Investigation of Doubly Nonlinear Parabolic Equation

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Abstract: We study the properties of solutions for a porous medium equation (PME) in non-divergent form with a source term. The PME is a fundamental model in various physical and biological processes, including fluid flow through porous media, heat transfer, and population dynamics. Unlike the classical heat equation, the PME exhibits nonlinear diffusion, leading to rich mathematical structures and solution behaviours. Our main focus is obtaining exact solutions using the separable variable method under certain parameter constraints. These solutions provide explicit representations of the evolving profile of the medium and provide insight into the dynamics of the equation. Additionally, we construct a self-similar Barenblatt-type solution, a fundamental tool for analysing long-time asymptotics and the spreading behaviour of solutions. Self-similar solutions provide insights into the scaling properties of the PME and the influence of the source term on solution evolution. Moreover, we have constructed a numerical scheme, calculated numerical results and based on numerical solutions shown graphs in some particular cases.

1 INTRODUCTION

We consider the nonlinear equation stated as follows:

$$\rho(x)u_t = u^q div\left(\rho(x)\nabla u^m\right) + \varepsilon\gamma(t)\rho(x)u^\beta, \quad (1)$$

$$u(x,0) = u_0(x) \geq 0, x \in R^N, \quad (2)$$

where $(x,t) \in D, D = \{(x,t) \mid x \in R^N, t > 0\}$, $m \geq 1, q \neq \{1 \pm m, \beta - m\}, \varepsilon = \pm 1, \rho(x) = |x|^{1-N}$, $0 < \gamma(t) \in C(R_+), \beta$ depends on the function $\gamma(t)$, and we consider a few cases throughout the paper.

Equation (1) includes many well-known equations, such as the Laplacian equation, heat equation, Leibenson equation, and the Boussinesq equation in the filtration of liquid and gas [1], [2], [3], [4]. Equation (1) is important for simulating a broad variety of physical processes. For example, curve shortening flow, resistive diffusion phenomena in force-free magnetic fields, diffusive processes found in biological species, and the spread of infectious diseases are among the many applications of (1) (see [5], [6], [7], [8] and references therein).

J.L. Díaz studied the following problem, which addresses the critical issue of modelling an aircraft fire extinguishing process within an engine nacelle [9]

$$u_t = \Delta u^m + |x|^\sigma u^p, \quad (3)$$

$$u(x,0) = u_0(x) \in L^\infty_{loc}(R^N), \quad (4)$$

where $m > 1, \sigma > 0, p < 1$.

J.L. Díaz used the nonlinear splitting method and the comparison principle [10]-[11] to obtain a self-similar solution in the following form:

$$u(x,t) = t^\alpha \left(A - \frac{m-1}{2m}\beta |x|^2 t^{-2\beta}\right)_+^{1/(m-1)},$$

where $\beta = \dfrac{1}{N(m-1)+2}, \alpha = \beta N, (d)_+ = max(d,0)$.

Moreover, J.L. Díaz showed that finite values for the model parameters p, m and σ exist, and that the combination of such values ensures the existence of global solutions

J.L. Vázquez studied [12] the PME and the existence of solutions in finite time or globally for the following problem:

$$u_t = \Delta u^m + f(x,t), \quad (5)$$

$$u(x,0)=u_0(x), x\in R^N, \quad (6)$$

where f is a reaction term.

J.L. Vázquez showed the decay rate of the solution such that $f=0, \|u(.,t)\|_{L^p}\le Ct^{-\gamma_p}$, $\gamma_p=N(p-1)/((m-1)N+2)$; on the other hand, $f=u^q, \|u(.,t)\|_\infty : (T_c-t)^{-1/(q-1)}, T_c>0$.

In the work [13] the existence of the solution and blow-up problems of the Cauchy problem are studied for a more general case.

Another significant work by D.G. Aronson [14] studied the PME using self-similar analysis [15], [16], the comparison principle, dynamical system behaviour, and other methods. D.G. Aronson obtained lower – upper estimates of the solution to the problem (5)-(6), and showed the decay rate of the solution.

M. Winkler showed that all global solutions of (5) with zero boundary data are uniformly bounded in the case $\sigma=0$ [17].

