A Novel Numerical Approach for Solving Initial Value Problems in Heat Equations Using Variational Regularization and Intelligent Particle Swarm Optimization

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Abstract: In this article, we use the variational approach as a regularization tool to solve the initial value problem that appears in a heat partial differential equation. Although the temperature obtained at time t=T>0 is known, the initial temperature distribution remains unknown. By using the separation of variables method, the partial differential equation is transformed into a Fredholm integral equation of the first kind. We then apply a discretization algorithm to reduce the integral equation to a system of linear algebraic equations, commonly referred to as an inverse linear operator problem. The variational regularization method is employed to obtain a regularized solution. We also present a fundamental analysis of this method for solving inverse problems. Furthermore, we describe the application of the Intelligent Particle Swarm Optimization (IPSO) technique to determine the optimal regularization parameter. Our results demonstrate that integrating particle swarm optimization with variational optimization is both effective and computationally feasible.

1 INTRODUCTION

Because a "small" change in the data might result in "large" mistakes in the solution, the Cauchy inverse problem of the heat equation is ill-posed. According to Jacques Hadamard, an issue is well-posed if and only if the following characteristics are true [1].

- There is a solution, or at least one solution exists;
- The solution is unique; there is only one solution in existence;
- The data (stability) is a constant determinant of the solution.

The ill-posedness problem cannot be solved with traditional numerical techniques [2] – [4]. It calls for specialized methods, such as regularization approaches [5]. Inverse heat transfer problems can now be solved more easily with numerical methods because to the advancement of fast personal computers [6]. Numerous writers have addressed the inverse problem of heat equation's theoretical ideas and computer implementation, and numerous approaches have been detailed [4], [6] – [9].

Several regularization techniques have been developed to address the ill-posed nature of inverse equations of the first kind [10]. Classical methods include Tikhonov regularization [11], Landweber iteration [12] or other iteration methods [13], and the discrepancy principle [14]. Regularization introduces a penalty term to prevent overfitting and ensure stability, especially in ill-posed inverse problems [7]. The discrepancy principle helps determine the optimal balance between fitting the noisy data and avoiding excessive regularization [15]. However, these methods often require careful selection of a regularization parameter, which can significantly influence the quality of the solution [16], [17]. Nature-inspired optimization algorithms have shown promise in automating the selection of regularization parameters for ill-posed problems [18], [19]. Genetic Algorithms [20], Particle Swarm Optimization [21], and Artificial Bee Swarm Optimization have been employed to explore the parameter space and identify optimal or near-optimal values [15], [22]. These algorithms offer advantages in handling complex, non-convex objective functions, which are often encountered in regularization problems.

Numerous important search techniques use population-based optimization. These techniques are part of a class of algorithms, such as particle swarm optimization, that are inspired by nature. The PSO [23] algorithm, created by Eberhart and Kennedy, is capable of finding precise approximations or global optimum solutions. Numerous inverse problem types that can be resolved. By proposing a dynamic parameter update strategy, offering a new appropriateness choosing methodology, and balancing the trade-off between investigation and exploitation searches, the PSO method was used in [24] to address the electromagnetic inverse problems. In order to prevent getting stuck in local optima, this new methodology also includes the development of a new position updating formula. In the work [25], inverse design problems for cylindrical thermal multilayer shielding and cloaking shells are solved via particle swarm optimization. These inverse problems are converted into associated control problems. Particle Swarm Optimization (PSO) is often used to solve inverse problems, however it does not immediately resolve typical security issues. Current studies of post-quantum cryptography, however, provide a more thorough comprehension of computational algorithm security. Research like [26] and [27], for instance, highlight the need for secure implementations and demonstrate how inadequate it is to rely solely on secure algorithms. Even methods that are mathematically valid, such as PSO, might be exposed by implementation errors. Therefore, the implementation of PSO-Tikhonov Regularization must be secured by employing methods such as input validation, safe random number generation, and meticulous code review. The correctness and reliability of the obtained results are ensured by these methods [28]. Security concerns are not limited to implementations and methods; they also include specific hazards that machine learning models must deal with [29], [30].

