Y by the equivalence $y_1 \leq_C y_2$ if and only if $y_2 - y_1 \in C$. We are using the vector approach to define the solution concept for the set-valued optimization problem such that the optimality is understood in the sense of Pareto, as given in the next definition.

Definition 3.1 (Pareto minimum) A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (\Omega \times Y)$ is called a local Pareto minimum point for F, or for problem (P), if there exists a neighborhood U of $\overline{x}$ such that

$$(F(U \cap \Omega) - \overline{y}) \cap (-C) \subseteq C. \tag{3.1}$$

Remark that we do not suppose that C is pointed (i.e., $C \cap (-C) = \{0\}$), for some technical reasons we will clarify a few lines below. This general framework was previously studied in literature (see, e.g., [2, 9]), and of course, under pointedness of C, the relation "$\leq_C$" becomes a partial order, and the definition of Pareto minimality for (P) is the classical one, that is, relation (3.1) collapses into:

$$(F(U \cap \Omega) - \overline{y}) \cap (-C) = \{0\}.$$

A natural question that arose in the context of Pareto minimality was the possibility to consider that the preorder relation given by the cone C is not fixed anymore. Consequently, other set-valued maps were considered for the appropriate definition of optimality. Let us denote these new, order-oriented set-valued maps by K and Q, respectively. At this point, two main approaches were developed: on the one hand, the set-valued map K is cone-valued and acts between the same spaces as F, i.e., $K : X \rightrightarrows Y$, and on the other hand, $Q : Y \rightrightarrows Y$, also cone-valued, gives a domination structure by the equivalence:

$$v \leq_Q y \Leftrightarrow y \in v + Q(v) \setminus (-Q(v)) \cup \{0\}. \tag{3.2}$$

The corresponding optimality concepts are as follows.

Definition 3.2 (nondominated solutions) Let $F : X \rightrightarrows Y$ be a set-valued map, $\Omega \subseteq X$ be a closed set, and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F \cap (\Omega \times Y)$.

(I) Consider $K : X \rightrightarrows Y$ and suppose that, for any $x \in X$, the set $K(x)$ is a closed, convex and proper cone in Y. One says that $(\overline{x}, \overline{y})$ is a local nondominated point of type I for F with respect to K on Ω if there is a neighborhood U of $\overline{x}$ such that, for every $x \in U \cap \Omega$,

$$(F(x) - \overline{y}) \cap (-K(x)) \subseteq K(x). \tag{3.3}$$

(II) Consider $Q : Y \rightrightarrows Y$ and suppose that, for any $y \in Y$, the set $Q(y)$ is a closed, convex and proper cone in Y. One says that $(\overline{x}, \overline{y})$ is a local nondominated point of type II for F with respect to Q on Ω if there is a neighborhood U of $\overline{x}$ such that, for every $x \in U \cap \Omega$,

$$\forall y \in F(x) \setminus \{\overline{y}\}, \overline{y} \notin y + Q(y) \setminus (-Q(y)) \cup \{0\}. \tag{3.4}$$

If $\Omega := X$, we say that $(\overline{x}, \overline{y})$ is a local nondominated point for F with respect to K or Q, respectively, and if the neighborhood U is the whole space, we obtain the global variants of the above definitions.

Remark 3.3 Notice that relation (3.4) is equivalent to

$$\forall y \in F(x) \setminus \{\overline{y}\}, \overline{y} \notin y + Q(y) \setminus (-Q(y)). \tag{3.5}$$

Remark 3.4 Let us point out some elements of comparison between the two types. For instance, note that, according to (3.3), the choice of the variable x influences how vectors in the space Y are compared via K, thereby intertwining the spaces X and Y. In contrast, under (3.4), any changes in the partial order on Y appear to be an intrinsic property of Y alone.

Remark 3.5 Observe that, in case that the cones given by $K(x)$ and $Q(y)$ are also pointed, the relations (3.3) and (3.4) become, respectively,

$$(F(x) - \overline{y}) \cap (-K(x)) \subseteq \{0\} \tag{3.6}$$

and

$$\forall y \in F(x) \setminus \{\overline{y}\}, \overline{y} \notin y + Q(y), \tag{3.7}$$

and this corresponds to the usual definitions from literature (see, e.g., [11, 15]). Furthermore, in the case of fixed ordering cone, under pointedness, both nondominations recover the classical definition of Pareto minima.

Remark 3.6 The main reason for which we choose to study the more general case is the possibility of reduction of type I nondominated solutions to the second type (see Remark 3.7), which is available only if one drops the pointedness of the involved cones in the definitions.

For references concerning optimization problems with respect to variable domination structure, see [28].

Remark that the first approach (the domination map acts between the same spaces as the objective map) was introduced in [11], and subsequently studied in [12, 27], while the second setting (when the domination map is defined and takes values in the output space of the objective map) was introduced by Yu [39, 40]. Yu defined a domination structure as a family of cones, whereas Engau [20] considered it as a set-valued map. Furthermore, problems with variable domination structure are investigated by Eichfelder and her collaborators (see, e.g., [13, 15, 17, 18]), and attracted more attention in the optimization community (see, e.g., [4, 8, 21, 26, 27, 30] and the references therein). We mention as well that for both approaches, a variety of possible applications are given in the literature (see [2–7, 14–16, 19, 22–24, 30, 31, 34, 35, 37, 38, 41]). At the first glance, these two variants seem to be independent, and, actually, this is the way they are treated, up to now, in literature. However, the main point of this work is to show that the each setting can be reduced to the other one, by means of the simple devices which we describe next.

Remark 3.7 Suppose that we are in the case (II) in Definition 3.2, i.e., $(\overline{x}, \overline{y})$ is a local nondominated point of type II for F with respect to Q.

If we consider $\widetilde{F}, \widetilde{K} : X \times Y \rightrightarrows Y$ the set-valued maps given by

$$\widetilde{F}(x, y) := \begin{cases} \{y\} & \text{if } y \in F(x) \\ \emptyset & \text{otherwise,} \end{cases} \tag{3.8}$$

$$\widetilde{K}(x, y) := Q(y) \ \text{ for all } y \in Y, \tag{3.9}$$

remark that $((\overline{x}, \overline{y}), \overline{y}) \in \operatorname{Gr} \widetilde{F}$ is a local nondominated point of type I for $\widetilde{F}$ with respect to $\widetilde{K}$ on $\Omega \times Y$ if there is a neighborhood $U \times V$ of $(\overline{x}, \overline{y})$ such that, for every $(x, y) \in (U \cap \Omega) \times V$,

$$\left(\widetilde{F}(x, y) - \overline{y}\right) \cap \left(-\widetilde{K}(x, y)\right) \subseteq \widetilde{K}(x, y),$$

so in the case $V = Y$, this reduces to: for any $x \in U \cap \Omega$,

$$\forall y \in F(x), \{y - \overline{y}\} \cap (-Q(y)) \subseteq Q(y),$$

which is exactly (3.4).

Remark 3.8 Suppose that we are in the case (I) in Definition 3.2, i.e., $(\overline{x}, \overline{y})$ is a local nondominated point of type I for F with respect to K.

If we consider $\overline{F} : X \rightrightarrows X \times Y$, $\overline{Q} : X \times Y \rightrightarrows X \times Y$ the set-valued maps given by

$$\overline{F}(x) := \{x\} \times F(x) \text{ for all } x \in X \tag{3.10}$$

$$\overline{Q}(x, y) := X \times K(x) \text{ for all } (x, y) \in X \times Y, \tag{3.11}$$

observe that, via Remark 3.3, $(\overline{x}, (\overline{x}, \overline{y})) \in \operatorname{Gr} \overline{F}$ is a local nondominated point of type II for $\overline{F}$ with respect to $\overline{Q}$ on Ω if there is a neighborhood U of $\overline{x}$ such that, for every $x \in U \cap \Omega$,

$$\forall (u, y) \in \overline{F}(x) \setminus \{(\overline{x}, \overline{y})\}, (\overline{x}, \overline{y}) \notin (u, y) + \overline{Q}(u, y) \setminus \left(-\overline{Q}(u, y)\right),$$

which is equivalent to: for any $x \in U \cap \Omega$,

$$\begin{aligned} \forall (x, y) \in (\{x\} \times F(x)) \setminus \{(\overline{x}, \overline{y})\}, (\overline{x}, \overline{y}) \notin (x, y) + (X \times K(x)) \setminus (X \times (-K(x))) \\ = (x, y) + X \times (K(x) \setminus (-K(x))). \end{aligned}$$

This is equivalent to

$$\forall y \in F(x), \overline{y} \notin y + K(x) \setminus (-K(x)),$$

which is exactly (3.3).

Of course, as one can immediately see from (3.11) in the previous reduction scheme, $\overline{Q}(x, y)$ is never pointed. Therefore, it is necessary to work in a general setting where the dominance structure is given by maps whose images are closed, convex and proper cones.

As a consequence of the previous remarks, it is enough to get optimality results for one type of nondominated solutions, and then one can extend these assertions to those of the other type. Moreover, there are several paths to study optimality conditions, and our concern is to investigate such possibilities, to compare the final conditions which are derived, and to cover and expand in both types I and II approaches the results from literature.

The first method to obtain necessary optimality conditions is to use, for type I nondominated solutions, the Extremal Principle (Theorem 2.1). On this basis, we can apply, by the reduction method given above, similar arguments that lead to optimality conditions for type II nondominated solutions. Moreover, the necessary optimality conditions for the type II nondominated solutions directly imply, by the reduction given by Remark 3.8, the corresponding ones for type I nondominated solutions.

The second method, used, for instance, in [9] for the case of classical Pareto efficiency, employs the incompatibility between openness of some set-valued map and optimality. Notice that this incompatibility is, actually, a principle, since it works in several contexts. In the case of vector optimization problems, this method can be traced back at least to the paper [36]. The adaptation of this method to the current settings involves a serious technical effort, and we propose two ways for getting our results.

The diagram from Fig. 1 underlies this program.

4 Optimality Conditions by Extremality

We start the investigation of optimality conditions by a technical lemma that asserts the positiveness of the argument of the Fréchet as well as the Mordukhovich coderivatives (see (2.3) and (2.4)) of the cone-valued map K (see [11, Lemma 4.9]).

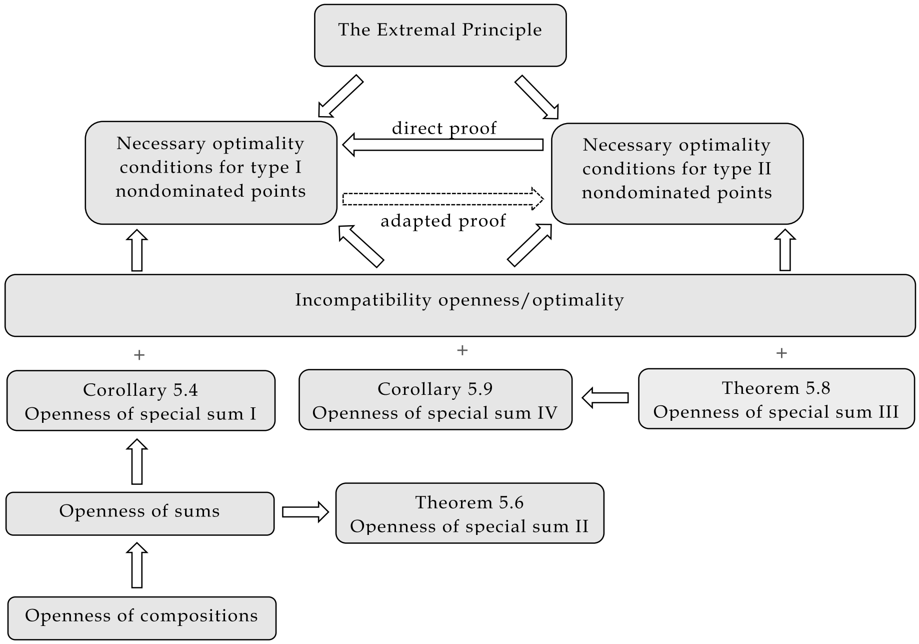

Fig. 1 General overview of the results

Lemma 4.1 *Let X, Y be Banach spaces, $K : X \rightrightarrows Y$ a cone-valued set-valued map and $(\overline{x}, \overline{k}) \in \operatorname{Gr} K$.*

(i) If $\widehat{D}^* K(\overline{x}, \overline{k})(k^*) \neq \emptyset$, then $k^* \in K(\overline{x})^+$.

(ii) If X, Y are Asplund spaces, $\operatorname{Gr} K$ *is closed around $(\overline{x}, \overline{k})$, K is lower semicontinuous at $(\overline{x}, \overline{k})$ and $D^* K(\overline{x}, \overline{k})(k^*) \neq \emptyset$, then $k^* \in K(\overline{x})^+$.*

Using the Extremal Principle in Theorem 2.1, we derive the following necessary condition for local nondominated points of type I.

Theorem 4.2 *Let X, Y be Asplund spaces, $\Omega \subseteq X$ be a closed set, $F, K : X \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}) \in X \times Y$ be a local nondominated point of type I for F with respect to K on Ω. Moreover, assume that:*

(i) there is a neighborhood U of $\overline{x}$ such that $\bigcap_{x \in (\Omega \cap U) \cap \operatorname{Dom} F} (K(x) \setminus (-K(x))) \neq \emptyset$;

(ii) K is SNC at $(\overline{x}, 0)$, and either Ω is SNC at $\overline{x}$, or F is PSNC at $(\overline{x}, \overline{y})$;

(iii) the following assumptions are satisfied:

$$D^* K(\overline{x}, 0)(0) = \{0\}, \tag{4.1}$$

$$D^*_M F(\overline{x}, \overline{y})(0) \cap (-N(\Omega, \overline{x})) = \{0\}. \tag{4.2}$$

Then there exists $y^* \in K(\overline{x})^+ \setminus \{0\}$ such that

$$0 \in D^* F(\overline{x}, \overline{y})(y^*) + D^* K(\overline{x}, 0)(y^*) + N(\Omega, \overline{x}). \tag{4.3}$$

Proof Let us prove that, in our assumptions, $(\overline{x}, \overline{y}, 0)$ is a local extremal point for the system $\{C_1, C_2, C_3\}$, where

$$C_1 := \left\{(x, y, k) \in X \times Y^2 \mid (x, y) \in \operatorname{Gr} F\right\},$$
$$C_2 := \left\{(x, y, k) \in X \times Y^2 \mid (x, k) \in \operatorname{Gr} K\right\},$$
$$C_3 := \left\{(x, y, k) \in X \times Y^2 \mid x \in \Omega, k = \overline{y} - y\right\}.$$

For this, observe that $\overline{x} \in (\Omega \cap U) \cap \operatorname{Dom} F$, and take $\overline{k} \in \bigcap_{x \in (\Omega \cap U) \cap \operatorname{Dom} F} (K(x) \setminus (-K(x)))$, where U is a neighborhood of $\overline{x}$ such that (i) and the nondomination condition hold. Since $(\overline{x}, \overline{y}, 0) \in C_1 \cap C_2 \cap C_3$, it suffices to show

$$C_1 \cap \left(C_2 + n^{-1}\left(0, 0, \overline{k}\right)\right) \cap C_3 \cap (U \times Y \times Y) = \emptyset, \quad \forall n \in \mathbb{N} \setminus \{0\}.$$

Indeed, if we suppose by contradiction that there exists (x, y, k) in the above intersection, then $x \in \Omega \cap U$, $y \in F(x)$, $k \in K(x) + n^{-1}\overline{k} \subseteq K(x)$, $k = \overline{y} - y$, hence

$$-k \in (F(x) - \overline{y}) \cap (-K(x)).$$

By the nondomination condition, $k = -K(x)$, which shows that $\overline{k} \in -K(x)$, a contradiction.

