Solitary Wave Structures of the One-Dimensional Mikhailov-Novikov-Wang System Using Kudryashov's New Function Method

Nadia Dahham Rashad, Mohammed Al-Amr and Ahmed Mohammed Fawze
Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, 41002 Mosul, Iraq
nadia.23csp148@student.uomosul.edu.iq, {alamr, aahmedamer68}@uomosul.edu.iq

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Abstract: Traveling waves and integrable equations are the most well-known features of nonlinear wave propagation phenomena. Analytical solutions to nonlinear integrable equations play an important role in examining the behaviour and structure of nonlinear systems. They offer valuable insights into how these systems evolve over time and under different conditions. Such solutions are essential for accurately describing a range of real-world phenomena. This study aims to derive closed-form traveling wave solutions of the (1+1)-dimensional Mikhailov-Novikov-Wang model by employing the Kudryashov's new function method. This system provides novel perspectives for understanding the connection between integrability and water wave phenomena. New solitary wave solutions are constructed in terms of hyperbolic functions by assigning particular values of the parameters. The study yields two types of solitons, including bell-shaped and singular soliton solutions. The solutions are simulated in 2D and 3D graphical representations to illustrate their physical features. The results highlight the effectiveness of the employed approach in constructing novel solutions which are essential to understand the dynamics of the governing system.

1 INTRODUCTION

Nonlinear integrable partial differential equations (PDEs) are fundamental in representing nonlinear physical phenomena in diverse fields such as applied mathematics, physics, and engineering [1], [2], [3]. The investigation of nonlinear physical behaviors often necessitates the use of both numerical and analytical methods by mathematicians and researchers for the analysis of nonlinear PDEs [4], [5], [6], [7], [8], [9], [10].

The Mikhailov-Novikov-Wang (MNW) system was introduced in 2006, and is given by [11]

$$\begin{cases} g_t = g_{xxxxx} - 20gg_{xxx} - 50g_xg_{xx} + 80g^2g_x + h_x, \\ h_t = -6hg_{xxx} - 2g_{xx}h_x + 96hgg_x + 16h_xg^2. \end{cases} \quad (1)$$

with $g(x,t)$ and $h(x,t)$ representing differentiable functions of velocity and height, respectively. This Boussinesq-type system consists of a couple of (1+1)-dimensional integrable fifth-order PDEs that represent nonlinear wave phenomena. Hence, the system is significant for examining spatial and temporal dynamics, as well as waves in physical environments like shallow water. When $h(x,t)=0$, (1) simplifies to the well-known Kaup-Kupershmidt (KK) equation [12]. Previous studies, including works by Mikhailov et al. [13] and Vojčák [14] have demonstrated its integrability. Traveling wave solutions of the MNW system were constructed by Cesar via the extended tanh method [15]. A zero-curvature representation of (1) was formulated by Sergyeyev [12]. Shan et al. [16] employed Lie algebra techniques to establish the Lax-integrability of the equations, further confirming the presence of Hamiltonian structures. The time-fractional MNW system was examined by Jiang et al. [17] using Lie symmetry analysis.

Exact solutions of nonlinear PDEs have significant importance for many researchers. Hence, numerous powerful and effective methods have been developed, including: the generalized exponential function method [18], the uniform method [19], the Kudryashov new function method [20], [21], Exp-function method [22], the homogeneous Balance Method [23], the generalized Kudryashov method [24] and many other analytical methods.

The Kudryashov's new function method is employed in this study to construct novel solitary wave structures of the (1+1) dimensional integrable

MNW system which models the propagation of nonlinear waves. This robust technique facilitates the discovery of analytical solutions by converting the mathematical model into ordinary differential equations via the wave transform.

Next, this study unfolds through the following sections: Section 2 presents a concise overview of the Kudryashov's new function method. In Section 3, new exact solutions of the MNW system are presented. Section 4 discusses the derived solutions. Finally, Section 5 concludes the work.

