# On Generalized-Convex Constrained Multi-Objective Optimization and Application in Location Theory

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To my family

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# Contents

## List of Symbols and Abbreviations (general theory)

| | |
|---|---|
| $\mathbb{N}$ | natural numbers, i.e., $\mathbb{N} := \{1, 2, 3, \ldots\}$, |
| $l, m, n \in \mathbb{N}$ | three specific natural numbers, |
| $\mathbb{R}$ | real numbers, |
| $\mathbb{R}_+$ | nonnegative real numbers, |
| $\mathbb{R}_{++}$ | positive real numbers, |
| $\emptyset$ | empty set, |
| $(\mathbf{E}, \mathcal{T})$ | real topological linear space $\mathbf{E}$ with underlying topology $\mathcal{T}$, |
| $0_{\mathbf{E}}$ | origin in $\mathbf{E}$ (however, in most cases we simply write 0), |
| $\dim \mathbf{E}$ | dimension of the linear space $\mathbf{E}$, |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space (for notational convenience, we use the notation $x = (x_1, \cdots, x_n)$ for a vector $x \in \mathbb{R}^n$), |
| $\Omega^1, \Omega^2$ | nonempty sets in $\mathbf{E}$, |
| $(\Omega^1)^c$ | complement of $\Omega^1$ in $\mathbf{E}$, i.e., $(\Omega^1)^c := \mathbf{E} \setminus \Omega^1$, |
| $\Omega^1 \subsetneq \Omega^2$ | $\Omega^1$ is a proper subset of $\Omega^2$, |
| $\Omega^1 \subseteq \Omega^2$ | $\Omega^1 \subsetneq \Omega^2$ or $\Omega^1 = \Omega^2$, |
| $\Omega^1 \nsubseteq \Omega^2$ | $\Omega^1$ is not a subset of $\Omega^2$, |
| $\Omega^1 \cup \Omega^2$ | unification of the sets $\Omega^1$ and $\Omega^2$, |
| $\Omega^1 \cap \Omega^2$ | intersection of the sets $\Omega^1$ and $\Omega^2$, |
| $\Omega^1 \setminus \Omega^2$ | set of all elements from $\Omega^1$ which do not belong to $\Omega^2$, |
| $\Omega^1 + \Omega^2$ | algebraic sum of two sets $\Omega^1$ and $\Omega^2$, i.e., $\Omega^1 + \Omega^2 := \{x^1 + x^2 \mid x^1 \in \Omega^1,\ x^2 \in \Omega^2\}$, where $\Omega^1 + \emptyset = \emptyset + \Omega^2 = \emptyset$, |
| $\alpha \cdot \Omega^1$ | multiplication of the set $\Omega^1$ with a scalar $\alpha \in \mathbb{R}$, i.e., $\alpha \cdot \Omega^1 := \{\alpha x \mid x^1 \in \Omega^1\}$, where $\alpha \cdot \emptyset = \emptyset$, |
| $\mathbb{R}_+ \cdot \Omega^1$ | $\mathbb{R}_+ \cdot \Omega^1 := \bigcup_{\alpha \in \mathbb{R}_+} \alpha \cdot \Omega^1$, |
| $\Omega^1 - \Omega^2$ | $\Omega^1 - \Omega^2 := \Omega^1 + (-\Omega^2) = \{x^1 - x^2 \mid x^1 \in \Omega^1,\ x^2 \in \Omega^2\}$, |
| $x + \Omega^1$ | $x + \Omega^1 := \{x\} + \Omega^1 = \{x + x^1 \mid x^1 \in \Omega^1\}$, $x \in \mathbf{E}$, |
| $\Omega^1 \times \Omega^2$ | Cartesian product of the sets $\Omega^1$ and $\Omega^2$, |
| $\operatorname{conv} \Omega^1$ | convex hull of the set $\Omega^1$, |
| $\operatorname{aff} \Omega^1$ | affine hull of the set $\Omega^1$, |
| $\operatorname{bd} \Omega^1$ | topological boundary of the set $\Omega^1$, |
| $\operatorname{int} \Omega^1$ | topological interior of the set $\Omega^1$, |
| $\operatorname{rint} \Omega^1$ | relative topological interior of the set $\Omega^1$, |

| | |
|---|---|
| $\operatorname{cl}\Omega^1$ | topological closure of the set $\Omega^1$, cl $\Omega^1 = (\operatorname{int}\Omega^1) \cup (\operatorname{bd}\Omega^1)$, |
| $\operatorname{cor}\Omega^1$ | algebraic interior of the set $\Omega^1$, i.e., $\operatorname{cor}\Omega^1 := \{x \in \Omega^1 \mid \forall\, v \in \mathbf{E}\ \exists\, \delta \in \mathbb{R}_{++} :\ x + [0,\delta] \cdot v \subseteq \Omega^1\}$, |
| $\operatorname{cone}\Omega^1$ | cone generated by the set $\Omega^1$, i.e., $\operatorname{cone}\Omega^1 := \mathbb{R}_+ \cdot \Omega^1$, |
| $\operatorname{card}\Omega^1$ | cardinality of the set $\Omega^1$, |
| $[x, x']$ | closed line segment between the points $x, x' \in \mathbf{E}$, i.e, $[x, x'] := \{\lambda x + (1-\lambda)x' \mid 0 \leq \lambda \leq 1\}$, |
| $]x, x'[$ | open line segment between the points $x, x' \in \mathbf{E}$, i.e., $]x, x'[ := [x, x'] \setminus \{x, x'\}$, |
| $[x, x'[$, $]x, x']$ | half open line segments between the points $x, x' \in \mathbf{E}$, i.e., $[x, x'[ := [x, x'] \setminus \{x'\}$ and $]x, x'] := [x, x'] \setminus \{x\}$, |
| $d$ | metric $d : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$, |
| $\lVert \cdot \rVert$ | norm $\lVert \cdot \rVert : \mathbf{E} \to \mathbb{R}$, |
| $\langle \cdot, \cdot \rangle$ | inner product $\langle \cdot, \cdot \rangle : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$, |
| $\lVert \cdot \rVert_1$ | Manhattan norm $\lVert \cdot \rVert_1 : \mathbb{R}^n \to \mathbb{R}$ (also known as $l_1$ norm or Lebesgue norm), |
| $\lVert \cdot \rVert_2$ | Euclidean norm $\lVert \cdot \rVert_2 : \mathbb{R}^n \to \mathbb{R}$ (also known as $l_2$ norm), |
| $\lVert \cdot \rVert_\infty$ | Maximum norm $\lVert \cdot \rVert_\infty : \mathbb{R}^n \to \mathbb{R}$ (also known as $l_\infty$ norm or Chebyshev norm), |
| $\mu_{\Omega^1}$ | Minkowski gauge associated to the set $\Omega^1$, $\mu_{\Omega^1}(\cdot) = \inf\{\lambda \in \mathbb{R}_+ \mid \cdot \in \lambda \cdot \Omega^1\}$, |
| $d_{\Omega^1}$ | Distance function with respect to the set $\Omega^1$, $d_X(\cdot) = \inf\{\lVert x^1 - \cdot \rVert \mid x^1 \in \Omega^1\}$, |
| $\triangle_{\Omega^1}$ | signed distance function or Hiriart-Urruty function, $\triangle_{\Omega^1} := d_{\Omega^1} - d_{\mathbf{E} \setminus \Omega^1}$, |
| $\varphi_{\Omega^1,k}$ | Tammer-Weidner scalarizing function, $\varphi_{\Omega^1,k}(\cdot) := \inf\{s \in \mathbb{R} \mid \cdot \in sk + \Omega^1\}$, $k \in \mathbf{E}$, |
| $I_{\Omega^1}$ | indicator function with respect to $\Omega^1$, i.e., $I_{\Omega^1}(x^1) := 0$ for $x^1 \in \Omega^1$, otherwise, $I_{\Omega^1}(x^1) := +\infty$ for $x^1 \in \mathbf{E} \setminus \Omega^1$, |
| $\operatorname{Proj}^{\lVert\cdot\rVert}_{\Omega^1}(x)$ | projection of $x \in \mathbf{E}$ onto $\Omega^1$ with respect to $\lVert \cdot \rVert$, i.e., $\operatorname{Proj}^{\lVert\cdot\rVert}_{\Omega^1}(x) := \operatorname{argmin}\{\lVert x^1 - x \rVert \mid x^1 \in \Omega^1\}$, |
| $\operatorname{Proj}^{\lVert\cdot\rVert}_{\Omega^1}(\Omega^2)$ | projection of $\Omega^2$ onto $\Omega^1$ with respect to $\lVert \cdot \rVert$, i.e., $\operatorname{Proj}^{\lVert\cdot\rVert}_{\Omega^1}(\Omega^2) := \bigcup_{x^2 \in \Omega^2} \operatorname{Proj}_{\Omega^1}(x^2)$, |
| **($\mathbf{E}$, $d$)** | metric space ($\mathbf{E}$, $d$), |
| **($\mathbf{E}$, $\lVert \cdot \rVert$)** | normed space ($\mathbf{E}$, $\lVert \cdot \rVert$), |
| $(\mathbf{E}, \langle \cdot, \cdot \rangle)$ | inner product space / pre Hilbert space $(\mathbf{E}, \langle \cdot, \cdot \rangle)$, |
| $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ | Hilbert space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ with $\langle x, x' \rangle := \sum_{i=1}^n x_i x_i'$ for $x := (x_1, \ldots, x_n),\ x' := (x_1', \ldots, x_n') \in \mathbb{R}^n$, |
| $V \subseteq \mathbf{E}$ | neighborhood of $x \in \mathbf{E}$ (relative to the topology $\mathcal{T}$), i.e., $\exists\, O \in \mathcal{T} :\ x \in O \subseteq V$, |
| $\mathcal{V}(x)$ | family of all neighborhoods of $x \in \mathbf{E}$, |
| $B_d(x, \varepsilon)$ | open unit ball in $(\mathbf{E}, d)$ at $x \in \mathbf{E}$ of radius $\varepsilon \in \mathbb{R}_{++}$, i.e., $B_d(x, \varepsilon) := \{x' \in \mathbf{E} \mid d(x, x') < \varepsilon\}$, |

| | |
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| $\overline{B}_d(x, \varepsilon)$ | closed unit ball in $(\mathbf{E}, d)$ at $x \in \mathbf{E}$ of radius $\varepsilon \in \mathbb{R}_{++}$, i.e., $\overline{B}(x, \varepsilon) := \{x' \in \mathbf{E} \mid d(x, x') \leq \varepsilon\}$, |
| $I_m$ | set of indices, $I_m := \{1, 2, \dots, m\}$, |
| $I$ | subset of indices of the set $I_m$, $\emptyset \neq I \subseteq I_m$, |
| $\mathcal{D}$ | nonempty set in $\mathbf{E}$, |
| $\Omega$ | nonempty subset of $\mathcal{D}$, |
| $Y$ | nonempty, convex subset of $\mathcal{D}$, |
| $X$ | nonempty, closed set $X \subsetneq Y$, |
| $D_1, \cdots, D_l$ | $D_i \subsetneq \mathbf{E}, i \in I_l$, are closed, convex sets with nonempty interiors, |
| $h$ | extended real-valued objective function $h : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, |
| $\operatorname{dom} h$ | effective domain of $h$, i.e., $\operatorname{dom} h := \{x \in \mathcal{D} \mid h(x) < +\infty\}$, |
| $\operatorname{epi}(\Omega, h)$ | epigraph of $h$ (with respect to the set $\Omega$), i.e., $\operatorname{epi}(\Omega, h) := \{(x, r) \in \Omega \times \mathbb{R} \mid h(x) \leq r\}$, |
| $L_{\leq}(\Omega, h, s)$ | lower-level set of $h$ to the level $s \in \mathbb{R}$, i.e., $L_{\leq}(\Omega, h, s) := \{x \in \Omega \mid h(x) \leq s)\}$, |
| $L_{<}(\Omega, h, s)$ | strict lower-level set of $h$ to the level $s \in \mathbb{R}$, i.e., $L_{<}(\Omega, h, s) := \{x \in \Omega \mid h(x) < s\}$, |
| $L_{=}(\Omega, h, s)$ | level line of $h$ to the level $s \in \mathbb{R}$, i.e., $L_{=}(\Omega, h, s) := \{x \in \Omega \mid h(x) = s\}$, |
| $L_{\geq}(\Omega, h, s)$ | upper-level set of $h$ to the level $s \in \mathbb{R}$, i.e., $L_{\geq}(\Omega, h, s) := \{x \in \Omega \mid h(x) \geq s\}$, |
| $L_{>}(\Omega, h, s)$ | strict upper-level set of $h$ to the level $s \in \mathbb{R}$, i.e., $L_{>}(\Omega, h, s) := \{x \in \Omega \mid h(x) > s\}$, |
| $\mathrm{Sol}(\Omega \mid \mathrm{h})$ | set of all minimal solutions of the problem $h(x) \to \min_{x \in \Omega}$, i.e., $\mathrm{Sol}(\Omega \mid \mathrm{h}) := \operatorname{argmin}_{x \in \Omega} h(x)$, |
| $\mathrm{Sol_u}(\Omega \mid \mathrm{h})$ | if $\mathrm{card}(\mathrm{Sol}(\Omega \mid \mathrm{h})) = 1$, then $\mathrm{Sol_u}(\Omega \mid \mathrm{h}) := \mathrm{Sol}(\Omega \mid \mathrm{h})$, otherwise $\mathrm{Sol_u}(\Omega \mid \mathrm{h}) := \emptyset$, |
| $f$ | vector-valued objective function $f = (f_1, \dots, f_m) : \mathcal{D} \to \mathbb{R}^m$, |
| $h \circ f$ | composition of the functions $f = (f_1, \dots, f_m) : \mathcal{D} \to \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathbb{R}$, |
| $f[\Omega]$ | image of $f$ over $\Omega$, i.e., $f[\Omega] := \{f(x) \in \mathbb{R}^m \mid x \in \Omega\}$, |
| $f_i$ | real-valued component function of $f$, $i \in I_m$, |
| $f_I$ | vector-valued objective function $f_I = (f_{i_1}, \dots, f_{i_k}) : \mathcal{D} \to \mathbb{R}^k$, where $I = \{i_1, \dots, i_k\} \subseteq I_m$ with $i_1 < \ldots < i_k$ and $k := \lvert I \rvert$, |
| $\phi$ | penalization function $\phi : \mathcal{D} \to \mathbb{R}$, |
| $f^{\oplus}$ | penalized vector-valued objective function $f^{\oplus} = (f, \phi) : \mathcal{D} \to \mathbb{R}^{m+1}$, |
| $f_I^{\oplus}$ | penalized vector-valued objective function $f_I^{\oplus} = (f_I, \phi) : \mathcal{D} \to \mathbb{R}^{k+1}$, |
| $S_{<}(\Omega, f, x)$ | intersection of strict lower-level sets of the component functions of $f$ at $x \in \Omega$, i.e., $S_{<}(\Omega, f, x) := \bigcap\limits_{i \in I_m} L_{<}(\Omega, f_i, f_i(x))$, |
| $S_{=}(\Omega, f, x)$ | intersection of level lines of the component functions of $f$ at $x \in \Omega$, i.e., $S_{=}(\Omega, f, x) := \bigcap\limits_{i \in I_m} L_{=}(\Omega, f_i, f_i(x))$, |
| $S_{\leq}(\Omega, f, x)$ | intersection of lower-level sets of the component functions of $f$ at $x \in \Omega$, i.e., $S_{\leq}(\Omega, f, x) := \bigcap\limits_{i \in I_m} L_{\leq}(\Omega, f_i, f_i(x))$, |

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| $(\mathcal{P}_\Omega)$ | constrained multi-objective optimization problem $f(x) \to \min_{x\in\Omega}$, |
| $(s_\lambda\mathcal{P}_\Omega)$ | scalar problem obtained by applying the Weighted Sum Scalarization Method to the multi-objective optimization problem $(\mathcal{P}_\Omega)$, $\lambda \in \mathbb{R}^m_+ \setminus \{0\}$, |
| $(\mathcal{P}_X)$ | constrained multi-objective optimization problem $f(x) \to \min_{x\in X}$ with not necessarily convex feasible set $X$, |
| $(\mathcal{P}_Y)$ | multi-objective optimization problem $f(x) \to \min_{x\in Y}$ with convex feasible set $Y$, |
| $(\mathcal{P}_Y^\oplus)$ | penalized multi-objective optimization problem $f^\oplus(x) \to \min_{x\in Y}$ with convex feasible set $Y$, |
| $F$ | nonempty subset of $\mathbb{R}^m$, |
| $K$ | pointed, convex cone in $\mathbb{R}^m$ (pointedness, i.e., $K \cap (-K) = \{0\}$; convexity, i.e., $K + K = K$; cone, i.e., $\mathbb{R}_+ \cdot K = K$), |
| $\preceq$ | partial ordering induced by the cone $K$, i.e., $\preceq := \{(x, x') \in \mathbb{R}^m \times \mathbb{R}^m \mid x' \in x + K\}$, |
| $\mathbb{R}^m_+$ | natural ordering cone (nonnegative orthant) in $\mathbb{R}^m$, i.e., $\mathbb{R}^m_+ := \{x \in \mathbb{R}^m \mid \forall i \in I_m :\ x_i \geq 0\}$, |
| $(\mathbb{R}^m, \preceq)$ | Euclidean space $\mathbb{R}^m$ endowed with the partial ordering $\preceq$, |
| $\mathrm{MIN}(F, K)$ | set of minimal elements of $F$ with respect to the cone $K$, i.e., $\mathrm{MIN}(F, K) := \{y \in F \mid (y - K) \cap F = \{y\}\}$, |
| $\mathrm{WMIN}(F, K)$ | set of weakly minimal elements of $F$ with respect to the cone $K$ (assume that $K$ is solid, i.e., $\mathrm{int}\, K \neq \emptyset$), i.e., $\mathrm{WMIN}(F, K) := \{y \in F \mid (y - \mathrm{int}\, K) \cap F = \emptyset\}$, |
| $\mathrm{Eff}\,(\Omega \mid f)$ | set of Pareto efficient solutions of $(\mathcal{P}_\Omega)$, i.e., $\mathrm{Eff}(\Omega \mid f) := \{x \in \Omega \mid f[\Omega] \cap (f(x) - (\mathbb{R}^m_+ \setminus \{0\})) = \emptyset\}$, |
| $\mathrm{WEff}\,(\Omega \mid f)$ | set of weakly Pareto efficient solutions of $(\mathcal{P}_\Omega)$, i.e., $\mathrm{WEff}(\Omega \mid f) := \{x \in \Omega \mid f[\Omega] \cap (f(x) - \mathrm{int}\, \mathbb{R}^m_+) = \emptyset\}$, |
| $\mathrm{SEff}\,(\Omega \mid f)$ | set of strictly Pareto efficient solutions of $(\mathcal{P}_\Omega)$, i.e., $\mathrm{SEff}(\Omega \mid f) := \{x \in \mathrm{Eff}(\Omega \mid f) \mid \mathrm{card}(\{x' \in \Omega \mid f(x') = f(x)\}) = 1\}$, |
| $\mathrm{PEff}\,(\Omega \mid f)$ | set of properly Pareto efficient solutions of $(\mathcal{P}_\Omega)$ (in the sense of Geoffrion). |

# List of Symbols and Abbreviations (location theory)

| | |
|---|---|
| $k, l, m, p \in \mathbb{N}$ | four specific natural numbers, |
| $a^1, \dots, a^m \in \mathbf{E}$ | existing (attraction) facilities, |
| $x \in \mathbf{E}$ | new facility, |
| $\mathcal{A}$ | set of all existing facilities, $\mathcal{A} := \{a^1, \dots, a^m\} \subseteq \mathbf{E}$, |
| $\mathcal{N}(\mathcal{A})$ | rectangular hull of the set $\mathcal{A} \subseteq \mathbf{E} = \mathbb{R}^2$ w.r.t. Maximum norm, |
| $B_1, \cdots B_m$ | $B_i \subsetneq \mathbf{E}$, $i \in I_m$, are closed, convex sets with $0 \in \operatorname{core} B_i$, |
| $\eta_1, \cdots, \eta_m$ | Minkowski gauges $\eta_i(\cdot) := \mu_{B_i}(\cdot) = \inf\{\lambda \in \mathbb{R}_+ \mid \cdot \in \lambda \cdot B_i\}$, $i \in I_m$, |
| $\eta_{\mathcal{A}}$ | $\eta_{\mathcal{A}}(\cdot) := (\eta_1(\cdot - a^1), \cdots, \eta_m(\cdot - a^m))$, |
| $h_1, \cdots, h_p$ | scalar (disutility) functions of the decision maker $h_1, \cdots, h_p : \mathbb{R}^m_+ \to \mathbb{R}$, |
| $g_1, \cdots, g_p$ | $g_i := h_i \circ \eta_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}$, $i \in I_p$, |
| $g_{\mathcal{A}}$ | vector-valued objective function $g_{\mathcal{A}} = (g_1, \cdots, g_p) : \mathbf{E} \to \mathbb{R}^p$, |
| $g^{\oplus}_{\mathcal{A}}$ | penalized vector-valued objective function $g^{\oplus}_{\mathcal{A}} = (g_1, \cdots, g_p, \phi) : \mathbf{E} \to \mathbb{R}^{p+1}$, |
| $(\mathrm{LP}_X(\mathcal{A}))$ | constrained multi-objective composite location problem $g_{\mathcal{A}}(x) \to \min_{x \in X}$, |
| $(\mathrm{LP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ | penalized unconstrained multi-objective composite location problem $g^{\oplus}_{\mathcal{A}}(x) \to \min_{x \in \mathbf{E}}$, |
| $(\mathrm{POLP}_X(\mathcal{A}))$ | constrained point-objective location problem, |
| $(\mathrm{s}_\lambda\mathrm{POLP}_X(\mathcal{A}))$ | generalized Fermat-Weber problem, |
| $(\mathrm{POLP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ | penalized unconstrained point-objective location problem, |
| $(\mathrm{POLP}^1_{\mathbf{E}}(\mathcal{A}))$ | unconstrained planar point-objective location problem involving the Manhattan norm, |
| $(\mathrm{s}_\lambda\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ | unconstrained generalized Fermat-Weber problem involving the Manhattan norm, |
| $(\mathrm{POLP}^1_X(\mathcal{A}))$ | constrained planar point-objective location problem involving the Manhattan norm, |
| $(\mathrm{POLP}^1_{\mathbf{E}}(\mathcal{A})^{\oplus})$ | penalized unconstrained planar point-objective location problem involving the Manhattan norm, |
| $(\mathrm{POLP}^2_X(\mathcal{A}))$ | constrained point-objective location problem involving a norm induced by a scalar product in a finite-dimensional Hilbert space, |
| $(\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A})^{\oplus_i}), i \in I_l$ | penalized unconstrained point-objective location problems involving a norm induced by a scalar product in a finite-dimensional Hilbert space, |
| $(\mathrm{MOMSLP}_X(\mathcal{A}))$ | constrained multi-objective min-sum location problem, |

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|---|---|
| $(\mathrm{MOMSLP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ | penalized unconstrained multi-objective min-sum location problem, |
| $(\mathrm{MOMMLP}_{X}(\mathcal{A}))$ | constrained multi-objective min-max location problem, |
| $(\mathrm{MOMMLP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ | penalized unconstrained multi-objective min-max location problem. |
| $(\mathrm{MOOMLP}_{X}(\mathcal{A}))$ | constrained multi-objective ordered median location problem, |
| $(\mathrm{MOOMLP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ | penalized unconstrained multi-objective ordered median location problem, |
| $b^1, \dots, b^k \in \mathbf{E}$ | existing repulsion facilities, |
| $\mathcal{B}$ | set of all existing repulsion facilities, $\mathcal{B} := \{b^1, \dots, b^k\} \subseteq \mathbf{E}$, |
| $(\mathrm{POLP}^2_X(\mathcal{A}, \mathcal{B}))$ | constrained point-objective location problem with attraction and repulsion involving a norm induced by a scalar product in a finite-dimensional Hilbert space, |
| $(\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A}, \mathcal{B})^{\oplus_i}), i \in I_l$ | penalized unconstrained point-objective location problems with attraction and repulsion involving a norm induced by a scalar product in a finite-dimensional Hilbert space. |

# Introduction

In *multi-objective optimization*, one considers an optimization problem that consists of minimizing a vector-valued objective function

$$f = (f_1, \cdots, f_m) : \mathbf{E} \to \mathbb{R}^m$$

over a nonempty feasible set $X \subseteq \mathbf{E}$, where $\mathbf{E}$ is a real topological linear space and $f_1, \cdots, f_m : \mathbf{E} \to \mathbb{R}$, $m \geq 2$, are the component functions of $f$. Usually one looks for so-called *Pareto efficient solutions.* A feasible point $x \in X$ of our initial multi-objective optimization problem

$$\begin{cases} f(x) = (f_1(x), \cdots, f_m(x)) \to \min \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in X \end{cases} \tag{$\mathcal{P}_X$}$$

is said to be a *Pareto efficient solution* in $X$ if

$$\nexists\, x' \in X \text{ subject to } \begin{cases} \forall\, i \in I_m : \; f_i(x') \leq f_i(x), \\ \exists\, j \in I_m : \; f_j(x') < f_j(x), \end{cases}$$

where $I_m = \{1, 2, \ldots, m\}$ consists of all indices of the component functions of $f$. The set of Pareto efficient solutions of the problem $(\mathcal{P}_X)$ is denoted by $\mathrm{Eff}(X \mid f)$. The solution concept of *Pareto efficiency* for multi-objective optimization problems is well-studied in the literature (see, e.g., Ehrgott [29], Eichfelder and Jahn [33], Göpfert et al. [50], and Jahn [64]) and dates back to the fundamental works by Edgeworth [28] (1881) and Pareto [96] (1896).

In this thesis, we are interested in computing the whole set of Pareto efficient solutions of our initial constrained multi-objective optimization problem. Clearly, this is a difficult task in general. In order to develop effective algorithms it is very important to use structural properties of the given problems. For that reason, we mainly focus on problems where each of the objective functions $f_1, \cdots, f_m : \mathbf{E} \to \mathbb{R}$ is generalized-convex and the feasible set $X$ is closed but not necessarily convex.

### Generalized-convexity in multi-objective optimization

*Convexity* plays a crucial role in optimization theory (see, e.g., the books of convex analysis by Hiriart-Urruty and Lemaréchal [63], Rockafellar [111] and Zălinescu [131]). In the last decades, several new classes of functions are obtained by preserving several fundamental properties of convex functions. The first generalization is probably due to De Finetti [23] (1949) who introduced the notion of *quasi-convexity.* Further generalizations of convexity are for instance due to Arrow and Enthoven [6] (1961), Avriel et al. [7] (1988), Fenchel [39] (1953), Hanson [61] (1964), Karamardian [70] (1967), Mangasarian [86] (1965) and Ponstein [101] (1967). An overview on generalized-convexity and optimization can be found in Cambini and Martein [17] and Giorgi *et al.* [49].

In the present thesis, the following classes of *generalized-convex functions* $f_i : \mathbf{E} \to \mathbb{R}$ will be of special interest:

- *Quasi-convex functions*: The level sets of $f_i$ are convex for each level (hence the set of minimal solutions of $f_i$ on $\mathbf{E}$ is a convex set in $\mathbf{E}$).
- *Semi-strictly quasi-convex functions*: Each local minimum point of $f_i$ on $\mathbf{E}$ is also a global minimum point.

- *Explicitly quasi-convex functions*: Each local maximum point of $f_i$ on $\mathbf{E}$ is actually a global minimum point (see Bagdasar and Popovici [8]).

Of course, as generalization of convexity, every convex function is quasi-convex, semi-strictly quasi-convex as well as explicitly quasi-convex. Generalized convexity assumptions appear in several branches of applications, e.g., *production theory*, *utility theory* or *location theory*. Cambini and Martein [17] pointed out important applications of generalized-convexity. For instance, there are certain relationships between the field of *generalized-convexity* and *fractional programming* (see [17, Th. 2.3.8, Ch. 6, Ch. 7]). Moreover, in [17, Sec. 2.4], examples of quasi-concave classes of homogeneous functions that appear frequently in *economics* (e.g., in *utility theory* and *production theory*) are provided. Since maximizing a generalized-concave function is equivalent to minimizing the negative of this function (a generalized-convex function), such functions from Economics (e.g., the well-known *Cobb-Douglas function*) are important examples for our work.

The area of multi-objective optimization has gained more and more interest, some authors studied the role of generalized-convexity in the framework of multi-objective optimization / vector optimization (see, e.g., Bagdasar and Popovici [9, 10], Flores-Bazán [41], Jahn and Sachs [65], Luc [81], Mäkelä, Eronen and Karmitsa [83, 84], Malivert and Boissard [85], Popovici [102, 103, 105], and Puerto and Rodríguez-Chía [110]). For certain classes of multi-objective optimization problems it is known how to compute the whole set of Pareto efficient solutions. In most cases, one considers a problem where the goal is to minimize a vector-valued componentwise convex function $f$ over a nonempty, closed, convex feasible set $X$. In particular, the case when the feasible set is given by the $n$-dimensional Euclidean space (i.e., $X = \mathbf{E} = \mathbb{R}^n$) is often considered in the literature since unconstrained problems can more easily be handled in comparison to constrained ones. However, depending on the application in practice, optimization problems often involve certain constraints.

### A new penalization approach in constrained multi-objective optimization

In the literature, there exist techniques for solving different classes of constrained multi-objective optimization problems by using corresponding unconstrained problems with an objective function that involve certain *penalization terms* in the component functions (see, e.g., Apetrii, Durea and Strugariu [4], and Ye [129]), and, respectively, additional *penalization functions* (see, e.g., Durea, Strugariu and Tammer [25], and Klamroth and Tind [72]).

In this thesis, we derive a new *penalization approach* for (generalized-convex) multi-objective optimization problems involving not necessarily convex constraints where the vector-valued objective function is acting between a real topological linear pre-image space and a finite-dimensional image space. Given a certain scalar-valued *penalization function* $\phi : \mathbf{E} \to \mathbb{R}$ (a penalty term concerning the set $X$), our aim is to study the relationships between the initial multi-objective optimization problem $(\mathcal{P}_X)$ with not necessarily convex feasible set $X$ and a penalized multi-objective optimization problem

$$\begin{cases} f^{\oplus}(x) = (f_1(x), \cdots, f_m(x), \phi(x)) \to \min \ \text{ w.r.t. } \mathbb{R}^{m+1}_+ \\ x \in Y \end{cases} \tag{$\mathcal{P}^{\oplus}_Y$}$$

with a new feasible set $Y \subseteq \mathbf{E}$ that is a convex upper set of the original feasible set $X$. We show that the set of Pareto efficient solutions of the multi-objective optimization problem $(\mathcal{P}_X)$ involving a nonempty, closed (not necessarily convex) feasible set $X$, can be computed completely by using at most two corresponding multi-objective optimization problems (namely problem $(\mathcal{P}_X)$ with $Y$ in the role of $X$ as well as problem $(\mathcal{P}^{\oplus}_Y)$) with a new convex feasible set $Y$ that fulfils $X \subseteq Y$. Our approach relies on the fact that the original feasible set $X$ can be described by using level sets of the penalization function $\phi$.

We characterize the set of Pareto efficient solutions of generalized-convex multi-objective optimization problems involving certain types of nonconvex constraints. In particular, we will consider a feasible set that is given by the whole pre-image space $\mathbf{E}$ excepting some forbidden regions that are given by convex sets (i.e., the feasible set is an intersection of so-called *reverse convex* sets). Such a feasible set is of nonconvex type and occurs often in (single-objective) optimization, for instance, in the applied field of *location theory* (see, e.g., Hamacher and Nickel [58] and Nickel and Puerto [90]).

### Multi-objective location theory

*Multi-objective location problems*, as well as their scalarizations, may be found in the literature in many variants. Indeed, the objective functions and the constraints depend on the specific practical applications, as for instance *urban development planning*, *engineering*, *logistics or economics* (see, e.g., Hamacher [57], Nickel and Puerto [90], Nickel, Puerto and Rodríguez-Chía [92, 93], Klamroth [71], and Schöbel [113]). Many authors considered unconstrained multi-objective problems (see, e.g., Wendell et al. [128], Chalmet, Francis and Kolen [21], Gerth and Pöhler [47], Pelegrín and Fernández [97], Durier and Michelot [27], Puerto and Rodríguez-Chía [109], Nickel et al. [90], and Alzorba, Günther and Popovici [2]). It is known that considering problems without any constraints is a rather inaccurate approximation in many real world location problems (see, e.g., Carrizosa et al. 1995). Constrained multi-objective location problems are considered for instance in the papers Carrizosa et al. [18], Carrizosa and Plastria [20] and Ndiaye and Michelot [88] for special types of convex objective functions and convex constraints. Jourani, Michelot and Ndiaye [67] considered a multi-objective location problem with a nonconvex objective function and a convex feasible set. Planar multi-objective location problems with nonconvex constraints are considered in the work by Carrizosa et al. [19]. However, Puerto and Rodríguez-Chía [110] noted that there is a lack of a common geometrical description of the solution sets for constrained versions of multi-objective location problems. Since in practical location problems, there often exist regions where it is forbidden to locate a new facility, it is interesting to study problems involving forbidden regions.

So, we emphasize the importance of our theoretical results derived in this thesis by applying it to special multi-objective location problems. In particular, we are interested in the well-known class of *point-objective location problems*.

Consider $m$ a priori given facilities located at the points $a^1, \cdots, a^m \in \mathbf{E}$. Our aim is to find a point $x \in X$ for a new facility such that the distances (induced by the norm $||\cdot|| : \mathbf{E} \to \mathbb{R}$) between $x$ and the given points $a^1, \cdots, a^m$ are to be simultaneously minimized. More precisely, we consider the multi-objective location problem

$$\begin{cases} g_{\mathcal{A}}(x) = \big(||x - a^1||, \cdots, ||x - a^m||\big) \to \min \ \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in X. \end{cases} \tag{POLP$_X(\mathcal{A})$}$$

Two particular cases of the problem (POLP$_X(\mathcal{A})$) will be of special interest:

1°: $X = \mathbf{E} = \mathbb{R}^2$ and $||\cdot|| : \mathbb{R}^2 \to \mathbb{R}$ represents the *Manhattan norm*;

2°: $\mathbf{E}$ is a finite-dimensional Hilbert space, $||\cdot|| : \mathbf{E} \to \mathbb{R}$ is the norm induced by the scalar product, $X$ is the whole space $\mathbf{E}$ excepting some forbidden regions that are given by open balls (defined with respect to $||\cdot||$).

Under the setting given in Case 1°, the problem (POLP$_X(\mathcal{A})$) is convex. In this thesis, we characterize the nonessential objectives and, by eliminating them, we develop an effective algorithm (the *Rectangular Decomposition Algorithm*) for generating the whole set of Pareto efficient solutions as the union of a special family of rectangles and line segments.

Assuming that the setting given in Case 2° holds, for the nonconvex problem (POLP$_X(\mathcal{A})$), under the assumption that the forbidden regions are pairwise disjoint, we completely characterize the set of Pareto efficient solutions by using our penalization approach as well as results derived by Jourani, Michelot and Ndiaye [67].

Figure I.1 (see the next page) shows an example problem (POLP$_X(\mathcal{A})$) where $||\cdot||$ is given by the *Euclidean norm* defined on $\mathbf{E} = \mathbb{R}^2$. One aim of this thesis is to construct the whole set of Pareto efficient solutions for the nonconvex location problem illustrated in the right part of Figure I.1. The construction will be given within Chapter 6.

### Outline of the thesis

The thesis is structured as follows. In **Chapter 1**, we present preliminary facts about linear topological spaces, semi-continuous and generalized-convex functions, and the class of Minkowski gauge functions. Moreover, we recall solution concepts for the vector-valued minimization in our initial constrained multi-objective optimization problem ($\mathcal{P}_X$).

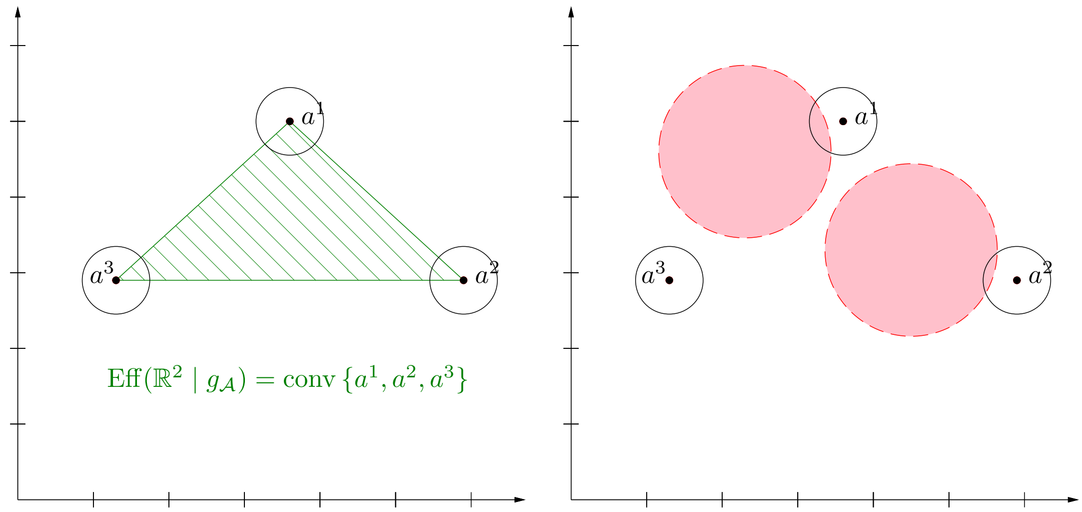


Figure I.1: The figure shows an example problem $(\mathrm{POLP}_X(\mathcal{A}))$ for the case $\mathbf{E} = \mathbb{R}^2$, $m = 3$, and $||\cdot||$ is given by the Euclidean norm. In the left part, one can see that in the case $X = \mathbf{E} = \mathbb{R}^2$ the set of Pareto efficient solutions is given by the convex hull of the points $a^1, a^2$ and $a^3$ (according to Durier and Michelot [27, Prop. 1.3]). In the right part, we add two forbidden regions (illustrated by two Euclidean balls that are red colored) such that the feasible set $X \subsetneq \mathbb{R}^2$ of the location problem becomes a nonconvex one. So, the question arises: *How to compute* $\mathrm{Eff}(X \mid g_{\mathcal{A}})$ *for this more complicated problem?*

In the first part of the thesis (**Chapters 2 and 3**), we derive a new penalization approach for constrained multi-objective optimization problems. In **Chapter 2**, we show relationships between the initial multi-objective optimization problem with generalized-convex objective functions involving a not necessarily convex feasible set, and two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set. In addition, we point out some useful relationships between single-objective and bi-objective optimization.

As a consequence of our penalization approach, in **Chapters 2 and 3**, we derive characterizations for the set of Pareto efficient solutions of special types of multi-objective optimization problems with nonconvex feasible set in terms of the sets of Pareto efficient solutions of some corresponding problems with convex feasible set. In particular, we analyze problem $(\mathcal{P}_X)$ for the cases that the objective funtions $f_1, \cdots, f_m$ are generalized-convex and

1°: $X$ is given by a system of inequalities with a finite number of constraint functions;

2°: $X$ is given by a finite union of closed, convex sets;

3°: $X$ is the whole space $\mathbf{E}$ excepting a finite number of forbidden regions that are given by convex sets.

In the second part of the thesis (**Chapters 4, 5 and 6**), we emphasize the importance of our results. In **Chapter 4**, we consider the general class of *multi-objective composite location problems* which includes several well-known classes of multi-objective location problems, e.g., *point-objective location problems*, *multi-objective min-sum location problems*, *multi-objective min-max location problems*, and *multi-objective ordered median location problems*. We give an overview on existing literature for each of the considered classes. In addition, we point out how the results derived in this thesis contribute to the development of algorithms for multi-objective location problems involving some constraints. **Chapter 5** is devoted to the study of a special planar point-objective location problem where the distances are defined by means of the Manhattan norm. In **Chapter 6**, we consider a nonconvex point-objective location problem where the distances are measured by a norm and the feasible set is given by the whole pre-image space (a finite-dimensional Hilbert space) excepting some forbidden regions that are given by open balls (defined with respect to the underlying norm).

We end the thesis with some conclusions and a summary of contributions.

# Chapter 1

# Preliminaries

In this chapter, we introduce some preliminary notions that will be used throughout the thesis. After giving a short introduction of generalized-convexity and semi-continuity properties, we recall solution concepts for the vector-valued minimization in our initial multi-objective optimization problem, and further, we present some facts about Minkowski gauge functions.

## 1.1 Preliminaries in real topological linear spaces

We are going to recall definitions and important facts from the field of convex analysis. Our main references in this section are the books by Barbu and Precupanu [11], Göpfert *et al.* [50], Jahn [64], and Zălinescu [131].

Throughout this thesis, let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ stand for the sets of positive integers, real numbers, non-negative and positive real numbers, respectively. The $m$-dimensional Euclidean space is denoted by $\mathbb{R}^m$.

First, we recall the definition of a *topological space*.

**Definition 1.1** ([64, Def. 1.29]) Let $\mathbf{E}$ be a nonempty set. A *topology* $\mathcal{T}$ on $\mathbf{E}$ is defined to be a set of subsets of $\mathbf{E}$ which satisfy the following axioms:

(i) every union of sets of $\mathcal{T}$ belongs to $\mathcal{T}$;

(ii) every finite intersection of sets of $\mathcal{T}$ belongs to $\mathcal{T}$;

(iii) $\emptyset \in \mathcal{T}$ and $\mathbf{E} \in \mathcal{T}$.

The pair $(\mathbf{E}, \mathcal{T})$ is called a *topological space* and the elements of $\mathcal{T}$ are called *open sets*.

A subset $\Omega$ of $\mathbf{E}$ is a *closed* set if and only if its *complement* $\Omega^c := \mathbf{E} \setminus \Omega$ is open.

An important class of topological spaces are so-called *metric spaces*, as given in the next definition.

**Definition 1.2** ([64, Def. 1.30]) Let $\mathbf{E}$ be a nonempty set. A function $d : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ is called *metric* on $\mathbf{E}$ if $d$ fulfills the following assertions for all $x, x', x'' \in \mathbf{E}$:

(i) $\quad d(x, x') = 0 \iff x = x' \quad$ (*definiteness*),

(ii) $\quad d(x, x') = d(x', x) \quad$ (*symmetry*),

(iii) $\quad d(x, x'') \leq d(x, x') + d(x', x'') \quad$ (*triangle inequality*).

The pair $(\mathbf{E}, d)$ is called *metric space*.

Next, we define a special class of topological spaces, namely real topological linear spaces.

**Definition 1.3** ([64], Def. 1.31) Let $\mathbf{E}$ be a real linear space and let $\mathcal{T}$ be a topology on $\mathbf{E}$. The pair $(\mathbf{E}, \mathcal{T})$ is called a *real topological linear space* if addition and multiplication with reals are continuous, i.e,

$$(x, x') \mapsto x + x' \quad \text{ with } x, x' \in \mathbf{E},$$
$$(\alpha, x) \mapsto \alpha x \quad \text{ with } \alpha \in \mathbb{R} \text{ and } x \in \mathbf{E}$$

are continuous on $\mathbf{E} \times \mathbf{E}$ and $\mathbb{R} \times \mathbf{E}$, respectively.

For notational convenience, we use $\mathbf{E}$ instead of $(\mathbf{E}, \mathcal{T})$ for a real topological linear space.

**Remark 1.4** Let $\mathbf{E}$ be a real linear space. Then, $\mathbf{E}$ can be seen as real topological linear space by using the *trivial topology* $\mathcal{T} := \{\emptyset, \mathbf{E}\}$. Moreover, using the *core convex topology* (generated by the *family of all semi-norms defined on* $\mathbf{E}$) the real linear space will be a *locally convex space* (see Kahn, Tammer and Zălinescu [68, Prop 6.3.1]), which is in fact a real topological linear space.

Throughout this thesis, we assume that

$$\mathbf{E} \text{ is a real topological linear space.}$$

A very important class of metric spaces as well as of topological linear spaces is given by the class of *normed spaces.*

**Definition 1.5** ([64, Def. 1.35]) A function $|| \cdot || : \mathbf{E} \to \mathbb{R}$ is called *norm* on a linear space $\mathbf{E}$ if $|| \cdot ||$ fulfils the following assertions for all $x, x' \in \mathbf{E}$ and for all $\alpha \in \mathbb{R}$:

(i) $||x|| = 0 \iff x = 0_{\mathbf{E}}$ (*definiteness*),

(ii) $||\alpha x|| = |\alpha| \, ||x||$ (*positive homogeneity*),

(iii) $||x + x'|| \leq ||x|| + ||x'||$ (*triangle inequality*),

where $0_{\mathbf{E}}$ denotes the *origin* in the linear space $\mathbf{E}$. The pair $(\mathbf{E}, || \cdot ||)$ is called *normed space.*

Assuming that $(\mathbf{E}, ||\cdot||)$ is *complete* (i.e., every *Cauchy sequence* in $\mathbf{E}$ converges to a well defined limit point that belongs to $\mathbf{E}$), then the space is called *Banach space.* One prominent example of a Banach space is given by the Euclidean space $(\mathbb{R}^m, || \cdot ||)$ with respect to a norm $|| \cdot || : \mathbb{R}^m \to \mathbb{R}$. Notice, in the case that $\mathbf{E}$ is a normed space, we assume that the topology $\mathcal{T}$ of $\mathbf{E}$ is generated by the metric induced by the norm $|| \cdot ||$.

We call a norm $||\cdot|| : \mathbf{E} \to \mathbb{R}$ *strictly convex* if, for any $x', x'' \in \Omega$, $x' \neq x''$, with $||x'|| = ||x''|| = 1$, it follows

$$]x', x''[ \subseteq \{x \in \mathbf{E} \mid ||x|| < 1\}.$$

A normed space $(\mathbf{E}, || \cdot ||)$ with underlying strictly convex norm $|| \cdot || : \mathbf{E} \to \mathbb{R}$ is called *strictly convex.* In addition, the normed space $(\mathbf{E}, || \cdot ||)$ is called *reflexive* if the *canonical embedding* of $\mathbf{E}$ into its *bidual space* $(\mathbf{E}^*)^*$ (where $\mathbf{E}^*$ is the *dual space* of $\mathbf{E}$), namely $J : \mathbf{E} \to (\mathbf{E}^*)^*$, defined, for any $x \in \mathbf{E}$, by

$$J(x)(x^*) = x^*(x), \; x^* \in \mathbf{E}^*,$$

is surjective. Every reflexive normed space is a Banach space, while each finite-dimensional Banach space is reflexive.

A significant class of strictly convex normed spaces are *inner product spaces* (in particular *Hilbert spaces*).

**Definition 1.6** ([64, Def. 1.37]) A function $\langle \cdot, \cdot \rangle : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ is called *inner product* on $\mathbf{E}$ if $\langle \cdot, \cdot \rangle$ fulfils the following assertions for all $x, x', x'' \in \mathbf{E}$ and for all $\alpha \in \mathbb{R}$:

(i) $\langle x, x \rangle > 0$ for $x \neq 0_{\mathbf{E}}$ (*positivity*),

(ii) $\langle x, x' \rangle = \langle x', x \rangle$ (*symmetry*),

(iii) $\langle \alpha x, x' \rangle = \alpha \langle x, x' \rangle$ (*positive homogeneity*),

(iv) $\langle x+x', x''\rangle = \langle x, x''\rangle + \langle x', x''\rangle$ (*additivity*).

The pair $(\mathbf{E}, \langle\cdot,\cdot\rangle)$ is called *inner product space.* Assuming that $(\mathbf{E}, \langle\cdot,\cdot\rangle)$ is complete, the space is called *Hilbert space.*

Notice that each inner product space is a normed space with underlying norm

$$||\cdot|| := \sqrt{\langle\cdot,\cdot\rangle}.$$

Moreover, each Hilbert space is an inner product space as well as a reflexive normed space. In contrast, an inner product space (a normed space) must not be a Hilbert space (an inner product space) in general. However, in the finite-dimensional case, each inner product space is a Hilbert space. Hence, the space $(\mathbb{R}^m, \langle\cdot,\cdot\rangle)$ with respect to an inner product defined by

$$\langle x, x'\rangle := \sum_{i=1}^{m} x_i x_i'$$

for all $x = (x_1, \cdots, x_m), x' = (x_1', \cdots, x_m') \in \mathbb{R}^m$, is a Hilbert space.

Considering a metric $d : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$, we define the *open ball* around $x \in \mathbf{E}$ of radius $\varepsilon \in \mathbb{R}_{++}$ by

$$B_d(x, \varepsilon) := \{x' \in \mathbf{E} \mid d(x, x') < \varepsilon\} = x + \varepsilon \cdot B_d(0_{\mathbf{E}}, 1),$$

while the *closed ball* around $x \in \mathbf{E}$ of radius $\varepsilon \in \mathbb{R}_{++}$ is given by

$$\overline{B}_d(x, \varepsilon) := \{x' \in \mathbf{E} \mid d(x, x') \leq \varepsilon\} = x + \varepsilon \cdot \overline{B}_d(0_{\mathbf{E}}, 1).$$

In the case that $d$ is induced by a norm $||\cdot||$ (i.e., $d(x, x') = ||x - x'||$ for any $x, x' \in \mathbf{E}$), we simply write $B_{||\cdot||}(x, \varepsilon)$ and $\overline{B}_{||\cdot||}(x, \varepsilon)$.

**Example 1.7** Now, let us recall some well-known norms for the special case $\mathbf{E} = \mathbb{R}^m$:

$$\begin{aligned}
x \mapsto ||x||_1 &:= \sum_{i=1}^{m} |x_i| && \textit{(Manhattan norm)},\\
x \mapsto ||x||_2 &:= \left(\sum_{i=1}^{m} |x_i|^2\right)^{\frac{1}{2}} = \sqrt{\langle x, x\rangle} && \textit{(Euclidean norm)},\\
x \mapsto ||x||_\infty &:= \max\{|x_i| \mid i = 1, \cdots, m\} && \textit{(Maximum norm)}.
\end{aligned}$$

Figure 1.1 shows for the special case $m = 2$ the closed unit balls

$$\overline{B}_{||\cdot||_i}(0_{\mathbb{R}^2}, 1) := \{x \in \mathbb{R}^2 \mid ||x||_i \leq 1\},\ i \in \{1, 2, \infty\},$$

of the norms given in this example.

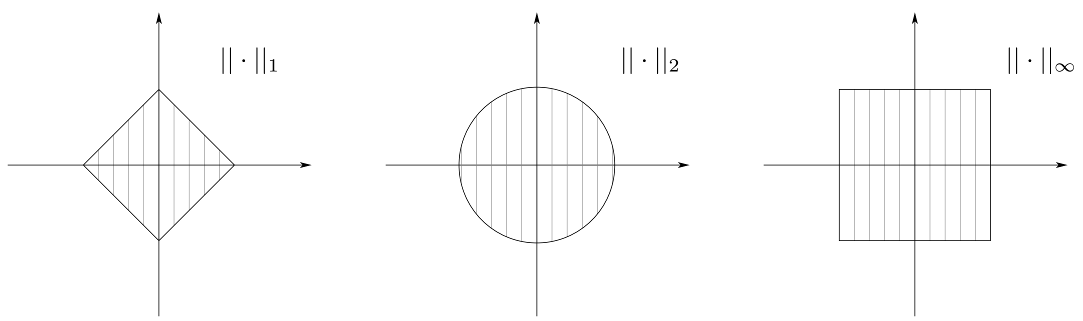


Figure 1.1: Unit balls of the norms $||\cdot||_1$, $||\cdot||_2$ and $||\cdot||_\infty$ on $\mathbb{R}^2$.

In order to define some notions in the topological framework, we define, for any $x \in \mathbf{E}$, the *family of all neighborhoods* by $\mathcal{V}(x)$. Recall that $V \subseteq \mathbf{E}$ is a *neighborhood* of $x \in \mathbf{E}$ (relative to the topology $\mathcal{T}$) if there exists an open set $O \in \mathcal{T}$ such that $x \in O \subseteq V$. A subset $\mathcal{V}_B(x)$ of $\mathcal{V}(x)$ is called a *base of neighborhoods* of $x \in \mathbf{E}$ (relative to the topology $\mathcal{T}$) if for every $V \in \mathcal{V}(x)$ there exists $V' \in \mathcal{V}_B(x)$ such that $V' \subseteq V$.

**Definition 1.8** For any set $\Omega \subseteq \mathbf{E}$, we define the *interior* (in the topological sense) of $\Omega$ by

$$\begin{aligned} \operatorname{int} \Omega &:= \bigcup \{\Omega' \subseteq \mathbf{E} \mid \Omega' \subseteq \Omega, \Omega' \text{ is open}\} \\ &= \{x \in \Omega \mid \exists\, V \in \mathcal{V}(x) :\, V \subseteq \Omega\}, \end{aligned}$$

while the *closure* of $\Omega$ (in the topological sense) is

$$\operatorname{cl} \Omega \;:=\; \bigcap \{\Omega' \subseteq \mathbf{E} \mid \Omega \subseteq \Omega', \Omega' \text{ is closed}\}.$$

In addition, we define the *boundary* of $\Omega$ (in the topological sense) by

$$\operatorname{bd} \Omega := (\operatorname{cl} \Omega) \setminus \operatorname{int} \Omega.$$

In the next two lemmata, we present some properties of the interior, closure, and boundary of a set $\Omega \subseteq \mathbf{E}$.

**Lemma 1.9** For any set $\Omega \subseteq \mathbf{E}$, the following assertions hold:

**1°.** $\operatorname{int} \Omega \subseteq \Omega \subseteq \operatorname{int} \Omega \cup \operatorname{bd} \Omega = \operatorname{cl} \Omega$.

