Robust Bayesian Estimation of Mixed Normal Dirichlet Models to Study the Effect of Some Climatic Factors on Evaporation

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Abstract: This study proposes and validates a robust Bayesian model based on a Dirichlet process mixture of normals (DMNM) for probability density estimation and missing data imputation in multivariate datasets. The primary focus is on addressing the challenge of incomplete data by providing a flexible and accurate estimation of their underlying probability density function. To fit the model, three Bayesian estimation algorithms are implemented and compared: the Expectation-Maximization (EM) algorithm, the Markov Chain Monte Carlo (MCMC) method, and a Traditional Bayesian (TB) algorithm. The framework is applied to real-world climatic data (temperature, humidity, wind speed, and evaporation) obtained from the Meteorological Service in Basra, Iraq, with artificially introduced missing values at rates of 10%, 20%, and 40%. Model performance is evaluated using two key metrics: the Mean Squared Error (MSE) for imputation accuracy and computational execution time. The results demonstrate that the EM algorithm achieves the highest estimation accuracy (lowest MSE), while the TB method is the most computationally efficient. This work provides a practical toolkit for the statistical analysis of incomplete multivariate data in fields such as environmental modeling, hydrology, and agriculture.

1 INTRODUCTION

The rapid development of data in the field of statistical analysis in recent decades has led to a significant increase in complexity, particularly in multivariate data, due to the overlapping influence of multiple variables simultaneously. Therefore, there is a real need for sophisticated statistical models such as Bayesian models, given that we are dealing with incomplete data.

The use of advanced, efficient, and highly accurate probabilistic models characterized by great flexibility to model multivariate data, such as Dirichlet mixture normal models, is an effective tool in many economic, environmental, medical, educational, and agricultural fields. It is essential to employ advanced strategic techniques to correct and address the problem of missing values in data. The unique features of DMNM are a tool for addressing these challenges.

The challenges posed by missing data in nonparametric models were highlighted when the journal Econometrics published this in 2023 [1].

There are many studies and researches that have addressed Dirichlet models. Among them, T. S. Ferguson (1973) discussed nonparametric methods and their importance for estimating probability distributions, where he was able to clarify the concept of the Dirichlet process. In 2017, an analytical study of traffic accident data was published by researcher S. Heydari, who was able to employ mixed Dirichlet process models to analyze these accidents in a unique way that helps us understand them. In recent years, specifically in 2024, a group of researchers (P. Cardoso and other) used Dirichlet processes mixture normal on data containing missing values. The model was used to select the best treatment for patients with type 2 diabetes, and this model was distinguished by its efficiency and capabilities [2], [3], [4].

The improvement in Bayesian analysis methods introduced in this study, when dealing with incomplete data, represents an important addition to scientific research to address all practical challenges. Consequently, this paper provides a new perspective on climate and evaporation studies using three comparative methods: the Markov chain Monte Carlo method, the traditional Bayesian method, and the expectation-maximization method.

Based on the above, this research aims to present an advanced Bayesian statistical framework based on mixed Dirichlet models for normal distributions, incorporating mechanisms to handle missing values and achieve robust multivariate probability density estimation. The research also seeks to demonstrate the effectiveness of this model in analyzing the impact of climate variables on evaporation, by comparing its performance with traditional and modern estimation methods. A list of the symbols and abbreviations used throughout the paper is provided in Appendix (Table A.1).

2 MATERIALS AND METHODS

2.1 Dirichlet Distribution

Used in Bayesian models, this distribution is a generalization of the beta distribution for more than two parameters, and is considered a continuous distribution.

The Dirichlet distribution is a multivariate probability distribution for vectors X = (x1, x2,…, xs) where $\sum_{r=1}^{s} X_r = 1$ and Xr ≥ 0 for all r.

The probability density function of the Dirichlet distribution is defined as follows [5]:

$$f(X/\tau)=\frac{\Gamma(\sum_{r=0}^{s}\tau_r)}{\prod\Gamma(\tau_r)}\prod_{r=1}^{s} X_r^{\tau_{r-1}}, \quad (1)$$

where:-

τ_r : shape parameters (concentration coefficient)

Γ(·) : Gamma function

The shape of the Dirichlet distribution can be illustrated by generating 1000 points in the R program that were created from a Dirichlet distribution with three shape parameters $\tau^3 = (\tau_1,\tau_2,\tau_3)$, where “Tau” (τ) is the Dirichlet’s concentration (shape) parameter vector, and $\tau_1, \tau_{2,}\tau_3$ are the shape parameters with the following values [6]:

- τ_1 Tau value = (1,1,1): then the dirichlet distribution becomes a uniform distribution.
- τ_2Tau value = (30,30,30) and (8,16,32): then values tend to be X_r centrered.
- τ_3Tau value = (0.2,0.2,0.2): then value tend to be broder centric.

