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Nematic liquid crystals: Ericksen-Leslie theory with general stress tensors

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Abstract

The Ericksen-Leslie model for nematic liquid crystal flows in case of an isothermal and incompressible fluid with general Leslie stress and anisotropic elasticity, i.e. with general Ericksen stress tensor, is shown for the first time to be strongly well-posed. Of central importance is a fully nonlinear boundary condition for the director field, which, in this generality, is necessary to guarantee that the system fulfills physical principles. The system is shown to be locally, strongly well-posed in the L_p-setting. More precisely, the existence and uniqueness of a local, strong L_p-solution to the general system is proved and it is shown that the director d satisfies $|d|_2 \equiv 1$ provided this holds for its initial data d_0. In addition, the solution is shown to depend continuously on the data. The results are proven without any structural assumptions on the Leslie coefficients and in particular without assuming Parodi's relation.

1. Introduction

In physics there are various ways of describing order parameters in liquid crystals: the Doi-Onsager -, Landau-De Gennes - and Ericksen-Leslie theories. These lead to mathematical theories at various levels. The Ericksen-Leslie model is a so-called vector model. Another type of model describing liquid crystal flows is the Q-tensor model, including the Landau-De Gennes theory. In contrast to vector models, it uses a traceless 3×3-matrix Q to describe the alignment of molecules, see e.g. [4] or [45]. We should also mention the micropolar Eringen model taking into account microstructure effects, see e.g. [16,17]. For a relation between the Ericksen-Leslie equations and Eringen's theory; see e.g. [20].

In this article we concentrate on the Ericksen-Leslie model with general Leslie and general Ericksen stress. In their pioneering works, Ericksen and Leslie [14, 15,32,33] developed during the 1960's the continuum theory of nematic liquid crystals. This theory models nematic liquid crystal flows from a hydrodynamical point of view and reduces to the Oseen-Frank theory in the static case. It describes the evolution of the complete system under the influence of the velocity u of the fluid and the orientation configuration d of rod-like liquid crystals; see also [3]. The original derivation [14,33] is based on the conservation laws for mass, linear and angular momentums as well as on certain very specific constitutive relations, which nowadays are called the *Leslie* and *Ericksen stress*. For a very thorough description and investigation of the Ericksen-Leslie model we refer to the monographs by Virga [48] and Sonnet and Virga [45].

Due to the complexity of these systems, certain simplified systems were investigated frequently in the past. In fact, the rigorous analysis of the Ericksen-Leslie system began with the work of Lin [34] and Lin and Liu [35], who introduced and studied the nowadays called simplified isothermal system, see also [22]. For well-posedness criteria concerning various simplifications and various assumptions on the Leslie as well as Ericksen coefficients, we refer to [10,18,36,50–52] as well to the survey articles [25,53] and the references therein.

It was a long outstanding open problem to decide whether the Ericksen-Leslie system subject to general Leslie and general Ericksen stress is well-posed in the weak or strong sense. First results in this direction go back to Wu, Xu and Liu [49,51] and Lin and Wang [36], who proved well-posedness results for the Ericksen-Leslie system under the assumptions that the Leslie coefficients are satisfying certain assumptions related to Parodi's relation and where the Oseen-Frank free energy ψ is simplified to the energy functional for harmonic maps, i.e., $I(d) = \int_\Omega \psi(d, \nabla d) = \int_\Omega |\nabla d|^2 dx$ due to the isotropy assumptions in the Oseen-Frank functional described in detail below.

In this article we give an affirmative answer to this problem for the general case with general Leslie stress and anisotropic elasticity, i.e. general Ericksen stress. For the description of the full system see, e.g., Section 4.7 of [25] or Section 2 below. Denoting by ϱ the density and by θ the temperature of the fluid, the free energy ψ is of the form $\psi = \psi(\varrho, \theta, d, \nabla d)$, where d is the director field. In the isothermal and incompressible situation, ψ is given by the classical Oseen-Frank free energy

$$\begin{aligned}\psi(d, \nabla d) = k_1(\operatorname{div} d)^2 + k_2|d \times (\nabla \times d)|_2^2 + k_3|d \cdot (\nabla \times d)|^2 \\ + (k_2 + k_4)[\operatorname{tr}(\nabla d)^2 - (\operatorname{div} d)^2],\end{aligned}$$

where k_i are the so-called *Frank coefficients*. Based on physical principles, the four Frank coefficients k_1, k_2, k_3, k_4 are all different in general, however, the first three ones should be strictly positive. The Frank coefficients are often assumed to satisfy the Ericksen inequalities

$$k_1 > 0, k_2 > 0, k_3 > 0, k_2 > |k_4|, 2k_1 > k_2 + k_4,$$

which are known to be necessary and sufficient for the inequality $\psi(d, \nabla d) \geq c|\nabla d|^2$ for all d and some constant $c > 0$, see [2].

The first three terms in ψ defined above describe the splay, twist and bend of the director field. The fourth term $k_2 + k_4$, the saddle-splay term, is a null Lagrangian meaning that $(k_2 + k_4) \int_\Omega \operatorname{tr}(\nabla d)^2 - (\operatorname{div} d)^2 dx$ depends only on the values of d

on the boundary $\partial\Omega$. Thus if $d_{|\partial\Omega}$ is prescribed, such as for Dirichlet boundary conditions, the term can be neglected with respect to energy considerations. However, this is *not* the case when $d_{|\partial\Omega}$ is only partially prescribed, as in the situation of weak anchoring boundary conditions or for *fully nonlinear* boundary conditions as in our case and described in detail below, see (1.4). For the general free energy $\psi(d, \nabla d)$, the general Ericksen stress tensor becomes

$$S_E = -\varrho \frac{\partial \psi}{\partial(\nabla d)}[\nabla d]^{\mathsf{T}}.$$

We remark that in the *isotropic* case, i.e. if $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$, the Oseen-Frank free energy $\psi(d, \nabla d)$ coincides with $|\nabla d|^2$, hence $\frac{\partial \psi}{\partial(\nabla d)} = 2\nabla d$, which simplifies the problem.

In [24], strong well-posedness for the Ericksen-Leslie system in the incompressible case with general Leslie but *isotropic* elasticity stress, i.e. if $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$ in the Oseen-Frank free energy ψ, was proved for the first time without assuming any structural conditions on the Leslie coefficients, as e.g. Parodi's relation [40]. It was possible to prove a result of this type, since the approach given in [24] is based on the theory of quasilinear evolution equations and not on energy estimates. It seems that the latter requires certain dissipation rates and therefore structural conditions on the Leslie coefficients are needed, when pursuing an approach based on energy estimates as in [49,51]. In [24,25] it is only assumed that the six Leslie coefficients are smooth functions and that the coefficient μ_s (corresponding to α_4 in [36]) associated with the usual Cauchy stress tensor is strictly positive, thus guaranteeing that the resulting Laplacian has the correct sign.

In the special case of $\Omega = \mathbb{R}^3$, Hong, Li and Xin [27] and Ma, Gong and Li [37] obtained a well-posedness result for anisotropic elasticity, however, for completely vanishing Leslie stress S_L and without stretching terms and with the assumption that $k_2 + k_4 = 0$.

For a modification of the Ericksen-Leslie equations with positive moment of inertia but without any dissipative terms, in particular without any stretching terms, Chechkin, Ratiu, Romanov and Samokhin [8,9] proved the existence and uniqueness of strong solutions in the two- and three-dimensional L_2-setting with either periodic, Dirichlet or (linear) Neumann boundary conditions, provided the Frank coefficients satisfy the particular conditions $k_1 = k_2 = k_3 = -k_4$ in [8] and $k_2 = k_3 = -k_4$ in [9]. For results on a simplified version of the Eringen model [17] we refer to [7].

It is the aim of this article to investigate for the first time the Ericksen-Leslie system with general Leslie stress S_L and *anisotropic* elasticity, i.e. with general Ericksen stress S_E (see (2.7) and (2.12)), in bounded domains $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega \in C^3$. We show that this system is strongly well-posed without any structural assumptions on the Leslie coefficients. For the Frank coefficients, we assume that

$$k_1 > 0, k_2 > 0, k_3 > 0 \text{ and for } 0 < \alpha \leq \min\{k_1, k_2, k_3\} \text{ let } k_4 = \alpha - k_2. \quad (1.1)$$

Moreover, we suppose that at least one of the conditions

$$9k_3 > k_1 \quad \text{or} \quad 2|k_1 - k_3| < \min\{k_2, k_3\} \quad (1.2)$$

is satisfied.

In the two-dimensional case, and if d is the heat flow of harmonic maps, solutions with finite time singularities are constructed in [6], which yields blow-up in the d-equation. Huang, Lin, Liu and Wang [28] were able to construct examples, in case of Ω being the unit ball in $\mathbb{R}^3$ and of initial data u_0, d_0 having sufficiently small energy and d_0 fulfilling a topological condition in the case of Dirichlet boundary conditions $d = (0, 0, 1)^\mathsf{T}$, for which one has finite time blow-up of (u, d). For a related result in two space dimensions we refer to [29].

On the other hand, recalling the results given in [24] and [25], we already noted that the Ericksen-Leslie system with isotropic elasticity and general Leslie coefficients, but classical Neumann boundary conditions for d is strongly well-posed. It was also shown in [23] and [25] that in this case the Ericksen-Leslie system subject to Neumann boundary conditions is thermodynamically consistent, meaning that it fulfills the second law of thermodynamics. In order to find boundary conditions for d in the case of general, anisotropic elasticity which respect the underlying physics, we refer for example to [23]. There it was shown that in the case of general elasticity certain fully nonlinear and natural boundary conditions for d are needed in order to ensure that the system is consistent with physical principles. More precisely, it is shown in [23, Section 15.2], that the entropy production of the Ericksen-Leslie system is nonnegative, i.e. the *second law of thermodynamics* is satisfied, provided that the energy flux Φ_e of the Ericksen-Leslie system is modeled by

$$\Phi_e = q + \pi u - Su - \frac{\partial \psi}{\partial (\nabla d)^\mathsf{T}} \mathcal{D}_t d.$$

Here u means velocity, π pressure, S extra stress (cf. (2.7)) and, q denotes the heat flux, while ψ is the Oseen-Frank free energy and $\mathcal{D}_t = \partial_t + u \cdot \nabla$ is the Lagrangian derivative.

At the boundary $\partial\Omega$, energy should be preserved, meaning that $(\Phi_e|\nu) = 0$, where ν denotes the unit normal vector field on $\partial\Omega$. Under the assumptions $(q|\nu) = 0$ and $u = 0$ at $\partial\Omega$, this readily implies

$$\left(\frac{\partial \psi}{\partial (\nabla d)} \cdot \nu \middle| \mathcal{D}_t d \right) = 0 \quad \text{on} \quad \partial\Omega. \tag{1.3}$$

As the director d has length 1, it holds that $P_d \mathcal{D}_t d = \mathcal{D}_t d$, where $P_d = I - d \otimes d$. Therefore, (1.3) is clearly valid, provided d satisfies the natural boundary condition

$$P_d \frac{\partial \psi}{\partial (\nabla d)} \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

For this reason we employ (1.4) throughout this paper. Let us emphasize that this type of boundary condition has already been investigated in detail in the book of E. Virga, see (3.116) in [48], using variational techniques and the Euler-Lagrange formalism. In [48], this type of boundary condition is called *no anchoring condition* for d. Moreover, citing [48, page 132], "it should be noted that boundary conditions are often responsible for the appearance of defects in liquid crystals". It is very interesting to see, that the *no anchoring boundary condition* obtained by variational principles in [48], coincides with the boundary condition (1.4), obtained by using the entropy principle and the principle of thermodynamical consistency.

Let us also note that we are investigating here the boundary condition (1.4) for the first time with respect to well-posedness of the Ericksen-Leslie system.

Observe that, in general, (1.4) is a *fully nonlinear* boundary condition for d; see Section 3.2 for details. On the contrary, in the isotropic case $k_1 = k_2 = k_3 = 1$, $k_4 = 0$, it holds that $\frac{\partial \psi}{\partial(\nabla d)} = 2\nabla d$, and hence in this case, pure Neumann boundary conditions for d are natural. For a detailed discussion of other possible boundary conditions for d we refer the reader to, e.g., [48, Section 3.5].

From a mathematical point of view it is very satisfactory to see that the natural (nonlinear) boundary condition (1.4), motivated originally by physical principles, yields the existence and uniqueness of strong solutions to the Ericksen-Leslie system subject to general Leslie stress and anisotropic elasticity, i.e., with general Ericksen stress, see Theorem 2.1 for details.

There are several major difficulties arising in the investigation of the general Ericksen-Leslie system. In a first key step, we will show that the associated Ericksen operator is normally elliptic in $\mathbb{R}^3$ under the assumption that the Frank coefficients $k_1, \ldots, k_4$ satisfy the above condition (1.1)-(1.2) but no other assumptions. Secondly, considering the situation of bounded domains, by physical principles, we are naturally lead to the above fully nonlinear boundary condition (1.4) for d, which, however, is analytically delicate. We master these difficulties by showing first that the linearized system satisfies the Lopatinskii-Shapiro condition and thus that the general Ericksen-Leslie system constitutes a normally elliptic boundary value problem in the sense of [42, Section 6]. On the technical side, we note that our proofs are using a broad range of methods ranging from operator-valued Fourier multipliers to Schur complements.

Our approach, based on modern quasilinear theory, yields strong local-in-time well-posedness of the general Ericksen-Leslie system. Of course, d satisfies $|d(t, x)|_2 = 1$ for all $t \in [0, T]$, for some $T > 0$ and all $x \in \Omega$, provided Ω is bounded. The approach also yields that the solution depends continuously on the data.

The rest of this article is organized as follows; in Section 2 we start with a precise description of the Ericksen-Leslie model including the involved stress tensors as well as its natural boundary conditions motivated by the second law of thermodynamics and we close Section 2 by stating the two main results of this article. Section 3 is devoted to the computation of the Ericksen operator, based on the general Oseen-Frank free energy functional, while in Section 4, we introduce the functional analytic setting for our approach. In Section 5 we show that the principal part of the linearized Ericksen operator subject to the principal part of the linearization of the boundary operator is strongly elliptic and satisfies the Lopatinskii-Shapiro condition, thus allowing an approach based on modern quasilinear theory. The results obtained in Section 5 are crucial for proving maximal regularity results in Section 6. In Section 7, we prove our two main results. Finally, in Section 8 we summarize the results of this paper in a conclusion.

To formulate our main result in the next section, we define time weighted spaces $L_{p,\mu}(J;E)$ for a UMD-space E, $J=(0,T)$, $0<T\le\infty$, $p\in(1,\infty)$, $\mu\in(\frac{1}{p},1]$ as

$$L_{p,\mu}(J;E):=\{u\colon J\to E\mid [t\mapsto t^{1-\mu}u(t)]\in L_p(J;E)\}$$

equipped with their natural norms $\|u\|_{L_{p,\mu}(J;E)}:=\|[t\mapsto t^{1-\mu}u(t)]\|_{L_p(J;E)}$. For $k\in\mathbb{N}_0$, the associated weighted Sobolev spaces are defined by

$$\begin{aligned} W^k_{p,\mu}(J;E) &= H^k_{p,\mu}(J;E):\\ &=\{u\in W^k_{1,\mathrm{loc}}(J;E)\mid u^{(j)}\in L_{p,\mu}(J;E),\quad j\in\{0,\dots,k\}\}\end{aligned}$$

and these spaces are equipped with their natural norms. For $s\in(0,1)$, the weighted Sobolev-Slobodeckii spaces $W^s_{p,\mu}(J;E)$ are defined as

$$W^s_{p,\mu}(J;E)=\{u\in L_{p,\mu}(J;E):\|u\|_{W^s_{p,\mu}(J;E)}<\infty\},$$

where

$$\|u\|_{W^s_{p,\mu}(J;E)}:=\|u\|_{L_{p,\mu}(J;E)}+[u]_{W^s_{p,\mu}(J;E)},$$

and

$$[u]_{W^s_{p,\mu}(J;E)}:=\left(\int_0^T\int_0^t\tau^{p(1-\mu)}\frac{\|u(t)-u(\tau)\|_E^p}{(t-\tau)^{sp+1}}d\tau dt\right)^{1/p};$$

see [13, Formula (1.5)]. Furthermore, we set $W^s_{p,\sigma}(\Omega;\mathbb{R}^3):=W^s_p(\Omega;\mathbb{R}^3)\cap L_{p,\sigma}(\Omega;\mathbb{R}^3)$ for any $s>0$, where $L_{p,\sigma}(\Omega;\mathbb{R}^3)$ denotes the space of solenoidal L_p-functions on Ω and $W^s_p(\Omega;\mathbb{R}^3)$ is a classical Sobolev-Slobodeckii space, see, e.g., [46,47].

2. The general Ericksen-Leslie model and Main Results

In their pioneering articles, Ericksen [14] and Leslie [33] developed a continuum theory for the flow of nematic liquid crystals based on the conservation laws for mass, linear and angular momentums as well as on certain constitutive relations.

The general incompressible Ericksen-Leslie model in the isothermal case reads as

$$\begin{cases} \partial_t u+(u\cdot\nabla)u=\operatorname{div}\sigma & \text{in } (0,T)\times\Omega,\\ \operatorname{div}u=0 & \text{in } (0,T)\times\Omega,\\ d\times\left(g+\operatorname{div}\left(\frac{\partial\psi}{\partial(\nabla d)}\right)-\nabla_d\psi\right)=0 & \text{in } (0,T)\times\Omega,\\ |d|_2=1 & \text{in } (0,T)\times\Omega,\\ (u,d)_{|t=0}=(u_0,d_0) & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here, u denotes the velocity of the fluid, d the so-called director, and σ the stress tensor, given by

$$\sigma := -\pi I - \frac{\partial\psi}{\partial(\nabla d)}\nabla d + \sigma_L,$$

where π is the fluid pressure and

$$\sigma_L := \alpha_1(d^T Dd)d\otimes d + \alpha_2 N\otimes d + \alpha_3 d\otimes N + \alpha_4 D + \alpha_5(Dd)\otimes d + \alpha_6 d\otimes(Dd) \tag{2.2}$$

denotes the Leslie stress. Here α_i, $i = 1,\dots,6$ denote the Leslie viscosities, $D = \frac{1}{2}([\nabla u]^\mathsf{T} + \nabla u)$ is the deformation tensor,

$$N := N(u,d) := \partial_t d + (u\cdot\nabla)d - Vd,$$

with $V = \frac{1}{2}(\nabla u - [\nabla u]^\mathsf{T})$ being the vorticity tensor, and $(a\otimes b)_{ij} := a_i b_j$ for $1 \le i, j \le 3$. The kinematic transport of d is denoted by g and is given by

$$g := g(u,d) := \lambda_1 N + \lambda_2 Dd, \tag{2.3}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. Moreover, the free energy ψ is given by the classical Oseen-Frank free energy defined by

$$\begin{aligned}\psi(d,\nabla d) = k_1(\operatorname{div} d)^2 + k_2|d\times(\nabla\times d)|_2^2 + k_3|d\cdot(\nabla\times d)|^2 \\ + (k_2+k_4)[\operatorname{tr}(\nabla d)^2 - (\operatorname{div} d)^2],\end{aligned} \tag{2.4}$$

where k_i are the so-called *Frank coefficients*. The system has to be completed by suitable initial and boundary conditions.

Following arguments from thermodynamics and employing the entropy principle, the above Ericksen-Leslie model (2.1) was extended in [23] to the non-isothermal situation and to the case of compressible fluids in a thermodynamically consistent way. Let us emphasize that these extended models *contain the classical Ericksen-Leslie model in its general form as a special case*.