2 OVERVIEW OF THE NEW FUNCTION METHOD

The methodology of Kudryashov's new function method will be exhibited in this section.

Suppose a nonlinear partial differential (2):

$$\psi(g, g_t, g_x, g_{tt}, g_{xt}, \ldots) = 0. \quad (2)$$

By employing the wave transformation

$$g(x,t) = G(\omega), \omega = kx - \beta t + \omega_0, \quad (3)$$

where β is the wave speed, and k and ω_0 are arbitrary constants, (2) is converted into a reduced ODE where φ is expressed as a polynomial function of the new function $G(\omega)$ and its derivatives as follows:

$$\varphi\left(G, -\beta G', kG', \beta^2 G'', \ldots\right) = 0. \quad (4)$$

where the prime symbol indicates differentiation with respect to ω.

The foremost steps of the new function method can be summarized in the following manner [20]:

Step 1: For identifying the pole order for the formal solution of (4), assume $G = y^{\rho}$ and compare the exponents of y associated with the highest-order derivative term and the highest nonlinear term subsequently.

Step 2: Assume the solution of (4) is expressed in the form:

$$G(\omega) = \sum_{i=0}^{N} b_i Q^i(\omega), \quad (5)$$

where N represents the pole order and $b_0, b_1, \ldots, b_N$ are constants that will be identified subsequently.

Kudryashov'–s new function is given by [25]

$$Q(\omega) = \frac{1}{pe^{\omega} + \left(\frac{\lambda}{4p}\right)e^{-\omega}}, \quad (6)$$

where p and λ are unknown parameters. Equation (6) satisfies the following auxiliary ODE:

$$\left(\frac{dQ}{d\omega}\right)^2 = Q^2(\omega)\left(1 - \lambda Q^2(\omega)\right). \quad (7)$$

It is easy to find distinct derivatives of $Q(\omega)$ as follows:

$$\frac{d^2Q}{d\omega^2} = Q - 2\lambda Q^3, \quad (8)$$

$$\frac{d^3Q}{d\omega^3} = Q_{\omega} - 6\lambda Q^2 Q_{\omega}, \quad (9)$$

$$\frac{d^4Q}{d\omega^4} = Q - 20\lambda Q^3 + 24\lambda^2 Q^5, \quad (10)$$

$$\frac{d^5Q}{d\omega^5} = Q_{\omega} - 60\lambda Q^2 Q\omega + 120\lambda^2 Q^4 Q_{\omega}. \quad (11)$$

The relations (8)-(11) are utilized to compute the derivatives of $Q(\omega)$ as follows:

$$\frac{dG}{d\omega} = \sum_{i=0}^{N} b_i i Q^{i-1} \frac{dQ}{d\omega}, \quad (12)$$

$$\frac{d^2G}{d\omega^2} = \sum_{i=0}^{N} b_i i^2 Q^i - \lambda i^2 Q^{i+2} - i\lambda Q^{i+2}, \quad (13)$$

$$\frac{d^3G}{d\omega^3} = \sum_{i=0}^{N} b_i \frac{dQ}{d\omega}\Big(i^3 Q^{i-1} - i^2(i+2)\lambda Q^{i+1} - i(i+2)\lambda Q^{i+1}\Big). \quad (14)$$

And so on.

Step 3: By substituting (5) together with (7) and (12)-(14) into (4), we obtain a polynomial in terms of Q and its derivatives.

Step 4: A system of algebraic equations results from setting the coefficients of various different powers of $Q^{\alpha}\left(\frac{dQ}{d\omega}\right)^{\beta}$ $\left(\alpha = 0,1,2,\ldots; \beta = 0,1\right)$ to zero. The unknown constants $b_i (i = 0,1,2,\ldots,N)$, β, and λ can be explicitly determined by solving this system.

Step 5: By substituting these values and the general solution (6) of (7) into (5), the traveling wave solutions of (2) is obtained.