R.G. Iagar and D.R. Munteanu [18] studied a very singular solution of (5) in an absorption case based on self-similar solutions and a dynamical system [19]. The authors showed the critical Fujita exponent [20] as follows:

$$p_F=m+(\sigma+2)/N.$$

Moreover, the authors proved that $u(x,t): C|x|^{-(\sigma+2)/(p-m)}$ as $|x|\to\infty$ while $p\ge p_F$, where C - some constant.

We seek the solution in several cases for the function $\gamma(t)$ and β, considering different functional forms of $\gamma(t)$ (such as constant, power-law, or polynomial) as well as the general case, and analyze how various choices of β influence the qualitative and quantitative properties of the solution, including decay rates, self-similarity, and asymptotic behaviour.

2 SEPARATION OF VARIABLES

We consider the following cases:

I. $\gamma=const>0$,

II. $\gamma(t)=A(T-\varepsilon t)^a$,

III. General case.

2.1 I Case

In this case, we consider $\beta=q+m$. First, using the separation method [21], we seek the solution to (1) as follows:

$$u(x,t)=(T-\varepsilon t)^{-\alpha}y(r), \quad (7)$$

where $r=|x|$ and $\alpha=\frac{1}{q+m-1}>0$.

We put the (7) into (1). Then we obtain:

$$\varepsilon\alpha r^{1-N}(T-\varepsilon t)^{-\alpha-1}y= (T-\varepsilon t)^{-\alpha(q+m)}r^{1-N}\left[y^q\left((y^m)'\right)+\varepsilon\gamma y^{q+m}\right]. \quad (8)$$

Therefore, we rewrite the equation as follows:

$$\left((y^m)'\right)-\varepsilon\alpha y^{1-q}+\varepsilon\gamma y^m=0, \quad (9)$$

or $$y''+(m-1)\frac{y'^2}{y}-\frac{\varepsilon}{m}(\alpha y^{2-q-m}-\gamma y)=0. \quad (10)$$

We denote $y(r)=z^{1/2}(r)$. Hence, we get

$$y'=\frac{1}{2}z^{-1/2}z'_r=\frac{1}{2}z'_y.$$

Consequently, we obtain the following equation:

$$\frac{1}{2}z'_y+(m-1)\frac{z}{y}-\frac{\varepsilon}{m}(\alpha y^{2-q-m}-\gamma y)=0. \quad (11)$$

We seek z as

$$z(y)=z_1(y)z_2(y), \quad (12)$$

and put it into (14)

$$\frac{1}{2}(z_1z_2)'+(m-1)\frac{z_1z_2}{y}-\frac{\varepsilon}{m}(\alpha y^{2-q-m}-\gamma y)=0. \quad (13)$$

We choose z_2 such that:

$$\frac{z_1}{2}\left(z_{2'}+\frac{2(m-1)z_2}{y}\right)=0,\ z_2(y)=y^{2-2m},$$

that implies:

$$z=\frac{2\alpha\varepsilon}{m(m-q+1)}y^{3-q-m}\left(1-\frac{(m-q+1)\gamma}{2m\alpha}y^{q+m-1}\right). \quad (14)$$

Hence, we get

$$z=\frac{2\alpha\varepsilon}{m(m-q+1)}y^{3-q-m}\left(1-\frac{(m-q+1)\gamma}{2m\alpha}y^{q+m-1}\right). \quad (15)$$

Therefore, we derive that

$$\sqrt{\frac{m(m-q+1)}{2\alpha}}\int\frac{y^{\frac{m+q-1}{2}-1}dy}{\sqrt{\varepsilon-\varepsilon\left(\sqrt{\frac{(m-q+1)\gamma}{2m\alpha}}y^{\frac{q+m-1}{2}}\right)^2}}=$$

$$\left|\begin{array}{l} y^{\frac{m+q-1}{2}}=\sqrt{\frac{2m\alpha}{(m-q+1)\gamma}}\varphi(Y), \\ \varphi'=\psi,\psi^2=\varepsilon(1-\varphi^2) \end{array}\right|=\frac{2m}{(m+q-1)\sqrt{\gamma}}\int\frac{\psi dY}{\sqrt{\varepsilon-\varepsilon\varphi^2}}$$

$$=\frac{2m}{(m+q-1)\sqrt{\gamma}}Y=r+\frac{2m}{(m+q-1)\sqrt{\gamma}}C_1, \qquad (19)$$

$$y(x)=\left[\sqrt{\frac{2m\alpha}{m-q+1}}\varphi\left(\frac{(m-q+1)\sqrt{\gamma}}{2m}|x|+C_1\right)\right]^{2\alpha},$$

where $\varphi(x)=\begin{cases} cos(x), & \text{if} \quad \varepsilon=1, \\ ch(x), & \text{if} \quad \varepsilon=-1. \end{cases}$ (20)