Given a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, with smooth boundary, the general Ericksen-Leslie model in the non-isothermal situation derived as in [23,25], reads as

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0 & \text{in } \Omega, \\ \rho\mathcal{D}_t u + \nabla\pi = \operatorname{div} S & \text{in } \Omega, \\ \rho\mathcal{D}_t\varepsilon + \operatorname{div} q = S:\nabla u - \pi\operatorname{div} u + \operatorname{div}(\rho\partial_{\nabla d}\psi\mathcal{D}_t d) & \text{in } \Omega, \\ \gamma\mathcal{D}_t d - \mu_V V d = P_d\big(\operatorname{div}(\rho\frac{\partial\psi}{\partial(\nabla d)}) - \rho\nabla_d\psi\big) + \mu_D P_d Dd & \text{in } \Omega, \\ u = 0, \quad q\cdot\nu = 0 & \text{on } \partial\Omega, \\ \rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) = \theta_0, \quad d(0) = d_0 & \text{in } \Omega. \end{cases} \tag{2.5}$$

Here the unknown variables ρ, u, π denote the density, velocity and pressure of the fluid, respectively, ε the internal energy and d the so called director, which must have modulus 1. Moreover, q denotes the heat flux, $\mathcal{D}_t = \partial_t + u\cdot\nabla$ the

Lagrangian derivative and P_d is defined as $P_d = I - d \otimes d$. These equations have to be supplemented by the thermodynamical laws

$$\varepsilon = \psi + \theta\eta, \quad \eta = -\partial_\theta \psi, \quad \kappa = \partial_\theta \varepsilon, \quad \pi = \rho^2 \partial_\rho \psi, \tag{2.6}$$

and by the constitutive laws

$$\begin{cases} S &= S_N + S_E + S_L, \\ S_N &= 2\mu_s D + \mu_b \operatorname{div} u\, I, \\ S_E &= -\rho \frac{\partial \psi}{\partial \nabla d} [\nabla d]^\mathsf{T}, \\ S_L &= S_L^{stretch} + S_L^{diss}, \\ S_L^{stretch} &= \frac{\mu_D + \mu_V}{2\gamma} \mathsf{n} \otimes d + \frac{\mu_D - \mu_V}{2\gamma} d \otimes \mathsf{n}, \quad \mathsf{n} = \mu_V V d + \mu_D P_d D d - \gamma \mathcal{D}_t d, \\ S_L^{diss} &= \frac{\mu_P}{\gamma} (\mathsf{n} \otimes d + d \otimes \mathsf{n}) + \frac{\gamma \mu_L + \mu_P^2}{2\gamma} (P_d D d \otimes d + d \otimes P_d D d) + \mu_0 (Dd|d) d \otimes d, \end{cases} \tag{2.7}$$

and

$$q = -\tilde{\alpha}_0 \nabla\theta - \tilde{\alpha}_1 (d|\nabla\theta) d. \tag{2.8}$$

Here all coefficients μ_j, $\tilde{\alpha}_j$ and γ are functions of $\rho, \theta, d, \nabla d$. For thermodynamical consistency the following conditions are required

$$\mu_s \geq 0, \quad 2\mu_s + n\mu_b \geq 0, \quad \tilde{\alpha}_0 \geq 0, \quad \tilde{\alpha}_0 + \tilde{\alpha}_1 \geq 0, \quad \mu_0, \mu_L \geq 0, \quad \gamma > 0. \tag{2.9}$$

We also recall from (1.4) that the natural boundary condition at $\partial\Omega$ for d becomes

$$P_d \frac{\partial \psi}{\partial (\nabla d)} \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{2.10}$$

which, in general, is *fully nonlinear*.

In the case of *isotropic elasticity* with constant density and constant temperature one has $k_1 = k_2 = k_3 = 1$ and $k_4 = 0$ and so the Oseen-Frank energy reduces to the Dirichlet energy, i.e.

$$\psi(d, \nabla d) = |\nabla d|^2,$$

and thus $\operatorname{div}(\frac{\partial \psi}{\partial (\nabla d)}) = 2\Delta d$. Then the Ericksen stress tensor simplifies to

$$S_E = -\lambda \nabla d [\nabla d]^\mathsf{T}, \tag{2.11}$$

where $\lambda = \rho \partial_\tau \psi$, $\tau = \frac{1}{2} |\nabla d|^2$, and the natural boundary condition at $\partial\Omega$ for d becomes the Neumann boundary condition $\partial_\nu d = 0$ on $\partial\Omega$.

We emphasize that in the case $\mu_V = \gamma$, our parameters $\mu_s, \mu_0, \mu_V, \mu_D, \mu_P, \mu_L$ are in *one-to-one correspondence* to the celebrated Leslie parameters $\alpha_1, \ldots, \alpha_6$ given in the Leslie stress σ_L defined as in (2.2), where D, N and V are defined as above. This shows that our model (2.5)-(2.7) contains the classical isothermic and incompressible Ericksen-Leslie model as a particular case.

It is the aim of this article to investigate the Ericksen-Leslie system with general Leslie stress and anisotropic elasticity, i.e. with general Ericksen stress tensor, analytically, in the isothermal situation and for incompressible fluids. Since the density $\rho > 0$ in this case is constant, we set $\rho = 1$ in the sequel, since this does not change the analysis.

The system then reads as

$$\begin{cases} \mathcal{D}_t u + \nabla\pi = \operatorname{div} S & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \gamma\mathcal{D}_t d - \mu_V V d = P_d\big(\operatorname{div}(\frac{\partial\psi}{\partial(\nabla d)}) - \nabla_d\psi\big) + \mu_D P_d D d & \text{in } \Omega, \\ P_d \frac{\partial\psi}{\partial(\nabla d)} \cdot \nu = 0 & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.12}$$

subject to initial data $(u(0), d(0)) = (u_0, d_0)$. Here $\mathcal{D}_t$, P_d are defined as above and S, ψ are defined as in (2.7) & (2.4). We show in Section 3.2 that the fully nonlinear boundary condition for d reads as

$$\begin{aligned} P_d \frac{\partial\psi}{\partial(\nabla d)} \cdot \nu = {} & 2k_3 \nabla d \cdot \nu + 2P_d(k_1 \operatorname{div} d \cdot I - k_3(\nabla d)^{\mathsf{T}}) \cdot \nu \\ & + 2(k_2 - k_3)(d \cdot \operatorname{curl} d)(d \times \nu) \\ & + 2(k_2 + k_4) P_d((\nabla d)^{\mathsf{T}} - \operatorname{div} d \cdot I) \cdot \nu. \end{aligned}$$

We note furthermore, that all coefficients μ_j, γ are functions of d, ∇d.

Let us state our assumptions on the coefficients in the Ericksen-Leslie model with general Leslie and Ericksen stress tensor.

(R) The Leslie coefficients μ_j for $j \in \{s, V, D, P, L, 0\}$ and the parameter γ may depend on d, ∇d and are assumed to be smooth,

(P) It holds that $\mu_s > 0$, $\gamma > 0$ and $\mu_j \geq 0$ for $j \in \{0, L\}$,

(F) The Frank coefficients satisfy $k_j > 0$ for $j \in \{1, 2, 3\}$ and for $0 < \alpha \leq \min\{k_1, k_2, k_3\}$ let $k_4 = \alpha - k_2$. Furthermore, let at least one of the conditions
- $9k_3 > k_1$,
- $2|k_1 - k_3| < \min\{k_2, k_3\}$,

be satisfied.

(B) The initial data d_0 satisfies the compatibility condition $\mathcal{B}_{d_0}(\nabla)d_0 = 0$ on $\partial\Omega$, where

$$\begin{aligned} \frac{1}{2}\mathcal{B}_{d_0}(\nabla)d_0 := {} & k_3 \nabla d_0 \cdot \nu + P_{d_0}(k_1 \operatorname{div} d_0 \cdot I - k_3(\nabla d_0)^{\mathsf{T}}) \cdot \nu \\ & + (k_2 - k_3)(d_0 \cdot \operatorname{curl} d_0)(d_0 \times \nu) \\ & + (k_2 + k_4) P_{d_0}((\nabla d_0)^{\mathsf{T}} - \operatorname{div} d_0 \cdot I) \cdot \nu. \end{aligned}$$

In particular, note that we do *not* assume Parodi's relations. We note furthermore, that assuming condition (F), the Oseen-Frank functional ψ is positive, i.e. we have $\psi(d, \nabla d) \geq 0$ for all $d \in \mathbb{R}^3$ with $|d|_2 = 1$.

Theorem 2.1. *Let $1 < p < \infty$, $\mu \in (\frac{1}{2} + \frac{5}{2p}, 1]$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, and assume that (R), (P) and (F) are satisfied. Then, given*

$$(u_0, d_0) \in W^{2\mu-2/p}_{p,\sigma}(\Omega; \mathbb{R}^3) \times W^{2\mu+1-2/p}_{p}(\Omega; \mathbb{R}^3),$$

satisfying $|d_0(x)|_2 = 1$ *for all* $x \in \Omega$*, the compatibility conditions (B) as well as* $u_0 = 0$ *on* $\partial\Omega$*, there exists* $T = T(u_0, d_0) > 0$ *such that problem* (2.12) *has a unique, strong solution*

$$u \in \mathbb{E}^u_{1,\mu}(0,T) := H^1_{p,\mu}((0,T); L_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^2_p(\Omega;\mathbb{R}^3)),$$
$$d \in \mathbb{E}^d_{1,\mu}(0,T) := H^1_{p,\mu}((0,T); H^1_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^3_p(\Omega;\mathbb{R}^3))$$
$$\nabla\pi \in L_{p,\mu}((0,T); L_p(\Omega;\mathbb{R}^3)).$$

Moreover, d *satisfies* $|d(t,x)|_2 = 1$ *for all* $(t,x) \in [0,T] \times \Omega$ *and the solution can be extended to a maximal solution with a maximal interval of existence* $[0, T_+(u_0, d_0))$.

Remarks 2.2.

a) Note that in the case of isotropic elasticity, i.e., if $k_1 = k_2 = k_3$ and $k_4 = 0$, the nonlinear boundary condition (2.10) reduces to the classical linear Neumann boundary condition $\partial_\nu d = 0$ on $\partial\Omega$.
b) We emphasize that the boundary condition (2.10) is not a mathematical construction but occurs as a natural condition for proving that the non-isothermal and compressible Ericksen-Leslie system is thermodynamically consistent meaning that the total energy is preserved, whereas in the isothermal situation, the available energy is nonincreasing, see Remark 2.4 below. Moreover, the associated entropy production rate of the system is nonnegative and thus satisfies the second law of thermodynamics. For more details we refer to [23] and [25].
c) To construct nontrivial initial data d_0 satisfying the compatibility condition $\mathcal{B}_{d_0}(\nabla)d_0 = 0$ on $\partial\Omega$ as well as $|d_0(x)|_2 = 1$ for $x \in \Omega$, take e.g. *any* $\tilde{d}_0 \in C^\infty(\Omega;\mathbb{R}^3)$ with compact support in Ω and such that the third component of $\tilde{d}_0$ is nonnegative. Then $\hat{d}_0 := \tilde{d}_0 + (0,0,1)^\mathsf{T}$ satisfies $|\hat{d}_0|_2 \geq 1$ and therefore the unit vector field $d_0 := \hat{d}_0/|\hat{d}_0|_2 \in W^{2\mu+1-2/p}_p(\Omega;\mathbb{R}^3)$ is well-defined. Since $\tilde{d}_0$ is compactly supported, we have $d_0 = (0,0,1)^\mathsf{T}$ in a tubular neighborhood of the boundary $\partial\Omega$, hence $\mathcal{B}_{d_0}(\nabla)d_0 = 0$ on $\partial\Omega$.
d) For the related system [8,9] with positive moment of inertia (with $k_2 = k_3 = -k_4$ for the Frank coefficients, without dissipative terms as stretching and with linear boundary conditions), the authors assume the regularity $u_0 \in W^2_2(\Omega;\mathbb{R}^3)$ and $d_0 \in W^3_2(\Omega;\mathbb{R}^3)$ for the initial data. Compared to our setting, Theorem 2.1 requires that $u_0 \in W^s_p(\Omega;\mathbb{R}^3)$ and $d_0 \in W^{s+1}_p(\Omega;\mathbb{R}^3)$ for any $s > 1 + 3/p$ and any $p > 5$. This shows that, making use of the time-weighted L_p-spaces, the regularity of the initial data can be decreased considerably. We refer to [43] for a general theory on quasilinear parabolic equations in the framework of weighted L_p-spaces.

We also prove that the solution of (2.12) depends continuously on the initial data (u_0, d_0). To formulate the result precisely, let us define the sets

$$X_{\gamma,\mu} := \{u \in W^{2\mu-2/p}_{p,\sigma}(\Omega;\mathbb{R}^3) : u = 0 \text{ on } \partial\Omega\} \times W^{2\mu+1-2/p}_p(\Omega;\mathbb{R}^3) \text{ and}$$

and

$$\mathcal{M} := \{(u,d) \in X_{\gamma,\mu} : |d|_2 = 1 \text{ in } \Omega \text{ and } \mathcal{B}_d(\nabla)d = 0 \text{ on } \partial\Omega\},$$

where $\mathcal{B}_d(\nabla)d$ is defined as in (B) but with d_0 being replaced by d. We also set that $\mathbb{E}_{1,\mu}(0,T) = \mathbb{E}^u_{1,\mu}(0,T) \times \mathbb{E}^d_{1,\mu}(0,T)$.

Theorem 2.3. *Let $1 < p < \infty$, $\mu \in (\frac{1}{2} + \frac{5}{2p}, 1]$, $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$, and assume that (R), (P) and (F) hold. For $(u_0, d_0) \in \mathcal{M}$ consider the unique solution of* (2.12) *with maximal time of existence $T_+(u_0, d_0) > 0$ given by Theorem 2.1.*

Then, for any $T \in (0, T_+(u_0, d_0))$, there exists $r > 0$ such that for each $(\tilde{u}_0, \tilde{d}_0) \in \mathcal{M} \cap \mathbb{B}_{X_{\gamma,\mu}}((u_0, d_0), r)$, the unique solution $(\tilde{u}, \tilde{d}, \tilde{\pi}) = (\tilde{u}(\cdot, (\tilde{u}_0, \tilde{d}_0)), \tilde{d}(\cdot, (\tilde{u}_0, \tilde{d}_0)), \tilde{\pi}(\cdot, (\tilde{u}_0, \tilde{d}_0)))$ of (2.12) *with initial value $(\tilde{u}_0, \tilde{d}_0)$ satisfies*

$$(\tilde{u}, \tilde{d}, \tilde{\pi}) \in \mathbb{E}_{1,\mu}(0,T) \times L_{p,\mu}((0,T); \dot{H}^1_p(\Omega))$$

and the mapping

$$\mathcal{M} \cap \mathbb{B}_{X_{\gamma,\mu}}((u_0, d_0), r) \to \mathbb{E}_{1,\mu}(0,T) \times L_{p,\mu}((0,T); \dot{H}^1_p(\Omega)),$$
$$(\tilde{u}_0, \tilde{d}_0) \mapsto \Big(\tilde{u}(\cdot, (\tilde{u}_0, \tilde{d}_0)), \tilde{d}(\cdot, (\tilde{u}_0, \tilde{d}_0)), \tilde{\pi}(\cdot, (\tilde{u}_0, \tilde{d}_0))\Big)$$

is continuous.

Remark 2.4. Let us comment on the thermodynamical consistency of (2.5) as well as (2.12). In the case of the non-isothermal system (2.5), the *total mass specific energy* is given by $e := |u|_2^2/2 + \varepsilon$. A short computation as in [23, Section 15.2] yields

$$\rho(\partial_t + u \cdot \nabla)e + \operatorname{div} \Phi_e = 0 \quad \text{in } \Omega,$$

where $\Phi_e := q + \pi u - Su - \frac{\partial \psi}{\partial(\nabla d)^{\mathsf{T}}} \mathcal{D}_t d$ is the energy flux. Consequently, if $q \cdot \nu = 0$, $u = 0$ on $\partial\Omega$ and if the natural boundary condition

$$P_d \frac{\partial \psi}{\partial(\nabla d)} \cdot \nu = 0 \quad \text{on } \partial\Omega \tag{2.13}$$

for the director d is satisfied, the *total energy*

$$\mathsf{E}(t) := \int_\Omega \rho(t,x) e(t,x) dx$$

is preserved for all $t \in [0, T]$.

On the other hand, in the isothermal situation, the temperature θ is constant and the equation for the internal energy ε in (2.5) is ignored. Then one cannot expect conservation of total energy, cf. [26, Section 2]. However, to handle the isothermal case, we define the *available energy* e_a by $e_a := |u|_2^2/2 + \psi$ with the free energy ψ. Since $\varepsilon = \psi + \theta\eta$ with the entropy η, it follows from the balances of internal energy and entropy that

$$\rho(\partial_t + u \cdot \nabla)e_a + \operatorname{div}(\Phi_e - \theta\Phi_\eta) = -\theta r - \rho\eta\mathcal{D}_t\theta - \Phi_\eta \cdot \nabla\theta,$$

with the entropy flux $\Phi_\eta := q/\theta$ and the *entropy production*

$$\theta r = [\alpha_0|\nabla\theta|_2^2 + \alpha_1(d|\nabla\theta)^2]/\theta + 2\mu_s|D|_2^2 + \mu_b|\operatorname{div} u|^2 \\ + \frac{1}{\gamma}|P_d(\mathsf{a} - \mu_P Dd)|_2^2 + \mu_L|P_d Dd|_2^2 + \mu_0(Dd|d)^2,$$

where $\mathsf{a} := \partial_i(\rho\nabla_{\partial_i d}\psi) - \rho\nabla_d\psi$, cf. [23, Section 15.2.8 & 15.3.3]. In case of constant θ, this reduces to

$$\rho(\partial_t + u\cdot\nabla)e_a + \operatorname{div}(\Phi_e - \theta\Phi_\eta) = -r_a,$$

with

$$r_a := \theta r = 2\mu_s|D|_2^2 + \mu_b|\operatorname{div} u|^2 \\ + \frac{1}{\gamma}|P_d(\mathsf{a} - \mu_P Dd)|_2^2 + \mu_L|P_d Dd|_2^2 + \mu_0(Dd|d)^2.$$

Consequently, by the natural boundary condition (2.13), the *total available energy*

$$\mathsf{E}_a(t) := \int_\Omega \rho(t,x)e_a(t,x)dx$$

satisfies

$$\frac{d}{dt}\mathsf{E}_a(t) = -\int_\Omega r_a(t,x)dx \le 0$$

for all $t \in [0,T]$, since $r_a(t,x) \ge 0$ by assumption (P) and if $\mu_b \ge 0$. Hence the total available energy is nonincreasing and serves as a Lyapunov functional.

3. The Ericksen operator

In this section we compute the nonlinear operator associated to $\operatorname{div}(\frac{\partial\psi}{\partial(\nabla d)})$ as well as its boundary condition given in (2.12).

3.1. Computation of the Ericksen-operator

In the following, We compute the nonlinear operator associated to $\operatorname{div}(\frac{\partial\psi}{\partial(\nabla d)})$ defined in (2.12), with ψ from (2.4). For a smooth vector field d, its derivative $\nabla d \in \mathbb{R}^{3\times 3}$ and rotation $\operatorname{curl} d \in \mathbb{R}^3$ are defined by $\nabla d = (\partial_j d_i)_{i,j}$ and

$$\operatorname{curl} d = (\partial_2 d_3 - \partial_3 d_2, \partial_3 d_1 - \partial_1 d_3, \partial_1 d_2 - \partial_2 d_1)^{\mathsf{T}}.$$

It follows that

$$\operatorname{Tr}(\nabla d)^2 = \sum_{i,k=1}^{3} \partial_k d_i\,\partial_i d_k,$$

where $\operatorname{Tr} A = \sum_{i=1}^{3} a_{ii}$ is the trace of a Matrix $A = (a_{ij}) \in \mathbb{R}^{3\times 3}$. This readily implies

$$\frac{\partial \operatorname{Tr}(\nabla d)^2}{\partial(\nabla d)} = 2(\nabla d)^\mathsf{T}.$$

Furthermore, we have

$$\frac{\partial(\operatorname{div} d)^2}{\partial(\nabla d)} = 2 \operatorname{div} d \cdot \frac{\partial \operatorname{div} d}{\partial(\nabla d)} = 2 \operatorname{div} d \cdot I_{3\times 3}.$$

For a differentiable matrix $A = (a_{ij}) \in \mathbb{R}^{3\times 3}$, its divergence $\operatorname{div} A \in \mathbb{R}^3$ is defined by $\operatorname{div} A = \left(\sum_{j=1}^{3} \partial_j a_{ij}\right)_{i=1,2,3}$. We note that this readily implies

$$\begin{aligned}\operatorname{div}\left(\frac{\partial \operatorname{Tr}(\nabla d)^2}{\partial(\nabla d)} - \frac{\partial(\operatorname{div} d)^2}{\partial(\nabla d)}\right) &= 2 \operatorname{div}(\nabla d)^\mathsf{T} - 2 \operatorname{div}(\operatorname{div} d \cdot I_{3\times 3}) \\ &= 2\nabla \operatorname{div} d - 2\nabla \operatorname{div} d = 0,\end{aligned}$$

hence

$$\operatorname{div}\left(\frac{\partial \psi}{\partial(\nabla d)}\right) = \operatorname{div}\left(\frac{\partial \tilde{\psi}}{\partial(\nabla d)}\right),$$

where

$$\tilde{\psi}(d, \nabla d) = k_1(\operatorname{div} d)^2 + k_2(d \cdot \operatorname{curl} d)^2 + k_3|d \times \operatorname{curl} d|^2.$$

For convenience, we will rewrite the expression for $\tilde{\psi}$ as follows:

$$\begin{aligned}\tilde{\psi}(d, \nabla d) &= k_1(\operatorname{div} d)^2 + k_2(d \cdot \operatorname{curl} d)^2 + k_3|d \times \operatorname{curl} d|^2 \\ &= k_1(\operatorname{div} d)^2 + k_3|\operatorname{curl} d|_2^2 \\ &\quad + (k_2 - k_3)(d \cdot \operatorname{curl} d)^2.\end{aligned}$$

Here we have used the property $|d|_2 = 1$ and the fact $|a \cdot b|^2 + |a \times b|_2^2 = |a|_2^2 \cdot |b|_2^2$ for all $a, b \in \mathbb{R}^3$.