**2°.** $\operatorname{cl} \Omega = \Omega$ if and only if $\Omega$ is closed.

**3°.** $\operatorname{int} \Omega = \Omega$ if and only if $\Omega$ is open.

**Remark 1.10** Considering a metric space $(\mathbf{E}, d)$ with metric $d : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$, then $V \subseteq \mathbf{E}$ is a neighborhood of $x \in \mathbf{E}$ (relative to the topology $\mathcal{T}$) if there is some $\varepsilon \in \mathbb{R}_{++}$ such that

$$B_d(x, \varepsilon) \subseteq V.$$

Notice that $B_d(x, \varepsilon)$ is an open set while $\overline{B}_d(x, \varepsilon)$ is a closed set.

**Lemma 1.11** Let $\Omega \subseteq \mathbf{E}$ be a set with $\emptyset \neq \Omega \neq \mathbf{E}$. Then, we have $\operatorname{bd} \Omega \neq \emptyset$.

*Proof.* Since each real topological linear space is *connected* (i.e., the space can not be divided into two disjoint nonempty open sets), the only subsets of $\mathbf{E}$ with empty boundary are $\mathbf{E}$ and $\emptyset$. □

The class of *convex sets* will be of special interest in this thesis.

**Definition 1.12** A set $\Omega \subseteq \mathbf{E}$ is called *convex* if

$$\lambda \cdot \Omega + (1 - \lambda) \cdot \Omega \subseteq \Omega \quad \text{for all } \lambda \in (0, 1).$$

Notice, for any $x \in \mathbf{E}$ and $\varepsilon \in \mathbb{R}_{++}$, the balls $B_{||\cdot||}(x, \varepsilon)$ and $\overline{B}_{||\cdot||}(x, \varepsilon)$ are convex sets in the normed space $(\mathbf{E}, ||\cdot||)$.

The next lemma collects some important properties of convex sets.

**Lemma 1.13** ([131, Th. 1.1.2]) Let $\Omega \subseteq \mathbf{E}$ be a convex set. Then, the following assertions hold:

1°. $\operatorname{int} \Omega$ and $\operatorname{cl} \Omega$ are convex.

2°. If $x \in \operatorname{int} \Omega$ and $x' \in \operatorname{cl} \Omega$, then $[x, x'[ \subseteq \operatorname{int} \Omega$.

We are also interested in considering so-called *reverse convex sets*.

**Definition 1.14** A set $\Omega \subseteq \mathbf{E}$ is called *reverse convex* if the complement of $\Omega$ (i.e., the set $\Omega^c := \mathbf{E} \setminus \Omega$) is a convex set in $\mathbf{E}$.

Clearly, the complements of the convex sets $B_{||\cdot||}(x, \varepsilon)$ and $\overline{B}_{||\cdot||}(x, \varepsilon)$ are reverse convex sets for every $x \in \mathbf{E}$ and $\varepsilon \in \mathbb{R}_{++}$.

In the following, we recall sufficient conditions which ensure that two convex sets can be *strictly separated by a hyperplane.*

**Proposition 1.15** ([11, Th. 1.44], *Seperation Theorem for Convex Sets*) Let $\mathbf{E}$ be a normed space. Consider two disjoint, nonempty, closed, convex sets $\Omega, \Omega' \subseteq \mathbf{E}$ such that at least one of them is compact, then there exists a continuous linear functional $\psi$ such that

$$\sup\{\psi(x) \mid x \in \Omega\} < \inf\{\psi(x) \mid x \in \Omega'\}.$$

So, the hyperplane $\{x \in \mathbf{E} \mid \psi(x) = k\}$ with

$$k \in ]\sup\{\psi(x) \mid x \in \Omega\}, \inf\{\psi(x) \mid x \in \Omega'\}[$$

*strictly separates* the convex sets $\Omega$ and $\Omega'$.

**Corollary 1.16** Let $\mathbf{E}$ be a normed space. Consider a nonempty, closed, convex set $\Omega$ and a point $x' \notin \Omega$. Then, there exists a continuous linear functional $\psi$ such that

$$\sup\{\psi(x) \mid x \in \Omega\} < \psi(x').$$

In the next definition, we recall two important types of hulls for a nonempty set in $\mathbf{E}$.

**Definition 1.17** ([131, Sec. 1.1]) The *affine hull* of a nonempty set $\Omega \subseteq \mathbf{E}$ is the intersection of all affine subspaces of $\mathbf{E}$ containing $\Omega$, i.e.,

$$\operatorname{aff}\Omega := \bigcap\{\Omega' \subseteq \mathbf{E} \mid \Omega \subseteq \Omega',\, \Omega' \text{ is an affine space}\}.$$

The *convex hull* of a nonempty set $\Omega \subseteq \mathbf{E}$ is the intersection of all convex sets containing $\Omega$, i.e.,

$$\operatorname{conv}\Omega := \bigcap\{\Omega' \subseteq \mathbf{E} \mid \Omega \subseteq \Omega',\, \Omega' \text{ is a convex set}\}.$$

In certain results, we will deal with the *algebraic interior* $\operatorname{cor}\Omega$ instead of the topological interior $\operatorname{int}\Omega$ of a set $\Omega \subseteq \mathbf{E}$. The definition of $\operatorname{cor}\Omega$ is given below.

**Definition 1.18** ([64, Def. 1.8]) Let $\Omega$ be a nonempty set in $\mathbf{E}$. The *algebraic interior* of $\Omega$ (or the *core* of $\Omega$) is given by

$$\operatorname{cor}\Omega := \{x \in \Omega \mid \forall\, v \in \mathbf{E}\ \exists\, \delta \in \mathbb{R}_{++} :\ x + [0, \delta] \cdot v \subseteq \Omega\}.$$

**Definition 1.19** A set $\Omega \subseteq \mathbf{E}$ is called *algebraically open* if $\operatorname{cor}\Omega = \Omega$.

The next lemma recalls known relationships between the topological interior and the algebraic interior of a nonempty set in $\mathbf{E}$.

**Lemma 1.20** ([11, Sec. 1.1.2], [64, Lem. 1.3.2]) Let $\Omega$ be a nonempty set in $\mathbf{E}$. Then, we have

$$\operatorname{int}\Omega \subseteq \operatorname{cor}\Omega \subseteq \Omega.$$

Moreover, assuming that $\Omega$ is convex, we have

$$\operatorname{int}\Omega = \operatorname{cor}\Omega$$

if one of the following conditions is satisfied:

(i) $\operatorname{int}\Omega \neq \emptyset$;

(ii) $\mathbf{E}$ is a Banach space and $\Omega$ is closed;

(iii) $\mathbf{E}$ is a finite-dimensional normed space.

It is an important fact that the topological interior can be a proper subset of the algebraic interior, as shown in the next example.

**Example 1.21** Consider $\mathbf{E} = \mathbb{R}^2$ and the nonconvex set

$$\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq x_1^2 \vee x_2 \leq 0\}.$$

Then, we have $(0, -1) \in \operatorname{int} \Omega$, i.e., $\operatorname{int} \Omega \neq \emptyset$. However, $0 \in \operatorname{cor} \Omega$ but $0 \notin \operatorname{int} \Omega$. So, we conclude $\emptyset \neq \operatorname{int} \Omega \subsetneq \operatorname{cor} \Omega$ in this example.

For two points $x, x' \in \mathbf{E}$, we define the closed, open, half-open line segments by

$$[x, x'] := \{(1 - \lambda)x + \lambda x' \mid \lambda \in [0, 1]\}, \qquad ]x, x'[ := [x, x'] \setminus \{x, x'\},$$
$$[x, x'[ := [x, x'] \setminus \{x'\}, \qquad ]x, x'] := [x, x'] \setminus \{x\}.$$

In the proofs of Lemma 1.56 and Theorem 2.51, we will use the following property for interior points of a nonempty set $\Omega$ in a real normed space $\mathbf{E}$.

**Lemma 1.22** Let $\Omega$ be a set in a real normed space $(\mathbf{E}, ||\cdot||)$ with $\operatorname{int} \Omega \neq \emptyset$. Consider $x \in \operatorname{int} \Omega$, i.e., it exists $\varepsilon > 0$ such that $B_{||\cdot||}(x, \varepsilon) \subseteq \Omega$. Then, for all $v \in \mathbf{E}$ with $||v|| = 1$ and all $\delta \in (0, \varepsilon)$, we have

$$[x - \delta v, x + \delta v] \subseteq B_{||\cdot||}(x, \varepsilon) \subseteq \Omega.$$

*Proof.* Noting that

$$||x \pm \delta v - x|| = \delta ||v|| = \delta < \varepsilon,$$

we get the assertion by the convexity of $B_{||\cdot||}(x, \varepsilon)$. □

**Remark 1.23** The assertions given in Lemmata 1.20 and 1.22 are not true in general metric spaces. Consider the metric space $(\mathbb{R}^2, d)$, where $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ represents the discrete metric on $\mathbb{R}^2$ that is defined by $d(x, x') = 1$ for all $x, x' \in \mathbb{R}^2$ with $x \neq x'$ and $d(x, x') = 0$ for $x = x'$. The feasible set is given by $\Omega := [-1, 1] \times [-1, 1]$. Now, it is easily seen that $x := (1, 1) \in \operatorname{int} \Omega$, since we have $B_d(x, \varepsilon) = \{x\} \subseteq \Omega$ for $\varepsilon \in ]0, 1[$. However, for $v := \frac{x' - x}{||x' - x||}$ with $x' := (2, 2) \neq x$, we have $x + \delta v \in \mathbb{R}^2 \setminus \Omega$ for all $\delta \in \mathbb{R}_{++}$.

It is important to mention that the metric space $(\mathbb{R}^2, d)$ with the discrete metric $d$ is a topological space (considering the discrete metric topology associated with $d$) but not a topological linear space as well as not a normed space ($d$ is not derived from a norm).

If $\mathbf{E}$ is a real linear space and $d$ is a metric on $\mathbf{E}$ that is invariant with respect to translation and homogeneous, then $d(\cdot, 0) =: ||\cdot|| : \mathbf{E} \to \mathbb{R}$ defines a norm on $\mathbf{E}$.

In certain results, we need the *relative interior* of a set $\Omega \subseteq \mathbf{E}$ that is given by

$$\operatorname{rint} \Omega := \{x \in \Omega \mid x \text{ is interior point of } \Omega \text{ w.r.t. the topology induced on } \operatorname{aff} \Omega\}.$$

If $\mathbf{E}$ is endowed with a metric $d : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$, then we have

$$\operatorname{rint} \Omega = \{x \in \Omega \mid \exists\, \varepsilon \in \mathbb{R}_{++} : \; B_d(x, \varepsilon) \cap \operatorname{aff} \Omega \subseteq \Omega\}.$$

The next lemma points out some important properties of the relative interior.

**Lemma 1.24** ([11, Ch. 1]) Let $\Omega$ be a nonempty set in $\mathbf{E}$. Then, the following assertions hold:

**1°.** $\operatorname{int} \Omega \subseteq \operatorname{rint} \Omega$.

**2°.** $\operatorname{rint} \Omega = \operatorname{int} \Omega$ if $\operatorname{aff} \Omega = \mathbf{E}$.

**3°.** $\operatorname{aff} \Omega = \mathbf{E}$ if $\operatorname{cor} \Omega \neq \emptyset$ or $\operatorname{int} \Omega \neq \emptyset$.

**4°.** If $\mathbf{E}$ is finite-dimensional and $\Omega$ is convex, then $\operatorname{rint} \Omega$ is a nonempty, convex set.

In preparation of the definition of the class of *partially ordered linear spaces*, it is convenient to recall the notion of a *cone* and corresponding cone properties.

**Definition 1.25** ([64, Ch. 1]) A nonempty set $\Omega \subseteq \mathbf{E}$ is called a *cone* if $\mathbb{R}_+ \cdot \Omega = \Omega$ (i.e., a cone contains the origin). The cone $\Omega \subseteq \mathbf{E}$ is said to be

- *nontrivial* if $\{0\} \neq \Omega \neq \mathbf{E}$;
- *pointed* if $\Omega \cap (-\Omega) = \{0\}$;
- *closed* if $\operatorname{cl} \Omega = \Omega$;
- *convex* if $\Omega = \Omega + \Omega$;
- *solid* if $\operatorname{int} \Omega \neq \emptyset$.

Endowing the linear space $\mathbf{E}$ with a *partial ordering* "$\preceq$" (i.e., a binary relation $\preceq \subseteq \mathbf{E} \times \mathbf{E}$ that is reflexive, transitive and compatible with the linear structure of the space) induced by a convex cone $\Omega \subseteq \mathbf{E}$ (a so-called *ordering cone*), we call $\mathbf{E}$ a *partially ordered linear space.* Then, for any $x, x' \in \mathbf{E}$, we define

$$x \preceq x' \quad :\Longleftrightarrow \quad x' \in x + \Omega.$$

If, in addition, $\Omega$ is pointed, then the partial ordering "$\preceq$" is antisymmetric. For more details about partially ordered linear spaces, we refer to the book by Jahn [64, Sec. 1.2].

**Example 1.26** In Section 1.5, we will consider a multi-objective optimization problem where the objective function is acting between a linear topological pre-image space $\mathbf{E}$ and the Euclidean space $\mathbb{R}^m$ as image space. In this case, the partial ordering "$\preceq$"of $\mathbb{R}^m$ can be induced by any pointed, convex cone in $\mathbb{R}^m$, for instance by the well-known *natural ordering cone* $\mathbb{R}^m_+$ that is given by

$$\mathbb{R}^m_+ := \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m \mid \forall\, i \in I_m : \, x_i \in \mathbb{R}_+\}.$$

For this example, "$\preceq$" is called *componentwise partial ordering* of $\mathbb{R}^m$ since

$$x \preceq x' \iff x' \in x + \mathbb{R}^m_+ \iff \forall i \in I_m : \, x_i \leq x'_i$$

for any two points $x = (x_1, \cdots, x_m), x' = (x'_1, \cdots, x'_m) \in \mathbb{R}^m$.

In this thesis, we will mainly work with two particular cones, namely, the *natural ordering cone* $\mathbb{R}^m_+$ in the Euclidean space $\mathbb{R}^m$, and *the cone generated by a set* $\Omega \subseteq \mathbf{E}$, as given in the next definition.

**Definition 1.27** ([64], Def. 1.15) For any nonempty set $\Omega \subseteq \mathbf{E}$, the set

$$\begin{aligned} \operatorname{cone} \Omega &:= \bigcap \{\Omega' \subseteq \mathbf{E} \mid \Omega \subseteq \Omega', \, \Omega' \text{ is a cone}\} \\ &= \{\lambda x \in \mathbf{E} \mid (\lambda, x) \in \mathbb{R}_+ \times \Omega\} \end{aligned}$$

is called *the cone generated by the set* $\Omega$.

**Remark 1.28** For any point $x$ of a nonempty, convex set $\Omega \subseteq \mathbf{E}$, the set $\operatorname{cl}(\operatorname{cone}(\Omega - x))$ stands for the *contingent cone* $T(\Omega, x)$ of $\Omega$ at the point $x$ (see, e.g., Jahn [64, Ch. 3]).

In the next lemma, we recall characterizations for the (algebraic) interior and the affine hull of any nonempty, convex set $\Omega \subseteq \mathbf{E}$ in terms of cones generated by some sets $\Omega' \subseteq \mathbf{E}$.

**Lemma 1.29** ([131, Sec. 1.1]) Let $\Omega$ be a nonempty, convex set in $\mathbf{E}$. Then, the following assertions hold:

**1°.** For any $x \in \Omega$, it holds that

$$x \in \operatorname{core} \Omega \iff \operatorname{cone}(\Omega - x) = \mathbf{E}.$$

**2°.** Suppose, in addition, that $\mathbf{E}$ is finite-dimensional and normed. Then, for any $x \in \Omega$, we have

$$x \in \operatorname{int}\Omega \iff \operatorname{cone}(\Omega - x) = \mathbf{E}.$$

**3°.** For any $x \in \Omega$, we have

$$\operatorname{aff}\Omega = x + \operatorname{cone}(\Omega - \Omega).$$

*Proof.* Follows by Lemma 1.24 and Zălinescu [131, Sec. 1.1]). □

Consider a nonempty set $\Omega \subseteq \mathbf{E}$ in a normed space $(\mathbf{E}, ||\cdot||)$. Given a point $x' \in \mathbf{E}$, the *set of points in* $\Omega$ *closest to* $x'$ with respect to the norm $||\cdot|| : \mathbf{E} \to \mathbb{R}$ is defined by

$$\operatorname{Proj}_{\Omega}^{||\cdot||}(x') := \operatorname{argmin}\{||x - x'|| \mid x \in \Omega\}.$$

In addition, for a nonempty set $\Omega' \subseteq \mathbf{E}$, we define the *projection of* $\Omega'$ *onto* $\Omega$ with respect to the norm $||\cdot||$ by

$$\operatorname{Proj}_{\Omega}^{||\cdot||}(\Omega') := \bigcup_{x' \in \Omega'} \operatorname{Proj}_{\Omega}^{||\cdot||}(x').$$

We end this section by recalling some crucial facts about the projection operator $\operatorname{Proj}_{\Omega}^{||\cdot||}(\cdot)$.

**Lemma 1.30** ([11, Sec. 3.3.2]) Consider a nonempty, convex set $\Omega \subseteq \mathbf{E}$ in a normed space $(\mathbf{E}, ||\cdot||)$ and assume that $x' \in \mathbf{E}$. Then, the following assertions hold:

**1°.** $\operatorname{Proj}_{\Omega}^{||\cdot||}(x')$ is a convex (possibly empty) set.

**2°.** Let $(\mathbf{E}, ||\cdot||)$ be reflexive, and let $\Omega$ be closed. Then, $\operatorname{Proj}_{\Omega}^{||\cdot||}(x')$ is a nonempty, convex set.

**3°.** Let $(\mathbf{E}, ||\cdot||)$ be strictly convex. Then, $\operatorname{Proj}_{\Omega}^{||\cdot||}(x')$ is a singleton set or the empty set.

**4°.** Let $(\mathbf{E}, ||\cdot||)$ be reflexive and strictly convex, and let $\Omega$ be closed. Then, $\operatorname{Proj}_{\Omega}^{||\cdot||}(x')$ is a singleton set.

**5°.** Let $\mathbf{E} = \mathbb{R}^m$, $||\cdot|| = ||\cdot||_2$, and let $\Omega$ be closed. Then, $\operatorname{Proj}_{\Omega}^{||\cdot||_2}(x')$ is a singleton set.

## 1.2 Semi-continuity and generalized-convexity properties

In this section, we recall some definitions and facts about generalized-convex and semi-continuous functions (see, e.g., Barbu and Precupanu [11, Ch. 2], Cambini and Martein [17], Giorgi, Guerraggio and Thierfelder [49], Löhne [76, Sec. 2.3], and Zălinescu[131]).

In order to operate with certain generalized-convexity and semi-continuity notions, we define, for any $(x, x') \in \mathbf{E} \times \mathbf{E}$, the function $l_{x,x'} : [0, 1] \to \mathbf{E}$,

$$l_{x,x'}(\lambda) := (1 - \lambda)x + \lambda x' \quad \text{for all } \lambda \in [0, 1].$$

Throughout this thesis, consider a nonempty set

$$\mathcal{D} \subseteq \mathbf{E}.$$

At first we will concentrate on the *class of extended real-valued functions* (i.e., functions that take values in $\mathbb{R} \cup \{+\infty\}$), later we will mainly work with the *class of real-valued functions* (i.e., functions that take values only in $\mathbb{R}$). We use the convention $(+\infty) + (-\infty) = +\infty$. Letting a so-called *extended real-valued function* $h : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ be given, we define the *effective domain* of $h$ by

$$\operatorname{dom} h := \{x \in \mathcal{D} \mid h(x) < +\infty\}.$$

In the next Definition 1.31, we recall some notions related to certain types of continuity.

**Definition 1.31** ([131], [76, Sec. 2.3]) Consider a nonempty set $\Omega \subseteq \mathcal{D}$. An extended real-valued function $h : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ is called

- *lower semi-continuous* at $x' \in \Omega$, if we have

$$h(x') \leq \sup_{V \in \mathcal{V}_B(x')} \inf_{x \in V \cap \Omega} h(x),$$

  where $\mathcal{V}_B(x')$ is a base of neighborhoods of $x'$ in $\mathbf{E}$.

- *upper semi-continuous* at $x' \in \Omega$, if we have

$$h(x') \geq \inf_{V \in \mathcal{V}_B(x')} \sup_{x \in V \cap \Omega} h(x).$$

- *upper (lower) semi-continuous* on $\Omega$, if $h$ is upper (lower) semi-continuous at every $x' \in \Omega$.

- *continuous* on $\Omega$, if $h$ is both lower and upper semi-continuous on $\Omega$.

- *upper (lower) semi-continuous along line segments* on $\Omega$ (assume that $\Omega$ is convex), if the composition of $h$ and $l_{x,x'}$,

$$h \circ l_{x,x'} : [0,1] \to \mathbb{R} \cup \{+\infty\},$$

  is upper (lower) semi-continuous on $[0,1]$ for all $x, x' \in \Omega$.

- *continuous along line segments* on $\Omega$ (assume that $\Omega$ is convex), if $h$ is both lower and upper semi-continuous along line segments on $\Omega$.

- *Lipschitz continuous* on $\Omega$ of rank $L_h > 0$ (assume that $\mathbf{E}$ is normed with norm $||\cdot||$), if $h$ is finite-valued on $\Omega$ and for all $x, x' \in \Omega$ we have

$$|h(x) - h(x')| \leq L_h ||x - x'||.$$

Now, let us recall the definition of *(strictly) convex functions* as often used in convex and extended real-valued analysis.

**Definition 1.32** ([11, Def. 2.1]) Consider a nonempty, convex set $\Omega \subseteq \mathcal{D}$. An extended real-valued function $h : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ is said to be

- *convex* on $\Omega$, if for all $x, x' \in \Omega \cap \operatorname{dom} h$ and for all $\lambda \in ]0,1[$ we have

$$h((1-\lambda)x + \lambda x') \leq (1-\lambda)h(x) + \lambda h(x').$$

- *strictly convex* on $\Omega$, if for all $x, x' \in \Omega \cap \operatorname{dom} h$, $x \neq x'$, and for all $\lambda \in ]0,1[$ we have

$$h((1-\lambda)x + \lambda x') < (1-\lambda)h(x) + \lambda h(x').$$

- *concave* (*strictly concave*) on $\Omega$, if $-h$ is convex (strictly convex) on $\Omega$.

Strictly convex norms are strictly convex functions which are in fact convex as well. One prominent example of an extended real-valued convex function is the so-called *indicator function* $I_\Omega$ with respect to a nonempty set $\Omega \subseteq \mathbf{E}$. In the next example, we recall the definition of $I_\Omega$ and present some useful properties.

**Example 1.33** Let $\Omega$ be a nonempty set in $\mathbf{E}$. The extended real-valued function $I_\Omega : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$, defined by

$$I_\Omega(x) := \begin{cases} 0 & x \in \Omega, \\ +\infty & \text{otherwise}, \end{cases}$$

is called *indicator function with respect to the set* $\Omega$ and has the following properties:

1°**.** Assume that $\Omega$ is a closed set in $\mathbf{E}$. Then, $I_\Omega$ is lower semi-continuous on $\Omega$ (see Barbu and Precupanu [11, Cor. 2.7]) and continuous on $\mathbf{E} \setminus \Omega$.

2°**.** The indicator function $I_\Omega$ is convex on $\mathbf{E}$ if and only if $\Omega$ is a convex set in $\mathbf{E}$ (see Barbu and Precupanu [11, Prop. 2.2]).

**Remark 1.34** In Example 1.33, we presented a well-known example of an extended real-valued function. However, in this thesis, we will mainly focus on real-valued functions as often considered in works related to *generalized-convexity* (see, e.g., Cambini and Martein [17], Mäkelä, Eronen and Karmitsa [83, 84], Malivert and Boissard [85], and Popovici [103, 104]).

Important notions of *generalized-convex functions* are recalled in the next definition.

**Definition 1.35** ([17, Ch. 2]) Consider a nonempty, convex set $\Omega \subseteq \mathcal{D}$. A real-valued function $h : \mathcal{D} \to \mathbb{R}$ is said to be

- *quasi-convex* on $\Omega$, if for all $x, x' \in \Omega$ and for all $\lambda \in ]0,1[$ we have
$$h((1-\lambda)x + \lambda x') \leq \max\{h(x), h(x')\}.$$
- *strictly quasi-convex* on $\Omega$, if for all $x, x' \in \Omega$, $x \neq x'$, and for all $\lambda \in ]0,1[$ we have
$$h((1-\lambda)x + \lambda x') < \max\{h(x), h(x')\}.$$
- *semi-strictly quasi-convex* on $\Omega$, if for all $x, x' \in \Omega$, $h(x) \neq h(x')$, and for all $\lambda \in ]0,1[$ we have
$$h((1-\lambda)x + \lambda x') < \max\{h(x), h(x')\}.$$
- *explicitly quasi-convex* on $\Omega$, if $h$ is both quasi-convex and semi-strictly quasi-convex on $\Omega$.

Moreover, a function $h : \mathcal{D} \to \mathbb{R}$ is called *quasi-concave* (respectively, *strictly quasi-concave*, *semi-strictly quasi-concave*, *explicitly quasi-concave*) on $\Omega$, if $-h$ is quasi-convex (respectively, strictly quasi-convex, semi-strictly quasi-convex, explicitly quasi-convex) on $\Omega$.

We say that a vector-valued function $f = (f_1, \cdots f_m) : \mathcal{D} \to \mathbb{R}^m$ is *componentwise lower semi-continuous along line segments* (respectively, *upper semi-continuous along line segments*, *continuous along line segments*, *convex*, *quasi-convex*, *semi-strictly quasi-convex*, *explicitly quasi-convex*, *semi-strictly quasi-convex or quasi-convex*) on $\Omega \subseteq \mathcal{D}$ if $f_i$ is lower semi-continuous along line segments (respectively, upper semi-continuous along line segments, continuous along line segments, convex, quasi-convex, semi-strictly quasi-convex, explicitly quasi-convex, semi-strictly quasi-convex or quasi-convex) on $\Omega$ for all $i \in I_m$.

**Remark 1.36** Notice that each real-valued convex function is explicitly quasi-convex and upper semi-continuous along line segments. Moreover, a semi-strictly quasi-convex function which is lower semi-continuous along line segments is explicitly quasi-convex. Counter-examples for the reverse implications are given in Example 1.37. The reader should pay attention to the differences between the concepts of semi-strict quasi-convexity and strict quasi-convexity. Real-valued convex functions are semi-strictly quasi-convex but not strictly quasi-convex in general. However, strict quasi-convexity implies semi-strict quasi-convexity.

**Example 1.37** Consider the set $\Omega := \mathbb{R}$. The function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^3$ for all $x \in \mathbb{R}$ is explicitly quasi-convex and continuous but not convex on $\Omega$. Furthermore, the function $h : \mathbb{R} \to \mathbb{R}$ given by
$$h(x) := \begin{cases} (x-1)^3 & \text{for all } x > 1, \\ 0 & \text{for all } x \in [-1, 1], \\ (x+1)^3 & \text{for all } x < -1 \end{cases}$$

is quasi-convex and continuous but not semi-strictly quasi-convex on $\Omega$. A semi-strictly quasi-convex function which is upper semi-continuous along line segments must not be quasi-convex (e.g., consider the function $h_1 : \mathbb{R} \to \mathbb{R}$ given in Example 1.48).

Next, we point out that the well-known *Cobb-Douglas function*, which is an important tool in some economic fields such as *production theory* or *utility theory* (see Cambini and Martein [17]), is in fact an example function that can be used in the framework of this thesis.

**Example 1.38** ([17, Sec. 2.4.1]) The *Cobb-Douglas function* $h : \mathbb{R}^n_+ \to \mathbb{R}$ is defined by

$$h(x) := \delta x_1^{\alpha_1} x_2^{\alpha_2} \cdot \ldots \cdot x_n^{\alpha_n} \quad \text{for all } x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+,$$

where $\alpha := (\alpha_1, \cdots, \alpha_n) \in \operatorname{int} \mathbb{R}^n_+$ and $\delta \in \mathbb{R}_{++}$. In *economics* one often tries to maximize $h$ over a nonempty feasible region in $\mathbb{R}^n_+$. For doing this it is important to know that the following properties hold (see Cambini and Martein [17, Sec. 2.4.1]):

- $h$ is quasi-concave on $\mathbb{R}^n_+$.
- $h$ is concave on $\mathbb{R}^n_+$ if and only if $||\alpha||_1 \leq 1$.
- $h$ is strictly concave on $\mathbb{R}^n_+$ if and only if $||\alpha||_1 < 1$.

It is an important fact that maximizing a generalized-concave function $h$ is equivalent to minimizing the negative of this function $-h$ (a generalized-convex function).

In the sequel, we will see that generalized-convex functions can be characterized by certain statements that involve notions of *level sets* and *level lines*. These notions will play a key role for proving the main results related to our penalization approach in Chapter 2.

**Definition 1.39** Let $h : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, let $\Omega$ be a nonempty subset of $\mathcal{D}$, and let $s \in \mathbb{R}$. We define the following notions:

$$\begin{array}{ll} L_{\leq}(\Omega, h, s) := \{x \in \Omega \mid h(x) \leq s\} & (\textit{lower-level set of } h \textit{ to the level } s); \\ L_{=}(\Omega, h, s) := \{x \in \Omega \mid h(x) = s\} & (\textit{level line of } h \textit{ to the level } s); \\ L_{<}(\Omega, h, s) := \{x \in \Omega \mid h(x) < s\} & (\textit{strict lower-level set of } h \textit{ to the level } s); \\ L_{\geq}(\Omega, h, s) := L_{\leq}(\Omega, -h, -s) & (\textit{upper-level set of } h \textit{ to the level } s); \\ L_{>}(\Omega, h, s) := L_{<}(\Omega, -h, -s) & (\textit{strict upper-level set of } h \textit{ to the level } s). \end{array}$$

**Remark 1.40** Notice, for any set $\Omega'$ with $\emptyset \neq \Omega \subseteq \Omega' \subseteq \mathcal{D}$, we have

$$L_{\sim}(\Omega, h, s) = L_{\sim}(\Omega', h, s) \cap \Omega \quad \text{for all } \sim \in \{\leq, =, <, \geq, >\}.$$

It is well-known that a function $h : \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ is convex on a convex set $\Omega \subseteq \mathcal{D}$ if and only if the *epigraph of* $h$ (with respect to $\Omega$), i.e., the set

$$\operatorname{epi}(\Omega, h) := \{(x, r) \in \Omega \times \mathbb{R} \mid h(x) \leq r\},$$

is a convex set in $\mathbf{E} \times \mathbb{R}$ (see, e.g., Barbu and Precupanu [11, Prop. 2.3]). Moreover, quasi-convex functions are characterized by the convexity of its (strict) lower-level sets, as stated in the next lemma.

**Lemma 1.41** ([49]) Let $h : \mathcal{D} \to \mathbb{R}$ be a function and $\Omega$ be a convex subset of $\mathcal{D}$. Then, the following assertions are equivalent:

1°. $h$ is quasi-convex on $\Omega$.

2°. $L_{\leq}(\Omega, h, s)$ is convex for all $s \in \mathbb{R}$.

3°. $L_{<}(\Omega, h, s)$ is convex for all $s \in \mathbb{R}$.

Next, we present a useful equivalent characterization of semi-strictly quasi-convexity in terms of strict lower-level sets and level lines.

**Lemma 1.42** Let $h : \mathcal{D} \to \mathbb{R}$ be a function and $\Omega$ be a convex subset of $\mathcal{D}$. Then, the following assertions are equivalent:

**1°.** $h$ is semi-strictly quasi-convex on $\Omega$.

**2°.** For all $s \in \mathbb{R}$, $x \in L_{=}(\Omega, h, s)$, $x' \in L_{<}(\Omega, h, s)$, we have $[x', x[\, \subseteq L_{<}(\Omega, h, s)$.

*Proof.* To prove the implication "$1° \implies 2°$", let $s \in \mathbb{R}$, $x \in L_{=}(\Omega, h, s)$ and $x' \in L_{<}(\Omega, h, s)$. If $h$ is semi-strictly quasi-convex on $\Omega$, then

$$h((1-\lambda)x + \lambda x') < \max\left\{h(x), h(x')\right\} = h(x) = s,$$

since $h(x) = s > h(x')$. Consequently, it follows $[x', x[\, \subseteq L_{<}(\Omega, h, s)$.

Now, we prove the reverse implication "$2° \implies 1°$". Let $a, b \in \Omega$ arbitrarily chosen and assume without loss of generality $h(b) < h(a)$. Define $s := h(a)$. Then, for $x := a$ and $x' := b$, we have

$$h((1-\lambda)a + \lambda b) < s = h(a) = \max\{h(a), h(b)\} \quad \text{for all } \lambda \in ]0, 1[.$$

So, $h$ is semi-strictly quasi-convex on $\Omega$, which completes the proof. □

In the forthcoming Chapters 2 and 3, we will see that the property for semi-strictly quasi-convex function given in the next Lemma 1.43 (discovered by Popovici [102, Prop. 2], see also Popovici [105, Prop. 2.1.2]) is essential for proving some of our main results.

**Lemma 1.43** ([102, Prop. 2]) Let $h : \mathcal{D} \to \mathbb{R}$ be a semi-strictly quasi-convex function on a nonempty, convex set $\Omega \subseteq \mathcal{D}$. Then, for every $(x, x') \in \Omega \times \Omega$, the set

$$L_{>}\left(]x, x'[, h, \max\{h(x), h(x')\}\right)$$

is either a singleton set or the empty set.

In Lemma 1.44, we recall useful equivalent characterizations of lower and upper semi-continuity by using lower-level and upper-level sets (see, e.g., Barbu and Precupanu [11, Prop. 2.5] and Löhne [76, Sec. 2.3]).

**Lemma 1.44** ([11, Prop. 2.5], [76, Sec. 2.3]) Let $h : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The following assertions are equivalent:

**1°.** $h$ is upper (lower) semi-continuous on $\mathbf{E}$.

**2°.** $L_{\geq}(\mathbf{E}, h, s)$ $(L_{\leq}(\mathbf{E}, h, s))$ is closed for all $s \in \mathbb{R}$.

Upper (lower) semi-continuity of a real-valued function on a nonempty, closed subset of $\mathbf{E}$ can be characterized as follows.

**Lemma 1.45** Let $h : \mathcal{D} \to \mathbb{R}$ be a real-valued function, and let $\Omega$ be a nonempty, closed subset of $\mathcal{D}$. Then, the following assertions are equivalent:

**1°.** $h$ is upper (lower) semi-continuous on $\Omega$.

**2°.** $L_{\geq}(\Omega, h, s)$ $(L_{\leq}(\Omega, h, s))$ is closed for all $s \in \mathbb{R}$.

*Proof.* We are going to prove "$1° \iff 2°$" for lower semi-continuous functions. Notice that the assertion concerning upper semi-continuity follows by the facts that $h$ is upper semi-continuous on $\Omega$ if and only if $-h$ is lower semi-continuous on $\Omega$, and $L_{\leq}(\Omega, -h, s) = L_{\geq}(\Omega, h, -s)$ for all $s \in \mathbb{R}$.

Let us consider the extended real-valued function

$$h_{\Omega} := h + I_{\Omega} : \mathbf{E} \to \mathbb{R} \cup \{+\infty\},$$

where $I_{\Omega}$ is the indicator function with respect to $\Omega$ (see Example 1.33). Notice that

$$\forall\, x \in \Omega: \; h(x) = h_{\Omega}(x) < +\infty. \tag{1.1}$$

First, observe that $h$ is lower semi-continuous on $\Omega$ if and only if $h_\Omega$ is lower semi-continuous on $\Omega$. Indeed, consider any $x' \in \Omega$, then by Definition 1.31, the function $h$ is lower semi-continuous at $x'$ if

$$h(x') \leq \sup_{V \in \mathcal{V}_B(x')} \inf_{x \in V \cap \Omega} h(x). \tag{1.2}$$

The condition (1.2) is equivalent to

$$h_\Omega(x') \leq \sup_{V \in \mathcal{V}_B(x')} \inf_{x \in V \cap \Omega} h_\Omega(x) = \sup_{V \in \mathcal{V}_B(x')} \inf_{x \in V} h_\Omega(x) \tag{1.3}$$

in view of (1.1) and taking into account that $h_\Omega(x) = +\infty$ for all $x \in \mathbf{E} \setminus \Omega$.

Now, we claim that $h_\Omega$ is lower semi-continuous on the closed set $\Omega$ if and only if $h_\Omega$ is lower semi-continuous on $\mathbf{E}$. The implication "$\Longrightarrow$" follows by the fact that $h_\Omega$ is constant $+\infty$ on the open set $\Omega^c = \mathbf{E} \setminus \Omega$, hence continuous on $\Omega^c$. So, $h_\Omega$ is lower semi-continuous on $\mathbf{E}$. The reverse implication is obvious since $\Omega \subseteq \mathbf{E}$.

Moreover, in view of the preceding Lemma 1.44, we get that $h_\Omega$ is lower semi-continuous on $\mathbf{E}$ if and only if $L_\leq(\mathbf{E}, h_\Omega, s)$ is a closed for all $s \in \mathbb{R}$.

Of course, because of $h_\Omega(x) = +\infty$ for all $x \notin \Omega$ we have $L_\leq(\mathbf{E}, h_\Omega, s) = L_\leq(\Omega, h, s)$ for every $s \in \mathbb{R}$, which ensures that $L_\leq(\mathbf{E}, h_\Omega, s)$ is closed for all $s \in \mathbb{R}$ if and only if $L_\leq(\Omega, h, s)$ is closed for all $s \in \mathbb{R}$.

This completes the proof of "$1° \Longleftrightarrow 2°$".

□

**Remark 1.46** The assertion of Lemma 1.45 does not hold if $\Omega$ is not supposed to be closed. Indeed, consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := 1$ for every $x \in \mathbb{R}$, and $\Omega := (0, 1)$, then the set $L_\leq(\Omega, f, 1) = \Omega$ is not closed.

In Section 2.6, we are interested in considering the function defined by the maximum of a finite number of scalar functions. In the next lemma, we recall some important properties of this function.

**Lemma 1.47** Let a family of functions $h_i : \mathcal{D} \to \mathbb{R}$, $i \in I_l$, $l \in \mathbb{N}$, be given. Define the maximum of $h_i$, $i \in I_l$, by

$$(\max\, h_i)(x) := \max\{h_1(x), \cdots, h_l(x)\} \quad \text{for all } x \in \mathcal{D}.$$

Suppose that $\Omega$ is a nonempty set in $\mathcal{D}$. Then, the following assertions hold:

**1°.** Let $\Omega$ be closed. If $h_i$, $i \in I_l$, are lower semi-continuous on $\Omega$, then $(\max\, h_i)(\cdot)$ is lower semi-continuous on $\Omega$.

**2°.** Let $\Omega$ be convex. If $h_i$, $i \in I_l$, are convex on $\Omega$, then $(\max\, h_i)(\cdot)$ is convex on $\Omega$.

**3°.** Let $\Omega$ be convex. If $h_i$, $i \in I_l$, are quasi-convex on $\Omega$, then $(\max\, h_i)(\cdot)$ is quasi-convex on $\Omega$.

*Proof.* **1°.** Due to

$$L_\leq(\Omega, (\max\, h_i)(\cdot), s) = \{x \in \Omega \mid (\max\, h_i)(x) \leq s\} = \bigcap_{j \in I_l} L_\leq(\Omega, h_j, s)$$

and the closedness of $L_\leq(\Omega, h_j, s)$ for all $j \in I_l$ and all $s \in \mathbb{R}$ (compare Lemma 1.45), we obtain the lower semi-continuity of $(\max\, h_i)(\cdot)$ on $\Omega$.

**2°.** Since convex functions are characterized by the convexity of their epigraphs, the assertion follows immediately by

$$\operatorname{epi}(\Omega, (\max\, h_i)(\cdot)) = \{(x, r) \in \Omega \times \mathbb{R} \mid (\max\, h_i)(x) \leq r\} = \bigcap_{j \in I_l} \operatorname{epi}(\Omega, h_j)$$

and the convexity of $\operatorname{epi}(\Omega, h_j) = \{(x, r) \in \Omega \times \mathbb{R} \mid h_j(x) \leq r\}$ for all $j \in I_l$.

**3°.** In view of Lemma 1.41, we get the assertion by

$$L_{\leq}(\Omega, (\max h_i)(\cdot), s) = \{x \in \Omega \mid (\max h_i)(x) \leq s\} = \bigcap_{j \in I_l} L_{\leq}(\Omega, h_j, s)$$

and the convexity of $L_{\leq}(\Omega, h_j, s)$ for all $j \in I_l$ and all $s \in \mathbb{R}$.

□

In the next example, we show that an analogous assertion to 2° and 3° of Lemma 1.47 does not hold for the concept of semi-strict quasi-convexity.

**Example 1.48** Consider the set $\Omega := \mathbb{R}$ and two functions $h_i : \mathbb{R} \to \mathbb{R}$, $i \in I_2$, defined by $h_i(x) := 0$ for all $x \in \Omega$, $x \neq i$, and $h_i(i) := 1$. Notice that $h_1$ and $h_2$ are semi-strictly quasi-convex on $\Omega$. Then, $(\max h_i)(\cdot) : \mathbb{R} \to \mathbb{R}$ is given by

$$(\max h_i)(x) = \begin{cases} 0 & \text{for all } x \in \mathbb{R} \setminus \{1, 2\}, \\ 1 & \text{for all } x \in \{1, 2\}. \end{cases}$$

Since $(\max h_i)(0) = 0 < 1 = (\max h_i)(1) = (\max h_i)(2)$, the function $(\max h_i)(\cdot)$ is not semi-strictly quasi-convex on $\Omega$.

Recall that a function $g : S \to \mathbb{R}$, defined on a nonempty set $S \subseteq \mathbb{R}^l$, is $\mathbb{R}^l_+$*-increasing on* $S$, if for any $y, y' \in S$,

$$y' \in y + \mathbb{R}^l_+ \implies g(y) \leq g(y'). \tag{1.4}$$

**Lemma 1.49** Let $h = (h_1, \cdots, h_l) : \mathcal{D} \to \mathbb{R}^l$, $l \in \mathbb{N}$, be componentwise convex on the nonempty, convex set $\Omega$ in $\mathcal{D}$. Consider a real-valued function $g : S \to \mathbb{R}$, defined on a set $S \subseteq \mathbb{R}^l$ with $h[\Omega] \subseteq S$, that is $\mathbb{R}^l_+$-increasing on $S$. Then, the following assertions hold:

**1°.** If $g$ is convex on $S$, then $g \circ h$ is convex on $\Omega$.

**2°.** If $g$ is quasi-convex on $S$, then $g \circ h$ is quasi-convex on $\Omega$.

**3°.** If $g$ is semi-strictly quasi-convex on $S$, then $g \circ h$ is semi-strictly quasi-convex on $\Omega$.

*Proof.* Consider $x, x' \in \Omega$ and $\lambda \in [0, 1]$. By the componentwise convexity of $h$ on $\Omega$, we have

$$\lambda h(x) + (1 - \lambda) h(x') \in h(\lambda x + (1 - \lambda) x') + \mathbb{R}^l_+. \tag{1.5}$$

So, assertion 1° follows by the fact that

$$\begin{aligned} (g \circ h)(\lambda x + (1 - \lambda) x') &\leq g(\lambda h(x) + (1 - \lambda) h(x')) && \text{(due to (1.4) and (1.5))} \\ &\leq \lambda g(h(x)) + (1 - \lambda) g(h(x')) && \text{(convexity of } g\text{)} \\ &= \lambda (g \circ h)(x) + (1 - \lambda)(g \circ h)(x'), \end{aligned}$$

while assertion 2° holds since

$$\begin{aligned} (g \circ h)(\lambda x + (1 - \lambda) x') &\leq g(\lambda h(x) + (1 - \lambda) h(x')) && \text{(due to (1.4) and (1.5))} \\ &\leq \max\{g(h(x)), g(h(x'))\} && \text{(quasi-convexity of } g\text{)} \\ &= \max\{(g \circ h)(x), (g \circ h)(x')\}. \end{aligned}$$

To show assertion 3°, assume $(g \circ h)(x) \neq (g \circ h)(x')$. Then, it holds that

$$\begin{aligned} (g \circ h)(\lambda x + (1 - \lambda) x') &\leq g(\lambda h(x) + (1 - \lambda) h(x')) && \text{(due to (1.4) and (1.5))} \\ &< \max\{g(h(x)), g(h(x'))\} && \text{(semi-strictly quasi-convexity of } g\text{)} \\ &= \max\{(g \circ h)(x), (g \circ h)(x')\}. \end{aligned}$$

□

By Lemma 1.49 we immediately get the well-known fact that the weighted sum of a finite number of convex functions is convex as well.

**Corollary 1.50** Let $h = (h_1, \cdots, h_l) : \mathcal{D} \to \mathbb{R}^l$, $l \in \mathbb{N}$, be componentwise convex on the nonempty, convex set $\Omega$ in $\mathcal{D}$. For any $\lambda \in \mathbb{R}^l_+$, the function $\langle \lambda, h(\cdot) \rangle$ is convex on $\Omega$.

It is known that the property given in Corollary 1.50 fails for quasi-convex functions. We conclude this section by noting that the property given in Corollary 1.50 also fails for semi-strictly quasi-convex functions, as to see in the next example.

**Example 1.51** Consider the functions $h_1$ and $h_2$ as well as the set $\Omega$ as defined in Example 1.48. Then, for $\lambda := (1, 1) \in \mathbb{R}^2_+$, we have

$$\langle \lambda, h(x) \rangle = h_1(x) + h_2(x) = (\max\ h_i)(x),$$

which shows that $\langle \lambda, h(\cdot) \rangle$ is not semi-strictly quasi-convex on $\Omega$, in view of Example 1.48.

## 1.3 Local versions of generalized-convexity

In classical definitions of generalized-convexity notions (see Definition 1.35), the involved set $\Omega \subseteq \mathcal{D}$ is always assumed to be convex. In order to use generalized-convexity notions also for nonconvex sets, one could define corresponding local versions of generalized-convexity notions. Notice that local concepts are already used in the literature of optimization theory for other notions (e.g., Lipschitz continuity).

In the next definition, for any normed space $\mathbf{E}$ equipped with the norm $|| \cdot || : \mathbf{E} \to \mathbb{R}$, we introduce local versions of semi-strict quasi-convexity and quasi-convexity for a real-valued function $h : \mathcal{D} \to \mathbb{R}$.

**Definition 1.52** ([56]) Let $(\mathbf{E}, || \cdot ||)$ be a normed space and let $\Omega \subseteq \mathcal{D}$ be open. A real-valued function $h : \mathcal{D} \to \mathbb{R}$ is called

- *locally semi-strictly quasi-convex* (*locally quasi-convex*) at a point $x \in \Omega$ if there exists $\varepsilon \in \mathbb{R}_{++}$ such that $h$ is semi-strictly quasi-convex (quasi-convex) on $B_{||\cdot||}(x, \varepsilon)$.

- *locally explicitly quasi-convex* at $x \in \Omega$ if it is both locally semi-strictly quasi-convex and locally quasi-convex at $x \in \Omega$.

The local concepts of generalized-convexity given in Definition 1.52 will be used in Lemma 2.50 and Theorem 2.51.

**Remark 1.53** Notice that the open ball $B_{||\cdot||}(x, \varepsilon)$ is an open and convex set in a normed space $(\mathbf{E}, || \cdot ||)$. Clearly, in view of Remark 1.36, if $h$ is locally semi-strictly quasi-convex at $x \in \Omega$ and lower semi-continuous along line segments on $B_{||\cdot||}(x, \varepsilon)$, then $h$ is locally quasi-convex at $x \in \Omega$.

In the following lemma, we present relationships between global and corresponding local versions of generalized-convexity.

**Lemma 1.54** ([56]) Let $(\mathbf{E}, || \cdot ||)$ be a normed space and let $\Omega \subseteq \mathcal{D}$ be open and convex. A function $h : \mathcal{D} \to \mathbb{R}$, which is semi-strictly quasi-convex (quasi-convex) on the set $\Omega$, is locally semi-strictly quasi-convex (locally quasi-convex) at every point $x \in \Omega$.

The reverse implications are not true, as shown in the next example.

**Example 1.55** For the function $h = (\max\ h_i)(\cdot) : \mathbb{R} \to \mathbb{R}$ considered in Example 1.48, we know that $h$ is not semi-strictly quasi-convex on $\Omega := \mathbb{R}$. However, $h$ is semi-strictly quasi-convex on $B_{||\cdot||}(x, \varepsilon) = ]x - \varepsilon, x + \varepsilon[$ for every $x \in \Omega$ and for $\varepsilon \in ]0, 1[$. Moreover, the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) := \begin{cases} x + 1 & \text{for all } x < -1, \\ 0 & \text{for all } x \in [-1, 1], \\ 1 - x & \text{for all } x > 1 \end{cases}$$

is not quasi-convex on $\Omega := \mathbb{R}$, but quasi-convex on $B_{||\cdot||}(x,\varepsilon) = ]x-\varepsilon, x+\varepsilon[$ for every $x \in \Omega$ and for $\varepsilon \in ]0,1[$.

A further relationship between global and corresponding local versions of generalized-convexity is given in the next lemma.

**Lemma 1.56** ([56]) Let $(\mathbf{E}, ||\cdot||)$ be a normed space, let $\Omega \subseteq \mathcal{D}$ be open and convex, and let $h : \mathcal{D} \to \mathbb{R}$ be upper semi-continuous along line segments on $\Omega$. Then, $h$ is semi-strictly quasi-convex on the set $\Omega$ if both of the following assertions are fulfilled:

1°. $h$ is locally explicitly quasi-convex at each point $x \in \Omega$.

2°. Every local minimum of $h$ is also global for each restriction on a line segment in $\Omega$.

*Proof.* Under the validity of 1° and 2°, we suppose that the contrary holds, i.e., $h$ is not semi-strictly quasi-convex on $\Omega$. Due to Lemma 1.42, there exist $s \in \mathbb{R}$, $x^0 \in L_=(\Omega, h, s)$ and $x^1 \in L_<(\Omega, h, s)$ such that $x^\lambda := l_{x^0,x^1}(\lambda) \in L_\geq(\Omega, h, s)$ for some $\lambda \in ]0,1[$. Since $h(x^\lambda) \geq h(x^0) > h(x^1)$ and $h \circ l_{x^0,x^1} : [0,1] \to \mathbb{R}$ is upper semi-continuous on $[0,1]$, we can choose

$$\lambda_{\max} \in \left\{ \overline{\lambda} \in ]0,1[ \mid h(l_{x^0,x^1}(\overline{\lambda})) = \max_{\lambda \in [0,1]} h(l_{x^0,x^1}(\lambda)) \right\}$$

by a well-known *Weierstrass-type Existence Theorem* (see, e.g., Aliprantis and Border [1, Th. 2.43]). Now, put $x^2 := x^{\lambda_{\max}}$. Consider $\varepsilon \in \mathbb{R}_{++}$ such that $h$ is explicitly quasi-convex on $B_\varepsilon := B_{||\cdot||}(x^2, \varepsilon)$. Thanks to Lemma 1.22, we get that $B_\delta := [x^2 - \delta v, x^2 + \delta v] \subseteq B_\varepsilon$ for $v := \frac{x^1 - x^0}{||x^1 - x^0||}$ (note that $x^1 \neq x^0$) and $\delta \in ]0, \varepsilon[$. Now, define

$$\overline{\delta} := \min\left\{\delta, ||x^2 - x^0||, ||x^2 - x^1||\right\} \in ]0, \delta]$$

and

$$B_{\overline{\delta}} := [x^2 - \overline{\delta} v, x^2 + \overline{\delta} v].$$

Consider $\delta', \delta'' \in \mathbb{R}_{++}$. It is easily seen that $x^2 + \delta' v = x^1$ implies $\delta' = ||x^2 - x^1||$, while $x^2 - \delta'' v = x^0$ implies $\delta'' = ||x^2 - x^0||$. Hence, we have $B_{\overline{\delta}} \subseteq [x^0, x^1]$ and $B_{\overline{\delta}} \subseteq B_\delta \subseteq B_\varepsilon$. Since $h(x^1) < s \leq h(x^2)$, in view of 2°, we know that $x^2 \in ]x^0, x^1[$ can not be a local minimum point of $h$ on the line segment $[x^0, x^1]$. So, there exists $x^3 \in B_{\overline{\delta}} \setminus \{x^2\}$ with $h(x^3) < h(x^2)$. Clearly, for the point $x^4 := x^2 + (x^2 - x^3) \in B_{\overline{\delta}}$, we have $h(x^4) \leq h(x^2)$. Now, we consider three cases:

*Case 1:* If $h(x^3) = h(x^4)$, then $x^2 \in ]x^3, x^4[ \subseteq L_\leq(B_\varepsilon, h, h(x^3))$ by the quasi-convexity of $h$ on $B_\varepsilon$, a contradiction to $h(x^3) < h(x^2)$.

*Case 2:* If $h(x^3) < h(x^4)$, then $x^2 \in ]x^3, x^4[ \subseteq L_<(B_\varepsilon, h, h(x^4))$ by the semi-strict quasi-convexity of $h$ on $B_\varepsilon$, a contradiction to $h(x^4) \leq h(x^2)$.

*Case 3:* If $h(x^4) < h(x^3)$, then $x^2 \in ]x^3, x^4[ \subseteq L_<(B_\varepsilon, h, h(x^3))$ by the semi-strict quasi-convexity of $h$ on $B_\varepsilon$, a contradiction to $h(x^3) < h(x^2)$.

In all cases we have a contradiction, which completes the proof. □

In the next theorem, we present a new characterization of semi-strictly quasi-convex functions.