Apply now the Extremal Principle (Theorem 2.1) to the system $\{C_1, C_2, C_3, (\overline{x}, \overline{y}, 0)\}$, and obtain that there exist the sequences $(x_{in}, y_{in}, k_{in}) \subset X \times Y^2$ and $\left(x_{in}^*, y_{in}^*, k_{in}^*\right) \subset X^* \times (Y^*)^2$ such that, for any $n \in \mathbb{N}$,

$$\begin{aligned}
&(x_{in}, y_{in}, k_{in}) \xrightarrow{C_i} (\overline{x}, \overline{y}, 0), i = 1, 2, 3\\
&\left(x_{1n}^*, -y_{1n}^*, k_{1n}^*\right) \in \widehat{N}(C_1, (x_{1n}, y_{1n}, k_{1n})) \Leftrightarrow x_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})\left(y_{1n}^*\right), k_{1n}^* = 0 \qquad (4.4)\\
&\left(x_{2n}^*, y_{2n}^*, -k_{2n}^*\right) \in \widehat{N}(C_2, (x_{2n}, y_{2n}, k_{2n})) \Leftrightarrow x_{2n}^* \in \widehat{D}^* K(x_{2n}, k_{2n})\left(k_{2n}^*\right), y_{2n}^* = 0\\
&\left(x_{3n}^*, y_{3n}^*, k_{3n}^*\right) \in \widehat{N}(C_3, (x_{3n}, y_{3n}, k_{3n})) \Leftrightarrow x_{3n}^* \in \widehat{N}(\Omega, x_{3n}), y_{3n}^* = k_{3n}^*,
\end{aligned}$$

and, moreover, satisfying the conditions

$$\left\|\left(x_{1n}^*, -y_{1n}^*, 0\right) + \left(x_{2n}^*, 0, -k_{2n}^*\right) + \left(x_{3n}^*, y_{3n}^*, y_{3n}^*\right)\right\| \leq n^{-1}, \qquad (4.5)$$

$$1 - n^{-1} \leq \left\|\left(x_{1n}^*, -y_{1n}^*, 0\right)\right\| + \left\|\left(x_{2n}^*, 0, -k_{2n}^*\right)\right\| + \left\|\left(x_{3n}^*, y_{3n}^*, y_{3n}^*\right)\right\| \leq 1 + n^{-1}. \qquad (4.6)$$

Condition (4.6) shows that all sequences are bounded, so we may suppose without loss of generality, since the unit ball in Asplund spaces is weakly* sequentially compact, that all sequences weakly* converge to corresponding limits, i.e., $\left(x_{1n}^*, -y_{1n}^*, 0\right) \xrightarrow{*} \left(x_1^*, -y_1^*, 0\right)$, $\left(x_{2n}^*, 0, -k_{2n}^*\right) \xrightarrow{*} \left(x_2^*, 0, -k_2^*\right)$, $\left(x_{3n}^*, y_{3n}^*, y_{3n}^*\right) \xrightarrow{*} \left(x_3^*, y_3^*, y_3^*\right)$ for $n \to \infty$. But this shows, also using (4.5), that

$$x_1^* \in D^* F(\overline{x}, \overline{y})\left(y_1^*\right), x_2^* \in D^* K(\overline{x}, 0)\left(k_2^*\right), x_3^* \in N(\Omega, \overline{x}),$$
$$x_1^* + x_2^* + x_3^* = 0, y_1^* = k_2^* = y_3^*.$$

It follows that condition (4.3) holds for $y^* := y_3^*$. It remains to prove that $y^* \in K(\overline{x})^+ \setminus \{0\}$.

Because K is SNC at $(\overline{x}, 0)$, and (4.1) holds, we deduce by Lemma 2.2 that K has the Aubin property around $(\overline{x}, 0)$, so it is lower semicontinuous at $(\overline{x}, 0)$. Hence, the fact that $y^* \in K(\overline{x})^+$ follows from Lemma 4.1.

Let us deduce that $y^* \neq 0$. If $y^* = 0$, then by (4.1) it follows $x_2^* = 0$, and since $(x_{2n}, k_{2n}) \xrightarrow{\operatorname{Gr} K} (\overline{x}, 0)$, $\left(x_{2n}^*, k_{2n}^*\right) \xrightarrow{*} (0, 0)$, $x_{2n}^* \in \widehat{D}^* K(x_{2n}, k_{2n})\left(k_{2n}^*\right)$, using the SNC property of K at $(\overline{x}, 0)$, it follows that $\left(x_{2n}^*, k_{2n}^*\right) \to (0, 0)$. By relation (4.5), we deduce that $y_{1n}^*, y_{3n}^* \to 0$, and hence

$$-x_3^* = x_1^* \in D_M^* F(\overline{x}, \overline{y})(0) \cap (-N(\Omega, \overline{x})).$$

Using now the qualification condition (4.2), we obtain $x_3^* = x_1^* = 0$.

If Ω is SNC at $\overline{x}$, since $x_{3n} \xrightarrow{\Omega} \overline{x}$, $x_{3n}^* \xrightarrow{*} x_3^* = 0$ and $x_{3n}^* \in \widehat{N}(\Omega, x_{3n})$, we obtain that $x_{3n}^* \to 0$, hence by (4.5) also $x_{1n}^* \to 0$, and this contradicts the left-hand side inequality in relation (4.6). If F is PSNC at $(\overline{x}, \overline{y})$, then since $(x_{1n}, y_{1n}) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y})$, $x_{1n}^* \xrightarrow{*} 0,$, $y_{1n}^* \to 0$, $x_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})\left(y_{1n}^*\right)$, we get that $x_{1n}^* \to 0$, so by (4.5) also $x_{3n}^* \to 0$, and again we contradict relation (4.6).

In conclusion, $y^* \neq 0$ and the proof is complete. □

In the unconstrained case (i.e., $\Omega := X$) we obtain the following consequence.

Corollary 4.3 *Let X, Y be Asplund spaces, $F, K : X \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}) \in X \times Y$ be a local nondominated point of type I for F with respect to K. Moreover, assume that:*

(i) there is a neighborhood U of $\overline{x}$ such that $\displaystyle\bigcap_{x \in U \cap \operatorname{Dom} F} (K(x) \setminus (-K(x))) \neq \emptyset$;

(ii) K is SNC at $(\overline{x}, 0)$;

(iii) $D^* K(\overline{x}, 0)(0) = \{0\}$.

Then there exists $y^* \in K(\overline{x})^+ \setminus \{0\}$ such that

$$0 \in D^* F(\overline{x}, \overline{y})(y^*) + D^* K(\overline{x}, 0)(y^*). \tag{4.7}$$

Remark 4.4 For an illustrative example of the optimality conditions we derived above for nondominated points of type I, we refer the reader to [11, Example 4.12].

Let us come back to the nondominated solutions of type II (see Definition 3.2 (II)). We start with a simple lemma, which asserts the formulae for the Fréchet and Mordukhovich coderivatives of the set-valued maps involved in the reduction technique described in Remark 3.7.

Lemma 4.5 *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$, $Q : Y \rightrightarrows Y$ be set-valued maps, and suppose that $\widetilde{F}, \widetilde{K} : X \times Y \rightrightarrows Y$ are given by (3.8), (3.9). We have:*

$$\left(x^*, y^*\right) \in \widehat{D}^* \widetilde{F}(x, y, y)(z^*) \Leftrightarrow x^* \in \widehat{D}^* F(x, y)(z^* - y^*), \tag{4.8}$$

$$\left(x^*, y^*\right) \in \widehat{D}^* \widetilde{K}(x, y, z)(z^*) \Leftrightarrow x^* = 0, y^* \in \widehat{D}^* Q(y, z)(z^*). \tag{4.9}$$

Moreover, if F and Q are closed-graph, then $\widetilde{F}, \widetilde{K}$ are also closed-graph, and

$$\left(x^*, y^*\right) \in D^* \widetilde{F}(x, y, y)(z^*) \Leftrightarrow x^* \in D^* F(x, y)(z^* - y^*), \tag{4.10}$$

$$\left(x^*, y^*\right) \in D_M^* \widetilde{F}(x, y, y)(z^*) \Leftarrow x^* \in D_M^* F(x, y)(z^* - y^*), \tag{4.11}$$

$$\left(x^*, y^*\right) \in D^* \widetilde{K}(x, y, z)(z^*) \Leftrightarrow x^* = 0, y^* \in D^* Q(y, z)(z^*). \tag{4.12}$$

Proof The formulae (4.8), (4.9), (4.12) and the direct implication in (4.10) immediately follow from definitions. For the converse implication in (4.10), take $x^* \in D^* F(x, y)(z^* - y^*)$. Then there exist $(x_n, y_n) \xrightarrow{\operatorname{Gr} F} (x, y)$, and $\left(x_n^*, v_n^*\right) \xrightarrow{*} (x^*, z^* - y^*)$ such that $x_n^* \in \widehat{D}^* F(x_n, y_n)(v_n^*)$, which imply that $(x_n, y_n, y_n) \xrightarrow{\operatorname{Gr} \widetilde{F}} (x, y, y)$, $\left(x_n^*, y^*, v_n^* + y^*\right) \xrightarrow{*} (x^*, y^*, z^*)$ and

$$\left(x_n^*, y^*\right) \in \widehat{D}^* \widetilde{F}(x_n, y_n, y_n)(v_n^* + y^*),$$

so the conclusion follows. For (4.11), from $v_n^* \to z^* - y^*$ it follows that $\left(y^*, v_n^* + y^*\right) \to (y^*, z^*)$. □

We obtain the following necessary condition for local nondominated points of type II.

Theorem 4.6 *Let X, Y be Asplund spaces, $\Omega \subseteq X$ be a closed set, $F : X \rightrightarrows Y$, $Q : Y \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}) \in X \times Y$ be a local nondominated point of type II for F with respect to Q on Ω. Moreover, assume that*:

(*i*) *there is a neighborhood U of $\overline{x}$ such that* $\bigcap_{y \in F(\Omega \cap U)} (Q(y) \setminus (-Q(y))) \neq \emptyset$;

(*ii*) *Q is SNC at $(\overline{y}, 0)$, and either Ω is SNC at $\overline{x}$, or F is PSNC at $(\overline{x}, \overline{y})$*;

(*iii*) *the following assumptions are satisfied*:

$$D^* Q(\overline{y}, 0)(0) = \{0\}, \tag{4.13}$$

$$D_M^* F(\overline{x}, \overline{y})(0) \cap (-N(\Omega, \overline{x})) = \{0\}. \tag{4.14}$$

Then there exist $z^* \in Q(\overline{y})^+ \setminus \{0\}$ and $y^* \in D^* Q(\overline{y}, 0)(z^*)$ such that

$$0 \in D^* F(\overline{x}, \overline{y})(z^* + y^*) + N(\Omega, \overline{x}). \tag{4.15}$$

Proof Taking into account Remark 3.7, we observe that $((\overline{x}, \overline{y}), \overline{y}) \in \operatorname{Gr} \widetilde{F}$ is a local nondominated point of type I for $\widetilde{F}$ (see (3.8)) with respect to $\widetilde{K}$ (see (3.9)) on $\Omega \times Y$, and suppose, without loss of generality, that the neighborhood involved in the definition is $U \times Y$, i.e., for any $x \in U \cap \Omega$, $y \in Y$

$$\left(\widetilde{F}(x, y) - \overline{y}\right) \cap \left(-\widetilde{K}(x, y)\right) \subseteq \widetilde{K}(x, y).$$

Observe that $\operatorname{Dom} \widetilde{F} = \operatorname{Gr} F$, hence

$$\bigcap_{(x,y) \in (\Omega \times Y) \cap (U \times Y) \cap \operatorname{Dom} \widetilde{F}} \left(\widetilde{K}(x, y) \setminus \left(-\widetilde{K}(x, y)\right)\right) \neq \emptyset \Leftrightarrow \bigcap_{y \in F(\Omega \cap U)} (Q(y) \setminus (-Q(y))) \neq \emptyset.$$

This allows us to deduce, as in the proof of Theorem 4.2, for the sets

$$C_1 = \left\{(x, y, y, k) \in X \times Y^3 \mid (x, y, y) \in \operatorname{Gr} \widetilde{F}\right\},$$

$$C_2 = \left\{(x, y, z, k) \in X \times Y^3 \mid (x, y, k) \in \operatorname{Gr} \widetilde{K}\right\},$$

$$C_3 = \left\{(x, y, z, k) \in X \times Y^3 \mid x \in \Omega, k = \overline{y} - y\right\}$$

and the point $((\overline{x},\overline{y}),\overline{y},0)$, the existence of $(x_{in},y_{in},z_{in},k_{in})\subset X\times Y^3$ and $\left(x_{in}^*,y_{in}^*,z_{in}^*,k_{in}^*\right)\subset X^*\times (Y^*)^3$ such that, for any $n\in\mathbb{N}$,

$$\begin{aligned}
&(x_{in},y_{in},z_{in},k_{in})\xrightarrow{C_i}(\overline{x},\overline{y},\overline{y},0)\,,i=1,2,3,\\
&\left(x_{1n}^*,y_{1n}^*,-z_{1n}^*,k_{1n}^*\right)\in\widehat{N}\left(C_1,(x_{1n},y_{1n},z_{1n},k_{1n})\right)\\
&\quad\Leftrightarrow x_{1n}^*\in\widehat{D}^*F\left(x_{1n},y_{1n}\right)\left(z_{1n}^*-y_{1n}^*\right),k_{1n}^*=0,\\
&\left(x_{2n}^*,y_{2n}^*,z_{2n}^*,-k_{2n}^*\right)\in\widehat{N}\left(C_2,(x_{2n},y_{2n},z_{2n},k_{2n})\right)\\
&\quad\Leftrightarrow x_{2n}^*=0,\,z_{2n}^*=0,\,y_{2n}^*\in\widehat{D}^*Q\left(y_{2n},k_{2n}\right)\left(k_{2n}^*\right),\\
&\left(x_{3n}^*,y_{3n}^*,z_{3n}^*,k_{3n}^*\right)\in\widehat{N}\left(C_3,(x_{3n},y_{3n},z_{3n},k_{3n})\right)\Leftrightarrow x_{3n}^*\in\widehat{N}\left(\Omega,x_{3n}\right),z_{3n}^*=0,\,y_{3n}^*=k_{3n}^*,
\end{aligned}\tag{4.16}$$

and, moreover, such that

$$\left\|\left(x_{1n}^*,y_{1n}^*,-z_{1n}^*,0\right)+\left(0,y_{2n}^*,0,-k_{2n}^*\right)+\left(x_{3n}^*,y_{3n}^*,0,y_{3n}^*\right)\right\|\le n^{-1},\tag{4.17}$$

$$\begin{aligned}
1-n^{-1}&\le\left\|\left(x_{1n}^*,y_{1n}^*,-z_{1n}^*,0\right)\right\|+\left\|\left(0,y_{2n}^*,0,-k_{2n}^*\right)\right\|+\left\|\left(x_{3n}^*,y_{3n}^*,0,y_{3n}^*\right)\right\|\\
&\le 1+n^{-1}.
\end{aligned}\tag{4.18}$$

Relations (4.17), (4.18) and the fact that X, Y are Asplund imply that $\left(x_{1n}^*,y_{1n}^*\right)\xrightarrow{*}\left(x_1^*,y_1^*\right)$, $z_{1n}^*\to 0$, $\left(y_{2n}^*,k_{2n}^*\right)\xrightarrow{*}\left(y_2^*,k_2^*\right)$, $\left(x_{3n}^*,y_{3n}^*\right)\xrightarrow{*}\left(x_3^*,y_3^*\right)$, hence

$$\begin{aligned}
&x_1^*\in D^*F\left(\overline{x},\overline{y}\right)\left(-y_1^*\right),\,y_2^*\in D^*Q\left(\overline{y},0\right)\left(k_2^*\right),\,x_3^*\in N\left(\Omega,\overline{x}\right),\\
&x_1^*+x_3^*=0,\,y_1^*+y_2^*+y_3^*=0,\,k_2^*=y_3^*.
\end{aligned}$$

Denoting $z^*:=y_3^*$ and $y^*:=y_2^*$, we get $y^*\in D^*Q(\overline{y},0)(z^*)$ and

$$0=x_1^*+x_3^*\in D^*F\left(\overline{x},\overline{y}\right)\left(z^*+y^*\right)+N\left(\Omega,\overline{x}\right).$$

Observe that, by (4.9), the SNC of $\widetilde{K}$ at $(\overline{x},\overline{y},0)$ is equivalent to the SNC of Q at $(\overline{y},0)$. Also, the SNC property of $\Omega\times Y$ at $(\overline{x},\overline{y})$ trivially reduces to the SNC property of Ω at $\overline{x}$. By Lemma 2.2, assumption (4.13) and the SNC property of Q at $(\overline{y},0)$ imply that Q is lower semicontinuous at $(\overline{y},0)$, hence $z^*\in Q(\overline{y})^+$ by Lemma 4.1.