Since $|\operatorname{curl} d|_2^2 = (\partial_2 d_3 - \partial_3 d_2)^2 + (\partial_3 d_1 - \partial_1 d_3)^2 + (\partial_1 d_2 - \partial_2 d_1)^2$, we obtain

$$\frac{\partial |\operatorname{curl} d|^2}{\partial(\nabla d)} = 2(\nabla d - (\nabla d)^\mathsf{T}).$$

This readily implies that

$$\begin{aligned}\operatorname{div}\left(\frac{\partial(k_1(\operatorname{div} d)^2 + k_3|\operatorname{curl} d|^2)}{\partial(\nabla d)}\right) &= 2k_1 \operatorname{div}(\operatorname{div} d \cdot I_{3\times 3}) + 2k_3 \operatorname{div}(\nabla d - (\nabla d)^\mathsf{T}) \\ &= 2k_1 \nabla \operatorname{div} d + 2k_3 \Delta d - 2k_3 \nabla \operatorname{div} d \\ &= 2k_3 \Delta d + 2(k_1 - k_3)\nabla \operatorname{div} d.\end{aligned}$$

Next, we will take care about the term $(d \cdot \operatorname{curl} d)^2$. In a first step, we obtain

$$\frac{\partial (d \cdot \operatorname{curl} d)}{\partial (\nabla d)} = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix}, \tag{3.1}$$

by employing the definition of $\operatorname{curl} d$. This in turn implies

$$\begin{aligned} \operatorname{div}\left(\frac{\partial (d \cdot \operatorname{curl} d)^2}{\partial (\nabla d)}\right) &= 2 \operatorname{div}\left((d \cdot \operatorname{curl} d)\frac{\partial (d \cdot \operatorname{curl} d)}{\partial (\nabla d)}\right) \\ &= 2(d \cdot \operatorname{curl} d) \operatorname{div}\left(\frac{\partial (d \cdot \operatorname{curl} d)}{\partial (\nabla d)}\right) + 2\frac{\partial (d \cdot \operatorname{curl} d)}{\partial (\nabla d)} \\ &\quad \cdot (\nabla (d \cdot \operatorname{curl} d))^{\mathsf{T}}. \end{aligned}$$

Here we have used the fact that $\operatorname{div}(f \cdot A) = f \cdot \operatorname{div} A + A \cdot (\nabla f)^{\mathsf{T}}$ for any differentiable matrix A and any differentiable scalar f with derivative $\nabla f = (\partial_1 f, \partial_2 f, \partial_3 f)$. A direct computation yields

$$\operatorname{div}\left(\frac{\partial (d \cdot \operatorname{curl} d)}{\partial (\nabla d)}\right) = -\operatorname{curl} d,$$

by (3.1), and hence

$$\begin{aligned} \operatorname{div}\left(\frac{\partial (d \cdot \operatorname{curl} d)^2}{\partial (\nabla d)}\right) &= 2 \operatorname{div}\left((d \cdot \operatorname{curl} d)\frac{\partial (d \cdot \operatorname{curl} d)}{\partial (\nabla d)}\right) \\ &= -2(d \cdot \operatorname{curl} d) \operatorname{curl} d + 2 \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \\ &\quad \cdot (\nabla (d \cdot \operatorname{curl} d))^{\mathsf{T}}. \end{aligned}$$

The last term can be rewritten as

$$\begin{aligned} &\begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \cdot (\nabla (d \cdot \operatorname{curl} d))^{\mathsf{T}} \\ &= (d \times \nabla)(d \cdot \operatorname{curl} d) \\ &= ((d \times \nabla) \otimes d) \cdot \operatorname{curl} d + ((d \times \nabla) \otimes \operatorname{curl} d) \cdot d, \end{aligned}$$

where $a \otimes b = (a_i b_j)_{i,j} \in \mathbb{R}^{3\times 3}$ for $a, b \in \mathbb{R}^3$ and $\nabla = (\partial_1, \partial_2, \partial_3)^{\mathsf{T}}$.

Altogether, this yields

$$\begin{aligned} \operatorname{div}\left(\frac{\partial \tilde{\psi}}{\partial (\nabla d)}\right) &= 2k_3 \Delta d + 2(k_1 - k_3)\nabla \operatorname{div} d + 2(k_2 - k_3)\left((d \times \nabla) \otimes \operatorname{curl} d\right) \cdot d \\ &\quad + 2(k_2 - k_3)\left((d \times \nabla) \otimes d\right) \cdot \operatorname{curl} d - 2(k_2 - k_3)(d \cdot \operatorname{curl} d) \operatorname{curl} d \end{aligned}$$

It remains to apply the projection $P_d = I - d \otimes d$. Since $|d|_2 = 1$, we obtain $\Delta d \cdot d = -|\nabla d|_2^2$ and hence

$$P_d(\Delta d) = \Delta d + |\nabla d|_2^2 d.$$

Furthermore, a direct computation yields

$$((d \times \nabla) \otimes \operatorname{curl} d) \cdot d \perp d \text{ and } ((d \times \nabla) \otimes d) \cdot \operatorname{curl} d \perp d. \tag{3.2}$$

Therefore

$$P_d\, (((d \times \nabla) \otimes \operatorname{curl} d) \cdot d) = ((d \times \nabla) \otimes \operatorname{curl} d) \cdot d$$

as well as

$$P_d\, (((d \times \nabla) \otimes d) \cdot \operatorname{curl} d) = ((d \times \nabla) \otimes d) \cdot \operatorname{curl} d.$$

Summarizing, we arrive at the expression

$$\begin{aligned} P_d \operatorname{div}\left(\frac{\partial \tilde{\psi}}{\partial(\nabla d)}\right) &= 2k_3 \Delta d + 2(k_1 - k_3) P_d \nabla \operatorname{div} d \\ &\quad + 2(k_2 - k_3)\, ((d \times \nabla) \otimes \operatorname{curl} d) \cdot d \\ &\quad + 2(k_2 - k_3)\, ((d \times \nabla) \otimes d) \cdot \operatorname{curl} d \\ &\quad - 2(k_2 - k_3)(d \cdot \operatorname{curl} d) P_d \operatorname{curl} d + 2k_3 |\nabla d|_2^2 d. \end{aligned}$$

Finally, we note that the principle part of the linearized Ericksen operator is then given by

$$\mathcal{A}_{\tilde{d}}(\nabla) d = 2k_3 \Delta d + 2(k_1 - k_3) P_{\tilde{d}} \nabla \operatorname{div} d + 2(k_2 - k_3) \Big((\tilde{d} \times \nabla) \otimes \operatorname{curl} d\Big) \cdot \tilde{d} \tag{3.3}$$

for some given $\tilde{d}$.

3.2. Boundary condition for the Ericksen operator

As explained in (1.4), the natural boundary condition for the Ericksen operator reads as

$$P_d \frac{\partial \psi}{\partial(\nabla d)} \cdot \nu = 0, \tag{3.4}$$

with ψ from (2.4). We first compute $\frac{\partial \psi}{\partial(\nabla d)} \cdot \nu$. The results in Section 3.1 yield

$$\begin{aligned} &\frac{\partial (\operatorname{div} d)^2}{\partial(\nabla d)} \cdot \nu = 2(\operatorname{div} d)\nu, \quad \frac{\partial \operatorname{Tr}(\nabla d)^2}{\partial(\nabla d)} \cdot \nu = 2(\nabla d)^{\mathsf{T}} \cdot \nu, \\ &\frac{\partial |\operatorname{curl} d|^2}{\partial(\nabla d)} \cdot \nu = 2(\nabla d - (\nabla d)^{\mathsf{T}}) \cdot \nu, \end{aligned}$$

and

$$\frac{\partial (d \cdot \operatorname{curl} d)^2}{\partial(\nabla d)} \cdot \nu = 2(d \cdot \operatorname{curl} d) \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix} \cdot \nu = 2(d \cdot \operatorname{curl} d)(d \times \nu).$$

Summarizing, this yields

$$\frac{\partial \psi}{\partial(\nabla d)} \cdot \nu = 2k_3 \nabla d \cdot \nu + 2(k_1 \operatorname{div} d \cdot I - k_3 (\nabla d)^{\mathsf{T}}) \cdot \nu \\ + 2(k_2 - k_3)(d \cdot \operatorname{curl} d)(d \times \nu) + 2(k_2 + k_4)((\nabla d)^{\mathsf{T}} - \operatorname{div} d \cdot I) \cdot \nu.$$

We observe that $P_d(\nabla d \cdot \nu) = \nabla d \cdot \nu$ since $d^{\mathsf{T}} \cdot \nabla d \cdot \nu = \nu^{\mathsf{T}} \cdot (\nabla d)^{\mathsf{T}} \cdot d = 0$, as $|d|_2 = 1$. Furthermore, we have $P_d(d \times \nu) = d \times \nu$. This finally yields

$$\begin{aligned} P_d \frac{\partial \psi}{\partial(\nabla d)} \cdot \nu = {} & 2k_3 \nabla d \cdot \nu + 2P_d(k_1 \operatorname{div} d \cdot I - k_3 (\nabla d)^{\mathsf{T}}) \cdot \nu \\ & + 2(k_2 - k_3)(d \cdot \operatorname{curl} d)(d \times \nu) \\ & + 2(k_2 + k_4) P_d((\nabla d)^{\mathsf{T}} - \operatorname{div} d \cdot I) \cdot \nu. \end{aligned} \tag{3.5}$$

We note that the principle part of the linearization $\mathcal{B}_{\tilde{d}}(\nabla)d$ of the right hand side of (3.5) reads

$$\mathcal{B}_{\tilde{d}}(\nabla)d = 2k_3 \nabla d \cdot \nu + 2P_{\tilde{d}}(k_1 \operatorname{div} d \cdot I - k_3 (\nabla d)^{\mathsf{T}}) \cdot \nu \\ + 2(k_2 - k_3)(\tilde{d} \cdot \operatorname{curl} d)(\tilde{d} \times \nu) + 2(k_2 + k_4) P_{\tilde{d}}((\nabla d)^{\mathsf{T}} - \operatorname{div} d \cdot I) \cdot \nu, \tag{3.6}$$

for some given $\tilde{d}$.

4. Functional analytic setting

Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega \in C^3$. For the velocity field u, we choose the base space

$$L_{p,\mu}((0, T); L_p(\Omega; \mathbb{R}^3)),$$

where $1 < p < \infty$ and $\mu \in (1/p, 1]$. In the equation for u, the terms of highest order are $\partial_t u$ for the variable $t \in (0, T)$ and $\partial_{x_i} \partial_{x_j} u$ for the variable $x \in \Omega$. Therefore, the optimal regularity class for u is given by

$$H^1_{p,\mu}((0, T); L_p(\Omega; \mathbb{R}^3)) \cap L_{p,\mu}((0, T); H^2_p(\Omega; \mathbb{R}^3)),$$

whereas

$$\nabla \pi \in L_{p,\mu}((0, T); L_p(\Omega; \mathbb{R}^3))$$

is optimal for π. Since $\mathcal{D}_t d$ is a part of S_L and since $\operatorname{div} S_L$ appears in the equation for u, it is natural to assume

$$d \in H^1_{p,\mu}((0, T); H^1_p(\Omega; \mathbb{R}^3)).$$

In Section 3.1, we explicitly computed the Ericksen operator $P_d \operatorname{div} \left(\frac{\partial \psi}{\partial(\nabla d)}\right)$, which acts as a quasilinear differential operator of second order. Hence, choosing

$$L_{p,\mu}((0, T); H^1_p(\Omega; \mathbb{R}^3))$$

as the base space for the d-equation, it follows that

$$H^1_{p,\mu}((0,T); H^1_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^3_p(\Omega;\mathbb{R}^3))$$

is the optimal regularity class for the function d.

We note that for $\ell \in \{0,1\}$, the embeddings

$$\begin{aligned} &H^1_{p,\mu}((0,T); H^\ell_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^{2+\ell}_p(\Omega;\mathbb{R}^3)) \\ &\quad \hookrightarrow C([0,T]; B^{2\mu+\ell-2/p}_{pp}(\Omega;\mathbb{R}^3)) \end{aligned}$$

imply that, necessarily,

$$u(0) \in B^{2\mu-2/p}_{pp}(\Omega;\mathbb{R}^3) \quad \text{and} \quad d(0) \in B^{2\mu+1-2/p}_{pp}(\Omega;\mathbb{R}^3).$$

Here $B^s_{pq}(\Omega;\mathbb{R}^3)$, $1 < p, q < \infty$, $s > 0$, denotes a classical Besov space, see e.g. [46,47].

With a view on the nonlinear boundary term $P_d \frac{\partial\psi}{\partial(\nabla d)} \cdot \nu$ computed in Section 3.2, we will also need the optimal regularity class for the trace of $\partial_j d$, $j \in \{1,2,3\}$ on $\partial\Omega$. Given

$$d \in H^1_{p,\mu}((0,T); H^1_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^3_p(\Omega;\mathbb{R}^3)),$$

it follows that

$$\partial_j d \in H^1_{p,\mu}((0,T); L_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^2_p(\Omega;\mathbb{R}^3))$$

and hence

$$\mathrm{tr}_{\partial\Omega}\, \partial_j d \in W^{1-1/2p}_{p,\mu}((0,T); L_p(\partial\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); W^{2-1/p}_p(\partial\Omega;\mathbb{R}^3)),$$

for each $j \in \{1,2,3\}$. For the remainder of this article, we will always assume that

$$1 < p < \infty \quad \text{and} \quad \frac{1}{2} + \frac{5}{2p} < \mu \le 1, \tag{4.1}$$

so that

$$W^{2\mu+\ell-2/p}_p(\Omega;\mathbb{R}^3) = B^{2\mu+\ell-2/p}_{pp}(\Omega;\mathbb{R}^3),$$

as $2\mu + \ell - 2/p \notin \mathbb{N}$ for $\ell \in \{0,1\}$. Moreover, (4.1) ensures the embedding

$$W^{2\mu+\ell-2/p}_p(\Omega;\mathbb{R}^3) \hookrightarrow C^{1+\ell}(\overline{\Omega};\mathbb{R}^3), \quad \ell \in \{0,1\}.$$

Setting $z := (u,d)$, we rewrite (2.12) together with its boundary and initial conditions in the equivalent form

$$\left\{ \begin{aligned} \partial_t u + A_u(z)u + R_1^{\mathsf{T}}(z)\partial_t d + \nabla\pi &= F_u(z) && \text{in} \quad (0,T)\times\Omega, \\ -R_0(z)u + \gamma(z)\partial_t d + A_d(z)d &= F_d(z) && \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} u &= 0 && \text{in} \quad (0,T)\times\Omega, \\ u &= 0 && \text{in} \quad (0,T)\times\partial\Omega, \\ \mathcal{B}_d(\nabla)d &= 0 && \text{in} \quad (0,T)\times\partial\Omega, \\ (u,d) &= (u_0,d_0) && \text{in} \quad \{0\}\times\Omega. \end{aligned} \right. \tag{4.2}$$

Here, for given

$$\tilde{z} = (\tilde{u}, \tilde{d}) \in W_p^{2\mu-2/p}(\Omega; \mathbb{R}^3) \times W_p^{2\mu+1-2/p}(\Omega; \mathbb{R}^3),$$

we set $-A_d(\tilde{z})d := \mathcal{A}_{\tilde{d}}(\nabla)d$, and

$$\mathcal{A}_{\tilde{d}}(\nabla)d := 2k_3\Delta d + 2(k_1 - k_3)P_{\tilde{d}}\nabla \operatorname{div} d + 2(k_2 - k_3)\left((\tilde{d} \times \nabla) \otimes \operatorname{curl} d\right) \cdot \tilde{d},$$

is the principle part of the linearized Ericksen operator defined in (3.3). Furthermore,

$$\begin{aligned}\mathcal{B}_{\tilde{d}}(\nabla)d :&= 2k_3\nabla d \cdot \nu + 2P_{\tilde{d}}(k_1 \operatorname{div} d \cdot I - k_3(\nabla d)^{\mathsf{T}}) \cdot \nu \\ &\quad + 2(k_2 - k_3)(\tilde{d} \cdot \operatorname{curl} d)(\tilde{d} \times \nu) \\ &\quad + 2(k_2 + k_4)P_{\tilde{d}}((\nabla d)^{\mathsf{T}} - \operatorname{div} d \cdot I) \cdot \nu,\end{aligned}$$

is defined as in (3.6) and, following Section 3.2, we have

$$\mathcal{B}_d(\nabla)d = P_d \frac{\partial \psi}{\partial(\nabla d)} \cdot \nu.$$

For the definition of the second order operator $A_u(\tilde{z})$ and of the first order operators $R_1(\tilde{z})$ and $R_0(\tilde{z})$ we refer to [26, pages 1454–1455], where, however, one has to replace u by $\tilde{z}$ and ∇_w by ∇ in that reference. We note further that $\operatorname{div}(\nabla u)^{\mathsf{T}} = 0$ since $\operatorname{div} u = 0$, so that the second term in the definition of $A_u(\tilde{z})$ in [26] vanishes. If $\tilde{z} = (\tilde{u}, \tilde{d}) \in \mathbb{R}^3 \times \mathbb{R}^3$ is constant, a detailed expression of the Fourier-symbols of $R_j(\tilde{z})$ an $A_u(\tilde{z})$, is given in the proof of Theorem 6.2 below. Finally, $F_u(z)$ as well as $F_d(z)$ collect all terms of lower order.

5. Properties of the linearized Ericksen operator

The aim of this section is twofold: we first prove that in $\mathbb{R}^3$, the principal part of the linearized Ericksen operator is strongly elliptic in the sense of [42, Section 6]. Secondly, we show that in half-spaces, the linearized Ericksen operator subject to the principal part of the linearized boundary condition satisfies the Lopatinskii-Shapiro condition. The results obtained in this section are crucial for proving maximal regularity results in the following Section 6.

