A thermodynamically consistent framework for finite third gradient elasticity and plasticity

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Summary in German (Zusammenfassung in deutscher Sprache)

Die vorliegende Arbeit trägt den Titel "Ein thermodynamisch konsistentes Rahmenwerk für finite Elastizität und Plastizität dritter Ordnung". Sie verwendet die Konzepte aus [Bertram 2015] und [Bertram 2014] und verallgemeinert diese.
Nach einer Einführung und einer Literaturübersicht zu Kontinua mit erstem und zweitem Deformationsgradienten werden benötigte Grundlagen und Notation aus Tensoranalysis, Differentialgeometrie, Funktionalanalysis und Kontinuumsmechanik kurz vorgestellt. Der darauf folgende Hauptteil der Arbeit besteht aus vier Teilen, denen je ein Kapitel gewidmet ist. Der erste Teil rekapituliert eine mathematische Methode zur Herleitung von verallgemeinerten Spannungstensoren der Ordnung zwei, drei und vier (fortan als Spannungstensoren bezeichnet), die leistungskonjugiert zum ersten, zweiten und dritten räumlichen Geschwindigkeitsgradienten sind. Aus der vorgestellten Methode lassen sich alle mechanischen Grundgleichungen inklusive dem Prinzip der virtuellen Leistung und die zugehörigen Randbedingungen für den vorliegenden Fall herleiten.
Im zweiten Teil wird ein Rahmenwerk für Elastizität erarbeitet. Es werden zuerst leistungskonjugierte, materielle Deformationsvariablen und Spannungsvariablen hergeleitet, und es wird ihr Transformationsverhalten bei Wechseln der Bezugsplatzierung bestimmt. Danach werden die grundlegenden Konzepte elastischer Isomorphie und materieller Symmetrie für den vorliegenden Fall verallgemeinert.
Im dritten Teil wird ein Rahmenwerk für Elastoplastizität entwickelt. Als erstes werden dazu grundlegende Begriffe und Konzepte wie elastische Bereiche, Fließgrenzen und elastische Isomorphie vorgestellt und auf den vorliegenden Fall verallgemeinert. Dann wird die plastische Dissipation berechnet, und Fließregeln werden für den vorliegenden Fall angepasst.
Im vierten Teil wird gezeigt, dass das erarbeitete Modell thermodynamisch konsistent ist. Dazu werden die oben genannten Konzepte nochmal unter Annahme des ersten Gesetzes der Thermodynamik und der Clausius-Duhem Ungleichung hergeleitet.
Der fünfte Teil der vorliegenden Arbeit stellt Ergebnisse einer numerischen Simulation mit finiten Elementen vor. Es wurde mittels Lagrange-Multiplikatoren ein elastisches Materialmodell für kleine Verformungen implementiert, das den zweiten und dritten Verschiebungsgradienten einbezieht. Dieses wird auf Polyeder mit Verschiebungsrandbedingungen an Ecken und Kanten angewendet. Die Lösungen zeigen in den meisten Fällen keinerlei Anzeichen von Singularitäten in den Verschiebungen oder der mechanischen Leistung. Es wird erklärt, warum eine Theorie dritter Ordnung notwendig ist, um diese Ergebnisse zu erhalten, was als eine der Motivationen für das Aufstellen des oben beschriebenen Rahmenwerkes gesehen werden kann.
In einem Anhang wird erklärt wie das Rahmenwerk in Teil eins bis vier mit einer anderen, äquivalenten Deformationvariable verwendet werden kann, und warum die Verwendung materieller Gradienten des rechten Cauchy-Green Tensors als Deformationsvariablen das Rahmenwerk unnötig verkomplizieren würde. Es stellt sich heraus, dass sog. Pullbacks der Deformationsgradienten sich am besten für das vorgestellte Rahmenwerk eignen.

Chapter 1

Introduction

1.1 A short review of the development of strain gradient theories in continuum mechanics

The amount of literature on mechanical theories that involve strain gradients of higher order or gradients of other quantities is enormous, even though this field of research is relatively young. In this section a very brief overview over the development of strain gradient theories is given, focusing on elasticity and plasticity. It is followed by a more detailed review of important publications on second gradient of strain theories, since the present work only examines this class of models.

In classical mechanics strain is defined as the spatial derivative of the displacement field. In 1827 Cauchy introduced the concept of the stress tensor [Cauchy 1827]. The so called constitutive equation defines the relationship between the stress and strain tensor. It is a property of the material. This approach is classified as a first-order theory. According to [Askes & Aifantis 2011] Cauchy himself was among the first to suggest an extension of the classical first-order theory. In 1850, he mentioned that the constitutive equation should involve spatial displacement gradients of higher order to model discrete lattices (see [Cauchy 1850 i, Cauchy 1850 ii, Cauchy 1851]). Such an approach is classified as a strain gradient theory of higher order, i.e. a theory involving strain gradients of any order. Another approach to leave the classical first-order framework, is the well known work by the brothers Cosserat. In [Cosserat & Cosserat 1909] they suggest the introduction of microrotations and couple stresses. This is not a strain gradient approach though. It is often mentioned in this context, because it clearly is one of the first major approaches that leaves the classical Cauchy continuum behind. At this point it should also be mentioned, that according to [dell'Isola et al. 2015] Gabrio Piola introduced strain gradients to continuum mechanics through the principle of virtual power as early as 1845. Unfortunately his contribution remained mainly unnoticed. So it happened that the 1960s became the decade of great innovations in strain gradient theories.

In 1962 Toupin developed in [Toupin 1962] a strain gradient elasticity theory for large deformations, where the elastic energy also depends on the strain gradient which yields so called couple stresses, i.e., additional stress tensors of order three. The additional strain tensors of order three in this work are derived from the spatial gradient of the right Cauchy-Riemann Tensor. Soon after that, in 1963, Toupin and Gazis explained in [Toupin & Gazis 1963] that the strain gradient can be set in relation to surface effects on an atomic lattice with its nearest and next-nearest neighbor interaction. One can say that Toupin's work triggered the development of research in strain gradient theory. His work was followed by publications by Mindlin, who developed a linear elasticity theory for continua under small deformations with a so called microstructure in [Mindlin 1964]. The term "continuum with microstructure" has been established in the research community for a continuum where in addition to a stress measure further tensor fields enter the balance equations. These additional tensor fields also describe the current state of the deformation. They are often interpreted as a measure of a micro deformation. It is up for discussion whether this interpretation, that distinguishes a micro and a macro level, can be justified for such a theory. In the literature one often finds the remark, that a strain gradient elasticity theory can be regarded as a special case of a continuum with microstructure, because the strain gradient can be chosen to represent the structure of the microdeformation. In the same year Green and Rivlin presented in [Green & Rivlin 1964] a first approach to generalize Toupin's work [Toupin 1962] to higher strain gradients of large deformations. In 1965 Mindlin published another very original work that had great impact. In [Mindlin 1965] he derives a linear elasticity theory for small deformations with second strain gradients and uses this theory to model surface effects. The aforementioned results by Toupin, Mindlin, Green and Rivlin also inspired Germain's work in 1972 [Germain 1972, Germain 1973] where he treats strain gradient continua and continua with microstructure for large and small deformations.

All the publications mentioned so far were focused on elasticity. Strain gradient plasticity was developed later but quickly became a complex research branch mainly concerned with dislocation phenomena. One of the first plasticity models incorporating strain gradients was suggested in 1970 by Dillon and Kratochvil in [Dillon & Kratovchvil 1970] for small strains. It builds upon Mindlin's elastic second strain gradient theory in [Mindlin 1965] and is motivated by dislocation interactions. The main aim of this model is to mimic nonuniform deformation patterns on microlevel in hardening metals. This publication uses an approach that can be classified as a constrained plasticity model. This term describes models where the higher-order plastic variables are determined by the plastic second-order tensor, e.g., defining the gradient of the plastic strain as the plastic third-order deformation tensor.

From the 1980s on Aifantis and his coworkers developed strain gradient theories with reduced complexity compared to the aforementioned theories from the 1960s and 1970s. The main feature of most of these models is, that they contain less components of the strain gradients and thus less material constants have to be introduced. This reduction of complexity makes them suitable for numerical solution techniques. In [Aifantis 1984, Aifantis 1987] constrained strain gradient plasticity models are developed for small and large deformations to determine the width of shear bands in soil and metals. Basically in these models the yield stress depends on the plastic strain and its gradients. This is motivated by certain dislocation mechanisms. An isotropic strain gradient elasticity model is developed (see [Aifantis 1992]), which makes use of the Laplacian of the strain and addresses small and large deformations. This model removes singularities of the strain at dislocation lines and crack tips, which is discussed in [Altan & Aifantis 1992, Ru & Aifantis 1993]. In [Askes & Aifantis 2006, Askes & Aifantis 2011] the already mentioned strain gradient elasticity models developed by Aifantes and coworkers are also extended to include higher-order inertia in dynamics, but focus on small deformations. Aifantis aimed at making gradient theories more accessible for application and his research has been expanded by many others with emphasis on different aspects. In plasticity theory a wide range of publications exists with size-dependence effects in crystal plasticity as one of the major fields of application. Fleck and Hutchinson apply strain gradient approaches in crystal plasticity of small deformations where the focus is on length scale effects, motivated by dislocation theory. It is shown, how J_2 theory can be generalized by introducing length scale parameters to model wire torsion and beam bending on the micron scale. Models for constrained plasticity [Fleck & Hutchinson 1993, Fleck & Hutchinson 2001] as well as for unconstrained plasticity [Fleck & Hutchinson 2001] are developed in this context. Gurtin developed a strain gradient plasticity theory for large deformations of single crystals in [Gurtin 2000, Gurtin 2002], which is formulated for small deformations in

[Cermelli & Gurtin 2002]. Both models are constraint plasticity theories. Further treatments of constrained strain gradient plasticity for small deformations that use a similar approach can be found in [Gurtin 2003, Gudmundson 2003]. In [Dahlberg & Faleskog 2013] a plane strain gradient theory for small deformations (based on [Gudmundson 2003]) is used together with a grain boundary mechanism to model polycrystaline microstructures, where the strain gradients theory is used to model grain size-dependent strengthening. An unifying thermomechanical framework for elastoviscoplastic constitutive equations is proposed in [Forest & Sievert 2003] for strain gradient continua (and continua with higher-grade microstructure), where small and large deformations are addressed. A multiplicative decomposition into an elastic and a plastic part is applied to the deformation gradient and an additive decomposition to the pullback of the second deformation gradient, which is an unconstrained plasticity approach. Then as an alternative it is suggested to use the gradient of the right Cauchy-Green tensor and to apply to it an additive decomposition into a plastic and an elastic part, depending on the gradient of the elastic and the plastic part of the deformation gradient respectively, which makes this a constrained approach. The aim of a unifying thermomechanical framework was further pursued in [Svendsen et al. 2009], where the first and second gradient of the deformation are used as strain variables to model large deformations. It is assumed that a free energy exists which is invariant under changes of the observer. The transformation rules of the strain variables under changes of the reference placement are derived and the principles of material isomorphisms and material symmetry are generalized. The multiplicative decomposition into an elastic and a plastic part is generalized in a constrained plasticity approach and a thermodynamical extension for such a strain gradient framework is suggested. A thermomechanical framework for strain gradient elastoplasticity was also published in [Bertram & Forest 2014] for small deformations but with an unconstrained plasticity approach. This elastoplastic framework is further generalized for large deformations in [Bertram 2015] and also further extended to thermoplasticity in [Bertram 2014]. The results of both publications were later integrated in [Bertram 2016]. These publications use material strain variables by pulling back the second deformation gradient to the reference placement. The transformation behavior of generalized stress tensors for changes of the reference placement and material isomorphisms is derived in these works. Furthermore plasticity is introduced in an unconstrained approach. Further aspects of this framework are elaborated in [Bertram & Glüge 2016, Glüge et al. 2016] where one finds generalizations of internal constraints and eigenmodes respectively.

Of course the field of strain gradient elasticity was also developed further after the mentioned publications in the 1960s and 1970s. Forest and Cordero apply Mindlin's ideas from [Mindlin 1965] in [Cordero et al. 2015] in a small deformations framework. They model size-dependent surface effects in the mechanical behavior of objects in the nano scale accounting for relaxation behavior of traction-free surfaces and provide a finite element (FE) implementation for Mindlin's second strain gradient elasticity (see also [Cordero et al. 2011] for such a FE implementation). In [Polizzotto 2012] a strain gradient elastictiy theory in line with Aifantis' approach is developed for small deformations. The classical linear and angular momentum equations are extended to include higher-order inertia and a wave dispersion problem for beams is solved. An extension to second strain gradient elasticity, taking into account velocity gradient inertia, is developed in [Polizzotto 2013] and is extended in [Polizzotto 2014] with focus on surface effects.

An important feature of strain gradient theories is their regularization property. As already mentioned the strain gradient model in [Aifantis 1992] is discussed with respect to its regularization property that no strain singularities occur. It is shown in [Lazar & Maugin 2006] that this model for small deformations still has singularities in the higher-order stresses. In [Lazar et al. 2006], published shortly after, it is shown for small deformations, that for dislocation problems in an infinite plane a second strain gradient theory produces no singularities at all. This result is confirmed with applications in dislocation analysis in [Lazar & Maugin 2006, Lazar 2013].

Another topic that plays an important role in strain gradient theories, is the generalization of Cauchy's tetrahedron argument for these theories. The question, what form generalized stresses, tractions or forces have, is inherent to strain gradient theories of any order. In [dell'Isola et al. 2016] it is shown that the tetrahedron argument can be generalized for strain gradient theories of any order by arguments similar to those used by Cauchy, see also [Dell'Isola & Seppecher 1997, Dell'Isola Seppecher 2012]. Most of the other papers mentioned in this section that derive the form of higher-order stresses, do so by applying the principle of virtual power.

In the field of fluid dynamics, gradient theories have been established as well. However higher strain gradients do not play the same role as in solid mechanics. E.g. in [Eremeyev & Altenbach 2014] a fluid is modeled so that its strain energy depends on the mass density and its spatial gradients (sometimes referred to as a Korteweg Model, see [Korteweg 1901]). This type of gradient models is not examined in the present work, since the focus lies on the role of spatial gradients of the strain tensor in solids. See [Cordero et al. 2011] for a comparison of Mindlin's approach with other first strain gradient capillarity models and Korteweg models.

Strain gradients have also been introduced in thermodynamical models, however the amount of publications in this field is smaller than in plasticity or elasticity and mainly deals with certain topics in thermoplasticity. Examples are [Polizzotto & Borino 1998] or [Polizzotto 2011] where a thermodynamically consistent strain gradient plasticity theory is developed (for small deformations). The later takes into account dislocation theory as also done in [Gurtin 2010]. In [Gurtin & Anand 2009] the question of thermodynamic consistency of the earlier mentioned plasticity models by Aifantis and by Fleck and Hutchinson is discussed. Another noteworthy work in this field is [Perzyna 1971]. Temperature gradients have been introduced therein with the result is that thy cancel out since they do not have a counterpart in the Clausius-Duhem inequality that allows them to make a contribution to the power.

1.2 A detailed review of recent advances in second strain gradient theories in mechanics

Second gradient of strain in elasticity

Leaving aside Piola's contribution, mentioned in [dell'Isola et al. 2015], the first publication on second gradient of strain and on higher-order strain gradient models was [Green & Rivlin 1964] which addressed large deformations. In this work the authors derive a very broad framework for a large deformation elasticity theory that incorporates strain gradients of arbitrary order.

They assume the existence of higher-order surface tractions and of higher-order body forces that are work conjugate to the matching gradient of the velocity field. After postulating a generalized thermodynamical balance equation they derive generalized local balances. Boundary conditions are only derived for the case of a first gradient of strain elasticity theory. The first publication to elaborate precisely how the second gradient of the strain can be introduced to continuum mechanics is Mindlin's work [Mindlin 1965], in which he works out a linear elasticity theory for small deformations that incorporates the first and second gradient of the strain. Mindlin assumes the existence of a potential elastic energy, that depends on the symmetric part of the strain as well as on the first and second gradient of the strain. The variation of this energy together with the surface divergence theorem leads to the boundary conditions and yields three stress tensors of order two, three and four, work conjugate to the symmetrized strain and the first and second gradient of strain respectively. In an appendix the boundary conditions on edges and corners are derived by the same means. Mindlin then considers a homogeneous, centrosymmetric and isotropic material. For the elastic energy density this yields a polynomial with the standard Lamé constants and 16 additional constants. From there a component of the fourth-order stress tensor is used to derive a punctual surface tension or a surface energy per unit area. For an elastic solid in the form of a half-plane he shows that this framework yields exponentially decreasing strain with distance from a body's surface. This result is related to results of strain in extremely thin boundary layers. Besides a short section on lattice models with next nearest and second next nearest neighbor interaction, Mindlin also addresses concentrated forces. Finally elastic liquids are examined by defining a potential elastic energy that depends on the infinitesimal dilatation (the divergence of the displacement field) and its first and second gradient. With this definition, plane and spherical surfaces of an elastic liquid are examined. Mindlin established three topics in his paper that were subject of many publications on second gradient of strain theories in the following decades:

1. The derivation of boundary conditions from the principle of virtual power,

2. surface effects in elastic solids and fluids in a boundary layer,

3. localized forces on edges and corners.

Mindlins approach is explained in [Wu 1992] where a concept for adhesion is introduced alongside a derivation of Mindlin's results with slightly different means. The solution to the displacement equation of equilibrium is deduced differently and Mindlin's results are further simplified. A thin film is analyzed with the concept of an interface phase with the result that the apparent Young's modulus for such a film is slightly higher than that of a plate. Another direct continuation of Mindlin's work can be found in [Cordero et al. 2011] and [Cordero et al. 2015]. Both publications focus on surface effects in nanoelasticity (for small deformations). In [Cordero et al. 2011] the focus lies on elastic fluids and the surface effects. It is explained, how the Korteweg theory can be included in a first gradient of strain theory. It is laid out why these theories can model capillarity effects at interfaces, but are not able to model internal stresses and strains in the boundary layer, which is due to the already mentioned fourth-order stress component that represents cohesive forces. The authors also explain how a second-order micromorphic framework lends itself to applying the FEM to second gradient of strain theories. The introduction of penalization methods or Lagrangian multipliers provides an amount of degrees of freedom that can be handled by FEM solvers. This approach is also used in the present work for FEM models of point and line displacements in Chapter 7. In [Cordero et al. 2015] the authors continue to elaborate Mindlin's ideas, focusing this time on isotropic elasticity of solids under small deformations. One of the main findings is that besides the already mentioned cohesion modulus, responsible for the "surface energy property" of the material, further coupling moduli in the elastic potential are responsible for surface stress effects. These two properties turn out to be independent of each other. The cohesion parameter, as introduced by Mindlin, controls the relaxation of a traction-free surface, while the so called "higher-order elastic moduli" (or coupling moduli) have a noticeable influence on the apparent elastic behavior of thin films and beams on the nano scale. The theoretical framework is applied to determine the size-dependent apparent shear modulus for thin strips under shear analytically. For numerical studies the FEM solution technique based on a micromorphic theory has been implemented, which has already been outlined in [Cordero et al. 2011]. It is used to determine the size-dependent apparent Young's modulus and Poisson's ratio of thin films on the nano scale as well as to simulate surface relaxation in a porous material. The authors deduce from their work, that for the case of shear a first gradient of strain theory is not sufficient to model size-dependent effects, but the second gradient of strain must be taken into account. In Mindlin's work as well as in [Cordero et al. 2011] and [Cordero et al. 2015] only the isotropic case is addressed and the authors point out that the generalization to the non-isotropic case remains an important challenge for future research. Another recent publication focuses on the other two points that were brought up by Mindlin. In [Javili et al. 2013] boundary conditions and localized forces are in the focus. The authors use Mindlin's approach for the derivation of boundary conditions of a body with edges and corners. This is done in an elastic framework for large deformations. These were already derived in the appendix of [Mindlin 1965]. The authors expand Mindlin's framework by equipping surfaces of a body with an elastic energy that depends on the strain and its gradient as well as edges with an elastic energy that depends on the strain. They derive the boundary conditions for this case by applying the concepts used by Mindlin. It is shown that for an elastic surface energy to depend on the strain, the elastic bulk energy must depend on the strain gradient. Similarly for the elastic curve energy to depend on the strain of the curve, the bulk energy must also depend on the gradient of strain. Another implication is that for a body to sustain point force on its corners, the elastic bulk energy must depend on the second gradient of strain. In [Aifantis 1992] small and large deformations are addressed, though emphasis lies on small deformations. The second strain gradient enters the model by introducing the Laplacian of the strain in the linear elastic law. In order to obtain a second-order stress tensor, it is scaled with a parameter and then subtracted from the strain. In the elastic law the stress is equal to the stiffness tensor (of order four) contracted twice with this new stress tensor. This way no new stiffness tensor must be introduced, which serves the already mentioned aim to reduce complexity in higher strain gradient models. In [Lazar et al. 2006, Lazar & Maugin 2006] second gradient of strain theories are applied for modeling dislocation phenomena in an elastic solid under small deformations. An infinitely extended medium is modeled, thus no boundary conditions are derived. It is assumed that dislocations or disclinations are present. In order to account for this, the strain is decomposed as the sum of the elastic and the plastic strain. The bend-twist tensor and the dislocation density tensor are defined by using the incompatible elastic strain and its spatial gradient. The Burgers vector is a closed line integral of the elastic strain. The potential strain energy in this case depends on the strain, the elastic strain and the first two gradients of each of these variables. Furthermore an isotropic elastic energy is introduced which resembles Mindlin's idea, but differs in the constitutive relation of the third-order stress tensor. It still only requires three model constants, and the stresses coincide with those in the model of nonlocal elasticity proposed by Eringen in [Eringen 1992, Eringen 2002]. The resulting partial differential equation is solved analytically for an infinite plane by using integral transformation techniques (e.g. Fourier transformation). The solutions for stress, strain, distortion, dislocation density and bend-twist tensor are given for the case of a straight screw dislocation and straight edge dislocation and are all free of singularities. The elastic energy from [Lazar et al. 2006] is also used in [Mousavi & Paavola 2014], where Kirchhoff's theory for small deformations of plates is extended by the first and second strain gradient. Boundary conditions are derived by a variation of the energy and analytical solutions for a stability and free vibration analysis of a simply supported rectangular plate are given. In [Polizzotto 2013] the author addresses localized forces on edges and corners in combination with the micromorphic approach introduced in [Mindlin 1964], [Germain 1973] for an elastic material with small deformations. The author uses a non-standard multi-cell homogenization procedure, where the body and its cells have edges and corners. He shows that this approach naturally leads to a second gradient of strain model, if one uses for the microstrains within a cell a Taylor approximation that is truncated after the second gradient. This idea stems from [Mindlin 1964, Germain 1973] where the truncation was set after the first gradient already. Polizzotto proceeds by first introducing a material with microstructure, where a microcell is attached to every spatial point of the body. The cells attached to inner points of the body have the form of a ball. Cells attached to regular points of the surface are the piece of a line, orthogonal to the surface. Cells at points on edges (that are not corner points) have the form of a circular sector of the plane, that is orthogonal to the edge. Cells at corner points are portions of a ball. The energy density at each point of the body is the average of the energy density of the attached cell where the microstrains within the cell are approximated by the truncated Taylor polynomial. The choice of the cell elements leads naturally to a form of the virtual power, that is very similar to the one derived in [Mindlin 1965]. By combining both approaches to derive the principle of virtual power, the author obtains a model that allows for surface effects which stem from a membrane-like boundary layer. In an appendix Polizotto gives a very pictorial interpretation of higher-order stresses, by using the idea of lever arms to describe the action of generalized stress tensors on a strain gradient continuum. This very pictorial section is an important contribution for understanding the abstract concepts that were introduced by Toupin and Mindlin.

Second gradient of strain in plasticity

In the field of plasticity the second gradient of strain was introduced soon after Mindlin came forward with his ideas. One of the first publications in second gradient of strain plasticity was [Dillon & Kratovchvil 1970], where a framework that is formally similar to Mindlin's is used to model perfect plasticity and linear work hardening under small strains. The motivation of this approach lies, as in many publications on gradient plasticity, in the interaction of dislocations. Even though clearly inspired by Mindlin's work, the authors suggest boundary conditions, that are different from Mindlin's result, by taking into account the gradient of the velocity at the surface instead of just the normal gradient. Plasticity is modeled by introducing a residual displacement and its first three gradients as internal variables which makes a constrained plasticity theory. The yield function is then defined to depend on the second-, third- and fourth-order stress tensors associated with the strain and its gradients. An important feature of this work, setting it apart from Mindlin's work, is the extension of the second law of thermodynamics in such a way, that it includes the power associated with the third-order and fourth-order stress tensors. Important thermodynamical properties of a generalized plasticity can be derived from this point. Finally perfect plasticity is addressed in a model, that does not contain strain gradients, while linear work hardening is modeled by introducing first and second gradients of strain to the free energy. In [Aifantis 1987] small and large deformations are covered. However the part on large deformations (mainly concerned with kinematic hardening) only covers plasticity without strain gradients but points out where these could enter. Shear bands are modeled by defining the flow stress as a function of plastic strain minus a term linear in the second gradient of the plastic strain which makes this a constrained plasticity model. This approach is elaborated in more detail in [Zbib & Aifantis 1987], where it is shown that this approach is capable of predicting width and growth of shear bands by solving a nonlinear partial differential equation (PDE) that describes the plastic strain in perpendicular direction to the shear band.

Polizzotto recently addressed surface effects in second gradient of strain plasticity in [Polizzotto 2014]. In this work small deformations are considered and thus the strain is split additively into an elastic and a plastic part. The virtual power is then assumed to include contractions of the first and second gradient of plastic strain with corresponding work conjugate stress tensors (of higher order). This constrained plasticity model is interpreted as a micromorphic continuum, where at each point a microcell is attached. The independent deformation of this microcell is described by the plastic part of the strain and its first two gradients. The virtual power is split up into an internal and external part. The internal power has the expected form, which can already be found in [Mindlin 1965] (but with plastic strain gradients) the external power on the surface of the body differs in its form from those terms derived by Mindlin. Polizotto introduces tractions in what he calls a "heuristical" appraoch. In this approach the tractions are two tensors of order two, contracted with the plastic strain and its partial derivative normal to the surface. This way one obtains two principles of virtual power: The classical one and an inner one that holds only for the plastic degrees of freedom. From there surface effects are derived. It turns out, that the model from [Aifantis 1987] can be cast into Polizotto's framework in order to model scalar hardening. Finally one has to mention, that many publications on both, elasticity and plasticity, with second gradients of strain originated from Aifanti's work (Altan, Askes, Mühlhaus, Triantafyllidis and Vadoulakis, to name a few authors). A good overview is given in [Askes et al. 2002] where the authors categorize publications on second gradient of strain models as either attempts to regularize elasticity or plasticity, as means of modelling damage mechanics, as micromechanical models or as homogenizations of a discrete medium.

Second gradient of strain in thermodynamics

Thermodynamic theories that contain the second gradient of strain are comparatively rare. One of the earlier publications in this field is [Dillon & Kratovchvil 1970] which has already been mentioned in the context of plasticity. The thermodynamical part in Chapter 6 of the present work uses the same methods that are applied therein to derive the the potential relations from the Clausius-Duhem inequality. A more recent publication in this field is [Polizzotto 2003] where strain gradients of arbitrary order are introduced but elasticity and plasticity frameworks are worked out for the setting where the strain and its first two gradients are taken into account. The frameworks are for small deformations. The framework for plasticity assumes that the free energy depends on the gradients of the norm of the plastic strain tensor. The thermodynamic restrictions of the constitutive laws are then derived from the Clausius-Duhem inequality. In the framework for the purely elastic case the free energy depends on the first two gradients of the strain tensor. The derivation of thermodynamic restrictions from the Clausius-Duhem inequality takes a different form than in the plastic case due to the integrability of the strain condition and is compared to the results in [Mindlin 1965]. The present work has a similar aim as [Dillon & Kratovchvil 1970] and [Polizzotto 2003] but suggests a unifying framework for elastoplasticity of large deformations.

1.3 Derivation of fundamental principles in mechanics through the virtual power functional

As already mentioned in the last sections, the approach of deriving the principles of mechanics from the virtual power functional plays an important role for the present work. Apriori it is neither clear how quantities such as surface tractions or the stress tensor should be generalized nor how this can be done for the principles of mechanics in cases where higher deformation gradients are involved. In Section 3.2 it is lined out, how these questions can be solved through the virtual power functional in the case of second strain gradient materials. The idea to apply the principle of virtual power to set up a non-classical continuum theory has been brought forward quite early. In [Cosserat & Cosserat 1909] it is used to include local rotations. In [Hellinger 1913, p.622] it is used to include higher strain gradients and therein Piola is cited as one of the first to advertise the principle of virtual power as a starting point for continuum mechanics. The present work applies the concepts presented in [Bertram & Forest 2007], where the principles of mechanics are derived from an objective power functional. It is stated therein though, that the idea of deriving equations of motions from an invariance requirement for the power has been proposed much earlier e.g. in [Noll 1963], [Green & Rivlin 1964], [Gurtin & Williams 1971], [Germain 1972], [Germain 1973], [Maugin 1980], [Gurtin 1981], [Bertram 1983], [Bertram 1989].

1.4 Motivation and structure of the present work

The literature review in Sections 1.1 and 1.2 shows, that a second gradient of strain framework in continuum mechanics has features that classical or first gradient of strain models cannot provide. It can model surface effects as discussed in [Mindlin 1965, Polizzotto 2014, Cordero et al. 2015], allows a body with corners to sustain point forces as explained in [Mindlin 1965, Javili et al. 2013, Polizzotto 2013] and has strong regularization properties as indicated in [Lazar et al. 2006]. The authors of [Cordero et al. 2015] showed, that for the case of shear a second gradient of strain must be taken into account and that a generalization of their work to the nonisotropic case is highly desirable. The literature review also shows that in continuum mechanics gradients of strain models are often taylored for a specific field of application, e.g., surface effects in elasticity or dislocation phenomena in plasticity. A unifying thermodynamically consistent elastoplastic framework for large deformations that can accommodate all these models would be desirable. In the case of first gradient of strain models this aim has been pursued in [Forest & Sievert 2003, Svendsen et al. 2009, Bertram 2014, Bertram 2015], see also [Bertram 2016] on developments in this field. A corresponding second gradient of strain framework does not exist yet and thus the aim of the present work lies in setting up such a framework. This is done by generalizing the concepts in [Bertram 2015], which builds upon those in [Forest & Sievert 2003, Svendsen et al. 2009, Bertram & Forest 2014, Bertram & Forest 2007]. Since one of the main advantages of a second gradient of strain theory is the fact that a body can sustain point and line forces on its corners and edges, it is then shown how the behavior of classical continua and first or second gradient of strain continua under point and line forces can be modeled with a FEM implementation. The present work is structured as follows. In Chapter 2 notation and some preliminaries in mathematics and mechanics are given. In Chapter 3 it is shown how the virtual power functional can be used to generalize the basic principles of mechanics (such as Euler's laws of motion, Cauchy's laws or the principle of virtual power) for a second gradient of strain framework. This is done by applying the results from [Mindlin 1965] and [Bertram & Forest 2007] to the case of a power functional that depends on the velocity and its first three gradients.

Next follows Chapter 4 which can be regarded as the core of the present work. In this chapter generalized material strain and stress measures are derived by pulling back the stress power to the reference placement. It is shown, that this procedure naturally yields two sets of material stress and strain variables for the envisaged framework. The present work relies on material variables because they do not require the introduction of objective time derivatives of the strain and stress variables. One set of variables is chosen to develop the framework in the following sections. In Appendix A it is shown, that this framework also works with the other set of stress and strain variables. In Appendix B it is laid out why gradients of the right Cauchy-Green-tensor, which were used for example by Toupin in [Toupin 1962], are less suitable for the framework in the present work. In the following sections the transformation behavior under changes of the reference placement of these strain and stress measures is investigated and the concepts of material isomorphy and symmetry are generalized. This allows later the generalization of elasticity and elastic isomorphisms. In the following Chapter 5 the generalization of plasticity for second gradient of strain materials as well as a split of the deformation power into an elastic and a plastic part is presented. A generalization of hardening rules and yield criteria is also presented and turns out to be a straightforward extension of these concepts in classical and first gradient of strain continuum theories. In Chapter 6 it is shown that second gradient of strain materials can be modeled in a thermodynamically consistent form. This is done by introducing the set thermodynamic variables that account for the first and second gradient of the strain and the Helmholtz free energy. The elastic and plastic behavior of a second gradient of strain material, described in Chapters 4 and 5 is then embedded into the thermodynamical framework. It turns out that the second law of thermodynamics in the form of the Clausius-Duhem inequality yields the thermoplastic potential for the generalized stresses as well as restrictions of the yield and hardening rules. It is also shown how elastic and plastic deformations contribute to changes of the temperature. Chapter 7 presents a FEM implementation of an elastic second gradient of strain continuum with prescribed point and line displacements. This Chapter has the aim of illustrating the advantages that a second gradient of strain continuum has, when concentrated point and line forces have to be modeled. Using Lagrangian multipliers the higher gradients of the strain are included in a standard FEM framework. A tetrahedron and a cube with prescribed point and line displacements is examined as a classical continuum as well as a first and second gradient of strain continuum. The results show that only the second gradient of strain continuum yields solutions where stress and strain measures do not tend to a solution with discontinuities as the mesh is refined. Throughout the present work some results are obtained from rather lengthy computations, which are presented in Appendix C.

Chapter 2

Preliminaries

2.1 Preliminaries in mathematics

2.1.1 Tensor analysis

First the tensor notation is introduced.

- **Tensorial quantities** are printed in bold letters. In most cases the order of the tensors will be denoted above it to avoid confusions. As an example $\overset{\langle 3\rangle}{\mathbf{A}}$, $\overset{\langle 4\rangle}{\mathbf{A}}$ denote two different tensors: one of order three and one of order four. If the context does not allow any confusion tensors will be denoted without the indication of their index above them. All tensors in the present work are real tensors over $\mathbb{R}^3$.

- **Tensor contractions** for tensors $\mathbf{A} = A_{i_1...i_n}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_n}$, $\mathbf{B} = B_{j_1...j_m}\mathbf{e}_{j_1} \otimes ... \otimes \mathbf{e}_{j_m}$, each of sufficiently high order n and m, respectively, are denoted as follows:

$$\mathbf{A} \cdot \mathbf{B} := A_{i_1...i_{n-1}a}B_{aj_2...j_m}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_{n-1}} \otimes \mathbf{e}_{j_2} \otimes ... \otimes \mathbf{e}_{j_m} \tag{2.1}$$

$$\mathbf{A} : \mathbf{B} := A_{i_1...i_{n-2}ab}B_{abj_3...j_m}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_{n-2}} \otimes \mathbf{e}_{j_2} \otimes ... \otimes \mathbf{e}_{j_m} \tag{2.2}$$

$$\mathbf{A} \vdots \mathbf{B} := A_{i_1...i_{n-3}abc}B_{abcj_4...j_m}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_{n-3}} \otimes \mathbf{e}_{j_2} \otimes ... \otimes \mathbf{e}_{j_m} \tag{2.3}$$

$$\mathbf{A} :: \mathbf{B} := A_{i_1...i_{n-4}abcd}B_{abcdj_5...j_m}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_{n-4}} \otimes \mathbf{e}_{j_2} \otimes ... \otimes \mathbf{e}_{j_m} \tag{2.4}$$

$$\mathbf{A} \underbrace{\cdot ... \cdot}_{\text{p times}} \mathbf{B} := A_{i_1...i_{n-p}k_1...k_p}B_{k_1...k_pj_{p+1}...j_m}\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_{n-p}} \otimes \mathbf{e}_{j_{p+1}} \otimes ... \otimes \mathbf{e}_{j_m} \tag{2.5}$$

  where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are an ONB of $\mathbb{R}^3$.

- For a tensor $\mathbf{A}$ of order n with $i < j \leq n$ one defines $\mathbf{A}^{[i,j]}$ as **the transposed** of $\mathbf{A}$ with respect to the i^{th} and j^{th} index. With respect to an orthonormal vector basis (ONB) one thus writes

$$\mathbf{A}^{[i,j]} = A_{k_1...k_j...k_i...k_n}\mathbf{e}_{k_1} \otimes ... \otimes \mathbf{e}_{k_i} \otimes ... \otimes \mathbf{e}_{k_j} \otimes ... \otimes \mathbf{e}_{k_n} \tag{2.6}$$

- $\mathbf{A}^R$ denotes the **right transposed** of the tensor $\mathbf{A}$ i.e. the interchange of the last two indices with respect to an orthonormal basis. Thus one can write $\mathbf{A}^R = \mathbf{A}^{[n-1,n]}$. For $n = 2$ $\mathbf{A}$ is a second-order tensor which yields $\mathbf{A}^T = \mathbf{A}^R$.

$\mathbf{A}^L$ denotes the **left transposed** of a tensor $\mathbf{A}$ i.e. the interchange of the first two indices with respect to an ONB. This yields $\mathbf{A}^L = \mathbf{A}^{[1,2]}$. For $n = 2$ $\mathbf{A}$ is a second-order tensor which yields $\mathbf{A}^T = \mathbf{A}^L$.

- **Symmetrisations** of a tensor $\mathbf{A}$ of order greater than two will be abbreviated as follows:

$$(2.7) \qquad 2\overset{[i,j]}{sym}\left[\mathbf{A}\right] := \mathbf{A} + \mathbf{A}^{[i,j]}$$

$$(2.8) \qquad 3\overset{[i,j][k,l]}{sym}\left[\mathbf{A}\right] := \mathbf{A} + \mathbf{A}^{[i,j]} + \mathbf{A}^{[k,l]}$$

$$(2.9) \qquad \overset{(i,j)}{skw}\left[\mathbf{A}\right] := \frac{1}{2}(\mathbf{A} - \mathbf{A}^{[i,j]})$$

If $\mathbf{A}$ is a second-order tensor one obtains the classic definition of the **symmetric part of a tensor**:

$$(2.10) \qquad sym[\mathbf{A}] = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{[1,2]}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$$

- For a second-order tensor $\mathbf{A}$, one defines its **axial vector** $axi(\mathbf{A})$ as the vector that fulfills

$$(2.11) \qquad axi(\mathbf{A}) \times \mathbf{w} = skw\left[\mathbf{A}\right] \cdot \mathbf{w}.$$

for every vector $\mathbf{w}$

- The **inverse** of a second-order tensor $\overset{\langle 2\rangle}{\mathbf{A}}$ is denoted by $\overset{\langle 2\rangle}{\mathbf{A}}^{-1}$. The components of $\overset{\langle 2\rangle}{\mathbf{A}}^{-1}$ with respect to an ONB are denoted by $\overset{\langle 2\rangle}{A}{}^{-1}_{ab}$. This is an abuse of notation since $\overset{\langle 2\rangle}{A}{}^{-1}_{ab}$ does not neccessarily equal $1/\overset{\langle 2\rangle}{A}_{ab}$. However confusion will be avoided by denoting the multiplicative inverse of a real number $x \in \mathbb{R}$ by $1/x$ or $\frac{1}{x}$. It is possible to define the inverse for tensors of even order, while for odd-order tensors this is not possible. Since only the inverse of second-order tensors is used in the present work, the definition of the inverse is only given for this case.

- The **second-order identity tensor** is denoted by $\overset{\langle 2\rangle}{\mathbf{I}}$ and the **fourth-order identity**

tensor is denoted by $\overset{\langle 4\rangle}{\mathbf{I}}$. (In general one could introduce even-order identity tensors. Identity tensors of odd order do not make sense.) For a vector $\mathbf{v}$ and a second-order tensor $\overset{\langle 2\rangle}{\mathbf{F}}$ one thus obtains

$$\overset{\langle 2\rangle}{\mathbf{I}} \cdot \mathbf{v} = \mathbf{v} \tag{2.12}$$

$$\overset{\langle 4\rangle}{\mathbf{I}} : \overset{\langle 2\rangle}{\mathbf{F}} = \overset{\langle 2\rangle}{\mathbf{F}} \tag{2.13}$$

The **zero tensor** of order n is denoted by $\overset{\langle n\rangle}{\mathbf{0}}$ such that for every n^{th}-order tensor $\overset{\langle n\rangle}{\mathbf{T}}$

$$\overset{\langle n\rangle}{\mathbf{0}} \underbrace{\cdot \ldots \cdot}_{\text{n times}} \overset{\langle n\rangle}{\mathbf{T}} = 0 \tag{2.14}$$

- The **determinant** of a second-order tensor $\overset{\langle 2\rangle}{\mathbf{F}}$ is denoted by $J_{\overset{\langle 2\rangle}{\mathbf{F}}}$.
- The **scalar product** between two tensors $\mathbf{A}, \mathbf{B}$ of order $n \in \mathbb{N}$ can also be denoted by $\langle \mathbf{A}, \mathbf{B} \rangle$ such that

$$\langle \mathbf{A}, \mathbf{B} \rangle := \mathbf{A} \underbrace{\cdot \ldots \cdot}_{\text{n times}} \mathbf{B} \tag{2.15}$$

This notation is introduced, since it allows compact notations for tensors of unknown or arbitrary order.

- **Important sets of tensors**

(2.16) $\mathscr{Inv}$ denotes the set of all invertible second-order tensors.

(2.17) $\mathscr{Sym}$ denotes the set of all symmetric, positive definite second-order tensors.

(2.18) $\mathscr{Orth}^{+}$ denotes the set of all orthogonal second-order tensors with positive determinant.

(2.19) $\mathscr{Unim}$ denotes the unimodular group, i.e., the group of all second-order tensors with determinant of absolute value one.

- **Gradients** are denoted as follows

(2.20)	$Grad()$	denotes the first gradient.
(2.21)	$Grad^{II}()$	denotes the second gradient.
(2.22)	$Grad^{III}()$	denotes the third gradient.
(2.23)	$Grad^{IV}()$	denotes the fourth gradient.

The order of the gradient is indicated by Roman numbers to avoid confusion with a transposition, which is indicated by Arabic numbers. A higher-order gradient of undetermined order n is denoted by $Grad^n$. Similarly, repeated application of the **divergence operator** to a tensor field $\mathbf{T}$ is denoted by

$$div^{II}(\mathbf{T}) := div(div(\mathbf{T})) \tag{2.24}$$

$$div^{III}(\mathbf{T}) := div(div(div(\mathbf{T}))) \tag{2.25}$$

where the roman numbers prevent confusion with an exponent.

- **Time derivatives**

  Time derivatives are indicated by a dot

$$()^{\bullet} := \frac{\partial()}{\partial t} \tag{2.26}$$

- For a second-order tensor $\mathbf{A}$ one defines

$$\overset{\langle 4 \rangle}{\mathbf{K}}_{\mathbf{A}} := \mathbf{A}^{-1} \cdot Grad^{II}(\mathbf{A}) \tag{2.27}$$

- The **Rayleigh product** is denoted by "$*$". For a second-order tensor $\mathbf{F}$ its action on a tensor basis element $\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_n}$ with respect to an ONB $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined as

$$\mathbf{F} * (\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_n}) := \mathbf{F} \cdot \mathbf{e}_{i_1} \otimes ... \otimes \mathbf{F} \cdot \mathbf{e}_{i_n} \tag{2.28}$$

If $\mathbf{F}$ is the differential of a diffeomorphism, the Rayleigh product can be interpreted as the pushforward of a contravariant n-th-order tensor.

- By "$\circ$" a product will be denoted, that is very similar to the Rayleigh product. For a second-order-tensor $\mathbf{F}$ its action on $\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_n}$ with respect to an ONB $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined as

$$\mathbf{F} \circ (\mathbf{e}_{i_1} \otimes ... \otimes \mathbf{e}_{i_n}) := \mathbf{F}^{-T} \cdot \mathbf{e}_{i_1} \otimes \mathbf{F} \cdot \mathbf{e}_{i_2} \otimes ... \otimes \mathbf{F} \cdot \mathbf{e}_{i_n} \tag{2.29}$$

  If $\mathbf{F}$ is the differential of a diffeomorphism, the $\circ$ product can be interpreted as the pushforward of a n-th-order tensor where the first entry is covariant and the others are contravariant.

- If not stated otherwise, $B(t)$ denotes a body in $\mathbb{R}^3$ with smooth surface $\partial B(t)$ and $\mathbf{v}(t)$ is a smooth velocity field on $B(t)$. The variable t stands for the time.

Remark 2.1.

Let $\underline{B}$ and B be two three-dimensional regions and $\kappa : \underline{B} \to B$ be a smooth mapping between them. Define $\overset{\langle 2\rangle}{\mathbf{P}} := \underline{Grad}(\kappa)$. All quantities in $\underline{B}$ are marked by underlining them. In index notation partial derivatives with respect to variables in $\underline{B}$ are denoted by ”$\underline{,}$”, partial derivatives with respect to variables in B are denoted by ”,”. Then the following equalities hold:

$$Grad(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}) = -\overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \left(\left(Grad(\overset{\langle 2\rangle}{\mathbf{P}}) \right)^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \right)^{[2,3]} \tag{2.30}$$

$$\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}) = \overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} - [\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}]^{[2,4]} \tag{2.31}$$

$$\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \tag{2.32}$$

$$Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}}) = -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \left(\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}) + \overset{[2,3][2,4]}{3sym} \left[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \right] \right) \tag{2.33}$$

$$\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = \overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \left(-\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} + \overset{[2,3][2,4]}{3sym} \left[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \right] \right) \tag{2.34}$$

Proof.

The proof will be given by computing components with respect to an ONB $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$$\overset{\langle 2\rangle}{\mathbf{I}} = \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{P}} \tag{2.35}$$

$$\Rightarrow \overset{\langle 2\rangle}{\mathbf{0}} = Grad(\overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{P}}) \tag{2.36}$$

$$\Rightarrow \overset{\langle 2\rangle}{\mathbf{0}} = [Grad(\overset{\langle 2\rangle}{\mathbf{P}}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}]^{[2,3]} + \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot Grad(\overset{\langle 2\rangle}{\mathbf{P}}) \tag{2.37}$$

$$\Leftrightarrow [Grad(\overset{\langle 2\rangle}{\mathbf{P}}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}]^{[2,3]} = - \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot Grad(\overset{\langle 2\rangle}{\mathbf{P}}) \tag{2.38}$$

$$\Leftrightarrow Grad(\overset{\langle 2\rangle}{\mathbf{P}}^{-1}) = - \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot [Grad(\overset{\langle 2\rangle}{\mathbf{P}})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1}]^{[2,3]} \tag{2.39}$$

$$\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}) = \underline{Grad}(\overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \underline{Grad}(\overset{\langle 2\rangle}{\mathbf{P}})) = (\overset{\langle 2\rangle}{P}^{-1}_{\alpha a} \overset{\langle 2\rangle}{P}_{a\beta\underline{,}\gamma})_{\underline{,}\delta} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_\delta \tag{2.40}$$

$$= \overset{\langle 2\rangle}{P}^{-1}_{\alpha a\underline{,}\delta} \overset{\langle 2\rangle}{P}_{a\beta\underline{,}\gamma} + \overset{\langle 2\rangle}{P}^{-1}_{\alpha a} \overset{\langle 2\rangle}{P}_{a\beta\underline{,}\gamma\delta}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_\delta \tag{2.41}$$

$$= \Big(\overset{\langle 2\rangle}{P}^{-1}_{\alpha\beta} \overset{\langle 2\rangle}{P}_{bc\underline{,}\delta} \overset{\langle 2\rangle}{P}^{-1}_{ca} \overset{\langle 2\rangle}{P}_{a\beta\underline{,}\gamma} + \overset{\langle 2\rangle}{P}^{-1}_{\alpha a} \overset{\langle 2\rangle}{P}_{a\beta\underline{,}\gamma\delta} \Big) \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_\delta \tag{2.42}$$

$$= - [\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}]^{[2,4]} + \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \cdot \underline{Grad}^{II}(\overset{\langle 2\rangle}{\mathbf{P}}) = \overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} - [\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}]^{[2,4]}$$

$$\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = \overset{\langle 2\rangle}{P}_{\alpha a} \cdot \overset{\langle 2\rangle}{P}^{-1}_{\alpha\beta,\gamma}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \tag{2.43}$$

$$= - \overset{\langle 2\rangle}{P}_{\alpha a} \overset{\langle 2\rangle}{P}^{-1}_{ab} \overset{\langle 2\rangle}{P}_{bc,\gamma} \overset{\langle 2\rangle}{P}^{-1}_{c\beta}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \tag{2.44}$$

$$= - \overset{\langle 2\rangle}{P}_{\alpha a} \overset{\langle 2\rangle}{P}^{-1}_{ab} \overset{\langle 2\rangle}{P}_{bc,d} \overset{\langle 2\rangle}{P}^{-1}_{c\beta} \overset{\langle 2\rangle}{P}^{-1}_{d\gamma}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \tag{2.45}$$

$$= \overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$$

$$Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}}) = - Grad(\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}) = -(\overset{\langle 2\rangle}{P}_{\alpha c\underline{,}d} \overset{\langle 2\rangle}{P}^{-1}_{d\gamma} \overset{\langle 2\rangle}{P}^{-1}_{c\beta})_{,\delta} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_\delta \tag{2.46}$$

$$= - (\overset{\langle 2\rangle}{P}_{\alpha c\underline{,}d} \overset{\langle 2\rangle}{P}^{-1}_{d\gamma} \overset{\langle 2\rangle}{P}^{-1}_{c\beta})_{\underline{,}e} \overset{\langle 2\rangle}{P}^{-1}_{e\delta}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_\gamma \otimes \mathbf{e}_\delta \tag{2.47}$$

$$
\begin{aligned}
(2.48)\qquad &=\Big(-\overset{\langle 2\rangle}{P}_{\alpha c,de}\overset{\langle 2\rangle^{-1}}{P}_{d\gamma}\overset{\langle 2\rangle^{-1}}{P}_{c\beta}\overset{\langle 2\rangle^{-1}}{P}_{e\delta}-\overset{\langle 2\rangle}{P}_{\alpha c,d}\overset{\langle 2\rangle^{-1}}{P}_{c\beta}\overset{\langle 2\rangle^{-1}}{P}_{df}\overset{\langle 2\rangle}{P}_{fg,e}\overset{\langle 2\rangle^{-1}}{P}_{g\gamma}\overset{\langle 2\rangle^{-1}}{P}_{e\delta}\\
&\quad-\overset{\langle 2\rangle}{P}_{\alpha c,d}\overset{\langle 2\rangle^{-1}}{P}_{d\gamma}\overset{\langle 2\rangle^{-1}}{P}_{cf}\overset{\langle 2\rangle}{P}_{fg,e}\overset{\langle 2\rangle^{-1}}{P}_{g\beta}\overset{\langle 2\rangle^{-1}}{P}_{e\delta}\Big)\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\otimes\mathbf{e}_\gamma\otimes\mathbf{e}_\delta\\
(2.49)\qquad &=\Big(-\overset{\langle 2\rangle}{P}_{\alpha f}\overset{\langle 2\rangle^{-1}}{P}_{fg}\overset{\langle 2\rangle}{P}_{gc,de}\overset{\langle 2\rangle^{-1}}{P}_{d\gamma}\overset{\langle 2\rangle^{-1}}{P}_{c\beta}\overset{\langle 2\rangle^{-1}}{P}_{e\delta}-\overset{\langle 2\rangle}{P}_{\alpha c,d}\overset{\langle 2\rangle^{-1}}{P}_{c\beta}\overset{\langle 2\rangle^{-1}}{P}_{df}\overset{\langle 2\rangle}{P}_{fg,e}\overset{\langle 2\rangle^{-1}}{P}_{g\gamma}\overset{\langle 2\rangle^{-1}}{P}_{e\delta}\\
&\quad-\overset{\langle 2\rangle}{P}_{\alpha c,d}\overset{\langle 2\rangle^{-1}}{P}_{d\gamma}\overset{\langle 2\rangle^{-1}}{P}_{cf}\overset{\langle 2\rangle}{P}_{fg,e}\overset{\langle 2\rangle^{-1}}{P}_{g\beta}\overset{\langle 2\rangle^{-1}}{P}_{e\delta}\Big)\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\otimes\mathbf{e}_\gamma\otimes\mathbf{e}_\delta\\
(2.50)\qquad &=-\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\Big)+\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\Big)\\
&\quad+\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\Big)^{[2,3]}
\end{aligned}
$$

with (2.31)

$$
\begin{aligned}
(2.51)\qquad &=-\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})+[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}]^{[2,4]}\Big)\\
&\quad+\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\Big)+\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\Big)^{[2,3]}\\
(2.52)\qquad &=-\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})+[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}]^{[2,4]}+\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\\
&\quad+\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}^{[2,3]}\Big)\\
(2.53)\qquad &=-\overset{\langle 2\rangle^{-T}}{\mathbf{P}}\circ\Big(\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})+\overset{[2,3][2,4]}{3sym}\Big[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\Big]\Big)
\end{aligned}
$$

For the proof of (2.34) one makes use of the fact that (2.30) can be rewritten as

$$
(2.54)\qquad Grad(\overset{\langle 2\rangle^{-1}}{\mathbf{P}})=-[(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\cdot\overset{\langle 2\rangle^{-1}}{\mathbf{P}})^{[2,3]}\cdot\overset{\langle 2\rangle^{-1}}{\mathbf{P}}]^{[2,3]}
$$

Using (2.54) one can write

$$
\begin{aligned}
(2.55)\qquad &\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle^{-1}}{\mathbf{P}}}=\overset{\langle 2\rangle}{\mathbf{P}}\cdot Grad(Grad(\overset{\langle 2\rangle^{-1}}{\mathbf{P}}))\\
(2.56)\qquad &=-\overset{\langle 2\rangle}{\mathbf{P}}\cdot Grad([(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\cdot\overset{\langle 2\rangle^{-1}}{\mathbf{P}})^{[2,3]}\cdot\overset{\langle 2\rangle^{-1}}{\mathbf{P}}]^{[2,3]})
\end{aligned}
$$

In index notation the components of $\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}}$ can thus be written as

$$(2.57) \qquad -\overset{\langle 2\rangle}{P}_{a\underline{A}}\,[\overset{\langle 3\rangle}{K}_{\overset{\langle 2\rangle}{P}_{\underline{ABC}}}\overset{\langle 2\rangle}{P}^{-1}_{\underline{C}c}\overset{\langle 2\rangle}{P}^{-1}_{\underline{B}b}]_{,d}$$

where in a slight abuse of notation the components of $\overset{\langle 2\rangle}{\mathbf{P}}^{-1}$ are denoted by $\overset{\langle 2\rangle}{P}^{-1}_{ij}$. One continues the equation by applying the product rule repetitively (as has been demonstrated in the other parts of this proof) and obtains

$$(2.58) \qquad \begin{aligned} &= -\overset{\langle 2\rangle}{P}_{a\underline{A}}\Big\{\Big[\overset{\langle 4\rangle}{K}_{P_{\underline{ABCD}}} - \overset{\langle 3\rangle}{K}_{P_{\underline{ADE}}}\overset{\langle 3\rangle}{K}_{P_{\underline{ECB}}}\Big]\overset{\langle 2\rangle}{P}^{-1}_{\underline{D}d}\overset{\langle 2\rangle}{P}^{-1}_{\underline{B}b}\overset{\langle 2\rangle}{P}^{-1}_{\underline{C}c} \\ &\quad - \overset{\langle 3\rangle}{K}_{P_{\underline{ABC}}}\overset{\langle 3\rangle}{K}_{P_{\underline{BDE}}}\overset{\langle 2\rangle}{P}^{-1}_{\underline{D}b}\overset{\langle 2\rangle}{P}^{-1}_{\underline{E}d}\overset{\langle 2\rangle}{P}^{-1}_{\underline{C}c} - \overset{\langle 3\rangle}{K}_{P_{\underline{ABC}}}\overset{\langle 3\rangle}{K}_{P_{\underline{BDE}}}\overset{\langle 2\rangle}{P}^{-1}_{\underline{C}c}\overset{\langle 2\rangle}{P}^{-1}_{\underline{D}b}\overset{\langle 2\rangle}{P}^{-1}_{\underline{E}d}\Big\} \end{aligned}$$

This yields

$$(2.59) \qquad \overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = \overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ\Big(-\overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} + 3\overset{[2,3][2,4]}{sym}\Big[\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}\Big]\Big)$$

□

Remark 2.2.

