Projection Method for Solving Systems of Linear Equations Using Wavelet Packet Decomposition of the Residual

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Abstract: The work is devoted to the problem of solving large systems of linear algebraic equations with irregular structure matrices. To solve them the variant of the projection method in the Petrov-Galerkin form is proposed. Most of the known projection methods is based on the use of bases of Krylov subspaces. The main difference of the proposed method is the choice of the basis from coefficients of wavelet packet decomposition of the residuals. In general, the wavelet transform can be adaptive due to the entropic criteria for the evaluation of elements of the wavelet tree. This distinguishes the proposed method from the known FOM method, the GMRES algorithm and other projection solvers. Conducted a series of computational experiments comparing the proposed algorithm with the main existing projection methods. The experiments showed that the proposed algorithm is competitive with the major existing projection type methods, and in some cases can exceed them.

1 INTRODUCTION

The development of computer technology causes transition to more complex models (three- and multi-dimensional geometry in arbitrary areas) in the form of systems of differential equations in partial derivatives and to their discrete analogues on unstructured grids. This leads to the necessity of solving large sparse systems of linear algebraic equations with irregular matrices, for example, in the analysis of three-dimensional scanning of complex shapes, tomography, etc.

The most efficient and robust among the iterative methods for the solution of such systems of equations are projective techniques, especially the class that is associated with the projection on the Krylov subspace (Saad, 1981). These methods have a number of advantages: they are stable, allow effective parallelization, works with different row and column formats and different types of preconditioners.

2 BACKGROUND

Consider the system $Ax = b$ and formulate the following problem. Let set some two subspaces $K \subset R^n$ and $L \subset R^n$. Is required to find a vector that would provide a solution to the original system optimum with respect to the subspace *L*, i.e. to satisfy the Petrov-Galerkin condition (Reddy, 2006):

$$\forall l \in L : (Ax, l) = (b, l) \tag{1}$$

Grouping both sides of the properties of the scalar product, and noting that $b - Ax = r_x$, condition (1) can be rewritten as:

$$\forall l \in L : (r_x, l) = 0 \tag{2}$$

i.e. $r_x = b - Ax \perp L$. This problem is called ***the problem of designing the solution x on the subspace K orthogonal to the subspace L***.

In a more general formulation of the problem is as follows. For the original system is known an approximation x_0 to the solution x_*. Required to clarify its amendment $\delta_x \in K$ so that $b - A(x_0 + \delta_x) \perp L$. The condition of Petrov-Galerkin (2) in this case takes the form:

$$\forall l \in L: (r_{x0+\delta x}, l) = ((b - Ax_0) - A\delta_x, l) = (r_0 - A\delta_x, l) = 0\ . \tag{3}$$

Let $\dim K = \dim L = m$. Let's enter in subspaces K and the L bases $\{v_j\}_{j=1}^{m}$ and $\{w_j\}_{j=1}^{m}$ respectively. It is easy to see that (3) holds if and only if:

$$\forall j (1 \le j \le m) : (r_0 - A\delta_x, w_j) = 0 \tag{4}$$

Introducing matrix notations $V = [v_1 \,|\, v_2 \,|\, ... \,|\, v_m]$ and $W = [w_1 \,|\, w_2 \,|\, ... \,|\, w_m]$ for bases, we can write $\delta_x = Vy$ where $y \in R^m$ – the vector of coefficients. Then (4) takes the form:

$$W^T (r_0 - AVy) = 0\ , \tag{5}$$

from whence $W^T AVy = W^T r_0$ and

$$y = (W^T AV)^{-1} W^T r_0\ . \tag{6}$$

Thus, the decision should be specified in accordance with the formula:

$$x_1 = x_0 + V(W^T AV)^{-1} W^T r_0\ , \tag{7}$$

from which immediately follows an important requirement: ***in practical implementations of projection methods subspace K and L and their bases should be chosen so that the matrix*** $W^T AV$ ***were either low dimensions, or had a simple structure, easy to inversion***.

Formula (7) comprises a wide class of iterative methods. The simplest situation is when the space K and L are one-dimensional. Let $K = span\{v\}$ and $L = span\{w\}$. Then (7) takes the form:

$$x_{k+1} = x_k + \gamma_k v_k\ , \tag{8}$$

and γ_k is easily found from the orthogonality conditions $r_k - A(\gamma_k v_k) \perp w_k$:

$$(r_k - \gamma_k Av_k, w_k) = (r_k, w_k) - \gamma_k (Av_k, w_k) = 0\ ,$$

from whence:

$$\gamma_k = (r_k, w_k) / (Av_k, w_k)\ .$$

Let $v_k = w_k = r_k$. Then (8) takes the form:

$$x_{k+1} = x_k + \frac{(r_k, r_k)}{(Ar_k, r_k)}\ .$$

Since the expression in the denominator represents the quadratic form $r_k^T Ar_k$, the process of convergence is guaranteed if the matrix A is symmetric and positive definite.

