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The equivariant spectral flow and bifurcation for functionals with symmetries: part I

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Abstract

We consider bifurcation of critical points from a trivial branch for families of functionals that are invariant under the orthogonal action of a compact Lie group. Based on a recent construction of an equivariant spectral flow by the authors, we obtain a bifurcation theorem that generalises well-established results of Smoller and Wasserman as well as Fitzpatrick, Pejsachowicz and Recht. Finally, we discuss first examples of strongly indefinite systems of differential equations where the mentioned classical approaches fail but an invariance under an orthogonal action of a compact group makes our methods applicable and yields the existence of bifurcation.

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1 Introduction

Let H be a real separable Hilbert space of infinite dimension and $f : I \times H \to \mathbb{R}$ a continuous map such that each $f_\lambda := f(\lambda, \cdot) : H \to \mathbb{R}$ is C^2 with derivatives depending

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continuously on the parameter $\lambda \in I:=[0,1]$. Let 0 be a critical point of all f_λ and consider the family of equations

$$(\nabla f_\lambda)(u) = 0,$$

which now has $u = 0$ as solution for all $\lambda \in I$. A parameter value $\lambda_0 \in I$ is called a bifurcation point (of critical points) if in every neighbourhood $U \subset I \times H$ of $(\lambda_0, 0)$ there is some (λ, u) such that $(\nabla f_\lambda)(u) = 0$ and $u \neq 0$. The existence of bifurcation points is a classical problem in nonlinear analysis that has been systematically studied for many decades. A central role is played by the second derivative $D_0^2 f_\lambda$ at the critical point $0 \in H$, which is a symmetric bounded bilinear form on H. By the Riesz-representation theorem it uniquely determines a selfadjoint operator L_λ on H such that

$$\langle L_\lambda u, v\rangle_H = (D_0^2 f_\lambda)(u, v), \quad u, v \in H, \tag{1}$$

which is called the Hessian of f at $0 \in H$. Note that it is an immediate consequence of the implicit function theorem that L_λ is not invertible if λ is a bifurcation point. Krasnoselskii considered in the sixties the case that $L_\lambda = I_H - \lambda K$, where I_H denotes the identity on H and K is a compact selfadjoint operator. He showed in a celebrated theorem that the bifurcation points of f are exactly those parameter values for which L_λ has a non-trivial kernel or, in other words, $\frac{1}{\lambda}$ is an element of the spectrum $\sigma(K)$ of the compact operator K. More generally, let us now assume that the selfadjoint operators L_λ are Fredholm, i.e., they have a finite dimensional kernel and a closed range. The following generalisation of Krasnoselskii's work is nowadays a common result in nonlinear analysis that can be found e.g. in the monographs [19, 23]. Henceforth, we denote by $\mu_-(S)$ the Morse index of a selfadjoint Fredholm operator S, i.e., the number of negative eigenvalues of S counted with multiplicities.

Theorem 1.1 *If $\mu_-(L_\lambda) < \infty$ for all $\lambda \in I$, L_0, L_1 are invertible and*

$$\mu_-(L_0) \neq \mu_-(L_1), \tag{2}$$

then there is a bifurcation point of critical points of f in $(0, 1)$.

It is readily seen that the invertibility of L_0 and L_1 cannot be lifted in this theorem (cf. [11, §12.2]). However, there is ample motivation to relax the assumption on the finiteness of the Morse indices and condition (2). Firstly, sometimes symmetries of the functionals affect the applicability of (2). For example, if the spectra of the operators L_λ are symmetric in some neighbourhood about 0, then the Morse indices are necessarily constant and the above theorem cannot be applied. Secondly, the finitness of the Morse indices excludes various important applications. For example, when studying solutions of Hamiltonian systems or non-cooperative elliptic systems of PDE as critical points of a suitable functional, the appearing operators L_λ will usually not meet this condition.

The obstacle caused by constant Morse-indices was treated by Smoller and Wasserman in their seminal work [31] as follows. Assume that G is a compact Lie group that acts orthogonally on H and that each functional f_λ is invariant under the action of G, i.e., $f_\lambda(gu) = f_\lambda(u)$ for all $g \in G$ and $u \in H$. Then the Hessians L_λ are readily seen to be G-equivariant, i.e., $L_\lambda(gu) = gL_\lambda u$. If now $\mu_-(L_\lambda) < \infty$, then the direct sum $E^-(L_\lambda)$ of all eigenspaces with respect to negative eigenvalues is of finite dimension. As $E^-(L_\lambda)$ is easily seen to be invariant under the action of G, this space is a finite dimensional representation of the Lie group G. The terminology of a nice Lie group, that was introduced in [31], will be recalled below in Sect. 3.

Theorem 1.2 *Assume that G is nice, L_0, L_1 are invertible and $\mu_-(L_\lambda) < \infty$ for all $\lambda \in I$. If*

$$E^-(L_0) \ncong E^-(L_1), \tag{3}$$

where $\cong$ stands for isomorphic representations of G, then there is a bifurcation point of critical points for f in $(0, 1)$.

Note that Theorem 1.1 follows from Theorem 1.2 as isomorphic representations are of the same dimension and $\mu_-(L_\lambda) = \dim(E^-(L_\lambda))$. Smoller and Wasserman applied Theorem 1.2 in [31] to study bifurcation of radial solutions of semilinear elliptic equations.

There have been various attempts to generalise Theorem 1.1 to the case when $\mu_-(L_\lambda) = \infty$. These were mostly tailored to specific applications like, e.g., bifurcation of branches of periodic solutions of Hamiltonian systems (cf. e.g. [20, 33]). A very general approach to this problem was introduced by Fitzpatrick, Pejsachowicz and Recht in [12]. The spectral flow is an integer-valued homotopy invariant that is defined for any path $L = \{L_\lambda\}_{\lambda \in I}$ of selfadjoint Fredholm operators that was introduced by Atiyah, Patodi and Singer in [5] in connection with the Atiyah-Singer Index Theorem. Roughly speaking, the spectral flow counts the net number of eigenvalues crossing 0 whilst the parameter λ of the path traverses the interval. We recall the construction of the spectral flow below in Sect. 2 and now just mention that if $\mu_-(L_\lambda) < \infty$ for all $\lambda \in I$, then

$$\mathrm{sf}(L) = \mu_-(L_0) - \mu_-(L_1).$$

Thus the following main theorem of [12] is a natural generalisation of Theorem 1.1 which is applicable to any family of functionals f_λ such that the associated Hessians L_λ in (1) are Fredholm operators.

Theorem 1.3 *If L_0, L_1 are invertible and*

$$\mathrm{sf}(L) \neq 0 \in \mathbb{Z}, \tag{4}$$

then there is a bifurcation point of critical points of f in $(0, 1)$.

Let us note that there are various efficient methods to compute the spectral flow, e.g., dimension reductions or crossing forms, that were in particular developed for applications in symplectic geometry (cf. e.g. [13, 29]). These have yielded several bifurcation theorems for various types of differential equations, e.g., [13, 27, 35, 37].

A natural question about Theorem 1.3 is if (4) can be further relaxed. For example, if the spectra of the operators L_λ are symmetric about 0 by some symmetry of the functionals f_λ, then necessarily $\mathrm{sf}(L)$ vanishes as net number of eigenvalues crossing through 0. But if in this case there are pairs of eigenvalues crossing the axis in opposite direction, one might still have the idea that there should be a bifurcation point. Interestingly, this is not necessarily the case by the following theorem of Alexander and Fitzpatrick from [1].

Theorem 1.4 *Let $L = \{L_\lambda\}_{\lambda\in I}$ be a path of selfadjoint Fredholm operators and $\lambda_0 \in (0, 1)$ such that L_λ is invertible for $\lambda \neq \lambda_0$. If $\mathrm{sf}(L) = 0$, then there exist an open interval $J \subset [0, 1]$ containing λ_0, an open ball $B \subset H$ and a continuous family $f : J \times B \to \mathbb{R}$ of C^2-functionals such that L_λ are the Hessians of f_λ at $0 \in H$ and $\nabla f_\lambda(0) = 0$ holds for $\lambda \in J$, but there is no bifurcation of critical points for f in J.*

The previous theorem suggests that Theorem 1.3 is optimal, and indeed the predicted phenomenon is not at all pathologic as there are natural examples of differential equations to which Theorem 1.4 applies [22].

The aim of this work is to introduce a result that generalises all previously mentioned theorems about the existence of bifurcation and which shows that despite of Theorem 1.4 there is still bifurcation under a suitable symmetry assumption on the functionals f_λ. The authors recently introduced in [17] the G-equivariant spectral flow $\mathrm{sf}_G(L)$ for paths of selfadjoint Fredholm operators $L = \{L_\lambda\}_{\lambda\in I}$ that are equivariant under the orthogonal action of a compact Lie group, i.e., $L_\lambda(gu) = g(L_\lambda u)$ for all $u \in H$ and $g \in G$. This novel homotopy invariant is an element of the representation ring $RO(G)$ that was introduced by Segal in [30]. It was shown in [17] that

$$F(\mathrm{sf}_G(L)) = \mathrm{sf}(L) \in \mathbb{Z}$$

by a natural homomorphism $F : RO(G) \to \mathbb{Z}$ that we recall below in Sect. 2. Thus even if the spectral flow $\mathrm{sf}(L) \in \mathbb{Z}$ vanishes, $\mathrm{sf}_G(L)$ can be non-trivial in $RO(G)$. Moreover, if $\mu_-(L_\lambda) < \infty$, then

$$\mathrm{sf}_G(L) = [E^-(L_0)] - [E^-(L_1)] \in RO(G), \tag{5}$$

where the square brackets stand for isomorphism classes of representations of the compact Lie group G. Consequently, if in this case $\mathrm{sf}_G(H) \neq 0 \in RO(G)$, then $E^-(L_0)$ and $E^-(L_1)$ are non-isomorphic G-representations and thus (3) holds. The main theorem of this work is Theorem 3.1 below, which states that if L_0, L_1 are invertible and $\mathrm{sf}_G(L)$ is non-trivial in the representation ring $RO(G)$ of the compact Lie group G, then there is a bifurcation point of critical points of f. As already said, this is a strong statement as it implies Theorem 1.2 and Theorem 1.3 (and thus ultimately also Theorem 1.1). The price to pay is that it is a rather challenging task to join the proofs of Theorem 1.2 and Theorem 1.3 and the majority of this paper is devoted to this issue. On the other hand, it turns out that $\mathrm{sf}_G(L)$ is quite applicable. Indeed, major tools for computing the classical spectral flow $\mathrm{sf}(L)$ in $\mathbb{Z}$ carry over to the G-equivariant case in $RO(G)$ and pave the way to various applications to Hamiltonian systems and PDEs that are invariant under actions of subgroups of the matrix groups $O(n)$ or $SO(n)$. However, to introduce these methods for computing $\mathrm{sf}_G(L)$ in detail and a thorough discussion of applications require a second part of this work. Here we instead focus on examples that shall give an impression on how the equivariant spectral flow works as bifurcation invariant, when the classical one in Theorem 1.3 vanishes and thus fails to show bifurcation. We firstly consider the finite Lie groups $\mathbb{Z}_2$ as well as $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and study bifurcation for critical points of even functionals f_λ as in [9]. Our first example is a Dirichlet problem of a system of nonlinear ODEs, where we illustrate that the particular choice of the group action can heavily affect the applicability of our main Theorem 3.1. Interestingly, the right choice in this example has no non-trivial fixed points, which thus makes clear that our Theorem 3.1 is in general not just a restriction of the original problem to a fixed-point space of the action. As second class of examples, we consider bifurcation of homoclinic solutions of even Hamiltonian systems. There have been many studies devoted to bifurcation for functionals that are invariant under orthogonal actions of compact Lie groups by degree theory (see e.g. [6] and ref. therein). As those methods only apply to functionals where the Hessians L_λ are compact perturbations of a fixed operator, homoclinic solutions are out of their scope. Finally, we consider strongly indefinite systems of elliptic PDEs that are invariant under an action of the continuous Lie group SO(2). In the non-equivariant case these systems have recently been studied by the second and the last author in [18]. It turns out that the obtained dimension reductions for the classical spectral flow work in the present setting as well and allow to compute the equivariant spectral flow for investigating bifurcation by Theorem 3.1. This shall particularly emphasize the strength of our findings and should be an appetizer for the study of continuous group actions in the upcoming second part.

The paper is structured as follows. In the next two sections we recap the classical spectral flow, introduce the G-equivariant spectral flow from [17] and state our main theorem. Section 4 provides a first pillar of the proof of our main theorem. We show the existence of a G-equivariant parametrix for any path of G-equivariant selfadjoint Fredholm operators, which allows to reduce a path of Hessians to a normal form. In Sect. 5 we use the result of Theorem 4.1 to prove Theorem 3.1, which we do in several steps. In the final section of this paper we discuss the announced examples which should stress the high applicability of our work.

2 The *G*-equivariant spectral flow

The aim of this section is to introduce the G-equivariant spectral flow, where we follow the authors recent work [17]. Let us first recap the definition of the classical spectral flow.

Let H be a real separable Hilbert space and let $\mathcal{FS}(H)$ be the set of all selfadjoint Fredholm operators on H with the norm topology. It was shown by Atiyah and Singer in [4] that $\mathcal{FS}(H)$ has three connected components

$$\mathcal{FS}^+(H):=\{T \in \mathcal{FS}(H) : \sigma_{ess}(T) \subset (0, +\infty)\}$$
$$\mathcal{FS}^-(H):=\{T \in \mathcal{FS}(H) : \sigma_{ess}(T) \subset (-\infty, 0)\},$$

and

$$\mathcal{FS}^i(H):=\mathcal{FS}(H) \setminus \mathcal{FS}^{\pm}(H),$$

where $\sigma_{ess}(T)$ denotes the essential spectrum, i.e., the set of all $\lambda \in \mathbb{R}$ such that $\lambda - T$ is not a Fredholm operator. The operators in $\mathcal{FS}^+(H)$ have a finite Morse index

$$\mu_-(T) = \dim\left(\oplus_{\mu<0}\{u \in H : Tu = \mu u\}\right), \tag{6}$$

i.e., they have at most finitely many negative eigenvalues including multiplicities. In general, for every $T \in \mathcal{FS}(H)$, there is a neighbourhood of 0 that either belongs to the resolvent set or it contains finitely many eigenvalues including multiplicities (cf. [34, 36]).

Let now $L = \{L_\lambda\}_{\lambda\in I}$ be a path in $\mathcal{FS}(H)$. As the spectra of the operators L_λ cannot accumulate at 0, it can be shown that there is a partition $0 = \lambda_0 < \ldots < \lambda_N = 1$ of the unit interval and $a_i > 0$, $i = 1, \ldots N$, such that $[\lambda_{i-1}, \lambda_i] \ni \lambda \mapsto \chi_{[-a_i,a_i]}(L_\lambda) \in \mathcal{L}(H)$ are continuous families of finite rank projections, where $\chi_{[a,b]}(T)$ denotes the spectral projection of a selfadjoint operator T with respect to the interval $[a, b] \subset \mathbb{R}$. Then, for $i = 1, \ldots, N$, the spaces $E(L_\lambda, [0, a_i]):=\operatorname{im}(\chi_{[0,a_i]}(L_\lambda))$, $\lambda_{i-1} \leq \lambda \leq \lambda_i$, are finite dimensional and the spectral flow of the path L was defined by Phillips in [26] by

$$\operatorname{sf}(L) = \sum_{i=1}^{N} (\dim(E(L_{\lambda_i}, [0, a_i])) - \dim(E(L_{\lambda_{i-1}}, [0, a_i]))) \in \mathbb{Z}. \tag{7}$$

Note that $E(L_\lambda, [0, a])$ is the direct sum of the eigenspaces of L_λ for eigenvalues in the interval $[0, a]$.

