Set optimization with respect to variable domination structures

Dissertation

zur Erlangung des Doktorgrades der Naturwissenschaften
(Dr. rer. nat.)

der

Naturwissenschaftlichen Fakultät II
Chemie, Physik und Mathematik

der Martin-Luther-Universität
Halle-Wittenberg

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Tag der Verteidigung: 02.10.2018

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Acknowledgments

First of all, I would like to express my deepest gratitude to my supervisor Prof. Dr. Christiane Tammer for all her help, guidance and kindness. She has shared her deep, immense knowledge with me and enthusiastically supported me since the first day I came to Halle. Thank you for being patient and helping me improve.

I would like to thank Dr. Elisabeth Köbis for her great collaboration, unconditional help, understanding, and encouragements during my difficult time. I am grateful to Dr. Behnam Soleimani, who spent emotional time when he lived in Halle and gave me great advice such that I could improve my work. I would like to thank Prof. Dr. Bao Truong for his valuable suggestions on my research and for helping my family during the time he stayed in Halle. Thank you, my friend, Bettina for the unforgettable memories we had together in our office.

I would like to thank all members of our working group Optimization and Stochastic: Alireza, Brian, Christian, Ernest, Elham, Niklas, Mahdi, and Marcus for their very nice friendship. I always feel great when seeing them and talking to them. I am very happy that I had an opportunity to work in this wonderful environment. Thanks to the secretary Mrs. Sauter, who has assisted me very much during my Ph.D. time.

My thanks also go to all my colleagues from University of Transport and Communications for their support during my work on this thesis. I'm grateful that this work is supported by the Minister of Education and training of Vietnam, Ministry Federal for Education and Research (Germany), and PhD Finalization Grant of the International Office of Martin-Luther-Universität Halle-Wittenberg (MLU) in association with DAAD.

There are not enough words to say thank you to my mother and my sister, who always care and love me. They are keen and happy to know what I was doing and how I was proceeding. I am grateful to my father whom I could not touch and see to say goodbye four years ago. I know that he always looks at me and wishes me all the best. I especially thank my husband Tuan, my children Quang and Anna for their limitless love, care and encouragements. Thank you for being the reason I smile every day.

Chapter 1

Introduction

Many real world problems can be formulated as vector- or set-valued optimization problems. This leads to an intensive development of vector as well as set optimization. In this work, we consider set optimization problems with respect to (w.r.t.) variable domination structures. Precisely, we study optimization problems whose objective mappings are set-valued and domination structures that induce certain set relations. Because the notion of set-valued mappings includes single-valued mappings, set optimization can be considered as an extension of vector optimization. Set-valued optimization problems are crucial and interesting not only from the mathematical but also from the practical point of view since they have many applications in production theory, radiotherapy treatment, game theory, welfare economics and uncertain optimization (see, for instance, [7, 8, 30, 36, 63, 72]).

One application of variable domination structures in the theory of consumer demand is illustrated in the following example:

Example 1.0.1. *In order to explain consumer behavior, John [58, 59] and references therein (compare [36]) studied two models of consumer preference: Local and global theory. We assume that the consumer faces a nonempty set of feasible alternatives $\Omega \subseteq \mathbb{R}^n$. By contrast with the global approach, a local preference only requires that the consumer is able to rank alternatives in a small neighborhood of a given bundle relative to that bundle. This idea can be represented by a continuous function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that y in the neighborhood of $\bar{y}$ is consider to be better than $\bar{y}$ if and only if $g(\bar{y})^T(y - \bar{y}) > 0$. In this case, we call $d := y - \bar{y}$ a preference direction from $\bar{y}$. As illustrated in [59], the choice set for the consumer is determined by:*

$$\mathcal{S}_g(\Omega) := \{\bar{y} \in \Omega | \; \forall y \in \Omega : \; g(\bar{y})^T(y - \bar{y}) \leq 0\}.$$

This means that the consumer will choose alternatives $\bar{y}$ such that for all $y \in \Omega$, y is not better than $\bar{y}$. This leads us to a mapping $\mathcal{D} : \; \Omega \rightrightarrows \mathbb{R}^n$ such that the image of each bundle $\bar{y}$ is its non-preference set, i.e., the variable domination structure $\mathcal{D}(\cdot)$ is given *by:*

$$\bar{y} \in \Omega, \ \mathcal{D}(\bar{y}) := \{d \in \mathbb{R}^n | \ g(\bar{y})^T d \le 0\}.$$

Now, we assume that the consumer is seeking for alternatives $\bar{y} \in \Omega$ *such that for all* $y \in \Omega \setminus \{\bar{y}\}$*:* $g(\bar{y})^T(y - \bar{y}) < 0$, *i.e.,* $g(\bar{y})^T(\bar{y} - y) > 0$, *which is equivalent to*

$$\begin{aligned} & \forall y \in \Omega \setminus \{\bar{y}\} : \ \bar{y} - y \notin \mathcal{D}(\bar{y}) \\ \iff \quad & \nexists y \in \Omega \setminus \{\bar{y}\} : \ \bar{y} \in y + \mathcal{D}(\bar{y}). \end{aligned}$$

This means that the consumer is looking for minimal elements of the set Ω *w.r.t. variable domination structure* $\mathcal{D}(\cdot)$*. For corresponding solution concepts, see Section 3.1, especially Definition 3.1.1.*

Another significant application of optimization w.r.t. variable domination structures is the intensity problem in radiotherapy treatment, see [36, Chapter 10]. In this problem, the treatment seeks for the goal dose to deliver to the patient such that the tumor is destroyed as much as possible while the normal organs are still protected. Eichfelder [36, Chapter 10] explained that in order to obtain an improvement of a critical organ, the doctor may accept a sacrifice in other critical organs. Therefore, the importance of each critical organ could be changed in a certain time. Furthermore, several advantages of vector- and set-valued optimization w.r.t. domination structures in vector variational inequality as well as behavior theory are discussed in [36, 41, 61, 63, 105].

There are three approaches to define solution concepts for a set-valued optimization problem, namely the vector approach, the set approach and the lattice approach. For the definitions of solution concepts and more details, we refer the reader to Khan, Tammer and Zălinescu [63].

In this dissertation, we consider set-valued optimization problems w.r.t. variable domination structures, where we use the vector approach and the set approach. Given a set-valued objective map $F : X \rightrightarrows Y$, where X and Y are Banach spaces. We study the following set-valued optimization problem w.r.t. a domination structure given by a set-valued mapping $\mathcal{Q} : X \rightrightarrows Y$:

$$\mathcal{Q} - \operatorname*{Min}_{x \in X} F(x), \tag{$P_{\mathcal{Q}}$}$$

where the solution concept is given by the vector approach. This means that the solution concept is defined on the graph of the mapping F: A point $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is called a nondominated solution of ($P_{\mathcal{Q}}$) if there is no point $(x, y) \in \operatorname{Gr} F \setminus \{(\bar{x}, \bar{y})\}$ such that $y \in \bar{y} - \mathcal{Q}(x)$. Although the vector approach is interesting for the theoretical perspective in set-valued optimization, it may not useful enough in practical applications, see [39, 56, 57, 63]. The reason is that a solution $(\bar{x}, \bar{y})$ is defined based on only one special point in the image set $F(\bar{x})$ ($\bar{y} \in F(\bar{x})$) and one does not care how the other

points in $F(\bar{x})$ perform. Generally, one takes the 'best' point $\bar{y}$ of $F(X)$ to imply the 'best' set $F(\bar{x})$ in the family of sets $\mathcal{F}(X) := \{F(x)|\ x \in X\}$ even $F(\bar{x})$ contains many inappropriate elements.
The set approach is a more natural way to define solutions of set-valued optimization problems and has important applications. This approach is equipped with a mapping $\mathcal{K} : Y \rightrightarrows Y$ to study the following problem:

$$\mathcal{K} - \underset{x \in X}{\mathrm{Min}}\, F(x). \tag{$P_{\mathcal{K}}$}$$

Using a certain set relation $\preceq^{\mathcal{K}}$, the solution concept of the problem ($P_{\mathcal{K}}$) is given by: A point $\bar{x} \in X$ is called a minimal solution of ($P_{\mathcal{K}}$) if

$$F(x) \preceq^{\mathcal{K}} F(\bar{x}) \Longrightarrow\ F(\bar{x}) \preceq^{\mathcal{K}} F(x).$$

It is important to mention that Young [110] introduced the set less relation, see [110, page 262]. Later on, other set relations have gained much attention of many researchers, see [17, 38, 39, 40, 57, 75, 76, 77]. Suppose that Y is a linear space, A and B are two nonempty subsets of Y, $K \subseteq Y$ is a proper, closed, convex, pointed cone, the lower set relation and upper set relation (see Kuroiwa [75, 76]; cf. Young [110]) are expressed by:

$$A \preceq_l^K B :\Longleftrightarrow B \subseteq A + K, \tag{1.1}$$

and

$$A \preceq_u^K B :\Longleftrightarrow A \subseteq B - K, \text{ respectively.} \tag{1.2}$$

This dissertation considers five aspects of set-valued optimization problems w.r.t. variable domination structures. Our main results are listed as follows:

- New relationships between the solution concepts of ($P_{\mathcal{K}}$) and ($P_{\mathcal{Q}}$).
- Characterizations of solutions of ($P_{\mathcal{K}}$) by new scalarizing functionals.
- Necessary optimality conditions for solutions of ($P_{\mathcal{K}}$) in terms of Mordukhovich's coderivatives using the new relationships between the solution concepts of ($P_{\mathcal{K}}$) and ($P_{\mathcal{Q}}$).
- Well-posedness property of ($P_{\mathcal{K}}$) based on new scalarizing functionals.
- Applications of set optimization problems w.r.t. variable domination structures in radiotherapy treatment, image registration problems and uncertain optimization.

This thesis is organized as follows: Chapter 2 recalls relevant mathematical concepts as well as several useful and important results. In particular, we present in detail the role of domin nation structures to compare vectors as well as sets, see Section 2.2. Chapter 3 concerns a special case of set-valued optimization, that is vector-valued optimization. We recall the definitions of nondominated solutions and minimal solutions for a vector-valued optimization problem w.r.t. variable domination structures. In addition, we present necessary optimality conditions for solutions of general vector optimization problems, especially vector-valued approximation problems.

Our main results will be presented in Chapters 4, 5, 6, 7 and 8. For the convenient of the reader, we outline them and illustrate several comparisons with regards to the literatures as follows:

In Chapter 4, we derive relationships between the solution concepts of $(P_{\mathcal{K}})$ and $(P_{\mathcal{Q}})$, see Section 4.3. These relationships show a bridge between the vector approach and the set approach in set-valued optimization. Using these results, it is possible to apply results known for solutions based on vector approach for deriving corresponding results for solutions based on set approach. To our best knowledge, there are only four current papers that explicitly investigate relationships between solution concepts based on these two approaches, see [39, 40, 57, 73]. It is important to note that these relationships have been studied in [57, 73] when $\mathcal{K}(\cdot)$ and $\mathcal{Q}(\cdot)$ are two constant mappings, and in [39] when both domination structures are set-valued mappings acting from Y to Y. However, this thesis studies these relationships for the case $\mathcal{K} : Y \rightrightarrows Y$ and $\mathcal{Q} : X \rightrightarrows Y$. The results in this chapter will be used in Chapter 6 to derive optimality conditions for solutions of $(P_{\mathcal{K}})$ based on the dual approach (where the Mordukhovich's coderivative of a set-valued map is used).

Chapter 5 follows the set approach to characterize solutions of a set-valued optimization problem w.r.t. a general variable domination structure, i.e., it is not necessary a cone-valued map as usual. For this aim, we utilize nonlinear scalarizing functionals extended from the well-known Gerstewitz functional. This functional is introduced and investigated in [42, 43, 44] and has been widely applied in publications, see, for instance, [20, 45, 46, 64, 66, 67, 72]. The authors in [20, 66, 67] used the Gerstewitz functional to characterize set relations, and the authors in [45, 46, 64] proposed nonlinear scalarizing functionals to investigate well-posedness properties of set-valued optimization problems. Recently, Kuwano and Tanaka [80] have proved the continuity of cone-convex set-valued maps by using nonconvex scalarization techniques for sets. In this chapter, we introduce for each set relation an appropriate corresponding scalarizing functional. This technique is beneficial for us to describe the comparison of given sets, see Section 5.1. Section 5.2 characterizes minimal elements of sets defined by set relations by means of the newly introduced functionals. In order to study well-posedness property of set optimization problems in Chapter 7, we introduce in Section 5.1 a new directional minimal time function, where the mapping $\mathcal{K}(\cdot)$ is involved. This function performs many desired properties for the proof of the equivalence between Tykhonov well-posedness of a scalar problem and well-posedness property of $(P_{\mathcal{K}})$, see Theorem 7.2.6. Section 5.4 presents a descent method for finding approximations of minimal solutions of a set-valued optimization problem equipped with a variable domination structure without convexity assumptions. In this numerical method, we use the scalarizing functional introduced and discussed in Section 5.1.1.

Chapter 6 deals with the dual approach to derive necessary optimality conditions for solutions of the set-valued optimization problem $(P_{\mathcal{K}})$. These conditions are derived in terms of Mordukhovich's coderivative for solutions of $(P_{\mathcal{K}})$ w.r.t. various set relations, see Section 6.2. Note that these results have not been investigated before. Bao and Mordukhovich [7] have shown necessary conditions for nondominated points of sets and nondominated solutions of vector optimization problems with variable ordering structures. Durea, Strugariu and Tammer [30] have investigated necessary conditions for solutions of $(P_{\mathcal{Q}})$ where the solution concepts are based on the vector approach. Khan, Soleimani and Tammer [62] considered weak solutions of $(P_{\mathcal{Q}})$ based on the vector approach and derived second order optimality conditions for these solutions. Recently, Eichfelder and Pilecka [40] have dealt with a primal approach (where the Bouligand derivative of a set-valued map is used) to derive optimality conditions for solutions of problem $(P_{\mathcal{K}})$. We take into account the paper [30] and apply the results presented in Chapter 4 to derive optimality conditions for solutions of the problem $(P_{\mathcal{K}})$ for the (possibly, certainly) lower less relation w.r.t. $\mathcal{K}(\cdot)$. For the (certainly) upper less relation w.r.t. $\mathcal{K}(\cdot)$, we utilize the sufficient conditions in terms of coderivatives for the openness of the composition of set-valued mappings (see [27, Theorem 4.2]).

Chapter 7 studies well-posedness properties for the problem $(P_{\mathcal{K}})$ by means of the directional minimal time function introduced in Chapter 5. The well-posedness properties of both vector-valued and set-valued optimization problems w.r.t. fixed cones have been studied intensively in the literature, see, for instance [31, 46, 64, 88, 89]. There are many publications investigating the parallelism between the well-posedness property of a vector optimization problem and the Tykhonov well-posedness property of a corresponding scalar problem; see, for example, [31, 89]. A similar result for set optimization problems w.r.t. fixed cones was first introduced in [112] and recently studied in [46, 64] and the references therein. Chapter 7 studies the lower set less relation which has been used widely in the literature and applied in many practical problems, see [45, 46, 64, 112]. We prove the equivalence between Tykhonov well-posedness of a scalar problem and well-posedness property of $(P_{\mathcal{K}})$, see Theorem 7.2.6. Based on this equivalence and two classes of well-posed scalar optimization problems given by Beer et al. [10] and Dontchev et al. [25], we derive two classes of well-posedness set optimization problems w.r.t. variable domination structures, see Theorems 7.2.11 and

7.2.14. Our approach can be considered as an extension of the results in [46] for fixed cones to variable domination structures. In addition, the assumption that the image of the objective mapping at the considered solution is cone-proper in [46] is relaxed since we utilize the advantage of our scalarizing functional.

In Chapter 8, we present three interesting applications of set-valued optimization w.r.t. variable domination structures in our real life. Section 8.1 concerns the inverse beam intensity problem in radiotherapy treatment. This problem has attracted many researchers in designing optimization models, methods, and theories concerning problems with fixed ordering structures. A detail survey on this field is presented in [32], where the geometry problem (selecting of beam angles), the intensity problem (computing of an intensity map for each selected beam angle), and the realization problem (finding a sequence of configurations of a multi-leaf collimator to deliver the treatment) are discussed. We also refer the reader to [15, 48, 81, 84] for more information about radiotherapy treatment as well as [49, 94] for guides on toxicology. Recently, Eichfelder [36, Chapter 10] has shown that it is more appropriate to consider the beam intensity problem w.r.t. variable ordering structures than using the component-wise partial ordering used so far. However, a challenging question is: *How should the variable ordering structures be?*. To answer this, we illustrate some dose respond curves for organs in lung cancer treatment and construct an appropriate variable ordering structure depending on the threshold doses. By using this structure, we formulate a beam intensity problem as a special case of approximation problems, see Section 8.1.3. Furthermore, necessary optimality conditions of this beam intensity problem are calculated in detail.

Section 8.2 investigates the image registration problem, where the decision maker has to compare two sets of data (images), see [35, 105, 108]. We formulate this problem as finding minimal solutions of a set-valued problem based on the set approach equipped with a variable ordering structure proposed by Wacker [105]. By means of the lower set relation w.r.t. $\mathcal{K}(\cdot)$, we characterize solutions of this problem by using the results derived in Section 5.1.1.

Section 8.3 shows characterizations of solutions of uncertain optimization problems. We explain the role of variable domination structures in studying these problems. Roughly speaking, because of the uncertainty of the data, it is likely that there exist undesired elements which may not be handled by using fixed ordering structures as usual. Therefore, variable dominations will help us to deal with these unexpected elements. By using the results derived in Section 7.1, we characterize optimistic and strictly optimistic solutions of uncertain optimization problems.

Chapter 2

Preliminaries

In this chapter, we present some necessary backgrounds and concepts which will be used throughout this dissertation. First, we recall the well-known notions of linear spaces, topological spaces, normed spaces and binary relations as well as cones defined on linear spaces. Section 2.2 illustrates how variable domination structures are used to compare vectors or sets. In particular, we recall several set relations, which are recently studied in [39, 40], and their relationships. Section 2.3 presents the concepts and properties of the Fenchel subdifferential, limiting normal cones, coderivatives, and the limiting subdifferential. Finally, we recall some compactness requirements and the openness, which play an important role in generating optimality conditions for solutions of set-valued optimization problems in Chapters 6 and 8.

2.1 Binary Relations

2.1.1 Linear Spaces, Topological Vector Spaces and Normed Spaces

As usual, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ present the sets of natural numbers, integers and real numbers, respectively. We denote the set of all nonnegative real numbers by $\mathbb{R}_+$, i.e., $\mathbb{R}_+:=\{u \in \mathbb{R}|\ u \geq 0\}$. In addition, we define $\mathbb{R}^n_+ := \{v = (v_1, \dots, v_n) \in \mathbb{R}^n|\ v_i \geq 0, i = 1, \dots, n\}$. We consider only real linear spaces throughout this dissertation, so the term *linear space* will refer to a linear space over the real field $\mathbb{R}$.

Definition 2.1.1. *Let X be a nonempty set. X is called to be a* ***linear space*** *if an addition (that is, a mapping $+ : X \times X \to X$) and a multiplication by scalars (that is, a mapping $\cdot : \mathbb{R} \times X \to X$) are defined satisfying the following conditions:*

(i) $\forall\ x, y, z \in X\ :\ (x + y) + z = x + (y + z)$ ***(associativity)****,*

(ii) $\forall\ x, y \in X\ :\ x + y = y + x$ ***(commutativity)****,*

(iii) $\exists\ 0 \in X,\ \forall\ x \in X\ :\ x + 0 = x$ ***(null element)****,*

(iv) $\forall\, x \in X,\ \exists\, x' \in X\ :\ x + x' = 0$; *we write* $x' = -x$,

(v) $\forall\, x, y \in X,\ \forall\, \lambda \in \mathbb{R}\ :\ \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$,

(vi) $\forall\, x \in X,\ \forall\, \lambda, \mu \in \mathbb{R}\ :\ (\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$,

(vii) $\forall\, x \in X,\ \forall\, \lambda, \mu \in \mathbb{R}\ :\ \lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$,

(viii) $\forall\, x \in X\ :\ 1 \cdot x = x$ ***(unity element)***.

From now on, for $\lambda \in \mathbb{R}, x \in X$ we write λx for the multiplication $\lambda \cdot x$. We define $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ with conventions $-\infty + a = -\infty$ for all $a \in \mathbb{R}$, $a(-\infty) = -\infty$, if $a > 0$ and $(+\infty) + (-\infty) = +\infty$. The multiplication of a scalar $\lambda \in \mathbb{R}$ with a set $S \subseteq X$ is defined as $\lambda S := \{\lambda s : s \in S\}$. In addition, we define algebraic sum of two sets S and U as: $S + U := \{s + u : s \in S, u \in U\}$. When S is singleton, $S = \{s\}$, we write $s + U$ instead of $\{s\} + U$.
We recall some algebraic properties of a set $S \subseteq X$ in a linear space as follows:

Definition 2.1.2. *Let S be a nonempty subset of a linear space X.*

*(i) The **algebraic interior** of S is denoted by* $\operatorname{core} S$ *and defined as*

$$\operatorname{core} S := \{\bar{x} \in S \mid \forall x \in X,\ \exists \bar{\lambda} > 0 :\ \bar{x} + \lambda x \in S \text{ for all } \lambda \in [0, \bar{\lambda}]\}.$$

*(ii) S is called **algebraically open** if* $S = \operatorname{core} S$.

*(iii) An element $\bar{x} \in X$ is called **linear accessible** from S if there is an $x \in S \setminus \{\bar{x}\}$ such that*

$$\lambda x + (1 - \lambda)\bar{x} \in S \text{ for all } \lambda \in (0, 1].$$

*The union of S and the set of all linear accessible elements from S is called the **algebraic closure** of S and it is denoted by* $\operatorname{lin} S$.

*(iv) S is called **algebraically closed** if* $S = \operatorname{lin} S$.

(v) The set of all elements in X which do not belong to $\operatorname{core} S$ *and* $\operatorname{core}(X \setminus S)$ *is called the **algebraic boundary** of S. We denoted this set by* $\operatorname{bd} S$.

We now consider the topological structure on the family of subsets of a nonempty set X.

Definition 2.1.3. *Let X be a nonempty set, and $\mathcal{T}$ be a family of subsets of X. We say that $(X, \mathcal{T})$ (we write X, for short) is a **topological space** if $\mathcal{T}$ satisfies the following conditions:*

(i) every union of sets of $\mathcal{T}$ belongs to $\mathcal{T}$,

(ii) *every finite intersection of sets of* $\mathcal{T}$ *belongs to* $\mathcal{T}$*,*

(iii) *the empty set* $\emptyset$ *and the whole set* X *belong to* $\mathcal{T}$*.*

The elements of $\mathcal{T}$ *are called* ***open*** *sets. A subset* S *of* X *is* ***closed*** *if and only if* $X \setminus S$ *is open.*

We consider the following topological notions:

Definition 2.1.4. *Let* S *be a subset of a topological space* X *and let some* $x \in X$ *be given.*

(i) *The set* S *is called a* ***neighborhood*** *of* x *if there is an open set* T *with* $x \in T \subset S$*.*

(ii) *The point* x *is called an* ***interior*** *element of* S *if there is a neighborhood* T *of* x *such that* $T \subseteq S$*. The set of all interior elements of* S *is called the interior of* S *and it is denoted by* $\operatorname{int} S$*.*

(iii) *The set of all elements of* X *for which every neighborhood meets the set* S *is called* ***closure*** *of* S *and it is denoted by* $\operatorname{cl} S$*.*

Some relationships between algebraic notions and the corresponding topological notions are illustrated as follows:

Proposition 2.1.5. *[50, p. 59] Let* S *be a nonempty convex set of a topological linear space* X*. If* $\operatorname{int} S \neq \emptyset$ *then the following assertions hold:*

(i) $\operatorname{int} S = \operatorname{core} S$;

(ii) $\operatorname{cl} S = \operatorname{cl}(\operatorname{int} S)$ *and* $\operatorname{int} S = \operatorname{int}(\operatorname{cl} \mathrm{S})$;

(iii) $\operatorname{cl} S = \operatorname{lin} S$.

Now, we recall the definition of a normed space and many special normed spaces which will be used in the next chapters.

Definition 2.1.6. *Let* X *be a linear space. A* ***norm*** *on* X *is a function* $\|\cdot\| : X \to \mathbb{R}$ *such that the following properties hold true for all* $x, y \in X$ *and for all* $\lambda \in \mathbb{R}$*:*

(i) $\|x\| = 0 \Longleftrightarrow x = 0$ *(**definiteness**);*

(ii) $\|\lambda x\| = |\lambda| \|x\|$ *(**positive homogeneity**);*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *(**the triangle inequality**).*

We call $(X, \|\cdot\|)$ *a* ***normed space****.*

We observe that a normed space is a metric space w.r.t. the metric d defined by

$$\forall\, x,\ y \in X:\ d(x,y) := \|x-y\|. \tag{2.1}$$

In addition, a normed space is a topological linear space, if the topology is generated by the metric d in (2.1). The normed space $(X, \|\cdot\|)$ is called a **Banach space** if every Cauchy sequence $\{x_n\} \subset X$ is convergent to an element of X. For instance, $\mathbb{R}^n$, l^∞ , l^p and $C[0,1]$ are Banach spaces. In a normed space $(X, \|\cdot\|)$, the **open ball** and the **closed ball** with the center x and the radius $r > 0$ are denoted by $B_X(x,r)$, $B_X[x,r]$ and respectively defined as

$$B_X(x,r) := \{y \in X|\ \|x-y\| < r\},$$

and

$$B_X[x,r] := \{y \in X|\ \|x-y\| \le r\}.$$

In addition, we denote by B_X the closed unit ball of X.

We introduce in the following some well-known norms in $\mathbb{R}^n$.

Example 2.1.7. *(a) The **Euclidean norm** of $\mathbb{R}^n$ presents the length of a vector x in $\mathbb{R}^n$ in the form:*

$$\|x\|_2 := \sqrt{x_1^2 + \dots x_n^2}.$$

*(b) The **Maximum norm** is defined for all $x \in \mathbb{R}^n$ by:*

$$\|x\|_\infty := \max\{|x_1|, \dots, |x_n|\}.$$

*(c) The **Manhattan (city block norm or rectangular) norm** is defined for all $x \in \mathbb{R}^n$ by:*

$$\|x\|_1 := |x_1| + \dots + |x_n|.$$

The relationships among three norms above are given by:

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1,$$

and

$$\|x\|_1 \le \sqrt{n}\|x\|_2 \le n\|x\|_\infty.$$

These norms have been used in several practical problems related to location theory, machine engineering, radiotherapy treatment, etc. For the theory and numerical methods, we refer to [32, 44, 81] and the references therein.

Definition 2.1.8. *[97, Definition 1.22] Let X be a Banach space. We say that X is **Asplund** if every continuous convex function defined on a nonempty open convex subset U of X is Fréchet differentiable at each point of some dense subset G of U; G is called a dense subset of U if every point $x \in U$ either belongs to G or is a limit point of G.*

The advantage of Asplund spaces is that they ensure a full calculus of the basic generalized differential constructions. It is known that the Banach spaces with separable dual and a reflexive Banach spaces are Asplund spaces. Some examples of Asplund spaces are c_0 and ℓ^p, $L^p[0,1]$ for $1 < p < +\infty$, see [90] for more detail.
In Chapter 8, we will present an approximation problem where the objective function is a vectorial norm. This kind of problem has several interesting applications, for example, in health care, location problems as well as in inverse problems (see [44]). The concept of vectorial norm is first introduced by Kantorovitch [60] who investigated the method of successive approximations. We recall it as follows:

Definition 2.1.9. *[54, Definition 1.35] Let X, Y be topological linear spaces and $C \subset Y$ be a proper, closed, convex cone. A function $\|\!|\cdot|\!\| : X \to C$ is called* ***vectorial norm*** *if for all $x, x_1, x_2 \in X$ and $\lambda \in \mathbb{R}$ the following conditions hold:*

(i) $\|\!|x|\!\| = 0 \Longleftrightarrow x = 0$;

(ii) $\|\!|\lambda x|\!\| = |\lambda| \; \|\!|x|\!\|$;

(iii) $\|\!|x_1 + x_2|\!\| \in \|\!|x_1|\!\| + \|\!|x_2|\!\| - C$.

Observe that if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$ then $\|\!|\cdot|\!\|$ becomes the norm $\|\cdot\|$ in X. We will study subdifferentials of vectorial norms and optimality conditions for approximation problems, where vectorial norms are involved in the next parts.

2.1.2 Order Relations on Linear Spaces

We suppose in this section that X be a linear spaces. When X is equipped with a topology we denote by X^* the topological dual space of X and w^* the weak star topology on X^*. In the following, we present a classical concept of binary relations.

Definition 2.1.10. *Let $A \subseteq X$ and $A \neq \emptyset$, the set of ordered pairs of elements of A is defined as*

$$A \times A := \{(x_1, x_2) |\; x_1, x_2 \in A\}.$$

A nonempty subset $\mathcal{R}$ of $A \times A$ is called a ***binary relation*** *on A. We denote by $x_1 \mathcal{R} x_2$ if $(x_1, x_2) \in \mathcal{R}$.*

Some important properties of binary relations are defined as follows:

Definition 2.1.11. *Let $\mathcal{R}$ be a binary relation on A. We say that $\mathcal{R}$ is*

(i) ***reflexive*** *if* $\forall\, x \in A$: $x\mathcal{R}x$;

(ii) ***transitive*** *if* $\forall\, x_1, x_2, x_3 \in A$: $\;x_1\mathcal{R}x_2,\; x_2\mathcal{R}x_3 \Longrightarrow x_1\mathcal{R}x_3$;

(iii) ***symmetric*** *if* $\forall\, x_1, x_2 \in A :\;\; x_1\mathcal{R}x_2 \Longrightarrow x_2\mathcal{R}x_1$;

(iv) ***antisymmetric*** *if* $\forall\ x_1, x_2 \in A:\ x_1\mathcal{R}x_2,\ x_2\mathcal{R}x_1 \Longrightarrow x_1 = x_2$.

Definition 2.1.12. *A binary relation $\mathcal{R}$ on A is said to be*

(i) *a* ***pre-order*** *if it is reflexive and transitive;*

(ii) *a* ***partial order*** *if it is reflexive, transitive and antisymmetric;*

(iii) *an* ***equivalence*** *if it is reflexive, transitive and symmetric;*

(iv) *a* ***linear*** *or* ***total order*** *if $\mathcal{R}$ is a partial order and any two elements of A are* ***comparable****, i.e., for all $x_1, x_2 \in A$ either $x_1\mathcal{R}x_2$ or $x_2\mathcal{R}x_1$.*

Example 2.1.13. (i) *Let $n \in \mathbb{N}, n \geq 2$. Consider a binary relation $\mathcal{R}$ defined on $\mathbb{R}^n$ by*

$$\mathcal{R} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n |\ y - x \in \mathbb{R}^n_+\}.$$

Obviously, $\mathcal{R}$ is a partial order but it is neither an equivalence nor a linear order.

(ii) *Let $A := \{(a, 2a)|\ a \in \mathbb{R}_+\}$ and the binary relation $\overline{\mathcal{R}}$ on A be defined by*

$$\overline{\mathcal{R}} := \{(u, v) \in A \times A|\ v - u \in \mathbb{R}^2_+\}.$$

Then, $\overline{\mathcal{R}}$ is a linear order on A but not an equivalence.

When the relation $\mathcal{R}$ is a pre-order (or a partial order), we say that A is a pre-ordered (partial ordered, respectively) set. In addition, a partially ordered linear space is a linear space equipped with a partial order.

Definition 2.1.14. *Let $\mathcal{R}$ be a binary relation on the linear space X; we say that $\mathcal{R}$ is* ***compatible*** *with the linear structure of X if two following properties hold true:*

$$\forall \lambda \geq 0, \forall x_1, x_2 \in X:\ x_1\mathcal{R}x_2 \Longrightarrow \lambda x_1\mathcal{R}\lambda x_2 \tag{2.2}$$

$$\forall x, x_1, x_2 \in X:\ x_1\mathcal{R}x_2 \Longrightarrow (x + x_1)\mathcal{R}(x + x_2) \tag{2.3}$$

Now we define minimal (maximal) elements of a set A relative to relation $\mathcal{R}$ as follows:

Definition 2.1.15. *Suppose that $\mathcal{R}$ is a binary relation on A. Let S be a nonempty subset of A. An element $\bar{x} \in S$ is said to be*

(i) *a minimal element of S relative to $\mathcal{R}$ if for all $x \in S : x\mathcal{R}\bar{x} \Longrightarrow \bar{x}\mathcal{R}x$.*

(ii) *a maximal element of S relative to $\mathcal{R}$ if for all $x \in S : \bar{x}\mathcal{R}x \Longrightarrow x\mathcal{R}\bar{x}$.*

We denote by $\mathrm{Min}(S, \mathcal{R})$ *and* $\mathrm{Min}(S, \mathcal{R})$ *the set of minimal and maximal elements of S relative to $\mathcal{R}$, respectively.*

Remark 2.1.16. *(a) If the binary relation $\mathcal{R}$ is antisymmetric, then $\bar{x} \in S$ is a minimal (maximal) element if and only if*

$$\forall\, x \in S: \ x\mathcal{R}\bar{x} \Rightarrow \bar{x} = x \quad (\forall\, x \in S: \ \bar{x}\mathcal{R}x \Rightarrow x = \bar{x}).$$

(b) Assume that $\mathcal{R}$ is a binary relation on A and S is a nonempty subset of A, then $\overline{\mathcal{R}} := \mathcal{R} \cap (S \times S)$ is a binary relation on S. In addition, if $\mathcal{R}$ is a pre-order (partial order, linear order) on A, then $\overline{\mathcal{R}}$ is a pre-order (partial order, linear order) on S. Therefore, $\bar{x} \in \mathrm{Min}(A, \mathcal{R})$ $(\bar{x} \in \mathrm{Max}(A, \mathcal{R}))$ if and only if $\bar{x} \in \mathrm{Min}(S, \overline{\mathcal{R}})$ $(\bar{x} \in \mathrm{Max}(S, \overline{\mathcal{R}}),$ respectively).

(c) When $\mathcal{R}$ is a partial order on A, a nonempty subset S of A may have zero, one or several maximal elements. However, if $\mathcal{R}$ is a linear order, then every subset S of A has at most one minimal (maximal) element. For instance, the set A in Example 2.1.13 has only one minimal element, that is $(0, 0)$.

2.1.3 Cone Properties

In this part, we remind a class of relations determined by cones in a linear space. These relations are compatible in the sense of Definition 2.1.14. We begin with the definition of a cone as follows:

Definition 2.1.17. *([44, Definition 2.1.11]) Let Y be a linear space. A set $C \subset Y$ is called a* ***cone*** *if $\lambda c \in C$ for all $c \in C$ and for all $\lambda \in \mathbb{R}_+$.*

Of course, if C is a cone and $C \neq \emptyset$, then $0 \in C$. Some important properties of C are defined as follows:

Definition 2.1.18. *([44, Definition 2.1.11]) Let C be a cone in Y. We said that C is*

(a) ***convex*** *if $\forall\, x_1, x_2 \in C: \ x_1 + x_2 \in C$;*

(b) ***nontrivial or proper*** *if $C \neq \emptyset$, $C \neq \{0_Y\}$ and $C \neq Y$;*

(c) ***pointed*** *if $C \cap (-C) = \{0\}$;*

(d) ***reproducing*** *if $C - C = Y$.*

Example 2.1.19. *The sets*

$$C_1 := \{(x_1, x_2) \in \mathbb{R}^2_+ |\ 0 \leq x_1 \leq x_2\},$$

and

$$C_2 := \{(x, y) \in \mathbb{R}^2 |\ x_1 \leq 0,\ x_1 \leq x_2\}$$

are proper, convex, pointed cones but they are not reproducing. In addition, $C_1 \cup C_2$ is a proper, pointed cone, but it is neither a convex cone nor a reproducing cone.

It is stated in [54, Lemma 1.11] that a cone C in a linear space is convex if and only if the following assertion holds

$$C + C \subseteq C.$$

Definition 2.1.20. *([54, Definition 1.10])*

*(a) Let C be a proper convex cone in a linear space Y. Suppose that U is a nonempty convex subset of C. We said that U is a **base** for C if each $c \in C \setminus \{0_Y\}$ has a unique representation of the form $c = \lambda u$ for some $\lambda > 0$ and some $u \in U$.*

(b) Let $S \subset Y$ be a nonempty set. We denote by $\operatorname{cone}(S)$ the cone generated by S and defined by

$$\operatorname{cone}(S) := \{\lambda s |\ \lambda \geq 0,\ s \in S\}.$$

Observe that if U is a base of a proper convex cone C then $\operatorname{cone}(U) = C$. A cone which admits a base is called based. In addition, a proper convex cone with a base is pointed. It is stated in [115] that if S is convex then the following assertion holds true

$$\operatorname{core} S = \{s \in S |\ \operatorname{cone}(S - s) = Y\}.$$

In addition, when S is a convex cone and $\operatorname{core} S \neq \emptyset$, it holds that $\operatorname{core} S = S + \operatorname{core} S$, see [54, Lemma 1.12]. Therefore, taking into account Proposition 2.1.5(i), we get the following result:

Proposition 2.1.21. *Let C be a convex cone in a linear space Y with a nonempty interior. Then, $\operatorname{int} C = C + \operatorname{int} C$.*

The following theorem presents the relationships between binary relations and cones. For the proof of this result, we refer the reader to [44, Theorem 2.1.13].

Theorem 2.1.22. *[44, Theorem 2.1.13] Let Y be a linear space and let C be a cone in Y. Then the binary relation $\leq^C$ given by*

$$\leq^C := \{(x, y) \in Y \times Y |\ y - x \in C\} \tag{2.4}$$

is reflexive and satisfies (2.2) and (2.3). Moreover, C is convex if and only if $\leq^C$ is transitive, and C is pointed if and only if $\leq^C$ is antisymmetric. Conversely, if $\mathcal{R}$ is a reflexive relation on X satisfying (2.2) and (2.3), then the set

$$C := \{y \in Y |\ 0_Y \mathcal{R} y\}$$

is a cone and $\mathcal{R} = \leq^C$.

Observe from the above result that when $\emptyset \neq C \subset Y$ the relation $\leq^C$ defined by (2.4) is a pre-order if and only if C is a convex cone. Furthermore, $\leq^C$ is a partial order if and only if C is a pointed convex cone.

Remark 2.1.23. *The sets C_1 and C_2 in Example 2.1.19 are ordering cones in $\mathbb{R}^2$, while the set $C_1 \cup C_2$ is not an ordering cone.*

When Y is equipped with the binary relation $\leq^C$, where C is a proper, closed, convex and pointed cone we have the following definition of Pareto efficient element of a given set, see, e.g., the books [44, 54, 85] for more details.

Definition 2.1.24. *Let $\emptyset \neq A \subseteq Y$, $C \subset Y$ be a proper, closed, convex and pointed cone. We say that $\bar{y} \in A$ is a* ***Pareto efficient point*** *of A w.r.t. the ordering cone C if there is no other point $y \in A \setminus \{\bar{y}\}$ such that $y \leq^C \bar{y}$, which is equivalent to*

$$A \cap (\bar{y} - C) = \{\bar{y}\}.$$

The definition, existence and necessary optimality conditions for many kinds of Pareto efficient points have been investigated in many publications, see for instance [4, 5, 6]. In particular, [6] derived necessary optimality conditions for these efficient points without pointedness assumption on ordering cones.

In the following, we recall concepts of a dual cone and the quasi-interior of the dual cone.

Definition 2.1.25. *Let Y be a topological linear space, $C \subset Y$ and let Y', Y^* denote the algebraic and topological dual space, respectively.*

(a) The ***(algebraic) dual cone*** *for C is denoted by C' and defined as:*

$$C' := \{y' \in Y' | \ \forall c \in C: \ y'(c) \geq 0\}.$$

(b) The ***(algebraic) quasi-interior*** *of the dual cone for C is denoted by $C^{\#}_{Y'}$ and determined as:*

$$C^{\#}_{Y'} := \{y' \in Y' | \ \forall c \in C \setminus \{0_Y\}: \ y'(c) > 0\}.$$

(c) The ***(topological) dual cone*** *for C is denoted by C^+ and defined as:*

$$C^+ := \{y^* \in Y^* | \ \forall c \in C: \ y^*(c) \geq 0\}.$$

(c) The ***(topological) quasi-interior*** *of the dual cone for C is denoted by $C^{\#}$ and defined as:*

$$C^{\#} := \{y^* \in Y^* | \ \forall c \in C \setminus \{0_Y\}: \ y^*(c) > 0\}.$$

Observe that the dual cone C^+ is always a convex cone, even if C is neither convex nor a cone. In addition, if $C^{\#} \neq \emptyset$ then C is pointed. When Y is a finite-dimensional space, this implication becomes an equivalent; for the proof and more detail, see [44]. The following proposition characterizes elements of a cone in a linear space by elements of its dual cone. The proof of this result is given in [54, Lemmas 1.26 and 3.21] and is therefore skipped here for the sake of shortness.

Proposition 2.1.26. *([54])Let C be a convex cone in a linear space Y. Then, the following assertions hold true:*

(i) $\operatorname{core} C \subseteq \big\{y \in Y |\ \forall c' \in C' \setminus \{0_{Y'}\}:\ c'(y) > 0\big\}$.

(ii) If Y is locally convex and C is closed then,

$$C = \{y \in Y |\ \forall c^* \in C^+:\ c^*(y) \geq 0\}.$$

(iii) If Y is a topological linear space and $\operatorname{int} C \neq \emptyset$ *then,*

$$\operatorname{int} C = \big\{y \in Y |\ \forall c^* \in C^+ \setminus \{0_{Y^*}\}:\ c^*(y) > 0\big\}.$$

We end this section with the definition of a normal cone which shows a connection between topology and order of the space Y.

Definition 2.1.27. *(Normal cone, [44, Definition 2.1.21]) Let C be a proper convex cone in a topological linear space Y. Then, C is called* ***normal*** *if the origin $0 \in Y$ has a neighborhood base formed by full sets w.r.t. C; a set $U \subseteq Y$ is full w.r.t. C if $U = (U + C) \bigcap (U - C)$.*

Observe that if Y is a topological linear space then C is normal if and only if $\operatorname{cl} C$ is normal [44, Theorem 2.1.22]. In addition, a proper convex cone $C \subset \mathbb{R}^n$ is normal if and only if $\operatorname{cl} C$ is pointed, see [44, Corollary 2.2.11].

2.2 Variable Domination Structures

In this section, we illustrate how variable domination structures have been used to compare vectors in a set and sets in a family of sets. These methods will be used in order to define several solution concepts for set optimization problems w.r.t. variable domination structures in Chapter 4.

2.2.1 Comparison of Vectors

In classical vector optimization, a special case of set optimization, one defines optimality concepts based on partial orderings. However, in many practical problems, it is necessary to consider general concepts to compare an element $y \in Y$ with some point $z \in Y$ w.r.t. a variable domination structure. We begin this section by recalling the concept of domination structures. This concept was first introduced by Yu [111] to investigate the decision-making problem where the objective function $f : X \to Y$ is a vector-valued function. In this problem, $X \subset \mathbb{R}^n$ is a set of possible decisions, called the decision space and $Y \subset \mathbb{R}^p$. Given two outcomes y^1 and y^2 in Y, $y^1 \neq y^2$, we denote by $y^1 \succ y^2$ if y^1 is preferred to y^2.

Definition 2.2.1. *[111, Definition 5.1] A nonzero vector $d \in \mathbb{R}^p$ is a* ***domination factor*** *for $y \in Y$ if $y' = y + d$ implies that $y \succ y'$. The set of all domination factors for y, together with the zero vector, will be denoted by $\mathcal{D}(y)$. The family $\{\mathcal{D}(y)|\ y \in Y\}$ is called the* ***domination structure*** *of the decision-making problem. For simplicity, the domination structure will be denoted by $\mathcal{D}(\cdot)$.*

In order to derive concepts of solutions for a vector optimization w.r.t. domination structures, one needs methods to compare elements. There exist two methods which are introduced by Chen et al. [18] and Yu[111] as follows: Let $\mathcal{D} : Y \rightrightarrows Y$ be a set-valued mapping such that for all $y \in Y$, $\mathcal{D}(y)$ is a nonempty set. We consider the two following binary relations where $\mathcal{D}(\cdot)$ is involved:

$$y \preceq_1 z \Longleftrightarrow z \in y + \mathcal{D}(y), \tag{2.5}$$

and

$$y \preceq_2 z \Longleftrightarrow z \in y + \mathcal{D}(z). \tag{2.6}$$

Yu [111] used the first one to find the so called nondominated elements of a certain set A w.r.t. a cone-valued mapping $\mathcal{D}(\cdot)$. In other words, the author looked for an element $\bar{y} \in A$ such that

$$\nexists y \neq \bar{y},\ y \in A :\ y \preceq_1 \bar{y},\ i.e.,\ A \cap (\bar{y} - \mathcal{D}(y) \setminus \{0\}) = \emptyset, \tag{2.7}$$

see [111, Definition 5.2]. Few years later, Bergstresser, Charnes and Yu [13] investigated these nondominated concepts for the case where the domination structure at each point is not a convex cone but a convex set.

The second method is utilized by Chen, Huang and Yang [18] in order to find a nondominated-like (minimal) element $\bar{y} \in A$ under the following sense:

$$\nexists y \neq \bar{y},\ y \in A :\ y \preceq_2 \bar{y},\ i.e.,\ A \cap (\bar{y} - \mathcal{D}(\bar{y}) \setminus \{0\}) = \emptyset, \tag{2.8}$$

see also [18, Definition 1.13]. In Chapter 3, we will study further properties and relationships between nondominated elements and minimal elements of a certain set.

Definition 2.2.2. *([36, Definition 1.8]) Let Y be a topological linear space, $\mathcal{D} : Y \rightrightarrows Y$ be a set-valued mapping such that $\mathcal{D}(y)$ is a proper, closed, convex cone for all $y \in Y$. The cone-valued map $\mathcal{D}(\cdot)$ is called an ordering map if elements in the space Y are compared using the binary relation (2.5) or (2.6). We say that $\mathcal{D}(\cdot)$ defines a* ***variable ordering*** *(structure) on Y.*

Remark 2.2.3. *Observe that, if for all $y \in Y$, $\mathcal{D}(y) = C$, where C is a convex cone, then both the relations (2.5) and (2.6) reduce to the relation $\leq^C$ given by (2.4).*

Some relationships between properties of $\mathcal{D}(\cdot)$ and the binary relations (2.5) and (2.6) are given as follows. For the proof of this result and more details, we refer the reader to [36].

Proposition 2.2.4. *([36, Lemma 1.10]) Let $\mathcal{D} : Y \rightrightarrows Y$ be an ordering map on a linear space Y.*

(a) The binary relations (2.5) and (2.6) are reflexive.

(b) The binary relation (2.5) is transitive if

$$\mathcal{D}(y+d) \subseteq \mathcal{D}(y), \text{ for all } y \in Y \text{ and for all } d \in \mathcal{D}(y).$$

(c) The binary relation (2.6) is transitive if

$$\mathcal{D}(y-d) \subseteq \mathcal{D}(y), \text{ for all } y \in Y \text{ and for all } d \in \mathcal{D}(y).$$

In Chapter 4, we will define optimality concepts for set optimization problems w.r.t. variable domination structures by using several set relations. The reflexivity and transitivity of these set relations are ensured under some appropriate properties of the general set-valued mappings. This will be illustrated in the following section.

2.2.2 Variable Set Relations

Let Y be a linear space and we denote the set of all nonempty subsets in Y by $\mathcal{P}(Y)$. In the following, we recall several set relations given in [39] under different names, where a domination mapping $\mathcal{K} : Y \rightrightarrows Y$ is involved. Notice that we do not require $\mathcal{K}(\cdot)$ to be a cone-valued map. These concepts will be used in Section 4.2 to define solutions for set optimization problems w.r.t. variable domination structures based on the set approach.

Definition 2.2.5. *Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued mapping such that for all $y \in Y$, $\mathcal{K}(y) \neq \emptyset$. We define the binary relations on $\mathcal{P}(Y)$ w.r.t. $\mathcal{K}(\cdot)$ as follows:*

*(i) The **lower less relation w.r.t.** $\mathcal{K}(\cdot)$ $\preceq_l^{\mathcal{K}}$ is defined by*

$$A \preceq_l^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)).$$

*(ii) The **upper less relation w.r.t.** $\mathcal{K}(\cdot)$ $\preceq_u^{\mathcal{K}}$ is defined by*

$$A \preceq_u^{\mathcal{K}} B \quad \Longleftrightarrow \quad A \subseteq \bigcup_{b \in B} (b - \mathcal{K}(b)).$$

*(iii) The **certainly lower less relation w.r.t.** $\mathcal{K}(\cdot)$ $\preceq_{cl}^{\mathcal{K}}$ is defined by*

$$A \preceq_{cl}^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \subseteq \bigcap_{a \in A} (a + \mathcal{K}(a)).$$

(iv) The ***certainly upper less relation w.r.t.*** *$\mathcal{K}(\cdot)$ $\preceq^{\mathcal{K}}_{cu}$ is defined by*

$$A \preceq^{\mathcal{K}}_{cu} B \quad \Longleftrightarrow \quad A \subseteq \bigcap_{b\in B}(b - \mathcal{K}(b)).$$

(v) The ***possibly lower less relation w.r.t.*** *$\mathcal{K}(\cdot)$ $\preceq^{\mathcal{K}}_{pl}$ is defined by*

$$A \preceq^{\mathcal{K}}_{pl} B \quad \Longleftrightarrow \quad B \cap \bigcup_{a\in A}(a + \mathcal{K}(a)) \neq \emptyset.$$

(vi) The ***possibly upper less relation w.r.t.*** *$\mathcal{K}(\cdot)$ $\preceq^{\mathcal{K}}_{pu}$ is defined by*

$$A \preceq^{\mathcal{K}}_{pu} B \quad \Longleftrightarrow \quad A \cap \bigcup_{b\in B}(b - \mathcal{K}(b)) \neq \emptyset.$$

Remark 2.2.6. *Observe that if for all $y \in Y$, $\mathcal{K}(y) = K$, where K is a convex pointed cone in Y, then the relations $\preceq^{\mathcal{K}}_{l}$ and $\preceq^{\mathcal{K}}_{u}$ reduce to the classical set relations introduced by (1.1) and (1.2).*

We derive the following proposition by directly using Definition 2.2.5.

Proposition 2.2.7. *Let $A, B \in \mathcal{P}(Y)$ and consider the relations (i)-(vi) given by Definition 2.2.5 . The following assertions hold true.*

(i) $A \preceq^{\mathcal{K}}_{u} B \quad \Longleftrightarrow \quad B \preceq^{-\mathcal{K}}_{l} A.$

(ii) $A \preceq^{\mathcal{K}}_{cu} B \quad \Longleftrightarrow \quad B \preceq^{-\mathcal{K}}_{cl} A.$

(iii) $A \preceq^{\mathcal{K}}_{pu} B \quad \Longleftrightarrow \quad B \preceq^{-\mathcal{K}}_{pl} A.$

(iv) $A \preceq^{\mathcal{K}}_{cl} B \quad \Longrightarrow \quad A \preceq^{\mathcal{K}}_{l} B \quad \Longrightarrow \quad A \preceq^{\mathcal{K}}_{pl} B.$

(v) $A \preceq^{\mathcal{K}}_{cu} B \quad \Longrightarrow \quad A \preceq^{\mathcal{K}}_{u} B \quad \Longrightarrow \quad A \preceq^{\mathcal{K}}_{pu} B.$

Proof. We present the proofs of part (i) and (iv) since the other cases can be done similarly.