The mapping $()^{-1}$ that maps an invertible tensor to its inverse is differentiable on the open set of all invertible second-order-tensors. For an invertible second-order-tensor $\overset{\langle 2\rangle}{\mathbf{F}}$ the following equality holds with respect to an orthonormal coordinate system

$$(2.60) \qquad \frac{\partial \overset{\langle 2\rangle}{F}^{-1}_{ac}}{\partial \overset{\langle 2\rangle}{F}_{ef}} = -\overset{\langle 2\rangle}{F}^{-1}_{ae}\overset{\langle 2\rangle}{F}^{-1}_{fc}$$

Proof.

$$(2.61) \qquad \overset{\langle 2\rangle}{F}^{-1}_{ax}\overset{\langle 2\rangle}{F}_{xc} = \delta_{ac}$$

$$(2.62)\quad \Rightarrow \frac{\partial \overset{\langle 2\rangle^{-1}}{F}_{ax}}{\partial \overset{\langle 2\rangle}{F}_{ef}} \overset{\langle 2\rangle}{F}_{xc} = -\overset{\langle 2\rangle^{-1}}{F}_{ax} \frac{\partial \overset{\langle 2\rangle}{F}_{xc}}{\partial \overset{\langle 2\rangle}{F}_{ef}}$$

$$(2.63)\quad \Leftrightarrow \frac{\partial \overset{\langle 2\rangle^{-1}}{F}_{ax}}{\partial \overset{\langle 2\rangle}{F}_{ef}} \underbrace{\overset{\langle 2\rangle}{F}_{xy} \overset{\langle 2\rangle^{-1}}{F}_{yc}}_{=\delta_{xc}} = -\overset{\langle 2\rangle^{-1}}{F}_{ax} \underbrace{\frac{\partial \overset{\langle 2\rangle}{F}_{xy}}{\partial \overset{\langle 2\rangle}{F}_{ef}}}_{=\delta_{xe}\delta_{yf}} \overset{\langle 2\rangle^{-1}}{F}_{yc}$$

$$(2.64)\quad \Leftrightarrow \frac{\partial \overset{\langle 2\rangle^{-1}}{F}_{ac}}{\partial \overset{\langle 2\rangle}{F}_{ef}} = -\overset{\langle 2\rangle^{-1}}{F}_{ae} \overset{\langle 2\rangle^{-1}}{F}_{fc}$$

□

2.1.2 Differential geometry

Definition 2.1. Tangential decomposition

On a smooth surface with normal vector $\mathbf{n}$, one can decompose the gradient of a smooth tensor field $\mathbf{A}$ into a tangential part $grad_t(\mathbf{A})$ and normal part $grad_n(\mathbf{A})$:

$$(2.65)\quad grad(\mathbf{A}) = \underbrace{grad(\mathbf{A}) \cdot (\mathbf{I} - [\mathbf{n} \otimes \mathbf{n}])}_{grad_t(\mathbf{A})} + \underbrace{grad(\mathbf{A}) \cdot [\mathbf{n} \otimes \mathbf{n}]}_{grad_n(\mathbf{A})}$$

This definition gives rise to a decomposition of the divergence

$$(2.66)\quad div(\mathbf{A}) = div_t(\mathbf{A}) + div_n(\mathbf{A})$$

Remark 2.3. Multiple application of Gauss' theorem

Gauss' theorem yields for a smooth tensor field $\overset{\langle s\rangle}{\mathbf{T}}$ of order s:

$$(2.67)\quad \int\limits_{B(t)} \langle \overset{\langle s\rangle}{\mathbf{T}}, grad^{s-1}(\mathbf{v})\rangle \, dv$$
$$= \int\limits_{\partial B(t)} \langle (\overset{\langle s\rangle}{\mathbf{T}} \cdot \mathbf{n}), grad^{s-2}(\mathbf{v})\rangle \, da \; - \int\limits_{B(t)} \langle div(\overset{\langle s\rangle}{\mathbf{T}}), grad^{s-2}(\mathbf{v})\rangle \, dv$$

This theorem can now be applied again to the integral over the body on the right hand side of Equation (2.67) above. Therefore by subsequent application a gradient term in the body is removed and a divergence term is created instead. At the same time a gradient term is added in the surface integral.

Theorem 2.1. Surface divergence theorem

For a smooth tensor field $\mathbf{A}$ of any order on a smooth, closed surface ∂B with normal $\mathbf{n}$ the following equation holds:

$$\int\limits_{\partial B(t)} div_t(\mathbf{A})\ da = \int\limits_{\partial B(t)} \underbrace{div_t(\mathbf{n})}_{=:\kappa_m} \mathbf{A} \cdot \mathbf{n}\ da \tag{2.68}$$

The term κ_m is the negative mean curvature.

Proof.

A proof can be found in [Brand 1947, pp.217]. □

Remark 2.4. Dropping tangential tensor components

Let $\mathbf{A}$ be a smooth tensor field of order s and $\mathbf{T}$ a smooth tensor field of order $s+1$. Then

$$\begin{aligned}
&\int\limits_{\partial B} \langle \mathbf{T}, grad(\mathbf{A}) \rangle\ da \\
&= \int\limits_{\partial B} \langle \mathbf{T}, grad_n(\mathbf{A}) \rangle + \langle \mathbf{T}, grad_t(\mathbf{A}) \rangle\ da && (2.69) \\
&= \int\limits_{\partial B} \langle \mathbf{T}, grad_n(\mathbf{A}) \rangle + div_t(\mathbf{A} \underbrace{\cdot \ldots \cdot}_{\text{s times}} \mathbf{T}) - \langle div_t(\mathbf{T}), \mathbf{A} \rangle\ da && (2.70)
\end{aligned}$$

Apply Theorem 2.1:

$$= \int\limits_{\partial B} \langle \mathbf{T}, grad_n(\mathbf{A}) \rangle + \underbrace{div_t(\mathbf{n})}_{\kappa_m}(\mathbf{A} \underbrace{\cdot \ldots \cdot}_{\text{s times}} \mathbf{T}) \cdot \mathbf{n} - \langle div_t(\mathbf{T}), \mathbf{A} \rangle\ da \tag{2.71}$$

2.1.3 Functional analysis

The following mathematical definitions will be needed in the present work.

Definition 2.2. Completion

A metric space M is **complete** if every Cauchy sequence in M is convergent in M. If M is not complete one defines the complete metric space $\overline{M}$, as the space that contains M as a dense subspace. $\overline{M}$ is called the **completion** of M.

Definition 2.3. L^P Spaces

Let $B \subset \mathbb{R}^3$ be a body and $1 \leq p \in \mathbb{R}$. One defines $L^p(B)$ as the space of all measurable, real-valued functions such that the Lebesgue integral

$$\int_B |f|^p \, d\mu \tag{2.72}$$

exists.

Definition 2.4. C^kFunctions

Functions for which the k-th derivative exists and is continuous are called C^k Functions. The set of all such functions on a body B is denoted by $C^k(B)$.

Definition 2.5. The weak derivative

Assume that $B \subset \mathbb{R}^3$ and $f \in L^2(B)$. Let $i = (i_1, ..., i_n)$ be a multiindex. Then $g \in L^2(B)$ is the i-th weak derivative of f if for every testfunction ϕ

$$\int_B g(x)\phi(x)dV = (-1)^{|i|} \int_B f(x) \frac{\partial^{|i|}\phi(x)}{\partial^{i_1}x_1 ... \partial^{i_n}x_1} \tag{2.73}$$

with $|i| = \sum\limits_{k=1}^{n} i_k$.

For a tensor field the weak derivative is defined by applying (2.73) to each component.

A detailed introduction on the topics of this section and the last sections can be found in [Brand 1947] and [Adams 1975].

2.2 Preliminaries in mechanics

Since the present work should be regarded as an extension of the classic elasticity and plasticity as presented in [Bertram 2005], this section introduces some basic concepts in the spirit of [Bertram 2005]. The present work deals with large deformations of a body which occupies the volume B_t at the time t. An **abstract material body** is a three-dimensional manifold with boundary denoted by B. The time-dependent, smooth map

$$\kappa_t : B \to \mathbb{R}^3,\ \kappa_t(B) = B_t \tag{2.74}$$

is called the **spatial placemen**t of B or also the **momentary placement**. One also introduces a **reference placement**

$$\kappa_0 : B \to \mathbb{R}^3,\ \kappa_0(B) = B_0 \tag{2.75}$$

The (large) **motion** of the body is described by the one parameter family of mappings

$$\chi_t : B_0 \to \mathbb{R}^3,\ \chi_t(B_0) = B_t \tag{2.76}$$

Coordinates in the reference placement are called material coordinates, coordinates in the spatial placement are called spatial coordinates. The reference placement is not unique. For two reference placements κ and $\underline{\kappa}$ one calls the composition $\kappa\underline{\kappa}^{-1}$ the change of reference placement. Figure 2.1 is a visualization of this setting. Gradients in the reference placements are denoted by $Grad()$, gradients in the spatial placement are denoted by $grad()$.

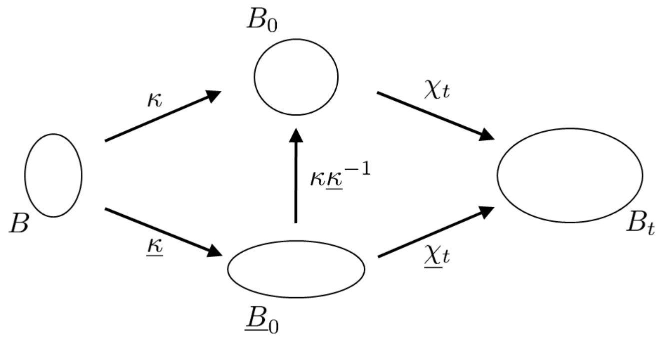

Figure 2.1: Spatial and reference placements of an abstract body manifold

In the present work the time parameter t will be suppressed sometimes. The velocity field is denoted by

$$\mathbf{v} := \chi^{\bullet} \tag{2.77}$$

Material gradients are denoted with a capital letter and spatial gradients with a lower case letter. The **gradient of the motion** is denoted by

$$\mathbf{F} := Grad(\chi) \tag{2.78}$$

The right Cauchy-Green tensor (a strain measure) is denoted by

$$\mathbf{C} := \mathbf{F}^T \cdot \mathbf{F} \tag{2.79}$$

This yields for the spatial velocity gradient

$$grad(v) = \mathbf{F}^{\bullet} \cdot \mathbf{F}^{-1} \tag{2.80}$$

The Cauchy stress tensor (which is a spatial tensor) is denoted by $\mathbf{T}$, the (material) **second Piola-Kirchhoff Tensor** is denoted by

$$\mathbf{T}^{2PK} := J_{\mathbf{F}} \mathbf{F}^{-1} \cdot \mathbf{T} \cdot \mathbf{F}^{-T} \tag{2.81}$$

The material stress tensor is defined as

$$(2.82)\quad \mathbf{S} := J_{\mathbf{F}}^{-1}\mathbf{T}^{2PK}$$

The (global) **stress power** in classical continuum mechanics is defined as

$$(2.83)\quad P := \int\limits_{B_t} \mathbf{T} : grad(\mathbf{v})dv = \int\limits_{B_0} \mathbf{S} : \mathbf{C}^{\bullet} dV = \int\limits_{B_0} p\ dV = \int\limits_{B_0} p\ \frac{1}{\rho_0} dm.$$

where dv and dV denote the spatial and material **volume element**, respectively, ρ and ρ_0 denote the spatial and material **mass density**, respectively, which implies

$$(2.84)\quad \rho = J_{\mathbf{F}}^{-1}\rho_0$$

Furthermore p is the **local stress power** in classical continuum mechanics. This definition of the stress power needs to be generalized in the present work.

Chapter 3

The virtual power in third-order continua

3.1 Chapter introduction

This chapter applies the concept presented in [Bertram & Forest 2007] for a generalization of the virtual power to third gradient of the strain theories. This approach allows to deduce what higher-order strain and stress measures look like. In the present work they occur as tensors of order two, three and four. The existence and form of generalized stress tensors is not a trivial topic. In the literature several approaches exist. Many authors such as [Mindlin 1965], [Polizzotto 2012] or [Dillon & Kratovchvil 1970] simply postulate that the stress power or virtual work has a certain form with stress tensors being work conjugate to strain gradient related terms. Similarly a certain elastic energy or an action functional can be postulated to have a certain form, as done in [Auffray et al. 2015], [dell'Isola et al. 2015] or [Javili et al. 2013]. Therein stress tensors are derived from the first variation of the functional. Boundary conditions and tractions can be derived once the virtual power or virtual work has been established. An alternative approach to derive the existence of generalized stress tensors for higher-order strain gradient materials can be found in [dell'Isola et al. 2016]. In this work the authors generalize Cauchy's tetrahedron argument. This requires to equip a body's surface with a surface structure of what the authors call "wedges" which are defined through additional normal vectors on the surface. However, the approach through the stress power or virtual work is the most common. In Section 3.2 it is assumed that the virtual power is a functional that is independent of the observer. These assumptions allow to derive the principles of classical mechanics, such as the principle of d'Alembert, the Newton-Euler laws of motion and a global form of the principle of virtual power. Then Riesz' representation theorem is applied to the virtual power functional which yields the generalized stress tensors of a third gradient of the strain theory. In Section 3.3 the boundary conditions for a second gradient of strain material are derived. This is done by applying the surface divergence theorem to the form of the power functional that Riesz' presentation theorem yields. This approach stems from [Toupin 1962] and [Mindlin 1965]. It has been presented in many works ever since such as [Germain 1972], [Dillon & Kratovchvil 1970], [Polizzotto 2013] or [Javili et al. 2013]. The boundary conditions are derived for a body that has a smooth surface. In [Mindlin 1965] the boundary conditions are also derived for a body with corners and edges and these results will be needed in Chapter 7. Since the presented procedure for the derivation of boundary conditions of a body with smooth surface or with edges and corners is well established in the literature it is only outlined for the case of a body with smooth surface. From this it becomes clear how the concept can be extended to the case of a surface with edges and corners. This section also contains a short review of the methods and results to derive the boundary conditions in a higher-order continuum.

3.2 Generalization of the virtual power for higher-order continua

One assumes that a body B with smooth surface ∂B has a smooth velocity field $\mathbf{v} = \mathbf{v}(t)$ at any time t. In certain parts of the present section two observers ϕ and ψ need to be distinguished. Two observers only differ by a rotation and a shift vector. The observer dependence of a quantity is indicated by an index. In those cases, where only one observer is considered, the index will be suppressed. Two observers see a motion χ_ϕ and χ_ψ respectively. Since in the present chapter only the momentary placement is of interest the position vectors $\mathbf{r}_\phi$ and $\mathbf{r}_\psi$ are used. For the observer ϕ one obtains

(3.1) the motion $\mathbf{r}_\phi := \chi_\phi$,

(3.2) the velocity $\mathbf{r}_\phi^{\bullet} := \mathbf{v}_\phi := \chi_\phi^{\bullet}$ and

(3.3) the acceleration $\mathbf{r}_\phi^{\bullet\bullet} := \mathbf{v}_\phi := \chi_\phi^{\bullet\bullet}$.

Basic assumptions

Assumption 3.1. Principle of determinism

For every motion of a body χ_ϕ w.r.t. to an observer ϕ there exists a **power functional** $\pi_\phi(\chi_\phi)$. Its value is zero if the momentary velocity is zero everywhere.

(3.4) $\pi_\phi(\chi_\phi, \mathbf{v}_\phi) = 0$ if $\mathbf{v}_\phi = \overset{\langle 1 \rangle}{\mathbf{0}}$

Assumption 3.2. Principle of objectivity

A motion is dynamically admissible if and only if the power functional for the body is invariant under changes of observer:

$$\pi_\phi(\chi_\phi) = \pi_\psi(\chi_\psi) \tag{3.5}$$

Corollary 3.1. Euclidean transformations

(3.4) and (3.5) imply that for two observers ϕ and ψ that differ by a rotation matrix $\mathbf{Q}(t)$ and a shift vector $\mathbf{c}(t)$

$$\mathbf{r}_\phi(t) = \mathbf{Q}(t) \cdot \mathbf{r}_\psi(t) + \mathbf{c}(t) \tag{3.6}$$

$$\mathbf{v}_\phi(t) = \mathbf{Q}(t) \cdot \mathbf{v}_\psi(t) + \underbrace{\mathbf{Q}^\bullet(t)\mathbf{Q}^t(t) \cdot (\mathbf{r}_\phi(t) - \mathbf{c}(t))}_{\omega\times[\mathbf{r}_\phi(t)-\mathbf{c}(t)]} \tag{3.7}$$

Definition of the space of virtual velocities

One is tempted to consider the power functional of an observer ϕ as a linear function of the velocity. This is not correct. One has to introduce first the linear space of virtual velocities:

Definition 3.1. Space of virtual velocities

The space of virtual velocities is denoted by δV_ϕ. It is defined as the space $C^k(B_t)$ i.e. all k-times continuously differentiable vector fields on B_t, equipped with a scalar product $\langle .,. \rangle$ defined for $\delta \mathbf{v}_\phi, \delta \mathbf{w}_\phi \in \delta V_\phi$

$$\begin{aligned} \langle \delta \mathbf{v}_\phi, \delta \mathbf{w}_\phi \rangle := \int_{B_t} & \delta \mathbf{v}_\phi \cdot \delta \mathbf{w}_\phi \\ & + \widetilde{grad}(\delta \mathbf{v}_\phi) : \widetilde{grad}(\delta \mathbf{w}_\phi) \\ & + \ldots \\ & + \widetilde{grad}^k(\delta \mathbf{v}_\phi) \underbrace{\cdot \ldots \cdot}_{\text{k+1 times}} \widetilde{grad}^k(\delta \mathbf{w}_\phi) dv \end{aligned} \tag{3.8}$$

The scalar product in (3.8) defines a norm on δV_ϕ

$$\text{(3.9)} \quad \|\delta \mathbf{v}_\phi\|_{k,2} := \left(\int_B |\delta \mathbf{v}_\phi|^2 + \left|\widetilde{grad}(\delta \mathbf{v}_\phi)\right|^2 + \left|\widetilde{grad}^{II}(\delta \mathbf{v}_\phi)\right|^2 + ... + \left|\widetilde{grad}^{k}(\delta \mathbf{v}_\phi)\right|^2 \; dv \right)^{1/2}$$

The weak gradient $\widetilde{grad}$ coincides with the strong gradient $grad$ since the fields in this definition are sufficiently smooth.

Virtual velocities transform like velocities:

$$\text{(3.10)} \quad \delta \mathbf{v}_\phi(\mathbf{P}, t) = \mathbf{Q}(t) \cdot \delta \mathbf{v}_\psi(\mathbf{P}, t) + \underbrace{\mathbf{Q}^\bullet(t)\mathbf{Q}^t(t) \cdot \Big(\mathbf{r}_\psi(\mathbf{P}, T) - \mathbf{c}(t)\Big)}_{=\omega \times \left(\mathbf{r}_\psi(\mathbf{P},T) - \mathbf{c}(t)\right)}.$$

Assuming that Equation (3.10) holds is fundamental in the definition of the space of virtual velocities. It equips the mathematical space with a mechanical property, that allows the deduction of the fundamental principles of mechanics in the following sections.

Definition of the virtual power functional

The completion of δV_ϕ with respect to the norm in (3.9) is denoted by $\overline{\delta V_\phi}$. Then $\overline{\delta V_\phi}$ equipped with the scalar product from (3.8) is a Hilbert space. The reason why $\overline{\delta V_\phi}$ can be equipped with such a scalar product is that the limit of a sequence in δV_ϕ lies in L^2. Such a limit might not be differentiable though! The virtual power can be extended to a functional on $\overline{\delta V_\phi}$:

Definition 3.2. The virtual power functional

The **virtual power** for an observer ϕ is a mapping $\delta\pi(\chi_\phi, \cdot) : \overline{\delta V_\phi} \to \mathbb{R}$ which

1. is continuous and linear,

2. generalizes the mechanical power

$$\text{(3.11)} \quad \delta\pi_\phi(\chi_\phi, \mathbf{v}_\phi) = \pi_\phi(\chi_\phi)$$

Definition 3.2 implies that the following transformation behavior can be assumed for a second observer if it is known for the first.

$$\delta\pi_\phi(\chi_\phi, \delta\mathbf{v}_\phi) = \delta\pi_\psi(\chi_\psi, \delta\mathbf{v}_\psi) + \pi_\phi(\chi_\phi) - \pi_\psi(\chi_\psi) \tag{3.12}$$

The definition of the virtual power functional does not ensure uniqueness. The question of existence is complicated, since Definition 3.2 prescribes the values of a functional on a certain set of vector fields and then assumes existence of a functional on δV that assumes these values on this set. Whether such a functional exists, depends on the set where its values were prescribed. In many cases it is reasonable to assume the existence of such a functional. (A detailed proof for which cases the functional exists lies beyond the scope of the present work.)

Forces and moments

Riesz' representation theorem ensures the existence of a **generalized force** $\underline{\mathbf{f}}_\phi \in \mathbb{R}^3$ and a **generalized moment** $\underline{\mathbf{m}}_{0\phi} \in \mathbb{R}^3$ such that

$$\delta\pi_\phi(\chi_\phi, \delta\mathbf{v}^c) = \underline{\mathbf{f}}_\phi \cdot \delta\mathbf{v}^c \text{ for constant } \delta\mathbf{v}^c \in \delta V_\phi \tag{3.13}$$

$$\delta\pi_\phi(\chi_\phi, \omega \times \mathbf{r}_\phi) = \underline{\mathbf{m}}_{0\phi} \cdot \omega \text{ for constant } \omega \in \delta V_\phi \tag{3.14}$$

This yields

Definition 3.3. Resultant forces and moments

$$\textbf{Resultant force:} \quad \mathbf{f}_\phi := \underline{\mathbf{f}}_\phi + \int_B \overset{\bullet\bullet}{\mathbf{r}}_\phi \, dm \tag{3.15}$$

$$\textbf{Resultant moment:} \quad \mathbf{m}_{0\phi} := \underline{\mathbf{m}}_{0\phi} + \int_B \mathbf{r}_\phi \times \overset{\bullet\bullet}{\mathbf{r}}_\phi \, dm \tag{3.16}$$

One then deduces the following

Theorem 3.1. Properties of the generalized force and moment

(3.17) $\pi_\phi(\chi_\phi) - \pi_\psi(\chi_\psi) = \underline{\mathbf{f}}_\phi \cdot \mathbf{v}_0 + \underline{\mathbf{m}}_{0\phi} \cdot \omega$

(3.18) $\underline{\mathbf{f}}_\phi = \mathbf{Q} \cdot \underline{\mathbf{f}}_\psi$ and $\underline{\mathbf{m}}_{0\phi} = \mathbf{Q} \cdot \underline{\mathbf{m}}_{0\psi}$

(3.19) $\underline{\mathbf{m}}_{0'} = \underline{\mathbf{m}}_0 + \overrightarrow{\mathbf{0'0}} \times \underline{\mathbf{f}}$

Here $\mathbf{Q}$ denotes the rotation by which the two observers ϕ and ψ differ and $\mathbf{0}$ and $\mathbf{0}'$ two points of reference for the moments. $\overrightarrow{\mathbf{0'0}}$ denotes the vector, that connects these two points.

Equation (3.18) shows that generalized forces and moments are objective and Equation (3.19) means that Varignon's principle holds.

The principles of classical mechanics

The main principles of classical mechanics now follow from Theorem 3.1 as

Corollary 3.2. The principles of classical mechanics
A motion is dynamically admissible if and only if for one observer (and thus for all) one of the following holds

(3.20) Principle of d'Alembert: $\underline{\mathbf{f}}_\phi = 0 \textit{ and } \underline{\mathbf{m}}_{0\phi} = 0$

(3.21) Newton-Euler laws of motion: $\mathbf{f}_\phi = \int\limits_B \mathbf{r}_\phi^{\bullet\bullet} dm$ and $\mathbf{m}_{0\phi} = \int\limits_B \mathbf{r}_\phi \times \mathbf{r}_\phi^{\bullet\bullet} dm$

(3.22) Global principle of virtual power: $\forall \delta\mathbf{v}_0, \delta\omega \in \mathbb{R}^3 : \underline{\mathbf{f}}_\phi \cdot \delta\mathbf{v}_0 + \underline{\mathbf{m}}_{0\phi} \cdot \delta\omega = 0$

Riesz' representation of the power functional

$\delta\pi_\phi$ is a continuous and linear functional on $\overline{\delta V_\phi}$. Therefore Riesz' representation theorem from [Adams 1975, p. 5, theorem 1.11] can be applied. It exists a unique vector field $\overset{\langle 1 \rangle}{\widetilde{\mathbf{T}}}_\phi \in \overline{\delta V_\phi}$

such that for every $\delta \mathbf{v}_\phi \in \overline{\delta V_\phi}$

$$(3.23)\quad \delta\pi_\phi(\delta\mathbf{v}_\phi) = \int\limits_{B(t)} \overset{\langle 1 \rangle}{\widetilde{\mathbf{T}}}_\phi \cdot \delta\mathbf{v}_\phi + \widetilde{grad}(\overset{\langle 1 \rangle}{\widetilde{\mathbf{T}}}_\phi) : \widetilde{grad}(\delta\mathbf{v}_\phi)$$

$$+ ... + \widetilde{grad}^k(\overset{\langle 1 \rangle}{\widetilde{\mathbf{T}}}_\phi)) \underbrace{\cdot ... \cdot}_{\text{k+1 times}} \widetilde{grad}^k(\delta\mathbf{v}_\phi) dv$$

Riesz' theorem says that in general the field $\overset{\langle 1 \rangle}{\widetilde{\mathbf{T}}}_\phi$ is only differentiable in the weak sense even if $\delta\mathbf{v}_\phi$ is differentiable in the classic sense. ($\widetilde{grad}$ denotes the weak gradient and coincides with *grad* in case of differentiability in the classic sense.) However [Adams 1975, Theorem 6.2 PART III, p. 144] says that in the case, which is considered here, $\overset{\langle 1 \rangle}{\widetilde{\mathbf{T}}}_\phi$ is a C^{k-2} function which means that the first k-2 gradients in (3.23) are derivatives in the classic sense for sufficiently large k.

The form of the power functional in (3.23) is not suitable for mechanical frameworks which will become clear later. In order to introduce the concept of stress tensors one needs a representation where the gradients of $\overset{\langle 1 \rangle}{\widetilde{\mathbf{T}}}_\phi$ can be replaced by arbitrary tensor fields of suitable order. From [Adams 1975, p. 48, theorem 3.8] one can see that the power functional can also be written in a different form: There exists a non-unique set of tensor fields $\{\overset{\langle 1 \rangle}{\mathbf{T}}_\phi, \overset{\langle 2 \rangle}{\mathbf{T}}_\phi, ..., \overset{\langle k \rangle}{\mathbf{T}}_\phi\}$, each with components in $L^2(B_t)$, such that

$$(3.24)\quad \delta\pi_\phi(\delta\mathbf{v}_\phi) = \int\limits_{B(t)} \overset{\langle 1 \rangle}{\mathbf{T}}_\phi \cdot \delta\mathbf{v}_\phi + \overset{\langle 2 \rangle}{\mathbf{T}}_\phi : \widetilde{grad}(\delta\mathbf{v}_\phi) + ... + \overset{\langle k \rangle}{\mathbf{T}}_\phi \underbrace{\cdot ... \cdot}_{\text{k+1 times}} \widetilde{grad}^k(\delta\mathbf{v}_\phi) dv$$

One has to keep in mind that in general the tensor fields $\{\overset{\langle 1 \rangle}{\mathbf{T}}_\phi, ..., \overset{\langle k \rangle}{\mathbf{T}}_\phi\}$ are not differentiable, neither in the classic nor in the weak sense. They are only L^2-integrable. One can show that $\{\overset{\langle 1 \rangle}{\mathbf{T}}_\phi, ..., \overset{\langle k \rangle}{\mathbf{T}}_\phi\}$ can always be assumed to fulfill (3.24) and

$$(3.25)\quad \overset{\langle i \rangle}{\mathbf{T}}_\phi \neq \widetilde{grad}^{i-1}(\overset{\langle 1 \rangle}{\mathbf{T}}_\phi) \quad for\ i \in \{2, ..., k\}$$

This means one can always represent the power functional $\delta\pi_\phi$ in the form (3.24) and choose if the tensor fields $\{\overset{\langle 2\rangle}{\mathbf{T}}_\phi, ..., \overset{\langle k\rangle}{\mathbf{T}}_\phi\}$ are weak gradients of $\overset{\langle 1\rangle}{\mathbf{T}}$ or not. Equation (3.24) is a mathematical form of the virtual power functional that allows the construction of higher-order field theories for continuum mechanics. The tensor fields $\{\overset{\langle 1\rangle}{\mathbf{T}}_\phi, ..., \overset{\langle k\rangle}{\mathbf{T}}_\phi\}$ will be interpreted as generalized stress tensors, which is the reason why it is desirable that they are independent of each other rather than gradient of a C^1 vector field $\overset{\langle 1\rangle}{\widetilde{\mathbf{T}}}$. The problem here is that in general $\overset{\langle 1\rangle}{\mathbf{T}}_\phi, ..., \overset{\langle k\rangle}{\mathbf{T}}_\phi$ are not differentiable in the classic sense. This property will be crucial in order to apply integration by parts and the divergence theorem which will allow not only to derive boundary conditions for higher-order theories but also to generalize the classic local balance equations of momentum and moment of momentum. Unfortunately from mathematical reasoning it is not possible to substitute $\overset{\langle 1\rangle}{\mathbf{T}}_\phi, ..., \overset{\langle k\rangle}{\mathbf{T}}_\phi$ by a set of differentiable tensors. The fact that C^∞ is dense in L^2 allows for any $\varepsilon > 0$ to find a set of C^∞-differentiable tensor fields $\overset{\langle 1\rangle}{\mathbf{T}}_{\varepsilon\phi}, ..., \overset{\langle k\rangle}{\mathbf{T}}_{\varepsilon\phi}$ such that

$$(3.26) \quad |\delta\pi_{\varepsilon\phi}(\delta\mathbf{v}_\phi) - \delta\pi_\phi(\delta\mathbf{v}_\phi)| \leq \varepsilon \left\|\delta\mathbf{v}_\phi\right\|_{k,2}$$

where $\delta\pi_{\varepsilon\phi}$ is defined as

$$(3.27) \quad \delta\pi_{\varepsilon\phi}(\delta\mathbf{v}_\phi) := \int\limits_{B(t)} \overset{\langle 1\rangle}{\mathbf{T}}_{\varepsilon\phi} \cdot\delta\mathbf{v}_\phi + \overset{\langle 2\rangle}{\mathbf{T}}_{\varepsilon\phi} : \widetilde{grad}(\delta\mathbf{v}_\phi) + ... + \overset{\langle k\rangle}{\mathbf{T}}_{\varepsilon\phi} \underbrace{\cdots\cdots}_{\text{k+1 times}} \widetilde{grad}^k(\delta\mathbf{v}_\phi)dv.$$

However (3.26) allows $|\delta\pi_{\varepsilon\phi}(\delta\mathbf{v}_\phi) - \delta\pi_\phi(\delta\mathbf{v}_\phi)|$ to become large if $\left\|\delta\mathbf{v}\right\|_{k,2}$ is large enough. This shows that the outlined approach to derive the existence of a power functional mathematically has its limitations. It does ensure the existence of the virtual power functional as well as tensors that can be interpreted as stress tensors. It does not provide the differentiability of those stress tensors which would be needed. Therefore the only options to set up a higher strain gradient framework is to follow the common approach to assume that the power functional has the form (3.24) and that stress tensors are differentiable.

Principle of invariance of the stress power under rigid body motions

Definition 3.4. Stress power

In the form (3.24) one defines

$$P := \int_{B(t)} \overset{\langle 2\rangle}{\mathbf{T}}_{\phi} : \widetilde{grad}(\delta \mathbf{v}_{\phi}) + ... + \overset{\langle k\rangle}{\mathbf{T}}_{\phi} \underbrace{\cdot ... \cdot}_{\text{k+1 times}} \widetilde{grad}^{k}(\delta \mathbf{v}_{\phi}) dv \tag{3.28}$$

as the **stress power** or **internal power**.

Assumption 3.3. Principle of invariance under rigid body motions

Superimposing a rigid body motion does not alter the virtual power of a motion χ. For a rotation matrix $\mathbf{Q}(t)$ and a shift vector $\mathbf{c}(t)$

$$P\Big(\chi(t)\Big) = P\Big(\mathbf{Q}(t) \cdot \chi(t) + \mathbf{c}(t)\Big). \tag{3.29}$$

It is important to note that assumption 3.3 is not equivalent to corollary 3.1. These are two independent concepts as explained in [Svendsen & Bertram 1999] and [Bertram & Svendsen 2001], where kinetic gases are given as an example of a material that does not obey the principle of superimposed rigid body motion. This distinction of material properties plays an important role in Section 4.4 for the derivation of reduced forms.

3.3 Derivation of boundary conditions for third-order continua

In [Mindlin 1965] the following procedure for obtaining the boundary conditions of the third-order theory is presented.

$$\delta\pi(\delta \mathbf{v}) = \int_{B(t)} \overset{\langle 1\rangle}{\mathbf{T}} \cdot \delta \mathbf{v} + \underbrace{\overset{\langle 2\rangle}{\mathbf{T}} : grad(\delta \mathbf{v}) + \overset{\langle 3\rangle}{\mathbf{T}} \,\vdots\, grad^{2}(\delta \mathbf{v}) + \overset{\langle 4\rangle}{\mathbf{T}} :: grad^{3}(\delta \mathbf{v})}_{\text{Transforms with Remark 2.3}} dv \tag{3.30}$$

$$= \int_{B(t)} \{\overset{\langle 1\rangle}{\mathbf{T}} - div(\overset{\langle 2\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 3\rangle}{\mathbf{T}}) - div^{III}(\overset{\langle 4\rangle}{\mathbf{T}})\} \cdot \delta \mathbf{v} \; dv \tag{3.31}$$

$$+ \int_{\partial B(t)} \{[\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n}\} \cdot \delta \mathbf{v}$$

$$+\underbrace{\{(\overset{\langle 3\rangle}{\mathbf{T}}-div(\overset{\langle 4\rangle}{\mathbf{T}}))\cdot\mathbf{n}\}:grad(\delta\mathbf{v})}_{\mathrm{I}}+\underbrace{\{\overset{\langle 4\rangle}{\mathbf{T}}\cdot\mathbf{n}\}\vdots grad^2(\delta\mathbf{v})}_{\mathrm{II}}\;da$$

Now Remark 2.4 is applied to each of the terms I and II

$$\begin{aligned}(3.32)\quad \delta\pi(\delta\mathbf{v}) = &\int_{B(t)}\{\overset{\langle 1\rangle}{\mathbf{T}}-div(\overset{\langle 2\rangle}{\mathbf{T}})+div^{II}(\overset{\langle 3\rangle}{\mathbf{T}})-div^{III}(\overset{\langle 4\rangle}{\mathbf{T}})\}\cdot\delta\mathbf{v}\;dv\\
+&\int_{\partial B(t)}\{[\overset{\langle 2\rangle}{\mathbf{T}}-div(\overset{\langle 3\rangle}{\mathbf{T}})+div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})]\cdot\mathbf{n}\}\cdot\delta\mathbf{v}\\
&+\{\kappa_m[\overset{\langle 3\rangle}{\mathbf{T}}-div(\overset{\langle 4\rangle}{\mathbf{T}})]:\mathbf{n}\otimes\mathbf{n}-div_t[(\overset{\langle 3\rangle}{\mathbf{T}}-div(\overset{\langle 4\rangle}{\mathbf{T}}))\cdot\mathbf{n}]\}\cdot\delta\mathbf{v}\\
&+\{[\overset{\langle 3\rangle}{\mathbf{T}}-div(\overset{\langle 4\rangle}{\mathbf{T}})]\cdot\mathbf{n}\}:grad_n(\delta\mathbf{v})\\
&+\underbrace{\{\kappa_m\,\overset{\langle 4\rangle}{\mathbf{T}}:\mathbf{n}\otimes\mathbf{n}\}:grad(\delta\mathbf{v})}_{\mathrm{III}}+\underbrace{(\overset{\langle 4\rangle}{\mathbf{T}}\cdot\mathbf{n})\vdots grad_n(grad(\delta\mathbf{v}))}_{\mathrm{IV}}\;da\end{aligned}$$

Again Remark 2.4 is applied to term III

$$\begin{aligned}(3.33)\quad \delta\pi(\delta\mathbf{v}) = &\int_{B(t)}\{\overset{\langle 1\rangle}{\mathbf{T}}-div(\overset{\langle 2\rangle}{\mathbf{T}})+div^{II}(\overset{\langle 3\rangle}{\mathbf{T}})-div^{III}(\overset{\langle 4\rangle}{\mathbf{T}})\}\cdot\delta\mathbf{v}\;dv\\
+&\int_{\partial B(t)}\{[\overset{\langle 2\rangle}{\mathbf{T}}-div(\overset{\langle 3\rangle}{\mathbf{T}})+div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})]\cdot\mathbf{n}\}\cdot\delta\mathbf{v}\\
&+\{\kappa_m[\overset{\langle 3\rangle}{\mathbf{T}}-div(\overset{\langle 4\rangle}{\mathbf{T}})]:\mathbf{n}\otimes\mathbf{n}-div_t[(\overset{\langle 3\rangle}{\mathbf{T}}-div(\overset{\langle 4\rangle}{\mathbf{T}}))\cdot\mathbf{n}]\}\cdot\delta\mathbf{v}\\
&+\{[\overset{\langle 3\rangle}{\mathbf{T}}-div(\overset{\langle 4\rangle}{\mathbf{T}})]\cdot\mathbf{n}\}:grad_n(\delta\mathbf{v})\\
&+\{(\kappa_m)^2\,\overset{\langle 4\rangle}{\mathbf{T}}\vdots\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}-div_t(\overset{\langle 4\rangle}{\mathbf{T}}:\mathbf{n}\otimes\mathbf{n})\}\cdot\delta\mathbf{v}\\
&+\underbrace{(\overset{\langle 4\rangle}{\mathbf{T}}\cdot\mathbf{n})\vdots grad_n(grad(\delta\mathbf{v}))}_{\mathrm{IV}}\;da\end{aligned}$$

Now $grad_n(grad(\delta\mathbf{v}))$ in term IV is decomposed additively

$$(3.34)\quad \delta\pi(\delta\mathbf{v}) = \int_{B(t)}\{\overset{\langle 1\rangle}{\mathbf{T}}-div(\overset{\langle 2\rangle}{\mathbf{T}})+div^{II}(\overset{\langle 3\rangle}{\mathbf{T}})-div^{III}(\overset{\langle 4\rangle}{\mathbf{T}})\}\cdot\delta\mathbf{v}\;dv$$

$$
\begin{aligned}
+ \quad \int_{\partial B(t)} & \{[\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n}\} \cdot \delta\mathbf{v} \\
& + \{\kappa_m[\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}})] : \mathbf{n}\otimes\mathbf{n} - div_t[(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}})) \cdot \mathbf{n}]\} \cdot \delta\mathbf{v} \\
& + \{[\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n}\} : grad_n(\delta\mathbf{v}) \\
& + \{(\kappa_m)^2 \overset{\langle 4\rangle}{\mathbf{T}} \,\vdots\, \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} - div_t(\overset{\langle 4\rangle}{\mathbf{T}} : \mathbf{n}\otimes\mathbf{n})\} \cdot \delta\mathbf{v} \\
& + \{\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n}\} \,\vdots\, grad_n(grad_n(\delta\mathbf{v})) \\
& + \underbrace{\{\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n}\} \,\vdots\, grad_t(grad_n(\delta\mathbf{v}))}_{\mathrm{V}} + \underbrace{\{\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n}\} \,\vdots\, grad_t(\delta\mathbf{v}) \cdot grad_t(\mathbf{n}\otimes\mathbf{n})}_{\mathrm{VI}} \; da
\end{aligned}
$$

The integrals of the terms V and VI can be transformed by applying the product rule and then Theorem 2.1 (surface divergence theorem):

$$
\begin{aligned}
\int_{\partial B(t)} \mathrm{V}\, da &= \int_{\partial B(t)} div_t(grad_n(\delta\mathbf{v}) : \overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n}) - div_t(\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n}) : grad_n(\delta\mathbf{v})\; da \\
&= \int_{\partial B(t)} \{\kappa_m \overset{\langle 4\rangle}{\mathbf{T}} : [\mathbf{n}\otimes\mathbf{n}] - div_t(\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n})\} : grad_n(\delta\mathbf{v})\; da \\
\int_{\partial B(t)} \mathrm{VI}\, da &= \int_{\partial B(t)} div_t(\delta\mathbf{v} \cdot grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} \cdot\mathbf{n}) - \delta\mathbf{v} \cdot div_t(grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} \cdot\mathbf{n}) da \\
&= \int_{\partial B(t)} \{[\kappa_m grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} : \mathbf{n}\otimes\mathbf{n}] - div_t(grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} \cdot\mathbf{n})\} \cdot \delta\mathbf{v}\; da.
\end{aligned}
$$

One obtains with these transformations

$$
\begin{aligned}
(3.35) \quad \delta\pi(\delta\mathbf{v}) = \quad & \int_{B(t)} \{\overset{\langle 1\rangle}{\mathbf{T}} - div(\overset{\langle 2\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 3\rangle}{\mathbf{T}}) - div^{III}(\overset{\langle 4\rangle}{\mathbf{T}})\} \cdot \delta\mathbf{v}\; dv \\
+ \quad \int_{\partial B(t)} & \{[\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n}\} \cdot \delta\mathbf{v} \\
& + \{\kappa_m[\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}})] : \mathbf{n}\otimes\mathbf{n} - div_t[(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}})) \cdot \mathbf{n}]\} \cdot \delta\mathbf{v} \\
& + \{[\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n}\} : grad_n(\delta\mathbf{v}) \\
& + \{(\kappa_m)^2 \overset{\langle 4\rangle}{\mathbf{T}} \,\vdots\, \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} - div_t(\overset{\langle 4\rangle}{\mathbf{T}} : \mathbf{n}\otimes\mathbf{n})\} \cdot \delta\mathbf{v} \\
& + \{\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n}\} \,\vdots\, grad_n(grad_n(\delta\mathbf{v}))
\end{aligned}
$$

$$+\{\kappa_m \overset{\langle 4\rangle}{\mathbf{T}} : [\mathbf{n}\otimes\mathbf{n}] - div_t(\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n})\} : grad_n(\delta\mathbf{v})$$

$$+\{[\kappa_m grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} : \mathbf{n}\otimes\mathbf{n}]$$

$$- div_t(grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} \cdot\mathbf{n})\}\cdot\delta\mathbf{v}\ da$$

After rearranging the terms of this equation one arrives at

The final form of the virtual power

$$\delta\pi(\delta\mathbf{v}) = \int\limits_{B(t)} \{\overset{\langle 1\rangle}{\mathbf{T}} - div\ \overset{\langle 2\rangle}{\mathbf{T}} + div^{II}\ \overset{\langle 3\rangle}{\mathbf{T}} - div^{III}\ \overset{\langle 4\rangle}{\mathbf{T}}\}\cdot\delta\mathbf{v}\ dv \tag{3.36}$$

$$+ \int\limits_{\partial B(t)} \{[\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})$$

$$- div_t(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR}) + div_t^2(\overset{\langle 4\rangle}{\mathbf{T}})]\cdot\mathbf{n}$$

$$- [(1+\kappa_m) div_t(\overset{\langle 4\rangle}{\mathbf{T}}) - \kappa_m grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR}$$

$$- \kappa_m(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}))] : \mathbf{n}\otimes\mathbf{n} + [(\kappa_m)^2\ \overset{\langle 4\rangle}{\mathbf{T}}] \vdots\ \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\}\cdot\delta\mathbf{v}$$

$$+\left\{\left[\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - 2div_t(\overset{\langle 4\rangle}{\mathbf{T}})\right]\cdot\mathbf{n} + 2(\kappa_m)\ \overset{\langle 4\rangle}{\mathbf{T}} : \mathbf{n}\otimes\mathbf{n}\right\} : grad_n(\delta\mathbf{v})$$

$$+\{\overset{\langle 4\rangle}{\mathbf{T}} \cdot\mathbf{n}\} \vdots\ grad_n grad_n(\delta\mathbf{v})\ da$$

In the form obtained above the virtual power functional yields generalized forms of the main principles of classical mechanics. These are presented in the next section.

3.4 Generalization of the principles of mechanics for third-order continua

Following the lines of [Bertram & Forest 2007] allows to obtain generalizations of the main principles of mechanics in the third-order theory. The results are given here as a few bullet points since the concept is explained in detail in [Bertram & Forest 2007] and its application is straightforward.

- **For a constant field $\delta\mathbf{v}{=}\mathbf{v}_0$ one obtains the generalized force $\underline{\mathbf{f}}$:**

$$\text{(3.37)}\quad \underline{\mathbf{f}} = \int_{B(t)} \underbrace{\Big(\overset{\langle 1\rangle}{\mathbf{T}} - div\ \overset{\langle 2\rangle}{\mathbf{T}} + div^{II}\ \overset{\langle 3\rangle}{\mathbf{T}} - div^{III}\ \overset{\langle 4\rangle}{\mathbf{T}} \Big)}_{=:\ \rho\underline{\mathbf{b}}\ =:\ \rho(\mathbf{b}-\ddot{r})} dv$$

$$+ \int_{\partial B(t)} \{[\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})$$

$$- div_t(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}{}^{LR}) + div_t^2(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n}$$

$$- [(1+\kappa_m) div_t(\overset{\langle 4\rangle}{\mathbf{T}}) - \kappa_m grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}{}^{LR}$$

$$- \kappa_m(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}))] : \mathbf{n}\otimes\mathbf{n} + [(\kappa_m)^2\ \overset{\langle 4\rangle}{\mathbf{T}}] \vdots\ \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\} da$$

where $\underline{\mathbf{b}}$ is called the specific generalized body force.

- **For $\delta\mathbf{v} = \delta\omega \times \mathbf{r}$ with constant $\delta\omega$ one obtains the generalized moment $\underline{\mathbf{m}}_0$:**

$$\text{(3.38)}\quad \underline{\mathbf{m}}_o = \int_{B(t)} \mathbf{r} \times \underline{\mathbf{b}}\ dm$$

$$+ \int_{\partial B(t)} \mathbf{r} \times \Big(\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}}) \Big) \cdot \mathbf{n} + 2axi\Big((\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}})) \cdot \mathbf{n} \Big)\ da$$

- **Integral form of balance of linear momentum and moment of momentum**

It follows from d'Alemberts Principle that a motion is dynamically admissible if and only if

$$\text{(3.39)}\quad 1.\quad \int_{B(t)} \ddot{\mathbf{r}}\ dm = \int_{B(t)} \mathbf{b}\ dm + \int_{\partial B(t)} \{[\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})$$

$$- div_t(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}{}^{LR}) + div_t^2(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n}$$

$$- [(1+\kappa_m) div_t(\overset{\langle 4\rangle}{\mathbf{T}}) - \kappa_m grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}{}^{LR} - \kappa_m(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}))] : \mathbf{n}\otimes\mathbf{n}$$

$$+ [(\kappa_m)^2\ \overset{\langle 4\rangle}{\mathbf{T}}] \vdots\ \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\} da$$

$$\text{(3.40)}\quad 2.\quad \int_{B(t)} \mathbf{r} \times \ddot{\mathbf{r}}\ dm = \int_{B(t)} \mathbf{r} \times \mathbf{b}\ dm$$

$$+\int\limits_{\partial B(t)} \mathbf{r}\times(\overset{\langle 2\rangle}{\mathbf{T}} - div\,\overset{\langle 3\rangle}{\mathbf{T}} + div^{II}\,\overset{\langle 4\rangle}{\mathbf{T}})\cdot\mathbf{n} + 2axi\Big((\overset{\langle 3\rangle}{\mathbf{T}} - div\,\overset{\langle 4\rangle}{\mathbf{T}})\cdot\mathbf{n}\Big)\ da$$

hold for the body for one and thus for all observers.

- **Extended Cauchy's Laws**

  A motion of a body is dynamically admissible if and only if

$$(3.41)\quad 1.\Big(div(\overset{\langle 2\rangle}{\mathbf{T}}) - div^{II}(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{III}(\overset{\langle 4\rangle}{\mathbf{T}})\Big) + \rho\mathbf{b} = \rho\ddot{\mathbf{r}} \text{ and}$$

$$(3.42)\quad 2.\ \overset{\langle 2\rangle}{\mathbf{T}} = \overset{\langle 2\rangle}{\mathbf{T}}^{T}$$

  almost everywhere in a body.