**Theorem 1.57** ([56]) Let $(\mathbf{E}, ||\cdot||)$ be a normed space, let $\Omega \subseteq \mathcal{D}$ be open and convex, and let $h : \mathcal{D} \to \mathbb{R}$ be continuous along line segments on $\Omega$. Then, $h$ is semi-strictly quasi-convex on $\Omega$ if and only if both of the following assertions hold:

1°. $h$ is locally semi-strictly quasi-convex at each point $x \in \Omega$.

2°. Every local minimum of $h$ is also global for each restriction on a line segment in $\Omega$.

*Proof.* First, we show the implication "$\Longleftarrow$". As mentioned in Remark 1.53, 1° together with the lower semi-continuity along line segments of $h$ on $\Omega$ imply the local explicit quasi-convexity of $h$ at every point $x \in \Omega$. So, in view of Lemma 1.56, we get the semi-strict quasi-convexity of $h$ on $\Omega$, taking into account 2° and the upper semi-continuity along line segments of $h$ on $\Omega$.

Now, we prove the reverse implication "$\Longrightarrow$". The validity of 1° follows by Lemma 1.54, while it is easily seen that the semi-strict quasi-convexity of $h$ on $\Omega$ ensures the validity of 2° (see Cambini and Martein [17, Th. 2.3.4]). □

The next corollary presents an analogous assertion as given in Theorem 1.58 for the concepts of (local) explicit quasi-convexity.

**Corollary 1.58** ([56]) Let $(\mathbf{E}, ||\cdot||)$ be a normed space, let $\Omega \subseteq \mathcal{D}$ be open and convex, and let $h : \mathcal{D} \to \mathbb{R}$ be continuous along line segments on $\Omega$. Then, $h$ is explicitly quasi-convex on $\Omega$ if and only if both of the following assertions hold:

1°. $h$ is locally explicitly quasi-convex at each point $x \in \Omega$.

2°. Every local minimum of $h$ is also global for each restriction on a line segment in $\Omega$.

**Remark 1.59** To the best of our knowledge, the results related to local versions of generalized-convexity notions presented in this section are novel. Later in Section 2.5 we will use these notions in order to derive sufficient conditions for the validity of some level set / level line conditions in our presented penalization approach (see Chapter 2, in particular Section 2.5).

It should be mentioned that there are some notions of *convexity and generalized-convexity at a point* (say $x' \in \Omega \subseteq \mathcal{D}$) known that can be seen as relaxations of the concept of generalized-convexity (see Cambini and Martein [17, Sec. 3.5]). These relaxed generalized-convexity notions also do not need the convexity of the set $\Omega$, however it is assumed that certain conditions at $x'$ are fulfilled for each point of the set $\Omega$. In contrast, in Definition 1.52, one considers only points that are belonging to a certain local ball around the point $x'$.

## 1.4 Minkowski gauges

In this section, we present some relationships between *convex sets* and *Minkowski gauges*. First, let us recall the definition of a *Minkowski gauge* which is defined on a linear topological space $\mathbf{E}$.

**Definition 1.60** ([131, Sec. 1.1]) Let $\Omega$ be a subset of $\mathbf{E}$ with $0 \in \operatorname{core}\Omega$ (i.e., $\Omega$ is an *absorbing* subset of $\mathbf{E}$). A *Minkowski gauge* $\mu_\Omega : \mathbf{E} \to \mathbb{R}$ *associated to the set* $\Omega$ is defined by

$$\mu_\Omega(x) := \inf\{\lambda \in \mathbb{R}_+ \mid x \in \lambda \cdot \Omega\} \quad \text{for all } x \in \mathbf{E}.$$

Notice that $\mu_\Omega$ is finite-valued for every $x \in \mathbf{E}$ since $\Omega$ is absorbing.

In the following lemma, we recall useful properties of the Minkowski gauge $\mu_\Omega$. In particular, we point out some relationships between (strict) level sets / level lines of $\mu_\Omega$, and topological / algebraic notions defined for the corresponding set $\Omega$.

**Lemma 1.61** ([131, Sec. 1.1], [68, Sec. 6.2], [112, Th. 1.35]) Let $\Omega$ be a subset of $\mathbf{E}$ with $0 \in \operatorname{cor}\Omega$ and $[0,1]\cdot\Omega = \Omega$, where $[0,1]\cdot\Omega := \{tx \mid t \in [0,1], x \in \Omega\}$. Then, the following assertions hold:

1°. It holds that

$$L_<(\mathbf{E}, \mu_\Omega, 1) \subseteq \Omega \subseteq L_\leq(\mathbf{E}, \mu_\Omega, 1).$$

2°. Assume that $\Omega$ is convex. Then, $\mu_\Omega$ is convex (hence explicitly quasi-convex) on $\mathbf{E}$,

$$\operatorname{core}\Omega = L_<(\mathbf{E}, \mu_\Omega, 1),$$

and

$$\mu_{\operatorname{core}\Omega} = \mu_\Omega = \mu_{L_\leq(\mathbf{E}, \mu_\Omega, 1)}.$$

3°. If $\Omega$ is convex and $0 \in \operatorname{int}\Omega$, then $\mu_\Omega$ is continuous on $\mathbf{E}$, and we have

$$\begin{aligned} L_<(\mathbf{E}, \mu_\Omega, 1) &= \operatorname{int}\Omega = \operatorname{core}\Omega, \\ L_\leq(\mathbf{E}, \mu_\Omega, 1) &= \operatorname{cl}\Omega, \\ L_=(\mathbf{E}, \mu_\Omega, 1) &= \operatorname{bd}\Omega. \end{aligned}$$

**Remark 1.62** Assume that $\Omega$ is a closed, convex subset of $\mathbf{E}$ with $0 \in \operatorname{int} \Omega$. Then, $L_{\leq}(\mathbf{E}, \mu_\Omega, 1) = \Omega$, i.e., the set $\Omega$ can be described by using a level set of the Minkowski gauge $\mu_\Omega$.

**Corollary 1.63** Let $\Omega$ be a closed, convex subset of $\mathbf{E}$ with $x' \in \operatorname{int} \Omega$. Consider the function $h : \mathbf{E} \to \mathbb{R}$ that is defined by

$$h(x) := \mu_\Omega(x - x') - 1 \quad \text{for all } x \in \mathbf{E}.$$

Then, $h$ is convex (hence explicitly quasi-convex) and continuous on $\mathbf{E}$, and we have

$$\begin{aligned} L_{<}(\mathbf{E}, h, 0) &= x' + L_{<}(\mathbf{E}, \mu_{-x'+\Omega}, 1) = \operatorname{int} \Omega = \operatorname{core} \Omega, \\ L_{\leq}(\mathbf{E}, h, 0) &= x' + L_{\leq}(\mathbf{E}, \mu_{-x'+\Omega}, 1) = \Omega, \\ L_{=}(\mathbf{E}, h, 0) &= x' + L_{=}(\mathbf{E}, \mu_{-x'+\Omega}, 1) = \operatorname{bd} \Omega. \end{aligned}$$

**Example 1.64** Figure 1.2 visualizes the process of determining function values of a Minkowski gauge $\mu_\Omega$, where the set $\Omega \subseteq \mathbb{R}^2$ is given by a polytope that is determined by the convex hull of four points $e^i \in \mathbb{R}^2$, $i \in I_4$, i.e., $\Omega = \operatorname{conv}\{e^i \mid i \in I_4\}$. Notice that $\Omega$ is a closed, convex set with $0 \in \operatorname{core} \Omega = \operatorname{int} \Omega$.

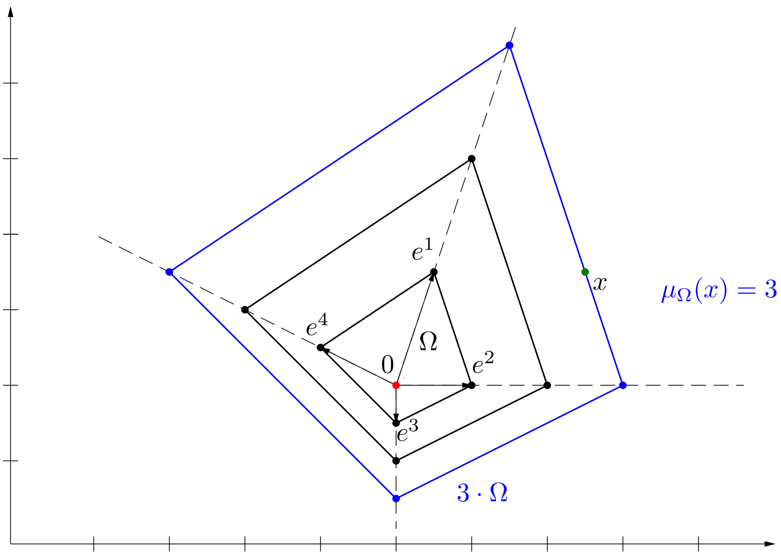


Figure 1.2: Determination of function values of a polyhedral Minkowski gauge.

**Remark 1.65** Under the assumption that $\Omega$ is a convex, absorbing, symmetric (i.e., $\Omega = -\Omega$) set in $\mathbf{E}$, the Minkowski gauge $\mu_\Omega$ is a *semi-norm* on $\mathbf{E}$ (i.e., (*ii*) and (*iii*) in Definition 1.5 hold) by Rudin [112, Th. 1.35]. Conversely, each semi-norm $\rho : \mathbf{E} \to \mathbb{R}$ can be represented by a Minkowski gauge $\mu_\Omega$ with corresponding convex, absorbing unit ball $\Omega := \{x \in \mathbf{E} \mid \rho(x) < 1\}$ (see Rudin [112, Th. 1.34]). By Lemma 1.61 (2°), we actually have $\rho = \mu_\Omega = \mu_{\operatorname{core} \Omega} = \mu_{L_{\leq}(\mathbf{E}, \mu_\Omega, 1)}$. Notice that each norm $|| \cdot || : \mathbf{E} \to \mathbb{R}$ is a semi-norm as well. Further interesting properties of Minkowski gauges can be found in Aliprantis and Border [1, Sec. 5.8].

A Minkowski gauge $\mu_\Omega$ associated to a polyhedral set $\Omega \subseteq \mathbf{E}$ (i.e., $\Omega$ is the intersection of finitely many half spaces) is a so-called *polyhedral Minkowski gauge* (e.g., $\mu_\Omega$ in Example 1.64). If $\Omega$ is polyhedral and $\mu_\Omega$ is a norm, then $\mu_\Omega$ is called *block norm*. Well-known examples of block norms (defined on the Euclidean space $\mathbb{R}^n$) are the *Manhattan norm* and the *Maximum norm*, as considered in Example 1.7. In the field of *location theory*, one often calls a strictly convex norm (defined on $\mathbf{E} = \mathbb{R}^2$) *round* since the unit ball of the norm has no flat spots on the boundary. An example of a strictly convex norm is given by the well-known *Euclidean norm* (see Example 1.7).

## 1.5 Multi-objective optimization

In the following section, we recall some basic facts from the field of *multi-objective optimization* that will be used throughout the thesis.

*Minimal elements* of an arbitrarily nonempty set $F \subseteq \mathbb{R}^m$ with respect to a pointed, convex cone $K \subseteq \mathbb{R}^m$ can be defined according to the next definition.

**Definition 1.66** ([64, Ch. 4]) Let $F \subseteq \mathbb{R}^m$ be a nonempty set and let $K \subseteq \mathbb{R}^m$ be a pointed, convex cone. The set of *minimal elements of $F$ with respect to $K$* is defined by

$$\begin{aligned} \mathrm{MIN}(F, K) &:= \{y \in F \mid (y - K) \cap F = \{y\}\} \\ &= \{y \in F \mid (y - K) \cap F \backslash \{y\} = \emptyset\}. \end{aligned}$$

Moreover, assuming that $K$ is solid, the set of *weakly minimal elements of $Y$ with respect to $K$* is defined by

$$\mathrm{WMIN}(F, K) := \{y \in F \mid (y - \operatorname{int} K) \cap F = \emptyset\}.$$

By Definition 1.66, when $K$ is solid, we obviously have

$$\mathrm{MIN}(F, K) \subseteq \mathrm{WMIN}(F, K) \subseteq F \cap \operatorname{bd} F. \tag{1.6}$$

Also, it is a simple exercise to check that

$$\mathrm{MIN}(F, K) = \mathrm{MIN}(F \backslash \{y\}, K) \quad \text{for all} \quad y \in F \backslash \mathrm{MIN}(F, K). \tag{1.7}$$

In *multi-objective optimization* (see, e.g., Ehrgott [29], Eichfelder and Jahn [33], Göpfert *et al.* [50], and Jahn [64]) one tries to minimize a vector-valued objective function

$$f = (f_1, \cdots, f_m) : \mathcal{D} \to \mathbb{R}^m,$$

where $\mathcal{D}$ is a nonempty subset of the linear topological space $\mathbf{E}$, over a nonempty subset $\Omega$ of $\mathcal{D}$. This means we are going to study the multi-objective optimization problem

$$\begin{cases} f(x) := (f_1(x), \cdots, f_m(x)) \to \min \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in \Omega, \end{cases} \tag{$\mathcal{P}_\Omega$}$$

where $\mathbb{R}^m$ is partially ordered by the natural ordering cone $\mathbb{R}^m_+$ (highlighted by the notation: with respect to (w. r. t. for short) $\mathbb{R}^m_+$).

In the next definition, we recall the concepts of *(strict, weak) Pareto efficiency* that will be used for the vector-valued minimization in the problem $(\mathcal{P}_\Omega)$. Let us denote, for any set $\Omega' \subseteq \mathbf{E}$, the *image* of $f$ over $\Omega'$ by

$$f[\Omega'] := \{f(x) \in \mathbb{R}^m \mid x \in \Omega'\},$$

and the *cardinality* of $\Omega'$ by

$$\operatorname{card} \Omega'.$$

**Definition 1.67** ([29, Ch. 2], [64, Ch. 11]) The set of *Pareto efficient solutions* of problem $(\mathcal{P}_\Omega)$ is defined by

$$\begin{aligned} \mathrm{Eff}(\Omega \mid f) &:= \{x \in \Omega \mid f(x) \in \mathrm{MIN}(f[\Omega], \mathbb{R}^m_+)\} \\ &= \{x \in \Omega \mid f[\Omega] \cap (f(x) - (\mathbb{R}^m_+ \setminus \{0\})) = \emptyset\}, \end{aligned}$$

while that of *weakly Pareto efficient solutions* is given by

$$\begin{aligned} \mathrm{WEff}(\Omega \mid f) &:= \{x \in \Omega \mid f(x) \in \mathrm{WMIN}(f[\Omega], \mathbb{R}^m_+)\} \\ &= \{x \in \Omega \mid f[\Omega] \cap (f(x) - \operatorname{int} \mathbb{R}^m_+) = \emptyset\}. \end{aligned}$$

The set of *strictly Pareto efficient solutions* of $(\mathcal{P}_\Omega)$ is defined by

$$\mathrm{SEff}(\Omega \mid f) := \{x \in \mathrm{Eff}(\Omega \mid f) \mid \mathrm{card}\,\{x' \in \Omega \mid f(x') = f(x)\} = 1\}.$$

It is easily seen that

$$\mathrm{SEff}(\Omega \mid f) \subseteq \mathrm{Eff}(\Omega \mid f) \subseteq \mathrm{WEff}(\Omega \mid f).$$

Notice that the following characterizations of (strictly, weakly) Pareto efficient solutions

$$\begin{aligned}
x \in \mathrm{Eff}(\Omega \mid f) &\iff \nexists\, x' \in \Omega \;\text{ s.t. } \begin{cases} \forall\, i \in I_m : \; f_i(x') \leq f_i(x), \\ \exists\, j \in I_m : \; f_j(x') < f_j(x); \end{cases} \\
x \in \mathrm{WEff}(\Omega \mid f) &\iff \nexists\, x' \in \Omega \;\text{ s.t. } \forall\, i \in I_m : \; f_i(x') < f_i(x); \\
x \in \mathrm{SEff}(\Omega \mid f) &\iff \nexists\, x' \in \Omega \setminus \{x\} \;\text{ s.t. } \forall\, i \in I_m : \; f_i(x') \leq f_i(x)
\end{aligned}$$

are often used in the literature of multi-objective optimization.

Let us consider an example where we illustrate the above definitions of minimality and efficiency, respectively.

**Example 1.68** Figure 1.3 shows an example problem $(\mathcal{P}_\Omega)$ with $m = 2$. For the point $f(x) \in f[\Omega]$, $x \in \Omega$, given in Figure 1.3 we have $f(x) \in \mathrm{MIN}(f[\Omega], \mathbb{R}^m_+)$, hence $x \in \mathrm{Eff}(\Omega \mid f)$.

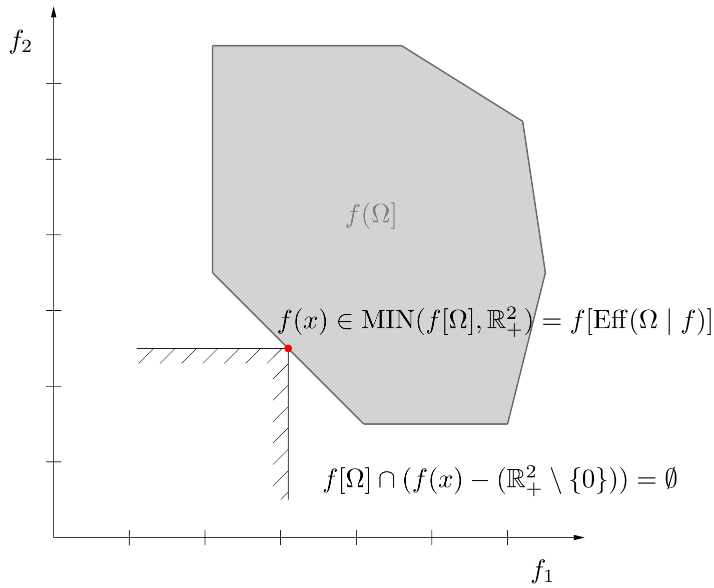


Figure 1.3: $x$ is a Pareto efficient solution of an example problem $(\mathcal{P}_\Omega)$ with $m = 2$.

In the next lemma, we recall useful characterizations of (strictly, weakly) Pareto efficient solutions by using certain level sets and level lines of the component functions of $f : \mathcal{D} \to \mathbb{R}^m$. Before, for any $x \in \Omega$, we define the intersections of (strict) lower-level sets / level lines by

$$\begin{aligned}
S_<(\Omega, f, x) &:= \bigcap_{i \in I_m} L_<(\Omega, f_i, f_i(x)); \\
S_\leq(\Omega, f, x) &:= \bigcap_{i \in I_m} L_\leq(\Omega, f_i, f_i(x)); \\
S_=(\Omega, f, x) &:= \bigcap_{i \in I_m} L_=(\Omega, f_i, f_i(x)).
\end{aligned}$$

**Lemma 1.69** ([29, Th. 2.30]) For any $x \in \Omega$, we have

$$\begin{aligned} x \in \operatorname{Eff}(\Omega \mid f) &\iff S_{\leq}(\Omega, f, x) \subseteq S_{=}(\Omega, f, x); \\ x \in \operatorname{WEff}(\Omega \mid f) &\iff S_{<}(\Omega, f, x) = \emptyset; \\ x \in \operatorname{SEff}(\Omega \mid f) &\iff S_{\leq}(\Omega, f, x) = \{x\}. \end{aligned}$$

**Remark 1.70** The equivalences given in Lemma 1.69 can be for instance found in Ehrgott [29, Th. 2.30] and Nickel [89, Th. 1.4.2]. Notice that these geometrical characterizations given in Lemma 1.69 were already used in the works by Durier and Michelot [27, Prop. 1.1] and Plastria [98] in the context of location theory.

The next example shows how one can use the characterizations of (strictly, weakly) Pareto efficient solutions from Lemma 1.69 that are given in terms of (strict) lower-level sets / level lines.

**Example 1.71** Let us consider the scalar functions $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}$ defined by $f_i(x) := ||x - a^i||_1 = |x_1 - a_1^i| + |x_2 - a_2^i|$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and all $i \in I_3$, where $a^i = (a_1^i, a_2^i) \in \mathbb{R}^2$, $i \in I_3$, are three points in the plane as given in Figure 1.4. In the left part of Figure 1.4, the level lines of $f_1$ and $f_2$ at the point $x \in \mathbb{R}^2$ are shown. Due to Lemma 1.69, we get

$$x \in \operatorname{Eff}(\mathbb{R}^2 \mid (f_1, f_2)) \setminus \operatorname{SEff}(\mathbb{R}^2 \mid (f_1, f_2)).$$

In the right part of Figure 1.4, by adding one additional function $f_3$ to the problem, we can see that the point $x$ is not longer a Pareto efficient solution for the problem with objective function $(f_1, f_2, f_3)$. However, we have

$$x \in \operatorname{WEff}(\mathbb{R}^2 \mid (f_1, f_2, f_3)) \setminus \operatorname{Eff}(\mathbb{R}^2 \mid (f_1, f_2, f_3)).$$

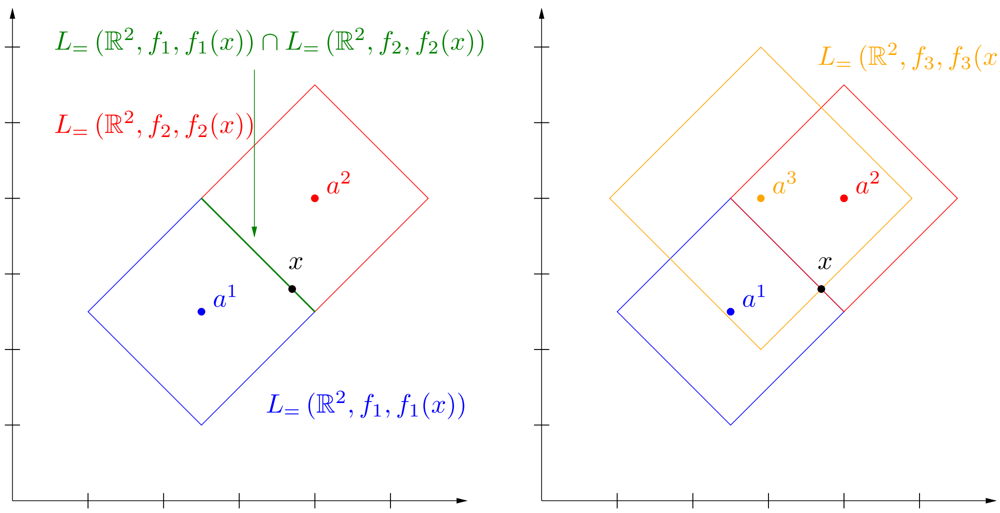


Figure 1.4: Geometric characterization of (weakly) Pareto efficient solutions.

A common approach in multi-objective optimization is the transformation of the initial problem $(\mathcal{P}_\Omega)$ into a scalar one and then to solve this scalarized problem by applying methods from single-objective optimization theory. Such approaches are known in the literature as *scalarization methods*. Probably the best known technique is the so-called *Weighted-Sum Scalarization Method* which we recall below. However, notice that there are several other known scalarization methods, for instance the *Tammer-Weidner Scalarization Method* based on a *nonlinear scalarization functional* (which includes as special cases the *$\varepsilon$-Constraint Method*, the *Pascoletti-Serafini Scalarization Method*, or the *Weighted Chebyshev Norm Approach*), the *Conic Scalarization Method* by

Kasimbeyli, or the Scalarization Method based on the *signed distance function* by Hiriart-Urruty (for more details, see the books by Ehrgott [29], Eichfelder [31, 32], Göpfert *et al.* [50], Jahn [64], and Khan, Tammer and Zălinescu [68]).

For any given $\lambda := (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m_+ \setminus \{0\}$, consider the problem

$$\begin{cases} \langle \lambda, f(x) \rangle := \sum\limits_{i=1}^{m} \lambda_i f_i(x) \to \min \\ x \in \Omega. \end{cases} \qquad (s_\lambda \mathcal{P}_\Omega)$$

Before presenting relationships between the problems $(\mathcal{P}_\Omega)$ and $(s_\lambda \mathcal{P}_\Omega)$, we recall a well-known definition of *proper Pareto efficiency* (in the sense of Geoffrion [44]).

**Definition 1.72** ([44], [29, Def. 2.39], [64, Def. 11.8]) A point $x \in \Omega$ is called a *properly Pareto efficient solution* of $(\mathcal{P}_\Omega)$ if $x \in \mathrm{Eff}(\Omega \mid f)$ and there is some $\rho \in \mathbb{R}_{++}$ so that for every $i \in I_m$ and every $x' \in \Omega$ with $f_i(x') < f_i(x)$ at least one $j \in I_m$ exists with $f_j(x') > f_j(x)$ and

$$\frac{f_i(x) - f_i(x')}{f_j(x') - f_j(x)} \leq \rho.$$

The *set of all properly Pareto efficient solutions* of $(\mathcal{P}_\Omega)$ is denoted by $\mathrm{PEff}(\Omega \mid f)$.

**Remark 1.73** There are also some other notions of *proper Pareto efficiency* known in the literature of *vector optimization* (for instance, concepts in the sense of Benson, Borwein, or Henig). Some useful notes on the literature of *proper Pareto efficiency* (respectively, *proper minimality*) are given by Jahn [64, p. 113-114]. Notice that properly efficient solutions, according to our Definition 1.72, are those efficient solutions that have bounded trade-offs between the objective functions (see, e.g., Ehrgott [29, Sec. 2.4]).

It can easily be seen that the following inclusions hold:

$$\mathrm{PEff}(\Omega \mid f) \subseteq \mathrm{Eff}(\Omega \mid f) \subseteq \mathrm{WEff}(\Omega \mid f).$$

In preparation of the next lemma, for any real-valued function $h : \mathcal{D} \to \mathbb{R}$, let the set of minimal solutions of the problem

$$\begin{cases} h(x) \to \min \\ x \in \Omega. \end{cases}$$

be denoted by

$$\mathrm{Sol}(\Omega \mid h) := \mathrm{argmin}\{h(x) \mid x \in \Omega\}.$$

Define $\mathrm{Sol}_\mathrm{u}(\Omega \mid h) := \emptyset$. Under the assumption that $\mathrm{Sol}(\Omega \mid h)$ is a singleton set, i.e., there exists a solution $x' \in \Omega$ such that $\{x'\} = \mathrm{Sol}(\Omega \mid h)$, we put

$$\mathrm{Sol}_\mathrm{u}(\Omega \mid h) := \{x'\}.$$

In the next lemma, we will see that a solution of the scalarized problem $(s_\lambda \mathcal{P}_\Omega)$ is a (weakly, properly, strictly) Pareto efficient solution of the multi-objective optimization problem $(\mathcal{P}_\Omega)$ under certain assumptions on the parameter $\lambda$.

**Lemma 1.74** ([29, Prop. 3.9], [64, Sec. 11.2.1]) The following assertions hold:

**1°.** If $\lambda \in \mathbb{R}^m_+ \setminus \{0\}$, then

$$\mathrm{Sol}(\Omega \mid \langle \lambda, f(\cdot) \rangle) \subseteq \mathrm{WEff}(\Omega \mid f).$$

**2°.** If $\lambda \in \mathrm{int}\, \mathbb{R}^m_+$, then

$$\mathrm{Sol}(\Omega \mid \langle \lambda, f(\cdot) \rangle) \subseteq \mathrm{PEff}(\Omega \mid f).$$

**3°.** If $\lambda \in \mathbb{R}^m_+ \setminus \{0\}$, then

$$\mathrm{Sol}_u(\Omega \mid \langle \lambda, f(\cdot) \rangle) \subseteq \mathrm{SEff}(\Omega \mid f).$$

Now, under convexity assumption on the problem $(\mathcal{P}_\Omega)$, we get important relationships between the solutions of the scalar optimization problem $(s_\lambda \mathcal{P}_\Omega)$ and properly and weakly Pareto efficient solutions of the multi-objective optimization problem $(\mathcal{P}_\Omega)$.

**Lemma 1.75** ([29, Sec. 3.2], [64, Sec. 11.2.1]) Assume that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise convex on the nonempty, convex set $\Omega \subseteq \mathcal{D}$. Then, the following equalities hold:

$$\mathrm{PEff}(\Omega \mid f) = \bigcup_{\lambda \in \operatorname{int} \mathbb{R}^m_+} \mathrm{Sol}(\Omega \mid \langle \lambda, f(\cdot) \rangle);$$

$$\mathrm{WEff}(\Omega \mid f) = \bigcup_{\lambda \in \mathbb{R}^m_+ \setminus \{0\}} \mathrm{Sol}(\Omega \mid \langle \lambda, f(\cdot) \rangle).$$

# Part I

# On Generalized-Convex Constrained Multi-Objective Optimization

# Chapter 2

# A new penalization approach in constrained multi-objective optimization

In this chapter, we derive a novel approach (a kind of *vectorial penalization approach*) for solving generalized-convex multi-objective optimization problems involving not necessarily convex constraints where the vector-valued objective function is acting between a topological linear pre-image space $\mathbf{E}$ and a finite-dimensional image space $\mathbb{R}^m$.

In scalar optimization theory, the famous *Exact Penalty Principle* (see Clarke [22, Prop. 2.4.3], Eremin [35], and Zangwill [132]) is based on the idea to replace the initial constrained single-objective optimization problem

$$\begin{cases} h(x) \to \min \\ x \in \Omega \end{cases}$$

with a nonempty feasible set $\Omega \subseteq \mathbf{E}$ by a penalized unconstrained single-objective optimization problem

$$\begin{cases} h(x) + \rho\phi(x) \to \min, \\ x \in \mathbf{E}, \end{cases}$$

where $\rho \in \mathbb{R}_{++}$ and $\phi : \mathbf{E} \to \mathbb{R}$ is a scalar-valued function satisfying

$$\begin{aligned} \phi(x) = 0 \quad &\iff \quad x \in \Omega; \\ \phi(x) > 0 \quad &\iff \quad x \in \mathbf{E} \setminus \Omega. \end{aligned}$$

*Clarke's Exact Penalty Principle* (more precisely an improved version by Ye [129, Th. 1.2]) can be stated as follows:

**Proposition 2.1** ([22, Prop. 2.4.3], [129, Th. 1.2]) Let $(\mathbf{E}, ||\cdot||)$ be a normed space, let $h : \mathbf{E} \to \mathbb{R}$ be a Lipschitz continuous function on $\mathbf{E}$ of rank $L_h \in \mathbb{R}_{++}$, and let $\Omega \subseteq \mathbf{E}$ be a nonempty set. Consider the function $\phi : \mathbf{E} \to \mathbb{R}$ defined by

$$\phi(x) := \inf\{||x' - x|| \mid x' \in \Omega\} \quad \text{for all } x \in \mathbf{E}.$$

Then, for any $L > L_h$, we have

$$\operatorname{argmin}_{x \in \Omega} h(x) = \operatorname{argmin}_{x \in \mathbf{E}} h(x) + L\phi(x).$$

Some authors (see Apetrii, Durea and Strugariu [4] and Ye [129]) extended the above *Exact Penalty Principle* to vector optimization. These approaches consists of adding a penalization term in each component function of the objective function $f : \mathcal{D} \to \mathbb{R}^m$ given in $(\mathcal{P}_\Omega)$. Our approach presented in this chapter is a kind of *vectorial penalization* and differs from the works by Ye [129] and Apetrii, Durea and Strugariu [4]. Many of the results presented in this chapter are based on Günther and Tammer [55, 56]. Further ideas for *vectorial penalization* can be found in Durea, Strugariu and Tammer [25], and Klamroth and Tind [72] (see Section 2.8 for more details).

## 2.1 Relationships between problem $(\mathcal{P}_X)$ with nonconvex feasible set $X$ and problem $(\mathcal{P}_Y)$ with convex feasible set $Y$

In this section, we suppose that the following assumptions hold:

$$\begin{cases} \text{Let } \mathbf{E} \text{ be a real topological linear space;} \\ \text{let } \mathcal{D} \subseteq \mathbf{E} \text{ be a nonempty set;} \\ \text{let } Y \subseteq \mathcal{D} \text{ be a convex set;} \\ \text{let } X \subseteq Y \text{ be a nonempty set with } X \neq Y. \end{cases} \tag{2.1}$$

In what follows, under the validity of the assumption (2.1), we study some relationships between two multi-objective optimization problems

$$\begin{cases} f(x) = (f_1(x), \cdots, f_m(x)) \to \min \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in X \end{cases} \tag{$\mathcal{P}_X$}$$

and

$$\begin{cases} f(x) = (f_1(x), \cdots, f_m(x)) \to \min \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in Y. \end{cases} \tag{$\mathcal{P}_Y$}$$

**Lemma 2.2** ([55, 56]) Assume that (2.1) is satisfied. Then, the following assertions hold:

**1°.** We have

$$\begin{aligned} X \cap \mathrm{Eff}(Y \mid f) &\subseteq \mathrm{Eff}(X \mid f); \\ X \cap \mathrm{WEff}(Y \mid f) &\subseteq \mathrm{WEff}(X \mid f); \\ X \cap \mathrm{SEff}(Y \mid f) &\subseteq \mathrm{SEff}(X \mid f). \end{aligned}$$

**2°.** Let $Y$ be convex. If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$, then

$$(\mathrm{cor}\, X) \setminus \mathrm{Eff}(Y \mid f) \subseteq (\mathrm{cor}\, X) \setminus \mathrm{Eff}(X \mid f); \tag{2.2}$$
$$(\mathrm{cor}\, X) \setminus \mathrm{WEff}(Y \mid f) \subseteq (\mathrm{cor}\, X) \setminus \mathrm{WEff}(X \mid f). \tag{2.3}$$

**3°.** Let $Y$ be convex. If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$, then

$$(\mathrm{cor}\, X) \setminus \mathrm{SEff}(Y \mid f) \subseteq (\mathrm{cor}\, X) \setminus \mathrm{SEff}(X \mid f).$$

*Proof.* **1°.** Follows easily by Lemma 1.69.

**2°.** Let us show the first inclusion. Consider $x^0 \in (\mathrm{cor}\, X) \setminus \mathrm{Eff}(Y \mid f)$. Since $x^0 \notin \mathrm{Eff}(Y \mid f)$, there exists $x^1 \in L_<(Y, f_j, f_j(x^0)) \cap S_\leq(Y, f, x^0)$ for some $j \in I_m$. We define the following two index sets

$$\begin{aligned} I^< &:= \{j \in I_m \mid x^1 \in L_<(Y, f_j, f_j(x^0))\}, \\ I^= &:= \{i \in I_m \mid x^1 \in L_=(Y, f_i, f_i(x^0))\}. \end{aligned}$$

Of course, we know that $I^< \neq \emptyset$ and $I^= \cup I^< = I_m$.

Clearly, for $x^1 \in X$, we get immediately $x^0 \in (\mathrm{cor}\, X) \setminus \mathrm{Eff}(X \mid f)$. Now, assume $x^1 \in Y \setminus X$. Since $x^0 \in \mathrm{cor}\, X$, by Lemma 1.20, we get $x^0 + [0, \delta] \cdot v \subseteq X$ for $v := x^1 - x^0 \neq 0$ and some $\delta \in \mathbb{R}_{++}$. Obviously, since $x^1 \notin X$, it follows $\delta \in ]0, 1[$. Hence, for $\lambda^* := \delta$, we have $x^\lambda := l_{x^0,x^1}(\lambda) \in X \cap ]x^0, x^1[$ for all $\lambda \in ]0, \lambda^*]$.

Now, for an arbitrarily $i \in I_m$, we consider two cases:

*Case 1:* Consider $i \in I^<$. The semi-strict quasi-convexity of $f_i$ on $Y$ implies $x^\lambda \in L_<(Y, f_i, f_i(x^0))$ for all $\lambda \in ]0, 1]$ by Lemma 1.42. Because of $x^\lambda \in X$ for all $\lambda \in ]0, \lambda^*]$, we get $x^\lambda \in L_<(X, f_i, f_i(x^0))$ for all $\lambda \in ]0, \lambda^*]$.

*Case 2:* Consider $i \in I^{=}$. This means that $f_i(x^1) = f_i(x^0)$. By Lemma 1.43 (applied for $\Omega := Y$), we infer

$$\operatorname{card} L_{>}(]x^0, x^1[, f_i, f_i(x^0)) \leq 1.$$

In the case that $\operatorname{card} L_{>}(]x^0, x^1[, f_i, f_i(x^0)) = 1$, we get that there exists $\lambda_i \in ]0,1[$ such that $f_i(l_{x^0,x^1}(\lambda_i)) > f_i(x^0)$. Otherwise, we define $\lambda_i := 2\lambda^*$.

For $\overline{\lambda} := \min\{\lambda^*, 0.5 \cdot \min\{\lambda_i \mid i \in I^{=}\}\}$, it follows that $x^{\overline{\lambda}} \in L_{\leq}(X, f_i, f_i(x^0))$ for all $i \in I^{=}$ as well as $x^{\overline{\lambda}} \in L_{<}(X, f_i, f_i(x^0))$ for all $i \in I^{<}$. So, we get $x^0 \in (\operatorname{cor} X) \setminus \operatorname{Eff}(X \mid f)$ by Lemma 1.69.

The proof of the second inclusion is analogous to the proof of the first inclusion in 2°. Notice that $I^{<} = I_m$ and $I^{=} = \emptyset$.

3°**.** Consider $x^0 \in (\operatorname{cor} X) \setminus \operatorname{SEff}(Y \mid f)$. Since $x^0 \notin \operatorname{SEff}(Y \mid f)$, it exists $x^1 \in Y \setminus \{x^0\}$ such that $x^1 \in S_{\leq}(Y, f, x^0)$. Of course, since $x^1 \in X$, we get $x^0 \in (\operatorname{cor} X) \setminus \operatorname{SEff}(X \mid f)$. Now, assume that $x^1 \in Y \setminus X$. Analogously to the proof of assertion 2° in this lemma, there exists $\lambda^* \in ]0,1[$ such that $x^\lambda := l_{x^0,x^1}(\lambda) \in X \cap ]x^0, x^1[$ for all $\lambda \in ]0, \lambda^*]$.

Let $i \in I_m$ and consider two cases:

*Case 1:* Let $f_i$ be semi-strictly quasi-convex on $Y$. Similar to the proof of assertion 2° of this lemma, there exists $\lambda_i \in ]0, \lambda^*]$ with $x^\lambda \in L_{\leq}(X, f_i, f_i(x^0))$ for all $\lambda \in ]0, \lambda_i]$.

*Case 2:* Let $f_i$ be quasi-convex on $Y$. By the convexity of the level sets of $f_i$, we conclude $[x^0, x^1] \subseteq L_{\leq}(Y, f_i, f_i(x^0))$ for $x^0, x^1 \in L_{\leq}(Y, f_i, f_i(x^0))$ by Lemma 1.41. We put $\lambda_i := \lambda^*$.

Hence, for $\overline{\lambda} := \min\{\lambda_i \mid i \in I_m\}$, it follows that $x^{\overline{\lambda}} \in S_{\leq}(X, f, x^0) \setminus \{x^0\}$. Finally, we get $x^0 \in (\operatorname{cor} X) \setminus \operatorname{SEff}(X \mid f)$ by Lemma 1.69.

□

**Remark 2.3** Assertions 2° and 3° in Lemma 2.2 also holds for the case that we remove the convexity assumption on the set $Y$ but, in addition, we assume that the set $\mathcal{D}$ is convex and $f$ is componentwise generalized-convex on $\mathcal{D}$.

**Remark 2.4** The inclusion

$$(\operatorname{int} X) \setminus \operatorname{WEff}(\mathbf{E} \mid f) \subseteq (\operatorname{int} X) \setminus \operatorname{WEff}(X \mid f) \tag{2.4}$$

(compare formula (2.3)) was already discussed in the proof of Theorem 3.1 in the paper by Puerto and Rodríguez-Chía [110] for the particular case that $Y = \mathbf{E} = \mathbb{R}^2$ and $X$ is a nonempty, closed convex set in $\mathbb{R}^2$. One argument in the proof of this theorem is the convexity of level sets. It is important to mention that the inclusion given by (2.4) may fails for some componentwise quasi-convex functions (see Example 2.5). Puerto and Rodríguez-Chía [110] supposed that the objective function $f$ is componentwise strictly quasi-convex on $\mathbb{R}^2$. However, it is well-known that strictly quasi-convex functions (or convex functions) are semi-strictly quasi-convex too (see Remark 1.36). Therefore, the inclusion (2.4) is true for the model discussed by Puerto and Rodríguez-Chía [110].

The semi-strict quasi-convexity assumption with respect to $f$ can not be replaced by a quasi-convexity assumption in 2° of Lemma 2.2, as shown in the next example.

**Example 2.5** Let $f_i : \mathbb{R}^2 \to \mathbb{R}$, for any $i \in I_2$, be defined by

$$f_i(x) := \begin{cases} 0 & \text{for } ||x - (i-1, 0)||_\infty < 1, \\ 1 & \text{for } ||x - (i-1, 0)||_\infty \geq 1 \end{cases}$$

for all $x \in \mathbb{R}^2$. Then, $f := (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}$ is componentwise quasi-convex on $\mathbb{R}^2$, since the set $L_{\leq}(\mathbb{R}^2, f_i, s)$ is convex for all $s \in \mathbb{R}$ and all $i \in I_2$. However, notice that $f$ is not componentwise semi-strictly quasi-convex on $\mathbb{R}^2$. Now, consider $x' := (0.5, 3) \in \operatorname{int} X$, and define $X := \{x \in \mathbb{R}^2 \mid ||x - x'||_\infty \leq 1\}$. By the definitions of $f$ and $X$, we get $\operatorname{Eff}(X \mid f) = \operatorname{WEff}(X \mid f) = X$. For the point $x'' := (0.5, 0) \notin X$, we have $f_i(x') = 1 > 0 = f_i(x'')$ for all $i \in I_2$. Consequently, it follows $x'' \in S_<(\mathbb{R}^2, f, x')$, but $S_<(\mathbb{R}^2, f, x') \cap X = \emptyset$. This means that $x' \in (\operatorname{int} X) \setminus \operatorname{WEff}(\mathbb{R}^2 \mid f)$, hence $x' \in (\operatorname{int} X) \setminus \operatorname{Eff}(\mathbb{R}^2 \mid f)$, but $x' \in \operatorname{WEff}(X \mid f) = \operatorname{Eff}(X \mid f)$. So, we can not replace the semi-strict quasi-convexity assumption with respect to $f$ by a quasi-convexity assumption in 2° of Lemma 2.2. The above example is illustrated in Figure 2.1.

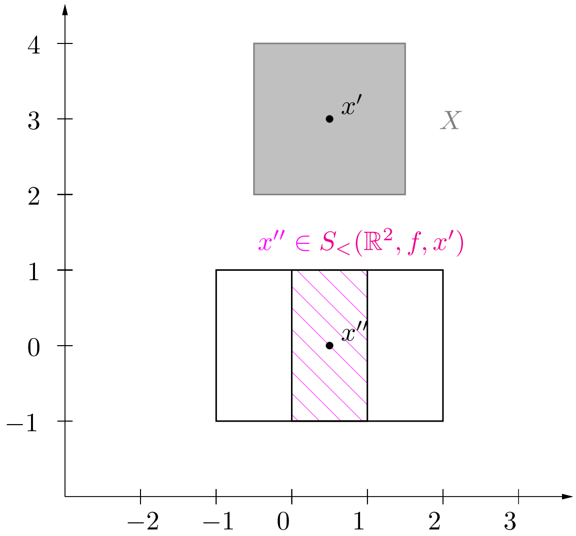


Figure 2.1: Counter-example for the validity of (2.2) and (2.3) for a quasi-convex function.

The following corollary gives useful bounds for the sets of (strictly, weakly) Pareto efficient solutions of the problem $(\mathcal{P}_X)$ under generalized-convexity assumption on $f$ with respect to the set $Y$ but without convexity assumption on the feasible set $X$.

**Corollary 2.6** ([55, 56]) Assume that (2.1) is satisfied. Then, the following assertions hold:

**1°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$, then

$$\begin{aligned} X \cap \operatorname{Eff}(Y \mid f) &\subseteq \operatorname{Eff}(X \mid f) \subseteq [X \cap \operatorname{Eff}(Y \mid f)] \cup \operatorname{bd} X; \\ X \cap \operatorname{WEff}(Y \mid f) &\subseteq \operatorname{WEff}(X \mid f) \subseteq [X \cap \operatorname{WEff}(Y \mid f)] \cup \operatorname{bd} X. \end{aligned}$$

**2°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$, then

$$X \cap \operatorname{SEff}(Y \mid f) \subseteq \operatorname{SEff}(X \mid f) \subseteq [X \cap \operatorname{SEff}(Y \mid f)] \cup \operatorname{bd} X.$$

**Corollary 2.7** ([56]) Assume that (2.1) is satisfied. In addition, suppose that the set $X$ is open. Then, the following assertions hold:

**1°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$, then

$$\begin{aligned} X \cap \operatorname{Eff}(Y \mid f) &= \operatorname{Eff}(X \mid f); \\ X \cap \operatorname{WEff}(Y \mid f) &= \operatorname{WEff}(X \mid f). \end{aligned}$$

**2°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$, then

$$X \cap \operatorname{SEff}(Y \mid f) = \operatorname{SEff}(X \mid f).$$

## 2.2 The penalized multi-objective optimization problem ($\mathcal{P}_Y^{\oplus}$) with convex feasible set $Y$

In what follows, we suppose that the following standard assumptions are fulfilled:

$$\begin{cases} \text{Let (2.1) be satisfied;} \\ \text{let } X \text{ be closed.} \end{cases} \tag{2.5}$$

**Remark 2.8** Notice, under the assumptions given in (2.5), we have $\operatorname{bd} X \neq \emptyset$ (see Lemma 1.11). For the case $\operatorname{bd} X = \emptyset$ (hence $X$ is open), we refer to Corollary 2.7.

In our approach, under the validity of the assumption (2.5), we add a real-valued *penalization function* $\phi : \mathcal{D} \to \mathbb{R}$ to the original objective function $f = (f_1, \cdots, f_m) : \mathcal{D} \to \mathbb{R}^m$ of the problem ($\mathcal{P}_Y$) as a new component function $f_{m+1} := \phi$. So, the new penalized multi-objective optimization problem can be stated as

$$\begin{cases} f^{\oplus}(x) := (f_1(x), \cdots, f_m(x), \phi(x)) \to \min \text{ w.r.t. } \mathbb{R}^{m+1}_+ \\ x \in Y. \end{cases} \tag{$\mathcal{P}_Y^{\oplus}$}$$

In the sequel, we will need in certain results some of the following assumptions concerning the lower-level sets / level lines of the penalization function $\phi$:

$$\forall x' \in \operatorname{bd} X : \ L_{\leq}(Y, \phi, \phi(x')) = X, \tag{A1}$$
$$\forall x' \in \operatorname{bd} X : \ L_{=}(Y, \phi, \phi(x')) = \operatorname{bd} X, \tag{A2}$$
$$\forall x' \in X : \ L_{=}(Y, \phi, \phi(x')) = L_{\leq}(Y, \phi, \phi(x')) = X, \tag{A3}$$
$$\forall x' \in X : \ L_{\leq}(Y, \phi, \phi(x')) \subseteq X, \tag{A4}$$
$$L_{\leq}(Y, \phi, 0) = X, \tag{A5}$$
$$L_{=}(Y, \phi, 0) = \operatorname{bd} X. \tag{A6}$$

In particular, Assumptions (A1) and (A2) (respectively, Assumption (A3)) will play a key role for proving our main results within this chapter.

**Remark 2.9** Notice, under both Assumptions (A1) and (A2), we have

$$\forall x' \in \operatorname{bd} X : \ L_{<}(Y, \phi, \phi(x')) = \operatorname{int} X,$$

while under Assumption (A3) it follows

$$\forall x' \in X : \ L_{<}(Y, \phi, \phi(x')) = \emptyset.$$

In the next two lemmata, we present some preliminary results related to the validity of the above assumptions.

**Lemma 2.10** Let (2.5) be satisfied. Then, we have:

1°. (A3) $\Longrightarrow$ (A1).

2°. If $\operatorname{int} X = \emptyset$, then (A1) $\wedge$ (A2) $\Longleftrightarrow$ (A3).

3°. (A5) $\wedge$ (A6) $\Longrightarrow$ (A1) $\wedge$ (A2).

4°. (A1) $\vee$ (A3) $\vee$ (A5) $\Longrightarrow$ (A4).

5°. $\phi$ fulfils (A1) and (A2) if and only if $\widehat{\phi} := h \circ \phi : \mathcal{D} \to \mathbb{R}$ fulfils (A1) and (A2) (with $\widehat{\phi}$ in the role of $\phi$), where $h : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function on the image set $\phi[\mathcal{D}]$.

6°. $\phi$ fulfils (A1) and (A2) if and only if $\widehat{\phi} := \phi - \phi(x')$, $x' \in \operatorname{bd} X$, fulfils (A1), (A2), (A5) and (A6) (with $\widehat{\phi}$ in the role of $\phi$).

We omit the proof of Lemma 2.10 since the assertions can easily be verified.

**Lemma 2.11** Let (2.5) be satisfied. Assume that $\phi : \mathcal{D} \to \mathbb{R}$ satisfies the Assumption $(A*) \in \{(A1), (A2), \cdots, (A6)\}$ with $\mathcal{D}$ in the role of $Y$. Then, $\phi$ satisfies the Assumption $(A*)$ for the set $Y \subseteq \mathcal{D}$ with $X \subseteq Y$.

*Proof.* Due to

$$L_{\sim}(Y, \phi, s) = L_{\sim}(\mathcal{D}, \phi, s) \cap Y$$

for all $\sim \in \{<, \leq, =\}$ and all $s \in \mathbb{R}$, and $\operatorname{bd} X \subseteq X \subseteq Y \subseteq \mathcal{D}$, the assertion follows immediately. $\square$

## 2.3 Examples for the penalization function $\phi$

In this section, we will present some examples for the penalization function that can be used in our vectorial penalization approach. Throughout this section, we suppose that $\phi$ is defined on the whole space, i.e., $\mathcal{D} = \mathbf{E}$.

**Example 2.12** Let $X \subseteq \mathbf{E}$ be a closed, convex set with $x' \in \operatorname{int} X$ and $X \neq \mathbf{E}$. Let a *Minkowski gauge* $\mu := \mu_B : \mathbf{E} \to \mathbb{R}$ with corresponding unit ball $B := -x' + X$ be given. Recall, in view of Section 1.4, we have

$$\mu(x) = \mu_B(x) = \inf\{\lambda \in \mathbb{R}_+ \mid x \in \lambda \cdot B\} \quad \text{for all } x \in \mathbf{E}.$$

Then, the function

$$\phi(\,\cdot\,) := \mu(\,\cdot\, - x')$$

fulfils Assumptions (A1) and (A2) for $Y = \mathbf{E}$. Indeed, the function

$$\widehat{\phi}(\,\cdot\,) := \mu(\,\cdot\, - x') - 1$$

fulfils (A5) and (A6) by Corollary 1.63, hence $\phi$ satisfies (A1) and (A2) by Lemma 2.10 (3°, 6°).

**Example 2.13** Let $X$ be a nonempty, closed set in a normed space $(\mathbf{E}, ||\cdot||)$ and let the *distance function with respect to* $X$, namely $d_X : \mathbf{E} \to \mathbb{R}$, be given by

$$d_X(x) := \inf\{||x - x'|| \mid x' \in X\} \quad \text{for all } x \in \mathbf{E}.$$

We recall some important properties of $d_X$ (see, e.g., Mordukhovich and Nam [87], and references therein):

- $d_X$ is Lipschitz continuous on $\mathbf{E}$ of rank 1;
- $d_X$ is convex on $\mathbf{E}$ if and only if $X$ is convex in $\mathbf{E}$;
- $L_{\leq}(\mathbf{E}, d_X, 0) = L_{=}(\mathbf{E}, d_X, 0) = X$.

Hence, the penalization function

$$\phi := d_X$$

fulfils Assumptions (A3) and (A5) for $Y = \mathbf{E}$.

**Example 2.14** Let $X$ be a nonempty, closed set in a normed space $(\mathbf{E}, ||\cdot||)$ with $X \neq \mathbf{E}$. Based on the distance function $d_X : \mathbf{E} \to \mathbb{R}$ (see Example 2.13), we consider a function $\triangle_X : \mathbf{E} \to \mathbb{R}$ that is defined by

$$\triangle_X(x) := d_X(x) - d_{\mathbf{E}\setminus X}(x) = \begin{cases} d_X(x) & \text{for } x \in \mathbf{E} \setminus X, \\ -d_{\mathbf{E}\setminus X}(x) & \text{for } x \in X. \end{cases}$$

The function $\triangle_X$ was introduced by Hiriart-Urruty [62] and is known in the literature as *signed distance function* or *Hiriart-Urruty function*. Next, we recall some well-known properties of $\triangle_X$ (see Hiriart-Urruty [62], Liu, Ng and Yang [75], and Zaffaroni [130]):

- $\triangle_X$ is Lipschitz continuous on $\mathbf{E}$ of rank 1;
- $\triangle_X$ is convex on $\mathbf{E}$ if and only if $X$ is convex in $\mathbf{E}$;
- $L_{\leq}(\mathbf{E}, \triangle_X, 0) = X$ and $L_{=}(\mathbf{E}, \triangle_X, 0) = \operatorname{bd} X$.

It follows that the penalization function

$$\phi := \triangle_X$$

fulfils Assumptions (A1), (A2), (A5) and (A6) for $Y = \mathbf{E}$.