Figure 1 shows the Dirichlet distribution, where 1000 points were generated using R software, and Table 1 shows the Dirichlet distribution.

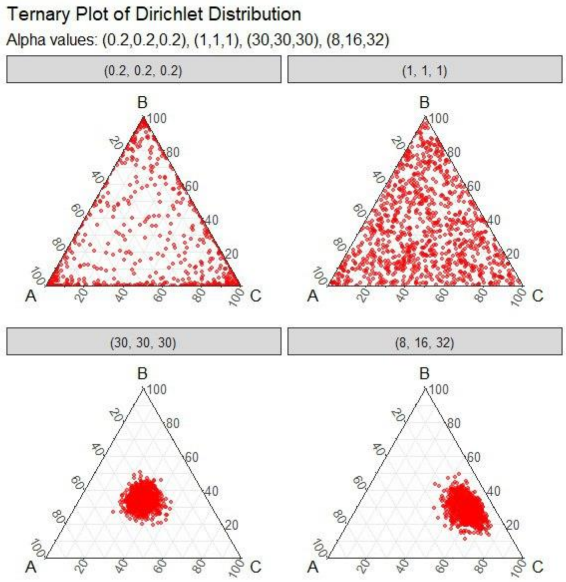

Figure 1: Dirichlet distribution generated from 1,000 sample points using R software.

Table 1: Some properties of the Dirichlet distribution.

Mean	$E[X_r]=\frac{\tau_r}{\tau_0}$ r=1,2, … s
Variance	$Var[X_r]=\frac{\tau_r(\tau_0-\tau_r)}{\tau_0^2(\tau_0+1)}$ r=1,2, … s
Covariance matrix	$Cov(X_r, X_q)=\frac{-\tau_r\tau_q}{\tau_0^2(\tau_0+1)}$ r,q=1,2, … s $and\ (r \neq q)$

2.2 Dirichlet Process

It is a stochastic process used in nonparametric Bayesian models, especially in Dirichlet mixture models. The Dirichlet process is a nonparametric generalization of the Dirichlet distribution, where it is not a fixed probability distribution over a finite number of classes, but rather is defined over the space of probability distributions that is, it is a probability distribution over the probability distributions themselves. Mathematically, it can be stated as follows [7]:

$$\text{Let } K \sim DP(\tau,K_0) \text{ and } X \sim K_0. \quad (2)$$

Where:

1) K: It is the distribution resulting from the Dirichlet process;
2) K_0: It is the basic distribution (the distribution around which the resulting distributions are centered);
3) τ: It is the Dirichlet parameter or concentration coefficient:

- If τ is small, the resulting distribution is more concentrated around specific values.
- If τ is large, the resulting distribution is closer to the underlying distribution.

2.2.1 The Basic Property of the Dirichlet Process

The Dirichlet process has a fundamental property known as the distributional consistency property. If we have a Dirichlet process (K) defined by a concentration parameter (τ) and a basic distribution (K0), then any subdivision of the probability space {B1,B2,…,Bs} of X produces a Dirichlet distribution [8]:

$$[K(B_1),K(B_2),..,K(B_s)] \sim \text{Dir}\ [\tau\ K_0\ (B_1),\ \tau\ K_0\ (B_2),..,\ \tau\ K_0\ (B_s)], \quad (3)$$

where:

- $K(A_r)$: It is the probability mass that K assigns to the region Ai of the probability space (the probability value of the class B_r).
- $\tau\ K_0\ (B_r)$: Weight assigned to category Br based on primary distribution.