(i) Suppose that $A \preceq^{\mathcal{K}}_{u} B$. This means

$$\begin{aligned} & A \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)) \\ \Longleftrightarrow\ & A \subseteq \bigcup_{b\in B}(b + (-\mathcal{K})(b)) \\ \Longleftrightarrow\ & B \preceq^{-\mathcal{K}}_{l} A. \end{aligned}$$

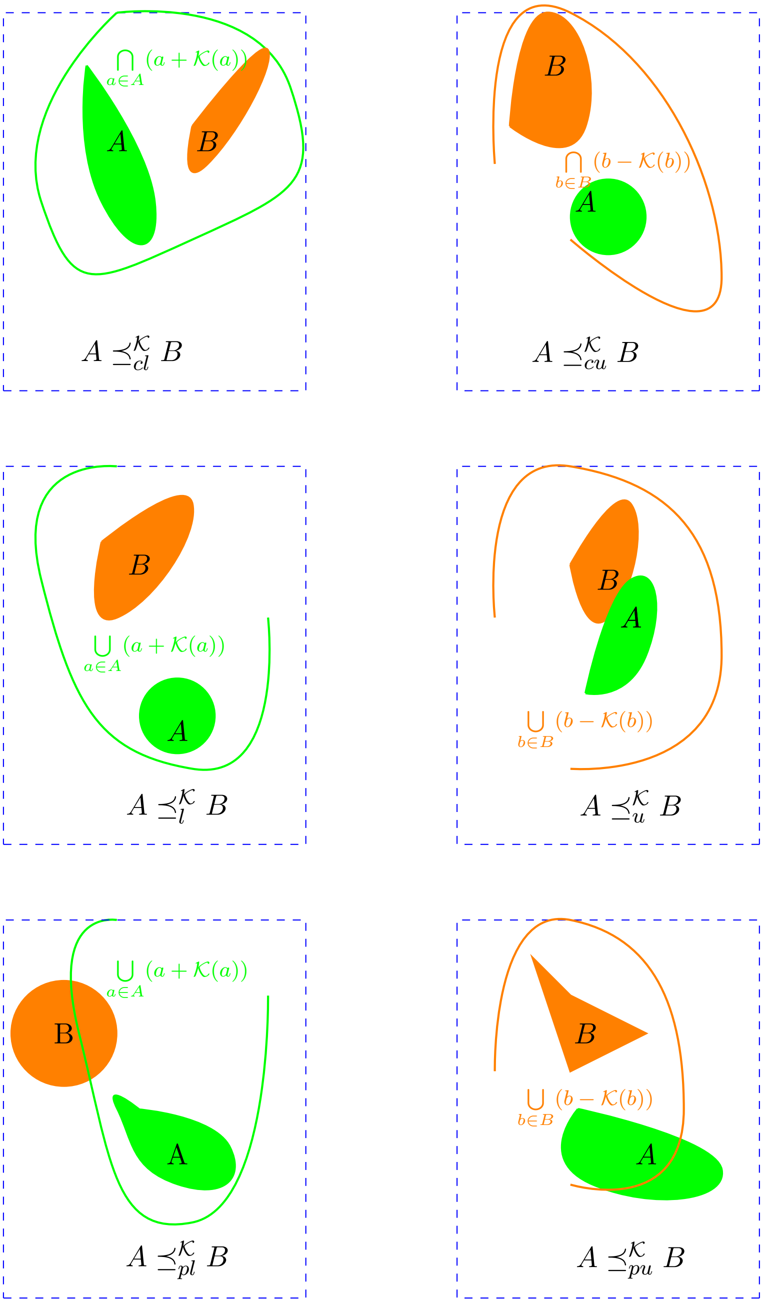


Figure 2.1: Set relations $\preceq^{\mathcal{K}}_t$, $t \in \{l, u, pl, pu, cl, cu\}$ in the sense of Definition 2.2.5.

Thus, $A \preceq_u^{\mathcal{K}} B \Longleftrightarrow B \preceq_l^{-\mathcal{K}} A$.

(iv) Assume that $A \preceq_{cl}^{\mathcal{K}} B$, i.e.,

$$\begin{aligned} & B \subseteq \bigcap_{a \in A} (a + \mathcal{K}(a)) \\ \Longrightarrow\ & B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)) \\ \Longleftrightarrow\ & A \preceq_l^{\mathcal{K}} B. \end{aligned}$$

Therefore, $A \preceq_{cl}^{\mathcal{K}} B \Rightarrow A \preceq_l^{\mathcal{K}} B$ holds true.

In addition, since $A \preceq_l^{\mathcal{K}} B$, i.e., $B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a))$ and $B \neq \emptyset$, it holds that

$$B \cap \bigcup_{a \in A} (a + \mathcal{K}(a)) \neq \emptyset.$$

This relation is equivalent to $A \preceq_{pl}^{\mathcal{K}} B$. The proof is complete. $\square$

Remark 2.2.8. *Each of the above relations has its own meaning in practical problems. For example, in uncertain optimization problems $\preceq_l^{\mathcal{K}}$ is used by a decision maker who is interested in minimizing the best case, and when the worst case is concerned, he will choose the relation $\preceq_u^{\mathcal{K}}$.*

In order to derive some properties of the relations given in Definition 2.2.5, some of the following properties of the domination structure $\mathcal{K} : Y \rightrightarrows Y$ will be used:

$$\forall y \in Y : 0 \in \mathcal{K}(y); \tag{2.9}$$

$$\forall y \in Y : \mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y); \tag{2.10}$$

$$\forall y \in Y, d \in \mathcal{K}(y) : \mathcal{K}(y + d) \subseteq \mathcal{K}(y); \tag{2.11}$$

$$\forall y \in Y, d \in \mathcal{K}(y) : \mathcal{K}(y - d) \subseteq \mathcal{K}(y); \tag{2.12}$$

$$\forall y \in Y : \mathcal{K}(y) \cap (-\mathcal{K}(y)) = \{0\}. \tag{2.13}$$

Obviously, if $\mathcal{K}(y)$ is a convex, pointed cone in Y, for all $y \in Y$ then $\mathcal{K}$ satisfies the properties (2.9), (2.10) and (2.13).

Remark 2.2.9. *The assumptions (2.10) and (2.11) of $\mathcal{K}(\cdot)$ can be fulfilled when $\mathcal{K}(y)$ is not necessarily given by a cone for all $y \in Y$. For instance, the mapping $\mathcal{K}(\cdot)$ given by*

$$\mathcal{K} : \ Y \rightrightarrows Y; \quad \mathcal{K}(y) = \mathbb{N}y, \ \textit{for all } y \in Y,$$

where $\mathbb{N}y := \{ny |\ n \in \mathbb{N}\}$ is not a cone.

The relations given in Definition 2.2.5 satisfy the following properties.

Proposition 2.2.10. *The following statements hold true:*

(i) If $\mathcal{K}(\cdot)$ satisfies property (2.9), then the binary relations $\preceq_l^{\mathcal{K}}$ and $\preceq_u^{\mathcal{K}}$ are reflexive.

(ii) If $\mathcal{K}(\cdot)$ satisfies properties (2.10) and (2.11), then the binary relations $\preceq_l^{\mathcal{K}}$ and $\preceq_{cl}^{\mathcal{K}}$ are transitive.

(iii) If $\mathcal{K}(\cdot)$ satisfies properties (2.10) and (2.12), then the binary relations $\preceq_u^{\mathcal{K}}$ and $\preceq_{cu}^{\mathcal{K}}$ are transitive.

(iv) If for all $y, z \in Y$, $\mathcal{K}(y) \cap \mathcal{K}(-z) = \{0\}$ then the binary relations $\preceq_{cl}^{\mathcal{K}}$ and $\preceq_{cu}^{\mathcal{K}}$ are antisymmetric.

Proof.

(i) Let $A \in \mathcal{P}(Y)$ arbitrary. Since for all $a \in A$, $0 \in \mathcal{K}(a)$, it holds that

$$A \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)) \text{ and } A \subseteq \bigcup_{a\in A}(a - \mathcal{K}(a)).$$

These two statements above are equivalent to $A \preceq_l^{\mathcal{K}} A$ and $A \preceq_u^{\mathcal{K}} A$, respectively. Therefore, $\preceq_l^{\mathcal{K}}$ and $\preceq_u^{\mathcal{K}}$ are reflexive.

(ii) Let $A, B, C \in \mathcal{P}(Y)$ such that $A \preceq_l^{\mathcal{K}} B$ and $B \preceq_l^{\mathcal{K}} C$. We will prove that $A \preceq_l^{\mathcal{K}} C$. Let $b \in B$ arbitrary. Since $A \preceq_l^{\mathcal{K}} B$, there exists $a_b \in A$ such that $b = a_b + d$, where $d \in \mathcal{K}(a_b)$. Taking into account (2.11), we get that $\mathcal{K}(b) = \mathcal{K}(a_b + d) \subseteq \mathcal{K}(a_b)$. This relation as well as $d \in \mathcal{K}(a_b)$ and (2.10) imply that

$$\begin{aligned} b + \mathcal{K}(b) = a_b + d + \mathcal{K}(a_b + d) &\subseteq a_b + \mathcal{K}(a_b) + \mathcal{K}(a_b) \\ &\subseteq a_b + \mathcal{K}(a_b). \end{aligned}$$

Therefore,

$$\bigcup_{b\in B}(b + \mathcal{K}(b)) \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

This, together with $B \preceq_l^{\mathcal{K}} C$, i.e., $C \subseteq \bigcup_{b\in B}(b + \mathcal{K}(b))$, shows that $C \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a))$, i.e., $A \preceq_l^{\mathcal{K}} C$.

The case of the relation $\preceq_{cl}^{\mathcal{K}}$ we can prove similarly.

(iii) We follow the same argument as in the proof of part (ii).

(iv) We prove for the relation $\preceq_{cl}^{\mathcal{K}}$ and the relation $\preceq_{cu}^{\mathcal{K}}$ can be derived similarly.

Let $A, B \in \mathcal{P}(Y)$ such that $A \preceq_{cl}^{\mathcal{K}} B$ and $B \preceq_{cl}^{\mathcal{K}} A$. We will prove that $A = B$. Take $\bar{a} \in A$ and $\bar{b} \in B$ arbitrary. Since $A \preceq_{cl}^{\mathcal{K}} B$, it holds that for all $a \in A$, $\bar{b} \in a + \mathcal{K}(a)$. This implies

$$\bar{b} \in \bar{a} + \mathcal{K}(\bar{a}). \tag{2.14}$$

Furthermore, because $B \preceq_{cl}^{\mathcal{K}} A$, it holds that $\bar{a} \in b + \mathcal{K}(b)$ for all $b \in B$. Then, we get that

$$\bar{a} \in \bar{b} + \mathcal{K}(\bar{b}). \tag{2.15}$$

Since (2.14) and (2.15), it holds that

$$\bar{b} - \bar{a} \in \mathcal{K}(\bar{a}) \cap (-\mathcal{K}(\bar{b})) = \{0\}.$$

This implies $\bar{a} = \bar{b}$ and thus $A = B$. The proof is complete. □

Remark 2.2.11. *Proposition 2.2.10 extends [39, Lemma 2.1], where $\mathcal{K}$ is a cone-valued map. Moreover, the necessary condition for the antisymmetry of the relation $\preceq^{\mathcal{K}}_{cl}$ and $\preceq^{\mathcal{K}}_{cu}$ in [39, Lemma 2.1] is replaced by $\mathcal{K}(Y) \cap (-\mathcal{K}(Y)) = \{0\}$, where $\mathcal{K}(Y) := \bigcup_{y \in Y} \mathcal{K}(y)$.*

Remark 2.2.12. *From now on, we denote by $\preceq^{\mathcal{K}}_{t}$ one of the relations $(i) - (vi)$ given in Definition 2.2.5, $t \in \{l, u, cl, cu, pl, pu\}$. If $A, B \in \mathcal{P}(Y)$ such that $A \preceq^{\mathcal{K}}_{t} B$ and $B \preceq^{\mathcal{K}}_{t} A$, we will write $A \sim B$. In addition, the set of all elements $B \in \mathcal{P}(Y)$ such that $B \sim A$ is denoted by $[A]$. Obviously, if $\preceq^{\mathcal{K}}_{t}$ is reflexive, then $A \sim A$ and $A \in [A]$.*

2.3 Concepts of generalized Differentiation

2.3.1 Subdifferentials of Extended Real-valued Convex Functions

In this section, we recall some concepts on subdifferentials of convex functionals as well as subdifferentials of cone-convex vector-valued functions. We suppose in this part that X, Y are Banach spaces and denote the linear space of the continuous linear maps from X to Y by $L(X, Y)$. Let $F : X \rightrightarrows Y$ be a set-valued mapping. As usual, we denote the graph and the domain of F by $\operatorname{Gr} F$ and $\operatorname{Dom} F$, respectively. They are defined as follows:

$$\operatorname{Dom} F := \{x \in X |\ F(x) \neq \emptyset\},$$

and

$$\operatorname{Gr} F := \{(x, y) \in X \times Y |\ y \in F(x)\}.$$

When Y is equipped with the binary relation $\leq^C$, where C is a convex cone in Y we define an element $+\infty_Y$ such that for all $y \in Y$ it holds that $y \leq^C +\infty_Y$. For the case of a vector-valued function $f : X \to Y \cup \{\pm\infty_Y\}$, we denote the domain of f by $\operatorname{dom} f := \{x \in X |\ f(x) \in Y \cup \{-\infty_Y\}\}$. The function f is called **proper** if $\operatorname{dom} f \neq \emptyset$ and $f(x) \in Y \cup \{+\infty_Y\}$ for all $x \in X$. Now we recall the definition of directional derivative in order to define the subdifferential of convex functionals, especially of the norm.

Definition 2.3.1. *Consider a function $f : X \to \mathbb{R}$. For each $\bar{x} \in X$, the **Gateaux derivative** of f in the direction $h \in X$ is denoted by $f'(\bar{x}, h)$ and defined as*

$$f'(\bar{x}, h) := \lim_{t \to 0} \frac{1}{t}(f(\bar{x} + th) - f(\bar{x})),$$

when the limit exists in $\overline{R}$.
If the limitation

$$f'_+(\bar{x}, h) := \lim_{t \to +0} \frac{1}{t}(f(\bar{x} + th) - f(\bar{x})$$

exists, then $f'_+(\bar{x}, h)$ is called the ***right-hand side directional derivative*** *of f in the direction h. Similarly, when taking $t \to 0^-$ in the limit, we have the definition of the* ***left-hand side directional derivative****.*

Obviously, if f is *Gateaux* differential, then it is right-hand side differential and left-hand side differential and $f'(\bar{x}, h) = f'_+(\bar{x}, h) = f'_-(\bar{x}, h)$. In addition, the function f is left-hand side differentiable at $\bar{x}$ in direction h, if and only if it is right-hand side differentiable at $\bar{x}$ in direction $-h$ and the equality $f'_-(\bar{x}, h) = f'_+(\bar{x}, -h)$ always holds.

The existence of right-hand side (left-hand side) directional derivative is given by [97] as follows:

Proposition 2.3.2. *[97, Lemma 1.2] Let X be a Banach space. If $\varphi : X \to \mathbb{R}$ is a convex function, then the right-hand side (left-hand side) directional derivative of φ exists at every point $x \in \operatorname{dom} \varphi$.*

In the following, we consider the definition of subdifferential in the sense of convex analysis (or Fenchel subdifferential) of a convex functional.

Definition 2.3.3. *Let X be a Banach space, $\varphi : X \to \overline{\mathbb{R}}$ be a proper convex function and $\bar{x} \in X$ be such that $\varphi(\bar{x}) \in \mathbb{R}$. The subdifferential or* ***Fenchel subdifferential*** *of φ at $\bar{x}$ is denoted by $\partial\varphi(\bar{x})$ and defined as*

$$\partial\varphi(\bar{x}) := \{x^* \in X^* | \ \forall x \in X : \varphi(x) - \varphi(\bar{x}) \geq x^*(x - \bar{x})\}.$$

If $|\varphi(\bar{x})| = +\infty$ one puts $\partial\varphi(\bar{x}) = \emptyset$. The function φ is said to be subdifferentiable at $\bar{x}$ if the set $\partial\varphi(\bar{x})$ is nonempty.

It follows from Definition 2.3.3 that $\varphi'_+(\bar{x}, \cdot) \in \partial\varphi(\bar{x})$. Thus, since Proposition 2.3.2, $\partial\varphi(\bar{x}) \neq \emptyset$ provided that φ is a proper convex functional and $|\varphi(\bar{x})| < +\infty$.

Some calculus rules for subdifferentials of convex functions are presented as follows:

Proposition 2.3.4. *[115, Theorem 2.4.2] Let $\varphi, \xi : X \to \overline{\mathbb{R}}$ be proper convex functionals on X, $x \in X$. The following assertions hold true*

(i) For any scalar λ, we have $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$.

(ii) $\partial\varphi(x) + \partial\xi(x) \subseteq \partial(\varphi + \xi)(x)$.

The equality holds if $x \in \operatorname{dom} \varphi \cap \operatorname{dom} \xi$ and one of the functionals is continuous.

We now recall the definition of a cone-convex function.

Definition 2.3.5. *Let X, Y be topological linear spaces and $C \subseteq Y$ be a proper convex cone. A function $f : X \to Y$ is said to be C-**convex** if for all $\lambda \in [0,1]$ and for all $x, y \in X$ it holds that*

$$f(\lambda x + (1-\lambda)y) \in \lambda f(x) + (1-\lambda)f(y) - C.$$

Now we define the subdifferential of a proper vector-valued function $f : X \to Y$.

Definition 2.3.6. *(Subdifferential of vector-valued function, [54]) Let X and Y be Banach spaces, $C \subset Y$ be a proper, pointed, convex cone in Y, and $f : X \to Y$ be a C-convex function. For an arbitrary $\bar{x} \in X$ and the binary relation $\leq^C$ given by (2.4), the set*

$$\partial^{\leq^C} f(\bar{x}) := \{T \in L(X,Y) |\ \forall h \in X : T(h) \leq^C f(\bar{x}+h) - f(\bar{x})\} \tag{2.16}$$

*is called the **subdifferential** of f at $\bar{x}$.*

It is obvious that $\partial^{\leq^C} f(\bar{x})$ is a convex subset of $L(X,Y)$. If C is pointed, and $f : X \to Y$ is a sublinear operator, i.e, $f(x_1) + f(x_2) \in f(x_1 + x_2) + C$, $f(0) = 0$, and $f(\alpha x) = \alpha f(x)$ for all $x_1, x_2, x \in X$ and $\alpha \in (0, \infty)$, the following formula holds true for all $x_0 \in \operatorname{dom} f$:

$$\partial^{\leq^C} f(x_0) = \{T \in \partial f(0) |\ T(x_0) = f(x_0)\}.$$

In the case that $Y = \mathbb{R}$, $C = \mathbb{R}_+$ and f is convex, (2.16) reduces to the Fenchel subdifferential given in Definition 2.3.3.

The following result concerns the nonempty property of subdifferential in the sense of Definition 2.16.

Proposition 2.3.7. *[63, Corollary 6.1.10] Let $f : X \to Y$ be a proper C-convex function, and $\bar{x}$ be a given point in $\operatorname{int}(\operatorname{dom} f)$. Then, $\partial^{\leq^C} f(\bar{x})$ is nonempty.*

We recall in the following the subdifferential of some special kinds of vector-valued functions.

Remark 2.3.8. *Let X, Y be Banach spaces, $C \subseteq Y$ be a pointed, closed, convex cone and $\leq^C$ be given by (2.4).*

(i) ([54, Example 2.22]) For the vector-valued norm function $\|\cdot\| : X \to Y$ given in Definition 2.1.9 and $\bar{x} \in X$, we have :

$$\partial^{\leq^C} \|\bar{x}\| = \{T \in L(X,Y) |\ T(\bar{x}) = \|\bar{x}\| \text{ and for all } x \in X : T(x) \leq^C \|\bar{x}\|\},$$

where C is a proper, pointed, closed, convex cone in Y.

(ii) ([44, Theorem 4.1.12]) Let $A \in L(X,Y)$ and A^* denotes the adjoint operator to A, $a \in Y$, $\bar{x} \in X$. Then,

$$\partial\|A(\cdot) - a\|(\bar{x}) = \{A^*T|\ T \in L(Y,\mathbb{R}),\ T(A\bar{x} - a) = \|A\bar{x} - a\| \text{ and } \|T\|_* \leq 1\},$$

where $\|\cdot\|$ is a norm in Y.

In Chapter 8, we will formulate a beam intensity problem as a vector problem whose objective function is a vector-valued function. In addition, we will take into account Remark 2.3.8(ii) to derive necessary optimality conditions for this problem.

2.3.2 Limiting Normal Cones, Coderivatives and Subdifferentials

In this section, let X, Y be Banach spaces. For $x \in X$, we denote the system of the neighborhoods of x by $\mathcal{V}(x)$. Let $F : X \rightrightarrows Y$ be a set-valued mapping. If $S \subseteq X$, we denote the image of S under F by $F(S) := \bigcup\limits_{x \in S} F(x)$ and the inverse set-valued mapping of F is $F^{-1} : Y \rightrightarrows X$ given by $(y,x) \in \operatorname{Gr} F^{-1}$ if and only if $(x,y) \in \operatorname{Gr} F$. In what follows, we introduce the definition of the lower semicontinuous property of a set-valued mapping which is utilized in the sum rule of limiting subdifferential. Furthermore, the lower semicontinuous property is also beneficial for Chapter 6 to find optimality conditions for solutions of a set-valued problem.

Definition 2.3.9. *(Lower semicontinuous mapping) The set-valued mapping $F : X \rightrightarrows Y$ is called **lower semicontinuous** (l.s.c., for short) at $\bar{x} \in \operatorname{Dom} F$ if for every sequence $\{x_n\} \to \bar{x}$ there exists a sequence $\{y_n\} \to \bar{y}$ with $y_n \in F(x_n)$ for every n. The set-valued mapping F is called l.s.c. if it is l.s.c. at every point $\bar{x} \in \operatorname{Dom} F$.*

Remark 2.3.10. *As shown in [1, Pages 40 and 42], F is l.s.c. if and only if the inverse image of any open subset is open. Moreover, if $\operatorname{Dom} F$ is closed, then F is l.s.c. if and only if the core of any closed subset is closed. We also have that F is l.s.c. at $x \in \operatorname{Dom} F$ if and only if $F(x) \subseteq \liminf\limits_{x' \to x} F(x')$.*

We recall in the following the definition of the Lipschitzianity of a vector-valued mapping. This property ensures calculus rules for the limiting subdifferential of locally Lipschitz functions on Asplund spaces. In addition, it will be used to derive the relationship between coderivative of a vector-valued function and subdifferential of its scalarization in the next part.

Definition 2.3.11. *(Lipschitz and Strictly Lipschitz function, [90]) Let $f : X \to Y$ be a vector-valued function between Banach spaces.*

*(i) f is **Lipschitz** on $U \subset X$ if $U \subset \operatorname{dom} f$ and there exists $l \geq 0$ such that*

$$\forall x, x' \in U : \ \|f(x) - f(x')\|_Y \leq l\|x - x'\|_X.$$

(ii) *f is said to be* ***Lipschitz around*** *$x \in X$ if there is a neighbourhood U_x of x such that f is Lipschitz on U_x.*

(iii) *f is said to be* ***locally Lipschitz*** *on a nonempty subset D of X, if f is Lipschitz around every point $x \in D$.*

(iv) *Suppose that f is Lipschitz continuous around $\bar{x} \in X$, then f is* ***strictly Lipschitz*** *at $\bar{x}$ if there is a neighborhood V of the origin in X such that the sequence*

$$y_k := \frac{f(x_k + t_k v) - f(x_k)}{t_k}, k \in \mathbb{N},$$

contains a norm convergent subsequence whenever $v \in V, x_k \to \bar{x}$ and $t_k \to 0$.

(v) *Suppose that f is Lipschitz continuous around $\bar{x}$, then f is f is* ***ω^*-strictly Lipschitz*** *at $\bar{x}$ if there is a neighborhood V of the origin in X such that for any $v \in X$ and any sequences $\{x_k\} \to \bar{x}$, $t_k \downarrow 0$, and $\{y_k^*\} \xrightarrow{w^*} 0$ one has $y_k^*(y_k) \to 0$ as $k \to \infty$, where $\{y_k\}$ are defined in (iv).*

It is shown in [90] that when Y is finite-dimensional, both (*iv*) and (*v*) of the Definition 2.3.11 reduce to the class of locally Lipschitz function $f : X \to \mathbb{R}^n$. It is known that every scalar, proper convex function in a finite-dimensional normed linear space is Lipschitz around any interior point of its domain, see [98, Theorem 10.4] . In addition, Tan et al. [86] proved that a convex vector function from a convex subset D of $\mathbb{R}^m$ to $\mathbb{R}^n$ is locally Lipschitz on $\operatorname{rint} D$. The following result shows the Lipschitz property of a norm-vector function.

Proposition 2.3.12. *([104, Lemma 5]) We assume that C is a proper, normal cone. If the vector-valued norm $|||\cdot||| : X \to C$ is continuous around a given point $x \in X$, then $|||\cdot|||$ is Lipschitz.*

Obviously, $\mathbb{R}^n_+$ is a closed and pointed cone, then Proposition 2.3.12 holds true for $C = \mathbb{R}^n_+$.

In Chapter 6, we will derive necessary optimality conditions of set-valued optimization problems by means of Mordukhovich's coderivative (see the book [90] by Mordukhovich for more details). Now, we present the main objects which will be used in the sequel.

Definition 2.3.13. *[90] Let Ω be a nonempty subset of a normed space X and let $\bar{x}$, $x \in \Omega$, $\epsilon \geq 0$.*

(i) *The set of ϵ-normals to Ω at x is defined by*

$$\hat{N}_\epsilon(\Omega, x) := \left\{ x^* \in X^* | \limsup_{u \xrightarrow{\Omega} x} \frac{x^*(u - x)}{\|u - x\|} \leq \epsilon \right\}, \tag{2.17}$$

where $u \xrightarrow{\Omega} x$ means that $u \to x$ and $u \in \Omega$. If $\epsilon = 0$, we call elements of (2.17) ***Fréchet normals*** *and their collection, denoted by $\hat{N}(\Omega, x)$, is the* ***Fréchet normal cone*** *to Ω at x.*

(ii) *The basic (or limiting, or Mordukhovich)* ***normal cone*** *to Ω at $\bar{x}$ is defined as*

$$N(\Omega, \bar{x}) := \{x^* \in X^* |\ \exists\ \epsilon_n \downarrow 0,\ x_n \xrightarrow{\Omega} \bar{x},\ x_n^* \xrightarrow{w^*} x^*,\ x_n^* \in \hat{N}_{\epsilon_n}(\Omega, x_n),\ \forall n \in \mathbb{N}\},$$

where $x_n^* \xrightarrow{w^*} x^*$ means that for all $x \in X$ we have that $x_n^*(x) \to x^*(x)$.

Remark 2.3.14. *[90] If X is an Asplund space and Ω is closed around $\bar{x}$ (i.e., there is a neighborhood V of $\bar{x}$ such that $\Omega \cap \operatorname{cl} V$ is closed), the formula for the basic normal cone looks as follows:*

$$N(\Omega, \bar{x}) = \{x^* \in X^* \mid \exists\ x_n \xrightarrow{\Omega} \bar{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(\Omega, x_n), \forall n \in \mathbb{N}\}.$$

The following definition presents concepts of coderivatives correspondingly to the concepts of normal cones for set-valued mappings.

Definition 2.3.15. *Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$.*

(i) *The* ***Fréchet coderivative*** *of F at $(\bar{x}, \bar{y})$ is the set-valued mapping $\hat{D}^*F(\bar{x}, \bar{y})$: $Y^* \rightrightarrows X^*$ defined by*

$$\hat{D}^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* |\ (x^*, -y^*) \in \hat{N}(\operatorname{Gr} F, (\bar{x}, \bar{y})\}.$$

(ii) *The* ***normal coderivative*** *of F at $(\bar{x}, \bar{y})$ is the set-valued mapping $D^*F(\bar{x}, \bar{y})$: $Y^* \rightrightarrows X^*$ given by*

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* |\ (x^*, -y^*) \in N(\operatorname{Gr} F, (\bar{x}, \bar{y})\}.$$

In the following, we recall the definition of limiting subdifferential, which becomes the subdifferential of convex analysis when the function is convex, see [90, Theorem 1.93].

Definition 2.3.16. *[90, Definition 1.77] The (basic, limiting, Mordukhovich) subdifferential for a given function $\varphi : X \to \overline{\mathbb{R}}$ at $\bar{x} \in X$ with $|\varphi(\bar{x})| < +\infty$ is defined by*

$$\partial_L \varphi(\bar{x}) := \{x^* \in X^* |\ (x^*, -1) \in N((\bar{x}, \varphi(\bar{x}), \operatorname{epi} \varphi)\};$$

where $\operatorname{epi} \varphi := \{(x, r) \in X \times \overline{\mathbb{R}} :\ \varphi(x) \le r\}$. If $|\varphi(\bar{x})| = +\infty$, we put $\partial_L \varphi(\bar{x}) = \emptyset$.

Now, we recall some calculus rules for the limiting subdifferential of locally Lipschitz functions on Asplund spaces (see Section 2.1.1). For the proof, we refer the reader to [90, Theorems 3.36, 3.41 and Corollary 3.43].

Proposition 2.3.17. *[90] Let X, Y be Asplund spaces.*

(i) (sum rule) Let $\varphi_i : X \to \overline{\mathbb{R}}$, $i = 1, 2 \dots n, n \geq 2$ be l.s.c. around $\bar{x}$ and let all but one of these functions be locally Lipschitz around $\bar{x}$. Then,

$$\partial_L(\varphi_1 + \varphi_2 + \dots \varphi_n)(\bar{x}) \subseteq \partial_L \varphi_1(\bar{x}) + \dots + \partial_L \varphi_n(\bar{x}).$$

The equality holds if each φ_i is convex (or strictly differentiable).

(ii) (chain rule) Let $\xi : X \to Y$ be strictly Lipschitz at $\bar{x}$, and $\varphi : Y \to \mathbb{R}$ be locally Lipschitz around $\xi(\varphi(\bar{x}))$. Then, one has

$$\partial_L(\varphi \circ \xi)(\bar{x}) \subseteq \bigcup_{y^* \in \partial_L \varphi(\xi(\bar{x}))} \partial_L(y^* \circ \xi)(\bar{x}).$$

When a vector-valued mapping is ω^*-strictly Lipschitz, we have the following relationship between its coderivative and the subdifferential of its scalarization.

Proposition 2.3.18. *([90, Theorem 3.28]) Let f be a mapping $f : X \to Y$ between an Asplund space X and a Banach space Y. Then, for all $y^* \in Y^*$ it holds that*

$$D^* f(\bar{x})(y^*) = \partial_L(y^* \circ f)(\bar{x}) \neq \emptyset$$

provided that f is ω^*-strictly Lipschitz at $\bar{x}$.

In order to provide specific optimality conditions for solutions of a beam intensity problem which will be studied in Chapter 8, we recall some results of normal cone to some special sets in $\mathbb{R}^n$. These results are given by Rockafelar and Wet in [99], so we omit their proofs in this thesis.

Proposition 2.3.19. *(Normal cones to product sets, [99, Theorem 6.41])*
Let C_i be closed subset of $\mathbb{R}^{n_i}, i = 1, .., k$ and $\mathbb{R}^n = \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$. If $C = C_1 \times ... \times C_k$, then at any $\bar{x} = (\bar{x}_1, ..., \bar{x}_k), \bar{x}_i \in C_i$, it holds that

$$N(C, \bar{x}) = N(C_1, \bar{x}_1) \times ... \times N(C_k, \bar{x}_k).$$

Proposition 2.3.20. *(Normal cones to boxes, [99, Example 6.10])*
Assume that $C = C_1 \times ... \times C_n$ in which C_i is a closed interval in $\mathbb{R}$. Then, at any $\bar{x} = (\bar{x}_1, \bar{x}_2, .., \bar{x}_n) \in C$, one has

$$N(C, \bar{x}) = N(C_1, \bar{x}_1) \times ... \times N(C_n, \bar{x}_n),$$

where

$$N(C_i, \bar{x}_i) = \begin{cases} [0, \infty) & \text{if } \bar{x}_i \text{ is (only) the right end point of } C_i, \\ (-\infty, 0] & \text{if } \bar{x}_i \text{ is (only) the left end point of } C_i, \\ \{0\} & \text{if } \bar{x}_i \text{ is an interior point of } C_i, \\ (-\infty, \infty) & \text{if } C_i \text{ is a one-point interval.} \end{cases}$$

2.3.3 Sequentially Normally Compactness and Openness

We begin this section by presenting two local properties of sets in Banach spaces and in Asplund spaces which ensure the equivalence between the weak* and norm convergence to zero of the ϵ-normals and the Fréchet normals introduced in Definition 2.3.13. These properties are automatic in finite-dimensional spaces while they are unavoidably needed in infinite dimensions because of the natural lack of compactness therein, see [6, 47, 113, 114]. In addition, they are included in the assumptions of calculus rules for various operations on sets and mappings.

Definition 2.3.21. *(i) Let X be a Banach space and $\Omega \subseteq X$ be a nonempty set. Then, Ω is said to be* ***sequentially normally compact*** *(SNC) at $\bar{x} \in \Omega$ if for any sequence $\{(\epsilon_n, x_n, x_n^*)\}$:*

$$[\epsilon_i \to 0^+, x_n \xrightarrow{\Omega} \bar{x},\ x_n^* \xrightarrow{w^*} 0,\ x_n^* \in \hat{N}_{\epsilon_n}(\Omega, x_n)] \Longrightarrow (x_n^* \to 0).$$

(ii) Let X be Asplund. A nonempty set $\Omega \subseteq X$ is said to be SNC at $\bar{x} \in \Omega$, if for any sequence $\{(x_n, x_n^*)\}$:

$$[x_n \xrightarrow{\Omega} \bar{x},\ x_n^* \xrightarrow{w^*} 0,\ x_n^* \in \hat{N}(\Omega, x_n)] \Longrightarrow (x_n^* \to 0).$$

It follows from Definition 2.3.21 that Ω is SNC at $\bar{x}$ if its closure is SNC at this point. In addition, every closed and convex cone Q with nonempty interior is SNC at 0 and every nonempty set in a finite-dimensional space is SNC at each of its points, see [90]. The corresponding property for a set-valued mapping $F : X \rightrightarrows Y$ is induced naturally by the concepts of sequential normal compactness of a set, that is, F is said to be sequentially normally compact (SNC) at $(\bar{x}, \bar{y})$ if its graph is SNC at this point, see [91]. However, the case of mappings allows us to consider also a weaker property defined as follows:

Definition 2.3.22. *[90] Let X, Y be Banach spaces, $\mathcal{Q} : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \operatorname{Gr} \mathcal{Q}$ be given. The map $\mathcal{Q}$ is said to be* ***partially sequentially normally compact*** *(PSNC) at $(\bar{x}, \bar{y})$, if for any sequence $\{(x_n, y_n, x_n^*, y_n^*)\}$:*

$$[(x_n, y_n) \xrightarrow{\operatorname{Gr} \mathcal{Q}} (\bar{x}, \bar{y}), x_n^* \xrightarrow{w^*} 0, y_n^* \to 0, (x_n^*, y_n^*) \in \hat{N}(\operatorname{Gr} \mathcal{Q}, (x_n, y_n))] \Rightarrow (x_n^* \to 0).$$

It is stated in [30] that if a set-valued mapping $\mathcal{Q} : X \rightrightarrows Y$ satisfying $\mathcal{Q}(x) = C$ with C being a closed, convex cone for all $x \in X$ is considered, then the (SNC) property of C at 0 is the (PSNC) property of $\mathcal{Q}^{-1}$ at $(0, \bar{x}) \in \operatorname{Gr} \mathcal{Q}^{-1}$. This could be obtained for instance when $\operatorname{int} C \neq \emptyset$ holds true. In addition, the PSNC property of a set-valued mapping $F : X \rightrightarrows Y$ is implied when F is Lipschitz-like (or Aubin). Recall that F is **Lipschitz-like** around $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ with some modulus $\gamma > 0$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x) \cap V \subseteq F(x') + \gamma\|x - x'\|B_Y, \ \forall x, x' \in U.$$

The Lipschitz-like property is fundamental in Nonlinear Analysis and Variational Analysis and it has some important relations with the linear openness which is defined as follows:

Definition 2.3.23. *$F : X \rightrightarrows Y$ is said to be* ***open at linear rate*** *$L > 0$ (or $L-$open) around $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ if there exist two neighborhoods $U \in \mathcal{V}(\bar{x}), V \in \mathcal{V}(\bar{y})$ and a positive number $\varepsilon > 0$ such that, for every $(x, y) \in \operatorname{Gr} F \cap (U \times V)$ and every $\rho \in (0, \varepsilon)$,*

$$B_Y(y, \rho L) \subseteq F(B_X(x, \rho)).$$

It is necessary to mention that the openness at linear rate is a stronger property than the openness. We say that F is open at $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ if the image through F of every neighborhood of $\bar{x}$ is a neighborhood of $\bar{y}$. The link between the Lipschitz-like and linear openness is illustrated by the following theorem (see, e.g., [53, Chap. 1], [90, Theorem 1.52] and [99, Theorem 9.43]).

Theorem 2.3.24. *Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$. Then, F is open at linear rate around $(\bar{x}, \bar{y})$ if and only if F^{-1} is Lipschitz-like at $(\bar{y}, \bar{x})$.*

The incompatibility between openness and optimality is a main tool for the proof of the necessary optimality conditions in terms of Mordukhovich's coderivative. In [27], the authors presented a new technique to obtain a proof of the openness at linear rate of composite set-valued mappings. In order to show necessary optimality conditions in terms of Mordukhovich's coderivative in Chapter 6, we need certain assumptions on the alliedness (or transversality) of sets, see [74, 83, 96] and the references therein for more details. This property is essential for the validity of qualification conditions in optimization as well as subdifferential, normal cone and coderivative calculus.

Definition 2.3.25. *(Allied sets) Let $S_1, S_2, ..., S_k$ be closed subsets of a normed vector space Z, $\bar{z} \in \bigcap_{i=1}^k S_i$. One says that they are* ***allied*** *at $\bar{z}$ whenever $\{z_{in}\} \subset S_i$, $\{z_{in}\} \to \bar{z}$, $z_{in}^* \in \hat{N}(S_i, z_{in})$, the relation $\sum_{i=1}^k z_{in}^* \to 0$ implies $\{z_{in}^*\} \to 0$ for all $i = 1, \ldots, k$.*

Notice that Definition 2.3.25 is equivalent to the definition of η-regularity introduced and characterized in [74, Definition 7, Proposition 10]: The sets $S_1, S_2, \ldots, S_k$ are η-regular at $\bar{z} \in S_1 \cap \cdots \cap S_k$ if there exist $\gamma,\ \delta > 0$ such that

$$\|\sum_{i=1}^k z_i^*\| \geq \gamma \sum_{i=1}^k \|z_i^*\|,$$

for every $z_i \in B(\bar{z}, \delta) \cap S_i$, $z_i^* \in \hat{N}(S_i, z_i)$, $i = 1, \ldots, k$. In addition, these notions also imply the metric inequality of $(S_1, \ldots, S_k)$ at $\bar{z}$ [27, Theorem 4.1], which is used as a main tool to establish chain rules for the limiting Fréchet subdifferentials [93].

In order to derive optimality conditions for the set optimization problem ($P_{\mathcal{Q}}$), one needs the alliedness property of the two following sets where the objective map F and the domination $\mathcal{Q}$ in ($P_{\mathcal{Q}}$) are involved (see Theorem 6.1.3):

$$C_1 := \{(x, y, k) | \ (x, y) \in \operatorname{Gr} F, \ k \in Y\},$$
$$\text{and} \quad C_2 := \{(x, y, k) | \ (x, k) \in \operatorname{Gr} \mathcal{Q}, \ y \in Y\}.$$

2.4 Nonlinear Scalarizing Functionals

In this part, we recall the well known Gerstewitz functional and some important properties of this functional. This functional will be used in Chapters 5, 7 and 8 to characterize solutions for set-optimization problems w.r.t. variable domination structures.

Let Y be a topological linear space. Let $D \subset Y$ be a proper closed set, and $k \in Y \setminus \{0\}$ satisfying

$$D + [0, +\infty)k \subseteq D. \tag{2.18}$$

We introduce the functional $z^{D,k} \colon Y \to \overline{\mathbb{R}}$ given by

$$z^{D,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - D\}, \tag{2.19}$$

where we use the convention that $\inf \emptyset = +\infty$. The functional $z^{D,k}$ assigns the smallest value t such that the property $y \in tk - D$ is fulfilled. The scalarizing functional $z^{D,k}$ was introduced by Gerstewitz (1983) [42] and used in [43] to prove separation theorems for not necessarily convex sets. This functional is intensively used in the so-called ϵ-constraint method, see [24, 34, 87]. In addition, Pascoletti-Serafini scalarization, where two parameters are allowed to vary arbitrarily, is a special case of this functional, see [34, 95]. Properties of $z^{D,k}$ were studied in [43, 44, 107]. First, let us recall the definition of C-monotonicity of a functional.

Definition 2.4.1. *Let Y be a topological linear space, $C \subset Y$, $C \neq \emptyset$. A functional $z \colon Y \to \overline{\mathbb{R}}$ is C-**monotone**, if the following implication holds*

$$\forall y_1, \ y_2 \in Y: \ y_1 \in y_2 - C \Longrightarrow z(y_1) \leq z(y_2) \, .$$

Below we provide some properties of the functional $z^{D,k}$ introduced in (2.19).

Theorem 2.4.2. *[43, 44] Let Y be a topological linear space, $C \subset Y$, $D \subset Y$ be a proper closed set, and let $k \in Y \setminus \{0\}$ be such that (2.18) is fulfilled. Then, the following properties hold for $z = z^{D,k}$:*

(a) z *is l.s.c.*

In addition, if $k \in Y \setminus \{0\}$ *such that* $D + (0, +\infty)k \subseteq \operatorname{int} D$*, then* z *is continuous.*

(b) z *is convex* $\Longleftrightarrow$ D *is convex,*

(c) z *is proper* $\Longleftrightarrow$ D *does not contain lines parallel to* k*, i.e.,* $\forall\, y \in Y,\ \exists\, r \in \mathbb{R}:\ y + rk \notin D$.

(d) z *is* C*-monotone* $\Longleftrightarrow$ $D + C \subset D$.

(e) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y) \leq r \Longleftrightarrow y \in rk - D$.

(f) z *is finite-valued* $\Longleftrightarrow$ D *does not contain lines parallel to* k *and* $\mathbb{R}k - D = Y$.

For the proof of Theorem 2.4.2, see [44, Theorem 2.3.1]. In the following, we recall the exact subdifferential formulas for the functional $z^{D,k}$. The first one is given in [31] where the authors computed the classical Fenchel sundifferential of $z^{D,k}$ under the assumption that D is a proper, closed, convex set. In addition, [31] also derived fuzzy optimality conditions for vectorial location problems and discussed an application of the function $z^{D,k}$ in stochastic finance.

Proposition 2.4.3. *[31, Theorem 2.2] Let* $D \subset Y$ *be a closed, convex set and* $k \in Y \setminus \{0\}$ *satisfying* (2.18) *and for every* $y \in Y$*, there is* $t \in \mathbb{R}$ *such that* $y + tk \notin D$*. Consider the functional* $z^{D,k}$ *given by* (2.19) *and let* $\bar{y} \in \operatorname{dom} z^{D,k}$*. Then,*

$$\partial z^{D,k}(\bar{y}) = \{y^* \in Y^* |\ y^*(k) = 1 \ \textit{and}\ \forall d \in D:\ y^*(d) + y^*(\bar{y}) - z^{D,k}(\bar{y}) \geq 0\}.$$

Further more, in the case that D does not enjoy conical or convex properties, [3] obtained the following result.

Proposition 2.4.4. *[3, Proposition 3.1] Let* Y *be Asplund space,* $D \subset Y$ *be a closed set and* $k \in Y \setminus \{0\}$ *sastisfying*

$$D + (0, +\infty)k \subseteq \operatorname{int} D$$

and $y \in \operatorname{dom} z^{D,k} = \mathbb{R}k - D$*. Then, we have:*

$$\partial_L z^{D,k}(y) = \{y^* \in Y^* |\ y^*(k) = 1 \ \textit{and}\ -y^* \in N(\operatorname{bd} D, z^{D,k}(y)k - y)\}.$$

There are several papers generating the functional $z^{D,k}$ for set optimization w.r.t. a fixed cone and for vector optimization equipped with variable domination structures. For more detail and further discussions, we refer the reader to [3, 16, 19, 45, 46, 72]. In Chapter 5, we will introduce some appropriated functionals based on $z^{D,k}$ in order to characterize minimizers of a family of set as well as minimal solutions of problem $(P_{\mathcal{K}})$.

Chapter 3

Vector Optimization w.r.t. Variable Domination Structures

Recently, vector optimization problems with variable ordering structures are studied intensively in the literature, see for instance, [7, 19, 35, 36, 37, 101] and references therein. This chapter studies the following vector optimization problem:

$$\mathcal{K} - \min_{x \in \Omega} f(x), \qquad (P^{vec}_{\mathcal{K}})$$

where $f : X \to Y$ is a continuous mapping between two Banach spaces, $\Omega \subseteq X$ is a nonempty closed set and $\mathcal{K} : Y \rightrightarrows Y$ is a cone-valued ordering map. We recall solution concepts for $(P^{vec}_{\mathcal{K}})$ where two assertions (2.7) and (2.8) are involved. Section 3.2 presents an optimality condition for nondominated solutions of $(P^{vec}_{\mathcal{K}})$ given by [7]. Using this result, we derive optimality conditions for an approximation problem w.r.t. a cone-valued mapping $\mathcal{K} : Y \rightrightarrows Y$. This will be used in Section 8.1 to derive optimality conditions for solutions of the beam intensity problem in radiotherapy treatment.

3.1 Solution Concepts

We begin this section with the definition of nondominated elements and minimal elements of a set in Banach spaces, which are introduced by Yu[111] and Chen et al. [18]. Note that this definition takes into account two assertions (2.7) and (2.8).

Definition 3.1.1. *[18, 111] Let A be a nonempty subset of a Banach space Y, $\bar{a} \in A$, $\mathcal{K} : Y \rightrightarrows Y$ be a cone-valued mapping. We say that:*

(i) $\bar{a}$ is a ***nondominated element*** *of A w.r.t. $\mathcal{K}(\cdot)$ if there is no $a \in A \setminus \{\bar{a}\}$ such that $\bar{a} \in a + \mathcal{K}(a)$ or equivalently $\bar{a} \notin \bigcup_{a \in A} (\{a\} + \mathcal{K}(a) \setminus \{0_Y\})$. The set of all nondominated elements of A w.r.t. $\mathcal{K}(\cdot)$ is denoted by* $\mathrm{ND}(A, \mathcal{K}(\cdot))$.

(ii) $\bar{a}$ *is a* ***minimal element*** *of* A *w.r.t.* $\mathcal{K}(\cdot)$ *if there is no* $a \in A \setminus \{\bar{a}\}$ *such that* $\bar{a} \in a + \mathcal{K}(\bar{a})$*, or equivalently* $(\{\bar{a}\} - \mathcal{K}(\bar{a})) \bigcap A = \{\bar{a}\}$*. The set of all minimal elements of* A *w.r.t.* $\mathcal{K}(\cdot)$ *is denoted by* $\mathrm{Min}(A, \mathcal{K}(\cdot))$.

Remark 3.1.2. *(i) Equivalently, we can define nondominated (minimal) elements of a set* A *by using two binary relations* $\leq_1$ *and* $\leq_2$ *given by* (2.5) *and* (2.6) *as:*

$$\bar{a} \in \mathrm{ND}(A, \mathcal{K}(\cdot)) \Longleftrightarrow \nexists a \in A \setminus \{\bar{a}\} : \ a \leq_1 \bar{a},$$

and

$$\bar{a} \in \mathrm{Min}(A, \mathcal{K}(\cdot)) \Longleftrightarrow \nexists a \in A \setminus \{\bar{a}\} : \ a \leq_2 \bar{a}.$$

(ii) Observe that if $\bar{a} \in \mathrm{Min}(A, \mathcal{K}(\cdot))$ *then* $\mathcal{K}(\bar{a})$ *is pointed. Indeed, suppose that there is* $b \in \mathcal{K}(\bar{a}) \bigcap (-\mathcal{K}(\bar{a}))$ *and* $b \neq 0$*. Let* $a' = \bar{a} + b$*. This implies* $a' \in \bar{a} - \mathcal{K}(\bar{a})$ *and* $a' \neq \bar{a}$*. This contradicts* $\bar{a} \in \mathrm{Min}(A, \mathcal{K}(\cdot))$*. The pointedness of* $\mathcal{K}(\bar{a})$ *does not always hold if* $\bar{a} \in \mathrm{ND}(A, \mathcal{K}(\cdot))$*.*

(iii) If $\mathcal{K}(\cdot) = C$*, where* C *is a proper, closed, convex, pointed cone of* Y*, the concepts of nondominated elements and minimal elements are identical, i.e., they are Pareto efficient points of the set* A *in the sense of Definition 2.1.24.*

(iv) Supppose that $\bar{a} \in A$ *and* $\mathcal{K}(\bar{a})$ *is a proper, closed, convex, pointed cone. Then,* $\bar{a}$ *is a minimal element of* A *w.r.t.* $\mathcal{K}(\cdot)$ *if and only if it is a Pareto efficient element of* A *w.r.t. the cone* $\mathcal{K}(\bar{a})$*, see also [36, Lemma 2.15].*

In addition, [36] shows a result concerning some relationships between the notions of nondominated elements and minimal elements. We present it in the following without proof.

Proposition 3.1.3. *[36, Lemma 2.11] Let* Y *be a linear space,* $A \in \mathcal{P}(Y)$ *and the set-valued mapping* $\mathcal{K} : Y \rightrightarrows Y$ *such that for all* $y \in Y$*,* $\mathcal{K}(y)$ *is a convex cone. The following assertions hold true:*

(i) If $\bar{y}$ *is a minimal element of* A *w.r.t.* $\mathcal{K}(\cdot)$ *and* $\mathcal{K}(y) \subseteq \mathcal{K}(\bar{y})$ *for all* $y \in A$*, then* $\bar{y}$ *is a nondominated element of* A *w.r.t.* $\mathcal{K}(\cdot)$*.*

(ii) If $\bar{y}$ *is a nondominated element of* A *w.r.t.* $\mathcal{K}(\cdot)$ *and* $\mathcal{K}(\bar{y}) \subseteq \mathcal{K}(y)$ *for all* $y \in A$*, then* $\bar{y}$ *is a minimal element of* A *w.r.t.* $\mathcal{K}(\cdot)$*.*

The following proposition presents some characterizations of minimal elements and nondominated elements. For the proof, we refer the reader to Lemmas 2.34, 2.38, 2.46 and 2.48 in [36].

Proposition 3.1.4. *[36] Let Y be a linear space, $A \in \mathcal{P}(Y)$ and the cone-valued mapping $\mathcal{K} : Y \rightrightarrows Y$ be given. Let*

$$A^{\mathcal{K}} := \bigcup_{y \in A} (y + \mathcal{K}(y)).$$

Then, the following assertions hold true:

(i) If $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{K}(\cdot)$ and $\bigcap_{y \in A} \mathcal{K}(y) \neq 0_Y$, then $\bar{y} \in \operatorname{bd} A$.

