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A note on the Morse homology for a class of functionals in Banach spaces involving the $2p$-area functional

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Abstract. In this paper we show how to construct Morse homology for an explicit class of functionals involving the $2p$-area functional. The natural domain of definition of such functionals is the Banach space $W_0^{1,2p}(\Omega)$, where $p > n/2$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently smooth boundary. As $W_0^{1,2p}(\Omega)$ is not isomorphic to its dual space,critical points of such functionals cannot be non-degenerate in the usual sense, and hence in the construction of Morse homology we only require that the second differential at each critical point be injective. Our result upgrades, in the case $p > n/2$, the results in Cingolani and Vannella (Ann Inst H Poincaré Anal Non Linéaire 2:271–292, 2003; Ann Mat Pura Appl 186:155–183, 2007), where critical groups for an analogous class of functionals are computed, and provides in this special case a positive answer to Smale's suggestion that injectivity of the second differential should be enough for Morse theory

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1. Introduction

Let $n \geq 2$. For $\Omega \subset \mathbb{R}^n$ bounded domain with sufficiently regular boundary, and for $p > n/2$, we consider the functional

$$f : X := W_0^{1,2p}(\Omega) \to \mathbb{R}, \quad f(u) := \frac{1}{2p} \int_\Omega \big(1 + |\nabla u|^2\big)^p \, \mathrm{d}x + \int_\Omega G(u) \, \mathrm{d}x, \tag{1.1}$$

where $G : \mathbb{R} \to \mathbb{R}$ is a function of class C^2 such that

$$|G(t)| \leq \beta |t|^\alpha + \delta, \quad \forall t \in \mathbb{R}, \tag{1.2}$$

for some $\alpha \in [0, 2p)$ and $\beta, \delta \geq 0$.

Condition (1.2) is needed to ensure that the functional f in (1.1) satisfies the Palais–Smale condition, a crucial property to do global critical point theory in infinite dimension. If one is interested in the computation of critical groups only, then such a condition can be removed, see [7] and references therein, where an analogous class of functionals is considered (we refer to the discussion after the statement of the main theorem for more details). Also, in this paper we are not interested in finding sharp conditions under which Morse homology can be defined. In this sense, Condition (1.2) as well as the condition $p > n/2$ can surely be relaxed.

In this paper we show that Morse homology for functionals as in (1.1) is well-defined provided that all critical points $\bar{u}$ of f are non-degenerate in the sense that the second differential of f at $\bar{u}$ defines an injective linear operator $\mathrm{d}^2 f(\bar{u}) : X \to X^*$. We shall stress the fact that such a condition is in general not enough to construct Morse homology (actually, not even to compute critical groups), as it does not even imply that the critical points are isolated, see e.g. [16].

Theorem 1.1. *Let $f : X \to \mathbb{R}$ be a functional as in (1.1) such that all critical points of f are non-degenerate in the sense that the second differential $\mathrm{d}^2 f(\bar{u}) : X \to X^*$ is injective for all $\bar{u} \in \operatorname{crit}(f)$. Then, Morse homology with $\mathbb{Z}_2$-coefficients for f is well-defined and isomorphic to the singular homology of X, i.e.*

$$HM_*(f;\mathbb{Z}_2) \cong H_*(X;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & * = 0, \\ 0 & * \geq 1. \end{cases}$$

In particular, it is independent of $p > n/2$.

To our best knowledge, the theorem above represents the first concrete instance in which Morse homology in a Banach space setting is defined. For an abstract construction we refer to [2]. Functionals as in (1.1) are interesting for at least two reasons: first, they are intimately related with the class of functionals considered in [7,8], whose critical points correspond to weak solutions of a quasi-linear problem (involving the p-Laplacian) which arises in the mathematical description of propagation phenomena of solitary waves. In fact, for this latter class of functionals the construction of Morse homology carries over word by word. Second, they are similar to the class of functionals of α-harmonic maps, $\alpha > 1$, introduced in [13] to prove the existence of harmonic maps by studying the convergence of α-harmonic maps as $\alpha \downarrow 1$. In fact, our approach should allow to define Morse homology for such class of functionals as well.

The strategy of the proof of Theorem 1.1 is the following: arguing as in [7, 8] one sees that the critical points of f are isolated and have finite Morse index. Also, the growth Condition (1.2) implies that f satisfies the Palais–Smale condition on X. Therefore, in order to apply the abstract theory developed in [2] one has to prove the existence of a C^2-smooth complete Morse (i.e. with only hyperbolic rest points) vector field F on X such that f is a Lyapounov function for the flow induced by F, see Sect. 2. The key ingredient here is a sort of uniform convexity of f in the positive direction determined by the second differential $\mathrm{d}^2 f$ at each critical point, see Lemma 2.4. It would be interesting to check such a condition in other concrete examples.

The fact that f is a Lyapounov function for F together with the fact that all critical points have finite Morse index (such a condition can be relaxed, see [2, Theorem 1.20]) implies that the stable resp. unstable manifold $W^s(\bar{u}, F)$ resp. $W^u(\bar{u}, F)$ of a rest point $\bar{u}$ of F (equivalently, of a critical point $\bar{u}$ of f) is an embedded C^2-submanifold homeomorphic to an open disc. Since critical points of f have finite Morse index and the pair (f, F) satisfies the Palais–Smale condition, after a generic perturbation of F we can achieve transverse intersection between stable and unstable manifolds of pairs of critical points whose Morse indices differ at most by two. Such intersections are therefore finite dimensional pre-compact embedded submanifolds of X of dimension equal the difference of the Morse indices. Now one argues as usually to define a Morse complex which is generated by critical points of f and whose boundary operator counts the number of gradient flow lines (modulo two) between pairs of critical points whose Morse indices differ by one. For more details about the abstract construction of the Morse complex we refer to [2, Sect. 2].