The main problem of all these methods is the choice of the dimension m of the space K.

3 SOLUTION

In (Esaulov, 2015) proposed an iterative algorithm using wavelet solutions. In the paper we propose to implement (7) using the basis V of elements different from that used in Krylov subspaces.

As a variant of the construction of the basis V can be the basis of containing levels of decomposition the R_0-residuals obtained using the wavelet transform (Chui, 1992). Wavelet theory offers a more flexible signal processing technique than the Fourier transform (Bracewell, 2000). It provides the possibility of analysis of the signal not only by its frequency components, but also localizes them. By using wavelet analysis for signal processing it is advisable to use the methods of multiresolution analysis and fast algorithm for finding the wavelet coefficients. Multiscale representation makes it possible to review the signal at different levels of decomposition.

One of the most well-known algorithms for multiresolution analysis is Mallat algorithm (Resnikoff, 1998). In this algorithm, two filters, a smoothing A and detailing the D, recursively used to obtain data for all available scales. As a rule, filters are of finite impulse response in which the samples of the analysed signal, trapped in a small window, are multiplied by a predetermined set of coefficients, the resulting values are summed, and the window is shifted to calculate the next value of the output. Flowchart of the Mallat algorithm shown in Figure 1.

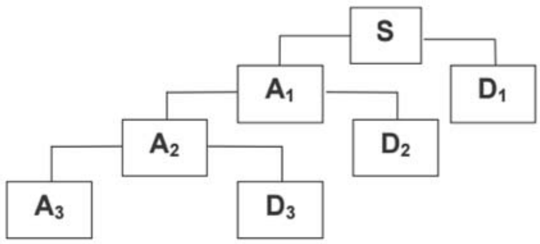

Figure 1: Flowchart of the Mallat algorithm wavelet analysis of a signal S.

It is also known wavelet packet decomposition (Coifman, 1992), which is characterized by repeated filtering of the detailing coefficients. Wavelet packet decomposition allows better control of the separation process of the original signal spectrum into parts, but significantly increases the computational complexity. In addition, the wavelet packet decomposition contains an excessive number of wavelet coefficients, which can be reduced if to organize the search for "best tree". Wavelet packet decomposition is adaptive, and is widely used for the signal compression and noise reduction. It can adapt more accurately to the characteristics of signals by selecting the appropriate optimal form of the tree decomposition, which provides a minimum number of wavelet coefficients for a given accuracy of reconstruction of the signal, and, thus deliberately excludes from the inverse wavelet transform insignificant, redundant information or unnecessary signal details.

A measure of optimality usually is the number of wavelet coefficients to reconstruct the signal with a given accuracy. It can be performed by entropy, evaluated as:

$$E = \exp(-\sum_n p_n \cdot \log(p_n)) \ , \ p_n = |x_n|^2 \|x\|^{-2} \quad (9)$$

Any averaging of coefficients increases the entropy. While tree analysis calculates the entropy of a node and its split parts. If the entropy is not reduced, when splitting a node, then further branching from this node does not make sense.

In accordance with the proposed working hypothesis the algorithm of solving of the system of linear equations, using wavelets may be formulated as following steps:

1. Set an initial approximation x_0.
2. Generate wavelet tree of residuals r_0 in accordance with the selected type of wavelets tree building algorithm and (9).
3. On the basis of the wavelet-tree builds subspaces *K* and *L*.
4. In accordance with (6-7) the clarification of solutions is carried out.

There are many ways to build bases *V* and *W*. Using of these methods in solving test problems does not give positive result, due to the bad conditioning of the main matrix in the (5). This occurs because of the proximity of the values of individual rows. In view of smoothing and detailing properties of the wavelet transform has been proposed the following approach.

Let there is a wavelet tree Ω. The basis of the subspace *K* is the set of nodes corresponding to the coefficient of approximation Ω_L. As the basis *W* of the subspace *L* selects the set of nodes Ω_r corresponding to detailing coefficients. This choice can be explained by the fact that in view of the approximation properties of the elements of the basis *V* it will display the most relevant information about the structure of residuals r_0.

The vectors corresponding to the elements of the basis of the subspace *W* of *L* are sparse in many cases. This fact can avoid orthogonalization of the basis *V*.