- **Principle of virtual power, integral version**

  A motion is dynamically admissible if and only if for every $\delta\mathbf{v}\in\delta V_\phi$ for one observer ϕ (and thus for all)

$$(3.43)\quad \int\limits_{B(t)} \overset{\langle 2\rangle}{\mathbf{T}} : sym\ grad(\delta\mathbf{v}) + \overset{\langle 3\rangle}{\mathbf{T}} \vdots grad^{II}(\delta\mathbf{v}) + \overset{\langle 4\rangle}{\mathbf{T}} :: grad^{III}(\delta\mathbf{v})$$

$$= \int\limits_{B(t)} \underline{\mathbf{b}}\cdot\delta\mathbf{v}\ dv$$

$$+ \int\limits_{\partial B(t)} \{[\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}})$$

$$- div_t(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR}) + div_t^2(\overset{\langle 4\rangle}{\mathbf{T}})]\cdot\mathbf{n}$$

$$- [(1+\kappa_m)div_t(\overset{\langle 4\rangle}{\mathbf{T}}) - \kappa_m grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} - \kappa_m(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}))] : \mathbf{n}\otimes\mathbf{n}$$

$$+ [(\kappa_m)^2\ \overset{\langle 4\rangle}{\mathbf{T}}] \vdots \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}\}\cdot\delta\mathbf{v}\ da$$

$$+ \int\limits_{\partial B(t)} \{\Big[\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - 2div_t(\overset{\langle 4\rangle}{\mathbf{T}})\Big]\cdot\mathbf{n} + 2(\kappa_m)\ \overset{\langle 4\rangle}{\mathbf{T}} : \mathbf{n}\otimes\mathbf{n}\} : grad_n(\delta\mathbf{v})\ da$$

$$+ \int\limits_{\partial B(t)} \{\overset{\langle 4\rangle}{\mathbf{T}}\cdot\mathbf{n}\} \vdots grad_n grad_n(\delta\mathbf{v})\ da$$

Using the fact that two vector fields $\partial_{\mathbf{n}}(\delta v)$ and $\partial^2_{\mathbf{n}}(\delta v)$ exist such that

$$grad_n(\delta\mathbf{v}) = \partial_{\mathbf{n}}(\delta\mathbf{v}) \otimes \mathbf{n} \qquad grad_n grad_n(\delta\mathbf{v}) = \partial^2_{\mathbf{n}}(\delta\mathbf{v}) \otimes \mathbf{n} \otimes \mathbf{n} \tag{3.44}$$

one defines three tensor fields that can be interpreted as **generalized surface tractions**:

1. the vector field of tractions,

$$\begin{aligned} \mathbf{t}_1 :=& [\overset{\langle 2\rangle}{\mathbf{T}} - div(\overset{\langle 3\rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4\rangle}{\mathbf{T}}) \\ & - div_t(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR}) + div_t^2(\overset{\langle 4\rangle}{\mathbf{T}})] \cdot \mathbf{n} \\ & - [(1+\kappa_m) div_t(\overset{\langle 4\rangle}{\mathbf{T}}) - \kappa_m grad_t(\mathbf{n}\otimes\mathbf{n}) : \overset{\langle 4\rangle}{\mathbf{T}}^{LR} \\ & - \kappa_m(\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}))] : \mathbf{n}\otimes\mathbf{n} + [(\kappa_m)^2 \overset{\langle 4\rangle}{\mathbf{T}}] \,\vdots\, \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} \end{aligned} \tag{3.45}$$

2. the tensor field of double tractions,

$$\mathbf{t}_2 := \{\Big[\overset{\langle 3\rangle}{\mathbf{T}} - div(\overset{\langle 4\rangle}{\mathbf{T}}) - 2div_t(\overset{\langle 4\rangle}{\mathbf{T}})\Big] \cdot \mathbf{n} + 2(\kappa_m)\overset{\langle 4\rangle}{\mathbf{T}} : \mathbf{n}\otimes\mathbf{n}\} \cdot \mathbf{n} \tag{3.46}$$

3. the tensor field of triple tractions,

$$\mathbf{t}_3 := \overset{\langle 4\rangle}{\mathbf{T}} \,\vdots\, \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n} \tag{3.47}$$

Therefore **Neumann boundary conditions** for the generalized tractions can be prescribed on the surface $\partial B(t)$ in the following form:

$$\mathbf{t}_1 = \mathbf{t}_1^{pre} \tag{3.48}$$

$$\mathbf{t}_2 = \mathbf{t}_2^{pre} \tag{3.49}$$

$$\mathbf{t}_3 = \mathbf{t}_3^{pre}. \tag{3.50}$$

- **Balance of power, integral version (obtained by setting $\delta \mathbf{v} = \mathbf{v}$)**

  If a motion is kinematically admissible then the balance of power states that the power of the external loads equals the change of the kinetic energy plus stress power:

$$
\begin{aligned}
&\left(\int\limits_{B(t)} \frac{1}{2}\rho \mathbf{v} \cdot \mathbf{v} dv \right)^{\bullet} + \int\limits_{B(t)} \overset{\langle 2 \rangle}{\mathbf{T}} : sym\ grad(\mathbf{v}) + \overset{\langle 3 \rangle}{\mathbf{T}} \vdots grad^{II}(\mathbf{v}) + \overset{\langle 4 \rangle}{\mathbf{T}} :: grad^{III}(\mathbf{v}) \\
&= \int\limits_{B(t)} \mathbf{b} \cdot \mathbf{v}\ dv \\
&+ \int\limits_{\partial B(t)} \{ [\overset{\langle 2 \rangle}{\mathbf{T}} - div(\overset{\langle 3 \rangle}{\mathbf{T}}) + div^{II}(\overset{\langle 4 \rangle}{\mathbf{T}}) \\
&\quad - div_t(\overset{\langle 3 \rangle}{\mathbf{T}} - div(\overset{\langle 4 \rangle}{\mathbf{T}}) - grad_t(\mathbf{n} \otimes \mathbf{n}) : \overset{\langle 4 \rangle}{\mathbf{T}}^{LR}) + div_t^2(\overset{\langle 4 \rangle}{\mathbf{T}})] \cdot \mathbf{n} \\
&\quad - [(1+\kappa_m) div_t(\overset{\langle 4 \rangle}{\mathbf{T}}) - \kappa_m grad_t(\mathbf{n} \otimes \mathbf{n}) : \overset{\langle 4 \rangle}{\mathbf{T}}^{LR} - \kappa_m(\overset{\langle 3 \rangle}{\mathbf{T}} - div(\overset{\langle 4 \rangle}{\mathbf{T}}))] : \mathbf{n} \otimes \mathbf{n} \\
&\quad + [(\kappa_m)^2 \overset{\langle 4 \rangle}{\mathbf{T}}] \vdots \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \} \cdot \mathbf{v}\ da \\
&+ \int\limits_{\partial B(t)} \{ \left[\overset{\langle 3 \rangle}{\mathbf{T}} - div(\overset{\langle 4 \rangle}{\mathbf{T}}) - 2 div_t(\overset{\langle 4 \rangle}{\mathbf{T}}) \right] \cdot \mathbf{n} + 2(\kappa_m) \overset{\langle 4 \rangle}{\mathbf{T}} : \mathbf{n} \otimes \mathbf{n} \} : grad_n(\mathbf{v})\ da \\
&+ \int\limits_{\partial B(t)} \{ \overset{\langle 4 \rangle}{\mathbf{T}} \cdot \mathbf{n} \} \vdots grad_n grad_n(\mathbf{v})\ da
\end{aligned}
\tag{3.51}
$$

Chapter 4

A material framework for third-order elasticity

4.1 Chapter introduction

In this chapter the foundations for a material third-order elastoplasticity framework are laid. This is done by extending the second-order framework proposed in [Bertram 2015], following the outline of this work. First the generalized material strain variables for a second gradient of strain theory are introduced in Section 4.2. These variables are obtained by pulling back the stress power to a reference placement. In turns out that this procedure naturally yields material strain variables for the classical and first gradient of strain theory but not for the second gradient of strain theory. In the second gradient of strain theory this procedure yields two sets of material strain variables. The section also reveals another novum in the second gradient of the strain theory. It is necessary to abolish the concept that for each material strain tensor a work conjugate material stress tensor exists. Instead the definition of work conjugacy has to be modified in such a way that one defines for a set containing a material second-, third- and fourth-order strain tensor the work conjugate set of a material second-, third- and fourth-order strain tensor.

One of the material variables that the pullback of the stress power yields is chosen to develop the envisaged framework for second strain gradient elastoplasticity, while the same framework is outlined for the other variable in Appendix A. In Appendix B it is shown that a third possible strain variable, the gradient of the right Cauchy-Green tensor exists but that it is less convenient to handle than the two other ones.

In Section 4.3 the material work conjugate stress tensors are derived for the chosen set of material strain tensors, and in Section 4.4 generalized stresses and strains are set in relation to each other by generalizing the concept of constitutive equations and the elastic energy. Throughout the chapter all concepts will be formulated for an elastic energy as well as for constitutive equations. Section 4.5 contains the transformation rules for these quantities under changes of the reference placement. An important result from Section 4.5 is that the transformation behavior (under changes of the reference placement) of the material fourth-order strain tensor depends on the current state of the material third-order strain tensor. Similarly the transformation behavior of the material third-order stress tensor is dependent on the state of the material fourth-order stress tensor and of the material third-order strain tensor. Such a rather complicated transformation behavior under changes of the reference placement does not occur in the first gradient of strain theory, where no stress or strain variable influences the transformation behavior of another stress or strain variable under changes of the reference placement. What follows are Sections 4.6 and 4.7 where the results obtained so far are applied to generalize the concepts of elastic isomorphy and material symmetry. This is a straightforward task but it is clearly more complicated than in the case of a first gradient of strain theory due to the complex transformation behavior under changes of the reference placement. The concept of elastic isomorphy allows to define what it means that two points of a material body exhibit the same elastic behavior and the concept of elastic symmetry allows to define symmetry groups of a material.

4.2 Introduction of material strain measures

In this section material strain measures are introduced and compared for first-, second- and third-order elasto-plasticity theories of finite deformations. In this context the order of a mechanical theory indicates up to which order gradients of the motion are involved. This means that the classic Cauchy theory is a first-order theory since the first gradient of χ determines the stresses and strains. Therefore one strain and one stress tensor is required in such a theory. For a second-order theory the stresses and strains are determined by the first and second gradient of the motion. Therefore two strain tensors and two stress tensors are required for such a theory. Accordingly in a third-order theory three stress and three strain tensors exist. The concept of work conjugacy plays an important role in this context. It will be generalized for second- and third-order theories in this section.

Definition 4.1. Work conjugacy in third-order theories
Let $\{\overset{\langle 2\rangle}{\mathbf{E}}_X, \overset{\langle 3\rangle}{\mathbf{E}}_X, \overset{\langle 4\rangle}{\mathbf{E}}_X\}$ be a set of strain measures and $\{\overset{\langle 2\rangle}{\mathbf{S}}_X, \overset{\langle 3\rangle}{\mathbf{S}}_X, \overset{\langle 4\rangle}{\mathbf{S}}_X\}$ a set of stress

measures. These two sets are **work conjugate** to each other if

$$\rho_0 p = \overset{\langle 2\rangle}{\mathbf{S}}_X : \overset{\langle 2\rangle}{\mathbf{E}}{}^{\bullet}_X + \overset{\langle 3\rangle}{\mathbf{S}}_X \vdots \overset{\langle 3\rangle}{\mathbf{E}}{}^{\bullet}_X + \overset{\langle 4\rangle}{\mathbf{S}}_X :: \overset{\langle 4\rangle}{\mathbf{E}}{}^{\bullet}_X \tag{4.1}$$

holds for arbitrary processes.

It is important to note that according to definition (4.1) in the third-order theories a single material strain tensor of order three or four does not have a work conjugate material stress tensor or vice versa. This is only the case in a first- or second-order theory. From the following sections it will become clear, why this definition of work conjugacy is required.

4.2.1 A natural strain measure in a first-order theory

The motivation of definition (4.1) lies in a transformation of the classic stress power.

$$P = \int_{B_t} \frac{1}{\rho} \overset{\langle 2\rangle}{\mathbf{T}} : grad(\mathbf{v}) dm = \int_{B_t} \frac{1}{\rho} \overset{\langle 2\rangle}{\mathbf{T}} : sym\big(grad(\mathbf{v})\big) dm = \int_{B_t} \frac{1}{\rho} \frac{1}{2} \underbrace{[\mathbf{F}^{-1} * \overset{\langle 2\rangle}{\mathbf{T}}]}_{=: J_{\mathbf{F}}^{-1} \overset{\langle 2\rangle}{\mathbf{S}}} : \mathbf{C}^{\bullet} dm \tag{4.2}$$

$$= \int_{B_0} \frac{1}{\rho_0} \frac{1}{2} \overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} dm \tag{4.3}$$

Equation (4.2) shows how the stress power as an integral over the body $B(t)$ can be transformed into an integral expression over the reference placement B_0. In the Cauchy continuum (i.e. a first-order theory) the most common material strain measure is the right Cauchy-Green tensor $\mathbf{C}$. It is the natural choice since Equation (4.2) holds for the stress power functional. In (4.2) one makes use of the fact that $\overset{\langle 2\rangle}{\mathbf{T}}$ is symmetric. This is why only the symmetric part of $grad(\mathbf{v})$ enters the scalar product in Equation (4.2). The reason for $\mathbf{C}$ being a natural choice is that $\frac{1}{2}\mathbf{C}^{\bullet}$ is the pullback of $sym\big(grad(\mathbf{v})\big)$. This fact allows to interpret the second Piola-Kirchhoff tensor $\overset{\langle 2\rangle}{\mathbf{S}}$ as the work conjugate stress tensor of $\frac{1}{2}\mathbf{C}^{\bullet}$.
It is very important to note that the symmetry of $\overset{\langle 2\rangle}{\mathbf{T}}$, which follows from the invariance of the virtual power under rotations of the observer, plays a crucial role here. If $\overset{\langle 2\rangle}{\mathbf{T}}$ wasn't symmetric

one would obtain for the stress power

$$(4.4) \qquad P = ... = \int_{B_0} \frac{1}{\rho_0}\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : [\mathbf{F}^T * grad(\mathbf{v})]dm = \int_{B_0} \frac{1}{\rho_0}\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : [\mathbf{F}^{-1} \cdot \mathbf{F}^{\bullet}]dm$$

There is no tensor field $\overset{\langle 2\rangle}{\mathbf{X}}$ that fullfills

$$(4.5) \qquad \overset{\langle 2\rangle}{\mathbf{X}}{}^{\bullet} = \mathbf{F}^{-1} \cdot \mathbf{F}^{\bullet}$$

such that in this case $\overset{\langle 2\rangle}{\mathbf{X}}$ could be used as a strain measure with $\overset{\langle 2\rangle}{\mathbf{S}}$ being power conjugate to $\overset{\langle 2\rangle}{\mathbf{X}}$.

This can be seen from the standard integrability condition which follows from Schwartz' theorem: Assuming that such a smooth tensor field $\overset{\langle 2\rangle}{\mathbf{X}}$ exists it must be a function of $\mathbf{F}$ which means $\overset{\langle 2\rangle}{\mathbf{X}}=\overset{\langle 2\rangle}{\mathbf{X}}(\mathbf{F})$ and one can write

$$(4.6) \qquad \frac{\partial X_{ab}}{\partial F_{cd}} F^{\bullet}_{cd} = X^{\bullet}_{ab}(\mathbf{F}) = F^{-1}_{ac}F^{\bullet}_{cb} = F^{-1}_{ac}\delta_{db}F^{\bullet}_{cd}$$

$$(4.7) \qquad \Rightarrow \frac{\partial X_{ab}}{\partial F_{cd}} = F^{-1}_{ac}\delta_{db}$$

According to the Schwarz theorem for sufficiently smooth $\overset{\langle 2\rangle}{\mathbf{X}}$ the second spatial derivative of $\overset{\langle 2\rangle}{\mathbf{X}}$ must have the following symmetry property:

$$(4.8) \qquad \frac{\partial X_{ab}}{\partial F_{cd}\partial F_{ef}} = \frac{\partial X_{ab}}{\partial F_{ef}\partial F_{cd}}$$

Using (4.7) to replace the gradient of $\overset{\langle 2\rangle}{\mathbf{X}}$ in (4.8) yields

$$(4.9) \qquad \frac{\partial[F^{-1}_{ac}\delta_{db}]}{\partial F_{ef}} = \frac{\partial[F^{-1}_{ae}\delta_{fb}]}{\partial F_{cd}}$$

Applying the product rule and then Remark 2.2 this transforms into

$$(4.10) \qquad F^{-1}_{ae}F^{-1}_{fc}\delta_{db} = F^{-1}_{ac}F^{-1}_{de}\delta_{fb}$$

Equation (4.10) does not hold in general which means that the tensor $\overset{\langle 2\rangle}{\mathbf{X}}$ does not exist.

4.2.2 A natural strain measure in a second-order theory

For a second-order theory one obtains the following equality for the stress power:

$$P = \int_{B_t} \frac{1}{\rho} \left(\overset{\langle 2\rangle}{\mathbf{T}} : grad(\mathbf{v}) + \overset{\langle 3\rangle}{\mathbf{T}} \,\vdots\, grad^{II}(\mathbf{v}) \right) dm \tag{4.11}$$

$$= \int_{B_t} \frac{1}{\rho} \Bigg(\frac{1}{2} \underbrace{[\mathbf{F}^{-1} * \overset{\langle 2\rangle}{\mathbf{T}}]}_{=:J_{\mathbf{F}}^{-1} \overset{\langle 2\rangle}{\mathbf{S}}} : \mathbf{C}^{\bullet} + \underbrace{[\mathbf{F}^{-1} \circ \overset{\langle 3\rangle}{\mathbf{T}}]}_{=:J_{\mathbf{F}}^{-1} \overset{\langle 3\rangle}{\mathbf{S}}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} \Bigg) dm \tag{4.12}$$

$$= \int_{B_0} \frac{1}{\rho_0} \left(\frac{1}{2} \overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} \right) dm \tag{4.13}$$

This shows that $\{\frac{1}{2}\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\}$ is a natural choice for a set of material strain measures because $\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet}$ is the pullback of $grad^{II}(\mathbf{v})$, which has been shown e.g. in [Bertram 2015]. The stress tensors $\overset{\langle 2\rangle}{\mathbf{S}}$ and $\overset{\langle 3\rangle}{\mathbf{S}}$ are work conjugate stress tensors for $\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{C}}$ and $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$, respectively, because they contribute to the power functional by working on $\frac{1}{2}\mathbf{C}^{\bullet}$ and $\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet}$ respectively. It is important to note that $\overset{\langle 2\rangle}{\mathbf{T}}$ in the first-order and the second-order theory takes different roles. In the first-order theory it represents the whole power functional while in the second-order theory it only represents a part of it. This is a consequence of the Riesz representation theorem.

Note: In the index notation the components of $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ will be denoted by $\overset{\langle 3\rangle}{K}_{abc}$. An indicator of the dependence on $\mathbf{F}$ is is not used in this case to avoid confusion with the indices of the components.

4.2.3 Strain measures in a third-order theory

The results in this section are a consequence of the following two remarks

Remark 4.1.

$$\mathbf{F}^T \circ grad^{III}(\mathbf{v}) = \left([\overset{\langle 3\rangle}{\mathbf{K}}\overset{\langle 3\rangle\bullet}{\mathbf{K}}]^{24} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}}\overset{\langle 3\rangle}{\mathbf{K}}]^{23} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}}\overset{\langle 3\rangle}{\mathbf{K}}] + Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}) \right) \tag{4.14}$$

Proof.

$$grad^{III}(\mathbf{v}) = Grad\left(grad^{II}(\mathbf{v})\right) \cdot \mathbf{F}^{-1} = Grad(\mathbf{F}^{-T} \circ \overset{\langle 3\rangle\bullet}{\mathbf{K}}) \cdot \mathbf{F}^{-1} \tag{4.15}$$

$$= Grad(\mathbf{F} \cdot [\overset{\langle 3\rangle\bullet}{\mathbf{K}} \cdot \mathbf{F}^{-1}]^{23} \cdot \mathbf{F}^{-1}) \cdot \mathbf{F}^{-1} \tag{4.16}$$

$$\begin{aligned} =& \big(F_{a\beta,\alpha} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\delta b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} + F_{a\beta} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta,\alpha} F^{-1}_{\delta b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} \\ &+ F_{a\beta} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\delta b,\alpha} F^{-1}_{\gamma c} F^{-1}_{\alpha d} + F_{a\beta} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\delta b} F^{-1}_{\gamma c,\alpha} F^{-1}_{\alpha d}\big) \mathbf{e}_a \otimes \mathbf{e}^b \otimes \mathbf{e}^c \otimes \mathbf{e}^d \end{aligned} \tag{4.17}$$

In the first term of the sum one substitutes $Grad(\mathbf{F}) = \mathbf{F} \cdot \mathbf{K}$. In the last two terms one applies $Grad(\mathbf{F}^{-1}) = \mathbf{F}^{-1} \cdot [Grad(\mathbf{F}) \cdot \mathbf{F}^{-1}]^{23}$.

$$\begin{aligned} =& \big(F_{a\epsilon} \overset{\langle 3\rangle}{K}_{\epsilon\beta\alpha} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\delta b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} + F_{a\beta} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta,\alpha} F^{-1}_{\delta b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} \\ &- F_{a\beta} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\delta\xi} F_{\xi\alpha,\nu} F^{-1}_{\nu b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} \\ &- F_{a\beta} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\delta b} F^{-1}_{\gamma\xi} F_{\xi\alpha,\nu} F^{-1}_{\nu c} F^{-1}_{\alpha d}\big) \mathbf{e}_a \otimes \mathbf{e}^b \otimes \mathbf{e}^c \otimes \mathbf{e}^d \end{aligned} \tag{4.18}$$

Rearranging some terms reveals the form of a pullback

$$\begin{aligned} =& \big(\overset{\langle 3\rangle}{K}_{\epsilon\beta\alpha} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F_{a\epsilon} F^{-1}_{\delta b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} \\ &+ \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta,\alpha} F_{a\beta} F^{-1}_{\delta b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} \\ &- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\delta\xi} F_{\xi\alpha,\nu} F_{a\beta} F^{-1}_{\nu b} F^{-1}_{\gamma c} F^{-1}_{\alpha d} \\ &- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F^{-1}_{\gamma\xi} F_{\xi\alpha,\nu} F_{a\beta} F^{-1}_{\delta b} F^{-1}_{\nu c} F^{-1}_{\alpha d}\big) \mathbf{e}_a \otimes \mathbf{e}^b \otimes \mathbf{e}^c \otimes \mathbf{e}^d \end{aligned} \tag{4.19}$$

$$= \overset{\langle 3\rangle}{K}_{\epsilon\beta\alpha} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} F_{a\epsilon} F^{-T}_{b\delta} F^{-T}_{c\gamma} F^{-T}_{d\alpha} \mathbf{e}_a \otimes \mathbf{e}^b \otimes \mathbf{e}^c \otimes \mathbf{e}^d \tag{4.20}$$

$$
\begin{aligned}
&+ \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta,\alpha}\, F_{a\beta} F^{-T}_{b\delta} F^{-T}_{c\gamma} F^{-T}_{d\alpha} \mathbf{e}_a \otimes \mathbf{e}^b \otimes \mathbf{e}^c \otimes \mathbf{e}^d \\
&- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta}\, F^{-1}_{\delta\xi} F_{\xi\alpha,\nu} F_{a\beta} F^{-T}_{b\nu} F^{-T}_{c\gamma} F^{-T}_{d\alpha} \mathbf{e}_a \otimes \mathbf{e}^b \otimes \mathbf{e}^c \otimes \mathbf{e}^d \\
&- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta}\, F^{-1}_{\gamma\xi} F_{\xi\alpha,\nu} F_{a\beta} F^{-T}_{b\delta} F^{-T}_{c\nu} F^{-T}_{d\alpha} \mathbf{e}_a \otimes \mathbf{e}^b \otimes \mathbf{e}^c \otimes \mathbf{e}^d
\end{aligned}
$$

$$
\begin{aligned}
&= \overset{\langle 3\rangle}{K}_{\epsilon\beta\alpha} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta}\, (\mathbf{F} \cdot \mathbf{e}_\epsilon \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\delta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\gamma \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha) \\
&+ \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta,\alpha}\, (\mathbf{F} \cdot \mathbf{e}_\beta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\delta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\gamma \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha) \\
&- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta}\, \mathbf{F}^{-1}_{\delta\xi} \mathbf{F}_{\xi\alpha,\nu} (\mathbf{F} \cdot \mathbf{e}_\beta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\nu \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\gamma \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha) \\
&- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta}\, \mathbf{F}^{-1}_{\gamma\xi} \mathbf{F}_{\xi\alpha,\nu} (\mathbf{F} \cdot \mathbf{e}_\beta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\delta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\nu \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha)
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
&= \overset{\langle 3\rangle}{K}_{\epsilon\beta\alpha} \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta}\, (\mathbf{F} \cdot \mathbf{e}_\epsilon \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\delta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\gamma \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha) \\
&+ \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta,\alpha}\, (\mathbf{F} \cdot \mathbf{e}_\beta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\delta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\gamma \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha) \\
&- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} \overset{\langle 3\rangle}{K}_{\delta\alpha\nu}\, (\mathbf{F} \cdot \mathbf{e}_\beta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\nu \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\gamma \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha) \\
&- \overset{\langle 3\rangle\bullet}{K}_{\beta\gamma\delta} \overset{\langle 3\rangle}{K}_{\gamma\alpha\nu}\, (\mathbf{F} \cdot \mathbf{e}_\beta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\delta \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\nu \otimes \mathbf{F}^{-T} \cdot \mathbf{e}^\alpha)
\end{aligned}
\tag{4.22}
$$

$$
= \mathbf{F}^{-T} \circ \left([\overset{\langle 3\rangle}{\mathbf{K}} \cdot \overset{\langle 3\rangle\bullet}{\mathbf{K}}]^{24} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}]^{23} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}] + Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}) \right)
\tag{4.23}
$$

Thus one can write

$$
\mathbf{F}^T \circ grad^{III}(\mathbf{v}) = \left([\overset{\langle 3\rangle}{\mathbf{K}} \overset{\langle 3\rangle\bullet}{\mathbf{K}}]^{24} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}} \overset{\langle 3\rangle}{\mathbf{K}}]^{23} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}} \overset{\langle 3\rangle}{\mathbf{K}}] + Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}) \right)
$$

$\square$

Remark 4.2.

$$
\overset{\langle 4\rangle\bullet}{\mathbf{K}}_{\mathbf{F}} = Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}) + [\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,4]} + [\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}]^{[2,4]}
\tag{4.24}
$$

Proof.

With respect to an ONB one can write

$$
Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}})
\tag{4.25}
$$

$$(4.26)\qquad = Grad(\mathbf{F}^{-1}\cdot Grad(\mathbf{F}))^{\bullet}$$

Applying the product rule and (2.30) one obtains

$$(4.27)\qquad = \Big(-F^{-1}_{\alpha b}F_{\alpha c\delta}F^{-1}_{ca}F_{a\beta,\gamma} + F^{-1}_{\alpha a}F_{a\beta,\gamma\delta}\Big)^{\bullet}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\otimes\mathbf{e}_\gamma\otimes\mathbf{e}_\delta$$

$$(4.28)\qquad = -[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{\bullet[2,4]} + \overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}$$

Applying the product rule to the first summand finally yields the stated formula

□

In a third-order theory one obtains the following equality for the stress power P from Riesz' theorem:

$$(4.29)\qquad P = \int_{B_t}\frac{1}{\rho}\Big(\overset{\langle 2\rangle}{\mathbf{T}} : grad(\mathbf{v}) + \overset{\langle 3\rangle}{\mathbf{T}} \vdots grad^{II}(\mathbf{v}) + \overset{\langle 4\rangle}{\mathbf{T}} :: grad^{III}(\mathbf{v})\Big)dm$$

$$(4.30)\qquad = \int_{B_t}\frac{1}{\rho}\Big(\frac{1}{2}\underbrace{[\mathbf{F}^{-1} * \overset{\langle 2\rangle}{\mathbf{T}}]}_{=:J_{\mathbf{F}}^{-1}\overset{\langle 2\rangle}{\mathbf{S}}} : \mathbf{C}^{\bullet} + \underbrace{[\mathbf{F}^{-1}\circ\overset{\langle 3\rangle}{\mathbf{T}}]}_{=:J_{\mathbf{F}}^{-1}\overset{\langle 3\rangle}{\mathbf{S}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} + \underbrace{[\mathbf{F}^{-1}\circ\overset{\langle 4\rangle}{\mathbf{T}}]}_{=:J_{\mathbf{F}}^{-1}\overset{\langle 4\rangle}{\mathbf{S}}} :: \underbrace{[\mathbf{F}^{T}\circ grad^{III}(\mathbf{v})]}_{\text{Use Remark 4.1}}\Big)dm$$

$$(4.31)\qquad = \int_{B_0}\frac{1}{\rho_0}\Big(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}$$

$$+ \overset{\langle 4\rangle}{\mathbf{S}} :: \Big\{\underbrace{Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}) + [\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}]^{[2,4]}}_{\text{Use Remark 4.2}} - 2\overset{[2,3]}{sym}\Big[\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big]\Big\}\Big)dm$$

$$(4.32)\qquad = \int_{B_0}\frac{1}{\rho_0}\Big(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \Big\{\overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} - 3\overset{[2,3][2,4]}{sym}\Big[\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big]\Big\}\Big)dm$$

This result reveals an important peculiarity of the second gradient of strain theory. The material fourth-order stress tensor $\overset{\langle 4\rangle}{\mathbf{S}}$ enters a scalar product with a fourth-order and a third-order strain tensor while in the first- and second-order theory each material stress tensor only enters a scalar product with the strain tensor of the same order. One has the choice between $\{\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\}$ and $\{\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\}$. This is the reason why in definition 4.1 work conjugacy must be defined for a set of stress tensors.

From (4.31) and (4.32) one can see that there is no natural choice for a set of material strain measures in the third-order theory. The reason for this is that the pullback of $grad^{III}(\mathbf{v})$ takes the form

$$\mathbf{F}^T \circ grad^{III}(\mathbf{v}) = Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}) + [\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}]^{[2,4]} - [\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,3]} - [\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}] \tag{4.33}$$

$$= [\mathbf{F}^{-1} \cdot Grad^{II}(\mathbf{F})]^{\bullet} - [\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,4]} - [\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,3]} - [\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}] \tag{4.34}$$

The right hand side is not integrable with respect to time because it violates standard integrability conditions similar to (4.10). This means that no fourth-order tensor field $\overset{\langle 4\rangle}{\mathbf{X}}$ exists such that

$$\mathbf{F}^T \circ grad^{III}(\mathbf{v}) = \overset{\langle 4\rangle}{\mathbf{X}}{}^{\bullet} \tag{4.35}$$

and can be seen as follows. Looking at (4.34) it becomes clear that, if a fourth-order tensor field $\overset{\langle 4\rangle}{\mathbf{X}}$ existed such that (4.35) holds, a fourth-order tensor $\overset{\langle 4\rangle}{\mathbf{Y}}$ would exist such that

$$\overset{\langle 4\rangle}{\mathbf{Y}}{}^{\bullet} = -[\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,4]} - [\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,3]} - [\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]. \tag{4.36}$$

One can assume that $\overset{\langle 4\rangle}{\mathbf{Y}}$ is a function of $\mathbf{F}, Grad(\mathbf{F}), Grad^{II}(\mathbf{F})$ such that

$$\overset{\langle 4\rangle}{Y}{}^{\bullet}_{abcd} = \overset{\langle 4\rangle}{Y}_{abcd}\left(\mathbf{F}, Grad(\mathbf{F}), Grad^{II}(\mathbf{F})\right)^{\bullet} \tag{4.37}$$

$$= -\,(F^{-1}_{a\alpha}F_{\alpha b,\beta})^{\bullet}F^{-1}_{\beta\gamma}F_{\gamma c,d} - (F^{-1}_{a\alpha}F_{\alpha d,\beta})^{\bullet}F^{-1}_{\beta\gamma}F_{\gamma c,b} - (F^{-1}_{a\alpha}F_{\alpha c,\beta})^{\bullet}F^{-1}_{\beta\gamma}F_{\gamma b,d} \tag{4.38}$$

$$\begin{aligned} = &-[(-F^{-1}_{a\delta}F^{\bullet}_{\delta\varepsilon}F^{-1}_{\varepsilon\alpha}F_{\alpha\beta,b} + F^{-1}_{a\alpha}F^{\bullet}_{\alpha b,\beta})F^{-1}_{\beta\gamma}F_{\gamma c,d}] \\ &-[(-F^{-1}_{a\delta}F^{\bullet}_{\delta\varepsilon}F^{-1}_{\varepsilon\alpha}F_{\alpha\beta,d} + F^{-1}_{a\alpha}F^{\bullet}_{\alpha d,\beta})F^{-1}_{\beta\gamma}F_{\gamma c,b}] \\ &-[(-F^{-1}_{a\delta}F^{\bullet}_{\delta\varepsilon}F^{-1}_{\varepsilon\alpha}F_{\alpha\beta,c} + F^{-1}_{a\alpha}F^{\bullet}_{\alpha c,\beta})F^{-1}_{\beta\gamma}F_{\gamma b,d}] \end{aligned} \tag{4.39}$$

Therefore one can deduce that

$$\frac{\partial \overset{\langle 4\rangle}{Y}_{abcd}}{\partial F_{\alpha x,\beta}}F^{\bullet}_{\alpha x,\beta} = -F^{-1}_{a\alpha}F^{-1}_{\beta\gamma}[F_{\gamma c,d} + F_{\gamma c,y}\delta_{yd} + F_{\gamma z,d}\delta_{cz}]F^{\bullet}_{\alpha b,\beta} \tag{4.40}$$

$$= -F^{-1}_{a\alpha} F^{-1}_{\beta\gamma} [F_{\gamma c,d} + F_{\gamma c,y}\delta_{yd} + F_{\gamma z,d}\delta_{cz}]\delta_{bx} \overset{\bullet}{F}_{\alpha x,\beta} \tag{4.41}$$

which reveals

$$\frac{\partial \overset{\langle 4\rangle}{Y}_{abcd}}{\partial F_{\alpha x,\beta}} = -F^{-1}_{a\alpha} F^{-1}_{\beta\gamma} [F_{\gamma c,d} + F_{\gamma c,y}\delta_{yd} + F_{\gamma z,d}\delta_{cz}]\delta_{bx} \tag{4.42}$$

Using the fact that $\frac{\partial F_{\gamma c,d}}{\partial F_{mn,o}} = \delta_{m\gamma}\delta_{nc}\delta_{od}$ one thus sees that the second derivative of $\overset{\langle 4\rangle}{\mathbf{Y}}$ with respect to $Grad(\mathbf{F})$ would take the form

$$\frac{\partial \overset{\langle 4\rangle}{Y}_{abcd}}{\partial F_{\alpha x,\beta} \partial F_{mn,o}} = -F^{-1}_{a\alpha} F^{-1}_{\beta\gamma} \delta_{bx} [\delta_{m\gamma}\delta_{nc}\delta_{od} + \delta_{m\gamma}\delta_{nc}\delta_{oy}\delta_{yd} + \delta_{m\gamma}\delta_{nz}\delta_{od}\delta_{cz}] \tag{4.43}$$

According to the Schwartz theorem the terms in (4.43) must be symmetric with respect to the index triplets (α, x, β) and (m, n, o) i.e. these triplets must be interchangeable. This is not the case. Thus a tensor $\overset{\langle 4\rangle}{\mathbf{Y}}$ that fulfills (4.35) does not exist. A very similar approach has been used in [Krawietz 2015a] and in [Krawietz 2015b] it is shown that even in the one-dimensional case such a variable does not exist.

Note: In the index notation the components of $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ will be denoted by $\overset{\langle 4\rangle}{K}_{abc}$. An indicator of the dependence on $\mathbf{F}$ is is not used in this case to avoid confusion with the indices of the components.

4.2.4 Comparison of strain measures for third-order theories

In Section 4.2.3 it has been shown that the approach via the virtual power functional, convenient as it is, does not yield a natural set of strain measures in third-order theory. It rather lets one choose between $\{\mathbf{C}, \mathbf{K}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\}$ or $\{\mathbf{C}, \mathbf{K}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\}$ since these sets appear in (4.31) and (4.32) respectively. In the present work it is shown that both sets can be used for a material elasto-plasticity framework that generalizes the approach in [Bertram 2015]. In the main part of the present work (i.e. the following sections) $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$ will be used as a strain measure. In appendix A it is outlined how $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ can be used as as strain measure. A comparison of both approaches shows that the same conceptual framework can be used in both cases. Differences only occur in notation and length of calculations. The framework with $\overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}}$ clearly yields more compact formulas and thus seems to be more suitable for further extension or applications of the presented third-order theory. This becomes particularly clear in Section 5.5 where the calculation of the plastic stress tensors is more convenient when $\overset{\langle 4\rangle}{\mathbf{K}_\mathbf{F}}$ is used. Of course there are also many other possible choices for strain measures.

One of them would be $\{\mathbf{C}, Grad(\mathbf{C}), Grad^{II}(\mathbf{C})\}$. This would be in the spirit of [Toupin 1962] where $Grad(\mathbf{C})$ has been introduced as a possible second-order strain measure. In appendix B some important aspects of such a framework are presented. This set of strain measures seems appealing at first since the set only contains material gradients of $\mathbf{C}$ and thus seems to be suitable for theories of arbitrary order. However appendix B shows that there is a problem with the transformation of the stress tensors under a change of reference placement. It turns out that the conjugate stress measures with respect to the power functional (4.29) are very hard to handle when it comes to changes of reference placement.

4.2.5 Kinematical variables

Definition 4.2. The space of configurations

The following sets of tensors will be used to define the space of configurations in third-order gradient elasticity and elastoplasticity where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are vector fields.

(4.44) $\mathscr{Sym}$ denotes the set of all symmetric, positive definite tensors of order two.

$$\mathscr{Conf}_3 := \{\overset{\langle 3\rangle}{\mathbf{P}} \,|(\overset{\langle 3\rangle}{\mathbf{P}} \cdot\mathbf{u})\cdot\mathbf{v} = (\overset{\langle 3\rangle}{\mathbf{P}} \cdot\mathbf{v})\cdot\mathbf{u}\}, \tag{4.45}$$

$$\begin{aligned}\mathscr{Conf}_4 := \{\overset{\langle 4\rangle}{\mathbf{P}} \,|((\overset{\langle 4\rangle}{\mathbf{P}} \cdot\mathbf{u})\cdot\mathbf{v})\cdot\mathbf{w} &= ((\overset{\langle 4\rangle}{\mathbf{P}} \cdot\mathbf{v})\cdot\mathbf{u})\cdot\mathbf{w} = ((\overset{\langle 4\rangle}{\mathbf{P}} \cdot\mathbf{w})\cdot\mathbf{v})\cdot\mathbf{u}\\ &= ((\overset{\langle 4\rangle}{\mathbf{P}} \cdot\mathbf{u})\cdot\mathbf{w})\cdot\mathbf{v}\}\end{aligned} \tag{4.46}$$

The **space of configurations** is then defined as

$$\mathscr{Config} := \mathscr{Sym} \times \mathscr{Conf}_3 \times \mathscr{Conf}_4 \tag{4.47}$$

The motivation of Definition 4.2 lies in the fact that

$$\{\mathbf{C},\mathbf{K},Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\}\in \mathscr{Config} \tag{4.48}$$

$$\{\mathbf{C},\mathbf{K},\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\}\in \mathscr{Config} \tag{4.49}$$

Thus, at a point in the reference placement the current state of deformation of a body is always described by a set of deformation tensors (generalized strain tensors)

$$\{\overset{\langle 2\rangle}{\mathbf{X}},\overset{\langle 3\rangle}{\mathbf{X}},\overset{\langle 4\rangle}{\mathbf{X}}\}\in \mathscr{Config}. \tag{4.50}$$

4.3 Derivation of material stress measures from the power functional

Starting from (4.31) one obtains

$$\begin{aligned} P = \int_{B_0} \frac{1}{\rho_0}\Big(&\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}} \,\vdots\, \overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}} \\ &+ \underbrace{\overset{\langle 4\rangle}{\mathbf{S}} :: \Big[[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}]^{[2,4]} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,3]} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}] + Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}})\Big]}_{*}\Big)dm \end{aligned} \tag{4.51}$$

Apply the following transformation to the term marked with "$*$"

$$\overset{\langle 4\rangle}{\mathbf{S}} :: \Big[[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}]^{24} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{23} - [\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}\cdot\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}] + Grad(\mathbf{K}^{\bullet})\Big]$$

$$= \overset{\langle 4\rangle}{S}_{abcd}\Big(\overset{\langle 3\rangle}{K}_{adx}\overset{\langle 3\rangle\bullet}{K}_{xcb} - \overset{\langle 3\rangle\bullet}{K}_{acy}\overset{\langle 3\rangle}{K}_{ybd} - \overset{\langle 3\rangle\bullet}{K}_{abz}\overset{\langle 3\rangle}{K}_{zcd} + \overset{\langle 3\rangle\bullet}{K}_{abc,d}\Big) \tag{4.52}$$

$$\begin{aligned} = \Big(&\overset{\langle 4\rangle}{S}_{abcd}\overset{\langle 3\rangle}{K}_{adx}\overset{\langle 3\rangle\bullet}{K}_{xcb} - \overset{\langle 4\rangle}{S}_{abcd}\overset{\langle 3\rangle\bullet}{K}_{acy}\overset{\langle 3\rangle}{K}_{ybd} - \overset{\langle 4\rangle}{S}_{abcd}\overset{\langle 3\rangle\bullet}{K}_{abz}\overset{\langle 3\rangle}{K}_{zcd} \\ &+ \overset{\langle 4\rangle}{S}_{abcd}\overset{\langle 3\rangle\bullet}{K}_{abc,d}\Big) \end{aligned} \tag{4.53}$$

$$= \overset{\langle 3\rangle}{K}{}^{[1,2]}_{xad}\overset{\langle 4\rangle}{S}_{adcb}\overset{\langle 3\rangle\bullet}{K}_{xcb} - \overset{\langle 4\rangle}{S}_{acdb}\overset{\langle 3\rangle}{K}{}^{[1,3]}_{dby}\overset{\langle 3\rangle\bullet}{K}_{acy} - \overset{\langle 4\rangle}{S}_{abdc}\overset{\langle 3\rangle}{K}{}^{[1,2]}_{dcz}\overset{\langle 3\rangle\bullet}{K}_{abz} + \overset{\langle 4\rangle}{S}_{abcd}\overset{\langle 3\rangle\bullet}{K}_{abc,d} \tag{4.54}$$

$$= \Big(\overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,2]}_{\mathbf{F}} : \overset{\langle 4\rangle}{\mathbf{S}} - \overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,3]}_{\mathbf{F}} - \overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,3]}_{\mathbf{F}}\Big) \,\vdots\, \overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}) \tag{4.55}$$

$$= \Big(\overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,2]}_{\mathbf{F}} : \overset{\langle 4\rangle}{\mathbf{S}} - \overset{\langle 4\rangle}{\mathbf{S}} : 2\,\overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,3]}_{\mathbf{F}}\Big) \,\vdots\, \overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: Grad(\overset{\langle 3\rangle\bullet}{\mathbf{K}}_{\mathbf{F}}) \tag{4.56}$$

to obtain

$$P = \int\limits_{B_0} \frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} + \left(\overset{\langle 3\rangle^{[1,2]}}{\mathbf{K}_{\mathbf{F}}} : \overset{\langle 4\rangle}{\mathbf{S}} - \overset{\langle 4\rangle}{\mathbf{S}} : 2\,\overset{\langle 3\rangle^{[1,3]}}{\mathbf{K}_{\mathbf{F}}}\right) \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet})\right) dm \tag{4.57}$$

With $\alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}}) := \overset{\langle 3\rangle^{[1,2]}}{\mathbf{K}_{\mathbf{F}}} : \overset{\langle 4\rangle}{\mathbf{S}} - \overset{\langle 4\rangle}{\mathbf{S}} : 2\,\overset{\langle 3\rangle^{[1,3]}}{\mathbf{K}_{\mathbf{F}}}$ one obtains

$$= \int\limits_{B_0} \frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \underbrace{[\overset{\langle 3\rangle}{\mathbf{S}} + \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})]}_{=:\,\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}} \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet})\right) dm \tag{4.58}$$

$$= \int\limits_{B_0} \frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet})\right) dm \tag{4.59}$$

Thus a set of three material stress and a set of three material strain measures has been defined and these sets are work conjugate to each other.

Stress measures: **Strain measures:**

$$\overset{\langle 2\rangle}{\mathbf{S}} := \mathbf{F}^{-1} * (J_{\mathbf{F}}\,\overset{\langle 2\rangle}{\mathbf{T}}) \qquad \mathbf{C} := \mathbf{F}^{T}\cdot\mathbf{F} \tag{4.60}$$

$$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} := \underbrace{\mathbf{F}^{-1}\circ(J_{\mathbf{F}}\,\overset{\langle 3\rangle}{\mathbf{T}})}_{=\overset{\langle 3\rangle}{\mathbf{S}}} + \underbrace{\overset{\langle 3\rangle^{[1,2]}}{\mathbf{K}_{\mathbf{F}}} : \overset{\langle 4\rangle}{\mathbf{S}} - \overset{\langle 4\rangle}{\mathbf{S}} : 2\,\overset{\langle 3\rangle^{[1,3]}}{\mathbf{K}_{\mathbf{F}}}}_{=\alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})} \qquad \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \tag{4.61}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} := \mathbf{F}^{-1}\circ(J_{\mathbf{F}}\,\overset{\langle 4\rangle}{\mathbf{T}}) \qquad Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \tag{4.62}$$

It is important to keep in mind that $\overset{\langle 3\rangle}{\mathbf{S}}$ is not a stress tensor but an auxillary term. $\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}$ is the third-order stress tensor of the third-order theory developed in the present work. It has been marked by the superscript $\widetilde{}$ in order to avoid confusion with the third-order tensor $\overset{\langle 3\rangle}{\mathbf{S}}$ which is the third-order stress tensor in the second-order theory.

4.4 Third-order elasticity

Definition 4.3. Third-order elastic material

A material is called a **third-order elastic material** if the stress tensors are functions of the motion χ, of $Grad(\chi)$, $Grad^{II}(\chi)$ and of $Grad^{III}(\chi)$.

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}\big(\chi, Grad(\chi), Grad^{II}(\chi), Grad^{III}(\chi)\big) \tag{4.63}$$

$$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = \overset{\langle 3\rangle}{\widetilde{f}}\big(\chi, Grad(\chi), Grad^{II}(\chi), Grad^{III}(\chi)\big) \tag{4.64}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\big(\chi, Grad(\chi), Grad^{II}(\chi), Grad^{III}(\chi)\big) \tag{4.65}$$

These constitutive equations can be reduced by taking into account the principle of invariance under rigid body motions (Assumption 3.3). It is important to note that the principle of euclidean invariance (Corollary 3.1) alone does not allow to deduce reduced forms of the constitutive equations as explained in [Bertram & Svendsen 2001]. Therein it is derived, that Assumption 3.3 yields the reduced forms, which are in the present case

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \tag{4.66}$$

$$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = \overset{\langle 3\rangle}{\widetilde{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \tag{4.67}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \tag{4.68}$$

Note: There is no constitutive equation for $\overset{\langle 3\rangle}{\mathbf{S}}$ because it is not a stress measure. It is a partial stress that helps making a comparison to the second order theory and facilitates the understanding of transformation rules.

Definition 4.4. Hyperelasticity

A material is called hyperelastic if there exists a specific elastic energy

$$w : \mathscr{Config} \mapsto \mathbb{R}$$

such that

$$p := \frac{1}{\rho_0}\left(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\bullet}\right) \tag{4.69}$$

$$= w\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right)^{\bullet} \tag{4.70}$$

$$= \frac{\partial w\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right)}{\partial \mathbf{C}} : \mathbf{C}^{\bullet} + \frac{\partial w\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right)}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} \tag{4.71}$$

$$+ \frac{\partial w\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right)}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\bullet}$$

A comparison with the components in (4.59) then reveals for all $\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right) \in \mathscr{Config}$

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right) = 2\rho_0 \frac{\partial w\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right)}{\partial \mathbf{C}} \tag{4.72}$$

$$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = \overset{\langle 3\rangle}{\widetilde{f}}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right) = \rho_0 \frac{\partial w\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right)}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \tag{4.73}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right) = \rho_0 \frac{\partial w\left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right)}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})} \tag{4.74}$$

In the rest of the present work all statements regarding elasticity will be made for the elastic energy without distinguishing elasticity and hyperelasticity. (A non-hyperelastic material would be unphysical.)

4.5 Changes of the reference placement

Theorem 4.1. Transformation of strain measures under a change of reference placement

Let κ and $\underline{\kappa}$ be two reference placements. The composition $\kappa \circ \underline{\kappa}^{-1}$ is referred to as the **change of the reference placement** (see Section 2.2). Its gradient is denoted by

$\mathbf{A} := \underline{Grad}(\kappa \circ \underline{\kappa}^{-1})$, the pullback of its gradient is denoted by $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} := \mathbf{A}^{-1} \cdot \underline{Grad}(\mathbf{A})$, the pullback of its second gradient is denoted by $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} := \mathbf{A}^{-1} \cdot Grad^{II}(\mathbf{A})$ and one defines $J_{\mathbf{A}} := det(\mathbf{A})$. Furthermore one defines

$\beta : \mathscr{Conf}_4 \times \mathscr{Conf}_3 \times \mathscr{Inv} \times \mathscr{Conf}_3 \times \mathscr{Conf}_4 \to \mathscr{Conf}_4$

$$\begin{aligned}(4.75)\quad & \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big) \\ & := \mathbf{A}^T \circ Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) + \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) - \Big[\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} \cdot \Big(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)\Big]^{[2,3]} \\ & \quad + 2\overset{[2,3]}{sym}\Big[(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Big]\end{aligned}$$

Then the generalized strain measures transform under a change of the reference placement as

$$(4.76)\qquad \underline{\mathbf{C}} = \mathbf{A}^T * \mathbf{C}$$

$$(4.77)\qquad \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}} = \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}$$

$$(4.78)\qquad \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}) = \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)$$

The function β has been introduced in order to facilitate notation.

Proof.

The detailed derivation of the transformation of $\mathbf{C}$ and $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ is given in [Bertram 2015]. With respect to an ONB $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the tensor $\underline{Grad}(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$ has the components

$$\begin{aligned} & (A^{-1}_{a\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta} A_{\delta b} A_{\gamma c})_{,\underline{d}} \\ (4.79)\quad & = A^{-1}_{a\alpha,\underline{d}} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta} A_{\delta b} A_{\gamma c} + A^{-1}_{a\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta,\underline{d}} A_{\delta b} A_{\gamma c} + A^{-1}_{a\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta} A_{\delta b,\underline{d}} A_{\gamma c} \\ & \quad + A^{-1}_{a\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta} A_{\delta b} A_{\gamma c,\underline{d}} \\ (4.80)\quad & = - A^{-1}_{a\epsilon} A_{\epsilon d,\underline{\mu}} A^{-1}_{\mu\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta} A_{\delta b} A_{\gamma c} + A^{-1}_{a\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta,\epsilon} A_{\epsilon d} A_{\delta b} A_{\gamma c} + A^{-1}_{a\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta} A_{\delta b,\underline{d}} A_{\gamma c} \\ & \quad + A^{-1}_{a\alpha} \overset{\langle 3\rangle}{K}_{\alpha\gamma\delta} A_{\delta b} A_{\gamma c,\underline{d}}\end{aligned}$$

Therefore one can write

$$
\begin{aligned}
&\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})\\
(4.81)\quad &=\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) + \underline{Grad}(\mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\\
(4.82)\quad &=\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) - \Big[\mathbf{A}^{-1}\cdot \underline{Grad}(\mathbf{A})\cdot \Big(\mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)\Big]^{[2,3][3,4]} + \mathbf{A}^T\circ Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\\
&\quad + \Big(\mathbf{A}^{-1}\cdot(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\mathbf{A})^{[2,3]}\cdot \underline{Grad}(\mathbf{A})\Big)^{[2,3]} + \Big(\mathbf{A}^{-1}\cdot(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\mathbf{A})^{[2,3]}\cdot \underline{Grad}(\mathbf{A})\Big)\\
(4.83)\quad &=\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) - \Big[\Big(\mathbf{A}^{-1}\cdot \underline{Grad}(\mathbf{A})\Big)\cdot \Big(\mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)\Big]^{[2,3][3,4]} + \mathbf{A}^T\circ Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\\
&\quad + 2\overset{[2,3]}{sym}\Big[\mathbf{A}^{-1}\cdot(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\mathbf{A})^{[2,3]}\cdot \underline{Grad}(\mathbf{A})\Big]\\
(4.84)\quad &=\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) - \Big[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\cdot\Big(\mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)\Big]^{[2,3]} + \mathbf{A}^T\circ Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\\
&\quad + 2\overset{[2,3]}{sym}\Big[\mathbf{A}^{-1}\cdot(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\mathbf{A})^{[2,3]}\cdot\mathbf{A}\cdot\mathbf{A}^{-1}\cdot \underline{Grad}(\mathbf{A})\Big]\\
(4.85)\quad &=\mathbf{A}^T\circ Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) + \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) - \Big[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\cdot\Big(\mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)\Big]^{[2,3]}\\
&\quad + 2\overset{[2,3]}{sym}\Big[(\mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Big]
\end{aligned}
$$

□

Remark 4.3.

Using the definition of β from Equation (4.75) in Equation (4.78) one obtains

$$
\begin{aligned}
(4.86)\quad \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}) =&\mathbf{A}^T\circ Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) + \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\\
&+ \mathbf{A}^T\circ\Big(-[\mathbf{A}\cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\cdot\mathbf{A}^{-1}\cdot\mathbf{A}^{-1}\cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,4]}\\
&+ [\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\mathbf{A}\cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\cdot\mathbf{A}^{-1}\cdot\mathbf{A}^{-1}]^{[2,3][2,4]} + [\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\cdot\mathbf{A}\cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\cdot\mathbf{A}^{-1}\cdot\mathbf{A}^{-1}]^{[3,4]}\Big).
\end{aligned}
$$

Equation (4.86) reveals that the transformation under a change of reference placement of $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$ is similar to the transformation of $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$. In both cases the tensors are pulled back by $\mathbf{A}^T$ and then a tensor is added. In the case of $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ the added tensor is $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}$. In the case of $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$ the added tensor depends not only on $\mathbf{A}$ and its second gradient but also on $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$. This means that the transformation under a change of reference placement for $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ only depends on the change of reference placement, while for $Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})$ it additionally depends on the current deformation. This is what sets this material third-order theory apart from the first- and second-order theories. In the following sections it will become clear that this feature requires non-straight-forward generalizations of basic concepts from material elasto-plasticity.