Let now G be a compact Lie group. A (real) representation of G is a pair (V, ρ) consisting of a finite dimensional (real) vector space V and a group homomorphism $\rho : G \to \mathrm{GL}(V)$. Two representations (V_1, ρ_1), (V_2, ρ_2) of G are isomorphic if there is an isomorphism $\alpha : V_1 \to V_2$ that is G-equivariant, i.e., $\rho_2(g)\circ\alpha = \alpha\circ\rho_1(g)$ for all $g \in G$. Two representations of G can be added by the direct sum and this turns the set of isomorphism classes of representations of G into a commutative monoid. The elements of the associated Grothendieck group $RO(G)$ are formal differences $[U] - [V]$ of isomorphism classes of G-representations modulo the equivalence relation generated by $[U]-[V] \sim [U \oplus W]-[V \oplus W]$. The neutral element in $RO(G)$ is $[V]-[V]$ for any G-representation V, and the inverse element of $[U] - [V]$ is $[V] - [U]$. $RO(G)$ was introduced by Segal in [30], who called it the representation ring of G as the tensor product of representations actually yields a ring structure on this Grothendieck group. We follow this terminology even though we will never use the ring structure of $RO(G)$ and consider it merely as an abelian group.

Let now G be a compact Lie group that acts orthogonally on H. We denote by $\mathcal{FS}(H)^G$ the set of G-equivariant selfadjoint Fredholm operators, i.e.,

$$T(gu) = g(Tu), \qquad u \in H, \quad g \in G,$$

and by $\mathcal{FS}^{\pm}(H)^G$ and $\mathcal{FS}^i(H)^G$ the corresponding subsets of the connected components of $\mathcal{FS}(H)$.

Let now $L = \{L_\lambda\}_{\lambda \in I}$ be a path in $\mathcal{FS}(H)^G$. As the operators L_λ are G-equivariant, it follows that the spaces $E(L_\lambda, [0, a])$ in (7) are G-invariant. Thus they define equivalence classes of G-representations and consequently the idea of (7) carries over to $RO(G)$ by setting

$$\mathrm{sf}_G(L) = \sum_{i=1}^{N} ([E(L_{\lambda_i}, [0, a_i])] - [E(L_{\lambda_{i-1}}, [0, a_i])]) \in RO(G). \tag{8}$$

Phillips proved in [26] that (7) only depends on the path L and not on the choices of the partition $0 = \lambda_0 < \ldots < \lambda_N = 1$ of the unit interval and the numbers $a_i > 0$. Recently, the authors showed in [17] that the same is true for (8) in $RO(G)$, and thus this equivariant spectral flow is well defined. Note that if G is trivial, then representations are isomorphic if and only if they are of the same dimension. Hence $RO(G) \cong \mathbb{Z}$ in this case and (8) can be identified with the ordinary spectral flow (7). In general, there is a canonical homomorphism

$$F : RO(G) \to \mathbb{Z}, \quad [U] - [V] \mapsto \dim(U) - \dim(V),$$

and it follows from (7) and (8) that

$$F(\mathrm{sf}_G(L)) = \mathrm{sf}(L). \tag{9}$$

Consequently, the classical spectral flow of L has to vanish if $\mathrm{sf}_G(L)$ is trivial. On the other hand, in [17] there is a simple example of a path of $G = \mathbb{Z}_2$-equivariant operators such that $\mathrm{sf}_G(L) \in RO(G) \cong \mathbb{Z} \oplus \mathbb{Z}$ is non-trivial even though $\mathrm{sf}(L) = 0 \in \mathbb{Z}$.

Finally, it was shown in [17] that all basic properties of the spectral flow hold mutatis mutandis for its G-equivariant generalisation (8), e.g.,

(i) If $L_\lambda \in \mathrm{GL}(H) \cap \mathcal{FS}(H)^G$ for all $\lambda \in I$, then

$$\mathrm{sf}_G(L) = 0 \in RO(G). \tag{10}$$

(ii) Let $H = H_1 \oplus H_2$, where H_1, H_2 are G-invariant and such that $L_\lambda \mid_{H_i} \in \mathcal{FS}(H_i)^G$ for $i = 1, 2$, $\lambda \in I$. Then

$$\mathrm{sf}_G(L) = \mathrm{sf}_G(L \mid_{H_1}) + \mathrm{sf}_G(L \mid_{H_2}) \in RO(G). \tag{11}$$

(iii) If L, L' are two paths in $\mathcal{FS}(H)^G$ such that $L_1 = L'_0$, then

$$\mathrm{sf}_G(L * L') = \mathrm{sf}_G(L) + \mathrm{sf}_G(L') \in RO(G), \tag{12}$$

where $L * L'$ denotes the concatenation of L' and L.

(iv) If the path L^- is defined by $L^-_\lambda = L_{1-\lambda}$, $\lambda \in I$, then

$$\mathrm{sf}_G(L^-) = -\,\mathrm{sf}_G(L) \in RO(G). \tag{13}$$

(v) If $h : I \times I \to \mathcal{FS}(H)^G$ is a homotopy such that $h(s, 0)$ and $h(s, 1)$ are invertible for all $s \in I$, then

$$\mathrm{sf}_G(h(0, \cdot)) = \mathrm{sf}_G(h(1, \cdot)) \in RO(G). \tag{14}$$

Moreover, let us recall the following proposition from [17, Prop. 3.2], where $E^-(T)$ denotes the direct sum of the eigenspaces with respect to negative eigenvalues of an operator $T \in \mathcal{FS}^+(H)$.

Proposition 2.1 *If $L = \{L_\lambda\}_{\lambda \in I}$ is a path in $\mathcal{FS}^+(H)^G$, then*

$$\mathrm{sf}_G(L) = [E^-(L_0)] - [E^-(L_1)] \in RO(G).$$

Finally, let us note that further properties of the G-equivariant spectral flow, and in particular a generalisation of (14) to homotopies with non-invertible endpoints, can be found in [17, §2.3]. The latter is necessary for the following proposition about compact perturbations in $\mathcal{FS}(H)^G$.

Proposition 2.2 *Let $L = \{L_\lambda\}_{\lambda \in I}$ and $L' = \{L'_\lambda\}_{\lambda \in I}$ be paths in $\mathcal{FS}(H)^G$ such that $L_0 = L'_0$, $L_1 = L'_1$ and $L_\lambda - L'_\lambda$ is compact for all $\lambda \in I$. Then*

$$\mathrm{sf}_G(L) = \mathrm{sf}_G(L').$$

Proof We set $K_\lambda = L_\lambda - L'_\lambda$ and first note that by assumption $K_0 = K_1 = 0$. Thus

$$h : I \times I \to \mathcal{FS}(H)^G, \qquad h(s, \lambda) = L'_\lambda + s K_\lambda$$

is a homotopy with fixed endpoints such that $h(0, \lambda) = L'_\lambda$ and $h(1, \lambda) = L_\lambda$. This shows the claimed equality as the spectral flow is invariant under homotopies with fixed endpoints by [17, Cor. 2.9]. □

We note that the previous proposition in particular applies to the case that $L_\lambda = A + K_\lambda$ and $L'_\lambda = A + K_0$, $\lambda \in I$, for some fixed $A \in \mathcal{FS}(H)^G$ and a closed path $\{K_\lambda\}_{\lambda \in I}$ of compact selfadjoint G-equivariant operators. Then $\mathrm{sf}_G(L) = \mathrm{sf}_G(L') = 0 \in RO(G)$ as it directly follows from the definition (8) that the spectral flow of a constant path vanishes.

3 Main theorem and corollaries

We consider equations of the type $\nabla f_\lambda(u) = 0$, where $f : I \times H \to \mathbb{R}$ is a family of C^2-functionals on an infinite dimensional real separable Hilbert space H, and we assume that $\nabla f_\lambda(0) = 0$ for all $\lambda \in I$, i.e. $0 \in H$ is a critical point of all functionals f_λ. A *bifurcation point* is a parameter value $\lambda^* \in I$ at which non-trivial critical points branch off from the trivial ones $I \times \{0\}$, i.e., in every neighbourhood of $(\lambda^*, 0) \in I \times H$ there is some (λ, u) such that $\nabla f_\lambda(u) = 0$ and $u \neq 0$. A crucial role for studying the existence of bifurcation points is played by the family of Hessians L_λ, which are bounded selfadjoint operators on H that are induced by the second derivatives $D_0^2 f_\lambda$ of f_λ at $0 \in H$ as in (1). It is a common assumption that the operators L_λ are Fredholm, i.e., they are elements of the space $\mathcal{FS}(H)$ introduced in the previous section. Let us now assume in addition that each f_λ is G-invariant, i.e., $f(gu) = f(u)$, $g \in G$, where G is a compact Lie group acting orthogonally on H. Under this assumption, the operators L_λ are G-equivariant, i.e. $L_\lambda(gu) = g(L_\lambda u)$ by [16, §1.3]. Thus the G-equivariant spectral flow $\mathrm{sf}_G(L)$ is defined.

Henceforth, we assume that G is *nice* in the sense of Smoller and Wasserman's work [31], i.e., any orthogonal representations E and F of G are isomorphic if the quotients $D(E)/S(E)$ and $D(F)/S(F)$ of the unit discs by the unit spheres have the same G-equivariant homotopy type. Let us stress that, e.g., if G_0 denotes the connected component of the identity in G, then G is nice if G/G_0 is trivial or a finite product of $\mathbb{Z}_2$ or $\mathbb{Z}_3$. Thus, in particular, S^1, $O(n)$ and $SO(n)$ are nice. The following theorem is the main result of this work.

Theorem 3.1 *If $L_\lambda \in \mathcal{FS}(H)^G$, $\lambda \in I$, L_0, L_1 are invertible and*

$$\mathrm{sf}_G(L) \neq 0 \in RO(G),$$

then there is a bifurcation point of critical points for f.

The following corollary is an immediate consequence of Proposition 2.1 and Theorem 3.1. It is the main result of Smoller and Wasserman's work [31] that we stated in the introduction in Theorem 1.2.

Corollary 3.2 *If $L_\lambda \in \mathcal{FS}^+(H)^G$, $\lambda \in I$, L_0, L_1 are invertible and $E^-(L_0)$ and $E^-(L_1)$ are not isomorphic as G-representations, then there is a bifurcation point of critical points for f.*

By (9), we also reobtain the main theorem of Fitzpatrick, Pejsachowicz and Recht's work [12] that we stated in the introduction in Theorem 1.3.

Corollary 3.3 *If $L_\lambda \in \mathcal{FS}(H)$, $\lambda \in I$, L_0, L_1 are invertible and* $\mathrm{sf}(L) \neq 0 \in \mathbb{Z}$, *then there is a bifurcation point of critical points for f.*

Finally, let us emphasize that Theorem 1.1 and Krasnoselskii's bifurcation theorem (of which we reminded above Theorem 1.1) are immediate consequences of both Corollary 3.2 and Corollary 3.3. Consequently, they ultimately are also covered by our Theorem 3.1.

4 The *G*-equivariant cogredient parametrix

We call an operator $Q \in \mathcal{L}(H)$ a symmetry if it is of the form $Q = P - (I_H - P) = 2P - I_H$ for an orthogonal projection P which has infinite dimensional kernel and range. Note that any symmetry Q satisfies $Q^2 = I_H$ and $Q \in \mathcal{FS}^i(H)$. Moreover, Q is G-equivariant if and only if $\operatorname{im}(P)$ and $\ker(P)$ are G-invariant subspaces of H.

The aim of this section is the proof of the following theorem, which is a pillar of the proof of Theorem 3.1. Henceforth we denote by $\mathcal{KS}(H)$ the space of all selfadjoint compact operators with the norm topology, and by $\mathcal{KS}(H)^G$ its G-equivariant elements. Similarly, $\mathrm{GL}(H)^G$ stands for the G-equivariant invertible operators.

Theorem 4.1 *Let $L = \{L_\lambda\}_{\lambda \in I}$ be a path in $\mathcal{FS}^i(H)^G$. Then there is a G-equivariant symmetry $Q \in \mathcal{FS}^i(H)^G$ and paths $M = \{M_\lambda\}_{\lambda \in I}$ in $\mathrm{GL}(H)^G$ and $K = \{K_\lambda\}_{\lambda \in I}$ in $\mathcal{KS}(H)^G$ such that*

$$M_\lambda^* L_\lambda M_\lambda = Q + K_\lambda, \qquad \lambda \in I.$$

The remainder of this section is devoted to the proof of this theorem. Let us first sketch the idea. We consider for a fixed G-equivariant symmetry $Q \in \mathcal{FS}^i(H)^G$ the map

$$\pi_Q : \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G \to \mathcal{FS}^i(H)^G, \quad \pi_Q(M, K) = MQM^* + K. \tag{15}$$

Note that $\operatorname{im}(\pi_Q)$ is indeed in $\mathcal{FS}^i(H)^G$ as G acts orthogonally and thus the adjoint M^* is G-equivariant as well. Clearly Theorem 4.1 is shown if we can prove that for some Q, the path L can be lifted to $\mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$, i.e., if there is a continuous map $\tilde{L} : I \to \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ such that $L_\lambda = \pi_Q \circ \tilde{L}$ for all $\lambda \in I$. In the non-equivariant case, this was done in [12] by showing that (15) is the projection of a fibre bundle. Then the desired lifting of L can be obtained from a global section of the pullback bundle of (15) by L, which exists as the latter bundle has a contractible base space.

Unfortunately, in our more general setting, π_Q is not necessarily surjective, which affects the argument of [12]. Indeed, we will see below that in general $\operatorname{im}(\pi_Q)$ is a union of connected components of $\mathcal{FS}^i(H)^G$ if G is non-trivial. Moreover, at the end of this section we provide an example where these components are not all of $\mathcal{FS}^i(H)^G$.

Before we begin the proof of Theorem 4.1, we note the following simple lemma about functional calculus that will be used throughout the rest of the paper.

Lemma 4.2 *Let $T \in \mathcal{L}(H)$ be selfadjoint and $f : \sigma(T) \to \mathbb{R}$ a continuous function on the spectrum of T. If T is G-equivariant, then so is $f(T)$.*

Proof Note that $p(T)$ is G-equivariant for every polynomial, and for every $\varepsilon > 0$ there is a polynomial such that $\|f - p\|_\infty < \varepsilon$ on $\sigma(T)$. Hence it follows for $g \in G$ that

$$\begin{aligned}\|f(T)g - gf(T)\| &\leq \|f(T)g - p(T)g\| + \|gp(T) - gf(T)\| \leq 2\|p(T) - f(T)\| \\ &\leq 2\|p - f\|_\infty \leq 2\varepsilon,\end{aligned}$$

which implies that $f(T)g = gf(T)$. □

The following quite technical proposition shows the existence of a local section of (15). Let us point out that even in the case of a trivial group action, the result is more general than the corresponding Lemma 2.2 in [12]. The latter only constructs a section in a neighbourhood of any symmetry in $\mathcal{FS}^i(H)^G$ which is not enough for our purposes due to the already mentioned lack of surjectivity of π_Q.