We finish this introduction discussing an easy consequence of Theorem 1.1.

Corollary 1.2. *Let $f : X \to \mathbb{R}$ be as in (1.1). Assume that G satisfies the growth Condition (1.2), and that f has two non-degenerate critical points. Then, f has a third critical point (possibly degenerate). In particular, f as in Theorem 1.1 has either one or at least three critical points.*

Proof. Assume that f has no other critical points. Then the assumptions of Theorem 1.1 are satisfied, and hence Morse homology for f is well-defined and isomorphic to $H_*(X; \mathbb{Z}_2)$. However, this is incompatible with f having only two critical points, as the second critical point cannot be canceled in homology. The second statement is an obvious consequence of the first one, observing that f has at least one critical point because it is bounded from below and satisfies the Palais–Smale condition. □

2. Construction of Morse homology

The differential of f as in (1.1) at u is given by

$$\mathrm{d}f(u)[v] = \int_\Omega \left(1 + |\nabla u|^2\right)^{p-1} \langle \nabla u, \nabla v \rangle \,\mathrm{d}x + \int_\Omega G'(u) v \,\mathrm{d}x. \tag{2.1}$$

If $\bar{u} \in X$ is a critical point of f, then the second differential of f at $\bar{u}$ is

$$\begin{aligned} \mathrm{d}^2 f(\bar{u})[v, w] = & \int_\Omega \left(1 + |\nabla \bar{u}|^2\right)^{p-1} \langle \nabla v, \nabla w \rangle \,\mathrm{d}x \\ & + 2(p-1) \int_\Omega (1 + |\nabla \bar{u}|^2)^{p-2} \langle \nabla \bar{u}, \nabla v \rangle \langle \nabla \bar{u}, \nabla w \rangle \,\mathrm{d}x \\ & + \int_\Omega G''(\bar{u}) v w \,\mathrm{d}x. \end{aligned}$$

Hereafter we assume that $\bar{u}$ is a *non-degenerate* critical point, in the sense that $\mathrm{d}^2 f(\bar{u}) : X \to X^*$ is injective. As easy examples show, such a condition is in general not sufficient to do Morse theory for abstract functionals on Banach manifolds, as it does not even imply that the critical point is isolated, see e.g. [16]. However, for the class of functionals in (1.1), critical groups can be defined in a similar way to [7] under such an assumption. In this paper, we upgrade such a result showing that full Morse homology can be defined.

Remark 2.1. Critical points of functionals as in (1.1) cannot be non-degenerate in the classical sense (i.e. $\mathrm{d}^2 f(\bar{u})$ isomorphism), since X is not isomorphic to its dual space X^*. On the other hand, injectivity of the second differential at an isolated critical point can be obtained by arbitrarily small finite dimensional Marino-Prodi type perturbations, as shown in [8, Theorem 1.6].

In the next theorem we construct a C^2-smooth complete Morse vector field on X for which f is a Lyapounov function. This is the crucial step in the definition of Morse homology.

Theorem 2.2. *Let f be a functional as in* (1.1) *having only non-degenerate critical points in the sense above. Then, there exists a C^2-smooth vector field F on X such that:*

(i) *F is complete,*
(ii) *f is a Lyapounov function for F,*
(iii) *F is Morse, i.e. the Jacobian of F at every critical point $\bar{u}$ of f is an hyperbolic operator on $T_{\bar{u}}X$,*
(iv) *(f, F) satisfies the Palais–Smale condition,*
(v) *F satisfies the Morse-Smale condition up to order two.*

Proof of Theorem 1.1. The growth Condition (1.2) implies that f is bounded from below. We define a chain complex $(C_*(f), \partial)$ by setting

$$C_k(f) := \bigoplus_{\mu(\bar{u})=k} \mathbb{Z}_2\langle \bar{u} \rangle,$$

where $\mu(\bar{u})$ denotes the Morse index of $\bar{u} \in \mathrm{crit}\,(f)$, and

$$\partial \bar{u} := \sum_{\mu(\bar{v})=\mu(\bar{u})-1} \Big(\big|\mathcal{M}(\bar{u}, \bar{v})\big| \bmod 2 \Big) \cdot \bar{v},$$

where $\mathcal{M}(\bar{u}, \bar{v})$ is the intersection between the unstable manifold $W^u(\bar{u}, F)$ of $\bar{u}$ and the stable manifold $W^s(\bar{v}, F)$ of $\bar{v}$. Conditions i)-iv) in Theorem 1.1 imply that $W^u(\bar{u}, F)$ resp. $W^s(\bar{v}, F)$ is a finite dimensional resp. codimensional embedded C^2-submanifold of X homeomorphic to a disk of dimension $\mu(\bar{u})$ resp. of codimension $\mu(\bar{v})$. The Morse-Smale condition up to order one then implies that $\mathcal{M}(\bar{u}, \bar{v})$ is a pre-compact one-dimensional embedded submanifold, and as such consists of only finitely many F-flow lines connecting $\bar{u}$ and $\bar{v}$. In particular, $\partial \bar{u}$ is well-defined as there can be only finitely many critical points of f of index $\mu(\bar{u}) - 1$ contained in $f^{-1}(-\infty, f(\bar{u}))$. This follows from the fact that f is bounded below and that the pair (f, F) satisfies the Palais–Smale condition. Finally, the Morse-Smale condition up to order two implies that