Reconstructing the source function of the Initial Value Problem IVP for the heat partial differential equation is the primary concept in this work. Through a series of phases, the starting value problem has been reduced to a linear inverse problem. Because of this, the solution is unstable because it does not rely on the data continuously. As a result, this problem is ill-posed. We will use variational regularization to obtain a well-posed problem. In this paper, we will use Intelligent Particle Swarm Optimization (IPSO) to solve the problem of choosing the optimal regularization parameter.

2 PROBLEM STATEMENT

[1] 's IVP taking the heat (1) as follows,

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial^2 w(x,t)}{\partial x^2}, \tag{1}$$

where $x \in [0,l]$,$t \in (0,T]$,

$$w(0,t) = 0, \quad w(l,t) = 0, \tag{2}$$

where $t \in (0,T]$,

$$w(x,0) = w(x), \tag{3}$$

where x $[0,l]$.

The boundary conditions are $w(0,t)$ and $w(l,t)$, and the beginning condition is $w(x)$ which must be determined. The partial differential equation above can be reduced to integral form using the separation of variables method, and the result is:

$$Aw(x) = \int_0^l P(x,y)w(y)dy = g(x). \tag{4}$$

Every step of separation was used in [2]. We must limit the sum of series to ten times when the kernel $P(x,y)$ is an infinite series because we are unable to handle unlimited sums.

$$P(x,y) = \frac{2}{l}\sum_{n=1}^{10} e^{\frac{-(n\pi)^2 T}{l^2}} \sin(\frac{n\pi x}{l})\sin(\frac{n\pi y}{l}) \tag{5}$$

For providing the estimated $w(x)$ We are able to transform the problem of initial value to $Aw = g$, liner operator form by applied discretization algorithm as defined and explained in [3].

$$A\begin{bmatrix} w(y_0) \\ w(y_2) \\ \vdots \\ w(y_{n-1}) \end{bmatrix} = \begin{bmatrix} g(x_0) \\ g(x_2) \\ \vdots \\ g(x_{n-1}) \end{bmatrix}, \tag{6}$$

where

$$A = \frac{l}{n}\begin{bmatrix} P(x_0,y_0) & P(x_0,y_1) & \dots & P(x_0,y_{n-1}) \\ P(x_1,y_0) & P(x_1,y_1) & \dots & P(x_1,y_{n-1}) \\ \vdots & \vdots & \dots & \vdots \\ P(x_{n-1},y_0) & P(x_{n-1},y_1) & \dots & P(x_{n-1},y_{n-1}) \end{bmatrix}, \tag{7}$$

The matrix A has ill-conditionally property and we must locate $w(x) \in L_2[0,1]$. The exact answer will be $w(x) = \sin \pi x$, where the value of x $0 \le x \le 1$. We created the input data as $w(x,T) = g(x)$, where $T = 0.01$ and $0 \le x \le 1$

3 VARIATIONAL REGULARIZATION METHOD

D. Phillips (1962) suggested this approach, which entails solving a variational problem [14]. Let consider (4). in liner operator form

$$Aw = g, \tag{8}$$

Suppose that A is a bounded, linear, injective operator with A^{-1} not continuous. The data are the elements $\{A, g_\delta, \delta\}$. If the noise level δ > 0 is given, the estimate $\|g_\delta - g\| \le \delta$ holds, and the noisy data y_δ is the δ−approximation of y.

Now, let's look at (8). The challenge is to identify the stable solution w_δ such that the error estimate $\|w_\delta - w\| \le \eta(\delta) \to 0$ as $\delta \to 0$ holds given $\{A, g_\delta, \delta\}$.

Let z be the minimal-norm solution to (8). By addressing the variational problem (minimization problem) and using a minimal-norm to create a stable approximation to the answer z, the variational regularization approach ensures that $z \perp N(A)$. Assume $\|A\| \le 1$ without losing generality, and then $\|A^*\| \le 1$. Assuming that $B = A^*A$, then $B \ge 1$ is a self-adjoint, bounded operator. The equation $Bw = q$, where $q = A^*g$, is equal to (8).