The mean distribution and its variance are respectively given by

$$E[K(B)] = K_0(B) \text{ and } Var[K(B)] = \frac{K_0(B)\left(1 - \tau K_0(B)\right)}{(\tau + 1)}.$$

2.3 Dirichlet Mixture Normal Model

It is a non-parametric statistical model used to identify an unlimited number of components in the data. When Dirichlet processes are combined with the normal distribution as a basic component, the model can adapt to data that follow a normal distribution flexibly and dynamically (the model's ability to adapt to changes in the data or requirements). Model formula [9]:

$$g_c(X \mid \vartheta\ ,\ \sigma\ )=\sum_{p=1}^{c} \vartheta_p \prod_{r=1}^{s} \frac{1}{\sigma_r}\ \varphi\left(\frac{x_r-M_{rp}}{\sigma_r}\right), \quad (4)$$

where:

- $\vartheta : (\tau_1,\ldots,\tau_m)$;
- $\sigma : (\sigma_1,\ldots,\sigma_s)$;
- $\varphi(\cdot)$: Density function of the standard normal distribution of a random variable;
- ϑ p: Mixture weights determined by Dirichlet operations.

Mrp,r=1,…,s ,p=1,..,c: Mr=(mr1,mr2,…,mrc)' are the c knote of variable Xr. We pice the knots for X1 first, M1=(m11,m12,…,m1c)' is where m11=min(X1), m1c=max(X1), and m1q=X1(uq) , q=2,…,c-1, which is the index of the ordered value for variable X1,uq=$[\frac{q-1}{c-1}v]$.

The parameters of each component in the mixture (σ_s^2) are generated from the underlying distribution K0 (representing the distribution of parameters) by the following:

$$(\sigma_s^2) = \vartheta_p,$$

$$(\sigma_s^2) \sim K0,$$

$$K0=\text{Inv-Gamma}(\sigma 2 \mid a0,b0).$$

then:

$$\vartheta \sim \text{Dirichlet}\ (\vartheta 1,\ldots,\vartheta c),$$

$$K0=\text{Inv-Gamma}(\sigma 2 \mid a0,b0).$$

2.4 Robust Bayesian Estimation

Bayesian analysis is named after Thomas Bayes, who introduced the concept of conditional probability as a basis for understanding how probabilities change when additional information becomes available. It is a statistical technique based on the principle of updating prior knowledge in light of new data. This type of analysis combines what we already know (i.e., prior information) with what we obtain from observations to arrive at more accurate and realistic conclusions. Bayesian analysis is characterized by its great flexibility in dealing with uncertainty, making it suitable in cases where data are incomplete or constantly changing [10] - [14].

Robust Bayesian estimation is an advanced statistical approach that aims to achieve accurate and reliable Bayesian inference even in the presence of data affected by confounding or outlier values and deviations from the underlying assumptions of the model. When studying the effect of climate factors on evaporation, this method is crucial due to the complex nature of the data, which often contains missing values [15] - [18].

In this research, we used three main techniques based on flexible probability distributions that can reduce the impact of extreme values [19] - [22].

2.4.1 Typical Bayesian Method (TB)

It is a traditional Bayesian algorithm based on normal Dirichlet models, and uses Gibbs sampling to update mixture weights and band sizes alternately until a numerical result for the prior Bayesian distribution is reached from which density properties can be inferred and missing values can be estimated when needed.

Algorithm 1. Typical Bayesian Mixture Density Estimation (TBMDE).

Suppose we have s dimensional complete data x1, ...xv, g(x|ϑ,σ) $=\sum_{p=1}^{c}\vartheta_p g_p(x|\sigma)$.

ϑ ~ Dir(τ1, ..., τc), $\sigma_r^2\sim$ InverseGamma(ar , br), r = 1, ..., s and ϑ,

{σr}1≤r≤s are independent.

Initialize $\vartheta^{(0)}$ = $(\frac{1}{c},\frac{1}{c},\ldots,\frac{1}{c})$ and $\sigma^{2(0)}$.

for iteration u = 1, 2, ... do.

Sample $\vartheta^{(u)}$ ~ $\vartheta|\sigma^{(u-1)}$using I

Sample $\sigma^{2(u)}$ ~ $\sigma^2|\vartheta^{(u)}$ using II

end for

I= (Sample $\boldsymbol{\vartheta}$ given $\boldsymbol{\sigma}$)

ϑ ~ Dir(τ1, ..., τc) and σ is fixed.

Initialize $\vartheta^{(0)}$ = $(\frac{1}{c},\frac{1}{c},\ldots,\frac{1}{c})$.

for iteration u = 1, 2, ... do.