$\partial^2 = 0$, so that $(C_*(f), \partial)$ is indeed a chain complex. The fact that the resulting Morse homology is isomorphic to the singular homology of X is proved in [2, Theorem 2.8]. □

To prove Theorem 2.2, the first step is to relate the notion of non-degeneracy above with a notion of non-degeneracy which is more convenient for Morse homology, namely the existence of a linear hyperbolic operator L on X such that f is a Lyapounov function for the linear flow defined by L in a neighborhood of $\bar{u}$, see e.g. [2,16,17].

Proposition 2.3. *Let $\bar{u} \in X$ be a non-degenerate critical point of f as in (1.1). Then, there exist a neighborhood $\mathcal{U}$ of $\bar{u}$ in X and a linear hyperbolic operator $L : T_{\bar{u}}X \to T_{\bar{u}}X$ such that, on $\mathcal{U}$, f is a Lyapounov function for the linear flow defined by L.*

To prove Proposition 2.3 we first recall some facts which are proved in [7] for a slightly different class of functionals, but all proofs go through with minor modifications to the setting of the present paper. Because of the embedding $X \hookrightarrow L^\infty(\Omega)$, the critical point $\bar{u}$ is obviously contained in $L^\infty(\Omega)$. The results in [14,15] then imply that $\bar{u} \in C^1(\overline{\Omega})$. Following [7], on $C_0^\infty(\Omega)$ we introduce the scalar product

$$\langle v, w\rangle_{\bar{u}}. := \int_\Omega \big(1 + |\nabla \bar{u}|^2\big)^{p-1} \langle \nabla v, \nabla w\rangle \,\mathrm{d}x \\ + 2(p-1) \int_\Omega \big(1 + |\nabla \bar{u}|^2\big)^{p-2} \langle \nabla \bar{u}, \nabla v\rangle \langle \nabla \bar{u}, \nabla w\rangle \,\mathrm{d}x,$$

and define the Hilbert space

$$\mathbb{H}_{\bar{u}} := \overline{C_0^\infty(\Omega)}^{\langle\cdot,\cdot\rangle_{\bar{u}}}.$$

It is easy to see that $\mathbb{H}_{\bar{u}}$ is isomorphic to $W_0^{1,2}(\Omega)$, and thus we have a continuous embedding $X \hookrightarrow \mathbb{H}_{\bar{u}}$. Moreover, $\mathrm{d}^2 f(\bar{u}) : X \to X^*$ extends to an invertible operator $H_{\bar{u}} : \mathbb{H}_{\bar{u}} \to \mathbb{H}_{\bar{u}}$ (where we have identified $\mathbb{H}_{\bar{u}}^*$ with $\mathbb{H}_{\bar{u}}$ using Riesz' representation theorem). Indeed,

$$H_{\bar{u}} = \mathrm{id} + K,$$

where $K : \mathbb{H}_{\bar{u}} \to \mathbb{H}_{\bar{u}}, v \mapsto Kv$, is the compact operator uniquely defined by

$$\langle Kv, w\rangle_{\bar{u}} = \int_\Omega G''(\bar{u}) v w \,\mathrm{d}x, \quad \forall w \in \mathbb{H}_{\bar{u}}.$$

Being $H_{\bar{u}}$ a compact perturbation of the identity, it has Fredholm index zero. Furthermore, $H_{\bar{u}}$ is self-adjoint and as such its spectrum is real and consists of the eigenvalue 1 (which has infinite multiplicity) and of eigenvalues different from 1 (with finite multiplicity) which accumulate to 1. Accordingly, we have a natural $\langle\cdot,\cdot\rangle_{\bar{u}}$-orthogonal decomposition

$$\mathbb{H}_{\bar{u}} = \mathbb{H}^- \oplus \mathbb{H}^0 \oplus \mathbb{H}^+,$$

where $\mathbb{H}^0 := \ker H_{\bar u}$ and

$$\mathbb{H}^- := \bigoplus_{\substack{\lambda\in\sigma(H_{\bar u})\\ \lambda<0}} \ker\big(\lambda\cdot \mathrm{id} - H_{\bar u}\big), \qquad \mathbb{H}^+ := \bigoplus_{\substack{\lambda\in\sigma(H_{\bar u})\\ \lambda>0}} \ker\big(\lambda\cdot \mathrm{id} - H_{\bar u}\big),$$

are the negative resp. positive eigenspace of $H_{\bar u}$. Clearly, the set of positive eigenvalues of $H_{\bar u}$ is uniformly bounded away from zero, thus we can find a constant $\mu > 0$ such that

$$\langle H_{\bar u} v, v\rangle_{\bar u} \geq \mu \|v\|^2_{\bar u}, \quad \forall v \in \mathbb{H}^+. \tag{2.2}$$