**Example 2.15** In this example, we consider a nonlinear function introduced as a scalarizing tool (the so-called *Gerstewitz function* or *Tammer-Weidner function*) in multi-objective optimization by Gerstewitz [45] (see also Gerstewitz and Iwanow [46], and Gerth and Weidner [48]). Let $(\mathbf{E}, ||\cdot||)$ be a normed space. Assume that $C \subseteq \mathbf{E}$ is a nontrivial, closed, convex cone, $k \in \operatorname{int} C$, and $X \subsetneq \mathbf{E}$ is a nonempty, closed set such that

$$X - (C \setminus \{0\}) = \operatorname{int} X.$$

The function $\varphi_{X,k} : \mathbf{E} \to \mathbb{R}$ defined, for any $x \in \mathbf{E}$, by

$$\varphi_{X,k}(x) := \inf\{s \in \mathbb{R} \,|\, x \in sk + X\}$$

is finite-valued and fulfils the important properties (see Kahn, Tammer and Zălinescu [68, Sec. 5.2]):

- $\varphi_{X,k}$ is Lipschitz continuous on $\mathbf{E}$;
- $\varphi_{X,k}$ is convex on $\mathbf{E}$ if and only if $X$ is convex in $\mathbf{E}$;
- $L_{\leq}(\mathbf{E}, \varphi_{X,k}, 0) = X$ and $L_{=}(\mathbf{E}, \varphi_{X,k}, 0) = \operatorname{bd} X$.

This means that the penalization function

$$\phi := \varphi_{X,k}$$

fulfils Assumptions (A1), (A2), (A5) and (A6) for $Y = \mathbf{E}$.

**Remark 2.16** Examples 2.13, 2.14 and 2.15 show that a nonconvex set $X$ can be considered in our approach. Let $X$ be an arbitrarily closed set with $\emptyset \neq X \neq \mathbf{E}$. In any normed space $(\mathbf{E}, ||\cdot||)$, we know that the *Hiriart-Urruty function* $\triangle_X$ fulfils Assumptions (A1) and (A2), and moreover, the *distance function* $d_X$ *with respect to* $X$ fulfils Assumption (A3).

## 2.4 Relationships between the multi-objective optimization problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

In this section, under the assumptions given in (2.5), we study the relationships between the initial problem $(\mathcal{P}_X)$ with not necessarily convex feasible set $X$ and two corresponding problems $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$ with convex feasible set $Y$.

### 2.4.1 Sets of Pareto efficient solutions of $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

Next, we present relationships between the sets of Pareto efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$. A first main result of this thesis is given in the next theorem where the penalization function $\phi$ satisfies Assumptions (A1) and (A2).

**Theorem 2.17** ([55, 56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumptions (A1) and (A2). Then, the following assertions hold:

**1°.** We have

$$[X \cap \mathrm{Eff}(Y \mid f)] \cup \big[(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})\big] \subseteq \mathrm{Eff}(X \mid f).$$

**2°.** In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{Eff}(Y \mid f)] \cup \big[(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})\big] \supseteq \mathrm{Eff}(X \mid f).$$

*Proof.* **1°.** The inclusion $X \cap \mathrm{Eff}(Y \mid f) \subseteq \mathrm{Eff}(X \mid f)$ follows by Lemma 2.2. Consider $x \in (\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})$. By Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$) and Assumptions (A1) and (A2), it follows

$$\begin{aligned} S_{\leq}(X, f, x) &= S_{\leq}(Y, f, x) \cap X \\ &= S_{\leq}(Y, f, x) \cap L_{\leq}(Y, \phi, \phi(x)) \\ &\subseteq S_{=}(Y, f, x) \cap L_{=}(Y, \phi, \phi(x)) \\ &= S_{=}(Y, f, x) \cap \mathrm{bd}\, X \\ &\subseteq S_{=}(Y, f, x) \cap X \\ &= S_{=}(X, f, x). \end{aligned}$$

Hence, we get $x \in \mathrm{Eff}(X \mid f)$ by Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$).

**2°.** Let $x \in \mathrm{Eff}(X \mid f) \subseteq X$. In the case $x \in X \cap \mathrm{Eff}(Y \mid f)$, the inclusion holds. So, we consider the case $x \in X \setminus \mathrm{Eff}(Y \mid f)$. If $\mathrm{int}\, X = \emptyset$, then clearly we have $x \in \mathrm{bd}\, X$. In the case $\mathrm{int}\, X \neq \emptyset$, we get $x \in \mathrm{bd}\, X$ from Corollary 2.6, taking into account the componentwise semi-strictly quasi-convexity of $f$ on $Y$. So, we have $x \in (\mathrm{bd}\, X) \setminus \mathrm{Eff}(Y \mid f)$. By Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$) and Assumption (A1), we can conclude

$$\begin{aligned} S_{\leq}(Y, f, x) \cap L_{\leq}(Y, \phi, \phi(x)) &= S_{\leq}(Y, f, x) \cap X \\ &= S_{\leq}(X, f, x) \\ &\subseteq S_{=}(X, f, x) \\ &= S_{=}(Y, f, x) \cap X. \end{aligned}$$

Now, we will prove the equation

$$S_{=}(Y, f, x) \cap X = S_{=}(Y, f, x) \cap \mathrm{bd}\, X. \tag{2.6}$$

In the case that $\mathrm{int}\, X = \emptyset$, (2.6) is obviously fulfilled. For the case $\mathrm{int}\, X \neq \emptyset$, it is sufficient to prove $S_{=}(Y, f, x) \cap \mathrm{int}\, X = \emptyset$ in order to get the validity of (2.6). Indeed, if we suppose that there exists $x' \in \mathrm{int}\, X$ with $x' \in S_{=}(Y, f, x)$, then we have to discuss following two cases:

*Case 1:* If $x' \in (\mathrm{int}\, X) \setminus \mathrm{Eff}(Y \mid f)$, then we get $x' \in (\mathrm{int}\, X) \setminus \mathrm{Eff}(X \mid f)$ by Lemma 2.2 under the assumption that $f$ is componentwise semi-strictly quasi-convex on $Y$. This implies $x \in X \setminus \mathrm{Eff}(X \mid f)$ because of $x' \in S_{=}(X, f, x)$, a contradiction to $x \in \mathrm{Eff}(X \mid f)$.

*Case 2:* If $x' \in \mathrm{Eff}(Y \mid f)$, then we get $x \in \mathrm{Eff}(Y \mid f)$ by $x' \in S_{=}(Y, f, x)$. This is a contradiction to $x \in X \setminus \mathrm{Eff}(Y \mid f)$.

This means that (2.6) holds.

Furthermore, since $x \in \mathrm{bd}\, X$ and (A2) holds, we have

$$S_{=}(Y, f, x) \cap \mathrm{bd}\, X = S_{=}(Y, f, x) \cap L_{=}(Y, \phi, \phi(x)).$$

From Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$), we conclude $x \in \mathrm{Eff}(Y \mid f^{\oplus})$. This means that $x \in (\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})$, which completes the proof of assertion 2°.

$\square$

**Corollary 2.18** Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumptions (A1) and (A2). In the case $\operatorname{int} X \neq \emptyset$, suppose additionally that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \operatorname{Eff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus})\right] = \operatorname{Eff}(X \mid f).$$

**Corollary 2.19** Assuming that the assumptions of Corollary 2.28 as well as the condition $X \cap \operatorname{Eff}(Y \mid f) = \emptyset$ are fulfilled, we have

$$\operatorname{Eff}(X \mid f) = (\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus}) \subseteq \operatorname{bd} X.$$

**Remark 2.20** The semi-strict quasi-convexity assumption with respect to $f$ in 2° of Theorem 2.17 can not be replaced by a quasi-convexity assumption (see Example 2.5 and the Example 2.30 in the next section). Moreover, the following both inclusions

$$\operatorname{Eff}(X \mid f) \subseteq X \cap \operatorname{Eff}(Y \mid f^{\oplus}); \tag{2.7}$$
$$(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f) \subseteq (\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus}) \tag{2.8}$$

do not hold under the assumptions supposed in Theorem 2.17 in general, as shown in the next Example 2.21.

**Example 2.21** For three given points in the plane $a^1 := (5,5), a^2 := (2,2.5), a^3 := (3.5,3.5) \in \mathbb{R}^2$, we consider a constrained convex multi-objective location problem with objective function $f = (f_1, f_2, f_3) : \mathbb{R}^2 \to \mathbb{R}^3$, where $f_i(x) := ||x - a^i||_1$ for all $x \in \mathbb{R}^2$ and all $i \in I_3$, and a feasible set $X := [2,5] \times [2.5,5]$. Let the penalization function be given by $\phi(\,\cdot\,) := \mu(\,\cdot\, - a^3)$ with $\mu = \mu_B$, where $B := -a^3 + X$ (see Example 2.12). It is easily seen that all assumptions of Corollary 2.18 are fulfilled in this example. The left part of Figure 2.2 illustrates that the given point $x' \in \operatorname{bd} X$ is contained in both sets $\operatorname{Eff}(X \mid f)$ and $\operatorname{Eff}(\mathbb{R}^2 \mid f)$. Notice that

$$\operatorname{Eff}(X \mid f) = \operatorname{Eff}(\mathbb{R}^2 \mid f) = ([2,3.5] \times [2.5,3.5]) \cup ([3.5,5] \times [3.5,5]),$$

as a result of our *Rectangular Decomposition Algorithm* formulated in Chapter 5. However, the right part of Figure 2.2 shows that $x' \notin \operatorname{Eff}(\mathbb{R}^2 \mid f^{\oplus})$, because $x'' \in (\operatorname{int} X) \cap S_{=}(\mathbb{R}^2, f, x')$. Hence, the inclusions (2.7) and (2.8) do not hold under the assumptions given in Corollary 2.18.

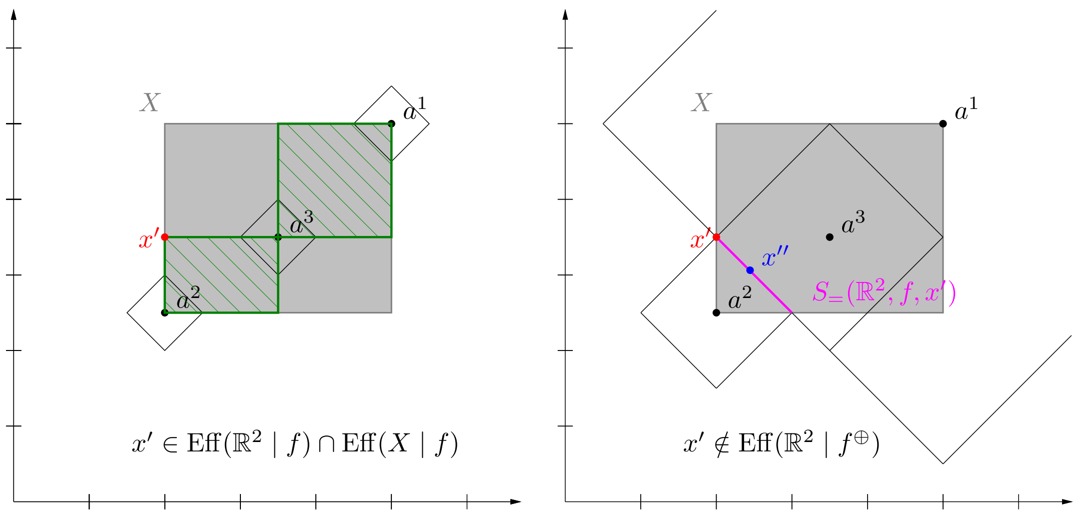


Figure 2.2: Counter-example for the inclusions (2.7) and (2.8) given in Remark 2.20

**Remark 2.22** Since the inclusion (2.8) is not true in general, we can not deduce the equality

$$[(\operatorname{int} X) \cap \operatorname{Eff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^\oplus)\right] = \operatorname{Eff}(X \mid f) \tag{2.9}$$

directly by applying Corollary 2.18. Notice that this is possible for the other solution concepts discussed in this thesis (see Remarks 2.31 and 2.44). Indeed, Example 2.21 shows that under the assumptions of Corollary 2.18 the equality in (2.9) does not hold (since $x' \in (\operatorname{bd} X) \cap \operatorname{Eff}(X \mid f)$ but $x' \notin \operatorname{Eff}(\mathbb{R}^2 \mid f^\oplus)$). However, since $\operatorname{int} X \subseteq X$ and because of 1° in Theorem 2.17, the inclusion "$\subseteq$" in (2.9) is true actually without assuming that $f$ is a generalized convex function on $Y$.

Under the assumptions of Corollary 2.18, it is possible to check whether a point $x \in X$ is efficient for the constrained problem $(\mathcal{P}_X)$ or not by using two problems with convex feasible set:

$$\begin{aligned}
x \in \operatorname{int} X: &\qquad x \in \operatorname{Eff}(Y \mid f) \iff x \in \operatorname{Eff}(X \mid f);\\
x \in \operatorname{bd} X: &\qquad x \in \operatorname{Eff}(Y \mid f) \vee x \in \operatorname{Eff}(Y \mid f^\oplus) \iff x \in \operatorname{Eff}(X \mid f).
\end{aligned}$$

In the next lemma, we present sufficient conditions for the fact that a solution $x \in \operatorname{Eff}(X \mid f)$ is belonging to $\operatorname{Eff}(Y \mid f^\oplus)$.

**Lemma 2.23** ([55, 56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A4). If $x \in \operatorname{Eff}(X \mid f)$ and $S_=(X, f, x) \subseteq L_=(Y, \phi, \phi(x))$, then $x \in X \cap \operatorname{Eff}(Y \mid f^\oplus)$.

*Proof.* Consider $x \in \operatorname{Eff}(X \mid f)$. The following assertions are equivalent:

- $S_=(X, f, x) \subseteq L_=(Y, \phi, \phi(x))$.
- $S_=(Y, f, x) \cap X \subseteq L_=(Y, \phi, \phi(x))$.
- $S_=(Y, f, x) \cap X \subseteq S_=(Y, f, x) \cap L_=(Y, \phi, \phi(x))$.

Hence, by Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$) and by Assumption (A4), it follows

$$\begin{aligned}
S_\leq(Y, f, x) \cap L_\leq(Y, \phi, \phi(x)) &\subseteq S_\leq(Y, f, x) \cap X\\
&= S_\leq(X, f, x)\\
&\subseteq S_=(X, f, x)\\
&= S_=(Y, f, x) \cap X\\
&\subseteq S_=(Y, f, x) \cap L_=(Y, \phi, \phi(x)).
\end{aligned}$$

Due to Lemma 1.69 (applied for $(\mathcal{P}_Y^\oplus)$ instead of $(\mathcal{P}_\Omega)$), we infer $x \in X \cap \operatorname{Eff}(Y \mid f^\oplus)$. ☐

In the next theorem, we present a second main result that holds under the assumption that the penalization function $\phi$ fulfils Assumption (A3).

**Theorem 2.24** ([56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A3). Then, the following assertions are true:

**1°.** It holds that

$$[X \cap \operatorname{Eff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^\oplus)\right] \subseteq \operatorname{Eff}(X \mid f) = X \cap \operatorname{Eff}(Y \mid f^\oplus).$$

**2°.** In the case $\operatorname{int} X \neq \emptyset$, suppose additionally that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \operatorname{Eff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^\oplus)\right] \supseteq \operatorname{Eff}(X \mid f).$$

*Proof.* 1°**.** We are going to show $\mathrm{Eff}(X \mid f) = X \cap \mathrm{Eff}(Y \mid f^{\oplus})$.

Let us prove the inclusion "$\supseteq$". Consider $x \in X \cap \mathrm{Eff}(Y \mid f^{\oplus})$. By Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$) and Assumption (A3), it follows

$$\begin{aligned} S_{\leq}(X, f, x) &= S_{\leq}(Y, f, x) \cap X \\ &= S_{\leq}(Y, f, x) \cap L_{\leq}(Y, \phi, \phi(x)) \\ &\subseteq S_{=}(Y, f, x) \cap L_{=}(Y, \phi, \phi(x)) \\ &= S_{=}(Y, f, x) \cap X \\ &= S_{=}(X, f, x). \end{aligned}$$

By Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$), we get $x \in \mathrm{Eff}(X \mid f)$.

Now, we prove the reverse inclusion "$\subseteq$". Let $x \in \mathrm{Eff}(X \mid f)$. Due to

$$\begin{aligned} S_{=}(X, f, x) &= S_{=}(Y, f, x) \cap X \\ &= S_{=}(Y, f, x) \cap L_{=}(Y, \phi, \phi(x)) \\ &\subseteq L_{=}(Y, \phi, \phi(x)), \end{aligned}$$

it follows $x \in X \cap \mathrm{Eff}(Y \mid f^{\oplus})$ by Lemma 2.23. Notice that (A3) implies (A4) by Lemma 2.10 (4°).

Moreover, the inclusion $X \cap \mathrm{Eff}(Y \mid f) \subseteq \mathrm{Eff}(X \mid f)$ follows by Lemma 2.2, while the second inclusion $(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus}) \subseteq \mathrm{Eff}(X \mid f)$ is a direct consequence of the equality $\mathrm{Eff}(X \mid f) = X \cap \mathrm{Eff}(Y \mid f^{\oplus})$ taking into account the closedness of $X$.

2°**.** The proof is analogous to the first part of the proof of 2° in Theorem 2.17. Notice, due to Assumption (A3), for any $x \in \mathrm{bd}\, X$, we have

$$S_{=}(Y, f, x) \cap X = S_{=}(Y, f, x) \cap L_{=}(Y, \phi, \phi(x)).$$

□

### 2.4.2 Sets of weakly Pareto efficient solutions of $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

In the first part of this section, we present some relationships between the sets of weakly Pareto efficient solutions of the multi-objective optimization problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$. The second part of this section is devoted to the study of the concept of *Pareto reducibility* for multi-objective optimizations problems that was introduced by Popovici [102, Def. 1].

#### 2.4.2.1 Relationships between the sets of solutions

Some first relationships between the sets of weakly Pareto efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$ are given in the next theorem.

**Theorem 2.25** ([55, 56]) Let (2.5) and Assumption (A4) be satisfied. Then, we have

$$X \cap \mathrm{WEff}(Y \mid f) \subseteq \mathrm{WEff}(X \mid f) \subseteq X \cap \mathrm{WEff}(Y \mid f^{\oplus}).$$

*Proof.* In view of Corollary 2.2, it follows $X \cap \mathrm{WEff}(Y \mid f) \subseteq \mathrm{WEff}(X \mid f)$. Now, let us prove the second inclusion $\mathrm{WEff}(X \mid f) \subseteq X \cap \mathrm{WEff}(Y \mid f^{\oplus})$.

Consider $x \in \mathrm{WEff}(X \mid f) \subseteq X$. By Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$) and by Assumption (A4), we get

$$\begin{aligned} \emptyset &= S_{<}(X, f, x) \\ &= S_{<}(Y, f, x) \cap X \\ &\supseteq S_{<}(Y, f, x) \cap L_{\leq}(Y, \phi, \phi(x)) \\ &\supseteq S_{<}(Y, f, x) \cap L_{<}(Y, \phi, \phi(x)). \end{aligned}$$

In view of Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$), it follows $x \in X \cap \mathrm{WEff}(Y \mid f^{\oplus})$. □

**Remark 2.26** Let (2.5) be satisfied and assume that $\operatorname{int} X \neq \emptyset$. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex on $Y$ and fulfils both Assumptions (A1) and (A2). Then, it follows

$$\forall\, x' \in \operatorname{bd} X \,\forall\, x'' \in \operatorname{int} X :\; [x'', x'[ \subseteq L_{<}(Y, \phi, \phi(x^1))$$

by Lemma 1.42. In particular, the function $\phi$ satisfies

$$\forall\, x' \in \operatorname{bd} X \,\exists\, x'' \in \operatorname{int} X :\; [x'', x'[ \subseteq L_{<}(Y, \phi, \phi(x')). \tag{A7}$$

Taking into account Remark 2.9, for any $x, x' \in \operatorname{bd} X$, we have

$$L_{<}(Y, \phi, \phi(x)) = L_{<}(Y, \phi, \phi(x')) = \operatorname{int} X.$$

Notice that $\phi$ fulfils (A7) if and only if $\widehat{\phi} := h \circ \phi : \mathcal{D} \to \mathbb{R}$ fulfils (A7) (with $\widehat{\phi}$ in the role of $\phi$), where $h : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function on the image set $\phi[\mathcal{D}]$.

The result given in Theorem 2.27 presents important relationships between the sets of weakly Pareto efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$.

**Theorem 2.27** ([55, 56]) Let (2.5) be satisfied. The following assertions are true:

$1^\circ$**.** Assume that $\operatorname{int} X \neq \emptyset$. Let $f : \mathcal{D} \to \mathbb{R}^m$ be componentwise upper semi-continuous along line segments on $Y$. Furthermore, we suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumptions (A1), (A2) and (A7). Then, we have

$$[X \cap \operatorname{WEff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{WEff}(Y \mid f^{\oplus})\right] \subseteq \operatorname{WEff}(X \mid f).$$

$2^\circ$**.** Let Assumption (A4) be fulfilled. In the case $\operatorname{int} X \neq \emptyset$, suppose additionally that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \operatorname{WEff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{WEff}(Y \mid f^{\oplus})\right] \supseteq \operatorname{WEff}(X \mid f).$$

*Proof.* Consider $i \in I_m$. Notice, in view Definition 1.31 and Lemma 1.45, the following assertions are equivalent:

- $f_i$ is upper semi-continuous on line segments in $Y$.
- $L_{\geq}\left([0,1], (f_i \circ l_{a,b}), s\right)$ is closed for all $s \in \mathbb{R}$ and all $a, b \in Y$.
- $L_{<}\left([0,1], (f_i \circ l_{a,b}), s\right) \cup (\mathbb{R} \setminus [0,1])$ is open for all $s \in \mathbb{R}$ and all $a, b \in Y$.

Now, we are going to prove both assertions $1^\circ$ and $2^\circ$:

$1^\circ$**.** In view of Corollary 2.2, it follows $X \cap \operatorname{WEff}(Y \mid f) \subseteq \operatorname{WEff}(X \mid f)$. Now, let us consider $x \in (\operatorname{bd} X) \cap \operatorname{WEff}(Y \mid f^{\oplus})$. By Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$), it follows

$$\emptyset = S_{<}(Y, f, x) \cap L_{<}(Y, \phi, \phi(x)). \tag{2.10}$$

We are going to prove that

$$S_{<}(Y, f, x) \cap L_{<}(Y, \phi, \phi(x)) = S_{<}(Y, f, x) \cap X. \tag{2.11}$$

Then, due to (2.10) and (2.11), we get $S_{<}(X, f, x) = \emptyset$, hence $x \in \operatorname{WEff}(X \mid f)$ by Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$).

By Assumption (A1), the inclusion "$\subseteq$" in (2.11) follows directly.

Let us prove the reverse inclusion "$\supseteq$" in (2.11). Assume the contrary holds, i.e., it exists $x' \in S_{<}(Y, f, x) \cap X$ such that $x' \notin L_{<}(Y, \phi, \phi(x))$. So, we have $x' \in L_{=}(Y, \phi, \phi(x)) = \operatorname{bd} X$ by Assumption (A1). By Assumption (A7) (see Remark 2.26), for the given points $x, x' \in \operatorname{bd} X$, there exists $x'' \in \operatorname{int} X$ such that

$$l_{x', x''}(\lambda) \in L_{<}(Y, \phi, \phi(x')) = L_{<}(Y, \phi, \phi(x)) \quad \text{for all } \lambda \in ]0, 1].$$

Consider any $i \in I_m$. Since $x' \in L_<(Y, f_i, f_i(x))$, we get

$$0 \in L_<([0,1], (f_i \circ l_{x',x''}), f_i(x)) \cup (\mathbb{R} \setminus [0,1]).$$

The openness of the set $L_<([0,1], (f_i \circ l_{x',x''}), f_i(x)) \cup (\mathbb{R} \setminus [0,1])$ implies that there exists some $\overline{\lambda}_i \in ]0,1[$ such that $f_i(l_{x',x''}(\lambda)) < f_i(x)$ for all $\lambda \in ]0, \overline{\lambda}_i]$.

So, we conclude that the point $\overline{x} := l_{x',x''}(\min\{\overline{\lambda}_i \mid i \in I_m\})$ fulfils $\overline{x} \in S_<(Y, f, x) \cap L_<(Y, \phi, \phi(x))$, a contradiction to (2.10).

Consequently, we infer that (2.11) holds.

**2°.** Consider $x \in \mathrm{WEff}(X \mid f) \subseteq X$. Of course, we can have $x \in \mathrm{WEff}(Y \mid f)$ and therefore $x \in X \cap \mathrm{WEff}(Y \mid f)$. Let us assume that $x \in X \setminus \mathrm{WEff}(Y \mid f)$. In view of Theorem 2.25, we know that $x \in \mathrm{WEff}(X \mid f)$ implies $x \in X \cap \mathrm{WEff}(Y \mid f^{\oplus})$.

Now, consider two cases:

*Case 1:* Let $\operatorname{int} X \neq \emptyset$. By Corollary 2.6, we get $x \in \operatorname{bd} X$ by the componentwise semi-strictly quasi-convexity of $f$ on $Y$.

*Case 2:* Let $\operatorname{int} X = \emptyset$. Obviously, we have $x \in X = \operatorname{bd} X$.

Finally, we get $x \in (\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid f^{\oplus})$.

□

**Corollary 2.28** Let (2.5) be satisfied and let $\operatorname{int} X \neq \emptyset$. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumptions (A1), (A2) and (A7). Furthermore, suppose that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on $Y$. Then, we have

$$\mathrm{WEff}(X \mid f) = [X \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid f^{\oplus})\right].$$

**Corollary 2.29** Assuming that the assumptions of Corollary 2.28 as well as the condition $(\operatorname{int} X) \cap \mathrm{WEff}(Y \mid f) = \emptyset$ are fulfilled, we have

$$\mathrm{WEff}(X \mid f) = (\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid f^{\oplus}) \subseteq \operatorname{bd} X.$$

The semi-strict quasi-convexity assumption with respect to $f$ in 2° of Theorem 2.27 and Theorem 2.27 can not be replaced by a quasi-convexity assumption, as shown in the next Example 2.30.

**Example 2.30** We consider the data given in Example 2.5. For the point $x' \in (\operatorname{int} X) \cap \mathrm{Eff}(X \mid f) \cap \mathrm{WEff}(X \mid f)$ we know that $x' \notin \mathrm{WEff}(\mathbb{R}^2 \mid f)$ and $x' \notin \mathrm{Eff}(\mathbb{R}^2 \mid f)$. Hence, the inclusions given in 2° of Theorem 2.17 and Theorem 2.27 are not fulfilled for the componentwise quasi-convex but not componentwise semi-strictly quasi-convex function $f$.

**Remark 2.31** Notice, in view of Theorem 2.25, we can also write

$$[(\operatorname{int} X) \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid f^{\oplus})\right] = \mathrm{WEff}(X \mid f)$$

in Corollary 2.28, where the union in the above formula is disjoint. Consequently, under the assumptions of Corollary 2.28, it is possible to check whether a point $x \in X$ is weakly efficient for the constrained problem $(\mathcal{P}_X)$ or not by using two problems with convex feasible set:

$$\begin{aligned} x \in \operatorname{int} X: &\qquad x \in \mathrm{WEff}(Y \mid f) \iff x \in \mathrm{WEff}(X \mid f); \\ x \in \operatorname{bd} X: &\qquad x \in \mathrm{WEff}(Y \mid f^{\oplus}) \iff x \in \mathrm{WEff}(X \mid f). \end{aligned}$$

**Theorem 2.32** ([56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A3). Then, we have

$$X \cap \mathrm{WEff}(Y \mid f^{\oplus}) = X.$$

*Proof.* The inclusion "$\subseteq$" is obvious. Let us prove the reverse inclusion "$\supseteq$".
Let $x \in X$. By Assumption (A3), it follows $L_<(Y, \phi, \phi(x)) = \emptyset$. So, we get

$$S_<(Y, f, x) \cap L_<(Y, \phi, \phi(x)) = \emptyset,$$

hence we infer $x \in X \cap \mathrm{WEff}(Y \mid f^{\oplus})$ by Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$). $\square$

**Remark 2.33** Assume that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A3). By 2° of Theorem 2.27 and by Theorem 2.32, we get

$$\begin{aligned}\mathrm{WEff}(X \mid f) &\subseteq [X \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid f^{\oplus})\right] \\ &= [(\mathrm{int}\, X) \cap \mathrm{WEff}(Y \mid f)] \cup \mathrm{bd}\, X.\end{aligned}$$

However, the reverse inclusion

$$\mathrm{WEff}(X \mid f) \supseteq [(\mathrm{int}\, X) \cap \mathrm{WEff}(Y \mid f)] \cup \mathrm{bd}\, X.$$

does not hold in general, since $\mathrm{bd}\, X \subseteq \mathrm{WEff}(X \mid f)$ is not true in general (see Example 5.28 in Chapter 5). Hence, it seems to be more appropriate to work with a penalization function $\phi$ that fulfils Assumptions (A1) and (A2) instead of Assumption (A3) in order to compute the set $\mathrm{WEff}(X \mid f)$.

#### 2.4.2.2 Pareto reducibility in multi-objective optimization

According to Popovici [102], the multi-objective optimization problem $(\mathcal{P}_X)$ is called *Pareto reducible* if the set of weakly Pareto efficient solutions of $(\mathcal{P}_X)$ can be represented as the union of the sets of Pareto efficient solutions of its subproblems.

Considering the objective function

$$f_I = (f_{i_1}, \cdots, f_{i_k}) : \mathcal{D} \to \mathbb{R}^k,$$

for a selection of indices $I = \{i_1, \ldots, i_k\} \subseteq I_{m+1}$, $i_1 < \cdots < i_k$, with cardinality $\mathrm{card}(I) = k \geq 1$, we define the problem

$$\begin{cases} f_I(x) = (f_{i_1}(x), \cdots, f_{i_k}(x)) \to \min \ \text{ w.r.t. } \mathbb{R}^k_+ \\ x \in X. \end{cases} \tag{2.12}$$

In fact, (2.12) is a single-objective optimization problem when $I$ is a singleton set, otherwise being a multi-objective one. Notice that $f_{I_m} = f$ and $f_{I_{m+1}} = f^{\oplus}$. If $\emptyset \neq I \subseteq I_m$, then (2.12) can be seen as a subproblem of the initial problem $(\mathcal{P}_X)$.

For any index set $I$ with $\emptyset \neq I \subseteq I_m$, we consider the function

$$f_I^{\oplus} := (f_I, \phi) : \mathcal{D} \to \mathbb{R}^{k+1}$$

and the following subproblem of the penalized problem $(\mathcal{P}_Y^{\oplus})$:

$$\begin{cases} f_I^{\oplus}(x) = (f_I(x), \phi(x)) = (f_{i_1}(x), \cdots, f_{i_k}(x), \phi(x)) \to \min \ \text{ w.r.t. } \mathbb{R}^{k+1}_+ \\ x \in Y. \end{cases}$$

Next, we present sufficient conditions for *Pareto reducibility* by recalling results derived by Popovici in [102, Prop. 4] and [104, Cor. 4.5]):

**Proposition 2.34** ([102, Prop. 4], [104, Cor. 4.5]) Assume that $\Omega$ is a nonempty, convex subset of $\mathcal{D} \subseteq \mathbf{E}$, where $\mathbf{E}$ is a linear topological space. If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex and upper semi-continuous along line segments on $\Omega$, then

$$\mathrm{WEff}(\Omega \mid f) = \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(\Omega \mid f_I).$$

In addition, if $\mathbf{E}$ is the $n$-dimensional Euclidean space $\mathbb{R}^n$ and $f$ is componentwise lower semi-continuous along line segments on $\Omega$, then

$$\mathrm{WEff}(\Omega \mid f) = \bigcup_{\substack{\emptyset \neq I \subseteq I_m;\\ \mathrm{card}\, I \leq n+1}} \mathrm{Eff}(\Omega \mid f_I).$$

In the next theorem, we present a result that is similar to the result given in Corollary 2.28, however the proof is now based on Theorem 2.17 and Popovici's Pareto reducibility result given in Proposition 2.34.

**Theorem 2.35** ([55]) Let (2.5) be satisfied, let $X$ be convex, and let $\mathrm{int}\, X \neq \emptyset$. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumptions (A1) and (A2). Furthermore, assume that $f^{\oplus} : \mathcal{D} \to \mathbb{R}^{m+1}$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on $Y$. Then, we have

$$\mathrm{WEff}(X \mid f) = [X \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid f^{\oplus})\right].$$

*Proof.* By Theorem 2.17, we have

$$[X \cap \mathrm{Eff}(Y \mid f_I)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f_I^{\oplus})\right] = \mathrm{Eff}(X \mid f_I) \tag{2.13}$$

for all $\emptyset \neq I \subseteq I_m$. Consider $x \in \mathrm{bd}\, X$ and $x' \in \mathrm{int}\, X$. Then, under the Assumptions (A1) and (A2), it follows

$$\mathrm{Eff}(Y \mid \phi) = \mathrm{argmin}_{x \in Y}\, \phi(x) \subseteq L_{\leq}(Y, \phi, \phi(x')) \subseteq L_{<}(Y, \phi, \phi(x)) = \mathrm{int}\, X,$$

hence

$$(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid \phi) = \emptyset. \tag{2.14}$$

By Proposition 2.34 (applied for the problems $(\mathcal{P}_X)$ with $\Omega = X$, $(\mathcal{P}_Y)$ with $\Omega = Y$, and $(\mathcal{P}_Y^{\oplus})$ with $\Omega = Y$ and $f^{\oplus}$ in the role of $f$), the assertion of this theorem follows immediately:

$$
\begin{aligned}
\mathrm{WEff}(X \mid f) &= \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(X \mid f_I) \\
&\overset{(2.13)}{=} \left[ X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I) \right] \cup \left[ (\operatorname{bd} X) \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I^{\oplus}) \right] \\
&\overset{(2.14)}{=} \left[ X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I) \right] \cup \left[ (\operatorname{bd} X) \cap \bigcup_{\substack{\emptyset \neq I \subseteq I_{m+1};\\ \{m+1\} \subseteq I}} \mathrm{Eff}(Y \mid f_I) \right] \\
&= \left[ X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I) \right] \\
&\cup \left[ (\operatorname{bd} X) \cap \left( \bigcup_{\substack{\emptyset \neq I \subseteq I_{m+1};\\ \{m+1\} \subseteq I}} \mathrm{Eff}(Y \mid f_I) \cup \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I) \right) \right] \\
&= \left[ X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I) \right] \cup \left[ (\operatorname{bd} X) \cap \bigcup_{\emptyset \neq I \subseteq I_{m+1}} \mathrm{Eff}(Y \mid f_I) \right] \\
&= [X \cap \mathrm{WEff}(Y \mid f)] \cup \left[ (\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid f^{\oplus}) \right].
\end{aligned}
$$

□

Now, we are able to present a Pareto reducibility type result for multi-objective optimization problems.

**Theorem 2.36** ([56]) Let (2.5) be satisfied and let $\operatorname{int} X \neq \emptyset$. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumptions (A1) and (A2). Moreover, assume that $f^{\oplus} : \mathcal{D} \to \mathbb{R}^{m+1}$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on $Y$. Then, we have

$$
\mathrm{WEff}(X \mid f) = \left[ X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I) \right] \cup \left[ (\operatorname{bd} X) \cap \bigcup_{\emptyset \neq I \subseteq I_{m+1}} \mathrm{Eff}(Y \mid f_I) \right].
$$

In addition, if $\mathbf{E}$ is the $n$-dimensional Euclidean space $\mathbb{R}^n$ and $f^{\oplus}$ is componentwise lower semi-continuous along line segments on $Y$, then

$$
\mathrm{WEff}(X \mid f) = \left[ X \cap \bigcup_{\substack{\emptyset \neq I \subseteq I_m;\\ \operatorname{card} I \leq n+1}} \mathrm{Eff}(Y \mid f_I) \right] \cup \left[ (\operatorname{bd} X) \cap \bigcup_{\substack{\emptyset \neq I \subseteq I_{m+1};\\ \operatorname{card} I \leq n+1}} \mathrm{Eff}(Y \mid f_I) \right].
$$

*Proof.* In view of Theorem 2.27 and Remark 2.26, we have

$$
\mathrm{WEff}(X \mid f) = [X \cap \mathrm{WEff}(Y \mid f)] \cup \left[ (\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid f^{\oplus}) \right].
$$

Due to Proposition 2.34 (applied for $(\mathcal{P}_Y)$ with $\Omega = Y$ as well as for $(\mathcal{P}_Y^{\oplus})$ with $\Omega = Y$ and $f^{\oplus}$ in the role of $f$), we get the desired equalities given in this theorem. □

**Remark 2.37** Under the assumption $Y = \mathcal{D} = \mathbf{E}$, Theorem 2.36 provides a representation for the set of weakly Pareto efficient solutions of the constrained problem $(\mathcal{P}_X)$ in terms of the sets of Pareto efficient solutions of families of unconstrained multi-objective optimization problems. In Lemma 2.55, we will see that the set $X$ given in Theorem 2.36 has to be a convex one if $\phi$ is semi-strictly quasi-convex on $X$ and satisfies the Assumption (A5) (i.e., $X = L_{\leq}(Y, \phi, 0)$).

**Theorem 2.38** ([56]) Let (2.5) be satisfied and let $X$ be convex. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A3). Moreover, assume that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on $Y$. Then, we have

$$\mathrm{WEff}(X \mid f) = X \cap \bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1};\\ \mathrm{card}\, I \geq 2}} \mathrm{Eff}(Y \mid f_I).$$

*Proof.* Due to Theorem 2.24, we have

$$X \cap \mathrm{Eff}(Y \mid f_I^{\oplus}) = \mathrm{Eff}(X \mid f_I) \quad \text{for all } \emptyset \neq I \subseteq I_m. \tag{2.15}$$

By Proposition 2.34 (applied for $(\mathcal{P}_X)$ with $\Omega = X$), it follows

$$\begin{aligned} \mathrm{WEff}(X \mid f) &= \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(X \mid f_I) \\ &\overset{(2.15)}{=} X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I^{\oplus}) \\ &= X \cap \bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1};\\ \mathrm{card}\, I \geq 2}} \mathrm{Eff}(Y \mid f_I). \end{aligned}$$

□

The reader should pay attention to the fact that the restriction card $I \geq 2$ concerning the index set $I$ in the assertion of Theorem 2.38 is essential. Indeed, since

$$\mathrm{Eff}(Y \mid f_{m+1}) = L_{=}(Y, \phi, \phi(x)) = X, \ x \in X,$$

under the Assumption (A3), we have

$$X \cap \bigcup_{\{m+1\} \subseteq I \subseteq I_{m+1}} \mathrm{Eff}(Y \mid f_I) = X.$$

### 2.4.3 Sets of strictly Pareto efficient solutions of $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

Some first relationships between the sets of strictly Pareto efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$ are given in the next theorem.

**Theorem 2.39** ([55, 56]) Let (2.5) and Assumption (A4) be satisfied. Then, we have

$$X \cap \mathrm{SEff}(Y \mid f) \subseteq \mathrm{SEff}(X \mid f) \subseteq X \cap \mathrm{SEff}(Y \mid f^{\oplus}).$$

*Proof.* By Corollary 2.2, we get $X \cap \mathrm{SEff}(Y \mid f) \subseteq \mathrm{SEff}(X \mid f)$. We now show the second inclusion.
Consider $x \in \mathrm{SEff}(X \mid f) \subseteq X$. In view of Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$) and the assumption (A4), we get

$$\begin{aligned} S_{\leq}(Y, f, x) \cap L_{\leq}(Y, \phi, \phi(x)) &\subseteq S_{\leq}(Y, f, x) \cap X \\ &= S_{\leq}(X, f, x) = \{x\}, \end{aligned}$$

Therefore, it follows $x \in X \cap \mathrm{SEff}(Y \mid f^{\oplus})$ by Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$). □

The following Theorem 2.40 presents important relationships between the sets of strictly Pareto efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$.

**Theorem 2.40** ([55, 56]) Let (2.5) be satisfied. The following assertions are true:

**1°.** If Assumption (A1) holds, then we have

$$[X \cap \mathrm{SEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus)\right] \subseteq \mathrm{SEff}(X \mid f).$$

**2°.** Assume that Assumption (A4) holds. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{SEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus)\right] \supseteq \mathrm{SEff}(X \mid f).$$

*Proof.* **1°.** By Corollary 2.2, we have $X \cap \mathrm{SEff}(Y \mid f) \subseteq \mathrm{SEff}(X \mid f)$. Consider $x \in (\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus)$. In view of Lemma 1.69 (applied for $(\mathcal{P}_Y^\oplus)$ instead of $(\mathcal{P}_\Omega)$) and Assumption (A1), we have

$$\begin{aligned} S_\leq(X, f, x) &= S_\leq(Y, f, x) \cap X \\ &= S_\leq(Y, f, x) \cap L_\leq(Y, \phi, \phi(x)) = \{x\}. \end{aligned}$$

From Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$), we get $x \in \mathrm{SEff}(X \mid f)$.

**2°.** Consider $x \in \mathrm{SEff}(X \mid f) \subseteq X$. If we have $x \in \mathrm{SEff}(Y \mid f)$, then $x \in X \cap \mathrm{SEff}(Y \mid f)$. We now suppose that $x \in X \setminus \mathrm{SEff}(Y \mid f)$. By Theorem 2.39, we immediately get $x \in X \cap \mathrm{SEff}(Y \mid f^\oplus)$.

Let us consider two cases:

*Case 1:* If $\mathrm{int}\, X \neq \emptyset$, then we conclude $x \in \mathrm{bd}\, X$ because of Corollary 2.6.

*Case 2:* If $\mathrm{int}\, X = \emptyset$, then clearly it follows $x \in \mathrm{bd}\, X$.

So, we infer that $x \in (\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus)$.

□

**Corollary 2.41** Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A1). In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$. Then, we have

$$\begin{aligned} \mathrm{SEff}(X \mid f) &= [X \cap \mathrm{SEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus)\right] \\ &= [(\mathrm{int}\, X) \cap \mathrm{SEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus)\right]. \end{aligned}$$

**Corollary 2.42** Assuming that the assumptions of Corollary 2.41 as well as the condition $(\mathrm{int}\, X) \cap \mathrm{SEff}(Y \mid f) = \emptyset$ are fulfilled, we have

$$\mathrm{SEff}(X \mid f) = (\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus) \subseteq \mathrm{bd}\, X.$$

**Remark 2.43** In contrast to 1° in Theorem 2.17 (Theorem 2.24) as well as 1° in Theorem 2.27, we only need the Assumption (A1) concerning the level sets of the function $\phi$ in 1° of Theorem 2.40. In accordance to 2° in Theorem 2.27, only Assumption (A4) concerning the level sets of $\phi$ must be fulfilled in 2° of Theorem 2.40. In 2° of Theorem 2.17 (Theorem 2.24) Assumptions (A1) and (A2) (Assumption (A3)) must be fulfilled.

**Remark 2.44** Notice, in view of Theorem 2.39, we can also write

$$[(\mathrm{int}\, X) \cap \mathrm{SEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^\oplus)\right] = \mathrm{SEff}(X \mid f)$$

in Corollary 2.41, where the union in the above formula is disjoint. Consequently, under the assumptions of Corollary 2.41, it is possible to check whether a point $x \in X$ is strictly Pareto efficient for the constrained problem $(\mathcal{P}_X)$ or not by using two problems with convex feasible set:

$$\begin{aligned} x \in \operatorname{int} X: &\qquad x \in \operatorname{SEff}(Y \mid f) \iff x \in \operatorname{SEff}(X \mid f);\\ x \in \operatorname{bd} X: &\qquad x \in \operatorname{SEff}(Y \mid f^{\oplus}) \iff x \in \operatorname{SEff}(X \mid f). \end{aligned}$$

Next, we present a corresponding result to the equality given in 1° of Theorem 2.24 for the concept of strict Pareto efficiency that holds under the assumption that the penalization function $\phi$ fulfils (A3).

**Theorem 2.45** ([56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A3). Then, we have

$$\operatorname{SEff}(X \mid f) = X \cap \operatorname{SEff}(Y \mid f^{\oplus}).$$

*Proof.* First, we show the inclusion "$\supseteq$", therefore consider $x \in X \cap \operatorname{SEff}(Y \mid f^{\oplus})$. Because of Lemma 1.69 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_\Omega)$) and Assumption (A3) it follows

$$\begin{aligned} S_{\le}(X, f, x) &= S_{\le}(Y, f, x) \cap X\\ &= S_{\le}(Y, f, x) \cap L_{\le}(Y, \phi, \phi(x)) = \{x\}. \end{aligned}$$

By Lemma 1.69 (applied for $(\mathcal{P}_X)$ instead of $(\mathcal{P}_\Omega)$), we have $x \in \operatorname{SEff}(X \mid f)$.

In view of assertion 1° in Theorem 2.39, we get immediately the reverse inclusion "$\subseteq$". Notice that (A3) implies (A4) by Lemma 2.10. □

Under the assumptions that $f$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$, and $\phi$ fulfils Assumption (A3), we easily infer that

$$(\operatorname{int} X) \cap \operatorname{SEff}(Y \mid f) = (\operatorname{int} X) \cap \operatorname{SEff}(Y \mid f^{\oplus})$$

taking into account Theorems 2.40 and 2.45.

## 2.5 Sufficient conditions for the validity of the Assumptions (A1) and (A2) based on (local) generalized-convexity concepts

As we have seen in Section 2.4, we need some additional assumptions concerning the level sets and level lines of the penalization function $\phi$ in order to obtain the main results. In particular, the Assumptions (A1) and (A2) play an important role in our penalization approach. We already know that under certain assumptions the *Minkowski gauge function* given in Example 2.12, the *Hiriart-Urruty function* given in Example 2.14, and the *Tammer-Weidner function* given in Example 2.15 fulfil these Assumptions (A1) and (A2). In this section, our aim is to identify further classes of functions that satisfy both Assumptions (A1) and (A2).

**Lemma 2.46** ([56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A5). Assume that $L_{<}(Y, \phi, 0)$ is a nonempty, open set. Then, $X$ has a nonempty interior, since

$$\emptyset \neq L_{<}(Y, \phi, 0) \subseteq \operatorname{int} X. \tag{2.16}$$

*Proof.* In view of (A5), we have

$$\emptyset \neq L_{<}(Y, \phi, 0) \subseteq L_{\le}(Y, \phi, 0) = X,$$

hence we conclude (2.16). □

**Lemma 2.47** Let (2.5) be satisfied, let $Y$ be open, and let $\mathcal{D} = \mathbf{E}$. Assume that $\phi : \mathbf{E} \to \mathbb{R}$ is upper semi-continuous on $\mathbf{E}$. Then, the set $L_{<}(Y, \phi, 0)$ is open.

*Proof.* Since $\phi : \mathbf{E} \to \mathbb{R}$ is upper semi-continuous on $\mathcal{D} = \mathbf{E}$, the set $L_{\geq}(\mathbf{E}, \phi, 0)$ is closed, hence $L_{<}(\mathbf{E}, \phi, 0)$ is open. By $L_{<}(Y, \phi, 0) = L_{<}(\mathbf{E}, \phi, 0) \cap Y$, we infer that $L_{<}(Y, \phi, 0)$ is open. □

However, the openness assumption concerning the set $L_{<}(Y, \phi, 0)$ in Lemma 2.46 is essential, as to see in Example 2.48.

**Example 2.48** Consider $\mathbf{E} = \mathbb{R}^2$ and define the function $\phi := ||\cdot||_\infty - 1$, where $||\cdot||_\infty : \mathbb{R}^2 \to \mathbb{R}$ represents the maximum norm. Notice that $\phi$ is convex on $\mathbb{R}^2$, hence explicitly quasi-convex as well as continuous on $\mathbb{R}^2$. Moreover, put $x := (0, 0)$, $x' := (1, 0)$ and $Y := \overline{B}_{||\cdot||_\infty}(x', 1)$. Then. we have

$$\begin{aligned} L_{\leq}(Y, \phi, 0) &= L_{\leq}(\mathbb{R}^2, ||\cdot||_\infty, 1) \cap Y \\ &= \overline{B}_{||\cdot||_\infty}(x, 1) \cap \overline{B}_{||\cdot||_\infty}(x', 1) \\ &= [0, 1] \times [-1, 1] \\ &=: X. \end{aligned}$$

In addition, it is easily seen that

$$\begin{aligned} L_{<}(Y, \phi, 0) &= L_{<}(\mathbb{R}^2, ||\cdot||_\infty, 1) \cap Y \\ &= B_{||\cdot||_\infty}(x, 1) \cap \overline{B}_{||\cdot||_\infty}(x', 1) \\ &= [(0, 1) \times (-1, 1)] \cup [\{0\} \times (-1, 1)] \\ &\supsetneq (0, 1) \times (-1, 1) \\ &= \operatorname{int} X, \end{aligned}$$

which shows that the inclusion given in (2.16) of Lemma 2.46 does not hold in this example. Hence, the openness assumption concerning the set $L_{<}(Y, \phi, 0)$ can not be removed in Lemma 2.46.

In the following, we are looking for conditions such that Assumptions (A1) and (A2) are fulfilled for the penalization function $\phi$.

**Lemma 2.49** ([56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A5). Assume that $L_{<}(Y, \phi, 0)$ is a nonempty, open set. Then, Assumption (A1) is fulfilled, and moreover, for every $x \in \operatorname{bd} X$, we have $\phi(x) = 0$.

*Proof.* Let $x \in \operatorname{bd} X$. We are going to show that $\phi(x) = 0$, hence Assumption (A1) follows by the fact that $X = L_{\leq}(Y, \phi, 0)$.

Assume the contrary, i.e., $\phi(x) < 0$. By Lemma 2.46, we get $x \in L_{<}(Y, \phi, 0) \subseteq \operatorname{int} X$, a contradiction to $x \in \operatorname{bd} X$. □

The next lemma uses the definition of local explicit quasi-convexity of the function $\phi$ (see Definition 1.52) and presents an auxiliary result that will be used for deriving sufficient conditions for the validity of the Assumptions (A1), (A2) and (A6) in Theorem 2.51.

**Lemma 2.50** ([56]) Let (2.5) be satisfied and let $(\mathbf{E}, ||\cdot||)$ be a normed space. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A5). Assume that $L_{<}(Y, \phi, 0)$ is a nonempty, open set. Consider two points $x^0 \in (\operatorname{int} X) \cap L_{=}(Y, \phi, 0)$ and $x^1 \in L_{<}(Y, \phi, 0)$. Let $\phi$ be explicitly quasi-convex on $B_{||\cdot||}(x^0, \varepsilon)$ for some $\varepsilon \in \mathbb{R}_{++}$. Then, there exists $x^2 \in \operatorname{int} X$ such that condition

$$x^2 \in B_{||\cdot||}(x^0, \varepsilon) \cap L_{<}(Y, \phi, 0). \tag{2.17}$$

holds, if one of the following assertions is true:

1°. Every local minimum point of $\phi$ on $\operatorname{int} X$ is also global.

2°. Assume that $X$ is convex. Every local minimum point of $\phi$ is also global for each restriction on a line segment in $\operatorname{int} X$.

*Proof.* Let 1° be fulfilled. Assume the contrary, i.e., there is no $x^2 \in \operatorname{int} X$ such that (2.17) holds. Then, $x^0$ is a local minimum point of $\phi$ on $\operatorname{int} X$, hence under 1° also global on $\operatorname{int} X$. This is a contradiction because we have $x^1 \in L_<(Y, \phi, 0) \subseteq \operatorname{int} X$ (see Lemma 2.46) and $\phi(x^1) < 0 = \phi(x^0)$.

Now, let 2° be satisfied. By Lemma 1.20, we have $x^0 \in \operatorname{cor} X$. For $v := x^0 - x^1 \neq 0$ there exists $\delta \in \mathbb{R}_{++}$ such that $x^0 + [0, \delta] \cdot v \subseteq X$. By $x^1 \in L_<(Y, \phi, 0) \subseteq \operatorname{int} X$ (see Lemma 2.46), $x^3 := x + \delta v \in X$, and the convexity of $X$, we know that $x^0 \in ]x^1, x^3[ \subseteq \operatorname{int} X$ by Lemma 1.13. Pick any $x^4 \in ]x^0, x^3[$. Assume the contrary, i.e., there is no $x^2 \in \operatorname{int} X$ such that (2.17) holds, hence $x^0$ is a local minimum point of $\phi$ on $\operatorname{int} X$. Then, $x^0 \in ]x^1, x^4[$ is also a local minimum point of $\phi$ on the line segment $[x^1, x^4] \subseteq \operatorname{int} X$. By 2° of this lemma, we infer that $x^0$ is also a global minimum point of $\phi$ on the line segment $[x^1, x^4]$, a contradiction to $\phi(x^1) < 0 = \phi(x^0)$. □

In the following theorem, we identify a further class of functions that fulfils the Assumptions (A1), (A2) and (A6).

**Theorem 2.51** ([56]) Let (2.5) be satisfied and let $(\mathbf{E}, ||\cdot||)$ be a normed space. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A5). Assume that $L_<(Y, \phi, 0)$ is a nonempty, open set. Let $\phi$ be locally explicitly quasi-convex on $\operatorname{int} X$. Then, Assumptions (A1), (A2) and (A6) are fulfilled, if one of the following assertions is true:

**1°.** Every local minimum point of $\phi$ on $\operatorname{int} X$ is also global.

**2°.** Assume that $X$ is convex. Every local minimum of $\phi$ is also global for each restriction on a line segment in $\operatorname{int} X$.

*Proof.* The validity of Assumption (A1) follows by Lemma 2.49. We are going to prove that Assumption (A6) holds.

For $x^0 \in \operatorname{bd} X$, we know that $\phi(x^0) = 0$ by Lemma 2.49, hence $x^0 \in L_=(Y, \phi, 0)$ is fulfilled. This shows $\operatorname{bd} X \subseteq L_=(Y, \phi, 0)$.