(ii) If $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{K}(\cdot)$ and $\mathcal{K}(\bar{y}) \neq 0_Y$, then $\bar{y} \in \operatorname{bd} A$.

(iii) If $\bar{y} \in A^{\mathcal{K}}$ is a nondominated element of $A^{\mathcal{K}}$ w.r.t. $\mathcal{K}(\cdot)$, then $\bar{y} \in A$ and $\bar{y}$ is also a nondominated element of A w.r.t. $\mathcal{K}(\cdot)$.

(iv) If $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{K}(\cdot)$, and if

$$\forall y \in A, \forall d \in \mathcal{K}(y) : \ \mathcal{K}(y + d) \subseteq \mathcal{K}(y)$$

then $\bar{y}$ is also a nondominated element of $A^{\mathcal{K}}$ w.r.t. $\mathcal{K}(\cdot)$.

(v) If $\bar{y} \in A$ is a minimal element of $A^{\mathcal{K}}$ w.r.t. $\mathcal{K}(\cdot)$, then it is also a minimal element of A w.r.t. $\mathcal{K}(\cdot)$.

(vi) If $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{K}(\cdot)$and if $\mathcal{K}(y) \subseteq \mathcal{K}(\bar{y})$ for all $y \in Y$ then $\bar{y}$ is also a minimal element of $A^{\mathcal{K}}$ w.r.t $\mathcal{K}(\cdot)$.

In addition, there are several references characterizing nondominated elements and minimal elements of a set by using scalarization techniques. For the sake of the shortness, we omit presenting them in this work. We refer the reader to [16, 19, 36] and references therein for further discussions.
Now, we consider problem ($P^{vec}_{\mathcal{K}}$) and define its corresponding solution concepts in the pre-image space by taking into account Definition 3.1.1 for $A = f(\Omega)$ in the image space.

Definition 3.1.5. *(Nondominated solutions and minimal solutions of a vector optimization problem w.r.t. a variable ordering structure)*
Consider the vector optimization problem ($P^{vec}_{\mathcal{K}}$) and $\bar{x} \in \Omega$. Then, $\bar{x}$ is said to be:

*(i) a **nondominated solution** of problem ($P^{vec}_{\mathcal{K}}$) if $f(\bar{x})$ is a nondominated element of the set $f(\Omega)$ w.r.t. $\mathcal{K}(\cdot)$.*

*(ii) a **minimal solution** of problem ($P^{vec}_{\mathcal{K}}$) if $f(\bar{x})$ is a minimal element of the set $f(\Omega)$ w.r.t. $\mathcal{K}(\cdot)$.*

Remark 3.1.6. *(i) When $\mathcal{K}(\cdot) = C$ where C is a proper, closed, convex and pointed cone of Y, the concepts of nondominated solutions and minimal solutions are identical. In this case, we called them* ***Pareto efficient solutions*** *of the problem $C - \underset{x\in\Omega}{\mathrm{Min}}\ f(x)$.*

(ii) If $\bar{x} \in \Omega$ is a minimal solution of problem $(P_{\mathcal{K}}^{vec})$, then it is also a Pareto efficient solution of the problem $\mathcal{K}(f(\bar{x})) - \underset{x\in\Omega}{\mathrm{Min}}\ f(x)$.

3.2 Necessary Optimality Conditions for Nondominated Solutions

This section presents optimality conditions for solutions of a vector optimization problem w.r.t. a cone-valued mapping. Recently, Eichfelder and Ha [37] have introduced scalarizing functionals and derived necessary and sufficient optimality conditions for solutions of $(P_{\mathcal{K}}^{vec})$ in the form of Fermat rule and Lagrange multiplier rule. Bao et al. [3] studied optimality conditions for minimal solutions of some nonconvex multiobjective location problems. Bao et al. [2] derived necessary conditions for approximate solutions and nondominated solutions of $(P_{\mathcal{K}}^{vec})$ using variational principles and the subdifferential calculus by Mordukhovich. The following result is shown by Bao and Mordukhovich [7], where the authors utilized the extremal principle as the main tool. We will adapt it to provide necessary conditions for beam intensity problems in Chapter 8.

Theorem 3.2.1. *([7, Theorem 4.12]) Let X, Y be Asplund spaces, $f : X \to Y$, $\mathcal{K} : Y \rightrightarrows Y$ and a nonempty closed subset $\Omega \subseteq X$. Let $\bar{x}$ be a nondominated solution of problem $(P_{\mathcal{K}}^{vec})$. Set $\bar{y} := f(\bar{x})$ and suppose that $\mathcal{K}(\cdot)$ satisfies the following conditions:*

(a) For all $y \in Y$, $\mathcal{K}(y)$ is a nonempty convex cone;

(b) There exists $e \in Y, e \neq 0$ such that $e \in \bigcap_{y\in Y} \mathcal{K}(y) \setminus (-\mathcal{K}(\bar{y}))$;

(c) There is a unique point y^* satisfying $-y^* \in D^*\mathcal{K}(\bar{y}, 0)(y^*)$.

Moreover, assume:

(i) $\mathcal{K}$ is SNC at $(\bar{y}, 0)$,

(ii) Either Ω is SNC at $\bar{x}$ or f is PSNC at $\bar{x}$,

(iii) $D^*f(\bar{x})(0) \cap (-N(\Omega, \bar{x})) = \{0\}$.

Then, there is $y^* \in Y^* \setminus \{0\}$ such that

$$0 \in D^*f(\bar{x})(y^* + D^*\mathcal{K}(f(\bar{x}); 0)(y^*)) + N(\Omega, \bar{x}).$$

Next, we utilize Theorem 3.2.1 to calculate optimality conditions for nondominated solutions of approximation problems equipped w.r.t. an ordering structure. This result will be applied in Section 8.1 to derive necessary conditions for solutions of beam intensity problems.

Let A_i be linear functions from $\mathbb{R}^m$ to $\mathbb{R}^{m_i}$, $a_i \in \mathbb{R}^{m_i}, i = 1, 2, ...n$, $\|\cdot\|_i$ be norms in $\mathbb{R}^{m_i}$. Given a nonempty closed set $\Omega \subseteq \mathbb{R}^m$ and a set-valued map $\mathcal{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfying $\mathcal{K}(y)$ is a closed, convex cone for each $y \in \mathbb{R}^n$. We consider the following problem:

$$\text{Minimize } f(x) \text{ subject to } x \in \Omega \text{ w.r.t. } \mathcal{K}(\cdot), \tag{$P_{\mathcal{K}}^{app}$}$$

where

$$f : \mathbb{R}^m \to \mathbb{R}^n,$$
$$f(x) := \begin{pmatrix} \|A_1 x - a_1\|_1 \\ \|A_2 x - a_2\|_2 \\ \cdots \\ \|A_n x - a_n\|_n \end{pmatrix}.$$

Now, we present necessary conditions for nondominated solutions of the vector approximation problem ($P_{\mathcal{K}}^{app}$). In Section 8.1.2, we will discuss the problem ($P_{\mathcal{K}}^{app}$) with a special ordering map based on threshold doses of organs in radiotherapy treatment.

Theorem 3.2.2. *We consider the problem ($P_{\mathcal{K}}^{app}$) w.r.t. a closed, convex, cone-valued mapping $\mathcal{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Suppose that $\bar{x} \in \Omega$ is a nondominated solution of ($P_{\mathcal{K}}^{app}$) and let $\bar{y} := f(\bar{x})$. We assume that the following conditions hold:*

(i) $\mathcal{K}(y)$ is a nonempty convex cone, for all $y \in \mathbb{R}^n$.

(ii) There exists $e \in \mathbb{R}^n$, $e \neq 0$ with $e \in \bigcap_{y \in \mathbb{R}^n} \mathcal{K}(y) \setminus (-\mathcal{K}(\bar{y}))$.

(iii) There is a unique point y^* satisfying $-y^* \in D^*\mathcal{K}(\bar{y}, 0)(y^*)$.

Then, there are $y^* \in \mathbb{R}^n \setminus \{0\}$ and corresponding $z^* \in (y^* + D^*\mathcal{K}(\bar{y}; 0)(y^*))$ and $T_i \in L(\mathbb{R}^{m_i}, \mathbb{R})$ satisfying $T_i(A_i(\bar{x}) - a^i) = \|A_i(\bar{x}) - a^i\|_i$ and $\|T_i\|_{i^*} \leq 1$ such that

$$0 \in \sum_{i=1}^{n} A_i^* z^* T_i + N(\Omega, \bar{x}). \tag{3.1}$$

Proof. Since f is Lipschitz and by using the relationship between coderivative of a vector function with subdiferential of its scalarization (Proposition 2.3.18), we get the following assertion

$$\forall\ y^* \in \mathbb{R}^n,\ \forall\ \bar{x} \in \Omega : D^* f(\bar{x})(y^*) = \partial(y^* \circ f)(\bar{x}).$$

This implies that $D^*f(\bar{x})(0) = \{0\}$ and thus $D^*f(\bar{x})(0) \cap (-N(\Omega, \bar{x})) = \{0\}$. Applying Theorem 3.2.1, there exists $y^* \in \mathbb{R}^n \setminus \{0\}$ such that

$$0 \in D^*f(\bar{x})(y^* + D^*\mathcal{K}(\bar{y}, 0)(y^*)) + N(\Omega, \bar{x}).$$

This means that there is $z^* \in (y^* + D^*\mathcal{K}(\bar{y}, 0)(y^*))$ satisfying

$$\begin{aligned} & 0 \in D^*f(\bar{x})(z^*) + N(\Omega, \bar{x}) \\ \iff & 0 \in \partial(z^* \circ f)(\bar{x}) + N(\Omega, \bar{x}). \end{aligned}$$

Taking into account the formulation of coderivative of a vector-valued norm function in Remark 2.3.8 we have that

$$\exists\, T_i \in L(\mathbb{R}^{m_i}, \mathbb{R}), T_i(A_i(\bar{x}) - a^i) = \|A_i(\bar{x}) - a^i\|_i$$

$$\text{and } \|T_i\|_{i^*} \leq 1, (i = 1, \ldots, n)$$

such that (3.1) holds, which completes the proof. □

In the following, we suppose that $\mathcal{K}(\cdot) \equiv K$, K is a proper, closed, convex and pointed cone in $\mathbb{R}^n$. This implies that $K \setminus (-K) = K \setminus \{0\}$. In this case, the problem $(P^{app}_{\mathcal{K}})$ will be denoted by (P^{app}_K). We present a corollary of Theorem 3.2.2 for optimality condition for Pareto efficient solutions of (P^{app}_K). Note that this result can be also implied by Bao et al. [9] and Durea et al. [26]. We will use this in Section 8.1 in order to derive necessary optimality conditions for minimal solutions of a mathematical formula of beam intensity problem.

Corollary 3.2.3. *Suppose that $\bar{x} \in \Omega$ is a Pareto efficient solution of (P^{app}_K), where K is a proper, closed, convex and pointed cone in $\mathbb{R}^n$, and let $\bar{y} := f(\bar{x})$. Then, there are $y^* \in -N(K, 0) \setminus \{0\}$ and $T_i \in L(\mathbb{R}^{m_i}, \mathbb{R})$ satisfying $T_i(A_i(\bar{x}) - a^i) = \|A_i(\bar{x}) - a^i\|_i$ and $\|T_i\|_{i^*} \leq 1$ such that (3.1) is fulfilled.*

Proof. The proof follows immediately by taking into account Theorem 3.2.2. □

Chapter 4

Set Optimization w.r.t. Variable Domination Structures

In this chapter, let X and Y be Banach spaces and a set-valued mapping $F : X \rightrightarrows Y$ be given. We follow the vector approach to define solution concepts based on $\mathcal{Q} : X \rightrightarrows Y$ for the problem

$$\mathcal{Q} - \operatorname*{Min}_{x \in X} F(x). \tag{$P_{\mathcal{Q}}$}$$

Furthermore, let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued mapping. The set approach will use this domination structure to define solution concepts for the problem

$$\mathcal{K} - \operatorname*{Min}_{x \in X} F(x). \tag{$P_{\mathcal{K}}$}$$

The aim of this chapter is investigating relationships between the solution concepts given by these two approaches. It is important to mention that, these relationships for the case the dominations $\mathcal{K}(\cdot)$ and $\mathcal{Q}(\cdot)$ are constant mappings have been studied in [57, 73]. Eichfelder and Pilecka [40] derive one connection between strictly minimal solutions of ($P_{\mathcal{K}}$) w.r.t the relation $\preceq_{pl}^{\mathcal{K}}$ and nondominated solutions of ($P_{\mathcal{Q}}$). In addition, Eichfelder and Pilecka [39] study some of these relationships in which the vector approach is equipped with a variable domination structure acting onto the image space of the objective mapping F. However, the present frame work investigates relationships between the solution concepts of ($P_{\mathcal{K}}$) and ($P_{\mathcal{Q}}$), where the domination $\mathcal{Q}(\cdot)$ has the same preimage space and image space as the mapping F. In particular, Section 4.1 introduces solution concepts for the problem ($P_{\mathcal{Q}}$). Section 4.2 defines minimal elements for a family of sets and properties of the sets of these elements. These concepts are related to set relations introduced in Section 2.2.2. Then, we follow the set approach to define solutions for the problem ($P_{\mathcal{K}}$). Section 4.3 illustrates relationships between solution concepts of ($P_{\mathcal{K}}$) and ($P_{\mathcal{Q}}$). These relationships will be used in Chapter 6 to derive necessary optimality conditions for solutions of the problem ($P_{\mathcal{K}}$) by taking into account corresponding results for solutions of the problem ($P_{\mathcal{Q}}$) (see [30]). The results presented within this chapter are based on Köbis, Le, Tammer and Yao [70].

4.1 Solution Concepts for Set Optimization Problems based on the Vector Approach

In Chapter 3, we have already defined the concepts of nondominated (minimal) solutions for problem ($P^{vec}_{\mathcal{K}}$) by using the relations $\leq_1$ and $\leq_2$ given in Chapter 2. Now, we consider problem ($P_{\mathcal{Q}}$), where the domination map $\mathcal{Q} : X \rightrightarrows Y$ is acting between the same spaces as the set-valued objective map $F : X \rightrightarrows Y$. We assume that $F(x) \neq \emptyset$, for all $x \in X$. The concept of solutions of ($P_{\mathcal{Q}}$) is defined as follows:

Definition 4.1.1. *Let $F : X \rightrightarrows Y$, $\mathcal{Q} : X \rightrightarrows Y$ such that for all $y \in Y$, $\mathcal{Q}(y)$ is a proper, closed, convex set. We say that a point $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is:*

*(i) a **nondominated solution** of the problem ($P_{\mathcal{Q}}$) w.r.t. $\mathcal{Q}(\cdot)$ if*

$$\bar{y} \notin \bigcup_{x \in X} \big(F(x) + \mathcal{Q}(x) \setminus \{0\}\big).$$

The set of all nondominated solutions of ($P_{\mathcal{Q}}$) w.r.t. $\mathcal{Q}(\cdot)$ is denoted by $\operatorname{ND}(F(X), \mathcal{Q})$.

*(ii) a **minimal solution** of the problem ($P_{\mathcal{Q}}$) w.r.t. $\mathcal{Q}(\cdot)$ if*

$$\bar{y} \notin F(X) + (\mathcal{Q}(\bar{x}) \setminus \{0\}).$$

We denote by $\operatorname{Min}(F(X), \mathcal{Q})$ the set of all minimal solutions of ($P_{\mathcal{Q}}$) w.r.t. $\mathcal{Q}(\cdot)$.

Obviously, if $\forall x \in X,\ \mathcal{Q}(x) = Q$, where Q is a proper, convex, pointed cone, the concepts of nondominated solutions and minimal solutions are identical. In this case, we call them minimizers of F w.r.t. the cone Q. Observe also that a minimal solution $\bar{x}$ of ($P_{\mathcal{Q}}$) is a minimizer of F w.r.t. $Q(\bar{x})$.

In the literature, there exist also solution concepts for set-valued optimization problems equipped with a domination mapping $\tilde{\mathcal{Q}} :\ Y \rightrightarrows Y$, see [39]. In this case, we denote by ($P_{\tilde{\mathcal{Q}}}$) the problem $\tilde{\mathcal{Q}} - \operatorname{Min}_{x \in X} F(x)$. For the purpose of completeness, we recall them as follows:

Definition 4.1.2. *[39, Definition 5.4] Let $F : X \rightrightarrows Y$ and $\tilde{\mathcal{Q}} : Y \rightrightarrows Y$ be given, such that for all $y \in Y$, $\tilde{\mathcal{Q}}(y)$ is a proper, closed, convex cone. Let $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ and consider the binary relations (2.5) and (2.6) introduced in Chapter 2.*

*(i) $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is called a **nondominated solution (in the sense of Eichfelder and Pilecka)** of the problem ($P_{\tilde{\mathcal{Q}}}$) w.r.t. $\tilde{\mathcal{Q}}$ if there is no $y \in F(X) \setminus \{\bar{y}\}$ such that*

$$y \leq_1 \bar{y} \Longrightarrow y \in \bar{y} - \tilde{\mathcal{Q}}(y).$$

(ii) $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ *is called a* ***minimal solution (in the sense of Eichfelder and Pilecka)*** *of the problem* $(P_{\tilde{\mathcal{Q}}})$ *w.r.t.* $\tilde{\mathcal{Q}}$ *if there is no* $y \in F(X) \setminus \{\bar{y}\}$ *such that*

$$y \leq_2 \bar{y} \Longrightarrow y \in \bar{y} - \tilde{\mathcal{Q}}(\bar{y}).$$

Eichfelder and Pilecka [39] investigated relationships between solutions of problem $(P_{\tilde{\mathcal{Q}}})$ (in the sense of Definition 4.1.2) and solutions of $(P_{\mathcal{K}})$ defined based on set approach, see [39, Theorem 5.1]. However, from now on we only study the vector approach for problem $(P_{\mathcal{Q}})$ with solution concepts given by Definition 4.1.1.

4.2 Solution Concepts for Set Optimization Problems based on the Set Approach

This section introduces different concepts for minimal elements of a family of sets and solution concepts for the problem $(P_{\mathcal{K}})$ introduced at the beginning of this chapter. These concepts are defined based on set relations introduced in Section 2.2.2. In addition, we will present relationships between the sets of different minimal elements.

Let $\mathcal{A}$ be a family of sets in $\mathcal{P}(Y)$ and $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$ be one of six set relations introduced in Definition 2.2.5. Some minimality notions of $\mathcal{A}$ w.r.t. $\preceq_t^{\mathcal{K}}$ are determined as follows:

Definition 4.2.1. *Let* $\mathcal{A}$ *be a family of nonempty subsets of* Y*,* $\mathcal{K} : Y \rightrightarrows Y$ *be a set-valued mapping, and* $t \in \{l, u, cl, cu, pl, pu\}$*.*

(a) A set $\bar{A} \in \mathcal{A}$ *is called a* ***minimal element*** *of* $\mathcal{A}$ *w.r.t.* $\preceq_t^{\mathcal{K}}$ *if*

$$A \in \mathcal{A}, A \preceq_t^{\mathcal{K}} \bar{A} \quad \Longrightarrow \quad \bar{A} \preceq_t^{\mathcal{K}} A.$$

(b) A set $\bar{A} \in \mathcal{A}$ *is called a* ***strongly minimal element*** *of* $\mathcal{A}$ *w.r.t.* $\preceq_t^{\mathcal{K}}$ *if*

$$\forall\ A \in \mathcal{A} \setminus \{\bar{A}\} : \ \bar{A} \preceq_t^{\mathcal{K}} A.$$

(c) A set $\bar{A} \in \mathcal{A}$ *is called a* ***strictly minimal element*** *of* $\mathcal{A}$ *w.r.t.* $\preceq_t^{\mathcal{K}}$ *if*

$$A \in \mathcal{A}, A \preceq_t^{\mathcal{K}} \bar{A} \quad \Longrightarrow \quad \bar{A} = A.$$

We denote respectively by $\operatorname{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$*,* $\operatorname{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$ *and* $\operatorname{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$ *the sets of all minimal, strongly minimal and strictly minimal elements of* $\mathcal{A}$ *w.r.t.* $\preceq_t^{\mathcal{K}}$*,* $t \in \{l, u, cl, cu, pl, pu\}$*. The index "Y" in* Min_Y*,* SoMin_Y*,* SiMin_Y *is used to mark that we consider concepts of minimality in the image space* Y*. In Definition 4.2.6, we will introduce additionally corresponding concepts in the pre-image space* X*.*

The following remark shows some relationships between the above definition and the definition of nondominated elements and minimal elements of a set in the sense of Definition 3.1.1.

Remark 4.2.2. *Let $\mathcal{A}$ be a family of singleton sets and $\mathcal{K}(y)$ be a closed, convex, pointed cone for each $y \in Y$. Then, the definition of strictly minimal elements of $\mathcal{A}$ w.r.t. $\preceq_l^{\mathcal{K}}$ reduces to the definition of nondominated elements of $\mathcal{A}$ w.r.t. $\mathcal{K}(\cdot)$, see Definition 3.1.1 (i). Moreover, the definition of strictly minimal elements of $\mathcal{A}$ w.r.t. $\preceq_u^{\mathcal{K}}$ reduces to the definition of minimal elements of $\mathcal{A}$ w.r.t. $\mathcal{K}(\cdot)$, see Definition 3.1.1 (ii).*

In the following, we illustrate some properties concerning the sets of (strictly, strongly) minimal elements of a family of sets. Note that some of them are derived by using Proposition 2.2.7.

Remark 4.2.3. *Let $t \in \{l, u, cl, cu, pl, pu\}$. We note the following properties by using the definitions of the sets $\mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$ and $\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$.*

(i) It is clear that if $\preceq_t^{\mathcal{K}}$ is transitive and $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$, then for all B such that $B \sim \bar{A}$, it holds that $B \in \mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$.

(ii) Obviously, we have the inclusions

$$\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}),$$

and

$$\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}).$$

(iii) We have that

$$\begin{aligned}\bar{A} \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_{cl}^{\mathcal{K}}) &\iff \forall\, A \in \mathcal{A} \setminus \{\bar{A}\} :\ \bar{A} \preceq_{cl}^{\mathcal{K}} A \\ &\overset{\text{Proposition 2.2.7}}{\Longrightarrow} \forall\, A \in \mathcal{A} \setminus \{\bar{A}\} :\ \bar{A} \preceq_{l}^{\mathcal{K}} A \\ &\Longrightarrow \bar{A} \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}}).\end{aligned}$$

Therefore, we get that $\mathrm{SoMin}_Y(\mathcal{A}, \preceq_{cl}^{\mathcal{K}}) \subseteq \mathrm{SoMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$. Similarly, we obtain that $\mathrm{SoMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}}) \subseteq \mathrm{SoMin}_Y(\mathcal{A}, \preceq_{pl}^{\mathcal{K}})$.
Furthermore, it yields from Definition 4.2.1(c) that

$$\begin{aligned}\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_{pl}^{\mathcal{K}}) &\iff \forall\, A \neq \bar{A} :\ A \not\preceq_{pl}^{\mathcal{K}} \bar{A} \\ &\overset{\text{Proposition 2.2.7}}{\Longrightarrow} \forall\, A \neq \bar{A} :\ A \not\preceq_{l}^{\mathcal{K}} \bar{A} \\ &\iff \bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}}).\end{aligned}$$

Thus, $\text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{pl}) \subseteq \text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{l})$.
Similarly, we have that $\text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{l}) \subseteq \text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{cl})$.
For the relations $\preceq^{\mathcal{K}}_{u}$, $\preceq^{\mathcal{K}}_{cu}$, *and* $\preceq^{\mathcal{K}}_{pu}$ *the following assertions also hold true:*

$$\text{SoMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{cu}) \subseteq \text{SoMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{u}) \subseteq \text{SoMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{pu})$$

and

$$\text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{pu}) \subseteq \text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{u}) \subseteq \text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{cu}).$$

The following example illustrates that neither $\text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{t}) \subseteq \text{SoMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{t})$ nor $\text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{t}) \subseteq \text{SoMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{t})$ always holds true.

Example 4.2.4. *Consider the four following sets:*
$A_1 := \{(y_1, y_2) \in \mathbb{R}^2 |\ 2 \le y_1, y_2 \le 3, y_1 + y_2 \le 5\}$,
$A_2 := \{(2, y_2) \in \mathbb{R}^2 |\ 2 \le y_2 \le 3\} \bigcup \{(y_1, 2) \in \mathbb{R}^2 |\ 2 \le y_1 \le 3\}$,
$A_3 := \{(5, 5)\}$,
$A_4 := \{(y_1, y_2) \in \mathbb{R}^2 |\ 3 \le y_1 \le 5, 0 \le y_2 \le 1\}$.
We define a set-valued mapping $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ *by:*

$$\mathcal{K}(y) := \begin{cases} \{(d_1, d_2) |\ 0 \le d_1 \le 2\ d_2\} & \text{if } y \in \mathbb{R}^2 \setminus \{(1, 3)\}, \\ \mathbb{R}^2_+ & \text{if } y = (1, 3). \end{cases}$$

It follows from the definition of $\preceq^{\mathcal{K}}_{l}$ *given in Definition 2.2.5 that:*

$$\begin{cases} A_1 \preceq^{\mathcal{K}}_{l} A_2,\ A_1 \preceq^{\mathcal{K}}_{l} A_3,\ A_1 \not\preceq^{\mathcal{K}}_{l} A_4, \\ A_2 \preceq^{\mathcal{K}}_{l} A_1,\ A_2 \preceq^{\mathcal{K}}_{l} A_3,\ A_2 \not\preceq^{\mathcal{K}}_{l} A_4, \\ A_3 \not\preceq^{\mathcal{K}}_{l} A_1,\ A_3 \not\preceq^{\mathcal{K}}_{l} A_2,\ A_3 \not\preceq^{\mathcal{K}}_{l} A_4, \\ A_4 \not\preceq^{\mathcal{K}}_{l} A_1,\ A_4 \not\preceq^{\mathcal{K}}_{l} A_2,\ A_4 \preceq^{\mathcal{K}}_{l} A_3. \end{cases}$$

Let $\mathcal{A} := \{A_1, A_2, A_3\}$. *We have that*

$$\text{Min}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{l}) = \{A_1, A_2\},\ \ \text{SoMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{l}) = \{A_1, A_2\},\ \ \text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{l}) = \emptyset.$$

Let $\mathcal{A}' := \{A_1, A_2, A_3, A_4\}$. *We get that*

$$\text{Min}_Y(\mathcal{A}', \preceq^{\mathcal{K}}_{l}) = \{A_1, A_2, A_4\},\ \ \text{SoMin}_Y(\mathcal{A}', \preceq^{\mathcal{K}}_{l}) = \emptyset,\ \ \text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{l}) = \{A_4\}.$$

Let $\mathcal{A}'' := \{A_3, A_4\}$. *It holds that*

$$\text{Min}_Y(\mathcal{A}'', \preceq^{\mathcal{K}}_{l}) = \text{SoMin}_Y(\mathcal{A}'', \preceq^{\mathcal{K}}_{l}) = \text{SiMin}_Y(\mathcal{A}'', \preceq^{\mathcal{K}}_{l}) = \{A_4\}.$$

For an illustration of this example, see Figure 4.1.

In addition, the sets $\text{SoMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{t})$ and $\text{SiMin}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_{t})$ have the following properties.

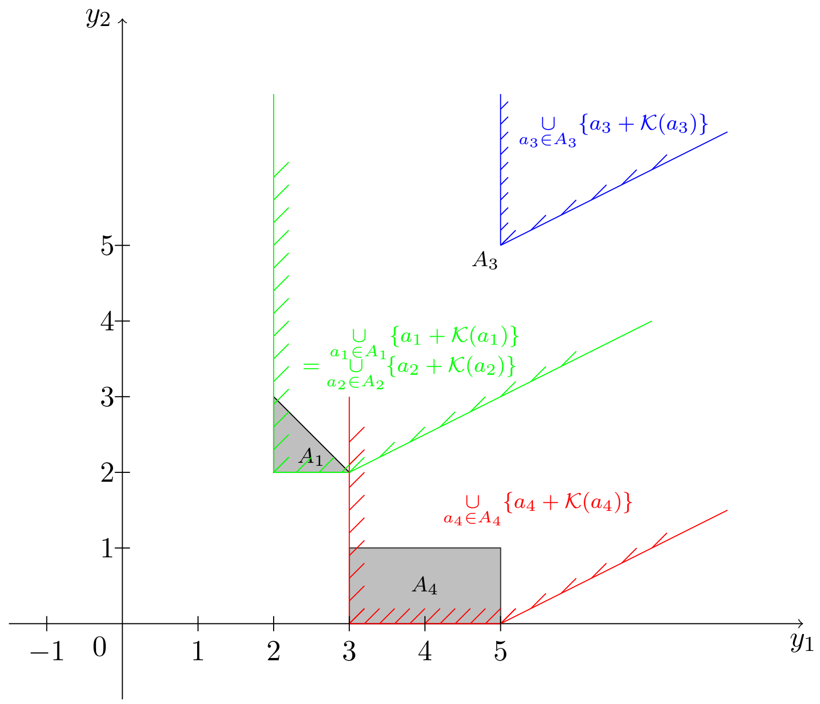


Figure 4.1: Illustration for Example 4.2.4.

Proposition 4.2.5. *[69, Proposition 2] Let $\mathcal{A}$ be a family of sets in $\mathcal{P}(Y)$, $S \in \mathcal{P}(Y)$ and let $|S|$ denote the number of elements of S. Then, for $t \in \{l, u, cl, cu, pl, pu\}$, it holds that*

(a) If $|\,\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})| > 1$ *then* $\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \emptyset$.

(b) If $|\,\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})| > 1$ *then* $\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \emptyset$.

(c) If $\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \bigcap \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$ *then*

$$\begin{cases} |\,\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})| = |\,\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})| = 1 \\ \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}). \end{cases}$$

Proof. **(a)** Suppose that $A_1, A_2 \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $A_1 \neq A_2$, and $\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$. Let $B \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$. If $B = A_1$ we have that $A_2 \preceq_t^{\mathcal{K}} B$. This implies $A_2 = B = A_1$ since $B \in \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$. This contradicts $A_1 \neq A_2$. The case $B = A_2$ is proved similarly.
Now, we assume that $B \neq A_1$ and $B \neq A_2$. It holds that

$$\begin{cases} A_1 \preceq_t^{\mathcal{K}} B, B \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A_1 = B, \\ A_2 \preceq_t^{\mathcal{K}} B, B \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A_2 = B. \end{cases}$$

Therefore $A_1 = A_2$, that is a contradiction.
(b) Let $B_1, B_2 \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $B_1 \neq B_2$ and suppose that $\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$. Let

$A \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$. If $A = B_1$, we have that $A \preceq_t^{\mathcal{K}} B_2$. This implies $A = B_2$ since $B_2 \in \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$. Therefore, $B_1 = B_2$, that is a contradiction. The case $A = B_2$ is proved similarly.

Now, suppose that $A \neq B_1$ and $A \neq B_2$. We have that

$$\begin{cases} A \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A \preceq_t^{\mathcal{K}} B_1 \Rightarrow A = B_1, \\ A \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A \preceq_t^{\mathcal{K}} B_2 \Rightarrow A = B_2. \end{cases}$$

Thus, $B_1 = B_2$, a contradiction.

(c) Part (a) and part (b) yield $|\,\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \cap \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})| \leq 1$. Therefore, if

$$\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \cap \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset,$$

then

$$|\,\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})| = |\,\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})| = 1.$$

The proof is complete. □

Now, we follow the set approach to define solutions of $(P_{\mathcal{K}})$, $\mathcal{K} : Y \rightrightarrows Y$, w.r.t. the relation $\preceq_t^{\mathcal{K}}$ introduced in Definition 2.2.5, where $t \in \{u, l, cu, cl, pu, pl\}$. Note that the solution concepts in the following definition are given in the preimage space X, whereas the solution concepts in Definition 4.2.1 are formulated in the image space Y.

Definition 4.2.6. *[39, 57] Let $F : X \rightrightarrows Y$, $\mathcal{K} : Y \rightrightarrows Y$ be two set-valued mappings such that $F(x)$ and $\mathcal{K}(y)$ are nonempty sets for all $x \in X$, $y \in Y$. Let $\preceq_t^{\mathcal{K}}$ be given in Definition 2.2.5, $t \in \{u, l, cu, cl, pu, pl\}$.*

*(a) A point $\bar{x} \in X$ is called a **minimal solution** of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$ if*

$$x \in X,\ F(x) \preceq_t^{\mathcal{K}} F(\bar{x}) \quad \Longrightarrow \quad F(\bar{x}) \preceq_t^{\mathcal{K}} F(x).$$

We denote by $\mathrm{Min}(F(X), \preceq_t^{\mathcal{K}})$ the set of all minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$.

*(b) A point $\bar{x} \in X$ is called a **strongly minimal solution** of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$ if*

$$\forall\, x \in X \setminus \{\bar{x}\} :\ F(\bar{x}) \preceq_t^{\mathcal{K}} F(x).$$

We denote by $\mathrm{SoMin}(F(X), \preceq_t^{\mathcal{K}})$ the set of all strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$.

*(c) A point $\bar{x} \in X$ is called a **strictly minimal solution** of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$ if*

$$x \in X,\ F(x) \preceq_t^{\mathcal{K}} F(\bar{x}) \ \text{ or } \ F(x) = F(\bar{x}) \Longrightarrow x = \bar{x}.$$

We denote by $\mathrm{SiMin}(F(X), \preceq_t^{\mathcal{K}})$ the set of all strictly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$.

Remark 4.2.7. *Let* $t \in \{l, u, cl, cu, pl, pu\}$.

(i) Observe that if $\preceq_t^{\mathcal{K}}$ *is transitive and* $\bar{x} \in \mathrm{Min}(F(X), \preceq_t^{\mathcal{K}})$, *then it also holds true for all* $x \in X$ *satisfying* $F(x) \sim F(\bar{x})$. *If the relation* $\preceq_t^{\mathcal{K}}$ *is reflexive, the definition of strictly solutions of* $(P_{\mathcal{K}})$ *is equivalent to*

$$x \in X, F(x) \preceq_t^{\mathcal{K}} F(\bar{x}) \Longrightarrow x = \bar{x}.$$

If for all $x \neq x', F(x) \neq F(x')$ *holds true, then the Definition 4.2.6(b) and (c) are equivalent to* $F(\bar{x}) \in \mathrm{SoMin}_Y(F(X), \preceq_t^{\mathcal{K}})$ *and* $F(\bar{x}) \in \mathrm{SiMin}_Y(F(X), \preceq_t^{\mathcal{K}})$, *respectively.*

Let $[F(\bar{x})] := \{F(x) \in F(X) |\ F(x) \sim F(\bar{x})\}$. *Then, it holds that*

$$\bar{x} \in \mathrm{SiMin}(F(X), \preceq_t^{\mathcal{K}}) \quad \Longrightarrow \quad [F(\bar{x})] = \{F(\bar{x})\}.$$

(ii) Definition 4.2.6 implies that $\mathrm{SiMin}(F(X), \preceq_t^{\mathcal{K}})$ *and* $\mathrm{SoMin}(F(X), \preceq_t^{\mathcal{K}})$ *are subsets of* $\mathrm{Min}(F(X), \preceq_t^{\mathcal{K}})$. *Furthermore, by using the same lines as in Remark 4.2.3(iii) the following relationships for the sets of minimal solutions of* $(P_{\mathcal{K}})$ *w.r.t the lower relations* $\preceq_l^{\mathcal{K}}$, $\preceq_{cl}^{\mathcal{K}}$, *and* $\preceq_{pl}^{\mathcal{K}}$ *hold true:*

$$\mathrm{SoMin}(F(X), \preceq_{cl}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preceq_l^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preceq_{pl}^{\mathcal{K}})$$

and

$$\mathrm{SiMin}(F(X), \preceq_{pl}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preceq_l^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preceq_{cl}^{\mathcal{K}}).$$

Similarly, we have the following relationships for the sets of minimal solutions of $(P_{\mathcal{K}})$ *w.r.t the upper relations* $\preceq_u^{\mathcal{K}}$, $\preceq_{cu}^{\mathcal{K}}$ *and* $\preceq_{pu}^{\mathcal{K}}$:

$$\mathrm{SoMin}(F(X), \preceq_{cu}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preceq_u^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preceq_{pu}^{\mathcal{K}})$$

and

$$\mathrm{SiMin}(F(X), \preceq_{pu}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preceq_u^{\mathcal{K}}) \subseteq \mathrm{SiMin}(F(X), \preceq_{cu}^{\mathcal{K}}).$$

4.3 Relationships between Solution Concepts based on the Set Approach and the Vector Approach

In this section, we derive relationships between solution concepts of $(P_{\mathcal{K}})$, $\mathcal{K} : Y \rightrightarrows Y$, and $(P_{\mathcal{Q}})$, $\mathcal{Q} : X \rightrightarrows Y$ given in Definitions 4.1.1 and 4.2.6. In the following theorems, we will utilize two set-valued mappings $\hat{\mathcal{Q}} : X \rightrightarrows Y$ and $\hat{\mathcal{Q}}' : X \rightrightarrows Y$ respectively determined by:

$$\forall x \in X : \ \hat{\mathcal{Q}}(x) := \bigcap_{y \in F(x)} \mathcal{K}(y), \tag{4.1}$$

and

$$\forall x \in X: \ \hat{\mathcal{Q}}'(x) := \bigcup_{y \in F(x)} \mathcal{K}(y). \tag{4.2}$$

Let us recall a result given by Eichfelder and Pilecka [40] about the relationships between strictly minimal solutions of ($P_{\mathcal{K}}$) w.r.t. the possibly lower less relation w.r.t. $\mathcal{K}(\cdot)$ ($\preccurlyeq^{\mathcal{K}}_{pl}$) introduced in Definition 2.2.5(v), and nondominated solutions of the set-valued optimization problem introduced in Definition 4.1.1(i).

Theorem 4.3.1. *([40, Lemma 5.1]) Consider problem ($P_{\mathcal{K}}$) w.r.t. $\preceq^{\mathcal{K}}_{pl}$, $\mathcal{K}: Y \rightrightarrows Y$ which satisfies that $\mathcal{K}(y)$ is a proper, convex cone for all $y \in Y$, and let some $\bar{x} \in \mathrm{SiMin}(F(X), \preceq^{\mathcal{K}}_{pl})$. Let $\hat{\mathcal{Q}}: X \rightrightarrows Y$ be given by (4.1). If there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y} \notin F(\bar{x}) + \hat{\mathcal{Q}}(\bar{x}) \setminus \{0\}$, then $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$.*

The following theorem derives a corresponding result for strongly minimal solutions of ($P_{\mathcal{K}}$) w.r.t. the lower less relation w.r.t. $\mathcal{K}(\cdot)$, i.e., $\preceq^{\mathcal{K}}_{l}$, introduced in Definition 2.2.5(i), and nondominated solutions of the set-valued optimization problem introduced in Definition 4.1.1 (i).

Theorem 4.3.2. *Consider problem ($P_{\mathcal{K}}$) w.r.t. $\preceq^{\mathcal{K}}_{l}$ and $\bar{x} \in \mathrm{SoMin}(F(X), \preceq^{\mathcal{K}}_{l})$. Suppose that there is $\bar{y} \in F(\bar{x})$ such that*

$$\forall \ y \in F(\bar{x}) \setminus \{\bar{y}\}: \ \bar{y} \notin y + \mathcal{K}(y). \tag{4.3}$$

Assume that $\mathcal{K}: Y \rightrightarrows Y$ satisfies properties (2.9)- (2.11) and $\mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y})) = \{0\}$. Let $\hat{\mathcal{Q}}: X \rightrightarrows Y$ be given by (4.1). Then, $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$.

Proof. Since $\bar{x} \in \mathrm{SoMin}(F(X), \preceq^{\mathcal{K}}_{l})$, it holds that

$$\forall \ x \in X \setminus \{\bar{x}\}: \ F(\bar{x}) \preceq^{\mathcal{K}}_{l} F(x).$$

Furthermore, it holds that

$$F(\bar{x}) \preceq^{\mathcal{K}}_{l} F(\bar{x}),$$

since $0 \in \mathcal{K}(y)$ for all $y \in Y$. Thus, we obtain

$$\forall \ x \in X: \ F(\bar{x}) \preceq^{\mathcal{K}}_{l} F(x),$$

which is equivalent to

$$\forall \ x \in X: F(x) \subseteq \bigcup_{y \in F(\bar{x})} (y + \mathcal{K}(y)). \tag{4.4}$$

Suppose by contradiction that $(\bar{x}, \bar{y}) \notin \mathrm{ND}(F(X), \hat{\mathcal{Q}})$. This means that

$$\exists x \in X: \ \bar{y} \in F(x) + \hat{Q}(x) \setminus \{0\}.$$

This implies that

$$\exists\, x \in X,\;\; y \in F(x) \setminus \{\bar{y}\} :\;\; \bar{y} \in y + \hat{\mathcal{Q}}(x) \subseteq y + \mathcal{K}(y).$$

From (4.4), taking into account that $\mathcal{K}(\cdot)$ satisfies (2.11), we have that

$$\exists\, \hat{y} \in F(\bar{x}) :\;\; y \in \hat{y} + \mathcal{K}(\hat{y}) \quad \Longrightarrow \quad \mathcal{K}(y) \subseteq \mathcal{K}(\hat{y}). \tag{4.5}$$

Therefore, we can conclude

$$\begin{aligned} &\bar{y} \in y + \mathcal{K}(y) \subseteq \hat{y} + \mathcal{K}(\hat{y}) + \mathcal{K}(y) \subseteq \hat{y} + \mathcal{K}(\hat{y}) + \mathcal{K}(\hat{y}) \subseteq \hat{y} + \mathcal{K}(\hat{y}) \\ \Longrightarrow\;\; &\bar{y} \in y + \mathcal{K}(y) \subseteq \hat{y} + \mathcal{K}(\hat{y}). \end{aligned} \tag{4.6}$$

Taking into account (4.3) we obtain that $\bar{y} = \hat{y}$. By (4.6), we get

$$y + \mathcal{K}(y) \subseteq \hat{y} + \mathcal{K}(\hat{y}) = \bar{y} + \mathcal{K}(\bar{y}).$$

Furthermore,

$$\bar{y} \in y + \mathcal{K}(y) \quad \Longrightarrow \quad \mathcal{K}(\bar{y}) \subseteq \mathcal{K}(y). \tag{4.7}$$

Since (4.5) and (4.7), it holds that $\mathcal{K}(y) \subseteq \mathcal{K}(\hat{y}) = \mathcal{K}(\bar{y}) \subseteq \mathcal{K}(y)$.
Thus,

$$\mathcal{K}(\bar{y}) = \mathcal{K}(y).$$

Taking into account (4.5), (4.7) and $\bar{y} = \hat{y}$, we have that

$$y - \bar{y} \in \mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y})).$$

Since $\mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y})) = \{0\}$, $y = \bar{y}$. This is a contradiction to $y \in F(x) \setminus \{\bar{y}\}$. Therefore, $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$. □

Observe from Remark 4.2.7(ii) that $\mathrm{SoMin}(F(X), \preceq_{cl}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(F(X), \preceq_{l}^{\mathcal{K}})$, i.e., if $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_{cl}^{\mathcal{K}})$, then $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_{l}^{\mathcal{K}})$. Therefore, we present the following corollary describing the relationships between strong minimal solutions of $(P_{\mathcal{K}})$ w.r.t. the certainly lower less relation w.r.t. $\mathcal{K}(\cdot)$, i.e., $\preccurlyeq_{cl}^{\mathcal{K}}$, introduced in Definition 2.2.5 (iii), and nondominated solutions of the set-valued optimization problem introduced in Definition 4.1.1 (i). We omit the proof of this result since it can be proved by using the same arguments as that one of Theorem 4.3.2.

Corollary 4.3.3. *Consider problem* $(P_{\mathcal{K}})$ *w.r.t.* $\preceq_{cl}^{\mathcal{K}}$, *and* $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_{cl}^{\mathcal{K}})$. *Suppose that there is* $\bar{y} \in F(\bar{x})$ *satisfying condition* (4.3). *Furthermore, assume that* $\mathcal{K} : Y \rightrightarrows Y$ *satisfies properties (2.9)- (2.11) and* $\mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y})) = \{0\}$. *Let* $\hat{\mathcal{Q}} : X \rightrightarrows Y$ *be given by (4.1). Then,* $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$.

Remark 4.3.4. *Observe that in [70, Theorem 4.2 and Corollary 4.3] the domination* $\mathcal{K}(\cdot)$ *is assumed to be such that* (2.13) *is fulfilled. Whilst, Theorem 4.3.2 and Corollary 4.3.3 show that this condition can be replaced by a weaker assumption, that is the pointedness of only the set* $\mathcal{K}(\bar{y})$*.*

On the other hand, under some assumptions we can derive a stronger result. The next theorem is given in [70], where the domination $\hat{\mathcal{Q}}'(\cdot)$ is supposed to be pointed at every point $x \in X$ (compare [70, Theorem 4.4]). However, we can weaken this assumption as follows:

Theorem 4.3.5. *Consider problem* ($P_{\mathcal{K}}$) *w.r.t.* $\preceq^{\mathcal{K}}_{cl}$*,* $\mathcal{K} : Y \rightrightarrows Y$*, where* $\mathcal{K}(\cdot)$ *satisfies properties (2.10) and (2.11), and* $\bar{x} \in \mathrm{SoMin}(F(X), \preceq^{\mathcal{K}}_{cl})$*. Assume that there exists* $\bar{y} \in F(\bar{x})$ *satisfying* $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$*. Let* $\hat{\mathcal{Q}}' : X \rightrightarrows Y$ *given by* (4.2) *such that*

$$\hat{\mathcal{Q}}'(\bar{x}) \cap (-\hat{\mathcal{Q}}'(\bar{x})) = \{0\}.$$

Then, $(\bar{x}, \bar{y}) \in \mathrm{Min}(F(X), \hat{\mathcal{Q}}')$*.*

Proof. Since $\bar{x} \in \mathrm{SoMin}(F(X), \preceq^{\mathcal{K}}_{cl})$, it holds that

$$\begin{aligned} & \forall\, x \in X \setminus \{\bar{x}\} : \; F(\bar{x}) \preceq^{\mathcal{K}}_{cl} F(x) \\ \iff & \forall\, x \in X \setminus \{\bar{x}\} : \; F(x) \subseteq \bigcap_{y \in F(\bar{x})} (y + \mathcal{K}(y)) \\ \iff & \forall\, x \in X \setminus \{\bar{x}\}, \; \forall\, y \in F(\bar{x}) : \; F(x) \subseteq y + \mathcal{K}(y). \end{aligned} \tag{4.8}$$

From the assumption $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$ and taking into account that $\mathcal{K}(\cdot)$ satisfies (2.11), it holds that

$$\forall\, y \in F(\bar{x}) : \; y \in \bar{y} + \mathcal{K}(\bar{y}) \quad \Longrightarrow \quad \mathcal{K}(y) \subseteq \mathcal{K}(\bar{y}).$$

This implies

$$y + \mathcal{K}(y) \subseteq \bar{y} + \mathcal{K}(\bar{y}) + \mathcal{K}(\bar{y}) \subseteq \bar{y} + \mathcal{K}(\bar{y}),$$

since $\mathcal{K}(\cdot)$ satisfies (2.10). Taking into account (4.8), we get

$$\forall x \in X \setminus \{\bar{x}\}, \; \forall\, y \in F(\bar{x}) : \; F(x) \subseteq y + \mathcal{K}(y) \subseteq \bar{y} + \mathcal{K}(\bar{y}).$$

Moreover, because of $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$, we obtain

$$F(X) \subseteq \bar{y} + \mathcal{K}(\bar{y}) \subseteq \bar{y} + \hat{\mathcal{Q}}'(\bar{x}).$$

Now, we claim that $(\bar{x}, \bar{y}) \in \mathrm{Min}(\mathrm{F(X)}, \hat{\mathcal{Q}}')$. Indeed, suppose that there is $x \in X$ satisfying

$$\begin{aligned} & \bar{y} \in F(x) + \hat{\mathcal{Q}}'(\bar{x}) \setminus \{0\} \\ \Longrightarrow \quad & \exists\, y \in F(x), \; t \in \hat{\mathcal{Q}}'(\bar{x}) \setminus \{0\} : \; \bar{y} = y + t. \end{aligned} \tag{4.9}$$

Since $F(X) \subseteq \bar{y} + \hat{\mathcal{Q}}'(\bar{x})$, it holds that

$$\exists\, t' \in \hat{\mathcal{Q}}'(\bar{x}) : \; y = \bar{y} + t'. \tag{4.10}$$

From (4.9) and (4.10), we get that

$$t = -t' \in \hat{\mathcal{Q}}'(\bar{x}) \setminus \{0\} \cap (-\hat{\mathcal{Q}}'(\bar{x})) = \emptyset,$$

which is a contradiction. The proof is complete. □

We illustrate the previous results by Figure 4.2.

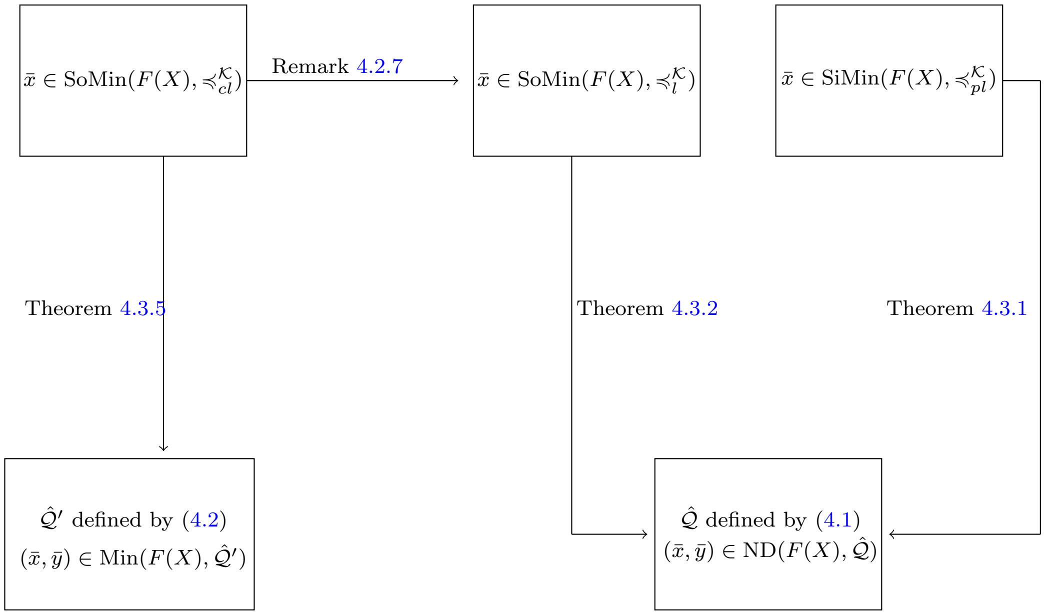


Figure 4.2: Illustration for Theorems 4.3.1, 4.3.2 and 4.3.5.

In order to derive some relationships between the solution concepts from Definitions 4.2.6 and 4.1.1 in the converse direction, it is necessary to introduce the following concepts of domination property of a family of sets $\mathcal{A} \subseteq \mathcal{P}(Y)$. Note that Luc [85] introduced the concept of domination property for a set A in a topological vector space equipped with a proper convex cone, see [85, Definition 4.1]. The following concepts are utilized in [39] in order to study relationships between optimal solutions according to the set approach and those ones according to the vector approach, where the solution concepts of the vector approach is given by Definition 4.1.2.