Also, $\dim \mathbb{H}^- \oplus \mathbb{H}^0 < +\infty$, and standard regularity theory implies that

$$\mathbb{H}^- \oplus \mathbb{H}^0 \subset X,$$

see [10, Theorems 8.15, 8.24, 8.29]. Consequently, we obtain a splitting

$$X = \mathbb{H}^- \oplus \mathbb{H}^0 \oplus W,$$

where $W := \mathbb{H}^+ \cap X$, and (2.2) implies that

$$\mathrm{d}^2 f(\bar u)[v, v] \geq \mu \|v\|^2_{\bar u}, \quad \forall v \in W. \tag{2.3}$$

Since by assumption $\mathrm{d}^2 f(\bar u) : X \to X^*$ is injective, we finally deduce that $L_{\bar u}$ is injective and thus an isomorphism, being Fredholm of index zero. In particular,

$$\mathbb{H}_{\bar u} = \mathbb{H}^- \oplus \mathbb{H}^+, \quad \text{and} \quad X = \mathbb{H}^- \oplus W.$$

In the next result we prove that (2.3) holds for any u in a sufficiently small neighborhood of $\bar u$, thus showing that f is locally uniformly convex around $\bar u$ in the W-direction with respect to the $\|\cdot\|_{\bar u}$-norm. The proof, which is analogous to the one of [7, Lemma 4.2], is included for the reader's convenience.

Lemma 2.4. *There exists $r > 0$ and $\mu > 0$ such that for any $u \in X$ with $\|u - \bar u\| < r$ we have*

$$\mathrm{d}^2 f(u)[v, v] \geq \mu \|v\|^2_{\bar u}, \quad \forall v \in W. \tag{2.4}$$

In particular, $\bar u$ is a strict minimum point of f in the W-direction.

Proof. Assume that there exist sequences $(u_n) \subset X$ and $(v_n) \subset W$ such that $u_n \to \bar u$ in X, $\|v_n\|_{\bar u} = 1$ for all $n \in \mathbb{N}$, and

$$\liminf_{n\to+\infty} \mathrm{d}^2 f(u_n)[v_n, v_n] \leq 0. \tag{2.5}$$

Up to taking a subsequence, v_n weakly converges (thus also strongly in L^2) to $v_\infty \in \mathbb{H}^+$. Noticing that

$$
\begin{aligned}
&\mathrm{d}^2 f(u_n)[v_n, v_n] \\
&\quad = \int_\Omega \left(1+|\nabla u_n|^2\right)^{p-1} |\nabla v_n|^2 \mathrm{d}x + 2(p-1)\int_\Omega \left(1+|\nabla u_n|^2\right)^{p-2} \langle \nabla u_n, \nabla v_n\rangle^2 \,\mathrm{d}x \\
&\qquad + \int_\Omega G''(u_n) v_n^2 \,\mathrm{d}x \\
&\quad \geq \|\nabla v_n\|_2^2 + \int_\Omega G''(u_n) v_n^2 \,\mathrm{d}x \\
&\quad \geq c\|v_n\|_{\bar u}^2 + \int_\Omega G''(u_n) v_n^2 \,\mathrm{d}x \\
&\quad = 1 + \int_\Omega G''(u_n) v_n^2 \,\mathrm{d}x,
\end{aligned}
$$

we infer that $v_\infty \neq 0$, as otherwise this would contradict (2.5). We now set

$$
h(x,u,v) := \left(1+|\nabla u|^2\right)^{p-1} |\nabla v|^2 + 2(p-1)\left(1+|\nabla u|^2\right)^{p-2} \langle \nabla u, \nabla v\rangle^2,
$$

so that

$$
\mathrm{d}^2 f(u_n)[v_n, v_n] = \int_\Omega h(x, u_n, v_n)\,\mathrm{d}x + \int_\Omega G''(u_n) v_n^2 \,\mathrm{d}x.
$$

Obviously, h is non-negative, continuous, and $v \mapsto h(x,u,v)$ is convex for every (x,u). Therefore, the result in [11] implies that

$$
(u,v) \mapsto \int_\Omega h(x,u,v)\,\mathrm{d}x
$$

is lower-semicontinuous with respect to the strong convergence in the u-direction and the weak convergence in the v-direction. Now, using Assumption (2.5), Equation (2.2), and the fact that $v_n \to v_\infty$ in L^2, we conclude

$$
\begin{aligned}
0 &\geq \liminf_{n\to+\infty}\ \mathrm{d}^2 f(u_n)[v_n, v_n] \\
&= \liminf_{n\to+\infty} \left(\int_\Omega h(x, u_n, v_n)\,\mathrm{d}x + \int_\Omega G''(u_n) v_n^2 \,\mathrm{d}x \right) \\
&\geq \int_\Omega h(x, \bar u, v_\infty)\,\mathrm{d}x + \int_\Omega G''(\bar u) v_\infty^2 \,\mathrm{d}x \\
&= \langle L_{\bar u} v_\infty, v_\infty\rangle_{\bar u} \\
&\geq \mu \|v_\infty\|_{\bar u}^2,
\end{aligned}
$$

clearly a contradiction, as $v_\infty \neq 0$. This shows that

$$
\liminf_{n\to+\infty}\ \mathrm{d}^2 f(u_n)[v_n, v_n] > 0
$$

for all sequences $(u_n) \subset X$ such that $u_n \to \bar u$, and all sequences $(v_n) \subset W$ such that $\|v_n\|_{\bar u} = 1$ for all $n \in \mathbb{N}$. We claim now that for all such sequences there exists $\mu > 0$ such that

$$
\liminf_{n\to+\infty}\ \mathrm{d}^2 f(u_n)[v_n, v_n] \geq \mu.
$$

Clearly, this finishes the proof of the lemma. Assuming this is not the case, for every $m \in \mathbb{N}$ we find sequences $(u_n^{(m)})$ and $(v_n^{(m)})$ as above such that

$$\liminf_{n\to+\infty} \mathrm{d}^2 f(u_n^{(m)})[v_n^{(m)}, v_n^{(m)}] < \frac{1}{m}.$$