4 EXPERIMENTS

As test problems was used examples from the Regularization Tools library for MATLAB (Hansen, 2007). The maximum number of iterations was set to 500, convergence error was assumed to be 10^{-9}.

As the first test problem was taken the system of linear algebraic equations of Fox & Goodwin problem (Baker, 1977). The order of the main matrix was set to 100. Table 1 shows values of relative errors of solutions for Fox & Goodwin problem: Transpose-free quasi-minimal residual method (tfqmr), Generalized minimal residual method (gmres), Conjugate gradients squared method (cgs), Quasi-minimal residual method (qmr).

Table 1: Solution of the Fox & Goodwin problem by MATLAB solvers.

Solver	Error, %
tfqmr	0.027
gmres	0.029
cgs	0.025
qmr	0.029

Table 2: Solution of the Fox & Goodwin problem by the proposed algorithm

Wavelet type	Error, %			
	Using the one-dimensional decomposition basis		Using the one-dimensional reconstruction basis	
	$V=A\Omega_L$, $W=\Omega_r$	$V=\Omega_L$, $W=\Omega_r$	$V=A\Omega_L$, $W=\Omega_r$	$V=\Omega_L$, $W=\Omega_r$
db1	0.085	0.066	0.018	0.087
db2	0.064	8.773	0.329	5.197
db8	0.324	5.911	0.125	7.485
sym2	0.064	4.378	0.329	0.356
sym8	0.324	6.172	0.335	6.167
coif5	0.136	23.527	0.427	6.415

Table 3: Solution of the Shaw problem by MATLAB solvers.

Solver	Error, %
tfqmr	0.592
gmres	0.644
cgs	0.602
qmr	0.650

Table 4: Solution of the Shaw problem by the proposed algorithm

Wavelet type	Error, %			
	Using the one-dimensional decomposition basis		Using the one-dimensional reconstruction basis	
	$V=A\Omega_L$, $W=\Omega_r$	$V=\Omega_L$, $W=\Omega_r$	$V=A\Omega_L$, $W=\Omega_r$	$V=\Omega_L$, $W=\Omega_r$
db1	0.920	0.901	31.117	31.117
db2	3.130	2.050	1.992	1.992
db8	1.828	7.002	3.299	3.269
sym2	3.130	3.276	0.570	0.570
sym8	2.869	5.125	3.115	3.115
coif5	1.923	5.141	3.236	3.225

Table 5: Solution of the Baart problem by MATLAB solvers.

Solver	Error, %
tfqmr	0.110
gmres	0.110
cgs	0.110
qmr	0.109

Table 6: Solution of the Baart problem by the proposed algorithm

Wavelet type	Error, %			
	Using the one-dimensional decomposition basis		Using the one-dimensional reconstruction basis	
	$V=A\Omega_L$, $W=\Omega_r$	$V=\Omega_L$, $W=\Omega_r$	$V=A\Omega_L$, $W=\Omega_r$	$V=\Omega_L$, $W=\Omega_r$
db1	1.017	0.054	0.083	0.083
db2	0.522	0.522	4.325	2.143
db8	4.741	11.96	3.858	3.687
sym2	0.522	0.522	3.079	1.254
sym8	6.463	3.670	4.252	24.235
coif5	4.726	4.086	4.531	4.795

The solution by the proposed algorithm using different types of wavelets shown in Table 2: Daubechies (db1, db2, db8), Symlets (sym2, sym8), Coiflets (coif5).

As the second test problem was taken the system of linear algebraic equations of Shaw problem (Shaw, 1972). The order of the main matrix was set to 256. Table 3 shows values of relative errors of MATLAB solvers solutions for Shaw problem. The solution by the proposed algorithm using different types of wavelets shown in Table 4.

As the third test problem was taken the system of linear algebraic equations of Baart problem (Baart, 1982).The order of the main matrix was set to 100. Table 5 shows values of relative errors of MATLAB solvers solutions for Baart problem. The solution by the proposed algorithm using different types of wavelets shown in Table 6.

5 CONCLUSIONS

The paper shows that projection methods using Krylov subspace is a promising method for solving systems of linear equations. Based on conducted analysis, it was formulated the hypothesis about the possibility of using elements of the wavelet decomposition and wavelet reconstruction of the residuals as an alternative to Krylov subspaces. Principles of wavelet analysis of one-dimensional signals using entropy criteria are formed.

Conducted computing experiments have shown that the proposed algorithm is competitive with the major existing projection type methods, and in some cases can exceed them. It is also shown that accuracy of the solution depends on the type of wavelet.

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