Definition 4.3.6. *[39, 57] Let a family $\mathcal{A} \subseteq \mathcal{P}(Y)$ and a relation $\preceq^{\mathcal{K}}_t$, $t \in \{u, l, cu, cl, pu, pl\}$ be given. We say that*

*(i) $\mathcal{A}$ has the **weak domination property** w.r.t. $\preceq^{\mathcal{K}}_t$ if for each set $A \in \mathcal{A}$ there exists a family of sets $\Gamma^A_{\mathcal{A}} \subseteq \mathcal{A}$ such that $\Gamma^A_{\mathcal{A}} \subseteq \mathrm{Min}_Y(\mathcal{A}, \preceq^{\mathcal{K}}_t)$ and*

$$\bigcup \{B \mid B \in \Gamma^A_{\mathcal{A}}\} \preceq^{\mathcal{K}}_t A.$$

(ii) $\bar{\mathcal{A}} \subseteq \mathcal{A}$ *has the* ***domination property*** *w.r.t.* $\preceq_t^{\mathcal{K}}$ *if* $\mathrm{Min}_Y(\bar{\mathcal{A}}, \preceq_t^{\mathcal{K}}) \neq \emptyset$ *and for each set* $A \in \mathcal{A}$ *there exists a set* $B \in \mathrm{Min}_Y(\bar{\mathcal{A}}, \preceq_t^{\mathcal{K}})$ *such that* $B \preceq_t^{\mathcal{K}} A$.

Observe that Definition 4.3.6(i) is weaker than Definition 4.3.6(ii) which is first introduced in [57, Definition 4.9] for constant ordering cones.

In the next theorem, we discuss the relationships between nondominated solutions of $(P_{\mathcal{Q}})$ (see Definition 4.1.1) and minimal solutions of a set-valued optimization problem where the solution concept is governed by the relation $\preceq_l^{\mathcal{K}}$ (see Definition 2.2.5(i)).

Theorem 4.3.7. *Consider problem* $(P_{\mathcal{Q}})$*,* $\mathcal{Q} : X \rightrightarrows Y$*, and* $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$*. Let* $\mathcal{K} : Y \rightrightarrows Y$ *be given by*

$$\mathcal{K}(y) := \begin{cases} \bigcap\limits_{x \in X : y \in F(x)} \mathcal{Q}(x) & \text{if } y \in F(X), \\ \{0\} & \text{if } y \notin F(X). \end{cases} \tag{4.11}$$

Suppose that $\mathrm{Min}(F(X), \preceq_l^{\mathcal{K}}) \neq \emptyset$ *and* $\mathcal{F}(X) := \{F(x) |\ x \in X\}$ *has the weak domination property w.r.t.* $\preceq_l^{\mathcal{K}}$*. Then, there exists* $x' \in \mathrm{Min}(F(X), \preceq_l^{\mathcal{K}})$ *provided that* $\bar{y} \in F(x')$*.*

Proof. Since $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$, it holds that

$$\bar{y} \notin \bigcup_{x \in X} (F(x) + (\mathcal{Q}(x) \setminus \{0\})). \tag{4.12}$$

Taking into account the weak domination property of $\mathcal{F}(X)$, it follows that there exists a family of set $\tilde{\mathcal{F}}(X) \subseteq \mathcal{F}(X)$, $\tilde{\mathcal{F}}(X) \subseteq \mathrm{Min}_Y(\mathcal{F}(X), \preceq_l^{\mathcal{K}})$ such that

$$\bigcup \{B |\ B \in \tilde{\mathcal{F}}(X)\} \preceq_l^{\mathcal{K}} F(\bar{x}). \tag{4.13}$$

Let $B \in \tilde{\mathcal{F}}(X)$. Then, there is $x^* \in X$ satisfying $B = F(x^*)$ and $x^* \in \mathrm{Min}(F(X), \preceq_l^{\mathcal{K}})$. We assume that $\bar{S}$ is a collection of such x^*. Therefore, $\bar{S} \subseteq \mathrm{Min}(F(X), \preceq_l^{\mathcal{K}}) \subseteq X$. Let $F(\bar{S}) := \bigcup \{B |\ B \in \tilde{\mathcal{F}}(X)\}$. Taking into account (4.13), it holds that $F(\bar{S}) \preceq_l^{\mathcal{K}} F(\bar{x})$. This is equivalent to

$$F(\bar{x}) \subseteq \bigcup_{y \in F(\bar{S})} (y + \mathcal{K}(y)).$$

Since $\bar{y} \in F(\bar{x})$, there is $y \in F(x')$, $x' \in \bar{S}$ such that

$$\bar{y} \in y + \mathcal{K}(y) = y + \bigcap_{\{x \in X :\ y \in F(x)\}} \mathcal{Q}(x) \subseteq y + \mathcal{Q}(x'). \tag{4.14}$$

(4.12) and (4.14) imply that $y = \bar{y} \in F(x')$.
In addition, $F(x') \in \mathrm{Min}_Y(F(X), \preceq_l^{\mathcal{K}})$, i.e., $x' \in \mathrm{Min}(F(X), \preceq_l^{\mathcal{K}})$, and the proof is complete. □

Note that the idea in the proof of Theorem 4.3.7 is similar to that one of [39, Theorem 5.1] where the authors used the solution concepts in the sense of Definition

4.1.2. To this end, we explain the relationships between nondominated solutions of $(P_{\mathcal{Q}})$ (see Definition 4.1.1) and minimal solutions of a set-valued optimization problem where the solution concept is governed by the relation $\preceq^{\mathcal{K}}_{cl}$ (see Definition 2.2.5, (iii)).

Theorem 4.3.8. *Assume that* $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$, $\mathcal{Q} : X \rightrightarrows Y$. *Suppose that* $\mathrm{Min}(F(X), \preceq^{\mathcal{K}}_{cl}) \neq \emptyset$ *and* $\mathcal{F}(X)$ *has the weak domination property w.r.t.* $\preceq^{\mathcal{K}}_{cl}$ *with* $\mathcal{K} : Y \rightrightarrows Y$ *given by (4.11). Then, there is* $\bar{S} \subseteq X$ *such that for all* $x \in \bar{S} : F(x) = \{\bar{y}\}$ *and* $x \in \mathrm{Min}(F(X), \preceq^{\mathcal{K}}_{cl})$.

Proof. By using the same arguments as in Theorem 4.3.7, it holds that there is $\bar{S} \subseteq \mathrm{Min}((F(X), \preceq^{\mathcal{K}}_{cl})$ such that $F(\bar{S}) \preceq^{\mathcal{K}}_{cl} F(\bar{x})$. Taking into account the definition of the relation $\preceq^{\mathcal{K}}_{cl}$, we get

$$F(\bar{x}) \subseteq \bigcap_{y \in F(\bar{S})} (y + \mathcal{K}(y)).$$

This yields

$$\forall x \in \bar{S}, \forall\ y \in F(x) : \bar{y} \in y + \mathcal{K}(y) \subseteq y + \mathcal{Q}(x).$$

Taking into account $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$, it holds that $\bar{y} = y$. Since this conclusion holds true for all $y \in F(x)$, we obtain for all $x \in \bar{S} : F(x) = \{\bar{y}\}$. □

The following figure illustrates Theorems 4.3.7 and 4.3.8.

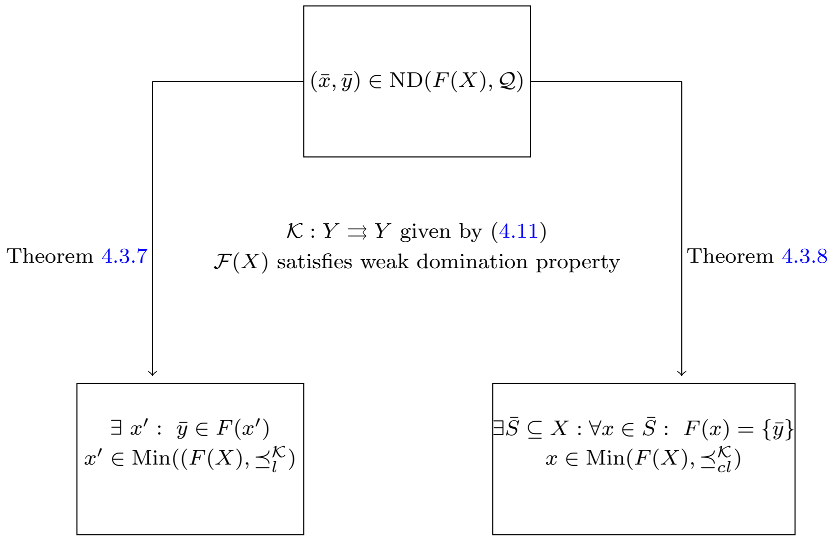


Figure 4.3: Illustration for Theorems 4.3.7 and 4.3.8.

Chapter 5

Characterizations of Solutions of Set Optimization Problems by means of Scalarization

In set optimization w.r.t. fixed cone-valued mappings, both linear and nonlinear scalarizing functionals have been used to characterize many set relations, see [40, 45, 46, 56, 65, 72]. It is also interesting to extend these methods to set-valued optimization problems equipped with variable domination structures. For vector optimization as well as set-valued optimization equipped with a Bishop-Phelp cone-valued mapping, Eichfelder [36] and Eichfelder and Pilecka [40] have characterized solutions of these problems by using nonlinear scalarizing functionals. In addition, Bouza and Tammer [16] have introduced a nonlinear scalarizing functional to characterize and compute minimal points of a set w.r.t. a variable domination structure.

This chapter presents characterizations of solutions of a set-valued optimization problem w.r.t. general variable domination structures via scalarization. For this aim, Section 5.1 introduces corresponding nonlinear scalarizing functionals for six certain set relations introduced in Definition 2.2.5. In this section, we utilize the proposed functionals to characterize these six relations. Section 5.2 uses the nonlinear functionals introduced in Section 5.1 to characterize minimal elements of a set in the sense of Definition 4.2.1. Section 5.3 introduces a minimal time function w.r.t. variable domination structures, which will be used for the main goal of Chapter 7. Section 5.4 derives a descent method to find approximate minimal solutions of a set-valued problem w.r.t. variable domination structures. The results presented within this chapter are based on [69] and [71].

5.1 Characterizations of Set Relations via Scalarization

Let Y be a linear topological space and $\mathcal{K} : Y \rightrightarrows Y$ be a given set-valued map. In the following, we introduce different kinds of nonlinear scalarizing functionals. These functionals are used to describe the set relations given in Definition 2.2.5 as well as to characterize minimal elements of a family of sets in the sense of Definition 4.2.1. Taking into account the assumptions in Theorem 2.4.2, we suppose in the whole chapter the following assumption:

(H_1) Let $A, B \in \mathcal{P}(Y), \mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that $\mathcal{K}(y)$ is closed for all $y \in Y$ and

$$\forall y \in Y : \ \mathcal{K}(y) + (0, +\infty)k^0 \subseteq \mathcal{K}(y).$$

5.1.1 Characterization of the Lower Set Less Relation

This part introduces appropriate scalarizing functionals generalizing the functional (2.19) for the relation $\preceq_l^{\mathcal{K}}$ given in Definition 4.2.1. Furthermore, we show how these functionals describe this set relation. For the following results, we define the following sets for a certain nonempty subset $A \subseteq Y$

$$\widetilde{\mathcal{K}}(A) := \bigcup_{a \in A} \mathcal{K}(a) \subset Y, \tag{5.1}$$

$$\text{and} \ \ \overline{\mathcal{K}}(A) := \bigcap_{a \in A} \mathcal{K}(a) \subset Y. \tag{5.2}$$

In addition, we will use two assumptions concerning the set-valued domination mapping $\mathcal{K} : Y \rightrightarrows Y$ (consequently for $\widetilde{\mathcal{K}}(A)$ and $\overline{\mathcal{K}}(A)$) and $k^0 \in Y \setminus \{0\}$ as follows:

(H_2) Assume that for all $A \in \mathcal{P}(Y)$, $\widetilde{\mathcal{K}}(A)$ is a proper closed set and

$$\forall A \in \mathcal{P}(Y) : \ \widetilde{\mathcal{K}}(A) + (0, +\infty)k^0 \subseteq \widetilde{\mathcal{K}}(A).$$

(H_3) Assume that for all $A \in \mathcal{P}(Y)$, $\overline{\mathcal{K}}(A)$ is a proper closed set and

$$\forall A \in \mathcal{P}(Y) : \ \overline{\mathcal{K}}(A) + (0, +\infty)k^0 \subseteq \overline{\mathcal{K}}(A).$$

In the next theorem, we give a characterization of $\preceq_l^{\mathcal{K}}$ by means of the functional (2.19) with $D = \widetilde{\mathcal{K}}(A)$ and $k = k^0$, i.e., we consider the functional

$$z^{\widetilde{\mathcal{K}}(A),k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - \widetilde{\mathcal{K}}(A)\}.$$

Theorem 5.1.1. *[69, Theorem 2] Let $A, B \in \mathcal{P}(Y)$, $\widetilde{\mathcal{K}}(A) \subset Y$ be given by (5.1) and $k^0 \in Y \setminus \{0\}$ such that (H_2) holds. Assume that $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$. Then, it holds that*

$$A \preceq_l^{\mathcal{K}} B \implies \inf_{a \in A} z^{\widetilde{\mathcal{K}}(A),k^0}(a) \leq \inf_{b \in B} z^{\widetilde{\mathcal{K}}(A),k^0}(b)\,.$$

Proof. Suppose that $A \preceq_l^{\mathcal{K}} B$ holds, i.e.,

$$\forall\, b \in B\ \exists\, a_b \in A:\ b \in a_b + \mathcal{K}(a_b).$$

This implies that

$$\forall\, b \in B\ \exists\, a_b \in A:\ b \in a_b + \widetilde{\mathcal{K}}(A).$$

Under the assumption $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$, the functional $z^{\widetilde{\mathcal{K}}(A),k^0}$ is $\widetilde{\mathcal{K}}(A)$-monotone because of Theorem 2.4.2 (d). Therefore,

$$\forall\, b \in B\ \exists\, a_b \in A:\ z^{\widetilde{\mathcal{K}}(A),k^0}(a_b) \le z^{\widetilde{\mathcal{K}}(A),k^0}(b),$$

which yields the assertion

$$\inf_{a\in A} z^{\widetilde{\mathcal{K}}(A),k^0}(a) \le \inf_{b\in B} z^{\widetilde{\mathcal{K}}(A),k^0}(b).$$

□

Remark 5.1.2. *Note that the property $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$ does not imply that $\widetilde{\mathcal{K}}(A)$ is a convex cone. Consider, for instance, the case $\widetilde{\mathcal{K}}(A) := \mathbb{N}$. Then, $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$ is fulfilled, but $\widetilde{\mathcal{K}}(A)$ is neither a cone nor a convex set.*

Now, we give a characterization of $\preceq_l^{\mathcal{K}}$ by means of the functional (2.19) with $D = a + \widetilde{\mathcal{K}}(A)$ or $D = a + \overline{\mathcal{K}}(A)$, where $A, B \in \mathcal{P}(Y)$ and $a \in A$ are given and $k = k^0$. In other words, we are using the following functionals:

$$z^{a+\widetilde{\mathcal{K}}(A),k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - (a + \widetilde{\mathcal{K}}(A))\},$$

and

$$z^{a+\overline{\mathcal{K}}(A),k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - (a + \overline{\mathcal{K}}(A))\}.$$

Theorem 5.1.3. *[69, Theorem 3] Let $A, B \in \mathcal{P}(Y)$, $\widetilde{\mathcal{K}}(A)$ and $\overline{\mathcal{K}}(A)$ are given by (5.1) and (5.2), respectively. In addition, let $k^0 \in Y \setminus \{0\}$ such that (H_2) and (H_3) are fulfilled. Then, it holds that*

(i) $A \preceq_l^{\mathcal{K}} B \implies \sup_{b\in B} \inf_{a\in A} z^{a+\widetilde{\mathcal{K}}(A),k^0}(-b) \le 0.$

(ii) Suppose that $\inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b)$ is attained for all $b \in B$. Then,

$$\sup_{b\in B} \inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \le 0 \quad \Longrightarrow \quad A \preceq_l^{\mathcal{K}} B.$$

Proof. **(i)** Suppose that $A \preceq_l^{\mathcal{K}} B$, i.e.,

$$\forall\, b \in B\ \exists\, a_b \in A:\ b \in a_b + \mathcal{K}(a_b).$$

This leads to

$$\forall\, b \in B \;\; \exists\, a_b \in A: \;\; a_b - b \in -\widetilde{\mathcal{K}}(A)\,.$$

Because of Theorem 2.4.2 (e), we get

$$\forall\, b \in B \;\; \exists\, a_b \in A: \quad z^{a_b+\widetilde{\mathcal{K}}(A),k^0}(-b) \leq 0$$

and this implies

$$\sup_{b\in B}\;\inf_{a\in A} z^{a+\widetilde{\mathcal{K}}(A),k^0}(-b) \leq 0\,.$$

(ii) Now, let $k^0 \in \overline{\mathcal{K}}(A) \setminus \{0\}$ be given such that

$$\sup_{b\in B}\inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \leq 0.$$

That is,

$$\forall\, b \in B: \;\; \inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \leq 0\,.$$

Because for all $b \in B$, $\inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b)$ is attained, we obtain

$$\forall\, b \in B\; \exists\, \bar{a}_b \in A: \;\; z^{a_b+\overline{\mathcal{K}}(A),k^0}(-b) = \inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \leq 0\,.$$

By Theorem 2.4.2 (e), we conclude

$$\forall\, b \in B\; \exists\, \bar{a}_b \in A: \;\; \bar{a}_b - b \in -\overline{\mathcal{K}}(A) \subseteq -\mathcal{K}(\bar{a}_b),$$

which implies that

$$\forall\, b \in B\; \exists\, \bar{a}_b \in A: \;\; b \in \bar{a}_b + \mathcal{K}(\bar{a}_b),$$

and this means that $A \preceq_l^{\mathcal{K}} B$. □

Remark 5.1.4. *Let us note that the property $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$ is not needed in Theorem 5.1.3.*

Remark 5.1.5. *It is obvious that*

$$\forall B \in \mathcal{P}(Y): \widetilde{\mathcal{K}}(B) \subseteq \widetilde{\mathcal{K}}(Y) \;\; \text{and} \;\; \overline{\mathcal{K}}(Y) \subseteq \overline{\mathcal{K}}(B).$$

Then, under the assumptions given by Theorem 5.1.3, we have that

(i)

$$A \preceq_l^{\mathcal{K}} B \Longrightarrow \sup_{b\in B}\;\inf_{a\in A} z^{a+\widetilde{\mathcal{K}}(Y),k^0}(-b) \leq 0.$$

(ii) Suppose that $\inf_{a\in A} z^{a+\overline{\mathcal{K}}(Y),k^0}(-b)$ is attained for all $b \in B$. Then,

$$\sup_{b\in B}\inf_{a\in A} z^{a+\overline{\mathcal{K}}(Y),k^0}(-b) \leq 0 \quad \Longrightarrow \quad A \preceq_l^{\mathcal{K}} B.$$

We introduce in the following a different scalarizing functional to obtain an equivalence between the set relation $\preceq_l^{\mathcal{K}}$ and properties of this functional mentioned in Theorem 5.1.3. Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$, $k^0 \in Y \setminus \{0\}$ such that (H_1) holds true. A new scalarizing functional is defined as follows:

$$\begin{aligned} g^{\preceq_l^{\mathcal{K}}} &: \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}}, \\ g^{\preceq_l^{\mathcal{K}}}(A, B) &:= \sup_{b \in B} \inf_{a \in A} z^{a+\mathcal{K}(a),k^0}(-b). \end{aligned} \tag{5.3}$$

In (5.3), we are using the functional (2.19) with $D = a + \mathcal{K}(a)$, $a \in A$ fixed, $k = k^0$, such that, for $b \in B$, the functional (2.19) has the form

$$z^{a+\mathcal{K}(a),k^0}(-b) = \inf\{t \in \mathbb{R} | \; -b \in tk^0 - (a + \mathcal{K}(a))\}.$$

In the following theorem, we show the relationships between the value of this new functional (5.3) for $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved w.r.t. the relation $\preceq_l^{\mathcal{K}}$. We also suppose that assumption (H_1) introduced at the beginning of this chapter holds true.

Theorem 5.1.6. *[69, Theorem 4] Consider $A, B \in \mathcal{P}(Y)$, a set-valued map $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following assertions hold true for all $r \in \mathbb{R}$:*

(a) $g^{\preceq_l^{\mathcal{K}}}(A, B) \leq r \Longrightarrow \bigcup_{t>r} (tk^0 + B) \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a))$, *i.e.*, $A \preceq_l^{\mathcal{K}} \bigcup_{t>r} (tk^0 + B)$.

(b) $rk^0 + B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)) \Longrightarrow g^{\preceq_l^{\mathcal{K}}}(A, B) \leq r$.

Proof. **(a)** Suppose that $g^{\preceq_l^{\mathcal{K}}}(A, B) \leq r$ holds true. Consider $\epsilon > 0$, arbitrarily, but fixed. We are using the functional $g^{\preceq_l^{\mathcal{K}}}(\cdot, \cdot)$ given by (5.3). Then, we have that

$$\begin{aligned} & \forall\, b \in B : \inf_{a \in A} z^{a+\mathcal{K}(a),k^0}(-b) < r + \epsilon \\ \Longleftrightarrow\; & \forall\, b \in B : \inf_{a \in A} \inf\{t \in \mathbb{R} : b + tk^0 \in a + \mathcal{K}(a)\} < r + \epsilon \\ \Longleftrightarrow\; & \forall\, b \in B : \exists\, a \in A, \exists\, l < r + \epsilon : b + lk^0 \in a + \mathcal{K}(a). \end{aligned}$$

This means that for all $b \in B$ there exist an element $a \in A$ and an element $l < r + \epsilon$ such that

$$b + (r + \epsilon)k^0 = b + lk^0 + (r + \epsilon - l)k^0 \in a + \mathcal{K}(a) + (r + \epsilon - l)k^0.$$

Taking into account the implication

$$r + \epsilon - l > 0 \Rightarrow (r + \epsilon - l)k^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a),$$

we get that

$$b + (r+\epsilon)k^0 \in a + \mathcal{K}(a).$$

Thus, $\forall\, \epsilon > 0, \forall\, b \in B,$ there is $a \in A$ such that $b + (r+\epsilon)k^0 \in a + \mathcal{K}(a), i.e.,$

$$\bigcup_{t>r}(tk^0 + B) \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

(b) Let $rk^0 + B \subseteq \bigcup\limits_{a\in A}(a+\mathcal{K}(a))$. It holds

$$\begin{aligned}
&\forall\, b \in B : rk^0 + b \in \bigcup_{a\in A}(a+\mathcal{K}(a))\\
\iff\ &\forall\, b\in B, \exists\, a_b \in A : rk^0 + b \in a_b + \mathcal{K}(a_b)\\
\implies\ &\forall\, b \in B :\ \inf_{a\in A} \inf\{t \in \mathbb{R} : tk^0 + b \in a + \mathcal{K}(a)\} \le r\\
\implies\ &\sup_{b\in B}\inf_{a\in A}\inf\{t\in\mathbb{R} : a - b \in tk^0 - \mathcal{K}(a)\} \le r\\
\iff\ &\sup_{b\in B}\inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-b) \le r
\end{aligned}$$

The last relation is also equivalent to $g^{\preceq_l^{\mathcal{K}}}(A,B) \le r$. □

Example 5.1.7. *[69, Example 2] Let us consider two sets in $\mathbb{R}^2$ given by*

$$A := \{(y_1,y_2)\in\mathbb{R}^2 |\ 1 \le y_1, y_2 \le 2\} \textit{ and } B := \left\{(y_1,y_2)\in\mathbb{R}^2 |\ 0\le y_1\le 1, 0 \le y_2 \le \frac{1}{2}\right\}.$$

Furthermore, a set-valued map $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is given by

$$\mathcal{K}(y) := \begin{cases} \mathbb{R}^2_+ & \text{if } y \in \mathbb{R}^2 \setminus \{(2,2)\},\\ \{(d_1,d_2)|\ d_1\in\mathbb{R},\ d_2 \ge 0\} & \text{if } y = (2,2).\end{cases}$$

We choose $k^0 := (1,1)$ and it satisfies (H_1). Obviously, we have that

$$1\cdot k^0 + B = \left\{(y_1,y_2)\in\mathbb{R}^2|\ 1\le y_1\le 2,\ 1 \le y_2 \le \frac{3}{2}\right\} \subseteq \bigcup_{a\in A}(a+\mathcal{K}(a)).$$

Taking into account Theorem 5.1.6 (b), it holds that

$$g^{\preceq_l^{\mathcal{K}}}(A,B) \le 1.$$

Moreover, from Theorem 5.1.6 (a) we get $\bigcup\limits_{t>1}(tk^0+B) \subseteq \bigcup\limits_{a\in A}(a+\mathcal{K}(a))$. See Figure 5.1 for an illustration of this example.

Remark 5.1.8. *Theorem 5.1.6 states that, if the functional $g^{\preceq_l^{\mathcal{K}}}$ given by (5.3) takes values that do not exceed r at (A,B), then the set A is smaller (w.r.t. $\preceq_l^{\mathcal{K}}$) than the union of all sets which are the movements of B along the direction tk^0, where $t > r$. However, the conversion is not always true.*

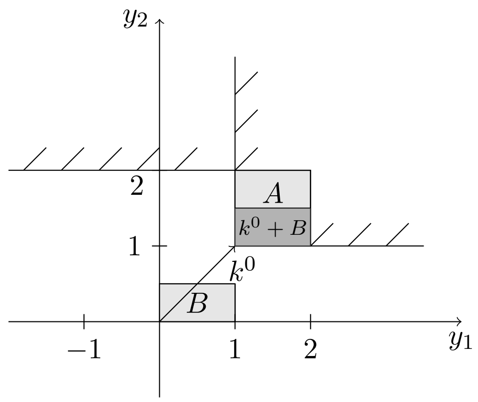


Figure 5.1: Illustration for Example 5.1.7.

In the following, we give an equivalence of the comparison $A \preceq_l^{\mathcal{K}} B$, where $A, B \in \mathcal{P}(Y)$ by means of the functional $g^{\preceq_l^{\mathcal{K}}}$ given by (5.3). Note that we again use the condition (H_1) introduced at the beginning of this chapter.

Theorem 5.1.9. *[69, Theorem 5] Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds true. Suppose that $\bigcup\limits_{a\in A}(a + \mathcal{K}(a))$ is closed. Then,*

$$A \preceq_l^{\mathcal{K}} B \Longleftrightarrow g^{\preceq_l^{\mathcal{K}}}(A, B) \leq 0.$$

Proof. Obviously,

$$A \preceq_l^{\mathcal{K}} B \Longleftrightarrow 0k^0 + B \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

By Theorem 5.1.6 (b), it holds that

$$g^{\preceq_l^{\mathcal{K}}}(A, B) \leq 0.$$

Now, we prove the sufficient condition. By Theorem 5.1.6 (a), we get that

$$\bigcup_{t>0}(tk^0 + B) \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

This means the following assertion holds for all $n > 0$

$$\frac{1}{n}k^0 + B \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

Taking the limit for $n \to +\infty$, we obtain

$$B \subseteq \mathrm{cl}\,\Big(\bigcup_{a\in A}(a + \mathcal{K}(a))\Big) = \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

Therefore,

$$B \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)),$$

i.e., $A \preceq_l^{\mathcal{K}} B$. The proof is complete. □

The following example indicates that even if $B = A$, $g^{\preceq_l^{\mathcal{K}}}(A, B) < 0$ can happen.

Example 5.1.10. *[69, Example 3]*
Let $B := \{(y_1, y_2) \in \mathbb{R}^2 | \ -1 \le y_1, y_2 \le 0$ *and* $y_1 + y_2 \ge -1\}$ *and a set-valued mapping* $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ *be determined as*

$$\forall y \in \mathbb{R}^2 : \ \mathcal{K}(y) := \begin{cases} \mathbb{R}^2_+ & \text{if } (y_1, y_2) \in \mathbb{R}^2 \setminus B, \\ \{(d_1, d_2) \in \mathbb{R}^2 | \ 0 \le d_1\} & \text{if } (y_1, y_2) \in B. \end{cases}$$

Choose $k^0 := (0, 1)$*. Then, for all* $b \in B$*,* $b - tk^0 \in b + \mathcal{K}(b)$ *holds true for all* $t \in \mathbb{R}$*. Therefore,* $g^{\preceq^{\mathcal{K}}_l}(B, B) = -\infty$ *since*

$$g^{\preceq^{\mathcal{K}}_l}(B, B) = \sup_{b \in B} \inf_{a \in B} \inf\{t \in \mathbb{R} | \ a - b \in tk^0 - \mathcal{K}(a)\} = -\infty.$$

See Figure 5.2 for an illustration of this example.

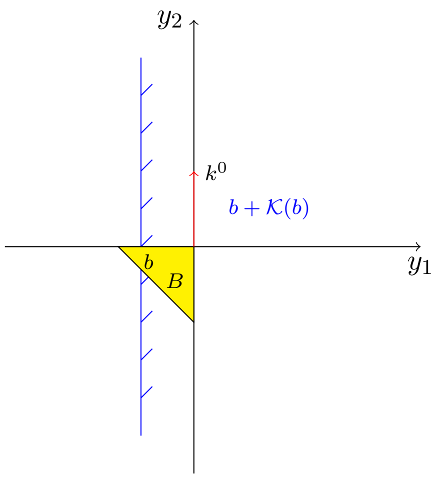


Figure 5.2: Illustration for Example 5.1.10.

In addition, if $B \sim A$, $g^{\preceq^{\mathcal{K}}_l}(A, B) < 0$ can hold. This is be illustrated by the following example.

Example 5.1.11. *[69, Example 4] Let* $A := \{(y_1, y_2) \in \mathbb{R}^2 | \ 1 \le y_1, y_2 \le 2$ *and* $y_1 + y_2 \le 3\}$ *and* $B := \{(2, y_2) | \ 2 \le y_2 \le 3\} \bigcup \{(y_1, 2) | \ 2 \le y_1 \le 3\}$*. Let* $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ *be determined as*

$$\forall y \in \mathbb{R}^2 : \ \mathcal{K}(y) := \begin{cases} \mathbb{R}^2_+ & \text{if } (y_1, y_2) \in \mathbb{R}^2 \setminus B, \\ \{(d_1, d_2) \in \mathbb{R}^2 | -1 \le d_1, -1 \le d_2\} & \text{if } (y_1, y_2) \in B. \end{cases} \tag{5.4}$$

It is clear that $A \sim B$*, since* A *and* B *are both subsets of the following set*

$$\bigcup_{a \in A} (a + \mathcal{K}(a)) = \bigcup_{b \in B} (b + \mathcal{K}(b)) = \{(d_1, d_2) \in \mathbb{R}^2 | \ d_1 \ge 1, d_2 \ge 1\}.$$

Let $k^0 := (1,1)$. Then,

$$g^{\preceq_l^{\mathcal{K}}}(A,B) = g^{\preceq_l^{\mathcal{K}}}(B,B) = -1 < 0.$$

For an illustration, see Figure 5.3.

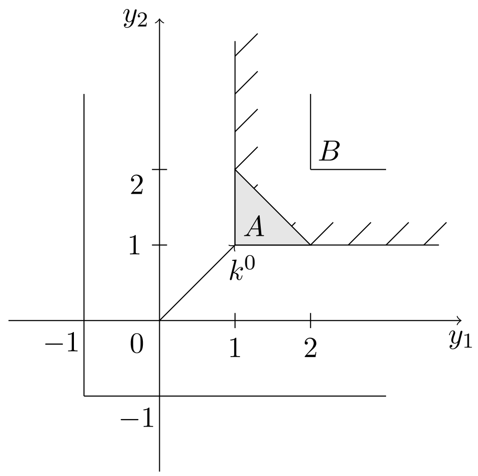


Figure 5.3: Illustration for Example 5.1.11.

5.1.2 Characterization of the Upper Set Less Relation

In the following theorem, we are using the functional (2.19) with $D = -b + \widetilde{\mathcal{K}}(B)$ or $D = -b + \overline{\mathcal{K}}(B)$, where $A, B \in \mathcal{P}(Y)$ and $b \in B$ are given and $k = k^0$. In addition, we again utilize conditions (H_2) and (H_3) introduced at the beginning of Section 5.1.1. The proof is similar to the one given for Theorem 5.1.3 and is therefore skipped.

Theorem 5.1.12. *[69, Theorem 6] Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_2) and (H_3) hold true. Then,*

(i) $A \preceq_u^{\mathcal{K}} B \implies \sup_{a\in A} \inf_{b\in B} z^{-b+\widetilde{\mathcal{K}}(B),k^0}(a) \le 0.$

(ii) Suppose that $\inf_{b\in B} z^{-b+\overline{\mathcal{K}}(B),k^0}(a)$ is attained for all $a \in A$. Then,

$$\sup_{a\in A} \inf_{b\in B} z^{-b+\overline{\mathcal{K}}(B),k^0}(a) \le 0 \quad \implies \quad A \preceq_u^{\mathcal{K}} B.$$

Remark 5.1.13. *It is obvious that*

$$\forall B \in \mathcal{P}(Y) : \widetilde{\mathcal{K}}(B) \subseteq \widetilde{\mathcal{K}}(Y) \text{ and } \overline{\mathcal{K}}(Y) \subseteq \overline{\mathcal{K}}(B).$$

Then, under the assumptions given by Theorem 5.1.12, we have that

(i)

$$A \preceq_u^{\mathcal{K}} B \Longrightarrow \sup_{a\in A}\inf_{b\in B} z^{-b+\tilde{\mathcal{K}}(Y),k^0}(a) \le 0.$$

(ii) If $\inf_{b\in B} z^{-b+\overline{\mathcal{K}}(Y),k^0}(a)$ *is attained for all* $a \in A$ *then,*

$$\sup_{a\in A}\inf_{b\in B} z^{-b+\overline{\mathcal{K}}(Y),k^0}(a) \le 0 \Longrightarrow A \preceq_u^{\mathcal{K}} B.$$

Observe that we cannot get an equivalent statement in Theorem 5.1.12. Therefore, we introduce in the following a different scalarizing functional to obtain an equivalence between the set relation $\preceq_u^{\mathcal{K}}$ and properties of this functional.

Now, let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. We consider the following functional:

$$g^{\preceq_u^{\mathcal{K}}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq_u^{\mathcal{K}}}(A, B) := \sup_{a\in A}\inf_{b\in B} z^{-b+\mathcal{K}(b),k^0}(a). \tag{5.5}$$

In (5.5), we are using the functional (2.19) with $D = -b + \mathcal{K}(b)$, $b \in B$ fixed, and $k = k^0$ such that, for $a \in A$, (2.19) has the form

$$z^{-b+\mathcal{K}(b),k^0}(a) = \inf\{t \in \mathbb{R} |\ a \in tk^0 - (\mathcal{K}(b) - b)\}.$$

Similarly to Theorem 5.1.6, we have the following relationships between the value of the function $g^{\preceq_u^{\mathcal{K}}}$ given by (5.5) at (A, B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets in which A and B are involved w.r.t. the relation $\preceq_u^{\mathcal{K}}$. Note that we again utilize condition (H_1) introduced at the beginning of this chapter.

Theorem 5.1.14. *[69, Theorem 7] Consider* $A, B \in \mathcal{P}(Y)$ *and let* $\mathcal{K} : Y \rightrightarrows Y$ *and* $k^0 \in Y \setminus \{0\}$ *such that* (H_1) *holds. Then, the following characterizations of the relation* $\preceq_u^{\mathcal{K}}$ *by means of the functional* $g^{\preceq_u^{\mathcal{K}}}$ *given by* (5.5) *hold true*

(a) $g^{\preceq_u^{\mathcal{K}}}(A, B) \le r \Longrightarrow \bigcup_{t>r}(A - tk^0) \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b))$, *i.e.,* $\bigcup_{t>r}(A - tk^0) \preceq_u^{\mathcal{K}} B$.

(b) $A - rk^0 \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)) \Longrightarrow g^{\preceq_u^{\mathcal{K}}}(A, B) \le r$.

Proof. **(a)** Let $g^{\preceq_u^{\mathcal{K}}}(A, B) \le r$, and $\epsilon > 0$ be arbitrary. We are using the functional $g^{\preceq_u^{\mathcal{K}}}(\cdot, \cdot)$ given by (5.5). It holds that

$$\begin{aligned}
&\forall\ a \in A : \inf_{b\in B} z^{-b+\mathcal{K}(b),k^0}(a) < r + \epsilon \\
\Longleftrightarrow\ &\forall\ a \in A : \inf_{b\in B} \inf\{t \in \mathbb{R} : a - b \in tk^0 - \mathcal{K}(b)\} < r + \epsilon \\
\Longleftrightarrow\ &\forall\ a \in A, \exists\ b \in B, \exists\ l < r + \epsilon : a - lk^0 \in b - \mathcal{K}(b).
\end{aligned}$$

This means that for all $a \in A$ there exist elements $b \in B$ and $l < r + \epsilon$ such that

$$\begin{aligned} a - (r+\epsilon)k^0 = &\ a - lk^0 - (r+\epsilon-l)k^0 \\ \in &\ b - (\mathcal{K}(b) + (r+\epsilon-l)k^0) \\ \subseteq &\ b - \mathcal{K}(b). \end{aligned}$$

Therefore,

$$\begin{aligned} &\forall\, \epsilon > 0, \forall\, a \in A, \exists\, b \in B : a - (r+\epsilon)k^0 \in b - \mathcal{K}(b) \\ \Longleftrightarrow\ &\bigcup_{t>r}(A - tk^0) \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)). \end{aligned}$$

(b) Let $A - rk^0 \subseteq \bigcup\limits_{b\in B}(b - \mathcal{K}(b))$. It holds that

$$\begin{aligned} &\forall\, a \in A : a - rk^0 \in \bigcup_{b\in B}(b - \mathcal{K}(b)) \\ \Longleftrightarrow\ &\forall\, a \in A, \exists\, b_a \in B : a - rk^0 \in b_a - \mathcal{K}(b_a) \\ \Longrightarrow\ &\forall\, a \in A, \inf_{b\in B} \inf\{t \in \mathbb{R} : a - tk^0 \in b - \mathcal{K}(b)\} \leq r \\ \Longrightarrow\ &\sup_{a\in A} \inf_{b\in B} \inf\{t \in \mathbb{R} : a - b \in tk^0 - \mathcal{K}(b)\} \leq r \\ \Longleftrightarrow\ &g^{\preceq^{\mathcal{K}}_u}(A, B) \leq r. \end{aligned}$$

The proof is complete. □

The following example depicts the above result for the case $r = 2$.

Example 5.1.15. *[69, Example 5] Let $A, B \in \mathcal{P}(\mathbb{R}^2)$ be determined by*

$$A := \{(y_1, y_2) \in \mathbb{R}^2 |\ (y_1 - 6)^2 + (y_2 - 1)^2 \leq 1\},$$

and

$$B := \{(y_1, y_2) \in \mathbb{R}^2 |\ 3 \leq y_1 \leq 5, 3 \leq y_2 \leq 4\}.$$

Let a set-valued mapping $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be given by

$$\mathcal{K}(y) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 |\ 0 \leq d_1 \leq d_2\} & \text{if } y \neq (5,3), \\ \mathbb{R}^2_+ & \text{if } y = (5,3). \end{cases}$$

Choose $k^0 = (1,1)$. Obviously,

$$\forall y \in \mathbb{R}^2,\ \forall t \in [0, +\infty) :\ tk^0 + \mathcal{K}(y) \subseteq \mathcal{K}(y).$$

We have that $A - 2k^0 \subseteq \bigcup\limits_{b\in B}(b - \mathcal{K}(b))$. By Theorem 5.1.14(b), $g^{\preceq^{\mathcal{K}}_u}(A, B) \leq 2$. Furthermore, the following assertion also holds true

$$\bigcup_{t>2}(A - tk^0) \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)).$$

For an illustration of this example, see Figure 5.4.

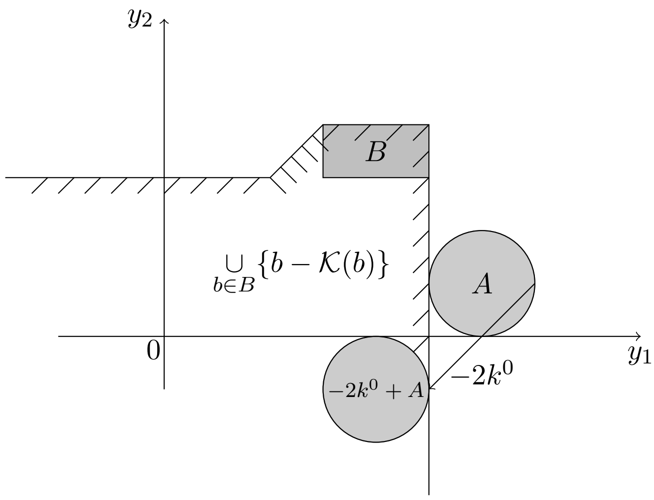


Figure 5.4: Illustration for Example 5.1.15.

The comparison $A \preceq_u^{\mathcal{K}} B$ can be described by an equivalent assertion by means of the functional $g^{\preceq_u^{\mathcal{K}}}$ given by (5.5) as follows:

Theorem 5.1.16. *[69, Theorem 8] Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Suppose that $\bigcup_{b\in B}(b - \mathcal{K}(b))$ is closed. Then, it holds that*

$$A \preceq_u^{\mathcal{K}} B \Longleftrightarrow g^{\preceq_u^{\mathcal{K}}}(A, B) \leq 0.$$

Proof. The necessary condition is a consequence of Theorem 5.1.14(b) with $r := 0$. Now, we prove the sufficient condition. Let $g^{\preceq_u^{\mathcal{K}}}(A, B) \leq 0$. By Theorem 5.1.14(a), it holds that

$$\bigcup_{t>0}(A - tk^0) \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)).$$

This means that for all $n > 0$, we have

$$(A - \frac{1}{n}k^0) \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)).$$

Taking the limit when $n \to +\infty$ we obtain

$$A \subseteq \operatorname{cl}\Big(\bigcup_{b\in B}(b - \mathcal{K}(b))\Big) = \bigcup_{b\in B}(b - \mathcal{K}(b)).$$

Therefore, $A \subseteq \bigcup_{b\in A}(b - \mathcal{K}(b))$, i.e., $A \preceq_u^{\mathcal{K}} B$. The proof is complete.

□

5.1.3 Characterization of the Certainly Lower Less Relation

Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) introduced at the beginning of this chapter holds. We consider a scalarizing functional

$$g^{\preceq_{cl}^{\mathcal{K}}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq_{cl}^{\mathcal{K}}}(A,B) := \sup_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b). \tag{5.6}$$

In (5.6), we are using the functional (2.19) with $D = a+\mathcal{K}(a)$, $a \in A$ fixed, and $k = k^0$. The following result describes the relationships between the value of the functional $g^{\preceq_{cl}^{\mathcal{K}}}$ given by (5.6) at (A,B), where $A, B \in \mathcal{P}(Y)$ and the comparison of sets where A, B are involved w.r.t. the relation $\preceq_{cl}^{\mathcal{K}}$.

Theorem 5.1.17. *[69, Theorem 9] Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following characterizations of the relation $\preceq_{cl}^{\mathcal{K}}$ by means of the functional $g^{\preceq_{cl}^{\mathcal{K}}}$ given by (5.6) hold true*

(a) $g^{\preceq_{cl}^{\mathcal{K}}}(A,B) \le r \Longrightarrow \bigcup_{t>r}(tk^0 + B) \subseteq \bigcap_{a\in A}(a+\mathcal{K}(a))$, *i.e.,* $A \preceq_{cl}^{\mathcal{K}} \bigcup_{t>r}(tk^0+B)$.

(b) $rk^0 + B \subseteq \bigcap_{a\in A}(a+\mathcal{K}(a)) \Longrightarrow g^{\preceq_{cl}^{\mathcal{K}}}(A,B) \le r$.

Proof. **(a)** Let $g^{\preceq_{cl}^{\mathcal{K}}}(A,B) \le r$ and $\epsilon > 0$, arbitrary. It holds that

$$\begin{aligned} & \forall\, b \in B : \sup_{a\in A} z^{a+\mathcal{K}(a),k^0}(-b) < r+\epsilon \\ \Longleftrightarrow\quad & \forall\, b \in B : \sup_{a\in A} \inf\{t \in \mathbb{R} : b + tk^0 \in a + \mathcal{K}(a)\} < r+\epsilon \\ \Longleftrightarrow\quad & \forall\, b \in B : \forall a \in A, \exists\, l < r+\epsilon : b + lk^0 \in a+\mathcal{K}(a). \end{aligned}$$

We have that

$$b + (r+\epsilon)k^0 = b + lk^0 + (r+\epsilon-l)k^0 \in a + \mathcal{K}(a) + (r+\epsilon-l)k^0.$$

Taking into account the implication

$$r+\epsilon-l > 0 \Longrightarrow (r+\epsilon-l)k^0) + \mathcal{K}(a) \subseteq \mathcal{K}(a),$$

we get that

$$b + (r+\epsilon)k^0 \in a + \mathcal{K}(a).$$

Thus, $\forall\, \epsilon > 0, \forall\, b \in B, \forall a \in A$ it holds that $b + (r+\epsilon)k^0 \in a + \mathcal{K}(a)$, *i.e.*,

$$\bigcup_{t>r}(tk^0 + B) \subseteq \bigcap_{a\in A}(a+\mathcal{K}(a)).$$

(b) Let $rk^0 + B \subseteq \bigcap_{a\in A} (a + \mathcal{K}(a))$. We have that

$$\begin{aligned}
& \forall\, b \in B: \ rk^0 + b \in \bigcap_{a\in A} (a + \mathcal{K}(a)) \\
\iff & \forall\, b \in B,\ \forall a \in A : rk^0 + b \in a + \mathcal{K}(a) \\
\implies & \forall\, b \in B,\ \sup_{a\in A} \inf\{t \in \mathbb{R}: \ tk^0 + b \in a + \mathcal{K}(a)\} \le r \\
\implies & \sup_{(a,b)\in A\times B} \inf\{t \in \mathbb{R}: \ a - b \in tk^0 - \mathcal{K}(a)\} \le r \\
\iff & \sup_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b) \le r.
\end{aligned}$$

Obviously, the last relation is equivalent to $g^{\preceq^{\mathcal{K}}_{cl}}(A, B) \le r$. □

We also get a similar result for the comparison $A \preceq^{\mathcal{K}}_{cl} B$ by means of the functional $g^{\preceq^{\mathcal{K}}_{cl}}$ given by (5.6) as follows:

Theorem 5.1.18. *[69, Theorem 10] Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, it holds that*

$$A \preceq^{\mathcal{K}}_{cl} B \iff g^{\preceq^{\mathcal{K}}_{cl}}(A, B) \le 0.$$

Proof. Since for all $a \in A$, $\mathcal{K}(a)$ is closed, $\bigcap_{a\in A} (a+\mathcal{K}(a))$ is closed. Therefore, we apply Theorem 5.1.17 and use the same arguments as in the proof of Theorem 5.1.9 to get the desired conclusion. □

5.1.4 Characterization of the Certainly Upper Less Relation

Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) introduced at the beginning of this chapter holds. We consider a scalarizing functional

$$g^{\preceq^{\mathcal{K}}_{cu}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq^{\mathcal{K}}_{cu}}(A, B) := \sup_{(a,b)\in A\times B} z^{-b+\mathcal{K}(b),k^0}(a). \tag{5.7}$$

In (5.7), we are using the functional (2.19) with $D = -b+\mathcal{K}(b)$, $b \in B$ fixed, and $k = k^0$. Similary to Theorem 5.1.17 and Theorem 5.1.18, we get the following relationships between the value of the function $g^{\preceq^{\mathcal{K}}_{cu}}$ given by (5.7) at (A, B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved w.r.t. the relation $\preceq^{\mathcal{K}}_{cu}$.

Theorem 5.1.19. *[69, Theorems 11 and 12] Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$, $k^0 \in Y \setminus \{0\}$ be given such that (H_1) holds. Then, the following characterizations of the relation $\preceq^{\mathcal{K}}_{cu}$ by means of the functional $g^{\preceq^{\mathcal{K}}_{cu}}$ given by (5.7) hold true*

(a) $g^{\preceq^{\mathcal{K}}_{cu}}(A,B) \leq r \Longrightarrow \bigcup_{t>r}(A - tk^0) \subseteq \bigcap_{b\in B}(b - \mathcal{K}(b))$, *i.e.,* $\bigcup_{t>r}(A - tk^0) \preceq^{\mathcal{K}}_{cu} B$.

(b) $A - rk^0 \subseteq \bigcap_{b\in B}(b - \mathcal{K}(b)) \Longrightarrow g^{\preceq^{\mathcal{K}}_{cu}}(A,B) \leq r$.

(c) $A \preceq^{\mathcal{K}}_{cu} B \iff g^{\preceq^{\mathcal{K}}_{cu}}(A,B) \leq 0$.

5.1.5 Characterization of the Possibly Lower Less Relation

Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) introduced at the beginning of this chapter is fulfilled. In this section, we consider the following scalarizing functional

$$g^{\preceq^{\mathcal{K}}_{pl}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq^{\mathcal{K}}_{pl}}(A,B) := \inf_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b). \tag{5.8}$$

In (5.8), we are using the functional (2.19) with $D = a + \mathcal{K}(a)$, $a \in A$ fixed, and $k = k^0$.

The following theorem illustrates the relationships between the value of the functional $g^{\preceq^{\mathcal{K}}_{pl}}$ given by (5.8) at (A,B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved w.r.t. the relation $\preceq^{\mathcal{K}}_{pl}$.

Theorem 5.1.20. *[69, Theorem 13] Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following assertions hold true*

(a) $g^{\preceq^{\mathcal{K}}_{pl}}(A,B) \leq r \Longrightarrow \bigcup_{t>r}\{tk^0\} \subseteq \bigcup_{(a,b)\in A\times B}(a + \mathcal{K}(a) - b)$.

(b) $rk^0 \in \bigcup_{(a,b)\in A\times B}(a + \mathcal{K}(a) - b) \Longrightarrow g^{\preceq^{\mathcal{K}}_{pl}}(A,B) \leq r$.