3 EXACT SOLUTIONS OF THE MNW SYSTEM

To derive the exact solutions of (1), the following transformation is applied:

$$\begin{cases} g(x,t) = G(\omega), \\ h(x,t) = H(\omega), \\ \omega = kx - \beta t + \omega_0. \end{cases} \quad (15)$$

Consequently, the system (15) is converted to a nonlinear ordinary differential system as follows:

$$\begin{cases} -\beta G' = k^5 G^{(5)} - 20k^3 G^{(3)} - 50k^3 G'G'' \\ \qquad + 80kG^2G' + kH', \\ -\beta H' = -6k^3 HG^{(3)} - 2k^3 G''H' + 96kHGG' \\ \qquad + 16kH'G^2, \end{cases} \quad (16)$$

thus,

$$\beta H' - 6K^3 HG^{(3)} - 2K^3 G''H' + 96KHGG' + 16KH'G^2 = 0. \quad (17)$$

The first equation in (16) can be expressed as

$$H' = -\frac{\beta}{k}G' - k^4 G^{(5)} + 20k^2 GG^{(3)} + 50k^2 G'G'' - 80G^2G'. \quad (18)$$

Integrating (18) once with respect to ω, while assuming the integration constant is zero, yields

$$H = -\frac{\beta}{k}G - k^4 G^{(4)} + 20k^2 GG'' + 15k^2 (G')^2 - \frac{80}{3}G^3. \quad (19)$$

By substituting (18) and (19) into the second equation of (16), the following ODE is obtained:

$$\begin{aligned} &-\frac{\beta^2}{k}G' - \beta k^4 G^{(5)} + 26\beta k^2 GG^{(3)} + 52\beta G'G'' \\ &+2k^7 G^{(5)}G^{(2)} - 160k^5 GG^{(2)}G^{(3)} - 90k^5 (G')^2 G^{(3)} \\ &-100k^5 G'(G'')^2 + 2880K^3 G'G''G^2 - 16k^5 G^{(5)}G^2 \\ &+480k^3 G^{(3)}G^3 - 3840kG^4G' + 1440k^3 G(G')^3 \\ &-96k^5 GG'G^{(4)} + 6k^7 G^{(3)}G^{(4)} - 192\beta G^2G' = 0. \end{aligned} \quad (20)$$

To determine the pole order, we substitute $G = y^{\rho}$ into (20), which results in $G''G^{(5)} \approx y^{2\rho-7}$ and $G^4G' \approx y^{5\rho-1}$. By balancing the exponents, we find $\rho = -2$, indicating that the pole order is $N = 2$. Consequently, we have

$$G = b_0 + b_1 Q(\omega) + b_2 Q^2(\omega). \quad (21)$$

With the help of Mathematica, substituting (21) and its derivatives into (20), and then collecting terms with identical powers of Q^{α} while setting the coefficients of Q^{α} $(\alpha = 0,1,2,\ldots,9)$ to zero, leads to a system of algebraic equations. This system is solved to obtain:

$$b_0 = \frac{k^2}{4},\ b_1 = 0, b_2 = -\frac{3}{4}k^2\lambda,\ \beta = -k^5. \quad (22)$$

By substituting these values into (21), the following exact solution is obtained:

$$\begin{cases} G(\omega) = \dfrac{k^2}{4} - \dfrac{3k^2\lambda}{4\left(pe^{\omega} + \dfrac{\lambda e^{-\omega}}{4p}\right)^2}, \\ H(\omega) = -\dfrac{k^6}{6}. \end{cases} \quad (23)$$

where $\omega = kx + k^5 t + \omega_0$.

We can randomly choose the parameters p and λ. Setting $\lambda = 4p^2$, we get the following solitary wave solution:

$$\begin{cases} G(\omega) = -\dfrac{k^2}{4}\left(3\operatorname{sech}^2(\omega) - 1\right), \\ H(\omega) = -\dfrac{k^6}{6}. \end{cases} \quad (24)$$

where $\omega = kx + k^5 t + \omega_0$.