$sample\ P_r^{(u)}\sim\frac{\vartheta_p^{(u-1)}g_p(x_r|\sigma)}{\sum_{p=1}^{c}\vartheta_p^{(u-1)}g_p(x_r|\sigma)}I(p_r=p).for\ r=1.2.\cdots.v$.

$sample\ \vartheta^{(u)}\sim Dir\left(v_1\left(P^{(u)}\right)+\tau_1.\cdots.v_c\left(P^{(u)}\right)+\tau_c\right)$.

$where\ v_p\left(P^{(u)}\right)=\sum_{i=1}^{v}I(\,P_r^{(u)}=p).for\ p=1.\cdots,c$.

end for.

II= (Sample $\boldsymbol{\sigma}$ given $\boldsymbol{\vartheta}$)

$\sigma_r^2\sim$ InverseGamma(ar , br), r = 1, ..., s,{σr}1≤r≤s are independent. and ϑ is fixed.

Initialize $\sigma^{2(0)}$

for iteration u = 1, 2, ... do.

$sample\ P_r^{(u)}\sim\frac{\vartheta_p g_p\left(x_r|\sigma^{(u-1)}\right)}{\sum_{p=1}^{c}\vartheta_p g_p\left(x_r|\sigma^{(u-1)}\right)}I(p_r=p).for\ r=1.2.\cdots.v$.

$$sample\ \sigma_r^{2(u)}\sim InverseGamma\left(\frac{v}{2}+a_r.\frac{\sum_{q=1}^{v}(x_{qr}-m_{rpq})^2}{2}+b_r\right).$$

$for\ r=1.\cdots.s$.

$where\ x_{qr}\ denotes\ the\ r^{th}\ variable\ of\ x_q$

end for

2.4.2 Markov Chain Monte Carlo Method (MCMC)

2.4.2.1 General Framework of MCMC

The MCMC method is a type of stochastic simulation. It is a method that allows approximation of complex and multi-dimensional integrals by using random sampling procedures from probability distributions. The MCMC method consists of two components, the Markov chain and the Monte Carlo integration. The importance of the Markov chain is to draw a series of samples from the target probability distribution to obtain balance or stability, due to the difficulty of obtaining random samples directly, and the greater the number of steps, the greater the convergence between the sample distribution and the actual distribution, then the second component, which is the Monte Carlo integration, is used to approximate the complex integration. There are many methods and algorithms used in MCMC, but in this research we used Gibbs Sampling (GS) algorithm [23].

2.4.2.2 Gibbs Sampling (GS) Algorithm

One important Bayesian algorithm estimates data density and compensates for missing values within a mixture of normal Dirichlet models. As shown below:

Algorithm 2. Gibbs Mixture Data Imputation (GMDI).

Same step (i) in the first algorithm

Initialize $\vartheta^{(0)}$ = $(\frac{1}{c},\frac{1}{c},\ldots,\frac{1}{c})$ and $\sigma^{2(0)}$.

for iteration u = 1, 2, ... do.

for r = 1, ..., v do.

for q ∈ Cr (Missing value index set of xr) do.

Sample $X_{rq.miss}^{(u)}\sim$ s(xrq,miss, q ∈ Cr |ϑ (u-1) , xrq,obs, q /∈ Cr).

end for.

end for.

Sample (ϑ (u) ,σ 2(u)) ~ (ϑ,σ 2)|Xobs,$X_{miss}^{(u)}$ using Algorithm 1.

end for.

2.4.3 Expectation Maximization Method (EM)

It is an iterative algorithm that transforms parameter estimates in the presence of hidden variables into a series of simple updates until stable estimates are reached. Therefore, it is an expectation-maximization algorithm applied to a mixture of normal Dirichlet models. It is used to find maximum-likelihood estimates (MLEs) of the model parameters ϑ (component weights) and σ (band sizes). The working mechanism of this algorithm can be explained as follows [24].

Algorithm 3. (EM).

As in step (i) of Algorithm 1

Initialize $\vartheta^{(0)}$ = $(\frac{1}{c},\frac{1}{c},\ldots,\frac{1}{c})$ and $\sigma^{2(0)}$.

for iteration u = 0,1, 2, ... do.