Proof. **(a)** Let $g^{\preceq^{\mathcal{K}}_{pl}}(A,B) \leq r$ and $\epsilon > 0$, arbitrary. It yields

$$\begin{aligned} & \exists\, b \in B : \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-b) < r + \epsilon \\ \iff & \exists\, b \in B : \inf_{a\in A} \inf\{t \in \mathbb{R} |\ b + tk^0 \in a + \mathcal{K}(a)\} < r + \epsilon \\ \iff & \exists\, b \in B, \exists a \in A, \exists\, l < r + \epsilon : b + lk^0 \in a + \mathcal{K}(a). \end{aligned}$$

Taking into account the implication $r + \epsilon - l > 0 \Rightarrow\ (r + \epsilon - l)k^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a)$, we get

$$\begin{aligned} & b + (r+\epsilon)k^0 = b + lk^0 + (r + \epsilon - l)k^0 \in a + \mathcal{K}(a) + (r + \epsilon - l)k^0 \\ \Longrightarrow\ & b + (r+\epsilon)k^0 \in a + \mathcal{K}(a). \end{aligned}$$

Thus, $\forall\, \epsilon > 0, \exists\, b \in B, \exists\, a \in A$ it holds that $b + (r+\epsilon)k^0 \in a + \mathcal{K}(a)$, *i.e.*,

$$\bigcup_{t>r}\{tk^0\} \subseteq \bigcup_{(a,b)\in A\times B}(a + \mathcal{K}(a) - b).$$

(b) Let $rk^0 \in \underset{(a,b)\in A\times B}{\cup}(a+\mathcal{K}(a)-b)$. It holds that

$$\begin{aligned}
& \exists\, (a,b) \in A \times B : rk^0 \in (a + \mathcal{K}(a) - b)\\
\Longleftrightarrow\quad & \exists\, (a,b) \in A \times B : tk^0 + b \in a + \mathcal{K}(a)\\
\Longrightarrow\quad & \inf_{(a,b)\in A\times B} \inf\{t \in \mathbb{R} : a - b \in tk^0 - \mathcal{K}(a)\} \leq r\\
\Longleftrightarrow\quad & \inf_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b) \leq r.
\end{aligned}$$

The last relation is equivalent to $g^{\preceq^{\mathcal{K}}_{pl}}(A,B) \leq r$. □

Furthermore, the comparison $A \preceq^{\mathcal{K}}_{pl} B$, where $A, B \in \mathcal{P}(Y)$ can be described by an equivalent statement as follows:

Theorem 5.1.21. *[69, Theorem 14] Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Suppose that for all $A, B \in \mathcal{P}(Y)$, $\bigcup_{(a,b)\in A\times B}(a + \mathcal{K}(a) - b)$ is closed. Then, we have the following characterization of the relation $\preceq^{\mathcal{K}}_{pl}$ be means of the functional $g^{\preceq^{\mathcal{K}}_{pl}}$ given by (5.8)*

$$A \preceq^{\mathcal{K}}_{pl} B \Longleftrightarrow g^{\preceq^{\mathcal{K}}_{pl}}(A,B) \leq 0.$$

Proof. Suppose that $A \preceq^{\mathcal{K}}_{pl} B$. Then, we get that

$$\exists\, (a,b) \in A \times B : b \in a + \mathcal{K}(a).$$

This is equivalent to $0k^0 \in \bigcup_{a\in A\times B}(a+\mathcal{K}(a)-b)$. Applying Theorem 5.1.20 (b), we have that

$$\inf_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b) \leq 0.$$

Conversely, assume that $\inf_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b) \leq 0$. Taking into account Theorem 5.1.20 (a), we get that

$$\bigcup_{t>0}\{tk^0\} \subseteq \bigcup_{(a,b)\in A\times B}(a + \mathcal{K}(a) - b).$$

Therefore, for all $t = \frac{1}{n}, n > 0$ it holds that

$$\frac{1}{n}k^0 \in \bigcup_{(a,b)\in A\times B}(a + \mathcal{K}(a) - b).$$

Let $n \to +\infty$, we have that

$$0 \in \operatorname{cl}\Big(\bigcup_{(a,b)\in A\times B} (a + \mathcal{K}(a) - b)\Big).$$

Taking into account $\bigcup_{(a,b)\in A\times B} (a + \mathcal{K}(a) - b)$ is closed, it holds that

$$0 \in \bigcup_{(a,b)\in A\times B} (a + \mathcal{K}(a) - b), \text{ i.e., } A \preceq^{\mathcal{K}}_{pl} B.$$

The proof is complete. □

5.1.6 Characterization of the Possibly Upper Less Relation

Consiser $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$, $k^0 \in Y \setminus \{0\}$ such that (H_1) introduced at the beginning of this chapter holds and set

$$g^{\preceq^{\mathcal{K}}_{pu}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq^{\mathcal{K}}_{pu}}(A, B) := \inf_{(a,b)\in A\times B} z^{-b+\mathcal{K}(b),k^0}(a). \tag{5.9}$$

In (5.9), we are using the functional (2.19) with $D = -b + \mathcal{K}(b)$, $b \in B$ fixed, and $k = k^0$.

The following results illustrate some relationships between the value of the function $g^{\preceq^{\mathcal{K}}_{pu}}$ given by (5.9) at (A, B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved w.r.t. the relation $\preceq^{\mathcal{K}}_{pu}$. Since their proofs are similar to that of Theorems 5.1.20 and 5.1.21, we skip them in this part.

Theorem 5.1.22. *[69, Theorems 15]Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following assertions hold true*

(a) $g^{\preceq^{\mathcal{K}}_{pu}}(A, B) \le r \Longrightarrow \bigcup_{t>r}\{tk^0\} \subseteq \bigcup_{(a,b)\in A\times B} (a - b + \mathcal{K}(b)).$

(b) $rk^0 \in \bigcup_{(a,b)\in A\times B} (a - b + \mathcal{K}(b)) \Longrightarrow g^{\preceq^{\mathcal{K}}_{pu}}(A, B) \le r.$

In addition, we obtain the equivalent condition for the assertion $A \preceq^{\mathcal{K}}_{pu} B$ by means of the functional $g^{\preceq^{\mathcal{K}}_{pu}}$ given in (5.9) as follows:

Theorem 5.1.23. *[69, Theorem 16] Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Suppose $\bigcup_{(a,b)\in A\times B} (a - b + \mathcal{K}(b))$ is closed. Then, we have the following characterization of the relation $\preceq^{\mathcal{K}}_{pu}$ be means of the functional $g^{\preceq^{\mathcal{K}}_{pu}}$ given by (5.9):*

$$A \preceq^{\mathcal{K}}_{pu} B \Longleftrightarrow g^{\preceq^{\mathcal{K}}_{pu}}(A, B) \le 0.$$

5.2 Characterizations of Minimal Elements defined by Set Relations

The preceding section shows how we characterize set relations using different nonlinear functionals of type (2.19). Since these relations are used to define minimal elements of a family of sets in the sense of Definition 4.2.1, in the following we characterize these minimal elements by means of the corresponding nonlinear functionals. In this part, we again utilize condition (H_1) introduced at the beginning of this chapter and consider the relation $\preceq_t^{\mathcal{K}}$ being reflexive and transitive, where $t \in \{l, u, cl, cu, pl, pu\}$.

Theorem 5.2.1. *[69, Theorem 17] Let $\mathcal{A} \subseteq \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds true. Assume that for $A \in \mathcal{A}$, $\bigcup\limits_{a \in A} (a + \mathcal{K}(a))$ is closed. Then,*

(a) $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if*

$$\forall A \in \mathcal{A}, A \not\sim \bar{A} : g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) > 0.$$

(b) $\bar{A} \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if*

$$\forall A \in \mathcal{A} : \ g^{\preceq_l^{\mathcal{K}}}(\bar{A}, A) \leq 0.$$

(c) $\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if*

$$\forall A \in \mathcal{A}, A \in \mathcal{A} \setminus \bar{A} : \ g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) > 0.$$

Proof. **(a)** Let $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$. We are using the definition of $g^{\preceq_l^{\mathcal{K}}}$ given by (5.3). Suppose by contradiction that

$$\exists\ A \in \mathcal{A}, A \not\sim \bar{A} : \ g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) \leq 0,$$

$$\text{i.e., } \exists\ A \in \mathcal{A}, A \not\sim \bar{A} : \sup_{d \in \bar{A}} \inf_{a \in A} z^{a + \mathcal{K}(a), k^0}(-d) \leq 0.$$

By Theorem 5.1.9, it holds that

$$\sup_{d \in \bar{A}} \inf_{a \in A} z^{a + \mathcal{K}(a), k^0}(-d) \leq 0 \Longrightarrow A \preceq_l^{\mathcal{K}} \bar{A}.$$

Since $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$, $\bar{A} \preceq_l^{\mathcal{K}} A$. Thus, $A \sim \bar{A}$, a contradiction.
Conversely, assume that

$$\forall\ A \in \mathcal{A}, A \not\sim \bar{A} : g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) > 0 \text{ and } \bar{A} \notin \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}}),$$

$$\text{i.e., } \forall\ A \in \mathcal{A}, A \not\sim \bar{A} : \sup_{d \in \bar{A}} \inf_{a \in A} z^{a + \mathcal{K}(a), k^0}(-d) > 0 \text{ and } \bar{A} \notin \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}}).$$

We have the following implication

$$\bar{A} \notin \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}}) \Longrightarrow \exists\ A \in \mathcal{A}, A \preceq_l^{\mathcal{K}} \bar{A} \text{ and } \bar{A} \not\preceq_l^{\mathcal{K}} A.$$

	$A \in \mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$	$A \in \mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$	$A \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$
$t = l$, $\bigcup_{a\in A}(a + \mathcal{K}(a))$ is closed			
$t = u$, $\bigcup_{b\in B}(b - \mathcal{K}(b))$ is closed			
$t = cl$			
$t = cu$	$\forall A \not\sim \bar{A}$: $g^{\preceq_t^{\mathcal{K}}}(A, \bar{A}) > 0$	$\forall A \in \mathcal{A}$: $g^{\preceq_t^{\mathcal{K}}}(\bar{A}, A) \leq 0$	$\forall A \in \mathcal{A} \setminus \{\bar{A}\}$: $g^{\preceq_t^{\mathcal{K}}}(A, \bar{A}) > 0$
$t = pl$, $\bigcup_{a\in A}(a + \mathcal{K}(a) - b)$ is closed			
$t = pu$, $\bigcup_{(a,b)\in A\times B}(a - b + \mathcal{K}(b))$ is closed			

Table 5.1: Characterizations of minimal elements defined by set relations w.r.t. variable domination structures [69, Table 1].

By Theorem 5.1.9, it holds that

$$A \preceq_l^{\mathcal{K}} \bar{A} \Longrightarrow \sup_{d\in\bar{A}} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-d) \leq 0, \text{ a contradiction.}$$

(b) It implies directly from the definition of $\preceq_l^{\mathcal{K}}$ and Theorem 5.1.9.

(c) This part is proved analogously to part (a). □

As for $\preceq_t^{\mathcal{K}}$, where $t \in \{u, cl, cu, pl, pu\}$ we can also obtain similar results as Theorem 5.2.1. We illustrate them by the following table, which describes characterizations for elements in $\mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$ and $\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$ by means of the corresponding scalarizing functionals. We again utilize condition (H_1) introduced at the beginning of this chapter.

Proposition 5.2.2. *[69, Proposition 3] Let Y be a linear topological space and $\mathcal{A}$ be a nonempty subset of $\mathcal{P}(Y)$. Suppose that $\mathcal{K} : Y \rightrightarrows Y$ and k^0 are given such that (H_1) is fulfilled. Assume $\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $\preceq_t^{\mathcal{K}}$ is reflexive, $t \in \{l, u, cl, cu, pl, pu\}$ and the corresponding closedness assumptions given in Table 5.1 are satisfied, then*

$$g^{\preceq_t^{\mathcal{K}}}(\bar{A}, \bar{A}) = \operatorname*{Min}_{V\in\mathcal{A}} g^{\preceq_t^{\mathcal{K}}}(V, \bar{A}).$$

Proof. Since $\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $t \in \{l, u, cl, cu, pl, pu\}$, we have that for all $A \neq \bar{A}$ the following assertion holds true:

$$g^{\preceq_t^{\mathcal{K}}}(A, \bar{A}) > 0.$$

On the other hand, because $\preceq_t^{\mathcal{K}}$ is reflexive, we have that $\bar{A} \preceq_t^{\mathcal{K}} \bar{A}$ and therefore $g^{\preceq_t^{\mathcal{K}}}(\bar{A}, \bar{A}) \leq 0$. Thus,

$$g^{\preceq_t^{\mathcal{K}}}(\bar{A}, \bar{A}) = \operatorname*{Min}_{V\in\mathcal{A}} g^{\preceq_t^{\mathcal{K}}}(V, \bar{A}).$$

□

Note that in vector optimization w.r.t. variable ordering structure, Tammer and Bouza in [16] have introduced a nonlinear scalarization which gets zero value at the minimal points of a given set. We also obtain this result by using our scalarizing functional for the relations $\preceq_l^{\mathcal{K}}$ and $\preceq_u^{\mathcal{K}}$ as follows:

Proposition 5.2.3. *[69, Proposition 4] Let Y be a linear topological space, $\mathcal{A} \subset Y$ be a nonempty set, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that for all $y \in Y$, $\mathcal{K}(y)$ is a closed convex pointed cone and (H_1) holds. Then,*

(a) If $\bar{y}$ is a minimal element of $\mathcal{A}$ and $k^0 \in \mathcal{K}(\bar{y}) \setminus \{0\}$, then

$$g^{\preceq_u^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) = 0.$$

(b) If $\bar{y}$ is a nondominated element of $\mathcal{A}$ and $k^0 \in \mathcal{K}(\bar{y}) \setminus \{0\}$, then

$$g^{\preceq_l^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) = 0.$$

Proof. We consider $\mathcal{A}$ is a family of singleton sets and we only prove part (a) since part (b) can be done by similar lines. By Remark 4.2.2, $\{\bar{y}\} \in \text{SiMin}_Y(\mathcal{A}, \preceq_u^{\mathcal{K}})$. Taking into account Theorem 5.1.16 and $\{\bar{y}\} \preceq_u^{\mathcal{K}} \{\bar{y}\}$, it hold that $g^{\preceq_u^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) \leq 0$. Now, we assume that $g^{\preceq_u^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) < 0$. Then, there exists $t < 0$ such that $\bar{y} - \bar{y} \in tk^0 - \mathcal{K}(\bar{y})$. This implies that $tk^0 \in \mathcal{K}(\bar{y})$. On the other hand, since $\mathcal{K}(\bar{y})$ is a cone, $k^0 \in \mathcal{K}(\bar{y})$ and $-t > 0$, we get that $-tk^0 \in \mathcal{K}(\bar{y})$. Therefore, $tk^0 \in \mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y}))$. Taking into account $\mathcal{K}(\bar{y})$ is pointed, it holds that $k^0 = 0$, a contradiction. □

Recently, [46, 64] have introduced some useful scalarizing functionals in order to investigate well-posedness properties for set optimization problems equipped with fixed cones. This will be further discussed in the next section as well as in Chapter 7.

5.3 A Directional Minimal Time Function w.r.t. Variable Domination Structures

This section introduces an other useful scalarizing functional and investigates several properties of this functional. We will show that this functional has an important property that the scalarizing functionals given in the previous parts do not have, that is, the value of this functional at a minimal point is zero. This property beneficial for us to derive the equivalence between well-posedness property of set-valued optimization problems and that of scalar optimization problems in Section 7.2. For the purpose of the shortness and the main goal in Chapter 7, we only concern the set relation $\preceq_l^{\mathcal{K}}$ in this part. The content of this section is based on Köbis, Le, Tammer and Yao [71].

Let $A, B \in \mathcal{P}(Y)$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) introduced at the beginning of this chapter holds true. We consider a scalarizing map

$$\varphi_{k^0} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$$

given by

$$\varphi_{k^0}(A, B) = \inf\{t \geq 0 |\ A \preceq_l^{\mathcal{K}} tk^0 + B\}, \tag{5.10}$$

where $\inf(\emptyset) = +\infty$.

If there is no confusion, from now, for k^0 satisfying (H_1) and $B \in \mathcal{P}(Y)$ fixed, we write

$$\varphi_{k^0,B}(A) := \varphi_{k^0}(A, B) = \inf\{t \geq 0 |\ A \preceq_l^{\mathcal{K}} tk^0 + B\}. \tag{5.11}$$

Remark 5.3.1. *Obviously, when $A = \{y\}$, $B = \{0\}$ and for all $z \in Y$, $\mathcal{K}(z) = Q$, where Q is a proper closed set in Y instead of taking the infimum over $t \in \mathbb{R}_+$ in (5.11), we take the infimum over $t \in \mathbb{R}$ of the set $\{A \preceq_l^{\mathcal{K}} tk^0 + B\}$ to receive the value $z^{Q,k^0}(y)$ determined by (2.19).*

It is important to mention that, if $A \in \mathcal{P}(Y)$, $B = \{y\}$, and for every $z \in Y$, $\mathcal{K}(z) = \{0\}$, then the scalarizing functional given by (5.10) becomes the directional minimal time function

$$T_{k^0}(A, y) := \varphi_{k^0}(A, y) = \inf\{t \geq 0 |\ tk^0 + y \in A\}, \tag{5.12}$$

which is introduced by Nam and Zălinescu in [92]. The functional (5.12) is called **directional minimal time function**. Moreover, the functional (5.12) has an interesting application in locational analysis, see [92] for more details. Recently, Durea, Pantiruc and Strugariu [28] have generalized the functional (5.12) to the case of a set of directions. As for the functional (5.10), we illustrate in the following another application in location problems of the functional $\varphi_{k^0,B}(A)$, where B is a fixed singleton set, $B = \{y\}$, and some uncertain conditions are involved.

Suppose that $A_1, ..., A_n$ are n concerned destinations. In addition, we denote by vector $y \in Y$ the producer who wants to deliver some products (clothes, food, furniture,...) to these destinations. Each destination A_i has its direction k^i, where $i \in \{1, 2, ..., n\}$. Assume that $\mathcal{K} : Y \rightrightarrows Y$ is a set-valued mapping which describes the changes acting on each point $z \in Y$ during the considered time. These changes often appear in many practical problems, for instance, traffic jams, renovation plans, weather conditions and so on. We suppose that the relation $y + tk^i \in \bigcup\limits_{a \in A_i} (a + \mathcal{K}(a))$, i.e., $A_i \preceq_l^{\mathcal{K}} y + tk^i$, means that the producer y delivers the products to the target A_i successfully, where $i \in \{1, 2, ..., n\}$. Then, the problem of finding the point $y \in \Omega$ such that the total time for the vector y to deliver products to the target sets $\{A_1, .., A_n\}$ can be modeled as follows:

$$\text{Minimize} \sum_{i=1,\ldots,n} \varphi_{k^i}(A_i, y) \text{ subject to } y \in \Omega.$$

Figure 5.5 depicts this model of location optimization for the case $n = 5$. We call that

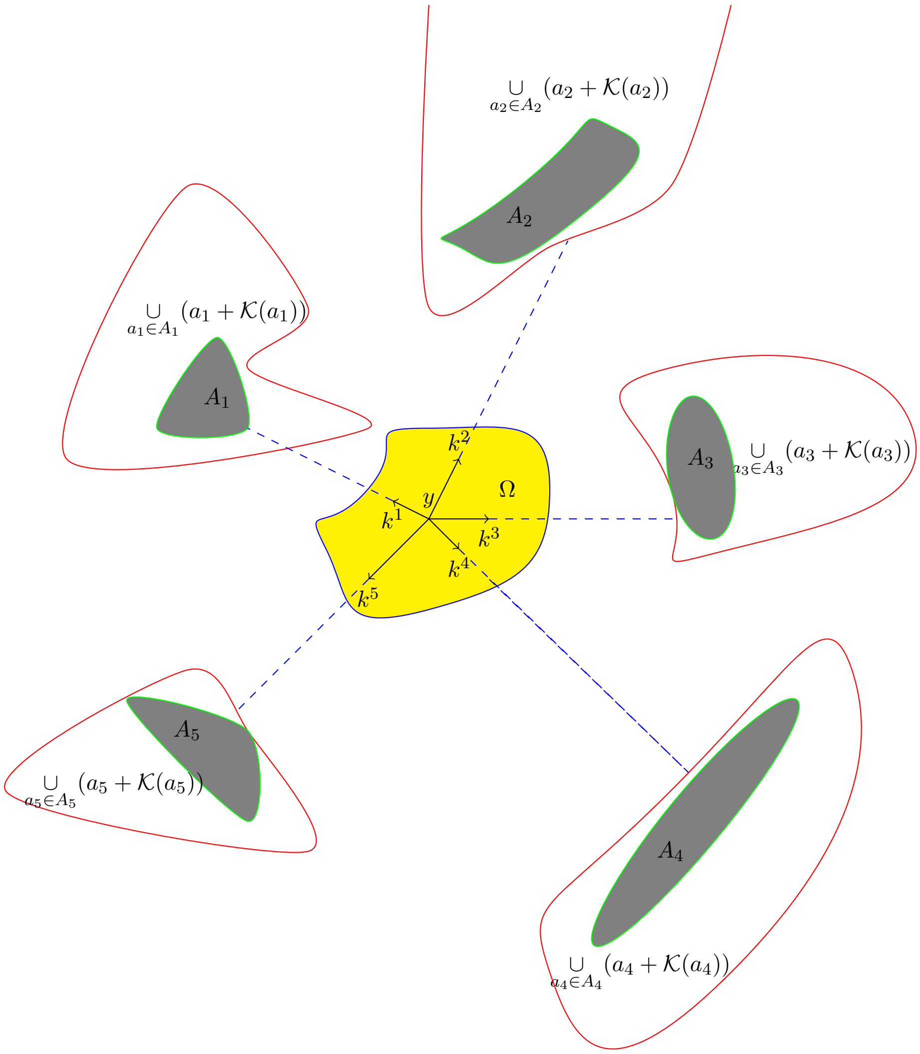


Figure 5.5: A model of location optimization w.r.t. variable domination structures.

$\varphi_{k^0,B}$ is $\preceq_l^{\mathcal{K}}$-monotone if

$$A_1, A_2 \in \mathcal{P}(Y), A_1 \preceq_l^{\mathcal{K}} A_2 \quad \Longrightarrow \quad \varphi_{k^0,B}(A_1) \le \varphi_{k^0,B}(A_2).$$

In the following theorem, we use condition (H_1) introduced at the beginning of this chapter and present several properties of the functional $\varphi_{k^0,B}$ given by (5.11).

Theorem 5.3.2. *[71, Theorem 3.8] Let $A, A_1, A_2, B \in \mathcal{P}(Y)$ and the set-valued map $\mathcal{K} : Y \rightrightarrows Y$ be given. Suppose that $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following properties of the functional $\varphi_{k^0,B}$ are satisfied.*

(a) *If $\mathcal{K}(\cdot)$ satisfies the conditions (2.10) and (2.11), then*

$$\varphi_{k^0,B} \text{ is } \preceq_l^{\mathcal{K}} \text{-monotone.}$$

In addition,

$$A_1 \sim A_2 \quad \Longrightarrow \quad \varphi_{k^0,B}(A_1) = \varphi_{k^0,B}(A_2).$$

(b) *If $\mathcal{K}(y+tk^0) = \mathcal{K}(y)$ for all $y \in Y$ and $t \in \mathbb{R}$, then $\varphi_{k^0,B}(A+rk^0) = \varphi_{k^0,B}(A)+r$ for all $r \in \mathbb{R}_+$.*

(c) *For all $r \in \mathbb{R}_+$, it holds that*

$$\varphi_{k^0,B}(A) \leq r \quad \Longleftrightarrow \quad \bigcup_{t>r}(tk^0 + B) \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

(d) *If $\mathcal{K}(\cdot)$ satisfies (2.9), then $\varphi_{k^0,B}(B)$=0.*

(e) *Suppose that for all $A \in \mathcal{P}(Y)$ the set $\bigcup_{a\in A}(a+\mathcal{K}(a))$ is closed and $\mathcal{K}(\cdot)$ satisfies (2.9), (2.10) and (2.11). Then,*

$$\varphi_{k^0,B}(A)=0 \quad \Longleftrightarrow \quad A \preceq_l^{\mathcal{K}} B.$$

(f) *Let $A, B \in \mathcal{K}(Y)$. Suppose that $\mathcal{K}(\cdot)$ satisfies (2.9), (2.10) and (2.11). Then,*

$$A \sim B \quad \Longleftrightarrow \quad \bigcup_{a\in A}(a + \mathcal{K}(a)) = \bigcup_{b\in B}(b + \mathcal{K}(b)).$$

(g) *If B is $\mathcal{K}(A)$-bounded and for all $r > 0$, it holds that $r \operatorname{int} \mathcal{K}(A) + \mathcal{K}(A) \subseteq \mathcal{K}(A)$, then $\varphi_{k^0,B}(A) < +\infty$ for all $k^0 \in \operatorname{int} \mathcal{K}(A)$.*

Proof. **(a)** Let $A_1, A_2 \in \mathcal{P}(Y)$ such that $A_1 \preceq_l^{\mathcal{K}} A_2$. It is sufficient to prove that

$$\{t \in \mathbb{R}_+ |\ A_1 \preceq_l^{\mathcal{K}} tk^0 + B\} \supseteq \{t \in \mathbb{R}_+ |\ A_2 \preceq_l^{\mathcal{K}} tk^0 + B\}.$$

The above assertion is obvious if $\{t \in \mathbb{R}_+ |\ A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} = \emptyset$. Now, we consider the case $\{t \in \mathbb{R}_+ |\ A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} \neq \emptyset$. Let $t \in \mathbb{R}_+$ such that $A_2 \preceq_l^{\mathcal{K}} tk^0 + B$. This implies $tk^0 + B \subseteq \bigcup_{a\in A_2}(a + \mathcal{K}(a))$, i.e., for arbitrary $b \in B$, there exists $a_b^2 \in A_2$ satisfying $tk^0 + b \in a_b^2 + \mathcal{K}(a_b^2)$. Since $A_1 \preceq_l^{\mathcal{K}} A_2$ and $a_b^2 \in A_2$, we obtain $\exists\ a_b^1 \in A_1$ such that $a_b^2 \in a_b^1 + \mathcal{K}(a_b^1)$, i.e., $\exists\ d_1 \in \mathcal{K}(a_b^1)$ satisfies $a_b^2 = a_b^1 + d_1$. We have that

$$\begin{aligned} tk^0 + b &\in a_b^1 + d_1 + \mathcal{K}(a_b^1 + d_1) \subseteq a_b^1 + \mathcal{K}(a_b^1) + \mathcal{K}(a_b^1 + d_1) \\ &\subseteq a_b^1 + \mathcal{K}(a_b^1) \subseteq \bigcup_{a\in A_1}(a + \mathcal{K}(a)). \end{aligned}$$

Therefore,

$$\begin{aligned} tk^0 + B &\subseteq \bigcup_{a\in A_1} (a + \mathcal{K}(a)) \\ \Longrightarrow\ & A_1 \preceq_l^{\mathcal{K}} tk^0 + B \\ \Longleftrightarrow\ & t \in \{t \in \mathbb{R}_+ |\ A_1 \preceq_l^{\mathcal{K}} tk^0 + B\}. \end{aligned}$$

Taking into account that t be arbitrarily chosen in $\mathbb{R}_+$ and $A_2 \preceq_l^{\mathcal{K}} tk^0 + B$, it holds that

$$\begin{aligned} &\{t \in \mathbb{R}_+ |\ A_1 \preceq_l^{\mathcal{K}} tk^0 + B\} \supseteq \{t \in \mathbb{R}_+ |\ A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} \\ \Longrightarrow\ & \inf\{t \in \mathbb{R}_+ |\ A_1 \preceq_l^{\mathcal{K}} tk^0 + B\} \le \inf\{t \in \mathbb{R}_+ |\ A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} \\ \Longleftrightarrow\ & \varphi_{k^0,B}(A_1) \le \varphi_{k^0,B}(A_2), \text{ i.e., } \varphi_{k^0,B} \text{is } \preceq_l^{\mathcal{K}}\text{-monotone.} \end{aligned}$$

Now, we prove the second assertion. Suppose that $A_1 \sim A_2$, by the observation that

$$A_1 \sim A_2 \quad \Longleftrightarrow \quad A_1 \preceq_l^{\mathcal{K}} A_2 \text{ and } A_2 \preceq_l^{\mathcal{K}} A_1,$$

and taking into account the $\preceq_l^{\mathcal{K}}$ −monotonicity of $\varphi_{k^0,B}$, it holds that

$$\varphi_{k^0,B}(A_1) \le \varphi_{k^0,B}(A_2) \text{ and } \varphi_{k^0,B}(A_2) \le \varphi_{k^0,B}(A_1), \text{ respectively.}$$

Hence, $\varphi_{k^0,B}(A_1) = \varphi_{k^0,B}(A_2)$.
(b) Let $\hat{t} \in \mathbb{R}_+$ such that $A \preceq_l^{\mathcal{K}} \hat{t}k^0 + B$. It holds that

$$\begin{aligned} \hat{t}k^0 + B &\subseteq \bigcup_{a\in A} (a + \mathcal{K}(a)) \\ \Longleftrightarrow\ & (\hat{t}+r)k^0 + B \subseteq \bigcup_{a\in A} (a + rk^0 + \mathcal{K}(a)) \\ \Longleftrightarrow\ & (\hat{t}+r)k^0 + B \subseteq \bigcup_{a\in A} (a + rk^0 + \mathcal{K}(a + rk^0)) \\ \Longleftrightarrow\ & (\hat{t}+r) \in \{t \in \mathbb{R}_+ |\ A + rk^0 \preceq_l^{\mathcal{K}} tk^0 + B\}. \end{aligned}$$

Therefore, $\{t \in \mathbb{R}_+ |\ A \preceq_l^{\mathcal{K}} tk^0 + B\} + r = \{t \in \mathbb{R}_+ |\ A + rk^0 \preceq_l^{\mathcal{K}} tk^0 + B\}$.
Taking the infimum over $t \in \mathbb{R}_+$, we get

$$\inf\{\{t \in \mathbb{R}_+ |\ A \preceq_l^{\mathcal{K}} tk^0 + B\} + r\} = \inf\{t \in \mathbb{R}_+ |\ A + rk^0 \preceq_l^{\mathcal{K}} tk^0 + B\}.$$

This yields $\varphi_{k^0,B}(A) + r = \varphi_{k^0,B}(A + rk^0)$.
(c) Suppose that $\varphi_{k^0,B}(A) = u$ and $r \in \mathbb{R}_+$ such that $u \le r$.
We prove that the following assertion holds true for all $t > u$:

$$tk^0 + B \subseteq \bigcup_{a\in A} (a + \mathcal{K}(a)).$$

By the definition of infimum and $\varphi_{k^0,B}(A)$, there is $\bar{t}$, $u \leq \bar{t} < t$ such that

$$A \preceq_l^{\mathcal{K}} \bar{t}k^0 + B, \text{ i.e., } \bar{t}k^0 + B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)).$$

Therefore,

$$tk^0 + B = \bar{t}k^0 + B + (t - \bar{t})k^0 \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)) + (t - \bar{t})k^0.$$

Taking into account (H_1) we get that

$$\bigcup_{a \in A} (a + \mathcal{K}(a)) + (t - \bar{t})k^0 \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)).$$

This implies

$$tk^0 + B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)), \text{ i.e., } A \preceq_l^{\mathcal{K}} tk^0 + B.$$

Now, let $t > r$ arbitrary. Since $r \geq u$, $t > u$ and thus $tk^0 + B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a))$. This implies $\bigcup_{t > r} (tk^0 + B) \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a))$, which finishes the proof of the necessary condition.

Now, we prove the sufficient condition. Assume by contradiction that

$$\bigcup_{t > r} (tk^0 + B) \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)) \text{ and } \varphi_{k^0,B}(A) = v > r.$$

Let $\epsilon := v - r > 0$ and $v' := r + \frac{\epsilon}{2}$. We have that

$$v > v' > r \text{ and } v'k^0 + B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)), \ i.e., \ A \preceq_l^{\mathcal{K}} v'k^0 + B.$$

Taking into account the definition of $\preceq_l^{\mathcal{K}}$, it holds that

$$\varphi_{k^0,B}(A) = \inf\{t \in \mathbb{R}_+ \mid \ A \preceq_l^{\mathcal{K}} tk^0 + B\} \leq v'.$$

Therefore, $\varphi_{k^0,B}(A) \leq v' < v$, a contradiction, and the proof of the sufficient condition is complete.

(d) Obviously, the following relations hold true for all $t > 0$

$$\begin{aligned} tk^0 + B = \bigcup_{b \in B} (b + 0 + tk^0) &\subseteq \bigcup_{b \in B} (b + \mathcal{K}(b) + tk^0) \\ &\subseteq \bigcup_{b \in B} (b + \mathcal{K}(b)). \end{aligned}$$

Then,

$$\bigcup_{t > 0} (tk^0 + B) \subseteq \bigcup_{b \in B} (b + \mathcal{K}(b)).$$

Taking into account part (c), we get that $\varphi_{k^0,B}(B) \leq 0$. In addition, since the definition of $\varphi_{k^0,B}(B)$, $\varphi_{k^0,B}(B) \geq 0$. Therefore, $\varphi_{k^0,B}(B) = 0$.

(e) The sufficient condition is a consequence of part (a) and part (d).

Conversely, if $\varphi_{k^0,B}(A)$=0, by part (c) it holds that

$$\bigcup_{t>0}(tk^0+B)\subseteq\bigcup_{a\in A}(a+\mathcal{K}(a)).$$

Take $b\in B$ arbitrary, it is clear that for all $n>0$ we have

$$\frac{1}{n}k^0+b\subseteq\bigcup_{a\in A}(a+\mathcal{K}(a)).$$

Therefore, taking the limit when $n\to+\infty$ we obtain

$$b\in\operatorname{cl}\Big(\bigcup_{a\in A}(a+\mathcal{K}(a))\Big)=\bigcup_{a\in A}(a+\mathcal{K}(a)).$$

Thus, $B\subseteq\bigcup\limits_{a\in A}(a+\mathcal{K}(a))$, i.e., $A\preceq_l^{\mathcal{K}}B$.

(f) $A\sim B$ implies that $A\preceq_l^{\mathcal{K}}B$, *i.e.,* $B\subseteq\bigcup\limits_{a\in A}(a+\mathcal{K}(a))$. Let $b\in B$ arbitrary. There exist $a_b\in A$ and $d_b\in\mathcal{K}(a_b)$ such that $b=a_b+d_b$. Since $\mathcal{K}$ satisfies (2.11), $\mathcal{K}(b)=\mathcal{K}(a_b+d_b)\subseteq\mathcal{K}(a_b)$. Taking into account that $\mathcal{K}$ satisfies (2.10), we have

$$\begin{aligned}b+\mathcal{K}(b)&=a_b+d_b+\mathcal{K}(b)\\&\subseteq a_b+\mathcal{K}(a_b)+\mathcal{K}(a_b)\\&\subseteq a_b+\mathcal{K}(a_b).\end{aligned}$$

Therefore, $b+\mathcal{K}(b)\subseteq\bigcup\limits_{a\in A}(a+\mathcal{K}(a))$. Because b is taken arbitrarily, it holds that

$$\bigcup_{b\in B}(b+\mathcal{K}(b))\subseteq\bigcup_{a\in A}(a+\mathcal{K}(a)).$$

Similarly, we get

$$\bigcup_{a\in A}(a+\mathcal{K}(a))\subseteq\bigcup_{b\in B}(b+\mathcal{K}(b)).$$

Therefore,

$$\bigcup_{a\in A}(a+\mathcal{K}(a))=\bigcup_{b\in B}(b+\mathcal{K}(b)).$$

Conversely, suppose that $\bigcup\limits_{a\in A}(a+\mathcal{K}(a))=\bigcup\limits_{b\in B}(b+\mathcal{K}(b))$. We will prove that $A\sim B$. Since $0\in\mathcal{K}(y)$ for all $y\in Y$, we have that

$$\begin{aligned}&A\subseteq\bigcup_{a\in A}(a+\mathcal{K}(a))=\bigcup_{b\in B}(b+\mathcal{K}(b))\\\Longrightarrow\ &A\subseteq\bigcup_{b\in B}(b+\mathcal{K}(b))\\\Longrightarrow\ &B\preceq_l^{\mathcal{K}}A.\end{aligned}$$

Similarly, $A \preceq_l^{\mathcal{K}} B$, and thus $A \sim B$.
(g) Since B is $\mathcal{K}(A)$-bounded and $\operatorname{int}(A) - k^0$ is a neighborhood of 0, there is $r > 0$ such that

$$\begin{aligned} & B \subseteq r(\operatorname{int} \mathcal{K}(A) - k^0) + \mathcal{K}(A) \subseteq -rk^0 + \mathcal{K}(A) \\ \Longrightarrow \; & B + rk^0 \subseteq \mathcal{K}(A) = \bigcup_{a \in A} (a + \mathcal{K}(a)) \\ \Longrightarrow \; & B + rk^0 \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)) \\ \Longrightarrow \; & \varphi_{k^0,B}(A) \leq r, \text{ i.e., } \varphi_{k^0,B}(A) < +\infty. \end{aligned}$$

The proof is complete. □

Remark 5.3.3. *(i) Theorem 5.3.2 (a)-(f) extends [46, Theorem 4.2], where $\mathcal{K}(y)$ is a constant convex cone $K \subset Y$ for all $y \in Y$. Note that even if B is not a K-proper set, i.e., $B + K = Y$, the assertion (d) holds true. However, $B + K \neq Y$ is needed in the proof of [46, Theorem 4.2] to obtain $\varphi_{k^0,B}(B) = 0$.*

(ii) Let $A, B \in \mathcal{P}(Y)$ such that $A \sim B$, $\bigcup_{a \in A} (a + \mathcal{K}(a))$ is closed and $\mathcal{K}(\cdot)$ satisfies (2.9) -(2.11). Then, it holds from Theorem 5.3.2(e) that $\varphi_{k^0,B}(A) = 0$.
Furthermore, by using the same lines in the proof of Theorem 5.3.2(e), we get the following assertion for all $\gamma \geq 0$ and $A, B \in \mathcal{P}(Y)$ under the assumption that $\bigcup_{a \in A} (a + \mathcal{K}(a))$ is closed:

$$\varphi_{k^0,B}(A) \leq \gamma \quad \Longleftrightarrow \quad \gamma k^0 + B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)), \;\; \textit{i.e., } A \preceq_l^{\mathcal{K}} \gamma k^0 + B.$$

(iii) If $\mathcal{K}(y) = K$, where K is a convex cone with nonempty interior, $\varphi_{k^0,B}(A) < +\infty$ for all K- bounded set B and $k^0 \in \operatorname{int} K$, see [64, Proposition 3.2].

Example 5.3.4. *An example for a set-valued map satisfying the condition in Theorem 5.3.2 (b), which is neither a constant map nor a cone-valued map, can be given as*

$$\mathcal{K}: \; Y \rightrightarrows Y; \quad \mathcal{K}(y) = \mathbb{Z}y + \mathbb{R}k^0, \; \textit{for all } y \in Y.$$

Indeed, we have that

$$\forall \; t \in \mathbb{R}: \; \mathcal{K}(y + tk^0) = \mathbb{Z}(y + tk^0) + \mathbb{R}k^0 = \mathbb{Z}y + \mathbb{R}k^0 = \mathcal{K}(y).$$

Therefore, $\mathcal{K}(y + tk^0) = \mathcal{K}(y)$ for all $y \in Y, \; t \in \mathbb{R}$.

Now, we briefly make a comparison between the scalarizing functional (5.11) and the scalarizing functional $g^{\preceq_l^{\mathcal{K}}}$ given by (5.3) for set optimization equipped with the relation $\preceq_l^{\mathcal{K}}$. Recall that the functional $g^{\preceq_l^{\mathcal{K}}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}}$ is defined as:

$$A, B \in \mathcal{P}(Y), \qquad g^{\preceq_l^{\mathcal{K}}}(A, B) \; := \sup_{b \in B} \inf_{a \in A} z^{a + \mathcal{K}(a), k^0}(-b),$$

where $k^0 \in Y \setminus \{0\}$ is taken such that (H_1) introduced at the beginning of this chapter is fulfilled. The following proposition shows the relationship between $\varphi_{k^0,B}(A)$ and $g^{\preceq_l^{\mathcal{K}}}(A, B)$, where $A, B \in \mathcal{P}(Y)$.

Proposition 5.3.5. *[71, Proposition 3.11] Let A, $B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ such that $g^{\preceq_l^{\mathcal{K}}}(A, B) \in \mathbb{R}_+$. Then, we have that: $\varphi_{k^0,B}(A) = g^{\preceq_l^{\mathcal{K}}}(A, B)$.*

Proof. Suppose that $g^{\preceq_l^{\mathcal{K}}}(A, B) = u \in \mathbb{R}_+$. By Theorem 5.1.6(a), it holds that

$$\bigcup_{t>u}(tk^0 + B) \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

Taking into account Theorem 5.3.2 (c), $\varphi_{k^0,B}(A) \leq u$. Assume by contradiction that $0 \leq \varphi_{k^0,B}(A) = v < u$. Therefore, there exists $w \in \mathbb{R}$ such that $v < w < u$. By Theorem 5.3.2 (c), it holds that

$$wk^0 + B \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a), \ i.e., \ A \preceq_l^{\mathcal{K}} wk^0 + B.$$

Taking into account Theorem 5.1.6(b), we get that $g^{\preceq_l^{\mathcal{K}}}(A, B) \leq w < u$, a contradiction. Therefore, $\varphi_{k^0,B}(A) = u = g^{\preceq_l^{\mathcal{K}}}(A, B)$. □

By using the functional $\varphi_{k^0,B}(\cdot)$, we can obtain characterizations for minimal elements of a family of sets as well as solutions of set-valued optimization problems. Since these results will be directly used in proving well-posedness property for set optimization, we will present them in Chapter 7, see Theorems 7.1.1 and 7.1.3.

5.4 A Descent Method for solving Set Optimization Problems w.r.t. Variable Domination Structures

This section presents a descent method for finding approximations of minimal solutions of a set-valued optimization problem equipped with a variable domination structure. For this aim, we utilize the scalarizing functional $g^{\preceq_l^{\mathcal{K}}}$ given by (5.3) in Section 5.1.1. In the literature, this method has been used in order to solve set-valued optimization problems w.r.t. fixed cones. Jahn [55] proposes a descent method that generates approximations of minimal elements of set-valued optimization problems under convexity assumptions on the considered sets. In [55], the set less order relation is characterized by means of linear functionals. More recently, in [66], the authors propose a similar descent method for obtaining approximations of minimal elements of set-valued optimization problems with a fixed domination structure. In this section, we consider this method for set optimization problems without any convexity assumptions and consider a domination structure which is variable. In addition to providing a numerical method, we show a convergence result. The results presented in this section are based on Köbis, Le and Tammer [69].

Now, we consider a set-valued optimization problem

$$\operatorname*{Min}_{x\in S} F(x), \tag{$\tilde{P}_{\mathcal{K}}$}$$

in the following setting: Let $Y = \mathbb{R}^m$, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map, $S \subseteq \mathbb{R}^n$, $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ such that $\mathcal{K}(y)$ is a proper closed set for each $y \in Y$.

From now on, let $k^0 \in Y \setminus \{0\}$ be given such that

$$\forall y \in Y : \; y+\mathcal{K}(y) + (0, +\infty)k^0 \subseteq \operatorname{int}(y + \mathcal{K}(y)). \tag{5.13}$$

This section provides a descent method to find approximations of minimal solutions of the problem ($\tilde{P}_{\mathcal{K}}$) equipped with the relation $\preceq_l^{\mathcal{K}}$. Note that we can modify this method in order to obtain approximate minimal solutions of ($\tilde{P}_{\mathcal{K}}$) w.r.t. $\preceq_t^{\mathcal{K}}$, where $t \in \{u, cl, cu, pl, pu\}$. However, we restrict ourselves to the relation $\preceq_l^{\mathcal{K}}$ for the sake of brevity.

In the following, we will use the minimality notion of ($\tilde{P}_{\mathcal{K}}$) in the sense of Definition 4.2.6 and we assume that the relation $\preceq_l^{\mathcal{K}}$ is reflexive and transitive. In addition, suppose that for all $x \in S$, $\bigcup_{y\in F(x)} (y + \mathcal{K}(y))$ is closed. Now, we define a functional $p : S \times S \to \mathbb{R}$ as:

$$p(z, x) := \sup_{b\in F(x)} \inf_{a\in F(z)} z^{a+\mathcal{K}(a),k^0}(-b) = g^{\preceq_l^{\mathcal{K}}}(F(z), F(x)),$$

where again $k^0 \in Y \setminus \{0\}$ such that (5.13) holds true.

Notice that the functional $z^{a+\mathcal{K}(a),k^0}(\cdot)$ is well-defined, as $a+\mathcal{K}(a)$ is a closed set. Furthermore, since $k^0 \in Y \setminus \{0\}$ and the condition (5.13) holds true, for each $a \in Y$, $z^{a+\mathcal{K}(a),k^0}(\cdot)$ is continuous (see, Theorem 2.4.2 (a)).

Obviously, it follows from Theorem 5.1.9 that:

$$p(z, x) \leq 0 \Longleftrightarrow F(z) \preceq_l^{\mathcal{K}} F(x).$$

In the following, we present a descent method for computing approximate minimal solutions of ($\tilde{P}_{\mathcal{K}}$) w.r.t. $\preceq_l^{\mathcal{K}}$, where $S = \mathbb{R}^n$. For one given starting point x^0 neighboring points x are tested whether the assertion $F(x) \preceq_l^{\mathcal{K}} F(x^0)$ holds. This can be done by evaluated the extremal term $p(x, x^0)$. Algorithm 1 approximates one minimal solution of problem ($\tilde{P}_{\mathcal{K}}$) w.r.t. $\preceq_l^{\mathcal{K}}$. To find more than one approximation of minimal solutions, one needs to vary the input parameters, such as choosing a different starting point $x^0 \in \mathbb{R}^n$, or modifying the vector k^0. Note that this algorithm is similar to [68, Algorithm 1] where the authors have dealt with set optimization problems w.r.t. a fixed cone.

Algorithm 1 (A descent method for finding an approximation of a minimal solution of the set-valued problem ($\tilde{P}_{\mathcal{K}}$))

Input: $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $S = \mathbb{R}^n$, $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, set relation $\preceq_l^{\mathcal{K}}$,
starting point $x^0 \in \mathbb{R}^n$, $k \in \mathbb{R}^m \setminus \{0\}$) satisfies (5.13), maximal number i_{max} of iterations, number of search directions n_s,
maximal number j_{max} of iterations for the determination of the step size,
initial step size h_0 and minimum step size $h_{\min}$, $\{\lambda_1, \dots, \lambda_N\} \subset [0, 1]$
for $p = 1 : 1 : N$ **do**
  % initialization for the descent method
  $i := 0$, $h := h_0$
  choose n_s points $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^{n_s}$ on the unit sphere around $0_{\mathbb{R}^n}$
  % iteration loop
  while $i \le i_{max}$ **do**
    check $F(x^i + h\tilde{x}^j) \preceq_l^{\mathcal{K}} F(x^i)$ for every $j \in \{1, \dots, n_s\}$ by evaluating extremal$_{term}$ (e. g. $p(x^i + h\tilde{x}^j, x^i) = g^{\preceq_l^{\mathcal{K}}}(F(x^i + h\tilde{x}^j), F(x^i))$.
    Choose the index $n_0 := j$ with the smallest function value extremal$_{term}$.
    if extremal$_{term} \le 0$ **then**
      $x^{i+1} := x^i + h\tilde{x}^{n_0}$ *% new iteration point*
      $j := 1$
      while $F(x^i + (j+1)h\tilde{x}^{n_0}) \preceq_l^{\mathcal{K}} F(x^i + jh\tilde{x}^{n_0})$ and $j \le j_{max}$ **do**
        $j := j + 1$
        $x^{i+1} := x^{i+1} + h\tilde{x}^{n_0}$ *% new iteration point*
      end while
    else
      $h := h/2$
      if $h \le h_{\min}$ **then**
        STOP. Output: $x := x^i$
      end if
    end if
    $i := i + 1$
  end while
end for
Output: A set of approximations x of minimal solutions of the set-valued problem ($\tilde{P}_{\mathcal{K}}$) w.r.t. $\preceq_l^{\mathcal{K}}$.

We now set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In order to show a convergence result for Algorithm 1, we need the following modifications in the algorithm:

(A) Assume that the pattern contains at least one direction of descent whenever a set $F(x^i)$ $(i \in \mathbb{N}_0)$ can be improved.

(B) Let some $\beta \in (0, 1)$ and an arbitrary null sequence $(\epsilon^i)_{i\in\mathbb{N}_0}$ with $\epsilon^i < 0$ for all $i \in \mathbb{N}_0$ be given. While $p(x^{i+1}, x^i) \geq \epsilon^i$, set $h := \beta^q h$ for $q := 0, 1, 2, \dots$ after line 26 of Algorithm 1.

Remark 5.4.1. *It is shown in [55] and [66] that the continuity of the function $p(\cdot, \cdot)$ is a sufficient condition such that assumption (A) holds true. However, we can utilize a weaker condition that ensures the fulfillment of the assumption (A), that is: If for every given element $a \in \mathbb{R}^n$, $p(\cdot, a)$ is continuous, then the assumption (A) also holds true. Indeed, when x^i is not the final iteration point, then there is a descent direction and a point $\bar{x} \in \mathbb{R}^n$ such that $p(\bar{x}, x^i) < 0$. By the continuity of $p(\cdot, x^i)$, it follows that there is some ball $B(\bar{x}, \delta)$ around $\bar{x}$ with radius δ such that for all $x \in B(\bar{x}, \delta)$, $p(x, x^i) < 0$. Therefore, we can get a descent direction by refining the grid, and the assumption (A) is fulfilled. It is also an interesting topic for further research to find other sufficient conditions for this assumption.*

Theorem 5.4.2. *[71, Theorem 18] Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a nonempty and compact-valued map. Let Algorithm 1 with the additional specifications (A) and (B) generate an iteration sequence $(x^i)_{i\in\mathbb{N}}$, where $x^0 \in \mathbb{R}^n$ denotes the initial iteration point. In addition, assume that $z^{y+\mathcal{K}(y),k^0}$ is finite-valued for all $y \in Y$ and k^0 satisfying (5.13). Then, $\limsup_{i\to+\infty} p(x^{i+1}, x^i) = 0$.*

Proof. Observe that for all $i \in \mathbb{N}, F(x^i) \preceq_l^{\mathcal{K}} F(x^0)$ since the relation $\preceq_l^{\mathcal{K}}$ is transitive.

Because $F(x^i)$ is compact for all $x^i \in \mathbb{R}^n$, there is $a^{i+1} \in F(x^{i+1})$, $b^i \in F(x^i)$ such that $p(x^{i+1}, x^i) = z^{a^{i+1}+\mathcal{K}(a^{i+1}),k^0}(-b^i)$. Thus, $\{p(x^{i+1}, x^i)\}$, $i \in \mathbb{N}_0$ is bounded. Consequently, there exists $\alpha := \limsup_{i\to+\infty} p(x^{i+1}, x^i)$. We now assume that $\alpha \neq 0$. By specification (A), it holds that $p(x^{i+1}, x^i) \leq 0$. Therefore,

$$\forall i \in \mathbb{N}_0 : p(x^{i+1}, x^i) \leq 0.$$

Taking into account $\limsup_{i\to+\infty} p(x^{i+1}, x^i) = \alpha \neq 0$, we get that $\alpha < 0$ and

$$\exists\, n_1 \in \mathbb{N} : \forall r > n_1,\ p(x^{i_r+1}, x^{i_r}) \leq \frac{\alpha}{2} < 0.$$

Let $\{\epsilon^i\}_{i\in\mathbb{N}_0}$ be a null sequence. Then, there is $n_2 \in \mathbb{N}$ such that

$$\forall r \geq n_2 :\ \frac{\alpha}{2} \leq \epsilon^{i_r} < 0.$$

Therefore, let $n = \max\{n_1, n_2\}$ it holds that: $\forall r \geq n :\ p(x^{i_r+1}, x^{i_r}) \leq \frac{\alpha}{2} \leq \epsilon^{i_r}$. This is a contradiction to specification (B). □

Chapter 6

Necessary Optimality Conditions for Set Optimization w.r.t. Variable Domination Structures

This chapter is devoted to derive necessary optimality conditions for solutions of set optimization problems w.r.t. variable domination structures. In the literature, necessary optimality conditions for solutions of set optimization problems based on the vector approach are derived by different methods (see, [7, 30] and references therein). Following the set approach, Eichfelder and Pilecka [40, Theorem 5.1] have presented necessary optimality conditions working on a primal approach where certain derivative concepts for set-valued maps are used. Moreover, necessary optimality conditions for solutions of set-valued optimization problems w.r.t. fixed ordering cones are shown by Dempe and Pilecka in [22, 23] by using special set differences, the so-called ℓ-difference and modified Demyanov difference. It is important to mention that all these given results are concerned with set optimization problems w.r.t. domination structures whose values are cones. We begin this chapter by recalling a result given by Durea, Strugariu and Tammer [30]. In this paper, the authors follow the vector approach to derive optimality conditions for solutions of $(P_{\mathcal{Q}})$ based on Mordukhovich coderivative. Section 6.2 studies necessary optimality conditions for solutions of set optimization problems defined by the set approach. For this aim, we utilize the relationships between solution concepts of $(P_{\mathcal{K}})$ and $(P_{\mathcal{Q}})$, which are derived in Chapter 4 to derive optimality conditions for solutions of $(P_{\mathcal{K}})$ w.r.t. the relations $\preceq_l^{\mathcal{K}}$, $\preceq_{cl}^{\mathcal{K}}$, and $\preceq_{pl}^{\mathcal{K}}$. For solutions of $(P_{\mathcal{K}})$ w.r.t. the relations $\preceq_u^{\mathcal{K}}$ and $\preceq_{cu}^{\mathcal{K}}$, we prove the sufficient conditions for the openness of a composition multifunctions where the objective mapping F and the domination $\mathcal{K}(\cdot)$ are involved. These conditions are derived in terms of Mordukhovich's coderivative and have been recently studied for general mappings in [27]. The following results with the exception of the first section are based on Köbis, Le, Tammer and Yao [70].