Proposition 4.3 *For any $S \in \mathcal{FS}^i(H)^G$ there is a G-equivariant symmetry Q_S, an open neighbourhood $\mathcal{U}_S$ of S in $\mathcal{FS}^i(H)^G$ and a map $\sigma_S : \mathcal{U}_S \to \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ such that*

$$(\pi_{Q_S} \circ \sigma_S)(T) = T \quad \textit{for all} \quad T \in \mathcal{U}_S.$$

Proof Let K be the orthogonal projection onto the kernel of S. Then $K \in \mathcal{KS}(H)^G$ as $\ker(S)$ is G-invariant and of finite dimension, and moreover

$$V{:=}S + K \in GL(H)^G. \tag{16}$$

Henceforth we denote by $P_+(V) = \chi_{(0,\infty)}(V)$ and $P_-(V) = \chi_{(-\infty,0)}(V)$ the projections on the positive and negative spectral subspaces of V. Note that these operators are G-equivariant by Lemma 4.2. We set

$$Q_S = 2P_+(V) - I_H \in \mathcal{FS}^i(H)^G \tag{17}$$

and choose a neighbourhood $\tilde{\mathcal{U}} \subset \mathcal{FS}^i(H)^G$ of Q_S that consists of invertible operators. As above, we let $P_+(T)$ and $P_-(T)$ denote the orthogonal projections onto the positive and negative spectral subspaces for $T \in \tilde{\mathcal{U}}$ which are again G-equivariant. As $P_\pm(T)$ continuously depend on $T \in \tilde{\mathcal{U}}$ and as $\mathrm{GL}(H)^G$ is open, there is a neighbourhood $\mathcal{U} \subset \tilde{\mathcal{U}}$ of Q_S such that

$$P_+(T)P_+(Q_S) + P_-(T)P_-(Q_S) \in GL(H)^G, \quad T \in \mathcal{U},$$

where we use that this operator is the identity for $T = Q_S$. Consequently, for each $T \in \mathcal{U}$, the restriction of $P_\pm(T)$ to $\mathrm{im}(P_\pm(Q_S))$ is a bijection onto $\mathrm{im}(P_\pm(T))$. Henceforth, we set $H_\pm{:=}\,\mathrm{im}(P_\pm(Q_S))$ to simplify notation.

We now consider for $T \in \mathcal{U}$ the bilinear form on H_+ defined by

$$B(T)(u, v) = \langle T P_+(T)u, P_+(T)v\rangle, \quad u, v \in H_+.$$

Then clearly $B(T)$ is bounded, symmetric and positive definite. Moreover, $B(T)$ is G-invariant as T and $P_+(T)$ are G-equivariant and G acts orthogonally. Let $\widetilde{T}$ be the Riesz-representation of $B(T)$, i.e., the unique selfadjoint operator such that

$$B(T)(u, v) = \langle \widetilde{T}u, v\rangle, \quad u, v \in H_+.$$

As $\widetilde{T}$ is unique and $B(T)$ is G-invariant, it is readily seen that $\widetilde{T}$ is G-equivariant. Thus, again by Lemma 4.2, the inverse square-root $S_+(T):=\widetilde{T}^{-\frac{1}{2}}$ is G-equivariant as well. Moreover,

$$\begin{aligned}\langle u,v\rangle &= \langle \widetilde{T}^{-1}\widetilde{T}u,v\rangle = \langle \widetilde{T}^{-\frac{1}{2}}\widetilde{T}u,\widetilde{T}^{-\frac{1}{2}}v\rangle = \langle \widetilde{T}\widetilde{T}^{-\frac{1}{2}}u,\widetilde{T}^{-\frac{1}{2}}v\rangle = B(T)(\widetilde{T}^{-\frac{1}{2}}u,\widetilde{T}^{-\frac{1}{2}}v)\\ &= \langle TP_+(T)\widetilde{T}^{-\frac{1}{2}}u, P_+(T)\widetilde{T}^{-\frac{1}{2}}v\rangle = \langle \widetilde{T}^{-\frac{1}{2}}P_+(T)TP_+(T)\widetilde{T}^{-\frac{1}{2}}u,v\rangle\\ &= \langle S_+(T)P_+(T)TP_+(T)S_+(T)u,v\rangle, \quad u,v\in H_+,\end{aligned}$$

which implies that

$$S_+(T)P_+(T)TP_+(T)S_+(T) = I_{H_+}. \tag{18}$$

In the same way, we can construct a family $S_- : \mathcal{U} \to \mathrm{GL}(H_-)^G$ such that for all $T \in \mathcal{U}$

$$-\langle u,v\rangle = \langle S_-(T)P_-(T)TP_-(T)S_-(T)u,v\rangle, \quad u,v\in H_-,$$

and thus

$$S_-(T)P_-(T)TP_-(T)S_-(T) = -I_{H_-}. \tag{19}$$

If we now define $S_0 : \mathcal{U} \to \mathrm{GL}(H)^G$ by

$$S_0(T) = P_+(T)S_+(T)P_+(Q_S) - P_-(T)S_-(T)P_-(Q_S),$$

then it follows from (18) and (19) that

$$S_0(T)^*TS_0(T) = P_+(Q_S) - P_-(Q_S) = Q_S, \quad T\in\mathcal{U}.$$

Consequently, the map

$$\sigma : \mathcal{U} \to \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G, \quad \sigma(T) = ((S_0(T)^{-1})^*, 0),$$

satisfies

$$(\pi_{Q_S}\circ\sigma)(T) = \pi_{Q_S}((S_0(T)^{-1})^*,0) = (S_0(T)^{-1})^*Q_SS_0(T)^{-1} = T, \quad T\in\mathcal{U}. \tag{20}$$

Let us recall that $\mathcal{U}$ is a neighbourhood of the symmetry Q_S that was defined in (17). Our next aim is to get a section σ_V in a neighbourhood $\mathcal{U}_V$ of V in (16). In [12] it was noted that $\mathcal{G}:=\mathrm{GL}(H)\times\mathcal{KS}(H)$ is a topological group with respect to

$$(M,K)\cdot(\tilde{M},\tilde{K}) = (M\tilde{M}, K + M\tilde{K}M^*), \tag{21}$$

and there is an action τ of $\mathcal{G}$ on $\mathcal{FS}^i(H)$ defined by

$$\tau_h(L) = MLM^* + K, \quad L \in \mathcal{FS}^i(H), \quad h = (M, K) \in \mathcal{G}.$$

Now $\mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ is a closed subgroup of $\mathcal{G}$ and the action τ restricts to an action of it on $\mathcal{FS}^i(H)^G$. We set $h{:=}(|V|^{\frac{1}{2}}, 0) \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ and see by functional calculus that

$$\tau_h(Q_S) = |V|^{\frac{1}{2}} Q_S |V|^{\frac{1}{2}} = V,$$

where (17) is used as well as the obvious equality $\sqrt{|x|}(2\chi_{(0,\infty)}(x) - 1)\sqrt{|x|} = x$, $x \in \mathbb{R}$. Thus $\mathcal{U}_V{:=}\tau_h(\mathcal{U})$ is an open neighbourhood of V in $\mathcal{FS}^i(H)^G$. We set

$$\sigma_V : \mathcal{U}_V \to \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G, \quad \sigma_V(T) = h \cdot \sigma(\tau_{h^{-1}}(T)), \quad T \in \mathcal{U}_V,$$

where $h = (|V|^{\frac{1}{2}}, 0) \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ as above and the group multiplication (21) is used. To show that $\pi_{Q_S} \circ \sigma_V = id\mid_{\mathcal{U}_V}$, first note that for any $h_1, h_2 \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$,

$$\pi_{Q_S}(h_1 \cdot h_2) = \tau_{h_1}(\pi_{Q_S}(h_2)),$$

which directly follows from the definition of τ and (21). Thus by (20)

$$\begin{aligned}(\pi_{Q_S} \circ \sigma_V)(T) &= \pi_{Q_S}(h \cdot \sigma(\tau_{h^{-1}}(T))) = \tau_h(\pi_{Q_S}(\sigma(\tau_{h^{-1}}(T))))\\ &= \tau_h(\tau_{h^{-1}}(T)) = T, \quad T \in \mathcal{U}_V,\end{aligned} \tag{22}$$

as claimed. Finally, we set $\tilde{h} = (I_H, -K) \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ and define $\mathcal{U}_S = \tau_{\tilde{h}}(\mathcal{U}_V)$ as well as

$$\sigma_S : \mathcal{U}_S \to \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G, \quad \sigma_S(T) = \tilde{h} \cdot \sigma_V(\tau_{\tilde{h}^{-1}}(T)), \quad T \in \mathcal{U}_S.$$

Then $\mathcal{U}_S$ is an open neighbourhood of S and the same computation as in (22) shows that indeed $\pi_{Q_S} \circ \sigma_S = id\mid_{\mathcal{U}_S}$, which ends our proof. □

The following lemma and its corollary concern the image of the map π_Q.

Lemma 4.4 *Let Q_1, Q_2 be symmetries and π_{Q_1}, π_{Q_2} the associated maps in* (15). *If* $\mathrm{im}(\pi_{Q_1}) \cap \mathrm{im}(\pi_{Q_2}) \neq \emptyset$, *then* $\mathrm{im}(\pi_{Q_1}) = \mathrm{im}(\pi_{Q_2})$.

Proof If $S \in \mathrm{im}(\pi_{Q_1}) \cap \mathrm{im}(\pi_{Q_2})$, then there are $(M_1, K_1), (M_2, K_2) \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ such that

$$S = \pi_{Q_1}(M_1, K_1) = \pi_{Q_2}(M_2, K_2).$$

Thus

$$S = M_1 Q_1 M_1^* + K_1 = M_2 Q_2 M_2^* + K_2,$$

which implies

$$Q_2 = M_2^{-1} M_1 Q_1 M_1^* (M_2^{-1})^* + M_2^{-1}(K_1 - K_2)(M_2^{-1})^*.$$

If we now set $\tilde{h} = (M_2^{-1} M_1, M_2^{-1}(K_1 - K_2)(M_2^{-1})^*) \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$, then a direct computation yields

$$\pi_{Q_2}(h) = \pi_{Q_1}(h \cdot \tilde{h}), \quad h \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G.$$

Thus $\mathrm{im}(\pi_{Q_2}) \subset \mathrm{im}(\pi_{Q_1})$, which actually shows the lemma by swapping Q_1 and Q_2 in the argument. □

Corollary 4.5 *For every symmetry Q, the image of π_Q is a union of connected components of $\mathcal{FS}^i(H)^G$.*

Proof We show that $\mathrm{im}(\pi_Q)$ is open and closed in $\mathcal{FS}^i(H)^G$. Firstly, if $S \in \mathrm{im}(\pi_Q)$, then by Proposition 4.3 there is an open neighbourhood $\mathcal{U}_S$ of S in $\mathcal{FS}^i(H)^G$ and a symmetry Q_S such that $\mathcal{U}_S \subset \mathrm{im}(\pi_{Q_S})$. Thus, by Lemma 4.4, $\mathcal{U}_S \subset \mathrm{im}(\pi_Q)$ showing that the latter set is open. Secondly, let $\{S_n\}_{n\in\mathbb{N}} \subset \mathrm{im}(\pi_Q)$ be a sequence that converges to some $S \in \mathcal{FS}^i(H)^G$. Again by Proposition 4.3, there is a neighbourhood $\mathcal{U}_S$ of S in $\mathcal{FS}^i(H)^G$ and a symmetry Q_S such that $\mathcal{U}_S \subset \mathrm{im}(\pi_{Q_S})$. Now $S_n \in \mathcal{U}_S$ for sufficiently large n, which by Lemma 4.4 implies that $\mathcal{U}_S \subset \mathrm{im}(\pi_Q)$ and thus $S \in \mathrm{im}(\pi_Q)$. Consequently, the image of π_Q is closed, which finishes the proof. □

In what follows we denote by B_Q the image of the map π_Q for a given symmetry Q. Moreover, note that if $Q \in \mathcal{FS}^i(H)^G$ is a symmetry and we set $S = Q$ in Proposition 4.3, then (16) and (17) show that $Q_Q = Q$. Consequently, for every symmetry $Q \in \mathcal{FS}^i(H)^G$ there is some $S \in \mathcal{FS}^i(H)^G$ such that $Q_S = Q$. Henceforth we simplify our notation by not specifying S anymore.

Proposition 4.6 *The map $\pi_Q : \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G \to B_Q$ is the projection of a locally trivial fibre-bundle with fibre given by the isotropy group of $Q \in \mathcal{FS}^i(H)^G$.*

Proof The proof is almost identical to [12, Prop. 2.4], but we sketch the argument for the convenience of the reader. By Proposition 4.3 there is an open subset $\mathcal{U}_Q \subset B_Q$ and a local section σ of π_Q on $\mathcal{U}_Q$. Then

$$\eta : \mathcal{U}_Q \times \pi_Q^{-1}(Q) \to \pi_Q^{-1}(\mathcal{U}_Q), \quad \eta(S, h) = \sigma(S) \cdot h$$

is a local trivialisation over $\mathcal{U}_Q$ with inverse

$$\eta^{-1}(h) = (\pi_Q(h), (\sigma \circ \pi_Q)(h))^{-1} \cdot h).$$

Now this trivialisation can be transported to any point $T \in B_Q$ as follows. As π_Q is surjective onto B_Q, there is some $\tilde{h} \in \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ such that $\pi_Q(\tilde{h}) =$

$\tau_{\tilde{h}}(Q) = T$. Then $\mathcal{U}:=\tau_{\tilde{h}}(\mathcal{U}_Q)$ is a neighbourhood of T and $\tau_{\tilde{h}} : \mathcal{U}_Q \to \mathcal{U}$ is a homeomorphism. Now a local trivialisation over $\mathcal{U}$ is given by

$$\eta' : \mathcal{U} \times \pi_Q^{-1}(Q) \to \pi_Q^{-1}(\mathcal{U}), \quad \eta'(S, h) = \tilde{h} \cdot \sigma(\tau_{\tilde{h}^{-1}}(S)) \cdot h,$$

which shows the claim of the proposition. □

Now we finally have everything at hand to prove Theorem 4.1 along the lines that we already sketched at the beginning of this section. Let $L = \{L_\lambda\}_{\lambda \in I}$ be a path in $\mathcal{FS}^i(H)^G$. Then the trace $L(I)$ of L is contained in a path component C of $\mathcal{FS}^i(H)^G$. Now let $S \in C$ be arbitrary and let Q be the associated proper symmetry by Propositon 4.3. By the previous proposition,

$$\pi_Q : \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G \to B_Q \tag{23}$$

is the projection of a locally trivial fibre-bundle and clearly $C \subset B_Q$ by Corollary 4.5. Let (E, I, π) be its pullback by L, i.e., the bundle having

$$E = \{(\lambda, h) \in I \times (\mathrm{GL}(H)^G \times \mathcal{KS}(H)^G) : L_\lambda = \pi_Q(h)\}$$

as total space and as bundle projection π the restriction of the projection onto the first component. Note that the projection onto the second component $I \times (\mathrm{GL}(H)^G \times \mathcal{KS}(H)^G) \to \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ yields a bundle map from E to the total space of (23). By composing with this map, sections of (E, I, π) yield liftings of L, and thus the desired map $\tilde{L} : I \to \mathrm{GL}(H)^G \times \mathcal{KS}(H)^G$ such that $L_\lambda = \pi_Q \circ \tilde{L}_\lambda$ for all $\lambda \in I$. Now (E, I, π) is a bundle over the contractible space I and thus trivial. As the triviality of the bundle implies the existence of a globally defined section, this proves Theorem 4.1.