Let us prove the reverse inclusion $\operatorname{bd} X \supseteq L_=(Y, \phi, 0)$. Consider some $x^0 \in L_=(Y, \phi, 0) \subseteq X$. Assume the contrary, i.e., $x^0 \in \operatorname{int} X$, hence there exists $\varepsilon' \in \mathbb{R}_{++}$ such that $B_{\varepsilon'} := B_{||\cdot||}(x^0, \varepsilon') \subseteq X$. Since $\phi$ be explicitly quasi-convex on $B_\varepsilon := B_{||\cdot||}(x^0, \varepsilon)$ for some $\varepsilon \in \mathbb{R}_{++}$, there exists

$$x^1 \in B_\varepsilon \cap L_<(Y, \phi, 0)$$

by Lemma 2.50. Obviously, we have $B_{\varepsilon''} := B_{||\cdot||}(x^0, \varepsilon'') \subseteq X$ for $\varepsilon'' := \min\{\varepsilon, \varepsilon'\}$. By Lemma 1.22, we know that

$$B_\delta := [x^0 - \delta v, x^0 + \delta v] \subseteq B_{\varepsilon''}$$

for $\delta \in ]0, \varepsilon''[$ and $v := \frac{x^1 - x^0}{||x^1 - x^0||}$ (note that $x^1 \neq x^0$). Due to the semi-strict quasi-convexity of $\phi$ on $B_\varepsilon$, and the fact that $x^0 \in L_=(Y, \phi, 0)$ and $x^1 \in L_<(Y, \phi, 0)$, we can choose $x^2 \in B_\delta \cap ]x^0, x^1]$ with $x^2 \in L_<(Y, \phi, 0)$. For $x^3 := x^0 + (x^0 - x^2)$, we have $x^3 \in B_\delta \subseteq B_{\varepsilon''} \subseteq X$ and $x^0 \in ]x^2, x^3[$.

Now, since we have $x^3 \in X = L_\leq(Y, \phi, 0)$, we can consider two cases:

*Case 1:* Let $x^3 \in L_=(Y, \phi, 0)$. Under the semi-strict quasi-convexity of $\phi$ on $B_\varepsilon$, we get $x^0 \in ]x^2, x^3[ \subseteq L_<(B_\varepsilon, \phi, 0)$. Since $]x^2, x^3[ \subseteq B_\delta$, it follows $x^0 \in L_<(B_\delta, \phi, 0) \subseteq L_<(Y, \phi, 0)$, a contradiction to $x^0 \in L_=(Y, \phi, 0)$.

*Case 2:* Let $x^3 \in L_<(Y, \phi, 0)$. Since $x^2, x^3 \in L_<(B_\varepsilon, \phi, 0)$, it follows $x^0 \in ]x^2, x^3[ \subseteq L_<(B_\varepsilon, \phi, 0)$ by the quasi-convexity of $\phi$ on $B_\varepsilon$. Because of $]x^2, x^3[ \subseteq B_\delta$, we have $x^0 \in L_<(B_\delta, \phi, 0) \subseteq L_<(Y, \phi, 0)$, again a contradiction to $x^0 \in L_=(Y, \phi, 0)$.

In both cases, we have a contradiction, which proves our claim $x^0 \in \operatorname{bd} X$.

We conclude that (A6) holds, which implies together with (A5) that (A1) and (A2) are true by Lemma 2.10. □

Notice that every local minimum point of a semi-strictly quasi-convex function on a convex set is also global (see, e.g., Bagdasar and Popovici [8]).

**Theorem 2.52** ([56]) Let (2.5) be satisfied. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ fulfils Assumption (A5). Assume that $L_<(Y, \phi, 0)$ is a nonempty, open set. If $\phi$ is explicitly quasi-convex on $Y$, then Assumptions (A1), (A2) and (A6) hold.

*Proof.* If, in addition, $\mathbf{E}$ is normed, then we get the assertion of this corollary by Theorem 2.51. Now, let us assume that $\mathbf{E}$ is not necessarily normed.

By Lemma 2.49, we know that Assumption (A1) is fulfilled. We are going to prove that Assumption (A2) holds. Similar to the proof of Theorem 2.51, we get $\operatorname{bd} X \subseteq L_{=}(Y, \phi, 0)$. In the following, we show $L_{=}(Y, \phi, 0) \cap \operatorname{int} X = \emptyset$.

Assume the contrary, i.e., we have $x^0 \in L_{=}(Y, \phi, 0) \cap \operatorname{int} X$. Consider $x^1 \in L_{<}(Y, \phi, 0)$. By Lemma 1.20, there exists $\delta \in \mathbb{R}_{++}$ such that $x^2 := x^0 + \delta(x^0 - x^1) \in X$. Notice that $x^0 \in ]x^1, x^2[$ and $x^2 \in X = L_{\leq}(Y, \phi, 0)$. Now, we look at two cases:

*Case 1:* Let $\phi(x^2) < 0$. Then, the quasi-convexity of $\phi$ on $Y$ implies $x^0 \in ]x^1, x^2[ \subseteq L_{<}(Y, \phi, 0)$ by Lemma 1.41, a contradiction to $\phi(x^0) = 0$.

*Case 2:* Let $\phi(x^2) = 0$. By the semi-strict quasi-convexity of $\phi$ on $Y$, we get $x^0 \in ]x^1, x^2[ \subseteq L_{<}(Y, \phi, 0)$ by Lemma 1.42, again a contradiction to $\phi(x^0) = 0$.

In both cases we have a contradiction, hence $L_{=}(Y, \phi, 0) \cap \operatorname{int} X = \emptyset$. The proof is complete. □

The openness assumption concerning the set $L_{<}(Y, \phi, 0)$ can not be removed in Lemma 2.49 and Theorems 2.51 and 2.52, as shown in the next example.

**Example 2.53** We consider again the problem in Example 2.48.

If we assume that Assumption (A1) holds, then we have $x' \notin L_{\leq}(Y, \phi, \phi(x)) = X$ for the point $x' \in \operatorname{bd} X$ since $\phi(x) = -1 < 0 = \phi(x')$, a contradiction.

Suppose that the Assumption (A2) is fulfilled for the convex problem considered in Example 2.48. Then, we have $L_{=}(Y, \phi, \phi(x')) = \operatorname{bd} X$ for the point $x' \in \operatorname{bd} X$. So, due to $\phi(x') = 0$, we must have $\phi(x'') = 0$ for all $x'' \in \operatorname{bd} X$. However, it is easily seen that $\phi(x) = -1$ for the point $x \in \operatorname{bd} X$, a contradiction.

Consequently, the Assumptions (A1) and (A2) do not hold for the problem given in Example 2.48. This means that the openness assumption concerning the set $L_{<}(Y, \phi, 0)$ can not be removed in Lemma 2.49 and Theorems 2.51 and 2.52.

**Corollary 2.54** ([56]) Let (2.5) be satisfied and let $Y = \mathcal{D} = \mathbf{E}$. Assume that $\phi : \mathbf{E} \to \mathbb{R}$ is semi-strictly quasi-convex as well as continuous on $\mathbf{E}$ and fulfils Assumption (A5). Let $L_{<}(\mathbf{E}, \phi, 0) \neq \emptyset$. Then, Assumptions (A1), (A2) and (A6) hold.

*Proof.* Directly follows by Lemma 2.47 and Theorem 2.52. □

The next lemma shows that under the validity of (A5) a (semi-strictly) quasi-convex function $\phi$ on $Y$ ensures that the feasible set $X$ is convex. Hence, in order to describe a nonconvex feasible set $X$ by using the level set $L_{\leq}(Y, \phi, 0)$, it is necessary that $\phi$ is not a (semi-strictly) quasi-convex function on $Y$.

**Lemma 2.55** ([56]) Let (2.5) be satisfied. Assume that $\phi : \mathcal{D} \to \mathbb{R}$ is quasi-convex or semi-strictly quasi-convex on the convex set $Y$ and fulfils Assumption (A5). Then, $X$ is a convex set in $\mathbf{E}$.

*Proof.* Since $X = L_{\leq}(Y, \phi, 0)$ by Assumption (A5), we know that the quasi-convexity of $\phi$ on $Y$ implies convexity of $X$.

Let $\phi$ be semi-strictly quasi-convex on $Y$. Assume the contrary, i.e., there exist $x^0, x^1 \in X$, $\lambda \in ]0, 1[$ such that $x^2 := l_{x^0,x^1}(\lambda) \notin X$. Consider the complement of $X = L_{\leq}(Y, \phi, 0)$, i.e, the set

$$X^c = \mathbf{E} \setminus X = L_{>}(Y, \phi, 0) \cup (\mathbf{E} \setminus Y). \tag{2.18}$$

The convexity of $Y$ ensures $x^2 \in ]x^0, x^1[ \subseteq Y$, and therefore,

$$x^2 \in L_{>}(Y, \phi, 0). \tag{2.19}$$

Since $X$ is closed, the set $X^c$ is open, hence by (2.18), (2.19) and Lemma 1.20, we get $x^2 \in \operatorname{cor} X^c$. Therefore, for $v := x^0 - x^2 \neq 0$, it exists $\delta \in \mathbb{R}_{++}$ such that $x^2 + [0, \delta] \cdot v \subseteq X^c$. Moreover, we have $x^2 + [0, 1] \cdot v = [x^2, x^0] \subseteq Y$. Hence, by (2.18), it follows

$$x^2 + [0, \overline{\delta}] \cdot v \subseteq L_{>}(Y, \phi, 0) \cap ]x^0, x^1[ \tag{2.20}$$

for $\overline{\delta} := \min\{\delta, 0.5\} > 0$. By Assumption (A5) and due to $x^0, x^1 \in X$, we have

$$\max\{\phi(x^0), \phi(x^1)\} \leq 0. \tag{2.21}$$

In view of (2.20) and (2.21), we get

$$x^2 + [0, \overline{\delta}] \cdot v \subseteq L_{>} \left( ]x^0, x^1[, \phi, 0 \right) \subseteq L_{>} \left( ]x^0, x^1[, \phi, \max\{\phi(x^0), \phi(x^1)\} \right). \tag{2.22}$$

Notice that

$$\operatorname{card} \left( x^2 + [0, \overline{\delta}] \cdot v \right) > 1. \tag{2.23}$$

However, under the semi-strict quasi-convexity of $\phi$ on the convex set $Y$, it follows

$$\operatorname{card} L_{>}(]x^0, x^1[, \phi, \max\{\phi(x^0), \phi(x^1)\}) \leq 1 \tag{2.24}$$

by Lemma 1.43. In view of (2.22), (2.23) and (2.24), we get a contradiction, which completes the proof. □

## 2.6 Problems involving constraints given by a system of inequalities

In the preceding part of this chapter, the feasible set $X \subseteq \mathcal{D} \subseteq \mathbf{E}$ was always represented by certain level sets of a penalization function $\phi : \mathcal{D} \to \mathbb{R}$ (see the Assumptions (A1), (A3), (A5)). However, in many cases, the feasible set $X$ is given by a system of inequalities, i.e., we have

$$X := \{x \in Y \mid g_1(x) \leq 0, \ldots, g_q(x) \leq 0\} = \bigcap_{i \in I_q} L_{\leq}(Y, g_i, 0) \tag{2.25}$$

for some constraint functions $g_1, \ldots, g_q : \mathcal{D} \to \mathbb{R}$, $q \in \mathbb{N}$, and a convex set $Y \subseteq \mathcal{D}$. For notational convenience, let us consider $g := (g_1, \ldots, g_q) : \mathcal{D} \to \mathbb{R}^q$ as the vector-valued constraint function.

In order to apply results from Section 2.4, for the penalization function $\phi$ considered in $(\mathcal{P}_Y^{\oplus})$, we put

$$\phi := \max\{g_1, \ldots, g_q\}.$$

Then, Assumption (A5) is satisfied, i.e., we have

$$X = \bigcap_{i \in I_q} L_{\leq}(Y, g_i, 0) = L_{\leq}(Y, \phi, 0). \tag{2.26}$$

For the special approach considered in this section, the assumptions given in (2.5) read as

$$\begin{cases} \text{Let } \mathbf{E} \text{ be a real topological linear space;} \\ \text{let } \mathcal{D} \subseteq \mathbf{E} \text{ be a nonempty set;} \\ \text{let } Y \subseteq \mathcal{D} \text{ be a convex set;} \\ \text{let } X = L_{\leq}(Y, \phi, 0) \text{ be nonempty and closed.} \end{cases} \tag{2.27}$$

Notice that under the assumptions that $Y$ is closed and $\phi$ is lower semi-continuous on $Y$, the set $X$ is closed too. In addition, due to Lemmata 1.45, 1.47 and 2.55, we get the following useful implications.

**Lemma 2.56** Let (2.27) be satisfied. Then, the following assertions hold:

**1°.** If $g : \mathcal{D} \to \mathbb{R}^q$ is componentwise convex (quasi-convex) on $Y$, then $\phi$ is convex (quasi-convex) on $Y$.

**2°.** If $\phi$ is quasi-convex or semi-strictly quasi-convex on $Y$, then the set $X$ is convex.

**3°.** Assume that $Y$ is closed. If $g$ is componentwise lower semi-continuous on $Y$, then $\phi$ is lower semi-continuous on $Y$.

In some results, we will need the well-known *Slater condition* that is given by

$$\bigcap_{i \in I_q} L_<(Y, g_i, 0) = L_<(Y, \phi, 0) \neq \emptyset. \tag{2.28}$$

Next, we present relationships between the initial problem $(\mathcal{P}_X)$ with feasible set $X$ and the objective function

$$f = (f_1, \dots, f_m),$$

and two related problems $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$ with convex feasible set $Y$ and the objective functions

$$f = (f_1, \dots, f_m),$$

and

$$f^{\oplus} = (f_1, \dots, f_m, \phi) = (f_1, \dots, f_m, \max\{g_1, \dots, g_q\}),$$

respectively.

**Theorem 2.57** ([56]) Let (2.27) and Slater's condition (2.28) be satisfied, and let $L_<(Y, \phi, 0)$ be an open set. Then, the following assertions hold:

**1°.** Let the Assumption (A2) be fulfilled. If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$, then

$$\mathrm{Eff}(X \mid f) = [X \cap \mathrm{Eff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})\right]. \tag{2.29}$$

**2°.** Assume that Assumptions (A2) and (A7) hold. If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on $Y$, then

$$\mathrm{WEff}(X \mid f) = [(\mathrm{int}\, X) \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid f^{\oplus})\right]. \tag{2.30}$$

**3°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$, then

$$\mathrm{SEff}(X \mid f) = [(\mathrm{int}\, X) \cap \mathrm{SEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^{\oplus})\right]. \tag{2.31}$$

*Proof.* The validity of Assumption (A1) follows by Slater's condition (2.28) and Lemma 2.49. Moreover, we have $\mathrm{int}\, X \neq \emptyset$ by Lemma 2.46. Hence, we get the assertion of this theorem by Theorems 2.17, 2.27 and 2.40, taking into account that (2.26) (i.e., (A5) holds) implies (A4) by Lemma 2.10. □

Under the assumption that the penalization function $\phi$ is explicitly quasi-convex on $Y$, we directly get the following result by Theorem 2.57.

**Corollary 2.58** ([56]) Let (2.27) and Slater's condition (2.28) be satisfied, and let $L_<(Y, \phi, 0)$ be an open set. Suppose that $\phi$ is explicitly quasi-convex on $Y$. Then, the following hold:

**1°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $Y$, then (2.29) holds.

**2°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on $Y$, then (2.30) holds.

**3°.** If $f : \mathcal{D} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $Y$, then (2.31) holds.

*Proof.* The Assumptions (A1) and (A2) are fulfilled by Theorem 2.52, while, in view of Remark 2.26, the Assumption (A7) is satsified for the semi-strictly quasi-convex function $\phi$ on $Y$. So, the assertion follows directly by Theorem 2.57. □

We conclude, for the special case $Y = \mathcal{D} = \mathbf{E}$, the following result by Corollary 2.58 and Remark 1.36.

**Corollary 2.59** ([56]) Let (2.27) be satisfied and let $Y = \mathcal{D} = \mathbf{E}$. Suppose that $\phi$ is semi-strictly quasi-convex and continuous on $\mathbf{E}$. Assume that Slater's condition (2.28) holds. Then, assertions $1^\circ$, $2^\circ$ and $3^\circ$ of Corollary 2.58 are fulfilled.

## 2.7 Some relationships between single-objective and bi-objective optimization

As a consequence of our results derived in Section 2.4, we present relationships between single-objective optimization and bi-objective optimization in this section. Let us consider a single-valued function $h : \mathcal{D} \to \mathbb{R}$ defined on a nonempty subset $\mathcal{D}$ of the linear topological space $\mathbf{E}$. Under the validity of our standard assumptions given in (2.5), we are interested in computing the solution set

$$\mathrm{Sol}(X \mid h) = \operatorname*{argmin}_{x \in X} h(x)$$

of the scalar constrained optimization problem

$$\begin{cases} h(x) \to \min \\ x \in X. \end{cases}$$

The next results show that techniques from bi-objective optimization can be used in order to solve single-objective optimization problems involving some constraints. As usual in this chapter, $\phi : \mathcal{D} \to \mathbb{R}$ represents the penalization function in our approach.

**Theorem 2.60** Let (2.5) be satisfied. Then, the following assertions are true:

**$1^\circ$.** Assume that both (A1) and (A2) hold or (A3) is satisfied. Then,

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid (h, \phi))] \subseteq \mathrm{Sol}(X \mid h).$$

**$2^\circ$.** Assume that both (A1) and (A2) hold or (A3) is satisfied. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid (h, \phi))] \supseteq \mathrm{Sol}(X \mid h).$$

**$3^\circ$.** Under Assumption (A3), it follows

$$X \cap \mathrm{Eff}(Y \mid (h, \phi)) = \mathrm{Sol}(X \mid h).$$

**$4^\circ$.** Assume that (A1), (A2) and (A7) hold. Suppose that $\mathrm{int}\, X \neq \emptyset$. Let $h : \mathcal{D} \to \mathbb{R}$ be upper semi-continuous along line segments on $Y$. Then,

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid (h, \phi))] \subseteq \mathrm{Sol}(X \mid h).$$

**$5^\circ$.** Assume that Assumption (A4) is fulfilled. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex on $Y$. Then, it holds that

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid (h, \phi))] \supseteq \mathrm{Sol}(X \mid h).$$

**$6^\circ$.** By Assumption (A1), we get

$$[X \cap \mathrm{Sol_u}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid (h, \phi))] \subseteq \mathrm{Sol_u}(X \mid h).$$

**$7^\circ$.** Assume that Assumption (A4) holds. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex or quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{Sol_u}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid (h, \phi))] \supseteq \mathrm{Sol_u}(X \mid h).$$

8°. Assume that Assumption (A3) holds. Then, we have

$$X \cap \mathrm{SEff}(Y \mid (h, \phi)) = \mathrm{Sol}_{\mathrm{u}}(X \mid h).$$

*Proof.* Follows directly by Theorems 2.17, 2.24, 2.27,2.40, and 2.45. □

Assuming that the penalization function $\phi$ is given by the *Hiriart-Urruty function* (i.e., we have $\phi := \triangle_X$; see Example 2.14), we get the following result.

**Corollary 2.61** Let (2.5) be satisfied. Suppose that $\mathbf{E}$ is a normed space and assume that $\phi : \mathcal{D} \to \mathbb{R}$ is given by the Hiriart-Urruty function, i.e., $\phi := \triangle_X$. Then, we have:

1°. It holds that

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid (h, \triangle_X))] \subseteq \mathrm{Sol}(X \mid h);$$
$$[X \cap \mathrm{Sol}_{\mathrm{u}}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid (h, \triangle_X))] \subseteq \mathrm{Sol}_{\mathrm{u}}(X \mid h).$$

2°. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid (h, \triangle_X))] \supseteq \mathrm{Sol}(X \mid h);$$
$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid (h, \triangle_X))] \supseteq \mathrm{Sol}(X \mid h).$$

3°. Suppose that $\mathrm{int}\, X \neq \emptyset$. Assume that (A7) holds (e.g., if $X$ is convex). Let $h : \mathcal{D} \to \mathbb{R}$ be upper semi-continuous along line segments on $Y$. Then, it holds that

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid (h, \triangle_X))] \subseteq \mathrm{Sol}(X \mid h).$$

4°. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex or quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{Sol}_{\mathrm{u}}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid (h, \triangle_X))] \supseteq \mathrm{Sol}_{\mathrm{u}}(X \mid h).$$

By taking the *distance function $d_X$ with respect to $X$* as penalization function (see Example 2.13), we can deduce the following relationships.

**Corollary 2.62** Let (2.5) be satisfied. Suppose that $\mathbf{E}$ is a normed space and assume that $\phi : \mathcal{D} \to \mathbb{R}$ is given by the distance function, i.e., $\phi := d_X$. Then, we have:

1°. It holds that

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid (h, d_X))] \subseteq \mathrm{Sol}(X \mid h);$$
$$[X \cap \mathrm{Sol}_{\mathrm{u}}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid (h, d_X))] \subseteq \mathrm{Sol}_{\mathrm{u}}(X \mid h);$$
$$X \cap \mathrm{Eff}(Y \mid (h, d_X)) = \mathrm{Sol}(X \mid h);$$
$$X \cap \mathrm{SEff}(Y \mid (h, d_X)) = \mathrm{Sol}_{\mathrm{u}}(X \mid h).$$

2°. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid (h, d_X))] \supseteq \mathrm{Sol}(X \mid h);$$
$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid (h, d_X))] \supseteq \mathrm{Sol}(X \mid h).$$

3°. In the case $\mathrm{int}\, X \neq \emptyset$, suppose additionally that $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex or quasi-convex on $Y$. Then,

$$[X \cap \mathrm{Sol}_{\mathrm{u}}(Y \mid h)] \cup [(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid (h, d_X))] \supseteq \mathrm{Sol}_{\mathrm{u}}(X \mid h).$$

In the case that $X$ is a closed, convex set with nonempty interior, we can use a penalization function that involves a certain *Minkowski gauge* $\mu$ (see Example 2.12).

**Corollary 2.63** Let (2.5) be satisfied. In addition, assume that $X$ is convex and $x' \in \operatorname{int} X$. Suppose that $\phi : \mathcal{D} \to \mathbb{R}$ is defined by $\phi(x) := \mu(x - x')$ for all $x \in \mathcal{D}$, where $\mu = \mu_B$ with $B := -x' + X$. Then, the following assertions hold:

**1°.** We have

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\operatorname{bd} X) \cap \mathrm{Eff}(Y \mid (h, \phi))] \subseteq \mathrm{Sol}(X \mid h);$$
$$[X \cap \mathrm{Sol}_{\mathrm{u}}(Y \mid h)] \cup [(\operatorname{bd} X) \cap \mathrm{SEff}(Y \mid (h, \phi))] \subseteq \mathrm{Sol}_{\mathrm{u}}(X \mid h).$$

**2°.** Let $h : \mathcal{D} \to \mathbb{R}$ be upper semi-continuous along line segments on $Y$. Then,

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid (h, \phi))] \subseteq \mathrm{Sol}(X \mid h).$$

**3°.** Let $h : \mathcal{D} \to \mathbb{R}$ be semi-strictly quasi-convex on $Y$. Then, we have

$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\operatorname{bd} X) \cap \mathrm{Eff}(Y \mid (h, \phi))] \supseteq \mathrm{Sol}(X \mid h);$$
$$[X \cap \mathrm{Sol}(Y \mid h)] \cup [(\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid (h, \phi))] \supseteq \mathrm{Sol}(X \mid h).$$

**4°.** If $h : \mathcal{D} \to \mathbb{R}$ is semi-strictly quasi-convex or quasi-convex on $Y$, then

$$[X \cap \mathrm{Sol}_{\mathrm{u}}(Y \mid h)] \cup [(\operatorname{bd} X) \cap \mathrm{SEff}(Y \mid (h, \phi))] \supseteq \mathrm{Sol}_{\mathrm{u}}(X \mid h).$$

## 2.8 Concluding remarks

In this chapter, we presented a new *vectorial penalization approach* for solving constrained multi-objective optimization problems. The results are mainly based on two papers Günther and Tammer [55, 56]. However, in this thesis, the objective function must not be defined on the whole space $\mathbf{E}$, but the objective function is defined on a nonempty subset $\mathcal{D}$ of $\mathbf{E}$. Because of this fact, we assume in several results of Section 2.5 that $L_<(Y, \phi, 0)$ is an open set. In Günther and Tammer [56] it is assumed that $Y$ is an open set and $\phi$ is upper semi-continuous on $\mathbf{E}$, hence the set $L_<(Y, \phi, 0) = L_<(\mathbf{E}, \phi, 0) \cap Y$ is open (compare Lemma 2.47).

By taking a look on the literature in single-objective optimization as well as multi-objective optimization, one can see that most authors (see, e.g., Apetrii, Durea and Strugariu [4], Durea, Strugariu and Tammer [25], Ye [129], and references therein) use a penalization function $\phi : \mathbf{E} \to \mathbb{R} \cup \{+\infty\}$ (penalty term concerning $X$) which fulfils Assumption (A3) for $Y = \mathcal{D} = \mathbf{E}$. This means, for $x' \in \mathbf{E}$, we have

$$x' \in X \iff \phi(x') = 0$$

and

$$x' \in \mathbf{E} \setminus X \iff \phi(x') > 0.$$

Such a penalization function $\phi$ can also be used in our approach (see Example 2.13 with $\phi = d_X$). Beside *Clarke's Exact Penalty Principle*, as considered at the beginning of this chapter, there is also the so-called *Exact Infinite Penalty Principle* known in the literature of convex analysis and scalar optimization which uses a penalization function $\phi$ defined by the indicator function $I_X$ (see Example 1.33).

It should be mentioned that there are relationships between Durea, Strugariu and Tammer [25, Prop. 3.1] and our Theorem 2.24. In Theorem 2.24, we characterized the set of of global Pareto efficient solutions,

$$\mathrm{Eff}(X \mid f) = X \cap \mathrm{Eff}(Y \mid f^{\oplus}),$$

under the assumption that $\phi$ fulfils Assumption (A3). Durea, Strugariu and Tammer [25] considered a general vector optimization problem with objective function $f : \mathbf{E} \to \overline{\mathbf{E}}$ and a nonempty, closed feasible set $X \subseteq \mathbf{E}$, where $\mathbf{E}$ is a normed space and $\overline{\mathbf{E}}$ is a linear space partially ordered by a proper, closed, convex, pointed cone $K \subseteq \overline{\mathbf{E}}$. Durea, Strugariu and Tammer [25] characterized local Pareto efficient solutions of the initial problem by local Pareto efficient solutions of the penalized problem with objective function $f^{\oplus} = (f, \phi)$ and feasible set $Y = \mathcal{D} = \mathbf{E}$. Notice that $x \in X$ is *local Pareto efficient solutions* for the problem $(\mathcal{P}_X)$ if

$$f(x) \in \mathrm{MIN}(f[X \cap B_{||\cdot||}(x, \varepsilon)], \mathbb{R}^m_+) \quad \text{for some } \varepsilon > 0.$$

Moreover, Durea, Strugariu and Tammer [25] studied for the above problem (here simplified for the case $\overline{\mathbf{E}} = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$) the special case that the feasible set $X$ is given by a system of inequalities (as given in (2.25) with $Y = \mathcal{D} = \mathbf{E}$). In [25, Prop. 3.2], it is shown that any point $x' \in \bigcap_{i \in I_q} L_=(\mathbf{E}, g_i, 0)$ fulfils the following assertion: $x'$ is a local Pareto efficient solution for the initial problem if and only if $x'$ is a local Pareto efficient solution for the penalized problem

$$\begin{cases} (f_1(x), \cdots, f_m(x), g_1(x), \cdots, g_q(x)) \to \min \ \text{ w.r.t. } \mathbb{R}^{m+q}_+ \\ x \in \mathbf{E}. \end{cases} \tag{2.32}$$

Notice that problem (2.32) is also discussed in Klamroth and Tind [72] for the single-objective case $m = 1$. By our approach presented in Section 2.6, we can give characterizations for global Pareto efficient solutions of the initial problem (without restriction to the set $\bigcap_{i \in I_q} L_=(\mathbf{E}, g_i, 0)$) by using the penalized problem

$$\begin{cases} (f_1(x), \cdots, f_m(x), \max\{g_1(x), \cdots, g_q(x)\}) \to \min \ \text{ w.r.t. } \mathbb{R}^{m+1}_+ \\ x \in \mathbf{E}. \end{cases} \tag{2.33}$$

In contrast to (2.32), with our approach in (2.33), the image space dimension is always increased only by one.

We note that there is a field of research in multi-objective optimization where additional objective functions are included in the formulation of the multi-objective optimization problem (see, e.g., Alzorba *et al.* [2], Fliege [40] and Mäkelä and Nikulin [82], and references therein).

At the end of this chapter, let us point out some important issues for further research:

- We aim to adapt the penalization approach for vector optimization problems involving general types of ordering cones. In the literature there exist interesting results (see, e.g, Dempe, Eichfelder and Fliege [24] and references therein) that point out relationships between vector optimization problems with respect to an ordering induced by a polyhedral cone, and multi-objective optimization problems with respect to the natural ordering.

- It is interesting to study corresponding relationships as presented in Section 2.4 for the concept of proper Pareto efficiency in the sense of Geoffrion [44]. Of course, for any nonempty set $\Omega \subseteq \mathcal{D}$, we know that

  $$\mathrm{PEff}(\Omega \mid f) \subseteq \mathrm{Eff}(\Omega \mid f) \subseteq \mathrm{WEff}(\Omega \mid f),$$

  and, under certain additional assumptions on $\Omega$ and $f$, one has the following chain of inclusions

  $$\mathrm{PEff}(\Omega \mid f) \subseteq \mathrm{Eff}(\Omega \mid f) \subseteq \mathrm{cl}(\mathrm{PEff}(\Omega \mid f)),$$

  which is known in the literature as *Arrow-Barankin-Blackwell-type Theorem* (named after the authors Arrow, Barankin and Blackwell of the famous article [5]). Under convexity assumptions on $f$ and $\Omega$, in view of Lemma 1.75, we have

  $$\mathrm{PEff}(\Omega \mid f) = \bigcup_{\lambda \in \mathrm{int}\, \mathbb{R}^m_+} \mathrm{Sol}(\Omega \mid \langle \lambda, f(\cdot) \rangle).$$

  However, for the proof of the relationships presented for the concepts of (strict, weak) Pareto efficiency, we used the geometrical characterizations given in Lemma 1.69. Unfortunately, we do not have such a characterization for the concept of proper Pareto efficiency (excepting some special situations, e.g., $\mathrm{Eff}(\Omega \mid f) = \mathrm{PEff}(\Omega \mid f)$; an example from the field of *location theory* is given in Chapter 5).

- The analysis of the relationships between single-objective and bi-objective optimization presented in Section 2.7 should be extended. May one can use our vectorial penalization approach as well as established techniques from the field of bi-objective optimization (see, e.g., Eichfelder [31]) in order to develop effective numerical procedures for solving single-objective constrained optimization problems.

- In view of the discussion presented above, it is interesting to look at relationships between local and corresponding global Pareto efficient solutions for multi-objective optimization problems. We present some details below:

  Consider a nonempty, open, convex set $Y \subseteq \mathcal{D}$ in a normed space $(\mathbf{E}, ||\cdot||)$. Pick some point $x' \in Y = \operatorname{int} Y$. For any $\varepsilon \in \mathbb{R}_{++}$, define $X(\varepsilon) := \overline{B}_{||\cdot||}(x', \varepsilon)$. Clearly, there exists $\varepsilon' \in \mathbb{R}_{++}$ such that $X(\varepsilon') \subseteq Y$. By our results derived in Chapter 2, we get relationships between local and corresponding global Pareto efficient solutions. Of course, if $x'$ is a global Pareto efficient solution of $(\mathcal{P}_Y)$, then $x'$ is a local Pareto efficient solution of $(\mathcal{P}_Y)$ as well. Conversely, consider a local Pareto efficient solution $x' \in \operatorname{Eff}(X(\varepsilon'') \mid f)$ for some $\varepsilon'' \in \mathbb{R}_{++}$. Notice, since $x' \in \operatorname{int} Y$, we can assume that $X(\varepsilon'') \subseteq Y$. Due to the special structure of $X(\varepsilon'')$, the penalization function $\phi : \mathbf{E} \to \mathbb{R}$, defined by

  $$\phi(x) := ||x - x'|| \quad \text{for all } x \in \mathbf{E},$$

  fulfils Assumptions (A1), (A2) and (A7). Now, since $x' \in \operatorname{int} X(\varepsilon')$, we get $x' \in \operatorname{Eff}(Y \mid f)$ under the componentwise semi-strict quasi-convexity of the vector-valued objective function $f : \mathcal{D} \to \mathbb{R}^m$ on $Y$ by Theorem 2.17. In other words, $x'$ is global Pareto efficient solution of the problem $(\mathcal{P}_Y)$. Analogous observations can be done for the concepts of strictly / weakly Pareto efficiency with the aid of Theorems 2.27 and 2.40.

  The analysis of this topic will be extended in a forthcoming work taking into account interesting results by Bagdasar and Popovici [8, 9, 10] and Durea, Strugariu and Tammer [25].

Chapter 3

# Special types of nonconvex multi-objective optimization problems

In this section, we apply our *vectorial penalization approach* presented in the previous chapter to special types of nonconvex multi-objective optimization problems. From the practical as well as theoretical point of view, it is interesting to study problems where the feasible set is not necessarily convex. In the following, we are interested in two particular types of feasible sets:

- $X$ is given by a finite union of closed, convex sets in the real linear topological space $\mathbf{E}$ (see Section 3.1);
- $X$ is given by the whole space $\mathbf{E}$ excepting a finite number of forbidden regions that are given by convex sets (see Section 3.2).

Throughout this chapter, let the objective function $f = (f_1, \cdots, f_m) : \mathcal{D} \to \mathbb{R}^m$ of the multi-objective optimization problem $(\mathcal{P}_X)$ be defined on the whole space, i.e, $\mathcal{D} = \mathbf{E}$. Consider a finite family of sets $D_1, \cdots, D_l$ where we assume that

$$D_1, \cdots, D_l \subsetneq \mathbf{E} \text{ are closed, convex sets with } \operatorname{int} D_i \neq \emptyset, i \in I_l, l \in \mathbb{N}. \tag{3.1}$$

Figure 3.1 shows an example where two sets $D_1$ and $D_2$ in $\mathbf{E} = \mathbb{R}^2$ are shown that fulfil the condition (3.1) for the case $l = 2$. In the left part of Figure 3.1, the feasible set $X$ is given by a union of two sets $D_1$ and $D_2$, while in the right part the feasible set $X$ is given by the intersection of two reverse convex sets $\mathbb{R}^2 \setminus \operatorname{int} D_1$ and $\mathbb{R}^2 \setminus \operatorname{int} D_1$.

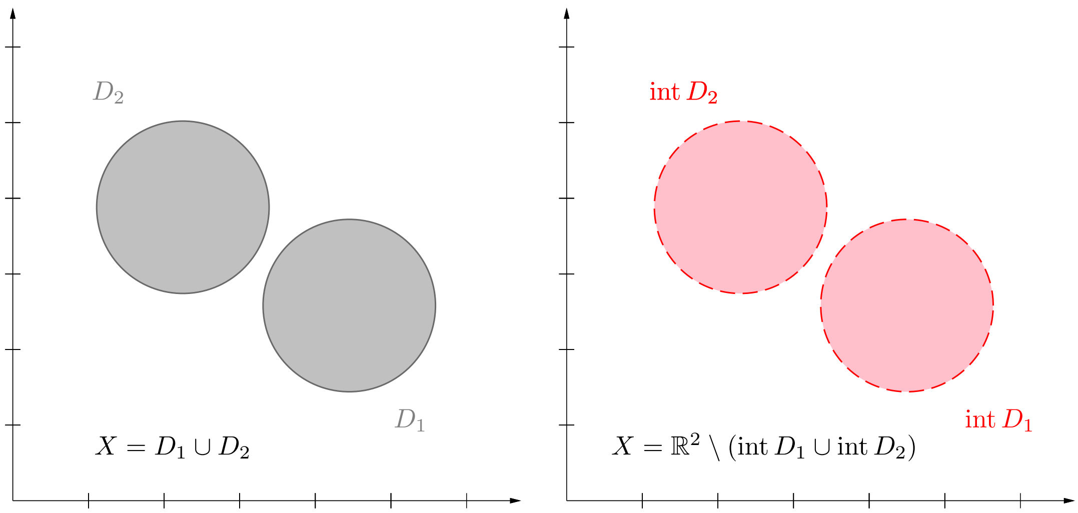


Figure 3.1: Nonconvex feasible sets.

The sets $D_i$, $i \in I_l$, are said to be pairwise disjoint if

$$D_i \cap D_j = \emptyset \quad \text{for all } i, j \in I_l, i \neq j. \tag{3.2}$$

Formula (3.2) implies that the sets $\operatorname{int} D_i$, $i \in I_l$, are pairwise disjoint, i.e.,

$$(\operatorname{int} D_i) \cap (\operatorname{int} D_j) = \emptyset \quad \text{for all } i, j \in I_l, i \neq j. \tag{3.3}$$

In the next corollary, we recall some important relationships between the constrained problem $(\mathcal{P}_X)$ and the corresponding unconstrained problems $(\mathcal{P}_{\mathbf{E}})$ (defined as $(\mathcal{P}_Y)$ with $Y = \mathbf{E}$) and $(\mathcal{P}_{\mathbf{E}}^{\oplus})$ (defined as $(\mathcal{P}_Y^{\oplus})$ with $Y = \mathbf{E}$).

**Corollary 3.1** Let $X \subsetneq \mathbf{E}$ be a nonempty, closed set with $\operatorname{int} X \neq \emptyset$. Suppose that $\phi : \mathbf{E} \to \mathbb{R}$ fulfils Assumptions (A1) and (A2). Then, the following assertions hold:

**1°.** It holds that

$$\begin{aligned} [X \cap \operatorname{Eff}(\mathbf{E} \mid f)] \cup \big[(\operatorname{bd} X) \cap \operatorname{Eff}(\mathbf{E} \mid f^{\oplus})\big] &\subseteq \operatorname{Eff}(X \mid f); \\ [X \cap \operatorname{SEff}(\mathbf{E} \mid f)] \cup \big[(\operatorname{bd} X) \cap \operatorname{SEff}(\mathbf{E} \mid f^{\oplus})\big] &\subseteq \operatorname{SEff}(X \mid f). \end{aligned}$$

**2°.** Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise upper semi-continuous along line segments on $\mathbf{E}$. Assume that $\phi : \mathbf{E} \to \mathbb{R}$ fulfils Assumption (A7). Then, we have

$$[X \cap \operatorname{WEff}(\mathbf{E} \mid f)] \cup \big[(\operatorname{bd} X) \cap \operatorname{WEff}(\mathbf{E} \mid f^{\oplus})\big] \subseteq \operatorname{WEff} X \mid f).$$

**3°.** If $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex on $\mathbf{E}$, then

$$\begin{aligned} [X \cap \operatorname{Eff}(\mathbf{E} \mid f)] \cup \big[(\operatorname{bd} X) \cap \operatorname{Eff}(\mathbf{E} \mid f^{\oplus})\big] &\supseteq \operatorname{Eff}(X \mid f); \\ [X \cap \operatorname{WEff}(\mathbf{E} \mid f)] \cup \big[(\operatorname{bd} X) \cap \operatorname{WEff}(\mathbf{E} \mid f^{\oplus})\big] &\supseteq \operatorname{WEff}(X \mid f). \end{aligned}$$

**4°.** If $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$, then

$$[X \cap \operatorname{SEff}(\mathbf{E} \mid f)] \cup \big[(\operatorname{bd} X) \cap \operatorname{SEff}(\mathbf{E} \mid f^{\oplus})\big] \supseteq \operatorname{SEff}(X \mid f).$$

*Proof.* Directly follows by Theorems 2.17, 2.27 and 2.40. □

Let us consider a finite number of nonempty, closed sets

$$X_1, \cdots, X_l \subseteq \mathbf{E}$$

with $X_i \neq \mathbf{E}$ for all $i \in I_l$. For each $i \in I_l$, we consider a penalization function $\phi_i : \mathbf{E} \to \mathbb{R}$ that fulfils the Assumptions (A1) and (A2) (with $\phi_i$ in the role of $\phi$, $X_i$ in the role of $X$, and $Y = \mathbf{E}$). Then, for any $i \in I_l$, we can define a new penalized multi-objective optimization problem by

$$\begin{cases} f^{\oplus_i}(x) := (f_1(x), \cdots, f_m(x), \phi_i(x)) \to \min \\ x \in \mathbf{E}. \end{cases} \tag{$\mathcal{P}_{\mathbf{E}}^{\oplus_i}$}$$

## 3.1 Problems with a feasible set given by a union of convex sets

In what follows, let the feasible set be given by a finite union of convex sets. More precisely, we assume that

$$X := \bigcup_{i \in I_l} X_i \text{ with } X_i := D_i, i \in I_l, \text{ where } D_i, i \in I_l, \text{ satisfy (3.1).} \tag{3.4}$$

Notice that the set $X$ given in (3.4) is a nonempty, closed set in $\mathbf{E}$.

**Lemma 3.2** Let (3.4) and (3.2) be satisfied. Then, we have

$$\operatorname{bd} X = \bigcup_{i \in I_l} \operatorname{bd} D_i. \tag{3.5}$$

*Proof.* First, we prove an auxiliary result

$$(\operatorname{bd} D_i) \cap (\operatorname{int} D_j)^c = \operatorname{bd} D_i \quad \text{for all } i, j \in I_l, i \neq j. \tag{3.6}$$

The inclusion "$\subseteq$" in (3.6) is obvious. To verify the opposite inclusion in (3.6), consider some $x \in \operatorname{bd} D_i$. Assume the contrary holds, i.e., we have $x \in \operatorname{int} D_j$ for some $j \in I_l \setminus \{i\}$. Let $d \in \operatorname{int} D_i$. Due to the convexity of $D_i$, we infer $[d, x[ \subseteq \operatorname{int} D_i$ by Lemma 1.13. By $x \in \operatorname{int} D_j$ and the convexity of $D_j$, there is some $\delta \in \mathbb{R}_{++}$ such that $x + [0, \delta] \cdot (d - x) \subseteq \operatorname{int} D_j$, in view of Lemmata 1.13 and 1.20. Hence, the point $x + \delta' \cdot (d - x)$ with $\delta' := \min\{\delta, 1\}$ belongs to $(\operatorname{int} D_i) \cap (\operatorname{int} D_j)$, a contradiction to (3.2). This shows (3.6).

Now, let us show a second auxiliary result,

$$\operatorname{int} X = \operatorname{int} \bigcup_{i \in I_l} D_i = \bigcup_{i \in I_l} \operatorname{int} D_i. \tag{3.7}$$

The inclusion "$\supseteq$" in (3.7) is easily seen. Now, let us prove the reverse inclusion "$\subseteq$". Pick some $x \in \operatorname{int} X$, i.e., there is $V \in \mathcal{V}(x)$ such that $V \subseteq X$. Moreover, we have $x \in D_i$ for some $i \in I_l$. Taking into account our assumption (3.2), it follows $x \notin D_j$ for all $j \in I_l \setminus \{i\}$, hence there is $V' \in \mathcal{V}(x)$ such that $V' \subseteq \mathbf{E} \setminus \bigcup_{j \in I_l \setminus \{i\}} D_j$. We conclude that $V'' := V \cap V' \subseteq D_i$, where it can easily be seen that $V'' \in \mathcal{V}(x)$ (a consequence of $(ii)$ in Definition 1.2). This implies $x \in \operatorname{int} D_i$, hence (3.7) is true.

Finally, (3.5) follows from (3.6) and (3.7), since

$$\begin{aligned}
\operatorname{bd} X &\overset{(3.7)}{=} \left( \bigcup_{i \in I_l} D_i \right) \setminus \bigcup_{j \in I_l} \operatorname{int} D_j \\
&= \left( \bigcup_{i \in I_l} D_i \right) \cap \bigcap_{j \in I_l} (\operatorname{int} D_j)^c \\
&= \bigcup_{i \in I_l} \left[ D_i \cap (\operatorname{int} D_i)^c \cap \bigcap_{j \in I_l \setminus \{i\}} (\operatorname{int} D_j)^c \right] \\
&= \bigcup_{i \in I_l} \left[ (\operatorname{bd} D_i) \cap \bigcap_{j \in I_l \setminus \{i\}} (\operatorname{int} D_j)^c \right] \\
&\overset{(3.6)}{=} \bigcup_{i \in I_l} \operatorname{bd} D_i.
\end{aligned}$$

□

**Remark 3.3** The equation (3.5) does not hold under the weaker assumption (3.3). For instance, consider the real intervals $D_1 = [0, 1]$ and $D_2 = [1, 2]$. Then, we have

$$\operatorname{bd} X = \{0, 2\} \subsetneq \{0, 1\} \cup \{1, 2\} = (\operatorname{bd} D_1) \cup (\operatorname{bd} D_2).$$

The next theorem provides outer and inner approximations (upper and lower bounds) for the sets of (strictly, weakly) Pareto efficient solutions of the constrained multi-objective optimization problem $(\mathcal{P}_X)$ involving a not necessarily convex feasible set $X$.

**Theorem 3.4** Let (3.4) be satisfied. Suppose that each function $\phi_i$, $i \in I_l$, fulfils Assumptions (A1) and (A2) (with $\phi_i$ in the role of $\phi$ and $X_i$ in the role of $X$). Then, the following hold:

**1°.** We have

$$\begin{aligned} X \cap \mathrm{Eff}(\mathbf{E} \mid f) \subseteq \mathrm{Eff}(X \mid f) &\subseteq \bigcup_{i \in I_l} \mathrm{Eff}(X_i \mid f); \\ X \cap \mathrm{WEff}(\mathbf{E} \mid f) \subseteq \mathrm{WEff}(X \mid f) &\subseteq \bigcup_{i \in I_l} \mathrm{WEff}(X_i \mid f); \\ X \cap \mathrm{SEff}(\mathbf{E} \mid f) \subseteq \mathrm{SEff}(X \mid f) &\subseteq \bigcup_{i \in I_l} \mathrm{SEff}(X_i \mid f). \end{aligned}$$

**2°.** If $f : \mathbf{E} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $\mathbf{E}$, then

$$\begin{aligned} \mathrm{Eff}(X \mid f) &\subseteq [X \cap \mathrm{Eff}(\mathbf{E} \mid f)] \cup \left[ (\mathrm{bd}\, X) \cap \bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{Eff}(\mathbf{E} \mid f^{\oplus_i}) \right]; \\ \mathrm{WEff}(X \mid f) &\subseteq [X \cap \mathrm{WEff}(\mathbf{E} \mid f)] \cup \left[ (\mathrm{bd}\, X) \cap \bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{WEff}(\mathbf{E} \mid f^{\oplus_i}) \right]. \end{aligned}$$

**3°.** If $f : \mathbf{E} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$, then

$$\mathrm{SEff}(X \mid f) \subseteq [X \cap \mathrm{SEff}(\mathbf{E} \mid f)] \cup \left[ (\mathrm{bd}\, X) \cap \bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{SEff}(\mathbf{E} \mid f^{\oplus_i}) \right].$$

Notice, if (3.2) holds, then $\mathrm{bd}\, D_i = (\mathrm{bd}\, D_i) \cap (\mathrm{bd}\, X)$ for every $i \in I_l$.

*Proof.* 1°. The inclusion $X \cap \mathrm{Eff}(\mathbf{E} \mid f) \subseteq \mathrm{Eff}(X \mid f)$ is obvious. Consider $x \in \mathrm{Eff}(X \mid f)$. Since $X = \bigcup_{i \in I_l} X_i$, there exist $j \in I_l$ such that $x \in X_j \subseteq X$. Due to $x \in X_j \cap \mathrm{Eff}(X \mid f) \subseteq \mathrm{Eff}(X_j \mid f)$, we infer $x \in \mathrm{Eff}(X_j \mid f) \subseteq \bigcup_{i \in I_l} \mathrm{Eff}(X_i \mid f)$.
The inclusions for the concepts of weak and strict Pareto efficiency follow analogously.

**2°.** By Corollary 2.6 (with $Y = \mathcal{D} = \mathbf{E}$) and assertion 1° of this theorem, it follows

$$\begin{aligned} \mathrm{Eff}(X \mid f) &\subseteq \left[ X \cap \mathrm{Eff}(\mathbf{E} \mid f) \cap \bigcup_{i \in I_l} \mathrm{Eff}(X_i \mid f) \right] \cup \left[ (\mathrm{bd}\, X) \cap \bigcup_{i \in I_l} \mathrm{Eff}(X_i \mid f) \right] \\ &= [X \cap \mathrm{Eff}(\mathbf{E} \mid f)] \cup \left[ (\mathrm{bd}\, X) \cap \bigcup_{i \in I_l} \mathrm{Eff}(X_i \mid f) \right]. \end{aligned}$$

Recalling Corollary 3.1 (3°), for any $i \in I_l$, we have

$$\mathrm{Eff}(X_i \mid f) \subseteq [X_i \cap \mathrm{Eff}(\mathbf{E} \mid f)] \cup \left[ (\mathrm{bd}\, X_i) \cap \mathrm{Eff}(\mathbf{E} \mid f^{\oplus_i}) \right].$$

Combining the above two conditions, we conclude

$$\mathrm{Eff}(X \mid f) \subseteq [X \cap \mathrm{Eff}(\mathbf{E} \mid f)] \cup \left[ (\mathrm{bd}\, X) \cap \bigcup_{i \in I_l} (\mathrm{bd}\, X_i) \cap \mathrm{Eff}(\mathbf{E} \mid f^{\oplus_i}) \right]$$

taking into account that, for any $i \in I_l$,

$$(\mathrm{bd}\, X) \cap X_i \cap \mathrm{Eff}(\mathbf{E} \mid f) \subseteq X \cap \mathrm{Eff}(\mathbf{E} \mid f).$$

Similarly, one can show that the inclusion for the concept of weak Pareto efficiency holds.

3°. Analogously to 2° of this theorem with the aid of Corollary 3.1 (4°).

□

**Remark 3.5** Let (3.4) be satisfied. Consider $i \in I_l$. By using a Minkowski gauge $\mu_i := \mu_{B_i} : \mathbf{E} \to \mathbb{R}$ associated to the set $B_i := -d^i + D_i$ with $d^i \in \operatorname{int} D_i$ (see Example 2.12), we define the penalization function $\phi_i : \mathbf{E} \to \mathbb{R}$ by

$$\widehat{\phi}_i(x) := \mu_i(x - d^i) - 1 = \inf\{\lambda \in \mathbb{R}_+ \mid x - d^i \in \lambda \cdot B_i\} - 1 \quad \text{for all } x \in \mathbf{E}.$$

Due to Corollary 1.63 (with $\widehat{\phi}_i$ in the role of $h$ and $B_i$ in the role of $\Omega$), the function $\widehat{\phi}_i$ is convex (hence explicitly quasi-convex) and continuous on $\mathbf{E}$, and fulfils Assumptions (A5) and (A6). By Remark 2.26, we know that (A7) (with $\widehat{\phi}_i$ in the role of $\phi$ and $D_i$ in the role of $X$) is satisfied. Thanks to Lemma 2.10 (6°) and Remark (2.26), the function $\phi_i : \mathbf{E} \to \mathbb{R}$, defined by

$$\phi_i(x) := \mu_i(x - d^i) \quad \text{for all } x \in \mathbf{E},$$

fulfils Assumptions (A1), (A2) and (A7) (with $\phi_i$ in the role of $\phi$ and $D_i$ in the role of $X$).

In the next lemma, we present some properties related to the penalization function $\widehat{\phi} : \mathbf{E} \to \mathbb{R}$ that is defined by

$$\widehat{\phi}(x) := \min\{\widehat{\phi}_i(x) \mid i \in I_l\} = \min\{\mu_i(x - d^i) - 1 \mid i \in I_l\} \quad \text{for all } x \in \mathbf{E}.$$

**Remark 3.6** The penalization function $\widehat{\phi} : \mathbf{E} \to \mathbb{R}$ is a minimum of a finite number of continuous functions $\widehat{\phi}_i(x)$, $i \in I_l$, hence it is continuous as well. However, notice that $\widehat{\phi}$ is not quasi-convex (hence not convex) on $\mathbf{E}$ in general, as to see in Example 3.7. Moreover, $\widehat{\phi}$ can not be semi-strictly quasi-convex, since otherwise $\widehat{\phi}$ would be quasi-convex in view of its continuity property.

**Example 3.7** Consider the particular case $\mathbf{E} = \mathbb{R}^2$ and $l = 2$. Define $d^1 := (0, 0)$, $D_1 := \overline{B}_{||\cdot||_2}(d^1, 1)$, $d^2 := (0, 2)$, and $D_2 := \overline{B}_{||\cdot||_2}(d^2, 1)$. So, we have $\mu_1 = \mu_2 = ||\cdot||_2$. Now, observe that $0 = \widehat{\phi}(d^1) = \widehat{\phi}(d^2)$, but for $x' := l_{d^1,d^2}(0.5) = (0, 1)$ we have $\widehat{\phi}(x') = \widehat{\phi}_1(x') = \widehat{\phi}_2(x') = 1$. This shows that $\widehat{\phi}$ is not quasi-convex on $\mathbb{R}^2$.

Next, we study the validity of the Assumptions (A5) and (A6) for the penalization function $\widehat{\phi}$.

**Lemma 3.8** Let (3.4) be satisfied. The following assertions hold:

1°. $\widehat{\phi}$ fulfils Assumption (A5) (with $\widehat{\phi}$ in the role of $\phi$).

2°. If (3.2) holds, then $\widehat{\phi}$ fulfils Assumptions (A5) and (A6) (with $\widehat{\phi}$ in the role of $\phi$).

*Proof.* 1°. We are going to show

$$L_{\leq}(\mathbf{E}, \widehat{\phi}, 0) = \bigcup_{i \in I_l} L_{\leq}(\mathbf{E}, \widehat{\phi}_i, 0) = \bigcup_{i \in I_l} D_i = X. \tag{3.8}$$

According to Remark 3.5, for any $i \in I_l$, we have $L_{\leq}(\mathbf{E}, \widehat{\phi}_i, 0) = D_i$. Now, we prove the first equality in (3.8).