Therefore, for every $m \in \mathbb{N}$ we can find $n(m) \in \mathbb{N}$ such that

$$\mathrm{d}^2 f(u_{n(m)}^{(m)})[v_{n(m)}^{(m)}, v_{n(m)}^{(m)}] < \frac{2}{m}.$$

The sequences $(u_{n(m)}^{(m)})_{m\in\mathbb{N}}$ and $(v_{n(m)}^{(m)})_{m\in\mathbb{N}}$ also satisfy

$$u_{n(m)}^{(m)} \to \bar{u}, \quad \|v_{n(m)}^{(m)}\|_{\bar{u}} = 1, \ \forall m \in \mathbb{N},$$

and by construction

$$\lim_{m\to+\infty} \mathrm{d}^2 f(u_{n(m)}^{(m)})[v_{n(m)}^{(m)}, v_{n(m)}^{(m)}] = 0,$$

a contradiction. □

Remark 2.5. A similar statement as in Lemma 2.4 holds also in the $\mathbb{H}^-$-direction, namely there exist $r > 0$ and $\mu > 0$ such that for any $u \in X$ with $\|u - \bar{u}\| < r$ we have

$$\mathrm{d}^2 f(u)[v, v] \leq -\mu \|v\|_{\bar{u}}^2, \quad \forall v \in \mathbb{H}^-. \tag{2.6}$$

However, the proof in this case is elementary since f is of class C^2 and $\mathbb{H}^-$ is finite dimensional, so that $\|\cdot\|_{\bar{u}}$ and $\|\cdot\|$ are equivalent on $\mathbb{H}^-$. The details are left to the reader.

Proof of Proposition 2.3. On

$$T_{\bar{u}}X \cong X = \mathbb{H}^- \oplus W$$

we define the linear operator $L = (\mathrm{id}, -\mathrm{id})$, that is

$$Lx := L(x_- + x_W) := x_- - x_W, \quad \forall x = x_- + x_W \in X.$$

The operator L is clearly hyperbolic. We claim that there exists a sufficiently small neighborhood $\mathcal{U}$ of $\bar{u}$ such that f is a Lyapounov function for the linear flow defined by L, meaning that, for every $x \neq 0 \in \mathcal{U} - \{\bar{u}\}$,

$$t \mapsto f(\bar{u} + e^{tL}x)$$

is strictly monotone decreasing, or, equivalenty, that

$$\mathrm{d}f(\bar{u} + x)[Lx] < 0, \quad \forall x \in \mathcal{U} \setminus \{0\}.$$

We have

$$\mathrm{d}f(\bar{u} + x)[\cdot] = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\big(f(\bar{u} + sx)\big)[\cdot]\, \mathrm{d}s = \int_0^1 \mathrm{d}^2 f(\bar{u} + sx)[\cdot, x]\, \mathrm{d}s,$$

choose $r, \mu > 0$ such that (2.4) and (2.6) hold, and compute for $x \in X$ with $\|x\| < r$:

$$\begin{aligned}
&\mathrm{d}f(\bar{u}+x)[Lx] \\
&\quad = \int_0^1 \mathrm{d}^2 f(\bar{u}+sx)[Lx, x]\,\mathrm{d}s \\
&\quad = \int_0^1 \mathrm{d}^2 f(\bar{u}+sx)[x_- - x_W, x_- + x_W]\,\mathrm{d}s \\
&\quad = \int_0^1 \Big(\mathrm{d}^2 f(\bar{u}+sx)[x_-, x_-] - \mathrm{d}^2 f(\bar{u}+sx)[x_W, x_W] \\
&\qquad + \underbrace{\mathrm{d}^2 f(\bar{u}+sx)[x_-, x_W] - \mathrm{d}^2 f(\bar{u}+sx)[x_W, x_-]}_{=0}\Big)\mathrm{d}s \\
&\quad \le -\mu\|x_-\|_{\bar{u}}^2 - \mu\|x_W\|_{\bar{u}}^2 \\
&\quad \le -\mu\|x\|_{\bar{u}}^2,
\end{aligned}$$

which implies the claim. □

We recall that $f : X \to \mathbb{R}$ of class C^1 is said to satisfy the *Palais–Smale condition*, if any sequence $(u_n) \subset X$ such that $f(u_n) \to c$, for some $c \in \mathbb{R}$, and $\mathrm{d}f(u_n) \to 0$ admits a converging subsequence. By the continuity of the differential, any limit point of a Palais–Smale sequence is a critical point of f.