Theorem 4.2. Transformation of elastic energies under a change of reference placement

A constant $\underline{w}_0 \in \mathbb{R}$ exists such that for a change of the reference placement the elastic energy transform as

$$\begin{aligned}(4.87)\quad w\Big(\overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})\Big) = J_{\mathbf{A}^{-1}}\underline{w}\Big(&\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{A}},\\ &\beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{A}})\big)\Big) + \underline{w}_0\end{aligned}$$

Proof. The transformation of the elastic energy follows directly from its definition. □

Theorem 4.3. Transformation of stress measures under a change of reference placement

One defines the auxillary function $\gamma : \mathscr{Conf}_3 \times \mathscr{Conf}_4 \times \mathscr{Inv} \times \mathscr{Conf}_3 \times \mathscr{Conf}_3 \mapsto \mathscr{Conf}_3$

$$\begin{aligned}(4.88)\quad &\gamma\big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}\big)\\ &:= (\mathbf{A}^{-1} \circ J_\mathbf{A}[\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \alpha(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})]) + (\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{A}})^{[1,2]} : (\mathbf{A}^{-1} \circ J_\mathbf{A}\, \overset{\langle 4\rangle}{\mathbf{S}})\\ &\quad - (\mathbf{A}^{-1} \circ J_\mathbf{A}\, \overset{\langle 4\rangle}{\mathbf{S}}) : 2(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{A}})^{[1,3]}\end{aligned}$$

Then the generalized stress tensors transform under changes of the reference placements as

$$(4.89)\quad \overset{\langle 2\rangle}{\underline{\mathbf{S}}} = J_{\underline{\mathbf{F}}}\underline{\mathbf{F}}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{T}} \cdot \underline{\mathbf{F}}^{-T} = \mathbf{A}^{-1} * (J_\mathbf{A}\, \overset{\langle 2\rangle}{\mathbf{S}})$$

$$(4.90)\quad \overset{\langle 3\rangle}{\underline{\mathbf{S}}} = \underline{\mathbf{F}}^{-1} \circ (J_{\underline{\mathbf{F}}}\, \overset{\langle 3\rangle}{\mathbf{T}}) = (\mathbf{A}^{-1} \cdot \mathbf{F}^{-1}) \circ (J_\mathbf{F} J_\mathbf{A}\, \overset{\langle 3\rangle}{\mathbf{T}}) = \mathbf{A}^{-1} \circ (J_A\, \overset{\langle 3\rangle}{\mathbf{S}}),$$

$$(4.91)\quad \underset{\sim}{\overset{\langle 3\rangle}{\underline{\mathbf{S}}}} = \gamma\big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \big)$$

$$(4.92)\quad \overset{\langle 4\rangle}{\underline{\mathbf{S}}} = \underline{\mathbf{F}}^{-1} \circ (J_{\underline{\mathbf{F}}} \overset{\langle 4\rangle}{\mathbf{T}}) = (\mathbf{A}^{-1} \cdot \mathbf{F}^{-1}) \circ (J_{\mathbf{F}} J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{T}}) = \mathbf{A}^{-1} \circ (J_A \overset{\langle 4\rangle}{\mathbf{S}})$$

Here one has to keep in mind that $\overset{\langle 3\rangle}{\mathbf{S}}$ is not a stress tensor in the third-order framework. Since it is used to calculate $\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}$ it has still been included in this list. The function γ has been introduced in order to facilitate notation.

Proof.

$$(4.93)\quad \underset{\sim}{\overset{\langle 3\rangle}{\underline{\mathbf{S}}}} = \overset{\langle 3\rangle}{\underline{\mathbf{S}}} + \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \overset{\langle 4\rangle}{\underline{\mathbf{S}}})$$

$$(4.94)\quad = \mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 3\rangle}{\mathbf{S}}) + \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}^{12} : \overset{\langle 4\rangle}{\underline{\mathbf{S}}} - \overset{\langle 4\rangle}{\underline{\mathbf{S}}} : 2\, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}^{13}$$

$$(4.95)\quad = \mathbf{A}^{-1} \circ \Big(J_{\mathbf{A}} [\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})] \Big) + \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}^{12} : \overset{\langle 4\rangle}{\underline{\mathbf{S}}} - \overset{\langle 4\rangle}{\underline{\mathbf{S}}} : 2\, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}^{13}$$

Apply transformation rules (4.90), (4.77) and (4.92)

$$(4.96)\quad = \mathbf{A}^{-1} \circ \Big(J_{\mathbf{A}} [\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})] \Big) + (\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{12} : \Big(\mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}}) \Big)$$
$$- \Big(\mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}}) \Big) : 2(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{13}$$

$$(4.97)\quad =: \gamma\big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \big)$$

□

Remark 4.4.

One can also bring (4.91) into a form similar to (4.78), i.e., a pullback plus an additional term, with $\alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}}) := \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}} - \overset{\langle 4\rangle}{\mathbf{S}} : 2\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}$ as defined in equation (4.57)

$$(4.98)\quad \underset{\sim}{\overset{\langle 3\rangle}{\underline{\mathbf{S}}}} = \mathbf{A}^{-1} \circ J_{\mathbf{A}} \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}$$
$$+ J_{\mathbf{A}} \Big(- \big(\mathbf{A}^{-1} \circ \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})\big) + (\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{[1,2]} : (\mathbf{A}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}})$$
$$- (\mathbf{A}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}) : 2(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{[1,3]} \Big).$$

This rearrangement shows two important things

1. One can make an observation similar to Remark 4.3. The relation (4.98) reveals another major difference between the third-order theory developed here and the second-order theory in [Bertram 2015]: While $\underset{\sim}{\overset{\langle 3\rangle}{\mathbf{S}}}$ in the second order theory is as a pullback of $\overset{\langle 3\rangle}{\mathbf{S}}$ with $\mathbf{A}$, its equivalent in the third-order theory, $\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}}$, is the pullback of $\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}$ plus an additional term, which is responsible for the fact, that the transformation behaviour of $\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}$ under changes of the reference placement depends on the the current placement, i.e. on $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$.

2. At every point of the body one can interpret γ as a bijective mapping from $\mathscr{Conf}_3$ to $\mathscr{Conf}_3$ by only considering the first argument of γ. (The other arguments can be seen as parameters). The reason for the bijectivity in this case is that $J_{\mathbf{A}} \neq 0$ for smooth deformations of the body.

Proof.

$$\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}} = \underline{\overset{\langle 3\rangle}{\mathbf{S}}} + \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \underline{\overset{\langle 4\rangle}{\mathbf{S}}})$$

$$= \mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 3\rangle}{\mathbf{S}}) + \overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{12} : \underline{\overset{\langle 4\rangle}{\mathbf{S}}} - \underline{\overset{\langle 4\rangle}{\mathbf{S}}} : 2\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{13} \tag{4.99}$$

$$= \mathbf{A}^{-1} \circ \Big(J_{\mathbf{A}} [\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})]\Big) + \overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{12} : \underline{\overset{\langle 4\rangle}{\mathbf{S}}} - \underline{\overset{\langle 4\rangle}{\mathbf{S}}} : 2\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{13} \tag{4.100}$$

Apply transformation rules (4.90), (4.77) and (4.92)

$$= \mathbf{A}^{-1} \circ \Big(J_{\mathbf{A}} [\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})]\Big) + (\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{12} : \Big(\mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}})\Big) \tag{4.101}$$
$$- \Big(\mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}})\Big) : 2(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{13}$$

$$=: \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \tag{4.102}$$

(4.103)

$\square$

Remark 4.5.

One can also bring the transformation of $\overset{\langle 3\rangle}{\underline{\widetilde{\mathbf{S}}}}$ into the form of a pullback.

$$\begin{aligned}(4.104)\quad \overset{\langle 3\rangle}{\underline{\widetilde{\mathbf{S}}}} = \mathbf{A}^{-1} \circ \Big(& J_A[\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})] \\ & + \big(\mathbf{A}^{-T} \cdot (\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{[1,2]} : J_{\mathbf{A}}(\mathbf{A}^T \otimes \mathbf{A}^{-1})^{[2,3]} : \overset{\langle 4\rangle}{\mathbf{S}}\ ) \\ & - \big(2J_{\mathbf{A}}\ \overset{\langle 4\rangle}{\mathbf{S}} : (\mathbf{A}^{-1} \otimes \mathbf{A}^{-1})^{[1,4]} : (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} + \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{[1,3]} \cdot \mathbf{A}^T\big)\Big)\end{aligned}$$

Proof.

$$(4.105)\quad \overset{\langle 3\rangle}{\underline{\widetilde{\mathbf{S}}}} = \overset{\langle 3\rangle}{\underline{\mathbf{S}}} + \underline{\alpha}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}}) = \mathbf{A}^{-1} \circ J_{\mathbf{A}}\ \overset{\langle 3\rangle}{\mathbf{S}} + \underbrace{\overset{\langle 3\rangle}{\mathbf{K}}{}^{12}_{\underline{\mathbf{F}}} : \overset{\langle 4\rangle}{\underline{\mathbf{S}}}}_{=*} - \underbrace{\overset{\langle 4\rangle}{\underline{\mathbf{S}}} : 2\ \overset{\langle 3\rangle}{\mathbf{K}}{}^{13}_{\underline{\mathbf{F}}}}_{=**}$$

$$(4.106)\quad * = \overset{\langle 3\rangle}{\mathbf{K}}{}^{12}_{\underline{\mathbf{F}}} : J_{\mathbf{A}}\ \overset{\langle 4\rangle}{S}_{\alpha\beta\gamma\delta}\ (\mathbf{A}^T \cdot \mathbf{e}^{\alpha}) \otimes (\mathbf{A}^{-1} \cdot \mathbf{e}_{\beta}) \otimes (\mathbf{A}^{-1} \cdot \mathbf{e}_{\gamma}) \otimes (\mathbf{A}^{-1} \cdot \mathbf{e}_{\delta})$$

$$(4.107)\quad = \overset{\langle 3\rangle}{\mathbf{K}}{}^{12}_{\underline{\mathbf{F}}} : J_{\mathbf{A}}\ \overset{\langle 4\rangle}{S}_{\alpha\beta\gamma\delta}\ A^T_{a\alpha} A^{-1}_{b\beta} A^{-1}_{k\gamma} A^{-1}_{l\delta} \mathbf{e}^a \otimes \mathbf{e}_b \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

$$(4.108)\quad = \overset{\langle 3\rangle}{\underline{K}}{}^{12}_{aij} J_{\mathbf{A}}\ \overset{\langle 4\rangle}{S}_{\alpha\beta\gamma\delta}\ A^T_{i\alpha} A^{-1}_{j\beta} A^{-1}_{k\gamma} A^{-1}_{l\delta} \mathbf{e}^a \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

$$(4.109)\quad = \delta_{ax} \overset{\langle 3\rangle}{\underline{K}}{}^{12}_{xij} J_{\mathbf{A}}\ \overset{\langle 4\rangle}{S}_{\alpha\beta\gamma\delta}\ A^T_{i\alpha} A^{-1}_{j\beta} A^{-1}_{k\gamma} A^{-1}_{l\delta} \mathbf{e}^a \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

Set $\delta_{ax} = A^T_{am} A^{-T}_{mx}$

$$(4.110)\quad = A^T_{am} A^{-T}_{mx} \overset{\langle 3\rangle}{\underline{K}}{}^{12}_{xij} J_{\mathbf{A}}\ \overset{\langle 4\rangle}{S}_{\alpha\beta\gamma\delta}\ A^T_{i\alpha} A^{-1}_{j\beta} A^{-1}_{k\gamma} A^{-1}_{l\delta} \mathbf{e}^a \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

Rearranging according to indices yields

$$(4.111)\quad = A^{-T}_{mx} \overset{\langle 3\rangle}{\underline{K}}{}^{12}_{xij} J_{\mathbf{A}} A^T_{i\alpha} A^{-1}_{j\beta}\ \overset{\langle 4\rangle}{S}_{\alpha\beta\gamma\delta}\ A^T_{am} A^{-1}_{k\gamma} A^{-1}_{l\delta} \mathbf{e}^a \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

$$(4.112)\quad = A^{-T}_{mx} \overset{\langle 3\rangle}{\underline{K}}{}^{12}_{xij} J_{\mathbf{A}} A^T_{i\alpha} A^{-1}_{j\beta}\ \overset{\langle 4\rangle}{S}_{\alpha\beta\gamma\delta}\quad \mathbf{A}^T \cdot \mathbf{e}^m \otimes \mathbf{A}^{-1} \cdot \mathbf{e}_{\gamma} \otimes \mathbf{A}^{-1} \cdot \mathbf{e}_{\delta}$$

$$(4.113)\quad = \mathbf{A}^{-1} \circ \left(\mathbf{A}^{-T} \cdot \overset{\langle 3\rangle}{\mathbf{K}}{}^{12}_{\underline{\mathbf{F}}} : J_{\mathbf{A}} \big(\mathbf{A}^T \otimes \mathbf{A}^{-1}\big)^{23} : \overset{\langle 4\rangle}{\mathbf{S}} \right)$$

$$(4.114)\quad \overset{\text{With (4.77)}}{=} \mathbf{A}^{-1} \circ \left(\mathbf{A}^{-T} \cdot (\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{12} : J_{\mathbf{A}} \big(\mathbf{A}^T \otimes \mathbf{A}^{-1}\big)^{23} : \overset{\langle 4\rangle}{\mathbf{S}} \right)$$

$$(4.115)\quad ** = \Big(2\mathbf{A}^{-1}\circ J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}}\Big) : \overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{13}$$

$$(4.116)\quad = 2J_{\mathbf{A}} \overset{\langle 4\rangle}{S}{}_{abcd}\,(\mathbf{A}^T\cdot\mathbf{e}^a)\otimes(\mathbf{A}^{-1}\cdot\mathbf{e}_b)\otimes(\mathbf{A}^{-1}\cdot\mathbf{e}_c)$$

$$\otimes(\mathbf{A}^{-1}\cdot\mathbf{e}_d) : \Big(\overset{\langle 3\rangle}{\underline{K}}{}^{13}_{cdf}\mathbf{e}_c\otimes\mathbf{e}^d\otimes\mathbf{e}^f\Big)$$

$$(4.117)\quad = 2J_{\mathbf{A}} \overset{\langle 4\rangle}{S}{}_{abcd}\,A^{-1}_{xc}A^{-1}_{yd}(\mathbf{A}^T\cdot\mathbf{e}^a)\otimes(\mathbf{A}^{-1}\cdot\mathbf{e}_b)\otimes\mathbf{e}_x\otimes\mathbf{e}_y : \Big(\overset{\langle 3\rangle}{\underline{K}}{}^{13}_{cdf}\mathbf{e}_c\otimes\mathbf{e}^d\otimes\mathbf{e}^f\Big)$$

$$(4.118)\quad = 2J_{\mathbf{A}} \overset{\langle 4\rangle}{S}{}_{abcd}\,A^{-1}_{xc}A^{-1}_{yd}\overset{\langle 3\rangle}{\underline{K}}{}^{13}_{xyf}(\mathbf{A}^T\cdot\mathbf{e}^a)\otimes(\mathbf{A}^{-1}\cdot\mathbf{e}_b)\otimes\mathbf{e}^f$$

Again one can extend the equations by using the Kronecker Symbol δ

$$(4.119)\quad = 2J_{\mathbf{A}} \overset{\langle 4\rangle}{S}{}_{abcd}\,A^{-1}_{xc}A^{-1}_{yd}\overset{\langle 3\rangle}{\underline{K}}{}^{13}_{xyf}\underbrace{A_{jf}A^{-1}_{ij}}_{=\delta_{fi}}(\mathbf{A}^T\cdot\mathbf{e}^a)\otimes(\mathbf{A}^{-1}\cdot\mathbf{e}_b)\otimes\mathbf{e}^i$$

Rearranging reveals the form of a pullback

$$(4.120)\quad = 2J_{\mathbf{A}} \overset{\langle 4\rangle}{S}{}_{abcd}\,A^{-1}_{xc}A^{-1}_{yd}\overset{\langle 3\rangle}{\underline{K}}{}^{13}_{xyf}A_{jf}(\mathbf{A}^T\cdot\mathbf{e}^a)\otimes(\mathbf{A}^{-1}\cdot\mathbf{e}_b)\otimes(A^{-1}_{ij}\mathbf{e}^i)$$

$$(4.121)\quad = \mathbf{A}^{-1}\circ\Big(2J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}} : (\mathbf{A}^{-1}\otimes\mathbf{A}^{-1})^{14} : \overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{13}\cdot\mathbf{A}^T\Big)$$

$$(4.122)\quad = \mathbf{A}^{-1}\circ\Big(2J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}} : (\mathbf{A}^{-1}\otimes\mathbf{A}^{-1})^{14} : \big(\mathbf{A}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\big)^{13}\cdot\mathbf{A}^T\Big)$$

Thus by substituting the expressions (*) and (**) one obtains

$$(4.123)\quad \overset{\langle 3\rangle}{\underline{\widetilde{\mathbf{S}}}} = \overset{\langle 3\rangle}{\underline{\mathbf{S}}} + \underline{\alpha} = \mathbf{A}^{-1}\circ J_{\mathbf{A}}\overset{\langle 3\rangle}{\mathbf{S}} + \underbrace{\overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{12} : \overset{\langle 4\rangle}{\underline{\mathbf{S}}}}_{=\,*} - \underbrace{\overset{\langle 4\rangle}{\underline{\mathbf{S}}} : 2\,\overset{\langle 3\rangle}{\mathbf{K}}{}_{\underline{\mathbf{F}}}^{13}}_{=\,**}$$

$$(4.124)\quad = \mathbf{A}^{-1}\circ J_{\mathbf{A}}\overset{\langle 3\rangle}{\mathbf{S}} + \mathbf{A}^{-1}\circ\Big(\mathbf{A}^{-T}\cdot(\mathbf{A}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{12} : J_{\mathbf{A}}\big(\mathbf{A}^T\otimes\mathbf{A}^{-1}\big)^{23} : \overset{\langle 4\rangle}{\mathbf{S}}\Big)$$

$$- \mathbf{A}^{-1}\circ\Big(2J_{\mathbf{A}}\overset{\langle 4\rangle}{\mathbf{S}} : (\mathbf{A}^{-1}\otimes\mathbf{A}^{-1})^{14} : \big(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} + \mathbf{A}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\big)^{13}\cdot\mathbf{A}^T\Big)$$

$$(4.125)\quad = \mathbf{A}^{-1}\circ\Big(J_{\mathbf{A}}\overset{\langle 3\rangle}{\mathbf{S}} + \big(\mathbf{A}^{-T}\cdot(\mathbf{A}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{12} : J_{\mathbf{A}}(\mathbf{A}^T\otimes\mathbf{A}^{-1})^{23} : \overset{\langle 4\rangle}{\mathbf{S}}\big)$$

$$-\big(2J_{\mathbf{A}}\overset{\langle 4\rangle}{\mathbf{S}} : (\mathbf{A}^{-1}\otimes\mathbf{A}^{-1})^{14} : (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} + \mathbf{A}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{13}\cdot\mathbf{A}^T\big)\Big)$$

$$(4.126)\quad = \mathbf{A}^{-1}\circ\Big(J_{\mathbf{A}}[\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \alpha(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{S}})]$$

$$+\big(\mathbf{A}^{-T}\cdot(\mathbf{A}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{12} : J_{\mathbf{A}}(\mathbf{A}^T\otimes\mathbf{A}^{-1})^{23} : \overset{\langle 4\rangle}{\mathbf{S}}\big)$$

$$-\Big(2J_{\mathbf{A}}\overset{\langle 4\rangle}{\mathbf{S}} : (\mathbf{A}^{-1}\otimes\mathbf{A}^{-1})^{14} : (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} + \mathbf{A}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{13}\cdot\mathbf{A}^{T})\Big)$$

□

In summary the following transformations for changes of reference placement have been obtained:

	Stress measures:	**Strain measures:**
(4.127)	$\underline{\overset{\langle 2\rangle}{\mathbf{S}}} = \mathbf{A}^{-1} * J_A \overset{\langle 2\rangle}{\mathbf{S}}$	$\underline{\mathbf{C}} = \mathbf{A}^{T} * \mathbf{C}$
(4.128)	$\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}} = \gamma\big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\big)$	$\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}} = \mathbf{A}^{T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}$
(4.129)	$\underline{\overset{\langle 4\rangle}{\mathbf{S}}} = \mathbf{A}^{-1}\circ J_A \overset{\langle 4\rangle}{\mathbf{S}}$	$\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})$
		$= \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)$

Remark 4.6. Transformation of stress and strain measures under two subsequent changes of the reference placement

Now it will be investigated how the stress and strain measures transform under two subsequent changes of reference placement. The results will be needed later for dealing with elastic isomorphy in Theorem 4.9. Three reference placements κ, $\underline{\kappa}$ and $\underline{\underline{\kappa}}$ are defined with $M := \underline{\underline{Grad}}(\underline{\kappa}\circ\underline{\underline{\kappa}}^{-1})$ and $N := \underline{Grad}(\kappa\circ\underline{\kappa}^{-1})$. This situation is sketched in Figure 4.1 below.

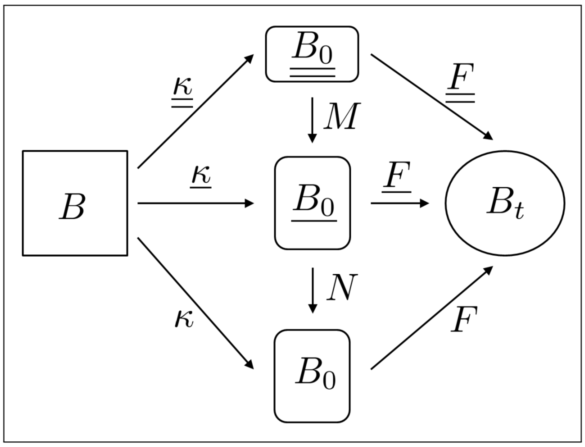

Figure 4.1: Change of the reference placement used in the definition of elastic isomorphy

Of course two subsequent changes of the reference placement can also be substituted by a single change of the reference placement. This fact lies behind the following relations:

$$\begin{aligned}
(4.130)\quad & \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{N}\cdot\mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}}, \underline{\underline{Grad}}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}})\big) \\
& = \beta\Big(\beta\big\{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{N}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}})\big\}, \mathbf{N}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \\
& \quad + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \underline{\underline{Grad}}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}})\Big) \\
(4.131)\quad & \gamma\Big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{N}\cdot\mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) \\
& = \gamma\Big(\gamma\Big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{N}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big), det(\mathbf{N})(\mathbf{N}^{-1}\circ \overset{\langle 4\rangle}{\mathbf{S}}), \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \mathbf{N}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}\Big)
\end{aligned}$$

Proof.

$$(4.132)\quad \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{N}\cdot\mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}}, \underline{\underline{Grad}}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}})\big)$$

Apply (4.78) with respect to the change of reference placement $\underline{\underline{\kappa}}^{-1}\circ\kappa$

$$(4.133)\quad =\underline{\underline{Grad}}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\underline{\mathbf{F}}}})$$

Apply (4.78) with respect to the change of reference placement $\underline{\underline{\kappa}}^{-1}\circ\underline{\kappa}$

$$(4.134)\quad =\beta\big(\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \underline{\underline{Grad}}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}})\big)$$

Now apply (4.78) to $\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})$ in the first argument and (4.77) to $\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}$ in the second argument, both times with respect to the change of reference placement $\underline{\kappa}^{-1}\circ\kappa$

$$(4.135)\quad =\beta\Big(\beta\big\{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{N}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}})\big\}, \mathbf{N}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \underline{\underline{Grad}}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}})\Big)$$

The proof of (4.131) follows a similar scheme:

$$(4.136)\quad \gamma\Big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{N}\cdot\mathbf{M}, Grad(\mathbf{N}\cdot\mathbf{M}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)$$

Apply (4.91) with respect to the change of reference placement $\underline{\underline{\kappa}}^{-1} \circ \kappa$

$$(4.137) \quad = \underline{\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}}}$$

Apply (4.91) with respect to the change of reference placement $\underline{\underline{\kappa}}^{-1} \circ \underline{\kappa}$

$$(4.138) \quad = \gamma\Big(\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}}, \underline{\overset{\langle 4\rangle}{\mathbf{S}}}, \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}\Big)$$

Now apply (4.91) to $\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}}$ in the first argument, apply (4.92) to $\underline{\overset{\langle 4\rangle}{\mathbf{S}}}$ in the second argument and (4.77) to $\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}$ in the last argument, each time with respect to the change of reference placement $\underline{\kappa}^{-1} \circ \kappa$

$$(4.139) \quad = \gamma\Big(\gamma\big(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{N}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\big), det(\mathbf{N})(\mathbf{N}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}), \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \mathbf{N}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}\Big)$$

$\square$

Remark 4.7. Transformation of elastic laws under a change of the reference placement

For two reference placements

- κ with strain measures $\mathbf{C}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$, $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$, an elastic energy w and stress tensors $\overset{\langle 2\rangle}{\mathbf{S}}$, $\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}$, $\overset{\langle 4\rangle}{\mathbf{S}}$
  with elastic laws $\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big)$, $\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = \overset{\langle 3\rangle}{\widetilde{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big)$,
  $\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big)$

- $\underline{\kappa}$ with strain measures $\underline{\mathbf{C}}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}$, $\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})$, an elastic energy $\underline{w}$ and stress tensors $\underline{\overset{\langle 2\rangle}{\mathbf{S}}}$, $\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}}$, $\underline{\overset{\langle 4\rangle}{\mathbf{S}}}$
  with elastic laws $\underline{\overset{\langle 2\rangle}{\mathbf{S}}} = \underline{\overset{\langle 2\rangle}{f}}\big(\underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})\big)$, $\underline{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}} = \underline{\overset{\langle 3\rangle}{\widetilde{f}}}\big(\underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})\big)$,

$$\underset{\sim}{\overset{\langle 4\rangle}{\mathbf{S}}} = \underset{\sim}{\overset{\langle 4\rangle}{f}}\big(\underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})\big)$$

$$\overset{\langle 2\rangle}{f}\ \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \tag{4.140}$$

$$= \mathbf{A} * J_{\mathbf{A}}^{-1}\underset{\sim}{\overset{\langle 2\rangle}{f}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big)$$

$$\overset{\langle 3\rangle}{\widetilde{f}}\ \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \tag{4.141}$$

$$= \gamma\Bigg(\underset{\sim}{\overset{\langle 3\rangle}{\widetilde{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big),$$

$$\underset{\sim}{\overset{\langle 4\rangle}{f}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big),$$

$$\mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\ \Bigg).$$

$$\overset{\langle 4\rangle}{f}\ \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \tag{4.142}$$

$$= \mathbf{A} \circ J_{\mathbf{A}}^{-1}\underset{\sim}{\overset{\langle 4\rangle}{f}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big)$$

Proof.

The transformation of the elastic laws for the second order stress tensor can be found in [Bertram 2015]. The fourth-order stress tensor transforms almost as the second-order stress tensor, one only has to replace the $*$ product by the $\circ$ product. Therefore the proof for the transformation of the fourth-order stress tensor can be obtained from the proof of the second-order stress tensor simply by substituting the product $*$ by the product $\circ$. The reason for this is that $\overset{\langle 2\rangle}{\mathbf{S}}$ and $\overset{\langle 3\rangle}{\mathbf{S}}$ both are pullbacks of the corresponding spatial tensors while the products denoted by $*$ and $\circ$ both can be interpreted as pullbacks. Only for the third-order stress tensor the newly introduced function γ lets the transformations look slightly different:

$$\underset{\sim}{\overset{\langle 3\rangle}{\widetilde{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big)$$

$$= \underset{\sim}{\overset{\langle 3\rangle}{\widetilde{f}}}\big(\underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})\big) = \underset{\sim}{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}} = \gamma\Big(\ \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\ \Big) \tag{4.143}$$

$$= \gamma\Big(\ \overset{\langle 3\rangle}{\widetilde{f}}\ \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big), \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\ \Big) \tag{4.144}$$

At this point one applies $\gamma\Big(\ \cdot\ , \overset{\langle 4\rangle}{\underline{\mathbf{S}}}, \mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}\Big)$ on the very left and very right side of the equation and obtains

$$
\begin{aligned}
\Leftrightarrow \gamma\Big(&\overset{\langle 3\rangle}{\underline{\widetilde{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big),\\
&\overset{\langle 4\rangle}{\underline{\mathbf{S}}}, \mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}\Big)\\
= \gamma\Big(&\gamma\Big(\overset{\langle 3\rangle}{\widetilde{f}}\ (\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})), \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big), \overset{\langle 4\rangle}{\underline{\mathbf{S}}}, \mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}\Big)
\end{aligned}
\tag{4.145}
$$

The next step is to make use of the fact that $\gamma\Big(\ \cdot\ , \overset{\langle 4\rangle}{\underline{\mathbf{S}}}, \mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}\Big)$ is the inverse of $\gamma\Big(\ \cdot\ , \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)$ and obtains

$$
\begin{aligned}
\Leftrightarrow \gamma\Big(&\overset{\langle 3\rangle}{\underline{\widetilde{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big),\\
&\overset{\langle 4\rangle}{\underline{\mathbf{S}}}, \mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}\Big) = \overset{\langle 3\rangle}{\widetilde{f}}\ (\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}))
\end{aligned}
\tag{4.146}
$$

As a last step one now applies to the left hand side of this equation the two relations $\overset{\langle 4\rangle}{\underline{\mathbf{S}}} = \overset{\langle 4\rangle}{\underline{f}}\big(\ \underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})\ \big) = \overset{\langle 4\rangle}{\underline{f}}\big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta(...)\ \big)$ and $\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}} = \mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}$ and obtains the result:

$$
\begin{aligned}
\Leftrightarrow \gamma\Big(&\overset{\langle 3\rangle}{\underline{\widetilde{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big),\\
&\quad \overset{\langle 4\rangle}{\underline{f}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})\big)\Big),\\
&\quad \mathbf{A}^{-1}, \underline{Grad}\mathbf{A}^{-1}, \mathbf{A}^T\circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Big)\\
= &\overset{\langle 3\rangle}{\widetilde{f}}\ (\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}))
\end{aligned}
\tag{4.147}
$$

$\square$

4.6 Elastic isomorphy

In this section the concept of elastic isomorphy is generalized for the third-order theory. The fundamental definition of elastic isomorphy is obtained by extending the corresponding definition in [Bertram 2015] with transformation rules for $\overset{\langle 3 \rangle}{\widetilde{f}}$ and $\overset{\langle 4 \rangle}{f}$.

Definition 4.5. Elastic isomorphy

Two elastic material points X and Y are called **elastically isomorphic** if one can find reference placements κ_X for X and κ_Y for Y such that

- In κ_X and κ_Y the mass densities are identical:

  (4.148) $\rho_{0X} = \rho_{0Y}$.

- With respect to κ_X and κ_Y the elastic energies are identical:

  (4.149) $w_x(\kappa_X, \cdot) = w_x(\kappa_X, \cdot)$

Remark 4.8. Definition of elastic isomorphy with elastic laws

One can also formulate Equation (4.149) with respect to the elastic laws. One then requires the elastic laws to fulfill

$$\overset{\langle 2 \rangle}{f}_X (\kappa_X, \cdot) = \overset{\langle 2 \rangle}{f}_Y (\kappa_Y, \cdot) \tag{4.150}$$

$$\overset{\langle 3 \rangle}{\widetilde{f}}_X (\kappa_X, \cdot) = \overset{\langle 3 \rangle}{\widetilde{f}}_Y (\kappa_Y, \cdot) \tag{4.151}$$

$$\overset{\langle 4 \rangle}{f}_X (\kappa_X, \cdot) = \overset{\langle 4 \rangle}{f}_Y (\kappa_Y, \cdot) \tag{4.152}$$

Theorem 4.4. Criterion for elastic isomorphy

Let X and Y be two elastic material points with arbitrary reference placements $\underline{\kappa}_X$ and $\underline{\kappa}_Y$ and w_X and w_Y the corresponding elastic energies. Then these two points are called elastically isomorphic if and only if there exist tensors $\overset{\langle 2 \rangle}{\mathbf{P}} \in \mathscr{Inv}$, $\overset{\langle 3 \rangle}{\mathbf{P}} \in \mathscr{Conf}_3$, $\overset{\langle 4 \rangle}{\mathbf{P}} \in \mathscr{Conf}_4$

such that

$$(4.153)\quad \rho_{0Y} = det(\overset{\langle 2\rangle}{\mathbf{P}})\rho_{0X}$$

$$(4.154)\quad \underline{w}_X\Big(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X})\Big)$$
$$= \underline{w}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\,\big)\Big) + \underline{w}_0$$

Proof.

The proof of (4.154) follows directly from the relations (4.72)-(4.74).

□

Note:

The tensor $\overset{\langle 2\rangle}{\mathbf{P}}$ can be interpreted as the gradient of a change of reference placement, the tensor $\overset{\langle 3\rangle}{\mathbf{P}}$ as $\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$ and the tensor $\overset{\langle 4\rangle}{\mathbf{P}}$ as $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})$. As long as only one material point is considered these tensors can be considered as independent which means they do not have to fulfill any integrability condition.

Remark 4.9. Criterion for elastic isomorphy for elastic laws

Theorem 4.4 can also be formulated for elastic laws. Let $\{\overset{\langle 2\rangle}{\underline{f}}_X, \overset{\langle 3\rangle}{\widetilde{\underline{f}}}_X, \overset{\langle 4\rangle}{\underline{f}}_X\}$ and $\{\overset{\langle 2\rangle}{\underline{f}}_Y, \overset{\langle 3\rangle}{\widetilde{\underline{f}}}_Y, \overset{\langle 4\rangle}{\underline{f}}_Y\}$ be the respective sets of elastic laws in the setting of Theorem 4.4. Then Equation 4.154 can be replaced by

$$(4.155)\quad \overset{\langle 2\rangle}{\underline{f}}_X(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}))$$
$$= \overset{\langle 2\rangle}{\mathbf{P}} * det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\ \overset{\langle 2\rangle}{f}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}},$$
$$\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\,\big)\Big)$$

$$(4.156)\quad \overset{\langle 3\rangle}{\widetilde{\underline{f}}}_X(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}))$$
$$= \gamma\Bigg(\overset{\langle 3\rangle}{\widetilde{\underline{f}}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\,\big)\Big),$$
$$\overset{\langle 4\rangle}{\underline{f}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\,\big)\Big),$$

$$\overset{\langle 2\rangle}{\mathbf{P}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 2\rangle}{\mathbf{P}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}} \Big)$$

$$(4.157)\quad \underline{\overset{\langle 4\rangle}{f}}_X\big(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X})\big)$$
$$= \overset{\langle 2\rangle}{\mathbf{P}} \circ det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\ \overset{\langle 4\rangle}{f}_Y \Big(\overset{\langle 2\rangle}{\mathbf{P}}^{T} * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}},$$
$$\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big)\Big)$$

Proof. A proof can be found in Appendix C.1. □

4.7 Material symmetry

Applying the concept of elastic isomorphy to only one reference point, i.e., assuming that X and Y are the same point in Definition 4.8 defines symmetry. In this case one can drop the notation for the reference point. As explained in Theorem 4.1 a change of the reference placement defines three tensors $\mathbf{A} \in \mathscr{Unim}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} \in \mathscr{Conf}_3$ and $\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) \in \mathscr{Conf}_4$, where $\overset{\langle 2\rangle}{\mathbf{A}}$ is the Jacobian of the change of the reference placement map. So in this case the isomorphism $\mathbf{A}$ becomes an automorphism since it maps the tangent space at a point onto itself. One defines

$$(4.158)\quad \overset{\langle 2\rangle}{\mathbf{A}} = \mathbf{A}, \qquad \overset{\langle 3\rangle}{\mathbf{A}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \qquad \overset{\langle 4\rangle}{\mathbf{A}} = \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})$$

Since the density must remain unaltered by this change of the reference placement $J_{\overset{\langle 2\rangle}{\mathbf{A}}} = 1$ must hold. This is why $\mathbf{A} \in \mathscr{Unim}$. If one assumes that $\overset{\langle 2\rangle}{\mathbf{A}}$ stems from change of the reference placement that fullfills $J_{\overset{\langle 2\rangle}{\mathbf{A}}} = 1$ everywhere, then this assumption implies

$$(4.159)\quad \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{A}} = 0$$
$$(4.160)\quad \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}} = 0$$

The considerations above motivate

Definition 4.6. Symmetry Transformation

For a third-order gradient elastic material a symmetry transformation is a triple $(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}) \in \mathscr{U}nim \times \mathscr{C}onf_3 \times \mathscr{C}onf_4$ that fulfills

$$\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{A}} = 0 \text{ and } \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}} = 0 \tag{4.161}$$

such that for the elastic energy

$$\begin{aligned} & w\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \\ &= w\Big(\overset{\langle 2\rangle}{\mathbf{A}}{}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}{}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big)\Big) \end{aligned} \tag{4.162}$$

Remark 4.10.

Definition 4.6 could also be set up without condition (4.161) as in

[Bertram 2015]. Condition (4.161) comes from the following reasoning. If one assumes that $\overset{\langle 2\rangle}{\mathbf{A}}$ is the Jacobian of a change of the reference placement with $J_{\mathbf{A}} = 1$ everywhere in the body, then

$$Grad(det(\overset{\langle 2\rangle}{\mathbf{A}})) = 0 \tag{4.163}$$

must hold everywhere. Since *det* is a differentiable matrix function one can write

$$0 = Grad(det(\overset{\langle 2\rangle}{\mathbf{A}})) \tag{4.164}$$

$$= \frac{d(det)}{d\overset{\langle 2\rangle}{\mathbf{A}}} : Grad(\overset{\langle 2\rangle}{\mathbf{A}}) \tag{4.165}$$

Applying Jacobi's formula (see Equation 1.39 in [Bertram 2005]) to the term $\frac{d(det)}{d\overset{\langle 2\rangle}{\mathbf{A}}}$ yields

$$0 = \overset{\langle 2\rangle}{\mathbf{I}} : \Big[\underbrace{J_{\overset{\langle 2\rangle}{\mathbf{A}}}}_{=1} \underbrace{\overset{\langle 2\rangle}{\mathbf{A}}{}^{-T} \cdot Grad(\overset{\langle 2\rangle}{\mathbf{A}})}_{=\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}}\Big] \tag{4.166}$$

$$(4.167)\quad = \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}$$

$$(4.168)\quad = \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{A}}$$

Furthermore since (4.167) holds everywhere in the body one obtains

$$(4.169)\quad 0 = Grad(\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) = \overset{\langle 2\rangle}{\mathbf{I}} : Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) = \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}}$$

In [Bertram 2015] symmetry transformations are defined without this condition.

Remark 4.11. Symmetry transformations of elastic laws

Definition 4.6 can also be formulated for the elastic laws $\overset{\langle 2\rangle}{f}, \overset{\langle 3\rangle}{\tilde{f}}, \overset{\langle 4\rangle}{f}$ One just has to substitute condition (4.162) by

$$(4.170)\quad \overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big)$$
$$= \overset{\langle 2\rangle}{\mathbf{A}} * J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 2\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big)\Big)$$

$$(4.171)\quad \overset{\langle 3\rangle}{\tilde{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big)$$
$$= \gamma\Big(\overset{\langle 3\rangle}{\tilde{f}}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big)\Big),$$
$$\overset{\langle 4\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big)\Big),$$
$$\overset{\langle 2\rangle}{\mathbf{A}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{A}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}\Big)$$

$$(4.172)\quad \overset{\langle 4\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big)$$
$$= \overset{\langle 2\rangle}{\mathbf{A}} \circ J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 4\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big)\Big)$$

for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})) \in \mathscr{Config}$.

Definition 4.7. Symmetry group of a material

The set of all symmetry transformations of an elastic energy w with respect to a specific reference placement is the **symmetry group** of a material. The **symmetry group** is an algebraic group. It is a subset of $\mathscr{Unim} \times \mathscr{Conf}_3 \times \mathscr{Conf}_4$ and the group operation is the composition

$$\left(\overset{\langle 2\rangle}{\mathbf{B}}, \overset{\langle 3\rangle}{\mathbf{B}}, \overset{\langle 4\rangle}{\mathbf{B}}\right)\left(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\right) := \tag{4.173}$$
$$\Bigg(\overset{\langle 2\rangle}{\mathbf{B}} \cdot \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^T \circ \overset{\langle 3\rangle}{\mathbf{B}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^T \circ \overset{\langle 4\rangle}{\mathbf{B}} + \overset{\langle 4\rangle}{\mathbf{A}} - \left[\overset{\langle 3\rangle}{\mathbf{A}} \cdot \left(\overset{\langle 2\rangle}{\mathbf{A}}^T \circ \overset{\langle 3\rangle}{\mathbf{B}}\right)\right]^{[2,3]}$$
$$+ 2\overset{[2,3]}{sym}\left[(\overset{\langle 2\rangle}{\mathbf{A}}^T \circ \overset{\langle 3\rangle}{\mathbf{B}}) \cdot \overset{\langle 3\rangle}{\mathbf{A}}\right]\Bigg)$$

The **neutral element** is defined as

$$\left(\overset{\langle 2\rangle}{\mathbf{I}}, \overset{\langle 3\rangle}{\mathbf{0}}, \overset{\langle 4\rangle}{\mathbf{0}}\right)$$

The **inverse element** is defined as

$$\left(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\right)^{-1} := \tag{4.174}$$
$$\left(\overset{\langle 2\rangle}{\mathbf{A}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{A}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{A}}, -\overset{\langle 2\rangle}{\mathbf{A}}^{-1} \circ \left(\overset{\langle 4\rangle}{\mathbf{A}} + 3\overset{[2,4][2,3]}{sym}\left[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\right]\right)\right)$$

Note: The group operation as well as the definition of the neutral and inverse elements follow from the definition of the strain variables. One has to imagine that two smooth transformations $\chi_{\underset{\mathbf{A}}{\langle 2\rangle}}$ and $\chi_{\underset{\mathbf{B}}{\langle 2\rangle}}$ exist with

$$\overset{\langle 2\rangle}{\mathbf{A}} = Grad(\chi_{\underset{\mathbf{A}}{\langle 2\rangle}}), \qquad \overset{\langle 3\rangle}{\mathbf{A}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\underset{\mathbf{A}}{\langle 2\rangle}}, \qquad \overset{\langle 4\rangle}{\mathbf{A}} = Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\underset{\mathbf{A}}{\langle 2\rangle}}) \tag{4.175}$$
$$\overset{\langle 2\rangle}{\mathbf{B}} = Grad(\chi_{\underset{\mathbf{B}}{\langle 2\rangle}}), \qquad \overset{\langle 3\rangle}{\mathbf{B}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\underset{\mathbf{B}}{\langle 2\rangle}}, \qquad \overset{\langle 4\rangle}{\mathbf{B}} = Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\underset{\mathbf{B}}{\langle 2\rangle}}) \tag{4.176}$$

Then the composition of the mappings $\chi_{\overset{\langle 2\rangle}{\mathbf{A}}}$ and $\chi_{\overset{\langle 2\rangle}{\mathbf{B}}}$ yields the definition of the group operation:

$$\left(\overset{\langle 2\rangle}{\mathbf{B}},\overset{\langle 3\rangle}{\mathbf{B}},\overset{\langle 4\rangle}{\mathbf{B}}\right)\left(\overset{\langle 2\rangle}{\mathbf{A}},\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 4\rangle}{\mathbf{A}}\right)=\left(\underbrace{Grad\big[\chi_{\overset{\langle 2\rangle}{\mathbf{A}}}\big(\chi_{\overset{\langle 2\rangle}{\mathbf{B}}}(\cdot)\big)\big]}_{=\overset{\langle 2\rangle}{\mathbf{B}}\cdot\overset{\langle 2\rangle}{\mathbf{A}}},\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{B}}\cdot\overset{\langle 2\rangle}{\mathbf{A}}},Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{B}}\cdot\overset{\langle 2\rangle}{\mathbf{A}}})\right) \tag{4.177}$$

In (4.177) one expresses $\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{B}}\cdot\overset{\langle 2\rangle}{\mathbf{A}}}$ and $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{B}}\cdot\overset{\langle 2\rangle}{\mathbf{A}}})$ in terms of the tensors in (4.175) and (4.176) and obtains (4.173). This also yields the definition of the neutral element. With the same approach the entries of the inverse element are calculated as

$$\left(\overset{\langle 2\rangle}{\mathbf{A}},\overset{\langle 3\rangle}{\mathbf{A}},\overset{\langle 4\rangle}{\mathbf{A}}\right)^{-1}=\left(\overset{\langle 2\rangle}{\mathbf{A}},\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}},Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}})\right)^{-1}=\left(\overset{\langle 2\rangle}{\mathbf{A}}{}^{-1},\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}{}^{-1}},Grad(\overset{\langle 2\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}{}^{-1}})\right) \tag{4.178}$$

Again $\overset{\langle 2\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{A}}}$, $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 3\rangle}{\mathbf{A}}})^{-1}$ are expressed in terms of the tensors in (4.175). This is explained in detail in Equations (2.30)-(2.34).

Definition 4.8. Undistorted states & solids

If for a certain reference placement the symmetry group is a subgroup of the orthogonal group in the first entry and zero in the other two entries then this reference placement is called an **undistorted state**. The elements of the symmetry group then have the form

$$\left(\overset{\langle 2\rangle}{\mathbf{Q}},\overset{\langle 3\rangle}{\mathbf{0}},\overset{\langle 4\rangle}{\mathbf{0}}\right) \tag{4.179}$$

with $\mathbf{Q} \in \mathcal{O}rth^{+}$ and can be interpreted as rotations. A material that has such an undistorted state is called a **solid**.

Definition 4.9. Isotropic material:

If the symmetry group contains the orthogonal group in the first entry and zero in the others then the material point is called **isotropic**. It is clear that for an isotropic material the elastic laws $\overset{\langle 2\rangle}{f}$ and $\overset{\langle 4\rangle}{f}$ are isotropic tensor functions: First one has to rearrange equations (4.170) and (4.172). Next one applies the fact that for isotropic materials $\overset{\langle 2\rangle}{\mathbf{A}}$

is orthogonal: $\overset{\langle 2\rangle}{\mathbf{A}}^{-1} = \overset{\langle 2\rangle}{\mathbf{A}}^{T}$. This yields $J_{\overset{\langle 2\rangle}{\mathbf{A}}} = 1$ and it yields that the product "$\circ$" can be replaced by the product "$*$" in Equation (4.172):

$$w\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) = w\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} * Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \tag{4.180}$$

$$\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) = \overset{\langle 2\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} * Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \tag{4.181}$$

$$\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 4\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) = \overset{\langle 4\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} * Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \tag{4.182}$$

The elastic law $\overset{\langle 3\rangle}{\widetilde{f}}$ is not an isotropic tensor function. The reason for this is the fact that $\overset{\langle 3\rangle}{\widetilde{f}}$ transforms with the function γ and not as a pull-back like the other elastic laws. This is another point where the third-order theory deviates remarkably from the first- and second-order theories.

Remark 4.12. Isotropic and centro-symmetric materials

A material is called **centro-symmetric** if it contains with a proper symmetry transformation $\Big(\overset{\langle 2\rangle}{\mathbf{Q}}, \overset{\langle 3\rangle}{\mathbf{0}}, \overset{\langle 4\rangle}{\mathbf{0}}\Big)$ also the improper one $\Big(-\overset{\langle 2\rangle}{\mathbf{Q}}, \overset{\langle 3\rangle}{\mathbf{0}}, \overset{\langle 4\rangle}{\mathbf{0}}\Big)$, which is equivalent to demanding that $\Big(-\overset{\langle 2\rangle}{\mathbf{I}}, \overset{\langle 3\rangle}{\mathbf{0}}, \overset{\langle 4\rangle}{\mathbf{0}}\Big)$ is an element of the symmetry group. A simple material is always centro-symmetric since

$$(-\overset{\langle 2\rangle}{\mathbf{A}}^{T}) * \overset{\langle 2\rangle}{\mathbf{C}} = (\overset{\langle 2\rangle}{\mathbf{A}}^{T}) * \overset{\langle 2\rangle}{\mathbf{C}} \tag{4.183}$$

implies for a symmetry transformation $\overset{\langle 2\rangle}{\mathbf{A}} \in \mathscr{I}_{nv}$

$$w\Big((-\overset{\langle 2\rangle}{\mathbf{A}}^{T}) * \mathbf{C}\Big) = w\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}\Big) = w\Big(\mathbf{C}\Big) \tag{4.184}$$

In the case of a second gradient of the strain theory this is not the case. This is due to the fact that

$$(-\overset{\langle 2\rangle}{\mathbf{A}}^{T}) \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} = -(\overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \tag{4.185}$$

$$\beta\Big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, -\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\Big) \neq \beta\Big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\Big). \tag{4.186}$$

Therefore in a second gradient of the strain framework one has to distinguish between symmetric and centro-symmetric materials. If one assumes a linear relationship between $\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)$ and $\Big(\overset{\langle 2\rangle}{\mathbf{S}}, \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}\Big)$ for a centro-symmetric gradient material, then this yields that $\overset{\langle 2\rangle}{\mathbf{S}}$ can only depend on $\mathbf{C}$ as well as that $\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}$ can only depend on $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$ and $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$.

Chapter 5

A material framework for third-order elastoplasticity

5.1 Chapter introduction

This chapter generalizes the concepts of elastoplasticity in [Bertram 2015]. Besides generalizing yield limits and criteria as well as flow or hardening rules, two important concepts are generalized. The concept of isomorphic elastic ranges governs the transformation of the elastic law during plastic deformation and yields internal variables. Furthermore the plastic dissipation of energy is specified, which turns out to be considerably more complicated than in the classic and first gradient of strain theory. This is again due to the more complicated transformation behavior of the stress and strain variables under changes of the reference placement. With respect to plasticity the concept of isomorphic elastic ranges sets the present work apart from many other publications where a multiplicative decomposition of the strain into a plastic and an elastic part is suggested instead. Usually this leads to constrained plasticity theories and limits the range of material behaviors that can be modeled. The so called concept of isomorphic elastic ranges is less restrictive, it includes the multiplicative decomposition and is thus more suitable for a unifying framework.

5.2 Elastic ranges

In the spirit of the definition of elastoplasticty in [Bertram 2015] we assume that during a plastic deformation the elastic law and elastic range of a material point changes continuously. In order to define this process rigourously, the definition of an elastic range from [Bertram 2015] is extended in this section in a straightforward manner to fit the current third-order framework.

Definition 5.1. Elastic range

An **elastic range** is a tupel $\{\mathscr{E}_P, w_P\}$ which consists of

1. a non-empty path-connected submanifold with boundary called $\mathscr{E}_P \subset \mathscr{Config}$ and an
2. elastic energy $w_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)$

such that after any continuation process $\{\mathbf{C}(\tau), \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}(\tau), Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})(\tau)\}|_{t_0}^{t}$ which remains entirely in $\mathscr{E}$ the stresses are determined by the final values of this process.

$$\overset{\langle 2\rangle}{\mathbf{S}}(t) = 2\rho_0 \frac{\partial w_P}{\partial \mathbf{C}} =: \overset{\langle 2\rangle}{f}\big(\mathbf{C}(t), \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}(t), Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})(t)\big) \tag{5.1}$$

$$\overset{\langle 3\rangle}{\tilde{\mathbf{S}}}(t) = \rho_0 \frac{\partial w_P}{\partial \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}} =: \overset{\langle 3\rangle}{\tilde{f}}\big(\mathbf{C}(t), \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}(t), Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})(t)\big) \tag{5.2}$$

$$\overset{\langle 4\rangle}{\mathbf{S}}(t) = \rho_0 \frac{\partial w_P}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})} := \overset{\langle 4\rangle}{f}\big(\mathbf{C}(t), \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}(t), Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})(t)\big) \tag{5.3}$$

Remark 5.1.

Definition 5.1 can also be reformulated such that the **elastic range** is a quadrupel $\{\mathscr{E}_P, \overset{\langle 2\rangle}{f}_P, \overset{\langle 3\rangle}{\tilde{f}}_P, \overset{\langle 4\rangle}{f}_P\}$ of the submanifold $\mathscr{E}_P$ and elastic laws $\overset{\langle 2\rangle}{f}_P, \overset{\langle 3\rangle}{\tilde{f}}_P, \overset{\langle 4\rangle}{f}_P$.

Remark 5.2.

It is important to note that the elastic laws are physically determined for configurations within one elastic range $\mathscr{E}$. In order to simplify things we extend the elastic laws to the entire set $\mathscr{Config}$. If one wants to define a material with elastic ranges one has to make the following

(5.4) **Assumption**: At any time an elastoplastic material point is associated with an elastic range.

Using the definition above one can describe the plastic deformation process by saying that the "material continously passes through different elastic ranges" (CH. 10 in [Bertram 2015]). This process is also referred to as **yielding**.