As we have pointed out before, the main difficulty in the above argument in comparison to [12] is that B_Q can be different from $\mathcal{FS}^i(H)^G$. The applications in this paper in Sect. 6 deal with the rather simple case of a $G = \mathbb{Z}_2$-action. We conclude this section by an example which shows that even in this case $B_Q \neq \mathcal{FS}^i(H)^G$ is possible. Let H be an infinite dimensional Hilbert space and consider on $H \oplus H$ the $\mathbb{Z}_2$-action which maps (u, v) to $(u, -v)$ by its non-trivial element. Then every equivariant operator is of diagonal form. If we now take the proper symmetry $Q(u, v) = (u, -v)$, then we obtain for $M = \mathrm{diag}(A, B) \in \mathrm{GL}(H \oplus H)^{\mathbb{Z}_2}$ and $K = \mathrm{diag}(C, D) \in \mathcal{KS}(H \oplus H)^{\mathbb{Z}_2}$

$$\pi_Q(M, K) = \begin{pmatrix} AA^* + C & 0 \\ 0 & -BB^* + D \end{pmatrix}$$

and thus every element in B_Q is of the form $\mathrm{diag}(S, T)$, where the essential spectrum of S is on the positive half-line and the essential spectrum of T is on the negative half-line. Thus the proper symmetry $\tilde{Q}:= - Q$, $\tilde{Q}(u, v) = (-u, v)$ is not an element of B_Q.

5 Proof of Theorem 3.1

We note at first that it suffices to prove the theorem in the case that the Hessians L_λ are strongly indefinite, i.e., $L_\lambda \in \mathcal{FS}^i(H)^G$, $\lambda \in I$. Indeed, if $f : I \times H \to \mathbb{R}$ is a family of G-invariant functionals as in Theorem 3.1 such that $L_\lambda \in \mathcal{FS}(H)^G$, $\lambda \in I$, then consider the family of functionals $\overline{f} : I \times H \times H \times H \to \mathbb{R}$ given by

$$\overline{f}_\lambda(w, u, v) = f_\lambda(u) + \frac{1}{2}\|w\|^2 - \frac{1}{2}\|v\|^2.$$

Clearly, this family has the same bifurcation points of critical points as f, and $\overline{f}$ is G-invariant under the orthogonal action $g(w, u, v) = (w, gu, v)$. Finally, the corresponding path of Hessians $\overline{L} = \{\overline{L}_\lambda\}_{\lambda\in I}$ has the same G-equivariant spectral flow by (11).

Thus we henceforth assume that $L_\lambda \in \mathcal{FS}^i(H)^G$, $\lambda \in I$, and obtain from Theorem 4.1 a G-equivariant cogredient parametrix for L, i.e., a path $M : I \to \mathrm{GL}(H)^G$ such that

$$M_\lambda^* L_\lambda M_\lambda = Q + K_\lambda, \qquad \lambda \in I, \tag{24}$$

where K_λ are G-equivariant and compact, and $Q \in \mathcal{FS}^i(H)^G$ is a G-equivariant symmetry. Note that the functionals of the family $\tilde{f} : I \times H \to \mathbb{R}$, $\tilde{f}_\lambda(u) = f_\lambda(M_\lambda u)$ are G-invariant and $\nabla \tilde{f}_\lambda(u) = M_\lambda^*(\nabla f_\lambda)(M_\lambda u)$ so that $\tilde{f}_\lambda$ and f_λ have the same bifurcation points of critical points. Moreover, the Hessians $\tilde{L}_\lambda$ of $\tilde{f}_\lambda$ are given by $\tilde{L}_\lambda = M_\lambda^* L_\lambda M_\lambda$. Note that $\tilde{L}_\lambda \in \mathcal{FS}(H)^G$, $\lambda \in I$ and thus the G-equivariant spectral flow is defined.

Lemma 5.1 *The paths of operators L and $\tilde{L}$ from above have the same G-equivariant spectral flow, i.e.,*

$$\mathrm{sf}_G(L) = \mathrm{sf}_G(\tilde{L}) \in RO(G).$$

Proof We note at first that $\tilde{L}$ is homotopic to the path $\{M_0^* L_\lambda M_0\}_{\lambda\in I}$ and the corresponding homotopy does not affect the spectral flow by (14) as $L_0, L_1 \in \mathrm{GL}(H)$ by the assumptions of Theorem 3.1. Now consider the polar decomposition $M_0 = UR$ of M_0, where $U = M_0(M_0^* M_0)^{-\frac{1}{2}}$ is orthogonal and $R = (M_0^* M_0)^{\frac{1}{2}}$ is selfadjoint and positive. Moreover, U and R are G-equivariant by Lemma 4.2. We have

$$M_0^* L_\lambda M_0 = RU^* L_\lambda U R, \quad \lambda \in I,$$

and see that the homotopy

$$\{((1-s)R + sI_H)U^* L_\lambda U((1-s)R + sI_H)\}_{(s,\lambda)\in I\times I}$$

deforms $\{M_0^* L_\lambda M_0\}_{\lambda\in I}$ into the path $\{U^* L_\lambda U\}_{\lambda\in I}$. Note that also this homotopy does not affect the spectral flow by (14) as $L_0, L_1 \in \mathrm{GL}(H)$ and $(1-s)R + sI_H \in GL(H)$

for all $s \in I$. Finally, for any $a > 0$,

$$U^* : E(L_\lambda, [0, a]) \to E(U^* L_\lambda U, [0, a]), \quad \lambda \in I,$$

is a G-equivariant isomorphism and thus

$$[E(L_\lambda, [0, a])] = [E(U^* L_\lambda U, [0, a])], \quad \lambda \in I.$$

Consequently, it follows from the definition (8) that $\{U^* L_\lambda U\}_{\lambda \in I}$ and L have the same G-equivariant spectral flow, which proves the lemma. □

In summary, we can henceforth assume without loss of generality that $L_\lambda = Q + K_\lambda$, $\lambda \in I$, where the operators K_λ are compact, selfadjoint and G-equivariant, and $Q = P - (I_H - P)$ for some G-equivariant orthogonal projection P having infinite dimensional kernel and range.

5.1 Reduction to finite dimensions I

We begin by a technical lemma on decomposing H into finite dimensional invariant subspaces. Note that in case of a trivial group action, this is merely the existence of an orthonormal basis. In the general case, our proof very much relies on the Kuratowski-Zorn lemma, as it is also used in [16, Cor. 5.4 (a)], and thus appears three times in our argument below.

Lemma 5.2 *There is a sequence of finite-dimensional G-invariant subspaces $H_n \subset H$, $n \in \mathbb{N}$, such that*

$$H_n \subset H_{n+1} \quad \textit{and} \quad P_n u \xrightarrow{n\to\infty} u \textit{ for all } u \in H,$$

where P_n denotes the orthogonal projection onto H_n.

Proof Let $\mathcal{F}$ be the set of all subsets $B \subset H$ such that

(i) $\|x\| = 1$ for all $x \in B$,
(ii) $\langle x, y\rangle = 0$ for all $x, y \in B$, $x \neq y$,
(iii) for all $x \in B$ there exists a subspace $V \subset H$ of finite dimension such that

$$Gx := \{gx : g \in G\} \subset V.$$

Note that $\mathcal{F}$ is not empty as every representation of G on an infinite dimensional Banach space has a finite dimensional subrepresentation by [16, Cor. 5.4 (a)]. We now partially order $\mathcal{F}$ by inclusion and aim to use the Kuratowski-Zorn lemma. If $\mathcal{E} \subset \mathcal{F}$ is totally ordered, then the union of all elements in $\mathcal{E}$ satisfies $(i) - (iii)$ from above and thus is an upper bound for $\mathcal{E}$. Consequently, there exists a maximal element B^* of $\mathcal{F}$. As B^* is orthonormal and H is separable, B^* is countable, say $B^* = \{e_1, e_2, \ldots\}$.

We now let H_n be the intersection of all G-invariant subspaces of H that contain $\{e_1, \dots, e_n\}$. Note that H_n is of finite dimension by (iii). Moreover,

$$U := \overline{\bigcup_{n=1}^{\infty} H_n}$$

is a G-invariant subspace of H. Now assume that $U \neq H$. Then $U^\perp$ is a G-invariant subspace and thus contains a finite dimensional subrepresentation. The latter claim is trivial if $U^\perp$ is of finite dimension, and otherwise again follows by [16, Cor. 5.4 (a)]. Now we take an element v, $\|v\| = 1$, of this finite dimensional subrepresentation of $U^\perp$. Then $B^* \cup \{v\} \in \mathcal{F}$ is larger than B^* which contradicts the maximality. Thus $U = H$, which in particular implies that $(P_n)_{n\in\mathbb{N}}$ weakly converges to the identity. □

Let us recall that $Q = P - (I_H - P)$ for some orthogonal projection P having infinite dimensional kernel and range.

Corollary 5.3 *There is a sequence of finite dimensional G-invariant subspaces $H_n \subset H$, $n \in \mathbb{N}$, such that*

$$H_n \subset H_{n+1}, \quad [P_n, Q] = 0,\ n \in \mathbb{N}, \quad \text{and} \quad P_n u \xrightarrow{n\to\infty} u \text{ for all } u \in H,$$

where P_n denotes the orthogonal projection onto H_n.

Proof We denote by H^+ the image of P and by H^- its kernel, which are both invariant under G and of infinite dimension. By Lemma 5.2, there are finite-dimensional G-invariant subspaces $H_n^\pm \subset H_{n+1}^\pm \subset H^\pm$, $n \in \mathbb{N}$, such that $P_n^\pm u \xrightarrow{n\to\infty} u$, $u \in H^\pm$, for the orthogonal projections $P_n^\pm$ onto $H_n^\pm$ in $H^\pm$. We set $H_n := H_n^+ \oplus H_n^-$ and note that $P_n := P_n^+ + P_n^-$ is the orthogonal projection onto H_n if we regard $P_n^\pm$ as orthogonal projection on H with kernel extended to $H^\mp$. Now the first and the third claimed property are satisfied. The remaining one follows from

$$\begin{aligned} P_n Q - Q P_n &= (P_n^+ + P_n^-)(P - (I_H - P)) - (P - (I_H - P))(P_n^+ + P_n^-) \\ &= P_n^+ P - P_n^-(I_H - P) - P P_n^+ + (I_H - P) P_n^- = 0, \end{aligned}$$

where we use that $P_1 P_2 = P_2 P_1 = P_1$ for orthogonal projections P_1, P_2 such that $\operatorname{im}(P_1) \subset \operatorname{im}(P_2)$. □

Note that as P_n commutes with Q by the previous lemma, it follows that $Q(H_n) = H_n$ as well as $Q(H_n^\perp) = H_n^\perp$.

Lemma 5.4 *There is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$*

(i) $(I_H - P_n)L_\lambda\,|_{H_n^\perp} \in \mathrm{GL}(H_n^\perp)$, $\quad \lambda \in I$,
(ii) $sL_\lambda + (1-s)((I_H - P_n)L_\lambda(I_H - P_n) + P_n L_\lambda P_n) \in \mathrm{GL}(H)$, $\quad \lambda = 0, 1,\ s \in I$.

Proof We note at first that

$$(I_H - P_n)L_\lambda \mid_{H_n^\perp} = Q + (I_H - P_n)K_\lambda \mid_{H_n^\perp}$$

is a compact perturbation of an invertible operator and thus Fredholm of index 0. Consequently, to prove the first assertion, we only need to show that $(I_H - P_n)L_\lambda \mid_{H_n^\perp}$ is injective.

Since $\{K_\lambda\}_{\lambda\in I}$ is a continuous family of compact operators, the set $\{K_\lambda(u) : \lambda \in I,\ \|u\| = 1\}$ is relatively compact. Therefore, as $I_H - P_n$ uniformly converges to 0 on compact subsets of H, there exists $n_0 \in \mathbb{N}$ such that

$$\|(I_H - P_n)K_\lambda u\| \leq \frac{1}{2}\|u\|, \quad u \in H,\ \lambda \in I,\ n \geq n_0.$$

Moreover, $\|Qu\| = \|u\|$, $u \in H$, as Q is orthogonal, and thus

$$\|(I_H - P_n)L_\lambda u\| = \|Qu + (I_H - P_n)K_\lambda u\| \geq \frac{1}{2}\|u\|, \quad u \in H_n^\perp,$$

showing the injectivity of $(I_H - P_n)L_\lambda \mid_{H_n^\perp}$.

To show (ii), we note at first that by a simple calculation

$$\begin{aligned} &sL_\lambda + (1-s)((I_H - P_n)L_\lambda(I_H - P_n) + P_n L_\lambda P_n) \\ &= Q + sK_\lambda + (1-s)((I_H - P_n)K_\lambda(I_H - P_n) + P_n K_\lambda P_n), \end{aligned}$$

which are all Fredholm operators of index 0. We now assume by contradiction that n_0 as in the assertion does not exist. Then there are sequences $(u_n)_{n\in\mathbb{N}}$, $\|u_n\| = 1$, and $(s_n)_{n\in\mathbb{N}}$ such that

$$Qu_n + s_n K_0 u_n + (1-s_n)((I_H - P_n)K_0(I_H - P_n)u_n + P_n K_0 P_n u_n) = 0, \quad n \in \mathbb{N}.$$

As K_0 is compact and P_n converges on compact subsets of H to the identity, we see that there is a convergent subsequence of (Qu_n). Henceforth, we denote this sequence by the same indices and assume as well that (s_n) converges to some $s^* \in I$. It follows from the invertibility of Q that (u_n) converges to some $u \in H$ of norm 1. Thus

$$\lim_{n\to\infty}(I_H - P_n)K_0(I_H - P_n)u_n = 0, \quad \lim_{n\to\infty} P_n K_0 P_n u_n = K_0 u,$$

and so

$$L_0 u = Qu + K_0 u = Qu + s^* K_0 u + (1-s^*)K_0 u = 0$$

in contradiction to the invertibility of L_0. Of course, the same argument applies to the invertible operator L_1. □

We now set $L^n_\lambda := P_n L_\lambda \mid_{H_n} : H_n \to H_n$ and note that these operators are G-equivariant. It follows from (10), (11) and Proposition 2.1 that for $n \geq n_0$

$$\mathrm{sf}_G(L) = \mathrm{sf}_G(L^n) = [E^-(L_1)] - [E^-(L_0)] \in RO(G) \tag{25}$$

and thus the Hessians are reduced to finite dimensions.

5.2 Reduction to finite dimensions II

For reducing the nonlinear problem to finite dimensions, we need the following technical lemma from [17, Lem. 3.7] that was shown in the non-equivariant case in [12, 24].