Remark 2.6. The Palais–Smale condition plays the role of compactness of sublevel sets, and as such is a crucial ingredient in infinite dimensional critical point theory. We shall however stress the fact that, when the Morse index and co-index of critical points is infinite, that is when f is *strongly indefinite*, the Palais–Smale condition alone is not enough to construct Morse homology, as the intersection between stable and unstable manifolds of pair of critical points might not be pre-compact, not even if finite dimensional. In such cases, stronger conditions are needed, see e.g. [1,3,4], where Morse homology for an abstract class of strongly indefinite functionals on a Hilbert manifold resp. for the Hamiltonian action in cotangent bundles is defined. Also, classical Morse theory is in such cases of no help, since the topology of sublevel sets does not change when crossing a critical point with infinite Morse index. Anyhow, this will not be the case in the present paper, since critical points of a functional as in (1.1) always have finite Morse index.

We show now that the growth condition (1.2) on the function G implies that the functional f in (1.1) satisfies the Palais–Smale condition on X.

Lemma 2.7. *Let $f : X \to \mathbb{R}$ be a functional as in* (1.1)*. Then f satisfies the Palais–Smale condition.*

Proof. Let $(u_n) \subset X$ be a Palais–Smale sequence for f.

Claim 1. (u_n) is bounded in X.

Suppose by contradiction that $\|u_n\| \to +\infty$. By the very definition of f, Equation (1.2), Poincaré inequality, and Hölder inequality we have for some constant $\gamma > 0$:

$$\begin{aligned}|f(u_n)| &= \left|\int_\Omega (1+|\nabla u_n|^2)^p \,\mathrm{d}x + \int_\Omega G(u_n)\,\mathrm{d}x\right| \\ &\geq \|\nabla u_n\|_{2p}^{2p} - \int_\Omega |G(u_n)|\,\mathrm{d}x \\ &\geq \gamma\|u_n\|^{2p} - \beta\|u_n\|_{L^\alpha}^\alpha - \delta\mu(\Omega) \\ &\geq \gamma\|u_n\|^{2p} - \beta\mu(\Omega)^{\alpha(2p-\alpha)/2p}\|u_n\|_{L^{2p}}^{\alpha^2/2p} - \delta\mu(\Omega) \\ &\geq \gamma\|u_n\|^{2p} - \beta\mu(\Omega)^{\alpha(2p-\alpha)/2p}\|u_n\|^{\alpha^2/2p} - \delta\mu(\Omega) \\ &\to +\infty,\end{aligned}$$

as by assumption $\alpha < 2p$. This is clearly a contradiction, since $f(u_n) \to c$ for some $c \in \mathbb{R}$.

Claim 2. (u_n) admits a converging subsequence.

Since (u_n) is bounded, up to a subsequence we can assume $u_n \rightharpoonup u$ for some $u \in X$, hence in particular $u_n \to u$ in $L^\infty(\Omega)$. In view of (2.1) we can write $\mathrm{d}f : X \to X^*$ as

$$\mathrm{d}f = D + K,$$

where

$$\begin{aligned} D &: X \to X^*, \quad D(u)[\cdot] := \int_\Omega (1+|\nabla u|^2)^{p-1}\langle \nabla u, \nabla \cdot\rangle\,\mathrm{d}x, \\ K &: X \to X^*, \quad K(u)[\cdot] := \int_\Omega G'(u) v\,\mathrm{d}x.\end{aligned}$$

As shown in [5, Appendix B], the non-linear operator D is invertible with continuous inverse D^{-1}. We claim that $K(u_n) \to K(u)$ in operator norm. This follows from the fact that K is sequentially compact, meaning that for any weakly converging sequence $w_n \rightharpoonup w$ in X the operators $K(w_n)$ converge in operator norm to $K(w)$. To see this we compute using Hölder inequality with conjugated exponents $q, 2p$:

$$\begin{aligned}\|K(w_n) - K(w)\| &= \sup_{\|v\|=1} |K(w_n)[v] - K(w)[v]| \\ &= \sup_{\|v\|=1} \left|\int_\Omega \big(G'(w_n) - G'(w)\big) v\,\mathrm{d}x\right| \\ &\leq \sup_{\|v\|=1} \left(\int_\Omega \left|G'(w_n) - G'(w)\right|^q \mathrm{d}x\right)^{1/q} \|v\|_{L^{2p}} \\ &\leq \left(\int_\Omega \left|G'(w_n) - G'(w)\right|^q \mathrm{d}x\right)^{1/q} \\ &\leq \mu(\Omega)^{1/q} \sup_{x\in\Omega} |G'(w_n(x)) - G'(w(x))| \\ &\to 0,\end{aligned}$$

as $w_n \to w$ in $L^\infty(\Omega)$ and G' is continuous. Now, since (u_n) is a Palais–Smale sequence we have

$$o(1) = \|\mathrm{d}f(u_n)\| = \|D(u_n) + K(u_n)\|,$$

thus $D(u_n) \to -K(u)$ in operator norm, and finally $u_n \to -D^{-1}(K(u))$ as D^{-1} is continuous. □

Proof of Theorem 2.2. Fix some $u_0 \in X \setminus \mathrm{crit}\,(f)$. Since $\mathrm{d}f(u_0) \neq 0$, we find $v \in T_{u_0}X$ such that

$$\mathrm{d}f(u_0)[v] \leq -\frac{1}{2}\|\mathrm{d}f(u_0)\|^2,$$

for some constant $\gamma(u_0) > 0$. By the continuity of $\mathrm{d}f$, we can find $r(u_0) > 0$ such that

$$\mathrm{d}f(u)[v] \leq -\frac{1}{4}\|\mathrm{d}f(u)\|^2, \quad \forall u \in B_{r(u_0)}(u_0). \tag{2.7}$$

Therefore, we set $V_{u_0}(u) \equiv v$.