5.3 Isomorphy of the elastic range

One should note that in Definition 4.8 isomorphy is a relation between two elastic points. But isomorphy can also be defined as a relation between two elastic laws at the same material point in a straightforward manner. The idea is to use the relations in Theorem 4.9 but to interpret each of the tensors $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{Unim}$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{Conf}_3$, $\overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{Conf}_4$ as part of a transformation between two elastic laws rather than changes of reference placement.

Definition 5.2. Isomorphy of elastic energies

Two different elastic energies w_1 and w_2 at one material point X are isomorphic if the point X with elastic energy w_1 is isomorphic to X with elastic energy w_2.

Theorem 5.1. Isomorphy criterion for elastic energies

Let X be a material point where a plastic deformation is applied. w_1 and w_2 denote two elastic energies at X. They are **isomorphic** if transformation tensors $\overset{\langle 2\rangle}{\mathbf{P}}_{12} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{P}}_{12} \in \mathscr{Conf}_3$, $\overset{\langle 4\rangle}{\mathbf{P}}_{12} \in \mathscr{Conf}_4$ exist such that

$$det(\overset{\langle 2\rangle}{\mathbf{P}}_{12}) = 1 \tag{5.5}$$

$$\begin{aligned} & w_1\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \\ & = w_2\Big(\overset{\langle 2\rangle}{\mathbf{P}}{}_{12}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}{}_{12}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}_{12}, \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \overset{\langle 4\rangle}{\mathbf{P}}_{12}\big)\Big) \end{aligned} \tag{5.6}$$

Proof.

This theorem is simply an application of Theorem 4.4 to the case $X = Y$. To validate the first requirement, $det(\overset{\langle 2\rangle}{\mathbf{P}}_{12}) = 1$, one has to remember that the reference placement stays the same when comparing a material point X to itself equipped with two different elastic laws. Thus the material densities are the same and the first isomorphy requirement $\rho_{0X} = det(\overset{\langle 2\rangle}{\mathbf{P}}_{12})\rho_{0Y}$ becomes $\rho_{0X} = det(\overset{\langle 2\rangle}{\mathbf{P}}_{12})\rho_{0X}$ which yields $det(\overset{\langle 2\rangle}{\mathbf{P}}_{12}) = 1 \Leftrightarrow \overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{Unim}$.

□

Theorem 5.2. Isomorphy criterion for elastic laws

Let X be a material point where a plastic deformation is applied. $\{\overset{\langle 2\rangle}{f}_1, \overset{\langle 3\rangle}{\tilde{f}}_1, \overset{\langle 4\rangle}{f}_1\}$ and $\{\overset{\langle 2\rangle}{f}_2, \overset{\langle 3\rangle}{\tilde{f}}_2, \overset{\langle 4\rangle}{f}_2\}$ denote two sets of elastic laws at X. These sets of elastic laws are **isomorphic** if transformation tensors $\overset{\langle 2\rangle}{\mathbf{P}}_{12} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{P}}_{12} \in \mathscr{Conf}_3$, $\overset{\langle 4\rangle}{\mathbf{P}}_{12} \in \mathscr{Conf}_4$ exist such that

$$det(\overset{\langle 2\rangle}{\mathbf{P}}_{12}) = 1 \tag{5.7}$$

$$\begin{aligned} &\overset{\langle 2\rangle}{f}_1 \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \\ &= \overset{\langle 2\rangle}{\mathbf{P}}_{12} * \Big[det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}}_{12})\ \overset{\langle 2\rangle}{f}_2 \Big(\overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}_{12}, \\ &\quad \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \overset{\langle 4\rangle}{\mathbf{P}}_{12}\big)\Big)\Big] \end{aligned} \tag{5.8}$$

$$\begin{aligned} &\overset{\langle 3\rangle}{\widetilde{f}}_1 \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \\ &= \gamma\Big(\overset{\langle 3\rangle}{\widetilde{f}}_2 \Big(\overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}_{12}, \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \overset{\langle 4\rangle}{\mathbf{P}}_{12}\big)\Big), \\ &\quad \overset{\langle 4\rangle}{f}_2 \Big(\overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}_{12}, \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \overset{\langle 4\rangle}{\mathbf{P}}_{12}\big)\Big), \\ &\quad \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1}_{12}, -\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T}_{12} \circ \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}_{12} \Big) \end{aligned} \tag{5.9}$$

$$\begin{aligned} &\overset{\langle 4\rangle}{f}_1 \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \\ &= \overset{\langle 2\rangle}{\mathbf{P}}_{12} \circ \Big[det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}}_{12})\ \overset{\langle 4\rangle}{f}_2 \Big(\overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}{}^T_{12} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}_{12}, \\ &\quad \overset{\langle 3\rangle}{\mathbf{P}}_{12}, \overset{\langle 4\rangle}{\mathbf{P}}_{12}\big)\Big)\Big] \end{aligned} \tag{5.10}$$

Proof.

The theorem can be regarded as a direct consequence of Theorem 5.1 or as an application of Remark 4.9

□

Now the definitions above will be used to describe the evolution of elastic energies and elastic laws during yielding. In general one has to consider two independent effects that occur during yielding:

1. Hardening (or softening) describes how the elastic range evolves.

2. Evolution of the elastic energies (or laws): The elastic energies (or laws), associated with the elastic ranges, evolve.

The second effect, the evolution of the elastic energy (or laws), is modeled in this framework via the concept of isomorphy from Theorem 5.2. For many materials it has been found, that the elastic behavior hardly changes under yielding, even if the deformations are very large (see [Silhavy & Kratochvil]). Therefore it is reasonable to make the following

Assumption 5.1. At one elastoplastic point the elastic energies (or laws) of all elastic ranges are isomorphic.

Assumption 5.1 is the one of the core features of the present framework since it allows to set up an unconstrained elastoplasticity framework. It has been introduced in [Bertram 1999] and implies that the elastic energies (or elastic laws) of all elastic ranges of a point are isomorphic to the elastic energy (or elastic laws respectively) of an arbitrarily chosen elastic range. This is because of the group property from Definition 4.7 of the ismorphy transformations. Therefore one can choose a so called **elastic reference energy** (or **elastic reference laws**) such that the elastic energy (or elastic laws) of the elastic range after any plastic deformation can be expressed as the transformation of the elastic reference energy (or elastic reference laws). This fact is formulated as

Theorem 5.3. Existence of an elastic reference energy

Under Assumption 5.1 one can always choose an elastic reference energy w_0 such that for any elastic range $\{\mathscr{E}_p, w_p\}$ there exist transformation tensors $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{Unim}$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{Conf}_3$, $\overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{Conf}_4$ with

$$
\begin{aligned}
(5.11)\quad & w_P\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\big) \\
&= w_0\Big(\underbrace{\overset{\langle 2\rangle}{\mathbf{P}}{}^T * \mathbf{C}}_{=:\mathbf{C}_e}, \underbrace{\overset{\langle 2\rangle}{\mathbf{P}}{}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}}_{=:\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}}, \underbrace{\beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big)}_{=:Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\Big).
\end{aligned}
$$

The index "0" indicates that the energy, an elastic law or a gradient is such an reference quantity. For gradients this means $Grad_0(...) = Grad(...)\cdot \overset{\langle 2\rangle}{\mathbf{P}}$

Theorem 5.4. Existence of elastic reference laws

Under Assumption 5.1 one can always choose a set $\{\overset{\langle 2\rangle}{f}_0, \overset{\langle 3\rangle}{\widetilde{f}}_0, \overset{\langle 4\rangle}{f}_0\}$ of elastic reference laws such that for any elastic range $\{\ \mathscr{E}_p, \overset{\langle 2\rangle}{f}_p, \overset{\langle 3\rangle}{\widetilde{f}}_p, \overset{\langle 4\rangle}{f}_p\ \}$ there exist transformation tensors $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{U}\!nim$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{C}\!onf_3$, $\overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{C}\!onf_4$ with

$$\begin{aligned}
&\overset{\langle 2\rangle}{f}_p \left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right) \\
&= \overset{\langle 2\rangle}{\mathbf{P}} * \Big[det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\ \overset{\langle 2\rangle}{f}_0 \Big(\underbrace{\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}}_{=:\mathbf{C_e}}, \underbrace{\overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}}_{=:\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F_e}}}, \underbrace{\beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\ \big)}_{=:Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F_e}})}\Big)\Big]
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
&\overset{\langle 3\rangle}{\widetilde{f}}_p \left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right) \\
&= \gamma\bigg(\overset{\langle 3\rangle}{\widetilde{f}}_0 \Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\ \big)\Big), \\
&\qquad \overset{\langle 4\rangle}{f}_0 \Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\ \big)\Big), \\
&\qquad \overset{\langle 2\rangle}{\mathbf{P}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}} \bigg)
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&\overset{\langle 4\rangle}{f}_p \left(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right) \\
&= \overset{\langle 2\rangle}{\mathbf{P}} \circ \Big[det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\ \overset{\langle 4\rangle}{f}_0 \Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\ \big)\Big)\Big]
\end{aligned} \tag{5.14}$$

Remark 5.3.

At this point it is very important to note that the theorems above transform the elastic energy or elastic laws, not the reference placement. The form of the transformations is the same as for a change of the reference placement but the underlying idea is completely different. $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{U}\!nim$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{C}\!onf_3$, $\overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{C}\!onf_4$ have the same form as kinematic quantities have but they are not kinematic quantities since in general they lack integrability. One interprets $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{U}\!nim$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{C}\!onf_3$, $\overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{C}\!onf_4$ as internal variables that describe how the elastic laws change during yielding. This becomes clear in Section 5.4.

Definition 5.3. Transformation by internal variables
Using the internal variables $\overset{\langle 2\rangle}{\mathbf{P}}\in \mathscr{U}nim$, $\overset{\langle 3\rangle}{\mathbf{P}}\in \mathscr{C}onf_3$, $\overset{\langle 4\rangle}{\mathbf{P}}\in \mathscr{C}onf_4$ that describe the change of an elastic law during yielding one can define auxillary stress and strain quantities as follows.

Transformed stresses

$$\overset{\langle 2\rangle}{f}_0\left(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})\right) = \overset{\langle 2\rangle}{\mathbf{S}}_0 := \overset{\langle 2\rangle}{\mathbf{P}}^{-1} * (J_{\mathbf{P}}\overset{\langle 2\rangle}{\mathbf{S}}) \tag{5.15}$$

$$\overset{\langle 3\rangle}{\widetilde{f}}_0\left(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})\right) = \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0 := \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \tag{5.16}$$

$$\overset{\langle 4\rangle}{f}_0\left(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})\right) = \overset{\langle 4\rangle}{\mathbf{S}}_0 := \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \circ (J_{\mathbf{P}}\overset{\langle 4\rangle}{\mathbf{S}}) \tag{5.17}$$

Transformed strains

$$\mathbf{C}_e = \overset{\langle 2\rangle}{\mathbf{P}}^{T} * \mathbf{C} \tag{5.18}$$

$$\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e} := \overset{\langle 2\rangle}{\mathbf{P}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}} \tag{5.19}$$

$$Grad_0(\mathbf{K}_{\mathbf{F}_e}) = \beta\Big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\Big) \tag{5.20}$$

5.4 Yield criteria

The derivation of the yield criteria is a straightforward extension of the concepts in [Bertram 2015].

Definition 5.4. Yield surface & yield criteria
Considering an elastic range $\left\{\mathscr{E}_P, w_P\right\}$ (or $\left\{\mathscr{E}_P, \overset{\langle 2\rangle}{f}_P, \overset{\langle 3\rangle}{\widetilde{f}}_P, \overset{\langle 4\rangle}{f}_P\right\}$) one decomposes $\mathscr{E}_P$ into its interior $\mathscr{E}_P^{\,0}$ and its boundary $\partial\,\mathscr{E}_P$. One calls $\partial\,\mathscr{E}_P$ the **yield surface**. The yield surface is assumed to be smooth enough such that it can be described by a level set

function:

$$\phi_P : \mathscr{Config} \to \mathbb{R}\ ,\ \Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \mapsto \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)$$

$$\partial\, \mathscr{E}_P = \Big\{\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \mid \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) = 0\Big\} \tag{5.21}$$

$$\mathscr{E}_P^{\,0} = \Big\{\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \mid \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) < 0\Big\} \tag{5.22}$$

$$\mathscr{Config} \setminus \mathscr{E}_P = \Big\{\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \mid \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) > 0\Big\} \tag{5.23}$$

The function ϕ_P is called a **yield criterion**. It is usually piecewise differentiable and a material property.

One can distinguish two phases of a deformation process:

1. **Purely elastic deformation**

   In this phase the strain measures either fulfill

   $$0 > \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \tag{5.24}$$

   or

   $$0 = \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \quad \wedge \quad 0 \geq \overset{\bullet}{\phi}_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big). \tag{5.25}$$

   The condition (5.24) means that the elastic process is in the set $\mathscr{E}_P^{\,0}$, while the conditions in (5.25) mean that the process is on the yield surface $\partial\, \mathscr{E}_P$ and either remains on the surface or is about to reenter the set $\mathscr{E}_P$.

2. **Elasto-plastic deformation**

   This phase is characterized by the fact that two equations must hold:

   $$0 = \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \tag{5.26}$$

   and

$$0 < \phi_P^{\bullet}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big) \tag{5.27}$$

$$= \frac{\partial \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)}{\partial \mathbf{C}} : \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} + \frac{\partial \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \frac{\partial \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}) \tag{5.28}$$

This means that the process is on the yield surface and about to enter the set $\mathscr{Config} \setminus \mathscr{E}_P$ thus leaving the set $\mathscr{E}_P$. The condition (5.26) is called the **yield condition** and the condition (5.27) is called the **loading condition**

The yield surface evolves during plastic yielding which means that ϕ_P depends on so called **hardening variables** which will be denoted by $\mathbf{Z}$. These hardening variables can be tensors of any order and characterize the softening or hardening of the material. A **general yield criterion** in the configuration space can then be denoted by

$$\phi\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \mathbf{Z}\Big) \tag{5.29}$$

During yielding this **general yield criterion** always fulfills

$$\phi\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \mathbf{Z}\Big) = 0 \tag{5.30}$$

which implies during yielding

$$\phi^{\bullet}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \mathbf{Z}\Big) = 0 \tag{5.31}$$

The **general loading condition** is

$$
(5.32)\quad 0 < \frac{\partial\phi\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)}{\partial\mathbf{C}} : \overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet} + \frac{\partial\phi\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}
$$
$$
+ \frac{\partial\phi\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}})
$$

5.5 Plastic dissipation

In this chapter the internal power during yielding is examined. The following abbreviations will be used:

$$
(5.33)\quad \overset{\langle 2\rangle}{\mathbf{G}} := \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1}
$$
$$
(5.34)\quad \overset{\langle 3\rangle}{\mathbf{G}} := -\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}
$$
$$
(5.35)\quad \overset{\langle 4\rangle}{\mathbf{G}} := -\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \circ \Big(Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}) + \overset{[2,4][2,3]}{3sym}\Big[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{P}}\Big]\Big)
$$

If one assumes that $\overset{\langle 2\rangle}{\mathbf{P}}$ is a sufficiently smooth tensor field with $\overset{\langle 3\rangle}{\mathbf{P}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$ and $\overset{\langle 4\rangle}{\mathbf{P}} = Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})$ then

$$
(5.36)\quad \overset{\langle 3\rangle}{\mathbf{G}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1}}
$$
$$
(5.37)\quad \overset{\langle 4\rangle}{\mathbf{G}} = Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1}})
$$

Let $\{\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\}$ be the internal variables that describe the change of the elastic law under yielding as defined in Section 5.3. Then $\{\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\}$ uniquely determine an alternative set of internal variables $\{\overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 4\rangle}{\mathbf{G}}\}$ as defined in (5.33) -(5.35). This will allow to abbreviate the notation in the following transformations. Since the chosen stress and strain measures are work conjugate one can write

$$\rho_0 l = \frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} \tag{5.38}$$

Now one can rewrite (5.38) by using the transformed stresses $\overset{\langle 2\rangle}{\mathbf{S}}_0, \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0$ and transformed strains $\mathbf{C}_0, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})$ from Definition 5.3. This means that one has to transform all stress and strain measures as explained in Theorem 5.4 and yields

$$\rho_0 l = \underbrace{\overset{\langle 2\rangle}{\mathbf{S}}_0 : \mathbf{C}_e^{\bullet} + \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0 \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_e}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_0 :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^{\bullet}}_{\text{elastic part}} + \underbrace{\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}} : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}} \vdots \overset{\langle 3\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} :: \overset{\langle 4\rangle}{\mathbf{G}}{}^{\bullet}}_{\text{plastic part}} \tag{5.39}$$

$$= w_0^{\bullet} + \underbrace{\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}} : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}} \vdots \overset{\langle 3\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} :: \overset{\langle 4\rangle}{\mathbf{G}}{}^{\bullet}}_{\text{plastic part}} \tag{5.40}$$

with **generalized plastic stress tensors**

$$\begin{aligned}
\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}} = {} & 2sym[\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{S}}_0 \cdot \mathbf{C}_e] + \overset{[1,2]}{2sym}\left[\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e} \cdot \overset{\langle 2\rangle}{\mathbf{G}}\right] : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}) \\
& - \Big((\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}) : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^{[1,2]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-1}\Big)^{T} \\
& - \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \cdot (\overset{\langle 2\rangle}{\mathbf{G}} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \vdots [\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})]^{[1,4]} + [(\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T})^{[1,2]} \vdots Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})]^{T} \\
& + \overset{[1,2]}{2sym}\Big[Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})\Big] \vdots \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} + \overset{\langle 4\rangle}{\mathbf{S}}_0 \vdots (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}})^{[1,3]} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^{[1,3]} \\
& + \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_e}^{[1,2]} : \Big((\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T})^{[3,4]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T}\Big) : \overset{\langle 3\rangle}{\mathbf{G}}{}^{[1,3]} + \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_e}^{[1,3]} : \Big((\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) : \overset{\langle 4\rangle}{\mathbf{S}}{}_0^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T}\Big) \\
& + (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \cdot \overset{\langle 3\rangle}{\mathbf{G}} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-1})^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}}_0 : \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_e}^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \\
& - 2\Big((\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}\Big)^{[1,2]} \vdots \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T}
\end{aligned} \tag{5.41}$$

$$\begin{aligned}
\overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}} = {} & \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}) + 2(\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}) : (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \\
& - (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) : (\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^{[1,3]}
\end{aligned} \tag{5.42}$$

$$\overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} = \overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0 \tag{5.43}$$

The step by step derivation of the generalized plastic stress tensors, starting from (5.38) is a straightforward but lengthy calculation. Therefore it is presented in Appendix C.2.

5.6 Flow and hardening rules

The evolution of the internal plastic variables $\overset{\langle 2\rangle}{\mathbf{P}}$, $\overset{\langle 3\rangle}{\mathbf{P}}$, $\overset{\langle 4\rangle}{\mathbf{P}}$, $\mathbf{Z}$ works exactly as in [Bertram 2015], one just has to add a flow rule for $\overset{\langle 4\rangle}{\mathbf{P}}$ and extend the set of variables accordingly. The result are three **flow rules**

$$\overset{\langle 2\rangle}{\mathbf{P}}{}^{\bullet} = \overset{\langle 2\rangle}{h}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\bullet}\Big) \tag{5.44}$$

$$\overset{\langle 3\rangle}{\mathbf{P}}{}^{\bullet} = \overset{\langle 3\rangle}{h}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\bullet}\Big) \tag{5.45}$$

$$\overset{\langle 4\rangle}{\mathbf{P}}{}^{\bullet} = \overset{\langle 4\rangle}{h}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}})\Big) \tag{5.46}$$

and a **hardening rule**

$$\mathbf{Z}^{\bullet} = h\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}})\Big) \tag{5.47}$$

The function h does not carry a superscript that indicates the tensor order, since the order of the hardening variable $\mathbf{Z}$ has not been fixed. The assumption of **rate-independency** leads to the introduction of a **plastic consistency parameter** $\lambda \geq 0$. It scales the functions that determine the direction of flow and hardening: $\overset{\langle 2\rangle}{h}{}^{\circ}, \lambda\, \overset{\langle 3\rangle}{h}{}^{\circ}, \lambda\, \overset{\langle 4\rangle}{h}{}^{\circ}$. This approach yields

$$\overset{\langle 2\rangle}{\mathbf{P}}{}^{\bullet} = \lambda\, \overset{\langle 2\rangle}{h}{}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}\Big) \tag{5.48}$$

$$\overset{\langle 3\rangle}{\mathbf{P}}{}^{\bullet} = \lambda\, \overset{\langle 3\rangle}{h}{}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}\Big) \tag{5.49}$$

$$\overset{\langle 4\rangle}{\mathbf{P}}{}^{\bullet} = \lambda\, \overset{\langle 4\rangle}{h}{}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}\Big) \tag{5.50}$$

$$\mathbf{Z}^{\bullet} = \lambda\, h^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}\Big) \tag{5.51}$$

with a scale parameter μ:

$$(5.52)\qquad \mu := \sqrt{\| \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} \|^2 + L^2 \| \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} \|^2 + \bar{L}^2 \| Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}) \|^2}$$

$$(5.53)\qquad \overset{\langle 2\rangle}{\mathbf{C}}^{\circ} := \frac{1}{\mu} \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet}, \qquad \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ} := \frac{1}{\mu} \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}, \qquad Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} := \frac{1}{\mu} Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet})$$

Here the constants L with dimension of a length and $\bar{L}$ with dimension of a squared length are constants that need to be introduced for dimensional consistency. They determine the influence that $\overset{\langle 2\rangle}{\mathbf{C}}^{\bullet}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet})$ have on yielding. During a purely elastic process the consistency parameter must be zero. During yielding the consistency parameter does not vanish, and it can be calculated by inserting Equations (5.48)-(5.51) into the yield condition (5.31):

$$
\begin{aligned}
(5.54)\qquad 0 =& \phi^{\bullet}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \mathbf{Z} \Big) \\
=& \frac{\partial \phi}{\partial \mathbf{C}} : \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} + \frac{\partial \phi}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \frac{\partial \phi}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}) + \frac{\partial \phi}{\partial \overset{\langle 2\rangle}{\mathbf{P}}} : \overset{\langle 2\rangle}{\mathbf{P}}^{\bullet} \\
&+ \frac{\partial \phi}{\partial \overset{\langle 3\rangle}{\mathbf{P}}} \vdots\, \overset{\langle 3\rangle}{\mathbf{P}}^{\bullet} + \frac{\partial \phi}{\partial \overset{\langle 4\rangle}{\mathbf{P}}} :: \overset{\langle 4\rangle}{\mathbf{P}}^{\bullet} + \Big\langle \frac{\partial \phi}{\partial \mathbf{Z}}, \mathbf{Z}^{\bullet} \Big\rangle \\
=& \frac{\partial \phi}{\partial \mathbf{C}} : \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} + \frac{\partial \phi}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \frac{\partial \phi}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}) \\
&+ \frac{\partial \phi}{\partial \overset{\langle 2\rangle}{\mathbf{P}}} : \lambda \overset{\langle 2\rangle}{h}^{\circ} \Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big) \\
&+ \frac{\partial \phi}{\partial \overset{\langle 3\rangle}{\mathbf{P}}} \vdots\, \lambda \overset{\langle 3\rangle}{h}^{\circ} \Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big) \\
&+ \frac{\partial \phi}{\partial \overset{\langle 4\rangle}{\mathbf{P}}} :: \lambda \overset{\langle 4\rangle}{h}^{\circ} \Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big) \\
&+ \Big\langle \frac{\partial \phi}{\partial \mathbf{Z}}, \lambda h^{\circ} \Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big) \Big\rangle
\end{aligned}
$$

Rearranging then yields

$$\lambda = - \Big(\frac{\partial \phi}{\partial \mathbf{C}} : \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} + \frac{\partial \phi}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \frac{\partial \phi}{\partial Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}) \Big) /$$

$$\Big(+ \frac{\partial \phi}{\partial \overset{\langle 2 \rangle}{\mathbf{P}}} : \lambda \overset{\langle 2 \rangle}{h}{}^{\circ} \Big(\overset{\langle 2 \rangle}{\mathbf{P}}, \overset{\langle 3 \rangle}{\mathbf{P}}, \overset{\langle 4 \rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2 \rangle}{\mathbf{C}}, \overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2 \rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3 \rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big)$$

$$+ \frac{\partial \phi}{\partial \overset{\langle 3 \rangle}{\mathbf{P}}} \vdots \lambda \overset{\langle 3 \rangle}{h}{}^{\circ} \Big(\overset{\langle 2 \rangle}{\mathbf{P}}, \overset{\langle 3 \rangle}{\mathbf{P}}, \overset{\langle 4 \rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2 \rangle}{\mathbf{C}}, \overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2 \rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3 \rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big)$$

$$+ \frac{\partial \phi}{\partial \overset{\langle 4 \rangle}{\mathbf{P}}} :: \lambda \overset{\langle 4 \rangle}{h}{}^{\circ} \Big(\overset{\langle 2 \rangle}{\mathbf{P}}, \overset{\langle 3 \rangle}{\mathbf{P}}, \overset{\langle 4 \rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2 \rangle}{\mathbf{C}}, \overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2 \rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3 \rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big)$$

$$+ \Big\langle \frac{\partial \phi}{\partial \mathbf{Z}}, \lambda h^{\circ} \Big(\overset{\langle 2 \rangle}{\mathbf{P}}, \overset{\langle 3 \rangle}{\mathbf{P}}, \overset{\langle 4 \rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2 \rangle}{\mathbf{C}}, \overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2 \rangle}{\mathbf{C}}{}^{\circ}, \overset{\langle 3 \rangle}{\mathbf{K}}{}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} \Big) \Big\rangle \Big)$$

Note:

The notation from (2.15) has been used for the scalar product since the variable $\mathbf{Z}$ comes from a vector space of arbitrary but finite dimension. The loading condition (5.27) implies that λ is positive during yielding and zero in all other cases. After substituting λ in (5.48) - (5.51) by (5.55) one obtains the **consistent flow and hardening rules**. At any time the **Kuhn-Tucker condition** holds:

(5.55) $\lambda\phi = 0$ with $\lambda \geq 0$ and $\phi \leq 0$

Chapter 6

A thermodynamical framework for third-order elasticity and elastoplasticity

6.1 Chapter introduction

It is the aim of this chapter to find a framework that is as large as possible, i.e., a framework that can accommodate the widest possible range of deformation processes and material behaviors. The approach used in this chapter stems from [Bertram & Krawietz 2012] and has already been applied in a similar manner in [Bertram 2014] and the present work is a straightforward extenion of the results therein. This chapter reintroduces the concepts that were introduced in the sections on elasticity and elastoplasticity but in a thermodynamical context. This means that the set of mechanical variables is extended by thermodynamical variables such as the temperature. Then the first and second law of thermodynamics and the Helmholtz free energy are introduced. From there potentials for the stresses and for thermodynamic quantities are derived for the elastic and the elastoplastic case. Concepts such as isomorphy and material symmetry are extended for the thermodynamical framework as well. This results in a framework where changes of the temperature can originate from elastic or plastic deformations as well as external heat sources.

6.2 Thermodynamical variables and basic concepts

Definition 6.1. Thermodynamic variables

The **specific internal energy** is denoted by ε.

The **heat supply** per unit mass and time by irradiation and conduction is denoted by Q.

The **spatial heat flux** per unit area and unit time in the current placement is denoted by $\mathbf{q}_E$.

The **material heat flux** per unit area and unit time in the reference placement is $\mathbf{q} := J_{\mathbf{F}}\mathbf{F}^{-1} \cdot \mathbf{q}_E$.

The **absolute temperature** is denoted by θ .

The **material temperature gradient** is denoted by $\mathbf{g} := Grad(\theta)$.

The **specific entropy** is denoted by η.

One could include higher gradients of the temperature as independent variables in Definition 6.1 but it has been shown in [Perzyna 1971] that defining higher (spatial) gradients of the temperature as independent variables would violate the second law of thermodynamics. The reason for this is, that they do not have a counterpart in the dissipation inequality and thus cancel out.

Remark 6.1. Changes of the reference placement
With the notation from Section 4.5 one obtains for changes of the reference placement:

$$\underline{\mathbf{g}} := \mathbf{A}^T \cdot \mathbf{g} \tag{6.1}$$

$$\underline{\mathbf{q}} := J_{\mathbf{A}} \mathbf{A}^{-1} \cdot \mathbf{q} \tag{6.2}$$

Definition 6.2. The Helmholtz free energy
The **Helmholtz free energy** is denoted by ψ and defined as

$$\psi := \varepsilon - \theta\eta \tag{6.3}$$

Assumption 6.1. Energy balance & Clausius-Duhem inequality

The **first law of thermodynamics** is assumed as

$$Q = \varepsilon^{\bullet} - p \tag{6.4}$$

The **second law of thermodynamics** is assumed in the form of the **Clausius-Duhem inequality**, which is a local and momentary restriction to all admissible thermodynamical processes:

$$p - \psi^{\bullet} - \theta^{\bullet}\eta - \frac{1}{\rho_0\theta}\mathbf{g} \cdot \mathbf{q} \geq 0. \tag{6.5}$$

In this inequality the **thermal dissipation** is defined as

$$\frac{1}{\rho_0\theta}\mathbf{g} \cdot \mathbf{q} \tag{6.6}$$

and the mechanical dissipation is

$$p - \psi^{\bullet} - \theta^{\bullet}\eta \tag{6.7}$$

Definition 6.3. Thermo-kinematical process

In this chapter the concept of the elastic or elasto-plastic process is extended to a **thermo-kinematical process**, described by the process

$$\{\chi(\tau), \mathbf{F}(\tau), Grad(\mathbf{F}), Grad^{II}(\mathbf{F}), \theta(\tau), grad(\theta)(\tau)\} \tag{6.8}$$

with $\tau \in [0, t]$. With the same arguments as layed out in [Bertram 2005] one can assume that this process determines the **caloro-dynamical state** at the end of the process, which is defined as

$$\{\overset{\langle 2\rangle}{\mathbf{T}}(t), \overset{\langle 3\rangle}{\mathbf{T}}(t), \overset{\langle 4\rangle}{\mathbf{T}}(t), \mathbf{q}_E(t), \varepsilon(t), \eta(t)\} \tag{6.9}$$

6.3 Third-order thermoelasticity

The concept of elasticity from Section 4.4 has to be extended in such a way that the current caloro-dynamical state is determined only by the current thermo-kinematical state. This means that the past process does not directly influence the current material behavior.

Definition 6.4. Third-order thermoelasticity

A material is called a **third-order thermoelastic material** if each of the quantities $\overset{\langle 2\rangle}{\mathbf{T}}$, $\overset{\langle 3\rangle}{\mathbf{T}}$, $\overset{\langle 4\rangle}{\mathbf{T}}$, $\mathbf{q}_E$, η and ε are functions of the set of variables $\{\chi$, $\mathbf{F}$, $Grad(\mathbf{F})$,$Grad^{II}(\mathbf{F})$, θ, $grad(\theta)\}$. Following the line of argumentation in Section 4.4 these constitutive equations can be reduced to the set of equations:

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{f}(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g}) \tag{6.10}$$

$$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = \overset{\langle 3\rangle}{\widetilde{f}}(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g}) \tag{6.11}$$

$$(6.12)\quad \overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{f}\,(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g})$$

$$(6.13)\quad \mathbf{q} = q(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g})$$

$$(6.14)\quad \varepsilon = \varepsilon(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g})$$

$$(6.15)\quad \eta = \eta(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g})$$

Definition 6.4 implies that in third-order thermoelasticity the free energy is a function of $\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta$ and $\mathbf{g}$ since

$$(6.16)\quad \psi(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g})$$

$$(6.17)\quad = \varepsilon(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g}) - \theta\eta(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g})$$

Theorem 6.1.

For a third-order thermoelastic material the Clausius-Duhem inequality (6.5) is fulfilled for every thermo-kinematical process if and only if

1. The free energy does not depend on the gradient of the temperature
2. The free energy acts as a potential for the generalized stresses and for the entropy
3. The heat conduction inequality holds: $\mathbf{q} \cdot \mathbf{g} \geq 0$

This shows that the thermoelastic behavior of a third-order material is completely determined by the two functions $\psi(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta)$ and $q(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g})$.

Proof.

Combining equations (6.16) and (6.5) one obtains

$$(6.18)\quad 0 \geq -\frac{1}{\rho_0}\Big(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^\bullet + \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} \;\vdots\; \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}} :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^\bullet\Big) + \partial_{\mathbf{C}}\psi : \mathbf{C}^\bullet + \partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi \;\vdots\; \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}$$

$$+ \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^\bullet + \partial_\theta\psi\theta^\bullet + \partial_{\mathbf{g}}\psi \cdot \mathbf{g}^\bullet + \theta^\bullet\eta + \frac{1}{\rho_0\theta}\mathbf{g} \cdot \mathbf{q}$$

$$(6.19)\quad = \Big(\partial_{\mathbf{C}}\psi - \frac{1}{2\rho_0}\overset{\langle 2\rangle}{\mathbf{S}}\Big) : \mathbf{C}^\bullet + \Big(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi - \frac{1}{\rho_0}\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}\Big) \;\vdots\; \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}$$

$$+ (\partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi - \frac{1}{\rho_0}\overset{\langle 4\rangle}{\mathbf{S}}) :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\bullet} + (\partial_\theta\psi + \eta)\theta^{\bullet} + \partial_{\mathbf{g}}\psi \cdot \mathbf{g}^{\bullet} + \frac{1}{\rho_0\theta}\mathbf{g}\cdot\mathbf{q}$$

By standard arguments this leads to the following thermoelastic relations:

$$\partial_{\mathbf{g}}\psi = \overset{\langle 1\rangle}{\mathbf{0}} \tag{6.20}$$

$$\overset{\langle 2\rangle}{\mathbf{S}} = 2\rho_0\partial_{\mathbf{C}}\psi \tag{6.21}$$

$$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = 2\rho_0\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi \tag{6.22}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = 2\rho_0\partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi \tag{6.23}$$

$$\eta = -\partial_\theta\psi \tag{6.24}$$

$$0 \geq \mathbf{g}\cdot\mathbf{q} \tag{6.25}$$

Now (6.20) means that the free energy is independent of the temperature gradient. Furthermore (6.21)-(6.23) shows that the free energy is a potential for the generalized stresses and (6.24) that it is a potential for the elastic part of the entropy. Finally (6.25) is the heat conduction inequality. □

6.4 Material Isomorphy and symmetry

The concept of elastic isomorphy in Section 4.6 can be extended for the case of thermoelasticity. One considers two thermoelastic points as isomorphic if their measurable thermoelastic behavior does not show any differences during arbitrary thermo-kinematical processes. Measurable quantities are the generalized stresses, the heat flux and the rate of the internal energy. The entropy and free energy are not considered as directly measurable quantities.

Definition 6.5. Thermoelastic isomorphy

Two thermoelastic points X and Y are **thermoelastically isomorphic** if two reference placements κ_X and κ_Y with constants $\eta_c \in \mathbb{R}$ and $\varepsilon_c \in \mathbb{R}$ exist such that

(6.26) $\rho_{0X} = \rho_{0Y}$

(6.27) $\psi_X(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) = \psi_Y(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) - \eta_c\theta + \varepsilon_c$

(6.28) $q_X(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta, \mathbf{g}) = q_Y(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta, \mathbf{g})$

for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}})) \in \mathscr{Config}$, $\theta \in \mathbb{R}$, $\mathbf{g} \in \mathbb{R}^3$.

Remark 6.2.

Definition 6.5 implies with the relations from Theorem 6.1

(6.29) $\overset{\langle 2\rangle}{f}_X(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) = \overset{\langle 2\rangle}{f}_Y(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta)$

(6.30) $\overset{\langle 3\rangle}{\widetilde{f}}_X(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) = \overset{\langle 3\rangle}{\widetilde{f}}_Y(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta)$

(6.31) $\overset{\langle 4\rangle}{f}_X(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) = \overset{\langle 4\rangle}{f}_Y(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta)$

(6.32) $\varepsilon_X(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) = \varepsilon_Y(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) + \varepsilon_c$

(6.33) $\eta_X(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) = \eta_Y(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_\mathbf{F}}), \theta) + \eta_c$

Definition 6.5 is derived from the following reasoning: For third-order thermoelastic materials the mechanical dissipation is zero

(6.34) $0 = p - \psi^\bullet - \theta^\bullet\eta$

Use (6.2)

(6.35) $= p - \varepsilon^\bullet + \theta\eta^\bullet$

Use (6.4)

(6.36) $= -Q + \theta\eta^\bullet$

This shows that the rate of the entropy is measurable since the temperature θ and the heat supply Q are measurable. In conclusion the entropy of two isomorphic points of a thermoelastic material can only differ by a constant η_c (which cannot be measured).

$$\eta_X(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \theta_X) = \eta_Y(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}), \theta_Y) + \eta_c \tag{6.37}$$

Integration with respect to the temperature then yields

$$\psi_X(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \theta_X) = \psi_Y(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}), \theta_Y) - \eta_c\theta + \varepsilon_c \tag{6.38}$$

where ε_c is another constant. Finally one can use these relations together with (6.3) to obtain

$$\begin{aligned}
&\varepsilon_X(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \theta_X) \\
&= \psi_X(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \theta_X) + \theta\eta_X(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \theta_X) && (6.39)\\
&= \psi_Y(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}), \theta_Y) + \theta\eta_Y(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}), \theta_Y) + \varepsilon_c && (6.40)\\
&= \varepsilon_Y(\mathbf{C}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_Y}), \theta_Y) + \varepsilon_c && (6.41)
\end{aligned}$$

which motivates Definition 6.5. From here it is clear that Theorem 4.4 can be generalized in the same manner.

Theorem 6.2. Criterion for thermoelastic isomorphy

Let X and Y be two thermoelastic material points with arbitrary reference placements κ_X and κ_Y. Let ψ_X, q_X and ψ_Y, q_Y be the corresponding thermoelastic laws. Then these two points are called **thermoelastically isomorphic** if and only if there exist tensors $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{Conf}_3$, $\overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{Conf}_4$ and two real constants ε_c, η_c such that

$$\rho_{0Y} = det(\overset{\langle 2\rangle}{\mathbf{P}})\rho_{0X} \tag{6.42}$$

$$\begin{aligned}
&\psi_X\Big(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \theta\Big) \\
&= \psi_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big), \theta\Big) \\
&\quad - \eta_c\theta + \varepsilon_c
\end{aligned} \tag{6.43}$$

$$(6.44)\quad det(\overset{\langle 2\rangle}{\mathbf{P}})q_X\Big(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\theta,\mathbf{g}_X\Big)$$
$$=\overset{\langle 2\rangle}{\mathbf{P}}*q_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 2\rangle}{\mathbf{P}},\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 4\rangle}{\mathbf{P}}\big),\theta,$$
$$\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{g}_X\Big)$$

for $(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}))\in\mathscr{Config}$, $\theta\in\mathbb{R}$, $\mathbf{g}\in\mathbb{R}^3$.

Remark 6.3.

Theorem 6.2 implies that the tensors $\overset{\langle 2\rangle}{\mathbf{P}}\in\mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{P}}\in\mathscr{Conf}_3$, $\overset{\langle 4\rangle}{\mathbf{P}}\in\mathscr{Conf}_4$ and two real constants ε_c, η_c determine the isomorphy transformation of the constitutive laws.
For $(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}))\in\mathscr{Config}$, $\theta\in\mathbb{R}$

$$(6.45)\quad \overset{\langle 2\rangle}{f}_X\big(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\theta\big)$$
$$=\overset{\langle 2\rangle}{\mathbf{P}}*det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\,\overset{\langle 2\rangle}{f}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},$$
$$\beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 2\rangle}{\mathbf{P}},\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 4\rangle}{\mathbf{P}}\big),\theta\Big)$$

$$(6.46)\quad \overset{\langle 3\rangle}{\widetilde{f}}_X\big(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\theta\big)$$
$$=\gamma\Bigg(\overset{\langle 3\rangle}{\widetilde{f}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 2\rangle}{\mathbf{P}},\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 4\rangle}{\mathbf{P}}\big),\theta\Big),$$
$$\overset{\langle 4\rangle}{f}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 2\rangle}{\mathbf{P}},\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 4\rangle}{\mathbf{P}}\big),\theta\Big),$$
$$\overset{\langle 2\rangle}{\mathbf{P}}^{-1},-\overset{\langle 2\rangle}{\mathbf{P}}^{-T}\circ\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}}\Bigg)$$

$$(6.47)\quad \overset{\langle 4\rangle}{f}_X\big(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\theta\big)$$
$$(6.48)\quad =\overset{\langle 2\rangle}{\mathbf{P}}\circ det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\,\overset{\langle 4\rangle}{f}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},$$
$$\beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 2\rangle}{\mathbf{P}},\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 4\rangle}{\mathbf{P}}\big),\theta\Big)$$

$$(6.49)\quad \varepsilon_X\big(\mathbf{C}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\theta\big)$$
$$=\varepsilon_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T*\mathbf{C}_X,\overset{\langle 2\rangle}{\mathbf{P}}^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}+\overset{\langle 3\rangle}{\mathbf{P}},\beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X},\overset{\langle 2\rangle}{\mathbf{P}},\overset{\langle 3\rangle}{\mathbf{P}},\overset{\langle 4\rangle}{\mathbf{P}}\big),\theta\Big)+\varepsilon_c$$

$$\eta_X\big(\mathbf{C}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \theta\big) \tag{6.50}$$
$$= \eta_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big), \theta\Big) + \eta_c$$

With these results the concept of symmetry transformations can be extended to thermoelastic materials in a straightforward manner.

Definition 6.6. Thermoelastic symmetry transformations

For a third-order thermoelastic material with material laws ψ and q a symmetry transformation is a triple $(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}) \in \mathscr{Unim} \times \mathscr{Conf}_3 \times \mathscr{Conf}_4$ that fulfills

$$\overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 3\rangle}{\mathbf{A}} = 0 \text{ and } \overset{\langle 2\rangle}{\mathbf{I}} : \overset{\langle 4\rangle}{\mathbf{A}} = 0 \tag{6.51}$$

such that for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})) \in \mathscr{Config}$, $\theta \in \mathbb{R}$, $\mathbf{g} \in \mathbb{R}^3$

$$\psi\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) \tag{6.52}$$
$$= \psi\Big(\overset{\langle 2\rangle}{\mathbf{A}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big)$$
$$q\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) \tag{6.53}$$
$$= \overset{\langle 2\rangle}{\mathbf{A}} * q\Big(\overset{\langle 2\rangle}{\mathbf{A}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big)$$

Remark 6.4. Symmetry transformations of thermoelastic laws

Definition 6.6 implies symmetry transformations for the thermoelastic laws.
For all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})) \in \mathscr{Config}$, $\theta \in \mathbb{R}$, $\mathbf{g} \in \mathbb{R}^3$

$$\overset{\langle 2\rangle}{f}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\big) \tag{6.54}$$
$$= \overset{\langle 2\rangle}{\mathbf{A}} * J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}}\, \overset{\langle 2\rangle}{f}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big)$$
$$\overset{\langle 3\rangle}{\widetilde{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\big) \tag{6.55}$$

$$
\begin{aligned}
&= \gamma\Big(\overset{\langle 3\rangle}{\widetilde{f}} \Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big),\\
&\qquad \overset{\langle 4\rangle}{f} \Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big),\\
&\qquad \overset{\langle 2\rangle}{\mathbf{A}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{A}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{A}} \Big)
\end{aligned}
$$

$$
\overset{\langle 4\rangle}{f} \big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \theta\big) \tag{6.56}
$$

$$
= \overset{\langle 2\rangle}{\mathbf{A}} \circ J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 4\rangle}{f} \Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big)
$$

$$
\varepsilon\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \theta\big) \tag{6.57}
$$

$$
= \varepsilon\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big)
$$

$$
\eta\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \theta\big) \tag{6.58}
$$

$$
= \eta\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{A}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}}\big), \theta\Big)
$$

The definitions of a symmetry group, undistorted states, a solid and of isotropy in Section 4.7 also apply to the thermoelastic case.

6.5 Thermoplasticity

Definition 6.7. Elastic range

An **elastic range** is a triple $\{\mathscr{E}_P, \psi_P, q_P\}$ which consists of

1. a non-empty path-connected submanifold with boundary called $\mathscr{E}_P \subset \mathscr{Config} \times \mathbb{R}^+ \times \mathbb{R}^3$ and
2. thermoelastic laws $\psi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \theta\Big)$, $q_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \theta, \mathbf{g}\Big)$

such that after any continuation process $\big\{\mathbf{C}(\tau), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}(\tau), Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})(\tau), \theta(\tau), \mathbf{g}(\tau)\big\}\big|_{t_0}^{t}$ which remains entirely in $\mathscr{E}_P$ the caloro-dynamical state is determined by the final values of this process:

$$\overset{\langle 2\rangle}{\mathbf{S}} = 2\rho_0 \partial_{\mathbf{C}} \psi_P \tag{6.59}$$

$$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = \rho_0 \partial_{\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}} \psi_P \tag{6.60}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \rho_0 \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})} \psi_P \tag{6.61}$$

$$\begin{aligned} \varepsilon = {} & \psi_P\big(\mathbf{C}(t), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}(t), Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})(t), \theta(t)\big) \\ & - \theta \partial_\theta \psi_P\big(\mathbf{C}(t), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}(t), Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})(t), \theta(t)\big) \end{aligned} \tag{6.62}$$

$$\eta = -\partial_\theta \psi_P\big(\mathbf{C}(t), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}(t), Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})(t), \theta(t)\big) \tag{6.63}$$

and all these functions are assumed to be continuously differentiable on $\mathscr{Config} \times \mathbb{R}^+ \times \mathbb{R}^3$.

In addition to Definition 6.7 one has to make the following

Assumption 6.2.

At the end of a thermokinematical process $\big\{\mathbf{C}(\tau), \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}(\tau), Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})(\tau), \theta(\tau), \mathbf{g}(\tau)\big\}|_{t_0}^t$ of a point in a third-order thermoelastic material there exists a thermoelastic range such that

- the final value of the process lies in $\mathscr{E}_P$,
- the caloro-dynamic state is determined by the thermoelastic laws ψ_P and q_P.

The concepts of Section 5.3 apply in the thermoelastic case as well, especially Assumption 5.1, which postulates isomorphy of the elastic ranges. One therefore obtains

Theorem 6.3. Existence of an thermoelastic reference energy

Let ψ_0 and q_0 be thermoelastic reference laws. For each thermoelastic range $\{\mathscr{E}_P, \psi_P, q_P\}$ a triple $\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{Unim} \times \mathscr{Conf}_3 \times \mathscr{Conf}_4$ and two real constants ε_c, η_c exist such that

$$\psi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}), \theta\Big) \tag{6.64}$$

$$= \psi_0\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big), \theta\Big) - \eta_c\theta + \varepsilon_c$$

$$(6.65)\quad q_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{g}\Big)$$

$$= \overset{\langle 2\rangle}{\mathbf{P}} * q_0\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big), \theta, \overset{\langle 2\rangle}{\mathbf{P}}^T * \mathbf{g}\Big)$$

holds for all $\{\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\} \in \mathscr{Config}$, $\theta \in \mathbb{R}^+, \mathbf{g} \in \mathbb{R}^3$.

In this thermoplastic framework $\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}} \in \mathscr{Unim} \times \mathscr{Conf}_3 \times \mathscr{Conf}_4$ and ε_c, η_c are the internal variables and determine the plastic transformations. Assumption 5.1, which postulates isomorphy of the elastic ranges, implies that ε_c and η_c cannot depend on $\{\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\}$.

6.6 Yield criteria

The concepts in Section 5.4 can be extended in a straightforward manner to the case of thermoplasticity. One just has to extend the set of variables by the temperature θ since no material is known, where the yield limit depends on the gradient of the temperature. Therefore it will be assumed in this section that the elastic range is defined as a tupel $\mathscr{E}_P := \{\mathscr{E}_P, \psi_P\} \subset \mathscr{Config} \times \mathbb{R}^+$.

Definition 6.8. Yield surface & yield criteria

Considering an elastic range $\{\mathscr{E}_P, \psi_P\}$ one decomposes $\mathscr{E}_P$ into its interior $\mathscr{E}_P^{\,0}$ and its boundary $\partial\mathscr{E}_P$. One calls $\partial\mathscr{E}_P$ the **yield surface**. The yield surface is assumed to be smooth enough such that it can be described by a level set function:

$$(6.66)\quad \phi_P : \mathscr{Config} \times \mathbb{R}^+ \to \mathbb{R}\ ,\ \Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) \mapsto \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big),$$

$$(6.67)\quad \partial\mathscr{E}_P = \Big\{\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) \mid \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) = 0\Big\}$$

$$(6.68)\quad \mathscr{E}_P^{\,0} = \Big\{\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) \mid \phi_P\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) < 0\Big\}$$

$$(6.69)\quad \mathscr{Config} \backslash \mathscr{E}_P = \Big\{\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) \mid \phi_P(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta\Big) > 0\Big\}$$

The function ϕ_P is called a **yield criterion** associated to the elastic range $\{\mathscr{E}_P, \psi_P\}$. It is usually piecewise differentiable and a material property.

The **yield condition**

$$(6.70) \quad \phi_P = 0$$

and the **loading condition**

$$(6.71) \quad \phi_P^{\bullet} > 0$$

remain the same.

The **hardening variables** will still be denoted by $\mathbf{Z}$, being tensors of any finite order. A **general yield criterion** in the configuration space can then be denoted by

$$(6.72) \quad \phi\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{Z}\Big)$$

During yielding this **general yield criterion** always fulfills

$$(6.73) \quad \phi\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{Z}\Big) = 0$$

which implies during yielding

$$(6.74) \quad \phi^{\bullet}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{Z}\Big) = 0$$

The **general loading condition** thus becomes

$$(6.75) \quad 0 < (\partial_{\mathbf{C}}\phi) : \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} + (\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\phi) \vdots \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + (\partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\phi) :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}) + (\partial_{\theta}\phi)\theta^{\bullet}$$

6.7 Flow and hardening rules

The rate-independent flow and hardening rules are obtained as in Section 5.6 with the only difference, that they depend on the temperature. The assumption of **rate-independency** lead to the introduction of a **plastic consistency parameter** $\lambda \geq 0$. It scales the functions that determine the direction of flow and hardening: $\overset{\langle 2\rangle}{h}^{\circ}$, $\overset{\langle 3\rangle}{h}^{\circ}$ and $\overset{\langle 4\rangle}{h}^{\circ}$. This approach now yields

$$\overset{\langle 2\rangle}{\mathbf{P}}^{\bullet} = \lambda\, \overset{\langle 2\rangle}{h}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \tag{6.76}$$

$$\overset{\langle 3\rangle}{\mathbf{P}}^{\bullet} = \lambda\, \overset{\langle 3\rangle}{h}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \tag{6.77}$$

$$\overset{\langle 4\rangle}{\mathbf{P}}^{\bullet} = \lambda\, \overset{\langle 4\rangle}{h}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \tag{6.78}$$

$$\mathbf{Z}^{\bullet} = \lambda\, h^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \overset{\langle 2\rangle}{\mathbf{C}}^{\circ}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \tag{6.79}$$

with a norming factor of the independent variables

$$\mu := \sqrt{\|\overset{\langle 2\rangle}{\mathbf{C}}^{\bullet}\|^2 + L^2\|\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}\|^2 + \bar{L}^2\|\mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet})\|^2 + \|\theta^{\bullet}\|^2/\theta_0^2} \tag{6.80}$$

and

$$\overset{\langle 2\rangle}{\mathbf{C}}^{\circ} := \frac{1}{\mu}\overset{\langle 2\rangle}{\mathbf{C}}^{\bullet}, \quad \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\circ} := \frac{1}{\mu}\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}, \quad \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ} := \frac{1}{\mu}\mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}), \quad \theta^{\circ} := \frac{1}{\mu}\theta^{\bullet}. \tag{6.81}$$

Here θ_0 is a reference temperature that can be chosen freely. The constants L with dimension of a length and $\bar{L}$ with dimension of a squared length are constants that need to be introduced for dimensional consistency. They determine the influence that $\overset{\langle 2\rangle}{\mathbf{C}}^{\bullet}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet})$ have on yielding. During a purely elastic process the consistency parameter must be zero. During yielding the consistency parameter does not vanish, and it can be calculated by inserting Equations (6.76)-(6.79) into the yield condition, which now takes the form

$$0 = \phi^{\bullet}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, \mathbf{Z}\Big) \tag{6.82}$$

$$\begin{aligned} &= (\partial_{\mathbf{C}}\phi) : \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} + (\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\phi) \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + (\partial_{\mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\phi) :: \mathit{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}) + (\partial_{\theta}\phi)\theta^{\bullet} \\ &\quad + (\partial_{\overset{\langle 2\rangle}{\mathbf{P}}}\phi) : \overset{\langle 2\rangle}{\mathbf{P}}^{\bullet} + (\partial_{\overset{\langle 3\rangle}{\mathbf{P}}}\phi) \,\vdots\, \overset{\langle 3\rangle}{\mathbf{P}}^{\bullet} + (\partial_{\overset{\langle 4\rangle}{\mathbf{P}}}\phi) :: \overset{\langle 4\rangle}{\mathbf{P}}^{\bullet} + \Big\langle \partial_{\mathbf{Z}}\phi, \mathbf{Z}^{\bullet}\Big\rangle \end{aligned} \tag{6.83}$$

$$
\begin{aligned}
(6.84)\qquad &= (\partial_{\mathbf{C}}\phi) : {\overset{\langle 2\rangle}{\mathbf{C}}}^{\bullet} + (\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\phi) \vdots\, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\bullet}_{\mathbf{F}} + (\partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\phi) :: Grad({\overset{\langle 3\rangle}{\mathbf{K}}}^{\bullet}_{\mathbf{F}}) + (\partial_\theta\phi)\theta^{\bullet} \\
&\quad + (\partial_{\overset{\langle 2\rangle}{\mathbf{P}}}\phi) : \lambda\, {\overset{\langle 2\rangle}{h}}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \\
&\quad + (\partial_{\overset{\langle 3\rangle}{\mathbf{P}}}\phi) \vdots\, \lambda\, {\overset{\langle 3\rangle}{h}}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \\
&\quad + (\partial_{\overset{\langle 4\rangle}{\mathbf{P}}}\phi) :: \lambda\, {\overset{\langle 4\rangle}{h}}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \\
&\quad + \Big\langle (\partial_{\mathbf{Z}}\phi), \lambda h^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big)\Big\rangle
\end{aligned}
$$

Rearranging then yields

$$
\begin{aligned}
(6.85)\qquad & \lambda\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\bullet}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\bullet}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\bullet}, \theta^{\bullet}\Big) \\
&= -\Bigg((\partial_{\mathbf{C}}\phi) : {\overset{\langle 2\rangle}{\mathbf{C}}}^{\bullet} + (\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\phi) \vdots\, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\bullet}_{\mathbf{F}} + (\partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\phi) :: Grad({\overset{\langle 3\rangle}{\mathbf{K}}}^{\bullet}_{\mathbf{F}}) + (\partial_\theta\phi)\theta^{\bullet}\Bigg) / \\
&\quad \Bigg(+ (\partial_{\overset{\langle 2\rangle}{\mathbf{P}}}\phi) : \lambda\, {\overset{\langle 2\rangle}{h}}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \\
&\quad + (\partial_{\overset{\langle 3\rangle}{\mathbf{P}}}\phi) \vdots\, \lambda\, {\overset{\langle 3\rangle}{h}}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \\
&\quad + (\partial_{\overset{\langle 4\rangle}{\mathbf{P}}}\phi) :: \lambda\, {\overset{\langle 4\rangle}{h}}^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big) \\
&\quad + \Big\langle (\partial_{\mathbf{Z}}\phi), \lambda h^{\circ}\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}), \theta, {\overset{\langle 2\rangle}{\mathbf{C}}}^{\circ}, {\overset{\langle 3\rangle}{\mathbf{K}}}^{\circ}_{\mathbf{F}}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{\circ}, \theta^{\circ}\Big)\Big\rangle \Bigg)
\end{aligned}
$$

where in contrast to Section 5.6 λ now also depends on the temperature and its time derivative. The **consistent flow and hardening rules** and the Kuhn-Tucker condition hold in the thermoplastic case as well.