E-step: for each r=1,…,v and p=1,…,c (compute responsibilities Ar,p)

let obsr⊆{1,…,s}

$$A_{rp}^{(u)}=\frac{\vartheta_p^{(u)}\prod_{q\in obsr}\frac{1}{\sigma_q^{(u)}}\varphi\left(\frac{x_{r.q}^{(u)}-m_{q.p}}{\sigma_q^{(u)}}\right)}{\sum_{p=1}^{c}\vartheta_p^{(u)}\prod_{q\in obsr}\frac{1}{\sigma_q^{(u)}}\emptyset\left(\frac{x_{r.q}^{(u)}-m_{q.p}}{\sigma_q^{(u)}}\right)}.$$

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Imputation: Fill in missing entries, For each sample r and each missing dimension q∈missr:
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$$x_{r.q}^{(u)} = \sum_{p=1}^{c} A_{r.p}^{(u)} m_{q.p}.$$

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end for.
end for.
M-step (update ϑ and σ) for each p=1,…,c and q=1,…,s
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$$\vartheta_p^{(u+1)} = \frac{1}{v}\sum_{r=1}^{v} A_{r.p}^{(u)}.$$

$$(\sigma_q^2)\,(u+1) = \frac{1}{v}\sum_{r=1}^{v}\sum_{p=1}^{c} A_{r.p}^{(u)}(x_{r.q}^{(u)} - c_{q.k})2, \quad \sigma_q^{(u+1)} = \sqrt{(\sigma_q^2)^{(u+1)}}$$

```
end for.
end for.
Convergence check
Stop when parameter changes ‖ϑ(u+1)−ϑ(u)‖ and ‖σ(u+1)−σ(u)‖ are below a preset threshold.
end for.
```

3 APPLICATION

In this section, a Bayesian estimation model was used to analyze the effect of temperature, humidity, and wind on evaporation. To obtain accurate estimates of the data probability density and address the missing data problem, we relied on three estimation methods: EM, MCMC, and TBMDE. When analyzing the data using missing ratios, a comprehensive comparison was conducted, allowing for a more accurate and specific assessment of the impact of climate factors. The applied results obtained by the model are considered important tools that can be utilized by the Meteorological Authority, the Water Resources Directorate, and the Agriculture Directorate to develop effective strategies for irrigation and agricultural systems to address the challenges posed by climate change, as well as when preparing academic research and studies.

3.1 Data Description

The data in this research were obtained from the Ministry of Transport/General Authority of Meteorology in Basra Governorate, Iraq. For the period 1/10/2023 to 24/4/2024. The research sample includes four variables: evaporation (X1) as a dependent variable, and temperature (X2), humidity (X3), and wind speed (X4) as independent variables. It consists of 207 observations. Random missingness values (0.1, 0.2, 0.4) were used in the data to analyze the effect of missing values on Bayesian models.

3.2 Results and Discussion

The analysis of the models was implemented using the R programming language, a leading tool for statistical computing, chosen for its efficiency in processing complex datasets. To evaluate the robustness of the Bayesian estimation methods – EM, GMDI, and TBMDE – against data incompleteness, their performance was tested on the original complete dataset after introducing three different rates of random missing values (0.1, 0.2, and 0.4). The evaluation was based on two primary criteria: estimation accuracy, measured by Mean Squared Error (MSE), and computational efficiency, assessed by execution time.

The following results were obtained as shown in Table 2.

The following Mean Squared Error (MSE) results are obtained based on the experimental results:

- MSE for variable X_1 at 0.1, 0.2, and 0.4. Whereas the best estimation method is EM.
- MSE for variable X_2 at 0.1. Whereas the best estimation method is TBMDE, at 0.2, and 0.4. Whereas the best estimation method is EM.
- MSE for variable X_3 at 0.1, 0.2, and 0.4. Whereas the best estimation method is EM.
- MSE for variable X_4 at 0.1, 0.2, and 0.4. Whereas the best estimation method is EM.

Figure 2 illustrates the influence of the climatic factors (X_2, X_3, and X_4) on X_1, highlighting how variations in these factors are associated with changes in X_1. Time (sec) ± SD: The fastest method in terms of execution time is TBMDE At 0.1, 0.2, and 0.4 , whereas the most efficient and accurate method is EM as shown in Table 3.