If $\bar{u} \in X$ is a critical point of f, then by Proposition 2.3 we have that there exist $\mu(\bar{u}), r(\bar{u}) > 0$ such that

$$\mathrm{d}f(\bar{u}+x)[V_{\bar{u}}(x)] \leq -\mu(\bar{u})\|x\|^2_{\bar{u}}, \quad \forall x \in X \text{ with } \|x\| < r(\bar{u}), \tag{2.8}$$

with $V_{\bar{u}}(x) := Lx$, where $L : T_{\bar{u}}X \to T_{\bar{u}}X$ is the hyperbolic operator given by the proposition. Without loss of generality we may also assume that the open sets $B_{r(\bar{u})}(\bar{u}),\ \bar{u} \in \mathrm{crit}\,(f)$, are pairwise disjoint.

We now consider the open covering of X given by

$$\mathfrak{U} := \left\{ B_{r(u_0)}(u_0) \;\middle|\; u_0 \in X \setminus \mathrm{crit}\,(f) \right\} \cup \left\{ B_{r(\bar{u})}(\bar{u}) \;\middle|\; \bar{u} \in \mathrm{crit}\,(f) \right\}.$$

By the paracompactness of X, there exists a locally finite refinement $\mathfrak{V} = \{\mathcal{V}_j \,|\, j \in J\}$ of the open covering $\mathfrak{U}$. Let $\Gamma : J \to X$ be a function such that $\mathcal{V}_j \subset B_{r(\Gamma(j))}(\Gamma(j))$ for all $j \in J$. Following [6,9], X admits C^2-smooth bump functions. Therefore, we can find a C^2-smooth partition of unity $\{\chi_j\}$ subordinated to the open covering $\mathfrak{V}$, and set

$$\tilde{F}(u) := \sum_{j \in J} \chi_j(u) V_{\Gamma(j)}(u), \quad \forall u \in X.$$

By construction $\tilde{F}$ is of class C^2, and the inequalities (2.7) and (2.8) imply that f is a Lyapounov function for $\tilde{F}$. Furthermore, we can make $\tilde{F}$ to a bounded vector field by multiplication by a suitable conformal factor: given a smooth monotonically decreasing function $\varphi : [0,+\infty) \to (0,+\infty)$ such that

$$\varphi \equiv 1 \;\text{ for } s \in [0,1], \quad \varphi(s) = \frac{1}{s} \;\text{ for } s \geq 2,$$

we set

$$F(u) := \varphi(\|\tilde{F}(u)\|) \cdot \tilde{F}(u), \quad \forall u \in X.$$

Clearly, F is complete, and f is a Lyapounov function for F as well. Moreover, in a neighborhood of each critical point the vector field F coincides with the linear vector field $x \mapsto Lx$. This implies (iii). We claim now that Palais–Smale sequences for (f, F) are also Palais–Smale sequences for f. Indeed, let $(u_n) \subset X$ be a sequence such that

$$f(u_n) \to c, \quad \mathrm{d}f(u_n)[F(u_n)] \to 0.$$

By Step 1 in the proof of Lemma 2.7 we have that (u_n) is a bounded sequence. If (u_n) admits a subsequence converging to a critical point of f then there is nothing to prove. So without loss of generality we can assume that, up to passing to a subsequence if necessary, (u_n) is contained in the complement of a open neighborhood of crit (f). By (2.7) we can find a constant $c > 0$ such that

$$\|\mathrm{d}f(u_n)\|^2 \leq -\frac{1}{c}\mathrm{d}f(u_n)[F(u_n)] = o(1), \quad \text{for } n \to +\infty.$$

This together with Lemma 2.7 implies (iv). Finally, since F is of class C^2, (v) can be achieved by a suitable generic perturbation as proved in Theorem 5.5 in [4]. As such a theorem deals with the much more delicate case of strongly indefinite functionals, we give a sketch of the proof here for the reader's convenience referring to [4] for the details. Whenever possible, we will also adopt the same notation as in [4]. The interested reader can also have a look at Theorem 2.20 in [2] for a transversality statement on Hilbert manifolds in the case of finite Morse indices where the perturbation remains within the class of gradient vector fields.

The Morse-Smale condition up to order 2 will follow from a version of the Sard-Smale theorem due to Quinn and Sard [12] which in the setting of the present paper can be formulated as follows: *let $\varphi : Y \to Z$ be a C^2-smooth σ-proper Fredholm map between the Banach spaces Y and Z with Fredholm index at most 2. Then, the set of regular values of φ is generic in Z.* Recall that φ is called σ-proper if Y is the countable union of open sets, on the closure of each of which φ is proper. σ-properness is required as the Banach spaces we are interested in do not satisfy the Lindelöf property, which is necessary for the original version of the Sard-Smale theorem. To explain how we apply the Sard-Smale theorem, let us start considering neighborhoods $\mathcal{U} \subset \mathcal{V} \subset X$ of the set crit(f) of critical points of f such that each critical point of f belongs to a different connected component of $\mathcal{V}$, and let $\mathfrak{C}$ be the space of vector fields C of class C^2 on X such that:

(B1) every $C \in \mathfrak{C}$ vanishes on $\mathcal{U}$.