We start by recalling from subsections 3.1 and 3.2 that the Ericksen operator is of the form

$$\begin{aligned}P_d \operatorname{div}\left(\frac{\partial \psi}{\partial(\nabla d)}\right) &= 2k_3\Delta d + 2(k_1 - k_3)P_d\nabla \operatorname{div} d \\ &\quad + 2(k_2 - k_3)\,((d \times \nabla) \otimes \operatorname{curl} d) \cdot d \\ &\quad + 2(k_2 - k_3)\,((d \times \nabla) \otimes d) \cdot \operatorname{curl} d \\ &\quad - 2(k_2 - k_3)(d \cdot \operatorname{curl} d)P_d \operatorname{curl} d + 2k_3|\nabla d|_2^2 d\end{aligned}$$

and that its natural (nonlinear) boundary condition reads as

$$P_d \frac{\partial \psi}{\partial(\nabla d)} \cdot \nu = 2k_3 \nabla d \cdot \nu + 2P_d(k_1 \operatorname{div} d \cdot I - k_3(\nabla d)^\mathsf{T}) \cdot \nu \\ + 2(k_2 - k_3)(d \cdot \operatorname{curl} d)(d \times \nu) \\ + 2(k_2 + k_4)P_d((\nabla d)^\mathsf{T} - \operatorname{div} d \cdot I) \cdot \nu.$$

We recall from Section 3 that for sufficiently smooth η, the principle part of the linearized Ericksen operator is given by

$$\mathcal{A}_d(\nabla)\eta := 2k_3\Delta\eta + 2(k_1 - k_3)P_d\nabla \operatorname{div} \eta + 2(k_2 - k_3)\left((d \times \nabla) \otimes \operatorname{curl} \eta\right) \cdot d, \tag{5.1}$$

and that the principle part of the linearized boundary condition reads as

$$\mathcal{B}_d(\nabla)\eta := 2k_3\nabla\eta \cdot \nu + 2P_d(k_1 \operatorname{div} \eta \cdot I - k_3(\nabla\eta)^\mathsf{T}) \cdot \nu \\ + 2(k_2 - k_3)(d \cdot \operatorname{curl} \eta)(d \times \nu) + 2(k_2 + k_4)P_d((\nabla\eta)^\mathsf{T} - \operatorname{div} \eta \cdot I) \cdot \nu, \tag{5.2}$$

with constant coefficients $d \in \mathbb{R}^3$ and with $|d|_2 = 1$.

5.1. Strong Ellipticity

We show in the following that the operator $-\mathcal{A}_d(\nabla)$ given by (5.1) is strongly elliptic. To this end, we denote by

$$\begin{aligned} m_d(\xi)\eta &= 2k_3|\xi|_2^2\eta + 2(k_1 - k_3)P_d(\xi \otimes \xi)\eta + 2(k_2 - k_3)\left((d \times \xi) \otimes (\xi \times \eta)\right) \cdot d \\ &= 2k_3|\xi|_2^2\eta + 2(k_1 - k_3)P_d(\xi \otimes \xi)\eta + 2(k_2 - k_3)(d \times \xi)[(\xi \times \eta) \cdot d] \\ &= 2k_3|\xi|_2^2\eta + 2(k_1 - k_3)P_d(\xi \otimes \xi)\eta + 2(k_2 - k_3)(d \times \xi)[(d \times \xi) \cdot \eta] \\ &= \left[2k_3|\xi|_2^2 I + 2(k_1 - k_3)P_d(\xi \otimes \xi) + 2(k_2 - k_3)(d \times \xi) \otimes (d \times \xi)\right]\eta \end{aligned}$$

the Fourier symbol of the operator $-\mathcal{A}_d(\nabla)$.

Proposition 5.1. (Strong ellipticity) *Assume (F) and let $d \in \mathbb{R}^3$ with $|d|_2 = 1$. Then the operator $-\mathcal{A}_d(\nabla)$ is strongly elliptic, i.e., there exists a constant $c > 0$ such that*

$$m_d(\xi)\eta \cdot \eta \geq c|\xi|_2^2|\eta|_2^2 \tag{5.3}$$

for all $\xi, \eta \in \mathbb{R}^3$.

Proof. We consider in a first step the condition $9k_3 > k_1$ from assumption (F). Let us denote by

$$m_d^{sym}(\xi) = 2k_3|\xi|_2^2 I + (k_1 - k_3)[P_d(\xi \otimes \xi) + (\xi \otimes \xi)P_d] \\ +2(k_2 - k_3)(d \times \xi) \otimes (d \times \xi)$$

the symmetric part of $m_d(\xi)$. We note that for $\xi \in \mathbb{R}^3$, $\xi \neq 0$, it holds that $m_d^{sym}(\xi) = |\xi|_2^2 \tilde{m}_d^{sym}(\zeta)$, where $\zeta = \xi/|\xi|_2$ and

$$\begin{aligned}\tilde{m}_d^{sym}(\zeta) &= 2k_3 I + (k_1 - k_3)[P_d(\zeta \otimes \zeta) + (\zeta \otimes \zeta)P_d] \\ &\quad +2(k_2 - k_3)(d \times \zeta) \otimes (d \times \zeta).\end{aligned}$$

We will prove that the symmetric matrix $\tilde{m}_d^{sym}(\zeta) \in \mathbb{R}^{3\times 3}$ is positive definite. To this end, we distinguish several cases.

Case I: Assume that $d \parallel \zeta$. Then $d \times \zeta = 0$ and $d = \zeta$ or $d = -\zeta$, since $|d|_2 = 1$. It follows that $\zeta \otimes \zeta = d \otimes d$ and therefore

$$(I - d \otimes d)(d \otimes d) = (d \otimes d)(I - d \otimes d) = (d \otimes d) - (d \otimes d) = 0,$$

where we made again use of the fact $|d|_2 = 1$. So, in this case,

$$\tilde{m}_d^{sym}(\zeta) = 2k_3 I,$$

wherefore all three eigenvalues coincide with $2k_3 > 0$.

Case II: Assume that $d \times \zeta \neq 0$. In this case, $d \times \zeta$ is an eigenvector of $\tilde{m}_d^{sym}(\zeta)$, since

$$\begin{aligned}\tilde{m}_d^{sym}(\zeta)(d \times \zeta) &= 2k_3(d \times \zeta) + 2(k_2 - k_3)|d \times \zeta|_2^2(d \times \zeta) \\ &= \left(2k_2|d \times \zeta|_2^2 + 2k_3(1 - |d \times \zeta|_2^2)\right)(d \times \zeta).\end{aligned}$$

Here we have used the fact that $\zeta, d \perp (d \times \zeta)$. The corresponding eigenvalue satisfies

$$2k_2|d \times \zeta|_2^2 + 2k_3(1 - |d \times \zeta|_2^2) \geq 2\min\{k_2, k_3\} > 0,$$

since $|d|_2 = |\zeta|_2 = 1$. We are now looking for other eigenvalues and eigenvectors and make the ansatz $\alpha\zeta + \beta d$, $\alpha, \beta \in \mathbb{R}$. Since

$$(\alpha\zeta + \beta d) \perp (d \times \zeta)$$

a short computation shows that

$$\begin{aligned}\tilde{m}_d^{sym}(\zeta)(\alpha\zeta + \beta d) &= [2k_3\alpha + (k_1 - k_3)(\alpha + \beta z + \alpha(1 - z^2))]\zeta \\ &\quad + [2k_3\beta - (k_1 - k_3)(\alpha z + \beta z^2)]d,\end{aligned}$$

where $z := (d|\zeta)$.

Case II.1: If $z = 0$, then ζ is an eigenvector with eigenvalue $2k_1 > 0$ ($(\alpha, \beta) = (1, 0)$) and also d is an eigenvector with eigenvalue $2k_3 > 0$ ($(\alpha, \beta) = (0, 1)$).

Case II.2: In case $z \neq 0$, we require that there exists λ (an eigenvalue) such that

$$2k_3\alpha + (k_1 - k_3)(\alpha + \beta z + \alpha(1 - z^2)) = \lambda\alpha$$

and

$$2k_3\beta - (k_1 - k_3)(\alpha z + \beta z^2) = \lambda\beta.$$

If $k_1 = k_3$, then ζ and d are eigenvectors to the same eigenvalue $2k_3$. In case $k_1 \neq k_3$, we solve the last equation for λ, to obtain (observe that $\beta = 0$ would yield $\alpha = 0$ in case $z \neq 0$ and $k_1 \neq k_3$)

$$\lambda = 2k_3 - (k_1 - k_3)(\alpha z/\beta + z^2), \tag{5.4}$$

and plug it into the other equation. This yields

$$2\alpha + \beta z + \alpha^2 z/\beta = 0.$$

Multiplying with β and dividing by $z \neq 0$ implies

$$\alpha^2 + \frac{2}{z}\alpha\beta + \beta^2 = 0.$$

This quadratic equation can be solved for α to the result

$$\alpha = -\frac{\beta}{z}(1 \pm \sqrt{1 - z^2}),$$

which in turn implies

$$\frac{\alpha z}{\beta} = -1 \mp \sqrt{1 - z^2}.$$

We insert this expression into the equation (5.4) for λ, to obtain

$$\begin{aligned}\lambda_\pm &= 2k_3 - (k_1 - k_3)(-1 \mp \sqrt{1 - z^2} + z^2)\\ &= 2k_3 + (k_1 - k_3)(1 - z^2 \pm \sqrt{1 - z^2}).\end{aligned}$$

This shows that, if $k_1 < k_3$, then $\lambda_+ < \lambda_-$ and

$$\lambda_+ = 2k_3 + (k_1 - k_3)(1 - z^2 + \sqrt{1 - z^2}) \geq 2k_3 + 2(k_1 - k_3) = 2k_1 > 0,$$

since

$$0 \leq 1 - z^2 + \sqrt{1 - z^2} \leq 2$$

for $|z| = |(d|\zeta)| \leq 1$, by the Cauchy-Schwarz inequality.

On the contrary, if $k_1 > k_3$, then $\lambda_+ > \lambda_-$ and

$$\lambda_- = 2k_3 + (k_1 - k_3)(1 - z^2 - \sqrt{1 - z^2}),$$

This time we use the estimate

$$1 - z^2 - \sqrt{1 - z^2} \geq -\frac{1}{4},$$

valid for all $|z| \leq 1$, to obtain

$$\lambda_- \geq 2k_3 - \frac{1}{4}(k_1 - k_3) = \frac{9k_3 - k_1}{4} > 0,$$

by the assumption $9k_3 > k_1$ from (F). This shows that the symmetric matrix $\tilde{m}_d^{sym}(\zeta)$ is positive definite, which in turn is equivalent to the fact that the matrix $\tilde{m}_d(\zeta)$ is positive definite.

Let us finally consider the condition $2|k_1 - k_3| < \min\{k_2, k_3\}$ from assumption (F). The considerations from above for the very special case $k_1 = k_3$ show that the symmetric matrix

$$M_d(\xi) := 2k_3|\xi|_2^2 I + 2(k_2 - k_3)(d \times \xi) \otimes (d \times \xi)$$

is positive definite and satisfies the estimate

$$M_d(\xi)\eta \cdot \eta \geq 2\min\{k_2, k_3\}|\xi|_2^2|\eta|_2^2.$$

Since $|d|_2 = 1$, it holds that

$$2|P_d(\xi \otimes \xi)\eta \cdot \eta| = 2|(I - d \otimes d)(\xi \otimes \xi)\eta \cdot \eta| \leq 4|\xi|_2^2|\eta|_2^2$$

by the Cauchy-Schwarz inequality, which implies that the matrix

$$M_d(\xi) + 2(k_1 - k_3)P_d(\xi \otimes \xi)$$

is positive definite, provided $2|k_1 - k_3| < \min\{k_2, k_3\}$. This completes the proof of Proposition 5.1. □

Remark 5.2. If, instead of condition (F), one merely assumes that the Frank coefficients satisfy $k_j > 0$ for $j \in \{1, 2, 3\}$, one can prove that all eigenvalues of the (non-symmetric) matrix-valued symbol $m_d(\xi)$ of $-\mathcal{A}_d(\nabla)$ are real and positive, i.e. the operator $-\mathcal{A}_d(\nabla)$ is *normally elliptic*. Let us emphasize that, in general, normal ellipticity *does not* imply strong ellipticity. However, in the proof of Theorem 6.2 below we need to know that $-\mathcal{A}_d(\nabla)$ is strongly elliptic.

5.2. The Lopatinskii-Shapiro condition

For given $d \in \mathbb{R}^3$ with $|d|_2 = 1$, we consider the pair $(\mathcal{A}_d(\nabla), \mathcal{B}_d(\nabla))$, defined in (5.1) and (5.2). Our aim is to prove that $(\mathcal{A}_d(\nabla), \mathcal{B}_d(\nabla))$ satisfies the Lopatinskii-Shapiro condition, formulated precisely in Proposition 5.4 below. To this end, let H denote a half space in $\mathbb{R}^3$, hence H may be written as a translation and/or rotation of the half space $\mathbb{R}^3_+$. We start with the following

Proposition 5.3. *Assume (F) and let $d \in \mathbb{R}^3$ with $|d|_2 = 1$. Then, for each $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda \geq 0$, *the only solution $\eta \in H_2^2(\mathsf{H}; \mathbb{C}^3)$ of the boundary value problem*

$$\lambda\eta - \mathcal{A}_d(\nabla)\eta = 0 \quad \text{in } \mathsf{H}, \quad \mathcal{B}_d(\nabla)\eta = 0 \quad \text{on } \partial\mathsf{H},$$

is $\eta = 0$. Moreover, for each $\eta_1 \in H_2^2(\mathsf{H}; P_d\mathbb{C}^3)$ with $\mathcal{B}_d(\nabla)\eta_1 = 0$, it holds that

$$-(\mathcal{A}_d(\nabla)\eta_1|\eta_1)_{L_2} \geq \alpha\|\nabla\eta_1\|^2_{L_2(\mathsf{H})},$$

with the constant α from assumption (F).

Proof. We split η and write $\eta = P_d\eta + (I - P_d)\eta =: \eta_1 + \eta_2$. Applying $(I - P_d)$ to the equation $\lambda\eta - \mathcal{A}_d(\nabla)\eta = 0$ yields

$$\lambda\eta_2 - 2k_3(I - P_d)\Delta(\eta_1 + \eta_2) = 0,$$

since $(I - P_d)P_d = 0$, as $|d|_2 = 1$, and $((d \times \nabla) \otimes \operatorname{curl}\eta) \cdot d \perp d$. Moreover, we have

$$(I - P_d)\Delta(\eta_1 + \eta_2) = \Delta(I - P_d)(\eta_1 + \eta_2) = \Delta\eta_2.$$

Applying $(I - P_d)$ to the boundary condition $\mathcal{B}_d(\nabla)\eta = 0$ yields

$$0 = 2k_3(I - P_d)\nabla(\eta_1 + \eta_2) \cdot \nu = 2k_3\nabla(I - P_d)(\eta_1 + \eta_2) \cdot \nu = 2k_3\nabla\eta_2 \cdot \nu.$$

Therefore, η_2 solves the problem

$$\lambda\eta_2 - 2k_3\Delta\eta_2 = 0 \quad \text{in } \mathsf{H}, \quad \nabla\eta_2 \cdot \nu = 0 \quad \text{on } \partial\mathsf{H}.$$

Taking the inner product with η_2 in $L_2(\mathsf{H}; \mathbb{C}^3)$, integrating by parts and taking real parts, yields

$$\operatorname{Re}\lambda\|\eta_2\|^2_{L_2(\mathsf{H})} + 2k_3\|\nabla\eta_2\|^2_{L_2(\mathsf{H})} = 0.$$

If $\operatorname{Re}\lambda > 0$, then $\eta_2 = 0$. If $\operatorname{Re}\lambda = 0$, then η_2 is constant, hence $\eta_2 = 0$, since otherwise $\eta_2 \notin L_2(\mathsf{H})$. This shows that $\eta = \eta_1 = P_d\eta$.

Taking the inner product of $-\mathcal{A}_d(\nabla)\eta_1$ with η_1 in $L_2(\mathsf{H}; \mathbb{C}^3)$, it follows that

$$\begin{aligned}(\mathcal{A}_d(\nabla)\eta_1|\eta_1)_{L_2} = {} & 2k_3(\Delta\eta_1|\eta_1)_{L_2} \\ & +2(k_1 - k_3)(\nabla \operatorname{div}\eta_1|\eta_1)_{L_2} \\ & +2(k_2 - k_3)(((d \times \nabla) \otimes \operatorname{curl}\eta_1) \cdot d|\eta_1)_{L_2},\end{aligned}$$

since $P_d^{\mathsf{T}} = P_d$ and $P_d\eta_1 = \eta_1$. We write

$$k_1(\nabla \operatorname{div}\eta_1|\eta_1)_{L_2} = k_1(\operatorname{div}(\operatorname{div}\eta_1 \cdot I)|\eta_1)_{L_2}$$

and

$$k_3(\nabla \operatorname{div}\eta_1|\eta_1)_{L_2} = k_3(\operatorname{div}(\nabla\eta_1)^{\mathsf{T}}|\eta_1)_{L_2}.$$

Since $d \in \mathbb{R}^3$ is constant, the latter term can be rewritten as

$$(((d \times \nabla) \otimes \operatorname{curl}\eta_1) \cdot d|\eta_1)_{L_2} = (\operatorname{div}(\mathsf{D}(d \cdot \operatorname{curl}\eta_1))|\eta_1)_{L_2},$$

where

$$\mathsf{D} := \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix}.$$

Integrating by parts and invoking the boundary condition $\mathcal{B}_d(\nabla)\eta_1 = 0$ yields

$$\begin{aligned}
&2k_3(\Delta\eta_1|\eta_1)_{L_2} + 2k_1(\operatorname{div}(\operatorname{div}\eta_1 \cdot I)|\eta_1)_{L_2} - 2k_3(\operatorname{div}(\nabla\eta_1)^{\mathsf{T}}|\eta_1)_{L_2}\\
&\quad + 2(k_2 - k_3)(\operatorname{div}(\mathsf{D}(d \cdot \operatorname{curl}\eta_1))|\eta_1)_{L_2}\\
&\;= -2k_3(\nabla\eta_1|\nabla\eta_1)_{L_2} - 2k_1\|\operatorname{div}\eta_1\|_{L_2}^2 + 2k_3((\nabla\eta_1)^{\mathsf{T}}|\nabla\eta_1)_{L_2}\\
&\quad - 2(k_2 - k_3)(\mathsf{D}(d \cdot \operatorname{curl}\eta_1)|\nabla\eta_1)_{L_2} - 2(k_2 + k_4)((\nabla\eta_1)^{\mathsf{T}} - \operatorname{div}\eta_1 \cdot I) \cdot \nu|\eta_1)_{L_{2,\partial\mathsf{H}}}
\end{aligned}$$

Another integration by parts yields, that the last (boundary) term can be written as

$$\begin{aligned}
&((\nabla\eta_1)^{\mathsf{T}} - \operatorname{div}\eta_1 \cdot I) \cdot \nu|\eta_1)_{L_{2,\partial\mathsf{H}}}\\
&\quad = ((\nabla\eta_1)^{\mathsf{T}} - \operatorname{div}\eta_1 \cdot I)|\nabla\eta_1)_{L_2} = ((\nabla\eta_1)^{\mathsf{T}}|\nabla\eta_1)_{L_2} - \|\operatorname{div}\eta_1\|_{L_2}^2\\
&\quad = ((\nabla\eta_1)^{\mathsf{T}} - \nabla\eta_1|\nabla\eta_1)_{L_2} + (\nabla\eta_1|\nabla\eta_1)_{L_2} - \|\operatorname{div}\eta_1\|_{L_2}^2,
\end{aligned}$$

since $\operatorname{div}(\nabla\eta_1)^{\mathsf{T}} - \operatorname{div}(\operatorname{div}\eta_1 \cdot I) = 0$.