Lemma 5.5 *Let H be a real Hilbert space and G a compact Lie group acting orthogonally on H. Let $U \subset H$ be an open invariant subset of H containing $0 \in U$ and $f : I \times U \to \mathbb{R}$ a continuous one-parameter family of G-invariant C^2-functionals. Let $F(\lambda, u) := (\nabla f_\lambda)(u)$ and assume that $F(\lambda, 0) = 0$ for all $\lambda \in I$. Suppose that there is an orthogonal decomposition $H = X \oplus Y$, where X is G-invariant and of finite dimension, and such that for*

$$F(\lambda, u) = (F_1(\lambda, x, y), F_2(\lambda, x, y)) \in X \oplus Y, \quad u = (x, y) \in X \oplus Y,$$

we have that $(D_y F_2)(\lambda, 0, 0) : Y \to Y$ is invertible for all $\lambda \in I$. Then:

(i) There are an open ball $B_X = B(0, \delta) \subset X$ and a unique continuous family of equivariant C^1-maps $\eta : I \times B_X \to Y$ such that $\eta(\lambda, 0) = 0$ for all $\lambda \in I$, and

$$F_2(\lambda, x, \eta(\lambda, x)) = 0, \quad (\lambda, x) \in I \times B_X. \tag{26}$$

(ii) Let the family of functionals $\overline{f} : I \times B_X \to \mathbb{R}$ and the map $\overline{F} : I \times B_X \to X$ be defined by

$$\overline{f}(\lambda, x) = f(\lambda, x, \eta(\lambda, x)), \qquad \overline{F}(\lambda, x) = F_1(\lambda, x, \eta(\lambda, x)).$$

Then $\overline{f}$ is a continuous family of G-invariant C^2-functionals on B_X and

$$\nabla \overline{f}(\lambda, x) = \overline{F}(\lambda, x), \qquad (\lambda, x) \in I \times B_X,$$

which is a G-equivariant map.

We now set $X = H_n$, $Y = H_n^\perp$ and consider the splitting $F = (F_1^n, F_2^n)$, where

$$F_1^n(\lambda, u, v) = P_n F(\lambda, u, v), \qquad F_2^n(\lambda, u, v) = (I_H - P_n) F(\lambda, u, v).$$

As $D_v F_2^n(\lambda, 0, 0) = (I_H - P_n) L_\lambda \mid_{H_n^\perp} : H_n^\perp \to H_n^\perp$ is an isomorphism for $n \geq n_0$ by Lemma 5.4, we obtain from the previous lemma a family of G-invariant functionals

$\overline{f}: I \times B_{H_n} \to \mathbb{R}$ for some ball $B_{H_n} \subset H_n$ such that every bifurcation point of critical points of $\overline{f}$ is a bifurcation point of f. Consequently, our aim is now to show that $\overline{f}$ has a bifurcation point of critical points from the trivial branch if (25) is non-trivial in $RO(G)$.

Proposition 5.6 *For the Hessians $\overline{L}^n_\lambda$ of the G-invariant functionals $\overline{f}_\lambda$ at $0 \in H_n$, there exists $n_1 \geq n_0$ such that $\overline{L}^n_\lambda$ is invertible and*

$$[E^-(\overline{L}^n_\lambda)] = [E^-(L^n_\lambda)], \quad \lambda = 0, 1,$$

for $n \geq n_1$.

Proof Let $\eta^n_\lambda : B_{H_n} \to H_n^\perp$ be the continuous family of C^1-maps from Lemma 5.5 for the splitting $H = H_n \oplus H_n^\perp$, and set $A^n_\lambda := D_0\eta^n_\lambda$. Note that A^n_λ is G-equivariant. By differentiating (26) implicitly, we obtain

$$A^n_\lambda = -(D_v F^n_2(\lambda, 0, 0))^{-1} D_u F^n_2(\lambda, 0, 0) = -((I_H - P_n)L_\lambda \mid_{H_n^\perp})^{-1}(I_H - P_n)L_\lambda \mid_{H_n} .$$

In the first part of the proof of Lemma 5.4 we obtained

$$\|(I_H - P_n)L_\lambda u\| \geq \frac{1}{2}\|u\|, \quad u \in H_n^\perp,\ n \geq n_0,$$

which shows that

$$\|((I_H - P_n)L_\lambda \mid_{H_n^\perp})^{-1}\| \leq 2, \quad n \geq n_0,\ \lambda = 0, 1.$$

Using once again that $L_\lambda = Q + K_\lambda$ and

$$\|(I_H - P_n)K_\lambda\| \to 0, \quad n \to \infty, \tag{27}$$

by the compactness of K_λ, this yields

$$\|A^n_\lambda\| \leq 2\|(I_H - P_n)(Q + K_\lambda) \mid_{H_n} \| \leq 2\|(I_H - P_n)K_\lambda\| \to 0, \quad n \to \infty, \tag{28}$$

which we note for later reference.

We now consider $\overline{L}^n_\lambda$ and note at first that

$$\overline{L}^n_\lambda = P_n L_\lambda (I_{H_n} + A^n_\lambda) = L^n_\lambda + P_n L_\lambda A^n_\lambda.$$

Let us introduce two paths $\{M^n_{(s,\lambda)}\}_{s\in I}, \lambda = 0, 1,$ of G-equivariant selfadjoint operators on H_n by

$$M^n_{(s,\lambda)} = L^n_\lambda + s P_n L_\lambda A^n_\lambda.$$

We now aim to find $n_1 \in \mathbb{N}$ such that $M^n_{(s,0)}$ and $M^n_{(s,1)}$ are invertible for all $s \in I$ and $n \geq n_1$.

We first note that there is $k_1 \in \mathbb{N}$ and a constant $C > 0$ such that for $\lambda = 0, 1$ and all $n \geq k_1$

$$\|L_\lambda^n u\| = \|P_n L_\lambda u\| \geq C\|u\|, \quad u \in H_n. \tag{29}$$

Indeed, as L_λ is invertible for $\lambda = 0, 1$, there is a constant $C > 0$ such that

$$\|L_\lambda u\| \geq 2C\|u\|, \quad u \in H,\ \lambda = 0, 1.$$

Now by direct computation

$$P_n L_\lambda u = L_\lambda u - (I_H - P_n)K_\lambda u, \quad u \in H_n,$$

and (27) implies that there is $k_1 \in \mathbb{N}$ such that for $n \geq k_1$

$$\|(I_H - P_n)K_\lambda u\| \leq C\|u\|, \quad u \in H,$$

which shows (29). Finally, by (28) there is $k_2 \in \mathbb{N}$ such that $\|L_\lambda\|\|A_\lambda^n\| \leq C$ for all $n \geq k_2$, $\lambda = 0, 1$. Consequently, if $n \geq n_1 := \max\{k_1, k_2\}$,

$$\|M_{(s,\lambda)}^n u\| \geq \|L_\lambda^n u\| - s\|P_n L_\lambda A_\lambda^n u\| \geq 2C\|u\| - \|L_\lambda\|\|A_\lambda^n\|\|u\| \geq C\|u\|$$

for $\lambda = 0, 1$ and $0 \leq s \leq 1$. Thus $M_{(s,\lambda)}^n : H_n \to H_n$ is injective and hence invertible on the finite dimensional space H_n. Note that the proposition is shown if we prove that

$$[E^-(M_{(0,\lambda)}^n)] = [E^-(M_{(1,\lambda)}^n)], \quad n \geq n_1, \tag{30}$$

for $\lambda = 0, 1$. As $M_{(s,\lambda)}^n$ is invertible for all $s \in I$, the maps $[0, 1] \ni s \mapsto \chi_{(-\infty,0)}(M_{(s,\lambda)}^n) \in \mathcal{L}(H_n)$ are continuous. Thus there is a partition $0 = s_0 \leq s_1 \leq \ldots \leq s_k = 1$ such that

$$\|\chi_{(-\infty,0)}(M_{(s_j,\lambda)}^n) - \chi_{(-\infty,0)}(M_{(s_{j-1},\lambda)}^n)\| < 1. \tag{31}$$

Moreover, these projections are G-equivariant as their images are invariant and G acts orthogonally. We now shorten our notation by setting $P := \chi_{(-\infty,0)}(M_{(s_j,\lambda)}^n)$, $Q := \chi_{(-\infty,0)}(M_{(s_{j-1},\lambda)}^n)$, and we claim that $\operatorname{im}(P)$ and $\operatorname{im}(Q)$ are isomorphic as G-representations. To prove this, we first note that the G-equivariant map $U := PQ + (I_H - P)(I_H - Q)$ maps $\operatorname{im}(P)$ into $\operatorname{im}(Q)$. Moreover, a direct computation yields

$$(QP + (I_H - Q)(I_H - P))U = I_H - (P - Q)^2.$$

As $\|P - Q\| < 1$ by (31), the right hand side is an isomorphism and consequently U is injective. Thus $U \mid_{\operatorname{im}(P)} : \operatorname{im}(P) \to \operatorname{im}(Q)$ is a G-equivariant isomorphism and so $[E^-(M_{(s_j,\lambda)}^n)] = [E^-(M_{(s_{j-1},\lambda)}^n)]$ for $j = 1, \ldots, k$. Thus (30) is shown, which eventually finishes the proof of the proposition. □

In conclusion, by (25) and the previous proposition, we have reduced Theorem 3.1 to finite dimensions, i.e., we only need to prove it under the additional assumption that $\dim(H) < \infty$.

5.3 Equivariant Conley index and end of the proof

The aim of this final step of the proof is to show Theorem 3.1 under the additional assumption that $\dim(H) < \infty$. The proof is based on the equivariant Conley index, for which we mainly follow Bartsch's monograph [7].

Let $\phi_\lambda : \mathbb{R} \times H \to H$ be the flow of the equation

$$u'(t) = -(\nabla f_\lambda)(u(t)) \tag{32}$$

and note that its stationary solutions are the critical points of f_λ. Here we assume without loss of generality that the flow is global, which can be achieved by multiplying f_λ by a smooth cut-off function in a neighbourhood of $0 \in H$ and this does not affect the existence of bifurcation of critical points from $0 \in H$. Note that $\varphi_\lambda(t, \cdot) : H \to H$ is equivariant.

For a G-invariant subset $U \subset H$ we denote by

$$\operatorname{inv}(U, \varphi_\lambda) = \{u \in H : \varphi_\lambda(t, u) \in U \text{ for all } t \in \mathbb{R}\}$$

the maximal (flow-)invariant subset of U, which clearly is G-invariant as well. A compact invariant set $S \subset H$ is called isolated if there is a compact G-invariant neighbourhood U of S such that $S = \operatorname{inv}(U, \varphi_\lambda)$ and $S \subset \operatorname{int} U$. In this case U is called an isolating neighbourhood of S. If $S \subset H$ is an isolated invariant set, then a pair (N_1, N_0) of compact G-invariant subsets $N_0 \subset N_1$ is called a G-index pair for S if

- $\overline{N_1 \setminus N_0}$ is an isolating neighbourhood of S,
- N_0 is positively invariant with respect to N_1, which means that if $u \in N_0$ and $\varphi_\lambda(t, u) \in N_1$ for all $0 \leq t \leq t'$, then $\varphi_\lambda(t, u) \in N_0$ for all $0 \leq t \leq t'$,
- N_0 is an exit set for N_1, which means that if $u \in N_1$ and $\varphi_\lambda(t, u) \notin N_1$ for some $t > 0$ then there is $t' \in [0, t)$ such that $\varphi_\lambda(t', u) \in N_0$ and $\varphi_\lambda([0, t'], u) \subset N_1$.

If $U \subset H$ is an isolating neighbourhood for the flow φ_λ, then there is a G-index pair for $S = \operatorname{inv} U$, and if (N_1, N_0), (N_1', N_0') are two G-index pairs for S, then the quotient spaces N_1/N_0 and N_1'/N_0' are homotopy equivalent by a base point preserving G-equivariant homotopy equivalence. Thus it is sensible to define the G-equivariant Conley index $\mathcal{C}(U, \varphi_\lambda)$ of S as the based G-homotopy type $[N_1/N_0, [N_0]]$, where (N_1, N_0) is any G-index pair for S. Finally, let us recall that by the continuation theorem for the Conley index $\mathcal{C}(U, \varphi_0) = \mathcal{C}(U, \varphi_1)$ if U is an isolating neighbourhood for φ_λ for all $\lambda \in I$.

Let us now come back to bifurcation of critical points and let us recall that $u \equiv 0 \in H$ is a stationary solution of (32) for all $\lambda \in I$. Suppose that there is no bifurcation point. Since any isolated critical point is an isolated invariant set, there exists $\epsilon > 0$ such that $U = D(0, \epsilon)$ is an isolating neighbourhood for all φ_λ, $\lambda \in I$. This implies that $\mathcal{C}(U, \varphi_0) = \mathcal{C}(U, \varphi_1)$. On the other hand, we know that $[E^-(L_0)] \neq [E^-(L_1)]$, i.e., these spaces are non-isomorphic G-representations. As G is nice, this implies that the quotients $D_0/\partial D_0$ and $D_1/\partial D_1$ are not G-homotopic, where D_λ denotes the unit disc of $E^-(L_\lambda)$ for $\lambda = 0, 1$. In our case $\mathcal{C}(U, \varphi_\lambda) = [D_\lambda/\partial D_\lambda, [\partial D_\lambda]]$ for $\lambda = 0, 1$. This contradicts the equality $\mathcal{C}(U, \varphi_0) = \mathcal{C}(U, \varphi_1)$, and consequently the assumption that there are no bifurcation points.

6 First examples of bifurcation of critical points

The aim of this section is to illustrate Theorem 3.1 by examples of functionals f_λ that are invariant under an action of the discrete groups $\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or the continuous group SO(2). Let us point out that an upcoming second part of our work is going to transfer various methods for computing the classical spectral flow (7) to its G-equivariant generalisation (8), which will yield more sophisticated examples. Here we restrict on rather elementary, but instructive ones, that show how the G-equivariant spectral flow (8) works: as for the G-equivariant degree (e.g. [6, 16]), its job is to decompose the space behind the scenes into G-equivariant subspaces and to find non-trivial solutions of the non-linear problem in them, even though the classical bifurcation invariant might fail by symmetry reasons. All our three examples focus on different aspects of this issue. We begin by an ODE, where we show that a wise choice of the group action may be needed to apply Theorem 3.1 successfully. Moreover, as this action has a trivial fixed-point space, this first example also shows in an elementary way that Theorem 3.1 is far more subtle than just restricting a bifurcation problem to a fixed point space of the action in an obvious way. Afterwards we discuss a closed path of Hamiltonian systems under homoclinic boundary conditions which explicitly shows that the G-equivariant spectral flow can be non-trivial for closed paths and thus in general does not only depend on the endpoints of the path. This is in strong contrast to Rybicki's G-equivariant degree (e.g. [15]). In both of our first two examples G is one of the simplest non-trivial groups, namely $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In our final example we instead consider a class of indefinite elliptic PDEs as in, e.g., [15, 27] or [18], and an action of the continuous group $SO(2)$, where we compute the equivariant spectral flow by a dimension reduction that has recently been studied for the classical spectral flow in [18]. This shall particularly be an appetizer for the study of continuous group actions in the upcoming second part of our work.

We now first recap some basics that are needed in the second and third example below. As all real irreducible representations of $\mathbb{Z}_2$ are one dimensional, every real k-dimensional representation is up to isomorphism a $k \times k$ diagonal matrix of the form $\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$. Thus we obtain an isomorphism $\phi : RO(\mathbb{Z}_2) \to \mathbb{Z} \oplus \mathbb{Z}$ of abelian groups by setting

$$\phi([E] - [F]) = (\dim(E) - \dim(F), \dim(E^G) - \dim(F^G)), \tag{33}$$

where $E^G \subset E$ and $F^G \subset F$ denote the spaces of fixed points under the group action.