Let $x \in L_{\leq}(\mathbf{E}, \widehat{\phi}, 0)$. Then, we have

$$0 \geq \widehat{\phi}(x) = \min\left\{\widehat{\phi}_i(x) \mid i \in I_l\right\} = \widehat{\phi}_j(x) \quad \text{for some } j \in I_l,$$

hence $x \in L_{\leq}(\mathbf{E}, \widehat{\phi}_j, 0)$.

Conversely, let $x \in L_{\leq}\left(\mathbf{E}, \widehat{\phi}_k, 0\right)$ for some $k \in I_l$. Then,

$$\widehat{\phi}(x) = \min\left\{\widehat{\phi}_i(x) \mid i \in I_l\right\} \leq \widehat{\phi}_k(x) \leq 0,$$

which shows $x \in L_{\leq}(\mathbf{E}, \widehat{\phi}, 0)$. The proof of assertion 1° is complete.

**2°.** Here we prove that

$$L_{=}(\mathbf{E}, \widehat{\phi}, 0) = \bigcup_{i \in I_l} L_{=}(\mathbf{E}, \widehat{\phi}_i, 0) = \bigcup_{i \in I_l} \operatorname{bd} D_i = \operatorname{bd} X. \tag{3.9}$$

By Remark 3.5, for any $i \in I_l$, we know that $L_{=}(\mathbf{E}, \widehat{\phi}_i, 0) = \operatorname{bd} D_i$, while $\bigcup_{i \in I_l} \operatorname{bd} D_i = \operatorname{bd} X$ follows by Lemma 3.2. Let us prove the remaining equality (the first one) in (3.9).

Consider $x \in L_{=}(\mathbf{E}, \widehat{\phi}, 0)$. So, we have

$$0 = \widehat{\phi}(x) = \min\left\{\widehat{\phi}_i(x) \mid i \in I_l\right\} = \widehat{\phi}_j(x) \quad \text{for some } j \in I_l,$$

hence $x \in L_{=}(\mathbf{E}, \widehat{\phi}_j, 0)$.

Now, let $x \in L_{=}(\mathbf{E}, \widehat{\phi}_k, 0) = \operatorname{bd} D_k$ for some $k \in I_l$. Then, it follows

$$\widehat{\phi}(x) = \min\left\{\widehat{\phi}_i(x) \mid i \in I_l\right\} \leq \widehat{\phi}_k(x) = 0. \tag{3.10}$$

Due to (3.3) (follows by the assumption (3.2)), for any $i \in I_l \setminus \{k\}$, we infer $x \notin \operatorname{int} D_i = L_{<}(\mathbf{E}, \widehat{\phi}_i, 0)$, or in other words,

$$\min\left\{\widehat{\phi}_i(x) \mid i \in I_l \setminus \{k\}\right\} \geq 0. \tag{3.11}$$

Combining (3.10) and (3.11), we get $x \in L_{=}(X, \widehat{\phi}, 0)$, which completes the proof of assertion 2°. $\square$

Consider the functions $\phi_i$, $i \in I_l$, as defined in Remark 3.5. Now, we turn our attention to the penalization function $\phi : \mathbf{E} \to \mathbb{R}$ that is defined by

$$\phi(x) := \min\{\phi_i(x) \mid i \in I_l\} = \min\{\mu_i(x - d^i) \mid i \in I_l\} \quad \text{for all } x \in \mathbf{E}.$$

The function $\phi$ satisfies the Assumptions (A1) and (A2), as to see in the next corollary.

**Corollary 3.9** Let (3.4) be satisfied. The following assertions hold:

**1°.** $\phi$ fulfils Assumption (A1).

**2°.** If (3.2) holds, then $\phi$ fulfils Assumptions (A1) and (A2).

*Proof.* Noting that $\phi = \widehat{\phi} + 1$, the assertion follows by Lemmata 3.8 and 2.10 (6°). $\square$

The vector-valued objective function $f^{\oplus}$ of the penalized problem $(\mathcal{P}^{\oplus}_{\mathbf{E}})$ is then given by

$$f^{\oplus}(x) = \left(f_1(x), \cdots, f_m(x), \min\{\mu_i(x - d^i) \mid i \in I_l\}\right) \quad \text{for all } x \in \mathbf{E}.$$

In the next lemma, we show that the penalization function $\phi : \mathbf{E} \to \mathbb{R}$ can be rewritten to a so-called *d.c function* (i.e., a function that can written as a difference of two convex functions).

**Lemma 3.10** Let (3.4) and (3.2) be satisfied. Then, the penalization function $\phi : \mathbf{E} \to \mathbb{R}$ admits the following representation

$$\phi(x) = \min\{\mu_i(x - d^i) \mid i \in I_l\} = \phi'(x) - \phi''(x) \quad \text{for all } x \in \mathbf{E},$$

where $\phi', \phi'' : \mathbf{E} \to \mathbb{R}$ are convex functions, defined, for any $x \in \mathbf{E}$, by

$$\phi'(x) := \sum_{i \in I_l} \mu_i(x - d^i),$$
$$\phi''(x) := \max_{i \in I_l} \sum_{j \in I_l \setminus \{i\}} \mu_j(x - d^j).$$

*Proof.* Let us prove the inequality $\phi \geq \phi' - \phi''$. Clearly, there exists $k \in I_l$ such that, for any $x \in \mathbf{E}$, we have

$$\begin{aligned}
\phi(x) = \min\{\mu_i(x - d^i) \mid i \in I_l\} &= \mu_k(x - d^k) \\
&= \sum_{i \in I_l} \mu_i(x - d^i) + \mu_k(x - d^k) - \sum_{j \in I_l} \mu_j(x - d^j) \\
&= \sum_{i \in I_l} \mu_i(x - d^i) - \sum_{j \in I_l \setminus \{k\}} \mu_i(x - d^j) \\
&\geq \sum_{i \in I_l} \mu_i(x - d^i) - \max_{i \in I_l} \sum_{j \in I_l \setminus \{i\}} \mu_j(x - d^j) = \phi'(x) - \phi''(x).
\end{aligned}$$

Now, we prove the reverse inequality $\phi' - \phi'' \geq \phi$. Consider any $x \in \mathbf{E}$. It is easily seen that there exists $k \in I_l$ such that

$$\max_{i \in I_l} \sum_{j \in I_l \setminus \{i\}} \mu_j(x - d^j) = \sum_{j \in I_l \setminus \{k\}} \mu_j(x - d^j).$$

Thus, we infer

$$\begin{aligned}
\phi'(x) - \phi''(x) &= \sum_{i \in I_l} \mu_i(x - d^i) - \max_{i \in I_l} \sum_{j \in I_l \setminus \{i\}} \mu_j(x - d^j) \\
&= \sum_{i \in I_l} \mu_i(x - d^i) - \sum_{j \in I_l \setminus \{k\}} \mu_j(x - d^j) \\
&= \mu_k(x - d^k) \\
&\geq \min\{\mu_i(x - d^i) \mid i \in I_l\} = \phi(x),
\end{aligned}$$

which completes the proof.

□

**Remark 3.11** Thanks to Lemma 3.10, we have a representation of $\phi$ as a difference of two convex functions $\phi'$ and $\phi''$. So, $\phi$ is a so-called *d.c. function*. A comprehensive duality theory for *d.c. optimization problems* was developed by Toland [118] and Singer [115]. Notice that Bozau [15] solved a special scalar nonconvex unconstrained location problem via a reformulation of a scalar function $h(\cdot) = \min\{|| \cdot - d^i||_2 \mid i \in I_l\}$, $l \in \mathbb{N}$, defined on $\mathbb{R}^n$, to a similar *d.c. function* as given in Lemma 3.10. Therefore, Lemma 3.10 is a simple generalization of the result by Bozau [15]. Further interesting works related to *d.c. optimization* and *Toland-Singer duality theory* are Löhne and Wagner [77], Wagner, Martinez-Legaz and Tammer [121], and Wagner [120].

## 3.2 Problems involving multiple forbidden regions

In this section, we consider a feasible set $X$ that is given by the whole pre-image space $\mathbf{E}$ excepting some forbidden regions that are given by convex sets. More precisely, we suppose that the following assumption is fulfilled:

$$\text{Let } X := \bigcap_{i \in I_l} X_i \text{ with } X_i := \mathbf{E} \setminus \operatorname{int} D_i, i \in I_l, \text{ where } D_i, i \in I_l, \text{ satisfy (3.1).} \tag{3.12}$$

Under the assumption (3.12), the feasible set $X$ is an intersection of closed, reverse convex sets $X_1, \cdots, X_l$. So, $X$ is a closed set too. We have $\operatorname{bd} D_i = \operatorname{bd} X_i$ for all $i \in I_l$. Notice that each of the conditions (3.2) and (3.3) implies

$$X \cap \operatorname{bd} X_i = \operatorname{bd} X_i = \operatorname{bd} D_i, \tag{3.13}$$

which is a direct consequence of the next result.

**Lemma 3.12** ([53]) Let (3.12) and (3.3) be satisfied. Then, we have

$$\operatorname{bd} X = \bigcup_{i \in I_l} \operatorname{bd} D_i.$$

*Proof.* Since $I_l$ is a finite index set, we can deduce that

$$\operatorname{int}\left(\bigcap_{i \in I_l} X_i\right) = \bigcap_{i \in I_l} \operatorname{int} X_i = \left(\bigcup_{i \in I_l} D_i\right)^c. \tag{3.14}$$

Now, we are going to prove that

$$(\operatorname{int} D_j)^c \cap D_i = D_i \quad \text{for every } i, j \in I_l, i \neq j. \tag{3.15}$$

Assume the contrary holds, i.e., there exists $x \in D_i \setminus (\operatorname{int} D_j)^c = D_i \cap (\operatorname{int} D_j)$ for some $i, j \in I_l$, $i \neq j$. Of course, in view of (3.3), we must have $x \in (\operatorname{bd} D_i) \cap (\operatorname{int} D_j)$. Consider some $d \in \operatorname{int} D_i$ (notice that $d \neq x$). Due to the convexity of $D_i$, we infer that $]x, d] \subseteq \operatorname{int} D_i$ by Lemma 1.13. This means, for every $\delta \in ]0, 1]$, we have $x + ]0, \delta] \cdot (d - x) \subseteq \operatorname{int} D_i$. Moreover, since $x \in \operatorname{int} D_j$ and $D_j$ is convex, we get $x + ]0, \delta'] \cdot (d - x) \subseteq \operatorname{int} D_j$ for some $\delta' \in ]0, 1]$. Hence, we have

$$\emptyset \neq x + ]0, \delta'] \cdot (d - x) \subseteq (\operatorname{int} D_i) \cap (\operatorname{int} D_j)$$

in contradiction to (3.3). So, (3.15) holds.

Consequently, we have

$$\begin{aligned}
\operatorname{bd} X = X \setminus \operatorname{int} X &= \left(\bigcap_{j \in I_l} X_j\right) \setminus \operatorname{int}\left(\bigcap_{i \in I_l} X_i\right) \\
&\overset{(3.14)}{=} \left(\bigcap_{j \in I_l} X_j\right) \setminus \left(\bigcup_{i \in I_l} D_i\right)^c \\
&= \left(\bigcap_{j \in I_l} X_j\right) \cap \left(\bigcup_{i \in I_l} D_i\right) \\
&= \bigcup_{i \in I_l} \bigcap_{j \in I_l} (\operatorname{int} D_j)^c \cap D_i \\
&= \bigcup_{i \in I_l} (\operatorname{bd} D_i) \cap \bigcap_{j \in I_l \setminus \{i\}} (\operatorname{int} D_j)^c \cap D_i \\
&\overset{(3.15)}{=} \bigcup_{i \in I_l} (\operatorname{bd} D_i) \cap D_i = \bigcup_{i \in I_l} \operatorname{bd} D_i.
\end{aligned}$$

□

### 3.2.1 Problems with one forbidden region ($l = 1$)

Let us analyze an important special case in which we have exactly one (i.e., $l = 1$) forbidden region. For notational convenience, we assume that $\phi := \phi_1$ and $D := D_1$.

In preparation of the next lemma, we define a new penalization function $\widehat{\phi} : \mathbf{E} \to \mathbb{R}$ by

$$\widehat{\phi} := -\phi.$$

**Lemma 3.13** ([53]) Let (3.4) be satisfied. Then, the following assertions are equivalent:

**1°.** $\phi$ fulfils the Assumptions (A5) and (A6).

**2°.** $\widehat{\phi}$ fulfils the Assumptions (A5) and (A6) with $\widehat{\phi}$ in the role of $\phi$ and $D$ in the role of $X$.

*Proof.* First, we are going to prove that

$$\operatorname{int}(\mathbf{E} \setminus \operatorname{int} D) = \mathbf{E} \setminus D. \tag{3.16}$$

Since $D$ is closed, we infer that $\mathbf{E} \setminus D$ is open. Then, then inclusion "$\supseteq$" in (3.16) follows by the fact that $\mathbf{E} \setminus D \subseteq \mathbf{E} \setminus \operatorname{int} D$. Now, we prove the reverse inclusion "$\subseteq$".

Assume that there is $x \in \operatorname{int}(\mathbf{E} \setminus \operatorname{int} D)$ with $x \notin \mathbf{E} \setminus D$, i.e., $x \in D$. Of course, since $x \in \mathbf{E} \setminus \operatorname{int} D$ we must have $x \in \operatorname{bd} D$. Consider $d \in \operatorname{int} D$. Due to the convexity of $D$, we infer that $]x, d] \subseteq \operatorname{int} D$ by Lemma 1.13. This means, for every $\delta \in ]0, 1]$, we have $x + ]0, \delta] \cdot (d - x) \subseteq \operatorname{int} D$. Hence, $x$ is no algebraic interior point of $\mathbf{E} \setminus \operatorname{int} D$, which implies $x \notin \operatorname{int}(\mathbf{E} \setminus \operatorname{int} D)$ by Lemma 1.20, a contradiction. We conclude that (3.16) holds.

So, we have

$$\begin{aligned}
& L_{<}(\mathbf{E}, \phi, 0) = \operatorname{int} X && (3.17)\\
\iff\ & L_{\geq}(\mathbf{E}, \phi, 0) = \mathbf{E} \setminus \operatorname{int} X \\
\iff\ & L_{\leq}(\mathbf{E}, \widehat{\phi}, 0) = \mathbf{E} \setminus \operatorname{int}(\mathbf{E} \setminus \operatorname{int} D) \\
\overset{(3.16)}{\iff}\ & L_{\leq}(\mathbf{E}, \widehat{\phi}, 0) = D && (3.18)
\end{aligned}$$

and

$$\begin{aligned}
& L_{\leq}(\mathbf{E}, \phi, 0) = X && (3.19)\\
\iff\ & L_{>}(\mathbf{E}, \phi, 0) = \mathbf{E} \setminus X \\
\iff\ & L_{<}(\mathbf{E}, \widehat{\phi}, 0) = \mathbf{E} \setminus (\mathbf{E} \setminus \operatorname{int} D) \\
\iff\ & L_{<}(\mathbf{E}, \widehat{\phi}, 0) = \operatorname{int} D. && (3.20)
\end{aligned}$$

Notice that (3.17) follows by (3.19) and

$$L_{=}(\mathbf{E}, \phi, 0) = \operatorname{bd} X, \tag{3.21}$$

while (3.17) and (3.19) imply (3.21). Analogously, (3.20) follows by (3.18) and

$$L_{=}(\mathbf{E}, \widehat{\phi}, 0) = \operatorname{bd} D, \tag{3.22}$$

while (3.20) and (3.18) imply (3.22). The proof is complete. □

**Lemma 3.14** ([53]) Let (3.4) be satisfied. Assume that $\widehat{\phi} = -\phi$ is a semi-strictly quasi-convex and continuous function on $\mathbf{E}$ which fulfils Assumption (A5) (with $\widehat{\phi}$ in the role of $\phi$ and $D$ in the role of $X$) and suppose that $L_{<}(\mathbf{E}, \widehat{\phi}, 0) \neq \emptyset$. Then, $\phi$ is a semi-strictly quasi-concave and continuous function and fulfils the Assumptions (A1), (A2), (A5) and (A6).

*Proof.* Follows immediately by Lemma 2.10 (3°) and Lemmata 2.54 and 3.13. □

**Example 3.15** By using a Minkowski gauge $\mu := \mu_B : \mathbf{E} \to \mathbb{R}$ associated to the set $B := -d + D$ with $d \in \operatorname{int} D$ (see Example 2.12), we consider the function $\phi : \mathbf{E} \to \mathbb{R}$ defined by

$$\widehat{\phi}(x) := \mu(x - d) - 1 = \inf\{\lambda \in \mathbb{R}_+ \mid x - d \in \lambda \cdot B\} - 1 \quad \text{for all } x \in \mathbf{E},$$

As we know from Corollary 1.63, $\widehat{\phi}$ is semi-strictly quasi-convex and continuous on $\mathbf{E}$ and fulfils Assumption (A5) (with $\widehat{\phi}$ in the role of $\phi$ and $D$ in the role of $X$). Moreover, we have $d \in L_<(\mathbf{E}, \widehat{\phi}, 0) \neq \emptyset$. So, in view of Lemma 3.14, we get that

$$\phi := -\widehat{\phi}(\,\cdot\,) = -\mu(\,\cdot\, - d) + 1$$

satisfies the Assumptions (A1), (A2), (A5) and (A6). In our considerations, the function

$$\overline{\phi}(\,\cdot\,) := -\mu(\,\cdot\, - d),$$

which fulfils the Assumptions (A1) and (A2) (with $\overline{\phi}$ in the role of $\phi$) by Lemma 2.10 (6°), will be of special interest.

In assertion 1° of Proposition 2.27, we need that the function $\phi$ fulfils the Assumption (A6). In the next lemma, we will show that the penalization function $\overline{\phi}$ given in Example 2.12 satisfies Assumption (A6) (with $\overline{\phi}$ in the role of $\phi$).

**Lemma 3.16** ([53]) Let (3.4) be satisfied. Consider any $x \in \operatorname{bd} X$ and $d \in \operatorname{int} D$ and define $x' := x + (x - d) \neq x$. Then, we have

$$[x', x[\, \subseteq L_<(\mathbf{E}, \overline{\phi}, \overline{\phi}(x)) = \operatorname{int} X.$$

Thus, $\overline{\phi}$ fulfils the Assumption (A6) (with $\overline{\phi}$ in the role of $\phi$).

*Proof.* First, notice that $\mu(x - d) = 1 > 0$ since $x \in \operatorname{bd} X = \operatorname{bd} D = L_=(\mathbf{E}, \overline{\phi}, -1)$. Hence, for any $\lambda \in ]0, 1]$, we have

$$\begin{aligned}
\overline{\phi}((1-\lambda)x + \lambda x') &= -\mu((1-\lambda)x + \lambda(2x - d) - d) \\
&= -\mu((\lambda + 1)(x - d)) \\
&= -(\lambda + 1)\mu(x - d) \\
&< -\mu(x - d) \\
&= \overline{\phi}(x).
\end{aligned}$$

The equality $L_<(\mathbf{E}, \overline{\phi}, \overline{\phi}(x)) = \operatorname{int} X$ follows by Remark 2.9 and Example 3.15. This shows the assertion in this lemma. □

In view of Theorems 2.25 and 2.39, we have the inclusions

$$\begin{aligned}
\operatorname{SEff}(X \mid f) &\subseteq \operatorname{SEff}(\mathbf{E} \mid f^{\oplus_1}), \\
\operatorname{WEff}(X \mid f) &\subseteq \operatorname{WEff}(\mathbf{E} \mid f^{\oplus_1}),
\end{aligned}$$

but in Example 2.21 we presented a counter-example for the convex case which shows that

$$\operatorname{Eff}(X \mid f) \subseteq \operatorname{Eff}(\mathbf{E} \mid f^{\oplus_1}) \tag{3.23}$$

does not hold in general. In the next example, we point out that inclusion (3.23) does not hold in our class of problems.

**Example 3.17** We consider a constrained convex multi-objective location problem with functions $f_1, f_2, f_3 : \mathbb{R}^2 \to \mathbb{R}$ defined by $f_i(x) := ||x - a^i||_1$ for all $x \in \mathbb{R}^2$ and all $i \in I_3$, where $a^1 := (5, 5), a^2 := (2, 2.5), a^3 := (3.5, 3.5) \in \mathbb{R}^2$. Consider the feasible set $X := \mathbb{R}^2 \setminus \operatorname{int} D$ with $D := [2, 3.5] \times [3.5, 5]$, and put $d := (3, 4) \in \operatorname{int} D$. Let the penalization function $\phi_1$ be given by the function $\overline{\phi}$ considered in Example 3.15. In the left part of Figure 3.2 one can see that the point $x' \in \operatorname{bd} X = \operatorname{bd} D$ is belonging to both sets $\operatorname{Eff}(X \mid f)$ and $\operatorname{Eff}(\mathbb{R}^2 \mid f)$. Notice that we have

$$\operatorname{Eff}(X \mid f) = \operatorname{Eff}(\mathbb{R}^2 \mid f) = ([2, 3.5] \times [2.5, 3.5]) \cup ([3.5, 5] \times [3.5, 5])$$

by the *Rectangular Decomposition Algorithm* (see Chapter 5). However, the right part of Figure

3.2 shows that $x' \notin \mathrm{Eff}(\mathbb{R}^2 \mid f^{\oplus_1})$ since $x'' \in (\operatorname{int} X) \cap S_=(\mathbb{R}^2, f, x')$. Consequently, the inclusion in (3.23) does not hold in this example.

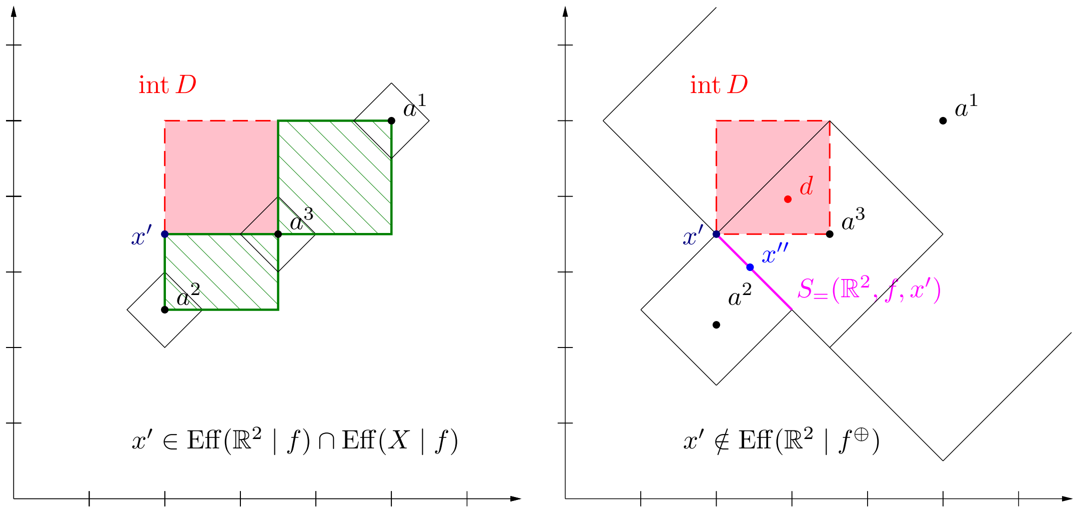


Figure 3.2: Counter-example for the inclusion (3.23).

### 3.2.2 Problems with multiple forbidden regions ($l > 1$)

The next theorem is related to the concept of Pareto efficiency and presents relationships between the initial constrained multi-objective optimization problem $(\mathcal{P}_X)$ and a finite family of unconstrained multi-objective optimization problems $(\mathcal{P}_{\mathbf{E}})$, $(\mathcal{P}_{\mathbf{E}}^{\oplus_i})$, $i \in I_l$.

**Theorem 3.18** ([53]) Let (3.4) be satisfied. Suppose that each function $\phi_i$, $i \in I_l$, fulfils Assumptions (A1) and (A2) (with $\phi_i$ in the role of $\phi$ and $X_i$ in the role of $X$). Then, we have:

**1°.** It holds that

$$X \cap \mathrm{Eff}(\mathbf{E} \mid f) \subseteq X \cap \bigcup_{i \in I_l} \mathrm{Eff}(X_i \mid f) \subseteq \mathrm{Eff}(X \mid f). \tag{3.24}$$

**2°.** Assume that (3.2) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex on $\mathbf{E}$. Then, we have

$$X \cap \bigcup_{i \in I_l} \mathrm{Eff}(X_i \mid f) \supseteq \mathrm{Eff}(X \mid f). \tag{3.25}$$

**3°.** Assume that (3.3) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise explicitly quasi-convex on $\mathbf{E}$. Then, (3.25) is true.

**4°.** We have

$$\mathrm{Eff}(X \mid f) \supseteq [X \cap \mathrm{Eff}(\mathbf{E} \mid f)] \cup \left[ \bigcup_{i \in I_l} X \cap (\operatorname{bd} D_i) \cap \mathrm{Eff}(\mathbf{E} \mid f^{\oplus_i}) \right]. \tag{3.26}$$

Now, suppose that (3.2) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex on $\mathbf{E}$. Then, we have

$$\mathrm{Eff}(X \mid f) = [X \cap \mathrm{Eff}(\mathbf{E} \mid f)] \cup \left[ \bigcup_{i \in I_l} (\operatorname{bd} D_i) \cap \mathrm{Eff}\left(\mathbf{E} \mid f^{\oplus_i}\right) \right]. \tag{3.27}$$

5°. Assume that (3.3) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise explicitly quasi-convex. Then, (3.27) is true.

*Proof.* 1°. Since $X \subseteq X_i \subseteq \mathbf{E}$ for all $i \in I_l$, we get (3.24) directly by Lemma 2.2 (1°).

2°. Consider $x \in \mathrm{Eff}(X \mid f)$. On one hand, we can have $x \in \mathrm{Eff}(\mathbf{E} \mid f)$, hence it follows $x \in \mathrm{Eff}(X_i \mid f)$ for some $i \in I_l$ by 1° of this theorem. On the other hand, we can have $x \notin \mathrm{Eff}(\mathbf{E} \mid f)$. Then, there exists $x^1 \in \mathbf{E} \setminus X = \bigcup_{i \in I_l} \operatorname{int} D_i$ with

$$x^1 \in L_<(\mathbf{E}, f_j, f_j(x)) \cap S_\leq(\mathbf{E}, f, x) \quad \text{for some } j \in I_m. \tag{3.28}$$

Without loss of generality, we assume $x^1 \in \operatorname{int} D_k$ for some $k \in I_l$. We are going to show that

$$\left[\bigcup_{i \in I_m} L_<(\mathbf{E}, f_i, f_i(x))\right] \cap S_\leq(\mathbf{E}, f, x) \subseteq \operatorname{int} D_k,$$

which implies $x \in \mathrm{Eff}(X_k \mid f)$.
Suppose that the contrary holds, i.e., there exists $x^2 \in \operatorname{int} D_{\overline{k}}$ with $\overline{k} \in I_l \setminus \{k\}$ such that

$$x^2 \in L_<(\mathbf{E}, f_{\overline{j}}, f_{\overline{j}}(x)) \cap S_\leq(\mathbf{E}, f, x) \quad \text{for some } \overline{j} \in I_m.$$

By (3.2) and the closedness of $D_i$, $i \in I_l$, we infer that the set $X \cap ]x^1, x^2[$ has an infinite number of elements. In particular, we have

$$\operatorname{card}\big(X \cap ]x^1, x^2[\big) \geq m + 2. \tag{3.29}$$

We are going to prove that

$$\exists\, x^3 \in ]x^1, x^2[ \colon\ x^3 \in L_<(X, f_j, f_j(x)) \cap S_\leq(X, f, x), \tag{3.30}$$

which implies $x \notin \mathrm{Eff}(X \mid f)$, a contradiction.
Since $\max\{f_i(x^1), f_i(x^2)\} \leq f_i(x)$ for every $i \in I_m$, we infer that

$$\operatorname{card}\left(\bigcup_{i \in I_m} L_>\big(]x^1, x^2[\,,\ f_i\,,\ f_i(x)\big)\right) \leq m \tag{3.31}$$

by Lemma 1.43. Now, for the specific index $j$ given in (3.28), we consider two cases:
*Case 1:* If $x^2 \in L_=(\mathbf{E}, f_j, f_j(x))$, then in view of Lemma 1.42 we get $[x^1, x^2[ \subseteq L_<(\mathbf{E}, f_j, f_j(x))$. By (3.29), it follows

$$\operatorname{card}\big(X \cap L_<\big(]x^1, x^2[\,,\ f_j\,,\ f_j(x)\big)\big) \geq m + 1. \tag{3.32}$$

*Case 2:* If $x^2 \in L_<(\mathbf{E}, f_j, f_j(x))$, then we have

$$\operatorname{card} L_>\big(]x^1, x^2[\,,\ f_j\,,\ s\big) \leq 1 \tag{3.33}$$

with $s := \max\{f_j(x^1), f_j(x^2)\} < f_j(x)$ by Lemma 1.43. Due to (3.29) and (3.33), it follows (3.32).

So, in both cases (3.32) holds. Consequently, we get the validity of (3.30) by (3.31) and (3.32). This completes the proof of assertion 2°.

3°. The proof is analogous to the proof of assertion 2°. By (3.3), we get $\operatorname{card}\big(X \cap ]x^1, x^2[\big) \geq 1$ instead of (3.29). Notice, for any $i \in I_m$, the conditions $x^1, x^2 \in L_\sim(\mathbf{E}, f_i, f_i(x))$ imply $]x^1, x^2[ \subseteq L_\sim(\mathbf{E}, f_i, f_i(x))$ for all $\sim \in \{<, \leq\}$ by the quasi-convexity of $f_i$ on $\mathbf{E}$. Consequently, it follows

$$\emptyset \neq X \cap ]x^1, x^2[ \subseteq L_<(X, f_j, f_j(x)) \cap S_\leq(X, f, x).$$

4°. By Corollary 3.1 (1°), for any $i \in I_l$, we have

$$[X_i \cap \mathrm{Eff}(\mathbf{E} \mid f)] \cup \big[(\operatorname{bd} X_i) \cap \mathrm{Eff}(\mathbf{E} \mid f^{\oplus_i})\big] \subseteq \mathrm{Eff}(X_i \mid f). \tag{3.34}$$

Notice that $\operatorname{int} X_i \neq \emptyset$ by Lemma 3.16. Then, due to 1° of this theorem, we get

$$\begin{aligned}
\operatorname{Eff}(X \mid f) &\overset{(3.24)}{\supseteq} X \cap \bigcup_{i \in I_l} \operatorname{Eff}(X_i \mid f) \\
&\overset{(3.34)}{\supseteq} X \cap \bigcup_{i \in I_l} \big( [X_i \cap \operatorname{Eff}(\mathbf{E} \mid f)] \cup [(\operatorname{bd} X_i) \cap \operatorname{Eff}(\mathbf{E} \mid f^{\oplus_i})] \big) \\
&= \left[ \bigcup_{i \in I_l} X \cap X_i \cap \operatorname{Eff}(\mathbf{E} \mid f) \right] \cup \left[ \bigcup_{i \in I_l} X \cap (\operatorname{bd} X_i) \cap \operatorname{Eff}(\mathbf{E} \mid f^{\oplus_i}) \right] \\
&= [X \cap \operatorname{Eff}(\mathbf{E} \mid f)] \cup \left[ \bigcup_{i \in I_l} X \cap (\operatorname{bd} D_i) \cap \operatorname{Eff}(\mathbf{E} \mid f^{\oplus_i}) \right],
\end{aligned}$$

where $X \cap X_i = X$ for every $i \in I_l$, which shows (3.26).

Assume that (3.2) holds. Let $f$ be componentwise semi-strictly quasi-convex on $\mathbf{E}$. By (3.25) and by Corollary 3.1 (3°), we get the reverse inclusion, which shows (3.27) in view of (3.13).

**5°.** This assertion follows by 1° and 3° of this theorem as well as by the ideas given in the proof of assertion 4°.

□

Notice that the assumptions (3.2) in 4° and (3.3) in 5° of Theorem 3.18 are essential for the validity of (3.27) (see Example 6.6 in Chapter 6).

In the next theorem, we derive relationships between the initial constrained multi-objective optimization problem $(\mathcal{P}_X)$ and the corresponding unconstrained problems $(\mathcal{P}_{\mathbf{E}})$ and $(\mathcal{P}_{\mathbf{E}}^{\oplus_i})$, $i \in I_l$, for the concept of weak Pareto efficiency.

**Theorem 3.19** ([53]) Let (3.4) be satisfied. Suppose that each penalization function $\phi_i$, $i \in I_l$, fulfils Assumptions (A1) and (A2) (with $\phi_i$ in the role of $\phi$ and $X_i$ in the role of $X$). Then, the following hold:

**1°.** We have

$$X \cap \operatorname{WEff}(\mathbf{E} \mid f) \subseteq X \cap \bigcup_{i \in I_l} \operatorname{WEff}(X_i \mid f) \subseteq \operatorname{WEff}(X \mid f).$$

**2°.** Assume that (3.2) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$. Then, we have

$$X \cap \bigcup_{i \in I_l} \operatorname{WEff}(X_i \mid f) \supseteq \operatorname{WEff}(X \mid f). \tag{3.35}$$

**3°.** Assume that (3.3) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise quasi-convex on $\mathbf{E}$. Then, (3.35) is true.

**4°.** Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise upper semi-continuous along line segments on $\mathbf{E}$. Assume that each function $\phi_i$, $i \in I_l$, fulfils Assumption (A6). Then, we have

$$\operatorname{WEff}(X \mid f) \supseteq [X \cap \operatorname{WEff}(\mathbf{E} \mid f)] \cup \left[ \bigcup_{i \in I_l} X \cap (\operatorname{bd} D_i) \cap \operatorname{WEff}(\mathbf{E} \mid f^{\oplus_i}) \right].$$

Now, suppose that (3.2) holds. In addition, assume that $f : \mathbf{E} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $\mathbf{E}$. Then, we have

$$\operatorname{WEff}(X \mid f) = [X \cap \operatorname{WEff}(\mathbf{E} \mid f)] \cup \left[ \bigcup_{i \in I_l} (\operatorname{bd} D_i) \cap \operatorname{WEff}\left(\mathbf{E} \mid f^{\oplus_i}\right) \right]. \tag{3.36}$$

**5°.** Suppose that (3.3) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise explicitly quasi-convex and upper semi-continuous along line segments on $\mathbf{E}$. Assume that each function $\phi_i$, $i \in I_l$, fulfils Assumption (A6). Then, (3.36) is true.

*Proof.* The proof uses similar ideas as given in the proof of Theorem 3.18.

**1°.** Follows by Lemma 2.2 (1°).

**2°.** Let $x \in \mathrm{WEff}(X \mid f)$. If $x \in \mathrm{WEff}(\mathbf{E} \mid f)$, then $x \in X \cap \mathrm{WEff}(X_j \mid f)$ for some $j \in I_l$ by 1° of this theorem. In what follows, we assume that $x \notin \mathrm{WEff}(\mathbf{E} \mid f)$. Consequently, there is $x^1 \in S_<(\mathbf{E}, f, x) \cap \operatorname{int} D_k$ for some $k \in I_l$. We show that $x \in \mathrm{WEff}(X_k \mid f)$.

Assume the contrary holds, i.e., $x \notin \mathrm{WEff}(X_k \mid f)$. Then, there exists $x^2 \in S_<(\mathbf{E}, f, x) \cap \operatorname{int} D_{\overline{k}}$ for some $\overline{k} \in I_l \setminus \{k\}$. Consider $i \in I_m$. If $f_i$ is semi-strictly quasi-convex on $\mathbf{E}$, then we get

$$\operatorname{card} L_{\geq}\left(]x^1, x^2[\,,\, f_i\,,\, f_i(x)\right) \leq 1$$

by Lemma 1.43. If $f_i$ is quasi-convex on $\mathbf{E}$, then it follows

$$\operatorname{card} L_{\geq}\left(]x^1, x^2[\,,\, f_i\,,\, f_i(x)\right) = 0.$$

So, we conclude

$$\operatorname{card}\left(\bigcup_{i \in I_m} L_{\geq}\left(]x^1, x^2[\,,\, f_i\,,\, f_i(x)\right)\right) \leq m. \tag{3.37}$$

By (3.29) and (3.37), we infer that there exists $x^3 \in ]x^1, x^2[$ such that $x^3 \in S_<(X, f, x)$. This shows $x \notin \mathrm{WEff}(X \mid f)$, a contradiction.

**3°.** The proof is analogous to the proof of assertion 2°. Notice that one has

$$\emptyset \neq X \cap ]x^1, x^2[ \subseteq S_<(X, f, x).$$

**4°.** The proof uses Corollary 2.27, Theorem 3.19 (1°, 2°), formula (3.13), and the ideas given in the proof of Theorem 3.18 (4°).

**5°.** This assertion follows by 1° and 3° of this theorem as well as by the ideas given in the proof of assertion 4°.

□

It is important to mention that the assumptions (3.2) in 4° and (3.3) in 5° of Theorem 3.19 are essential for the validity of (3.36) (see Example 6.6 in Chapter 6).

We now present similar relationships for the concept of strict Pareto efficiency.

**Theorem 3.20** ([53]) Let (3.4) be satisfied. Suppose that each function $\phi_i$, $i \in I_l$, fulfils Assumptions (A1) and (A2) (with $\phi_i$ in the role of $\phi$ and $X_i$ in the role of $X$). Then, the following hold:

**1°.** We have

$$X \cap \mathrm{SEff}(\mathbf{E} \mid f) \subseteq X \cap \bigcup_{i \in I_l} \mathrm{SEff}(X_i \mid f) \subseteq \mathrm{SEff}(X \mid f).$$

**2°.** Assume that (3.2) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$. Then, we have

$$X \cap \bigcup_{i \in I_l} \mathrm{SEff}(X_i \mid f) \supseteq \mathrm{SEff}(X \mid f). \tag{3.38}$$

**3°.** We have

$$\mathrm{SEff}(X \mid f) \supseteq [X \cap \mathrm{SEff}(\mathbf{E} \mid f)] \cup \left[ \bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{SEff}(\mathbf{E} \mid f^{\oplus_i}) \right].$$

Now, suppose that (3.2) holds. In addition, assume that $f : \mathbf{E} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$. Then, we have

$$\mathrm{SEff}(X \mid f) = [X \cap \mathrm{SEff}(\mathbf{E} \mid f)] \cup \left[ \bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{SEff}\left(\mathbf{E} \mid f^{\oplus_i}\right) \right]. \tag{3.39}$$

*Proof.* The proof uses similar ideas as given in the proof of Theorem 3.18.

**1°.** Follows by Lemma 2.2 (1°).

**2°.** Consider $x \in \mathrm{SEff}(X \mid f)$. In the case that $x \in \mathrm{SEff}(\mathbf{E} \mid f)$, we conclude $x \in X \cap \mathrm{SEff}(X_j \mid f)$ for some $j \in I_l$ by 1° of this theorem. In the second case, we can have $x \notin \mathrm{SEff}(\mathbf{E} \mid f)$, hence there exists $x^1 \in S_{\leq}(\mathbf{E}, f, x) \cap \mathrm{int}\, D_k$ for some $k \in I_l$. Now, we are going to prove that $x \in \mathrm{SEff}(X_k \mid f)$. Assume the contrary holds, i.e., $x \notin \mathrm{SEff}(X_k \mid f)$. Then, there exists a point $x^2 \in S_{\leq}(\mathbf{E}, f, x) \cap \mathrm{int}\, D_{\overline{k}}$ for some $\overline{k} \in I_l \setminus \{k\}$.
Let $i \in I_m$. If $f_i$ is semi-strictly quasi-convex on $\mathbf{E}$, then we get

$$\mathrm{card}\, L_{>}\left( ]x^1, x^2[\, , \, f_i \, , \, f_i(x) \right) \leq 1$$

by Lemma 1.43. If $f_i$ is quasi-convex on $\mathbf{E}$, then it follows

$$\mathrm{card}\, L_{>}\left( ]x^1, x^2[\, , \, f_i \, , \, f_i(x) \right) = 0.$$

Hence, we infer

$$\mathrm{card} \left( \bigcup_{i \in I_m} L_{>}\left( ]x^1, x^2[\, , \, f_i \, , \, f_i(x) \right) \right) \leq m. \tag{3.40}$$

Taking into account (3.29) and (3.40), we get that there exists $x^3 \in ]x^1, x^2[ \setminus \{x\}$ such that $x^3 \in S_{\leq}(X, f, x)$. This implies $x \notin \mathrm{SEff}(X \mid f)$, a contradiction.

**3°.** The proof uses Corollary 2.40, Theorem 3.20 (1°, 2°), formula (3.13), and the ideas given in the proof of Theorem 3.18 (4°).

□

**Remark 3.21** Consider the points $x, x^1, x^2 \in \mathbf{E}$ as given in the proof of 2° in Theorem 3.20. Under the weaker assumption (3.3) (in comparison to (3.2)) and the componentwise quasi-convexity of $f$, we get

$$\emptyset \neq X \cap ]x^1, x^2[ \subseteq S_{\leq}(X, f, x).$$

We notice, however, that $X \cap ]x^1, x^2[$ can be a singleton set. Hence, in the proof of 2° in Theorem 3.20, we can not ensure that we have

$$X \cap ]x^1, x^2[ \neq \{x\}. \tag{3.41}$$

For the concepts of Pareto efficiency and weak Pareto efficiency, we know that there is $x^3 \in X \cap ]x^1, x^2[$ such that $x^3 \in L_{<}(\mathbf{E}, f_j, f_j(x)) \cap S_{\leq}(\mathbf{E}, f, x)$ for some $j \in I_m$, hence (3.41) holds.

The assumption (3.2) in 3° of Theorem 3.20 is essential for the validity of (3.39), as shown in Example 6.6 in Chapter 6.

We end this section by considering a specific type of penalization functions $\phi_i$, $i \in I_l$, that fulfils the Assumptions (A1), (A2) and (A6) (with $\phi_i$ in the role of $\phi$ and $X_i$ in the role of $X$, see Example 3.15 and Lemma 3.16).

**Corollary 3.22** ([53]) Assume that (3.4) holds. Let each penalization function $\phi_i$, $i \in I_l$, be defined by

$$\phi_i(x) := -\inf\{\lambda \in \mathbb{R}_+ \mid x - d^i \in \lambda \cdot (-d^i + D_i)\} \quad \text{for all } x \in \mathbf{E},$$

where $d^i \in \operatorname{int} D_i$. Then, the following hold:

**1°.** We have

$$\operatorname{SEff}(X \mid f) \supseteq [X \cap \operatorname{SEff}(\mathbf{E} \mid f)] \cup \left[\bigcup_{i \in I_l} X \cap (\operatorname{bd} X_i) \cap \operatorname{SEff}(\mathbf{E} \mid f^{\oplus_i})\right];$$

$$\operatorname{Eff}(X \mid f) \supseteq [X \cap \operatorname{Eff}(\mathbf{E} \mid f)] \cup \left[\bigcup_{i \in I_l} X \cap (\operatorname{bd} X_i) \cap \operatorname{Eff}(\mathbf{E} \mid f^{\oplus_i})\right].$$

Suppose that $f : \mathbf{E} \to \mathbb{R}^m$ is componentwise upper semi-continuous along line segments on $\mathbf{E}$. Then, it follows

$$\operatorname{WEff}(X \mid f) \supseteq [X \cap \operatorname{WEff}(\mathbf{E} \mid f)] \cup \left[\bigcup_{i \in I_l} X \cap (\operatorname{bd} X_i) \cap \operatorname{WEff}(\mathbf{E} \mid f^{\oplus_i})\right].$$

Moreover, under the validity of (3.2) or (3.3), one can replace $X \cap (\operatorname{bd} X_i)$ by $\operatorname{bd} D_i$ for every $i \in I_l$.

**2°.** If $f$ is componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$, then we have

$$\operatorname{SEff}(X \mid f) \subseteq [X \cap \operatorname{SEff}(\mathbf{E} \mid f)] \cup \operatorname{bd} X.$$

**3°.** If $f : \mathbf{E} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on $\mathbf{E}$, then

$$\operatorname{Eff}(X \mid f) \subseteq [X \cap \operatorname{Eff}(\mathbf{E} \mid f)] \cup \operatorname{bd} X;$$
$$\operatorname{WEff}(X \mid f) \subseteq [X \cap \operatorname{WEff}(\mathbf{E} \mid f)] \cup \operatorname{bd} X.$$

**4°.** Assume that (3.2) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$. Then, we have

$$\operatorname{SEff}(X \mid f) = [X \cap \operatorname{SEff}(\mathbf{E} \mid f)] \cup \left[\bigcup_{i \in I_l} (\operatorname{bd} D_i) \cap \operatorname{SEff}\left(\mathbf{E} \mid f^{\oplus_i}\right)\right].$$

**5°.** Assume that (3.3) holds. Let $f : \mathbf{E} \to \mathbb{R}^m$ be componentwise explicitly quasi-convex. Then, we have

$$\operatorname{Eff}(X \mid f) = [X \cap \operatorname{Eff}(\mathbf{E} \mid f)] \cup \left[\bigcup_{i \in I_l} (\operatorname{bd} D_i) \cap \operatorname{Eff}\left(\mathbf{E} \mid f^{\oplus_i}\right)\right].$$

In addition, suppose that $f : \mathbf{E} \to \mathbb{R}^m$ is componentwise upper semi-continuous along line segments on $\mathbf{E}$. Then, it follows

$$\operatorname{WEff}(X \mid f) = [X \cap \operatorname{WEff}(\mathbf{E} \mid f)] \cup \left[\bigcup_{i \in I_l} (\operatorname{bd} D_i) \cap \operatorname{WEff}\left(\mathbf{E} \mid f^{\oplus_i}\right)\right].$$

*Proof.* Follows by Corollary 2.6, Theorems 3.18, 3.19 and 3.20 and formula (3.13). $\square$

## 3.3 Concluding remarks

In Section 3.1, we considered the problem $(\mathcal{P}_X)$ with a feasible set given by a union of a finite number of pairwise disjoint, closed, convex sets with nonempty interiors. We succeded to find a penalization function $\phi : \mathbf{E} \to \mathbb{R}$, defined by

$$\phi(x) := \min\{\mu_i(x - d^i) \mid i \in I_l\} = \sum\nolimits_{i \in I_l} \mu_i(x - d^i) - \max_{i \in I_l} \sum\nolimits_{j \in I_l \setminus \{i\}} \mu_j(x - d^j)$$

for all $x \in \mathbf{E}$. This function $\phi$ is a difference of two convex functions (a so-called *d.c. function*) and fulfils the important Assumptions (A1) and (A2) (see Corollary 3.9, Lemma 3.10 and Remark 3.11). In forthcoming works, we aim to extend the analysis of multi-objective optimization problems where the feasible set is given by a finite union of convex sets. It seems to be attractive to use the decomposition of $\phi$ as a *d.c. function* in order to characterize the sets of (strictly, weakly) Pareto efficient solutions of certain multi-objective optimization problems. In Theorem 3.4, we presented some outer approximations for the sets of (strictly, weakly) Pareto efficient solutions of problem $(\mathcal{P}_X)$ . It is an open question whether it is possible to find characterizations for these sets of solutions that possibly involve at least a part of the sets of (strictly, weakly) Pareto efficient solutions of the unconstrained problems $(\mathcal{P}_\mathbf{E})$, $(\mathcal{P}_\mathbf{E}^{\oplus_i})$, $i \in I_l$.

It should be mentioned that the idea to use a closed, convex decomposition of the closed, nonconvex, feasible set $X \subsetneq \mathbf{E}$ (as considered in Section 3.1, see the assumption in (3.4)) is also used by Carrizosa *et al.* [19] in order to generate the set of weakly Pareto efficient solutions of a special class of constrained planar multi-objective location problems. Moreover, Hansen, Peeters, Thisse [60] considered a scalar location problem involving a feasible set given by a union of convex polygons.

The results given in Section 3.2 are based on the paper Günther [53]. It is interesting to study the question whether it is possible to characterize the sets of (strictly, weakly) Pareto efficient solutions of problem $(\mathcal{P}_X)$, under similar assumptions as in Corollary 3.22), but without assuming that (3.2) or (3.3) hold.

In Chapter 6, we apply our results from Section 3.2 to a special class of nonconvex multi-objective location problems involving multiple forbidden regions.

# Part II

# Application in Location Theory

# Chapter 4

# Multi-objective location theory

In this chapter, we demonstrate that the results derived in the previous chapters are very useful for application to *constrained multi-objective locational analysis*. Consider $m$ *a priori given facilities* located at the points $a^1, \cdots, a^m$ in a real linear topological space $\mathbf{E}$. For notational convenience, we define the *set of all existing facilities* by

$$\mathcal{A} := \{a^1, \cdots, a^m\}.$$

To each *existing facility* $a^i$, $i \in I_m$, we associate a certain Minkowski gauge $\eta_i : \mathbf{E} \to \mathbb{R}$. Recall that a Minkowksi gauge is already introduced in Section 1.4, and now, for any $i \in I_m$, we put

$$\eta_i(x) := \mu_{B_i}(x) = \inf\{\lambda > 0 \,|\, x \in \lambda \cdot B_i\} \quad \text{for all } x \in \mathbf{E},$$

where $B_i \subsetneq \mathbf{E}$ is a closed, convex unit ball with $0 \in \operatorname{core} B_i$. So, for any $i \in I_m$, the distance between the *new facility* $x \in \mathbf{E}$ and an *existing facility* $a^i$ is given by $\eta_i(x - a^i)$.

Let us define a vector-valued function $\eta_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^m$ associated to the set $\mathcal{A}$ by

$$\eta_{\mathcal{A}}(x) := (\eta_1(x - a^1), \cdots, \eta_m(x - a^m)) \quad \text{for all } x \in \mathbf{E}.$$

In order to describe the preferences of the decision maker, one could introduce additional scalar functions $h_1, \cdots, h_p : \mathbb{R}^m_+ \to \mathbb{R}$, $p \in \mathbb{N}$. Under the assumption that $h_1, \cdots, h_p$ are $\mathbb{R}^m_+$-increasing on $\mathbb{R}^m_+$, it is usual in economics to call them *disutility functions*. Then, the functions $h_1, \cdots, h_p$ and $\eta_{\mathcal{A}}$ can be used to define a vector-valued objective function

$$g_{\mathcal{A}} = (g_1, \cdots, g_p) : \mathbf{E} \to \mathbb{R}^p$$

associated to the set $\mathcal{A}$, where its components are given by the composite functions $g_i := h_i \circ \eta_{\mathcal{A}}$, $i \in I_p$. In what follows, our principal goal is to minimize $g_{\mathcal{A}}$ over a nonempty, closed feasible set $X \subseteq \mathbf{E}$, i.e., we consider the *multi-objective composite location problem*

$$\begin{cases} g_{\mathcal{A}}(x) = ((h_1 \circ \eta_{\mathcal{A}})(x), \cdots, (h_p \circ \eta_{\mathcal{A}})(x)) \to \min \text{ w.r.t. } \mathbb{R}^p_+ \\ x \in X. \end{cases} \tag{LP$_X(\mathcal{A})$}$$

**Remark 4.1** Plastria [99] was probably the first author who considered this general location model given in problem (LP$_X(\mathcal{A})$) for the single-objective case (i.e., $p = 1$), where he assumed that $X = \mathbf{E} = \mathbb{R}^n$, $h_1$ is a lower semi-continuous and $\mathbb{R}^m_+$-increasing function on $\mathbb{R}^m_+$, and $\eta_1, \cdots, \eta_m$ are norms on $\mathbb{R}^n$. In addition, the single-objective case is also considered by Gromicho [51], Nickel, Puerto and Rodríguez-Chía [91], Puerto and Fernández [108], Schöbel [114, Sec. 7.2] and Wanka, Boţ and Vargyas [124]. In fact (LP$_X(\mathcal{A})$) can be seen as a special case of a *multi-objective composite optimization problem*, studied, among others, by Boţ [13], Boţ, Vargyas and Wanka [14], Jeyakumar and Yang [66], Vargyas [119], and Wanka, Boţ and Vargyas [123]. This observation justifies the name "*multi-objective composite location problem*" for (LP$_X(\mathcal{A})$).

By imposing additional assumptions on the scalar functions $h_i$, $i \in I_p$, we can ensure that $g_{\mathcal{A}}$ is a componentwise convex (quasi-convex, semi-strictly quasi-convex) function on $\mathbf{E}$ by Lemma 1.49.

**Lemma 4.2** Assume that $h_i : \mathbb{R}^m_+ \to \mathbb{R}$, $i \in I_p$, are $\mathbb{R}^m_+$-increasing functions on $\mathbb{R}^m_+$. Then, the following assertions hold:

**1°.** If $h_i$, $i \in I_p$, are convex on $\mathbb{R}^m_+$, then $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ is componentwise convex on $\mathbf{E}$.

**2°.** If $h_i$, $i \in I_p$, are quasi-convex on $\mathbb{R}^m_+$, then $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ is componentwise quasi-convex on $\mathbf{E}$.

**3°.** If $h_i$, $i \in I_p$, are semi-strictly quasi-convex on $\mathbb{R}^m_+$, then $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ is componentwise semi-strictly quasi-convex on $\mathbf{E}$.