A short computation shows that $(\mathsf{D}(d \cdot \operatorname{curl}\eta_1)|\nabla\eta_1)_{L_2} = \|d \cdot \operatorname{curl}\eta_1\|_{L_2}^2$. In summary, this yields

$$\begin{aligned}
-(\mathcal{A}_d(\nabla)\eta_1|\eta_1)_{L_2} &= 2k_1\|\operatorname{div}\eta_1\|_{L_2}^2 + 2k_3(\nabla\eta_1 - (\nabla\eta_1)^{\mathsf{T}}|\nabla\eta_1)_{L_2}\\
&\quad + 2(k_2 - k_3)\|d \cdot \operatorname{curl}\eta_1\|_{L_2}^2\\
&\quad + 2(k_2 + k_4)[((\nabla\eta_1)^{\mathsf{T}} - \nabla\eta_1|\nabla\eta_1)_{L_2} + \|\nabla\eta_1\|_{L_2}^2 - \|\operatorname{div}\eta_1\|_{L_2}^2]
\end{aligned}$$

We observe that

$$(\nabla\eta_1 - (\nabla\eta_1)^{\mathsf{T}}|\nabla\eta_1)_{L_2} = \|\operatorname{curl}\eta_1\|_{L_2}^2,$$

whence

$$\begin{aligned}
-(\mathcal{A}_d(\nabla)\eta_1|\eta_1)_{L_2} &= 2(k_1 - k_2 - k_4)\|\operatorname{div}\eta_1\|_{L_2}^2 + 2(k_3 - k_2 - k_4)\|\operatorname{curl}\eta_1\|_{L_2}^2\\
&\quad + 2(k_2 - k_3)\|d \cdot \operatorname{curl}\eta_1\|_{L_2}^2 + (k_2 + k_4)\|\nabla\eta_1\|_{L_2}^2.
\end{aligned}$$

Since $|d|_2 = 1$, we may write

$$\|\operatorname{curl}\eta_1\|_{L_2}^2 = \|d \times \operatorname{curl}\eta_1\|_{L_2}^2 + \|d \cdot \operatorname{curl}\eta_1\|_{L_2}^2,$$

which in turn yields

$$\begin{aligned}
-(\mathcal{A}_d(\nabla)\eta_1|\eta_1)_{L_2} &= 2(k_1 - k_2 - k_4)\|\operatorname{div}\eta_1\|_{L_2}^2 + 2(k_3 - k_2 - k_4)\|d \times \operatorname{curl}\eta_1\|_{L_2}^2\\
&\quad - k_4\|d \cdot \operatorname{curl}\eta_1\|_{L_2}^2 + (k_2 + k_4)\|\nabla\eta_1\|_{L_2}^2.
\end{aligned}$$

For any $0 < \alpha \leq \min\{k_1, k_2, k_3\}$, it follows from (F) that $k_4 = \alpha - k_2$, hence

$$k_j - k_2 - k_4 = k_j - \alpha \geq 0, \;\; j \in \{1, 3\}, \quad \text{and} \quad -k_4 \geq 0,$$

which implies

$$-(\mathcal{A}_d(\nabla)\eta_1|\eta_1)_{L_2} \geq \alpha\|\nabla\eta_1\|_{L_2(\mathsf{H})}^2.$$

This is the claimed estimate.

Finally, taking the inner product of $\lambda\eta_1 - \mathcal{A}_d(\nabla)\eta_1 = 0$ with η_1 in $L_2(\mathsf{H}; \mathbb{C}^3)$, and taking real parts, we obtain

$$0 = \operatorname{Re}\lambda\|\eta_1\|^2_{L_2(\mathsf{H})} - (\mathcal{A}_d(\nabla)\eta_1|\eta_1)_{L_2} \geq \operatorname{Re}\lambda\|\eta_1\|^2_{L_2(\mathsf{H})} + \alpha\|\nabla\eta_1\|^2_{L_2(\mathsf{H})},$$

which in turn implies that $\eta_1 = 0$. □

By the same strategy, we are able to prove the following result:

Proposition 5.4. (Lopatinskii-Shapiro condition) *Assume (F) and let* $d \in \mathbb{R}^3$ *with* $|d|_2 = 1$. *Then, for all* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq 0$ *and all* $\xi, \nu \in \mathbb{R}^3$ *with* $\xi \perp \nu$ *and* $(\lambda, \xi) \neq (0, 0)$*, the only solution* $\eta \in H^2_2(\mathbb{R}_+; \mathbb{C}^3)$ *of the initial value problem*

$$\lambda\eta - \mathcal{A}_d(i\xi + \nu\partial_y)\eta = 0, \quad y > 0, \quad \mathcal{B}_d(i\xi + \nu\partial_y)\eta = 0, \quad y = 0,$$

is $\eta = 0$. *Moreover, for each* $\eta_1 \in H^2_2(\mathbb{R}_+; P_d\mathbb{C}^3)$ *with* $\mathcal{B}_d(i\xi + \nu\partial_y)\eta_1 = 0$*, it holds that*

$$-(\mathcal{A}_d(i\xi + \nu\partial_y)\eta_1|\eta_1)_{L_2} \geq \alpha(|\xi|^2\|\eta_1\|^2_{L_2(\mathbb{R}_+)} + \|\partial_y\eta_1\|^2_{L_2(\mathbb{R}_+)}),$$

with the constant α *from assumption (F).*

Proof. We use the same strategy as in the proof of the preceding proposition, however now the differential operator ∇ is being replaced by $i\xi + \nu\partial_y$. The splitting $\eta = \eta_1 + \eta_2$ with $\eta_1 = P_d\eta$ and integration by parts with respect to the variable $y > 0$ yields

$$\operatorname{Re}\lambda\|\eta_2\|^2_{L_2(\mathbb{R}_+)} + 2k_3(|\xi|^2\|\eta_2\|^2_{L_2(\mathbb{R}_+)} + \|\partial_y\eta_2\|^2_{L_2(\mathbb{R}_+)}) = 0,$$

which shows that $\eta_2 = (I - P_d)\eta = 0$, hence $\eta = \eta_1 = P_d\eta$. Taking the inner product of $-\mathcal{A}_d(i\xi + \nu\partial_y)\eta_1$ with η_1 in $L_2(\mathbb{R}_+)^3$ and integrating by parts with respect to $y > 0$, implies the desired estimate. In fact, the technical steps from the proof of Proposition 5.3 can be mimicked and are therefore omitted. Finally, taking the inner product of the equation $\lambda\eta_1 - \mathcal{A}_d(i\xi + \nu\partial_y)\eta_1 = 0$ with η_1 in $L_2(\mathbb{R}_+; \mathbb{C}^3)$ yields $\eta_1 = 0$, hence $\eta = 0$. □

Remark 5.5. Since the state space $\mathbb{C}^3$ of η in Proposition 5.4 is finite dimensional, strong ellipticity of $-\mathcal{A}_d(\nabla)$ implies that the Lopatinskii-Shapiro condition is equivalent to the fact that for each $g \in \mathbb{C}^3$, the problem

$$\lambda\eta - \mathcal{A}_d(i\xi + \nu\partial_y)\eta = 0, \quad y > 0, \quad \mathcal{B}_d(i\xi + \nu\partial_y)\eta = g, \quad y = 0$$

has a unique solution $\eta \in H^2_q(\mathbb{R}_+; \mathbb{C}^3)$ for *any* $1 < q < \infty$. We refer to [42, Remark 6.2.2 (iv) & Section 6.2.2] for details.

6. Linearized problems and maximal regularity

We start by considering a linearized problem for the function d, which reads as

$$\begin{cases} \gamma\partial_t d - \mathcal{A}_{\tilde{d}}(\nabla)d = f & \text{in} \quad (0,T)\times\Omega, \\ \mathcal{B}_{\tilde{d}}(\nabla)d = g & \text{in} \quad (0,T)\times\partial\Omega, \\ d = d_0 & \text{in} \quad \{0\}\times\Omega, \end{cases} \tag{6.1}$$

where $\mathcal{A}_{\tilde{d}}(\nabla)d$ and $\mathcal{B}_{\tilde{d}}(\nabla)d$ are defined as in Section 4.

Proposition 6.1. *Let* $1 < p < \infty$, $\mu \in (\frac{1}{2} + \frac{5}{2p}, 1]$, $\gamma > 0$ *and assume (F). Suppose that* $T > 0$, $\Omega \in \{\mathbb{R}^3, \mathbb{R}^3_+\}$ *and let* $\tilde{d} \in \mathbb{R}^3$ *with* $|\tilde{d}|_2 = 1$. *Then, for any* $T > 0$, *the problem* (6.1) *admits a unique solution*

$$d \in H^1_{p,\mu}((0,T); H^1_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^3_p(\Omega;\mathbb{R}^3)),$$

if and only if the data are subject to the following conditions.

(1) $f \in L_{p,\mu}((0,T); H^1_p(\Omega;\mathbb{R}^3))$,
(2) $g \in W^{1-1/2p}_{p,\mu}((0,T); L_p(\partial\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); W^{2-1/p}_p(\partial\Omega;\mathbb{R}^3))$,
(3) $d_0 \in W^{2\mu+1-2/p}_p(\Omega;\mathbb{R}^3)$
(4) $\mathcal{B}_{\tilde{d}}(\nabla)d_0 = g(t=0)$.

Proof. The assertion follows from [42, Section 6.1.5 (i) & Proof of Theorem 6.3.3], since $-\mathcal{A}_{\tilde{d}}(\nabla)$ is strongly elliptic by Proposition 5.1 and since the pair $(\mathcal{A}_{\tilde{d}}(\nabla), \mathcal{B}_{\tilde{d}}(\nabla))$ satisfies the Lopatinskii-Shapiro condition by Proposition 5.4. □

Next, we linearize the problem (4.2) for (u, d) at some given

$$\tilde{z} = (\tilde{u}, \tilde{d}) \in W^{2\mu-2/p}_p(\Omega;\mathbb{R}^3) \times W^{2\mu+1-2/p}_p(\Omega;\mathbb{R}^3),$$

and drop all terms of lower order. This yields the *principal linearization* of equation (4.2)

$$\begin{cases} \partial_t u + A_u(\tilde{z})u + R_1^{\mathsf{T}}(\tilde{z})\partial_t d + \nabla\pi = f_u & \text{in} \quad (0,T)\times\Omega, \\ -R_0(\tilde{z})u + \gamma(\tilde{z})\partial_t d + A_d(\tilde{z})d = f_d & \text{in} \quad (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in} \quad (0,T)\times\Omega, \\ u = 0 & \text{in} \quad (0,T)\times\partial\Omega, \\ \mathcal{B}_{\tilde{d}}(\nabla)d = g & \text{in} \quad (0,T)\times\partial\Omega, \\ (u,d) = (u_0,d_0) & \text{in} \quad \{0\}\times\Omega. \end{cases} \tag{6.2}$$

For the system (6.2), we prove the following maximal regularity result.

Theorem 6.2. *Let* $1 < p < \infty$, $\mu \in (\frac{1}{2} + \frac{5}{2p}, 1]$ *and assume (F),(P) and (R). Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with boundary* $\partial\Omega \in C^3$ *and let*

$$\tilde{z} = (\tilde{u}, \tilde{d}) \in W^{2\mu-2/p}_p(\Omega;\mathbb{R}^3) \times W^{2\mu+1-2/p}_p(\Omega;\mathbb{R}^3),$$

with $|\tilde{d}(x)|_2 = 1$, $x \in \Omega$. *Then, for any* $T \in (0, \infty)$, *the problem* (6.2) *admits a unique solution*

$$\begin{aligned} u &\in H^1_{p,\mu}((0,T); L_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^2_p(\Omega;\mathbb{R}^3)),\\ \nabla\pi &\in L_{p,\mu}((0,T); L_p(\Omega;\mathbb{R}^3)),\\ d &\in H^1_{p,\mu}((0,T); H^1_p(\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^3_p(\Omega;\mathbb{R}^3)), \end{aligned}$$

if and only if the data are subject to the following conditions.

(1) $f_u \in L_{p,\mu}((0,T); L_p(\Omega;\mathbb{R}^3))$,
(2) $f_d \in L_{p,\mu}((0,T); H^1_p(\Omega;\mathbb{R}^3))$,
(3) $g \in W^{1-1/2p}_{p,\mu}((0,T); L_p(\partial\Omega;\mathbb{R}^3)) \cap L_{p,\mu}((0,T); W^{2-1/p}_p(\partial\Omega;\mathbb{R}^3))$,
(4) $u_0 \in W^{2\mu-2/p}_p(\Omega;\mathbb{R}^3)$,
(5) $\operatorname{div} u_0 = 0$,
(6) $u_0 = 0$ *on* $\partial\Omega$,
(7) $d_0 \in W^{2\mu+1-2/p}_p(\Omega;\mathbb{R}^3)$,
(8) $\mathcal{B}_{\tilde{d}}(\nabla)d_0 = g(t=0)$ *on* $\partial\Omega$.

Proof. The proof is subdivided into several steps.
Step1: Let $\Omega = \mathbb{R}^3$ and consider the case of constant coefficients, i.e., we linearize at a constant vector $\tilde{z} = (\tilde{u}, \tilde{d}) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|\tilde{d}|_2 = 1$. By Proposition 6.1 for $\Omega = \mathbb{R}^3$ and [42, Theorem 7.1.1], we may first reduce (6.2) to the case $(u_0, d_0) = (0, 0)$. Note that the inhomogeneities (f_u, f_d) then have to replaced by some modified data in the right regularity classes.

Denoting by ξ the Fourier variable in space and by λ the Laplace variable in time, the Laplace-Fourier symbol of the differential operator defined by the left side of $(6.2)_1$–$(6.2)_3$ is given by

$$\mathcal{L}_{(\tilde{u},\tilde{d})}(\lambda, i\xi) = \begin{pmatrix} M_u(\lambda,\xi) & i\xi^\top & i\lambda R_1(\xi)^\top \\ i\xi & 0 & 0 \\ -iR_0(\xi) & 0 & M_d(\lambda,\xi) \end{pmatrix},$$

where $iR_j(\xi)$ are the Fourier-symbols of the first order differential operators $R_j(\tilde{z})$, $M_u(\lambda,\xi)$ is the Laplace-Fourier-symbol of

$$\partial_t + A_u(\tilde{z}),$$

and $M_d(\lambda,\xi)$ is the Laplace-Fourier-symbol of

$$\gamma(\tilde{z})\partial_t + A_d(\tilde{z}) = \gamma(\tilde{z})\partial_t - \mathcal{A}_{\tilde{d}}(\nabla).$$

In particular, it holds that

$$
\begin{aligned}
M_d(\lambda,\xi) &= \gamma\lambda I + m_{\tilde d}(\xi),\\
m_{\tilde d}(\xi) &= 2k_3|\xi|^2 I + 2(k_1-k_3)(I-\tilde d\otimes\tilde d)\xi\otimes\xi + 2(k_2-k_3)(\tilde d\times\xi)\otimes(\tilde d\times\xi),\\
R_0(\xi) &= \frac{\mu_D+\mu_V}{2}P_{\tilde d}\xi\otimes\tilde d + \frac{\mu_D-\mu_V}{2}(\xi|\tilde d)P_{\tilde d},\\
R_1(\xi) &= R_\mu(\xi) - R_0(\xi),\\
R_\mu(\xi) &= \mu_+ P_{\tilde d}\xi\otimes\tilde d + \mu_-(\xi|\tilde d)P_{\tilde d},\\
M_u(\lambda,\xi) &= m_u(\lambda,\xi)I + \mu_0(\xi|\tilde d)^2\tilde d\otimes\tilde d + \frac{\mu_L}{4}R(\xi)^{\mathsf T}R(\xi)\\
&\quad + \frac{1}{4\gamma}R_\mu(\xi)^{\mathsf T}R_\mu(\xi) + \frac{\mu_P\mu_V}{2\gamma}(\xi|\tilde d)(R(\xi)-R^{\mathsf T}(\xi)),\\
R(\xi) &= P_{\tilde d}\xi\otimes\tilde d + (\xi|\tilde d)P_{\tilde d},\\
m_u(\lambda,\xi) &= \lambda + \mu_s|\xi|^2,
\end{aligned}
$$

where $P_{\tilde d} = I - \tilde d\otimes\tilde d$ and $\mu_\pm = \mu_D \pm \mu_V + \mu_P$. Let us remark that, apart from $M_d(\lambda,\xi)$, the definitions of the above symbols coincide with those in Section 5 of [24] or [26] .

For the time being consider, the purely parabolic part $\mathcal L^0$ of the symbol $\mathcal L$

$$
\mathcal L^0_{(\tilde u,\tilde d)}(z, i\xi) = \begin{pmatrix} M_u(\lambda,\xi) & i\lambda R_1(\xi)^{\mathsf T}\\ -iR_0(\xi) & M_d(\lambda,\xi)\end{pmatrix},
$$

which results from $\mathcal L_{(\tilde u,\tilde d)}$ by dropping the pressure gradient and divergence equation. For $J(u,d) := (u,\lambda d)$ and $v := (u,d)$, a computation shows that

$$
\begin{aligned}
\mathrm{Re}(\mathcal L^0_{(\tilde u,\tilde d)}v|Jv) &= \mathrm{Re}\, m_u(\lambda,\xi)|u|_2^2 + \mu_0(\xi|\tilde d)^2|(\tilde d|u)|^2 + \frac{\mu_L}{4}|Ru|_2^2\\
&\quad + \frac{1}{4\gamma}|R_\mu u|_2^2 + \mathrm{Re}[i\lambda(d|R_\mu u)] + \gamma|\lambda|^2|d|_2^2 + \mathrm{Re}\,\lambda(m_{\tilde d}(\xi)d|d).
\end{aligned}
$$

By strong ellipticity of $-\mathcal A_{\tilde d}(\nabla)$, see (5.3), we obtain

$$
(m_{\tilde d}(\xi)d|d) \ge c|\xi|_2^2|d|_2^2
$$

for some constant $c > 0$. Furthermore,

$$
\begin{aligned}
\frac{1}{4\gamma}|R_\mu u|_2^2 + \mathrm{Re}[i\lambda(d|R_\mu u)] + \gamma|\lambda|^2|d|_2^2 &\ge \frac{1}{4\gamma}|R_\mu u|_2^2 - |\lambda||d|_2|R_\mu u|_2 + \gamma|\lambda|^2|d|_2^2\\
&= \left(\sqrt{\gamma}|\lambda||d|_2 - \frac{1}{2\sqrt{\gamma}}|R_\mu u|_2\right)^2,
\end{aligned}
$$

by the Cauchy-Schwarz inequality. Assumption (P) then yields the estimate

$$
\mathrm{Re}(\mathcal L^0_{(\tilde u,\tilde d)}v|Jv) \ge (\mathrm{Re}\,\lambda + \mu_s|\xi|_2^2)|u|_2^2, \tag{6.3}
$$

provided $\mathrm{Re}\,\lambda \ge 0$. Next, we consider the equation

$$
-iR_0(\xi)u + M_d(\lambda,\xi)d = f_d
$$

and solve it for d. Let us note that for $\operatorname{Re}\lambda \geq 0$ and $(\lambda, \xi) \neq (0, 0)$, the matrix $M_d(\lambda, \xi)$ is invertible by (5.3). Therefore, we obtain

$$d = M_d(\lambda, \xi)^{-1} (f_d + i R_0(\xi) u) .$$

For $f_d = 0$, this yields the *Schur complement*

$$M(\lambda, i\xi) := M_u(\lambda, \xi) - \lambda R_1(\xi)^{\mathsf{T}} M_d(\lambda, \xi)^{-1} R_0(\xi)$$

for u and (6.3) implies the estimate

$$\operatorname{Re}(M(\lambda, i\xi)u|u) \geq (\operatorname{Re}\lambda + \mu_s|\xi|_2^2)|u|_2^2,$$

since Schur reduction preserves positive (semi) definiteness. Therefore, we are in a position to apply the techniques from [42, Section 7.1] to prove maximal L_p-regularity of the corresponding generalized Stokes problem for u in the full space $\mathbb{R}^3$. In fact, in [42, Section 7.1] one has to replace $\lambda + \mathcal{A}(\xi)$ by $M(\lambda, i\xi)$.

Having a unique solution u of the generalized Stokes equation in its optimal regularity class, it follows that

$$R_0(\tilde{z})u \in L_{p,\mu}((0, T); H_p^1(\mathbb{R}^3; \mathbb{R}^3)),$$

since $R_0(\tilde{z})$ is a first order differential operator. Hence, solving the equation for d by Proposition 6.1 we obtain a unique solution d which belongs to its optimal regularity class. This completes the proof for the case $\Omega = \mathbb{R}^3$.

Step 2: Let $\Omega = \mathbb{R}^3_+$ and consider the case of constant coefficients, i.e. we linearize at a constant vector $\tilde{z} = (\tilde{u}, \tilde{d}) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $|\tilde{d}|_2 = 1$. We first reduce (6.2) to the case $(u_0, d_0, g) = (0, 0, 0)$, by applying Proposition 6.1 for the case $\Omega = \mathbb{R}^3_+$ and [42, Theorem 7.2.1] for the case of no-slip boundary conditions.