On $\mathfrak{C}$ we can introduce a norm $\|\cdot\|_{\mathfrak{C}}$ which induces the topology of C^2_{loc}-convergence and such that:

(B2) for every $C \in \mathfrak{C}$ with $\|C\|_{\mathfrak{C}} \leq 1$, the set of rest points of $F + C$ coincides with crit(f), f is a Lyapounov function for $F+C$, and $(f, F+C)$ satisfies the Palais–Smale condition.

For instance, pick a smooth function $\chi : [0, +\infty) \to \mathbb{R}$ such that

$$0 < \chi(\rho) < \frac{1}{2} \inf_{B_\rho(0)\setminus\mathcal{U}} -\mathrm{d}f[F], \quad \forall \rho \geq 0,$$

where $B_\rho(0)$ denotes the open ball with radius ρ around the origin in X, and define for every $C \in \mathfrak{C}$

$$\|C\|_{\mathfrak{C}} := \|\chi^{-1} \cdot C\|_{C^2}.$$

The straightforward proof that (B2) is satisfied is left to the reader. Notice that $\mathfrak{C}$ also satisfies:

(B4) $\mathfrak{C}$ is closed under multiplication by a vector space of functions which includes bump functions,

(B5) $\{C(u) \,|\, C \in \mathfrak{C}\} = T_uX \cong X$, for all $u \in X \setminus \mathcal{V}$.

We also notice that Properties (B1) and (B2) imply that

$$W^u(v;F+C)\cap W^s(w;F+C)$$

is pre-compact for all $\|C\|_{\mathfrak{C}}\le 1$ and all $v,w\in \mathrm{crit}(f)$.

Assume now that $v,w\in\mathrm{crit}(f)$ are such $\mu(v)-\mu(w)\le 2$ and

$$W^u(v;F)\cap W^s(w;F)\neq\emptyset.$$

We define the Banach space

$$K:=C^1_{v,w}(\mathbb{R},X):=\big\{\varphi:\mathbb{R}\to X\ \big|\ \varphi(t)\overset{t\to\pm\infty}{\longrightarrow} v,w,\ \dot\varphi(t)\overset{t\to\pm\infty}{\longrightarrow}0\big\}$$

and observe that the tangent space to K at each $\varphi\in K$ can be identified with

$$T_\varphi K\cong C^1_{0,0}(\mathbb{R},X)\subset B:=C^0_0(\mathbb{R},X).$$

Finally, denoting with $\mathfrak{C}_1$ the unit ball of $\mathfrak{C}$ we set

$$\Phi:\mathfrak{C}_1\times K\to B,\quad (C,\varphi)\mapsto\varphi'-(F+C)\circ\varphi,$$

so that

$$\mathcal{Z}:=\Phi^{-1}(0)=\bigcup_{C\in\mathfrak{C}_1}\Big(W^u(v;F+C)\cap W^s(w;F+C)\Big).$$

The fact that F is of class C^2 together with the fact that the topology on $\mathfrak{C}$ coincides with the topology of C^2_{loc}-convergence implies that Φ is of class C^2. Standard Fredholm theory (see [4, Lemma 5.6] and references therein) implies that, for $(C,\varphi)\in\mathcal{Z}$, $\mathrm{d}_\varphi\Phi(C,\varphi)$ is Fredholm with Fredholm index $\mu(v)-\mu(w)$, and it is onto if and only if $W^u(v;F+C)$ and $W^s(w;F+C)$ meet transversally along φ. This together with Properties (B4) and (B5) implies that 0 is a regular value for Φ, so that $\mathcal{Z}$ is a C^2-submanifold of $\mathfrak{C}_1\times K$; see Lemma 5.7 in [4] for the details.

Let now $\mathcal{S}\subset\mathcal{U}$ be a small smooth sphere centered at v and transversal to the flow of F (hence, also to the flow of $F+C$ for every $C\in\mathfrak{C}$ by Property (B1)). We denote by $\mathcal{Z}_0\subset\mathcal{Z}$ the codimension-one C^2-submanifold given by pairs $(C,\varphi)\in\mathcal{Z}$ such that $\varphi(0)\in\mathcal{S}$, and by

$$\pi:\mathcal{Z}_0\to\mathfrak{C}_1,\quad (C,\varphi)\mapsto C,$$

the projection onto the first factor. One readily sees that π is Fredholm of index $\mu(v)-\mu(w)-1$, and that $C\in\mathfrak{C}_1$ is a regular value of π if and only if $W^u(v;F+C)$ and $W^s(w;F+C)$ have transverse intersection. The Palais–Smale condition finally implies that π is σ-proper. This fact is proved in full details in [4, Proposition 5.9]; we shall however stress that the proof given there is much more involved as it deals with the case of infinite Morse index, case in which the Palais–Smale condition alone is actually not sufficient.

We are now in position to apply the Sard-Smale theorem, thus obtaining that the set $\mathfrak{C}_1(v,w)$ of regular values of the map π is generic in $\mathfrak{C}_1$. Since the set $\mathrm{crit}(f)$ is at most countable, the intersection

$$\mathfrak{C}_1^{MS}:=\bigcap\Big\{\mathfrak{C}_1(v,w)\ \Big|\ v\neq w\in\mathrm{crit}(f),\ \mu(v)-\mu(w)\le 2\Big\}$$

is also a generic subset of $\mathfrak{C}_1$, and by construction, for every $C\in\mathfrak{C}_1^{MS}$, the vector field $F+C$ satisfies the Morse-Smale property up to order 2. □

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