Moreover, we have

$$u(x,t)=\begin{cases}\left(\sqrt{\frac{2m\alpha}{m-q+1}}\frac{cos^2(\frac{m-q+1}{2m}\sqrt{\gamma}|x|+C_1)}{T-t}\right)^{\alpha}, \varepsilon=1, \\ \left(\sqrt{\frac{2m\alpha}{m-q+1}}\frac{ch^2(\frac{m-q+1}{2m}\sqrt{\gamma}|x|+C_2)}{T+t}\right)^{\alpha}, \varepsilon=-1. \end{cases} \qquad (16)$$

It is easy to see that, $u_1(x,t)\to\infty$ as $t\to T^-$ in (16). This kind of solution is called a blow-up solution and is intensively studied in the works [22]-[23], [24]-[25]. Similar results were also obtained in the work [26]-[27] in the case $q=0$, and using the method of separation of variables, alternative solutions were found in some groups by A.D. Polyanin and A.I. Zhurov [28].

2.2 II Case

In this case, we consider $\gamma(t)$ as follows

$$\gamma(t)=A(T-\varepsilon t)^a, \qquad (17)$$

where $A,a>0$, and $\beta=1+(a+1)(q+m-1)$.

We seek the solution $u(x,t)$ as follows:

$$u(x,t)=B(T-\varepsilon t)^{-b}g(r) \qquad (18)$$

where $b=\frac{a+1}{\beta-1}, B=(bA)^{-\frac{1}{\beta-1}}$.

After inserting the notation (18) into (1) it yields following

$$Cg^q\left((g^m)'\right)-\varepsilon g+\varepsilon g^{\beta}=0, \qquad (19)$$

which is invariant to the following equation under the mapping $r\to\frac{r}{\sqrt{C}}$

$$g^q\left((g^m)'\right)-\varepsilon g+\varepsilon g^{\beta}=0, \qquad (20)$$

where $C=B^{q+m-\beta}/A$.

Using a similar approach as in (10)-(15), we deduce that

$$\sqrt{\frac{m(m-q+1)}{2}}\int\frac{g^{\frac{m+q-3}{2}}dg}{\sqrt{\varepsilon-\varepsilon\frac{m-q+1}{\beta-m-q}g^{\beta-1}}}=\left|g^{\beta-1}=\frac{\beta-m-q}{m-q+1}z\right|$$

$$=\frac{\sqrt{m\left((\beta-m-q)(m-q+1)^a\right)^{\frac{1}{a+1}}}}{\sqrt{2}(\beta-1)}\cdot\int(\varepsilon(1-z))^{-\frac{1}{2}}z^{\frac{1}{2(a+1)}-1}dz$$

$$=C_3\cdot J_{\varepsilon}\left(z;\frac{1}{2},\frac{1}{2(a+1)}\right)=r+C_3C_4, \qquad (21)$$

where $C_4\geq 0, J_{\varepsilon}(z;a,b)=\int(\varepsilon(1-z))^{a-1}z^{b-1}dz$,

$$C_3=\frac{\sqrt{m\left((\beta-m-q)(m-q+1)^a\right)^{\frac{1}{a+1}}}}{\sqrt{2}(\beta-1)},$$

It is easy to see that

$$\frac{r}{C_3}+C_4=J_1\left(z;\frac{1}{2},\frac{1}{2(a+1)}\right)= C_5+B\left(z;\frac{1}{2},\frac{1}{2(a+1)}\right), \quad \text{for } 0\leq z\leq 1, \qquad (22)$$

where $B(z;a,b)$ is an incomplete beta function and C_5 some constant.

Furthermore, we rewrite (22) as follows:

$$\frac{|x|}{C_3}+C_4-C_5=I_z\left(\frac{1}{2},\frac{1}{2(a+1)}\right)\cdot B\left(\frac{1}{2},\frac{1}{2(a+1)}\right), \qquad (23)$$

where $B\left(\frac{1}{2},\frac{1}{2(a+1)}\right)$ and $I_z\left(\frac{1}{2},\frac{1}{2(a+1)}\right)$ - beta and regularized incomplete beta functions, respectively.