In the half space $\mathbb{R}^3_+$, we replace the spatial co-variable ξ by $\xi - i\nu\partial_y$, where $y > 0$ and $\xi \perp \nu$. As in Step 1, we extract the Schur complement of u. To this end, we consider the differential operator

$$M_d(\lambda, \xi - i\nu\partial_y)d = \gamma\lambda d + m_{\tilde{d}}(\xi - i\nu\partial_y)d = \gamma\lambda d - \mathcal{A}_{\tilde{d}}(i\xi + \nu\partial_y)d,$$

supplemented with the homogeneous boundary condition

$$\mathcal{B}_{\tilde{d}}(i\xi + \nu\partial_y)d = 0 \text{ for } y = 0.$$

We claim that for $\operatorname{Re}\lambda \geq 0$, $(\lambda, \xi) \neq (0, 0)$ with $\xi \perp \nu$, the operator M_d is invertible. Indeed, consider the equation

$$M_d(\lambda, \xi - i\nu\partial_y)\eta = f$$

for given f. In a first step, we extend f from $\mathbb{R}_+$ to a function $\tilde{f}$ on $\mathbb{R}$ and solve the full space problem

$$\gamma\lambda\tilde{\eta} - \mathcal{A}_{\tilde{d}}(i\xi + \nu\partial_y)\tilde{\eta} = \tilde{f}, \quad y \in \mathbb{R},$$

by the classical Mihklin multiplier theorem with respect to the variable y and with the help of (5.3). This yields a unique solution $\tilde{\eta}$. Next, we consider the boundary value problem

$$\begin{cases} \gamma\lambda\hat{\eta} - \mathcal{A}_{\tilde{d}}(i\xi + \nu\partial_y)\hat{\eta} = 0 & y > 0, \\ \mathcal{B}_{\tilde{d}}(i\xi + \nu\partial_y)\hat{\eta} = h & y = 0, \end{cases} \tag{6.4}$$

with $h := -\mathcal{B}_{\tilde{d}}(i\xi + \nu\partial_y)\tilde{\eta}$, to obtain a unique solution $\hat{\eta}$ by Remark 5.5. This shows that M_d is invertible, which in turn implies that

$$d = M_d(\lambda, \xi - i\nu\partial_y)^{-1}(f_d + iR_0(\xi - i\nu\partial_y)u).$$

Therefore, the Schur complement of u is given by

$$M_u(\lambda, \xi - i\nu\partial_y) - \lambda R_1(\xi - i\nu\partial_y)^{\mathsf{T}} M_d(\lambda, \xi - i\nu\partial_y)^{-1} R_0(\xi - i\nu\partial_y).$$

We will now show that the *Lopatinskii-Shapiro condition* is satisfied. To be precise, this means that, for all $\operatorname{Re}\lambda \geq 0$, $(\lambda, \xi) \neq (0, 0)$ with $\xi \perp \nu$, the problem

$$\begin{cases} M_u(\lambda, \xi - i\nu\partial_y)u + i\lambda R_1(\xi - i\nu\partial_y)^{\mathsf{T}} d = 0 & y > 0, \\ -iR_0(\xi - i\nu\partial_y)u + M_d(\lambda, \xi - i\nu\partial_y)d = 0 & y > 0, \\ u = 0 & y = 0, \\ \mathcal{B}_{\tilde{d}}(i\xi + \nu\partial_y)d = 0 & y = 0, \end{cases} \tag{6.5}$$

admits only the trivial solution $u = d = 0$ in $L_2(\mathbb{R}_+)$. Let us split $d = d_1 + d_2$, where $d_1 = P_{\tilde{d}}d$ and $d_2 = (I - P_{\tilde{d}})d$. Since

$$(I - P_{\tilde{d}})R_0(\xi - i\nu\partial_y)u = 0,$$

by the definition of R_0, we may conclude from equation $(6.5)_2$ that $d_2 = 0$. Indeed, this can be seen as in the proof of Proposition 5.4. Therefore, we may replace d by $d_1 = P_{\tilde{d}}d$ in (6.5). As in step 1, we will test $(6.5)_1$ with $\bar{u}$ and $(6.5)_2$ with $\bar{\lambda}\bar{d}_1$ and integrate by parts with respect to the variable $y > 0$. Assumption (P) then yields the estimate

$$\begin{aligned} 0 \geq \operatorname{Re}\lambda[\|u\|^2_{L_2(\mathbb{R}_+)} + (m_{\tilde{d}}(\xi - i\nu\partial_y)d_1|d_1)_{L_2(\mathbb{R}_+)}] \\ +|\lambda|^2\|d_1\|^2_{L_2(\mathbb{R}_+)} + |\xi|_2^2\|u\|^2_{L_2(\mathbb{R}_+)} + \|\partial_y u\|^2_{L_2(\mathbb{R}_+)}. \end{aligned}$$

Since, by Proposition 5.4,

$$\begin{aligned} (m_{\tilde{d}}(\xi - i\nu\partial_y)d_1|d_1)_{L_2(\mathbb{R}_+)} &= -(\mathcal{A}_{\tilde{d}}(i\xi + \nu\partial_y)d_1|d_1)_{L_2(\mathbb{R}_+)} \\ &\geq \alpha(|\xi|^2\|d_1\|^2_{L_2(\mathbb{R}_+)} + \|\partial_y d_1\|^2_{L_2(\mathbb{R}_+)}), \end{aligned}$$

we may conclude that $\partial_y u = 0$, hence $u = 0$ as $u \in L_2(\mathbb{R}_+)$. Inserting this information into $(6.5)_2$, the boundary condition $(6.5)_4$ and Proposition 5.4 imply $d_1 = 0$. This shows that the Lopatinskii-Shapiro condition is satisfied.

We may now employ half-space theory for u by the methods in [5, Section 6] or [41, Section 2] or [42, Section 7.2] to prove maximal L_p-regularity for the half space $\mathbb{R}^3_+$. Having a unique solution u in its optimal regularity class, it follows that

$$R_0(\tilde{z})u \in L_{p,\mu}((0, T); H^1_p(\mathbb{R}^3_+; \mathbb{R}^3)).$$

Hence, we solve the equation for d by Proposition 6.1 to obtain a unique solution d which belongs to its optimal regularity class. This completes the proof for the case $\Omega = \mathbb{R}^3_+$.
Step 3: The results of Steps 1 and 2 extend by perturbation arguments to a bent half-space and to the case of variable coefficients with small deviation from constant ones. We then may apply a localization procedure to cover the case of general domains with boundary of class C^3 and variable coefficients. For details we refer at this point to, e.g., [42, Sections 6.3 & 7.3]. This completes the proof of Theorem 6.2. □

We will now rewrite (4.2) in a more abstract form. To this end, let

$$X_0 = L_{p,\sigma}(\Omega; \mathbb{R}^3) \times H^1_p(\Omega; \mathbb{R}^3),$$

where $L_{p,\sigma}(\Omega; \mathbb{R}^3) := \mathbb{P}_H L_p(\Omega; \mathbb{R}^3)$ and $\mathbb{P}_H$ denotes the Helmholtz projection. Furthermore, let

$$X_1 := \{u \in H^2_{p,\sigma}(\Omega; \mathbb{R}^3) \mid u = 0 \text{ on } \partial\Omega\} \times H^3_p(\Omega; \mathbb{R}^3),$$

where $H^2_{p,\sigma}(\Omega; \mathbb{R}^3) := H^2_p(\Omega; \mathbb{R}^3) \cap L_{p,\sigma}(\Omega; \mathbb{R}^3)$ and let

$$X_{\gamma,\mu} := (X_0, X_1)_{\mu-1/p,p}$$

be the space of the initial data. In fact, it holds that

$$X_{\gamma,\mu} = \{u \in W^{2\mu-2/p}_{p,\sigma}(\Omega; \mathbb{R}^3) \mid u = 0 \text{ on } \partial\Omega\} \times W^{2\mu+1-2/p}_p(\Omega; \mathbb{R}^3). \tag{6.6}$$

Observe next that

$$X_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega}; \mathbb{R}^3) \times C^2(\overline{\Omega}; \mathbb{R}^3),$$

by our assumption (4.1) on p and μ. Given any $\tilde{z} = (\tilde{u}, \tilde{d}) \in X_{\gamma,\mu}$, we define

$$\mathsf{A}(\tilde{z}) := \begin{pmatrix} \mathbb{P}_H A_u(\tilde{z}) + \frac{1}{\gamma(\tilde{z})}\mathbb{P}_H R_1(\tilde{z})^{\mathsf{T}} R_0(\tilde{z}) & -\frac{1}{\gamma(\tilde{z})}\mathbb{P}_H R_1(\tilde{z})^{\mathsf{T}} A_d(\tilde{z}) \\ -\frac{1}{\gamma(\tilde{z})} R_0(\tilde{z}) & \frac{1}{\gamma(\tilde{z})} A_d(\tilde{z}) \end{pmatrix} \tag{6.7}$$

and, for sufficiently smooth $z = (u, d)$, we introduce, following (3.6), the boundary operator $\mathsf{B}(\tilde{z})$ by

$$\begin{aligned} \mathsf{B}(\tilde{z})z := \mathcal{B}_{\tilde{d}}(\nabla)d = 2k_3 \nabla d \cdot \nu &+ 2P_{\tilde{d}}(k_1 \operatorname{div} d \cdot I - k_3(\nabla d)^{\mathsf{T}}) \cdot \nu \\ &+ 2(k_2 - k_3)(\tilde{d} \cdot \operatorname{curl} d)(\tilde{d} \times \nu) \\ &+ 2(k_2 + k_4)P_{\tilde{d}}((\nabla d)^{\mathsf{T}} - \operatorname{div} d \cdot I) \cdot \nu, \end{aligned} \tag{6.8}$$

Finally, let

$$\mathsf{F}(z) := \begin{pmatrix} \mathbb{P}_H F_u(z) - \frac{1}{\gamma(\tilde{z})}\mathbb{P}_H R_1^{\mathsf{T}}(z) F_d(z) \\ \frac{1}{\gamma(\tilde{z})} F_d(z) \end{pmatrix}. \tag{6.9}$$

We note that the definition of $\mathsf{A}(\tilde{z})$ as well as of $\mathsf{F}(z)$ results from (4.2) by applying the Helmholtz projection to the first equation and by substituting $\partial_t d$ from the second equation into the first equation of (4.2).

With these definitions, system (4.2) can be rewritten as

$$\begin{cases} \partial_t z + \mathsf{A}(z)z = \mathsf{F}(z) & \text{in} \quad (0,T)\times\Omega, \\ \mathsf{B}(z)z = 0 & \text{in} \quad (0,T)\times\partial\Omega, \\ z(0) = z_0 & \text{in} \quad \{0\}\times\Omega. \end{cases} \tag{6.10}$$

For the sake of readability, for $0 < T < \infty$, we further define that $J_T = [0, T]$,

$$\mathbb{E}_{0,\mu}(J_T) := L_{p,\mu}(J_T; X_0), \quad \mathbb{E}_{1,\mu}(J_T) := H^1_{p,\mu}(J_T; X_0) \cap L_{p,\mu}(J_T; X_1)$$

and

$$\mathbb{F}_\mu(J_T) := W^{1-1/2p}_{p,\mu}(J_T; L_p(\partial\Omega; \mathbb{R}^3)) \cap L_{p,\mu}(J_T; W^{2-1/p}_p(\partial\Omega; \mathbb{R}^3)).$$

Moreover, we set

$$\hat{X}_{\gamma,\mu} := \{z = (u,d) \in X_{\gamma,\mu} \mid |d(x)|_2 = 1,\ x \in \Omega\} \tag{6.11}$$

and, for $\tilde{z} \in X_{\gamma,\mu}$,

$$\mathbb{D}_\mu(\tilde{z}, T) := \{(f, g, z_0) \in \mathbb{E}_{0,\mu}(J_T) \times \mathbb{F}_\mu(J_T) \times X_{\gamma,\mu} \mid \mathsf{B}(\tilde{z})z_0 = \mathrm{tr}_{t=0}\, g\}.$$

Then the following result for the linearized system

$$\begin{cases} \partial_t z + \mathsf{A}(\tilde{z})z = f & \text{in} \quad (0,T)\times\Omega, \\ \mathsf{B}(\tilde{z})z = g & \text{in} \quad (0,T)\times\partial\Omega, \\ z(0) = z_0 & \text{in} \quad \{0\}\times\Omega, \end{cases} \tag{6.12}$$

is a direct consequence of Theorem 6.2.

Corollary 6.3. *Let the assumptions of Theorem 6.2 be satisfied. Then, for any $\tilde{z} \in \hat{X}_{\gamma,\mu}$ and all $(f, g, z_0) \in \mathbb{D}_\mu(\tilde{z}, T)$, the linear problem (6.12) admits a unique solution $z \in \mathbb{E}_{1,\mu}(J_T)$.*

In the situation of Corollary 6.3, for any $\tilde{z} \in \hat{X}_{\gamma,\mu}$, the mapping

$$L(\tilde{z}) = (\partial_t + \mathsf{A}(\tilde{z}), \mathsf{B}(\tilde{z}), \mathrm{tr}_{t=0}) : \mathbb{E}_{1,\mu}(J_T) \to \mathbb{D}_\mu(\tilde{z}, T)$$

is linear, bounded and invertible. Denoting by

$$S(\tilde{z}) = L(\tilde{z})^{-1} \tag{6.13}$$

the inverse operator, the open mapping theorem implies that $S(\tilde{z}) : \mathbb{D}_\mu(\tilde{z}, T) \to \mathbb{E}_{1,\mu}(J_T)$ is bounded. Therefore, there exists a constant $C = C(T) > 0$ such that the unique solution z of (6.12) satisfies

$$\|z\|_{\mathbb{E}_{1,\mu}(J_T)} \le C \left(\|f\|_{\mathbb{E}_{0,\mu}(J_T)} + \|g\|_{\mathbb{F}_\mu(J_T)} + \|z_0\|_{X_{\gamma,\mu}} \right). \tag{6.14}$$

With the help of extension-restriction arguments one can prove that in case $z_0 = 0$, the constant $C = C(T)$ is uniform in $T \in (0, T_*]$ for some given and fixed $T_* \in (0, \infty)$, see e.g. [13, Proposition 4.1 (b)].

We remark that by assumptions (R) and (4.1), the mapping

$$X_{\gamma,\mu} \ni \tilde{z} \mapsto L(\tilde{z}) \in \mathcal{L}({}_0\mathbb{E}_{1,\nu}(J_T), \mathbb{E}_{0,\nu}(J_T) \times {}_0\mathbb{F}_\nu(J_T))$$

is continuous, where the lower left subscript 0 means that the trace at $t = 0$ vanishes. Indeed, for A continuity follows from a direct calculation and concerning B, one may use the fact that the space $\mathbb{F}_\mu(J_T)$ is a Banach algebra, cf. also [13, Lemma B.1]. Corollary 6.3 then implies that the mapping

$$\hat{X}_{\gamma,\mu} \ni \tilde{z} \mapsto S(\tilde{z}) \in \mathcal{L}(\mathbb{E}_{0,\nu}(J_T) \times {}_0\mathbb{F}_\nu(J_T), {}_0\mathbb{E}_{1,\nu}(J_T))$$

is continuous, since $\hat{X}_{\gamma,\mu}$ is a subset of $X_{\gamma,\mu}$, see (6.11), and since inversion is smooth. Here the solution operator S is defined in (6.13), but restricted to trivial initial data.

Next, let us define nonlinear mappings $(\mathcal{A}, \mathcal{B})$ by

$$\mathcal{A}(z) = \mathsf{A}(z)z \quad \text{and} \quad \mathcal{B}(z) = \mathsf{B}(z)z,$$

where (A, B) are given in (6.7) and (6.8). Let further F be given as in (6.9). Then the functions $\mathcal{A}$, $\mathcal{B}$ and F enjoy the following regularity properties:

Lemma 6.4. *Let* $1 < p < \infty$, $\mu \in (\frac{1}{2} + \frac{5}{2p}, 1]$ *and assume (P), (R). Then*

$$\mathcal{A}, \mathsf{F} \in C^1(\mathbb{E}_{1,\mu}(J_T); \mathbb{E}_{0,\mu}(J_T)),\ \mathcal{B} \in C^1(\mathbb{E}_{1,\mu}(J_T); \mathbb{F}_\mu(J_T)),$$

with

$$\mathcal{A}'(z_*)z = \mathsf{A}(z_*)z + [\mathsf{A}'(z_*)z]z_*,\ \mathcal{B}'(z_*)z = \mathsf{B}(z_*)z + [\mathsf{B}'(z_*)z]z_*,$$

where $z, z_* \in \mathbb{E}_{1,\mu}(J_T)$.

Moreover, given $T_0, M > 0$, *then for any* $T \in (0, T_0]$ *and any* $z_* = (u_*, d_*) \in \mathbb{E}_{1,\mu}(J_T)$, $z \in \mathbb{E}_{1,\mu}(J_T)$ *with* $z(0) = 0$ *satisfying*

$$\|\operatorname{tr}_{\partial\Omega} \nabla^j d_*\|_{\mathbb{F}_\mu(J_T)},\ \|z_*\|_{C(J_T;X_{\gamma,\mu})},\ \|z_*\|_{\mathbb{E}_{1,\mu}(J_T)},\ \|z\|_{\mathbb{E}_{1,\mu}(J_T)} \le M,$$

for $j \in \{0, 1\}$, *the estimate*

$$\|\mathcal{H}(z_* + z) - \mathcal{H}(z_*) - \mathcal{H}'(z_*)z\|_{\mathbb{X}} \le \varepsilon(\|z\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)}$$

holds for $(\mathcal{H}, \mathbb{X}) \in \{(\mathcal{A}, \mathbb{E}_{0,\mu}(J_T)), (\mathsf{F}, \mathbb{E}_{0,\mu}(J_T)), (\mathcal{B}, \mathbb{F}_\mu(J_T))\}$. *Here* $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous such that* $\varepsilon(r) \to 0$ *as* $r \to 0$.

If in addition $\bar{z} = (\bar{u}, \bar{d}) \in \mathbb{E}_{1,\mu}(J_T)$ *with* $\bar{z}(0) = z_*(0)$ *satisfies*

$$\|\operatorname{tr}_{\partial\Omega} \nabla^j \bar{d}\|_{\mathbb{F}_\mu(J_T)},\ \|\bar{z}\|_{C(J_T;X_{\gamma,\mu})},\ \|\bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)} \le M,$$

for $j \in \{0, 1\}$, *then the following estimate holds*

$$\|\mathcal{H}'(z_*)z - \mathcal{H}'(\bar{z})z\|_{\mathbb{X}} \le \varepsilon(\|z_* - \bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|z\|_{\mathbb{E}_{1,\mu}(J_T)},$$

where the choice of $(\mathcal{H}, \mathbb{X})$ *is as above.*

Proof. For the proof one may follow the strategy of [13, Proof of Proposition B.3 and Lemma B.2]. Indeed, by (4.1), the space $\mathbb{F}_\mu(J_T)$ is a Banach algebra and the nonlinearities in $\mathcal{B}$ are of the form

$$(d \otimes d) \cdot \phi(\nabla d) \quad \text{or} \quad (d|\,\mathrm{curl}\, d)d,$$

where

$$\phi(\nabla d) = (\nabla d)^{\mathsf{T}} \quad \text{or} \quad \phi(\nabla d) = \mathrm{div}\, d \cdot I = \mathrm{Tr}(\nabla d) \cdot I.$$

□

For the proof of local well-posedness in the next section, we consider the following nonautonomous problem

$$\begin{cases} \partial_t z + \mathcal{A}'(z_*(t))z - \mathsf{F}'(z_*(t))z = f & \text{in} \quad (0, T) \times \Omega, \\ \mathcal{B}'(z_*(t))z = g & \text{in} \quad (0, T) \times \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \{0\} \times \Omega. \end{cases} \tag{6.15}$$

Here,

$$\mathcal{A}(z) := \mathsf{A}(z)z \quad \text{and} \quad \mathcal{B}(z) := \mathsf{B}(z)z,$$

and $z_* \in \mathbb{E}_{1,\mu}(J_T)$ is a given function. By Lemma 6.4, $\mathcal{A}$, $\mathcal{B}$ and F are continuously differentiable, with

$$\mathcal{A}'(z_*)z = \mathsf{A}(z_*)z + [\mathsf{A}'(z_*)z]z_*, \; \mathcal{B}'(z_*)z = \mathsf{B}(z_*)z + [\mathsf{B}'(z_*)z]z_*.$$

Proposition 6.5. *Suppose that the conditions of Theorem 6.2 are satisfied. Then, for any $z_* \in \mathbb{E}_{1,\mu}(J_T)$ with $z_*(t) \in \hat{X}_{\gamma,\mu}$, $t \in J_T$, and all*

$$(f, g, z_0) \in \mathbb{E}_{0,\mu}(J_T) \times \mathbb{F}_\mu(J_T) \times X_{\gamma,\mu}$$

such that $\mathcal{B}'(z_*(0)) = \mathrm{tr}_{t=0}\, g$, the linear problem (6.15) admits a unique solution $z \in \mathbb{E}_{1,\mu}(J_T)$ satisfying the estimate (6.14).