If q, $\|q - q_\delta\| \le \|T^*\| \delta \le \delta$, then q_δ is provided instead. Given that $N(T) = N(B)$ $z \perp N(B)$ and $\|(B+\alpha)^{-1}\| \le \frac{1}{\alpha}$, Examine the issue of determining the functional minimum.

$$F(w) = \|Aw - g_\delta\|^2 + \alpha \|w\|^2 = \min, \tag{9}$$

where $\alpha > 0$ parameter of regularization and $F(w)$ is a function for α, and δ. Minimizers are solutions to the variational problem (9). As a result, the next formula provides the unique solution to (10).

$$w = (A^*A + \alpha)^{-1} A^* g_\delta. \tag{10}$$

4 IPSO ALGORTHIM

The IPOS method for choosing the variational regularization parameter α without knowing the noise level δ is examined in the following.

The following is a concise statement of IPSO's core idea [5]. Every swarm member, referred to as a "particle," is a potential solution; each particle adjusts its position within the search domain and updates its velocity based on its own and its "neighbors'" flying experiences at each iteration, aiming for a better position for itself as long as it satisfies specific fitness requirements.

The "current best position" (*cbest*), the site with the highest engagement rate across all affiliates from the beginning to the end of the search, is effective and is iteratively taught by the algorithm. Every particle additionally commits to memory its best experienced position, or personal best position (*pbest*). The distance each particle travels to reach its subsequent place is calculated using the equation $\Delta x = e\Delta t$, where e is the velocity. At iteration t, it is computed as follows for particle I:

$$e_i^{t+i} = \omega e_i^t + c_1 r_1 d_c^{i,t} + c_2 r_2 d_p^{i,t}, \tag{11}$$

$$d_p^{i,t} = \left[x_i^t - pbest^{i,t} \right], \tag{12}$$

$$d_c^{i,t} = \left[x_i^t - cbest^t \right]. \tag{13}$$

A particle's best individual and global positions are separated by $d_c^{i,t}$ and $d_p^{i,t}$, respectively, from its current position. The symbol ω is well-defined as inertia weight, r_1 & r_2 are numbers defined randomly, and c_1, c_2 are coefficients for acceleration. Between any two following repetitions, $\Delta t = 1$ between any two subsequent iterations (i.e., $\Delta t = (t+1) - t = 1$), Consequently, at iteration, particle I will be in the following location $(t+1)$:

$$a_i^{t+1} = a_i^t + v_i^{t+1}. \tag{14}$$

The procedures that follow now apply the IPSO calculation algorithms to solve the problem.

Step 1: Describe the IPSO and the issue. The IPSO and integral equations of the first kind are defined in this section. We develop the "cost function" or fitness function (Cost_Fun). Next, we must configure the following function and variables:

- n - size of domain;
- Cost function:

```
Cost_Fun (w, A, g, n)
{
```
$I = eye(n,n)$

$w = inv(A^{*}A + \alpha I) * A^{*}g$

return $\|Aw - g_{\delta}\|^{2} + \|\alpha\|\|w\|^{2}$
```
}
```

```
n_Var; / number of variables for unknown decisions;
Var_Size = [1, n_Var]; / choice variables' matrix size;
Var_Min;/choice variables' lowest bound;
Var_Max;/choice variables' upper bound.
```

Step 2: Parameters:
```
Max_It
n_Pop /population size
w, c1, c2
```

Step 3: Population Initialize:

```
arr_Pos = [ ]; arr_Vel = [ ]; arr_Cost = [ ];
arr_Best.Pos = [ ]; arr_Best.Cost = [ ];
arr_Cur_Best = inf;
Loop over (n_Pop)
arr_Pos = unifrnd (Var_Max, Var_Size, Var_Min);
arr_Vel = zeros (Var_Size);
arr_Cost = Cost_Fun(arr_Pos);
"Update" arr_Best.Pos and arr_Best.Cost = arr_Cost;
"Update "
If arr_Best.Cost < arr_Cur.Best Cost
arr_Curt.Best = arr_Best.Pos and arr_Best.Cost = arr_Cost
Over n_Pop end loop
```

Step 4: Main loop of IPSO:
- Loop over iterations it=1: Max_It;
- Loop under population size (n_Pop);
- Update velocity:

```
arr_Vel = w * arr_Vel + c1*rand (Var_Size) .* arr_Best.Pos - arr_Pos+ c2*rand(Var_Size) .* arr_Cur.Best.Pos - arr_Pos);
```

- Position updating:

```
arr_Pos = arr_Pos + arr_Vel;
```

- Assessment:

```
arr_Cost = Cost_Fun(arr_Pos);
```

- Update:

```
if arr_Cost < arr_Best.Cost
      arr_Best.Pos = arr_Pos;
      arr_ Best.Cost = arr_Cost;
```

- Update:

```
if arr_Best.Cost < arr_Cur.Best.Cost
      arr_Curt.Best = Best;
      End
End
```

- Loop n_Pop ending;
- Store the best cost value arr_Best.Costs(it) = arr_Cur.Best.Cost;
- End loop over iterations Max_It.