It remains to prove that $z^*\neq 0$. Suppose, by contradiction, that $z^*=0$. Then by (4.13), $y^*=0$, and since $(y_{2n},k_{2n})\xrightarrow{\operatorname{Gr}Q}(\overline{y},0)$, $\left(y_{2n}^*,k_{2n}^*\right)\xrightarrow{*}(0,0)$, $y_{2n}^*\in\widehat{D}^*Q\left(y_{2n},k_{2n}\right)\left(k_{2n}^*\right)$, by the fact that Q is SNC at $(\overline{y},0)$, we get $\left(y_{2n}^*,k_{2n}^*\right)\to(0,0)$. Using (4.17), we know that

$$\begin{aligned}
&x_{1n}^*+x_{3n}^*\to 0,\\
&y_{1n}^*+y_{2n}^*+y_{3n}^*\to 0,\\
&-k_{2n}^*+y_{3n}^*\to 0,
\end{aligned}$$

hence we deduce that $y_{3n}^*\to 0$ and, furthermore, that $y_{1n}^*\to 0$. Then, since $z_{1n}^*-y_{1n}^*\to 0$, we obtain by (4.14) that

$$x_1^*=-x_3^*\in D_M^*F(\overline{x},\overline{y})(0)\cap\left(-N\left(\Omega,\overline{x}\right)\right)=\{0\},$$

hence $x_{1n}^*,x_{3n}^*\xrightarrow{*}0$.

If Ω is SNC at $\overline{x}$, since $x_{3n} \xrightarrow{\Omega} \overline{x}$, $x_{3n}^* \xrightarrow{*} 0$ and $x_{3n}^* \in \widehat{N}(\Omega, x_{3n})$, we obtain that $x_{3n}^* \to 0$, hence $x_{1n}^* \to 0$, a contradiction to the left-hand side inequality in (4.18). If F is PSNC at $(\overline{x}, \overline{y})$, since $(x_{1n}, y_{1n}) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y}), x_{1n}^* \xrightarrow{*} 0, z_{1n}^* - y_{1n}^* \to 0$, $x_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})\left(z_{1n}^* - y_{1n}^*\right)$, we get that $x_{1n}^* \to 0$, hence also $x_{3n}^* \to 0$, again a contradiction to (4.18). The conclusion holds. □

Remark 4.7 A variant of the previous result, for single-valued maps, is given in [4, Theorem 4.2]. Notice that, in the case of set-valued maps, Theorem 4.6 substantially improves the corresponding results from [30] and [27] in the sense that our result avoids the use of auxiliary objects, employing directly the maps defining the problem under study. Notice as well that in [27] necessary conditions are derived also for solution concepts defined with respect to set relations.

Remark 4.8 Observe that PSNC property of F at $(\overline{x}, \overline{y})$ does not imply the PSNC property of $\widetilde{F}$ at $(\overline{x}, \overline{y}, \overline{y})$, and also that relation (4.14) does not imply

$$D_M^* \widetilde{F}(\overline{x}, \overline{y}, \overline{y})(0) \cap (-N(\Omega \times Y, (\overline{x}, \overline{y}))) = \{(0, 0)\}.$$

So, Theorem 4.2 cannot be applied directly to obtain the conclusion of Theorem 4.6 in full generality. The assumptions in Theorem 4.6 should be modified, respectively, in F to be SNC at $(\overline{x}, \overline{y})$, and $D^* F(\overline{x}, \overline{y})(0) \cap (-N(\Omega, \overline{x})) = \{0\}$. This was the reason for repeating above some parts of the proof of Theorem 4.2.

Now, we study the way the optimality conditions for type II nondominated solutions obtained in Theorem 4.6 imply, by the reduction from Remark 3.8, the optimality conditions for type I in Theorem 4.2. In order to do that, we present in a lemma the relationships between the coderivatives of F and K and those of $\overline{F}$ and $\overline{Q}$, respectively.

Lemma 4.9 *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$, $K : X \rightrightarrows Y$ be set-valued maps, and suppose that $\overline{F} : X \rightrightarrows X \times Y$, $\overline{Q} : X \times Y \rightrightarrows X \times Y$ are given by* (3.10), (3.11). *We have*:

$$x^* \in \widehat{D}^* \overline{F}(x, x, y)(u^*, y^*) \Leftrightarrow x^* - u^* \in \widehat{D}^* F(x, y)(y^*), \tag{4.19}$$

$$\left(x^*, y^*\right) \in \widehat{D}^* \overline{Q}(x, y, u, v)(u^*, v^*) \Leftrightarrow y^* = 0, u^* = 0, x^* \in \widehat{D}^* K(x, v)(v^*). \tag{4.20}$$

Moreover, if F and K are closed-graph, then $\overline{F}$, $\overline{Q}$ are also closed-graph, and

$$x^* \in D^* \overline{F}(x, x, y)(u^*, y^*) \Leftrightarrow x^* - u^* \in D^* F(x, y)(y^*), \tag{4.21}$$

$$x^* \in D_M^* \overline{F}(x, x, y)(u^*, y^*) \Leftrightarrow x^* - u^* \in D_M^* F(x, y)(y^*), \tag{4.22}$$

$$\left(x^*, y^*\right) \in D^* \overline{Q}(x, y, u, v)(u^*, v^*) \Leftrightarrow y^* = 0, u^* = 0, x^* \in D^* K(x, v)(v^*). \tag{4.23}$$

Proof The equivalences (4.19), (4.20), (4.23) and the implications in (4.21), (4.22) immediately follow from definitions. For the reverse implication in (4.21), take $x^* - u^* \in D^* F(x, y)(y^*)$, which means that there exist $(x_n, y_n) \xrightarrow{\operatorname{Gr} F} (x, y)$, and $\left(x_n^*, y_n^*\right) \xrightarrow{*} (x^* - u^*, y^*)$ such that $x_n^* \in \widehat{D}^* F(x_n, y_n)(y_n^*)$, which imply that $(x_n, x_n, y_n) \xrightarrow{\operatorname{Gr} \overline{F}} (x, x, y)$, $\left(x_n^* + u^*, u^*, y_n^*\right) \xrightarrow{*} (x^*, u^*, y^*)$ and

$$x_n^* + u^* \in \widehat{D}^* \overline{F}(x_n, x_n, y_n)(u^*, y_n^*).$$

For the reverse implication in (4.22), the proof is similar to the above, since $\left(u^*, y_n^*\right) \to (u^*, y^*)$. This ends the proof. □

Proof of Theorem 4.2 by Theorem 4.6 If $(\overline{x}, \overline{y})$ is a local nondominated point of type I for F with respect to K, using Remark 3.8, it follows that $(\overline{x}, (\overline{x}, \overline{y})) \in \operatorname{Gr} \overline{F}$ is a local nondominated point of type II for $\overline{F}$ with respect to $\overline{Q}$ on Ω.

Observe that:

- The following equivalences hold

$$\bigcap_{(x,y)\in F(\Omega\cap U)} \left(\overline{Q}(x, y) \setminus \left(-\overline{Q}\,(x, y)\right)\right) \neq \emptyset \Leftrightarrow$$

$$\bigcap_{x\in \operatorname{Dom} F\cap(\Omega\cap U)} ((X \times K\,(x)) \setminus (X \times (-K\,(x)))) \neq \emptyset \Leftrightarrow$$

$$\bigcap_{x\in \operatorname{Dom} F\cap(\Omega\cap U)} (K\,(x) \setminus (-K\,(x))) \neq \emptyset.$$

- Using (4.23), one has that K is SNC at $(\overline{x}, 0)$ iff $\overline{Q}$ is SNC at $(\overline{x}, \overline{y}, 0, 0)$.
- PSNC property of F at $(\overline{x}, \overline{y})$ implies the PSNC property of $\overline{F}$ at $(\overline{x}, \overline{x}, \overline{y})$.

Indeed, suppose that we have $(x_n, x_n, y_n) \overset{\operatorname{Gr}\overline{F}}{\to} (\overline{x}, \overline{x}, \overline{y})$, $x_n^* \overset{*}{\to} 0$, $(u_n^*, y_n^*) \to (0, 0)$, such that $x_n^* \in \widehat{D}^*\overline{F}(x_n, x_n, y_n)(u_n^*, y_n^*)$. This means that $(x_n, y_n) \overset{\operatorname{Gr}F}{\to} (\overline{x}, \overline{y})$, $x_n^* - u_n^* \overset{*}{\to} 0$, $y_n^* \to 0$ and $x_n^* - u_n^* \in \widehat{D}^*F(x_n, y_n)(y_n^*)$. By the PSNC property of F at $(\overline{x}, \overline{y})$, we have that $x_n^* - u_n^* \to 0$, and since $u_n^* \to 0$, we finally obtain that $x_n^* \to 0$.

- By (4.23), it follows

$$D^*K(\overline{x}, 0)(0) = \{0\} \Leftrightarrow D^*\overline{Q}\,(\overline{x}, \overline{y}, 0, 0)\,(0, 0) = \{(0, 0)\}\,.$$

- Using (4.22), we have

$$D_M^*F(\overline{x}, \overline{y})(0) \cap (-N\,(\Omega, \overline{x})) = \{0\} \Leftrightarrow D_M^*\overline{F}(\overline{x}, \overline{x}, \overline{y})(0, 0) \cap (-N\,(\Omega, \overline{x})) = \{0\}\,.$$

Hence, all the assumptions of Theorem 4.6 are satisfied, so there exist $(u^*, v^*) \in \overline{Q}(\overline{x}, \overline{y})^+ \setminus \{(0, 0)\} = \{0\} \times \left(K\,(\overline{x})^+ \setminus \{0\}\right)$ and $(x^*, y^*) \in D^*\overline{Q}(\overline{x}, \overline{y}, 0, 0)(u^*, v^*)$ such that

$$0 \in D^*\overline{F}(\overline{x}, \overline{x}, \overline{y})(u^* + x^*, v^* + y^*) + N\,(\Omega, \overline{x})\,.$$

This means that $u^* = 0$, $y^* = 0$, $x^* \in D^*K\,(\overline{x}, 0)\,(v^*)$ and there exists $t^* \in D^*\overline{F}(\overline{x}, \overline{x}, \overline{y})(x^*, v^*)$, hence by (4.21) $t^* - x^* \in D^*F(x, y)(v^*)$, such that $-t^* \in N\,(\Omega, \overline{x})$. Finally, we get that there is $v^* \in K\,(\overline{x})^+ \setminus \{0\}$ with

$$\begin{aligned} 0 \in t^* + N\,(\Omega, \overline{x}) &= \left(t^* - x^*\right) + x^* + N\,(\Omega, \overline{x}) \\ &\subseteq D^*F(x, y)(v^*) + D^*K\,(\overline{x}, 0)\left(v^*\right) + N\,(\Omega, \overline{x})\,. \end{aligned}$$

The proof is complete. □

Remark 4.10 Notice that the proof of Theorem 4.2 by Theorem 4.6 is much more direct than the opposite implication, and this shows, once more, the utility of considering nonpointed preordering cones.

In the unconstrained case, we have the following necessary condition for local nondominated points of type II (Definition 3.2 (II)).

Corollary 4.11 *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$, $Q : Y \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}) \in X \times Y$ be a local nondominated point of type II for F with respect to Q. Moreover, assume that:*

(i) there is a neighborhood U of $\overline{x}$ such that $\bigcap_{y \in F(\Omega \cap U)} (Q(y) \setminus (-Q(y))) \neq \emptyset$;

(ii) Q is SNC at $(\overline{y}, 0)$;

(iii) $D^*Q(\overline{y}, 0)(0) = \{0\}$.

Then there exist $z^* \in Q(\overline{y})^+ \setminus \{0\}$ and $y^* \in D^*Q(\overline{y}, 0)(z^*)$ such that

$$0 \in D^*F(\overline{x}, \overline{y})(z^* + y^*). \tag{4.24}$$

Both Theorems 4.2 and 4.6 reduce, in case that the cone-valued map K or Q is a constant cone, to the well-known necessary optimality conditions for Pareto minima (see, e.g., [2, Theorem 5.3]).

Corollary 4.12 *Let X, Y be Asplund spaces, $\Omega \subseteq X$ be a closed set, $F : X \rightrightarrows Y$ be a closed-graph set-valued map, $C \subseteq Y$ be a proper closed and convex cone and $(\overline{x}, \overline{y}) \in X \times Y$ be a local Pareto minimal point for F on Ω. Moreover, assume that (4.2) holds, that C is SNC at 0, and either Ω is SNC at $\overline{x}$, or F is PSNC at $(\overline{x}, \overline{y})$. Then there exists $y^* \in C^+ \setminus \{0\}$ such that*

$$0 \in D^*F(\overline{x}, \overline{y})(y^*) + N(\Omega, \overline{x}). \tag{4.25}$$

Furthermore, in the unconstrained case, the previous corollary reduces to the well-known necessary optimality conditions for Pareto solutions (see, e.g., [2, Theorem 5.1], [9, Theorem 3.11]).

At the end of this section, we illustrate the optimality conditions for nondominated points of type II.

Example 4.13 Consider the objective map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ given by

$$F(t) := [(0, 0), (\cos t, \sin t)].$$

As in [11, Example 4.12], we have that

$$N(\operatorname{Gr} F, (0, 0, 0)) = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0,\ y \leq 0\}.$$

Moreover, following [4, p. 364], we particularize the Bishop-Phelps ordering structure by considering $Q : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ given by

$$Q(a, b) := \left\{(u, v) \in \mathbb{R}^2 \mid |u| + |v| \leq 2\left(e^{|a|} \cdot u + e^{|b|} \cdot v\right)\right\}.$$

Take $\overline{x} := 0$, $\overline{y} := (0, 0)$. Remark that $(\overline{x}, \overline{y})$ is a nondominated point of type II for F with respect to Q. We have

$$Q(\overline{y}) = \left\{(u, v) \in \mathbb{R}^2 \mid |u| + |v| \leq 2(u + v)\right\},$$

$$Q(\overline{y})^+ = \left\{(u, v) \in \mathbb{R}^2 \mid u \leq 3v, v \leq 3u\right\},$$

and we consider $z^* := (2,2) \in Q(\overline{y})^+ \setminus \{(0,0)\}$. Moreover, $z^* \in \mathbb{R}_+((2,2) + S_{\mathbb{R}^2})$, so, according to [4, Proposition 3.12],

$$D^* Q(\overline{y}, (0,0))\left(z^*\right) = \{(0,0)\}.$$

Now it is obvious that

$$0 \in D^* F(\overline{x}, \overline{y})(z^* + y^*),$$

with $y^* = (0,0)$. Of course, the assumptions (i)-(iii) of Corollary 4.11 are satisfied.

5 Openness

In this section, we derive the openness results fitted in our context, in the sense that, putted in relationship with the nondominated solutions of first and second type (Definition 3.2 (I) and (II)), they provide, by the incompatibility principle between openness and nondomination, the necessary optimality conditions.

We remark that there are at least two possible ways to follow this program. First of all, we extend a linear openness result for compositions of two set-valued maps given in [10] to a finite number of set-valued maps, and then we particularize it for sum maps. Next, we prove a new openness result, which gives yet another way to proceed in getting the optimality conditions and, moreover, gives some new alternatives in the formulation of the assumptions of the main results, which are not obvious in the context of the other variants (see Remark 6.6 and Corollary 6.7).

5.1 Openness of Compositions and Sums

If $S_1, \ldots, S_p$ are subsets of a normed vector space X, closed around $\overline{x} \in S_1 \cap \cdots \cap S_p$, one says that they are allied at $\overline{x}$ (for the Fréchet normal cones) whenever $(x_{in}) \xrightarrow{S_i} \overline{x}$, $x_{in}^* \in \widehat{N}(S_i, x_{in})$, $i = \overline{1,p}$, the relation $(x_{1n}^* + \cdots + x_{pn}^*) \to 0$ implies $(x_{in}^*) \to 0$ for every $i = \overline{1,p}$ (for more details and historical comments, see [10]).