Again, setting $\lambda = -4p^2$, we get the following solitary wave solution:

$$\begin{cases} G(\omega) = \dfrac{k^2}{4}\left(3\operatorname{csch}^2(\omega) + 1\right), \\ H(\omega) = -\dfrac{k^6}{6}. \end{cases} \quad (25)$$

where $\omega = kx + k^5 t + \omega_0$.

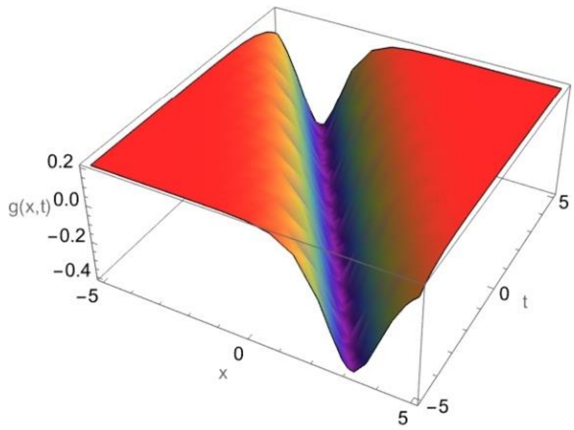

Figure 1: Solitary wave profile (24) at $k = 0.9,$ and $\omega_0 = 0.1$ in 3D plot.

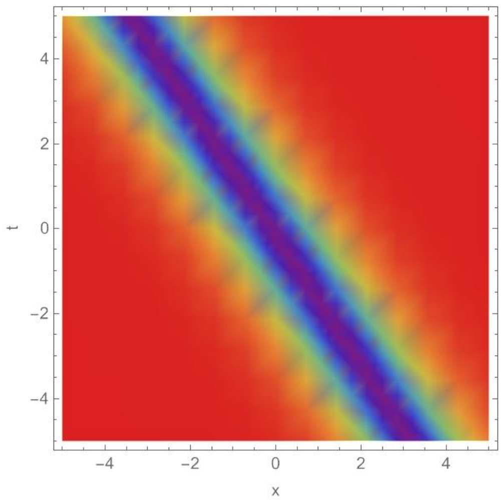

Figure 2: Solitary wave profile (24) at $k = 0.9,$ and $\omega_0 = 0.1$ in 2D density plot.

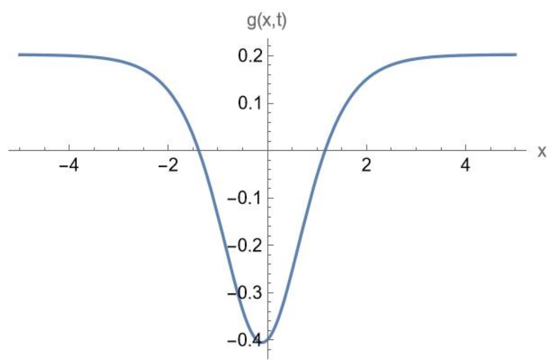

Figure 3: Solitary wave profile (24) at $t = 0, k = 0.9,$ and $\omega_0 = 0.1$ in 2D plot.

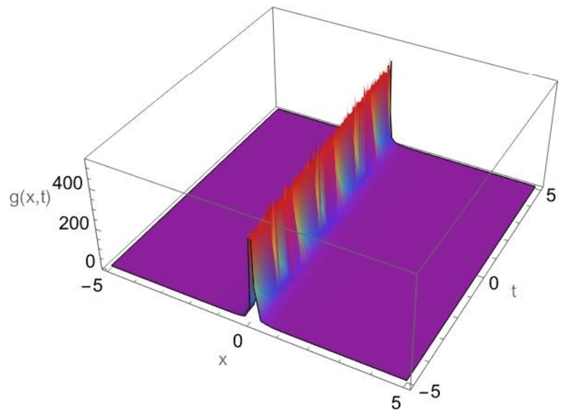

Figure 4: Solitary wave profile (25) at $k = 0.5,$ and $\omega_0 = 0.1$ in 3D plot.