Lemma 6.1 *Let H be a real separable Hilbert space on which $G = \mathbb{Z}_2$ acts orthogonally. Then for every path $L = \{L_\lambda\}_{\lambda\in I}$ in $\mathcal{FS}(H)^G$*

$$\phi(\mathrm{sf}_G(L)) = (\mathrm{sf}(L), \mathrm{sf}(L\mid_{H^G})) \in \mathbb{Z}\oplus\mathbb{Z}, \tag{34}$$

where H^G is the fixed point set of the action of G. Moreover,

$$\mathrm{sf}(L) = \mathrm{sf}(L\mid_{H^G}) + \mathrm{sf}(L\mid_{(H^G)^\perp}). \tag{35}$$

Proof Note that H^G reduces the operators L_λ and thus we indeed obtain a path of selfadjoint operators $L\mid_{H^G} = \{L_\lambda\mid_{H^G}\}_{\lambda\in I}$ that all have finite dimensional kernels. Moreover, $\mathrm{im}(L_\lambda\mid_{H^G}) = \mathrm{im}(L_\lambda\mid_{H^G\cap(\ker L_\lambda)^\perp})$ and the latter set is closed in $\mathrm{im}(L_\lambda)$ as $L_\lambda\mid_{(\ker L_\lambda)^\perp}\colon (\ker L_\lambda)^\perp \to \mathrm{im}(L_\lambda)$ is a homeomorphism. Consequently, $\mathrm{im}(L_\lambda\mid_{H^G})$ is closed in H and thus in H^G. Therefore the operators $L_\lambda\mid_{H^G}$ are in $\mathcal{FS}(H^G)$ and so $\mathrm{sf}(L\mid_{H^G})$ is defined. Likewise the restriction $L\mid_{(H^G)^\perp}$ to the invariant subspace $(H^G)^\perp$ is an element of $\mathcal{FS}((H^G)^\perp)$, and now (35) follows from (11).

Finally, (34) is a simple consequence of (7), (8) and (33) when noting that

$$E(L_\lambda,[0,a])^G = H^G\cap E(L_\lambda,[0,a]) = E(L_\lambda\mid_{H^G},[0,a])$$

for any $a > 0$. □

For applying (34) below we also need a common method to compute the classical spectral flow (7) (see [29, 35]). Let $L = \{L_\lambda\}_{\lambda\in I}$ be a path in $\mathcal{FS}(H)$ that is continuously differentiable in the parameter λ. We call $\lambda\in I$ a crossing if $\ker(L_\lambda)\neq\{0\}$, and the associated crossing form is the quadratic form defined by

$$\Gamma(L,\lambda)[u] = \langle \dot{L}_\lambda u, u\rangle, \quad u\in\ker(L_\lambda),$$

where $\dot{L}_\lambda$ denotes the derivative with respect to λ. A crossing $\lambda\in I$ is regular if $\Gamma(L,\lambda)$ is non-degenerate. Regular crossings are isolated and thus every path L parametrised by a compact interval I can only have finitely many of them. Finally, if $L = \{L_\lambda\}_{\lambda\in I}$ has only regular crossings and if L_0, L_1 are invertible, then the spectral flow (7) is given by

$$\mathrm{sf}(L) = \sum_{\lambda\in I}\mathrm{sgn}(\Gamma(L,\lambda)), \tag{36}$$

where $\mathrm{sgn}(\Gamma(L,\lambda))$ denotes the signature of the quadratic form $\Gamma(L,\lambda)$ (see [35, Thm. 2.7]).

Finally, let us note that the irreducible representations of

$$SO(2) = \left\{\begin{pmatrix}\cos(\phi) & -\sin(\phi)\\ \sin(\phi) & \cos(\phi)\end{pmatrix} : \phi\in[0,2\pi]\right\}$$

are the one-dimensional trivial representation ρ and the non-trivial representations ρ^j, $j \in \mathbb{N}$, given by

$$\rho^j(g) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}^j = \begin{pmatrix} \cos(j\phi) & -\sin(j\phi) \\ \sin(j\phi) & \cos(j\phi) \end{pmatrix}, \quad g \in \mathrm{SO}(2). \tag{37}$$

Note that the representation ρ^{-j} is isomorphic to ρ^j. If we build formal differences in the Grothendieck group, it is readily seen that $RO(\mathrm{SO}(2))$ is isomorphic (as abelian group) to the polynomial ring $\mathbb{Z}[x]$. Moreover, if $\mathrm{sf}_{SO(2)}(L) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{Z}[x]$ under this identification for some path L in $\mathcal{FS}(H)^{\mathrm{SO}(2)}$, then it follows from (9) that $\mathrm{sf}(L) = a_0 + 2\sum_{n=1}^{\infty} a_n \in \mathbb{Z}$.

6.1 An ODE example

Let us denote by γ the generator of the group $\mathbb{Z}_2$ and by α, β the generators of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. In what follows,

- V is the 1-dimensional real representation of $\mathbb{Z}_2$ such that $\gamma v = -v$ for every $v \in V$;
- V_α is the 1-dimensional real representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ such that $\alpha v = -v$ and $\beta v = v$ for every $v \in V_\alpha$;
- V_β is the 1-dimensional real representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ such that $\alpha v = v$ and $\beta v = -v$ for every $v \in V_\beta$.

Note that V_α and V_β are non-isomorphic representations of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. To shorten notation, we henceforth use the same letter Z for $\mathbb{R}^2$, $V \oplus V$ and $V_\alpha \oplus V_\beta$.

Now let $\mathcal{F}\colon [0,1] \times Z \to \mathbb{R}$ be defined by

$$\mathcal{F}(\lambda, u, v) = \lambda \left(e^{v^2} \sin(u^2) - \sin(v^2) \right).$$

We see at once that $\mathcal{F}$ is $\mathbb{Z}_2$-invariant if $Z = V \oplus V$, and it is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-invariant if $Z = V_\alpha \oplus V_\beta$. Consequently, $\nabla \mathcal{F}_z\colon [0,1] \times Z \to Z$ is an equivariant map.

Now consider the following family of strongly indefinite ODEs

$$\begin{cases} -u'' = 2\lambda u e^{v^2} \cos(u^2) & \text{in } (0,\pi), \\ \quad v'' = 2\lambda v \left(e^{v^2} \sin(u^2) - \cos(v^2) \right) & \text{in } (0,\pi), \\ u(0) = u(\pi) = v(0) = v(\pi) = 0. \end{cases} \tag{38}$$

The solutions of (38) are critical points of the functional $f_\lambda\colon H_0^1((0,\pi), Z) \to \mathbb{R}$ defined by

$$f_\lambda(u, v) = \frac{1}{2} \int_0^\pi \left(u'^2(t) - v'^2(t) \right) dt + \int_0^\pi \mathcal{F}(\lambda, u(t), v(t)) dt.$$

Obviously,

$$(u(t), v(t)) = (0, 0), \ \ t \in [0, \pi],$$

is a solution of (38) for every $\lambda \in [0, 1]$. The second derivative of f_λ at $(0, 0)$ is given by

$$\begin{aligned} D_0^2 f_\lambda(w_1, w_2) &= \int_0^\pi \big(u'(t)p'(t) - v'(t)q'(t)\big)\, dt \\ &\quad + \int_0^\pi (u(t), v(t)) \begin{pmatrix} 2\lambda & 0 \\ 0 & -2\lambda \end{pmatrix} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} dt, \end{aligned}$$

where $w_1 = (u, v)$, $w_2 = (p, q) \in H_0^1((0, \pi), Z)$.

We now aim to find an explicit formula for the Riesz representation L_λ of $D_0^2 f_\lambda$, which we need to compute the equivariant spectral flow. The set of elements

$$\{(\sin(mt), 0) : \ m \in \mathbb{N}\} \cup \{(0, \sin(nt)) \colon n \in \mathbb{N}\}$$

is an orthogonal Schauder basis of $H_0^1((0, \pi), Z)$. Let H_1 and H_2 be the closures in $H_0^1((0, \pi), Z)$ of the subspaces spanned by

$$\{(\sin(mt), 0) : \ m \in \mathbb{N}\} \text{ and } \{(0, \sin(nt)) \colon n \in \mathbb{N}\},$$

so that $H_0^1((0, \pi), Z) \cong H_1 \oplus H_2$. Note that, if $Z = V \oplus V$, then $H_1 \oplus H_2$ is an orthogonal representation of $\mathbb{Z}_2$ and $\gamma(x, y) = (-x, -y)$ for all $(x, y) \in H_1 \oplus H_2$. If, on the other hand, $Z = V_\alpha \oplus V_\beta$, then $H_1 \oplus H_2$ is an orthogonal representation of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ such that $\alpha(x, y) = (-x, y)$ and $\beta(x, y) = (x, -y)$ for all $(x, y) \in H_1 \oplus H_2$.

Let now $j \colon H_0^1((0, \pi), Z) \to L^2((0, \pi), Z)$ be the canonical inclusion and let $j^* \colon L^2((0, \pi), Z) \to H_0^1((0, \pi), Z)$ be its conjugate. Then the Riesz representation

$$L_\lambda \colon H_1 \oplus H_2 \to H_1 \oplus H_2$$

of $D_0^2 f_\lambda$, is given by

$$L_\lambda(u, v) = (u - 2\lambda j^* u, -v + 2\lambda j^* v) \tag{39}$$

for every $(u, v) \in H_1 \oplus H_2$. Note that $L_\lambda = T + K_\lambda$ for an isomorphism T and a path $K = \{K_\lambda\}_{\lambda \in I}$ of compact operators. Moreover,

$$L_\lambda(\sin(mt), 0) = \left(\sin(mt) - \frac{2\lambda}{m^2} \sin(mt), 0\right) = \left(1 - \frac{2\lambda}{m^2}\right) (\sin(mt), 0)$$

for all $m \in \mathbb{N}$, and

$$L_\lambda\, (0, \sin(nt)) = \left(0, -\sin(nt) + \frac{2\lambda}{n^2} \sin(nt)\right) = \left(-1 + \frac{2\lambda}{n^2}\right) (0, \sin(nt))$$

for all $n \in \mathbb{N}$. Thus L_λ is an isomorphism for every $\lambda \neq \frac{1}{2}$ and $\dim\ker L_{\frac{1}{2}} = 2$. These findings allow to compute the equivariant spectral flows under the group actions from above.

We just need to note that, if $Z = V \oplus V$, then every 1-dimensional subspace of $H_1 \oplus H_2$ is isomorphic to the representation V of $\mathbb{Z}_2$. Moreover, if $Z = V_\alpha \oplus V_\beta$ then every 1-dimensional subspace of H_1 is isomorphic to the representation V_α of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and every 1-dimensional subspace of H_2 is isomorphic to the representation V_β of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Consequently, we obtain for the path $L = \{L_\lambda\}_{\lambda \in I}$

$$\mathrm{sf}(L) = 0 \text{ if } Z = \mathbb{R}^2,$$

$$\mathrm{sf}_{\mathbb{Z}_2}(L_\lambda) = [V] - [V] = 0 \in RO(\mathbb{Z}_2) \text{ if } Z = V \oplus V,$$

and

$$\mathrm{sf}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}(L_\lambda) = [V_\beta] - [V_\alpha] \neq 0 \text{ in } RO(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \text{ if } Z = V_\alpha \oplus V_\beta.$$

Hence $\lambda^* = \frac{1}{2}$ is a bifurcation point for the parametrised system (38), which would not have been found by applying the classical Theorem 1.3. Furthermore, also our equivariant version of this theorem fails if the symmetry of the functionals in Theorem 3.1 is chosen too naively.

Finally, let us have a closer look at this example. Denote by (α) and (β) the subgroups of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by α and β, respectively. Then

$$(H_1 \oplus H_2)^{(\alpha)} = \{0\} \times H_2 = H_2$$

and

$$(H_1 \oplus H_2)^{(\beta)} = H_1 \times \{0\} = H_1.$$

As ∇f_λ respects symmetries of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, one has $\nabla f_\lambda^{(\alpha)} \colon H_2 \to H_2$ and $\nabla f_\lambda^{(\beta)} \colon H_1 \to H_1$, which shows that

$$\nabla f_\lambda(H_1) \subset H_1 \text{ and } \nabla f_\lambda(H_2) \subset H_2.$$

Since

$$\nabla \left(f_\lambda|_{H_i}\right) = P_{H_i} \circ \nabla f_\lambda|_{H_i},$$

where P_{H_i} is the orthogonal projection onto H_i, $i = 1, 2$,

$$\nabla \left(f_\lambda|_{H_i}\right) = (\nabla f_\lambda)\,|_{H_i}, \;\; i = 1, 2.$$

Thus, in order to find critical points of f_λ in H_i, it is enough to look for zeroes of $\nabla \left(f_\lambda|_{H_i}\right)$, $i = 1, 2$. An easy computation shows that the Riesz representation of

$D^2_0(f_\lambda|_{H_i})$ is $L^i_\lambda\colon H_i \to H_i$, where $L^i_\lambda = L_\lambda|_{H_i}$, $i = 1, 2$. Since $\mathrm{sf}(L^1) = -1$ and $\mathrm{sf}(L^2) = 1$, we have bifurcation of critical points in H_1 and H_2 by Theorem 1.3. This gives us two families of critical points of f_λ that bifurcate from $(\frac{1}{2}, 0) \in [0, 1] \times H^1_0((0, \pi), \mathbb{R}^2)$. In fact, applying the Krasnoselskii-Rabinowitz theorem (see [2]), we actually obtain two branches of global bifurcation.

6.2 Homoclinics of Hamiltonian systems

Note that the operators L_λ in the previous section are of the type $L_\lambda = T + K_\lambda$ for a fixed $T \in \mathcal{FS}(H)^G$ and compact operators K_λ. Thus for (38) the spectral flow of the corresponding Hessians L_λ in (39) actually only depends on the endpoints L_0 and L_1 by Proposition 2.2. In particular, if we would consider in such setting equations, where the corresponding paths of Hessians L are closed, then $\mathrm{sf}_G(L) = 0$ as we explained below Proposition 2.2. The aim of this section is to construct a $G = \mathbb{Z}_2$-invariant family of functionals f such that the Hessians L are a loop in $\mathcal{FS}(H)^G$ having a non-vanishing G-equivariant spectral flow. Thus by Theorem 3.1 there is a bifurcation point of critical points that could not have been found by any invariant that only depends on the endpoints L_0, L_1 of the path such as [6, 10] and [15]. Moreover, our example also has the feature that $\mathrm{sf}(L) = 0$ and consequently Theorem 1.3 fails as well.