Step 5: Create the results.
- α= arr_Best.Costs(it);
- The inverse problem's approximate solution can be found by: $w = inv(A^{*}A + \alpha I) * A^{*}g,$.

5 NUMERICAL EXPERIMENTS

We created the inverse problem's input data as $w(x,T) = g(x), 0 \le x \le 1$ and $T = 0.01$. The IPSO values for the common parameters for all testing with noise-free and nose-level are as follows:

n_Var = 1, Var_Size = [1, n_Var], Var_Min = 0, Var_Max = 1, c1 = c2 = 2.

Figures 1, 2, and 3 display the Initial Value test. A situation with substantial measurement noise (δ = 0.2) and (δ = 0.1) is shown in Figures 1 and 2, respectively. In spite of this noise, the IPSO method finds a good approximation (IPSO solution) and determines the ideal (α) parameter. In comparison, the noise-free scenario (δ = 0) is depicted in Figure 3. In this case, zero is the ideal α value.

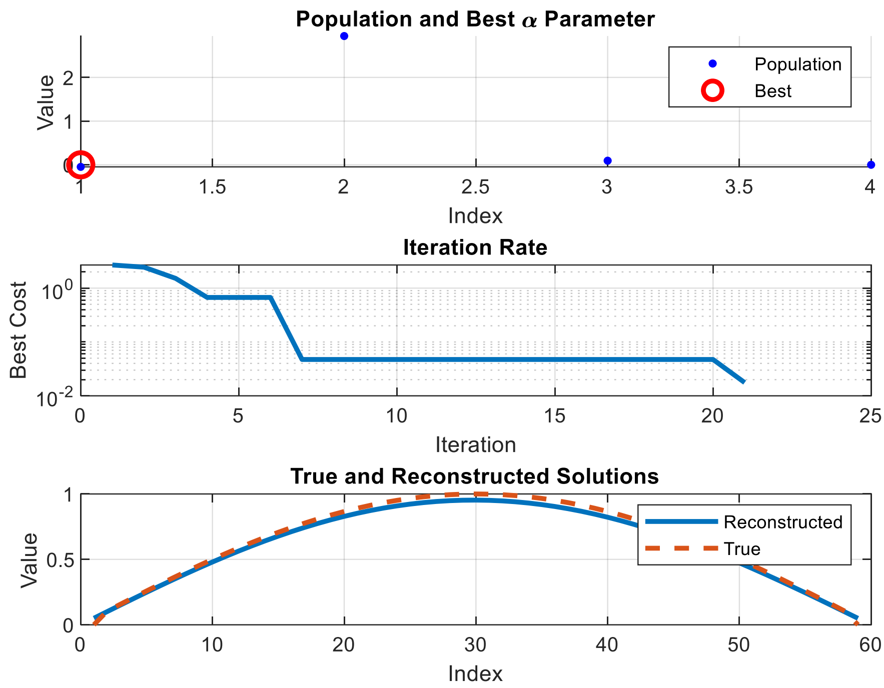

Figure 1: Initial value with δ=0.2 and best α=0.018.