In what follows, we recall four well-known special cases of the *multi-objective composite location problem* $(\mathrm{LP}_X(\mathcal{A}))$ for locating a new facility with respect to a set of fixed facilities $\mathcal{A}$, namely

$(C1)$ the class of *point-objective location problems* (see Section 4.1)

$$\begin{cases} p = m, \\ \forall i \in I_m : \ \lambda^i := 0_{\mathbb{R}^m} \ \text{and} \ \lambda^i_i := 1, \\ \forall i \in I_m \, \forall y \in \mathbb{R}^m_+ : \ h_i(y) := \langle \lambda^i, y \rangle; \end{cases} \tag{4.1}$$

$(C2)$ the class of *multi-objective min-sum location problems* (see Section 4.2)

$$\begin{cases} \forall i \in I_p : \ \lambda^i = (\lambda^i_1, \cdots, \lambda^i_m) \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\}, \\ \forall i \in I_p \, \forall y \in \mathbb{R}^m_+ : \ h_i(y) := \langle \lambda^i, y \rangle; \end{cases} \tag{4.2}$$

$(C3)$ the class of *multi-objective min-max location problems* (see Section 4.3)

$$\begin{cases} \forall i \in I_p : \ \lambda^i = (\lambda^i_1, \cdots, \lambda^i_m) \in \mathbb{R}^m_+ \setminus \{0_{\mathbb{R}^m}\}, \\ \forall i \in I_p \, \forall y \in \mathbb{R}^m_+ : \ h_i(y) := \max_{j \in I_m} \lambda^i_j \cdot y_j; \end{cases} \tag{4.3}$$

$(C4)$ the class of *multi-objective ordered median location problems* (see Section 4.4)

$$\begin{cases} \forall i \in I_p : \ \lambda^i = (\lambda^i_1, \cdots, \lambda^i_m) \in \mathbb{R}^m, \\ \forall i \in I_p : \ w^i = (w^i_1, \cdots, w^i_m) \in \mathbb{R}^m, \\ \forall i \in I_p \, \forall y \in \mathbb{R}^m_+ : \ h_i(y) := \langle \lambda^i, \mathrm{sort}_m(\mathrm{weight}_{w^i}(y)) \rangle, \end{cases} \tag{4.4}$$

where

- the function $\mathrm{sort}_m : \mathbb{R}^m \to \mathbb{R}^m$ is defined, for any $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, by

$$\mathrm{sort}_m(x) := (x_{j_1}, \cdots, x_{j_m}),$$

  and $x_{j_1}, \cdots, x_{j_m}$ is a enumeration of $x_1, \cdots, x_m$ such that $x_{j_1} \leq \cdots \leq x_{j_m}$;
- the function $\mathrm{weight}_w : \mathbb{R}^m \to \mathbb{R}^m$, $w = (w_1, \cdots, w_m) \in \mathbb{R}^m$, is defined, for any $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, by

$$\mathrm{weight}_w(x) := (w_1 x_1, \cdots, w_m x_m).$$

**Remark 4.3** Notice that the class $(C1)$ is included in the classes $(C2)$ and $(C3)$, while $(C2)$ and $(C3)$ are part of $(C4)$. There are also some further classes of multi-objective location problems that can be seen as subclasses of $(C4)$ (see Nickel and Puerto [90]).

In preparation of the next lemma, for any any $\lambda \in \mathbb{R}^m$, the function

$$\mathrm{OMF}_\lambda := \langle \lambda, \mathrm{sort}_m(\cdot) \rangle : \mathbb{R}^m \to \mathbb{R}$$

stands for the so-called *Ordered Median Function* (see Nickel and Puerto [90, Sec. 1.3]). Hence, for any $i \in I_p$, we have $h_i = \mathrm{OMF}_{\lambda^i} \circ \mathrm{weight}_{w^i}$ under the assumption (4.4) given for class $(C4)$.

**Lemma 4.4** The following assertions hold:

1°. Consider a problem of class $(C1)$, $(C2)$ or $(C3)$. Then, the function $g_{\mathcal{A}}$ given in $(\mathrm{LP}_X(\mathcal{A}))$ is componentwise convex on $\mathbf{E}$.

2°. Consider a problem of class $(C4)$. If $\lambda^i = (\lambda_1^i, \cdots, \lambda_m^i) \in \mathbb{R}_+^m$, $\lambda_1^i \leq \cdots \leq \lambda_m^i$ and $w^i = (w_1^i, \cdots, w_m^i) \in \mathbb{R}_+^m$ for all $i \in I_p$, then $g_{\mathcal{A}}$ is componentwise convex on $\mathbf{E}$.

*Proof.* Assertion 1° follows by Lemma 4.2 since it can easily be verified that $h_i$, $i \in I_p$, are convex and $\mathbb{R}_+^m$-increasing functions on $\mathbb{R}_+^m$.

Now, let us prove assertion 2°. Consider any $i \in I_p$. By Nickel and Puerto [90, Prop. 1.1], $\mathrm{OMF}_{\lambda^i}$ is convex on $\mathbb{R}^m$ if and only if $0 \leq \lambda_1^i \leq \cdots \leq \lambda_m^i$. So, since $\mathrm{OMF}_{\lambda^i}$ is convex and $\mathbb{R}_+^m$-increasing on $\mathbb{R}_+^m$ under our assumptions in assertion 2°, and $\mathrm{weight}_{w^i}$ is componentwise convex on $\mathbb{R}_+^m$, the convexity of $h_i = \mathrm{OMF}_{\lambda^i} \circ \mathrm{weight}_{w^i}$ on $\mathbb{R}_+^m$ follows by Lemma 1.49. Then, the functions $h_i$, $i \in I_p$, are convex and $\mathbb{R}_+^m$-increasing on $\mathbb{R}_+^m$ under our assumptions in 2°, hence $g_{\mathcal{A}}$ is componentwise convex on $\mathbf{E}$ by Lemma 4.2. □

In order to apply our penalization approach from the previous chapters, we have to think about an approriate penalization function $\phi : \mathbf{E} \to \mathbb{R}$. There are several possibilities, depending on the structure of the feasible set:

- $\phi(\cdot) := \mu(\cdot - d)$
  (assume that $X$ is convex, $d \in \operatorname{int} X$, and the unit ball of the Minkowski gauge $\mu = \mu_B$ is $B := -d + X$; see Example 2.12);
- $\phi(\cdot) := d_X(\cdot)$
  (distance function with respect to $X$; see Example 2.13);
- $\phi(\cdot) := \triangle_X(\cdot)$
  (Hiriart-Urruty function with respect to $X$; see Example 2.14);
- $\phi(\cdot) := \varphi_{X,k}(\cdot)$
  (Tammer-Weidner scalarizing function; see Example 2.15);
- $\phi(\cdot) := -\mu(\cdot - d)$
  (assume that $X := \mathbf{E} \setminus \operatorname{int} D$ for a closed, convex set $D \subseteq \mathbf{E}$ with $d \in \operatorname{int} D$, and the unit ball of the Minkowski gauge $\mu = \mu_B$ is $B := -d + X$; see Example 3.15);
- $\phi(\cdot) := \min\{\mu_i(\cdot - d^i) \mid i \in I_l\} = \sum_{i \in I_l} \mu_i(\cdot - d^i) - \max_{i \in I_l} \sum_{j \in I_l \setminus \{i\}} \mu_j(\cdot - d^j)$
  (assume that $X$ is given by a union of closed, convex, pairwise-disjoint sets $D_1, \cdots, D_l \subseteq \mathbf{E}$ with $d^i \in \operatorname{int} D_i$, $i \in I_l$, as used in (3.4); see Section 3.1).

So, we are able to introduce the following unconstrained multi-objective location problem:

$$\begin{cases} g_{\mathcal{A}}^{\oplus}(x) := ((h_1 \circ \eta_{\mathcal{A}})(x), \cdots, (h_p \circ \eta_{\mathcal{A}})(x), \phi(x)) \to \min \text{ w.r.t. } \mathbb{R}_+^{p+1} \\ x \in \mathbf{E}. \end{cases} \tag{$\mathrm{LP}_{\mathbf{E}}(\mathcal{A})^{\oplus}$}$$

In the following two propositions, we collect some relationships between the problems $(\mathrm{LP}_X(\mathcal{A}))$ and $(\mathrm{LP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ that follow from results derived in Section 2.4.

**Proposition 4.5** Suppose that $\phi : \mathbf{E} \to \mathbb{R}$ fulfils Assumptions (A1) and (A2). Then, the following assertions hold:

1°. We have

$$[X \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \left[(\operatorname{bd} X) \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\right] \subseteq \mathrm{Eff}(X \mid g_{\mathcal{A}});$$
$$[X \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \left[(\operatorname{bd} X) \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\right] \subseteq \mathrm{SEff}(X \mid g_{\mathcal{A}}).$$

**2°.** Suppose that $\operatorname{int} X \neq \emptyset$. Let $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ be componentwise upper semi-continuous along line segments on $\mathbf{E}$. Assume that $\phi : \mathbf{E} \to \mathbb{R}$ fulfils Assumption (A6). Then, it holds that

$$[X \cap \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \subseteq \mathrm{WEff} X \mid g_{\mathcal{A}}).$$

**3°.** In the case that $\operatorname{int} X \neq \emptyset$, assume additionally that $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ is componentwise semi-strictly quasi-convex on $\mathbf{E}$. Then, we have

$$[X \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \supseteq \mathrm{Eff}(X \mid g_{\mathcal{A}});$$
$$[X \cap \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \supseteq \mathrm{WEff}(X \mid g_{\mathcal{A}}).$$

**4°.** If $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ be componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$, then

$$[X \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \supseteq \mathrm{SEff}(X \mid g_{\mathcal{A}}).$$

**Proposition 4.6** Suppose that $\phi : \mathbf{E} \to \mathbb{R}$ fulfils Assumption (A3). Then, the following hold:

**1°.** It holds that

$$[X \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \subseteq \mathrm{Eff}(X \mid g_{\mathcal{A}}) = X \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus});$$
$$[X \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \subseteq \mathrm{SEff}(X \mid g_{\mathcal{A}}) = X \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus}).$$

**2°.** In the case that $\operatorname{int} X \neq \emptyset$, assume additionally that $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ is componentwise semi-strictly quasi-convex on $\mathbf{E}$. Then, we have

$$[X \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \supseteq \mathrm{Eff}(X \mid g_{\mathcal{A}}).$$

**3°.** In the case that $\operatorname{int} X \neq \emptyset$, assume additionally that $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^p$ is componentwise semi-strictly quasi-convex or quasi-convex on $\mathbf{E}$. Then, we have

$$[X \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}})] \cup \big[(\operatorname{bd} X) \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus})\big] \supseteq \mathrm{SEff}(X \mid g_{\mathcal{A}}).$$

Taking into account the literature in multi-objective location theory (see Sections 4.1, 4.2, 4.3, 4.4), one can see that most works concentrate on the case that the distances $\eta_i(x - a^i)$, $i \in I_m$, are induced by round norms or block norms (or more general polyhedral Minkowski gauges), and the feasible set $X$ is assumed to be closed and convex. In particular, the case $X = \mathbf{E}$ is well-studied in the literature. In contrast to that, the nonconvex case is less studied.

Our above observations stand in accordance with important remarks from the literature:

**1995:** Carrizosa *et al.* [19]: "The planar point-objective location problem has attracted considerable interest among Location Theory researchers. The result has been a number of papers giving properties or algorithms for particular instances of the problem. However, **most of these results are only valid when the feasible region where the facility is to be located is the whole space $\mathbb{R}^2$, which is a rather inaccurate approximation in many real world location problem**."

**1996:** Carrizosa and Plastria [20]: "A number of papers have been devoted to the search of efficient points of the problem above, known in the literature as the pointobjective location problem, **but mostly in the unconstrained case**, i.e., under the assumption that the facility can be placed at any point in the plane. **Although this assumption has been widely questioned, only some partial results have been obtained in the presence of constraints**."

**2005:** Nickel *et al.* [94]: "If only one objective has to be taken into account, a broad range of models is available in the literature. In contrast to that, **only a few papers have looked at more realistic models for facility location, where multiple objectives are involved**."

**2007:** Ohsawa *et al.* [95]: "Much attention has been given to ordered median location models, **but relatively little to multi-objective formulations, in particular, non-convex cases**."

**2008:** Puerto and Rodríguez-Chía [110]: "Scanning the literature, we can see that the multicriteria location problem has received special attention in the last years. However, there is **a lack of a common geometrical description of the nondominated solution set for the constrained version of these problems**."
(Notice that in [110] the *nondominated solution set* stands for the set of Pareto efficient solutions.)

**2009:** Jourani, Michelot and Ndiaye [67]: "**Only a limited number of papers deal with multiple criteria in a continuous setting.** Most of them addressed a bi-criteria problem which is to locate a semi-obnoxious facility with the two objectives of maximizing a utility function which measures the benefits provided by the facility and of minimizing the undesirable effects induced, see [...]."

**2010:** Farahani, SteadieSeifi and Asgari [37]: "We saw the literature on multi-criteria facility location problems has been growing increasingly. The growing attention and interest into these problems, is due to the recognition of the **need to consider more criteria in order to achieve closer solutions to reality**."

## 4.1 The class of point-objective location problems

In this section, we study the *constrained point-objective location problem involving mixed Minkowski gauges* that consists of minimizing the vector-valued function $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^m$ (under the assumption (4.1)) over the feasible set $X$:

$$\begin{cases} g_{\mathcal{A}}(x) = \big(\eta_1(x - a^1), \cdots, \eta_m(x - a^m)\big) \to \min \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in X. \end{cases} \tag{POLP$_X(\mathcal{A})$}$$

Hence, our aim is to find a point $x \in X$ for a *new facility* such that the distances between $x$ and the given points $a^1, \cdots, a^m$ are minimized simultaneously (in the sense of multi-objective optimization). By applying the *Weighted-Sum Scalarization Method* to the problem (POLP$_X(\mathcal{A})$) (see the end of Section 1.5), for any $\lambda := (\lambda_1, \cdots, \lambda_m) \in \operatorname{int} \mathbb{R}^m_+$, we infer that minimal solutions of the *generalized Fermat-Weber problem*

$$\begin{cases} \langle \lambda, g_{\mathcal{A}}(x) \rangle = \sum_{i=1}^m \lambda_i \eta_i(x - a^i) \to \min \\ x \in X \end{cases} \tag{s$_\lambda$POLP$_X(\mathcal{A})$}$$

are actually Pareto efficient solutions for the problem (POLP$_X(\mathcal{A})$), i.e., we have

$$\underset{x \in X}{\operatorname{argmin}} \sum_{i=1}^m \lambda_i \eta_i(x - a^i) \subseteq \operatorname{Eff}(X \mid g_{\mathcal{A}}).$$

In the original formulation of one of the most famous location problems, *Pierre de Fermat (1607 - 1665)* considered the problem (s$_\lambda$POLP$_X(\mathcal{A})$) and assumed that

$$X = \mathbf{E} = \mathbb{R}^2, m = 3, \lambda = (1, 1, 1), \text{ and } \eta_i = ||\cdot||_2 \text{ for all } i \in I_3.$$

The *Fermat problem* was first solved by *Evangelista Torricelli (1608 - 1647)* via a geometrical approach. The solution of the Fermat problem is known as *Fermat-Torricelli point.* Later *Alfred Weber (1868 - 1958)* [126] generalized the *Fermat problem* (by allowing $m > 3$ and by considering weights not necessarily equal to one) and interpreted the model from an economical point of view.

### 4.1.1 Literature review

As the reader can see, the formulation of the classical *point-objective location problem* given in (POLP$_X(\mathcal{A})$) is easy to understand but it has received the attention of many researchers in the last half century. In the following, we give a brief overview on some results (ordered by the year of publication) that are known for the problem ($\mathrm{POLP}_X(\mathcal{A})$):

**1922:** Fejer [38] proved $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{conv}\,\mathcal{A}$
($X = \mathbf{E} = \mathbb{R}^2$, $\eta_i = ||\cdot||_2$ for all $i \in I_m$).

**1967:** Kuhn [74] showed $\mathrm{Eff}(\mathbb{R}^n \mid g_{\mathcal{A}}) = \mathrm{conv}\,\mathcal{A}$
($X = \mathbf{E} = \mathbb{R}^n$ and $\eta_i = ||\cdot||_2$ for all $i \in I_m$).

**1973:** Wendell and Hurter [127] studied the characterization of (weakly) Pareto efficient solutions ($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||$ for all $i \in I_m$, where $||\cdot||$ denotes a norm).

**1977:** Wendell, Hurter and Lowe [128] proposed a geometric algorithm
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||_1$ for all $i \in I_m$).

**1981:** Chalmet, Francis and Kolen [21] proposed the *Row Algorithm*
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||_1$ for all $i \in I_m$).

**1984:** Thisse, Ward and Wendell [117] proved $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{conv}\,\mathcal{A}$
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||$ for all $i \in I_m$, where $||\cdot||$ is a round norm).

**1984:** Plastria [99] showed $\mathrm{WEff}(\mathbb{R}^n \mid g_{\mathcal{A}}) = \mathrm{conv}\,\mathcal{A}$
($X = \mathbf{E} = \mathbb{R}^n$ and $\eta_i = ||\cdot||_2$ for all $i \in I_m$, where $||\cdot||_2$ can be replaced by any linearly equivalent norm).

**1986:** Durier and Michelot [27, Prop. 1.3] showed

$$\mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}) = \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}) = \mathrm{conv}\,\mathcal{A}$$

($(\mathbf{E}, ||\cdot||)$ is a Hilbert space, $X = \mathbf{E}$, and $\eta_i = ||\cdot||$ for all $i \in I_m$; or $(\mathbf{E}, ||\cdot||)$ has dimension two, $||\cdot||$ is strictly convex, $X = \mathbf{E}$, and $\eta_i = ||\cdot||$ for all $i \in I_m$).
In addition, in [27, Th. 4.3], it is shown that the set of weakly Pareto efficient solutions $\mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}})$ coincides with an intersection of certain half spaces
($(\mathbf{E}, ||\cdot||)$ has dimension two, $X = \mathbf{E}$, and $\eta_i = ||\cdot||$ for all $i \in I_m$).
Moreover, in [27, Sec. 3.3], Durier and Michelot characterized the sets of (strictly, weakly) Pareto efficient solutions by using certain half spaces
($X = \mathbf{E} = \mathbb{R}^n$ and either $\eta_i = ||\cdot||_1$ for all $i \in I_m$ or $\eta_i = ||\cdot||_\infty$ for all $i \in I_m$).
In particular, Durier and Michelot [27, Rem. 3.1] showed that the set of strictly Pareto efficient solutions $\mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ coincides with the rectangular hull of the set $\mathcal{A} \subseteq \mathbb{R}^2$ considered by Love and Morris [78]
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||_1$ for all $i \in I_m$).

**1988:** Pelegrin and Fernández [97] developed an algorithm for solving the problem ($\mathrm{POLP}_X(\mathcal{A})$) ($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||$ for all $i \in I_m$, where $||\cdot||$ is a polyhedral norm).

**1988:** Gerth and Pöhler [47] succeeded to derive (by applying duality theory) a geometrical description for the set of Pareto efficient solutions of problem ($\mathrm{POLP}_X(\mathcal{A})$)
($X = \mathbf{E} = \mathbb{R}^2$ and either $\eta_i = ||\cdot||_\infty$ for all $i \in I_m$ or $\eta_i = ||\cdot||_1$ for all $i \in I_m$).

**1990:** Durier [26] presented ideas for solving the problem ($\mathrm{POLP}_X(\mathcal{A})$)
($X = \mathbf{E} = \mathbb{R}^n$ and $\eta_i$ represents a polyhedral Minkowski gauge with bounded unit ball for every $i \in I_m$).

**1991:** Tammer and Tammer [116] derived duality results for a more general convex vector approximation problem with linear restrictions
($\mathbf{E}$ is a Banach space, $X \subseteq \mathbf{E}$ is defined by a system of linear inequalities, $\eta_i$ is a general Minkowski gauge with convex unit ball for every $i \in I_m$).

**1993:** Carrizosa *et al.* [18, Th. 2, Th. 3] showed that

$$\mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{conv}\,\mathcal{A})$$

and

$$\mathrm{PEff}(X \mid g_{\mathcal{A}}) = (\mathcal{A} \cap X) \cup \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{rint}(\mathrm{conv}\,\mathcal{A}))$$

($\mathbf{E} = \mathbb{R}^n$, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i = ||\cdot||_2$ for all $i \in I_m$).

**1995:** Carrizosa *et al.* [19] considered the problem ($\mathrm{POLP}_X(\mathcal{A})$) and used a geometrical construction based on the concept of a closed and convex decomposition of the not necessarily convex feasible set (representation as a finite union of polytopes) in order to obtain a characterization for the set of weakly Pareto efficient solutions
($\mathbf{E} = \mathbb{R}^2$, $X \subseteq \mathbf{E}$ is a nonempty, closed set, and $\eta_i = ||\cdot||_2$ for all $i \in I_m$).

**1995:** Nickel [89] presented an algorithm for computing the set $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ based on results in Durier and Michelot [27]
($X = \mathbf{E} = \mathbb{R}^2$, and $\eta_i$ represents a polyhedral Minkowski gauge for every $i \in I_m$).

**1996:** Carrizosa and Plastria [20, Th. 1] showed that

$$\mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X^{||\cdot||}(\mathrm{conv}\,\mathcal{A})$$

($\mathbf{E} = \mathbb{R}^2$, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i = ||\cdot||$ for all $i \in I_m$, where $||\cdot||$ is a strictly convex norm).

**1997:** Benker, Hamel and Tammer [12] derived a proximal point algorithm for vectorial control approximation problems, where the problem ($\mathrm{POLP}_X(\mathcal{A})$) is a special case of this model
(($\mathbf{E}, ||\cdot||$) is a Hilbert space or $\mathbf{E} = \mathbb{R}^n$, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i = ||\cdot||$ for all $i \in I_m$).

**1998:** Ndiaye and Michelot [88, Prop. 3.5, Cor. 4.2] proved

$$\mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}})$$

(($\mathbf{E}, ||\cdot||$) is a strictly convex normed space, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i = ||\cdot||$ for all $i \in I_m$),
while under the additional assumption that $\mathbf{E}$ is a Hilbert space one gets

$$\mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X^{||\cdot||}(\mathrm{conv}\,\mathcal{A})$$

(($\mathbf{E}, ||\cdot||$) is a Hilbert space, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i = ||\cdot||$ for all $i \in I_m$).
By [88, Prop 4.3] one has

$$\mathrm{WEff}(X \mid g_{\mathcal{A}}) \supseteq \mathrm{Proj}_X^{||\cdot||}(\mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}))$$

(($\mathbf{E}, ||\cdot||$) is a normed space, $X$ is a nonempty, closed, convex set, and $\eta_i = ||\cdot||$ for all $i \in I_m$).
Moreover, Ndiaye and Michelot [88, Th. 4.2] showed

$$\mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X^{||\cdot||}(\mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}))$$

(($\mathbf{E}, ||\cdot||$) is a two-dimensional normed space, $X$ is a nonempty, closed, convex set, and $\eta_i = ||\cdot||$ for all $i \in I_m$).
Ndiaye and Michelot [88] observed that this *projection property* concerning the set of weakly Pareto efficient solutions fails for the other concepts of Pareto efficiency and strict Pareto efficiency under the assumption that $||\cdot||$ is not strictly convex. Moreover, the *projection property* concerning the set of weakly Pareto efficient solutions does not hold for dimension of $\mathbf{E}$ three or higher in general.

**2000:** Wanka [122] showed duality statements for a control approximation problem, where the problem ($\mathrm{POLP}_X(\mathcal{A})$) is a special case of the model

(($\mathbf{E}, ||\cdot||$) is a normed space, $X \subseteq \mathbf{E}$ is defined by a system of linear inequalities, and $\eta_i = ||\cdot||$ for all $i \in I_m$).

**2002:** Puerto and Rodríguez-Chía [109] considered a more general problem with convex objective functions (the problem (POLP$_X(\mathcal{A})$) is a special case of this model) and presented a characterization for the set of weakly Pareto efficient solutions
($X = \mathbf{E} = \mathbb{R}^2$).

**2008:** Puerto and Rodríguez-Chía [110] considered a more general problem with strictly quasi-convex / convex objective functions (the problem (POLP$_X(\mathcal{A})$) is a special case) and presented a characterization for the set of weakly Pareto efficient solutions
($\mathbf{E} = \mathbb{R}^2$ and $X$ is given by a nonempty, closed, convex set in $\mathbb{R}^2$).

**2009:** Jourani, Michelot and Ndiaye [67] extended the problem (POLP$_X(\mathcal{A})$) to a problem with attraction and repulsion demand points (a nonconvex problem) and derived geometrical characterizations for the sets of (strictly, weakly) Pareto efficient solutions of the unconstrained problem. For the problem with closed, convex constraints, Jourani, Michelot and Ndiaye [67] proved a sufficient condition for weak Pareto efficiency
(($\mathbf{E}, ||\cdot||$) is a finite-dimensional Hilbert space, $X$ is a nonempty, closed, convex feasible set in $\mathbf{E}$, and $\eta_i = ||\cdot||$ for all $i \in I_m$).

**2015:** Kaiser [69] proposed an algorithm for solving the problem (POLP$_X(\mathcal{A})$)
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i$ represents a polyhedral Minkowski gauge with bounded unit ball for every $i \in I_m$).

### 4.1.2 Contributions of this thesis

In the following, we emphasize the fact that our penalization approach derived in Chapter 2 can be used for the computation of the sets of (strictly, weakly) Pareto efficient solutions in situations in which the *projection property* (i.e., $\mathrm{Eff}(X \mid g_{\mathcal{A}})$ can be obtained by projecting $\mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}})$ onto $X$; analogous for the sets $\mathrm{SEff}(X \mid g_{\mathcal{A}})$ and $\mathrm{WEff}(X \mid g_{\mathcal{A}})$) for these sets do not hold.

In the next remark, we discuss the validity of the *projection property* for the sets of (strictly, weakly) Pareto efficient solutions of problem (POLP$_X(\mathcal{A})$).

**Remark 4.7** Let ($\mathbf{E}, ||\cdot||$) be a normed space, let $X \subsetneq \mathbf{E}$ be a nonempty, closed, convex set, and let $\eta_i = ||\cdot||$ for all $i \in I_m$. By Ndiaye and Michelot [88], Carrizosa *et al.* [18], and Carrizosa and Plastria [20], we get the following results:

- If $\mathbf{E}$ is a Hilbert space, then

$$\mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X^{||\cdot||}(\mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}})) = \mathrm{Proj}_X^{||\cdot||}(\mathrm{conv}\,\mathcal{A}). \tag{4.5}$$

- We have

$$\mathrm{WEff}(X \mid g_{\mathcal{A}}) \supseteq \mathrm{Proj}_X^{||\cdot||}(\mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}})),$$

  and if $\mathbf{E}$ has dimension two, then

$$\mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X^{||\cdot||}(\mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}})).$$

- The projection property concerning the set of weakly Pareto efficient solutions does not hold for dimension of $\mathbf{E}$ greater or equal to three in general.
- If $\mathbf{E} = \mathbb{R}^2$ is strictly convex, then (4.5) holds.
- The projection property fails for the sets of strictly Pareto efficient solutions and Pareto efficient solutions in general if the norm $||\cdot||$ is not strictly convex.
- If $\mathbf{E} = \mathbb{R}^n$ and $||\cdot|| = ||\cdot||_2$, then

$$\mathrm{PEff}(X \mid g_{\mathcal{A}}) = (\mathcal{A} \cap X) \cup \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{rint}(\mathrm{conv}\,\mathcal{A})).$$

Let us consider a nonempty, closed, convex feasible set $X$ in a linear topological space $\mathbf{E}$ with $d \in \operatorname{int} X$. In the case that the penalization function is given by $\phi(\cdot) := \mu(\cdot - d)$ (where $B := -d + X$ is the unit ball of $\mu = \mu_B$), the problem $(\mathrm{LP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ is given by an unconstrained multi-objective location problem

$$\begin{cases} g^{\oplus}_{\mathcal{A}}(x) = \big(\eta_1(x - a^1), \cdots, \eta_m(x - a^m), \mu(x - d)\big) \to \min \text{ w.r.t. } \mathbb{R}^{m+1}_+ \\ x \in \mathbf{E}. \end{cases} \tag{$\mathrm{POLP}_{\mathbf{E}}(\mathcal{A})^{\oplus}$}$$

Notice that $(\mathrm{POLP}_{\mathbf{E}}(\mathcal{A})^{\oplus})$ is equivalent to the *unconstrained point-objective location problem involving mixed Minkowski gauges* given by

$$\begin{cases} g_{\mathcal{A}'}(x) = \big(\eta_1(x - a^1), \cdots, \eta_{m+1}(x - a^{m+1})\big) \to \min \text{ w.r.t. } \mathbb{R}^{m+1}_+ \\ x \in \mathbf{E}, \end{cases} \tag{$\mathrm{POLP}_{\mathbf{E}}(\mathcal{A}')$}$$

where

$$\begin{aligned} a^{m+1} &:= d, \\ \mathcal{A}' &:= \mathcal{A} \cup \{a^{m+1}\}, \\ \eta_{m+1} &:= \mu. \end{aligned}$$

This means that we have to solve at most two unconstrained point-objective location problems $(\mathrm{POLP}_{\mathbf{E}}(\mathcal{A}))$ and $(\mathrm{POLP}_{\mathbf{E}}(\mathcal{A}'))$ in order to solve the initial constrained (convex) point-objective location problem $(\mathrm{POLP}_X(\mathcal{A}))$ taking into account Proposition 4.5.

In the next example, we show how to compute the whole set of Pareto efficient solutions of problem $(\mathrm{POLP}_X(\mathcal{A}))$ involving a strictly convex norm $||\cdot||$ but without using the *projection property*.

**Example 4.8** Let us consider a planar point-objective location problem involving the Euclidean norm (i.e., we put $\eta_i(\cdot) = ||\cdot||_2$ for all $i \in I_m$) with three given points $a^1, a^2, a^3 \in \mathbb{R}^2$ and a feasible set $X$ represented by a closed Euclidean ball centered at $x' \in \mathbb{R}^2$ with positive radius, as shown in Figure 4.1. Furthermore, Fig. 4.1 illustrates the location problem and shows the procedure for computing the set $\mathrm{Eff}(X \mid g_{\mathcal{A}})$. Notice that $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \operatorname{conv}\{a^1, a^2, a^3\}$ and $\mathrm{Eff}(\mathbb{R}^2 \mid g^{\oplus}_{\mathcal{A}}) = \operatorname{conv}\{a^1, a^2, a^3, x'\}$ (see Thisse, Ward and Wendell [117]). Moreover, the example shows that

$$\mathrm{Eff}(X \mid g_{\mathcal{A}}) = \big[X \cap \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})\big] \cup \big[(\operatorname{bd} X) \cap \mathrm{Eff}(\mathbb{R}^2 \mid g^{\oplus}_{\mathcal{A}})\big] = \mathrm{Proj}_X^{||\cdot||_2}(\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})),$$

where the equality $\mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X(\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}))$ is known for the constrained point-objective location problem involving the Euclidean norm (see Remark 4.7).

In Chapter 5, we will study a planar point-objective location problem involving the Manhattan norm in detail. Notice that the Manhattan norm is not strictly convex. So, in view of Remark 4.7, the projection property concerning the set of Pareto efficient solutions does not hold for the problem in general. Furthermore, in Chapter 6, we consider nonconvex multi-objective location problems. To the best of our knowledge, it is unknown how to compute the set of (strictly, weakly) Pareto efficient solutions of the problem $(\mathrm{POLP}_X(\mathcal{A}))$ (e.g., for $\mathbf{E} = \mathbb{R}^n$ and $\eta_i = ||\cdot||_2$ for all $i \in I_m$) involving forbidden regions. Since in practical problems, there often exist regions where it is forbidden to locate a new facility, it is interesting to study the classical problem $(\mathrm{POLP}_X(\mathcal{A}))$ in the presence of forbidden regions. For single-objective location problems involving forbidden regions, we refer the reader to the works by Brimberg and Juel [16], Hamacher [57], Hamacher and Nickel [58], Nickel [89], and Nickel and Puerto [90].

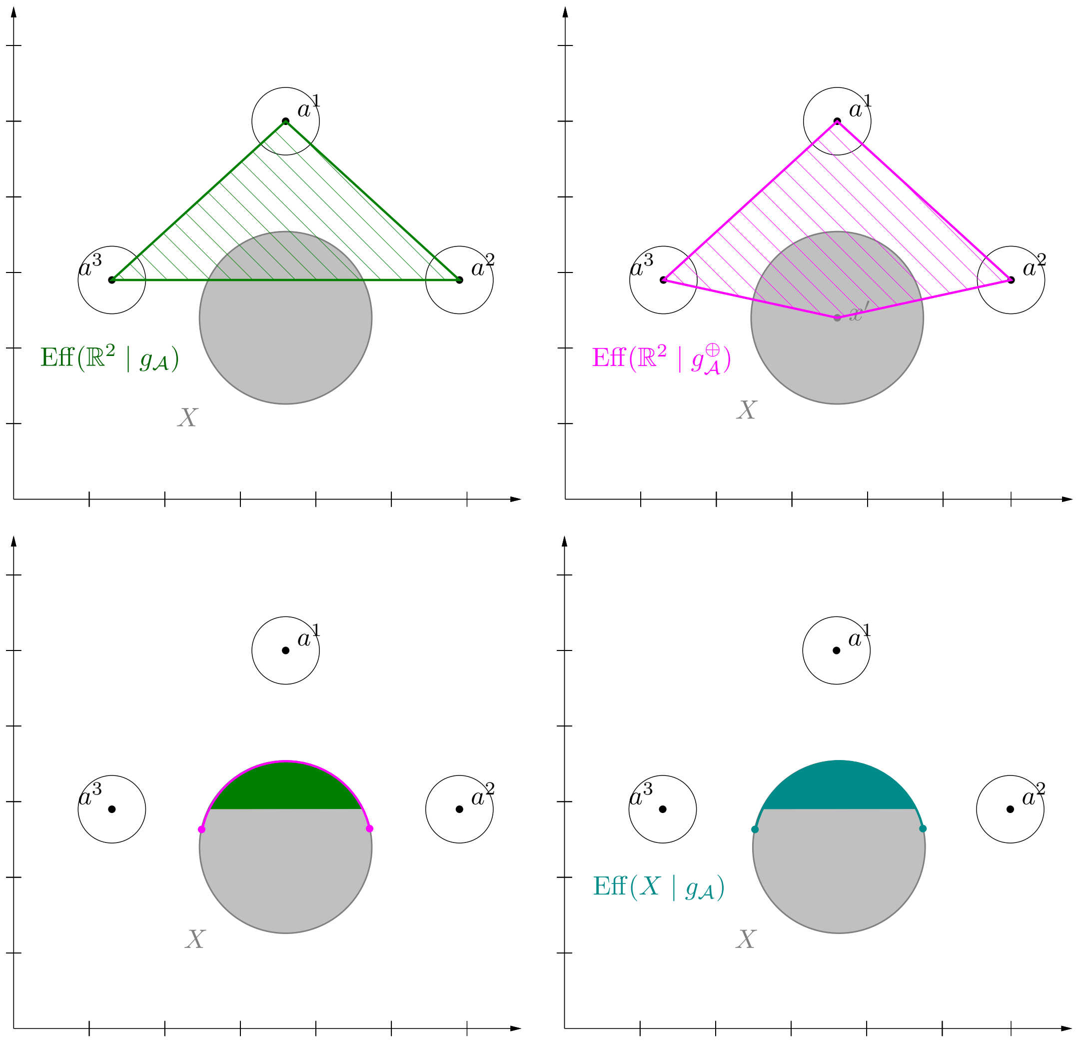


Figure 4.1: Construction of the set of Pareto efficient solutions of the problem ($\mathrm{POLP}_X(\mathcal{A})$), where $\mathbf{E} = \mathbb{R}^2$, $X \subseteq \mathbb{R}^2$ is given by an Euclidean ball, and $\eta_i(\cdot) = ||\cdot||_2$ for all $i \in I_m$.

The list of literature given in Section 4.1.1 can be extended by our results as follows:

**2016:** Günther and Tammer [55] derived relationships between constrained and unconstrained multi-objective optimization
($\mathbf{E} = \mathbb{R}^n$ and $X \subseteq \mathbf{E}$ is given by a closed, convex set with nonempty interior).

**2017:** Alzorba *et al.* [3] proposed the *Rectangular Decomposition Algorithm*
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||_1$ for all $i \in I_m$).

**2017:** Günther and Tammer [56] extended the results in [55] to a more general framework
($\mathbf{E}$ is a linear topological space and $X \subseteq \mathbf{E}$ is given by a nonempty, closed set).

**2018:** Günther [53] characterized the sets of (strictly, weakly) Pareto efficient solutions of the problem ($\mathrm{POLP}_X(\mathcal{A})$) involving multiple forbidden regions
(($\mathbf{E}$, $||\cdot||$) is a finite-dimensional Hilbert space, $X \subseteq \mathbf{E}$ is given by the whole space $\mathbf{E}$ excepting some forbidden regions that are given by open balls with respect to $||\cdot||$, and $\eta_i = ||\cdot||$ for all $i \in I_m$).

## 4.2 The class of multi-objective min-sum location problems

Another important class is given by *multi-objective min-sum location problems involving mixed Minkowski gauges*. The aim is to minimize the vector-valued function $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^m$ (under the assumption (4.2)) over a nonempty, closed feasible set $X \subseteq \mathbf{E}$:

$$\begin{cases} g_{\mathcal{A}}(x) = \left(\sum_{i\in I_m} \lambda_i^1 \eta_i(x-a^i), \cdots, \sum_{i\in I_m} \lambda_i^p \eta_i(x-a^i)\right) \to \min \text{ w.r.t. } \mathbb{R}^p_+ \\ x \in X. \end{cases} \qquad (\text{MOMSLP}_X(\mathcal{A}))$$

### 4.2.1 Literature review

In what follows, we give a short overview on some results (ordered by the year of publication) that are known for the *multi-objective min-sum location problem* ($\text{MOMSLP}_X(\mathcal{A})$):

**1995:** Nickel [89] proposed an algorithm for finding $\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ ($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = \mu_P$ for every $i \in I_m$, where $\mu_P$ represents a polyhedral Minkowski gauge with bounded unit ball $P \subseteq \mathbb{R}^2$).

**1996:** Hamacher and Nickel [59] characterized $\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ ($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i$ is given by the square of the Euclidean norm $||\cdot||_2^2$ for every $i \in I_m$). Moreover, Hamacher and Nickel [59] proposed an algorithm for finding $\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ ($X = \mathbf{E} = \mathbb{R}^2$, $p = 2$, and $\eta_i = ||\cdot||_1$ for every $i \in I_m$).

**1998:** Puerto and Fernández [106] approximated the set $\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ by using sequences of sets of Pareto efficient solutions of a similar problem with polyhedral norms ($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||$ for every $i \in I_m$, where $||\cdot||$ is a strictly convex norm).

**1999:** Puerto and Fernández [107] proposed an algorithm for computing $\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ ($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||$ for every $i \in I_m$, where $||\cdot||$ is a polyhedral norm).

**2002:** Puerto and Rodríguez-Chía [109] considered a more general problem with convex objective functions (the problem ($\text{MOMSLP}_X(\mathcal{A})$) is a special case of this model) and presented a characterization for the set of weakly Pareto efficient solutions ($X = \mathbf{E} = \mathbb{R}^2$).

**2002:** Klamroth and Wiecek [73] considered ($\text{MOMSLP}_X(\mathcal{A})$) with a line barrier and decomposed the problem to a family of multi-objective componentwise convex subproblems ($X = \mathbf{E} = \mathbb{R}^2$, $p = 2$, and $\eta_i = ||\cdot||$ for every $i \in I_m$, where $||\cdot||$ is a norm).

**2003:** Wanka, Boţ and Vargyas [123] analyzed a more general problem (the problem ($\text{MOMSLP}_X(\mathcal{A})$) is a special case of the model) from a duality theory point of view ($\mathbf{E}$ is a normed space, $X = \mathbf{E}$, and $\eta_i$ is a norm on $\mathbf{E}$ for every $i \in I_m$).

**2005:** Nickel *et al.* [94] considered the general class of multi-objective planar ordered median problems (the problem ($\text{MOMSLP}_X(\mathcal{A})$) is a special case of the model) and succeeded to characterize the set of Pareto efficient solutions ($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i$ represents a polyhedral Minkowski gauge with bounded unit ball for every $i \in I_m$).

**2007:** Ohsawa *et al.* [95] developed an algorithm for solving quadratic ordered median location problems (the problem ($\text{MOMSLP}_X(\mathcal{A})$) is a special case of this model) ($\mathbf{E} = \mathbb{R}^2$, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i$ is given by the square of the Euclidean norm $||\cdot||_2^2$ for every $i \in I_m$).

**2008:** Puerto and Rodríguez-Chía [110] considered a more general problem with strictly quasi-convex / convex objective functions (the problem ($\text{MOMSLP}_X(\mathcal{A})$) is a special case of this model) and presented a characterization for the set of weakly Pareto efficient solutions ($\mathbf{E} = \mathbb{R}^2$ and $X$ is given by a closed, convex set in $\mathbb{R}^2$).

### 4.2.2 Contributions of this thesis

The vectorial penalization approach derived in Chapter 2 can be used to solve multi-objective min-sum location problems involving some constraints. In particular, for the case that $X$ is a closed, convex set with $d \in \operatorname{int} X \neq \emptyset$, and $\mu = \mu_B$ is the Minkowski gauge associated to the set $B := -d + X$, we can reformulate the penalized multi-objective optimization problem

$$\begin{cases} g_{\mathcal{A}}^{\oplus}(x) = \left( \sum\limits_{i \in I_m} \lambda_i^1 \eta_i(x - a^i), \cdots, \sum\limits_{i \in I_m} \lambda_i^p \eta_i(x - a^i), \mu(x - d) \right) \to \min \text{ w.r.t. } \mathbb{R}_+^{p+1} \\ x \in \mathbf{E} \end{cases} \tag{MOMSLP$_{\mathbf{E}}(\mathcal{A})^{\oplus}$}$$

to an *unconstrained multi-objective min-sum location problem*

$$\begin{cases} g_{\mathcal{A}'}(x) = \left( \sum\limits_{i \in I_{m+1}} \lambda_i^1 \eta_i(x - a^i), \cdots, \sum\limits_{i \in I_{m+1}} \lambda_i^{p+1} \eta_i(x - a^i) \right) \to \min \text{ w.r.t. } \mathbb{R}_+^{p+1} \\ x \in \mathbf{E} \end{cases} \tag{MOMSLP$_{\mathbf{E}}(\mathcal{A}')$}$$

by defining

$$\begin{aligned} a^{m+1} &:= d, \\ \mathcal{A}' &:= \mathcal{A} \cup \{a^{m+1}\}, \\ \eta_{m+1} &:= \mu, \\ \lambda_{m+1}^{p+1} &:= 1, \\ \lambda_i^{p+1} &:= 0 \quad \text{for all } i \in I_m, \\ \lambda_{m+1}^j &:= 0 \quad \text{for all } j \in I_p. \end{aligned}$$

Hence, in view of Proposition 4.5, one has to solve at most two unconstrained multi-objective min-sum location problems (MOMSLP$_{\mathbf{E}}(\mathcal{A})$) and (MOMSLP$_{\mathbf{E}}(\mathcal{A}')$) in order to solve the initial constrained (convex) multi-objective min-sum location problem (MOMSLP$_X(\mathcal{A})$).

The list of literature can be extended by our results as follows:

**2016:** Günther and Tammer [55] derived relationships between constrained and unconstrained multi-objective optimization
($\mathbf{E} = \mathbb{R}^n$ and $X \subseteq \mathbf{E}$ is given by a closed, convex set with nonempty interior).

**2017:** Günther and Tammer [56] extended the results in [55] to a more general framework
($\mathbf{E}$ is a linear topological space and $X \subseteq \mathbf{E}$ is given by a nonempty, closed set).

**2018:** Günther [53] derived useful results for computing solutions of (MOMSLP$_X(\mathcal{A})$) involving multiple forbidden regions
($\mathbf{E}$ is a linear topological space, $X \subseteq \mathbf{E}$ is given by the whole space $\mathbf{E}$ excepting some forbidden regions that are given by convex sets).

## 4.3 The class of multi-objective min-max location problems

In this section, we apply our approach to the class of *multi-objective min-max location problems involving mixed Minkowski gauges.* Under the assumption (4.3), we consider the problem

$$\begin{cases} g_{\mathcal{A}}(x) = \left( \max\limits_{i \in I_m} \lambda_i^1 \eta_i(x - a^i), \cdots, \max\limits_{i \in I_m} \lambda_i^p \eta_i(x - a^i) \right) \to \min \text{ w.r.t. } \mathbb{R}_+^p \\ x \in X, \end{cases} \tag{MOMMLP$_X(\mathcal{A})$}$$

where $X \subseteq \mathbf{E}$ is a nonempty, closed set.

### 4.3.1 Literature review

In what follows, we give a brief overview on some results (ordered by the year of publication) that are known for the *multi-objective min-max location problem* ($\text{MOMMLP}_X(\mathcal{A})$):

**1995:** Nickel [89] proposed an algorithm for finding $\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$
($X = \mathbf{E} = \mathbb{R}^2$, $p = 2$, and $\eta_i = \mu_P$ for every $i \in I_m$, where $\mu_P$ represents a polyhedral Minkowski gauge with bounded unit ball $P \subseteq \mathbb{R}^2$).

**1996:** Hamacher and Nickel [59] presented ideas for computing $\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i = ||\cdot||_\infty$ for every $i \in I_m$).

**2002:** Puerto and Rodríguez-Chía [109] considered a more general problem with convex objective functions (the problem ($\text{MOMMLP}_X(\mathcal{A})$) is a special case of this model) and presented a characterization for the set of weakly Pareto efficient solutions
($X = \mathbf{E} = \mathbb{R}^2$).

**2003:** Wanka, Boţ and Vargyas [123] analyzed a more general problem (the problem ($\text{MOMMLP}_X(\mathcal{A})$) is a special case of the model) from a duality theory point of view
($\mathbf{E}$ is a normed space, $X = \mathbf{E}$, and $\eta_i$ is a norm on $\mathbf{E}$ for every $i \in I_m$).

**2005:** Nickel *et al.* [94] considered the general class of multi-objective planar ordered median problems (the problem ($\text{MOMMLP}_X(\mathcal{A})$) is a special case of the model) and succeeded to characterize the set of Pareto efficient solutions
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i$ represents a polyhedral Minkowski gauge with bounded unit ball for every $i \in I_m$).

**2007:** Ohsawa *et al.* [95] developed an algorithm for solving quadratic ordered median location problems (the problem ($\text{MOMMLP}_X(\mathcal{A})$) is a special case of this model)
($\mathbf{E} = \mathbb{R}^2$, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i$ is given by the square of the Euclidean norm $||\cdot||_2^2$ for every $i \in I_m$).

**2008:** Puerto and Rodríguez-Chía [110] considered a more general problem with strictly quasi-convex / convex objective functions (the problem ($\text{MOMMLP}_X(\mathcal{A})$) is a special case of this model) and presented a characterization for the set of weakly Pareto efficient solutions
($\mathbf{E} = \mathbb{R}^2$ and $X$ is given by a closed, convex set in $\mathbb{R}^2$).

### 4.3.2 Contributions of this thesis

In order to solve the constrained multi-objective min-max location problem ($\text{MOMMLP}_X(\mathcal{A})$), we can use our penalization approach derived in Chapter 2. Consider the particular case that $X$ is a closed, convex set with $d \in \operatorname{int} X \neq \emptyset$, and $\mu = \mu_B$ is the Minkowski gauge associated to the set $B := -d + X$. Then, we can consider the penalized multi-objective optimization problem

$$\begin{cases} g_{\mathcal{A}}^{\oplus}(x) = \left( \max\limits_{i \in I_m} \lambda_i^1 \eta_i(x - a^i), \cdots, \max\limits_{i \in I_m} \lambda_i^p \eta_i(x - a^i), \mu(x - d) \right) \to \min \text{ w.r.t. } \mathbb{R}_+^{p+1} \\ x \in \mathbf{E}. \end{cases} \tag{$\text{MOMMLP}_{\mathbf{E}}(\mathcal{A})^{\oplus}$}$$

Similar to Section 4.2.2, we can reformulate the problem ($\text{MOOMLP}_{\mathbf{E}}(\mathcal{A})^{\oplus}$) to an *unconstrained multi-objective min-max location problem*

$$\begin{cases} g_{\mathcal{A}'}(x) = \left( \max\limits_{i \in I_{m+1}} \lambda_i^1 \eta_i(x - a^i), \cdots, \max\limits_{i \in I_{m+1}} \lambda_i^{p+1} \eta_i(x - a^i) \right) \to \min \text{ w.r.t. } \mathbb{R}_+^{p+1} \\ x \in \mathbf{E} \end{cases} \tag{$\text{MOMMLP}_{\mathbf{E}}(\mathcal{A}')$}$$

by defining

$$\begin{aligned}
a^{m+1} &:= d,\\
\mathcal{A}' &:= \mathcal{A} \cup \{a^{m+1}\},\\
\eta_{m+1} &:= \mu,\\
\lambda^{p+1}_{m+1} &:= 1,\\
\lambda^{p+1}_{i} &:= 0 \quad \text{for all } i \in I_m,\\
\lambda^{j}_{m+1} &:= 0 \quad \text{for all } j \in I_p.
\end{aligned}$$

So, by Proposition 4.5 we have to solve at most two unconstrained multi-objective min-max location problems (MOMMLP$_{\mathbf{E}}(\mathcal{A})$) and (MOMMLP$_{\mathbf{E}}(\mathcal{A}')$) in order to solve the initial constrained (convex) multi-objective min-max location problem (MOMMLP$_X(\mathcal{A})$).

The list of literature can be extended by our results as follows:

**2016:** Günther and Tammer [55] derived relationships between constrained and unconstrained multi-objective optimization
($\mathbf{E} = \mathbb{R}^n$ and $X \subseteq \mathbf{E}$ is given by a closed, convex set with nonempty interior).

**2017:** Günther and Tammer [56] extended the results in [55] to a more general framework
($\mathbf{E}$ is a linear topological space and $X \subseteq \mathbf{E}$ is given by a nonempty, closed set).

**2018:** Günther [53] derived useful results for computing solutions of (MOMMLP$_X(\mathcal{A})$) involving multiple forbidden regions
($\mathbf{E}$ is a linear topological space, $X \subseteq \mathbf{E}$ is given by the whole space $\mathbf{E}$ excepting some forbidden regions that are given by convex sets).

## 4.4 The class of multi-objective ordered median location problems

Let us consider *multi-objective ordered median location problems involving mixed Minkowski gauges*. So, the aim is to minimize the vector-valued function $g_{\mathcal{A}} : \mathbf{E} \to \mathbb{R}^m$ (under the assumptions (4.4)) over a nonempty, closed feasible set $X \subseteq \mathbf{E}$:

$$\begin{cases} g_{\mathcal{A}}(x) = ((\mathrm{OMF}_{\lambda^1} \circ \mathrm{weight}_{w^1} \circ \eta_{\mathcal{A}})(x), \cdots, (\mathrm{OMF}_{\lambda^p} \circ \mathrm{weight}_{w^p} \circ \eta_{\mathcal{A}})(x)) \to \min \text{ w.r.t. } \mathbb{R}^p_+ \\ x \in X. \end{cases} \tag{MOOMLP$_X(\mathcal{A})$}$$

### 4.4.1 Literature review

A comprehensive overview on *ordered median location problems* can be found in Nickel and Puerto [90]. Now, we give a brief overview on some results (ordered by the year of publication) that are known for the *multi-objective ordered median location problem* (MOOMLP$_X(\mathcal{A})$):

**2002:** Puerto and Rodríguez-Chía [109] considered a more general problem with convex objective functions and presented a characterization for the set of weakly Pareto efficient solutions
($X = \mathbf{E} = \mathbb{R}^2$).

**2005:** Nickel *et al.* [94] considered convex multi-objective planar ordered median problems and succeeded to characterize the set of Pareto efficient solutions
($X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i$ represents a polyhedral Minkowski gauge with bounded unit ball for every $i \in I_m$).

**2007:** Ohsawa *et al.* [95] developed an algorithm for solving quadratic ordered median location problems
($\mathbf{E} = \mathbb{R}^2$, $X \subseteq \mathbf{E}$ is a nonempty, closed, convex set, and $\eta_i$ is given by the square of the Euclidean norm $||\cdot||_2^2$ for every $i \in I_m$).

**2008:** Puerto and Rodríguez-Chía [110] considered a more general problem with strictly quasi-convex / convex objective functions and presented a characterization for the set of weakly Pareto efficient solutions
($\mathbf{E} = \mathbb{R}^2$ and $X$ is given by a closed, convex set in $\mathbb{R}^2$).

### 4.4.2 Contributions of this thesis

Our aim is to apply the penalization approach derived in Chapter 2 to constrained convex multi-objective ordered median location problem ($\text{MOOMLP}_X(\mathcal{A})$). Assume that $\lambda^i, w^i \in \mathbb{R}^m_+ \setminus \{0\}$, $\lambda^i_1 \leq \cdots \leq \lambda^i_m$, for all $i \in I_p$. Consider the particular case that $X$ is a closed, convex set with $d \in \operatorname{int} X$, and $\mu = \mu_B$ is the gauge associated to the set $B := -d + X$. Then, we can consider the penalized multi-objective optimization problem

$$\begin{cases} g^{\oplus}_{\mathcal{A}}(x) = ((\text{OMF}_{\lambda^1} \circ \text{weight}_{w^1} \circ \eta_{\mathcal{A}})(x), \cdots, (\text{OMF}_{\lambda^p} \circ \text{weight}_{w^p} \circ \eta_{\mathcal{A}})(x), \mu(x-d)) \to \min \text{ w.r.t. } \mathbb{R}^{p+1}_+ \\ x \in \mathbf{E}. \end{cases} \tag{$\text{MOOMLP}_{\mathbf{E}}(\mathcal{A})^{\oplus}$}$$

Analogously to Section 4.2.2, we can reformulate the problem ($\text{MOOMLP}_{\mathbf{E}}(\mathcal{A})^{\oplus}$) to an *unconstrained multi-objective ordered median location problem*

$$\begin{cases} g_{\mathcal{A}'}(x) = \Big((\text{OMF}_{(\lambda')^1} \circ \text{weight}_{(w')^1} \circ \eta_{\mathcal{A}'})(x), \cdots, (\text{OMF}_{(\lambda')^{p+1}} \circ \text{weight}_{(w')^{p+1}} \circ \eta_{\mathcal{A}'})(x)\Big) \to \min \text{ w.r.t. } \mathbb{R}^{p+1}_+ \\ x \in \mathbf{E} \end{cases} \tag{$\text{MOOMLP}_{\mathbf{E}}(\mathcal{A}')$}$$

by defining

$$\begin{aligned} a^{m+1} &:= d, \\ \eta_{m+1} &:= \mu, \\ \mathcal{A}' &:= \mathcal{A} \cup \{a^{m+1}\}, \\ \eta_{\mathcal{A}'}(\cdot) &:= (\eta_{\mathcal{A}}(\cdot), \eta_{m+1}(\cdot - a^{m+1})), \\ (\lambda')^i &:= (0, \lambda^i) \quad \text{ for all } i \in I_p, \\ (w')^i &:= (w^j, 0) \quad \text{ for all } i \in I_p, \\ (\lambda')^{p+1} &:= (0_{\mathbb{R}^m}, 1), \\ (w')^{p+1} &:= (0_{\mathbb{R}^m}, 1). \end{aligned}$$

Due to Proposition 4.5, we have to solve at most two unconstrained multi-objective ordered median location problems ($\text{MOOMLP}_{\mathbf{E}}(\mathcal{A})$) and ($\text{MOOMLP}_{\mathbf{E}}(\mathcal{A}')$) in order to solve the initial constrained (convex) multi-objective ordered median location problem ($\text{MOOMLP}_X(\mathcal{A})$).