In what follows, we extend a linear openness result for compositions of two set-valued maps from [10] to a corresponding result for a finite number of set-valued maps defined on T in order to apply it to our current framework.

Let $T, Y_1, \ldots, Y_p, Z$ be normed vector spaces, and consider $F_i : T \rightrightarrows Y_i$, $i = \overline{1,p}$ and $\Psi : Y_1 \times \cdots \times Y_p \rightrightarrows Z$, $(\overline{t}, \overline{y}_1, \ldots, \overline{y}_p, \overline{z}) \in T \times Y_1 \times \cdots \times Y_p \times Z$ such that $(\overline{t}, \overline{y}_i) \in \operatorname{Gr} F_i$, $i = \overline{1,p}$, $((\overline{y}_1, \ldots, \overline{y}_p), \overline{z}) \in \operatorname{Gr} \Psi$. We denote

$$C_i := \{(t, y_1, \ldots, y_p, z) \in T \times Y_1 \times \cdots \times Y_p \times Z \mid y_i \in F_i(t)\}, i = \overline{1,p}, \tag{5.1}$$
$$C_{p+1} := \{(t, y_1, \ldots, y_p, z) \in T \times Y_1 \times \cdots \times Y_p \times Z \mid z \in \Psi(y_1, \ldots, y_p)\},$$

and remark that $(\overline{t}, \overline{y}_1, \ldots, \overline{y}_p, \overline{z}) \in \bigcap_{i=1}^{p+1} C_i$. Also, we consider the set-valued map $H : T \rightrightarrows Z$ given by

$$H(t) := \bigcup_{\substack{y_i \in F_i(t) \\ i=\overline{1,p}}} \Psi(y_1, \ldots, y_p). \tag{5.2}$$

If all the involved graphs are closed around the reference points, the alliedness property of $C_1, \ldots, C_{p+1}$ at $(\overline{t}, \overline{y}_1, \ldots, \overline{y}_p, \overline{z})$ reads as follows: for any sequences $(t_{in}, y_{in}) \xrightarrow{\operatorname{Gr} F_i} (\overline{t}, \overline{y}_i), i = \overline{1, p}$, $(b_{1n}, \ldots, b_{pn}, c_n) \xrightarrow{\operatorname{Gr} \Psi} (\overline{y}_1, \ldots, \overline{y}_p, \overline{z})$ and every $t_{in}^* \in \widehat{D}^* F_i(t_{in}, y_{in})(y_{in}^*)$, $i = \overline{1, p}$, $(b_{1n}^*, \ldots, b_{pn}^*) \in \widehat{D}^* \Psi(b_{1n}, \ldots, b_{pn}, c_n)(c_n^*)$, the relations $(t_{1n}^* + \cdots + t_{pn}^*) \to 0$, $\left(y_{in}^* + b_{in}^*\right) \to 0, i = \overline{1, p}$, $\left(c_n^*\right) \to 0$ imply

$$(t_{in}^*) \to 0, (y_{in}^*) \to 0, (b_{in}^*) \to 0, i = \overline{1, p}.$$

Theorem 5.1 *Suppose* $T, Y_1, \ldots, Y_p$ *and* Z *are Asplund spaces,* $F_i : T \rightrightarrows Y_i, i = \overline{1, p}$, $\Psi : Y_1 \times \cdots \times Y_p \rightrightarrows Z$ *are closed-graph set-valued maps, and* $(\overline{t}, \overline{y}_1, \ldots, \overline{y}_p, \overline{z}) \in T \times Y_1 \times \cdots \times Y_p \times Z$ *is such that* $\overline{z} \in \Psi(\overline{y}_1, \ldots, \overline{y}_p), \overline{y}_i \in F_i(\overline{t}), i = \overline{1, p}$. *Assume that the sets* $C_1, \ldots, C_{p+1}$ *defined by* (5.1) *are allied at* $(\overline{t}, \overline{y}_1, \ldots, \overline{y}_p, \overline{z})$ *and*

$$0 < c := \liminf_{\substack{\delta \downarrow 0,\ (u_i, v_i) \xrightarrow{\operatorname{Gr} F_i} (\overline{t}, \overline{y}_i),\ i = \overline{1,p} \\ (y_1, \ldots, y_p, w) \xrightarrow{\operatorname{Gr} \Psi} (\overline{y}_1, \ldots, \overline{y}_p, \overline{z})}} \left\{ \|t_1^* + \cdots + t_p^*\| \;\middle|\; \begin{array}{l} t_i^* \in \widehat{D}^* F_i(u_i, v_i)(y_i^*), i = \overline{1, p}, \\ (z_1^* + y_1^*, \ldots, z_p^* + y_p^*) \\ \quad \in \widehat{D}^* \Psi(y_1, \ldots, y_p, w)(w^*), \\ \|w^*\| = 1, \|z_i^*\| < \delta, i = \overline{1, p} \end{array} \right\}. \tag{5.3}$$

Then for every $a \in (0, c)$, H *given by* (5.2) *is* a*-open at* $(\overline{t}, \overline{z})$.

Since the proof of the above result is similar to the one of [10, Theorem 4.2], we leave it for the interested reader.

In the context of summation, i.e., all the spaces Y_i and Z coincide, and the set-valued map $\Psi : Y^p \to Y$ is the sum $\Psi(y_1, \ldots, y_p) := y_1 + \cdots + y_p$, the alliedness of the sets $C_1, \ldots, C_{p+1}$ defined by (5.1) translates into the alliedness of the sets

$$D_i := \{(t, y_1, \ldots, y_p) \in T \times Y^p \mid y_i \in F_i(t)\}, i = \overline{1, p}. \tag{5.4}$$

Indeed, in this context,

$$\widehat{D}^* \Psi(y_1, \ldots, y_p, w)(w^*) = \left(w^*, \ldots, w^*\right),$$

so if $(b_{1n}^*, \ldots, b_{pn}^*) \in \widehat{D}^* \Psi(b_{1n}, \ldots, b_{pn}, c_n)(c_n^*)$ and $\left(c_n^*\right) \to 0$, we have successively that $(b_{in}^*) \to 0, i = \overline{1, p}$, and then $(y_{in}^*) \to 0, i = \overline{1, p}$. So the alliedness property reduces to: for any sequences $(t_{in}, y_{in}) \xrightarrow{F_i} (\overline{t}, \overline{y}_i), i = \overline{1, p}$ and every $t_{in}^* \in \widehat{D}^* F_i(t_{in}, y_{in})(y_{in}^*), i = \overline{1, p}$, the relations $(t_{1n}^* + \cdots + t_{pn}^*) \to 0$, $\left(y_{in}^*\right) \to 0, i = \overline{1, p}$, imply $(t_{in}^*) \to 0, i = \overline{1, p}$. But this is exactly the alliedness of the sets $D_1, \ldots, D_p$ given by (5.4).

Also, the set-valued map $H : T \rightrightarrows Y$ becomes the sum set-valued map

$$H(t) = \sum_{i=1}^{p} F_i(t). \tag{5.5}$$

Consequently, the previous theorem becomes the following one (see, for the case $p = 2$, [11, Theorem 4.6]).

Theorem 5.2 *Suppose* T *and* Y *are Asplund spaces,* $F_i : T \rightrightarrows Y, i = \overline{1, p}$ *are closed-graph set-valued maps, and* $(\overline{t}, \overline{y}_1, \ldots, \overline{y}_p) \in T \times Y^p$ *is such that* $\overline{y}_i \in F_i(\overline{t}), i = \overline{1, p}$. *Assume that* *the sets* $D_1, \ldots, D_p$ *defined by* (5.4) *are allied at* $(\bar{t}, \bar{y}_1, \ldots, \bar{y}_p)$ *and*

$$0 < c := \liminf_{(u_i,v_i) \overset{\operatorname{Gr} Fi}{\to} (\bar{t}, \bar{y}_i),\ i=\overline{1,p},\ \delta \downarrow 0} \left\{ \|t_1^* + \cdots + t_p^*\| \;\middle|\; \begin{array}{l} t_i^* \in \widehat{D}^* F_i(u_i, v_i)(w^* - z_i^*), \\ \|w^*\| = 1, \|z_i^*\| < \delta, i = \overline{1,p} \end{array} \right\}. \quad (5.6)$$

Then for every $a \in (0, c)$, H *given by* (5.5) *is* a*-open at* $(\bar{t}, \bar{y}_1 + \cdots + \bar{y}_p)$.

Remark 5.3 The openness asserted in the previous two theorems can be extended to around-point linear openness, using the so-called composition stability, or sum stability of the involved set-valued maps, respectively. For more details, see [10, 11].

Let us formulate the result for the particular case when $F_1 := F$, $F_2 := K$, $F_3 := \Delta_\Omega$.

Corollary 5.4 *Let* X, Y *be Asplund spaces*, $\Omega \subseteq X$ *be a closed set*, $F : X \rightrightarrows Y$, $K : X \rightrightarrows Y$ *be closed-graph set-valued maps, and* $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$ *such that* $\bar{x} \in \Omega$, $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ *and* $(\bar{x}, \bar{z}) \in \operatorname{Gr} K$. *Suppose that the sets*

$$\begin{aligned} D_1 &:= \{(x, y, z, w) \in X \times Y^3 \mid y \in F(x)\}, \\ D_2 &:= \{(x, y, z, w) \in X \times Y^3 \mid z \in K(x)\}, \\ D_3 &:= \{(x, y, z, w) \in X \times Y^3 \mid w \in \Delta_\Omega(x)\} \end{aligned} \quad (5.7)$$

are allied at $(\bar{x}, \bar{y}, \bar{z}, 0)$, *and*

$$0 < c := \liminf_{\substack{\delta \downarrow 0,\ (x_1,y_1) \overset{\operatorname{Gr} F}{\to} (\bar{x},\bar{y}), \\ (x_2,y_2) \overset{\operatorname{Gr} K}{\to} (\bar{x},\bar{z}),\ x_3 \overset{\Omega}{\to} \bar{x}}} \left\{ \|x_1^* + x_2^* + x_3^*\| \;\middle|\; \begin{array}{l} x_1^* \in \widehat{D}^* F(x_1, y_1)(w^* - z_1^*), \\ x_2^* \in \widehat{D}^* K(x_2, y_2)(w^* - z_2^*), \\ x_3^* \in \widehat{N}(\Omega, x_3), \\ \|w^*\| = 1, \|z_i^*\| < \delta, i = 1, 2 \end{array} \right\}. \quad (5.8)$$

Then for every $a \in (0, c)$, H *given by*

$$H(x) := \begin{cases} F(x) + K(x) & \text{if } x \in \Omega \\ \emptyset & \text{otherwise} \end{cases} \quad (5.9)$$

is a*-open at* $(\bar{x}, \bar{y} + \bar{z})$.

An interesting observation is that Theorem 5.2 can furnish an openness result even in the case when the set-valued maps involved in summation are defined on different spaces, using a similar idea to the one of extending the type I nondominated solutions to the second type, as described next.

Suppose X, Y are normed vector spaces, and consider the closed set $\Omega \subseteq X$ and the set-valued maps $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Y$. Then we take $T := X \times Y$, and we denote by $F_1, F_2, F_3 : X \times Y \rightrightarrows Y$ the set-valued maps given by

$$\begin{aligned} F_1(x, y) &:= \begin{cases} \{y\} & \text{if } y \in F(x) \\ \emptyset & \text{otherwise,} \end{cases} \\ F_2(x, y) &:= G(y) \text{ for all } y \in Y, \\ F_3(x, y) &:= \Delta_\Omega(x) \text{ for all } x \in X. \end{aligned} \quad (5.10)$$

Then the set-valued map H defined by (5.5) becomes

$$H(x,y)=\begin{cases}\{y\}+G(y) & \text{if } y\in F(x), x\in\Omega\\ \emptyset & \text{otherwise.}\end{cases} \tag{5.11}$$

In this case, the Fréchet coderivatives of F_1, F_2, F_3 in the corresponding points from the graphs become

$$\begin{aligned}&\left(x^*,y^*\right)\in\widehat{D}^*F_1(x,y,y)(z^*)\Leftrightarrow x^*\in\widehat{D}^*F(x,y)(z^*-y^*),\\ &\left(x^*,y^*\right)\in\widehat{D}^*F_2(x,y,z)(z^*)\Leftrightarrow x^*=0, y^*\in\widehat{D}^*G(y,z)(z^*),\\ &\left(x^*,y^*\right)\in\widehat{D}^*F_3(x,y,0)(z^*)\Leftrightarrow x^*\in\widehat{N}(\Omega,x), y^*=0, z^*\in Y^*.\end{aligned} \tag{5.12}$$

Moreover, the sets D_i (for $i=1,2,3$) given by (5.4) are

$$\begin{aligned}E_1&=\{(x,y,z,w,v)\in X\times Y^4\mid z\in F_1(x,y)\}=\{(x,y,y,w,v)\in X\times Y^4\mid y\in F(x)\},\\ E_2&=\{(x,y,z,w,v)\in X\times Y^4\mid w\in F_2(x,y)\}\\ &=\{(x,y,z,w,v)\in X\times Y^4\mid w\in G(y)\},\\ E_3&=\{(x,y,z,w,v)\in X\times Y^4\mid v\in F_3(x,y)\}=\{(x,y,z,w,0)\in X\times Y^4\mid x\in\Omega\}\end{aligned} \tag{5.13}$$

and their alliedness takes a simpler form, as proven in the next result.

Proposition 5.5 *The sets E_1, E_2, E_3 are allied at $(\overline{x},\overline{y},\overline{y},\overline{z},0)$ if and only if the sets*

$$\begin{aligned}A_1&:=\{(x,y,z)\in X\times Y^2\mid y\in F(x)\},\\ A_2&:=\{(x,y,z)\in X\times Y^2\mid z\in G(y)\},\\ A_3&:=\{(x,y,z)\in X\times Y^2\mid x\in\Omega\}\end{aligned} \tag{5.14}$$

are allied at $(\overline{x},\overline{y},\overline{z})$.

Proof The alliedness of E_1, E_2, E_3 at $(\overline{x},\overline{y},\overline{y},\overline{z},0)$ means that for any $(x_{in}, y_{in}, z_{in}, w_{in}, v_{in})\xrightarrow{E_i}(\overline{x},\overline{y},\overline{y},\overline{z},0), i=1,2,3$ and every $\left(x_{in}^*, y_{in}^*, z_{in}^*, w_{in}^*, v_{in}^*\right)\in\widehat{N}(E_i, (x_{in}, y_{in}, y_{in}, w_{in}, v_{in}))$ the relation $(x_{1n}^*+x_{2n}^*+x_{3n}^*, y_{1n}^*+y_{2n}^*+y_{3n}^*, z_{1n}^*+z_{2n}^*+z_{3n}^*, w_{1n}^*+w_{2n}^*+w_{3n}^*, v_{1n}^*+v_{2n}^*+v_{3n}^*)\to(0,0,0,0,0)$ imply

$$(x_{in}^*)\to 0, (y_{in}^*)\to 0, (z_{in}^*)\to 0, (w_{in}^*)\to 0, (v_{in}^*)\to 0, i=1,2,3.$$

Using (5.12), we have $y_{1n}=z_{1n}$, $v_{3n}=0$, and $\left(x_{1n}^*, y_{1n}^*, z_{1n}^*, w_{1n}^*, v_{1n}^*\right)\in\widehat{N}(E_1,(x_{1n}, y_{1n}, y_{1n}, w_{1n}))$ if and only if $w_{1n}^*=v_{1n}^*=0$ and $\left(x_{1n}^*, y_{1n}^*\right)\in\widehat{D}^*F_1(x_{1n}, y_{1n}, y_{1n})\left(-z_{1n}^*\right)$, which means that $x_{1n}^*\in\widehat{D}^*F(x_{1n}, y_{1n})(-z_{1n}^*-y_{1n}^*)$. Moreover, $\left(x_{2n}^*, y_{2n}^*, z_{2n}^*, w_{2n}^*, v_{2n}^*\right)\in\widehat{N}(E_2,(x_{2n}, y_{2n}, z_{2n}, w_{2n}, v_{2n}))$ iff $z_{2n}^*=v_{2n}^*=0$, $\left(x_{2n}^*, y_{2n}^*\right)\in\widehat{D}^*F_2(x_{2n}, y_{2n}, z_{2n})\left(-w_{2n}^*\right)$, hence $x_{2n}^*=0$, $y_{2n}^*\in\widehat{D}^*G(y_{2n}, z_{2n})(-w_{2n}^*)$, and $(v_{3n}^*)\to 0$. Furthermore, $\left(x_{3n}^*, y_{3n}^*, z_{3n}^*, w_{3n}^*, v_{3n}^*\right)\in\widehat{N}(E_3,(x_{3n}, y_{3n}, z_{3n}, w_{3n}, 0))$ iff $x_{3n}^*\in\widehat{N}(\Omega, x_{3n})$, $y_{3n}^*=z_{3n}^*=w_{3n}^*=0$. It follows that $\left(z_{1n}^*\right)\to 0$, $\left(w_{2n}^*\right)\to 0$.