6.1 Necessary Optimality Conditions for Solutions based on the Vector Approach

In this part, we will recall a theorem about necessary optimality conditions for nondominated solutions of set-valued optimization problems given in [30]. We use the following assumption on the domination structure $\mathcal{Q} : X \rightrightarrows Y$ by which the original problem $(P_{\mathcal{Q}})$ is equipped:

$$\forall x \in X : \ \mathcal{Q}(x) \text{ is a closed, convex, pointed, proper cone.}$$

Chapter 4 has derived relationships between solution concepts of $(P_{\mathcal{K}})$ and $(P_{\mathcal{Q}})$ where two mappings $\hat{\mathcal{Q}}(\cdot)$ and $\hat{\mathcal{Q}}'(\cdot)$ given respectively by (4.1) and (4.2) are involved. Therefore, it is necessary to discuss this assumption concerning $\hat{\mathcal{Q}}(\cdot)$ and $\hat{\mathcal{Q}}'(\cdot)$. We show in the following that when $\mathcal{K}(y)$ is not necessarily a cone-valued map for all $y \in Y$, the above requirement can be fulfilled, cf. [70, Examples 5.4 and 5.5].

Example 6.1.1. *Let $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be given as*

$$\forall (x_1, x_2) \in \mathbb{R}^2 : F(x_1, x_2) := \{(d_1, d_2) |\ 0 \le d_1 \le |x_1|,\ 0 \le d_2 \le |x_2|\}$$

and $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be determined by

$$\forall (d_1, d_2) \in \mathbb{R}^2 : \ \mathcal{K}(d_1, d_2) := \begin{cases} \{(y_1, y_2) |\ y_2 \ge \frac{d_2}{d_1} y_1\} \cup \{(d_1, 0)\} & \text{if } d_1 \ne 0, \\ \{(y_1, y_2) |\ y_1 \le 0, y_2 \ge 0\} & \text{if } d_1 = 0. \end{cases}$$

Then, for all $(x_1, x_2) \in \mathbb{R}^2$ it holds

$$\hat{\mathcal{Q}}(x_1, x_2) = \bigcap_{(d_1, d_2) \in F(x)} \mathcal{K}(d_1, d_2) = \{(y_1,\ y_2) |\ y_1 \le 0,\ y_2 \ge 0\}.$$

It is obvious that for all $(x_1, x_2) \in \mathbb{R}^2$ we have that $\hat{\mathcal{Q}}(x_1, x_2)$ is a closed, convex pointed cone. However, $\mathcal{K}(y_1, y_2)$ is not a cone for all $(y_1, y_2) \in \mathbb{R}^2 \setminus \{0\}$. See Figure 6.1 for the illustration of this example, where the image spaces of $F(\cdot)$, $\mathcal{K}(\cdot)$ and $\hat{\mathcal{Q}}(\cdot)$ are combined.

Example 6.1.2. *Let $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined as*

$$\forall x \in \mathbb{R} : F(x) = \{(d_1, d_2) \in \mathbb{R}^2 |\ d_2 = |x| d_1\}$$

and $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be determined by

$$\forall (d_1, d_2) \in \mathbb{R}^2 : \ \mathcal{K}(d_1, d_2) := \begin{cases} \{(y_1, y_2) |\ |d_2| \le y_2 \le \frac{|d_2|}{|d_1|} y_1\}, & \text{if } d_1 \ne 0 \\ \{(y_1, y_2) |\ y_1 \ge 0,\ y_2 = 0\} & \text{if } d_1 = 0. \end{cases}$$

Then, it holds:

$$\forall x \in \mathbb{R} : \hat{\mathcal{Q}}'(x) = \bigcup_{(d_1, d_2) \in F(x)} \mathcal{K}(d_1, d_2) = \{(y_1, y_2) |\ 0 \le y_2 \le |x| y_1\}.$$

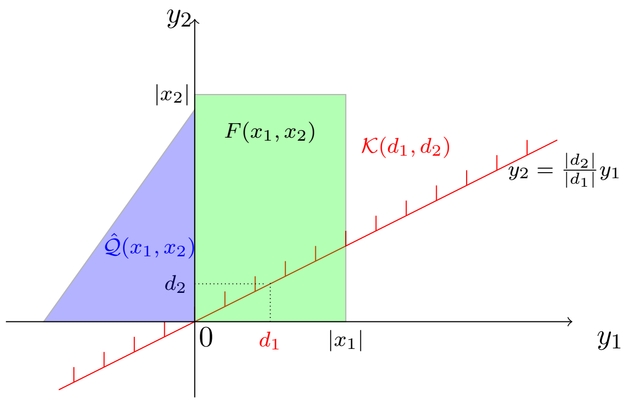


Figure 6.1: Illustration for Example 6.1.1.

This is a closed, convex and pointed cone. However, $\mathcal{K}(d_1, d_2)$ *is not a cone for all* $(d_1, d_2) \in F(x) \setminus \{0\}$*. For an illustration, see Figure 6.2, where the image spaces of* $F(\cdot)$*,* $\mathcal{K}(\cdot)$ *and* $\hat{\mathcal{Q}}'(\cdot)$ *are combined.*

Thus, from now on, we consider the solution concepts of problem $(P_{\mathcal{Q}})$ based on the vector approach with the assumption that for all $x \in X$, $\mathcal{Q}(x)$ ($\mathcal{Q}(x) := \hat{\mathcal{Q}}(x)$ or $\mathcal{Q}(x) := \hat{\mathcal{Q}}'(x)$) is a closed, convex, pointed, proper cone in Y.

Recently, Khan et al. [62] have followed the vector approach to derive second-order optimality conditions solutions of set-valued optimization problems w.r.t. variable domination structures using a second order tangential derivative. Eichfelder and Pilecka [40] also worked on primal approach to derive optimality conditions for minimal points of a set optimization problem w.r.t. the relation $\preceq^{\mathcal{K}}_{pl}$. In this part, we recall a necessary condition for nondominated solutions of $(P_{\mathcal{Q}})$ (see Definition 4.1.1) given by Durea, Strugariu and Tammer in [30, Theorem 4.10]. The main idea in the proof of the necessary condition in [30, Theorem 4.10] is the incompatibility between openness and optimality (previously developed in [29]). It is interesting to mention that for the proof of [30, Theorem 4.10] Ekeland's Variational Principle is involved by the application of sufficient conditions in terms of coderivatives for the openness of the composition of multifunctions in [27, Theorem 4.2]). Observe that the method utilized in [30] is different from the method used in [7] to derive necessary optimality conditions for problem $(P^{vec}_{\mathcal{K}})$ given in Section 3.2.

Theorem 6.1.3. *([30, Theorem 4.10]) Let* X*,* Y *be Asplund spaces,* F*,* $\mathcal{Q} : X \rightrightarrows Y$ *be two set-valued maps such that for all* $x \in X$*,* $F(x) \neq \emptyset$*. Consider the set-valued optimization problem* $(P_{\mathcal{Q}})$ *and* $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \mathcal{Q})$*. Furthermore, assume that* $\mathrm{Gr}\, F$ *and* $\mathrm{Gr}\, \mathcal{Q}$ *are closed around* $(\bar{x}, \bar{y})$ *and* $(\bar{x}, 0)$*, respectively. In addition, suppose that the following assumptions hold:*

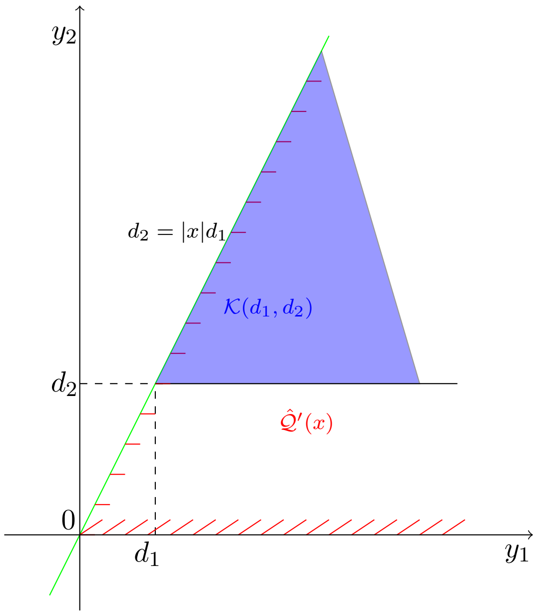


Figure 6.2: Illustration for Example 6.1.2.

(a) The two following sets are allied at $(\bar{x}, \bar{y}, 0)$*:*

$$C_1 = \{(x, y, k)|\ (x, y) \in \operatorname{Gr} F,\ k \in Y\},$$

and

$$C_2 = \{(x, y, k)|\ (x, k) \in \operatorname{Gr} \mathcal{Q},\ y \in Y\}.$$

(b) $\bigcap_{x \in X} \mathcal{Q}(x) \neq \{0\}$;

(c) $\mathcal{Q}$ *is lower semicontinuous at* $\bar{x}$;

(d) F^{-1} *is (PSNC) at* $(\bar{y}, \bar{x})$ *or* $\mathcal{Q}^{-1}$ *is (PSNC) at* $(0, \bar{x})$*.*

Then, there exists $y^* \in \mathcal{Q}(\bar{x})^+ \setminus \{0\}$ *such that*

$$0 \in D^* F(\bar{x}, \bar{y})(y^*) + D^* \mathcal{Q}(\bar{x}, 0)(y^*).$$

The above result will be used in the next section in order to derive optimality conditions for solutions of problem ($P_{\mathcal{Q}}$) w.r.t. the relations $\preceq_t^{\mathcal{K}}$, $t \in \{l, pl, cl\}$ introduced in Definition 2.2.5.

6.2 Necessary Optimality Conditions for Solutions based on the Set Approach

This section is devoted to necessary optimality conditions for solutions of problem ($P_{\mathcal{K}}$) based on the set approach in the sense of Definition 4.2.6. These necessary optimality conditions are derived in terms of Mordukhovich's coderivative for optimal solutions of set-valued problems w.r.t. various set relations introduced in Definition 2.2.5.

First, we show a necessary optimality condition for strong minimal solutions of ($P_{\mathcal{K}}$) w.r.t. the lower less relation w.r.t. $\mathcal{K}(\cdot)$ $\preceq_l^{\mathcal{K}}$, introduced in Definition 2.2.5, (i). The following result is given in [70, Theorem 5.7] with a stronger assumption on $\mathcal{K}(\cdot)$, that is for all $y \in Y$, $\mathcal{K}(y)$ is pointed.

Theorem 6.2.1. *Consider problem ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_l^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_l^{\mathcal{K}})$. Assume that there is an element $\bar{y} \in F(\bar{x})$ satisfying*

$$\forall y \in F(\bar{x}) \setminus \{\bar{y}\} : \bar{y} \notin y + \mathcal{K}(y).$$

Furthermore, suppose that $\mathcal{K}(\cdot)$ satisfies (2.9) -(2.11) and $\mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y})) = \{0\}$. Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ be given by (4.1) and assume that the assumptions in Theorem 6.1.3 hold true for the two multifunctions $F, \hat{\mathcal{Q}}$. Then, there exists $y^* \in \hat{\mathcal{Q}}(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

Proof. Consider $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_l^{\mathcal{K}})$. Theorem 4.3.2 implies that $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}})$ for $\bar{x}$ and $\bar{y} \in F(\bar{x})$ satisfying $\forall y \in F(\bar{x}) \setminus \{\bar{y}\} : \bar{y} \notin y + \mathcal{K}(y)$. Since all assumptions in Theorem 6.1.3 hold true for F and $\hat{\mathcal{Q}}$, it holds that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

The proof is complete. □

Remark 6.2.2. *When $\mathcal{K}(\cdot) \equiv K$, where K is a closed, pointed, convex cone in Y we have that $\hat{\mathcal{Q}}(\cdot) \equiv K$ and $D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*) = \{0\}$. In this case, the optimality condition for a strong minimal solution $\bar{x}$ of ($P_{\mathcal{K}}$) in Theorem 6.2.1 reduces to: $0 \in D^*F(\bar{x}, \bar{y})(y^*)$. This is also the optimality condition for nondominated solutions of problem ($P_{\mathcal{Q}}$) when $\hat{\mathcal{Q}}(\cdot) \equiv K$, which is given in [30, Corollary 4.13].*

The following result is an assertion about a necessary optimality condition for strict minimal solutions of ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_{pl}^{\mathcal{K}}$, introduced in Definition 2.2.5, (v).

Theorem 6.2.3. *Consider problem ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_{pl}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, which satisfies that $\mathcal{K}(y)$ is a proper convex cone for all $y \in Y$, and $\bar{x} \in \mathrm{SiMin}(F(X), \preceq_{pl}^{\mathcal{K}})$. Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ be determined by (4.1). Suppose that there is $\bar{y} \in F(\bar{x})$ satisfying $\bar{y} \notin F(\bar{x}) + \hat{\mathcal{Q}}(\bar{x}) \setminus \{0\}$. Assume that the two multifunctions $F, \hat{\mathcal{Q}}$ satisfy the assumptions given in Theorem 6.1.3. Then, there exists $y^* \in \hat{\mathcal{Q}}(\bar{x})^+ \setminus \{0\}$ such that*

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

Proof. We follow the line of the proof of Theorem 6.2.1. □

Now, we show a necessary optimality condition for strong minimal solutions of ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_{cl}^{\mathcal{K}}$, introduced in Definition 2.2.5 (iii).

The following theorem is a consequence of Theorem 6.2.1. It is proved by directly applying Corollary 4.3.3 and Theorem 6.1.3 and therefore, the proof is skipped.

Theorem 6.2.4. *Consider problem ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_{cl}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_{cl}^{\mathcal{K}})$. Assume that there is an element $\bar{y} \in F(\bar{x})$ satisfying*

$$\forall y \in F(\bar{x}) \setminus \{\bar{y}\} : \bar{y} \notin y + \mathcal{K}(y).$$

Suppose that $\mathcal{K} : Y \rightrightarrows Y$ satisfies properties (2.9) -(2.11) and $\mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y})) = \{0\}$. Let $\hat{\mathcal{Q}} : X \rightrightarrows Y$ given by (4.1) and assume that the assumptions in Theorem 6.1.3 hold true for the two multifunctions $F, \hat{\mathcal{Q}}$. Then, there exists $y^* \in \hat{\mathcal{Q}}(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*) + D^*\hat{\mathcal{Q}}(\bar{x}, 0)(y^*).$$

In addition, we obtain a stronger necessary optimality condition for strong minimal solutions of ($P_{\mathcal{K}}$) w.r.t. $\preceq_{cl}^{\mathcal{K}}$. The following result is given in [70, Theorem 5.10] with a stronger assumption on $\hat{\mathcal{Q}}'(\cdot)$, that is for all $x \in X$, $\hat{\mathcal{Q}}'(x)$ is pointed.

Theorem 6.2.5. *Consider problem ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_{cl}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_{cl}^{\mathcal{K}})$, where $\mathcal{K} : Y \rightrightarrows Y$ satisfies properties (2.10) and (2.11). Suppose that there exists $\bar{y} \in F(\bar{x})$ satisfying $F(\bar{x}) \subseteq \bar{y} + \mathcal{K}(\bar{y})$. Let $\hat{\mathcal{Q}}' : X \rightrightarrows Y$ be given by (4.2) such that*

$$\hat{\mathcal{Q}}'(\bar{x}) \cap (-\hat{\mathcal{Q}}'(\bar{x})) = \{0\}.$$

Assume in addition that the following assumptions are fulfilled

(i) $\hat{\mathcal{Q}}'(\bar{x}) \neq \{0\}$ and $\hat{\mathcal{Q}}'(\bar{x})$ closed ;

(ii) F^{-1} is (PSNC) at $(\bar{y}, \bar{x})$ or $\hat{\mathcal{Q}}'^{-1}$ is (PSNC) at $(0, \bar{x})$.

Then, there exists $y^* \in \hat{\mathcal{Q}}'(\bar{x})^+ \setminus \{0\}$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(y^*).$$

Proof. By Theorem 4.3.5 it holds that $(\bar{x}, \bar{y}) \in \mathrm{Min}(F(X), \hat{\mathcal{Q}}')$. Thus, $(\bar{x}, \bar{y}) \in \mathrm{ND}(F(X), \hat{\mathcal{Q}}'_*)$, where $\hat{\mathcal{Q}}'_* : X \rightrightarrows Y$ is defined by

$$\forall x \in X : \hat{\mathcal{Q}}'_*(x) := \hat{\mathcal{Q}}'(\bar{x}).$$

Now, we will prove that F and $\hat{\mathcal{Q}}'_*$ satisfy all assumptions supposed in Theorem 6.1.3.

$$\begin{aligned} \hat{D}^*\hat{\mathcal{Q}}'_*(u,k)(k^*) &= \{x^* \in X^* |\ (x^*, -k^*) \in \hat{N}(\mathrm{Gr}\,\hat{\mathcal{Q}}'_*, (u,k))\} \\ &= \{x^* \in X^* |\ (x^*, -k^*) \in \hat{N}(X \times \hat{\mathcal{Q}}'(\bar{x}), (u,k)\} \\ &= \{x^* \in X^* |\ (x^*, -k^*) \in \hat{N}(X, u) \times \hat{N}(\hat{\mathcal{Q}}'(\bar{x}), k)\} \\ &= \{0\}. \end{aligned} \tag{6.1}$$

The last equation is obtained by using [99, Proposition 6.41] and $\hat{N}(X,u) = \{0\}$. Therefore, the alliedness property of $(C_1,\ C_2)$ trivially holds (see [30]).
We have that $\bigcap_{x\in X} \hat{\mathcal{Q}}'_*(x) = \hat{\mathcal{Q}}'(\bar{x}) \neq \{0\}$, i.e., assumption (b) in Theorem 6.1.3 is fulfilled. Since $\hat{\mathcal{Q}}'_*(\bar{x}) = \liminf_{x'\to\bar{x}} \hat{\mathcal{Q}}'_*(x') = \hat{\mathcal{Q}}'(\bar{x})$ it holds that $\hat{\mathcal{Q}}'_*$ is l.s.c. at $\bar{x}$ (Remark 2.3.10). Now, we apply Theorem 6.1.3 and get that there exists $y^* \in \hat{\mathcal{Q}}'_*(\bar{x})^+\setminus\{0\} = \hat{\mathcal{Q}}'(\bar{x})^+\setminus\{0\}$ such that

$$0 \in D^*F(\bar{x},\bar{y})(y^*) + D^*\hat{\mathcal{Q}}'_*(\bar{x},0)(y^*).$$

By using the same lines to obtain (6.1), it also holds that $D^*\hat{\mathcal{Q}}'_*(\bar{x},0)(y^*) = \{0\}$ and thus we get the desired conclusion as follows:

$$\exists\ y^* \in \hat{\mathcal{Q}}'(\bar{x})^+ \setminus \{0\} \text{ such that } 0 \in D^*F(\bar{x},\bar{y})(y^*).$$

□

We are going to derive necessary optimality conditions for the minimal solutions of problem $(P_{\mathcal{K}})$ w.r.t. the upper set less relations $(\preccurlyeq_u^{\mathcal{K}}$ and $\preccurlyeq_{cu}^{\mathcal{K}})$. Observe that we can not analogously obtain corresponding results by using the previous results. The reason is that the definitions of nondominated and minimal solutions of $(P_{\mathcal{Q}})$ are given similarly when we concern to the "best case", whereas the upper set less relations $(\preccurlyeq_u^{\mathcal{K}}, \preccurlyeq_{cu}^{\mathcal{K}}$ and $\preccurlyeq_{pu}^{\mathcal{K}})$ are related to the "worst case". Therefore, we utilize another approach to obtain necessary optimality conditions for solutions of problem $(P_{\mathcal{K}})$ w.r.t. $(\preccurlyeq_u^{\mathcal{K}}$ and $\preccurlyeq_{cu}^{\mathcal{K}})$. It is necessary to mention that in [30] the authors derive optimality conditions for solutions of problem $(P_{\mathcal{Q}})$ by means of the sufficient conditions for the openness of a sum valued-mappings, that is $F + \mathcal{Q}$. However, we are concerning the set-valued problem $(P_{\mathcal{K}})$, where $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$, i.e., $F(\cdot)$ and $\mathcal{K}(\cdot)$ have different pre-image spaces. Furthermore, the definitions of solutions of $(\tilde{P}_{\mathcal{K}})$ w.r.t. the relations $\preccurlyeq_u^{\mathcal{K}}$ and $\preccurlyeq_{cu}^{\mathcal{K}}$ are related to composition of multifunctions $F(\cdot)$ and $\mathcal{K}(\cdot)$. For that reason, we study the sufficient conditions for the openness of a composition multifunctions contributed from these two mappings.

Now, we recall a result for general mappings which is invectigated by Durea, Huynh, Nguyen and Strugariu in [27].
Let $F_1 : X \rightrightarrows Y_1$, $F_2 :\ X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ be set-valued mappings, where X, Y_1, Y_2, Z are Asplund spaces. Consider the following composition multifunctions $H : X \rightrightarrows Z$ defined as

$$H(x) := \bigcup_{\substack{y_2\in F_2(x)\\ y_1\in F_1(x)}} G(y_1,y_2). \tag{6.2}$$

The next statement gives sufficient conditions in terms of coderivatives for the openness of the composition of set-valued mappings (see [27, Theorem 4.2]), where Ekeland's Variational Principle is the main tool in the proof.

Theorem 6.2.6. *([27, Theorem 4.2]) Let X, Y_1, Y_2, Z be Asplund spaces. Suppose that $F_1 : X \rightrightarrows Y_1$, $F_2 : X \rightrightarrows Y_2$ and $G : Y_1 \times Y_2 \rightrightarrows Z$ are closed-graph multifunctions and $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z}) \in X \times Y_1 \times Y_2 \times Z$ be such that $\bar{z} \in G(\bar{y}_1, \bar{y}_2)$, $(\bar{y}_1, \bar{y}_2) \in F_1(\bar{x}) \times F_2(\bar{x})$. Assume that the following sets are allied at $(\bar{x}, \bar{y}_1, \bar{y}_2, \bar{z})$:*

$$
\begin{aligned}
\hat{C}_1 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z : \ y_1 \in F_1(x)\}, \\
\hat{C}_2 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z : \ y_2 \in F_2(x)\}, \\
\hat{C}_3 &:= \{(x, y_1, y_2, z) \in X \times Y_1 \times Y_2 \times Z : \ z \in G(y_1, y_2)\}.
\end{aligned} \tag{6.3}
$$

Suppose that there exists $c > 0$ such that

$$
c < \liminf_{\substack{(t_1,t_2,w) \xrightarrow{\mathrm{Gr}\, G} (\bar{y}_1,\bar{y}_2,\bar{z}),\delta\downarrow 0 \\ (u_1,v_1) \xrightarrow{\mathrm{Gr}\, F_1} (\bar{x},\bar{y}_1),\ (u_2,v_2) \xrightarrow{\mathrm{Gr}\, F_2} (\bar{x},\bar{y}_2)}} \left\{ \|x_1^* + x_2^*\| : \begin{cases} x_1^* \in \hat{D}^* F_1(u_1, v_1)(t_1^*) \\ x_2^* \in \hat{D}^* F_2(u_2, v_2)(t_2^*) \\ (z_1^* + t_1^*, z_2^* + t_2^*) \in \hat{D}^* G(t_1, t_2, w)(w^*) \\ \|w^*\| = 1,\ \|z_1^*\| < \delta,\ \|z_2^*\| < \delta \end{cases} \right\}. \tag{6.4}
$$

Then, for every $L \in (0, c)$, H (given by (6.2)) is L-open at $(\bar{x}, \bar{z})$.

In order to apply the Theorem 6.2.6 to our problem $(P_{\mathcal{K}})$, it is necessary to determine appropriate set-valued maps as follows:
Let Y_1, Y_2 and Z be equal to the space Y, and suppose that the set-valued mappings $F_1,\ F_2 : X \rightrightarrows Y$ and $G : Y \times Y \rightrightarrows Y$ are respectively determined by

$$
\forall x \in X : \ F_1(x) := F(x),
$$

$$
\forall\ x \in X : \ F_2(x) := \{0\},
$$

$$
\forall\ (y_1, y_2) \in Y \times Y : \ G(y_1, y_2) := (I - \mathcal{K})(y_1) = y_1 - \mathcal{K}(y_1).
$$

Because G only depends on y_1, instead of studying G, we invest the following set-valued map:

$$
\hat{G} : Y \rightrightarrows Y,
$$

such that

$$
\forall\ y \in Y : \ \hat{G}(y) := (I - \mathcal{K})(y) = y - \mathcal{K}(y).
$$

Let $\hat{H} : X \rightrightarrows Y$ defined by

$$
\hat{H}(x) := \bigcup_{y \in F(x)} \hat{G}(y) = \bigcup_{y \in F(x)} (y - \mathcal{K}(y)).
$$

From the setting of $F_1, F_2, \hat{G}$, the allied property of $(\hat{C}_1, \hat{C}_2, \hat{C}_3)$ in (6.3) becomes the allied property of (E_1, E_2) given as

$$
E_1 := \{(x, y, z) \in X \times Y \times Y : \ y \in F(x)\},
$$

$$
E_2 := \{(x, y, z) \in X \times Y \times Y : \ z \in \hat{G}(y)\}.
$$

Proposition 6.2.7. *[70, Proposition 5.12] Consider problem* ($P_{\mathcal{K}}$) *w.r.t. the relation* $\preceq_u^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$ *satisfying* (2.9), $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_u^{\mathcal{K}})$ *and* $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, F$. *Suppose that there is a neighborhood* U *of* $\bar{x}$ *such that* $\hat{H}(U) - \bigcap_{x \in U} \hat{H}(x)$ *is a proper cone. Then,* $\hat{H}$ *is not open at* $(\bar{x}, \bar{y})$.

Proof. Since for all $y \in Y$, $0 \in \mathcal{K}(y)$, we get that $F(\bar{x}) \preccurlyeq_u^{\mathcal{K}} F(\bar{x})$. Taking into account that $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_u^{\mathcal{K}})$, it holds

$$\forall x \in X : \; F(\bar{x}) \preccurlyeq_u^{\mathcal{K}} F(x) \quad \Longleftrightarrow \quad F(\bar{x}) \subseteq \bigcup_{y \in F(x)} (y - \mathcal{K}(y)). \tag{6.5}$$

Let $\bar{y} \in F(\bar{x})$ be arbitrarily given. Then, (6.5) implies that

$$\forall x \in X : \; \bar{y} \in \bigcup_{y \in F(x)} (y - \mathcal{K}(y)) = \hat{H}(x).$$

Suppose, by contradiction, that $\hat{H}$ is open at $(\bar{x}, \bar{y})$. Then, for the given neighborhood U of $\bar{x}$ there is an open set $V (\bar{y} \in V)$ such that $V \subseteq \hat{H}(U)$, which is equivalent to

$$V \subseteq \bigcup_{y \in F(U)} (y - \mathcal{K}(y)).$$

Let us choose $y \in V$ arbitrarily. Then, there is $x \in U$ such that

$$y \in \bigcup_{y \in F(x)} (y - \mathcal{K}(y)) = \hat{H}(x).$$

Therefore,

$$\begin{aligned} y - \bar{y} \in \hat{H}(x) - \bar{y} \subseteq \hat{H}(x) - \bigcap_{x \in U} \hat{H}(x) \\ \subseteq \hat{H}(U) - \bigcap_{x \in U} \hat{H}(x). \end{aligned}$$

This implies

$$V - \bar{y} \subseteq \hat{H}(U) - \bigcap_{x \in U} \hat{H}(x).$$

Since the first set is absorbing and the second one is a cone, it follows that

$$Y \subseteq \hat{H}(U) - \bigcap_{x \in U} \hat{H}(x),$$

contradicting the fact that $\hat{H}(U) - \bigcap_{x \in U} \hat{H}(x)$ is proper. □

Now, we show a necessary optimality condition for strong minimal solutions of ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_u^{\mathcal{K}}$, introduced in Definition 2.2.5, (ii).

Theorem 6.2.8. *[70, Theorem 5.13] Let X, Y be Asplund spaces. Consider problem ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_u^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \mathrm{SoMin}(F(X), \preceq_u^{\mathcal{K}})$. Assume that $(\bar{x}, \bar{y}) \in \mathrm{Gr}\, F$, F and $\hat{G} := I - \mathcal{K}$ be closed graph multifunctions. Suppose in addition that the following assertions hold true*

(i) $\forall y \in Y : \ 0 \in \mathcal{K}(y)$;

(ii) there is a neighborhood U of $\bar{x}$ such that $\hat{H}(U) - \bigcap_{x \in U} \hat{H}(x)$ is a proper cone;

(iii) $\{E_1, E_2\}$ *are allied at* $(\bar{x}, \bar{y}, \bar{y})$;

(iv) F^{-1} *is (PSNC) at* $(\bar{y}, \bar{x})$ *and* $\hat{G}^{-1}$ *is (PSNC) at* $(\bar{y}, \bar{y})$;

(v) $D^* \hat{G}(\bar{y}, \bar{y})(0) = \{0\}$.

Then, for all $\bar{y} \in F(\bar{x})$ there exist $w^* \in Y^* \setminus \{0\}$ and $t^* \in D^* \hat{G}(\bar{y}, \bar{y})(w^*)$ such that

$$0 \in D^* F(\bar{x}, \bar{y})(t^*).$$

Proof. We have from Proposition 6.2.7 that $\hat{H}$ is not open at $(\bar{x}, \bar{y})$, hence it is not linearly open at this point. Since the other conditions from Theorem 6.2.6 are satisfied, the condition (6.4) does not hold true. Consequently, there exist sequences $(u_n, v_n) \xrightarrow{\mathrm{Gr}\, F} (\bar{x}, \bar{y})$, $(t_n, w_n) \xrightarrow{\mathrm{Gr}\, \hat{G}} (\bar{y}, \bar{y})$, $(w_n^*) \subset S_{Y^*}$, $(x_n^*) \in X^*$, $z_n^* \to 0$ such that

$$\forall\ n : \ x_n^* \in \hat{D}^* F(u_n, v_n)(t_n^*), z_n^* + t_n^* \in \hat{D}^* \hat{G}(t_n, w_n)(w_n^*) \text{ and } \|x_n^*\| \to 0. \tag{6.6}$$

Now, we prove that (t_n^*) is bounded. Suppose the contradiction and by $z_n^* \to 0$ we get that for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ sufficiently large such that

$$n < \|t_{kn}^*\| + \|z_{kn}^*\|. \tag{6.7}$$

For the reason of keeping the notation simple, we denote the subsequences (t_{kn}^*), (z_{kn}^*) by (t_n^*), (z_n^*), respectively. Because of the positive homogeneity of the Fréchet coderivatives, we have that

$$\frac{x_n^*}{n} \in \hat{D}^* F(u_n, v_n)(\frac{t_n^*}{n}),$$

and

$$\frac{1}{n}(z_n^* + t_n^*) \in \hat{D}^* \hat{G}(t_n, w_n)(\frac{w_n^*}{n}).$$

It yields

$$(\frac{x_n^*}{n}, \frac{-t_n^*}{n}) \in \hat{N}(\mathrm{Gr}\, F, (u_n, v_n)) \text{ and } (\frac{z_n^* + t_n^*}{n}, \frac{-w_n^*}{n}) \in \hat{N}(\mathrm{Gr}\, \hat{G}, (t_n, w_n)).$$

Thus,

$$(\frac{x_n^*}{n}, \frac{-t_n^*}{n}, 0) \in \hat{N}(E_1, (u_n, v_n, \bar{y})) \text{ and } (0, \frac{z_n^* + t_n^*}{n}, \frac{-w_n^*}{n}) \in \hat{N}(E_2, (\bar{x}, t_n, w_n)).$$

Since $w_n^* \in S_{Y^*}$, $z_n^* \to 0$ and $\|x_n^*\| \to 0$, it holds

$$(\frac{x_n^*}{n}, \frac{-t_n^*}{n}, 0) + (0, \frac{z_n^* + t_n^*}{n}, \frac{-w_n^*}{n}) = (\frac{x_n^*}{n}, \frac{z_n^*}{n}, \frac{-w_n^*}{n}) \to 0.$$

Then, the alliedness of the sets (E_1, E_2) implies $\frac{1}{n}(z_n^* + t_n^*) \to 0$, which is impossible in virtue of relation (6.7).

Consequently, since Y is Asplund and (t_n^*) is bounded, we get that there is a subsequence of (t_n^*) which weak* converges to $t^* \in Y^*$. Also, since $(w_n^*) \subset S_{Y^*}$, (w_n^*) contains a weak* convergent subsequence to an element w^*. For simplicity, we denote this subsequence also by (w_n^*).

We claim that $t^* = w^* = 0$ does not hold true. Indeed, suppose that $t^* = w^* = 0$, i.e., $t_n^* \xrightarrow{w^*} 0$ and $w_n^* \xrightarrow{w^*} 0$. Taking into account F^{-1} is (PSNC) at $(\bar{y}, \bar{x})$, and $x_n^* \to 0$ it holds $t_n^* \to 0$. From $z_n^* \to 0$ we get $(t_n^* + z_n^*) \to 0$. In addition, $w_n^* \xrightarrow{w^*} 0$ and $\hat{G}^{-1}$ is (PSNC) at $(\bar{y}, \bar{y})$, it holds $w_n^* \to 0$, which contradicts the fact that $(w_n^*) \subset S_{Y^*}$.

Moreover, taking into account (6.6), there exist some $w^* \in Y^*$ and $t^* \in D^*\hat{G}(\bar{y}, \bar{y})(w^*)$ satisfying $0 \in D^*F(\bar{x}, \bar{y})(t^*)$.

It is obvious from the assumption $D^*\hat{G}(\bar{y}, \bar{y})(0) = \{0\}$ that if $w^* = 0$, then $t^* = 0$, a contradiction. Then, $w^* \neq 0$. The proof is complete. □

Now, we consider the certainly upper less relation w.r.t. $\mathcal{K}(\cdot)$ $\preceq_{cu}^{\mathcal{K}}$. It holds from Remark 4.2.7(ii) that if $\bar{x} \in \text{SoMin}(F(X), \preceq_{cu}^{\mathcal{K}})$, then we also have that $\bar{x} \in \text{SoMin}(F(X), \preceq_{u}^{\mathcal{K}})$. Therefore, from Theorem 6.2.8 we have the following result.

Theorem 6.2.9. *[70, Theorem 5.14] Let X, Y be Asplund spaces. Consider problem ($P_{\mathcal{K}}$) w.r.t. the relation $\preceq_{cu}^{\mathcal{K}}$, $\mathcal{K} : Y \rightrightarrows Y$, and $\bar{x} \in \text{SoMin}(F(X), \preceq_{cu}^{\mathcal{K}})$. Assume that $(\bar{x}, \bar{y}) \in \text{Gr}\, F$, F and $\hat{G} := I - \mathcal{K}$ are closed graph multifunctions. Suppose in addition that the assumptions $(i) - (v)$ in Theorem 6.2.8 are fulfilled. Then, for all $\bar{y} \in F(\bar{x})$ there exist $w^* \in Y^* \setminus \{0\}$ and $t^* \in D^*\hat{G}(\bar{y}, \bar{y})(w^*)$ such that*

$$0 \in D^*F(\bar{x}, \bar{y})(t^*).$$

Proof. Since $\bar{x} \in \text{SoMin}(F(X), \preceq_{cu}^{\mathcal{K}})$, we get $\bar{x} \in \text{SoMin}(F(X), \preceq_{u}^{\mathcal{K}})$. Since all assumptions given in Theorem 6.2.8 hold true, we follow the same line of its proof to obtain that: for all $\bar{y} \in F(\bar{x})$ there exist $w^* \in Y^* \setminus \{0\}$ and $t^* \in D^*\hat{G}(\bar{y}, \bar{y})(w^*)$ such that

$$0 \in D^*F(\bar{x}, \bar{y})(t^*).$$

□

Chapter 7

Pointwise Well-posed Set Optimization Problems w.r.t. Variable Domination Structures

This chapter is devoted to the well-posedness property of a set-valued optimization problem with constraints

$$\mathcal{K} - \operatorname*{Min}_{x \in S} F(x), \tag{$\bar{P}_{\mathcal{K}}$}$$

where $F:\ X \rightrightarrows Y$ is a set-valued mapping acting between two linear topological spaces, $S \subseteq X$ is the feasible set and $\mathcal{K}:\ Y \rightrightarrows Y$. As shown in [31, 89], linear scalarization is not useful in deriving the equivalence between vector and scalar well-posedness notions even in the convex case. Therefore, many authors have used nonlinear scalarizing functionals to investigate well-posedness property of vector optimization and set optimization, see [31, 46, 64, 89, 88]. This chapter will use the directional minimal time function (5.11) given in Chapter 5 as the main tool. By means of this functional, we characterize minimal solutions for a family of sets and solutions of problem $(\overline{P}_{\mathcal{K}})$ in Section 7.1. Section 7.2 proves the parallelism between the well-posedness property of a set-valued optimization problem and the Tykhonov well-posedness property of a scalar problems in which the objective map of the original problem is involved. Moreover, two classes of pointwise well-posed set optimization problems w.r.t. a cone-valued ordering structure are also identified. The results presented within this chapter are based on Kobis, Le, Tammer and Yao [71].

7.1 Characterizations for Solutions of Set Optimization w.r.t. Variable Domination Structures via the Directional Minimal Time Function

This section presents characterizations of minimal and strictly minimal solutions of problem ($\bar{P}_{\mathcal{K}}$) by using the scalarizing functional given by (5.11). Recall that this function has the following formula:

$$\varphi_{k^0,B}(A) = \inf\{t \geq 0 |\ A \preceq_l^{\mathcal{K}} tk^0 + B\},$$

where $A, B \subset Y$, $k^0 \in Y \setminus \{0\}$ satisfies (H_1), i.e.,

$$\forall y \in Y: \ \mathcal{K}(y) + (0, +\infty)k^0 \subseteq \mathcal{K}(y).$$

We assume in this part that $\mathcal{K}(\cdot)$ satisfies the condition (2.9)-(2.11). As having promised at the end of Section 5.3, we now characterize (strictly) minimal elements of a family of sets in the sense of Definition 4.2.1. Let $\mathcal{A}$ be a nonempty subset of $\mathcal{P}(Y)$. In the following theorem, we are using the function (5.11) with $B = \bar{A}$, and $k^0 \in Y \setminus \{0\}$ such that the assumption (H_1) given at the beginning of Chapter 5 holds true.

Theorem 7.1.1. *[71, Theorem 4.1] The following assertions are satisfied.*

(a) Assume that $\bigcup_{a \in A} (a + \mathcal{K}(a))$ *is closed for all* $A \in \mathcal{A}$. *Then,* $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if* $\varphi_{k^0,\bar{A}}(A) > 0$ *for all* $A \in \mathcal{A}, A \not\sim \bar{A}$.

(b) Assume that $\bigcup_{a \in A} (a + \mathcal{K}(a))$ *is closed for all* $A \in \mathcal{A}$. *Then,* $\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if* $\varphi_{k^0,\bar{A}}(A) > 0$ *for all* $A \in \mathcal{A} \setminus \{\bar{A}\}$.

Proof. **(a)** Consider $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ and suppose that there exists $A \in \mathcal{A}, A \not\sim \bar{A}$ satisfying $\varphi_{k^0,\bar{A}}(A) = 0$. Taking into account Theorem 5.3.2(e), it holds that $A \preceq_l^{\mathcal{K}} \bar{A}$. Since $\bar{A} \in \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$, $\bar{A} \preceq_l^{\mathcal{K}} A$ and thus $A \sim \bar{A}$. This is a contradiction.
Conversely, assume that $\varphi_{k^0,\bar{A}}(A) > 0$ for all $A \in \mathcal{A}, A \not\sim \bar{A}$ and $\bar{A}$ is not a minimal element of $\mathcal{A}$. Then, from the definition of minimal elements of $\mathcal{A}$ there exists a set $A \in \mathcal{A}, A \preceq_l^{\mathcal{K}} \bar{A}$ and $\bar{A} \not\preceq_l^{\mathcal{K}} A$. Using Theorem 5.3.2(a), it holds that $\varphi_{k^0,\bar{A}}(A) \leq \varphi_{k^0,\bar{A}}(\bar{A})$. In addition, by Theorem 5.3.2(d) we get $\varphi_{k^0,\bar{A}}(\bar{A})$=0. Therefore, $\varphi_{k^0,\bar{A}}(A) \leq 0$, a contradiction. Thus, the assumption $\bar{A} \notin \mathrm{Min}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ is false and the proof of the sufficient condition is complete.

(b) Suppose that $\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$ and there is $A \in \mathcal{A} \setminus \{\bar{A}\}$ such that $\varphi_{k^0,\bar{A}}(A) = 0$. By Theorem 5.3.2(e), we have that $A \preceq_l^{\mathcal{K}} \bar{A}$. Since $\bar{A} \in \mathrm{SiMin}_Y(\mathcal{A}, \preceq_l^{\mathcal{K}})$, it yields $A = \bar{A}$, which is a contradiction.
Let us prove the sufficient condition. By contradiction, assume that $\varphi_{k^0,\bar{A}}(A) > 0$ for all $A \in \mathcal{A} \setminus \{\bar{A}\}$ and $\bar{A} \notin \mathrm{SiMin}_{\mathrm{Y}}(\mathcal{A}, \preceq_{\mathrm{l}}^{\mathcal{K}})$. Using the definition of strictly minimal

elements of $\mathcal{A}$, there exists $A \in \mathcal{A}$ such that $A \preceq_l^{\mathcal{K}} \bar{A}$ and $A \neq \bar{A}$. Taking into account parts (d) and (e) of Theorem 5.3.2, it holds that

$$\varphi_{k^0,\bar{A}}(A) \leq \varphi_{k^0,\bar{A}}(\bar{A}){=}0.$$

This implies $\varphi_{k^0,\bar{A}}(A){=}0$, which is a contradiction. □

Remark 7.1.2. *A similar result as Theorem 7.1.1 is generated in Chapter 5, where the scalarizing functional $g^{\preceq_l^{\mathcal{K}}}$ is used. If for all $y \in Y$, $\mathcal{K}(y) = K$, where K is a convex cone in Y, Theorem 7.1.1 reduces to [46, Theorem 4.3].*

In the following, we utilize the functional given by (5.11) to characterize (strictly) minimal solutions of problem $(\bar{P}_{\mathcal{K}})$ with assumptions that $F(x) \neq \emptyset$ for all $x \in S$ and $\mathcal{K} : Y \rightrightarrows Y$ such that the relation $\preceq_l^{\mathcal{K}}$ is reflexive. Remind that the descent method in Section 5.4 has used the functional $g^{\preceq_l^{\mathcal{K}}}$, given by (5.3), to find approximation solutions of this problem for the case $S = X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$.

Theorem 7.1.3. *[71, Theorem 4.3] Let $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$ be set-valued maps such that $\bigcup_{y \in F(x)} (y + \mathcal{K}(y))$ is closed for each $x \in X$ and the conditions (2.9)- (2.11) are fulfilled. Consider problem $(\bar{P}_{\mathcal{K}})$ and $\bar{x} \in X$. Then, the following assertions hold true.*

(a) $\bar{x}$ is a minimal solution of $(\bar{P}_{\mathcal{K}})$ if and only if there is a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \bigcup \{+\infty\}$ being $\preceq_l^{\mathcal{K}}$-monotone such that

$$x \in S, \quad F(x) \sim F(\bar{x}) \quad \Longleftrightarrow \quad G(F(x)){=}0. \tag{7.1}$$

(b) $\bar{x}$ is a strictly minimal solution of $(\bar{P}_{\mathcal{K}})$ if and only if there is a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \bigcup \{+\infty\}$ being $\preceq_l^{\mathcal{K}}$-monotone such that

$$x \in S, \quad G(F(x)){=}0 \quad \Longleftrightarrow \quad x = \bar{x}. \tag{7.2}$$

Proof. The idea of this proof is as similar as that in [46, Theorem 4.4], where $\mathcal{K}(\cdot) = K$, K is a convex cone in Y. We illustrate in the following for the case the domination structure is variable and the scalarizing functional is given by (5.11).

(a) Suppose that $\bar{x}$ is a minimal solution of $(\bar{P}_{\mathcal{K}})$. Let $k^0 \in Y$ such that (H_1) holds and define the following functional as:

$$G: \ \operatorname{Im} F \to \mathbb{R} \cup \{+\infty\}$$

$$G(F(x)) := \varphi_{k^0,F(\bar{x})}(F(x)),$$

where $\varphi_{k^0,F(\bar{x})}$ given by (5.11) with $B = F(\bar{x})$ is involved.
From Theorem 5.3.2(a), we get that G is $\preceq_l^{\mathcal{K}}$-monotone. Let us now prove that

$$x \in S,\ F(x) \sim F(\bar{x}) \quad \Longleftrightarrow \quad G(F(x))=0.$$

Taking into account Theorem 5.3.2(a) and (d), it holds that

$$\begin{aligned} F(x) \sim F(\bar{x}) &\Longrightarrow F(x) \preceq_l^{\mathcal{K}} F(\bar{x}) \\ &\Longrightarrow G(F(x)) \leq G(F(\bar{x})) = \varphi_{F(\bar{x})}F(\bar{x})=0 \\ &\Longrightarrow G(F(x))=0. \end{aligned}$$

Now, if we suppose that $F(x) \not\sim F(\bar{x})$, by Theorem 7.1.1 (b), it holds that $G(F(x)) > 0$. Therefore, if $G(F(x))=0$, we have that $F(x) \sim F(\bar{x})$.
Reciprocally, suppose that there exists a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying (7.1) and G is $\preceq_l^{\mathcal{K}}$-monotone. Let $x \in S$ such that $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$. It is sufficient to prove that $F(\bar{x}) \preceq_l^{\mathcal{K}} F(x)$. Since $\mathcal{K}$ satisfies (2.9), the relation $\preceq_l^{\mathcal{K}}$ is reflexive and thus $F(\bar{x}) \sim F(\bar{x})$. Taking into account (7.1), we get that $G(F(\bar{x}))=0$. Since G is $\preceq_l^{\mathcal{K}}$-monotone and $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$, it yields

$$\begin{aligned} &0 \leq G(F(x)) \leq G(F(\bar{x}))=0 \\ \Longrightarrow\ &G(F(x))=0 \end{aligned}$$

Taking into account (7.1) we get that $F(x) \sim F(\bar{x}) \Rightarrow F(\bar{x}) \preceq_l^{\mathcal{K}} F(x)$, which is the desired conclusion.

(b) Let $\bar{x}$ be a strictly minimal solution of problem $(\bar{P}_{\mathcal{K}})$ and the functional G defined as in part (a), that is $G(F(x)) = \varphi_{k^0,F(\bar{x})}(F(x))$. Because $\bar{x}$ is a strictly minimal solution of $(\bar{P}_{\mathcal{K}})$, it yields that

$$\forall\ x \neq \bar{x} :\ F(x) \not\preceq_l^{\mathcal{K}} F(\bar{x}).$$

Now, we suppose that $G(F(x))=0$. Taking into account Theorem 5.3.2(e), it holds that $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$. This implies $x = \bar{x}$. Therefore, if $G(F(x))=0$, then $x = \bar{x}$. On the other hand,

$$x = \bar{x} \Longrightarrow G(F(x)) = G(F(\bar{x})) = \varphi_{k^0,F(\bar{x})}F(\bar{x})=0.$$

Thus, the conclusion (7.2) holds true.
Now, we prove the sufficient condition. Suppose that there exists a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying (7.2) and G is $\preceq_l^{\mathcal{K}}$-monotone. Let $x \in S$ such that $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$. Since (7.2) holds true, it yields

$$\begin{aligned} F(x) \preceq_l^{\mathcal{K}} F(\bar{x}) &\Longrightarrow 0 \leq G(F(x)) \leq G(F(\bar{x})) \leq 0 \\ &\Longrightarrow G(F(x))=0 \\ &\Longrightarrow x = \bar{x}. \end{aligned}$$

The last equation states that $\bar{x}$ is a strictly minimal solution of $(\bar{P}_{\mathcal{K}})$. □

Remark 7.1.4. *(i) Since $G : X \to \mathbb{R}_+ \cup \{+\infty\}$, we can rewrite (7.1) and (7.2) respectively by*

$$\operatorname{argmin}(G \circ F, S) = \{x \in S|\ F(x) \sim F(\bar{x})\},$$

and

$$\operatorname{argmin}(G \circ F, S) = \{\bar{x}\}.$$

(ii) If for all $y \in Y$, $\mathcal{K}(y) = K$, where K is a convex cone in Y, $F(x)+K$ is closed for all $x \in S$ and $F(\bar{x})$ is K-proper, i.e., $F(\bar{x}) + K \neq Y$ then Theorem 7.1.3 reduces to [46, Theorem 4.4]

7.2 Pointwise Well-posedness for Set Optimization w.r.t. Variable Domination Structures

This section presents some results concerning well-posedness properties for set optimization problems w.r.t. variable domination structures. For the case of fixed domination structures, this problem has attracted many authors in the literature not only on vector but also on set optimization. Usually, one proves the equivalence between the well-posedness property of the concerned problem and the Tykhonov well-posedness property of a corresponding scalar problem. Then, by using many classical results related to this property of the scalar problem, one can derive some classes of well-posed vector (set) optimization problems for the concerned problem, see [31, 46, 64, 88, 89, 112]. In this section, we will show that under some appropriate conditions, we also obtain this equivalence for a set-valued optimization w.r.t. a variable domination structure. Moreover, we will find two sets of points at which a set-valued optimization problem is well-posed. Throughout this part, we suppose that the following assumption is fulfilled.
Assumption (P):

- $\mathcal{K} : Y \rightrightarrows Y$ is a set-valued map such that for all $y \in Y$, $\mathcal{K}(y)$ is a proper, closed, convex cone in Y and $\operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y) \neq \emptyset$.
- $F : X \rightrightarrows Y$ is a set-valued map between two real topological vector spaces, $S \subseteq X$ and for all $x \in S$, $\bigcup_{y \in F(x)} (y + \mathcal{K}(y))$ is closed.
- k^0 is taken in Y such that $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$.

We begin this section by recalling the notion of well-posedness property of an extended real-valued function (see [25]).