Figure 3 presents the density estimation results under varying missing data rates, comparing the performance of three methods: Expectation-Maximization (EM), Gaussian Mixture Density Imputation (GMDI), and Tree-Based Missing Data Estimation (TBMDE). The figure illustrates how each method responds to different levels of missingness, highlighting their relative accuracy and robustness in density estimation.

Table 2: MSE values for each variable on real data. Bold values indicate the best (lowest) MSE in each column/row.

r	Methods	X1	X2	X3	X4
0.1	TBMDE	0.2940	0.0387	0.0523	0.0252
	GMDI	0.8294	0.1057	0.1317	0.0460
	EM	0.1763	0.0396	0.0473	0.0202
0.2	TBMDE	0.2349	0.0414	0.0425	0.0223
	GMDI	0.6531	0.0880	0.0865	0.0569
	EM	0.1606	0.0413	0.0422	0.02226
0.4	TBMDE	0.1515	0.0478	0.0653	0.02174
	GMDI	0.3779	0.0983	0.1099	0.0437
	EM	0.1457	0.04646	0.04647	0.02167

Table 3: Comparison of the execution rate of each estimation method in seconds and the standard deviation of (r = 0.1, 0.2, and 0.4).

r	Methods	Time (sec) ± SD
0.1	TBMDE	0.0076 ± 0.0022
	GMDI	0.0147 ± 0.0064
	EM	0.0422 ± 0.0109
0.2	TBMDE	0.0071 ± 0.0029
	GMDI	0.0122 ± 0.0042
	EM	0.0326 ± 0.0085
0.4	TBMDE	0.0065 ± 0.0013
	GMDI	0.0082 ± 0.0026
	EM	0.0244 ± 0.0076

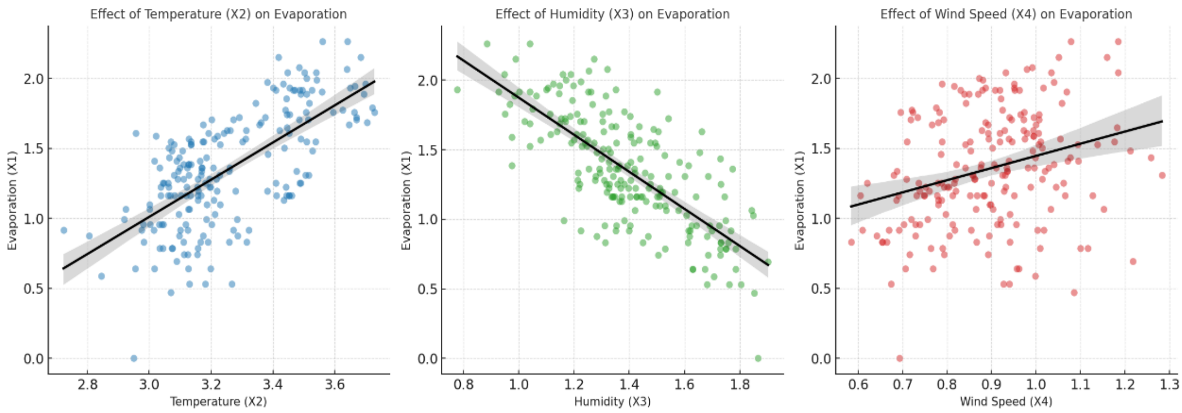

Figure 2: The effect of climatic factors (X2, X3, X4) on (X1).

The results indicate a clear positive linear relationship between temperature (X2) and evaporation (X1), with an increase in temperature leading to higher evaporation rates. In contrast, humidity (X3) shows the opposite effect, with evaporation values decreasing with increasing humidity. Wind (X4) also had a positive effect, but it was relatively weaker compared to the effect of temperature.

The graphs compare the probability density functions of the studied variables after introducing different missingness ratios (10%, 20%, and 40%) and using three methods to estimate the missing values. The EM method showed strong performance in preserving the distributional properties, while GMDI showed increasing sensitivity with higher missingness ratios, especially in the tails. TBMDE provided a conservative estimate centered around the mean, with partial loss of distributional properties in the tails. The results indicate that varying missingness ratios clearly affect the accuracy of the estimation, highlighting the importance of choosing the most appropriate method based on the nature of the data and the level of missingness.