From (23), we conclude that g is a solution to the following equation:

$$I_{\frac{m-q+1}{\beta-m-q}g^{\beta-1}(r)}\left(\frac{1}{2},\frac{1}{2(a+1)}\right)=\frac{r}{\sqrt{C}C_3B\left(\frac{1}{2},\frac{1}{2(a+1)}\right)}+C_6. \qquad (24)$$

where $C_6 = \left(C_4 - C_5\right) / B\left(\frac{1}{2}, \frac{1}{2(a+1)}\right)$.

Now, we consider $\varepsilon = -1$ case and compute following integral

$$\frac{r}{C_3} + C_4 = J_{-1}\left(z; \frac{1}{2}, \frac{1}{2(a+1)}\right) = \int (z-1)^{-\frac{1}{2}} z^{\frac{1}{2(a+1)}-1} dz$$

$$= \left| z = \frac{1}{1-\theta} \right| = \int (1-\theta)^{\frac{a}{2(a+1)}-1} \theta^{-\frac{1}{2}-1} d\theta = C_7 +$$

$$I_{1-1/z}\left(\frac{1}{2}, \frac{a}{2(a+1)}\right) \cdot B\left(\frac{1}{2}, \frac{a}{2(a+1)}\right), \quad \text{for } z > 1. \quad (25)$$

Analogously, we derive that:

$$I_{1-\frac{\beta-m-q}{m-q+1} g^{1-\beta}(r)}\left(\frac{1}{2}, \frac{a}{2(a+1)}\right) = \frac{r}{\sqrt{C} C_3 B\left(\frac{1}{2}, \frac{a}{2(a+1)}\right)} + C_8. \quad (26)$$

where $C_8 = \left(C_4 - C_7\right) / B\left(\frac{1}{2}, \frac{a}{2(a+1)}\right)$ and C_7 some constant.

We are interested only in non-negative solutions of the (1). Thus, we have skipped the cases: $z > 1$ in (27) and $z < 1$ in (25). We can find g explicitly [29] using the inverse of the incomplete beta function.

2.3 III Case

We rewrite the (1) as follows:

$$u_t = u^q \left(u^m\right)_{rr} + \varepsilon \gamma(t) u^\beta, \quad (27)$$

where $r = |x|$.

Using the nonlinear splitting method, we seek a solution to the (27) as follows

$$u(r,t) = u_1(t) w(\tau(t), r), \quad (28)$$

where $\tau(t) = \int u_1^{q+m-1}(t) dt$,

$$u_1(t) = \begin{cases} \left(T_1 + \varepsilon(\beta-1)\int_0^t \gamma(y) dy\right)^{-1/(\beta-1)}, & \beta \neq 1, \\ T_2 \exp(-\varepsilon \int_0^t \gamma(y) dy), \ \beta = 1, \ T_{1,2} = const \geq 0 \end{cases},$$

We put the (33) into (32), then the following yields:

$$w_\tau = w^q \left(w^m\right)_{rr} + \frac{\varepsilon u_2(t)}{\tau(t)} (w^\beta + w), \quad (29)$$

where $u_2(t) = \gamma(t) \tau(t) u_1^{\beta-1-m}(t)$.

Now, we seek $w(\tau, r)$ as follows:

$$w_\tau = w^q \left(w^m\right)_{rr} + \frac{\varepsilon u_2(t)}{\tau(t)} (w^\beta + w), \quad (30)$$

After substituting (30) into (29) we obtain the following results:

$$f^q \frac{d}{d\xi}\left(\frac{df^m}{d\xi}\right) + \frac{\xi}{2} \frac{df}{d\xi} + \varepsilon u_2(t)(f^\beta + f) = 0. \quad (31)$$

or $$L_1 f + L_2 f = 0, \quad (32)$$

where $L_1 f = f^q \frac{d}{d\xi}\left(\frac{df^m}{d\xi} + \frac{\xi f^{1-q}}{2(1-q)}\right)$,

$$L_2 f = f\left(\varepsilon u_2(t)\left(f^{\beta-1} + 1\right) - \frac{1}{2(1-q)}\right).$$

We introduce a new function

$$z(x,t) = u_1(t) f_1(\xi), \quad (33)$$

where $f_1(\xi) = A_1\left(\xi_0^2 - \xi^2\right)_+^{\frac{1}{q+m-1}}$,

$$A_1 = \left(\frac{q+m-1}{4m(1-q)}\right)^{\frac{1}{q+m-1}}, \xi_0 = const \geq 0.$$

Now, based on the comparison principle and $z(x,t)$ function, we estimate the solution of the problem (1)-(2).