Definition 7.2.1. *Let $f : X \to \overline{\mathbb{R}}$ be an extended real-valued function and consider problem*

$$\underset{x \in S}{\mathrm{Min}} f(x). \qquad (\mathcal{P}')$$

We say that problem ($\mathcal{P}'$) is:

(i) ***Tykhonov well-posed*** *if it has a unique solution $\bar{x} \in S$ and*

$$\{x_n\} \subset S,\ f(x_n) \to f(\bar{x}) \text{ implies } \{x_n\} \to \bar{x}.$$

(ii) ***generalized well-posed*** *if* $\arg\min(f, S) \neq \emptyset$ *and*

$$\{x_n\} \subset S,\ f(x_n) \to f(\bar{x}) \text{ implies } \exists\ \{x_{n_k}\} \subseteq \{x_n\} :\ \{x_{n_k}\} \to \bar{x}.$$

Remark 7.2.2. *Observe that ($\mathcal{P}'$) is Tykhonov well-posed if and only if it is generalized well-posed and the set* $\arg\min(f, S)$ *is a singleton.*

The following definition introduces some notions in order to investigate well-posedness property for the set-valued problem ($\bar{P}_{\mathcal{K}}$). Observe that this definition extends [46, Definition 5.1] where the authors investigated set-valued problems w.r.t. fixed domination structures.

Definition 7.2.3. *Let $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$ and $\bar{x}$ be a minimal solution of problem ($\bar{P}_{\mathcal{K}}$).*

(a) A sequence $\{x_n\} \subset S$ is said to be ***k^0-minimizing*** *for ($\bar{P}_{\mathcal{K}}$) at $\bar{x}$ if*

$$\exists\ \{\epsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}, \{\epsilon_n\} \to 0 :\ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \epsilon_n k^0, \forall n.$$

(b) ($\bar{P}_{\mathcal{K}}$) is said to be ***k^0-well-posed*** *at $\bar{x}$ if every k^0-minimizing sequence at $\bar{x}$ converges to $\bar{x}$.*

(c) $\{x_n\} \subset S$ is said to be ***minimizing*** *at $\bar{x}$ if*

$$\exists \{d_n\} \subset \bigcap_{y \in Y} \mathcal{K}(y) \setminus \{0\},\ \{d_n\} \to 0 :\ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n, \forall n.$$

(d) ($\bar{P}_{\mathcal{K}}$) is said to be ***well-posed*** *at $\bar{x}$ if $\bar{x}$ is a strictly minimal solution and for all minimizing $\{x_n\}$ at $\bar{x}$ it holds that $\{x_n\} \to \bar{x}$.*

The following lemma, given by Durea [31], will be used in the next proposition which states that Definitions 7.2.3(a) and (c) are equivalent.

Lemma 7.2.4. *[31, Lemma 2.2] Let $K \subseteq Y$ be a proper, closed, convex cone with nonempty interior and $\{k_n\}$ be a sequence of elements from Y that converges to 0. Then, for every $k \in \operatorname{int} K$ there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\{\alpha_n\} \to 0$ and $\alpha_n k - k_n \in \operatorname{int} K$ for every natural number n.*

Proposition 7.2.5. *[71, Proposition 5.5] Let* $\{x_n\} \subset S$*,* $k^0 \in \text{int} \bigcap_{y \in Y} \mathcal{K}(y)$ *and* $\bar{x}$ *be a minimal solution of problem* $(\bar{P}_{\mathcal{K}})$*. Then, the two following assertions are equivalent:*

(i) $\{x_n\}$ *is* k^0*-minimizing for* $(\bar{P}_{\mathcal{K}})$ *at* $\bar{x}$*.*

(ii) $\{x_n\}$ *is minimizing for* $(\bar{P}_{\mathcal{K}})$ *at* $\bar{x}$*.*

Proof.
$[(i) \to (ii)]$ Since $\{x_n\}$ is k^0-minimizing for $(\bar{P}_{\mathcal{K}})$ at $\bar{x}$, we have that

$$\exists \{\epsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}, \{\epsilon_n\} \to 0: \ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \epsilon_n k^0, \forall n.$$

Let $d_n := \epsilon_n k^0$, $\forall n$. It holds that

$$\{d_n\} \subset \bigcap_{y \in Y} \mathcal{K}(y) \setminus \{0\}, \{d_n\} \to 0 \text{ and } F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n.$$

Taking into account the definition of minimizing property, we get that $\{x_n\}$ is minimizing for $(\bar{P}_{\mathcal{K}})$ at $\bar{x}$, i.e., (ii) holds true.
$[(ii) \to (i)]$ Suppose that $\{x_n\}$ is minimizing for $(\bar{P}_{\mathcal{K}})$ at $\bar{x}$, i.e.,

$$\exists \{d_n\} \subset \bigcap_{y \in Y} \mathcal{K}(y) \setminus \{0\}, \ \{d_n\} \to 0: \ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n, \ \forall n.$$

We have that

$$\begin{aligned} F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n \iff & F(\bar{x}) + d_n \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) \\ \iff & F(\bar{x}) \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + (-d_n). \end{aligned} \tag{7.3}$$

Let $K := \bigcap_{y \in Y} \mathcal{K}(y)$. Since for all $y \in Y$, $\mathcal{K}(y)$ is a conex cone, $\mathcal{K}(y) + K \subseteq \mathcal{K}(y)$. Therefore, for all $n \in \mathbb{N}$, it holds that

$$\begin{aligned} \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + \text{int}\, K \subseteq & \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + K \\ \subseteq & \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)). \end{aligned} \tag{7.4}$$

By Assumption (P), K is a proper, closed, convex cone with $\text{int}\, K \neq \emptyset$. Taking into account $k^0 \in \text{int}\, K$, $\{d_n\} \xrightarrow{Y} 0$ and applying Lemma 7.2.4, we obtain that

$$\exists \{\alpha_n\} \subseteq \mathbb{R}_+ \setminus \{0\}, \{\alpha_n\} \to 0: \ \alpha_n k^0 - d_n \in \text{int}\, K, \ \forall n \in \mathbb{N}.$$

This implies that $-d_n \in -\alpha_n k^0 + \text{int}\, K$. Taking into account (7.3), it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned} & F(\bar{x}) \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) - \alpha_n k^0 + \text{int}\, K \\ \iff & F(\bar{x}) + \alpha_n k^0 \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + \text{int}\, K. \end{aligned}$$

Taking into account (7.4), we get that

$$F(\bar{x}) + \alpha_n k^0 \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)),\ \forall n \in \mathbb{N}$$
$$\iff F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \alpha_n k^0,\ \forall n \in \mathbb{N}. \tag{7.5}$$

The relation (7.5) ensures that $\{x_n\}$ is k^0-minimizing for the problem $(\bar{P}_{\mathcal{K}})$ at $\bar{x}$. The proof is complete. □

Now, we present an important result of this chapter, that is, we prove that there exists a class of scalar problems whose the Tykhonov well-posedness property is equivalent to the well-posedness of the original set optimization problem $(\bar{P}_{\mathcal{K}})$. Observe that [46, 64] have investigated this for the case the domination structures are fixed. However, it is not easy to directly extend these results for problem $(\bar{P}_{\mathcal{K}})$. The reason is that we need to choose an appropriate scalarizing functional and some additional properties on the mapping $\mathcal{K}(\cdot)$. The following result takes into account Theorem 7.1.3, where we use the directional time function given by (5.11).

Theorem 7.2.6. *[71, Theorem 5.6] Suppose that $\mathcal{K} : Y \rightrightarrows Y$ satisfies (2.11) and $\bar{x}$ is a strictly minimal solution of problem $(\bar{P}_{\mathcal{K}})$. Consider the scalar problem*

$$\operatorname{Min}\{\varphi_{k^0,F(\bar{x})}(F(x)) \mid\ x \in S\}, \tag{$P_{\varphi_{k^0,F(\bar{x})}}$}$$

where the functional $\varphi_{k^0,F(\bar{x})}$ given by (5.11) with $B = F(\bar{x})$. Then, the following statements are equivalent:

(a) Problem $(\bar{P}_{\mathcal{K}})$ is well-posed at $\bar{x}$.

(b) For every $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$, problem $(P_{\varphi_{k^0,F(\bar{x})}})$ is Tykhonov well-posed.

(c) There is $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$ such that problem $(P_{\varphi_{k^0,F(\bar{x})}})$ is Tykhonov well-posed.

Proof. [(a) ⇒ (b)]: Let $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$ arbitrary. Taking into account Theorem 7.1.3(b), we have that

$$\varphi_{k^0,F(\bar{x})}(F(\bar{x}))=0 \text{ and for all } x \neq \bar{x} : \varphi_{k^0,F(\bar{x})}(F(x)) > 0.$$

Thus $\operatorname{argmin}_{x\in S} \varphi_{k^0,F(\bar{x})}(F(x)) = \{\bar{x}\}$, i.e, $\bar{x}$ is a unique solution of $(P_{\varphi_{k^0,F(\bar{x})}})$. Now, we take $\{x_n\} \subseteq S$ such that $\varphi_{k^0,F(\bar{x})}(F(x_n)) \to \varphi_{k^0,F(\bar{x})}(F(\bar{x}))$. It is sufficient to prove that $\{x_n\} \to \bar{x}$.
Let $\bar{t}_n := \varphi_{k^0,F(\bar{x})}(F(x_n))$, and $\epsilon_n := \varphi_{k^0,F(\bar{x})}(F(x_n)) + \frac{1}{n}$. It holds that

$$\{\epsilon_n\} \to 0,\ \epsilon_n > \bar{t}_n \text{ and } F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \epsilon_n k^0.$$

By the last relation, we get that $\{x_n\}$ is k^0-minimizing and thus, a minimizing sequence for $(\bar{P}_{\mathcal{K}})$. Since $(\bar{P}_{\mathcal{K}})$ is well-posed, $\{x_n\} \to \bar{x}$.

$[(b) \Rightarrow (c)]$ This implication is obvious.

$[(c) \Rightarrow (a)]$ Suppose that (c) holds true, we will prove that (a) is fulfilled. Let $\{x_n\}$ is a minimal solution sequence for problem $(\bar{P}_{\mathcal{K}})$ at $\bar{x}$. By Proposition 7.2.5, there is a sequence $\{\epsilon_n\} \to 0^+$ and

$$\forall\ n : F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \epsilon_n k^0 \Longrightarrow \varphi_{k^0,F(\bar{x})}(F(x_n)) \leq \epsilon_n.$$

Taking into account $\bar{x}$ is a strictly minimal solution of $(\bar{P}_{\mathcal{K}})$, it holds that

$$\forall x_n \neq \bar{x} : \ \varphi_{k^0,F(\bar{x})}(F(x_n)) > 0.$$

Thus, we get that

$$\{\varphi_{k^0,F(\bar{x})}(F(x_n))\} \to 0 = \varphi_{k^0,F(\bar{x})}(F(\bar{x})).$$

Since $(P_{\varphi_{k^0,F(\bar{x})}})$ is Tykhonov well-posed, it holds that $\{x_n\} \to \bar{x}$, i.e., problem $(\bar{P}_{\mathcal{K}})$ is well-posed at $\bar{x}$.

□

Remark 7.2.7. *Observe that Theorem 7.1.3 supposes that* (2.9) *and* (2.10) *hold true for the mapping* $\mathcal{K}(\cdot)$. *However, these conditions are automatically fulfilled when we use Assumption (P) introduced at the beginning of this section. That is why we omit these conditions in Theorem 7.2.6 and later on.*

Now, we are finding some classes of well-posed set optimization problems. We recall the two following classical results of well-posed scalar optimization problems, which will be utilized in the sequel.

Theorem 7.2.8. *[10, Theorem 2.1] Let X be a locally compact metric space. Suppose $f : X \to \overline{\mathbb{R}}$ is a proper lower semicontinuous and quasiconvex function on X. The following conditions are equivalent:*

(a) Problem $(\mathcal{P}')$ is generalized well-posed;

(b) $\operatorname{argmin}(f, X)$ *is nonempty and compact.*

Proposition 7.2.9. *[25, Example 6] Let X be a normed vector space, $S \subset X$ be a compact set and $f : X \to \overline{\mathbb{R}}$ be a proper and lower semicontinuous function on X. Suppose that* $\operatorname{argmin}(f, S)$ *has a unique element. Then, problem $(\mathcal{P}')$ is Tykhonov well-posed.*

In the following proposition, we show the sufficient conditions which ensure the lower-semicontinuous property of the function $\varphi_{k^0,B} \circ F$, where $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$ and $B \in \mathcal{P}(Y)$.

Proposition 7.2.10. *[71, Proposition 5.9] Suppose that $F : X \rightrightarrows Y$ satisfies that $S(F, \preceq_l^{\mathcal{K}}, rk^0 + A) := \{x \in X | \ F(x) \preceq_l^{\mathcal{K}} rk^0 + A\}$ is closed for all $A \in \mathcal{P}(Y)$ and $r \geq 0$. In addition, assume that $\mathcal{K}(\cdot)$ satisfies (2.11). Then, $\varphi_{k^0,B} \circ F : X \to \mathbb{R}_+ \bigcup\{+\infty\}$ is lower semicontinuous on S for all $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$.*

Proof. We prove that for all $\gamma \in \mathbb{R}$, the set $S(\varphi_{k^0,B} \circ F, \gamma)$ is closed. This holds true when $\gamma < 0$ since $S(\varphi_{k^0,B} \circ F, \gamma) = \emptyset$. We prove that $S(\varphi_{k^0,B} \circ F, \gamma) = S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B)$ if $\gamma \geq 0$.

Let $x \in S(\varphi_{k^0,B} \circ F, \gamma)$. Taking into account Remark 5.3.3 (ii), we have that

$$\begin{aligned} \varphi_{k^0,B} F(x) \leq \gamma &\Longrightarrow F(x) \preceq_l^{\mathcal{K}} \gamma k^0 + B \\ &\Longrightarrow x \in S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B). \end{aligned}$$

Therefore,

$$S(\varphi_{k^0,B} \circ F, \gamma) \subseteq S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B). \tag{7.6}$$

Conversely, let $x \in S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B)$, i.e., $F(x) \preceq_l^{\mathcal{K}} \gamma k^0 + B$. By the definition (5.11), it holds that $\varphi_{k^0,B} F(x) \leq \gamma \Longrightarrow x \in S(\varphi_{k^0,B} \circ F, \gamma)$.Therefore,

$$S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B) \subseteq S(\varphi_{k^0,B} \circ F, \gamma), \tag{7.7}$$

(7.6) together with (7.7) imply that $S(\varphi_{k^0,B} \circ F, \gamma) = S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B)$. □

Now, we present the first class of well-posed set-valued optimization problems w.r.t. variable domination structures.

Theorem 7.2.11. *[71, Theorem 5.10] Let X be a normed vector space and Y be a linear topological space. Consider problem $(\bar{P}_{\mathcal{K}})$ with the mappings $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$ satisfy all the assumptions given in Proposition 7.2.10. Let $\bar{x}$ be a strictly minimal solution of problem $(\bar{P}_{\mathcal{K}})$ and S be a compact subset of X. Then, $(\bar{P}_{\mathcal{K}})$ is well-posed at $\bar{x}$.*

Proof. Let $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$. By Proposition 7.2.10, $\varphi_{k^0,F(\bar{x})} \circ F$ is lower semicontinuous. Furthermore, by Theorem 7.1.3 (b), it holds that $\operatorname{argmin}(\varphi_{k^0,F(\bar{x})} \circ F, S) = \{\bar{x}\}$. Therefore, according to Proposition 7.2.9, problem $(P_{\varphi_{k^0,F(\bar{x})}})$ is Tykhonov well-posed. Applying Theorem 7.2.6, we have that problem $(\bar{P}_{\mathcal{K}})$ is well-posed at $\bar{x}$. □

Before deriving the second class of well-posed set optimization problems w.r.t. variable domination structures, we introduce a $\mathcal{K}$-quasiconvex mapping.

Definition 7.2.12. *The set-valued mapping $F : X \rightrightarrows Y$ is said to be $\mathcal{K}$-**quasiconvex** w.r.t. $\preceq_l^{\mathcal{K}}$ on a nonempty, convex set $S \subseteq X$ if for all $x_1, x_2 \in S$ and $\lambda \in [0, 1]$ it holds that*

$$F(\lambda x_1 + (1 - \lambda)x_2) \preceq_l^{\mathcal{K}} \bigcup_{y \in F(x_1)} (y + \mathcal{K}(y)) \cap \bigcup_{y \in F(x_2)} (y + \mathcal{K}(y)). \tag{7.8}$$

Note that Definition 7.2.12 reduces to the classical definition of a quasiconvex real valued-function if F is single-valued and $\mathcal{K}(\cdot) = \mathbb{R}_+$. In addition, if $\mathcal{K}(\cdot) = K$, and K is a convex cone in Y with nonempty interior, Definition 7.2.12 becomes the definition of K-quasiconvex set-valued mapping, see [12, 46, 64, 79]. In the following, we show that the $\mathcal{K}$-quasiconvex property can be inherited via scalarizing functional given by (5.11).

Proposition 7.2.13. *[71, Proposition 5.12] If $F : X \rightrightarrows Y$ is $\mathcal{K}$-quasiconvex w.r.t. $\preceq_l^{\mathcal{K}}$ on a nonempty convex set $S \subseteq X$, then $\varphi_{k^0,B} \circ F$ is a quasiconvex function on S for all $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$ and $B \in \mathcal{P}(Y)$. Furthermore, the converse statement is true if $\mathcal{K}(\cdot)$ satisfies (2.11).*

Proof. Let $x_1, x_2 \in S$ be two arbitrary elements. We have to show that for all $\lambda \in [0, 1]$ it holds that

$$\varphi_{k^0,B} \circ F(\lambda x_1 + (1 - \lambda)x_2) \leq \max\{\varphi_{k^0,B} \circ F(x_1), \varphi_{k^0,B} \circ F(x_2)\}.$$

Obviously, this holds for either $\varphi_{k^0,B} \circ F(x_1) = +\infty$ or $\varphi_{k^0,B} \circ F(x_2) = +\infty$. We now suppose that both $\varphi_{k^0,B} \circ F(x_1)$ and $\varphi_{k^0,B} \circ F(x_2)$ are real numbers. We will prove that the set $S(\varphi_{k^0,B} \circ F, \gamma)$ is convex for all $\gamma \in \mathbb{R}$. This assertion is trivial when $\gamma < 0$ since $S(\varphi_{k^0,B} \circ F, \gamma) = \emptyset$. Now, we suppose that $\gamma \geq 0$, $\varphi_{k^0,B}F(x_1) \leq \gamma$ and $\varphi_{k^0,B}F(x_2) \leq \gamma$. Let $\alpha_1 := \varphi_{k^0,B}F(x_1)$ and $\alpha_2 := \varphi_{k^0,B}F(x_2)$. Take $\bar{\alpha} := \max\{\alpha_1, \alpha_2\} \leq \gamma$ and $\epsilon > 0$ arbitrary.
Since Theorem 5.3.2 (c), it holds that

$$F(x_1) \preceq_l^{\mathcal{K}} (\bar{\alpha} + \epsilon)k^0 + B$$

and

$$F(x_2) \preceq_l^{\mathcal{K}} (\bar{\alpha} + \epsilon)k^0 + B.$$

Therefore,

$$(\bar{\alpha} + \epsilon)k^0 + B \subseteq \bigcup_{y \in F(x_1)} (y + \mathcal{K}(y)) \cap \bigcup_{y \in F(x_2)} (y + \mathcal{K}(y)).$$

Taking into account Definition 7.2.12, we get that

$$(\bar{\alpha} + \epsilon)k^0 + B \subseteq \bigcup_{y \in F(\lambda x_1 + (1-\lambda)x_2)} (y + \mathcal{K}(y)).$$

Therefore,

$$\varphi_{k^0,B}F(\lambda x_1 + (1 - \lambda)x_2) \leq \bar{\alpha} + \epsilon, \text{for all} \epsilon > 0.$$

Thus,

$$\varphi_{k^0,B}F(\lambda x_1 + (1 - \lambda)x_2) \leq \bar{\alpha} \leq \gamma,$$

i.e.,

$$\lambda x_1 + (1 - \lambda)x_2 \in S(\varphi_{k^0,B} \circ F, \gamma),$$

or $S(\varphi_{k^0,B} \circ F, \gamma)$ is convex.

Conversely, suppose that $\varphi_{k^0,B} \circ F$ is quasiconvex, we prove that (7.8) is fulfilled for all $x_1, x_2 \in S$ and $\lambda \in [0,1]$.

Take $z \in \bigcup_{y \in F(x_1)} (y+\mathcal{K}(y)) \cap \bigcup_{y \in F(x_2)} (y+\mathcal{K}(y))$, arbitrarily. This is equivalent to $F(x_i) \preceq_l^{\mathcal{K}} \{z\}$, for $i = 1, 2$.

Therefore, by Theorem (5.3.2)(e), $\varphi_{k^0,\{z\}}(F(x_i)) = 0$. Since $\varphi_{k^0,\{z\}} \circ F$ is quasiconvex, $\varphi_{k^0,\{z\}} \circ F(\lambda x_1 + (1-\lambda)x_2) \leq 0$.

By Theorem (5.3.2)(e), $F(\lambda x_1 + (1-\lambda)x_2) \preceq_l^{\mathcal{K}} \{z\}$, that is,

$$z \in \bigcup_{y \in F(\lambda x_1 + (1-\lambda)x_2)} (y+\mathcal{K}(y)) \text{ for all } z \in \bigcup_{y \in F(x_1)} (y+\mathcal{K}(y)) \cap \bigcup_{y \in F(x_2)} (y+\mathcal{K}(y)),$$

which completes the proof. □

Observe that when $\mathcal{K}(\cdot) = K$, where K is a proper, convex cone with $\operatorname{int} K \neq \emptyset$ the first statement of Proposition 7.2.13 becomes [64, Proposition 3.4] and [46, Proposition 6.3]. In addition, [46, Proposition 6.3] assumed that B is a K-proper set.

Now, we are ready to present the second class of well-posed set optimization problems where the objective map $F : X \rightrightarrows Y$ is $\mathcal{K}$-quasiconvex.

Theorem 7.2.14. *[71, Theorem 5.13] Let X be a locally compact metric space and S be a convex subset of X. Suppose that $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$ satisfy all the assumptions given in Proposition 7.2.10 and F is $\mathcal{K}$-quasiconvex w.r.t. $\preceq_l^{\mathcal{K}}$ on S. Let $\bar{x}$ be a strictly minimal solution of problem $(\bar{P}_{\mathcal{K}})$. Then, $(\bar{P}_{\mathcal{K}})$ is well-posed at $\bar{x}$.*

Proof. Let $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$. By Propositions 7.2.10 and 7.2.13, $\varphi_{k^0,F(\bar{x})} \circ F$ is lower semicontinuous and quasiconvex. Taking into account $\operatorname{argmin}(\varphi_{k^0,F(\bar{x})} \circ F, S) = \{\bar{x}\}$ and Theorem 7.2.8, the problem $(P_{\varphi_{k^0,F(\bar{x})}})$ is generalized well-posed and also is Tykhonov well-posed. Applying Theorem 7.2.6, we have that the problem $(\bar{P}_{\mathcal{K}})$ is well-posed at $\bar{x}$. The proof is complete. □

Remark 7.2.15. *Theorems 7.2.11 and 7.2.14 respectively extend [64, Theorem 4.5] and [64, Theorem 4.6], in which the authors used the domination $\mathcal{K}(y) \equiv C$, where $C \subseteq Y$ is a convex cone such that $\operatorname{int} C \neq \emptyset$. Note that in this case (2.11) holds true and thus one can get [64, Theorem 4.5] and [64, Theorem 4.6] without the fulfillment of this condition.*

Chapter 8

Applications in Radiotherapy Treatment, Medical Image Registration and Uncertain Optimization

This chapter presents applications of our results given in the previous parts. In Section 8.1, we propose a variable ordering structure which is satisfied many conflict goals in radiotherapy treatment based on the threshold doses of many organs. After proving several important properties of this ordering structure, we derive necessary conditions for the goal dose of the beam intensity problem. This desired dose is concerned as a minimal (or nondominated) solution of a vector approximation optimization problem, which is studied in Section 3.2. Section 8.2 characterizes solutions of an Image Registration Problems, where the decision maker wants to compare two sets of data (images). Applying the results given in Chapter 5, we calculate this characterization in detail for the domination structure $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$. Section 8.3 presents an application of results given in Chapter 7 for uncertain optimization. We introduce a concept of *optimistic* solutions of an uncertain multiobjective optimization problem, where the domination structure is equipped with a variable ordering. Then, we characterize (strictly) optimistic solutions of multiobjective problems that are contaminated with uncertain data in general setting. The results presented within this chapter are based on [69, 71, 82].

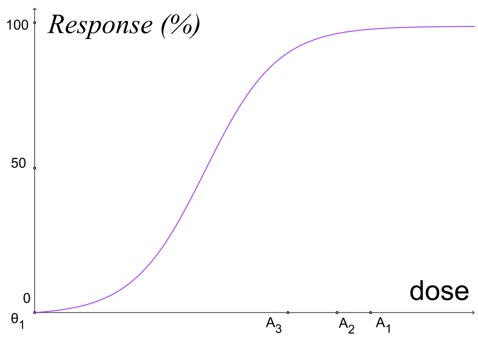
8.1 An Application in Radiotherapy Treatment

We begin this part by illustrating the importance of studying Intensity Modulated Radiotherapy Treatment (IMRT) w.r.t. variable domination structures. This treatment is an advancement in radiotherapy that allows modulating radiation intensity across a beam. Currently, it is being used to treat cancers of the prostate, head and neck, breast, lung as well as certain types of sarcomas. The basic idea of IMRT is to reduce the intensity of rays going through particularly sensitive critical structures and to increase the intensity of those rays seeing primarily the target volume.

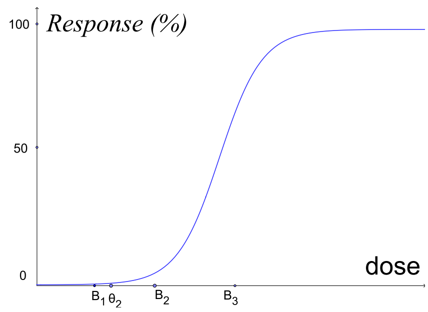
The problem of calculating those intensities based on dose prescription in the target volume and the surrounding critical structures is called inverse planning. This problem is modeled as a multiobjective optimization problem with an objective function depending on the specific goal that the treatment planner wants to achieve. In general, a level dose of radiation in the cancer organ should be closed to desired dose but at the same time the surrounding organs are still protected. This inverse problem w.r.t. a constant cone is studied by several authors and can be divided into two categories, namely the multiobjective nonlinear programming and the multiobjective linear programming. For more detail, we refer the reader to [32] and the references therein. However, by illustrating a treatment of a prostate cancer tumor, Eichfelder [36] showed that it may seem more appropriate to concern this inverse problem as a multiobjective optimization problem w.r.t. a variable ordering structure. Similarly, we explain the role of variable domination structures by presenting a special problem in radiotherapy treatment in the following.

We consider the treatment of a lung cancer, where lung is the most sensitive organ to radiotherapy damage. The dose delivered to lung is limited by spinal cord and heart (critical organs). Thus, to reduce side effects, the doses delivered to spinal cord and heart have to be minimized. A dose response curve describes the change in effect on an organ caused by differing levels of doses delivered to it. We suppose that the dose response curves for lung, spinal cord and heart in lung cancer treatment are illustrated in Fig1.

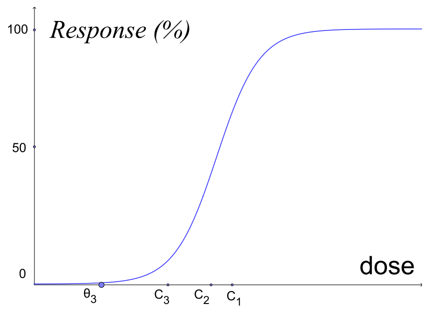
These curves can be used to estimate a threshold dose for each organ. The threshold dose is defined as the dose of radiation, below which the organism does not suffer from any effect. In mathematical point of view, it is the dose, below which the response is zero and above which it is nonzero, see [49, 94]. In this case, we assume that θ_1, θ_2 and θ_3 are respectively the threshold doses of lung, spinal cord and heart. We now have a look at three treatment plans $(A_1, B_1, C_1), (A_2, B_2, C_2),$ and (A_3, B_3, C_3), where A_i, B_i, C_i are the doses delivered to lung, spinal cord and heart respectively, $i = 1, 2, 3$. From a practical point of view, if the variations of dose imply a small effect on a certain organ, a rise of the dose delivered to that organ in order to improve the value for another organ is preferred, see [36, Chapter 10]. In more detail, beside the goal of an improvement on the dose level in lung, spinal cord and heart, we also prefer the changing of dose delivered to spinal cord from B_1 to B_2 for reducing the dose amount



(a) Dose response curve for lung.



(b) Dose response curve for spinal cord.



(c) Dose response curve for heart.

Figure 8.1: Dose response curves in lung cancer treatment.

in heart, for instance, from C_1 to C_2. The reason is that a large improvement in the respond on heart is reached by changing the dose to C_2 while the effects on lung and spinal cord are changed mildly.

We assume that all treatment plans is a subset of $\mathbb{R}^3$ and consider a closed convex cone $\mathcal{C} \subset \mathbb{R}^3$. Suppose that we derive a mathematical model for this problem w.r.t $\mathcal{C}$. We denote $(A_2, B_2, C_2) \leq_{\mathcal{C}} (A_1, B_1, C_1)$ if $d := (A_1, B_1, C_1) - (A_2, B_2, C_2) \in \mathcal{C}$. Since $\mathcal{C}$ is a cone, $\lambda d \in \mathcal{C}$ for all $\lambda > 0$ and therefore if (A_3, B_3, C_3) satisfies $(A_2, B_2, C_2) - (A_3, B_3, C_3) = \beta d$ with $\beta > 0$ we have $(A_3, B_3, C_3) \leq_{\mathcal{C}} (A_2, B_2, C_2)$ i.e., (A_3, B_3, C_3) is "better" than (A_2, B_2, C_2).

On the other hand, having a look at the dose response curve of spinal cord, the increase in the effect for spinal cord is large by changing the dose from B_2 to B_3. Therefore, (A_3, B_3, C_3) might not be a preferred solution from a practical point of view. Thus, the choice of variable ordering cone depending on the actual doses in this circumstance seems to be more appropriate.

8.1.1 A Variable Ordering Cone relevant to Radiotherapy Treatment

This part introduces an appropriate variable ordering structure for the beam intensity problem as well as its properties. At first, we find out how to derive a mathematical model for the beam intensity optimization problem. As illustrated in [32], the beam is discretized into p bixels or beamlets. The 3D volume of patient is divided into l voxels which included l_T tumor voxels, l_C critical organ voxels ($l = l_T + l_C$) in which T represents the tumor, C represents critical organs. The dose deposited in voxel i at unit intensity for bixel j is denoted by $a_{ij} \in \mathbb{R}$. We assume that the dose deposition matrix $A = (a_{ij}) \in \mathbb{R}^{l \times p}$ is given. We denote the beam intensity by $x \in \mathbb{R}^p$. The relationship between the beam intensity and the dose is illustrated as follows:

$$d = Ax,$$

where $d \in \mathbb{R}^l$ is a dose vector and its element d_i correspond to the dose deposited in voxel i. We assume that A can be partitioned and reordered into sub-matrices $A_T \in \mathbb{R}^{l_T \times p}$ and $A_C \in \mathbb{R}^{l_N \times p}$ whose rows corresponding to tumor and normal voxels (Figure 8.1.1). Moreover, A_C can be divided into $A_{C_1}, ..., A_{C_k}$ according to the doses deposited in k different critical organs $C_1, ..., C_k$.

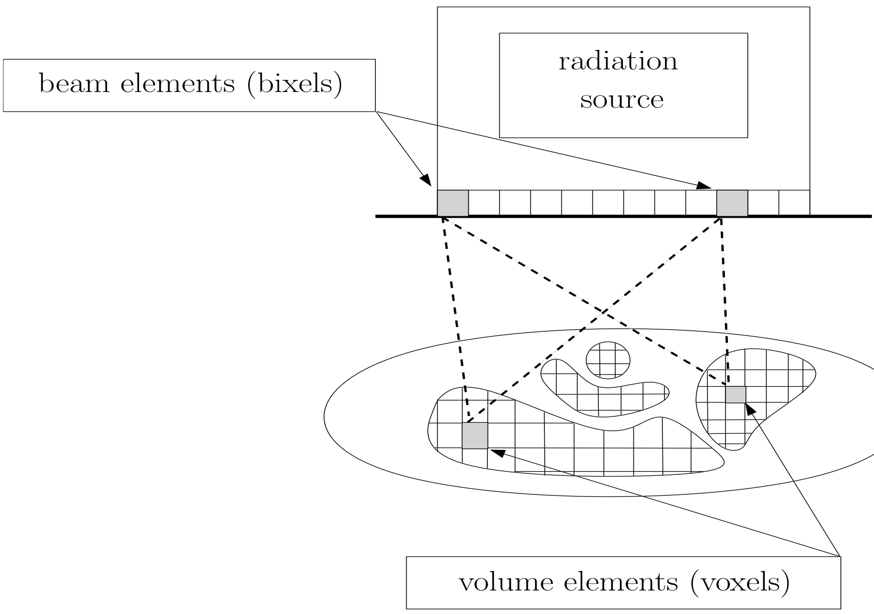


Figure 8.2: Discretization of patient into voxels and of beam into bixels, [32].

It is obvious that the dose delivered to tumor and critical organ voxels are $A_T x$, $A_{C_1}x, \ldots, A_{C_k}x$, respectively. Because different tissues can tolerate different amounts of radiation, the radiation oncologist need to determine a “prescription dose” which consists of the target dose for the tumor $TG \in \mathbb{R}^{l_T}$, the lower bounds and upper bounds on the dose to tumor voxels $TLB, TUB \in \mathbb{R}^{l_T}$, the upper bounds on the dose to different critical organs $C_1UB, C_2UB, ..., C_kUB$. In radiation treatment, threshold dose is defined as the amount of radiation that is required to cause a specific tissue effect.

As has been outlined before, a vector optimization w.r.t. a variable ordering cone models for radiotherapy treatment is more appropriate than that one w.r.t. a constant cone. Therefore, it is necessary to construct a suitable ordering structure in order to find the desired dose for our beam intensity problem. From a practical perspective, a dose delivered to a critical organ should be reduced when it exceeds the threshold dose of that organ. Otherwise, we can increase this dose in favor of an improvement in the value of another critical organ. This leads to a variable ordering structure in the space $\mathbb{R}^n$ determined as follows:

Given $\theta \in \mathbb{R}^n$, for every $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ we set

$$I^{>}(y) := \{i \in \{1, 2, ..., n\} |\ y_i > \theta_i\},$$

and

$$I_{\leq}(y) := \{i \in \{1, 2, ..., n\} |\ y_i \leq \theta_i\}.$$

Obviously, for each $y \in \mathbb{R}^n$, it holds that $I^{>}(y) \cup I_{\leq}(y) = \{1, 2, ..., n\}$.

We define the variable ordering map $\mathcal{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as follows:

$$\forall\ y \in \mathbb{R}^n,\ \mathcal{K}(y) := \begin{cases} \{d \in \mathbb{R}^n |\ d_i \geq 0 \text{ for } i \in I^{>}(y)\} & \text{if } I^{>}(y) \neq \emptyset, \\ \mathbb{R}^n & \text{if } I^{>}(y) = \emptyset. \end{cases} \tag{8.1}$$

This set-valued mapping will be used Section 8.1.3 to construct an intensity problem in radiotherapy treatment when θ is chosen appropriately. In the following, we present some properties of the proposed variable ordering cone $\mathcal{K}(\cdot)$ given by (8.1).

Proposition 8.1.1. *[82, Proposition 3.4] Let $\theta \in \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ be given and the variable ordering structure $\mathcal{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be determined by (8.1). Then, the following assertions hold true:*

(i) For each $y \in \mathbb{R}^n$, $\mathcal{K}(y)$ is a closed and convex cone and $\mathbb{R}^n_+ \subseteq \mathcal{K}(y)$. In addition, $\mathcal{K}(y)$ is pointed if and only if $y_i > \theta_i,\ \forall\ i = 1, 2, \ldots, n$.

(ii) For all $y^1, y^2 \in \mathbb{R}^n$, we have that

$$y^1 - y^2 \in \mathbb{R}^n_+ \Longrightarrow \mathcal{K}(y^1) \subseteq \mathcal{K}(y^2).$$

(iii) If $\bar{y} \in \Omega$ satisfies $I^{>}(\bar{y}) \neq \emptyset$, then there exists $e \neq 0$ such that

$$e \in \bigcap_{y \in \Omega} \mathcal{K}(y) \setminus (-\mathcal{K}(\bar{y})).$$

(iv) $\operatorname{Gr}\mathcal{K}$ *is a closed subset of* $\mathbb{R}^n \times \mathbb{R}^n$.

Proof. **(i)** Obviously, for all $y \in \mathbb{R}^n$ we have that $\mathcal{K}(y)$ is a closed and convex cone. $\mathcal{K}(y)$ is pointed if and only if $\mathcal{K}(y) \cap (-\mathcal{K}(y)) = \{0\}$. By the definition of $\mathcal{K}(\cdot)$, it holds that

$$\mathcal{K}(y) \cap (-\mathcal{K}(y)) = \{d \in \mathbb{R}^n |\ d_i = 0 \text{ with } i \in I^{>}(y)\}.$$

Thus, $\mathcal{K}(y)$ is pointed if and only if $I^{>}(y) = \{1, 2, \dots, n\}$. This condition also means that $y_i > \theta_i,\ \forall\ i = 1, 2, \dots, n$.

(ii) It follows from $y^1 - y^2 \in \mathbb{R}^n_+$ that $y_i^1 \geq y_i^2$ for all $i = 1, 2, \dots, n$. Therefore, for all $i \in I^{>}(y^2)$ we have $y_i^1 \geq y_i^2 > \theta_i$, i.e., $i \in I^{>}(y^1)$. Thus, $I^{>}(y^2) \subseteq I^{>}(y^1)$ and we obtain that $\mathcal{K}(y^1) \subseteq \mathcal{K}(y^2)$.

(iii) Assume that $i_0 \in I^{>}(\bar{y})$, i.e., $\bar{y}_{i_0} > \theta_{i_0}$. It follows from the definition of $\mathcal{K}(\cdot)$ that if $d = (d_1, \dots, d_n) \in (-\mathcal{K}(\bar{y}))$ then $d_{i_0} \leq 0$. Take $e := (e_1, \dots, e_n)$, where $e_i > 0$ for all $i = 1, 2, \dots, n$. This implies that $e \in \mathbb{R}^n_+$. Taking into account $\mathbb{R}^n_+ \subseteq \bigcap_{y \in \Omega} \mathcal{K}(y)$, it yields $e \in \bigcap_{y \in \Omega} \mathcal{K}(y)$. Since $e_{i_0} > 0$, $e \notin (-\mathcal{K}(\bar{y}))$ and thus $e \in \bigcap_{y \in \Omega} \mathcal{K}(y) \setminus (-\mathcal{K}(\bar{y}))$.

(iv) Consider a consequence $\{(y^k, d^k)\} \subset \operatorname{Gr}\mathcal{K}$ which convergences to (y, d) when $k \to \infty$. We need to show that $(y, d) \in \operatorname{Gr}\mathcal{K}$.

Suppose that

$$(y^k, d^k) = (y_1^k, y_2^k, \dots, y_n^k, d_1^k, d_2^k, \dots, d_n^k)$$

and

$$(y, d) = (y_1, y_2, \dots, y_n, d_1, d_2, \dots, d_n).$$

To proceed, we consider the following cases.

Case 1: $I^{>}(y) \neq \emptyset$. Let $i \in I^{>}(y)$ be arbitrary.

Since

$$y_i > \theta_i \text{ and } \{y_i^k\} \to y_i \text{ when } k \to \infty,$$

it holds that

$$\exists\ k_0 \in \mathbb{N} \text{ such that for all } k \geq k_0 : y_i^k > \theta_i.$$

Taking into account $(y_1^k, y_2^k, \dots, y_n^k, d_1^k, \dots, d_n^k) \in \operatorname{Gr}\mathcal{K}$, we get that

$$d_i^k \geq 0,\ \forall\ k \geq k_0.$$

Since $d_i^k \to d_i$ when $k \to \infty$, it yields $d_i \geq 0$. Thus, $(y, d) \in \operatorname{Gr}\mathcal{K}$.

Case 2: $I^{>}(y) = \emptyset$, i.e., $y_i \leq \theta_i,\ \forall\ i = 1, 2, \dots, n$. It follows directly from the definition of $\mathcal{K}(\cdot)$ that $(y, d) \in \operatorname{Gr}\mathcal{K}$. The proof is complete. □

Observer that properties (i) and (iii) of Theorem 8.1.1 ensure the fulfillment of conditions $(i) - (ii)$ in Theorem 3.2.2. This is beneficial for us to apply the necessary optimality condition for nondominated solutions of problem $(P_{\mathcal{K}}^{vec})$ in Theorem 3.2.2 for a vector approximation problem equipped with $\mathcal{K}(\cdot)$ given by (8.1). This will be illustrated in the following section.

8.1.2 Optimality Conditions for Vector Approximation Problems w.r.t. Variable Ordering Structures

In this part, we investigate necessary optimality conditions for solutions of problem ($P_{\mathcal{K}}^{app}$) equipped with $\mathcal{K}(\cdot)$ given by (8.1). This means that we consider a special cone-valued mapping useful in radiotherapy treatment. We again assume that A_i are linear functions from $\mathbb{R}^m$ to $\mathbb{R}^{m_i}$, $a_i \in \mathbb{R}^{m_i}$, $\|\cdot\|_i$ be norms in $\mathbb{R}^{m_i}, i = 1, 2, \ldots, n$. For convenient of the reader, we rewrite and denote it as follows:

$$\mathcal{K} - \text{Min} f(x) \tag{$\tilde{P}_{\mathcal{K}}^{app}$}$$

where

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

$$f(x) := \begin{pmatrix} \|A_1 x - a_1\|_1 \\ \|A_2 x - a_2\|_2 \\ \dots \\ \|A_n x - a_n\|_n \end{pmatrix},$$

and $\mathcal{K}(\cdot)$ is determined by (8.1).

Observe that Section 3.2 has already derived optimality conditions for nondominated solutions of ($P_{\mathcal{K}}^{app}$), which is a general formula of ($\tilde{P}_{\mathcal{K}}^{app}$). Now, we calculate a necessary optimality condition for nondominated solutions of problem ($\tilde{P}_{\mathcal{K}}^{app}$). Suppose that $\bar{x}$ is a nondominated solution of ($\tilde{P}_{\mathcal{K}}^{app}$), $\bar{y} = f(\bar{x})$. We determine $D^*\mathcal{K}(\bar{y}, 0)(y^*)$ by calculating the normal cone $N(\text{Gr}\,\mathcal{K}, (\bar{y}, 0))$. Suppose that W is a subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We consider the associated distance function

$$\text{dist}(x, W) := \inf_{u \in W} \|x - u\|,$$

and define the Euclidean projector of x to W by

$$P(x, W) := \{\omega \in W | \; \|x - W\| = \text{dist}(x, W)\}, \tag{8.2}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. The following theorem describes the formulation of the basic normal cone to a subset $W \subseteq \mathbb{R}^n$ which is locally closed around $\bar{x} \in W$.

Theorem 8.1.2. *[90, Theorem 1.6] Let $W \subseteq \mathbb{R}^n$ be locally closed around $\bar{x} \in W$. Then, it holds that*

$$N(W, \bar{x}) = \limsup_{x \to \bar{x}} \hat{N}(W, x),$$

and

$$N(W, \bar{x}) = \limsup_{x \to \bar{x}} [\text{cone}(x - P(x; W))].$$

In order to compute $N(\operatorname{Gr}\mathcal{K},(\bar{y},0))$, we rewrite the graph of mapping $\mathcal{K}(\cdot)$ as follows: For each $I \subseteq \{1,2,\ldots,n\}$ we set:

$$U_I := \{y \in \mathbb{R}^n |\ I^>(y) = I\},$$

and

$$\mathbb{R}^n_I := \{d \in \mathbb{R}^n |\ d_i \geq 0,\ \forall\ i \in I\}.$$

Obviously, if $y \in U_I$ then $\mathcal{K}(y) = \mathbb{R}^n_I$. Therefore, we obtain

$$\operatorname{Gr}\mathcal{K} = \bigcup_{I \subseteq \{1,2,\ldots,n\}} U_I \times \mathbb{R}^n_I.$$

Since $\operatorname{Gr}\mathcal{K}$ is closed (Proposition 8.1.1) and taking into account Theorem 8.1.2, we have that

$$N(\operatorname{Gr}\mathcal{K},(\bar{y},0)) = \limsup_{(y,d)\to(\bar{y},0)} \left[\operatorname{cone}((y,d) - P((y,d); \bigcup_{I \subseteq \{1,2,\ldots,n\}} U_I \times \mathbb{R}^n_I))\right].$$

This analysis leads to the question if we can provide the results of Euclidean projector to graph of $\mathcal{K}(\cdot)$ which gives the formulation of the normal cone to its graph by using Theorem 8.1.2. This is discussed in the following theorem.

Theorem 8.1.3. *[82, Theorem 4.4] Given a point $\theta = (\theta_1,\ldots,\theta_n) \in \mathbb{R}^n$ and the set-valued mapping $\mathcal{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ determined by (8.1). For each element $(y,d) \in \mathbb{R}^n \times \mathbb{R}^n$ we set*

$$J^{\geq}(d) := \{i \in \{1,2,\ldots,n\} |\ d_i \geq 0\},$$

and

$$I^>(y) := \{i \in \{1,2,\ldots,n\} |\ y_i > \theta_i\}.$$

Then, it holds for the Euclidean projector given by (8.2) that

(i) If $I \not\subseteq I^>(y)$ then $P((y,d); U_I \times \mathbb{R}^n_I) = \emptyset$.

(ii) If $I \subseteq I^>(y)$ then

(a)

$$P((y,d); U_I \times \mathbb{R}^n_I) = \{(y^I, d^I) \in U_I \times \mathbb{R}^n_I\},$$

where $(y^I, d^I) := (y^I_1,\ldots,y^I_n, d^I_1,\ldots,d^I_n)$ determined by:

$$\begin{aligned} d^I_i &= d_i, \forall\ i \in (\{1,2,\ldots,n\} \setminus I) \cup J^{\geq}(d),\\ d^I &= 0,\ \forall\ i \in I \setminus J^{\geq}(d),\\ y^I_i &= \theta_i,\ \forall\ i \in I^>(y) \setminus I,\\ y^I_i &= y_i,\ \forall\ i \in (\{1,2,\ldots,n\} \setminus I^>(y)) \cup I. \end{aligned}$$

(b) $\operatorname{dist}((y,d), U_I \times \mathbb{R}^n_I) = \sqrt{\sum_{i \in I^>(y) \setminus I} (y_i - \theta_i)^2 + \sum_{i \in I \setminus J^{\geq}(d)} (d_i)^2}.$

(iii) $P((y,d); \operatorname{Gr} \mathcal{K}) = \bigcup_I P((y,d); U_I \times \mathbb{R}^n_I)$, *where*

$$I = \operatorname{argmin}_{I \subseteq I^>(y)} dist((y,d), U_I \times \mathbb{R}^n_I).$$

Proof. Let I and I' be two arbitrary subsets of $\{1, 2, \ldots, n\}$. We have that

$$(U_I \times \mathbb{R}^n_I) \cap (U_{I'} \times \mathbb{R}^n_{I'}) = \emptyset \text{ with } I \neq I'.$$

Therefore,

$$P\big((y,d); \bigcup_{I \subseteq \{1,2,\ldots,n\}} U_I \times \mathbb{R}^n_I\big) = \operatorname{argmin}_{(y^I, d^I) \in U_I \times \mathbb{R}^n_I} \big(\|(y,d) - (y^I, d^I)\|^2\big).$$

Thus, for each $I \subset \{1, 2, \ldots, n\}$, we need to find $P((y,d); U_I \times \mathbb{R}^n_I)$.
(i) Suppose by contradiction that $P((y,d); U_I \times \mathbb{R}^n_I) \neq \emptyset$, i.e., there is $(y^I, d^I) \in U_I \times \mathbb{R}^n_I$ such that

$$\|(y^I, d^I) - (y,d)\|^2 = \inf_{(\omega,\gamma) \in U_I \times \mathbb{R}^n_I} \|(\omega, \gamma) - (y,d)\|^2.$$

Since $I \not\subseteq I^>(y)$, $\exists\, i_0 \in I$ but $i_0 \notin I^>(y)$. We assume that

$$(y^I, d^I) = (y^I_1, y^I_2, \ldots, y^I_n, d^I_1, d^I_2, \ldots, d^I_n),$$

and $y^I_{i_0} = \theta_{i_0} + \epsilon$ with $\epsilon > 0$ (because $y^I \in U_I$ and $i_0 \in I$).
We consider the point $(y^*, d^*) := (y^*_1, y^*_2, \ldots, y^*_n, d^*_1, d^*_2, \ldots, d^*_n)$, determined by

$$y^*_{i_0} = \theta_{i_0} + \frac{\epsilon}{2}, y^*_i = y^I_i \text{ for } i \in \{1, 2, \ldots, n\} \setminus \{i_0\},$$

and $d^*_k = d^I_k, k = 1, 2, \ldots, n$. Obviously, $(y^*, d^*) \in U_I \times \mathbb{R}^n_I$. Now, we get

$$\begin{aligned}
\|(y^*, d^*) - (y,d)\|^2 &= \sum_{i=1}^n ((y^*_i - y_i)^2 + (d^*_i - d_i)^2) \\
&= \sum_{i \in \{1,2,\ldots,n\} \setminus \{i_0\}} (y^*_i - y_i)^2 + (y^*_{i_0} - y_{i_0})^2 + \sum_{i=1}^n (d^*_i - d_i)^2 \\
&= \sum_{i \in \{1,2,\ldots,n\} \setminus \{i_0\}} (y^I_i - y_i)^2 + (\theta_{i_0} + \frac{\epsilon}{2} - y_{i_0})^2 + \sum_{i=1}^n (d^I_i - d_i)^2 \\
&< \sum_{i \in \{1,2,\ldots,n\} \setminus \{i_0\}} (y^I_i - y_i)^2 + (\theta_{i_0} + \epsilon - y_{i_0})^2 + \sum_{i=1}^n (d^I_i - d_i)^2 \\
&\quad \text{(because } y^I_{i_0} = \theta_{i_0} + \epsilon) \\
&= \sum_{i \in \{1,2,\ldots,n\} \setminus \{i_0\}} (y^I_i - y_i)^2 + (y^I_{i_0} - y_{i_0})^2 + \sum_{i=1}^n (d^I_i - d_i)^2 \\
&= \sum_{i=1}^n ((y^I_i - y_i)^2 + (d^I_i - d_i)^2) \\
&= \|(y^I, d^I) - (y,d)\|^2.
\end{aligned}$$

Thus, $(y^*, d^*) \in U_I \times \mathbb{R}^n_I$ and $\|(y^*, d^*) - (y, d)\|^2 < \|(y^I, d^I) - (y, d)\|^2$, this is a contradiction with the definition of (y^I, d^I):

$$\|(y^I, d^I) - (y, d)\|^2 = \inf_{(\omega,\gamma)\in U_I\times\mathbb{R}^n_I} \|(\omega, \gamma) - (y, d)\|^2.$$

(ii) (a) Let $I \subseteq I^>(y)$ and take an arbitrary element $(y^I, d^I) \in U_I \times \mathbb{R}^n_I$. It holds that

$$\begin{aligned}
\|(y^I, d^I) - (y, d)\|^2 &= \sum_{i=1}^{n}(y_i^I - y_i)^2 + \sum_{i=1}^{n}(d_i^I - d_i)^2 \\
&= \sum_{i\in I}(y_i^I - y_i)^2 + \sum_{i\in I^>(y)\setminus I}(y_i^I - y_i)^2 + \sum_{i\in\{1,2,\dots,n\}\setminus I^>(y)}(y_i^I - y_i)^2 \\
&+ \sum_{i\in(\{1,2,\dots,n\}\setminus\{I\cup J^\geq(d)\})}(d_i^I - d_i)^2 + \sum_{i\in I\setminus J^\geq(d)}(d_i^I - d_i)^2 \\
&+ \sum_{i\in J^\geq(d)\setminus I}(d_i^I - d_i)^2 + \sum_{I\cap J^\geq(d)}(d_i^I - d_i)^2 \\
&\geq \sum_{i\in I^>(y)\setminus I}(y_i^I - y_i)^2 + \sum_{i\in I\setminus J^\geq(d)}(0 - d_i)^2 \\
&\geq \sum_{i\in I^>(y)\setminus I}(\theta_i - y_i)^2 + \sum_{i\in I\setminus J^\geq(d)}(0 - d_i)^2.
\end{aligned}$$

The last conclusion is obtained since: $\begin{cases}\forall\ i \in I^>(y)\setminus I : y_i > \theta_i \text{ and } y_i^I \leq \theta_i,\\ \forall\ i \in I \setminus J^\geq(d) : d_i^I \geq 0 \text{ and } d_i < 0.\end{cases}$

Therefore,

$$\|(y^I, d^I) - (y, d)\|^2 \geq \sum_{i\in I^>(y)\setminus I}(\theta_i - y_i)^2 + \sum_{i\in I\setminus J^\geq(d)}(0 - d_i)^2,$$

and the equation holds true if we choose

$$\begin{aligned}
d_i^I &= d_i,\ \forall\ i \in (\{1, 2, \dots, n\} \setminus I) \cup J^\geq(d),\\
d^I &= 0,\ \forall\ i \in I \setminus J^\geq(d),\\
y_i^I &= \theta_i,\ \forall\ i \in I^>(y) \setminus I,\\
y_i^I &= y_i,\ \forall\ i \in (\{1, 2, \dots, n\} \setminus I^>(y)) \cup I.
\end{aligned}$$

(b) It is obviously that if $I \subseteq I^>(y)$ then

$$\operatorname{dist}((y, d), U_I \times \mathbb{R}^n_I) = \sqrt{\sum_{i\in I^>(y)\setminus I}(y_i - \theta_i)^2 + \sum_{i\in I\setminus J^\geq(d)}(d_i)^2}.$$

(iii) Since $\operatorname{Gr}\mathcal{K}$ is a closed set, $P((y, d), \operatorname{Gr}\mathcal{K}) \neq \emptyset$, see [99, Example 1.20]. Suppose that

$$(\hat{y}, \hat{d}) \in P((y, d), \operatorname{Gr}\mathcal{K}),$$

then

$$\exists\ J \subset \{1, 2, \dots, n\} \text{ such that } (\hat{y}, \hat{d}) \in U_J \times \mathbb{R}^n_J.$$

It holds that

$$\begin{aligned} d((y,d),(\hat{y},\hat{d}) &= d((y,d),\operatorname{Gr}\mathcal{K}) \\ &\le d((y,d),U_J\times\mathbb{R}^n_J) \\ &\le d((y,d),(\hat{y},\hat{d})). \end{aligned}$$

The equation holds true if $(\hat{y},\hat{d}) = P((y,d),U_J\times\mathbb{R}^n_J)$. Taking into account (i) and (ii), we get $J\subseteq I^>(y)$ and this completes the proof. □

The following remark shows how one obtains the Euclidean projector of an arbitrary point in $\mathbb{R}^n\times\mathbb{R}^n$ to the graph of the mapping $\mathcal{K}(\cdot)$, the normal cone to its graph as well as its coderivative, cf. [82, Remark 4.4].