6.8 Thermodynamic consistency

The constants ε_c and η_c can depend on the internal variables $\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}$ since these are constant during purely elastic processes. Using the notation from Theorem 5.3 one can thus write

$$(6.86)\quad \psi = \psi_0\Big(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}), \theta\Big) + \underbrace{\varepsilon_c\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}\Big) - \theta\eta_c\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}\Big)}_{=:\psi_c\left(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}, \theta\right)}$$

$$(6.87)\quad \varepsilon = \varepsilon_0\Big(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}), \theta\Big) + \varepsilon_c\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}\Big)$$

$$(6.88)\quad \eta = \eta_0\Big(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}), \theta\Big) + \eta_c\Big(\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}, \mathbf{Z}\Big)$$

where ψ_c is the plastic part of the free energy.

Therefore the time derivative of the free energy can be written as

$$\begin{aligned}(6.89)\quad \psi^\bullet =& \partial_{\mathbf{C}_e}\psi_0 : \mathbf{C}_e^\bullet + \partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}}\psi_0 \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}^{\bullet} + \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\psi_0 : Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^\bullet + \partial_\theta\psi_0\theta^\bullet \\ &+ \partial_{\overset{\langle 2\rangle}{\mathbf{P}}}\psi_c : \overset{\langle 2\rangle}{\mathbf{P}}^\bullet + \partial_{\overset{\langle 3\rangle}{\mathbf{P}}}\psi_c \,\vdots\, \overset{\langle 3\rangle}{\mathbf{P}}^\bullet + \partial_{\overset{\langle 4\rangle}{\mathbf{P}}}\psi_c :: \overset{\langle 4\rangle}{\mathbf{P}}^\bullet + \partial_{\mathbf{Z}}\psi_c : \mathbf{Z}^\bullet + \underbrace{\partial_\theta\psi_c}_{=-\eta_c}\theta^\bullet\end{aligned}$$

Due to the subsymmetries of $\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ only those parts of $\partial_{\mathbf{C}_e}\psi_0, \partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}}\psi_0, \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\psi_0$ with the same subsymmetries enter Equation (6.89). Thus it can be assumed that

$$(6.90)\quad \{\partial_{\mathbf{C}_e}\psi_0, \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\psi_0, \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\psi_0\} \in \mathscr{Config}$$

Now one can apply the approach and notation from Section 5.5 and Appendix C.2 (but with a pushforward rather than a pullback) to eliminate $(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}))$ in Equation (6.89). Thus one can write

$$\begin{aligned}(6.91)\quad \psi^\bullet =& (\overset{\langle 2\rangle}{\mathbf{P}} * \partial_{\mathbf{C}_e}\psi_0) : \mathbf{C}^\bullet \\ &+ \gamma(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}}\psi_0, \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\psi_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}) \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^\bullet \\ &+ (\overset{\langle 2\rangle}{\mathbf{P}} \circ \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\psi_0) :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^\bullet \\ &+ (\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P'}} + \partial_{\overset{\langle 2\rangle}{\mathbf{P}}}\psi_c) : \overset{\langle 2\rangle}{\mathbf{P}}^\bullet + (\overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P'}} + \partial_{\overset{\langle 3\rangle}{\mathbf{P}}}\psi_c) \,\vdots\, \overset{\langle 3\rangle}{\mathbf{P}}^\bullet + (\overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P'}} + \partial_{\overset{\langle 4\rangle}{\mathbf{P}}}\psi_c) :: \overset{\langle 4\rangle}{\mathbf{P}}^\bullet\end{aligned}$$

$$+\,\partial_{\mathbf{Z}}\psi_c : \mathbf{Z}^{\bullet} + (\partial_\theta\psi_0 - \eta_c)\theta^{\bullet}$$

with

$$
\begin{aligned}
(6.92)\quad \overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P'}} =\, & 2sym[\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} \cdot \partial_{\mathbf{C}}\psi \cdot \mathbf{C}] \\
& + 2\overset{[1,2]}{sym}\left[\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 2\rangle}{\mathbf{P}}\right] : \gamma(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi, \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \\
& - \big((\overset{\langle 2\rangle}{\mathbf{P}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) : \gamma(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi, \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{[1,2]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1}\big)^{T} \\
& - \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \cdot (\overset{\langle 2\rangle}{\mathbf{P}} \circ \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi) \,\vdots\, [\overset{\langle 2\rangle}{\mathbf{P}}{}^{T} \circ Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})]^{[1,4]} \\
& + [(\partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T})^{[1,2]} \,\vdots\, Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})]^{T} \\
& + 2\overset{[1,2]}{sym}\left[Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\right] \vdots\, \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \\
& + \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi \,\vdots\, (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}})^{[1,3]} \cdot (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{[1,3]} \\
& + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}{}^{[1,2]} : \big((\partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T})^{[3,4]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T}\big) : \overset{\langle 3\rangle}{\mathbf{P}}{}^{[1,3]} \\
& + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}{}^{[1,3]} : \big((\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}}) : \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T}\big) \\
& + (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \cdot \overset{\langle 2\rangle}{\mathbf{P}} \cdot \overset{\langle 3\rangle}{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1})^{[1,2]} : \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}{}^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \\
& - 2\big((\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\big)^{[1,2]} \,\vdots\, \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \\
(6.93)\quad \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P'}} =\, & \Big[\gamma(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\psi, \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) + 2(\overset{\langle 2\rangle}{\mathbf{P}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) : (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} \circ \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi) \\
& - (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} \circ \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi) : (\overset{\langle 2\rangle}{\mathbf{P}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^{[1,3]}\Big] \\
(6.94)\quad \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P'}} =\, & \Big[\overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} \circ \partial_{Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}\psi\Big]
\end{aligned}
$$

Now one substitutes Equation (6.91) and the definition of the stress power in to the Clausius-

Duhem inequality (6.5) to obtain

$$
\begin{aligned}
(6.95) \quad 0 \geq & \Big(-\frac{1}{2\rho_0} \overset{\langle 2\rangle}{\mathbf{S}} + \overset{\langle 2\rangle}{\mathbf{P}} * \partial_{\mathbf{C}_e}\psi_0\Big) : \mathbf{C}^\bullet \\
& + \Big(-\frac{1}{\rho_0} \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} + \gamma\big(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}} \psi_0, \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})} \psi_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}\big)\Big) \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}}^{\bullet} \\
& + \Big(-\frac{1}{\rho_0} \overset{\langle 4\rangle}{\mathbf{S}} + \overset{\langle 2\rangle}{\mathbf{P}} \circ \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})} \psi_0\Big) :: Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})^\bullet + \frac{1}{\rho_0\theta}\mathbf{q}_0\mathbf{g}_0 \\
& + \big(\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}'} + \partial_{\overset{\langle 2\rangle}{\mathbf{P}}}\psi_c\big) : \overset{\langle 2\rangle}{\mathbf{P}}{}^{\bullet} + \big(\overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}'} + \partial_{\overset{\langle 3\rangle}{\mathbf{P}}}\psi_c\big) \vdots \overset{\langle 3\rangle}{\mathbf{P}}{}^{\bullet} + \big(\overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}'} + \partial_{\overset{\langle 4\rangle}{\mathbf{P}}}\psi_c\big) :: \overset{\langle 4\rangle}{\mathbf{P}}{}^{\bullet} \\
& + \partial_{\mathbf{Z}}\psi_c : \mathbf{Z}^\bullet + (\partial_\theta\psi_0 + \eta_0)\theta^\bullet
\end{aligned}
$$

First one can exploit Equation (6.95) for the case where no yielding takes place, i.e., $\overset{\langle 2\rangle}{\mathbf{P}} = \overset{\langle 2\rangle}{\mathbf{0}}$, $\overset{\langle 3\rangle}{\mathbf{P}} = \overset{\langle 3\rangle}{\mathbf{0}}$, $\overset{\langle 4\rangle}{\mathbf{P}} = \overset{\langle 4\rangle}{\mathbf{0}}$, $\mathbf{Z}^\bullet = \overset{\langle n\rangle}{\mathbf{0}}$. This allows to deduce the thermoelastic relations as in Theorem 6.1.

$$
(6.96) \quad \overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{\mathbf{P}} * \partial_{\mathbf{C}_e}\psi_0
$$

$$
(6.97) \quad \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} = \gamma\big(\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}} \psi_0, \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})} \psi_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}\big)
$$

$$
(6.98) \quad \overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{\mathbf{P}} \circ \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})} \psi_0
$$

$$
(6.99) \quad \eta_0 = -\partial_\theta\psi_0
$$

$$
(6.100) \quad 0 \geq \mathbf{g} \cdot \mathbf{q}
$$

Furthermore one can examine the case of yielding, which allows to deduce in addition to the thermoelastic relations the **residual dissipation inequality**

$$
(6.101) \quad 0 \geq \big(\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}'} + \partial_{\overset{\langle 2\rangle}{\mathbf{P}}}\psi_c\big) : \overset{\langle 2\rangle}{\mathbf{P}}{}^{\bullet} + \big(\overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}'} + \partial_{\overset{\langle 3\rangle}{\mathbf{P}}}\psi_c\big) \vdots \overset{\langle 3\rangle}{\mathbf{P}}{}^{\bullet} + \big(\overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}'} + \partial_{\overset{\langle 4\rangle}{\mathbf{P}}}\psi_c\big) :: \overset{\langle 4\rangle}{\mathbf{P}}{}^{\bullet} + \partial_{\mathbf{Z}}\psi_c : \mathbf{Z}^\bullet
$$

which pose a restriction on the flow and hardening rule. An important consequence, that was already pointed out in [Bertram 2005] for the classic case, is that yield against the stress is possible. One can formulate these results as

Theorem 6.4.

For a thermoplastic third-order material the Clausius-Duhem inequality (6.5) is fulfilled for every thermo-kinematical process if and only if the free energy does not depend on the gradient of the temperature and if conditions (6.96)-(6.100) as well as (6.101) are fulfilled.

This implies for the stress power:

$$(6.102)\quad p = \underbrace{\partial_{\mathbf{C}_e}\psi_0 : \mathbf{C}_e^\bullet + \partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}}\psi_0 \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_e}^\bullet + \partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\psi_0 :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^\bullet}_{=\psi_0^\bullet+\eta_0\theta^\bullet} - \overset{\langle 2\rangle}{\mathbf{S}}_{P'} : \overset{\langle 2\rangle}{\mathbf{P}}{}^\bullet - \overset{\langle 3\rangle}{\mathbf{S}}_{P'} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{P}}{}^\bullet - \overset{\langle 4\rangle}{\mathbf{S}}_{P'} :: \overset{\langle 4\rangle}{\mathbf{P}}{}^\bullet$$

Similar to the case of plastic dissipation in Section 5.5 on can observe a split of the stress power into a part that is stored in ψ_0 and part that is dissipated, working on $\overset{\langle 2\rangle}{\mathbf{P}}{}^\bullet, \overset{\langle 3\rangle}{\mathbf{P}}{}^\bullet, \overset{\langle 4\rangle}{\mathbf{P}}{}^\bullet$ during yielding.

6.9 Changes of the temperature

Starting from the first law of thermodynamics (6.4) one obtains for the heat supply Q

$$(6.103)\quad Q = \varepsilon^\bullet - p$$

Using equations (6.87) and (6.102) yields

$$(6.104)\quad = \varepsilon_0^\bullet + \varepsilon_c^\bullet - \psi_0^\bullet - \eta_0\theta^\bullet + \overset{\langle 2\rangle}{\mathbf{S}}_{P'} : \overset{\langle 2\rangle}{\mathbf{P}}{}^\bullet + \overset{\langle 3\rangle}{\mathbf{S}}_{P'} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{P}}{}^\bullet + \overset{\langle 4\rangle}{\mathbf{S}}_{P'} :: \overset{\langle 4\rangle}{\mathbf{P}}{}^\bullet$$

Now one uses the definition of the Helmholtz free energy and obtains

$$(6.105)\quad = \theta\eta_0^\bullet + \varepsilon_c^\bullet + \overset{\langle 2\rangle}{\mathbf{S}}_{P'} : \overset{\langle 2\rangle}{\mathbf{P}}{}^\bullet + \overset{\langle 3\rangle}{\mathbf{S}}_{P'} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{P}}{}^\bullet + \overset{\langle 4\rangle}{\mathbf{S}}_{P'} :: \overset{\langle 4\rangle}{\mathbf{P}}{}^\bullet$$

This allows to define the **thermoelastic heat generation** as

$$(6.106)\quad Q_e := \theta\eta_0^\bullet = -\Big(\overset{\langle 2\rangle}{\mathbf{R}} : \mathbf{C}_e^\bullet + \overset{\langle 3\rangle}{\mathbf{R}} \,\vdots\, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_e}^\bullet + \overset{\langle 4\rangle}{\mathbf{R}} :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^\bullet\Big)\theta + c\theta^\bullet$$

which contains the definitions of

- **stress-temperature tensors** $\overset{\langle 2\rangle}{\mathbf{R}}, \overset{\langle 3\rangle}{\mathbf{R}}, \overset{\langle 4\rangle}{\mathbf{R}}$
- and the **specific heat** c

with

$$\overset{\langle 2\rangle}{\mathbf{R}}\left(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}), \theta\right) := -\partial_{\mathbf{C}_e}\eta_0 \tag{6.107}$$

$$\overset{\langle 3\rangle}{\mathbf{R}}\left(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}), \theta\right) := -\partial_{\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}}\eta_0 \tag{6.108}$$

$$\overset{\langle 4\rangle}{\mathbf{R}}\left(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}), \theta\right) := -\partial_{Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})}\eta_0 \tag{6.109}$$

$$c\left(\mathbf{C}_e, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}, Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e}), \theta\right) := \theta\partial_\theta\eta_0 \tag{6.110}$$

Furthermore one can define the **plastic heat generation** as

$$Q_P := \varepsilon_c^\bullet + \overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}'} : \overset{\langle 2\rangle}{\mathbf{P}}^\bullet + \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}'} \vdots \overset{\langle 3\rangle}{\mathbf{P}}^\bullet + \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}'} :: \overset{\langle 4\rangle}{\mathbf{P}}^\bullet \tag{6.111}$$

An equation that describes the change of the temperature can now be obtained by rearranging Equation (6.105).

$$\begin{aligned} \theta^\bullet = \frac{1}{c}\Big[Q + \Big(& \overset{\langle 2\rangle}{\mathbf{R}} : \mathbf{C}_e^\bullet + \overset{\langle 3\rangle}{\mathbf{R}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^\bullet_{\mathbf{F}_e} + \overset{\langle 4\rangle}{\mathbf{R}} :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^\bullet\Big)\theta \\ &- \varepsilon_c^\bullet - \overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}'} : \overset{\langle 2\rangle}{\mathbf{P}}^\bullet - \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}'} \vdots \overset{\langle 3\rangle}{\mathbf{P}}^\bullet - \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}'} :: \overset{\langle 4\rangle}{\mathbf{P}}^\bullet \Big]. \end{aligned} \tag{6.112}$$

Equation (6.112) allows to obtain the temperature at the end of an elasto-plastic process by integrating the time derivative of the temperature along the process. One sees that changes of the temperature can be caused by heat supply through irradiation and conduction, this change is described by Q. Temperature can also change through thermoelastic deformations, this change is then described by

$$\Big(\overset{\langle 2\rangle}{\mathbf{R}} : \mathbf{C}_e^\bullet + \overset{\langle 3\rangle}{\mathbf{R}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^\bullet_{\mathbf{F}_e} + \overset{\langle 4\rangle}{\mathbf{R}} :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_e})^\bullet\Big)\theta \tag{6.113}$$

Furthermore temperature can change through the heat Q_P generated during yielding and hardening, governed by the flow and hardening rules (6.76)-(6.79).

Chapter 7

Finite element analysis of polyhedra under point and line forces in second strain gradient elasticity

7.1 Chapter Introduction

The majority of results in this chapter has been published under the same title in colaboration with Ivan Giorgio [1] in [Reiher, Giorgio, Bertram 2016]. The concepts of point forces and force distributions along lines are often used in mechanics. However, the common approach of a Cauchy continuum, i.e., a continuum equipped with an elastic energy that depends on the gradient of the displacement, cannot sustain such point and line forces. A prescription of such boundary conditions along lines or on points results in singularities of the displacement field. If one wants to marry the idea of a continuum with that of point and line forces (or point and line displacements) one has to generalize the concept of the Cauchy continuum. Extending the elastic energy of the continuum to second and third gradients of the displacement clearly lends itself to this purpose. From Mindlin's and Germain's work [Mindlin 1965, Germain 1973] and further contributions that build upon these ideas such as [Javili et al. 2013, Alibert et al. 2003, Seppecher et al. 2011, dell'Isola et. al. 2015, Carcaterra et al. 2015] it is very clear why the introduction of the first and second strain gradient allow a continuum to sustain boundary conditions on vertices and edges of a body. In this chapter a finite element approach is presented, that allows the integration of displacement gradients up to the order of three. This method is then used to examine how different polyhedrons react to line and point displacements applied to their edges and corners. This is done for small deformations such that there is no need to distinguish a reference and a momentary placement. The notation in this chapter deviates from the other chapters to underline the fact that only a very special case is considered.

7.2 Implementation in a FEM software

Standard finite element methods are designed for application in first-order problems. Derivatives of order greater than one should be avoided. Therefore a Hellinger-Reissner type variational principle has been applied. Such an approach has been used for implementing strain gradient theories with the FEM in publications before, e.g. see [Cordero et al. 2011] and the

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references therein. This allows to bring the problem into a form that is suitable for a standard FEM environment. The key idea is to introduce additional kinematical fields and Lagrange multipliers. These allow a formulation where the elastic energy only depends on first-order derivatives of the variables. This approach increases the number of kinematical descriptors but it is possible to use lower-order polynomials as interpolating shape functions. In detail, micromorphic tensors are introduced in the standard FEM code:

(7.1) $\overset{\langle 2\rangle}{\mathbf{Q}}$, which is constrained to be $grad\ \mathbf{u}$ and

(7.2) $\overset{\langle 3\rangle}{\mathbf{Q}}$, which is constrained to be $grad\ \overset{\langle 2\rangle}{\mathbf{Q}}$

as well as micromorphic constraints, which are introduced by using Lagrange multipliers $\mathbf{\Lambda_1}$ and $\mathbf{\Lambda_2}$ (both are tensors). The **third gradient elastic energy** for the variables $\mathbf{E} := sym[grad(\mathbf{u})]$, $\overset{\langle 2\rangle}{\mathbf{Q}}$ and $\overset{\langle 3\rangle}{\mathbf{Q}}$, and under the assumption of a homogenous and isotropic material, is defined as

$$W_{grad^3}(\mathbf{E}, \overset{\langle 2\rangle}{\mathbf{Q}}, \overset{\langle 3\rangle}{\mathbf{Q}}) := \underbrace{\frac{1}{2}\Big(2\mu\mathbf{E} : \mathbf{E} + \lambda\ tr(\mathbf{E})^2\Big)}_{=:W_I(\mathbf{E})} + \underbrace{\frac{1}{2}\lambda_1\ grad(\overset{\langle 2\rangle}{\mathbf{Q}}) \vdots grad(\overset{\langle 2\rangle}{\mathbf{Q}})}_{=:W_{II}(\overset{\langle 2\rangle}{\mathbf{Q}})}$$
$$+ \underbrace{\frac{1}{2}\lambda_2\ grad(\overset{\langle 3\rangle}{\mathbf{Q}}) :: grad(\overset{\langle 3\rangle}{\mathbf{Q}})}_{=:W_{III}(\overset{\langle 3\rangle}{\mathbf{Q}})}$$
$$+ \mathbf{\Lambda}_1 \vdots \Big(\overset{\langle 2\rangle}{\mathbf{Q}} - grad(\mathbf{u})\Big) + \mathbf{\Lambda_2} :: \Big(\overset{\langle 3\rangle}{\mathbf{Q}} - grad(\overset{\langle 2\rangle}{\mathbf{Q}})\Big) \tag{7.3}$$

λ and μ are the Lamé parameters. λ_1 and λ_2 can be regarded as the second gradient stiffness and the third gradient stiffness, respectively.
The **second gradient elastic energy** for the variables $\mathbf{E}$, $\overset{\langle 2\rangle}{\mathbf{Q}}$ is specified as

$$W_{grad^2}(\mathbf{E}, \overset{\langle 2\rangle}{\mathbf{Q}}) := W_I(\mathbf{E}) + W_{II}(\overset{\langle 2\rangle}{\mathbf{Q}}) + \mathbf{\Lambda}_1 \vdots (\overset{\langle 2\rangle}{\mathbf{Q}} - grad\ \mathbf{u}) \tag{7.4}$$

The energies have been brought into a dimensionless form by choosing a reference length scale l_{ref} and a reference Lamé constant λ_{ref}. This means that the real material constants are related to the reference constants:

$$\lambda = \frac{\lambda_{real}}{\lambda_{ref}},\ \mu = \frac{\mu_{real}}{\lambda_{ref}},\ \lambda_1 = \frac{\lambda 1_{real}}{\lambda_{ref} f^{2}_{ref}},\ \lambda_2 = \frac{\lambda 2_{real}}{\lambda_{ref} f^{4}_{ref}}. \tag{7.5}$$

The problem of solving the boundary value problem of a second or third gradient material now reduces to the variational problem of finding an extremum of W_{grad^3} or W_{grad^2} on a certain set of shape functions subject to constraints. For all simulations $\lambda = 1$ and $\mu = 0.08$ has been set. Except for the section with the parameter study on λ_1 and λ_2 we set $\lambda_1 = 0.04$ and $\lambda_2 = 0.0015$. This choice is purely academic and based on dimensional reasoning. It ensures that the gradient effects occur in a boundary layer of size $\ell = 0.2\, l_{ref}$ which can be captured by the meshes that have been used.

The software package COMSOL Multiphysics [Comsol 2016] has been used to implement the approach described beforehand. This is done by using the weak form feature of the software that allows the user to enter the variational problem directly. The software solves for the fields u_i, $\overset{\langle 2\rangle}{Q}_{ij}$, $\overset{\langle 3\rangle}{Q}_{ijk}$ and the Lagrange multipliers $\Lambda_{1_{ij}}$ and $\Lambda_{2_{ijk}}$ in the third gradient cases or for the fields u_i, $\overset{\langle 2\rangle}{Q}_{ij}$ and the Lagrange multiplier $\Lambda_{1_{ij}}$ in the second gradient cases. Further Lagrange multipliers are needed to implement displacement boundary conditions. These multipliers are chosen to be quadratic Lagrange shape functions in order to be consistent with the before mentioned assumptions. Since the tools in this approach were not designed for higher gradient problems, in future research more suitable numerical tools could be investigated such as those in [Fischer et al. 2010, Greco & Cuomo 2014, Greco & Cuomo 2016, Cazzani et al. 2015, Cazzani et al. 2014, della Corte et al. 2016]. In what follows the deformation for polyhedrons equipped with the introduced gradient energy is presented. The grayscale scheme in all figures represents the stored elastic energy.

7.3 The tetrahedron

The tetrahedron has been chosen for extensive numerical studies, since it is the simplest polyhedron to demonstrate the effect of the second strain gradient in the elastic energy.

7.3.1 A tetrahedron with one point displacement and one fixed surface

A tetrahedron with each side of length l_{ref} is subjected to a displacement of magnitude $0.05\, l_{ref}$ at the tip to satisfy the small deformation assumption. The displacement vector is orthogonal to the surface opposite to the vertex to which the displacement is applied. In Figure 7.1 one can see, that for the classic first gradient energy the displacement field and the energy are both mesh-dependent. This spurious mesh dependence is apparent from the fact that the induced displacement and energy for any mesh are concentrated in the cell at the tip and vanish everywhere else. The energy density grows unbounded as can be seen on the scale bar on the right of the bodies. In the solutions for the second gradient energy in Figure 7.2 spurious mesh dependence can only be observed for the energy. In contrast to the case with a first-order energy, the displacement field does not show any indication of mesh-dependence. The displacement field clearly tends to a solution with a discontinuity at the tip with a singularity of the elastic energy. Again the spurious mesh dependence of the second gradient energy is apparent from the fact, that it is confined to the finite element at the tip, and that its density maximum grows with each mesh refinement. Thus the solutions tend to a limit where the displacement is continuous but the second gradient energy has a singularity at the tip. Only the solution for a third gradient energy in Figure 7.3 can be regarded as mesh-independent since both the elastic energy and the displacement undergo negligible changes when the mesh is refined.

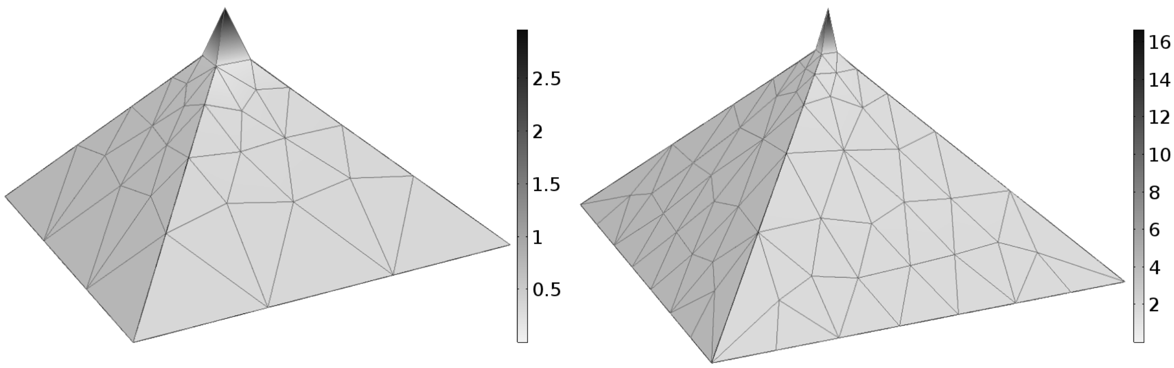

Figure 7.1: Mesh-dependent solution for tip displacement with first-order elastic energy. The mesh on the right has been refined around the tip.

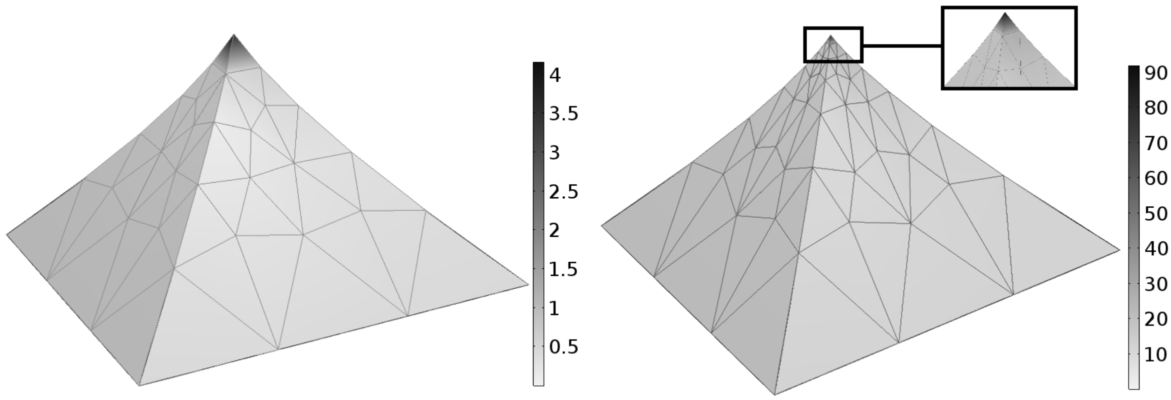

Figure 7.2: Mesh-dependent solution for tip displacement with second gradient elastic energy. The mesh on the right has been refined around the tip.

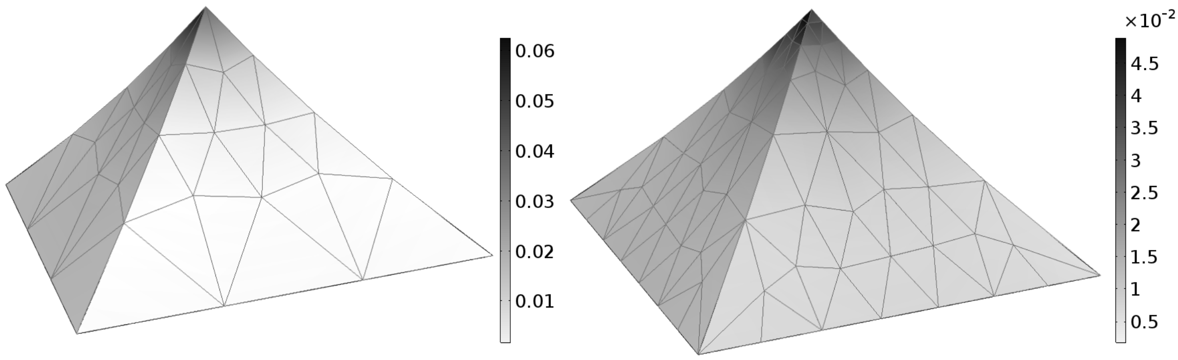

Figure 7.3: Mesh-independent solution for tip displacement with third gradient elastic energy. The mesh on the right has been refined around the tip.

Mesh refinement study for third gradient energy

A mesh refinement study has been conducted for the case of a third gradient energy by applying several tetrahedral meshes. Refinement has mainly been concentrated on the tip where the displacement is prescribed, as shown in Figure 7.4. In Figure 7.5 one can see that the values for the components of the elastic energy vary in a small range as the mesh is refined. These results suggest that a mesh with a number of elements that lies in the middle of the evaluated range is sufficient for further numerical studies.

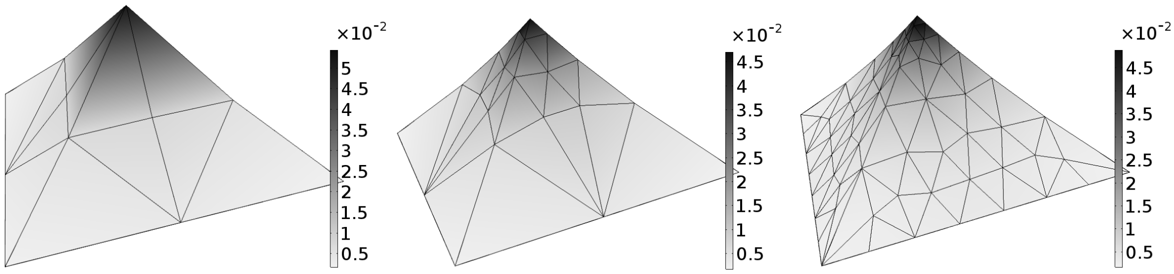

Figure 7.4: Third gradient energy: solutions for meshes with 46, 176 and 340 elements (from left to right)

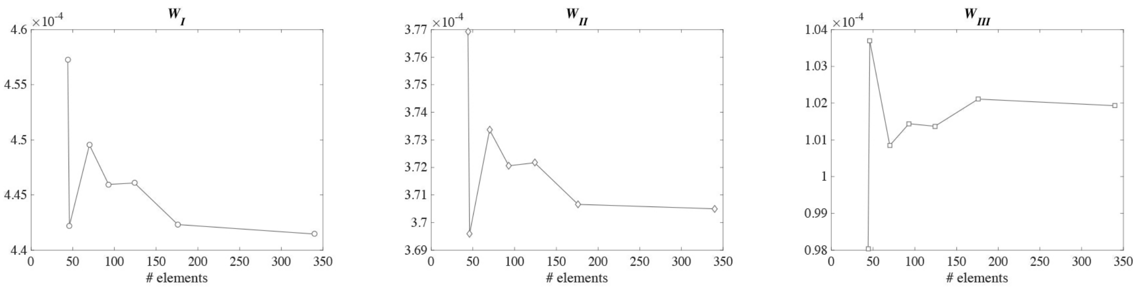

Figure 7.5: Value of the components W_I, W_{II}, W_{III} of the stored elastic third gradient energy W_{grad^3} for meshes with different number of elements

Parameter Study for λ_1 and λ_2

In order to understand the dependence of the third gradient energy on the choice of the dimensionless material parameters λ, λ_1, λ_2, a parameter study has been set up with $\lambda = 1$ and $\mu = 0.08$ as already mentioned. In Figures 7.6–7.9 the dependence of the third gradient energy W_{grad^3} and of its three components (W_I, W_{II} and W_{III}) on the parameter λ_1 and λ_2 is visualized. The plots show that in this case λ_1 has a greater influence on W_{grad^3} than λ_2.

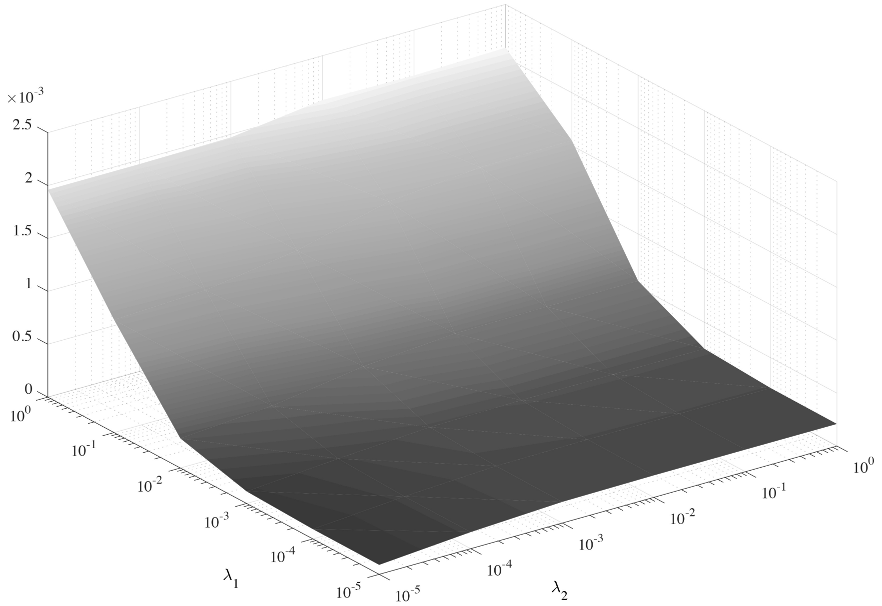

Figure 7.6: The stored elastic third gradient energy W_{grad^3} plotted over different ranges of λ_1 and λ_2. This plot has been created with kind support of Ivan Giorgio for [Reiher, Giorgio, Bertram 2016].

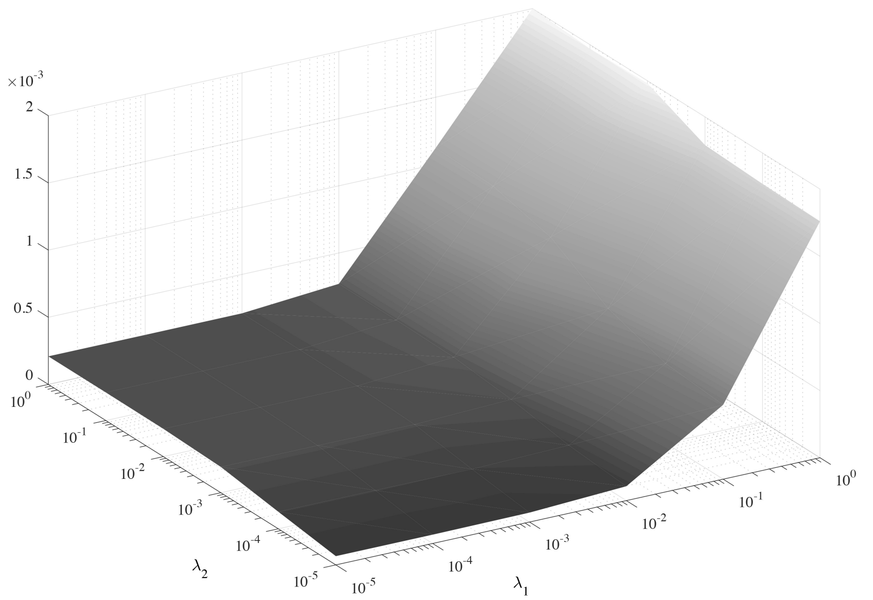

Figure 7.7: The stored energy component W_I plotted over different ranges of λ_1 and λ_2. This plot has been created with kind support of Ivan Giorgio for [Reiher, Giorgio, Bertram 2016].

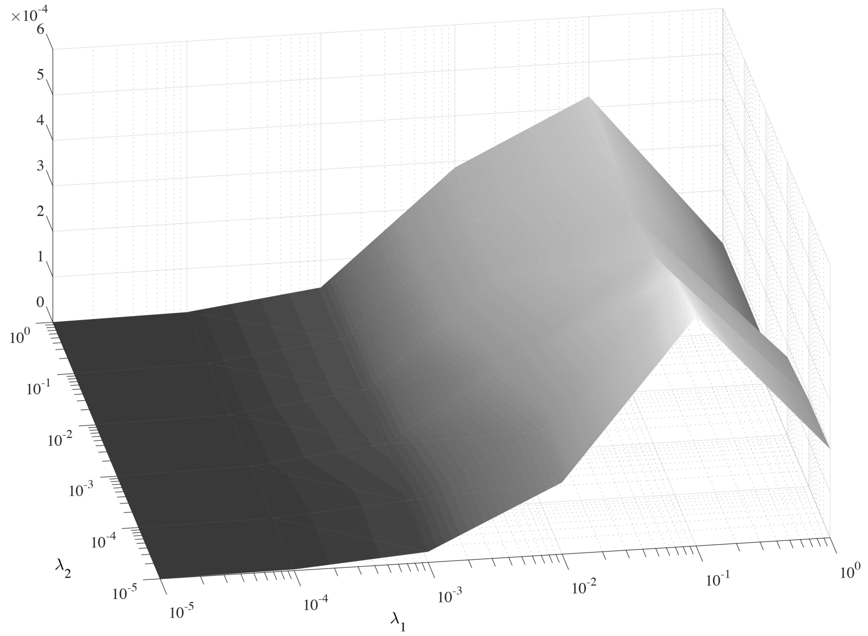

Figure 7.8: The stored energy component W_{II} plotted over different ranges of λ_1 and λ_2. This plot has been created with kind support of Ivan Giorgio for [Reiher, Giorgio, Bertram 2016].

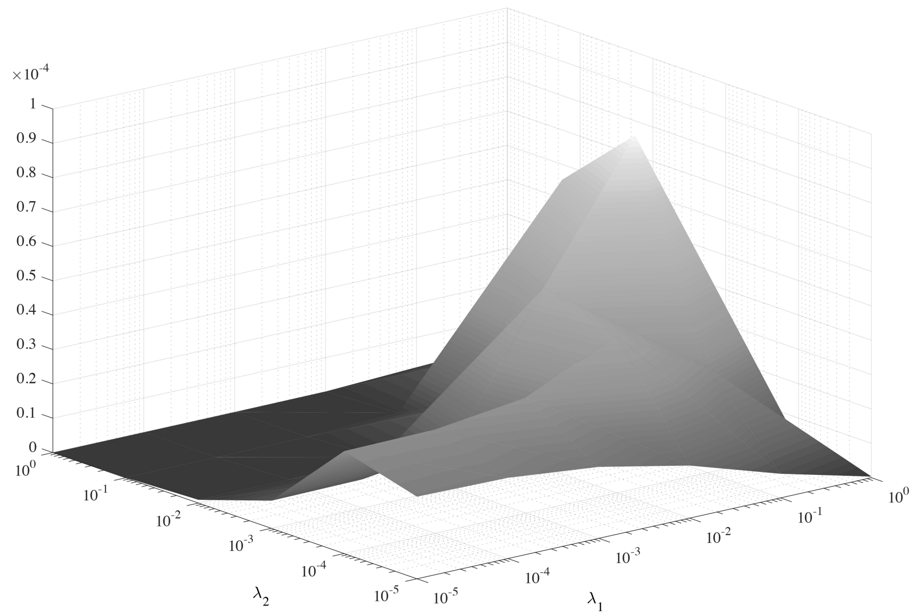

Figure 7.9: The stored energy component W_{III} plotted over different ranges of λ_1 and λ_2. This plot has been created with kind support of Ivan Giorgio for [Reiher, Giorgio, Bertram 2016].

Equivalent forces for a tetrahedron with point displacement

In the case of the prescribed tip displacement for a tetrahedron with a fixed bottom surface, the surface tractions have been calculated. This can be done by evaluating the Lagrange multipliers assigned to the displacement boundary conditions. These values are dimensionless since the energy has been non-dimensionalised. In order to obtain meaningful values in the first gradient energy, the tip displacement has been approximated by the exponential function. For a third gradient energy the force at the tip is 18 times higher than for the third gradient case. (Absolute values of the forces are not of interest in this case since no real material is modeled.) The traction field on the fixed bottom surface for both cases is shown in Figure 7.10. This discrepancy shows that a higher gradient model requires careful calibration in order to render meaningful values. This, however, does not lie within the scope of the present work. The same holds for the traction field that has very different forms for the cases of first- and third-order energies as can been seen in Figure 7.10.

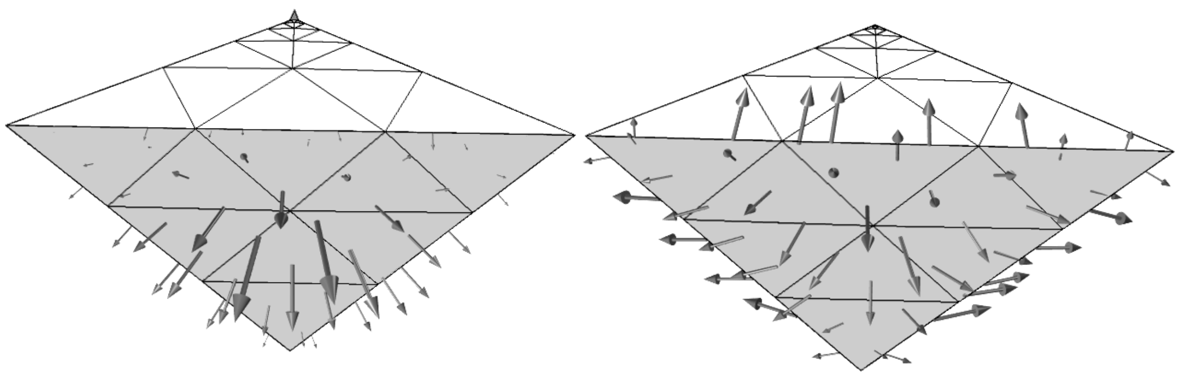

Figure 7.10: Traction fields on the fixed bottom surface for a first gradient (left) and third gradient (right) solution of a tetrahedron with a tip displacement. The arrows on the right plot are magnified by a factor of three compared to the left plot. This figure shows the deformed tetrahedron from the bottom surface where a zero displacement boundary condition has been prescribed. The deformation around the tip of the tetrahedron is hardly recognizable at the top of the figure due to the viewing angle.

7.3.2 A tetrahedron with one line displacement and one fixed surface

The displacements are prescribed along one edge of the tetrahedron. The displacements increase linearly along the edge starting from zero at one point and reaching $0.05\, l_{ref}$ at the other point as depicted in Figure 7.11. The surface at the bottom is fixed. In the case of the first gradient energy the solutions for the displacement in Figure 7.12 are not mesh dependent and tend to a limit that is continuous. The solutions for the energy are mesh-dependent and tend to a line singularity of the energy, i.e., all points on the line with the prescribed displacement become singularities of the energy. This spurious mesh dependence shows that a first gradient energy does not allow the bulk to sustain a line displacement. Figure 7.13 shows, that the solutions for a second gradient energy also tend to a case with no discontinuities in the displacement field. Furthermore Figure 7.13 gives evidence, that in the limit the energy is continuous except for the tip where it has a discontinuity. Along the rest of the line it appears that in the limit the second gradient energy is continuous. Of course it could also be that in the limit all points on the line become discontinuities of the energy. The third gradient energy in Figure 7.14 clearly yields solutions that only vary to a small extend with variations of the mesh. This indicates mesh independence which means that no discontinuities in the displacement or the third gradient energy are present. Therefore it confirms that a third gradient energy allows the bulk to sustain line displacements. Comparing Figure 7.13 and Figure 7.14 the most noticable difference is, that in the second gradient case the energy is focused in a narrower region.

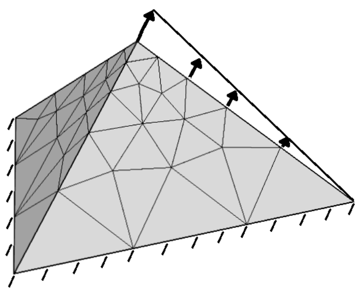

Figure 7.11: Prescribed line displacement along an edge of a tetrahedron

Figure 7.12-Figure 7.14 show the solutions of the tetrahedron for the first, second and third gradient elastic energy with the prescribed line displacement from Figure 7.11.

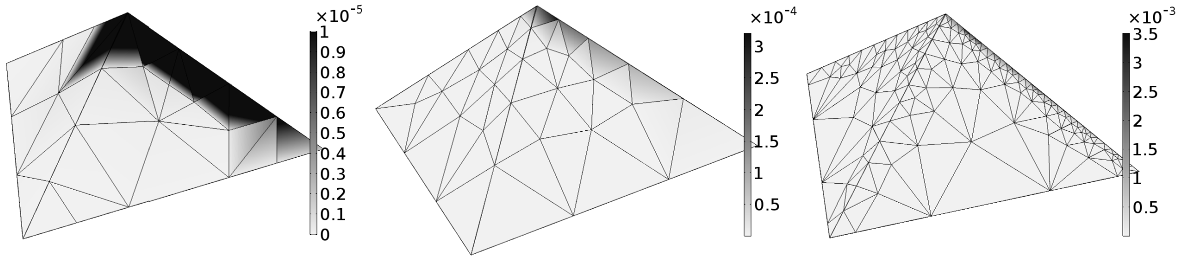

Figure 7.12: Solution for a tetrahedron with prescribed non-constant line displacement and fixed bottom surface equipped with the first gradient energy (mapped by grayscale scheme). From the left to the right the mesh becomes finer around the edge with the prescribed line displacement.

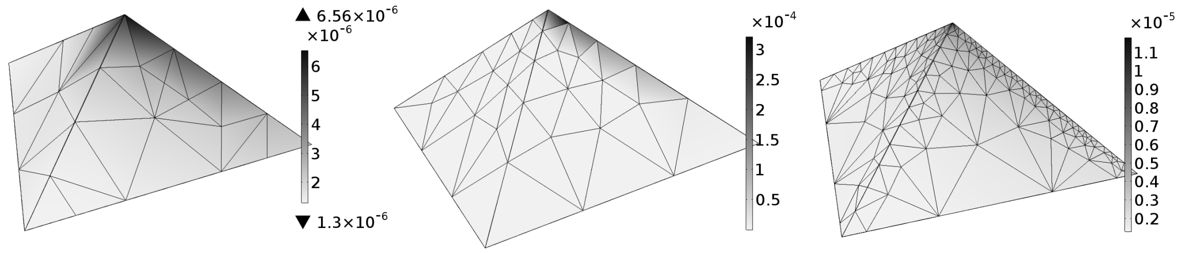

Figure 7.13: Solution for a tetrahedron with the prescribed non-constant line displacement and fixed bottom surface equipped with the second gradient energy (mapped by grayscale scheme). From the left to the right the mesh becomes finer around the edge with the prescribed line displacement.

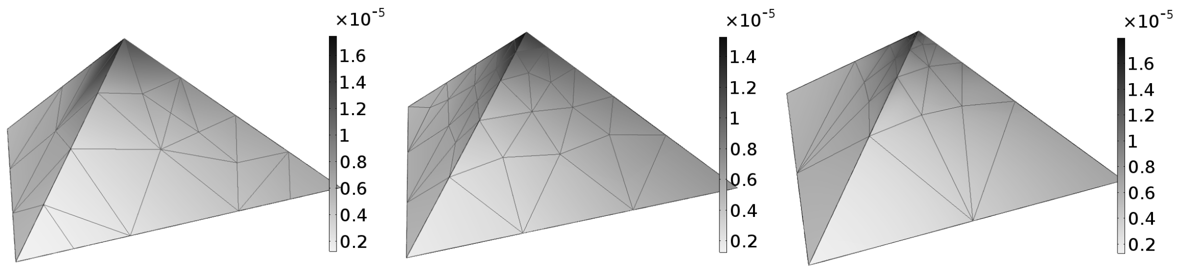

Figure 7.14: Solution for a tetrahedron with prescribed non-constant line displacement and fixed bottom surface equipped with the third gradient energy (mapped by grayscale scheme). From the left to the right the mesh becomes finer around the edge with the prescribed line displacement.

7.4 The cube

7.4.1 A cube with one prescribed, shear-like line displacement and one fixed surface

A cube has been chosen to demonstrate the effect of a line displacement. The bottom of the cube has been fixed and a displacement of $0.05\, l_{ref}$ parallel to the bottom surface is prescribed as illustrated in Figure 7.15.

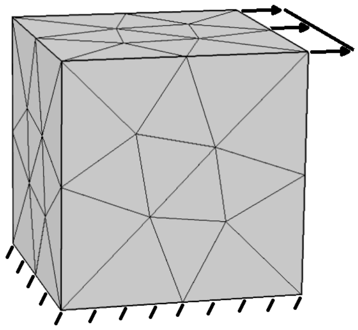

Figure 7.15: A cube with fixed bottom surface and prescribed displacement at one edge

For this geometry the influence of the higher gradients is even more pronounced than for a tetrahedron. The solutions have similar properties as those in Figures 7.12 - 7.14. One can see in Figure 7.16 that the first gradient energy produces solutions that tend to a continuous displacement field with a line singularity of the energy along the edge with the prescribed displacement. The second gradient material in Figure 7.17 yields results, that are similar to those obtained for the tetrahedron in Figure 7.13. In the limit the displacement field is clearly continuous while the second gradient energy has discontinuities at the end points of the edge with the prescribed line displacement. Figure 7.18 shows, that the third gradient material can sustain the line displacement with no singularities or discontinuities, neither in the displacement field nor in the energy.