Theorem 1: let one of the following inequalities hold:

- $\varepsilon > 0, u_2(t) \leq \frac{1}{A_2}, A_2 = 2(1-q)\left(1 + (A_1 a^{\frac{1}{q+m-1}})^{\beta-1}\right)$;
- $\varepsilon < 0$;

and $u_0(x) \leq z(x,0), x \in R^N$. Then the solution to the problem (1)-(2) has the estimate in D

$$u(x,t) \leq z(x,t),$$

and for the front the following estimate

$$|x| \geq x_f(t) = \xi_0 \tau^{1/2} \text{ hold.}$$

Proof. According to the definition, the function f_1 is nonzero if $\xi < \xi_0$, and equal to zero if $\xi \geq \xi_0$ or $|x| \geq \xi_0 \tau^{1/2}$. $|x| = \xi_0 \tau^{1/2}$ is called front (free boundary [10], [12]).

Now we show that $L_1 f_1 + L_2 f_1 \leq 0$. It is easy to see that f_1 satisfies the equality $L_1 f_1 = 0$. Therefore, from (27)-(31) it suffices to show that $L_2 f_1 \leq 0$ in order to hold $u(x,t) \leq z(x,t)$ until $u_0(x) \leq z(x,0)$. Since f_1 is a non-negative function, the following inequality yields:

$$\varepsilon u_2(t)(f_1^{\beta-1}+1)\le\frac{1}{2(1-q)}. \qquad (34)$$

Obviously, if $\varepsilon<0$ then inequality (34) holds. On the other hand, if $\varepsilon>0$:

$$u_2(t)(f_1^{\beta-1}+1)\le u_2(t)(\max(f_1^{\beta-1})+1)=$$
$$u_2(t)((A_1 a^{1/(q+m-1)})^{\beta-1}+1),$$
$$\text{or } \frac{A_2}{2(1-q)}u_2(t)\le\frac{1}{2(1-q)}.$$

The proof of the Theorem 1 is completed.

3 NUMERICAL ANALYSIS

In order to perform numerical calculations, we need to construct numerical grids Ω_x and Ω_t respectively

$$\Omega_x=\{x\,|\,x_i=a+ih_x, h_x=\frac{b-a}{N_x}; b>a\},$$

$$\Omega_t=\{t\,|\,t_j=jh_t, h_t=\frac{T}{N_t}, T>0\}.$$

Now, we rewrite the (27) on the grid $\Omega=\Omega_x\times\Omega_t$ as explicit scheme form with $O(h_t+h_x^2)$ approximation error

$$\begin{cases}\dfrac{y_{i,j}-y_{i,j-1}}{h_t}=a(y_{i+1})\dfrac{y_{i+1,j}-y_{i,j}}{h_x^2}-a(y_i)\dfrac{y_{i,j}-y_{i-1,j}}{h_x^2}\\ +\varepsilon\gamma(t_{j-1})(y_{i,j-1})^{\beta}, \qquad i=\overline{1,N_x-1}, j=\overline{1,N_t},\\ y_{i,0}=u_0(x_i), i=\overline{0,N_x},\\ y_{0,j}=\phi_1(t_j),\; y_{N_x,j}=\phi_2(t_j),\; j=\overline{0,N_t},\end{cases} \qquad (35)$$

where $\phi_1(t_j)=z(x_0,t_j), \phi_2(t_j)=z(x_{N_x},t_j)$,

$$a(y_i)=\begin{cases} m\cdot(y_{i-1,j})^{q}\left(0.5y_{i,j}-0.5y_{i-1,j}\right)^{m-1},\\ 0.5m\cdot(y_{i-1,j})^{q}\left((y_{i,j})^{m-1}-(y_{i-1,j})^{m-1}\right),\end{cases}$$