Let $\mathcal{H} : I \times \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth map and consider the Hamiltonian systems

$$\begin{cases} Ju'(t) + \nabla_u \mathcal{H}_\lambda(t, u(t)) = 0, & t \in \mathbb{R} \\ \lim_{t \to \pm\infty} u(t) = 0, \end{cases} \tag{40}$$

where $\lambda \in I$ and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{41}$$

is the standard symplectic matrix. In what follows, we assume that $\mathcal{H}$ is of the form

$$\mathcal{H}_\lambda(t, u) = \frac{1}{2}\langle A(\lambda, t)u, u\rangle + R(\lambda, t, u), \tag{42}$$

where $A : I \times \mathbb{R} \to \mathcal{L}(\mathbb{R}^{2n})$ is a family of symmetric matrices, $R(\lambda, t, u)$ vanishes up to second order at $u = 0$, and there are $p > 0$, $C \geq 0$ and $r \in H^1(\mathbb{R}, \mathbb{R})$ such that

$$|D^2_u R(\lambda, t, u)| \leq r(t) + C|u|^p.$$

Moreover, we suppose that $A_\lambda := A(\lambda, \cdot) : \mathbb{R} \to \mathcal{L}(\mathbb{R}^{2n})$ converges uniformly in λ to families

$$A_\lambda(+\infty) := \lim_{t\to\infty} A_\lambda(t), \quad A_\lambda(-\infty) := \lim_{t\to-\infty} A_\lambda(t), \quad \lambda \in I, \tag{43}$$

and that the matrices $JA_\lambda(\pm\infty)$ are hyperbolic, i.e. they have no eigenvalues on the imaginary axis. Note that by (42), $\nabla_u \mathcal{H}_\lambda(t,0) = 0$ for all $(\lambda, t) \in I \times \mathbb{R}$, so that $u \equiv 0$ is a solution of (40) for all $\lambda \in I$.

Let us now briefly recall the variational formulation of the equations (40) from [25, §4]. The bilinear form $b(u,v) = \langle Ju', v\rangle_{L^2(\mathbb{R},\mathbb{R}^{2n})}$, $u, v \in H^1(\mathbb{R}, \mathbb{R}^{2n})$, extends to a bounded form on the well known fractional Sobolev space $H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n})$. Under the assumption (42), the map $f : I \times H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n}) \to \mathbb{R}$ given by

$$f_\lambda : H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n}) \to \mathbb{R}, \quad f_\lambda(u) = \frac{1}{2} b(u,u) + \frac{1}{2}\int_{-\infty}^{\infty} \langle A(\lambda, t)u(t), u(t)\rangle\, dt$$
$$+ \int_{-\infty}^{\infty} R(\lambda, t, u(t))\, dt$$

is C^2. Moreover, it was shown in [25] that its critical points are the classical solutions of (40) and each sequence of critical points that converges to a bifurcation point actually converges in $C^1(\mathbb{R}, \mathbb{R}^{2n})$. Finally, the second derivative of f_λ at the critical point $0 \in H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n})$ is given by

$$D_0^2 f_\lambda(u,v) = b(u,v) + \int_{-\infty}^{\infty} \langle A(\lambda, t)u(t), v(t)\rangle\, dt \tag{44}$$

and, by using the hyperbolicity of $JA_\lambda(\pm\infty)$, it can be shown that the corresponding Riesz representations $L_\lambda : H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n}) \to H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^{2n})$ are Fredholm (cf. [25], [29]). Consequently, the operators L_λ are selfadjoint Fredholm operators, and it follows by elliptic regularity that the kernel of L_λ consists of the classical solutions of the linear differential equation

$$\begin{cases} Ju'(t) + A(\lambda, t)u(t) = 0, \quad t \in \mathbb{R} \\ \lim\limits_{t \to \pm\infty} u(t) = 0. \end{cases} \tag{45}$$

The stable and the unstable subspaces of (45) are

$$E^s(\lambda, 0) = \{u(0) \in \mathbb{R}^{2n} : \; Ju'(t) + A(\lambda, t)u(t) = 0, \; t \in \mathbb{R}; \, u(t) \to 0, t \to \infty\},$$
$$E^u(\lambda, 0) = \{u(0) \in \mathbb{R}^{2n} : \; Ju'(t) + A(\lambda, t)u(t) = 0, \; t \in \mathbb{R}; \, u(t) \to 0, t \to -\infty\},$$

and it is clear that (45) has a non-trivial solution if and only if $E^s(\lambda, 0)$ and $E^u(\lambda, 0)$ intersect non-trivially.

Denote by g the non-trivial element of $G = \mathbb{Z}_2$. We set

$$\rho(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{46}$$

and consider Hamitonian systems in $\mathbb{R}^4$ (cf. [3, 9]), where

$$A(\lambda,t)=\begin{pmatrix} a_\lambda(t) & 0 & c_\lambda(t) & 0\\ 0 & b_\lambda(t) & 0 & d_\lambda(t)\\ c_\lambda(t) & 0 & e_\lambda(t) & 0\\ 0 & d_\lambda(t) & 0 & h_\lambda(t)\end{pmatrix} \tag{47}$$

is equivariant under the action of G for any functions $a, b, c, d, e, h : I \times \mathbb{R} \to \mathbb{R}$. Now the fixed point space of our action is

$$H^G = \{(u_1, u_2, u_3, u_4) \in H^{\frac{1}{2}}(\mathbb{R}, \mathbb{R}^4) : u_2 = u_4 = 0\}$$

and it follows from (44) that the kernel of $L_\lambda\mid_{H^G}$ is made of the solutions of the Hamiltonian systems

$$\begin{cases} J\begin{pmatrix} u_1' \\ u_3' \end{pmatrix} + \begin{pmatrix} a_\lambda(t) & c_\lambda(t) \\ c_\lambda(t) & e_\lambda(t)\end{pmatrix}\begin{pmatrix} u_1 \\ u_3\end{pmatrix} = 0, \quad t\in\mathbb{R}\\ \lim\limits_{t\to\pm\infty} u(t) = 0, \end{cases} \tag{48}$$

in $\mathbb{R}^2$, and likewise the kernel of $L_\lambda\mid_{(H^G)^\perp}$ consists of the solutions of

$$\begin{cases} J\begin{pmatrix} u_2' \\ u_4' \end{pmatrix} + \begin{pmatrix} b_\lambda(t) & d_\lambda(t) \\ d_\lambda(t) & h_\lambda(t)\end{pmatrix}\begin{pmatrix} u_2 \\ u_4\end{pmatrix} = 0, \quad t\in\mathbb{R}\\ \lim\limits_{t\to\pm\infty} u(t) = 0. \end{cases} \tag{49}$$

We now use an example of Pejsachowicz from [25] to construct a loop of operators $L = \{L_\lambda\}_{\lambda\in I}$ such that $\mathrm{sf}(L) = 0$ but $\mathrm{sf}_G(L) \in RO(\mathbb{Z}_2)$ is non-trivial. To keep our formulas as simple as possible, we use instead of $I = [0, 1]$ as parameter interval $[-\pi, \pi]$ and consider for $\lambda \in [-\pi, \pi]$ the matrix family

$$\widetilde{A}(\lambda,t)=\begin{pmatrix} a_\lambda(t) & c_\lambda(t) \\ c_\lambda(t) & e_\lambda(t)\end{pmatrix} = \begin{cases} (\arctan t)JS_\lambda, & t\geq 0\\ (\arctan t)JS_0, & t<0, \end{cases}, \tag{50}$$

where

$$S_\lambda = \begin{pmatrix} \cos(\lambda) & \sin(\lambda)\\ \sin(\lambda) & -\cos(\lambda)\end{pmatrix}.$$

Note that $\tilde{A}(-\pi, t) = \tilde{A}(\pi, t)$ for all $t \in \mathbb{R}$.
The space $\mathbb{R}^2$ is symplectic with respect to the canonical symplectic form $\omega_0(u, v) = \langle Ju, v\rangle_{\mathbb{R}^2}$. As the matrices (50) converge uniformly in λ to families of hyperbolic matrices for $t \to \pm\infty$, it can be shown that the stable and unstable spaces $E^s(\lambda, 0)$, $E^u(\lambda, 0)$ are Lagrangian subspaces of $\mathbb{R}^2$ (cf. e.g. [35, Lemma 4.1]). This implies in particular that $E^s(\lambda, 0)$ and $E^u(\lambda, 0)$ are one-dimensional for all $\lambda \in [-\pi, \pi]$.

To find non-trivial solutions of (48), we now consider $E^u(\lambda, 0) \cap E^s(\lambda, 0) \neq \{0\}$. By a direct computation it can be checked that

$$u_-(t) = \sqrt{t^2+1}\, e^{-t \arctan(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; t \leq 0,$$
$$u_+(t) = \sqrt{t^2+1}\, e^{-t \arctan(t)} \begin{pmatrix} \cos\left(\frac{\lambda}{2}\right) \\ \sin\left(\frac{\lambda}{2}\right) \end{pmatrix}, \; t \geq 0,$$

are solutions of (48) on the negative and positive half-line, respectively. As they extend to global solutions and since $t \arctan(t) \to \infty$ as $t \to \pm\infty$, we see that $u_-(0) \in E^u(\lambda, 0)$ and $u_+(0) \in E^s(\lambda, 0)$. As $u_+(0)$ and $u_-(0)$ are linearly dependent if and only if $\lambda = 0$, we conclude that (48) has a non-trivial solution if and only if $\lambda = 0$, and the kernel of $L_0 \mid_{H^G}$ is the span of

$$u_*(t) = \sqrt{t^2+1}\, e^{-t \arctan(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Next we compute the spectral flow of $L \mid_{H^G}$ by a crossing form (36). We need to consider

$$\Gamma(L \mid_{H^G}, 0)[u_*] = \int_{-\infty}^{\infty} \left\langle \dot{\tilde{A}}(0,t) u_*(t), u_*(t) \right\rangle dt,$$

where

$$\dot{\tilde{A}}(0,t) = \begin{cases} (\arctan t) J \dot{S}_0, & t \geq 0 \\ 0, & t < 0, \end{cases}$$

and

$$\dot{S}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consequently,

$$\begin{aligned} \Gamma(L \mid_{H_G}, 0)[u_*] &= \int_0^{\infty} \left\langle \dot{\tilde{A}}(0,t) u_*(t), u_*(t) \right\rangle dt + \int_{-\infty}^{0} \left\langle \dot{\tilde{A}}(0,t) u_*(t), u_*(t) \right\rangle dt \\ &= \int_0^{\infty} \arctan(t) \langle J \dot{S}_0 u_*(t), u_*(t) \rangle \, dt \\ &= -\int_0^{\infty} \arctan(t)(t^2+1) e^{-2t \arctan(t)} \, dt < 0, \end{aligned}$$

which shows that $\Gamma(L \mid_{H_G}, 0)$ is non-degenerate and of signature -1 as quadratic form on the one-dimensional kernel of $L_0 \mid_{H^G}$. Therefore, by (36), $\mathrm{sf}(L \mid_{H^G}) = -1$ and so $\mathrm{sf}_{\mathbb{Z}_2}(L)$ is non-trivial in $RO(\mathbb{Z}_2)$ by (34). Thus for these functions a, c and e

there is a bifurcation point of critical points of f by Theorem 3.1, and consequently also bifurcation of solutions of (40) from the trivial solution. Let us once again point out, that this bifurcation point cannot be found by invariants that only depend on the endpoints of the path L.
Note that we have not yet chosen functions b, d and h in (47), which we now do in a way such that $\operatorname{sf}(L) = 0 \in \mathbb{Z}$ to obtain an example where also Theorem 1.3 is not applicable. Let us firstly point out that it readily follows from (7) that the spectral flow changes its sign if we reverse the orientation of the path of operators. We now set for $t \in \mathbb{R}$ and $\lambda \in [-\pi, \pi]$

$$b_\lambda(t) = a_{-\lambda}(t), \quad h_\lambda(t) = e_{-\lambda}(t), \quad d_\lambda(t) = c_{-\lambda}(t).$$

Then $L_\lambda \mid_{(H^G)^\perp} = L_{-\lambda} \mid_{H^G}$ and thus $\operatorname{sf}(L \mid_{(H^G)^\perp}) = -\operatorname{sf}(L \mid_{H^G}) = 1$. It follows from (35) that $\operatorname{sf}(L) = 0$ and so our example has all the required properties.

6.3 An indefinite system of elliptic PDE

In this final section we consider the non-discrete compact Lie group SO(2) and an indefinite system of PDEs that has been studied in this setting by the equivariant degree, e.g., in [15].

For a smooth bounded domain $\Omega \subset \mathbb{R}^n$, we are looking for bifurcation of solutions of the family of equations

$$\begin{cases} A\Delta u(x) = \nabla_u \mathcal{F}(\lambda, u(x)) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{51}$$

where $\lambda \in I := [0, 1]$, $A := \operatorname{diag}\{a_1, .., a_p\} \in \operatorname{Mat}(p, \mathbb{R})$, $a_i \in \{\pm 1\}$, $i = 1, \ldots, p$ and

$$\mathcal{F} : I \times \mathbb{R}^p \to \mathbb{R}$$

is a C^2-map such that

(A1) 0 is a critical point of $\mathcal{F}_\lambda := \mathcal{F}(\lambda, \cdot) : \mathbb{R}^p \to \mathbb{R}$ for all $\lambda \in I$. In what follows, we set

$$B_\lambda := \nabla_u^2 \mathcal{F}(\lambda, 0) \in \operatorname{Mat}(p, \mathbb{R}).$$

(A2) There exist $C > 0$ and $1 \leq s < (n+2)(n-2)^{-1}$ if $n \geq 3$ such that

$$|\nabla_u^2 \mathcal{F}(\lambda, u)| \leq C(1 + |u|^{s-1}).$$

If $n = 2$, we instead require that $s \in [1, \infty)$, and for $n = 1$ we do not impose any growth condition on $\mathcal{F}$.