Remark 8.1.4. *(i) We get the projection of (y,d) to $\operatorname{Gr}\mathcal{K}$ through these following steps:*

Step 1: Determine $I^>(y)$.

Step 2: For each $I\subseteq I^>(y)$, calculate $d((y,d),U_I\times\mathbb{R}^n_I)=\sigma_I$ and

$$P((y,d);U_I\times\mathbb{R}^n_I) = \{(y^I,d^I)\in U_I\times\mathbb{R}^I : d((y^I,d^I),U_I\times\mathbb{R}^n_I)=\sigma_I\}.$$

Step 3: Find $\sigma := min_{I\subseteq\{1,2,\ldots,n\}}\{\sigma_I\}$ and

$$P((y,d),\operatorname{Gr}\mathcal{K}) = \bigcup_I P((y,d);U_I\times\mathbb{R}^n_I),$$

where I satisfies $I\subseteq I^>(y)$ and $d((y,d),U_I\times\mathbb{R}^n_I)=\sigma$.

(ii) From the Theorem 8.1.3 above we obtain that

$$N(\operatorname{Gr}\mathcal{K},(\bar{y},0)) = \limsup_{(y,d)\to(\bar{y},0)} \operatorname{cone}((y,d)-P((y,d),\operatorname{Gr}\mathcal{K}), \tag{8.3}$$

where $P((y,d),\operatorname{Gr}\mathcal{K})$ is determined in Theorem 8.1.3 (iii). In addition, it holds that

$$D^*\mathcal{K}(\bar{y},0)(y^*) = \{x^*\in\mathbb{R}^n|\ (x^*,-y^*)\in N(\operatorname{Gr}\mathcal{K},(\bar{y},0))\}, \tag{8.4}$$

where $N(\operatorname{Gr}\mathcal{K},(\bar{y},0))$ given by (8.3).

Now, we utilize the result above and Theorem 3.2.2 to derive a necessary optimality condition for nondominated solutions of the problem $(\tilde{P}^{app}_{\mathcal{K}})$.

Theorem 8.1.5. *[82, Theorem 4.5] Let $\mathcal{K}:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ be a set-valued map given by (8.1). Suppose that $\bar{x}\in\Omega$ is a nondominated solution of the problem $(\tilde{P}^{app}_{\mathcal{K}})$, $\bar{y}:=f(\bar{x})$ and the following assertions hold true:*

(i) $I^>(\bar{y})\neq\emptyset$.

(ii) There is a unique point y^* such that $-y^* \in D^*\mathcal{K}(\bar{y}, 0)(y^*)$.

Then, there exist $y^* \in \mathbb{R}^n \setminus \{0\}$ and corresponding $z^* \in (y^* + D^*\mathcal{K}(\bar{y}; 0)(y^*))$ and $T_i \in L(\mathbb{R}^{m_i}, \mathbb{R})$ satisfying

$$T_i(A_i(\bar{x}) - a^i) = \|A_i(\bar{x}) - a^i\|_i \text{ and } \|T_i\|_{i^*} \leq 1 (i = 1, 2, \ldots, n) \text{ such that}$$

$$0 \in \sum_{i=1}^{n} A_i^* z^* T_i + N(\Omega, \bar{x}),$$

where $D^*\mathcal{K}(\bar{y}, 0)(y^*)$ is determined by (8.3).

Proof. For every $y \in \mathbb{R}^n$, $\mathcal{K}(y)$ is a closed convex cone (Proposition 8.1.1 (i)). Taking into account $I^>(\bar{y}) \neq \emptyset$ and Proposition 8.1.1(iii), it holds that

$$\exists\, e \in \mathbb{R}^n, e \neq 0 : e \in \bigcap_{y \in \mathbb{R}^n} \mathcal{K}(y) \setminus (-\mathcal{K}(\bar{y})).$$

Applying directly Theorem 3.2.2 and the formulation of $D^*\mathcal{K}(\bar{y}, 0)(y^*)$ given in Remark 8.1.4(ii), we obtain the desired conclusion. □

The following result provides specific optimality conditions for minimal solutions of $(\tilde{P}_{\mathcal{K}}^{app})$, cf. [82, Theorem 4.6]. Note that in the proof we utilize Propositions 2.3.19 and 2.3.20 given in Chapter 2.

Theorem 8.1.6. *Let $\mathcal{K} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map given by (8.1).*

(i) Let $\bar{x} \in \Omega$ and $\bar{y} := f(\bar{x})$. Then, the normal cone to $\mathcal{K}(\bar{y})$ at 0 is given by:

$$N(\mathcal{K}(\bar{y}), 0) = N_1 \times \ldots \times N_n,$$

where for $i = 1, 2, \ldots, n$,

$$\begin{cases} N_i := (-\infty, 0] & \text{with } i \in I^>(\bar{y}), \\ N_i := \{0\} & \text{with } i \notin I^>(\bar{y}). \end{cases} \tag{8.5}$$

(ii) Suppose that $\bar{x}$ is a minimal solution of $(\tilde{P}_{\mathcal{K}}^{app})$ w.r.t. $\mathcal{K}(\cdot)$. Then, there exist $y^* \in \mathbb{R}^n_+ \setminus \{0\}$ and $T_i \in L(\mathbb{R}^{m_i}, \mathbb{R})$ satisfying

$$T_i(A_i(\bar{x}) - a^i) = \|A_i(\bar{x}) - a^i\|_i \text{ and } \|T_i\|_{i^*} \leq 1 \text{ such that}$$

$$0 \in \sum_{i=1}^{n} A_i^* y^* T_i + N(\Omega, \bar{x}).$$

Proof. ***(i)*** By the definition of $\mathcal{K}(\cdot)$, we get that $\mathcal{K}(\bar{y}) = K_1 \times \ldots \times K_n$, where for $i = 1, 2, \ldots, n$,

$$\begin{cases} K_i := [0, +\infty) & \text{with } i \in I^>(\bar{y}), \\ K_i := \mathbb{R} & \text{with } i \notin I^>(\bar{y}). \end{cases} \tag{8.6}$$

Taking into account Proposition 2.3.20 and the formula of K_i in (8.6), it holds that

$$N(\mathcal{K}(\bar{y}),0) = N(K_1,0) \times N(K_2,0) \times \ldots \times N(K_n,0) = N_1 \times \ldots \times N_n,$$

where for $i = 1, 2, \ldots, n$,

$$\begin{cases} N_i := (-\infty, 0] & \text{if } K_i = [0,+\infty), \\ N_i := \{0\} & \text{if } K_i = \mathbb{R}. \end{cases} \tag{8.7}$$

Thus, from (8.6) and (8.7) it yields that (8.5) is fulfilled.

(ii) Suppose that $\bar{x}$ is a minimal solution of the problem $(\tilde{P}^{app}_{\mathcal{K}})$, then, $\bar{x}$ is a Pareto efficient solution of $(\tilde{P}^{app}_{\mathcal{K}(\bar{y})})$. This implies that $\mathcal{K}(\bar{y})$ is pointed. Since Theorem 8.1.1, $\mathcal{K}(\bar{y}) = \mathbb{R}^n_+$. Taking into account Corollary 3.2.3 and (8.7), we have that there exist $y^* \in \mathbb{R}^n_+ \setminus \{0\}$ and $T_i \in L(\mathbb{R}^{m_i}, \mathbb{R})$ satisfying

$$T_i(A_i(\bar{x}) - a^i) = \|A_i(\bar{x}) - a^i\|_i \text{ and } \|T_i\|_{i^*} \leq 1 (i = 1, 2, \ldots, n) \text{ such that}$$

$$0 \in \sum_{i=1}^{n} A_i^* y^* T_i + N(\Omega, \bar{x}).$$

□

8.1.3 Necessary Optimality Conditions for the Beam Intensity Problem

We begin this part by deriving a mathematical formulation of beam intensity optimization problem. As mentioned in Section 8.1.1, $A_T, A_{C_1}, \ldots, A_{C_k}$ denote the dose depositions corresponding to tumor T and critical organs $C_1, \ldots, C_k$. Assume that θ_{C_i} is given threshold dose of critical organ i, where $i \in \{1, \ldots, k\}$. Since the deviation from the dose delivered to tumor organ to the target dose is always nonnegative and should be minimized, we set $\theta := (0, \theta_{C_1}, \ldots, \theta_{C_k}) \in \mathbb{R}^{k+1}$. The set of bound conditions for beam intensity is given by

$$\Omega := \{x \in \mathbb{R}^p |\ 0 \leq x, TLB \leq A_T x \leq TUB, A_{C_i} x \leq C_i UB \text{ for } i = 1, \ldots, k\}.$$

The problem of finding beam intensity in radiotherapy treatment is denoted by $(P^{imrt}_{\mathcal{K}})$ and formulated as follows:

$$\text{Minimize } f(x) \text{ subject to } x \in \Omega \text{ w.r.t. } \mathcal{K}(\cdot), \tag{$P^{imrt}_{\mathcal{K}}$}$$

where

$$f : \mathbb{R}^p \to \mathbb{R}^{k+1}$$

$$f(x) := \begin{pmatrix} \|A_T x - TG\|_\infty \\ \|A_{C_1} x\|_\infty \\ \ldots \\ \|A_{C_k} x\|_\infty \end{pmatrix},$$

where the variable ordering mapping $\mathcal{K}(\cdot)$ given by (8.1) and $\|\cdot\|_\infty$ is maximum norm.

The first criterion can be interpreted as the deviation from the prescribed dose to the tumor. $\|A_{C_i}x\|_\infty$ is the dose to the critical organ i ($i = 1, \dots, k$). The objective function can be constructed by using Euclidean norm. However, this norm allows the averaging out of large deviations on a small tissue by small or no deviation on a large tissue, see [81]. Therefore, it seems to be more reasonable to use the maximum norm. Observe that ($P_{\mathcal{K}}^{imrt}$) is a special case of ($\tilde{P}_{\mathcal{K}}^{app}$). To be precise, ($\tilde{P}_{\mathcal{K}}^{app}$) reduces to ($P_{\mathcal{K}}^{imrt}$) when we choose: $\theta = (0, \theta_{C_1}, \dots, \theta_{C_k})$; $m = p; n = k + 1; A_1 = A_T, A_{j+1} = A_{C_j};\ j = 1, \dots, k;$ $a_1 = TG, a_2 = \dots = a_{k+1} = 0$ and $\|\cdot\|_i = \|\cdot\|_\infty,\ \forall\ i = 1, \dots, n$.

Now, we explain how the decision maker could use our proposed ordering structure $\mathcal{K}(\cdot)$ given by (8.1) in radiotherapy treatment. Assume that we are at a present beam intensity $\bar{x}$, $\bar{y} = f(\bar{x})$ and $\bar{y}_i > \theta_i, i \in I^>(\bar{y})$. The doctor can seek for a "better"element $x',\ y' = f(x')$ in the sense that he/she increases the amount of dose delivered to the tumor organ and decreases the quantity of dose delivered to critical organ $C_i, i \in I^>(\bar{y})$. In this case, we have that $y' \in \bar{y} - \mathcal{K}(\bar{y})$. We call $\bar{x}$ a desired beam intensity type I if it does not exist another beam intensity being better than $\bar{x}$, i.e., there is no $y \in f(\Omega)\setminus\{\bar{y}\}$ such that $y \in \bar{y} - \mathcal{K}(\bar{y})$. This means that $\bar{x}$ is a minimal solution of the problem ($P_{\mathcal{K}}^{imrt}$) w.r.t. $\mathcal{K}(\cdot)$ in the sense of Definition 3.1.5.

On the other hand, we consider that $\bar{x}$ is "worse"than a beam intensity $\tilde{x}$ if the change of dose determined by $\bar{x}$ compared to that determined by $\tilde{x}$ is described as: the dose delivered to tumor organ is decreased and the critical organ $C_i, i \in I^>(\tilde{y})$ will receive a larger amount of dose. In this case, we get that $\bar{y} \in \tilde{y} + \mathcal{K}(\tilde{y})$, where $\tilde{y} = f(\tilde{x})$. We call $\bar{x}$ a desired beam intensity type II if it is not worse than every other beam intensity $x \in \Omega \setminus \{\bar{x}\}$, i.e., there is no $y \in f(\Omega) \setminus \{\bar{y}\}$ such that $\bar{y} \in y + \mathcal{K}(y)$. This implies that $\bar{x}$ is a nondominated solution of the problem ($P_{\mathcal{K}}^{imrt}$) w.r.t. $\mathcal{K}(\cdot)$ in the sense of Definition 3.1.5.

The following remark shows a property of the desired beam intensity type I and II, cf. [82, Remark 5.5].

Remark 8.1.7. *From the practical point of view, we can see that if $\bar{x}$ is a desired beam intensity type I (or II) and $\bar{y} := f(\bar{x})$ then $I^>(\bar{y}) \neq \emptyset$. Indeed, suppose that $I^>(\bar{y}) = \emptyset$, i.e., $\bar{y}_1 \leq 0$ and $\bar{y}_i \leq \theta_{C_i},\ \forall\ i = 1, 2, \dots, k$. Since $\bar{y}_1 = \|A_T\bar{x} - TG\| \geq 0$, it yields $\bar{y}_1 = 0$. This condition means that the dose $A_T\bar{x}$ delivered to the tumor is equal to the target dose TG. Because of this large dose, some other critical organs will suffer from some effects. From this circumstance, there exists $i \in \{1, 2, \dots, k\}$ such that $\bar{y}_i > \theta_{C_i}$. Thus, we arrive at a contradiction to $I^>(\bar{y}) = \emptyset$.*

To this end, we present a corollary about the conditions for the desired beam intensity type I and II which we search when dealing with the inverse problem in IMRT. This result is concerned as a direct consequence of Theorems 8.1.5 and 8.1.6 (ii). Since the proof is mostly similar to that of these two results with the only exception being the condition $I^{>}(\bar{y}) \neq \emptyset$ is relaxed, we omit it in this thesis.

Corollary 8.1.8. *Consider the beam intensity problem* $(P_{\mathcal{K}}^{imrt})$ *with* θ_{C_i} *is the threshold dose of critical organ* C_i*,* $i = 1 \dots k$*. Let* $\bar{x} \in \Omega$ *be given,* $\bar{y} = f(\bar{x})$*.*

(i) If $\bar{x}$ *is a desired beam intensity type I then there are* $y^* \in \mathbb{R}^n_+ \backslash \{0\}$*,* $Z_1 \in L(\mathbb{R}^{l_T}, \mathbb{R})$*, and* $Z_i \in L(\mathbb{R}^{l_{C_{i-1}}}, \mathbb{R})$*,* $i = 2, \dots, k+1$ *satisfying*

$$Z_1(A_T\bar{x} - TG) = \|A_T\bar{x} - TG\|_\infty, Z_i(A_{C_{i-1}}\bar{x}) = \|A_{C_{i-1}}\bar{x}\|_\infty, i = 2, \dots, k+1 \tag{8.8}$$

and

$$\|Z_j\|_\infty \leq 1 \text{ for all } j = 1, \dots, k+1 \tag{8.9}$$

such that

$$0 \in \sum_{j=1}^{k+1} A_j^* y^* Z_j + N(\Omega, \bar{x}).$$

(ii) If $\bar{x}$ *is a desired beam intensity type II and there is a unique point* y^* *such that* $-y^* \in D^*\mathcal{K}(\bar{y}, 0)(y^*)$ *then there are* $y^* \in \mathbb{R}^n \setminus \{0\}$*,* $z^* \in (y^* + D^*\mathcal{K}(\bar{y}; 0)(y^*))$*,* $Z_1 \in L(\mathbb{R}^{l_T}, \mathbb{R})$*, and* $Z_i \in L(\mathbb{R}^{l_{C_{i-1}}}, \mathbb{R})$*,* $i = 2, \dots, k+1$ *satisfying* (8.8) *and* (8.9) *such that*

$$0 \in \sum_{j=1}^{k+1} A_j^* z^* Z_j + N(\Omega, \bar{x}).$$

8.2 An Application in Medical Image Registration

In this part, we characterize solutions of a medical image registration problem by using the results given in Chapter 5. Medical image registration has been used widely in medical treatment, for instance in radiotherapy (treatment verification, treatment planning, treatment guidance), orthopaedic surgery and surgical microscope. The problem of image registration is finding a transformation matching two given sets of data (images). Suppose that T is a subset of transformations. H and K are two images obtained by X-Ray or Angiography (2D image) or Computed Tomography (CT), Magnetic Resonance Tomography (MRT) and PET (Positron Emission Tomography)(3D image). Assume that $H, K \subseteq Y$, where $Y = \mathbb{R}^2$ or $Y = \mathbb{R}^3$. For each $t \in T$, a set of comparison mappings $\{f_i(t, H, K) \subset \mathbb{R}, i = 1, ..., m\}$ is calculated, where $f_i : (T, H, K) \rightrightarrows \mathbb{R}$. Note that f_i can be different distance functions used to compare H and K, $i = 1, .., m$. Wacker [105] proposed a multiobjective problem for this by consider an objective function $f := (f_1, \dots, f_i)$. However, we choose the set-valued mappings $f_i : (T, H, K) \rightrightarrows \mathbb{R}$,

$i = 1, ..., m$, since the fact that there may exist some movements of the patient during the time his images are taken, which can lead to perturbed data. For each point $y \in \mathbb{R}^m$, we attach a weight $\omega(y) := (\omega_1(y), \omega_2(y), ..., \omega_m(y)) \in \mathbb{R}^m_+$, which is chosen by the decision maker, see [35, 105] for more detail. To formulate our Image registration problem w.r.t. a variable ordering structure, we utilize a set-valued map $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ introduced in [35] and given as

$$\begin{aligned} \mathcal{K} :\ & \mathbb{R}^m \rightrightarrows \mathbb{R}^m, \\ \forall\ y \in \mathbb{R}^m, \mathcal{K}(y) :&= \{d \in \mathbb{R}^m | \sum_{i=1}^{m} \text{sign}(d_i)\omega_i(y) \geq 0\}, \end{aligned} \tag{8.10}$$

where

$$\text{sign}(d_i) := \begin{cases} 1 & \text{if } d_i > 0, \\ 0 & \text{if } d_i = 0, \\ -1 & \text{if } d_i < 0. \end{cases}$$

Obviously, for each $y \in \mathbb{R}^m$, $\mathcal{K}(y)$ is a cone satisfying $\mathbb{R}^m_+ \subseteq \mathcal{K}(y)$. We denote by $(P^{mir}_{\mathcal{K}})$ a mathematical model of Medical Image Registration and it is defined as:

$$\mathcal{K} - \underset{t \in T}{\text{Min}}(f_1(t, H, K), ..., f_m(t, H, K)). \tag{$P^{mir}_{\mathcal{K}}$}$$

Here, we use the set relation $\preceq^{\mathcal{K}}_l$ since this relation is often used when the decision maker concerns the best case. Observe that if we set

$$\hat{\mathcal{A}} := \{(f_1(t, H, K), ..., f_m(t, H, K)), t \in T\},$$

then each element of $\hat{\mathcal{A}}$ is a set $A \subseteq \mathbb{R}^m$. We are looking for a transformation $\bar{t} \in T$ such that

$$\bar{A} := (f_1(\bar{t}, H, K), ..., f_m(\bar{t}, H, K)) \in \text{Min}(\hat{\mathcal{A}}, \preceq^{\mathcal{K}}_l).$$

Motivated by Theorem 5.2.1, we obtain the following corollary. Since its proof is similar to that of Theorem 5.2.1, we omit it in this part. We again suppose that $k^0 \in Y \setminus \{0\}$ satisfies (H_1) determined in Section 2.4.

Corollary 8.2.1. *Let $\hat{A} \subset \mathcal{P}(\mathbb{R}^m)$, $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ and $k^0 \in \mathbb{R}^m \setminus \{0\}$ such that (H_1) is satisfied. Assume that for all $A \in \hat{\mathcal{A}}$, $\bigcup_{a \in A}(a + \mathcal{K}(a))$ is closed. Then, $\bar{t} \in T$ is a solution of the problem $(P^{mir}_{\mathcal{K}})$ if $\bar{A} := (f_1(\bar{t}, H, K), ..., f_m(\bar{t}, H, K)))$ is a minimal element of $\hat{\mathcal{A}}$ w.r.t. $\preceq^{\mathcal{K}}_l$, i.e*

$$\forall A \in \hat{\mathcal{A}}, A \not\sim \bar{A} : g^{\preceq^{\mathcal{K}}_l}(A, \bar{A}) > 0. \tag{8.11}$$

Observe that $g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) = \sup_{d \in \bar{A}} \inf_{a \in A} z^{a+\mathcal{K}(a),k^0}(-d)$, where $z^{a+\mathcal{K}(a),k^0}$ is given by (2.19) with $D = a + \mathcal{K}(a)$. It holds for all $a \in A$ and $d \in \bar{A}$ that

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) &= \inf\{r \in \mathbb{R} | \ a - d \in rk^0 - \mathcal{K}(a)\} \\ &= \inf\{r \in \mathbb{R} | \ d + rk^0 - a \in \mathcal{K}(a)\} \\ &= \inf\{r \in \mathbb{R} | \ \sum_{i=1}^{m} \operatorname{sign}(d + rk^0 - a)_i \omega_i(a) \geq 0\}. \end{aligned} \tag{8.12}$$

Now, we apply the characterization of minimal elements of $\hat{\mathcal{A}}$ which is derived in Corollary 8.2.1 to the image registration problem $(P_{\mathcal{K}}^{mir})$ when $m = 2$. The following proposition calculates $z^{a+\mathcal{K}(a),k^0}(-d)$ in detail.

Proposition 8.2.2. *[69, Proposition 5] Let $\hat{\mathcal{A}} \subset \mathcal{P}(\mathbb{R}^2)$, $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ and $k^0 \in \mathbb{R}^2 \backslash \{0\}$ such that (H_1) is satisfied. Assume that for all $A \in \hat{\mathcal{A}}$, $\bigcup_{a \in A} (a + \mathcal{K}(a))$ is closed. Then, $\bar{t} \in T$ is a solution of the problem $(P_{\mathcal{K}}^{mir})$ if $\bar{A} := (f_1(\bar{t}, H, K), f_2(\bar{t}, H, K))$ is a minimal element of $\hat{\mathcal{A}}$ w.r.t. $\preceq_l^{\mathcal{K}}$, i.e.,*

$$\forall A \in \hat{\mathcal{A}}, A \not\sim \bar{A} : g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) > 0,$$

where

$$g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) = \sup_{d \in \bar{A}} \inf_{a \in A} z^{a+\mathcal{K}(a),k^0}(-d),$$

in which $z^{a+\mathcal{K}(a),k^0}(-d)$, $a \in A, \bar{d} \in \bar{A}$ is determined by

$$z^{a+\mathcal{K}(a),k^0}(-d) = \begin{cases} \frac{a_1 - d_1}{k_1^0} & \text{if } \omega_1(a) > 0, \omega_2(a) = 0 \ \text{ or } \ \omega_1(a) \geq \omega_2(a) > 0, \\ \frac{a_2 - d_2}{k_2^0} & \text{if } \omega_1(a) = 0, \omega_2(a) > 0 \ \text{ or } \ \omega_2(a) \geq \omega_1(a) > 0. \end{cases}$$

Proof. As shown in [35], for each $\omega = (\omega_1, \omega_2) \in \mathbb{R}_+^m$ one has for each $a \in A$,

$$\mathcal{K}(a) = \begin{cases} \{d \in \mathbb{R}^2 | \ d_1 \geq 0, \ d_2 \in \mathbb{R}\} & \text{if } \omega_1(a) > 0, \ \omega_2(a) = 0, \\ \{d \in \mathbb{R}^2 | \ d_1 \in \mathbb{R}, \ d_2 \geq 0\} & \text{if } \omega_1(a) = 0, \ \omega_2(a) > 0, \\ \{d \in \mathbb{R}^2 | \ (d_1 \geq 0, \ d_2 \geq 0) \ \text{ or } \ (d_1 < 0, \ d_2 > 0)\} & \text{if } \omega_2(a) \geq \omega_1(a) > 0, \\ \{d \in \mathbb{R}^2 | \ (d_1 \geq 0, \ d_2 \geq 0) \ \text{ or } \ (d_1 > 0, \ d_2 < 0)\} & \text{if } \omega_1(a) \geq \omega_2(a) > 0. \end{cases}$$

In the following, we illustrate the formulation of $z^{a+\mathcal{K}(a),k^0}(-d)$, $a \in A, d \in \bar{A}$ in case m=2 for all possible weight $\omega(a) := (\omega_1(a), \omega_2(a))$.

(a) $\omega_1(a) > 0, \omega_2(a) = 0$. We have that $\mathcal{K}(a) = \{d \in \mathbb{R}^2 | \ d_1 \geq 0, d_2 \in \mathbb{R}\}$. Now, (8.12) becomes

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) &= \inf\{r \in \mathbb{R} | \ rk^0 - (a - d) \in \mathcal{K}(a)\} \\ &= \inf\{r \in \mathbb{R} | \ (d_1 + rk_1^0 - a_1) \geq 0\}. \end{aligned} \tag{8.13}$$

Since $\forall \gamma > 0 : \gamma k^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a)$ and $0 \in \mathcal{K}(a)$, we get that

$$\forall \gamma > 0 : \gamma k^0 \in \mathcal{K}(a) \Longleftrightarrow \gamma k_1^0 \geq 0, \forall \gamma > 0.$$

Therefore, $k_1^0 \geq 0$. We choose k^0 such that $k_1^0 > 0$. By (8.13), it holds that

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ r \geq \frac{a_1 - d_1}{k_1^0}\} = \frac{a_1 - d_1}{k_1^0}.$$

(b) $\omega_1 = 0, \omega_2(a) > 0$. We can prove this part analogously to part (a) and obtain:

$$z^{a+\mathcal{K}(a),k^0}(-d) = \frac{a_2 - d_2}{k_2^0}.$$

(c) $\omega_2(a) \geq \omega_1(a) > 0$. Then,

$$\mathcal{K}(a) = \{d \in \mathbb{R}^2|\ (d_1 \geq 0, d_2 \geq 0) \text{ or } (d_1 < 0, d_2 > 0)\}.$$

Now, (8.12) becomes

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ ((d_1 + rk_1^0 - a_1) \geq 0, (d_2 + rk_2^0 - a_2) \geq 0) \\ \text{or } ((d_1 + rk_1^0 - a_1) < 0, (d_2 + rk_2^0 - a_2) > 0)\}. \quad (8.14)$$

Since

$$\forall\ t > 0 :\ tk^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a) \text{ and } 0 \in \mathcal{K}(a),$$

it holds that

$$(tk_1^0 \geq 0, tk_2^0 \geq 0) \text{ or } (tk_1^0 < 0, tk_2^0 > 0) \\ \Longleftrightarrow (k_1^0 \geq 0, k_2^0 \geq 0) \text{ or } (k_1^0 < 0, k_2^0 > 0).$$

Now, we consider two cases:
Case 1: $(k_1^0 \geq 0, k_2^0 \geq 0)$, we choose $k_1^0 > 0$ and $k_2^0 > 0$, then (8.14) becomes:

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ (r \geq \frac{a_1 - d_1}{k_1^0}, r \geq \frac{a_2 - d_2}{k_2^0}) \\ \text{or } (r < \frac{a_1 - d_1}{k_1^0}, r > \frac{a_2 - d_2}{k_2^0})\}.$$

If $\frac{a_1 - d_1}{k_1^0} \leq \frac{a_2 - d_2}{k_2^0}$, then

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ r \geq \frac{a_2 - d_2}{k_2^0}\} = \frac{a_2 - d_2}{k_2^0}.$$

If $\frac{a_1 - d_1}{k_1^0} > \frac{a_2 - d_2}{k_2^0}$, then

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ r \geq \frac{a_1 - d_1}{k_1^0} \text{ or } r > \frac{a_2 - d_2}{k_2^0}\} \\ = \frac{a_2 - d_2}{k_2^0}.$$

Case 2: $(k_1^0 < 0, k_2^0 > 0)$, then (8.14) becomes:

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ (r \le \frac{a_1 - d_1}{k_1^0}, r \ge \frac{a_2 - d_2}{k_2^0}) \\ \text{or } (r > \frac{a_1 - d_1}{k_1^0}, r > \frac{a_2 - d_2}{k_2^0})\}.$$

If $\frac{a_1-d_1}{k_1^0} < \frac{a_2-d_2}{k_2^0}$, then

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ r > \frac{a_2 - d_2}{k_2^0}\} = \frac{a_2 - d_2}{k_2^0}.$$

If $\frac{a_1-d_1}{k_1^0} \ge \frac{a_2-d_2}{k_2^0}$ then,

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ \frac{a_2 - d_2}{k_2^0} \le r \le \frac{a_1 - d_1}{k_1^0} \text{ or } r > \frac{a_1 - d_1}{k_1^0}\} \\ = \frac{a_2 - d_2}{k_2^0}.$$

From the two above cases, we conclude that

$$z^{a+\mathcal{K}(a),k^0}(-d) = \frac{a_2 - d_2}{k_2^0}.$$

(d) $\omega_1(a) \ge \omega_2(a) > 0$. Then,

$$\mathcal{K}(a) = \{d \in \mathbb{R}^2|\ (d_1 \ge 0, d_2 \ge 0) \text{ or } (d_1 > 0, d_2 < 0)\}.$$

Now, (8.12) becomes

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R}|\ ((d_1 + rk_1^0 - a_1) \ge 0, (d_2 + rk_2^0 - a_2) \ge 0) \\ \text{or } ((d_1 + rk_1^0 - a_1) > 0, (d_2 + rk_2^0 - a_2) < 0)\}. \tag{8.15}$$

By using the same arguments like in part (c), we obtain:

$$z^{a+\mathcal{K}(a),k^0}(-d) = \frac{a_1 - d_1}{k_1^0}.$$

The proof is complete.

□

Remark 8.2.3. *Analogously, it is possible to characterize solutions of problem ($P_{\mathcal{K}}^{mir}$) by taking into account Theorem 7.1.1 and the functional given by (5.11). We are going to utilize them in the next section for uncertain multiobjective optimization problems.*

8.3 An Application in Uncertain Optimization

In this section, we will introduce a concept of *optimistic* solutions for an uncertain multiobjective optimization problem, where the domination structure is equipped with a variable ordering. Moreover, we characterize these optimistic solutions based on the results derived in Chapter 7. Uncertain data contaminate most optimization problems in various applications ranging from science and engineering to industry and thus represent an essential component in optimization. From a mathematical point of view, many problems can be modeled as optimization problems and be solved, but in real life, having exact data is very rare and seems almost impossible. Due to a lack of complete information, uncertain data can highly affect solutions and thus influence the decision making process. Hence, it is crucial to address this important issue in optimization theory. Many examples for uncertain data in optimization problems can be found in the field of market analysis, share prices, transportation science, timetabling and location theory.

Since the data of uncertainty is not usually completed known before the optimization problem with uncertainty is solved, it is very important to estimate the effects of this uncertainty on optimal solutions. There are many methods to deal with this question such as sensitivity analysis, stochastic programming and robust optimization techniques, see [14, 33, 51, 52, 63, 78, 100] and references therein. In this section, we concern robust multiobjective optimization, which is recently observed in [51, 52] as an important application of set optimization. Different approaches to robust multiobjective optimization with a fixed domination structure were examined in [51, 52]. Recently, Khan et al. [63] have illustrated in detail the relationships between robust counterparts of uncertain vector-valued optimization problems and set optimization based on the set approach. The field of robust optimization dates back to the 1940ies, where Wald [106] investigated worst case analysis in decision theory. Since the groundbreaking work by Ben-Tal, El Ghaoui, and Nemirovski in the 1990ies (see, for instance, [11]) robust optimization has been of great interest in the optimization community. The first robust counterpart concepts for uncertain vector-valued optimization problems was introduced by Deb and Gupta [21]. The authors define robustness as some sensitivity against disturbances in the decision space. They call a solution to a problem robust if small perturbations in the decision space result in only small disturbances in the objective space. Kuroiwa and Lee [78] presented the first scenario-based approach by directly transferring the main idea of robust scalar optimization to multiobjective optimization. This concept was generalized by Ehrgott et al. [33] who implicitly used a set-order relation to define robust solutions for uncertain multiobjective optimization problems.

Now, we recall some notation of uncertain multiobjective optimization introduced in Ehrgott et al. [33] (see also [52]) which will be used throughout this section. Let Y be a linear topological space, X be a linear space, $S \subseteq X$ be a nonempty set, and let $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}^N$ be an uncertainty set. The uncertainty set $\mathcal{U}$ contains all possible parameter values that the uncertain parameter may attain. Let $f : S \times \mathcal{U} \to Y$ be the function that is to be minimized. Our goal is to obtain solutions that are *optimistic*, i.e., that perform well in the best-case scenario. For the scalar case $Y = \mathbb{R}$, this would mean to minimize the functional $\inf_{\xi \in \mathcal{U}} f(x, \xi)$ on S. Of course, if f is vector-valued, this scalar approach cannot be easily transferred to vector optimization. Due to the absence of a total order on Y, we need to define the meaning of an optimal solution.

We define for $x \in S$ the outcome set

$$f_{\mathcal{U}}(x) := \{f(x, \xi) | \ \xi \in \mathcal{U}\},$$

i.e., the image of f under $\mathcal{U}$. For a fixed $\xi \in \mathcal{U}$, the vector optimization problem is denoted by

$$\min_{x \in S} f(x, \xi). \tag{$P(\xi)$}$$

The family of all problems $\bigcup_{\xi \in \mathcal{U}} (P(\xi))$, is called **uncertain optimization problem**, and is denoted by $P(\mathcal{U})$. Furthermore, the family of all sets $f_{\mathcal{U}}(x)$, $x \in S$, is denoted by $\mathcal{A}$. In contrast to the original robustness concepts, our "optimistic" concept uses the lower set less order relation equipped with a variable domination structure according to Definition 4.2.6. This kind of optimality focuses on the lower bound of a set $f_{\mathcal{U}}(x)$. Contrary to the traditional robustness approach, we are therefore not interested in a worst-case concept but a best-case concept. Thus, this approach is suitable for a decision maker who is not considered to be risk averse but rather risk affine and has positive expectations about the future.

Definition 8.3.1. *Let $P(\mathcal{U})$ be an uncertain optimization problem and let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map satisfying (2.9).*

*(i) $\bar{x} \in S$ is called an **optimistic solution** of problem $P(\mathcal{U})$ if $f_{\mathcal{U}}(\bar{x})$ is a minimal element of $\mathcal{A}$ in terms of Definition 4.2.6 (a).*

*(ii) $\bar{x}$ is called a **strictly optimistic solution** of problem $P(\mathcal{U})$ if $f_{\mathcal{U}}(\bar{x})$ is a strictly minimal element of $\mathcal{A}$ in terms of Definition 4.2.6 (b).*

Now, we discuss the role of the variable domination structure. For simplicity, we consider the case $Y = \mathbb{R}^2$, i.e., we consider an uncertain bicriteria optimization problem. Assume that the data of a vector $a \in \mathbb{R}^2$ is perturbed by uncertain data and only an approximation $A \subset \mathbb{R}^2$ is known (see Figure 8.3 (a)). Similarly, the data of a vector $\widetilde{b}$ is disturbed and only an estimated set $\widetilde{B}$ can be generated. In order to compare the set A to the set $\widetilde{B}$, the lower set less order relation $\preceq_l^Q$ with the fixed ordering cone

$Q = \mathbb{R}^2_+$ shall be used, such that $\widetilde{B} \subseteq A + Q \Longleftrightarrow A \preceq_l^Q \widetilde{B}$. This relation ensures that the lower bounds of $\widetilde{B}$ are not "worse" than those of A. Since the data are uncertain, it seems likely that there exist undesired elements located far from where most uncertain data is found. If there exists such an element $\bar{b} \notin \widetilde{B}$ which is located far away from $\widetilde{B}$, then the relation $A \preceq_l^Q B$, where $B := \widetilde{B} \cup \{\bar{b}\}$, may not hold anymore (see Figure 8.3 (b)). In order to still include $\bar{b}$ in the analysis but to obtain the result that the set A is, for the most part, preferred to B, a planner can introduce a variable domination structure in the following way: Let $\underline{a} \in A$ and $\mathcal{K} : Y \rightrightarrows Y$ with

$$\mathcal{K}(y) := \begin{cases} K & \text{if } y = \underline{a}, \\ \mathbb{R}^2_+ & \text{else,} \end{cases}$$

where K is a cone which fulfills $\bar{b} \in \{\underline{a}\} + K$ $(K := \mathcal{K}(\underline{a}))$, see Figure 8.3, (b)). Then, we have $A \preceq_l^{\mathcal{K}} B$. This ensures that all estimated elements are taken into account, as undesired elements can be handled by using variable domination structures.

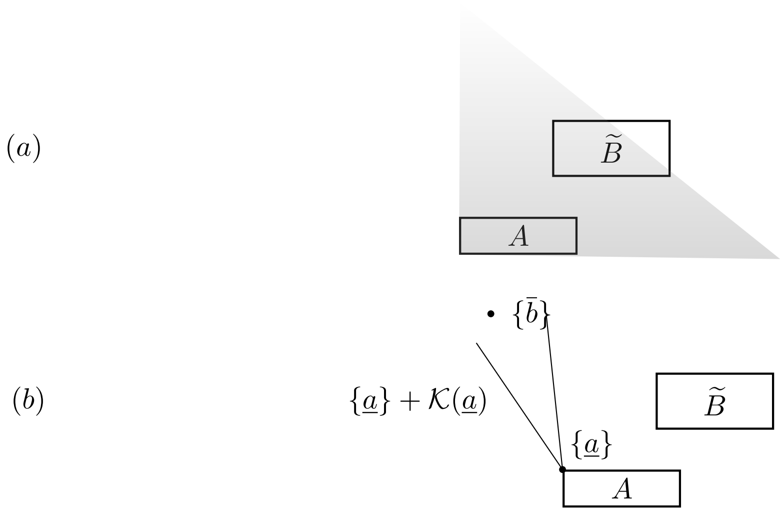


Figure 8.3: Visualization of two outcome sets $A, \widetilde{B} \subset \mathbb{R}^2$ of a uncertain bicriteria optimization problem with undesired elements.

Now, we are ready to apply the characterizations of solutions of set optimization problems w.r.t. variable domination structures, which were derived in Section 7.1, to the uncertain optimization problem $P(\mathcal{U})$.

Corollary 8.3.2. *[71, Corollary 6.2] Let $k^0 \in Y \setminus \{0\}$ be given such that the inclusion (H_1) is satisfied. Then, the following assertions hold.*

(a) Assume that $\bigcup_{y \in f_{\mathcal{U}}(x)} (y + \mathcal{K}(y))$ is closed for all $f_{\mathcal{U}}(x) \in \mathcal{A}$. Then, $\bar{x} \in S$ is an optimistic solution of problem $P(\mathcal{U})$ if and only if $\varphi_{k^0, f_{\mathcal{U}}(\bar{x})}(f_{\mathcal{U}}(x)) > 0$ for all $f_{\mathcal{U}}(x) \in \mathcal{A}, f_{\mathcal{U}}(x) \not\preceq f_{\mathcal{U}}(\bar{x})$.

(b) *Assume that* $\bigcup_{y\in f_{\mathcal{U}}(x)} (y+\mathcal{K}(y))$ *is closed for all* $f_{\mathcal{U}}(x) \in \mathcal{A}$*. Then,* $\bar{x} \in S$ *is a strictly optimistic solution of problem* $P(\mathcal{U})$ *if and only if* $\varphi_{k^0, f_{\mathcal{U}}(\bar{x})}(f_{\mathcal{U}}(x)) > 0$ *for all* $f_{\mathcal{U}}(x) \in \mathcal{A} \setminus \{f_{\mathcal{U}}(\bar{x})\}$.

In the next corollary, we denote $\operatorname{Im} f_{\mathcal{U}} := \{f_{\mathcal{U}}(x) \mid x \in S \text{ and } f_{\mathcal{U}}(x) \neq \emptyset\}$.

Corollary 8.3.3. *[71, Corollary 6.3] Let* $k^0 \in Y \setminus \{0\}$ *be given such that the inclusion* (H_1) *is satisfied and let* $\mathcal{K} : Y \rightrightarrows Y$ *be a set-valued map such that* $\bigcup_{y\in F(x)} (y+\mathcal{K}(y))$ *is closed for each* $x \in S$ *and the conditions (2.9)-(2.11) are fulfilled. Consider* $\bar{x} \in S$*. The following assertions hold true.*

(a) $\bar{x}$ *is an optimistic solution of problem* $P(\mathcal{U})$ *if and only if there is a functional* $G : \operatorname{Im} f_{\mathcal{U}} \to \mathbb{R}_+ \bigcup\{+\infty\}$ *being* $\preceq_l^{\mathcal{K}}$*-monotone such that*

$$x \in S, \quad f_{\mathcal{U}}(x) \sim f_{\mathcal{U}}(\bar{x}) \quad \Longleftrightarrow \quad G(f_{\mathcal{U}}(x)) = 0.$$

(b) $\bar{x}$ *is a strictly optimistic solution of problem* $P(\mathcal{U})$ *if and only if there is a functional* $G : \operatorname{Im} f_{\mathcal{U}} \to \mathbb{R}_+ \bigcup\{+\infty\}$ *being* $\preceq_l^{\mathcal{K}}$*-monotone such that*

$$x \in S, \quad G(f_{\mathcal{U}}(x)) = 0 \quad \Longleftrightarrow \quad x = \bar{x}.$$

Chapter 9

Conclusion and Out Look

In this dissertation, we derived new results concerning set-valued optimization problems w.r.t. variable domination structures. For an overview, we highlight some of the new results in the following:

1. We equipped each set relation $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$ defined in Chapter 2 with a suitable scalarizing functional to characterize these relations, see Sections 5.1 and 5.3. In addition, we characterized minimal elements of a family of sets as well as minimal solutions of problem ($P_{\mathcal{K}}$) by means of different nonlinear scalarizing functionals, see Sections 5.2 and 7.1.

2. We investigated important relationships between solution concepts of set-valued optimization problems w.r.t. variable domination structures based on the vector approach and on the set approach, see Theorems 4.3.2, 4.3.5, 4.3.7 and 4.3.8. Based on these connections, it is possible to apply known results on optimality conditions for solutions based on the vector approach in order to derive corresponding results for solutions based on the set approach.

3. Using the dual approach and the results presented in Section 4.3, we derived necessary optimality conditions for solutions of set-valued optimization problems w.r.t. the set relations $\preceq_l^{\mathcal{K}}$, $\preceq_{pl}^{\mathcal{K}}$ and $\preceq_{cl}^{\mathcal{K}}$, see Theorems 6.2.1, 6.2.3, 6.2.4 and 6.2.5.
Moreover, we proved some results related to the openness of a composition of set-valued functions in which the domination structure $\mathcal{K}(\cdot)$ and objective mapping $F(\cdot)$ are involved. Furthermore, we derived optimality conditions for solutions of set-valued optimization where we used the set relations $\preceq_u^{\mathcal{K}}$ and $\preceq_{cu}^{\mathcal{K}}$, see Theorems 6.2.8 and 6.2.9.

4. We investigated the relationships between the well-posedness property of a set-valued problem and the Tykhonov well-posedness property of the scalarized problem. To do that, we utilized the directional time functional introduced in Section 5.3. Furthermore, we identified two classes of well-posed set optimization problems w.r.t. variable domination structures, see Theorems 7.2.11 and 7.2.14.

5. Section 8.1 constructed an appropriate ordering structure which is suitable to

many goals in radiotherapy treatment for beam intensity problem. We formulated this problem as an approximation problem $(P_{\mathcal{K}}^{imrt})$ w.r.t. the proposed ordering. In addition, we calculated in detail necessary optimality conditions for the goal dose, which is concerned as minimal (nondominated) solutions of problem $(P_{\mathcal{K}}^{imrt})$. Moreover, we investigated applications in dealing with Medical Image Registration problems and Uncertain Optimization, see Sections 8.2 and 8.3.

During this present work, we have discovered many interesting topics for future investigations. Some of these topics are listed as follow:

1. It is of interest to derive existence results for set-valued optimization problems with variable domination structures. This is an interesting problem ensuring the validity of some results related to minimizers of set optimization problems given in this thesis. For this aim, we could follow the same approach given in [57], where the authors investigated a set-valued problem equipped with a general set relation. Another possibility is adapting results given for minimizers defined based on the lower (upper) set less relations of set optimization problems in [76].

2. The descent method we derived in Chapter 5 can be analogously performed for the set relations $\preceq_t^{\mathcal{K}}$, $t \in \{u, pl, pu, cl, cu\}$. Furthermore, this method can be applied to real-world applications, for example taking into account uncertainties in economic, radiotherapy treatment and behavioral sciences.

3. The well-known Jahn-Graef-Younes method, which was introduced in the dissertation by Younes [109] (see also Jahn [54, Section 12.4]), determines minimal elements in the vector-valued case, where $Y = \mathbb{R}^n$. This method is also used by Eichfelder [36] to formulate corresponding algorithms for vector-valued problems with a variable ordering structure. In addition, [68] extends this method to set optimization, where algorithms that deal with minimal solutions of a family of sets $\mathcal{F}(S)$ are proposed. Therefore, this approach can be performed for solutions of set-valued optimization problems w.r.t. variable domination structures.

4. Since the notion of well-posedness is closely related to the stability of an optimization problem, we can use our proposed scalarizing functionals to investigate stability results for vector-valued optimization w.r.t. variable domination structures. One possibility is extending the stability results for the case of vector optimization problems equipped with fixed cones given by Sterna-Karwat [102] and Tammer [103].

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List of Symbols and Abbreviations

$(P_{\mathcal{K}})$	set-valued optimization problem w.r.t. $\mathcal{K} : Y \rightrightarrows Y$	40
$(P_{\mathcal{Q}})$	set-valued optimization problem w.r.t. $\mathcal{Q} : X \rightrightarrows Y$	40
$(P_{\mathcal{K}}^{vec})$	vector optimization problem w.r.t. $\mathcal{K} : Y \rightrightarrows Y$	34
$(\overline{P}_{\mathcal{K}})$	set-valued optimization problem with constraints	96
$(P_{\mathcal{K}}^{imrt})$	mathematical model of beam intensity problem	120
$(P_{\mathcal{K}}^{mir})$	mathematical model of Medical Image Registration	123
$\mathrm{Min}(S, \mathcal{R})$	set of minimal elements of S relative to $\mathcal{R}$	12
$\mathrm{Min}(S, \mathcal{R})$	set of maximal elements of S relative to $\mathcal{R}$	12
$\mathrm{Min}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$	set of all minimal elements of $\mathcal{A}$ w.r.t $\preceq_t^{\mathcal{K}}$	42
$\mathrm{SoMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$	set of all strongly minimal elements of $\mathcal{A}$ w.r.t. $\preceq_t^{\mathcal{K}}$	42
$\mathrm{SiMin}_Y(\mathcal{A}, \preceq_t^{\mathcal{K}})$	set of all strictly minimal elements of $\mathcal{A}$ w.r.t. $\preceq_t^{\mathcal{K}}$	42
$\mathrm{Min}(F(X), \preceq_t^{\mathcal{K}})$	set of all minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$	46
$\mathrm{SoMin}(F(X), \preceq_t^{\mathcal{K}})$	set of all strongly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$	46
$\mathrm{SiMin}(F(X), \preceq_t^{\mathcal{K}})$	set of all strictly minimal solutions of $(P_{\mathcal{K}})$ w.r.t. $\preceq_t^{\mathcal{K}}$	46
$\mathrm{Min}(F(X), \mathcal{Q})$	set of all minimal solutions of $(P_{\mathcal{Q}})$	41
$\mathrm{ND}(F(X), \mathcal{Q})$	set of all nondominated solutions of $(P_{\mathcal{Q}})$	41
l.s.c.	lower semicontinuous	26
w.r.t.	with respect to	1
SNC	sequentially normally compact	30
PSNC	partially sequentially normally compact	30

Selbständigkeitserklärung

Hiermit erkläre ich Le Thanh Tam an Eides statt, dass ich die vorliegende Dissertation selbständig und ohne fremde Hilfe angefertigt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht.

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