To solve (35) we use iteration processes, so

$$\begin{cases}\dfrac{y_{i,j}^{s}-y_{i,j-1}^{s-1}}{h_t}=a(y_{i+1}^{s-1})\dfrac{y_{i+1,j}^{s}-y_{i,j}^{s}}{h_x^2}-a(y_i^{s-1})\dfrac{y_{i,j}^{s}-y_{i-1,j}^{s}}{h_x^2}\\ +\varepsilon\gamma(t_{j-1})(y_{i,j-1}^{s-1})^{\beta}, \quad i=\overline{1,N_x-1}, j=\overline{1,N_t}, s=1,2,\dots\\ y_{i,0}=u_0(x_i), i=\overline{0,N_x},\\ y_{0,j}=\phi_1(t_j),\; y_{N_x,j}=\phi_2(t_j),\; j=\overline{0,N_t},\end{cases} \qquad (36)$$

The iteration process continues until the following condition is satisfied:

$$\max_{i,j}\left|y_{i,j}^{s}-y_{i,j}^{s-1}\right|<\varepsilon,$$

where ε precision and we took it as 10^{-6}.

We denote

$$A_{i,j}^{s}=\frac{h_t}{h_x^2}a(y_{i+1}^{s-1}),\; B_{i,j}^{s}=\frac{h_t}{h_x^2}a(y_i^{s-1}),\; C_{i,j}^{s}=A_{i,j}^{s}+B_{i,j}^{s}+1,$$

$$F_{i,j}^{s}=y_{i,j-1}^{s-1}+\varepsilon h_t\gamma(t_{j-1})(y_{i,j-1}^{s-1})^{\beta} \qquad (37)$$

Applying this notation, we rewrite (36) as follows:

$$\hat{A}_i\hat{y}_{i+1}-\hat{C}_i\hat{y}_i+\hat{B}_i\hat{y}_{i-1}=-\hat{F}_i, \qquad (38)$$

where $\hat{y}_i=y_{i,j}^{s}, \hat{A}_i=A_{i,j}^{s}, \hat{B}_i=B_{i,j}^{s}, \hat{C}_i=C_{i,j}^{s}, \hat{F}_i=F_{i,j}^{s}$.

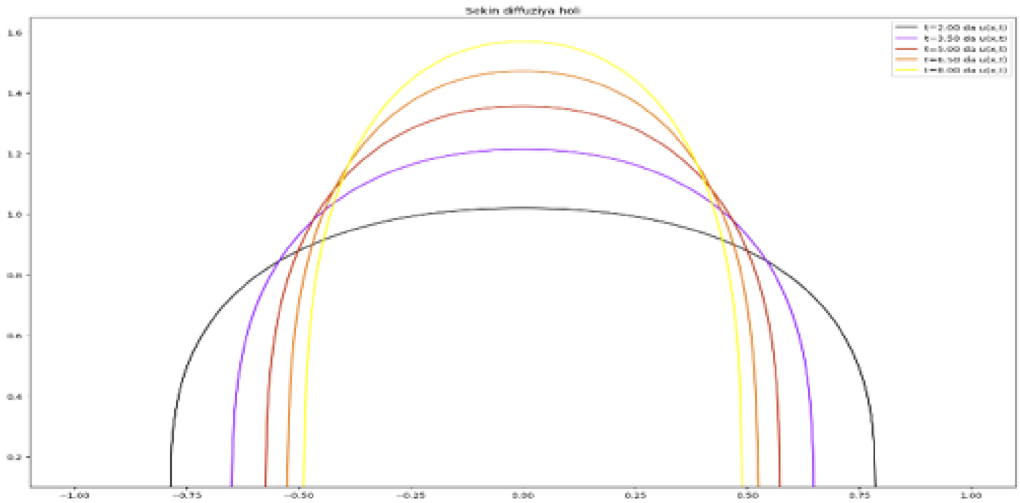

Figure 1: The numerical solution $q=0.5, m=2.2, \beta=2.01, \varepsilon=1, \gamma(t)=1$.