In conclusion, the alliedness of E_1, E_2, E_3 reduces to: for any $(x_{1n}, y_{1n})\xrightarrow{\operatorname{Gr}F}(\overline{x},\overline{y})$, $(y_{2n}, z_{2n})\xrightarrow{\operatorname{Gr}G}(\overline{y},\overline{z})$, and any $x_{1n}^*\in\widehat{D}^*F(x_{1n}, y_{1n})(-z_{1n}^*-y_{1n}^*)$, $y_{2n}^*\in\widehat{D}^*G(y_{2n}, z_{2n})(-z_{2n}^*)$,

$x_{3n}^* \in \widehat{N}(\Omega, x_{3n})$, given the relations $(x_{1n}^* + x_{3n}^*) \to 0$, $\left(y_{1n}^* + y_{2n}^*\right) \to 0$, $\left(z_{in}^*\right) \to 0$, $i = 1, 2, 3$ one must prove that $(x_{in}^*) \to 0$, $(y_{in}^*) \to 0$, $i = 1, 2, 3$ but this means the alliedness of the sets A_1, A_2, A_3 at $(\overline{x}, \overline{y}, \overline{z})$. □

Furthermore, in the case $\Omega := X$, the alliedness of the sets A_1, A_2, A_3 at $(\overline{x}, \overline{y}, \overline{z})$ trivially reduces to the alliedness of A_1, A_2 at $(\overline{x}, \overline{y}, \overline{z})$. We derive the next openness result.

Theorem 5.6 *Suppose X, Y are Asplund spaces, $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Y$ are closed-graph set-valued maps, and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ is such that $\overline{x} \in \Omega$, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $(\overline{y}, \overline{z}) \in \operatorname{Gr} G$. Assume that the sets A_1, A_2, A_3 defined by (5.14) are allied at $(\overline{x}, \overline{y}, \overline{z})$ and*

$$\begin{aligned} 0 < c \\ := \liminf_{\substack{\delta \downarrow 0,\ (x_1, y_1) \overset{\operatorname{Gr} F}{\to} (\overline{x}, \overline{y}), \\ (y_2, z_2) \overset{\operatorname{Gr} G}{\to} (\overline{y}, \overline{z}),\ x_3 \overset{\Omega}{\longrightarrow} \overline{x}}} \left\{ \left\| \left(x_1^* + x_3^*, y_1^* + y_2^*\right) \right\| \;\middle|\; \begin{array}{l} x_1^* \in \widehat{D}^* F(x_1, y_1)(w^* - y_1^* - z_1^*), \\ y_2^* \in \widehat{D}^* G(y_2, z_2)(w^* - z_2^*), \\ x_3^* \in \widehat{N}(\Omega, x_3), \\ \|w^*\| = 1, \|z_i^*\| < \delta, i = 1, 2 \end{array} \right\}. \end{aligned} \tag{5.15}$$

Then for every $a \in (0, c)$, H given by (5.11) is a-open at $(\overline{x}, \overline{y}, \overline{y} + \overline{z})$.

Again, in case $\Omega := X$, the openness result becomes the following one.

Corollary 5.7 *Suppose X, Y are Asplund spaces, $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Y$ are closed-graph set-valued maps, and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ is such that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $(\overline{y}, \overline{z}) \in \operatorname{Gr} G$. Assume that the sets A_1, A_2 defined by (5.14) are allied at $(\overline{x}, \overline{y}, \overline{z})$ and*

$$0 < c := \liminf_{\substack{(x_1, y_1) \overset{F}{\to} (\overline{x}, \overline{y}), \\ (y_2, z_2) \overset{G}{\to} (\overline{y}, \overline{z}),\ \delta \downarrow 0}} \left\{ \left\| \left(x_1^*, y_1^* + y_2^*\right) \right\| \;\middle|\; \begin{array}{l} x_1^* \in \widehat{D}^* F(x_1, y_1)(w^* - y_1^* - z_1^*), \\ y_2^* \in \widehat{D}^* G(y_2, z_2)(w^* - z_2^*), \\ \|w^*\| = 1, \|z_i^*\| < \delta, i = 1, 2 \end{array} \right\}. \tag{5.16}$$

Then for every $a \in (0, c)$, H given by (5.11) is a-open at $(\overline{x}, \overline{y}, \overline{y} + \overline{z})$.

5.2 Openness of a Special Sum

In the following, as explained at the beginning of the section, we explore another useful set-valued map. More exactly, for $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Y$, we consider the set-valued map $R : X \rightrightarrows Y$ given by

$$R(x) := \begin{cases} \{y + z \mid y \in F(x), z \in G(y)\} & \text{if } x \in \Omega \\ \emptyset & \text{otherwise.} \end{cases} \tag{5.17}$$

Note that the set-valued map R given by (5.17) is somehow similar to H given by (5.11), with the subtle and important difference that acts between different spaces.

Applying Ekeland Variational Principle, we show the following openness result concerning the set-valued map R.

Theorem 5.8 *Let X, Y be Asplund spaces, $\Omega \subseteq X$ be a closed set, $F : X \rightrightarrows Y$, $G : Y \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ such that $\overline{x} \in \Omega$, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $(\overline{y}, \overline{z}) \in \operatorname{Gr} G$. Suppose that the sets A_1, A_2, A_3 given by (5.14) are allied at*

$(\overline{x}, \overline{y}, \overline{z})$, *and there exist* $c, r > 0$ *such that for every* $(x_1, y_1) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, $(y_2, z_2) \in \operatorname{Gr} G \cap [B(\overline{y}, r) \times B(\overline{z}, r)]$, $x_3 \in \Omega \cap B(\overline{x}, r)$ *and every* $z^* \in S_{Y^*}$, $t^* \in 2cB_{Y^*}$, $y^* \in \widehat{D}^* G(y_2, z_2)(z^*)$, $x^* \in \widehat{D}^* F(x_1, y_1)(z^* + y^* + t^*)$, $u^* \in \widehat{N}(\Omega, x_3)$

$$c \left\| z^* + t^* \right\| \leq \left\| x^* + u^* \right\|. \tag{5.18}$$

Then for every $a \in (0, c)$, R *given by* (5.17) *is* a*-open at* $(\overline{x}, \overline{y} + \overline{z})$.

Proof Fix $a \in (0, c)$. Our aim is to prove that there exists $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$,

$$B(\overline{y} + \overline{z}, \rho a) \subseteq R\left(B\left(\overline{x}, \rho\right)\right). \tag{5.19}$$

Surely, we can find $b > 0$ and $\varepsilon > 0$ such that $b^{-1} a \varepsilon < r$ such that

$$(a+1)^{-1} a < b < b + \varepsilon < (c+1)^{-1} c. \tag{5.20}$$

Fix $\rho \in (0, \varepsilon)$ and take $w \in B(\overline{y} + \overline{z}, \rho a)$. Denote $A := A_1 \cap A_2 \cap A_3$. We endow the space $X \times Y \times Y$ with the sum norm and define the function

$$f : A \to \mathbb{R}, \quad f(x, y, z) := \|y + z - w\|.$$

Using the closedness of Ω, $\operatorname{Gr} F$ and $\operatorname{Gr} G$, it follows that the set A is closed. We apply the Ekeland Variational Principle for f and $(\overline{x}, \overline{y}, \overline{z}) \in \operatorname{dom} f$ to find $(x_b, y_b, z_b) \in A$ such that

$$\|y_b + z_b - w\| \leq \|\overline{y} + \overline{z} - w\| - b(\|\overline{x} - x_b\| + \|\overline{y} - y_b\| + \|\overline{z} - z_b\|) \tag{5.21}$$

and

$$\|y_b + z_b - w\| \leq \|y + z - w\| + b(\|x - x_b\| + \|y - y_b\| + \|z - z_b\|) \text{ for all } (x, y, z) \in A. \tag{5.22}$$

From (5.21), we have

$$\|\overline{x} - x_b\| + \|\overline{y} - y_b\| + \|\overline{z} - z_b\| \leq b^{-1} \|\overline{y} + \overline{z} - w\| < b^{-1} a \rho \leq b^{-1} a \varepsilon, \tag{5.23}$$

hence $(x_b, y_b, z_b) \in B\left(\overline{x}, r\right) \times B\left(\overline{y}, r\right) \times B\left(\overline{z}, r\right)$. If $w = y_b + z_b$, then

$$\begin{aligned} b \|\overline{x} - x_b\| &\leq \|\overline{y} + \overline{z} - w\| - b(\|\overline{y} - y_b\| + \|\overline{z} - z_b\|) \\ &\leq \|\overline{y} + \overline{z} - w\| - b \|(\overline{y} + \overline{z}) - (y_b + z_b)\| \\ &= (1 - b) \|\overline{y} + \overline{z} - w\| \\ &< (1 - b) a \rho < b \rho, \end{aligned}$$

hence $x_b \in B(\overline{x}, \rho)$, and since $x_b \in \Omega$, we obtain

$$w = y_b + z_b \in \{y_b + G\left(y_b\right) \mid y_b \in F\left(x_b\right)\} = R\left(x_b\right) \subseteq R(B(\overline{x}, \rho)),$$

which is exactly the conclusion (5.19).

We want to prove that $w = y_b + z_b$ is the only possible situation. For this, suppose that $w \neq y_b + z_b$ and consider the function

$$h : X \times Y \times Y \to \mathbb{R}, \; h(x, y, z) := \|y + z - w\| + b(\|x - x_b\| + \|y - y_b\| + \|z - z_b\|).$$

From the second relation of the Ekeland Variational Principle (5.22), we have that the point (x_b, y_b, z_b) is a minimum point for h on the set A, or, equivalently, (x_b, y_b, z_b) is a global minimum point for the function $h + \delta_A$. Applying the generalized Fermat rule, we have that

$$(0,0,0) \in \widehat{\partial}(h(\cdot,\cdot,\cdot,\cdot) + \delta_A(\cdot,\cdot,\cdot,\cdot))(x_b, y_b, z_b).$$

Using the fact that h is Lipschitz and δ_A is lower semicontinuous, we can apply the fuzzy calculus rule for the Fréchet subdifferential (see [33, Sect. 2.4]). Since, from (5.23), we have that $(x_b, y_b, z_b) \in B(\overline{x}, b^{-1}a\varepsilon) \times B(\overline{y}, b^{-1}a\varepsilon) \times B(\overline{z}, b^{-1}a\varepsilon)$, we can choose $\gamma \in (0, 2^{-1}\rho)$ such that

$$\begin{aligned}
\overline{B}(x_b, 2\gamma) &\subseteq B(\overline{x}, b^{-1}a\varepsilon),\\
\overline{B}(y_b, 2\gamma) &\subseteq B(\overline{y}, b^{-1}a\varepsilon),\\
\overline{B}(z_b, 2\gamma) &\subseteq B(\overline{z}, b^{-1}a\varepsilon),\\
w &\notin \overline{B}(y_b + z_b, 4\gamma)
\end{aligned}$$

and obtain that there exist

$$\begin{aligned}
(x_1, y_1, z_1) &\in \overline{B}(x_b, \gamma) \times \overline{B}(y_b, \gamma) \times \overline{B}(z_b, \gamma),\\
(x_2, y_2, z_2) &\in A \cap \left[\overline{B}(x_b, \gamma) \times \overline{B}(y_b, \gamma) \times \overline{B}(z_b, \gamma)\right]
\end{aligned}$$

such that

$$(0,0,0) \in \widehat{\partial} h(x_1, y_1, z_1) + \widehat{\partial}\delta_A(x_2, y_2, z_2) + \gamma(\overline{B}_{X^*} \times \overline{B}_{Y^*} \times \overline{B}_{Y^*}). \tag{5.24}$$

Observing that h is the sum of four convex functions, Lipschitz on $X \times Y \times Y$, $\widehat{\partial} h$ coincides with the sum of the Fenchel subdifferentials. Also, defining the linear operator $\Phi : Y \times Y \to Y$ by $\Phi(y, z) := y + z$, we obtain that

$$\partial \left\| \cdot + \cdot - w \right\| (y^1, z^1) = \Phi^*(\partial \left\| \cdot - w \right\| (y^1 + z^1)),$$

where $\Phi^* : Y^* \to Y^* \times Y^*$ denotes the adjoint of Φ and ∂ the Fenchel subdifferential. Remarking also that $w \neq y_1 + z_1 \in \overline{B}(y_b + z_b, 2\gamma)$ and using that $\Phi^*(y^*) = (y^*, y^*)$ for every $y^* \in Y^*$, we obtain

$$\partial \left\| \cdot + \cdot - w \right\| (y_1, z_1) = \{(y^*, y^*) \mid y^* \in S_{Y^*}, y^*(y_1 + z_1 - w) = \|y_1 + z_1 - w\|\}.$$

Consequently, we have from (5.24) that

$$\begin{aligned}
(0,0,0) \in\ &\{0\} \times \left\{(y^*, y^*) \mid y^* \in S_{Y^*}\right\}\\
&+ b\overline{B}_{X^*} \times \{0\} \times \{0\} + \{0\} \times b\overline{B}_{X^*} \times \{0\} + \{0\} \times \{0\} \times b\overline{B}_{Y^*}\\
&+ \widehat{N}(A_1 \cap A_2 \cap A_3, (x_2, y_2, z_2)) + \gamma(\overline{B}_{X^*} \times \overline{B}_{Y^*} \times \overline{B}_{Y^*})\\
=\ &\{0\} \times \left\{(y^*, y^*) \mid y^* \in S_{Y^*}\right\}\\
&+ \widehat{N}(A_1 \cap A_2 \cap A_3, (x_2, y_2, z_2)) + (b + \gamma)(\overline{B}_{X^*} \times \overline{B}_{Y^*} \times \overline{B}_{Y^*}).
\end{aligned}$$

Now, use the alliedness of A_1, A_2, A_3 at $(\overline{x}, \overline{y}, \overline{z})$ to get that

$$\begin{aligned}\widehat{N}(A_1 \cap A_2 \cap A_3, (x_2, y_2, z_2)) \subseteq \widehat{N}(A_1, (x_3, y_3, z_3)) + \widehat{N}(A_2, (x_4, y_4, z_4)) \\ + \widehat{N}(A_3, (x_5, y_5, z_5)) + \gamma(\overline{B}_{X^*} \times \overline{B}_{Y^*} \times \overline{B}_{Y^*}),\end{aligned}$$

where

$$\begin{aligned}(x_3, y_3, z_3) &\in [\overline{B}(x_2, \gamma) \times \overline{B}(y_2, \gamma) \times \overline{B}(z_2, \gamma)] \cap A_1 \\ &\subseteq \left[\overline{B}(x_b, 2\gamma) \times \overline{B}(y_b, 2\gamma) \times \overline{B}(z_b, 2\gamma)\right] \cap A_1 \\ &\subseteq \left\{\left[\overline{B}(\overline{x}, r) \times \overline{B}(\overline{y}, r)\right] \cap \operatorname{Gr} F\right\} \times Y, \\ (x_4, y_4, z_4) &\in [\overline{B}(x_2, \gamma) \times \overline{B}(y_2, \gamma) \times \overline{B}(z_2, \gamma)] \cap A_2 \\ &\subseteq X \times \left\{\left[\overline{B}(\overline{y}, r) \times \overline{B}(\overline{z}, r)\right] \cap \operatorname{Gr} G\right\}, \\ (x_5, y_5, z_5) &\in [\overline{B}(x_2, \gamma) \times \overline{B}(y_2, \gamma) \times \overline{B}(z_2, \gamma)] \cap A_3 \\ &\subseteq \left[\overline{B}(\overline{x}, r) \cap \Omega\right] \times Y \times Y.\end{aligned} \tag{5.25}$$