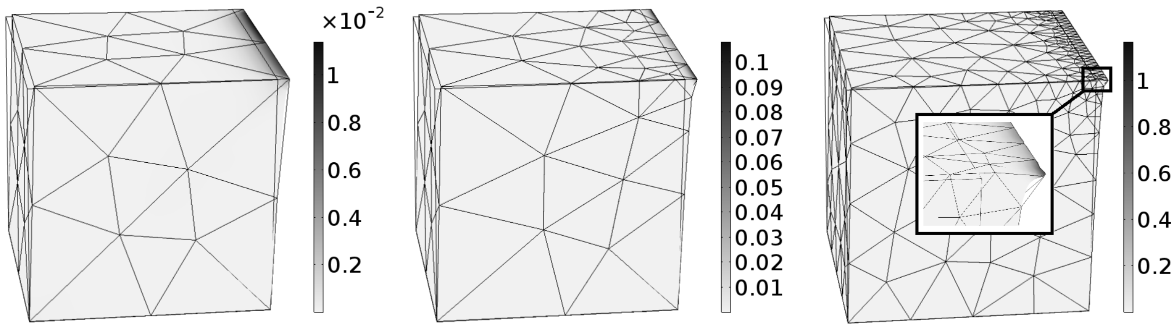

Figure 7.16: A deformed cube with fixed bottom surface and prescribed displacement at one edge with the first gradient energy. From the left to the right the mesh becomes finer around the edge with the prescribed line displacement.

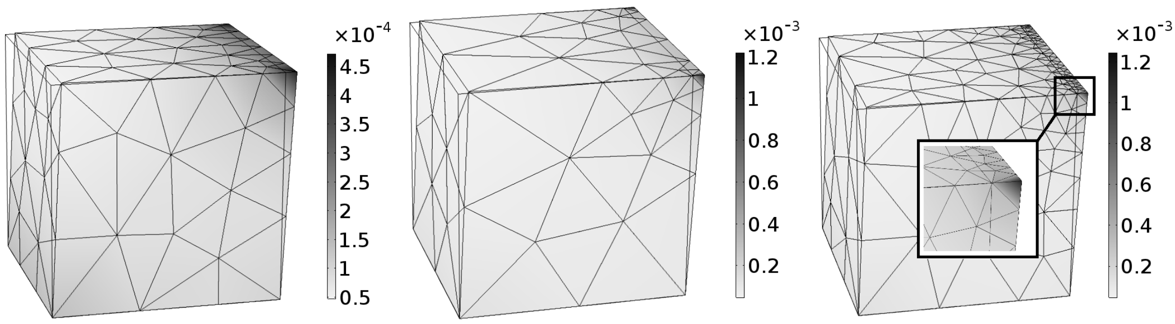

Figure 7.17: A deformed cube with fixed bottom surface and prescribed displacement at one edge with the second gradient energy. From the left to the right the mesh becomes finer around the edge with the prescribed line displacement.

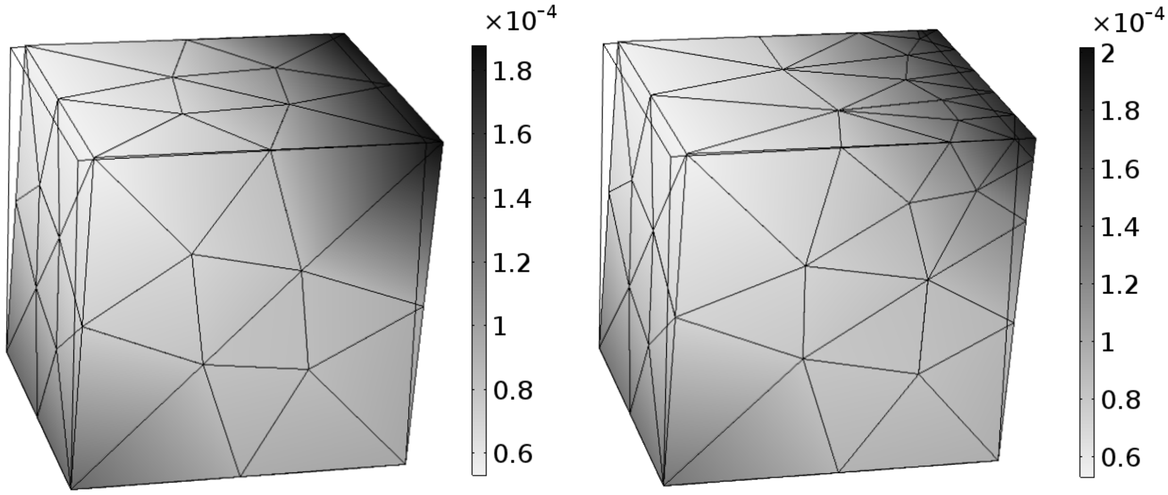

Figure 7.18: A deformed cube with fixed bottom surface and prescribed displacement at one edge with the third gradient energy. From the left to the right the mesh becomes finer around the edge with the prescribed line displacement.

7.4.2 A cube with point displacements

Figure 7.19 and 7.20 show two cases of prescribed point displacements on a cube. On Figure 7.19 a tip displacement has been applied in combination with a zero displacement boundary condition at one surface of the cube. The displacement with magnitude $0.05l_{ref}$ is chosen in direction of the space diagonal, while the fixed surface is in the the xy-plane. In Figure 7.20 displacements of magnitude $0.05l_{ref}$ have been prescribed at four vertices, each displacement in direction of the space diagonal corresponding to the vertex where it is applied. In both cases it is clear that a first-order material leads to a discontinuity in the displacement with a singularity in the energy. A second-order material results in continuous displacements with singularities in the energy. The third-order material clearly yields a continuous solutions for both, the displacement and the energy.

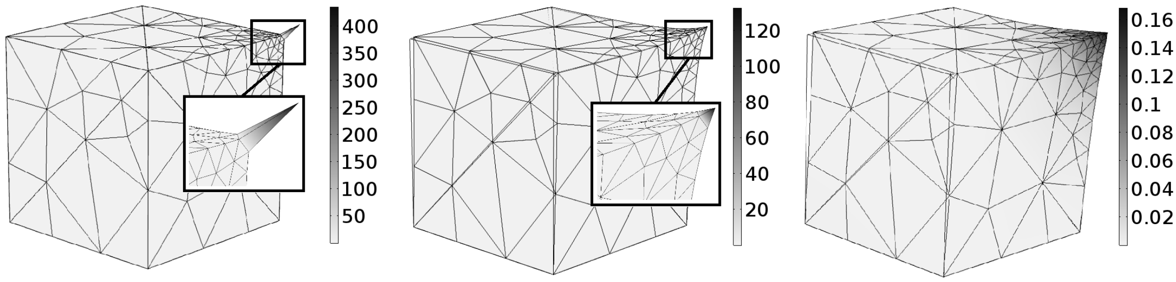

Figure 7.19: From left to right: Plot of the corresponding elastic energy for a first, second, and third-order material

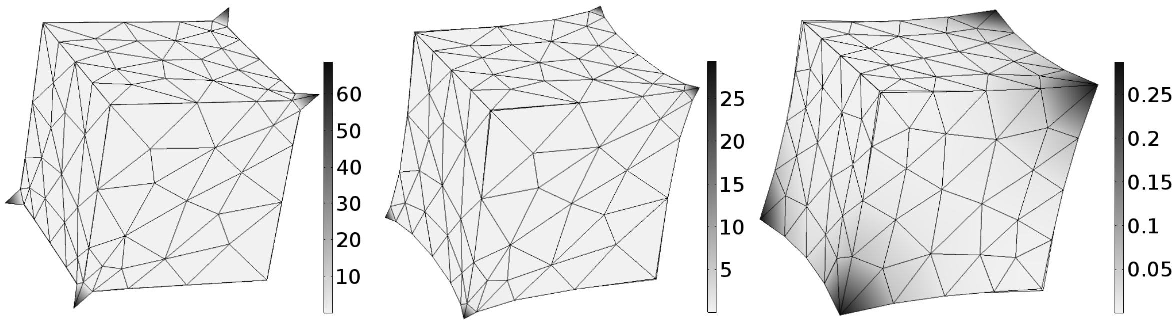

Figure 7.20: From left to right: Plot of the corresponding elastic energy for a first, second, and third-order material

7.5 The cylinder

For a cylinder of radius l_{ref} and height $0.5l_{ref}$ the bottom surface is fixed and at the upper edge a displacement of $0.05l_{ref}$ in direction orthogonal to the flat surface is prescribed (see Figure 7.21).

Here one would expect that the second gradient energy yields a solution without singularities. However Figure 7.21 suggests that this is not true. This case should be investigated with more suitable numeric tools.

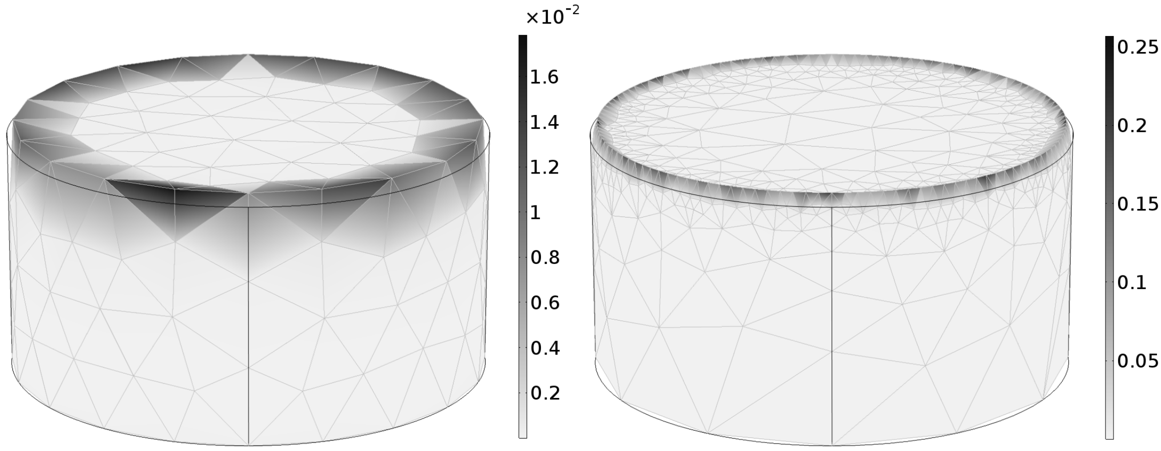

Figure 7.21: Solution of a cylinder with fixed bottom surface and prescribed displacement at the upper edge for the second gradient energy. The grayscale plot maps the (second gradient) elastic energy

7.6 Chapter conclusions

A numerical study has been carried out concerning different geometries and with various boundary conditions for a material with a constitutive laws that involve the second and third gradient of the displacement. The results show that only third gradient materials can sustain point forces while first- and second gradient continua are not able do so. It has been shown that a second gradient material under prescribed line displacements yields a continuous solution for the displacement field with discontinuities or possibly even singularities at vertices. Further research involving more suitable tools is required in the field, e.g. to investigate the case of the cylinder with a line displacement, where unexpected singularities occured.

Chapter 8

Concluding remarks and outlook

The present work develops a material, thermodynamically consistent second strain gradient framework in the spirit of [Svendsen et al. 2009], [Bertram 2014] and [Bertram 2015] for large deformations. (Also referred to as a third-order theory.) One novelty in the third-order theory is that a pair of two equivalent generalized material strain measures of order four exist: $\overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}}$ and $Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})$. In the lower-order theories the (generalized) unique material strain measures can be derived from the internal power. In the literature the material gradients of $\mathbf{C}$ have been suggested as suitable generalized strain measures but the present work shows that $\overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}}$ or $Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})$ are more convenient to handle since they include gradients of the anti-symmetric part of $\mathbf{F}$, which are not included in $Grad(\mathbf{C})$. The tensor $\overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}}$ has turned out to be the most convenient to handle for a material third-order theory.

Another unexpected novelty in the third-order theory is, that the generalized material stress tensor of order three depends on the the generalized spatial stress tensors of order three and four. This leads to a comparatively complicated transformation behavior of the stress and strain measures under changes of the reference placement, which is reflected by the introduction of the transformation functions α, β and γ. From there the further generalization of the concepts in elasticity, elastoplasticity and thermodynamics are straightforward but the derivation of the generalized plastic stress tensors becomes considerably more complicated.

From these facts it becomes evident that with this scheme it is possible to set up a material framework for strain gradients of n-th order. First one has to pull back the spatial velocity gradients and the generalized Cauchy stresses in the stress power. This can be expected to yield material strain measures of the form $\mathbf{F}^{-1} \cdot Grad^n(\mathbf{F})$. Each generalized material stress tensor of order greater than two will probably depend on the generalized Cauchy stress of the same order and maybe of those of higher order. (These conjectures have to be checked for each case though.) Then one has to determine the transformation behavior of the generalized stress and strain measures under changes of the reference placement by introducing further functions that generalize α, β and γ. Once this is done the further steps follow exactly the same scheme as in the present work.

Of course it would be desirable to find a general scheme how to determine a n-th-order framework, i.e., a formula for the generalized stress tensors and for the transformation rules under changes of the reference placement. This will be a very demanding problem to solve which starts with the boundary conditions. So far it has not been possible to derive from the second- and third-order boundary conditions a conjecture for a formula for the n-th-order boundary conditions. Similarly a conjecture for a formula for the generalized material stress tensors of order n seems very hard to pose leaving aside the corresponding transformations under changes of the reference placement. So far one has to calculate all these quantities separately for each n. This leads to the question up to which order it makes sense to work out strain gradient frameworks.

From an academic point of view it would be desirable to know the governing equations of a n-th-order framework to better understand the mathematics of higher strain gradient materials. From a more applied point of view no material models have been applied yet that include second or higher gradients of the strain. The enormous amount of material parameters required by such models is a major disadvantage with respect to complexity.

A strong motivation for introducing higher strain gradients is their ability to model point and line forces on vertices and edges respectively. In the present work it is laid out why no more than two gradients of the strain are required for this task. In the light of these results in seems questionable if the development of frameworks that include more than the first two gradients of the strain is a reasonable aim at this point. Further research on first and second gradient of strain frameworks seems to make more sense. With respect to numerics an implementation of large deformations and plasticity would be the next step. Furthermore the investigation of more complex geometries as the cylinder, which caused problems in the present work, is required. An isogeometric analysis could be an appropriate tool for further research especially since it could bring down computing times. In this context another field that needs to be considered for further research are cases where highly localized strains or stresses occur which could be modeled by point and line distributions. For all of these tasks the framework in the present work can be used as a foundation.

Appendix A

A second strain gradient elastoplasticity framework with $\overset{\langle 4 \rangle}{\mathbf{K}}_{\mathbf{F}}$

A.1 Derivation of material stress measures from the power functional

In this appendix a material framework for third-order plasticity is outlined, where $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ is used as the strain tensor of order four. This is done by reformulating the crucial results from Chapters 4 and 5 for $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$. In those cases, where only the notation changes, the results are not written down again but only the changes in notation are pointed out.

Starting from (4.32) one obtains

$$P = \int\limits_{B_0} \frac{1}{\rho_0}\Big(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \underbrace{\overset{\langle 3\rangle}{\mathbf{S}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \Big[\overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} - \overset{[2,4][2,3]}{3sym}\Big[\overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big]\Big]}_{*}\Big)dm \tag{A.1}$$

Due to to the subsymmetries of $\overset{\langle 4\rangle}{\mathbf{S}}$ the term marked with $*$

can be written with respect to an ONB as

$$\begin{aligned} &= \overset{\langle 3\rangle}{S}_{abc}\overset{\langle 3\rangle}{K}{}^{\bullet}_{abc} + \overset{\langle 4\rangle}{S}_{abcd}\overset{\langle 4\rangle}{K}{}^{\bullet}_{abcd} - \overset{\langle 4\rangle}{S}_{abcd}\overset{\langle 3\rangle}{K}{}^{[1,3]}_{cdx}\overset{\langle 3\rangle}{K}{}^{\bullet}_{abx} - \overset{\langle 4\rangle}{S}{}^{[2,3]}_{acbd}\overset{\langle 3\rangle}{K}{}^{[1,3]}_{bdx}\overset{\langle 3\rangle}{K}{}^{\bullet}_{acx} \\ &\quad - \overset{\langle 4\rangle}{S}{}^{[2,4]}_{adcb}\overset{\langle 3\rangle}{K}{}^{[1,3]}_{cbx}\overset{\langle 3\rangle}{K}{}^{\bullet}_{adx} \end{aligned} \tag{A.2}$$

$$= (\overset{\langle 3\rangle}{\mathbf{S}} - 3\,\overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 4\rangle}{\mathbf{K}}{}^{[1,3]}_{\mathbf{F}}) \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \tag{A.3}$$

which yields

$$P = \int\limits_{B_0} \frac{1}{\rho_0}\Big(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + (\overset{\langle 3\rangle}{\mathbf{S}} - \underbrace{3\,\overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,3]}_{\mathbf{F}}}_{=:\widehat{\alpha}(\overset{\langle 4\rangle}{\mathbf{S}},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})}) \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}\Big)dm \tag{A.4}$$

$$= \int\limits_{B_0} \frac{1}{\rho_0}\Big(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \underbrace{\Big(\overset{\langle 3\rangle}{\mathbf{S}} - \widehat{\alpha}(\overset{\langle 4\rangle}{\mathbf{S}},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})\Big)}_{=:\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}\Big)dm \tag{A.5}$$

$$= \int\limits_{B_0} \frac{1}{\rho_0}\Big(\frac{1}{2}\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\widehat{\mathbf{S}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}}\Big)dm \tag{A.6}$$

Thus a set of three material stress and a set of three material strain measures has been defined and these sets are work conjugate to each other.

	Stress measures	Strain measures
(A.7)	$\overset{\langle 2\rangle}{\mathbf{S}} := \mathbf{F}^{-1} * (J_{\mathbf{F}}\ \overset{\langle 2\rangle}{\mathbf{T}})$	$\mathbf{C} := \mathbf{F}^T \cdot \mathbf{F}$
(A.8)	$\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} := (\overset{\langle 3\rangle}{\mathbf{S}} - \underbrace{3\ \overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}}_{=:\widehat{\alpha}(\overset{\langle 4\rangle}{\mathbf{S}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})})$	$\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$
(A.9)	$\overset{\langle 4\rangle}{\mathbf{S}} := \mathbf{F}^{-1} \circ (J_{\mathbf{F}}\ \overset{\langle 4\rangle}{\mathbf{T}})$	$\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$

A.2 Third-order elasticity

One defines constitutive functions by extending the definitions in [Bertram 2015] as follows:

$$\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{\widehat{f}}\ (\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \tag{A.10}$$

$$\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} = \overset{\langle 3\rangle}{\widehat{f}}\ (\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \tag{A.11}$$

$$\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{\widehat{f}}\ (\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \tag{A.12}$$

Note: There is no constitutive equation for $\overset{\langle 3\rangle}{\mathbf{S}}$ because it is not a generalized stress measure but a partial stress. It should be regarded as a quantity that helps making a comparison to the second-order theory and it makes some transformations shorter.

Definition A.1. Hyperelasticity

A material is called hyperelastic if there exists a specific elastic energy

$$\widehat{w} : \mathscr{Config} \mapsto \mathbb{R}$$

such that

$$p := \frac{1}{\rho_0}\left(\frac{1}{2}\ \overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\widehat{\mathbf{S}}} \ \vdots\ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}\right) \tag{A.13}$$

$$
\begin{aligned}
&\text{(A.14)} \qquad =\widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)^{\bullet}\\
&\text{(A.15)} \qquad =\frac{\partial \widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)}{\partial \mathbf{C}} : \mathbf{C}^{\bullet} + \frac{\partial \widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet} + \frac{\partial \widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)}{\partial \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}} :: \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}^{\bullet}
\end{aligned}
$$

A comparison with the components in (A.6) then reveals for all $\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) \in \mathscr{Config}$

$$
\begin{aligned}
&\text{(A.16)} \quad \overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{\widehat{f}}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) = 2\rho_0 \frac{\partial \widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)}{\partial \mathbf{C}}\\
&\text{(A.17)} \quad \overset{\langle 3\rangle}{\widehat{\mathbf{S}}} = \overset{\langle 3\rangle}{\widehat{f}}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) = \rho_0 \frac{\partial \widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)}{\partial \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}}\\
&\text{(A.18)} \quad \overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{\widehat{f}}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) = \rho_0 \frac{\partial \widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)}{\partial \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}}
\end{aligned}
$$

A.3 Changes of the reference placement

The transformation of $\mathbf{C}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$, $\overset{\langle 2\rangle}{\mathbf{S}}$ and $\overset{\langle 4\rangle}{\mathbf{S}}$ is derived in Section 4.5. Therefore only the transformation of $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ and $\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}$ is derived here.

Theorem A.1. Transformation under changes of the reference placement

Again one defines auxillary functions

$$
\begin{aligned}
&\text{(A.19)} \quad \widehat{\beta} : \mathscr{Conf}_4 \times \mathscr{Conf}_3 \times \mathscr{Inv} \times \mathscr{Conf}_3 \times \mathscr{Conf}_4 \to \mathscr{Conf}_4,\\
&\text{(A.20)} \quad \widehat{\gamma} : \mathscr{Conf}_3 \times \mathscr{Conf}_4 \times \mathscr{Inv} \times \mathscr{Conf}_3 \times \mathscr{Conf}_3 \mapsto \mathscr{Conf}_3,\\
&\text{(A.21)} \quad \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big) = \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} + \overset{[2,4][2,3]}{3sym} \Big[\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Big],\\
&\text{(A.22)} \quad \widehat{\gamma}\big(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \mathbf{A}^{-1} \circ \Big[J_{\mathbf{A}}\big(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} + 3\, \overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}{}^{[1,3]}\big)\Big]\\
&\qquad\qquad - \Big(\mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}})\Big) : [\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}]^{[1,3]}.
\end{aligned}
$$

such that

$$\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}} = \widehat{\beta}(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}) \tag{A.23}$$

$$\overset{\langle 3\rangle}{\widehat{\underline{\mathbf{S}}}} = \widehat{\gamma}(\ \overset{\langle 3\rangle}{\widehat{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\ ) \tag{A.24}$$

Proof.

First (A.23) is proved.

$$\overset{\langle 4\rangle}{\underline{\mathbf{K}}} = \Big(\underline{\mathbf{F}}^{-1} \cdot Grad^{II}(\underline{\mathbf{F}})\Big) = \Big(\mathbf{A}^{-1} \cdot \mathbf{F}^{-1} \cdot Grad(Grad(\mathbf{F} \cdot \mathbf{A}) \cdot \mathbf{A}) \cdot \mathbf{A}\Big) \tag{A.25}$$

Thus, with respect to an ONB $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ one can write the components of $\overset{\langle 4\rangle}{\underline{\mathbf{K}}} := \overset{\langle 4\rangle}{\mathbf{K}}_{\underset{\underline{\mathbf{F}}}{\langle 2\rangle}}$ as

$$
\begin{aligned}
& A^{-1}_{\alpha a} F^{-1}_{ab} [(F_{bc} A_{c\beta})_{,d} A_{d\gamma}]_{,e} A_{e\delta} \\
&= A^{-1}_{\alpha a} F^{-1}_{ab} [F_{bc,d} A_{c\beta} A_{d\gamma} + F_{bc} A_{c\beta,d} A_{d\gamma}]_{,e} A_{e\delta} && \text{(A.26)} \\
&= A^{-1}_{\alpha a} F^{-1}_{ab} [F_{bc,de} A_{c\beta} A_{d\gamma} + F_{bc,d} A_{c\beta,e} A_{d\gamma} + F_{bc,d} A_{c\beta} A_{d\gamma,e} && \text{(A.27)} \\
&\quad + F_{bc,e} A_{c\beta,d} A_{d\gamma} + F_{bc} A_{c\beta,de} A_{d\gamma} + F_{bc} A_{c\beta,d} A_{d\gamma,e}] A_{e\delta} \\
&= A^{-1}_{\alpha a} F^{-1}_{ab} [F_{bc,de} A_{c\beta} A_{d\gamma} A_{e\delta} + F_{bc,d} A_{c\beta\underline{,}f} A^{-1}_{fe} A_{d\gamma} A_{e\delta} + F_{bc,d} A_{c\beta} A_{d\gamma\underline{,}f} A^{-1}_{fe} A_{e\delta} && \text{(A.28)} \\
&\quad + F_{bc,e} A_{c\beta\underline{,}f} A^{-1}_{fd} A_{d\gamma} A_{e\delta} + F_{bc} A_{c\beta,de} A_{d\gamma} A_{e\delta} + F_{bc} A_{c\beta\underline{,}g} A^{-1}_{gd} A_{d\gamma\underline{,}f} A^{-1}_{fe} A_{e\delta}] \\
&= A^{-1}_{\alpha a} F^{-1}_{ab} [F_{bc,de} A_{c\beta} A_{d\gamma} A_{e\delta} + F_{bc,d} A_{c\beta\underline{,}\delta} A_{d\gamma} + F_{bd,c} A_{c\beta} A_{d\gamma\underline{,}\delta} && \text{(A.29)} \\
&\quad + F_{bc,e} A_{c\beta\underline{,}\gamma} A_{e\delta} + F_{bc} A_{c\beta,de} A_{d\gamma} A_{e\delta} + F_{bc} A_{c\beta\underline{,}g} A^{-1}_{gd} A_{d\gamma\underline{,}\delta}]
\end{aligned}
$$

In the next step one makes use of $A_{c\beta,de} = A_{c\beta\underline{,}gf} A^{-1}_{gd} A^{-1}_{fe} - A_{c\beta\underline{,}g} A^{-1}_{gx} A_{xf\underline{,}y} A^{-1}_{yd} A^{-1}_{fe}$, which can be easily verified by applying Remark 2.1

$$
\begin{aligned}
&= A^{-1}_{\alpha a} F^{-1}_{ab} \Big[F_{bc,de} A_{c\beta} A_{d\gamma} A_{e\delta} + F_{bc,d} A_{c\beta\underline{,}\delta} A_{d\gamma} + F_{bd,c} A_{c\beta} A_{d\gamma\underline{,}\delta} && \text{(A.30)} \\
&\quad + F_{bc,e} A_{c\beta\underline{,}\gamma} A_{e\delta} + F_{bc} [A_{c\beta\underline{,}gf} A^{-1}_{gd} A^{-1}_{fe} - A_{c\beta\underline{,}g} A^{-1}_{gx} A_{xf\underline{,}y}] A^{-1}_{yd} A^{-1}_{fe} A_{d\gamma} A_{e\delta} \\
&\quad + F_{bc} A_{c\beta\underline{,}g} A^{-1}_{gd} A_{d\gamma\underline{,}\delta} \Big] \\
&= A^{-1}_{\alpha a} F^{-1}_{ab} F_{bc,de} A_{c\beta} A_{d\gamma} A_{e\delta} + A^{-1}_{\alpha a} F^{-1}_{ab} F_{bc,d} A_{c\beta\underline{,}\delta} A_{d\gamma} + A^{-1}_{\alpha a} F^{-1}_{ab} F_{bd,c} A_{c\beta} A_{d\gamma\underline{,}\delta} && \text{(A.31)}
\end{aligned}
$$

$$+ A^{-1}_{\alpha a} F^{-1}_{ab} F_{bc,e} A_{c\beta_{\underline{,}}\gamma} A_{e\delta} + A^{-1}_{\alpha a} A_{a\beta_{\underline{,}}\gamma\delta}$$

In notation without indices one therefore obtains

$$\text{(A.32)} \quad \overset{\langle 4\rangle}{\underline{\mathbf{K}}_{\mathbf{F}}} = \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{A}}} + \overset{[2,4][2,3]}{3sym} \left[\mathbf{A}^{-1} \cdot (\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} \cdot \mathbf{A})^{[2,3]} \cdot \underline{Grad}(\mathbf{A})\right]$$

$$\text{(A.33)} \quad = \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{A}}} + \overset{[2,4][2,3]}{3sym} \left[(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{A}}}\right]$$

Transformation (A.24) is proved similarly:

$$\text{(A.34)} \quad \overset{\langle 3\rangle}{\widehat{\underline{\mathbf{S}}}} = \overset{\langle 3\rangle}{\underline{\mathbf{S}}} - \overset{\langle 4\rangle}{\underline{\mathbf{S}}} : \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}^{[1,3]} = \mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 3\rangle}{\mathbf{S}}) - \left(\mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}})\right) : [\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{A}}}]^{[1,3]}$$

$$\text{(A.35)} \quad = \mathbf{A}^{-1} \circ \left[J_{\mathbf{A}}(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} + \widehat{\alpha}(\overset{\langle 4\rangle}{\mathbf{S}}, \overset{\langle 3\rangle}{\mathbf{K}}))\right] - \left(\mathbf{A}^{-1} \circ (J_{\mathbf{A}} \overset{\langle 4\rangle}{\mathbf{S}})\right) : [\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{A}}}]^{[1,3]}$$

$$\text{(A.36)} \quad =: \widehat{\gamma}(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{A}}}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})$$

□

In summary the following transformations for changes of reference placement have been obtained:

	Stress Measures:	**Strain measures:**
(A.37)	$\overset{\langle 2\rangle}{\underline{\mathbf{S}}} = \mathbf{A}^{-1} * J_A \overset{\langle 2\rangle}{\mathbf{S}}$	$\underline{\mathbf{C}} = \mathbf{A}^T * \mathbf{C}$
(A.38)	$\overset{\langle 3\rangle}{\widehat{\underline{\mathbf{S}}}} = \widehat{\gamma}(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{A}}}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})$	$\overset{\langle 3\rangle}{\mathbf{K}_{\underline{\mathbf{F}}}} = \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}} + \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{A}}}$
(A.39)	$\overset{\langle 4\rangle}{\underline{\mathbf{S}}} = \mathbf{A}^{-1} \circ J_A \overset{\langle 4\rangle}{\mathbf{S}}$	$\overset{\langle 4\rangle}{\mathbf{K}_{\underline{\mathbf{F}}}} = \widehat{\beta}(\overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{A}}}, \overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{A}}})$

At this point one should note that the transformations $\widehat{\beta}$ and $\widehat{\gamma}$ from the framework with $\overset{\langle 4\rangle}{\mathbf{K}_{\mathbf{F}}}$ are more compact than β and γ from the framework with $Grad(\overset{\langle 3\rangle}{\mathbf{K}_{\mathbf{F}}})$.

Remark A.1. Transformation of stress and strain measures under two subsequent changes of reference placement

Again it will be investigated how the stress and strain measures transform under two subsequent changes of reference placement. The results will be needed later for dealing with elastic isomorphy. Three reference placements κ, $\underline{\kappa}$ and $\underline{\underline{\kappa}}$ are defined with $M := \underline{\underline{Grad}}(\underline{\kappa} \circ \underline{\underline{\kappa}}^{-1})$ and $N := \underline{Grad}(\kappa \circ \underline{\kappa}^{-1})$. This situation is sketched in Figure 4.1. Two subsequent changes of reference placement can also be substituted by a single change of reference placement. This fact lies behind the following relations:

$$\text{(A.40)}\quad \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{N}\cdot\mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}} \big)$$

$$= \widehat{\beta}\Big(\widehat{\beta}\{ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{N}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}})\}, \mathbf{N}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{M}} \Big)$$

$$\text{(A.41)}\quad \widehat{\gamma}\Big(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{N}\cdot\mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}\cdot\mathbf{M}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \Big)$$

$$= \widehat{\gamma}\Big(\widehat{\gamma}\Big(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}, \overset{\langle 4\rangle}{\mathbf{S}}, \mathbf{N}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \Big), det(\mathbf{N})(\mathbf{N}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}), \mathbf{M}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{M}}, \mathbf{N}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{N}} \Big)$$

Proof.

The proof follows exactly the lines of the proof of Remark 4.6 and is therefore left to the reader. □

Remark A.2. Transformation of elastic energies and elastic laws under a change of reference placement

Using the transformation rules for strain and stress measures, one can deduce the following. For two reference placements

- κ with strain measures $\mathbf{C}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}$, $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$, an elastic energy $\widehat{w}$ and stress tensors $\overset{\langle 2\rangle}{\mathbf{S}}$, $\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}$, $\overset{\langle 4\rangle}{\mathbf{S}}$

  with elastic laws $\overset{\langle 2\rangle}{\mathbf{S}} = \overset{\langle 2\rangle}{\widehat{f}}\,(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})$, $\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} = \overset{\langle 3\rangle}{\widehat{f}}\,(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})$, $\overset{\langle 4\rangle}{\mathbf{S}} = \overset{\langle 4\rangle}{\widehat{f}}\,(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}})$

- $\underline{\kappa}$ with strain measures $\underline{\mathbf{C}}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}$, $\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}$, an elastic energy $\underline{\widehat{w}}$ and stress tensors $\underline{\overset{\langle 2\rangle}{\mathbf{S}}}$, $\underline{\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}}$, $\underline{\overset{\langle 4\rangle}{\mathbf{S}}}$

  with elastic laws $\underline{\overset{\langle 2\rangle}{\mathbf{S}}} = \underline{\overset{\langle 2\rangle}{\widehat{f}}}(\underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})$, $\underline{\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}} = \underline{\overset{\langle 3\rangle}{\widehat{f}}}(\underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})$, $\underline{\overset{\langle 4\rangle}{\mathbf{S}}} = \underline{\overset{\langle 4\rangle}{\widehat{f}}}(\underline{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}})$

a constant $\underline{\widehat{w}}_0 \in \mathbb{R}$ exists such that the elastic energies transform as

$$\widehat{w}\Big(\overset{\langle 2\rangle}{\mathbf{C}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) = \underline{\widehat{w}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big)\Big) + \underline{\widehat{w}}_0 \tag{A.42}$$

The elastic laws transform as

$$\begin{aligned}
&\overset{\langle 2\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) \\
&= \mathbf{A} * J_{\mathbf{A}}^{-1}\, \underline{\overset{\langle 2\rangle}{\widehat{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big)\Big)
\end{aligned} \tag{A.43}$$

$$\begin{aligned}
&\overset{\langle 3\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) \\
&= \widehat{\gamma}\Bigg(\underline{\overset{\langle 3\rangle}{\widehat{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big)\Big), \\
&\qquad \underline{\overset{\langle 4\rangle}{\widehat{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big)\Big), \\
&\qquad \mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Bigg)
\end{aligned} \tag{A.44}$$

$$\begin{aligned}
&\overset{\langle 4\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) \\
&= \mathbf{A} \circ J_{\mathbf{A}}^{-1}\, \underline{\overset{\langle 4\rangle}{\widehat{f}}}\Big(\mathbf{A}^T * \mathbf{C}, \mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big)\Big)
\end{aligned} \tag{A.45}$$

Proof.

The proof follows exactly the lines of the proof from Remark 4.7. □

A.4 Elastic Isomorphy

The definition of elastic isomorphy can be obtained from Definition 4.5 by simply substituting w by $\widehat{w}$ or $\{\overset{\langle 2\rangle}{f}, \overset{\langle 3\rangle}{\widetilde{f}}, \overset{\langle 4\rangle}{f}\}$ by $\{\overset{\langle 2\rangle}{\widehat{f}}, \overset{\langle 3\rangle}{\widehat{f}}, \overset{\langle 4\rangle}{\widehat{f}}\}$. Therefore it is not stated here.

Theorem A.2. Criterion for elastic isomorphy

Let X and Y be two elastic material points with arbitrary reference placements $\underline{\kappa}_X$ and $\underline{\kappa}_Y$ and elastic energies w_X and w_Y. Let $\{\underline{\overset{\langle 2\rangle}{\widehat{f}}}_X, \underline{\overset{\langle 3\rangle}{\widehat{f}}}_X, \underline{\overset{\langle 4\rangle}{\widehat{f}}}_X\}$ and $\{\underline{\overset{\langle 2\rangle}{\widehat{f}}}_Y, \underline{\overset{\langle 3\rangle}{\widehat{f}}}_Y, \underline{\overset{\langle 4\rangle}{\widehat{f}}}_Y\}$ be the respective sets of elastic laws. Then these two points are called elastically isomorphic if and only if there exist tensors $\overset{\langle 2\rangle}{\mathbf{P}} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{P}} \in \mathscr{Conf}_3$, $\overset{\langle 4\rangle}{\widehat{\mathbf{P}}} \in \mathscr{Conf}_4$ such that

$$\text{(A.46)}\quad \rho_{0Y} = det(\overset{\langle 2\rangle}{\mathbf{P}})\rho_{0X}$$

$$\text{(A.47)}\quad \underline{\widehat{w}}_X\Big(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}\Big)$$

$$= \underline{\widehat{w}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\widehat{\mathbf{P}}} \big) \Big) + \underline{\widehat{w}}_0$$

$$\text{(A.48)}\quad \overset{\langle 2\rangle}{\underline{\widehat{f}}}_X(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X})$$

$$= \overset{\langle 2\rangle}{\mathbf{P}} * det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\ \overset{\langle 2\rangle}{\widehat{f}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\widehat{\mathbf{P}}} \big)\Big)$$

$$\text{(A.49)}\quad \overset{\langle 3\rangle}{\underline{\widehat{f}}}_X(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X})$$

$$\begin{aligned} = \widehat{\gamma}\Big(&\overset{\langle 3\rangle}{\underline{\widehat{f}}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\widehat{\mathbf{P}}} \big) \Big), \\ &\overset{\langle 4\rangle}{\underline{\widehat{f}}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\widehat{\mathbf{P}}} \big) \Big), \\ &\overset{\langle 2\rangle}{\mathbf{P}}^{-1}, - \overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}} \Big) \end{aligned}$$

$$\text{(A.50)}\quad \overset{\langle 4\rangle}{\underline{\widehat{f}}}_X(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X})$$

$$= \overset{\langle 2\rangle}{\mathbf{P}} \circ det^{-1}(\overset{\langle 2\rangle}{\mathbf{P}})\ \overset{\langle 4\rangle}{\widehat{f}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\widehat{\mathbf{P}}} \big)\Big)$$

Note:

The tensor $\overset{\langle 2\rangle}{\mathbf{P}}$ can be interpreted as the gradient of a change of reference placement, the tensor $\overset{\langle 3\rangle}{\mathbf{P}}$ as $\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$ and the tensor $\overset{\langle 4\rangle}{\widehat{\mathbf{P}}}$ as $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{P}}$. As long as only one material point is considered these tensors can be considered as independent which means they do not have to fullfill any integrability condition.

Proof.

The proof follows exactly the lines of the proof from Theorem 4.9. $\square$

A.5 Material symmetry

Applying the concept of elastic isomorphy to only one point, i.e. setting $X = Y$ in Definition 4.5 defines symmetry. In this case one can drop the notation for the reference point. As explained in Theorem A.1 a change of reference placement defines three tensors $\mathbf{A} \in \mathscr{Inv}$, $\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} \in \mathscr{Conf}_3$ and $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} \in \mathscr{Conf}_4$. So in this case the isomorphism $\mathbf{A}$ becomes an automorphism since it maps the tangent space at a point onto itself. One can set

$$\overset{\langle 2\rangle}{\mathbf{A}} = \mathbf{A} \tag{A.51}$$

$$\overset{\langle 3\rangle}{\mathbf{A}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} \tag{A.52}$$

$$\overset{\langle 4\rangle}{\widehat{\mathbf{A}}} = \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} \tag{A.53}$$

The tensors $\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}}$ can then be considered as independent from each other because they are only considered at one point. The behavior around this point (necessary for derivatives) is not of interest. In the following definition of symmetry the idea is to express the fact that a certain change of the reference placement at a point does not change the elastic law at this point.

Definition A.2. Symmetry Transformation

For a gradient elastic material a symmetry transformation is a triple $(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}}) \in \mathscr{Config}$ such that

$$\widehat{w}\Big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) = \widehat{w}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}}\big)\Big) \tag{A.54}$$

For the elastic laws this means

$$\begin{aligned} &\overset{\langle 2\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) \\ &= \overset{\langle 2\rangle}{\mathbf{A}} * J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}} \overset{\langle 2\rangle}{\widehat{f}}\Big(\overset{\langle 2\rangle}{\mathbf{A}}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}}\big)\Big) \end{aligned} \tag{A.55}$$

(A.56)
$$
\begin{aligned}
&\overset{\langle 3\rangle}{\widehat{f}}\,\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\,\big)\\
&= \widehat{\gamma}\Big(\overset{\langle 3\rangle}{\widehat{f}}\,\Big(\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}}\,\big)\Big),\\
&\qquad \overset{\langle 4\rangle}{\widehat{f}}\,\Big(\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}}\,\big)\Big),\\
&\qquad \overset{\langle 2\rangle}{\mathbf{A}}{}^{-1}, - \overset{\langle 2\rangle}{\mathbf{A}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}\Big)
\end{aligned}
$$

(A.57)
$$
\begin{aligned}
&\overset{\langle 4\rangle}{\widehat{f}}\,\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\,\big)\\
&= \overset{\langle 2\rangle}{\mathbf{A}} \circ J^{-1}_{\overset{\langle 2\rangle}{\mathbf{A}}}\, \overset{\langle 4\rangle}{\widehat{f}}\,\Big(\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{A}}, \widehat{\beta}\big(\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}}\,\big)\Big)
\end{aligned}
$$

for all $(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \in \mathscr{Config}$.

Definition A.3. Symmetry group of a third-order material

The set of all symmetry transformations is the **symmetry group** of a material. The **symmetry group** is an algebraic group under composition:

The **composition** is defined as

(A.58)
$$
\begin{aligned}
&\Big(\overset{\langle 2\rangle}{\mathbf{B}}, \overset{\langle 3\rangle}{\mathbf{B}}, \overset{\langle 4\rangle}{\mathbf{B}} \Big)\Big(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{A}} \Big) :=\\
&\Big(\overset{\langle 2\rangle}{\mathbf{B}} \cdot \overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{B}} + \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} \circ \overset{\langle 4\rangle}{\mathbf{B}} + \overset{\langle 4\rangle}{\mathbf{A}} + \overset{[2,3][2,4]}{3sym} \Big[(\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{B}}) \cdot \overset{\langle 3\rangle}{\mathbf{A}} \Big]\Big)
\end{aligned}
$$

The **neutral element** is defined as

(A.59)
$$
\Big(\overset{\langle 2\rangle}{\mathbf{I}}, \overset{\langle 3\rangle}{\mathbf{0}}, \overset{\langle 4\rangle}{\mathbf{0}} \Big)
$$

The **inverse element** is defined as

(A.60)
$$
\Big(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}} \Big)^{-1} := \Big(\overset{\langle 2\rangle}{\mathbf{A}}{}^{-1}, - \overset{\langle 2\rangle}{\mathbf{A}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{-T} \circ \Big(- \overset{\langle 4\rangle}{\mathbf{A}} + \overset{[2,4][2,3]}{3sym} \Big[\overset{\langle 3\rangle}{\mathbf{A}} \cdot \overset{\langle 3\rangle}{\mathbf{A}} \Big]\Big)\Big)
$$

The entries of the inverse element are calculated as

$$(\overset{\langle 2\rangle}{\mathbf{A}}, \overset{\langle 3\rangle}{\mathbf{A}}, \overset{\langle 4\rangle}{\widehat{\mathbf{A}}})^{-1} = \big(\mathbf{A}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}}\big)^{-1} = \big(\mathbf{A}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}^{-1}}\big) \tag{A.61}$$

by using Equations (2.30)-(2.34).

One can observe here that the formulas in Definition A.3 are more compact than those in Definition 4.7.

Definition A.4. Undistorted states & solids

If for a certain reference placement the symmetry group is a subgroup of the orthogonal group in the first entry and zero in the other two entries then this reference placement is called an **undistorted state**. The elements of the symmetry group can be interpreted as rotations. A material that has such an undistorted state is called a **solid**.

Definition A.5. Isotropic material:

If the symmetry group is the orthogonal group in the first entry and zero in the others then the material point is called **isotropic**. It is clear that for an isotropic material the elastic laws $\overset{\langle 2\rangle}{\widehat{f}}$ and $\overset{\langle 4\rangle}{\widehat{f}}$ are isotropic tensor functions: First one has to rearrange Equations (A.55) and (A.57). Next one applies the fact that for isotropic materials $\overset{\langle 2\rangle}{\mathbf{A}}$ is orthogonal: $\overset{\langle 2\rangle}{\mathbf{A}}{}^{-1} = \overset{\langle 2\rangle}{\mathbf{A}}{}^{T}$. This yields $J_{\overset{\langle 2\rangle}{\mathbf{A}}} = 1$ and it yields that the product "$\circ$" can be replaced by the product "$*$" in Equation (A.57).

$$\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \overset{\langle 2\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \overset{\langle 2\rangle}{\widehat{f}}\Big(\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) \tag{A.62}$$

$$\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \overset{\langle 4\rangle}{\widehat{f}}\big(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\big) = \overset{\langle 4\rangle}{\widehat{f}}\Big(\overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \mathbf{C}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 2\rangle}{\mathbf{A}}{}^{T} * \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}\Big) \tag{A.63}$$

The elastic law $\overset{\langle 3\rangle}{\widehat{f}}$ is not an isotropic tensor function. The reason for this is the fact that $\overset{\langle 3\rangle}{\widehat{f}}$ transforms with the function $\widehat{\gamma}$ and not as a pull-back like the other elastic laws.

Remark 4.12 also applies in this case, one just has to substitute the according stress and strain measures.

A.6 Plastic dissipation

In this chapter the internal power during yielding is examined. The following abbreviations will be used:

$$\overset{\langle 2\rangle}{\mathbf{G}} := \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \tag{A.64}$$

$$\overset{\langle 3\rangle}{\mathbf{G}} := -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}} \tag{A.65}$$

$$\overset{\langle 4\rangle}{\widehat{\mathbf{G}}} := \overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \left(-\overset{\langle 4\rangle}{\mathbf{P}} + 3\overset{[2,4][2,3]}{sym}\left[\overset{\langle 3\rangle}{\mathbf{P}} \cdot \overset{\langle 3\rangle}{\mathbf{P}}\right]\right) \tag{A.66}$$

If one assumes that $\overset{\langle 2\rangle}{\mathbf{P}}$ is a sufficiently smooth tensor field with $\overset{\langle 3\rangle}{\mathbf{P}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$ and $\overset{\langle 4\rangle}{\widehat{\mathbf{P}}} = \overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$ then

$$\overset{\langle 3\rangle}{\mathbf{G}} = \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} \tag{A.67}$$

$$\overset{\langle 4\rangle}{\widehat{\mathbf{G}}} = \overset{\langle 4\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} \tag{A.68}$$

Let $\{\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\widehat{\mathbf{P}}}\}$ be the internal variables that describe the change of the elastic law under yielding. Then $\{\overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\widehat{\mathbf{P}}}\}$ uniquely determines an alternative set of internal variables $\{\overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 4\rangle}{\widehat{\mathbf{G}}}\}$. This will allow to abbreviate the notation in the following transformations. Since the chosen stress and strain measures are work conjugate one can write

$$\rho_0 l = \frac{1}{2}\overset{\langle 2\rangle}{\widehat{f}}\,(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) : \mathbf{C}^{\bullet} + \overset{\langle 3\rangle}{\widehat{f}}\,(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} + \overset{\langle 4\rangle}{\widehat{f}}\,(\mathbf{C}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}) :: \overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}} \tag{A.69}$$

Now one can rewrite (A.69) by using the reference laws $\{\overset{\langle 2\rangle}{\widehat{f}}_0, \overset{\langle 3\rangle}{\widehat{f}}_0, \overset{\langle 4\rangle}{\widehat{f}}_0\}$. This means that one has to transform all stress and strain measures in a similar way as in Appendix C.2. One obtains

$$\rho_0 l = \underbrace{\overset{\langle 2\rangle}{\mathbf{S}}_0 : \mathbf{C}_e^{\bullet} + \overset{\langle 3\rangle}{\widehat{\mathbf{S}}}_0 \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}_e} + \overset{\langle 4\rangle}{\mathbf{S}}_0 :: \overset{\langle 4\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}_e}}_{\text{elastic part}} + \underbrace{\overset{\langle 2\rangle}{\widehat{\mathbf{S}}}_{\mathrm{P}} : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 3\rangle}{\widehat{\mathbf{S}}}_{\mathrm{P}} \vdots \overset{\langle 3\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} :: \overset{\langle 4\rangle}{\mathbf{G}}{}^{\bullet}}_{\text{plastic part}} \tag{A.70}$$

$$= \widehat{w}_0^{\bullet} + \overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}} : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}} \vdots \overset{\langle 3\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} :: \overset{\langle 4\rangle}{\mathbf{G}}{}^{\bullet} \tag{A.71}$$

with

$$
\begin{aligned}
\overset{\langle 2\rangle}{\widehat{\mathbf{S}}}_{\mathrm{P}} :=& \Bigg[\Bigg(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \Big(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} + (\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}})\Big) \vdots \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \cdot \overset{\langle 4\rangle}{\mathbf{S}}_{0} \Big)^{[1,2]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-1} \Bigg)^{T} \\
&+ \Bigg(\Big(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \big(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} + \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}} \big) \Big)^{[1,2]} \vdots \overset{\langle 4\rangle}{\mathbf{S}}_{0} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \Bigg) \\
&+ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}}_{0} : \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \big(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} + \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}} \big) \Big)^{[1,2]} \\
&+ \Big(6\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}}_{0} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{T} \Big) - \Big(\overset{\langle 4\rangle}{\mathbf{S}}_{0} \vdots (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,2]} \Big) + \Big(3\, \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,2]} \vdots \overset{\langle 4\rangle}{\mathbf{S}}_{0} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \Big) \\
&- 3 \Big(\overset{\langle 4\rangle}{\mathbf{S}}_{0} \cdot \big(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \big)^{[1,2]} \Big)^{[2,4]} \vdots \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \cdot \big(\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \cdot \overset{\langle 3\rangle}{\mathbf{G}}^{[1,2]} \big)^{[1,2]} \Big) \Bigg] \\
\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}_{\mathrm{P}} :=& \Big[\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \Big(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}_{0} + 3\, \overset{\langle 4\rangle}{\mathbf{S}}_{0} : \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,3]} \Big) - \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_{0} \Big) : \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} + \overset{\langle 3\rangle}{\mathbf{G}} \Big)^{[1,3]} \\
&+ 3 \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \Big)^{[1,2]} : \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_{0} \Big) \Big] \\
\overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} :=& \Big[\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_{0} \Big]
\end{aligned}
$$

A.7 Variables to be substituted in other sections

The section on gradient elastoplasticity, flow and hardening rules, isomorphy of the elastic ranges, yield criteria and on flow and hardening rules can be directly transferred to the case where $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ is the fourth-order strain tensor. One just has to substitute

$Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$ by $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$

$Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})$ by $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}_0}$

β and γ by $\widehat{\beta}$ and $\widehat{\gamma}$,

$\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_{0}$ by $\overset{\langle 3\rangle}{\widehat{\mathbf{S}}}_{0}$,

w by $\widehat{w}$

w_0 and w_P by $\widehat{w}_0$ and $\widehat{w}_P$

$\mathscr{E}, \overset{0}{\mathscr{E}}, \partial\mathscr{E}$ by $\widehat{\mathscr{E}}, \overset{0}{\widehat{\mathscr{E}}}, \partial\widehat{\mathscr{E}}$.

$\{\overset{\langle 2\rangle}{\widetilde{f}}, \overset{\langle 3\rangle}{\widetilde{f}}, \overset{\langle 4\rangle}{\widetilde{f}}\}$ by $\{\overset{\langle 2\rangle}{\widehat{f}}, \overset{\langle 3\rangle}{\widehat{f}}, \overset{\langle 4\rangle}{\widehat{f}}\}$ $\overset{\langle 4\rangle}{\mathbf{P}}$ by $\overset{\langle 4\rangle}{\widehat{\mathbf{P}}}$

$\mathbf{Z}$ by $\tilde{\mathbf{Z}}$

$\overset{\langle 2\rangle}{h}, \overset{\langle 3\rangle}{h}, \overset{\langle 4\rangle}{h}, h$ by $\overset{\langle 2\rangle}{\widehat{h}}, \overset{\langle 3\rangle}{\widehat{h}}, \overset{\langle 4\rangle}{\widehat{h}}, \widehat{h}$

λ by $\widehat{\lambda}$

$\mu, \bar{L}$ by $\widehat{\mu}, \widehat{\bar{L}}$

ϕ and ϕ_P by $\widehat{\phi}$ and $\widehat{\phi}_P$.

From the results obtained so far it becomes clear that the thermodynamical framework can be obtained in the same manner as the elastic and the elastoplastic frameworrk with $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$.

A.8 Comparison of the generalized strain measures

In order to asses the suitability of the introduced sets of generalized strain measures one has to compare the transformation behavior under changes of the reference placement of stress and strain measures. (Abbreviated by the functions β and γ or $\widehat{\beta}$ and $\widehat{\gamma}$.