To solve the algebraic (38) we apply the tridiagonal algorithm [30]

$$\hat{y}_i = \alpha_i \hat{y}_{i+1} + \beta_i \qquad (39)$$

where $\alpha_i = \dfrac{\hat{A}_i}{\hat{D}_i}$, $\beta_i = \dfrac{\hat{F}_i + \beta_{i-1}\hat{B}_i}{\hat{D}_i}$, $\hat{D}_i = \hat{C}_i - \alpha_{i-1}\hat{B}_i$, $i = \overline{1, N_x - 1}$, and the initial values are $\alpha_0 = \dfrac{\hat{A}_1}{\hat{C}_1}, \beta_0 = \dfrac{\hat{F}_1 - \phi_1 \hat{B}_1}{\hat{C}_1}$.

To satisfy the diagonal dominant condition, which guarantees the stability of an algorithm or numerical scheme, the following condition must be held:

$$|\hat{C}_i| \geq |\hat{A}_i| + |\hat{B}_i|. \qquad (40)$$

According to the definition $\hat{C}_i$ in (37) the inequality (40) is valid.

Below we illustrate graphics of the numerical solution with Figures 1-4.

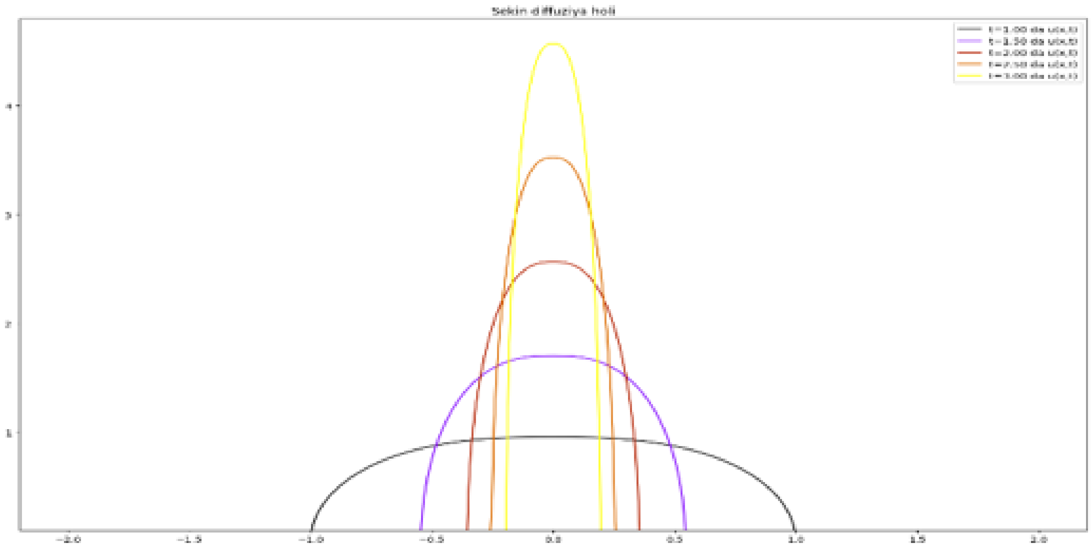

Figure 2: The numerical solution $q = -0.1, m = 3.2, \beta = 1.5, \varepsilon = -1, \gamma(t) = t^{0.1}$.