Hence,

$$\begin{aligned}(0,0,0) \in \{0\} \times \left\{\left(y^*, y^*\right) \mid y^* \in S_{Y^*}\right\} + \widehat{N}(A_1, (x_3, y_3, z_3)) \\ + \widehat{N}(A_2, (x_4, y_4, z_4)) + \widehat{N}(A_3, (x_5, y_5, z_5)) + (b + 2\gamma)\,(\overline{B}_{X^*} \times \overline{B}_{Y^*} \times \overline{B}_{Y^*}).\end{aligned}$$

In conclusion, there exist

$$\begin{aligned}&y_0^* \in S_{Y^*}, \\ &(x_3^*, y_3^*, 0) \in \widehat{N}(A_1, (x_3, y_3, z_3)) \Leftrightarrow x_3^* \in \widehat{D}^* F(x_3, y_3)(-y_3^*), \\ &(0, y_4^*, z_4^*) \in \widehat{N}(A_2, (x_4, y_4, z_4)) \Leftrightarrow y_4^* \in \widehat{D}^* G(y_4, z_4)(-z_4^*), \\ &(x_5^*, 0, 0) \in \widehat{N}(A_3, (x_5, y_5, z_5)) \Leftrightarrow x_5^* \in \widehat{N}(\Omega, x_5) \\ &(x_6^*, y_6^*, z_6^*) \in \overline{B}_{X^*} \times \overline{B}_{Y^*} \times \overline{B}_{Y^*}\end{aligned}$$

such that

$$\begin{aligned}x_3^* + x_5^* + (b + 2\gamma)\, x_6^* &= 0, \\ y_0^* + y_3^* + y_4^* + (b + 2\gamma)\, y_6^* &= 0, \\ y_0^* + z_4^* + (b + 2\gamma)\, z_6^* &= 0.\end{aligned}$$

Observe that

$$\left\| y_0^* + (b + 2\gamma) z_6^* \right\| \geq \left\| y_0^* \right\| - (b + 2\gamma)\left\| z_6^* \right\| \geq 1 - (b + 2\gamma) > 0,$$

hence by denoting

$$\begin{aligned}v^* &:= y_0^* + (b + 2\gamma) z_6^*, \\ z^* &:= \left\| v^* \right\|^{-1} v^*,\end{aligned}$$

$$y^* := \left\|v^*\right\|^{-1} y_4^*,$$
$$t^* := \left\|v^*\right\|^{-1} (b+2\gamma)\left(y_6^* - z_6^*\right),$$
$$x^* := \left\|v^*\right\|^{-1} x_3^*,$$
$$u^* := \left\|v^*\right\|^{-1} x_5^*,$$

we have

$$z^* \in S_{Y^*},$$
$$x^* \in \widehat{D}^* F(x_3, y_3)\left(z^* + y^* + t^*\right)$$
$$y^* \in \widehat{D}^* G(y_4, z_4)\left(z^*\right),$$
$$u^* \in \widehat{N}(\Omega, x_5),$$
$$\left\|t^*\right\| \le \frac{(b+2\gamma)}{1-(b+2\gamma)}\left\|y_6^* - z_6^*\right\| < 2c.$$

Using now (5.25) and the assumption (5.18), we obtain

$$\left\| -\left\|v^*\right\|^{-1}(b+2\gamma)x_6^*\right\| = \left\|x^*+u^*\right\| \ge c\left\|z^*+t^*\right\|$$
$$= c\left\|\left\|v^*\right\|^{-1}\left(v^* + (b+2\gamma)\left(y_6^* - z_6^*\right)\right)\right\|,$$
$$b+\varepsilon > b+2\gamma \ge \left\|-(b+2\gamma)x_6^*\right\| \ge c\left\|v^* + (b+2\gamma)\left(y_6^* - z_6^*\right)\right\|$$
$$\ge c\left(1-(b+2\gamma)\right) > c\left(1-(b+\varepsilon)\right),$$

in contradiction with (5.20). □

Another interesting aspect is that the set-valued map R defined by (5.17), via reduction given by Remark 3.8, has a form very close to the sum of F, K and Δ_Ω. More precisely, given $F, K : X \rightrightarrows Y$, without supposing K to be cone-valued, if one constructs the set-valued maps $\overline{F} : X \rightrightarrows X \times Y$ and $\overline{Q} : X \times Y \rightrightarrows X \times Y$ by relations (3.10) and (3.11), then the associated $\overline{R} : X \rightrightarrows X \times Y$ by (5.17) is

$$\overline{R}(x) = X \times (F + K + \Delta_\Omega)(x), \text{ for all } x \in X. \tag{5.26}$$

In this way, Theorem 5.8 gives yet another way to obtain sufficient conditions for the linear openness of the sum.

Corollary 5.9 *Let X, Y be Asplund spaces, $\Omega \subseteq X$ be a closed set, $F : X \rightrightarrows Y$, $K : X \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}, \overline{z}) \in X \times Y \times Y$ such that $\overline{x} \in \Omega$, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ and $(\overline{x}, \overline{z}) \in \operatorname{Gr} K$. Suppose that the sets D_1, D_2, D_3 given by (5.7) are allied at $(\overline{x}, \overline{y}, \overline{z}, 0)$, and there exist $c, r > 0$ such that for every $(x_1, y_1) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, $(x_2, y_2) \in \operatorname{Gr} K \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, $x_3 \in \Omega \cap B(\overline{x}, r)$ and every $z^* \in S_{Y^*}$, $(t^*, v^*) \in 2cB_{X^* \times Y^*}$, $x_1^* \in \widehat{D}^* F(x_1, y_1)(z^* + v^*)$, $x_2^* \in \widehat{D}^* K(x_2, y_2)(z^*)$, $x_3^* \in \widehat{N}(\Omega, x_3)$*

$$c\left\|\left(t^*, z^* + v^*\right)\right\| \le \left\|x_1^* + x_2^* + x_3^* + t^*\right\|.$$

Then for every $a \in (0, c)$, H given by (5.9) is a-open at $(\overline{x}, \overline{y} + \overline{z})$.

Proof. First of all, observe that the sets A_1, A_2, A_3 given by (5.14), corresponding to $\overline{F}$, $\overline{Q}$ and Ω are

$$\begin{aligned} B_1 &:= \{(x, u, v, p, q) \in X \times (X \times Y)^2 \mid u = x, y \in F(x)\} \\ B_2 &:= \{(x, u, v, p, q) \in X \times (X \times Y)^2 \mid q \in K(u)\} \\ B_3 &:= \{(x, u, v, p, q) \in X \times (X \times Y)^2 \mid x \in \Omega\}. \end{aligned} \tag{5.27}$$

Their alliedness means that for arbitrary $(x_{in}, u_{in}, v_{in}, p_{in}, q_{in}) \xrightarrow{B_i} (\overline{x}, \overline{x}, \overline{y}, \overline{z}, 0)$ and

$$\left(x_{in}^*, u_{in}^*, v_{in}^*, p_{in}^*, q_{in}^*\right) \in \widehat{N}\left(B_i, (x_{in}, u_{in}, v_{in}, p_{in}, q_{in})\right), i = 1, 2, 3$$

such that

$$\begin{aligned} &(x_{1n}^* + x_{2n}^* + x_{3n}^*, u_{1n}^* + u_{2n}^* + u_{3n}^*, v_{1n}^* + v_{2n}^* + v_{3n}^*, p_{1n}^* + p_{2n}^* + p_{3n}^*, q_{1n}^* + q_{2n}^* + q_{3n}^*) \\ &\quad \to (0, 0, 0, 0, 0), \end{aligned}$$

we have $\left(x_{in}^*, u_{in}^*, v_{in}^*, p_{in}^*, q_{in}^*\right) \to (0, 0, 0, 0, 0)$, $i = 1, 2, 3$.

According to Lemma 4.9 and usual calculus rules, one has that $p_{1n}^* = 0$, $q_{1n}^* = 0$, $x_{2n}^* = 0$, $v_{2n}^* = 0$, $p_{2n}^* = 0$, $u_{3n}^* = 0$, $v_{3n}^* = 0$, $p_{3n}^* = 0$, $q_{3n}^* = 0$, and $x_{1n}^* + u_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})(-v_{1n}^*)$, $u_{2n}^* \in \widehat{D}^* K(u_{2n}, q_{2n})(-q_{2n}^*)$, $x_{3n}^* \in \widehat{N}(\Omega, x_{3n})$. So, the alliedness of B_1, B_2, B_3 reduces to

$$(x_{1n}^* + x_{3n}^*, u_{1n}^* + u_{2n}^*) \to (0, 0) \Rightarrow x_{1n}^*, x_{3n}^*, u_{1n}^*, u_{2n}^* \to 0.$$

Under the assumption that the sets D_1, D_2, D_3 given by (5.7) are allied at $(\overline{x}, \overline{y}, \overline{z}, 0)$, by denoting $\overline{x}_{1n}^* := x_{1n}^* + u_{1n}^*$, $\overline{x}_{2n}^* := u_{2n}^*$, $\overline{x}_{3n}^* := x_{3n}^*$, since

$$\overline{x}_{1n}^* + \overline{x}_{2n}^* + \overline{x}_{3n}^* = x_{1n}^* + u_{1n}^* + u_{2n}^* + x_{3n}^* \to 0,$$

it follows that $x_{1n}^* + u_{1n}^* \to 0$, $u_{2n}^* \to 0$, $x_{3n}^* \to 0$. Using also $u_{1n}^* + u_{2n}^* \to 0$, we have $u_{1n}^* \to 0$ and $x_{1n}^* \to 0$.

The sufficient conditions from Theorem 5.8, adapted for $\overline{F}$ and $\overline{Q}$ are: there exist $c, r > 0$ such that for every $(x_1, x_1, y_1) \in \operatorname{Gr} \overline{F} \cap [B(\overline{x}, r) \times B(\overline{x}, r) \times B(\overline{y}, r)]$, $(u_2, v_2, x_2, y_2) \in \operatorname{Gr} \overline{Q} \cap [B(\overline{x}, r) \times B(\overline{y}, r) \times B(\overline{x}, r) \times B(\overline{y}, r)]$, $x_3 \in \Omega \cap B(\overline{x}, r)$ and every $(u^*, z^*) \in S_{X^* \times Y^*}$, $(t^*, v^*) \in 2cB_{X^* \times Y^*}$, $\left(x_2^*, y_2^*\right) \in \widehat{D}^* \overline{Q}(u_2, v_2, x_2, y_2)(u^*, z^*)$, $x_1^* \in \widehat{D}^* \overline{F}(x_1, x_1, y_1)((u^*, z^*) + \left(x_2^*, y_2^*\right) + (t^*, v^*))$, $x_3^* \in \widehat{N}(\Omega, x_3)$

$$c \left\| \left(u^*, z^*\right) + \left(t^*, v^*\right) \right\| \le \left\| x_1^* + x_3^* \right\|.$$

Using again Lemma 4.9, and also noting that $u^* = 0$, $y_2^* = 0$, this reduces to: there exist $c, r > 0$ such that for every $(x_1, y_1) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, $(u_2, y_2) \in \operatorname{Gr} K \cap [B(\overline{x}, r) \times B(\overline{y}, r)]$, $x_3 \in \Omega \cap B(\overline{x}, r)$ and every $z^* \in S_{Y^*}$, $(t^*, v^*) \in 2cB_{X^* \times Y^*}$, $x_2^* \in \widehat{D}^* K(u_2, y_2)(z^*)$, $x_1^* - x_2^* - t^* \in \widehat{D}^* F(x_1, y_1)(z^* + v^*)$, $x_3^* \in \widehat{N}(\Omega, x_3)$

$$c \left\| \left(t^*, z^* + v^*\right) \right\| \le \left\| x_1^* + x_3^* \right\|.$$

The obtained form is equivalent to the assumption, hence by Theorem 5.8, the set-valued map $\overline{R}$ given by (5.26) is a-open at $(\overline{x}, \overline{y} + \overline{z})$, which is equivalent to the conclusion. □

Remark 5.10 Observe that the sufficient conditions of Corollary 5.9 are slightly different than those from Corollary 5.4.

6 Optimality Conditions Using the Incompatibility Openness-Nondomination

6.1 First Strategy

The first method is to apply the incompatibility between openness and nondomination, in both cases, and then to apply Corollary 5.4 for type I and Theorem 5.8 for type II.

6.1.1 Nondominated Points of Type I

The next result asserts the incompatibility between openness of a sum of set-valued maps and nondomination of type I (Definition 3.2 (I)). Some results of this type are given in [9, 11], the novelty here being the fact that the third set-valued map is also involved.

Lemma 6.1 *Suppose that exists a neighborhood U of $\overline{x}$ such that $\bigcap_{x\in(\Omega\cap U)\cap \operatorname{Dom} F} (K(x)\setminus (-K(x))) \neq \emptyset$. If $(\overline{x},\overline{y})$ is a local nondominated point of type I for F with respect to K on Ω, then the set-valued map $F+K+\Delta_\Omega$ is not open at $(\overline{x},\overline{y})$.*

Proof We suppose that for every $x\in\Omega\cap U$,

$$(F(x)-\overline{y})\cap(-K(x))\subseteq K(x).$$

This is equivalent to the fact that for every $x\in U$,

$$(F(x)+K(x)+\Delta_\Omega(x)-\overline{y})\cap(-K(x))\subseteq K(x).$$

Suppose, by contradiction, that $F+K+\Delta_\Omega$ is open at $(\overline{x},\overline{y})$. Then, for the neighborhood U chosen before, there is an open set V such that $\overline{y}\in V\subseteq(F+K+\Delta_\Omega)(U)$. Choose $y\in V$. Then there is $u\in U$ such that $y\in(F+K+\Delta_\Omega)(u)$, hence $u\in\Omega$ and $u\in\operatorname{Dom} F$, otherwise the right-hand side set would be empty, and

$$y-\overline{y}\in(F(u)+K(u)+\Delta_\Omega(u)-\overline{y})\subseteq K(u)\cup(Y\setminus -K(u))$$
$$\subseteq K(u)\cup\left(Y\setminus\bigcap_{x\in(\Omega\cap U)\cap\operatorname{Dom} F}-K(x)\right).$$

But this means that $V-\overline{y}\subseteq K(u)\cup\left(Y\setminus\bigcap_{x\in(\Omega\cap U)\cap\operatorname{Dom} F}-K(x)\right)$, and since the left-hand side set is absorbing, and right-hand side set is a cone, we deduce that

$$Y=K(u)\cup\left(Y\setminus\bigcap_{x\in(\Omega\cap U)\cap\operatorname{Dom} F}-K(x)\right),$$

which implies $\bigcap_{x\in(\Omega\cap U)\cap\operatorname{Dom} F}-K(x)\subseteq K(u)$. Take $k\in\bigcap_{x\in(\Omega\cap U)\cap\operatorname{Dom} F}(K(x)\setminus(-K(x)))$. Then $-k\in\bigcap_{x\in(\Omega\cap U)\cap\operatorname{Dom} F}-K(x)$ and $-k\notin K(u)$, and this is a contradiction. □

The next, novel result is crucial in the sequel, showing that the alliedness property required in the openness theorem (that is, Theorem 5.2) is satisfied under the assumptions of Theorem 4.2.