Moreover, in case $z_0 = 0$, the constant $C = C(T)$ in (6.14) is uniform in $T \in (0, T_*]$ for some given and fixed $T_* \in (0, \infty)$.

Proof. In a first step, one proves the assertion for the nonautonomous problem

$$\begin{cases} \partial_t z + \mathsf{A}(z_*(t))z = f & \text{in} \quad (0, T) \times \Omega, \\ \mathsf{B}(z_*(t))z = g & \text{in} \quad (0, T) \times \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \{0\} \times \Omega. \end{cases} \tag{6.16}$$

To this end, we follow the strategy of the proof of Proposition 4.2 in [13], take into account that for any *fixed* $s \in J_T$, problem (6.12) with $\tilde{z} = z_*(s) \in \hat{X}_{\gamma,\mu}$ has the property of L_p-maximal regularity by Corollary 6.3 and that $\mathbb{F}_\mu(J_T)$ is a Banach algebra by (4.1). The result then follows via a suitable decomposition of the interval $[0, T]$.

In a second step, we include the terms $[\mathsf{A}'(z_*)z]z_*$, $[\mathsf{B}'(z_*)z]z_*$ and $\mathsf{F}'(z_*)z$ by a perturbation argument as in the proof of Proposition 4.3 in [13]. The details of this procedure hence are omitted. □

7. Proofs of the Main Results

We start with the unique strong solvability of the abstract system (6.10).

Proposition 7.1. *Let* $1 < p < \infty$, $\mu \in (\frac{1}{2} + \frac{5}{2p}, 1]$ *and assume (F),(P) and (R). Then, for every* $z_0 \in \hat{X}_{\gamma,\mu}$ *satisfying* $\mathsf{B}(z_0)z_0 = 0$, *there exists* $T = T(z_0) > 0$ *such that problem* (6.10) *has a unique solution* $z \in \mathbb{E}_{1,\mu}(J_T)$.

Proof. We claim first that there exists $T_0 > 0$ and $z_* \in \mathbb{E}_{1,\mu}(J_{T_0})$ satisfying $z_*(t) \in \hat{X}_{\gamma,\mu}$ for all $t \in J_{T_0}$ and such that $z_*(0) = z_0$. Indeed, for given

$$u_0 \in \{u \in W^{2\mu-2/p}_{p,\sigma}(\Omega; \mathbb{R}^3) \mid u = 0 \text{ on } \partial\Omega\}$$

we obtain from [42, Theorem 7.3.1] a function

$$u_* \in H^1_{p,\mu}(\mathbb{R}_+; L_{p,\sigma}(\Omega; \mathbb{R}^3)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_{p,\sigma}(\Omega; \mathbb{R}^3))$$

such that $u_*(0) = u_0$. Concerning $d_0 \in W^{2\mu+1-2/p}_p(\Omega; \mathbb{R}^3)$, we first extend d_0 to some $\tilde{d}_0 \in W^{2\mu+1-2/p}_p(\mathbb{R}^3; \mathbb{R}^3)$ and solve the full space problem

$$\begin{cases} \partial_t \tilde{d} + \omega \tilde{d} - \Delta \tilde{d} = 0 & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^3, \\ \tilde{d}(0) = \tilde{d}_0 & \text{in} \quad \{0\} \times \mathbb{R}^3, \end{cases} \tag{7.1}$$

for some $\omega > 0$, to obtain a unique solution

$$\tilde{d} \in H^1_{p,\mu}(\mathbb{R}_+; H^1_p(\mathbb{R}^3; \mathbb{R}^3)) \cap L_{p,\mu}(\mathbb{R}_+; H^3_p(\mathbb{R}^3; \mathbb{R}^3)),$$

by [42, Theorem 6.1.11 & Section 6.1.5 (i)]. The restricted function $\hat{d} := \tilde{d}|_\Omega$ then satisfies

$$\hat{d} \in H^1_{p,\mu}(\mathbb{R}_+; H^1_p(\Omega; \mathbb{R}^3)) \cap L_{p,\mu}(\mathbb{R}_+; H^3_p(\Omega; \mathbb{R}^3))$$

and $\hat{d}(0) = d_0$. Since $z_0 = (u_0, d_0) \in \hat{X}_{\gamma,\mu}$, we know that $|d_0|_2 = 1$. Let us recall that

$$\hat{d} \in C(\mathbb{R}_+; C^2(\overline{\Omega}; \mathbb{R}^3)) \tag{7.2}$$

by our assumption (4.1) on p and μ. Therefore, by continuity and compactness, there exist $T_0 > 0$ and $\alpha, \beta > 0$ such that

$$0 < \alpha \le |\hat{d}(t, x)|_2 \le \beta < \infty$$

for all $(t, x) \in [0, T_0] \times \overline{\Omega}$. This estimate and (7.2) in turn imply that the function

$$d_*(t, x) := \frac{1}{|\hat{d}(t, x)|_2} \hat{d}(t, x)$$

satisfies

$$d_* \in H^1_{p,\mu}(J_{T_0}; H^1_p(\Omega; \mathbb{R}^3)) \cap L_{p,\mu}(J_{T_0}; H^3_p(\Omega; \mathbb{R}^3))$$

with $|d_*(t,x)|_2 = 1$ for all $(t,x) \in J_{T_0} \times \overline{\Omega}$ and $d_*(0) = \frac{1}{|d_0|_2} d_0 = d_0$. Hence, $z_* := (u_*, d_*)$ has the desired properties.

For the remainder of the proof, we define

$$\mathsf{A}_*(t)z := \mathcal{A}'(z_*(t))z - \mathsf{F}'(z_*(t)), \quad \mathsf{B}_*(t)z := \mathcal{B}'(z_*(t))z$$

and solve in a first step the linear problem

$$\begin{cases} \partial_t z + \mathsf{A}_*(t)z = \mathcal{A}'(z_*)z_* - \mathcal{A}(z_*) + \mathsf{F}(z_*) - \mathsf{F}'(z_*)z_* & \text{in} \quad (0,T_0) \times \Omega, \\ \mathsf{B}_*(t)z = \mathcal{B}'(z_*)z_* - \mathcal{B}(z_*) & \text{in} \quad (0,T_0) \times \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \{0\} \times \Omega, \end{cases} \tag{7.3}$$

by Proposition 6.5 to obtain a unique solution $w \in \mathbb{E}_{1,\mu}(J_{T_0})$ of (7.3). We note that the compatibility condition

$$\mathsf{B}_*(0)z_0 = \mathcal{B}'(z_0)z_0 - \mathcal{B}(z_0)$$

at $t = 0$ is satisfied, since $\mathcal{B}(z_0) = \mathsf{B}(z_0)z_0$ by definition of $\mathcal{B}$ and $\mathsf{B}(z_0)z_0 = 0$ by assumption.

Given any $R_0 > 0$, we define, for $(T, R) \in (0, T_0] \times (0, R_0]$ the set

$$\Sigma(T,R) := \{z \in \mathbb{E}_{1,\mu}(J_T) \mid \|z - w\|_{\mathbb{E}_{1,\mu}(J_T)} \leq R, \ \mathrm{tr}_{t=0}\, z = z_0\},$$

which is a closed subset of $\mathbb{E}_{1,\mu}(J_T)$. For a given $\hat{z} \in \Sigma(T,R)$, we solve the linear problem

$$\begin{cases} \partial_t z + \mathsf{A}_*(t)z = \mathcal{A}'(z_*)\hat{z} - \mathcal{A}(\hat{z}) + \mathsf{F}(\hat{z}) - \mathsf{F}'(z_*)\hat{z} & \text{in} \quad (0,T) \times \Omega, \\ \mathsf{B}_*(t)z = \mathcal{B}'(z_*)\hat{z} - \mathcal{B}(\hat{z}) & \text{in} \quad (0,T) \times \partial\Omega, \\ z(0) = z_0 & \text{in} \quad \{0\} \times \Omega, \end{cases} \tag{7.4}$$

to obtain a unique solution $z = \mathcal{T}(\hat{z}) \in \mathbb{E}_{1,\mu}(J_T)$ by Proposition 6.5. We note that the compatibility condition $\mathsf{B}_*(0)z_0 = \mathcal{B}'(z_0)z_0 - \mathcal{B}(z_0)$ at $t = 0$ is again satisfied.

We see that $z \in \Sigma(T,R)$ solves (6.10) if and only if z is a fixed point of $\mathcal{T}$, i.e. $\mathcal{T}(z) = z$. For the latter purpose, we will employ the contraction principle. It follows from (6.14), (7.3) and (7.4) that there exists a constant $C > 0$, not depending on $T \in (0, T_0]$, such that

$$\begin{aligned} \|\mathcal{T}(\hat{z}) - w\|_{\mathbb{E}_{1,\mu}(J_T)} \leq C(&\|\mathcal{A}(\hat{z}) - \mathcal{A}(z_*) - \mathcal{A}'(z_*)(\hat{z} - z_*)\|_{\mathbb{E}_{0,\mu}(J_T)} \\ &+ \|\mathsf{F}(\hat{z}) - \mathsf{F}(z_*) - \mathsf{F}'(z_*)(\hat{z} - z_*)\|_{\mathbb{E}_{0,\mu}(J_T)} \\ &+ \|\mathcal{B}(\hat{z}) - \mathcal{B}(z_*) - \mathcal{B}'(z_*)(\hat{z} - z_*)\|_{\mathbb{F}_\mu(J_T)}), \end{aligned}$$

since $\mathrm{tr}_{t=0}(\mathcal{T}(\hat{z}) - w) = 0$. Observing that

$$\|\hat{z} - z_*\|_{\mathbb{E}_{1,\mu}(J_T)} \leq \|\hat{z} - w\|_{\mathbb{E}_{1,\mu}(J_T)} + \|w - z_*\|_{\mathbb{E}_{1,\mu}(J_T)} \leq R + \|w - z_*\|_{\mathbb{E}_{1,\mu}(J_T)},$$

Lemma 6.4 yields the estimate

$$\begin{aligned} \|\mathcal{T}(\hat{z}) - w\|_{\mathbb{E}_{1,\mu}(J_T)} &\leq \varepsilon(\|\hat{z} - z_*\|_{\mathbb{E}_{1,\mu}(J_T)})\|\hat{z} - z_*\|_{\mathbb{E}_{1,\mu}(J_T)} \\ &\leq \varepsilon(R + \|w - z_*\|_{\mathbb{E}_{1,\mu}(J_T)})(R + \|w - z_*\|_{\mathbb{E}_{1,\mu}(J_T)}) \leq R, \end{aligned}$$

provided T, R are chosen sufficiently small. Here we used the fact that $w, z_* \in \mathbb{E}_{1,\mu}(J_{T_0})$ are fixed functions and

$$\|w - z_*\|_{\mathbb{E}_{1,\mu}(J_T)} \to 0$$

as $T \to 0$. Furthermore, for $\hat{z}, \bar{z} \in \Sigma(T, R)$, we obtain

$$\begin{aligned}\|\mathcal{T}(\hat{z}) - \mathcal{T}(\bar{z})\|_{\mathbb{E}_{1,\mu}(J_T)} \leq & C(\|\mathcal{A}(\hat{z}) - \mathcal{A}(\bar{z}) - \mathcal{A}'(z_*)(\hat{z} - \bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)} \\ & + \|\mathsf{F}(\hat{z}) - \mathsf{F}(\bar{z}) - \mathsf{F}'(z_*)(\hat{z} - \bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)} \\ & + \|\mathcal{B}(\hat{z}) - \mathcal{B}(\bar{z}) - \mathcal{B}'(z_*)(\hat{z} - \bar{z})\|_{\mathbb{F}_{\mu}(J_T)}).\end{aligned}$$

Estimating

$$\begin{aligned}\|\mathcal{A}(\hat{z}) - \mathcal{A}(\bar{z}) - \mathcal{A}'(z_*)(\hat{z} - \bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)} \leq & \|\mathcal{A}(\hat{z}) - \mathcal{A}(\bar{z}) - \mathcal{A}'(\hat{z})(\hat{z} - \bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)} \\ & + \|(\mathcal{A}'(z_*) - \mathcal{A}'(\hat{z}))(\hat{z} - \bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)}\end{aligned}$$

and applying again Lemma 6.4 we verify that

$$\|\mathcal{A}(\hat{z}) - \mathcal{A}(\bar{z}) - \mathcal{A}'(\hat{z})(\hat{z} - \bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)} \leq \varepsilon(\|\hat{z} - \bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|\hat{z} - \bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)}$$

and

$$\|(\mathcal{A}'(z_*) - \mathcal{A}'(\hat{z}))(\hat{z} - \bar{z})\|_{\mathbb{E}_{0,\mu}(J_T)} \leq \varepsilon(\|z_* - \hat{z}\|_{\mathbb{E}_{1,\mu}(J_T)})\|\hat{z} - \bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)}.$$

Here we used the fact that there exists a constant $M > 0$ such that for all $(T, R) \in (0, T_0] \times (0, R_0]$ and for every $\hat{z} = (\hat{u}, \hat{d}) \in \Sigma(T, R)$ we have

$$\|\operatorname{tr}_{\partial\Omega} \nabla^j \hat{d}\|_{\mathbb{F}_\mu(J_T)},\ \|\hat{z}\|_{C(J_T; X_{\gamma,\mu})},\ \|\hat{z}\|_{\mathbb{E}_{1,\mu}(J_T)} \leq M, \quad j \in \{0, 1\}.$$

We further see that

$$\|\hat{z} - \bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)} \leq \|\hat{z} - w\|_{\mathbb{E}_{1,\mu}(J_T)} + \|w - \bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)} \leq 2R$$

and

$$\|z_* - \hat{z}\|_{\mathbb{E}_{1,\mu}(J_T)} \leq \|z_* - w\|_{\mathbb{E}_{1,\mu}(J_T)} + \|w - \hat{z}\|_{\mathbb{E}_{1,\mu}(J_T)} \leq \|z_* - w\|_{\mathbb{E}_{1,\mu}(J_T)} + R.$$

Analogous estimates for the terms involving F and $\mathcal{B}$ finally imply

$$\|\mathcal{T}(\hat{z}) - \mathcal{T}(\bar{z})\|_{\mathbb{E}_{1,\mu}(J_T)} \leq \frac{1}{2}\|\hat{z} - \bar{z}\|_{\mathbb{E}_{1,\mu}(J_T)},$$

provided that T and R are chosen sufficiently small. Hence, $\mathcal{T} : \Sigma(T, R) \to \Sigma(T, R)$ is a contraction and we obtain the existence and uniqueness of a fixed point $z \in \Sigma(T, R)$ of $\mathcal{T}$, which is then the unique solution of (6.10). □

It follows from the definition of the Helmholtz projection $\mathbb{P}_H$ that any solution to (6.10) is a solution of (2.12) or (4.2). Indeed, given $v \in L_p(\Omega; \mathbb{R}^3)$, we have $\mathbb{P}_H v = v - \nabla\pi$, where $\pi \in \dot{H}^1_p(\Omega)$ solves the weak Neumann problem

$$(\nabla\pi|\nabla\phi)_{L_2(\Omega)} = (v|\nabla\phi)_{L_2(\Omega)}, \quad \phi \in \dot{H}^1_{p'}(\Omega),$$

with $p' = p/(p-1)$. Hence, setting $v := -u \cdot \nabla u + \operatorname{div} S$, it follows that

$$\partial_t u - v + \nabla\pi = \partial_t u - \mathbb{P}_H v = 0.$$

Vice versa, any solution of (2.12) or (4.2) is also a solution of (6.10). This follows by applying the Helmholtz projection to the first equation of (2.12) or (4.2).

Proof of Theorem 2.1. Note first that existence and regularity of a unique solution for (2.12) follows directly from Proposition 7.1.

Concerning the property $|d(t,x)|_2 = 1$, we recall the differential equation for d

$$\gamma\partial_t d + \gamma u \cdot \nabla d = P_d(\operatorname{div}(\frac{\partial\psi}{\partial\nabla d}) - \nabla_d\psi) + \mu_V V d + \mu_D P_d D d. \tag{7.5}$$

Let us further recall from Section 3.1 that

$$\begin{aligned} P_d \operatorname{div}\left(\frac{\partial\psi}{\partial(\nabla d)}\right) &= 2k_3(\Delta d + |\nabla d|_2^2 d) + 2(k_1 - k_3) P_d \nabla \operatorname{div} d \\ &\quad + 2(k_2 - k_3)\,((d \times \nabla) \otimes \operatorname{curl} d) \cdot d \\ &\quad + 2(k_2 - k_3)\,((d \times \nabla) \otimes d) \cdot \operatorname{curl} d \\ &\quad - 2(k_2 - k_3)(d \cdot \operatorname{curl} d) P_d \operatorname{curl} d. \end{aligned}$$

Our goal is to derive an initial boundary value problem for the function $\varphi := |d|_2^2 - 1$ by testing (7.5) and the boundary condition for d with d and to show that $\varphi = 0$. To this end, we observe that $(\Delta d|d) = \frac{1}{2}\Delta\varphi - |\nabla d|_2^2$ and hence

$$(\Delta d + |\nabla d|_2^2 d|d) = \frac{1}{2}\Delta\varphi + |\nabla d|_2^2\varphi.$$

Furthermore, $(P_d \nabla \operatorname{div} d|d) = (\nabla \operatorname{div} d|P_d d) = -(\nabla \operatorname{div} d|d)\varphi$ and

$$(d \cdot \operatorname{curl} d)(P_d \operatorname{curl} d|d) = (d \cdot \operatorname{curl} d)(\operatorname{curl} d|P_d d) = -(d \cdot \operatorname{curl} d)(\operatorname{curl} d|d)\varphi,$$

since $P_d^\top = P_d$ and $P_d d = -\varphi d$. Therefore, by (3.2), we obtain

$$\begin{aligned} \Big(P_d \operatorname{div}\Big(\frac{\partial\psi}{\partial(\nabla d)}\Big)\Big|d\Big) &= k_3\Delta\varphi + 2k_3|\nabla d|_2^2\varphi - 2(k_1 - k_3)(\nabla \operatorname{div} d|d)\varphi \\ &\quad + 2(k_2 - k_3)(d \cdot \operatorname{curl} d)(\operatorname{curl} d|d)\varphi. \end{aligned}$$

For the remaining terms on the right side in (7.5) we have

$$(P_d \nabla_d \psi | d) = -(\nabla_d \psi | d)\varphi, \quad (P_d Dd | d) = -(Dd|d)\varphi \text{ and } (Vd|d) = 0,$$

since V is skew-symmetric. The terms on the left side of (7.5) are treated as

$$(\partial_t d|d) = \frac{1}{2}\partial_t \varphi \quad \text{and} \quad (u \cdot \nabla d|d) = \frac{1}{2}(u|\nabla\varphi).$$

Finally, we consider the contributions by the boundary condition for φ by d. Let us recall from (3.5) that

$$0 = P_d \frac{\partial \psi}{\partial(\nabla d)} \cdot \nu = k_3 \nabla d \cdot \nu + P_d(k_1 \operatorname{div} d \cdot I - k_3(\nabla d)^{\mathsf{T}}) \cdot \nu$$
$$+ (k_2 - k_3)(d \cdot \operatorname{curl} d)(d \times \nu) + (k_2 + k_4) P_d((\nabla d)^{\mathsf{T}} - \operatorname{div} d \cdot I) \cdot \nu,$$

which, multiplying by d, yields

$$0 = \frac{1}{2}k_3 \partial_\nu \varphi - (k_1 \operatorname{div} d \cdot \nu - k_3 (\nabla d)^{\mathsf{T}} \cdot \nu | d)\varphi - (k_2 + k_4)((\nabla d)^{\mathsf{T}} \cdot \nu - \operatorname{div} d \cdot \nu | d)\varphi.$$