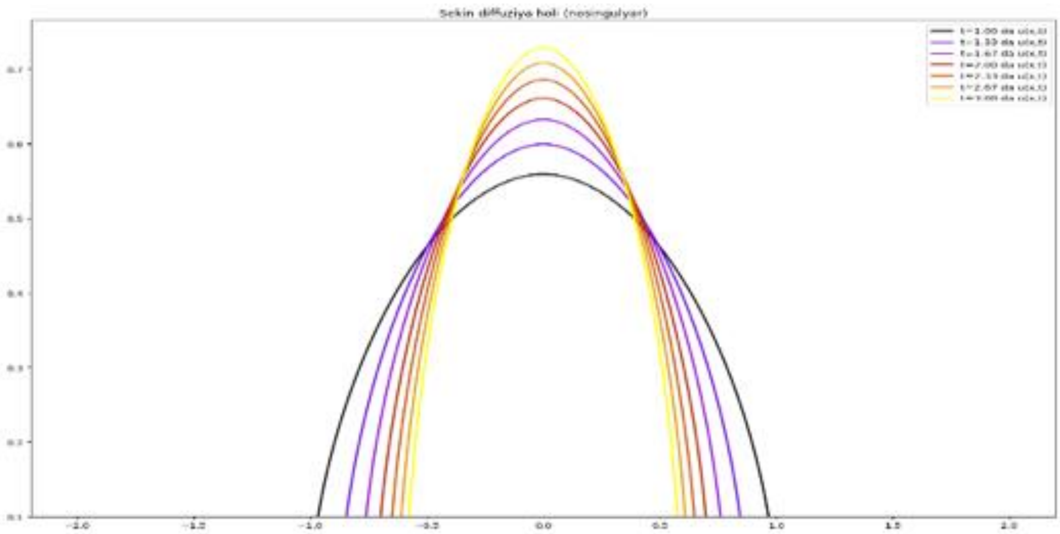

Figure 3: The numerical solution $q = 0.1, m = 2.2, \beta = 2.1, \varepsilon = -1, \gamma(t) = t^{0.5}$.

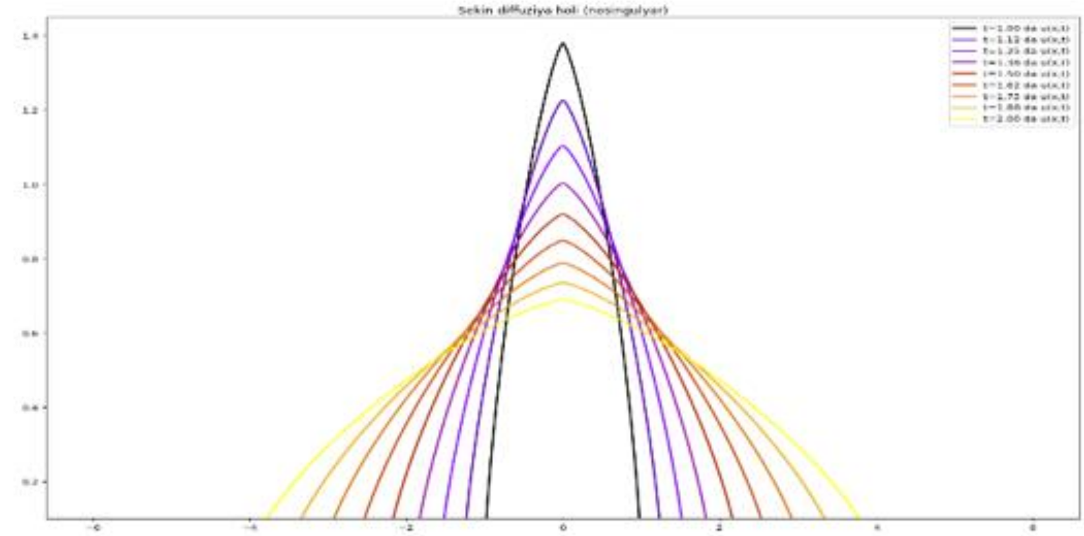

Figure 4: The numerical solution $q = 0.1, m = 2.25, \beta = 1.1, \varepsilon = 1, \gamma(t) = 0.1.$

4 CONCLUSIONS

In this work, we examine the porous medium equation written in non-divergence form with a spatially varying source term to represent an inhomogeneous medium. First, we derived exact solutions in special parameter regimes by separation of variables, yielding closed-form evolution laws that serve as benchmarks. Second, we discovered a novel explicit solution expressible in terms of the inverse regularized Beta function. Third, we constructed a family of Barenblatt-type self-similar profiles and applied a comparison-principle argument to bound the general solution by these profiles, thereby establishing sharp asymptotic estimates. Finally, we developed a fully implicit finite-difference scheme and solved the resulting tridiagonal linear systems via the Thomas algorithm. To perform quantitative analysis, we develop a finite-difference algorithm: we discretize the source term in (35) using the Samarskii–Sobol scheme at the origin and employ the Thomas algorithm to advance the solution in time. Two performance metrics - eps for accuracy and s for convergence - show marked improvement over our previous results (eps=10^(-3), s=5 and eps=10^(-3), s=4 in [5] and [11], respectively). The numerical experiments corroborate the analytical findings and illustrate representative behaviors (compact support, blow-up, interface motion) for general initial data.

The results have direct implications for physical and applied models. Equations of this doubly-nonlinear form arise in diverse contexts: for example, they describe heat conduction in media with temperature-dependent conductivity, resistive diffusion in magnetized plasmas, flow through heterogeneous porous materials, and density-dependent transport in biological media.

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