Lemma 6.2 *Let X, Y be Asplund spaces, $\Omega \subseteq X$ be a closed set, $F, K : X \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}, 0) \in X \times Y \times Y$ such that $\overline{x} \in \Omega$, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $(\overline{x}, 0) \in \operatorname{Gr} K$. Moreover, suppose that assumptions (ii) and (iii) in Theorem 4.2 hold. Then the sets D_1, D_2, D_3 given by (5.7) are allied at $(\overline{x}, \overline{y}, 0, 0)$.*

Proof Consider arbitrary $(x_{in}, y_{in}, z_{in}, w_{in}) \xrightarrow{D_i} (\overline{x}, \overline{y}, 0, 0)$ and

$$\left(x_{in}^*, y_{in}^*, z_{in}^*, w_{in}^*\right) \in \widehat{N}\left(D_i, (x_{in}, y_{in}, z_{in}, w_{in})\right), i = 1, 2, 3$$

such that

$$(x_{1n}^* + x_{2n}^* + x_{3n}^*, y_{1n}^* + y_{2n}^* + y_{3n}^*, z_{1n}^* + z_{2n}^* + z_{3n}^*, w_{1n}^* + w_{2n}^* + w_{3n}^*) \to (0, 0, 0, 0)\,. \tag{6.1}$$

Remark that

$$\begin{aligned}
&\left(x_{1n}^*, y_{1n}^*, z_{1n}^*, w_{1n}^*\right) \in \widehat{N}\left(D_1, (x_{1n}, y_{1n}, z_{1n}, w_{1n})\right)\\
&\quad \Leftrightarrow x_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})(-y_{1n}^*), z_{1n}^* = w_{1n}^* = 0,\\
&\left(x_{2n}^*, y_{2n}^*, z_{2n}^*, w_{2n}^*\right) \in \widehat{N}\left(D_2, (x_{2n}, y_{2n}, z_{2n}, w_{2n})\right)\\
&\quad \Leftrightarrow x_{2n}^* \in \widehat{D}^* K(x_{2n}, z_{2n})(-z_{2n}^*), y_{2n}^* = w_{2n}^* = 0,\\
&\left(x_{3n}^*, y_{3n}^*, z_{3n}^*, w_{3n}^*\right) \in \widehat{N}\left(D_3, (x_{3n}, y_{3n}, z_{3n}, w_{3n})\right)\\
&\quad \Leftrightarrow x_{3n}^* \in \widehat{N}\left(\Omega, x_{3n}\right), y_{3n}^* = z_{3n}^* = 0, w_{3n}^* \in Y^*.
\end{aligned} \tag{6.2}$$

By (6.1) and (6.2), it follows immediately that $y_{in}^* \to 0$, $z_{in}^* \to 0$, $w_{in}^* \to 0$, $i = 1, 2, 3$.

Using condition (4.1) and the fact that K is SNC at $(\overline{x}, 0)$, we know that K has the Aubin property around $(\overline{x}, 0)$, hence, since $z_{2n}^* \to 0$, we deduce that $x_{2n}^* \to 0$. So, we know that $x_{1n}^* + x_{3n}^* \to 0$, and it remains to prove that $x_{1n}^* \to 0$ and $x_{3n}^* \to 0$.

If one of the sequences $\left(x_{1n}^*\right)$, $\left(x_{3n}^*\right)$ is bounded, since $x_{1n}^* + x_{3n}^* \to 0$, we obtain that the other one is bounded, and because X is Asplund, $\left(x_{1n}^*\right)$, $\left(x_{3n}^*\right)$ admit some subsequences, denoted the same, weak* convergent to x_1^*, x_3^*, respectively. We obtain $x_1^* \in D_M^* F(\overline{x}, \overline{y})(0)$, $x_3^* \in N(\Omega, \overline{x})$, $x_1^* + x_3^* = 0$, hence by (4.2), $x_1^* = x_3^* = 0$. Now, if Ω is SNC at $\overline{x}$, from $x_{3n}^* \xrightarrow{*} 0$ we get $x_{3n}^* \to 0$, so $x_{1n}^* \to 0$, and similarly, if F is PSNC at $(\overline{x}, \overline{y})$, from $x_{1n}^* \xrightarrow{*} 0$ and $y_{1n}^* \to 0$ we get $x_{1n}^* \to 0$, hence also $x_{3n}^* \to 0$.

Suppose, by contradiction, that both sequences $\left(x_{1n}^*\right)$, $\left(x_{3n}^*\right)$ are unbounded, and hence, on a subsequence denote the same, $r_n := \max\left\{\left\|x_{1n}^*\right\|, \left\|x_{3n}^*\right\|\right\} \to \infty$. Denoting

$$u_{1n}^* := \frac{x_{1n}^*}{r_n}, u_{3n}^* := \frac{x_{3n}^*}{r_n}, v_{1n}^* := \frac{y_{1n}^*}{r_n},$$

the sequences $\left(u_{1n}^*\right)$, $\left(u_{3n}^*\right)$ are bounded, $v_{1n}^* \to 0$, $u_{1n}^* + u_{3n}^* \to 0$, and $u_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})(v_{1n}^*)$, $u_{3n}^* \in \widehat{N}(\Omega, x_{3n})$, so this situation reduces to the case previously analyzed, hence $u_{1n}^* \to 0$, $u_{3n}^* \to 0$, contradicting the fact that $\max\left\{\left\|u_{1n}^*\right\|, \left\|u_{3n}^*\right\|\right\} = 1$. □

Remark 6.3 The end of the proof is similar to the proof of [32, Proposition 16].

We are ready to provide a third proof for Theorem 4.2. We remark that, instead of alliedness, one may use, at least in some instances, some weaker variants of relations between the involved sets, as done, for instance, in [32], where the concept of synergetic collection of sets is used. We leave the details for the interested reader.

Third proof of Theorem 4.2 By Lemma 6.1, we know that $F + K + \Delta_\Omega$ is not open at $(\overline{x}, \overline{y})$, so it is not linearly open at $(\overline{x}, \overline{y})$. By Lemma 6.2, in the assumptions of Theorem 4.2, we have that the sets given by (5.7), corresponding to the three set-valued maps $F_1 := F$, $F_2 := K$, $F_3 := \Delta_\Omega$, are allied at $(\overline{x}, \overline{y}, 0, 0)$. Hence, by Corollary 5.4, relation (5.8) cannot hold for the set-valued maps mentioned before. Hence, for any $n \in \mathbb{N} \setminus \{0\}$, we find $(x_{1n}, y_{1n}) \overset{\operatorname{Gr} F}{\to} (\overline{x}, \overline{y})$, $(x_{2n}, y_{2n}) \overset{\operatorname{Gr} K}{\to} (\overline{x}, 0)$, $x_{3n} \overset{\Omega}{\to} \overline{x}$, and also $w_n^* \in S_{Y^*}$, $y_{1n}^*, y_{2n}^*, y_{3n}^* \in n^{-1}\overline{B}_{Y^*}$, $x_{1n}^* \in \widehat{D}^* F(x_{1n}, y_{1n})(w_n^* - y_{1n}^*)$, $x_{2n}^* \in \widehat{D}^* K(x_{2n}, y_{2n})(w_n^* - y_{2n}^*)$, $x_{3n}^* \in \widehat{N}(\Omega, x_{3n})$ such that

$$x_{1n}^* + x_{2n}^* + x_{3n}^* \to 0.$$

As above, using the assumptions made, we have that K has the Aubin property around $(\overline{x}, 0)$, and since $(w_n^* - y_{2n}^*)$ is bounded, we deduce that (x_{2n}^*) is bounded.

By contradiction, suppose that both sequences (x_{1n}^*), (x_{3n}^*) are unbounded. Then, for every n, there exists $k_n \in \mathbb{N}$ sufficiently large such that

$$n < \min \left\{ \left\| x_{1k_n}^* \right\|, \left\| x_{3k_n}^* \right\| \right\}. \tag{6.3}$$

For simplicity, we denote the sequences $(x_{1k_n}^*)$, $(x_{3k_n}^*)$ by (x_{1n}^*), (x_{3n}^*), respectively. We observe now that

$$n^{-1}\left(w_n^* - y_{1n}^*\right) \to 0, n^{-1}\left(w_n^* - y_{2n}^*\right) \to 0, n^{-1}\left(w_n^* - y_{3n}^*\right) \to 0,$$

and using the positive homogeneity of the Fréchet coderivatives, we have that

$$\begin{aligned} n^{-1}x_{1n}^* &\in \widehat{D}^* F(x_{1n}, y_{1n})\left(n^{-1}\left(w_n^* - y_{1n}^*\right)\right), \\ n^{-1}x_{2n}^* &\in \widehat{D}^* K(x_{2n}, y_{2n})\left(n^{-1}\left(w_n^* - y_{2n}^*\right)\right), \\ n^{-1}x_{3n}^* &\in \widehat{D}^* \Delta_\Omega(x_{3n}, 0)\left(n^{-1}\left(w_n^* - y_{3n}^*\right)\right), \end{aligned}$$

and also

$$n^{-1}x_{1n}^* + n^{-1}x_{2n}^* + n^{-1}x_{3n}^* \to 0.$$

By the alliedness of the sets (5.7), we get $n^{-1}x_{1n}^* \to 0$, $n^{-1}x_{2n}^* \to 0$ and $n^{-1}x_{3n}^* \to 0$, and this contradicts relation (6.3).

As a consequence, at least one sequence from (x_{1n}^*), (x_{3n}^*) is bounded, and since (x_{2n}^*) is bounded and $x_{1n}^* + x_{2n}^* + x_{3n}^* \to 0$, we obtain that all three sequences are bounded.

Because X, Y are Asplund spaces, we deduce that (x_{1n}^*), (x_{2n}^*), (x_{3n}^*), $(w_n^* - y_{1n}^*)$, $(w_n^* - y_{2n}^*)$ are weak* convergent to some elements $x_1^*, x_2^*, x_3^*, y_1^*, y_2^*$, respectively. Because (y_{1n}^*), (y_{2n}^*) strongly converge to 0, it follows $y_1^* = y_2^* = y^*$. We get

$$\begin{aligned} &x_1^* \in D^* F\left(\overline{x}, \overline{y}\right)\left(y^*\right), x_2^* \in D^* K\left(\overline{x}, \overline{y}\right)\left(y^*\right), x_3^* \in N\left(\Omega, \overline{x}\right), \\ &x_1^* + x_2^* + x_3^* = 0, \end{aligned}$$

so the conclusion of Theorem 4.2 follows if we show that $y^* \in K\left(\overline{x}\right)^+ \setminus \{0\}$.

The fact that $y^* \in K(\overline{x})^+$ follows, as above, from Lemma 4.1.

It remains to prove that $y^* \neq 0$. Suppose, by contradiction, that $y^* = 0$. By (4.1) it follows $x_2^* = 0$, and, repeating the reasoning from the proof of Theorem 4.2, $\left(x_{2n}^*, w_n^* - y_{2n}^*\right) \to (0,0)$, which means that $w_n^* \to 0$, but this is impossible since $\left\|w_n^*\right\| = 1$ for any n. □

6.1.2 Nondominated Points of Type II

Now, we prove also an incompatibility result in the framework of nondominated points of type II, for a set-valued map of the form given by (5.17).

Lemma 6.4 *Suppose that* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *is a local nondominated point of type II for* F *with respect to* Q *on* Ω, *and there exists a neighborhood* V *of* $\overline{x}$ *such that* $\bigcap\limits_{y \in F(V\cap\Omega)} (Q(y) \setminus (-Q(y))) \neq \emptyset$. *Then the set-valued map* $R : X \rightrightarrows Y$ *given by* (6.4) *for* $G := Q$

$$R(x) = \begin{cases} \{y + z \mid y \in F(x), z \in Q(y)\} & \text{if } x \in \Omega \\ \emptyset & \text{otherwise.} \end{cases} \tag{6.4}$$

is not open at $(\overline{x}, \overline{y})$.

Proof We only give a sketch of the proof. Suppose, by contradiction, that for $\varepsilon > 0$ such that $\bigcap\limits_{y \in F(B(\overline{x},\varepsilon)\cap\Omega)} (Q(y) \setminus (-Q(y))) \neq \emptyset$ and (3.4) holds for $U := B(\overline{x}, \varepsilon)$, there exists $\delta > 0$ such that $B(\overline{y}, \delta) \subseteq R(B(\overline{x}, \varepsilon))$. Then for every $v \in B(\overline{y}, \delta)$, there exist $x \in B(\overline{x}, \varepsilon) \cap \Omega$, $y \in F(x)$ such that

$$v \in y + Q(y). \tag{6.5}$$

We obtain that

$$v - \overline{y} \in Q(y) \cup (Y \setminus (-Q(y))) \subseteq Q(y) \cup \left(Y \setminus \bigcap_{t \in F(B(\overline{x},\varepsilon)\cap\Omega)} -Q(t) \right). \tag{6.6}$$

Indeed,

$$v - \overline{y} = (v - y) + (y - \overline{y}).$$

We know that $v - y \in Q(y)$. If $y - \overline{y} \in Q(y)$, then $v - \overline{y} \in Q(y)$. Consider now that $y - \overline{y} \notin Q(y)$, i.e., $\overline{y} - y \notin -Q(y)$. In this case, we prove that $v - \overline{y} \in Y \setminus (-Q(y))$. If it would not be the case, then $\overline{y} - v \in Q(y)$, and since $v - y \in Q(y)$, then $\overline{y} - y \in Q(y)$, hence $\overline{y} - y \in Q(y) \setminus (-Q(y))$, contradicting the nondomination property (3.4). From this point, we obtain, similarly to the proof of Lemma 6.1, that $\bigcap\limits_{t \in F(B(\overline{x},\varepsilon)\cap\Omega)} -Q(t) \subseteq Q(y)$, and the contradiction follows. □

Similar to Lemma 6.2, one proves that assumptions (ii) and (iii) in Theorem 4.6 imply the alliedness of the sets E_1, E_2, E_3 given by (5.13), which in turn by Proposition 5.5 is equivalent to the alliedness of the sets A_1, A_2, A_3 defined by (5.14).

Lemma 6.5 *Let X, Y be Asplund spaces, $\Omega \subseteq X$ be a closed set, $F : X \rightrightarrows Y$, $Q : Y \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}, 0) \in X \times Y \times Y$ such that $\overline{x} \in \Omega$, $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, $(\overline{y}, 0) \in \operatorname{Gr} Q$. Moreover, assume that (ii) and (iii) in Theorem 4.6 hold. Then the sets A_1, A_2, A_3 defined by (5.14) are allied at $(\overline{x}, \overline{y}, 0)$.*

Proof Take $(x_{1n}, y_{1n}, z_{1n}) \xrightarrow{\operatorname{Gr} F \times Y} (\overline{x}, \overline{y}, 0)$, $(x_{2n}, y_{2n}, z_{2n}) \xrightarrow{X \times \operatorname{Gr} Q} (\overline{x}, \overline{y}, 0)$, $(x_{3n}, y_{3n}, z_{3n}) \xrightarrow{\Omega \times Y \times Y} (\overline{x}, \overline{y}, 0)$, $\left(x_{in}^*, y_{in}^*, z_{in}^*\right) \in \widehat{N}\left(A_i, (x_{in}, y_{in}, z_{in})\right)$ for $i = 1, 2, 3$ such that

$$\left(x_{1n}^* + x_{2n}^* + x_{3n}^*, y_{1n}^* + y_{2n}^* + y_{3n}^*, z_{1n}^* + z_{2n}^* + z_{3n}^*\right) \to (0, 0, 0). \tag{6.7}$$

This means that

$$(x_{1n}, y_{1n}) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y}), (y_{2n}, z_{2n}) \xrightarrow{\operatorname{Gr} Q} (\overline{y}, 0), x_{3n} \xrightarrow{\Omega} \overline{x}$$
$$z_{1n}^* = 0, x_{2n}^* = 0, y_{3n}^* = z_{3n}^* = 0,$$
$$x_{1n}^* \in \widehat{D}^* F(x_n, y_{1n})(-y_{1n}^*), y_{2n}^* \in \widehat{D}^* Q(y_{2n}, z_n)(-z_{2n}^*), x_{3n}^* \in \widehat{N}(\Omega, x_{3n}),$$

hence from (6.7) we have that $z_{2n}^* \to 0$. Moreover, since Q has Aubin property around $(\overline{y}, 0)$, using [33, Theorem 1.43] we get that for any $\alpha > \operatorname{lip} Q(\overline{y}, 0)$,

$$\left\| y_{2n}^* \right\| \leq \alpha \left\| -z_{2n}^* \right\|,$$

hence $y_{2n}^* \to 0$, and since $y_{1n}^* + y_{2n}^* \to 0$, also $y_{1n}^* \to 0$.