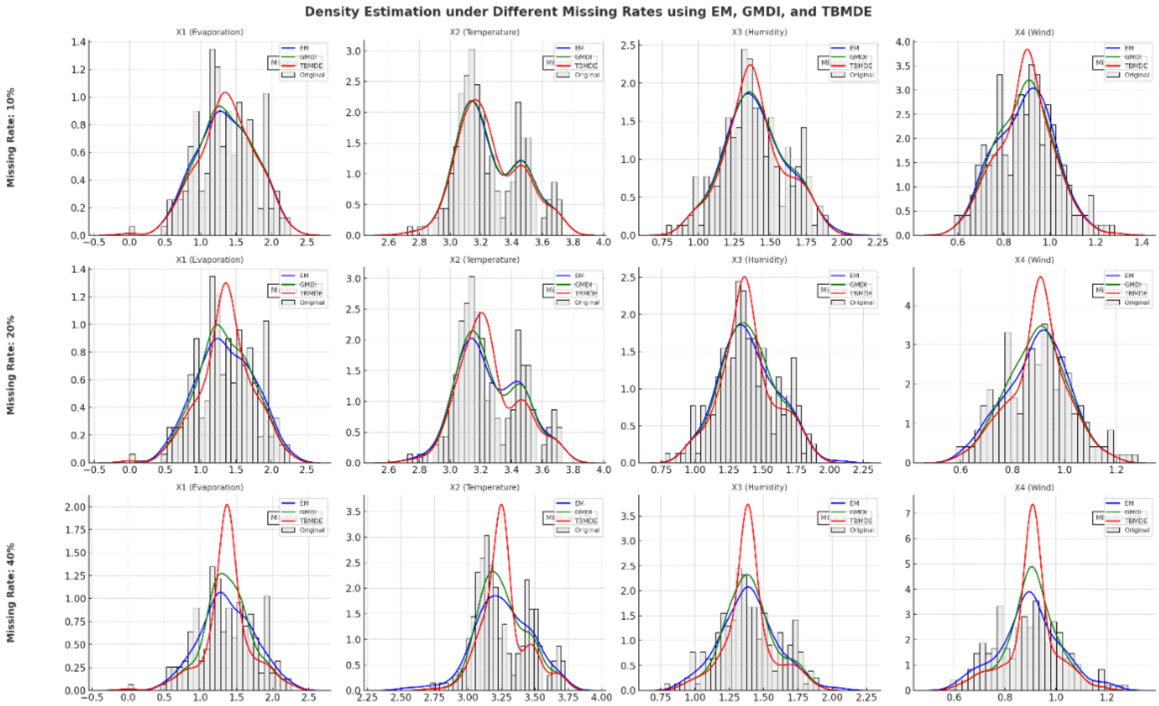

Figure 3: Density estimation under different missing rates using EM, GMDI and TBMDE.

4 CONCLUSIONS

This research seeks to present an integrated Bayesian framework for estimating probability density for multivariate data with missing data, using mixture models of normal distributions and three estimation methods: EM, GMDI, and TBMDE. The key points are as follows. The best estimation method that has shown high efficiency in estimation is EM. The method that is characterized by its rapid implementation is TBMDE, which is useful for researchers, especially if the sample size is very large. Regarding the effect of climatic factors (X2, X3, X4) on evaporation (X1), the most influential climatic factor on evaporation is maximum temperature, followed by minimum relative humidity (but with the opposite effect), and then wind speed, which is a lesser contributing factor. These graphs provide an important visual input that supports the construction of interpretive or predictive statistical models of evaporation rates, highlighting the importance of studying these variables within more complex analytical frameworks such as multivariate or Bayesian models. Based on the results, it is clear that the EM method is used to achieve high accuracy, while the TBMDE method is preferred if the goal is speedy implementation.

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APPENDIX

Table A.1: The key symbols and abbreviations.

Symbol	Meaning
X1	Evaporation rate (mm/day)
X2	Maximum daily temperature (C)
X3	Minimum daily humidity (%)
X4	Wind speed (m/s)
DP	Dirichlet process
DMNM	Dirichlet mixture Normal model
TB	Typical Bayesian
TBMDE	Typical Bayesian Mixture Density Estimation
GMDI	Gibbs Mixture Data Imputation
MCMC	Markov Chain Monte Carlo
EM	Expectation Maximization
GS	Gibbs Sampling