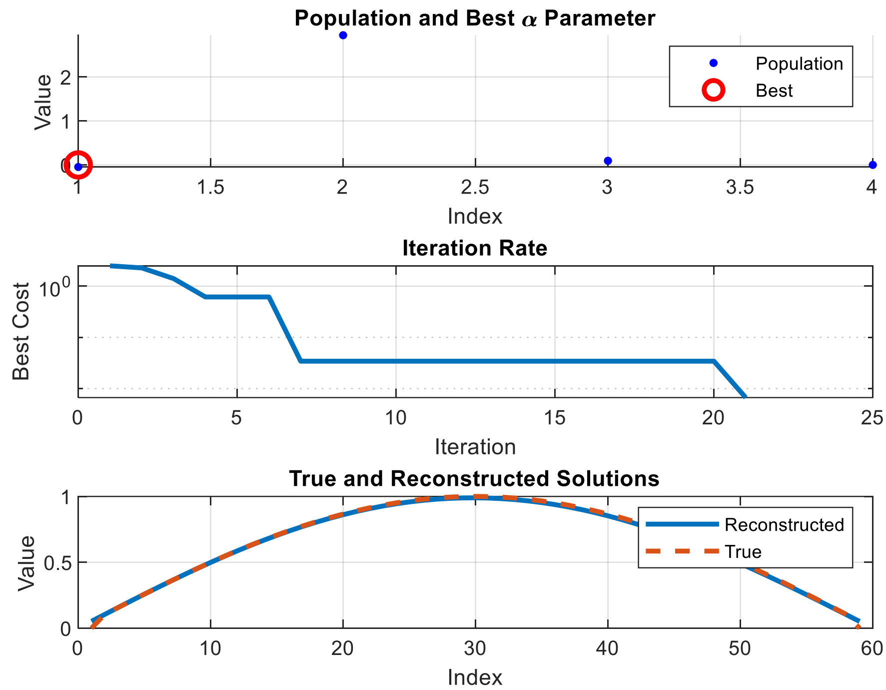

Figure 2: Initial value with δ=0.1, and best α=0.006.

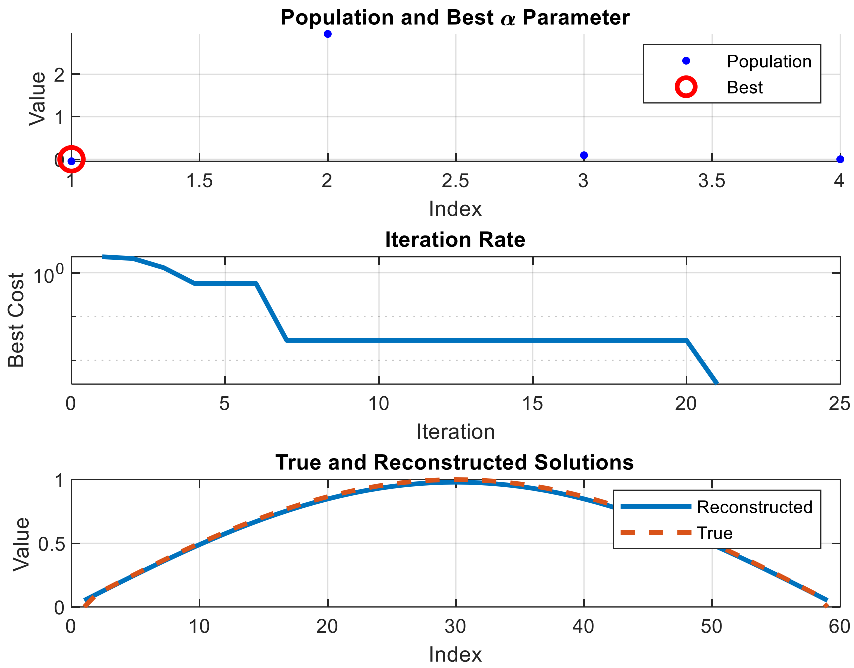

Figure 3: Initial value with δ=0, best α=0.

6 CONCLUSIONS

In order to solve the heat equation's initial value problem with both noisy and noise-free data, we examined the variational regularization technique and Intelligent Particle Swarm Optimization (IPSO) in this research. We sought to discover a stable solution to the problem, which was converted from a partial differential equation to an inverse linear operator equation. The regularization parameter in the variational regularization method was chosen using a novel technique to guarantee convergence to an appropriate approximation. IPSO played a crucial role in optimizing this parameter by searching for the best value that minimized the variational equation associated with the Tikhonov method. Numerical experiments, including test cases such as the Phillips and Gravity problems, demonstrated the effectiveness of integrating Tikhonov regularization with IPSO. The results confirm that this hybrid approach enhances solution accuracy and stability, making it a promising method for addressing inverse problems in heat equations.

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