As in the end of the proof of Lemma 6.2, one uses assumptions (iii) and (iv) from Theorem 4.6 to obtain that at least one of the sequences (x_{1n}^*), (x_{3n}^*) is bounded, and to deduce the conclusion. □

The next theorem gives another variant of necessary optimality conditions for nondominated points of second type.

Second proof of Theorem 4.6 Using Lemma 6.5, we know that the sets A_1, A_2, A_3 defined by (5.14) are allied at $(\overline{x}, \overline{y}, 0)$. By Lemma 6.4, R is not open at $(\overline{x}, \overline{y})$, hence it is not linearly open at $(\overline{x}, \overline{y})$. It follows that the final assumption from Theorem 5.8 is not satisfied, which means that for any $n \in \mathbb{N} \setminus \{0\}$, we can find $(x_{1n}, y_{1n}) \in \operatorname{Gr} F \cap [B(\overline{x}, n^{-1}) \times B(\overline{y}, n^{-1})]$, $(y_{2n}, z_{2n}) \in \operatorname{Gr} Q \cap [B(\overline{y}, n^{-1}) \times B(\overline{z}, n^{-1})]$, $x_{3n} \in \Omega \cap B(\overline{x}, n^{-1})$, $z_n^* \in S_{Y^*}$, $t_n^* \in 2n^{-1}\overline{B}_{Y^*}$, $y_n^* \in \widehat{D}^* Q(y_{2n}, z_{2n})(z_n^*)$, $x_n^* \in \widehat{D}^* F(x_{1n}, y_{1n})(z_n^* + y_n^* + t_n^*)$, $u_n^* \in \widehat{N}(\Omega, x_{3n})$ such that

$$n^{-1} \left\| z_n^* + t_n^* \right\| > \left\| x_n^* + u_n^* \right\|.$$

We get

$$(x_{1n}, y_{1n}) \xrightarrow{\operatorname{Gr} F} (\overline{x}, \overline{y}), (y_{2n}, z_{2n}) \xrightarrow{\operatorname{Gr} Q} (\overline{y}, 0), x_{3n} \xrightarrow{\Omega} \overline{x}$$
$$t_n^* \to 0, x_n^* + u_n^* \to 0.$$

By the Aubin property of Q, it follows that $\left(y_n^*\right)$ is bounded. Since Y is an Asplund space, the sequences $\left(y_n^*\right)$, (z_n^*) are weakly* convergent (on a common subsequence, if necessary) towards $y^*, z^* \in Y^*$, respectively.

As in the third proof of Theorem 4.2, using the alliedness of the sets A_1, A_2, A_3, we obtain that the sequences (x_n^*), (u_n^*) are also bounded, so we may suppose that they weakly* converge (on subsequences, denoted the same) to some x^*, u^*, respectively.

This means that

$$x^* \in D^*F(\overline{x}, \overline{y})(z^* + y^*),$$
$$y^* \in D^*Q(\overline{y}, 0)(z^*),$$
$$u^* \in N(\Omega, \overline{x}),$$

and by the uniqueness of the limit, $x^* + u^* = 0$, hence (4.15) holds. By Lemma 4.1, $z^* \in Q(\overline{y})^+$.

If we suppose, by contradiction, that $z^* = 0$, we obtain from (4.13) that $y^* = 0$, and because Q is SNC at $(\overline{y}, 0)$, it follows that $z_n^* \to 0$, which is impossible because $(z_n^*) \subset S_{Y^*}$. □

Remark 6.6 If one replaces the assumptions (ii)-(iv) from Theorem 4.6 by the following ones:

(ii') Q has the Aubin property around $(\overline{y}, 0)$ with modulus $\operatorname{lip} Q(\overline{y}, 0) < 1$;
(iii') either Ω is SNC at $\overline{x}$ and F^{-1} is PSNC at $(\overline{y}, \overline{x})$, or F is SNC at $(\overline{x}, \overline{y})$;
(iv') the following assumption is satisfied:

$$D^*F(\overline{x}, \overline{y})(0) \cap (-N(\Omega, \overline{x})) = \{0\}, \tag{6.8}$$

the conclusion of the theorem becomes as follows:

There exist $z^* \in Q(\overline{y})^+$ and $y^* \in D^*Q(\overline{y}, 0)(z^*) \cap B_{Y^*}$ such that $z^* + y^* \neq 0$ and (4.15) holds.

Indeed, in this case, by assumption (ii'), we can find $\alpha \in (\operatorname{lip} Q(\overline{y}, 0), 1)$. In the notations above, we get, using [33, Theorem 1.43], that $\|y_n^*\| \leq \alpha$. Using the fact that the norm is weakly* lower semicontinuous and $\|y_n^*\| \leq \alpha$, $y_n^* \xrightarrow{*} y^*$, it follows that $\|y^*\| \leq \alpha < 1$.

If we suppose, by contradiction, that $z^* + y^* = 0$, we obtain

$$x^* = -u^* \in D^*F(\overline{x}, \overline{y})(0) \cap (-N(\Omega, \overline{x})) = \{0\},$$

hence $x_n^* \xrightarrow{*} 0$, $u_n^* \xrightarrow{*} 0$. Using the SNC property of Ω, it follows that (u_n^*) strongly converges to 0, and because $x_n^* + u_n^* \to 0$, (x_n^*) converges to 0. But this implies that $y_n^* + z_n^* + t_n^* \xrightarrow{*} 0$ and $x_n^* \to 0$, and by the PSNC property of F^{-1}, we get that $y_n^* + z_n^* + t_n^* \to 0$, so $y_n^* + z_n^* \to 0$. But $\|y_n^* + z_n^*\| \geq \|z_n^*\| - \|y_n^*\| \geq 1 - \alpha > 0$, contradiction. The case when F is SNC at $(\overline{x}, \overline{y})$ is similar.

In the unconstrained case, we obtain the following corollary, which is slightly different from Corollary 4.11.

Corollary 6.7 *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$, $Q : Y \rightrightarrows Y$ be closed-graph set-valued maps, and $(\overline{x}, \overline{y}) \in X \times Y$ be a local nondominated point of type II for F with respect to Q. Moreover, assume that:*

(i) there is a neighborhood U of $\overline{x}$ such that $\bigcap_{y \in F(U)} (Q(y) \setminus (-Q(y))) \neq \{0\}$;
(ii) Q has the Aubin property around $(\overline{y}, 0)$ with modulus $\operatorname{lip} Q(\overline{y}, 0) < 1$;
(iii) F^{-1} is PSNC at $(\overline{y}, \overline{x})$.

Then there exist $z^* \in Q(\overline{y})^+$ and $y^* \in D^*Q(\overline{y}, 0)(z^*) \cap B_{Y^*}$ such that $z^* + y^* \neq 0$ and

$$0 \in D^*F(\overline{x}, \overline{y})(z^* + y^*).$$

6.2 Second Strategy

The second strategy is to use the reductions given by Remarks 3.7, 3.8, and then to use Corollary 5.9 (for type I) and Theorem 5.6 (for type II).

6.2.1 Nondominated Points of Type I

Here, we firstly use the reduction of type I nondomination for F with respect to K on Ω to the second type nondomination for $\overline{F}$ with respect to $\overline{Q}$ on Ω, and discover in another way that the sum set-valued map $F+K+\Delta_\Omega$ is not open. Moreover, we show that the alternative openness result given by Corollary 5.9 can be successfully used instead of Corollary 5.4.

Fourth proof of Theorem 4.2 Using the reduction given by $\overline{F}$ and $\overline{Q}$, we know that $(\overline{x},(\overline{x},\overline{y}))$ is a type II local nondominated point for $\overline{F}$ with respect to $\overline{Q}$ on Ω. Then, the corresponding set-valued map $\overline{R}$ given by (5.26) is not open at $(\overline{x},(\overline{x},\overline{y}))$, which is equivalent to the fact that $F+K+\Delta_\Omega$ is not open at $(\overline{x},\overline{y})$. So, it is not linearly open at $(\overline{x},\overline{y})$. Next, one can proceed as in the third proof of 4.2, using Corollary 5.4. Another way, which we prove next, is to use Corollary 5.9. Indeed, since by Lemma 6.2 the sets given by (5.7) are allied at $(\overline{x},\overline{y},0,0)$, and F, K are closed-graph, Ω is closed, for any $n\in\mathbb{N}\setminus\{0\}$, we find $(x_{1n},y_{1n})\overset{\operatorname{Gr}F}{\to}(\overline{x},\overline{y})$, $(x_{2n},y_{2n})\overset{\operatorname{Gr}K}{\to}(\overline{x},0)$, $x_{3n}\overset{\Omega}{\to}\overline{x}$, and also $z_n^*\in S_{Y^*}$, $\left(t_n^*,v_n^*\right)\in 2n^{-1}B_{X^*\times Y^*}$ and $x_{1n}^*\in\widehat{D}^*F(x_{1n},y_{1n})(z_n^*+v_n^*)$, $x_{2n}^*\in\widehat{D}^*K(x_{2n},y_{2n})(z_n^*)$, $x_{3n}^*\in\widehat{N}(\Omega,x_{3n})$ such that

$$n^{-1}\left\|\left(t_n^*,z_n^*+v_n^*\right)\right\|>\left\|x_{1n}^*+x_{2n}^*+x_{3n}^*+t_n^*\right\|,$$

hence $x_{1n}^*+x_{2n}^*+x_{3n}^*\to 0$. The rest of the proof is almost the same as in the third proof of Theorem 4.2. □

6.2.2 Nondominated Points of Type II

We use the set-valued maps F_1, F_2, F_3 given by (5.10) for $G:=Q$, and the fact that the nondomination of type II for F with respect to Q on Ω implies the nondomination of type I of F_1 with respect to F_2 on $\Omega\times Y$, as observed in Remark 3.7. We are able to provide a third proof for Theorem 4.6.

Third proof of Theorem 4.6 Since $(\overline{x},\overline{y})\in X\times Y$ be a local nondominated point of type II for F with respect to Q on Ω, we deduce that $((\overline{x},\overline{y}),\overline{y})$ is a local nondominated point of type I for F_1 with respect to F_2 on $\Omega\times Y$ (see relation (5.10), for $G:=Q$). Using Lemma 6.1, we see that the set-valued map $F_1+F_2+\Delta_{\Omega\times Y}$ is not open at $((\overline{x},\overline{y}),\overline{y})$. Observe that $F_1+F_2+\Delta_{\Omega\times Y}$ coincides to the set-valued map $H:X\times Y\rightrightarrows Y$ given by (5.11). Since the alliedness of the sets A_1, A_2, A_3 defined by (5.14) at $(\overline{x},\overline{y},0)$ is given by Lemma 6.5, we deduce that the condition (5.15) from Theorem 5.6 is not satisfied. Therefore, for any $n\in\mathbb{N}\setminus\{0\}$, there exists $(x_{1n},y_{1n})\overset{\operatorname{Gr}F}{\to}(\overline{x},\overline{y})$, $(y_{2n},z_{2n})\overset{\operatorname{Gr}Q}{\to}(\overline{y},0)$, $x_{3n}\overset{\Omega}{\to}\overline{x}$, and also $w_n^*\in S_{Y^*}$, $z_{1n}^*,z_{2n}^*\in n^{-1}\overline{B}_{Y^*}$, $y_{1n}^*\in Y^*$ and $x_{1n}^*\in\widehat{D}^*F(x_{1n},y_{1n})(w_n^*-y_{1n}^*-z_{1n}^*)$, $y_{2n}^*\in\widehat{D}^*Q(y_{2n},z_{2n})(w_n^*-z_{2n}^*)$, $x_{3n}^*\in\widehat{N}(\Omega,x_{3n})$ such that

$$x_{1n}^*+x_{3n}^*\to 0\quad\text{and}\quad y_{1n}^*+y_{2n}^*\to 0.$$

We observe again that the conditions of Theorem 4.6 imply the Aubin property of Q around $(\overline{y},0)$, so the boundedness of $\left(w_n^*-z_{2n}^*\right)$ implies the boundedness of $\left(y_{2n}^*\right)$, which in turn, since $y^*_{1n} + y^*_{2n} \to 0$, show the boundedness of $\left(y^*_{1n}\right)$. As in the second proof of Theorem 4.6 in the previous section, we may deduce the boundedness of the sequences $\left(x^*_{1n}\right)$ and $\left(x^*_{3n}\right)$.

The Asplund spaces assumption implies, as above, that (x^*_{1n}), (x^*_{3n}), $\left(y^*_{1n}\right)$, $\left(y^*_{2n}\right)$, $(w^*_n - y^*_{1n} - z^*_{1n})$, $(w^*_n - z^*_{2n})$ are weak* convergent to some elements x^*_1, x^*_3, y^*_1, y^*_2, w^*_1, w^*_2, respectively, with $w^*_1 + y^*_1 = w^*_2$. This shows that

$$x^*_1 \in D^*F(\overline{x},\overline{y})(w^*_2 - y^*_1),\ y^*_2 \in D^*Q(\overline{y},0)(w^*_2),\ x^*_3 \in N(\Omega,\overline{x}),$$
$$x^*_1 + x^*_3 = 0,\ y^*_1 + y^*_2 = 0,$$

so (4.15) holds for $z^* := w^*_2$ and $y^* := y^*_2$. Using the lower semicontinuity of Q, we get by Lemma 4.1 that $z^* \in Q(\overline{y})^+$.

If we suppose, by contradiction, that $z^* = 0$, we get, by assumption (4.13), that $y^* = 0$. By the SNC property of Q at $(\overline{y}, 0)$, using $(y_{2n}, z_{2n}) \overset{\text{Gr}\,Q}{\to} (\overline{y}, 0)$, $y^*_{2n} \in \widehat{D}^*Q(y_{2n}, z_{2n})(w^*_n - z^*_{2n})$, $\left(y^*_{2n}, w^*_n - z^*_{2n}\right) \overset{*}{\to} (0, 0)$, we get $w^*_n - z^*_{2n} \to 0$ and, since $z^*_{2n} \to 0$, that $w^*_n \to 0$, which is impossible because $\left\| w^*_n \right\| = 1$ for any n. □

7 Conclusions

In this paper, we studied the relationships between solution concepts for vector problems with respect to variable domination structures, that is, the situation where the solution concepts are given with respect to set-valued maps $K : X \rightrightarrows Y$ (nondominated solutions of type I), as well as $Q : Y \rightrightarrows Y$ (nondominated solutions of type II). Taking into account these relationships, it is sufficient to derive optimality results for nondominated solutions of one type, and to use these assertions for deriving corresponding results for solutions of the other type.

Notice that the solution concepts studied in this paper follow the vector approach, for which the image set is considered as a set of elements of the output space. However, the solution concept of the set-valued optimization problem could also be considered by establishing a set less relation (see, e.g., [29]) between the sets of the output space. In order to show optimality conditions for solutions based on the set approach, an appropriate calculus needs to be developed. Another way is to use the method proposed in [25, 27] to employ the relationships between solution concept based on the vector approach and the solution concept given by a set less relation. With these relationships, we could derive necessary optimality conditions for solutions given by a set less relation from corresponding conditions for solutions based on vector approach. In a forthcoming paper, we will derive necessary optimality conditions for solutions of set optimization problems based on the set approach from the results for solutions based on the vector approach shown in our paper. Also, for further research, it would be interesting to derive Ekeland's type variational principles for nondominated solutions of type I and II, equivalent assertions like fixed point theorems of Kirk-Caristi type and existence assertions in the sense of Takahashi employing this unifying approach.

Moreover, it is of interest to derive solution procedures for generating nondominated solutions of type I and II based on the necessary optimality conditions and the relationships between the solution concepts developed in our paper.

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Declarations

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