Note that the constant function $u \equiv 0$ is a solution of (51) for all $\lambda \in I$ and thus it is sensible to ask for bifurcation from this trivial branch of solutions. This problem has recently been studied by the second and fourth author in [18], where a spectral flow formula was obtained that can show the existence of bifurcation by Theorem 1.3. In this section we modify the setting by assuming an invariance under a natural action of $SO(2)$ that yields bifurcation by our main Theorem 3.1. Let us first briefly recall the variational setting, where $H^1_0(\Omega)$ is the standard Sobolev space and $H:=\bigoplus_{i=1}^p H^1_0(\Omega)$ is a Hilbert space with respect to

$$\langle u, v\rangle_H := \sum_{i=1}^{p} \langle u_i, v_i\rangle_{H^1_0(\Omega)}, \quad u=(u_1,\ldots,u_p),\ v=(v_1,\ldots,v_p)\in H.$$

We consider the map $f : I \times H \to \mathbb{R}$ given by

$$f(\lambda, u):=\frac{1}{2}\int_\Omega \sum_{i=1}^{p}(-a_i|\nabla u_i(x)|^2)\,\mathrm{d}x - \int_\Omega \mathcal{F}(\lambda, u(x))\,\mathrm{d}x. \tag{52}$$

It follows from assumption $(A2)$ that there exists $h \in \mathcal{C}^2(I\times\mathbb{R}^p,\mathbb{R})$ such that

$$\mathcal{F}(\lambda, u) = \frac{1}{2}\langle B_\lambda u, u\rangle + h(\lambda, u)$$

and for every $\lambda \in I$, we have $\nabla_u h(\lambda, 0) = 0$ as well as $\nabla_u^2 h(\lambda, 0) = 0$.
Now f is in $\mathcal{C}^2(I \times H, \mathbb{R})$ under the assumptions $(A1) - (A2)$ (cf. [28]) and the gradient of $f(\lambda,\cdot) : H \to \mathbb{R}$ is of the form

$$\nabla_u f(\lambda, u) = Tu + K_\lambda u - \nabla_u \eta(\lambda, u).$$

Here $T : H \to H$ is the selfadjoint invertible operator $Tu = -Au$, $K_\lambda : H \to H$ is the selfadjoint compact operator which implicitly is given by

$$\langle K_\lambda u, v\rangle_H = -\int_\Omega \langle B_\lambda u(x), v(x)\rangle_{\mathbb{R}^p}\ \mathrm{d}x, \tag{53}$$

and $\eta : I \times H \to \mathbb{R}$ is the C^2-map defined by

$$\eta(\lambda, u) = \int_\Omega h(\lambda, u(x))\,\mathrm{d}x,$$

where $\nabla_u\eta(\lambda, 0) = 0$ as well as $\nabla_u^2\eta(\lambda, 0) = 0$ for all $\lambda \in I$. The critical points of $f_\lambda:=f(\lambda,\cdot) : H \to \mathbb{R}$ are the weak solutions of (51), and thus in particular $0 \in H$ is a critical point of all f_λ, $\lambda \in I$. Moreover, the Hessians $\nabla_u^2 f_\lambda(0)$ at the critical point $0 \in H$ are the selfadjoint operators

$$L_\lambda:=T + K_\lambda. \tag{54}$$

Note that these are compact perturbations of an invertible operator and hence Fredholm. Accordingly $L:=\{L_\lambda\}_{\lambda\in I}$ is a path of selfadjoint Fredholm operators so that the spectral flow of L is defined. Moreover, the kernel of L_λ consists of the solutions of the linearised equations

$$\begin{cases} A\Delta u(x) = B_\lambda u(x) & \text{in } \Omega \\ u \quad\quad\quad = 0 & \text{on } \partial\Omega, \end{cases} \tag{55}$$

where B_λ is the Hessian matrix of $\mathcal{F}(\lambda, \cdot) : \mathbb{R}^p \to \mathbb{R}$ at 0 as in $(A1)$. Thus L_λ is invertible if and only if (55) has no non-trivial solution.
Now let us consider an orthogonal group action of SO(2) on $\mathbb{R}^p$ and the corresponding induced action on H by

$$(gu)(x) = gu(x), \quad x \in \Omega.$$

Henceforth we assume

(A3) The map $\mathcal{F}$ is G invariant, i.e., $\mathcal{F}(\lambda, (gu)(x)) = \mathcal{F}(\lambda, u(x))$, $x \in \Omega$, $g \in G$.

We now firstly consider the case $p = 4$, which allows to get a group action of SO(2) on $H = H_0^1(\Omega, \mathbb{R}^4)$ by

$$gu(x) := \left(\begin{pmatrix} \rho^s(g) & 0 \\ 0 & \rho^t(g) \end{pmatrix} u \right)(x), \tag{56}$$

where $s, t \in \mathbb{N}$, $s \neq t$, and ρ^s, ρ^t denote the corresponding irreducible representations in (37). By a simple computation, we see that for $A = \operatorname{diag}\{-1, -1, 1, 1\}$

$$\langle gu, gv \rangle_H = \langle u, v \rangle_H \quad \text{and} \quad \langle Agu, gu \rangle_H = \langle Au, u \rangle_H, \quad \text{for all} \quad u, v \in H,$$

and with $(A3)$, this implies that the functional f in (52) is invariant under G with respect to the action (56). The assumption $(A3)$ implies that B_λ is equivariant with respect to (56). For simplicity we henceforth assume as in [15] that

$$B_\lambda = \lambda I_4 \in \operatorname{Mat}(4, \mathbb{R}). \tag{57}$$

Let us now consider the equivariant spectral flow $\operatorname{sf}_{SO(2)}(L)$ for the path of Hessians L in (54). As in [18], we consider the subspaces $H_k \subset H$, $k \in \mathbb{N}$, where

$$H_k = \operatorname{span}\{f_k e_i : i = 1, \ldots, 4\},$$

$e_i, i = 1, \ldots, 4$, are the standard basis vectors of $\mathbb{R}^4$ and $(f_k)_{k\in\mathbb{N}}$ are the eigenfunctions of the scalar Dirichlet problem

$$\begin{cases} -\Delta u(x) = \alpha u(x) & \text{in } \Omega \\ \quad\quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us recall that there is a countable number of eigenvalues $(\alpha_k)_{k\in\mathbb{N}}$, which are all positive and called the Dirichlet eigenvalues of the domain Ω. We order these eigenvalues by $\alpha_k \leq \alpha_l$ for $k \leq l$ and assume that f_k is the eigenfunction of α_k. As shown in [18], the spaces H_k, $k \in \mathbb{N}$, are an orthogonal decomposition of H and there is some $k_0 \in \mathbb{N}$ such that L_λ is invertible on

$$V := \left(\bigoplus_{k=1}^{k_0} H_k \right)^\perp .$$

As all spaces H_k, and thus V as well as

$$U := \bigoplus_{k=1}^{k_0} H_k,$$

are invariant under the action (56) and under the operators L_λ, we see by (11) and (10) that $\mathrm{sf}_{\mathrm{SO}(2)}(L) = \mathrm{sf}_{\mathrm{SO}(2)}(L\mid_U)$. Since U is of finite dimension, and by the invariance of the H_k under the action and operators we can further decompose by Proposition 2.1 and (11) to obtain

$$\begin{aligned} \mathrm{sf}_{\mathrm{SO}(2)}(L) &= \mathrm{sf}_{\mathrm{SO}(2)}(L\mid_U) = [E^-(L_0\mid_U)] - [E^-(L_1\mid_U)] \\ &= \sum_{k=1}^{k_0} [E^-(L_0\mid_{H_k})] - [E^-(L_1\mid_{H_k})] = \sum_{k=1}^{k_0} [E^-(L_0^k)] - [E^-(L_1^k)], \end{aligned} \tag{58}$$

where $E^-(L_\lambda\mid_U)$, $\lambda \in I$, denotes the direct sum of eigenspaces with respect to negative eigenvalues of $L_\lambda\mid_U$ as in Proposition 2.1, and where we denote by L_λ^k the matrix representation of the restriction of L_λ to the invariant subspace H_k with respect to the basis $\{f_k e_i : i = 1, \ldots, 4\}$. By (57) and a direct calculation we obtain that

$$L_0^k = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad L_1^k = \begin{pmatrix} (1 - \frac{1}{\alpha_k})I_2 & 0 \\ 0 & (-1 - \frac{1}{\alpha_k})I_2 \end{pmatrix}$$

and note that -1 is the only negative eigenvalue of L_0^k and

$$E^-(L_0^k) = \mathrm{span}\{f_k e_i : i = 3, 4\}.$$

Moreover, $E^-(L_0^k)$ is a two-dimensional real SO(2)-representation given by $\rho_{E^-(L_0^k)} = \rho^t$.

To compute the $SO(2)$-equivariant spectral flow, we now need to inspect the matrices L_1^k. Let us firstly consider those $k \in \mathbb{N}$ with Dirichlet eigenvalues $\alpha_k \geq 1$ of Ω. Then $-1 - \frac{1}{\alpha_k}$ is the only negative eigenvalue of L_1^k and

$$E^-(L_1^k) = \mathrm{span}\{f_k e_i : i = 3, 4\}.$$

Thus, comparing with L_0^k, we see that $E^-(L_0^k) = E^-(L_1^k)$ and both spaces are the same SO(2)-representations, i.e.,

$$[E^-(L_0^k)] - [E^-(L_1^k)] = 0 \in RO(\mathrm{SO}(2)). \tag{59}$$

Secondly, we consider those $k \in \mathbb{N}$ with Dirichlet eigenvalues $\alpha_k < 1$ of Ω. Now all eigenvalues of L_1^k are negative and thus $E^-(L_1^k) = H_k$. Moreover, the corresponding SO(2)-representation of $E^-(L_1^k)$ is $\rho_{E^-(L_1^k)} = \rho^s \oplus \rho^t$. Thus, again comparing with L_0^k, we see that

$$[E^-(L_0^k)] - [E^-(L_1^k)] = [\rho^t] - [\rho^s \oplus \rho^t] = -[\rho^s] \neq 0 \in RO(\mathrm{SO}(2)) \tag{60}$$

and finally obtain from (58), (59) and (60) that

$$\mathrm{sf}_{\mathrm{SO}(2)}(L) = \sum_{k=1}^{k_0} [E^-(L_0^k)] - [E^-(L_1^k)] = -|\{k \in \mathbb{N} : \alpha_k < 1\}|\,[\rho^s], \tag{61}$$

where $|\{\ldots\}|$ stands for the cardinality of a set. Thus we have a non-trivial SO(2)-equivariant spectral flow on every domain having a Dirichlet eigenvalue less than 1 as, e.g., any two dimensional disc of radius greater than $\sqrt{6}$ (see [18, §6]). Note that by applying the map F in (9), we see from (61) that

$$\mathrm{sf}(L) = -2\,|\{k \in \mathbb{N} : \alpha_k < 1\}|. \tag{62}$$

This implies that $\mathrm{sf}_{\mathrm{SO}(2)}(L) \neq 0 \in RO(\mathrm{SO}(2))$ if and only if $\mathrm{sf}(L) \neq 0 \in \mathbb{Z}$ and thus bifurcation points could have also been found by applying Theorem 1.3.
Let us now further elaborate on this example and consider the path $\widetilde{L} = \{\widetilde{L}_\lambda\}_{\lambda \in I}$ on $H \times H = H_0^1(\Omega, \mathbb{R}^8)$, where $\widetilde{L}_\lambda = (L_\lambda) \oplus (-L_\lambda)$. Moreover, we consider for $s, t \in \mathbb{N}$, $s \neq t$, the action of SO(2) on $H \times H$

$$gu(x) := \left(\begin{pmatrix} \rho^s(g) & 0 & 0 & 0 \\ 0 & \rho^s(g) & 0 & 0 \\ 0 & 0 & \rho^t(g) & 0 \\ 0 & 0 & 0 & \rho^t(g) \end{pmatrix} u \right)(x)\,, \tag{63}$$

where ρ^s and ρ^t are the irreducible representations (37). If we follow the argument used to obtain (58) in this case, there is again some $k_0 \in \mathbb{N}$ such that

$$\mathrm{sf}_{\mathrm{SO}(2)}(\widetilde{L}) = \sum_{k=1}^{k_0} [E^-(\widetilde{L}_0^k)] - [E^-(\widetilde{L}_1^k)], \tag{64}$$

where we now denote by $\widetilde{L}^k_\lambda$ the matrix representation of the restriction of $\widetilde{L}_\lambda$ to the invariant subspace $H_k \times H_k$ with respect to the bases $\{f_k e_i : i = 1, \ldots, 4\}$ in H_k. By a direct calculation we obtain that

$$\widetilde{L}^k_0 = \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & -I_2 & 0 & 0 \\ 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & I_2 \end{pmatrix}, \quad \widetilde{L}^k_1 = \begin{pmatrix} (1-\frac{1}{\alpha_k})I_2 & 0 & 0 & 0 \\ 0 & (-1-\frac{1}{\alpha_k})I_2 & 0 & 0 \\ 0 & 0 & (-1+\frac{1}{\alpha_k})I_2 & 0 \\ 0 & 0 & 0 & (1+\frac{1}{\alpha_k})I_2 \end{pmatrix}.$$

Again we note that -1 is the only negative eigenvalue of $\widetilde{L}^k_0$, where now

$$E^-(\widetilde{L}^k_0) = \operatorname{span}\{f_k e_i : i = 3, 4, 5, 6\}.$$

Moreover, $E^-(\widetilde{L}^k_0)$ is the four-dimensional representation of SO(2) given by

$$\rho_{E^-(\widetilde{L}^k_0)} = \rho^s \oplus \rho^t. \tag{65}$$

Let us now consider the matrices $\widetilde{L}^k_1$ and again firstly focus on the Dirichlet eigenvalues $\alpha_k \geq 1$. Then $-1-\frac{1}{\alpha_k}, -1+\frac{1}{\alpha_k}$ are the only negative eigenvalues, $E^-(\widetilde{L}^k_1) = E^-(\widetilde{L}^k_0)$ and $\rho_{E^-(\widetilde{L}^k_1)}$ is as in (65). If, secondly, we consider the Dirichlet eigenvalues $\alpha_k < 1$ of Ω, then $1-\frac{1}{\alpha_k}, -1-\frac{1}{\alpha_k}$ are the negative eigenvalues,

$$E^-(\widetilde{L}^k_1) = \operatorname{span}\{f_k e_i : i = 1, 2, 3, 4\},$$

and $E^-(\widetilde{L}^k_1)$ is the four-dimensional representation

$$\rho_{E^-(\widetilde{L}^k_1)} = 2 \cdot \rho^s. \tag{66}$$

To summarise, we obtain for $k \in \mathbb{N}$ with $\alpha_k \geq 1$

$$[E^-(\widetilde{L}^k_0)] - [E^-(\widetilde{L}^k_1)] = [\rho^s \oplus \rho^t] - [\rho^s \oplus \rho^t] = 0_{RO(\mathrm{SO}(2))},$$

whereas (65) and (66) yield for $k \in \mathbb{N}$ with $\alpha_k < 1$

$$[E^-(\widetilde{L}^k_0)] - [E^-(\widetilde{L}^k_1)] = [\rho^s \oplus \rho^t] - [2 \cdot \rho^s] = [\rho^t] - [\rho^s] \neq 0_{RO(\mathrm{SO}(2))}.$$

Thus by (64) the SO(2)-equivariant spectral flow of the path $\widetilde{L}$ is given by

$$\begin{aligned} \mathrm{sf}_{\mathrm{SO}(2)}(\widetilde{L}) &= \sum_{k=1}^{k_0} [E^-(\widetilde{L}^k_0)] - [E^-(\widetilde{L}^k_1)] \\ &= |\{k \in \mathbb{N} : \alpha_k < 1\}| \, ([\rho^t] - [\rho^s]) \in RO(\mathrm{SO}(2)) \end{aligned}$$

and yields bifurcation for (51) if the underlying domain Ω has a Dirichlet eigenvalue less than 1. Note that instead Theorem 1.3 is not directly applicable, as by (9)

$$\mathrm{sf}(\widetilde{L}) = F(\mathrm{sf}_{\mathrm{SO}(2)}(\widetilde{L})) = |\{k \in \mathbb{N} : \alpha_k < 1\}| \, (2-2) = 0 \in \mathbb{Z},$$

which is a consequence of the implemented symmetry in the definition of the operators $\widetilde{L}_\lambda$, $\lambda \in I$. On the other hand, any cyclic group $\mathbb{Z}_k$ can be regarded as a subgroup of SO(2). If we assume that s and t are relatively prime and consider $\mathbb{Z}_s$, $\mathbb{Z}_t$, then $H_1 := (H \times H)^{\mathbb{Z}_s} = H \times \{0\}$ and $H_2 := (H \times H)^{\mathbb{Z}_t} = \{0\} \times H$ yield a decomposition $H = H_1 \oplus H_2$. Now the same argument as in the final paragraph of Sect. 6.1 shows that the critical points of f_λ in H_i are given by the zeroes of $\nabla(f\mid_{H_i})$, $i = 1, 2$. As $\tilde{L}_\lambda \mid_{H_1} = L_\lambda$ and $\tilde{L}_\lambda \mid_{H_2} = -L_\lambda$, $\lambda \in I$, it follows from (62) that there is bifurcation of critical points in H_1 and H_2 by Theorem 1.3.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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