In the framework with $Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}})$ the transformations under changes of the reference placement are

$$\text{(A.72)}\quad \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}) = \mathbf{A}^T \circ \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) + \underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}) - \Big[\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}} \cdot \Big(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}\Big)\Big]^{[2,3]} + 2\overset{[2,3]}{sym}\Big[(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Big]$$

$$\text{(A.73)}\quad \overset{\langle 3\rangle}{\underline{\widetilde{\mathbf{S}}}} = (\mathbf{A}^{-1} \circ J_{\mathbf{A}}[\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} - \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}} - \overset{\langle 4\rangle}{\mathbf{S}} : 2\,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]}]) + (\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{[1,2]} : (\mathbf{A}^{-1} \circ J_{\mathbf{A}}\,\overset{\langle 4\rangle}{\mathbf{S}}) - (\mathbf{A}^{-1} \circ J_{\mathbf{A}}\,\overset{\langle 4\rangle}{\mathbf{S}}) : 2(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}})^{[1,3]}$$

In the framework with $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ the transformations under changes of the reference placement are

$$\text{(A.74)}\quad \overset{\langle 4\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}} = \mathbf{A}^T \circ \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{A}} + 3\overset{[2,3][2,4]}{sym}\Big[(\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}\Big]$$

$$\text{(A.75)}\quad \overset{\langle 3\rangle}{\underline{\widehat{\mathbf{S}}}} = \mathbf{A}^{-1} \circ \Big[J_{\mathbf{A}}(\overset{\langle 3\rangle}{\widehat{\mathbf{S}}} + 3\,\overset{\langle 4\rangle}{\mathbf{S}} : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}^{[1,3]})\Big] - \Big(\mathbf{A}^{-1} \circ (J_{\mathbf{A}}\,\overset{\langle 4\rangle}{\mathbf{S}})\Big) : [\mathbf{A}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}}]^{[1,3]}$$

A direct comparison shows that the formulas that govern a change of the reference placement in the framework with $\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}$ have the advantage of being more compact. Both frameworks are equivalent though, since one can be obtained from the other by using the formula

$$\text{(A.76)}\quad Grad(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}) = [\overset{\langle 4\rangle}{\mathbf{K}}_{\mathbf{F}}] - [\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}}]^{[2,4]}$$

Appendix B

Transformation of stress and strain measures in a second order framework with C and $Grad(\mathrm{C})$

In this appendix it will be shown, that a strain gradient elastoplasticity framework with $\mathbf{C}$ and $Grad(\mathbf{C})$ as generalized strain measures yields transformation laws for the change of the reference placement that are more complicated than those in the framework with $\overset{\langle 3 \rangle}{\mathbf{K}_\mathbf{F}}$. Therefore one can deduce that a second-strain-gradient elastoplasticity framework with $\mathbf{C}$, $Grad(\mathbf{C})$ and $Grad^{II}(\mathbf{C})$ will yield even more complicated laws. This shows that gradients of $\mathbf{C}$ are not a good choice for higher-gradient elastoplasticity frameworks. One could use the following relation (from [Krawietz 1993], [Hwang et al. 2002]) in order to show this.

$$\text{(B.1)} \qquad \overset{\langle 3 \rangle}{\mathbf{K}_\mathbf{F}} = \mathbf{C}^{-1} \cdot Sym\big(Grad(\mathbf{C})\big)$$

with

$$\text{(B.2)} \qquad Sym\big(Grad(\mathbf{C})\big) := \frac{1}{2}\Big(\big(Grad(\mathbf{C})\big) + \big(Grad(\mathbf{C})\big)^{[2,3]} - \big(Grad(\mathbf{C})\big)^{[1,3]}\Big)$$

However in this chapter an approach similar to the one in Section 4.2.2 is used to derive material stress and strain measures and their transformation behavior under changes of the reference placement. This is done in order to facilitate comparison of these approaches. The notation and quantities introduced in Chapter 4 will be used in this chapter as well.

B.1 Relation between material and spatial stress tensors

In Chapter 4.2.2 the strain measure $\overset{\langle 3 \rangle}{\mathbf{K}_\mathbf{F}}$ is derived as the pullback of $grad(\mathbf{v})$, and $\overset{\langle 3 \rangle}{\mathbf{S}}$ is the pullback of $\overset{\langle 3 \rangle}{\mathbf{T}}$. If one wants to introduce $Grad(\mathbf{C})$ into this framework one has to proceed as follows. One starts with the assumption that the material stress power density can be written as

$$\text{(B.3)} \qquad \rho_0 l = \overset{\langle \mathbf{C}2 \rangle}{\mathbf{S}} : \overset{\langle 2 \rangle}{\mathbf{C}}{}^{\bullet} + \overset{\langle \mathbf{C}3 \rangle}{\mathbf{S}} \,\vdots\, Grad(\overset{\langle 2 \rangle}{\mathbf{C}}{}^{\bullet})$$

This assumption is valid since $\overset{\langle 2 \rangle}{\mathbf{C}}{}^{\bullet}$ and $Grad(\overset{\langle 2 \rangle}{\mathbf{C}}{}^{\bullet})$ can be derived from $\mathbf{F}^{\bullet}$ and $Grad(\mathbf{F})^{\bullet}$. Therefore generalized stress tensors $\overset{\langle \mathbf{C}2 \rangle}{\mathbf{S}}$ and $\overset{\langle \mathbf{C}3 \rangle}{\mathbf{S}}$ exist, which allow to express the power functional in terms of these variables. One should note that (B.3) implies that

$$\overset{\langle \mathbf{C}2\rangle}{\mathbf{S}} = \overset{\langle \mathbf{C}2\rangle}{\mathbf{S}}^{T} \tag{B.4}$$

$$\overset{\langle \mathbf{C}3\rangle}{\mathbf{S}} = 2\overset{[1,2]}{sym}\left[\overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\right]. \tag{B.5}$$

One can now calculate the pushforward of each stress and strain variable in (B.3).

Lemma B.1. Pushforward of the strain measures

$$\mathbf{F}^{-T} \circ \overset{\langle 2\rangle}{\mathbf{C}}^{\bullet} = 2sym\Big(grad(\mathbf{v})\Big) \tag{B.6}$$

$$\begin{aligned}&\mathbf{F}^{-T} \circ Grad(\overset{\langle 2\rangle}{\mathbf{C}}^{\bullet}) \\ &= 2\overset{[1,2]}{sym}\Big[grad^{II}(\mathbf{v}) + 2sym\Big(grad(\mathbf{v})\Big) \cdot [grad(\mathbf{F})^{[2,3]} \cdot \mathbf{F}^{-1}]^{[2,3]}\Big]\end{aligned} \tag{B.7}$$

Proof.

The derivation of (B.6) can be found e.g. in [Bertram 2015]. For the derivation of (B.7) the following identity will be used in this proof

$$grad^{II}(\mathbf{v}) = [grad(\mathbf{F}^{\bullet})^{[2,3]} \cdot \mathbf{F}^{-1}]^{[2,3]} - grad(\mathbf{v}) \cdot [grad(\mathbf{F})^{[2,3]} \cdot \mathbf{F}^{-1}]^{[2,3]} \tag{B.8}$$

With respect to an ONB the components of $\mathbf{F}^{-T} \circ \mathbf{C}$ are

$$\begin{aligned}&F^{-T}_{\alpha b} F^{T\bullet}_{ba,d} F_{ac} F^{-1}_{c\beta} F^{-1}_{d\gamma} + F^{-T}_{\alpha b} F^{T}_{ba,d} F^{\bullet}_{ac} F^{-1}_{c\beta} F^{-1}_{d\gamma} \\ &+ F^{-T}_{\alpha b} F^{T\bullet}_{ba} F_{ac,d} F^{-1}_{c\beta} F^{-1}_{d\gamma} + F^{-T}_{\alpha b} F^{T}_{ba} F^{\bullet}_{ac,d} F^{-1}_{c\beta} F^{-1}_{d\gamma}\end{aligned} \tag{B.9}$$

$$\begin{aligned}=&F^{-T}_{\alpha b} F^{T\bullet}_{b\beta,d} F^{-1}_{d\gamma} + F^{-T}_{\alpha b} F^{T}_{ba,d} v_{a,\beta} F^{-1}_{d\gamma} \\ &+ v_{a,\alpha} F_{ac,d} F^{-1}_{c\beta} F^{-1}_{d\gamma} + F^{\bullet}_{\alpha c,d} F^{-1}_{c\beta} F^{-1}_{d\gamma}\end{aligned} \tag{B.10}$$

$$\begin{aligned}=&F^{\bullet}_{\beta b,d} F^{-1}_{b\alpha} F^{-1}_{d\gamma} + F^{-T}_{\alpha b} F_{ab,d} v_{a,\beta} F^{-1}_{d\gamma} \\ &+ v_{a,\alpha} F_{ac,d} F^{-1}_{c\beta} F^{-1}_{d\gamma} + F^{\bullet}_{\alpha c,d} F^{-1}_{c\beta} F^{-1}_{d\gamma}\end{aligned} \tag{B.11}$$

Thus

$$\text{(B.12)}\quad \mathbf{F}^{-T}\circ\mathbf{C} = 2\overset{[1,2]}{sym}\Big[[grad(\mathbf{F}^{\bullet})^{[2,3]}\cdot\mathbf{F}^{-1}]^{[2,3]} + grad(\mathbf{v})^{T}\cdot[grad(\mathbf{F})^{[2,3]}\cdot\mathbf{F}^{-1}]^{[2,3]}\Big]$$

with the identity (B.8)

$$\begin{aligned}
&\text{(B.13)}\quad &&= 2\overset{[1,2]}{sym}\Big[grad^{II}(\mathbf{v}) + grad(\mathbf{v})\cdot[grad(\mathbf{F})^{[2,3]}\cdot\mathbf{F}^{-1}]^{[2,3]}\\
& &&\quad + grad(\mathbf{v})^{T}\cdot[grad(\mathbf{F})^{[2,3]}\cdot\mathbf{F}^{-1}]^{[2,3]}\Big]\\
&\text{(B.14)}\quad &&= 2\overset{[1,2]}{sym}\Big[grad^{II}(\mathbf{v}) + 2sym\big(grad(\mathbf{v})\big)\cdot[grad(\mathbf{F})^{[2,3]}\cdot\mathbf{F}^{-1}]^{[2,3]}\Big]
\end{aligned}$$

□

Theorem B.1. Relation between spatial and material stress tensors

$$\text{(B.15)}\quad \overset{\langle 2\rangle}{\mathbf{T}} = \big(2J_{\mathbf{F}}^{-1}\mathbf{F}\circ\overset{\langle\mathbf{C}2\rangle}{\mathbf{S}}\big) + 4J_{\mathbf{F}}^{-1}sym\big(Grad(\mathbf{F}):\overset{\langle\mathbf{C}3\rangle}{\mathbf{S}}\cdot\mathbf{F}^{T}\big)$$

$$\text{(B.16)}\quad \overset{\langle 3\rangle}{\mathbf{T}} = 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ\overset{\langle\mathbf{C}3\rangle}{\mathbf{S}}\big)$$

Proof. For the stress power one obtains

$$\begin{aligned}
&\text{(B.17)}\quad P &&= \int_{B_0}\frac{1}{\rho_0}\Big[\overset{\langle\mathbf{C}2\rangle}{\mathbf{S}}:\overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet} + \overset{\langle\mathbf{C}3\rangle}{\mathbf{S}}\;\vdots\;Grad(\overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet})\Big]dm\\
&\text{(B.18)} &&= \int_{B}\frac{1}{\rho}J_{\mathbf{F}}^{-1}\Big[\big(\mathbf{F}\circ\overset{\langle\mathbf{C}2\rangle}{\mathbf{S}}\big):\big(\mathbf{F}^{-T}\circ\overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet}\big) + \big(\mathbf{F}\circ\overset{\langle\mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots\big(\mathbf{F}^{-T}\circ Grad(\overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet})\big)\Big]dm\\
&\text{(B.19)} &&= \int_{B}\frac{1}{\rho}\Big[\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ\overset{\langle\mathbf{C}2\rangle}{\mathbf{S}}\big):\big(\mathbf{F}^{-T}\circ\overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet}\big) + \big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ\overset{\langle\mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots\big(\mathbf{F}^{-T}\circ Grad(\overset{\langle 2\rangle}{\mathbf{C}}{}^{\bullet})\big)\Big]dm
\end{aligned}$$

Apply (B.6) and (B.7)

$$\begin{aligned}
&\text{(B.20)}\quad &&= \int_{B}\frac{1}{\rho}\Big[\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ\overset{\langle\mathbf{C}2\rangle}{\mathbf{S}}\big):2sym\big(grad(\mathbf{v})\big)\\
& &&\quad + \big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ\overset{\langle\mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots\,2\overset{[1,2]}{sym}\Big[grad^{II}(\mathbf{v})
\end{aligned}$$

$$+ 2sym\Big(grad(\mathbf{v})\Big) \cdot [grad(\mathbf{F})^{[2,3]} \cdot \mathbf{F}^{-1}]^{[2,3]}\Big]\Big] dm$$

Due to the symmetries from (B.4) and (B.5) this can be written as

$$\text{(B.21)} \qquad = \int_B \frac{1}{\rho}\Big[\big(2J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}2\rangle}{\mathbf{S}}\big) : grad(\mathbf{v}) + 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots grad^{II}(\mathbf{v})$$

$$+ 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots\Big[2sym\Big(grad(\mathbf{v})\Big) \cdot [grad(\mathbf{F})^{[2,3]} \cdot \mathbf{F}^{-1}]^{[2,3]}\Big]\Big] dm$$

$$\text{(B.22)} \qquad = \int_B \frac{1}{\rho}\Big[\big(2J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}2\rangle}{\mathbf{S}}\big) : grad(\mathbf{v}) + 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots grad^{II}(\mathbf{v})$$

$$+ \big[[grad(\mathbf{F})^{[2,3]} \cdot \mathbf{F}^{-1}]^{[2,3]} : \big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)^{23}\big] : 4sym\Big(grad(\mathbf{v})\Big)\Big] dm$$

$$\text{(B.23)} \qquad = \int_B \frac{1}{\rho}\Big[\big(2J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}2\rangle}{\mathbf{S}}\big) : grad(\mathbf{v}) + 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots grad^{II}(\mathbf{v})$$

$$+ J_{\mathbf{F}}^{-1}\big[Grad(\mathbf{F}) : \big(\mathbf{F} \cdot (\overset{\langle \mathbf{C}2\rangle}{\mathbf{S}} \cdot \mathbf{F} \cdot \mathbf{F}^{-T})^{[2,3]} \cdot \mathbf{F}^{-T}\big)^{[2,3]}\big] : 4sym\big(grad(\mathbf{v})\big)$$

$$\text{(B.24)} \qquad = \int_B \frac{1}{\rho}\Big[\big(2J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}2\rangle}{\mathbf{S}}\big) : grad(\mathbf{v}) + 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots grad^{II}(\mathbf{v})$$

$$+ 4J_{\mathbf{F}}^{-1}sym\big(Grad(\mathbf{F}) : \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}} \cdot \mathbf{F}^{T}\big) : grad(\mathbf{v})\Big] dm$$

$$\text{(B.25)} \qquad = \int_B \frac{1}{\rho}\Big[\Big(\big(2J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}2\rangle}{\mathbf{S}}\big) + 4J_{\mathbf{F}}^{-1}sym\big(Grad(\mathbf{F}) : \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}} \cdot \mathbf{F}^{T}\big)\Big) : grad(\mathbf{v})$$

$$+ 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)\vdots grad^{II}(\mathbf{v})\Big] dm$$

This reveals for the Cauchy type stress tensors $\overset{\langle 2\rangle}{\mathbf{T}}$ and $\overset{\langle 3\rangle}{\mathbf{T}}$ from Section 4.2.2

$$\text{(B.26)} \qquad \overset{\langle 2\rangle}{\mathbf{T}} = \big(2J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}2\rangle}{\mathbf{S}}\big) + 4J_{\mathbf{F}}^{-1}sym\big(Grad(\mathbf{F}) : \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}} \cdot \mathbf{F}^{T}\big)$$

$$\text{(B.27)} \qquad \overset{\langle 3\rangle}{\mathbf{T}} = 2\big(J_{\mathbf{F}}^{-1}\mathbf{F}\circ \overset{\langle \mathbf{C}3\rangle}{\mathbf{S}}\big)$$

$\square$

B.2 Transformation of strain and stress tensors under changes of the reference placement

For a change of the reference placement as introduced in Chapter 4 one obtains

Theorem B.2. Transformation of $Grad(\mathbf{C})$ under changes of the reference placement

$$\underline{Grad}(\underline{\mathbf{C}}) = \mathbf{A} \circ Grad(\mathbf{C}) + 2\overset{[1,2]}{sym}\left[\mathbf{A}^T \cdot \mathbf{C} \cdot \underline{Grad}(\mathbf{A})\right] \tag{B.28}$$

Proof.

$$\underline{Grad}(\underline{\mathbf{C}})$$

$$= \underline{Grad}(\underline{\mathbf{F}} \cdot \underline{\mathbf{F}}) \tag{B.29}$$

$$= \underline{Grad}(\mathbf{A}^T \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot \mathbf{A}) \tag{B.30}$$

$$= \underline{Grad}(\mathbf{A}^T \cdot \mathbf{C} \cdot \mathbf{A}) \tag{B.31}$$

$$= [\underline{Grad}(\mathbf{A}) \cdot \mathbf{C} \cdot \mathbf{A}]^{[2,3]} + \mathbf{A}^T \cdot \left[\underline{Grad}(\mathbf{C})^{[2,3]} \cdot \mathbf{A}\right]^{[2,3]} + \mathbf{A}^T \cdot \mathbf{C} \cdot \underline{Grad}(\mathbf{A}) \tag{B.32}$$

$$= [\underline{Grad}(\mathbf{A}) \cdot \mathbf{C} \cdot \mathbf{A}]^{[2,3]} + \mathbf{A}^T \cdot \left[[Grad(\mathbf{C}) \cdot \mathbf{A}]^{[2,3]} \cdot \mathbf{A}\right]^{[2,3]} + \mathbf{A}^T \cdot \mathbf{C} \cdot \underline{Grad}(\mathbf{A}) \tag{B.33}$$

$$= \mathbf{A} \circ Grad(\mathbf{C}) + 2\overset{[1,2]}{sym}\left[\mathbf{A}^T \cdot \mathbf{C} \cdot \underline{Grad}(\mathbf{A})\right] \tag{B.34}$$

$\square$

Theorem B.3. Transformation of stress measures under changes of the reference placement

$$\begin{aligned}\overset{\langle \mathbf{C2} \rangle}{\underline{\mathbf{S}}} &= J_{\mathbf{A}}\mathbf{A}^{-1} \circ \overset{\langle \mathbf{C2} \rangle}{\mathbf{S}} \\ &\quad - 2J_{\mathbf{A}}\mathbf{A}^{-1} \circ \mathbf{F}^{-1} \circ sym\Big(\Big[\big(\mathbf{F} \cdot \mathbf{C}^{-1} \cdot Sym\big[Grad(\mathbf{C})\big] \cdot \mathbf{A}\big)^{[2,3]} \cdot \mathbf{A} \\ &\quad + \mathbf{F} \cdot \underline{Grad}(\mathbf{A})\Big] \cdot \mathbf{A} : J_{\mathbf{A}}(\mathbf{A}^{-1} \circ \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}}) \cdot \mathbf{A}^T\mathbf{F}^T\big) \\ &\quad + \mathbf{F} \cdot \mathbf{C}^{-1} \cdot Sym\big(Grad(\mathbf{C})\big) : \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}} \cdot \mathbf{F}^T\Big)\end{aligned} \tag{B.35}$$

$$\text{(B.36)} \qquad \overset{\langle \mathbf{C3} \rangle}{\underline{\mathbf{S}}} = J_{\mathbf{A}} \mathbf{A}^{-1} \circ \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}}$$

Proof.

Equation (B.36) can be obtained by rearranging Equation (B.16). For deriving the transformation of $\overset{\langle \mathbf{C2} \rangle}{\mathbf{S}}$ one uses (B.15) to write

$$\begin{aligned}
\text{(B.37)} \qquad & \Big(\big(2 J_{\mathbf{F}}^{-1} \mathbf{F} \circ \overset{\langle \mathbf{C2} \rangle}{\mathbf{S}} \big) + 4 J_{\mathbf{F}}^{-1} sym\big(Grad(\mathbf{F}) : \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}} \cdot \mathbf{F}^T \big) \Big) \\
& = \overset{\langle 2 \rangle}{\mathbf{T}} = \Big(\big(2 J_{\underline{\mathbf{F}}}^{-1} \underline{\mathbf{F}} \circ \overset{\langle \mathbf{C2} \rangle}{\underline{\mathbf{S}}} \big) + 4 J_{\underline{\mathbf{F}}}^{-1} sym\big(\underline{Grad}(\underline{\mathbf{F}}) : \overset{\langle \mathbf{C3} \rangle}{\underline{\mathbf{S}}} \cdot \underline{\mathbf{F}}^T \big) \Big) \\
\text{(B.38)} \qquad & \Leftrightarrow \big(2 J_{\underline{\mathbf{F}}}^{-1} \underline{\mathbf{F}} \circ \overset{\langle \mathbf{C2} \rangle}{\underline{\mathbf{S}}} \big) \\
& = \big(2 J_{\mathbf{F}}^{-1} \mathbf{F} \circ \overset{\langle \mathbf{C2} \rangle}{\mathbf{S}} \big) - 4 J_{\underline{\mathbf{F}}}^{-1} sym\big(\underline{Grad}(\underline{\mathbf{F}}) : \overset{\langle \mathbf{C3} \rangle}{\underline{\mathbf{S}}} \cdot \underline{\mathbf{F}}^T \big) \\
& \quad + 4 J_{\mathbf{F}}^{-1} sym\big(Grad(\mathbf{F}) : \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}} \cdot \mathbf{F}^T \big)
\end{aligned}$$

with $\underline{\mathbf{F}} = \mathbf{F} \cdot \mathbf{A}$ and (B.36)

$$\begin{aligned}
\text{(B.39)} \qquad & \Leftrightarrow 2 J_{\mathbf{F}}^{-1} J_{\mathbf{A}}^{-1} \mathbf{F} \circ \mathbf{A} \circ \overset{\langle \mathbf{C2} \rangle}{\underline{\mathbf{S}}} \\
& = \big(2 J_{\mathbf{F}}^{-1} \mathbf{F} \circ \overset{\langle \mathbf{C2} \rangle}{\mathbf{S}} \big) - 4 J_{\mathbf{F}}^{-1} J_{\mathbf{A}}^{-1} sym\Big(\big[\big(Grad(\mathbf{F}) \cdot \mathbf{A} \big)^{[2,3]} \cdot \mathbf{A} + \mathbf{F} \cdot \underline{Grad}(\mathbf{A}) \big] \\
& \quad \cdot \mathbf{A} : J_{\mathbf{A}} (\mathbf{A}^{-1} \circ \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}}) \cdot \mathbf{A}^T \mathbf{F}^T \Big) + 4 J_{\mathbf{F}}^{-1} sym\big(Grad(\mathbf{F}) : \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}} \cdot \mathbf{F}^T \big) \\
\text{(B.40)} \qquad & \Leftrightarrow \overset{\langle \mathbf{C2} \rangle}{\underline{\mathbf{S}}} = J_{\mathbf{A}} \mathbf{A}^{-1} \circ -2 J_{\mathbf{A}} \mathbf{A}^{-1} \circ \mathbf{F}^{-1} \circ sym\Big(\big[\big(Grad(\mathbf{F}) \cdot \mathbf{A} \big)^{[2,3]} \cdot \mathbf{A} \\
& \qquad + \mathbf{F} \cdot \underline{Grad}(\mathbf{A}) \big] \cdot \mathbf{A} : J_{\mathbf{A}} (\mathbf{A}^{-1} \circ \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}}) \cdot \mathbf{A}^T \mathbf{F}^T \big) + Grad(\mathbf{F}) : \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}} \cdot \mathbf{F}^T \Big)
\end{aligned}$$

Using (B.1) finally yields

$$\begin{aligned}
\text{(B.41)} \qquad & = J_{\mathbf{A}} \mathbf{A}^{-1} \circ \overset{\langle \mathbf{C2} \rangle}{\mathbf{S}} \\
& \quad - 2 J_{\mathbf{A}} \mathbf{A}^{-1} \circ \mathbf{F}^{-1} \circ sym\Big(\Big[\big(\mathbf{F} \cdot \mathbf{C}^{-1} \cdot Sym\big[Grad(\mathbf{C}) \big] \cdot \mathbf{A} \big)^{[2,3]} \cdot \mathbf{A} \\
& \quad + \mathbf{F} \cdot \underline{Grad}(\mathbf{A}) \Big] \cdot \mathbf{A} : J_{\mathbf{A}} (\mathbf{A}^{-1} \circ \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}}) \cdot \mathbf{A}^T \mathbf{F}^T \big) \\
& \quad + \mathbf{F} \cdot \mathbf{C}^{-1} \cdot Sym\big(Grad(\mathbf{C}) \big) : \overset{\langle \mathbf{C3} \rangle}{\mathbf{S}} \cdot \mathbf{F}^T \Big)
\end{aligned}$$

$\square$

The transformation of $\overset{\langle \mathbf{C}2 \rangle}{\mathbf{S}}$ under changes of the reference placement in Equation (B.35) has two inherent disadvantages over the transformation of $\overset{\langle 2 \rangle}{\mathbf{S}}$.

1. The transformation of $\overset{\langle \mathbf{C}2 \rangle}{\mathbf{S}}$ is much more complicated than the transformation of $\overset{\langle 2 \rangle}{\mathbf{S}}$ since it cannot be written as a pushforward.

2. It involves the tensor $\mathbf{F}$ which cannot be determined from $\mathbf{C}$. All transformations of strain and stress tensors under changes of the reference placement in Chapter 4 only involve the strain variables that are used in the framework:

$$\underline{\mathbf{C}}, \overset{\langle 3 \rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}, \underline{Grad}(\overset{\langle 3 \rangle}{\mathbf{K}}_{\underline{\mathbf{F}}}), \overset{\langle 2 \rangle}{\underline{\mathbf{S}}}, \overset{\langle 3 \rangle}{\underline{\tilde{\mathbf{S}}}}, \overset{\langle 4 \rangle}{\underline{\mathbf{S}}}$$

   only depend on

$$\mathbf{C}, \overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}, Grad(\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}}), \overset{\langle 2 \rangle}{\mathbf{S}}, \overset{\langle 3 \rangle}{\tilde{\mathbf{S}}}, \overset{\langle 4 \rangle}{\mathbf{S}} \text{ as well as } \mathbf{A} \text{ and its first two gradients.}$$

   This is not surprising since in a gradient theory the rotational parts of $\mathbf{F}$ affect the spatial variables. Thus, a framework, that uses $\mathbf{C}$ and its spatial gradients, must always be endowed with extra variables that carry information on the antisymmetric part of $\mathbf{F}$ since these are not included in $\mathbf{C}$. The material variable $\overset{\langle 3 \rangle}{\mathbf{K}}_{\mathbf{F}} := \mathbf{F}^{-1} \cdot Grad(\mathbf{F})$ has the symmetric and antisymmetric parts of $\mathbf{F}$ already "built in". Together with the very complicated transformation behavior of $\overset{\langle \mathbf{C}2 \rangle}{\mathbf{S}}$ under changes of the reference placement in (B.35) it appears that $\{\mathbf{C}, Grad(\mathbf{C})\}$ is not a convenient choice for the spatial variables in a framework for strain gradient elastoplasticity. In the cases of second strain gradient elastoplasticity with $\mathbf{C}$ and its gradients the transformations would become even more complicated so that the disadvantages remain the same.

Appendix C

Further proofs

C.1 Proof of Theorem 4.9

The proof for $\overset{\langle 2\rangle}{f}$ can be found [Bertram 2015] and the proof for $\overset{\langle 4\rangle}{f}$ follows exactly the same scheme. So only the proof for $\overset{\langle 3\rangle}{\widetilde{f}}$ is presented here. The transformation rule (4.141) for $\overset{\langle 3\rangle}{\widetilde{f}}_X(...)$ with respect to $\kappa_X \circ \underline{\kappa}_X^{-1}$ and $\mathbf{A}_X := \underline{Grad}(\kappa_X \circ \underline{\kappa}_X^{-1})$ says that

$$
\begin{aligned}
&\overset{\langle 3\rangle}{\underline{\widetilde{f}}}_X\big(\underline{\mathbf{C}}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X})\big)\\
=&\gamma\Bigg(\overset{\langle 3\rangle}{\widetilde{f}}_X\Big(\mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_X^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},\\
&\qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}})\big)\Big),\\
&\quad \overset{\langle 4\rangle}{f}_X\Big(\mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_X^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},\\
&\qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}})\big)\Big),\\
&\quad \mathbf{A}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X}, \mathbf{A}_X^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}\Bigg)
\end{aligned} \tag{C.1}
$$

Now one applies the isomorphy conditions $\overset{\langle 3\rangle}{\widetilde{f}}_X(...) = \overset{\langle 3\rangle}{\widetilde{f}}_Y(...)$ and $\overset{\langle 4\rangle}{f}_X(...) = \overset{\langle 4\rangle}{f}_Y(...)$

$$
\begin{aligned}
=&\gamma\Bigg(\overset{\langle 3\rangle}{\widetilde{f}}_Y\Big(\mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_X^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},\\
&\qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}})\big)\Big),\\
&\quad \overset{\langle 4\rangle}{\underline{f}}_Y\Big(\mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_X^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},\\
&\qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}, Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}})\big)\Big),\\
&\quad \mathbf{A}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X}, \mathbf{A}_X^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}\Bigg)
\end{aligned} \tag{C.2}
$$

Next, the transformation rules (4.141) and (4.142) are applied again, this time for $\overset{\langle 3\rangle}{\widetilde{f}}_Y$ and $\overset{\langle 4\rangle}{f}_Y$ with respect to $\kappa_Y \circ \underline{\kappa}_Y^{-1}$ and $\mathbf{A}_Y := \underline{Grad}(\kappa_Y \circ \underline{\kappa}_Y^{-1})$

$$
=\gamma\Bigg(\gamma\Bigg(\overset{\langle 3\rangle}{\underline{\widetilde{f}}}_Y\Big(\mathbf{A}_Y^{T} * \mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_Y^{T} \circ \mathbf{A}_X^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \mathbf{A}_Y^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y}, \tag{C.3}
$$

$$
\begin{aligned}
&\beta\big[\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X},\mathbf{A}_X^{-1},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}})\big),\\
&\quad\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},\mathbf{A}_Y,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y})\big]\Big),\\
&\overset{\langle 4\rangle}{\underline{f}}_Y\Big(\mathbf{A}_Y^T*\mathbf{A}_X^{-T}*\underline{\mathbf{C}}_X,\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},\\
&\quad\beta\big[\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X},\mathbf{A}_X^{-1},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}})\big),\\
&\quad\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},\mathbf{A}_Y,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y})\big]\Big),\\
&\mathbf{A}_Y^{-1},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y^{-1}},\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y}\Big),\\
&\mathbf{A}_Y\circ J_{\mathbf{A}_Y}^{-1}\overset{\langle 4\rangle}{\underline{f}}_Y\Big(\mathbf{A}_Y^T*\mathbf{A}_X^{-T}*\underline{\mathbf{C}}_X,\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},\\
&\quad\beta\big[\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X},\mathbf{A}_X^{-1},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},Grad_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}})\big),\\
&\quad\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}},\mathbf{A}_Y,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},\underline{Grad}(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y})\big]\Big),\\
&\mathbf{A}_X,\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X},\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}\Big)
\end{aligned}
$$

Next the transformation rule (4.130) is applied where $\mathbf{A}_X^{-1}$ takes the role of $\mathbf{N}$ and $\mathbf{A}_Y$ takes the role of $\mathbf{M}$.

$$
\begin{aligned}
\text{(C.4)}\quad =\gamma\Bigg(\gamma\Big(&\overset{\langle 3\rangle}{\widetilde{\underline{f}}}_Y\Big(\mathbf{A}_Y^T*\mathbf{A}_X^{-T}*\underline{\mathbf{C}}_X,\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},\\
&\quad\beta\big[\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X},\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y,\underline{Grad}_X(\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y),\\
&\qquad Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y})\big]\Big),\\
&\overset{\langle 4\rangle}{\underline{f}}_Y\Big(\mathbf{A}_Y^T*\mathbf{A}_X^{-T}*\underline{\mathbf{C}}_X,\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},\\
&\quad\beta\big[\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}),\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X},\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y,\underline{Grad}_X(\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y),\\
&\qquad Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y})\big]\Big),\\
&\mathbf{A}_Y^{-1},\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y^{-1}},\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y}\Big),\\
&\mathbf{A}_Y\circ J_{\mathbf{A}_Y}^{-1}\overset{\langle 4\rangle}{\underline{f}}_Y\Big(\mathbf{A}_Y^T*\mathbf{A}_X^{-T}*\underline{\mathbf{C}}_X,\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}+\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}+\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},
\end{aligned}
$$

$$\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}\cdot\mathbf{A}_Y, \underline{Grad}_X(\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y),$$
$$Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}))\Big),$$
$$\mathbf{A}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X}, \mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}}\Bigg)$$

Transformation rule (4.131) is applied where $\mathbf{A}_Y^{-1}$ takes the role of $\mathbf{N}$ and $\mathbf{A}_X$ takes the role of $\mathbf{M}$

$$\text{(C.5)}\quad =\gamma\Bigg(\overset{\langle 3\rangle}{\underline{\tilde{f}}}_Y\Big(\mathbf{A}_Y^T * \mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},$$
$$\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}\cdot\mathbf{A}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}))\Big),$$
$$\overset{\langle 4\rangle}{\underline{f}}_Y\Big(\mathbf{A}_Y^T * \mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y},$$
$$\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}\cdot\mathbf{A}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}))\Big),$$
$$\mathbf{A}_Y^{-1}\cdot\mathbf{A}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y^{-1}\cdot\mathbf{A}_X}, \mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y}\Bigg)$$

Now one applies the transformations $\mathbf{A}_Y^T * \mathbf{A}_X^{-T} * (...) = \mathbf{A}_Y^T\cdot\mathbf{A}_X^{-T} * (...)$, $\mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ(...) = \mathbf{A}_Y^T\cdot\mathbf{A}_X^{-T}\circ(...)$ and $\mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y} = \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}$

$$\text{(C.6)}\quad =\gamma\Bigg(\overset{\langle 3\rangle}{\underline{\tilde{f}}}_Y\Big(\mathbf{A}_Y^T\cdot\mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_Y^T\cdot\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y},$$
$$\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}\cdot\mathbf{A}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}))\Big),$$
$$\overset{\langle 4\rangle}{\underline{f}}_Y\Big(\mathbf{A}_Y^T\cdot\mathbf{A}_X^{-T} * \underline{\mathbf{C}}_X, \mathbf{A}_Y^T\cdot\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y},$$
$$\beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \mathbf{A}_X^{-1}\cdot\mathbf{A}_Y, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}\cdot\mathbf{A}_Y}))\Big),$$
$$\mathbf{A}_Y^{-1}\cdot\mathbf{A}_X, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y^{-1}\cdot\mathbf{A}_X}, \mathbf{A}_Y^T\circ\mathbf{A}_X^{-T}\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \mathbf{A}_Y^T\circ\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_X^{-1}} + \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{A}_Y}\Bigg)$$

Now one introduces the notation $\overset{\langle 2\rangle}{\mathbf{P}} := \mathbf{A}_X^{-1} \cdot \mathbf{A}_Y$ and obtains

$$
\begin{aligned}
\text{(C.7)} \quad =\gamma\Big(& \overset{\langle 3\rangle}{\widetilde{\underline{f}}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, \\
& \qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})\big)\Big), \\
& \overset{\langle 4\rangle}{\underline{f}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, \\
& \qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})\big)\Big), \\
& \overset{\langle 2\rangle}{\mathbf{P}}^{-1}, \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \Big)
\end{aligned}
$$

Next one uses the relation $\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}^{-1}} = -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$ which is derived in (2.32).

$$
\begin{aligned}
\text{(C.8)} \quad =\gamma\Big(& \overset{\langle 3\rangle}{\widetilde{\underline{f}}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, \\
& \qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})\big)\Big), \\
& \overset{\langle 4\rangle}{\underline{f}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, \\
& \qquad \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})\big)\Big), \\
& \overset{\langle 2\rangle}{\mathbf{P}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}} \Big)
\end{aligned}
$$

Introducing the notation $\overset{\langle 3\rangle}{\mathbf{P}} := \overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}}$ and $\overset{\langle 4\rangle}{\mathbf{P}} := Grad_Y(\overset{\langle 3\rangle}{\mathbf{K}}_{\overset{\langle 2\rangle}{\mathbf{P}}})$ yields

$$
\begin{aligned}
\text{(C.9)} \quad =\gamma\Big(& \overset{\langle 3\rangle}{\widetilde{\underline{f}}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big)\Big), \\
& \overset{\langle 4\rangle}{\underline{f}}_Y\Big(\overset{\langle 2\rangle}{\mathbf{P}}^T * \underline{\mathbf{C}}_X, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}}, \beta\big(\underline{Grad}_X(\overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}), \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X}, \overset{\langle 2\rangle}{\mathbf{P}}, \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 4\rangle}{\mathbf{P}}\big)\Big), \\
& \overset{\langle 2\rangle}{\mathbf{P}}^{-1}, -\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{P}}, \overset{\langle 2\rangle}{\mathbf{P}}^T \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\underline{\mathbf{F}}_X} + \overset{\langle 3\rangle}{\mathbf{P}} \Big)
\end{aligned}
$$

C.2 Derivation of the plastic stress tensors

$$\rho_0 l = \frac{1}{2}\underbrace{\overset{\langle 2\rangle}{\mathbf{S}} : \mathbf{C}^\bullet}_{\mathrm{I}} + \underbrace{\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^\bullet_{\mathbf{F}}}_{\mathrm{II}} + \underbrace{\overset{\langle 4\rangle}{\mathbf{S}} :: \overset{\langle 4\rangle}{\mathbf{K}}{}^\bullet_{\mathbf{F}}}_{\mathrm{III}} \tag{C.10}$$

$$\mathrm{I} = \overset{\langle 2\rangle}{\mathbf{S}} : (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \cdot \mathbf{C}_0 \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1})^\bullet \tag{C.11}$$

$$= \overset{\langle 2\rangle}{\mathbf{S}}{}_0 : \mathbf{C}_0^\bullet + \overset{\langle 2\rangle}{\mathbf{S}} : (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T\bullet} \cdot \mathbf{C}_0 \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1}) + \overset{\langle 2\rangle}{\mathbf{S}} : (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \cdot \mathbf{C}_0 \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1\bullet}) \tag{C.12}$$

$$= \overset{\langle 2\rangle}{\mathbf{S}}{}_0 : \mathbf{C}_0^\bullet + \overset{\langle 2\rangle}{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{S}}{}_0 \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{T} : (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T\bullet} \cdot \mathbf{C}_0 \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1}) + \overset{\langle 2\rangle}{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{S}}{}_0 \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{T} : (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \cdot \mathbf{C}_0 \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1\bullet}) \tag{C.13}$$

$$= \overset{\langle 2\rangle}{\mathbf{S}}{}_0 : \mathbf{C}_0^\bullet + \overset{\langle 2\rangle}{\mathbf{S}}{}_0 : \overset{\langle 2\rangle}{\mathbf{P}}{}^{T} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-T\bullet} \cdot \mathbf{C}_0 + \overset{\langle 2\rangle}{\mathbf{S}}{}_0 : \mathbf{C}_0 \cdot \mathbf{P}^{-1\bullet}\, \overset{\langle 2\rangle}{\mathbf{P}} \tag{C.14}$$

$$= \overset{\langle 2\rangle}{\mathbf{S}}{}_0 : \mathbf{C}_0^\bullet + 2sym[\overset{\langle 2\rangle}{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{S}}{}_0 \cdot \mathbf{C}_0] : \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1\bullet} \tag{C.15}$$

$$= \overset{\langle 2\rangle}{\mathbf{S}}{}_0 : \mathbf{C}_0^\bullet + 2sym[\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{S}}{}_0 \cdot \mathbf{C}_0] : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} \tag{C.16}$$

$$\mathrm{II} = \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}} \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^\bullet_{\mathbf{F}} \tag{C.17}$$

Using Equation (5.34) that defines $\overset{\langle 3\rangle}{\mathbf{G}}$ one obtains

$$= \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}{}_0, \overset{\langle 4\rangle}{\mathbf{S}}{}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0}) \vdots (\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0} + \overset{\langle 3\rangle}{\mathbf{G}})^\bullet \tag{C.18}$$

$$= \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}{}_0, \overset{\langle 4\rangle}{\mathbf{S}}{}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0}) \vdots \Big((\overset{\langle 2\rangle}{\mathbf{P}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0})^\bullet + \overset{\langle 3\rangle}{\mathbf{G}}{}^\bullet \Big) \tag{C.19}$$

$$\begin{aligned} = \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}{}_0, \overset{\langle 4\rangle}{\mathbf{S}}{}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0}) \vdots \Big(& \overset{\langle 2\rangle}{\mathbf{P}}{}^{\bullet} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}{}^\bullet_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} \\ & + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1\bullet})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1} + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}{}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}{}^{-1\bullet} + \overset{\langle 3\rangle}{\mathbf{G}}{}^\bullet \Big) \end{aligned} \tag{C.20}$$

Using $\overset{\langle 2\rangle}{\mathbf{P}}^{\bullet} = -\,\overset{\langle 2\rangle}{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1\bullet} \cdot \overset{\langle 2\rangle}{\mathbf{P}}$ yields

$$
\begin{aligned}
&= \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \vdots \Big(-\,\overset{\langle 2\rangle}{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1\bullet} \cdot \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \\
&\quad + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{\bullet} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1} + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1\bullet})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \\
&\quad + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1\bullet} + \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} \Big)
\end{aligned} \tag{C.21}
$$

$$
\begin{aligned}
&= \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \\
&\quad \vdots \Big(\overset{\langle 2\rangle}{\mathbf{P}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} - \overset{\langle 2\rangle}{\mathbf{P}} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1\bullet} \cdot \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1} \\
&\quad + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1\bullet})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1} + \overset{\langle 2\rangle}{\mathbf{P}} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{P}}^{-1\bullet} \Big)
\end{aligned} \tag{C.22}
$$

After applying the definitions in Equations (5.33) and (5.34) and some rearranging one obtains

$$
\begin{aligned}
&= \overset{\langle 2\rangle}{\mathbf{G}} \circ \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \vdots \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{\bullet} + \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \vdots \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} \\
&\quad + \Big(2\overset{(12)}{sym} \Big[\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \Big] : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \\
&\quad - \big(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot (\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{G}})^{[2,3]} \cdot \overset{\langle 2\rangle}{\mathbf{G}} : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,2]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-1} \big)^{T} \Big) : \overset{\langle 2\rangle}{\mathbf{G}}^{\bullet}
\end{aligned} \tag{C.23}
$$

$$
\begin{aligned}
&= \overset{\langle 2\rangle}{\mathbf{G}} \circ \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \vdots \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{\bullet} + \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \vdots \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} \\
&\quad + \Big(2\overset{(12)}{sym} \Big[\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \Big] : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \\
&\quad - \big((\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,2]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-1} \big)^{T} \Big) : \overset{\langle 2\rangle}{\mathbf{G}}^{\bullet}
\end{aligned} \tag{C.24}
$$

$$
\text{III} = (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \beta(Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}), \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 4\rangle}{\mathbf{G}})^{\bullet} \tag{C.25}
$$

$$
\begin{aligned}
&= (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \\
&\quad :: \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) + \overset{\langle 4\rangle}{\mathbf{G}} - \overset{\langle 3\rangle}{\mathbf{G}} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) + 2\overset{[2,3]}{sym} \Big[(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \cdot \overset{\langle 3\rangle}{\mathbf{G}} \Big] \Big)^{\bullet}
\end{aligned} \tag{C.26}
$$

$$
\begin{aligned}
&=(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})\Big)^{\bullet} + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \overset{\langle 4\rangle}{\mathbf{G}}^{\bullet} \\
&\quad - (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(\overset{\langle 3\rangle}{\mathbf{G}} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})\Big)^{\bullet} \\
&\quad + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(2\overset{[2,3]}{sym} \Big[(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \cdot \overset{\langle 3\rangle}{\mathbf{G}} \Big]\Big)^{\bullet}
\end{aligned} \tag{C.27}
$$

$$
\begin{aligned}
&=(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})\Big)^{\bullet} + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \overset{\langle 4\rangle}{\mathbf{G}}^{\bullet} \\
&\quad - (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(\overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})\Big) - (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(\overset{\langle 3\rangle}{\mathbf{G}} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet}\Big) \\
&\quad + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(2(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet} \cdot \overset{\langle 3\rangle}{\mathbf{G}} \Big) + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(2(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \cdot \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} \Big)
\end{aligned} \tag{C.28}
$$

$$
\begin{aligned}
&=(\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})\Big)^{\bullet} + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \overset{\langle 4\rangle}{\mathbf{G}}^{\bullet} \\
&\quad - \Big((\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) : (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,3]}\Big) \vdots\, \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} - (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(\overset{\langle 3\rangle}{\mathbf{G}} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet}\Big) \\
&\quad + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \Big(2(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet} \cdot \overset{\langle 3\rangle}{\mathbf{G}} \Big) + \Big(2(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) : (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0)\Big) \vdots\, \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet}
\end{aligned} \tag{C.29}
$$

Now one applies the product rule to all terms that involve a time derivative of the "$\circ$" product.

$$
\begin{aligned}
&= \overset{\langle 4\rangle}{\mathbf{S}}_0 :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet} + \Big(- \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \cdot (\overset{\langle 2\rangle}{\mathbf{G}} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \vdots [\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})]^{[1,4]} \\
&\quad + [(\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T})^{[1,2]} \vdots Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})]^T + 2\overset{[1,2]}{sym} \Big[Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})\Big] \vdots\, \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \Big) : \overset{\langle 2\rangle}{\mathbf{G}}^{\bullet} \\
&\quad + (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) :: \overset{\langle 4\rangle}{\mathbf{G}}^{\bullet} \\
&\quad - \Big((\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) : (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,3]}\Big) \vdots\, \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} \\
&\quad - \Big(\overset{\langle 2\rangle}{\mathbf{G}} \circ \big(\overset{\langle 3\rangle}{\mathbf{G}}^{[1,2]} : (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0)\big)\Big) \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{\bullet} \\
&\quad + \Big((\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \cdot \overset{\langle 3\rangle}{\mathbf{G}} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-1})^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}}_0 \vdots \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \\
&\quad - 2\Big((\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \Big)^{[1,2]} \vdots\, \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \Big) : \overset{\langle 2\rangle}{\mathbf{G}}^{\bullet} \\
&\quad + 2\Big(\overset{\langle 2\rangle}{\mathbf{G}} \circ \big((\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) : \overset{\langle 3\rangle}{\mathbf{G}}^{[1,3]} \big)\Big) \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{\bullet} \\
&\quad + \Big(\overset{\langle 4\rangle}{\mathbf{S}}_0 \vdots (\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}})^{[1,3]} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,3]}
\end{aligned} \tag{C.30}
$$

$$
\begin{aligned}
&+ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,2]} : \big((\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T})^{[3,4]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T}\big) : \overset{\langle 3\rangle}{\mathbf{G}}^{[1,3]} \\
&+ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,3]} : \big((\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) : \overset{\langle 4\rangle}{\mathbf{S}}_0^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T}\big)\Big) : \overset{\langle 2\rangle}{\mathbf{G}}^{\bullet} \\
&+ \Big(2(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) : (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0)\Big) \vdots\, \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet}
\end{aligned}
$$

One rearranges the terms in $\rho_0 l = \mathrm{I} + \mathrm{II} + \mathrm{III}$ and obtains

$$
\begin{aligned}
\text{(C.31)}\quad \rho_0 l = {} & \overset{\langle 2\rangle}{\mathbf{S}}_0 : \mathbf{C}_0^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_0 :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet} \\
&+ \Big[\, \overset{\langle 2\rangle}{\mathbf{G}} \circ \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) - \overset{\langle 2\rangle}{\mathbf{G}} \circ \big(\, \overset{\langle 3\rangle}{\mathbf{G}}^{[1,2]} : (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0)\big) \\
&+ 2\, \overset{\langle 2\rangle}{\mathbf{G}} \circ \big((\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) : \overset{\langle 3\rangle}{\mathbf{G}}^{[1,3]}\big)\Big] \vdots\, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{\bullet} \\
&+ \Big[2sym[\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{S}}_0 \cdot \mathbf{C}_0] + 2\overset{(12)}{sym} \Big[\, \overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \,\Big] : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \\
&- \big((\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,2]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-1}\big)^{T} \\
&- \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \cdot (\overset{\langle 2\rangle}{\mathbf{G}} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \vdots [\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})]^{[1,4]} \\
&+ [(\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T})^{[1,2]} \vdots Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})]^{T} + 2\overset{[1,2]}{sym} \Big[Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})\Big] \vdots\, \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \\
&+ \overset{\langle 4\rangle}{\mathbf{S}}_0 \vdots (\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}})^{[1,3]} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,3]} \\
&+ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,2]} : \big((\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T})^{[3,4]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T}\big) : \overset{\langle 3\rangle}{\mathbf{G}}^{[1,3]} \\
&+ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,3]} : \big((\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) : \overset{\langle 4\rangle}{\mathbf{S}}_0^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T}\big) \\
&+ (\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \cdot \overset{\langle 3\rangle}{\mathbf{G}} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-1})^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}}_0 : \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}^{[1,3]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \\
&- 2\big((\overset{\langle 2\rangle}{\mathbf{G}}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}\big)^{[1,2]} \vdots\, \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}^{-T} \,\Big] : \overset{\langle 2\rangle}{\mathbf{G}}^{\bullet} \\
&+ \Big[\gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) + 2(\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) : (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \\
&- (\overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) : (\overset{\langle 2\rangle}{\mathbf{G}}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,3]}\Big] \vdots\, \overset{\langle 3\rangle}{\mathbf{G}}^{\bullet} \\
&+ \Big[\, \overset{\langle 2\rangle}{\mathbf{G}}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0 \,\Big] :: \overset{\langle 4\rangle}{\mathbf{G}}^{\bullet}
\end{aligned}
$$

$$
\begin{aligned}
\text{(C.32)}\quad &= \overset{\langle 2\rangle}{\mathbf{S}}_0 : \mathbf{C}_0^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_0 :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet} + \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0 \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}_0} \\
&\quad + \Big[2sym[\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \cdot \overset{\langle 2\rangle}{\mathbf{S}}_0 \cdot \mathbf{C}_0] + 2\overset{(12)}{sym} \Big[\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \Big] : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \\
&\quad - \big((\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) : \gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,2]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \big)^{T} \\
&\quad - \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \cdot (\overset{\langle 2\rangle}{\mathbf{G}} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \vdots [\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})]^{[1,4]} \\
&\quad + [(\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T})^{[1,2]} \vdots Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})]^{T} + 2\overset{[1,2]}{sym} \Big[Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) \Big] \vdots \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \\
&\quad + \overset{\langle 4\rangle}{\mathbf{S}}_0 \vdots (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}})^{[1,3]} \cdot (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,3]} \\
&\quad + \overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,2]}_{\mathbf{F}_0} : \big((\overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T})^{[3,4]} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \big) : \overset{\langle 3\rangle}{\mathbf{G}}{}^{[1,3]} \\
&\quad + \overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,3]}_{\mathbf{F}_0} : \big((\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) : \overset{\langle 4\rangle}{\mathbf{S}}{}^{[1,3]}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \big) \\
&\quad + (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \cdot \overset{\langle 2\rangle}{\mathbf{G}} \cdot \overset{\langle 3\rangle}{\mathbf{G}} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-1})^{[1,2]} : \overset{\langle 4\rangle}{\mathbf{S}}_0 : \overset{\langle 3\rangle}{\mathbf{K}}{}^{[1,3]}_{\mathbf{F}_0} \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \\
&\quad - 2\big((\overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \circ \overset{\langle 3\rangle}{\mathbf{G}}) \cdot \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0} \big)^{[1,2]} \vdots \overset{\langle 4\rangle}{\mathbf{S}}_0 \cdot \overset{\langle 2\rangle}{\mathbf{G}}{}^{-T} \Big] : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} \\
&\quad + \Big[\gamma(\overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0, \overset{\langle 4\rangle}{\mathbf{S}}_0, \overset{\langle 2\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{G}}, \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) + 2(\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0}) : (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) \\
&\quad - (\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0) : (\overset{\langle 2\rangle}{\mathbf{G}}{}^{T} \circ \overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{[1,3]} \Big] \vdots \overset{\langle 3\rangle}{\mathbf{G}}{}^{\bullet} \\
&\quad + \Big[\overset{\langle 2\rangle}{\mathbf{G}}{}^{-1} \circ \overset{\langle 4\rangle}{\mathbf{S}}_0 \Big] :: \overset{\langle 4\rangle}{\mathbf{G}}{}^{\bullet} \\
\text{(C.33)}\quad &= \underbrace{\overset{\langle 2\rangle}{\mathbf{S}}_0 : \mathbf{C}_0^{\bullet} + \overset{\langle 3\rangle}{\widetilde{\mathbf{S}}}_0 \vdots \overset{\langle 3\rangle}{\mathbf{K}}{}^{\bullet}_{\mathbf{F}_0} + \overset{\langle 4\rangle}{\mathbf{S}}_0 :: Grad_0(\overset{\langle 3\rangle}{\mathbf{K}}_{\mathbf{F}_0})^{\bullet}}_{\text{elastic part}} + \underbrace{\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}} : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}} \vdots \overset{\langle 3\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} :: \overset{\langle 4\rangle}{\mathbf{G}}{}^{\bullet}}_{\text{plastic part}} \\
\text{(C.34)}\quad &= w_0^{\bullet} + \underbrace{\overset{\langle 2\rangle}{\mathbf{S}}_{\mathrm{P}} : \overset{\langle 2\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 3\rangle}{\mathbf{S}}_{\mathrm{P}} \vdots \overset{\langle 3\rangle}{\mathbf{G}}{}^{\bullet} + \overset{\langle 4\rangle}{\mathbf{S}}_{\mathrm{P}} :: \overset{\langle 4\rangle}{\mathbf{G}}{}^{\bullet}}_{\text{plastic part}}
\end{aligned}
$$

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