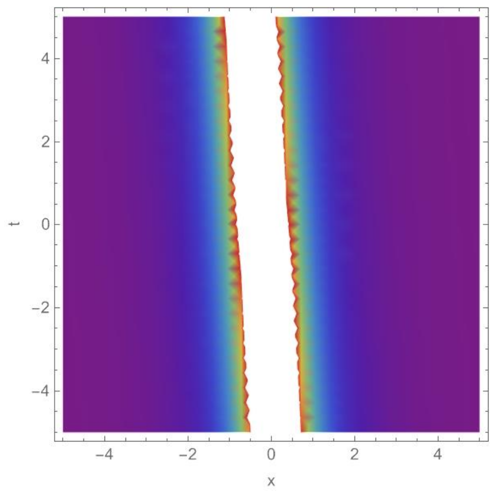

Figure 5: Solitary wave profile (25) at $k = 0.5,$ and $\omega_0 = 0.1$ in 2D density plot.

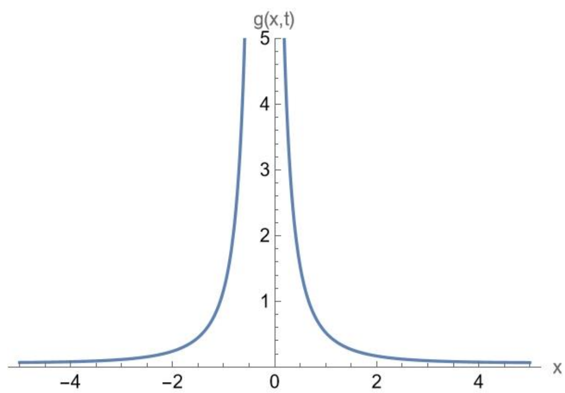

Figure 6: Solitary wave profile (25) at $t = 0, k = 0.5,$ and $\omega_0 = 0.1$ in 2D plot.

4 RESULTS AND DISCUSSION

We demonstrate the physical behavior of solutions by assigning particular values for the arbitrary parameters. These parameter choices play a crucial role in interpreting the physical features and properties of the derived solutions. Figures 1, 2 and 3 depict the 3D, 2D density and 2D profiles of the anti-bell-type soliton solution (24) respectively, for selected parameter values at $k = 0.9,$ and $\omega_0 = 0.1$. Figures 4, 5 and 6 depict the 3D, 2D density and 2D graphs of the singular soliton solution (25) respectively for specific parameters at $k = 0.5,$ and $\omega_0 = 0.1$.

A comparison between the solitary wave solutions obtained in this study for the governing system and those reported in earlier literature [15] confirms that the solutions presented here are new and have not appeared before. The results will significantly contribute to investigate numerous phenomena that arise in nature and across various nonlinear medium.

In this research, we employed advanced symbolic computation techniques via Mathematica to conduct both symbolic mathematical analysis and numerical simulations. Mathematica has demonstrated its effectiveness and robustness as a tool for deriving solitary wave solutions.

5 CONCLUSIONS

This paper examined the use of the Kudryashov new function technique to analyze the (1+1) dimensional Mikhailov-Novikov-Wang system. By applying the wave transform, the original system is reduced to a fifth-order ODE. Consequently, this analysis has yielded new solitary wave solutions, including bell-shaped and singular soliton solutions. These solutions are considered completely new and have not been reported before. The obtained solutions offer important information about the wave dynamics of the model. They are especially useful for researchers working on nonlinear processes in areas like fluid mechanics and plasma physics. These findings show that the Kudryashov new function method provides an efficient and reliable approach for analyzing complex dynamical systems. This approach serves as an effective and robust tool for analyzing and interpreting the dynamic behavior of the system. It not only enhances the understanding of complex mathematical models, but also promising for broader future applications in various disciplines. In the future, one can further extend our results to solve other Boussinesq-type equations due to their significance in modeling nonlinear physical phenomena.

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