Here we used again the identity $P_d d = -\varphi d$. Thus, we obtain the following problem for φ:

$$\begin{cases} \gamma \partial_t \varphi = 2k_3 \Delta \varphi + G_1 \varphi & \text{in} \quad \Omega, \\ k_3 \partial_\nu \varphi = G_0 \varphi & \text{on} \quad \partial\Omega, \\ \varphi(0) = 0 & \text{in} \quad \Omega, \end{cases} \tag{7.6}$$

where

$$G_1 \varphi := 2k_3 |\nabla d|_2^2 \varphi + (\nabla_d \psi | d)\varphi - \mu_D (Dd|d)\varphi - \gamma(u|\nabla\varphi)$$
$$- 2(k_1 - k_3)(\nabla \operatorname{div} d | d)\varphi + 2(k_2 - k_3)(d \cdot \operatorname{curl} d)(\operatorname{curl} d | d)\varphi,$$

and

$$G_0 \varphi := (k_2 + k_4 - k_3)((\nabla d)^{\mathsf{T}} \cdot \nu | d)\varphi - (k_2 + k_4 - k_1) \operatorname{div} d (\nu | d)\varphi.$$

Our aim is now to show that $\varphi = 0$ is the unique solution to (7.6). To this end, we observe that if

$$d \in H^1_{p,\mu}(J_T; H^1_p(\Omega; \mathbb{R}^3)) \cap L_{p,\mu}(J_T; H^3_p(\Omega; \mathbb{R}^3)),$$

then

$$\varphi \in H^1_{p,\mu}(J_T; L_p(\Omega)) \cap L_{p,\mu}(J_T; H^2_p(\Omega)) =: \mathbb{E}^\varphi_{1,\mu}(J_T),$$

since $d \in C(J_T; C^2(\overline{\Omega}))$ by our assumption (4.1) on p and μ. Furthermore, we have $u \in C(J_T; C^1(\overline{\Omega}))$. Hence, there exists a constant $C_1 = C_1(T) > 0$ such that for any $\tau \in (0, T]$ it holds that

$$\|G_1\varphi\|_{L_p(J_\tau \times \Omega)} \le C_1 \|\varphi\|_{L_p(J_\tau; H^1_p(\Omega))} \le \tau^{1/p} C_1 \|\varphi\|_{L_\infty(J_\tau; H^1_p(\Omega))}$$
$$\le \tau^{1/p} C_1 M_1 \|\varphi\|_{\mathbb{E}^\varphi_{1,\mu}(J_\tau)},$$

with some constant $M_1 > 0$, not depending on τ, since $\varphi(0) = 0$ and

$$\mathbb{E}^{\varphi}_{1,\mu}(J_\tau) \hookrightarrow C([0,\tau]; W_p^{2\mu-2/p}(\Omega)) \hookrightarrow C([0,\tau]; H_p^1(\Omega)),$$

by the assumption (4.1) on p and μ. Furthermore, the trace space for $\partial_\nu\varphi$ on $\partial\Omega$ is

$$\mathbb{G}_\mu(J_T) := W^{1/2-1/2p}_{p,\mu}(J_T; L_p(\partial\Omega)) \cap L_{p,\mu}(J_T; W_p^{1-1/p}(\partial\Omega)).$$

It follows from [13, Lemma A.5 (ii)] that there exists a constant $C_2 = C_2(T) > 0$, such that, for any $\tau \in (0, T]$, the estimate

$$\|G_0\varphi\|_{\mathbb{G}_\mu(J_\tau)} \le C_2\|\beta\|_{\mathbb{G}_\mu(J_\tau)}\|\varphi\|_{\mathbb{F}^{\varphi}_\mu(J_\tau)} \le C_2 M_2\|\beta\|_{\mathbb{G}_\mu(J_\tau)}\|\varphi\|_{\mathbb{E}^{\varphi}_{1,\mu}(J_\tau)}$$

is valid, with some constant $M_2 > 0$ being independent of τ, since $\varphi(0) = 0$. Here we have used the fact that

$$\mathbb{F}^{\varphi}_\mu(J_T) = \mathrm{tr}_{\partial\Omega}\,\mathbb{E}^{\varphi}_{1,\mu}(J_T) = W^{1-1/2p}_{p,\mu}(J_T; L_p(\partial\Omega)) \cap L_{p,\mu}(J_T; W_p^{2-1/p}(\partial\Omega))$$

for any $T \in (0, \infty]$. Finally, let us note that $\|\beta\|_{\mathbb{G}_\mu(J_\tau)} \to 0$ as $\tau \to 0$, by the absolute continuity of the integral.

Since by assumptions (R) & (P), $\gamma = \gamma(d, \nabla d) \in C(J_T \times \overline{\Omega})$ and $\gamma > 0$, maximal L_p-regularity for the Neumann-Laplacian with inhomogeneous boundary conditions and a perturbation argument imply that $\|\varphi\|_{\mathbb{E}^{\varphi}_{1,\mu}(J_\tau)} = 0$ provided $\tau > 0$ is sufficiently small. This in turn implies $\varphi(t) = 0$ in its trace space

$$X^{\varphi}_{\gamma,\mu} := W_p^{2\mu-2/p}(\Omega)$$

for all $t \in [0, \tau]$. We define that

$$\tau_* := \sup\{\tau \in [0, T] \mid \forall\, t \in [0,\tau] : \varphi(t) = 0 \text{ in } X^{\varphi}_{\gamma,\mu}\}.$$

Suppose that $\tau_* < T$. Then we may solve (7.6) with initial time τ_* and initial value $\varphi(\tau_*) = 0$. Applying again the maximal L_p-regularity for the Neumann-Laplacian and the above perturbation argument, we obtain the existence of some $\tau > 0$ such that $\varphi(t) = 0$ in $X^{\varphi}_{\gamma,\mu}$ for all $t \in [\tau_*, \tau_* + \tau]$, which is a contradiction to the maximality of τ_*. This yields $\varphi(t) = 0$ in $X^{\varphi}_{\gamma,\mu}$ for all $t \in [0, T]$. Hence, $\varphi(t, x) = 0$ for all $(t, x) \in [0, T] \times \overline{\Omega}$, by the embedding $X^{\varphi}_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega})$ and (4.1). Note that $\varphi \equiv 0$ if and only if $|d|_2 \equiv 1$.

Finally, the fact that the solution (u, d) of (2.12) can be extended to a maximal interval of existence follows by an iterated application of Proposition 7.1. □

Proof of Theorem 2.3. For the proof, one may literally follow the lines of the proof of [13, Proof of Theorem 5.1 (b)]. We consider the equivalent system (6.10) with the definitions (6.7)-(6.9). With a view on [13, Proof of Theorem 5.1 (b)], we need to know that

$$[z \mapsto \mathcal{B}(z) = \mathsf{B}(z)z] \in C^1(X_{\gamma,\mu}, Y_{\gamma,\mu}), \tag{7.7}$$

and that, for any $z_0 \in \mathcal{M}$,

$$\mathcal{B}'(z_0) \in \mathcal{L}(X_{\gamma,\mu}, Y_{\gamma,\mu})$$

has a continuous right inverse, where

$$Y_{\gamma,\mu} := \operatorname{tr}_{t=0} \mathbb{F}_\mu(J_T) = W_p^{2\mu-3/p}(\partial\Omega; \mathbb{R}^3)$$

is the trace space of $\mathbb{F}_\mu(J_T)$ at $t = 0$.

Adopting the strategy of the proof of [13, Lemma B.3 (a)], (7.7) follows readily. The existence of a continuous right inverse for $\mathcal{B}'(z_0)$ can be proven as in [38, Proof of Proposition 2.5.1] in combination with Propositions 5.1, 5.4 and [42, Section 6.3.5 (iv)]. □

8. Conclusion

In this paper we considered the Ericksen-Leslie model (2.12) for nematic liquid crystal flows in case of an isothermal and incompressible fluid with *general Leslie stress and anisotropic elasticity*. The free energy ψ is given by the classical Oseen-Frank free energy

$$\begin{aligned}\psi(d, \nabla d) = k_1(\operatorname{div} d)^2 + k_2|d \times (\nabla \times d)|_2^2 + k_3|d \cdot (\nabla \times d)|^2 \\ + (k_2 + k_4)[\operatorname{tr}(\nabla d)^2 - (\operatorname{div} d)^2],\end{aligned}$$

where k_i are the so-called *Frank coefficients*. For the director field d we prescribe the *fully nonlinear* boundary condition

$$P_d \frac{\partial \psi}{\partial(\nabla d)} \cdot \nu = 0,$$

which is a natural boundary condition ensuring that the non-isothermal and compressible Ericksen-Leslie system is thermodynamically consistent meaning that the total energy is preserved, whereas in the isothermal situation, the available energy is nonincreasing, see Remark 2.4. Moreover, the associated entropy production rate of the system is nonnegative and thus satisfies the second law of thermodynamics.

Under the conditions (R), (P), (F) and (B), we have proven in Theorem 2.1 the existence and uniqueness of a local-in-time strong solution

$$\begin{aligned} u &\in H^1_{p,\mu}((0,T); L_p(\Omega; \mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^2_p(\Omega; \mathbb{R}^3)), \\ d &\in H^1_{p,\mu}((0,T); H^1_p(\Omega; \mathbb{R}^3)) \cap L_{p,\mu}((0,T); H^3_p(\Omega; \mathbb{R}^3)) \\ \nabla\pi &\in L_{p,\mu}((0,T); L_p(\Omega; \mathbb{R}^3)) \end{aligned}$$

of (2.12). The use of the time-weighted $L_{p,\mu}$-spaces thereby allows to decrease the regularity of the initial data

$$(u_0, d_0) \in W^{2\mu-2/p}_{p,\sigma}(\Omega; \mathbb{R}^3) \times W^{2\mu+1-2/p}_{p}(\Omega; \mathbb{R}^3)$$

by adjusting the weight $\mu \in (1/2 + 5/2p, 1]$, where $p > 5$. We have furthermore proven that the condition $|d_0(x)|_2 = 1$ for all $x \in \Omega$ on the initial value d_0 is preserved, i.e. for all $t \in [0, T]$ and all $x \in \Omega$ it holds that $|d(t, x)|_2 = 1$. The proof of Theorem 2.1 is heavily based on the fact that the principal part of the linearized Ericksen operator subject to the principal part of the linearization of the boundary operator (cf. Sections 3.1 & 3.2) is strongly elliptic and satisfies the Lopatinskii-Shapiro condition (see Propositions 5.1 & 5.4), thus allowing an approach based on modern quasilinear theory.

We have furthermore proven in Theorem 2.3 that the solution (u, d, π) to (2.12) depends continuously on the initial data (u_0, d_0) and that the local existence time $T > 0$ in Theorem 2.1 is locally uniform.

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Declarations

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References

1. Amann, H.: *Linear and Quasilinear Parabolic Problems*, Monographs in Mathematics **89**, Birkhäuser 1995
2. Ball, J.: Mathematics and liquid crystals. *Mol. Cryst. Liq. Cryst.* **647**, 1–27, 2017
3. Ball, J., Zarnescu, A.: Orientability and energy minimization in liquid crystal models. *Arch. Rational Mech. Anal.* **202**, 493–535, 2011

4. Beris, A., Edwards, B.: *Thermodynamics of Flowing Systems*, Oxford University Press, 1994.
5. Bothe, D., Prüss, J.: L_p-theory for a class of non-Newtonian fluids. *SIAM J. Math. Anal.* **39**, 379–421, 2007
6. Chang, K., Ding, W., Ye, R.: Finite time blow up of th heat flow of harmonic maps from surfaces. *J. Diff. Geom.* **36**, 507–515, 1992
7. Chechkin, G.A., Ratiu, T.S., Romanov, M.S.: On the Eringen model for nematic liquid crystals. *C.R. Mec.* **349**, 21–27, 2021
8. Chechkin, G.A., Ratiu, T.S., Romanov, M.S., Samokhin, V.N.: Existence and uniqueness theorems for the two-dimensional Ericksen-Leslie system. *J. Math. Fluid Mech.* **18**, 571–589, 2016
9. Chechkin, G.A., Ratiu, T.S., Romanov, M.S., Samokhin, V.N.: Existence and uniqueness theorems for the full three-dimensional Ericksen-Leslie system. *Math. Models Methods Appl. Sci.* **27**, 807–843, 2017
10. Coutand, D., Shkoller, S.: Well-posedness of the full Ericksen-Leslie model of nematic liquid crystals, *C. R. Acad. Sci. Paris Sér. I Math.***333**, 919-924, 2001
11. DeGennes, P.G., Prost, J.: *The Physics of Liquid Crystals*, Oxford University Press, 1995
12. Denk, R., Dore, G., Hieber, M., Prüss, J., Venni, A.: New thoughts on old results of R.T. Seeley. *Math. Ann.***328**, 545-583, 2004
13. Du, H., Shao, Y., Simonett, G.: On a thermodynamically consistent model for magnetoviscoelastic fluids in 3D. *J. Evol. Equ.* **24**, 9, 2024
14. Ericksen, J.L.: Hydrostatic theory of liquid crystals. *Arch. Rational Mech. Anal.* **9**, 371–378, 1962
15. Ericksen, J.L.: Inequalities in liquid crystal theory. *Phys. Fluids* **9**, 1205–1207, 1966
16. Eringen, A.C.: An assessment of director and micropolar theories of liquid crystals. *lnt. J. Eng. Sci.* **31**, 605–616, 1993
17. Eringen, A.C.: A unified continuum theory of liquid crystals. *ARI Int. J. Phys. Eng. Sci.* **50**, 73–84, 1997
18. Feireisl, E., Frémond, M., Rocca, E., Schimperna, G.: A new approach to non-isothermal models for nematic liquid crystals. *Arch. Rational Mech. Anal.* **205**, 651–672, 2012
19. Frank, F.C.: On the theory of liquid crystals. *Discussions Faraday Soc.* **25**, 19–28, 1958
20. Gay-Balmaz, F., Ratiu, T.S., Tronci, C.: Equivalent theories of liquid crystal dynamics. *Arch. Rational Mech. Anal.* **210**, 773–811, 2013
21. Hieber, M., Hussein, A., Wrona, M.: Strong well-posedness of the Q-tensor model with general tumbling and alignments effects. *Archive Ration. Mech. Anal.* **248**, 40, 2024
22. Hieber, M., Nesensohn, M., Prüss, J., Schade, K.: Dynamics of nematic liquid crystal flow : the quasilinear approach, *Ann. Inst. H. Poincaré C Anal. Non Linéaire***33**, 379-408, 2016
23. Hieber, M., Prüss, J.: *Thermodynamic consistent modeling and analysis of nematic liquid crystal flows*, In: Math. Fluid Dynamics, Present and Future, Springer Proc. Math., Statistics, 433–460, 2014
24. Hieber, M., Prüss, J.: Dynamics of the Ericksen-Leslie Equations with General Leslie Stress I: The Incompressible Isotropic Case. *Math. Ann.* **369**, 977–996, 2017
25. Hieber, M., Prüss, J.: Modeling and analysis of the Ericksen-Leslie equations for nematic liquid crystal flows. *Handbook of Mathematical Analyis in Mechanics of Viscous Fluids,* pp. 1075-1134. Springer, Berlin 2018
26. Hieber, M., Prüss, J.: Dynamics of the Ericksen-Leslie Equations with General Leslie Stress II: The Compressible Isotropic Case. *Arch. Rational Mech. Anal.* **233**, 1441–1468, 2019
27. Hong, M., Li, J., Xin, Z.: Blow-up criteria of strong solutions to the Ericksen-Leslie system in $\mathbb{R}^3$. *Comm. Partial Differential Equ.* **39**, 1284–1328, 2014

28. Huang, T., Lin, F., Liu, C., Wang, C.: Finite time singularities of the nematic liquid crystal flow in dimension three. *Arch. Rational Mech. Anal.* **221**, 1223–1254, 2016
29. Lai, C., Lin, F., Wang, C., Wei, J.: Finite time blow up for the nematic liquid crystal flow in dimension two. *Comm. Pure Appl. Math.* **75**, 128–196, 2022
30. Latushkin, Y., Prüss, J., Schnaubelt, R.: Stable and unstable manifolds for quasilinear parabolic systems with fully nonlinear boundary conditions. *J. Evol. Equ.* **6**, 537–576, 2006
31. LeCrone, J., Prüss, J., Wilke, M.: On quasilinear parabolic evolution equations in weighted L_p-spaces II. *J. Evol. Equ.* **14**(3), 509–533, 2014
32. Leslie, F.M.: Some constitutive equations for aniostropic fluids. *Quart. J. Mech. Appl. Anal.* **19**, 357–370, 1966
33. Leslie, F.M.: Some constitutive equations for liquid crystals. *Arch. Rational Mech. Anal.* **28**, 265–283, 1968
34. Lin, F.: On nematic liquid crystals with variable degree of freedom. *Comm. Pure Appl. Math.* **44**, 453–468, 1991
35. Lin, F., Liu, Ch.: Nonparabolic dissipative systems modeling the flow of liquid crystals. *Comm. Pure Appl. Math.* **48**, 501–537, 1995
36. Lin, F., Wang, C.: Global existence of weak solutions of the nematic liquid crystal flow in dimension three. *Comm. Pure Appl. Math.* **69**, 1532–1571, 2016
37. Ma, W., Gong, H., Li, J.: Global strong solutions to incompressible Ericksen-Leslie system in $\mathbb{R}^3$. *Nonlinear Anal.* **109**, 230–235, 2014
38. Meyries, M.: Maximal Regularity in Weighted Spaces, Nonlinear Boundary Conditions, and Global Attractors. *Ph.D. thesis, Karlsruher Institut für Technologie*, 2010, https://publikationen.bibliothek.kit.edu/1000021198
39. Oseen, C.W.: The theory of liquid crystals. *Trans. Faraday Soc.* **29**, 883–899, 1933
40. Parodi, O.: Stress tensor for a nematic liquid crystal. *J. Physique* **31**, 581–584, 1970
41. Prüss, J.: H^∞-calculus for generalized Stokes operators. *J. Evol. Eq.* **18**, 1543–1574, 2018
42. Prüss, J., Simonett, G.: *Moving Interfaces and Quasilinear Parabolic Evolution Equations*. Monographs in Mathematics **105**, Birkhäuser 2016
43. Prüss, J., Simonett, G., Wilke, M.: Critical spaces for quasilinear parabolic evolution equations and applications. *J. Differential Equ.* **264**, 2028–2074, 2018
44. Prüss, J., Wilke, M.: Addendum to the paper "On quasilinear parabolic evolution equations in weighted L_p-spaces II". *J. Evol. Equ.* **17**, 1381–1388, 2017
45. Sonnet, A., Virga, E.: Dissipative Ordered Fluids: Theories of Liquid Crystals. Springer, Berlin (2012)
46. Triebel, H.: *Interpolation theory, function spaces, differential operators*. North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York, 1978
47. Triebel, H.: *Theory of function spaces*. Monographs in Mathematics, **78**, Birkhäuser Verlag, Basel, 1983
48. Virga, E.G.: Variational Theories for Liquid Crystals. Chapman-Hall, London (1994)
49. Wu, H., Xu, X., Liu, Ch.: On the general Ericksen-Leslie Ssystem: Parodi's relation, well-posedness and stability. *Arch. Ration. Mech. Anal.* **208**, 59–107, 2013
50. Wang, C.: Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data. *Arch. Ration. Mech. Anal.* **200**, 1–19, 2011
51. Wang, W., Zhang, P., Zhang, Z.: Well-posedness of the Ericksen-Leslie system. *Arch. Ration. Mech. Anal.* **210**, 837–855, 2013
52. Wang, W., Zhang, P., Zhang, Z.: Rigorous derivation from Landau-De Gennes theory to Ericksen-Leslie theory. *SIAM J. Math. Anal.* **47**, 127–158, 2015
53. Wang, W., Zhang, L., Zhang, P.: Modelling and computation of liquid crystals. *Acta Numer* **30**, 765–851, 2021

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