The list of literature given in Section 4.4.1 can be extended as follows:

**2016:** Günther and Tammer [55] derived relationships between constrained and unconstrained multi-objective optimization
($\mathbf{E} = \mathbb{R}^n$ and $X \subseteq \mathbf{E}$ is given by a closed, convex set with nonempty interior).

**2017:** Günther and Tammer [56] extended the results in [55] to a more general framework
($\mathbf{E}$ is a linear topological space and $X \subseteq \mathbf{E}$ is given by a nonempty, closed set).

**2018:** Günther [53] derived useful results for computing solutions of ($\text{MOOMLP}_X(\mathcal{A})$) involving multiple forbidden regions
($\mathbf{E}$ is a linear topological space, $X \subseteq \mathbf{E}$ is given by the whole space $\mathbf{E}$ excepting some forbidden regions that are given by convex sets).

## 4.5 Concluding remarks

As we have seen in this chapter, it is possible to solve general classes of multi-objective location problems (e.g., *point-objective location problems*, *multi-objective min-sum location problems*, *multi-objective min-max location problems*, or *multi-objective ordered median location problems*) with convex constraints by using our results derived in Chapter 2 and corresponding algorithms for the unconstrained case. So, it is an important task to design effective procedures for computing the set of (strictly, weakly) Pareto efficient solutions of unconstrained multi-objective location problems involving mixed Minkowski gauges. Clearly, our penalization approach presented in Chapter 2 can also be very useful for location problems with nonconvex constraints.

For further information about *location theory*, the reader should take a look on books by Farahani and Hekmatfar [36], Hamacher [57], Klamroth [71] and Schöbel [113]. In particular, for further overviews on literature in *multi-objective location theory* we refer to Farahani, SteadieSeifi and Asgari [37], Nickel and Puerto [90, Ch. 11], Nickel, Puerto and Rodríguez-Chía [92, 93].

At the end of this chapter, we note that there is MATLAB-based software tool

**F**acility **L**ocation **O**ptimizer (**FLO**)

developed by Günther *et al.* [54] that can be used for solving special types of single- as well as multi-objective location problems involving different distances measures. The tool can be downloaded for free. For more information, see

`http://www.project-flo.de`.

Figure 4.2 shows a screen capture of the Software *Facility Location Optimizer (FLO)*.

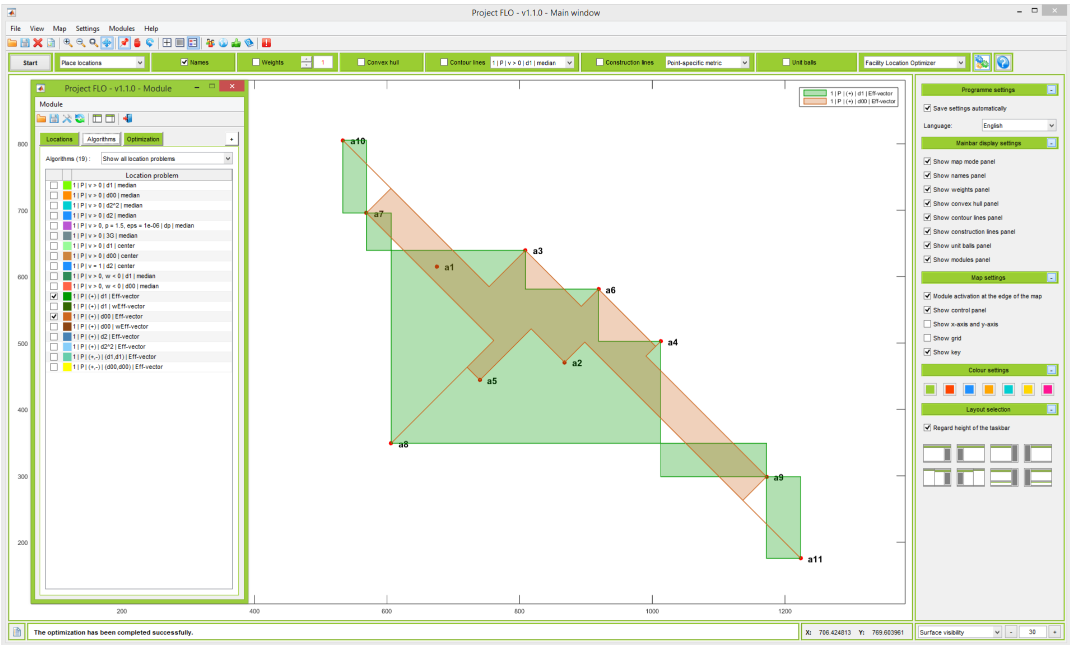


Figure 4.2: Screenshot of the Software *Facility Location Optimizer (FLO)*.

# Chapter 5

# Point-objective location problems in the plane

In this chapter, we consider *planar point-objective location problems involving the Manhattan norm.* Let us consider a finite family $\mathcal{A} = \{a^1, \ldots, a^m\}$ of points in the plane $\mathbf{E} = \mathbb{R}^2$,

$$a^i := (a_1^i, a_2^i) \quad \text{for all } i \in I_m,$$

representing some a priori given facilities. Throughout this chapter we assume that $\operatorname{card} \mathcal{A} = m \geq 2$, i.e., $a^1, \ldots, a^m$ are pairwise distinct. The multi-objective location problem associated to $\mathcal{A}$ consists in finding new facilities, i.e., points of $\mathbb{R}^2$, which minimize (simultaneously) the distances to all given points of $\mathcal{A}$. We assume that the distances are induced by the Manhattan norm $\|\cdot\|_1$. We start by studying the *unconstrained point-objective location problem* associated to $\mathcal{A}$ that is formulated as

$$\begin{cases} g_{\mathcal{A}}(x) = \big(\|x - a^1\|_1, \cdots, \|x - a^m\|_1\big) \to \min \\ x \in \mathbb{R}^2. \end{cases} \qquad (\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$$

**Remark 5.1** Notice that $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ is a special case of the problem $(\mathrm{POLP}_X(\mathcal{A}))$ considered in Section 4.1 (put $X = \mathbf{E} = \mathbb{R}^2$ and $\eta_i(\cdot) = ||\cdot||_1$ for all $i \in I_m$). Later on, in Section 5.5, we will study corresponding constrained problems, i.e., we minimize the vector-valued objective function $g_{\mathcal{A}} : \mathbb{R}^2 \to \mathbb{R}^m$ in the presence of some constraints $X \subsetneq \mathbb{R}^2$.

Two main types of algorithms are known in the literature for solving the problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$:

- Type (T1): algorithms that compute the boundary of the set of all Pareto efficient solutions (e.g., Pelegrin and Fernández [97] and Wendell, Hurter and Lowe [128]);
- Type (T2): algorithms that generate the whole set of Pareto efficient solutions as a union of rectangles and line segments, whose extreme points are adjacent intersection points of the grid composed by horizontal and vertical lines passing through each of the given location points $a^1, \ldots, a^m$ (e.g., Chalmet, Francis and Kolen [21], Nickel *et al.* [90], and Puerto and Fernández [107]).

The aim of this chapter is twofold: first, to characterize the *nonessential objectives* (i.e., the objective associated to those points among $a^1, \ldots, a^m$ which can be removed without changing the set of Pareto efficient solutions) and second, to develop an effective algorithm, which generates the whole set of Pareto efficient solutions as the union of a special family of axis-parallel rectangles and line segments. Our algorithm is neither of type (T1) nor of type (T2), since it does not require the computation of the boundary of the efficient solutions' set and it decomposes the whole set of Pareto efficient solutions into a reduced number of rectangles and line segments (each of our rectangles containing in general several "boxes" considered by Chalmet, Francis and Kolen [21]). This special feature of our approach is important for further practical applications, as for instance the maximization of certain quasi-convex functions (e.g., distances to some obnoxious additional location points) over the set of all Pareto efficient solutions of the original location problem (see, e.g., Alzorba, Günther and Popovici [2]).

In what follows, we will apply the classical *Weighted-Sum Scalarization Method* for generating Pareto efficient solutions of (POLP$^1_{\mathbb{R}^2}(\mathcal{A})$) (see Section 1.5). For every "weight" vector $\lambda := (\lambda_1, \dots, \lambda_m) \in \operatorname{int} \mathbb{R}^m_+$ let us consider the scalar optimization problem

$$\langle \lambda, g_{\mathcal{A}}(x) \rangle = \sum_{i=1}^{m} \lambda_i \|x - a^i\|_1 \to \min_{x \in \mathbb{R}^2} . \qquad (\mathrm{s}_\lambda \mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$$

In view of Lemma 1.74, any minimal solution of (s$_\lambda$POLP$^1_{\mathbb{R}^2}(\mathcal{A})$) is a properly Pareto efficient solution of (POLP$^1_{\mathbb{R}^2}(\mathcal{A})$), i.e., we have

$$\mathrm{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\mathcal{A}}(\cdot) \rangle) \subseteq \mathrm{PEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) \subseteq \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}). \tag{5.1}$$

In preparation for the next lemma, for each $k \in I_2$, we introduce the set

$$\mathcal{A}_k := \pi_k(\mathcal{A}) = \{a^1_k, \dots, a^m_k\},$$

i.e., the canonical projection of the set $\mathcal{A}$ on the $k$-th coordinate. Denoting $m_k := \operatorname{card} \mathcal{A}_k \leq \operatorname{card} \mathcal{A} = m$ for each $k \in I_2$, there exist uniquely determined numbers $u_1 < \dots < u_{m_1}$ and $v_1 < \dots < v_{m_2}$ such that

$$\mathcal{A}_1 = \{u_1, \dots, u_{m_1}\} \text{ and } \mathcal{A}_2 = \{v_1, \dots, v_{m_2}\}. \tag{5.2}$$

Then, the Cartesian product $\mathcal{A}_1 \times \mathcal{A}_2 := \{(u_i, v_j) \mid i \in I_{m_1} \text{ and } j \in I_{m_2}\}$ can be seen as a set of grid points in $\mathbb{R}^2$, such that

$$\mathcal{A} \subseteq \mathcal{A}_1 \times \mathcal{A}_2. \tag{5.3}$$

The following result (see Hamacher [57, Satz 2.3]) plays an important role for deriving properties of the set of properly/weakly Pareto efficient solutions of the problem (POLP$^1_{\mathbb{R}^2}(\mathcal{A})$) (see Corollary 5.7) and for the proof of the correctness of the proposed algorithm formulated in Section 5.3.

**Lemma 5.2** For every weight vector $\lambda \in \operatorname{int} \mathbb{R}^m_+$, the set $\mathrm{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\mathcal{A}}(\cdot) \rangle)$ is an axis-parallel rectangle, whose vertices are adjacent grid points, which may be degenerated into a line segment or a singleton. More precisely, $\mathrm{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\mathcal{A}}(\cdot) \rangle)$ has one of the following four forms:

**1°.** $\{(u_i, v_j)\}$ for some $(i, j) \in I_{m_1} \times I_{m_2}$;

**2°.** $[u_i, u_{i+1}] \times \{v_j\}$ for some $(i, j) \in I_{m_1 - 1} \times I_{m_2}$ $(m_1 \geq 2)$;

**3°.** $\{u_i\} \times [v_j, v_{j+1}]$ for some $(i, j) \in I_{m_1} \times I_{m_2 - 1}$ $(m_2 \geq 2)$;

**4°.** $[u_i, u_{i+1}] \times [v_j, v_{j+1}]$ for some $(i, j) \in I_{m_1 - 1} \times I_{m_2 - 1}$ $(m_1, m_2 \geq 2)$.

## 5.1 Structure of the sets of (weakly, properly) Pareto efficient solutions

The geometrical and topological structure of the sets of weakly and (properly) Pareto efficient solutions of certain multi-objective convex optimization problems–including multi-objective location problems, has been intensively studied (see, e.g., Durier and Michelot [27], Lowe *et al.* [79], Luc [80] or Popovici [102], and references therein). Characterizations of weakly and (properly) Pareto efficient solutions of multi-objective location problems can be found for instance in the papers by Chalmet, Francis and Kolen [21], Gerth and Pöhler [47], Lowe *et al.* [79], or Wendell, Hurter and Lowe [128]. In order to characterize the sets $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ and $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$, we will follow the approach proposed in [47], which is based on the dual of the Manhattan norm, namely the Maximum norm.

**Definition 5.3** Let $D \subseteq \mathbb{R}^2$ be a nonempty, bounded set. The set

$$\mathcal{N}(D) := \bigcap \left\{ \overline{B}_{||\cdot||_\infty}(x, r) \;\middle|\; x \in \mathbb{R}^2,\, r \in \mathbb{R}_{++},\, D \subseteq \overline{B}_{||\cdot||_\infty}(x, r) \right\}$$

is called the *rectangular hull* of $D$ (w.r.t. Maximum norm).

The following two results are counterparts of similar results obtained by Alzorba, Günther and Popovici [2, Lemma 4.2 and Theorem 4.3] (where the authors considered the rectangular hull w.r.t. Manhattan norm instead of Maximum norm).

**Lemma 5.4** ([3]) If $D \subseteq \mathbb{R}^2$ is nonempty and compact, then we have

$$\mathcal{N}(D) = \bigcup_{x', x'' \in D} \mathcal{N}(\{x', x''\}).$$

**Theorem 5.5** ([3]) The set of all weakly Pareto efficient solutions of the multi-objective location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ is given by

$$\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathcal{N}(\mathcal{A}).$$

In order to adapt the approach proposed by Gerth and Pöhler [47] for characterizing the set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$, we define certain sets related to the structure of the subdifferential of the Manhattan norm. For each $i \in I_m$, let

$$\begin{aligned}
s_1(a^i) &:= \{x \in \mathbb{R}^2 \mid x = (x_1, x_2),\ x_1 < a^i_1,\ x_2 < a^i_2\},\\
s_2(a^i) &:= \{x \in \mathbb{R}^2 \mid x = (x_1, x_2),\ x_1 > a^i_1,\ x_2 > a^i_2\},\\
s_3(a^i) &:= \{x \in \mathbb{R}^2 \mid x = (x_1, x_2),\ x_1 > a^i_1,\ x_2 < a^i_2\},\\
s_4(a^i) &:= \{x \in \mathbb{R}^2 \mid x = (x_1, x_2),\ x_1 < a^i_1,\ x_2 > a^i_2\},
\end{aligned}$$

and then, for every $r \in I_4$, consider the (possibly empty) set

$$S_r := \{x \in \mathcal{N}(\mathcal{A}) \mid \exists\, k \in I_m\, :\, x \in s_r(a^k)\} = \mathcal{N}(\mathcal{A}) \cap \bigcup_{k \in I_m} s_r(a^k). \tag{5.4}$$

The next result is a counterpart of the characterization of Pareto efficient solutions obtained by Gerth and Pöhler [47] (where the authors considered multi-objective location problems defined by the Maximum norm instead of the Manhattan norm).

**Theorem 5.6** ([47]) The set of Pareto efficient solutions of the multi-objective location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ is given by

$$\begin{aligned}
\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = &\left[(\mathrm{cl}\, S_1) \cap (\mathrm{cl}\, S_2)) \cup \left((\mathcal{N}(\mathcal{A}) \backslash S_1) \cap (\mathcal{N}(\mathcal{A}) \backslash S_2)\right)\right]\\
&\cap \left[((\mathrm{cl}\, S_3) \cap (\mathrm{cl}\, S_4)) \cup \left((\mathcal{N}(\mathcal{A}) \backslash S_3) \cap (\mathcal{N}(\mathcal{A}) \backslash S_4)\right)\right].
\end{aligned}$$

We conclude this section by highlighting some useful properties of the solution sets $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ and $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$.

**Corollary 5.7** ([3]) The following assertions hold:

**1°.** $\mathcal{A} \subseteq \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$, i.e., the given facilities are Pareto efficient solutions of $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$.

**2°.** $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \bigcup_{\lambda \in \mathrm{int}\, \mathbb{R}^m_+} \mathrm{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\mathcal{A}}(\cdot) \rangle)$, hence every efficient solution of $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ is properly efficient.

**3°.** $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ can be represented as a finite union of (possibly degenerated) axis-parallel rectangles.

**4°.** $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathcal{N}(\mathcal{A}) = [u_1, u_{p_1}] \times [v_1, v_{p_2}]$ with $u_1, u_{m_1}, v_1, v_{m_2}$ given by (5.2), where the intervals may degenerate into singletons when $m_1 = 1$ or $m_2 = 1$.

*Proof.* Assertion 1° follows from Definition 1.67, taking into account that $\{a^i\} = \mathrm{argmin}_{x \in \mathbb{R}^2} ||x - a^i||_1$ for all $i \in I_m$.

The equality stated at 2° is a classical result. Actually, the inclusion "$\supseteq$" follows from (5.1), while the inclusion "$\subseteq$" holds since the Manhattan norm is polyhedral, as pointed out by Wendell, Hurter and Lowe [128].

Assertion 3° is a straightforward consequence of Lemma 5.2, which shows actually that the set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ can be represented as a union of (at most) $(m_1-1)(m_2-1)+(m_1-1)m_2+m_1(m_2-1)+m_1m_2$ axis-parallel rectangles.

By Definition 5.3 and (5.3) we can easily deduce that

$$\mathcal{N}(\mathcal{A}) = [u_1, u_{m_1}] \times [v_1, v_{m_2}],$$

hence assertion 4° follows from Theorem 5.5. □

## 5.2 Reducing the number of objectives

In certain real-life applications the number $m$ of objectives is very large. In order to develop an effective algorithm for solving such location problems, we will first identify the nonessential objectives. Recall that an objective is said to be *nonessential* (or *redundant*) if the set of Pareto efficient solutions does not change when it is removed from the multi-objective optimization problem (see, e.g., the recent paper by Gal and Hanne [42] or the pioneering paper by Gal and Leberling [43]).

In this section, we will characterize the largest set of points which can be eliminated from $\mathcal{A}$ without altering the set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ of all Pareto efficient solutions of the multi-objective location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$. By removing all nonessential objectives we obtain a reduced location problem, which can be seen as a subproblem of $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$. Therefore, the results presented by us in this section may open the way for further investigations, concerning the decomposition of multi-objective problems into subproblems (see, e.g., Ehrgott and Nickel [30], Engau and Wiecek [34], Popovici [102], Ward [125], or Wendell *et al.* [128]).

**Lemma 5.8** ([3]) If $a^i \in \mathcal{A}$ $(i \in I_m)$ satisfies the relation

$$a^i \in \bigcap_{r \in I_4} \mathrm{cl}\, s_r(a^{j_r})$$

for some $a^{j_1}, \ldots, a^{j_4} \in \mathcal{A}\backslash\{a^i\}$ $(j_1, \ldots, j_4 \in I_m\backslash\{i\})$, then the following hold:

**1°.** $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$.

**2°.** $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$.

*Proof.* In order to prove 1° observe that, according to Theorem 5.5, we have

$$\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathcal{N}(\mathcal{A}) \text{ and } \mathcal{N}(\mathcal{A}\backslash\{a^i\}) = \mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}}).$$

Thus, it suffices to prove that $\mathcal{N}(\mathcal{A}) = \mathcal{N}(\mathcal{A}\backslash\{a^i\})$.

The inclusion $\mathcal{N}(\mathcal{A}) \supseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ holds by Lemma 5.4, since $\mathcal{A}\backslash\{a^i\} \subseteq \mathcal{A}$.

In order to prove the inclusion $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ consider an arbitrary $x \in \mathcal{N}(\mathcal{A})$. Thanks to Lemma 5.4, there exist two points $a^{t_1}, a^{t_2} \in \mathcal{A}$, i.e., $t_1, t_2 \in I_m$, such that $x \in \mathcal{N}(\{a^{t_1}, a^{t_2}\})$. We distinguish four possible cases:

*Case 1*: If $a^{t_1} \neq a^i \neq a^{t_2}$, then we have $\{a^{t_1}, a^{t_2}\} \subseteq \mathcal{A}\backslash\{a^i\}$, hence $x \in \mathcal{N}(\{a^{t_1}, a^{t_2}\}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ by Lemma 5.4.

*Case 2*: If $a^{t_1} = a^i = a^{t_2}$, then we have $x \in \mathcal{N}(\{a^{t_1}, a^{t_2}\}) = \{a^i\}$, i.e., $x = a^i$. Since $x = a^i \in \mathrm{cl}\, s_1(a^{j_1}) \cap \mathrm{cl}\, s_2(a^{j_2}) = \mathcal{N}(\{a^{j_1}, a^{j_2}\})$ and $\{a^{j_1}, a^{j_2}\} \subseteq \mathcal{A}\backslash\{a^i\}$, we get $x \in \mathcal{N}(\{a^{j_1}, a^{j_2}\}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ by Lemma 5.4.

*Case 3*: If $a^{t_1} \neq a^i = a^{t_2}$, then we have $x \in \mathcal{N}(\{a^{t_1}, a^i\})$ and there exists $r \in I_4$ such that $a^i \in \mathrm{cl}\, s_r(a^{t_1})$. Without loss of generality, we can suppose that $r = 1$. By hypothesis, there exists a point $a^{j_2} \in \mathcal{A}\backslash\{a^i\}$ such that $a^i \in \mathrm{cl}\, s_2(a^{j_2})$. We observe that $a^i \in \mathrm{cl}\, s_1(a^{t_1}) \cap \mathrm{cl}\, s_2(a^{j_2})$, and therefore the inclusion $\mathcal{N}(\{a^{t_1}, a^i\}) \subseteq \mathcal{N}(\{a^{t_1}, a^{j_2}\})$ holds. Recalling that $a^{t_1}, a^{j_2} \in \mathcal{A}\backslash\{a^i\}$, it follows that $\mathcal{N}(\{a^{t_1}, a^{j_2}\}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ by Lemma 5.4. Consequently, we have $x \in \mathcal{N}(\mathcal{A}\backslash\{a^i\})$.

*Case 4*: If $a^{t_1} = a^i \neq a^{t_2}$, then we deduce that $x \in \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ similarly to the previous case.

Thus, in all cases we have $x \in \mathcal{N}(\mathcal{A}\backslash\{a^i\})$, hence $\mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$.

In order to prove 2° we will apply Theorem 5.6 twice, for $\mathcal{A}$ and for $\mathcal{A}\backslash\{a^i\}$. Actually, since $\mathcal{N}(\mathcal{A}) = \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ (as already seen in the proof of 1°), it suffices to show that $S_r = \widehat{S}_r$ for all

$r \in I_4$, where $S_r$ is given by (5.4) while

$$\widehat{S}_r := \mathcal{N}(\mathcal{A}\backslash\{a^i\}) \cap \bigcup_{k\in I_m\backslash\{i\}} s_r(a^k). \tag{5.5}$$

Let $r \in I_4$ be arbitrarily chosen. The inclusion $\widehat{S}_r \subseteq S_r$ is obvious.

For proving the inclusion $S_r \subseteq \widehat{S}_r$, consider any $x \in S_r$. By (5.4) there exists $j \in I_m$ such that $x \in s_r(a^j)$. If $j \neq i$, then we obviously have $x \in \widehat{S}_r$ by (5.5). In what follows assume that $j = i$. By hypothesis there exists some $j_r \in I_m\backslash\{i\}$ such that $a^i \in \operatorname{cl} s_r(a^{j_r})$. Recalling the construction of the sets $s_r(\cdot)$, we deduce $x \in s_r(a^i) \subseteq \operatorname{cl} s_r(a^{j_r})$. Since the set $s_r(a^i)$ is open, it follows that $x \in s_r(a^i) \subseteq \operatorname{int}\operatorname{cl} s_r(a^{j_r}) = s_r(a^{j_r})$. Thus, we have $S_r \subseteq \widehat{S}_r$. $\square$

Consider now four pointed, convex cones of $\mathbb{R}^2$, namely the quadrants of the usual coordinate system, labelled as follows:

$$K_1 := \mathbb{R}_+ \times \mathbb{R}_+\,; \quad K_2 := \mathbb{R}_- \times \mathbb{R}_-\,;$$
$$K_3 := \mathbb{R}_- \times \mathbb{R}_+\,; \quad K_4 := \mathbb{R}_+ \times \mathbb{R}_-\,.$$

Notice that $K_2 = -K_1$ and $K_4 = -K_3$. Thus, it will be convenient to introduce the permutation $\psi : I_4 \to I_4$, defined by

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \psi(1) & \psi(2) & \psi(3) & \psi(4) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

It is easy to see that, for all $i \in I_m$ and $r \in I_4$, the following hold:

$$s_r(a^i) = a^i - \operatorname{int} K_r = a^i + \operatorname{int} K_{\psi(r)}; \tag{5.6}$$
$$\operatorname{cl} s_r(a^i) = a^i - K_r = a^i + K_{\psi(r)}. \tag{5.7}$$

In what follows, we will consider the minimal elements of $\mathcal{A} = \{a^1, \ldots, a^m\}$ with respect to the ordering cones $K_1, \ldots, K_4$. Notice that $\operatorname{MIN}(\mathcal{A}, K_r)$ is nonempty for every $r \in I_4$, since $\mathcal{A}$ is nonempty and finite.

**Lemma 5.9** ([3]) Let $i \in I_m$. The following assertions are equivalent:

**1°.** There exist $j_1, \ldots, j_4 \in I_m\backslash\{i\}$ such that

$$a^i \in \bigcap_{r\in I_4} \operatorname{cl} s_r(a^{j_r}).$$

**2°.** There exist $k_1, \ldots, k_4 \in I_m\backslash\{i\}$ such that $a^{k_r} \in \operatorname{cl}\, s_r(a^i)$ for all $r \in I_4$.

**3°.** There is no $r \in I_4$ such that $a^i \in \operatorname{MIN}(\mathcal{A}, K_r)$, i.e.,

$$a^i \in \mathcal{A} \setminus \bigcup_{r\in I_4} \operatorname{MIN}(\mathcal{A}, K_r).$$

*Proof.* We will prove the chain of implications $1° \Longrightarrow 2° \Longrightarrow 3° \Longrightarrow 1°$.

First, assume that 1° holds and consider any $j_1, \ldots, j_4 \in I_m\backslash\{i\}$ such that $a^i \in \bigcap_{r\in I_4} \operatorname{cl} s_r(a^{j_r})$. Then, by (5.6) we have $a^i \in \operatorname{cl} s_r(a^{j_r}) = a^{j_r} - K_r$, hence $a^{j_r} \in a^i + K_r = a^i - K_{\psi(r)} = \operatorname{cl} s_{\psi(r)}(a^i)$, for any $r \in I_4$. Since $\psi$ is a permutation on $I_4$, we infer that $a^{j_{\psi(r)}} \in \operatorname{cl} s_{\psi(\psi(r))}(a^i)$ for any $r \in I_4$. Taking into account that $\psi^{-1} = \psi$, i.e., $\psi(\psi(r)) = r$ for all $r \in I_4$, it follows that $a^{k_r} \in \operatorname{cl} s_r(a^i)$, where $k_r := j_{\psi(r)} \in I_m\backslash\{i\}$, for any $r \in I_4$. Thus 2° holds.

Now, assume that 2° holds and suppose to the contrary that 3° does not hold. Then, there exists some $r \in I_4$ such that $a^i \in \operatorname{MIN}(\mathcal{A}, K_r)$. By 2° we can choose an index $k_r \in I_m\backslash\{i\}$ (i.e., $a^{k_r} \in \mathcal{A}\backslash\{a^i\}$), such that $a^{k_r} \in \operatorname{cl}\, s_r(a^i)$. By (5.6), we infer that $a^{k_r} \in (a^i - K_r) \cap (\mathcal{A}\backslash\{a^i\})$. In view of Definition 1.66, it follows that $a^i \notin \operatorname{MIN}(\mathcal{A}, K_r)$, a contradiction.

Finally, assume that 3° holds. Then, for every index $r \in I_4$ we have $a^i \notin \mathrm{MIN}(\mathcal{A}, K_r)$, hence $(a^i - K_r) \cap (\mathcal{A}\backslash\{a^i\}) \neq \emptyset$. In other words, for each $r \in I_4$ there is $j_r \in I_m\backslash\{i\}$ such that $a^{j_r} \in (a^i - K_r) \cap (\mathcal{A}\backslash\{a^i\})$, which shows that $a^i \in a^{j_r} + K_r = a^{j_r} - K_{\psi(r)} = \mathrm{cl}\, s_{\psi(r)}(a^{j_r})$, according to (5.6). Therefore we have $a^i \in \bigcap_{r\in I_4} \mathrm{cl}\, s_{\psi(r)}(a^{j_r})$. Taking into account that $\psi$ is a permutation on $I_4$, we infer 1°.

□

**Example 5.10** Consider the set $\mathcal{A} = \{a^1, \dots, a^8\} \subseteq \mathbb{R}^2$, where

$$\begin{array}{llll} a^1 = (14, 12); & a^2 = (10, 9); & a^3 = (18, 10); & a^4 = (12, 4); \\ a^5 = (6, 2); & a^6 = (4, 8); & a^7 = (2, 6); & a^8 = (8, 7). \end{array}$$

By using Definition 1.66 for $F := \mathcal{A}$ and each cone $K \in \{K_1, \dots, K_4\}$, it is a simple exercise to check that

$$\begin{array}{ll} \mathrm{MIN}(\mathcal{A}, K_1) = \{a^5, a^7\}; & \mathrm{MIN}(\mathcal{A}, K_2) = \{a^1, a^3\}; \\ \mathrm{MIN}(\mathcal{A}, K_3) = \{a^3, a^4, a^5\}; & \mathrm{MIN}(\mathcal{A}, K_4) = \{a^1, a^2, a^6, a^7\}. \end{array}$$

Therefore we have

$$\mathcal{A} \setminus \bigcup_{r\in I_4} \mathrm{MIN}(\mathcal{A}, K_r) = \{a^8\}. \tag{5.8}$$

By (5.8) and Lemma 5.9 ($3° \Longrightarrow 1°$) there exist $j_1, \dots, j_4 \in I_7$ such that

$$a^8 \in \bigcap_{r\in I_4} \mathrm{cl}\, s_r(a^{j_r}). \tag{5.9}$$

Indeed, as we can see in Figure 5.1, we can choose the indices $j_1 = 1$, $j_2 = 7$, $j_3 = 6$ and $j_4 = 4$, which satisfy (5.9). Therefore, by the proof of Lemma 5.9 ($1° \Longrightarrow 2°$) it follows that $a^{k_r} \in \mathrm{cl}\ s_r(a^i)$ for all $r \in I_4$, where $k_1 := j_2 = 7$, $k_2 := j_1 = 1$, $k_3 := j_4 = 4$ and $k_4 := j_3 = 6$. This property is illustrated in Figure 5.2. On the other hand, by Corollary 5.7 (4°) and Lemma 5.8 (1°), we have

$$\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^8\}}) = [2, 18] \times [2, 12].$$

Moreover, by Lemma 5.8 (2°) we have $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^8\}})$. This set will be computed later on in Example 5.26 (see also Figure 5.6).

**Theorem 5.11** ([3]) Let $i \in I_m$. Then the following equivalence is true:

$$a^i \in \mathcal{A} \setminus \bigcup_{r\in I_4} \mathrm{MIN}(\mathcal{A}, K_r) \iff \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}}).$$

*Proof.* The implication "$\Longrightarrow$" follows from Lemma 5.8 and Lemma 5.9.

In order to prove the implication "$\Longleftarrow$", define the sets $S_r$ and $\widehat{S}_r$ for all $r \in I_4$ by (5.4) and (5.5), respectively. Assume that $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$ and suppose to the contrary that there exists $r \in I_4$ such that $a^i \in \mathrm{MIN}(\mathcal{A}, K_r)$. Without loss of generality, we can consider $r = 1$, hence $a^i \in \mathrm{MIN}(\mathcal{A}, K_1)$. In view of Definition 1.66 and (5.6), it follows that

$$(a^i - K_1) \cap (\mathcal{A}\backslash\{a^i\}) = (\mathrm{cl}\, s_1(a^i)) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset. \tag{5.10}$$

Consider the following two index sets (associated to $a^i \in \mathcal{A}$):

$$J_1 := \{j \in I_m \mid a_1^j > a_1^i\} \text{ and } J_2 := \{j \in I_m \mid a_2^j > a_2^i\}.$$

Notice that these sets are nonempty. Indeed, suppose to the contrary that $J_1 = \emptyset$. Then, we should have $\mathcal{A} \subseteq ]-\infty, a_1^i] \times \mathbb{R}$, which together with (5.10) yields $\mathcal{A}\backslash\{a^i\} \subseteq ]-\infty, a_1^i] \times ]a_2^i, +\infty[$. Since $\mathcal{A}\backslash\{a^i\}$ is a finite set, we infer that $\mathcal{N}(\mathcal{A}\backslash\{a^i\}) \subseteq ]-\infty, a_1^i] \times ]a_2^i, +\infty[$, hence $a^i \notin \mathcal{N}(\mathcal{A}\backslash\{a^i\})$.

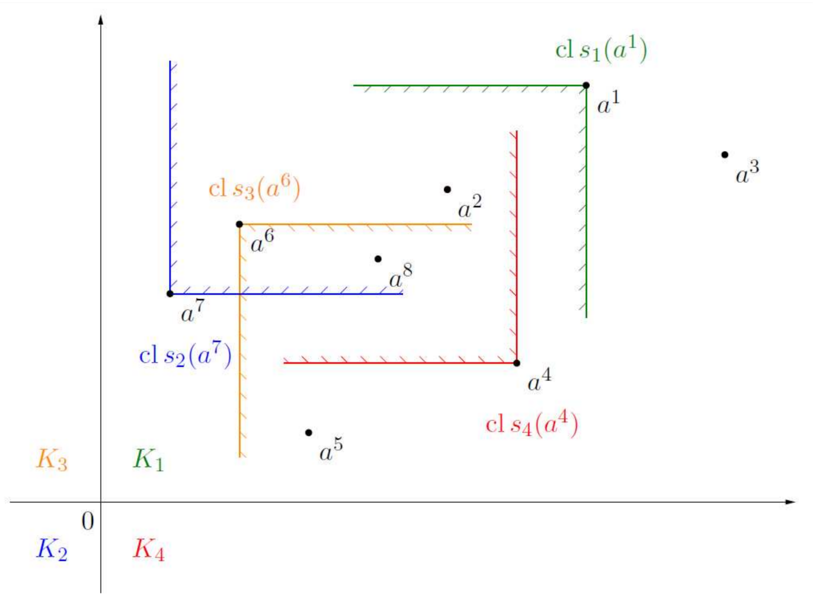


Figure 5.1: Illustration of Lemma 5.9 (1°) for the set $\mathcal{A}$ defined in Example 5.10.

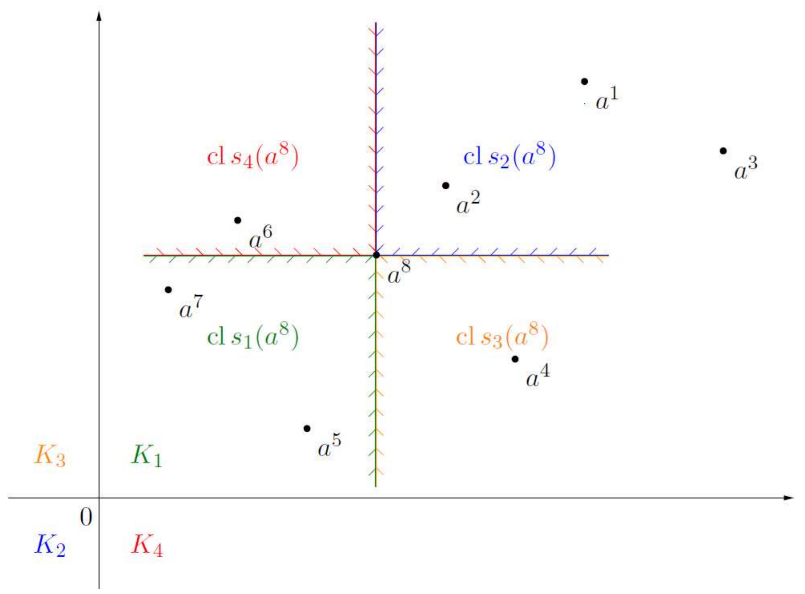


Figure 5.2: Illustration of Lemma 5.9 (2°) for the set $\mathcal{A}$ defined in Example 5.10.

However, by Corollary 5.7 (1°) and Theorem 5.6 (applied for $\mathcal{A}\backslash\{a^i\}$ in the role of $\mathcal{A}$) we deduce that $a^i \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$, a contradiction. Similarly we can prove that $J_2$ is nonempty.

Since both $J_1$ and $J_2$ are nonempty and finite, we can define the point

$$\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2) := \left(\frac{\delta_1 + a_1^i}{2}, \frac{\delta_2 + a_2^i}{2}\right) \in \mathbb{R}^2$$

by means of the real numbers

$$\delta_1 := \min\{a_1^j \mid j \in J_1\} = \min\{a_1^j \mid a^j \in \mathcal{A},\, a_1^j > a_1^i\};$$
$$\delta_2 := \min\{a_2^j \mid j \in J_2\} = \min\{a_2^j \mid a^j \in \mathcal{A},\, a_2^j > a_2^i\}.$$

By the construction of $\widetilde{x}$ and recalling (5.10), we can easy deduce that:

$$\widetilde{x} \in s_2(a^i); \tag{5.11}$$

$$(\widetilde{x} - K_1) \cap (\mathcal{A} \backslash \{a^i\}) = \emptyset. \tag{5.12}$$

In particular, (5.12) shows that there is no point $a^k \in \mathcal{A}\backslash\{a^i\}$ (i.e., $k \in I_m \backslash \{i\}$) such that $\widetilde{x} \in a^k + K_1 = a^k - K_2 = \operatorname{cl} s_2(a^k)$. Consequently, we have

$$\widetilde{x} \notin \operatorname{cl} \widehat{S}_2. \tag{5.13}$$

The following cases may occur:

*Case 1*: If $s_2(a^i) \cap (\mathcal{A}\backslash\{a^i\}) \neq \emptyset$, then there is $j \in I_m\backslash\{i\}$ (i.e., $a^j \in \mathcal{A}\backslash\{a^i\}$) such that $a^j \in s_2(a^i)$. In this case, by the definition of $\widetilde{x}$, we have

$$\widetilde{x} \in s_1(a^j). \tag{5.14}$$

Now, we distinguish two subcases:

*Subcase 1.1*: If there exist $a^h, a^k \in \mathcal{A}\backslash\{a^i\}$ (i.e., $h, k \in I_m\backslash\{i\}$) such that $a^h \in \operatorname{cl} s_3(a^i)$ and $a^k \in \operatorname{cl} s_4(a^i)$, then (5.10) yields $a^h, a^k \notin \operatorname{cl} s_1(a^i)$. Moreover, by definition of $\widetilde{x}$ it follows that $\widetilde{x}_1 < a_1^h$ and $\widetilde{x}_2 > a_2^h$ as well as $\widetilde{x}_1 > a_1^k$ and $\widetilde{x}_2 < a_2^k$. By Lemma 5.4 and relations (5.11) and (5.14), we deduce that $\widetilde{x} \in \mathcal{N}(\{a^h, a^k\}) \subseteq \mathcal{N}(\mathcal{A})$ and $\widetilde{x} \in s_4(a^h) \cap s_3(a^k) \cap s_2(a^i) \cap s_1(a^j)$. Therefore we have $\widetilde{x} \in S_1 \cap S_2 \cap S_3 \cap S_4$. By Theorem 5.6 we conclude that $\widetilde{x} \in \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$. On the other hand, since $a^h, a^k \in \mathcal{A}\backslash\{a^i\}$, we also have $\widetilde{x} \in \mathcal{N}(\{a^h, a^k\}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$ according to Lemma 5.4 (applied for $\mathcal{A}\backslash\{a^i\}$). Since $a^j \in \mathcal{A}\backslash\{a^i\}$, we infer from (5.14) that $\widetilde{x} \in \widehat{S}_1$. The latter relation together with (5.13) allows us to deduce by Theorem 5.6 (applied for $\mathcal{A}\backslash\{a^i\}$) that $\widetilde{x} \notin \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$. We conclude that $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) \neq \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$, a contradiction.

*Subcase 1.2*: If $(\operatorname{cl} s_3(a^i)) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$ or $(\operatorname{cl} s_4(a^i)) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$, then we can suppose without any loss of generality that $(\operatorname{cl} s_4(a^i)) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$. Then, by relation (5.10) we deduce that $((\operatorname{cl} s_1(a^i)) \cup (\operatorname{cl} s_4(a^i))) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$. It follows that $\mathcal{A}\backslash\{a^i\} \subseteq ]a_1^i, +\infty[ \times \mathbb{R}$, hence $a_1^i < a_1^k$ for all $k \in I_m\backslash\{i\}$. In particular, we have $a_1^i < a_1^j$. Consequently, we have $a^i \notin \mathcal{N}(\mathcal{A}\backslash\{a^i\})$. By Theorem 5.6 (applied for $\mathcal{A}\backslash\{a^i\}$), we conclude that $a^i \notin \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$. However, by assertion 1° of Corollary 5.7, we infer $a^i \in \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$, contradicting the hypothesis that $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$.

*Case 2*: If $s_2(a^i) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$, then we consider two subcases:

*Subcase 2.1*: If there exist $a^h, a^k \in \mathcal{A}\backslash\{a^i\}$ (i.e., $h, k \in I_m\backslash\{i\}$) such that $a^h \in \operatorname{cl} s_3(a^i)$ and $a^k \in \operatorname{cl} s_4(a^i)$, then $a^h, a^k \notin \operatorname{cl} s_1(a^i)$ by (5.10). Also, we have $\widetilde{x} \in \mathcal{N}(\{a^h, a^k\}) \subseteq \mathcal{N}(\mathcal{A})$. In view of (5.11), it follows that $\widetilde{x} \in \mathcal{N}(\mathcal{A}) \cap s_2(a^i)$, hence $\widetilde{x} \in S_2$. Also, since $s_2(a^i) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$, we have $\widetilde{x} \notin S_1$. Therefore, we can conclude by Theorem 5.6 that $\widetilde{x} \notin \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$. On the other hand, since $a^h, a^k \in \mathcal{A}\backslash\{a^i\}$, we have $\widetilde{x} \in \mathcal{N}(\{a^h, a^k\}) \subseteq \mathcal{N}(\mathcal{A}\backslash\{a^i\})$. Thus, $\widetilde{x} \notin S_1$ entails $\widetilde{x} \notin \widehat{S}_1$. Furthermore, similarly to Subcase 1.1, it is easy to check that $\widetilde{x} \in s_3(a^h) \cap s_4(a^k)$. Hence, the relation $\widetilde{x} \in \widehat{S}_3 \cap \widehat{S}_4$ holds. Recalling (5.13), we deduce that $\widetilde{x} \in (\mathcal{N}(\mathcal{A}\backslash\{a^i\})\backslash\widehat{S}_1) \cap (\mathcal{N}(\mathcal{A}\backslash\{a^i\})\backslash\widehat{S}_2) \cap \widehat{S}_3 \cap \widehat{S}_4$. By Theorem 5.6 we infer $\widetilde{x} \in \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$, which contradicts again the hypothesis that $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$.

*Subcase 2.2*: If $(\operatorname{cl} s_3(a^i)) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$ or $(\operatorname{cl} s_4(a^i)) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$, then observe that one of the sets $(\operatorname{cl} s_3(a^i)) \cap (\mathcal{A}\backslash\{a^i\})$ and $(\operatorname{cl} s_4(a^i)) \cap (\mathcal{A}\backslash\{a^i\})$ should be nonempty, because of the assumption of Case 2 and the fact that $\operatorname{card} \mathcal{A} \geq 2$. Without loss of generality, we can suppose that $(\operatorname{cl} s_4(a^i)) \cap (\mathcal{A}\backslash\{a^i\}) = \emptyset$, hence there exists a point $a^h \in \operatorname{cl} s_3(a^i) \cap (\mathcal{A}\backslash\{a^i\})$. It is easy to check that $a_1^i < a_1^h$ and $a^i \notin \mathcal{N}(\mathcal{A}\backslash\{a^i\})$. Therefore, following the same lines as in Subcase 1.2, we arrive to a contradiction.

□

**Remark 5.12** In contrast to Theorem 5.11, the equivalence

$$a^i \in \mathcal{A} \setminus \bigcup_{r \in I_4} \operatorname{MIN}(\mathcal{A}, K_r) \iff \operatorname{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \operatorname{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}\backslash\{a^i\}})$$

may be false for some $i \in I_m$, although the implication "$\Longrightarrow$" is always true by Lemma 5.8 and Lemma 5.9. As a counterexample for the implication "$\Longleftarrow$", consider the set $\mathcal{A} := \{a^1, a^2, a^3\} \subseteq \mathbb{R}^2$, where

$$a^1 := (0,0),\ a^2 := (0,1)\ \text{ and }\ a^3 := (1,1).$$

We have $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = [0,1] \times [0,1] = \mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}\setminus\{a^2\}})$, but $a^2 \in \mathrm{MIN}(\mathcal{A}, K_4)$.

In what follows, we consider the *reduced multi-objective location problem* ($\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})$), associated to the following nonempty subset of the initial set $\mathcal{A}$:

$$\widetilde{\mathcal{A}} := \bigcup_{r \in I_4} \mathrm{MIN}(\mathcal{A}, K_r). \tag{5.15}$$

**Corollary 5.13** ([3]) The multi-objective location problems ($\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})$) and ($\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}})$) have the same Pareto efficient solutions, i.e.,

$$\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}). \tag{5.16}$$

*Proof.* Since (5.16) trivially holds when $\mathcal{A} = \widetilde{\mathcal{A}}$, we just have to study the case when $\mathcal{A}\setminus\widetilde{\mathcal{A}} \neq \emptyset$. Assume that $\mathcal{A}\setminus\widetilde{\mathcal{A}} =: \{a^{i_1}, \dots, a^{i_n}\}$ with $i_1, \dots, i_n \in I_m$ and $\mathrm{card}(\mathcal{A}\setminus\widetilde{\mathcal{A}}) = n \geq 1$ (i.e., $a^{i_1}, \dots, a^{i_n}$ are pairwise distinct).

We will apply Theorem 5.11 recursively. First note that $a^{i_1} \in \mathcal{A}\setminus\widetilde{A}$ means

$$a^{i_1} \in \widetilde{\mathcal{A}} \setminus \bigcup_{r \in I_4} \mathrm{MIN}(\mathcal{A}, K_r), \tag{5.17}$$

according to (5.15). By applying Theorem 5.11 for $i := i_1$, we deduce that

$$\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\setminus\{a^{i_1}\}}). \tag{5.18}$$

If $n = 1$, then (5.18) becomes the desired relation (5.16). Otherwise, if $n \geq 2$, then consider the point $a^{i_2} \in (\mathcal{A}\setminus\{a^{i_1}\})\setminus\widetilde{\mathcal{A}} = (\mathcal{A}\setminus\{a^{i_1}\}) \setminus \bigcup_{r \in I_4} \mathrm{MIN}(\mathcal{A}, K_r)$. Relation (5.17) shows that, for all $r \in I_4$, we have $a^{i_1} \in \mathcal{A}\setminus\mathrm{MIN}(\mathcal{A}, K_r)$, hence $\mathrm{MIN}(\mathcal{A}, K_r) = \mathrm{MIN}(\mathcal{A}\setminus\{a^{i_1}\}, K_r)$, in view of (1.7). Therefore we have

$$a^{i_2} \in (\mathcal{A}\setminus\{a^{i_1}\}) \setminus \bigcup_{r \in I_4} \mathrm{MIN}(\mathcal{A}\setminus\{a^{i_1}\}, K_r).$$

By applying Theorem 5.11 again, this time for $\mathcal{A}\setminus\{a^{i_1}\}$ in the role of $\mathcal{A}$ and $i := i_2 \in I_m\setminus\{i_1\}$, we deduce that

$$\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\setminus\{a^{i_1}\}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\setminus\{a^{i_1}, a^{i_2}\}}). \tag{5.19}$$

If $n = 2$, then (5.16) follows from (5.18) and (5.19). Otherwise, by a similar argument as above we deduce that

$$\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\setminus\{a^{i_1}, \dots, a^{i_{m-1}}\}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}\setminus\{a^{i_1}, \dots, a^{i_m}\}\}}) \tag{5.20}$$

for every $m \in \{3, \dots, n\}$. Finally, we conclude (5.16) from (5.18)–(5.20).

□

**Remark 5.14** In the proof of Corollary 5.13, we have used the implication "$\Longrightarrow$" of Theorem 5.11. Similarly, by using Lemmas 5.8 and 5.9, we can prove that the sets of weakly Pareto efficient solutions of problems ($\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})$) and ($\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}})$) coincide, i.e.,

$$\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{WEff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}).$$

On the other hand, the implication "$\Longleftarrow$" of Theorem 5.11 shows that a further reduction of $(\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}}))$ is not possible, since the points of $\mathcal{A}$ that can be eliminated without altering the set of Pareto efficient solutions should belong to $\widetilde{\mathcal{A}}$. Moreover, Theorem 5.11 will play a key role in Section 5.3, for proving that the *Rectangular Decomposition Algorithm* generates all Pareto efficient solutions of $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ (see Theorem 5.25).

In Section 5.3, we will develop a new algorithm for computing the set of all Pareto efficient solutions of the multi-objective location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ via the reduced problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}}))$. As a part of the algorithm, we need an effective procedure for checking whether a given point $x \in \mathbb{R}^2$ is efficient for the reduced problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}}))$ or not. In the following, we will derive equivalent characterizations for the statement $x \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$ using the characterization given in Theorem 5.6.

For notational convenience, let

$$T_r := \mathrm{MIN}(\mathcal{A}, K_r) \tag{5.21}$$

be the set of all minimal elements of the finite set $\mathcal{A} = \{a^1, \dots, a^m\}$ with respect to the cone $K_r$, for every $r \in I_4$. Hence, the set of all existing facilities $\widetilde{A}$ of the reduced problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}}))$ can be represented by

$$\widetilde{\mathcal{A}} = \{\widetilde{a}^1, \dots, \widetilde{a}^q\} := \bigcup_{r \in I_4} T_r, \tag{5.22}$$

where $\widetilde{a}^1 = (\widetilde{a}^1_1, \widetilde{a}^1_2), \dots, \widetilde{a}^q = (\widetilde{a}^q_1, \widetilde{a}^q_2)$ and $q := \mathrm{card}(\bigcup_{r \in I_4} T_r)$.

For every $k \in I_2$ let

$$\widetilde{\mathcal{A}}_k = \pi_k(\widetilde{\mathcal{A}}) = \{\widetilde{a}^1_k, \dots, \widetilde{a}^q_k\} \tag{5.23}$$

be the canonical projection of $\widetilde{\mathcal{A}}$ on the $k$-th coordinate.

We define the (possibly empty) set

$$\widetilde{T}_r := T_{\psi(r)} \cap (e^r + \mathrm{int}\, K_r) \tag{5.24}$$

for every $r \in I_4$, where

$$e^1 := (\min \widetilde{\mathcal{A}}_1, \min \widetilde{\mathcal{A}}_2); \quad e^2 := (\max \widetilde{\mathcal{A}}_1, \max \widetilde{\mathcal{A}}_2);$$
$$e^3 := (\max \widetilde{\mathcal{A}}_1, \min \widetilde{\mathcal{A}}_2); \quad e^4 := (\min \widetilde{\mathcal{A}}_1, \max \widetilde{\mathcal{A}}_1).$$

In order to derive a practical test of efficiency (i.e., $x \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$), we describe the set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$ by using Theorem 5.6 (for $\widetilde{\mathcal{A}}$ in the role of $\mathcal{A}$) as

$$\begin{aligned} \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}) \;=\; & \Big[((\mathrm{cl}\, \widetilde{S}_1) \cap (\mathrm{cl}\, \widetilde{S}_2)) \cup \big(\mathcal{N}(\widetilde{\mathcal{A}}) \setminus (\widetilde{S}_1 \cup \widetilde{S}_2)\big)\Big] \\ & \cap \Big[((\mathrm{cl}\, \widetilde{S}_3) \cap (\mathrm{cl}\, \widetilde{S}_4)) \cup \big(\mathcal{N}(\widetilde{\mathcal{A}}) \setminus (\widetilde{S}_3 \cup \widetilde{S}_4)\big)\Big], \end{aligned} \tag{5.25}$$

where, for every $r \in I_4$,

$$\widetilde{S}_r := \mathcal{N}(\widetilde{\mathcal{A}}) \cap \bigcup_{i \in I_q} s_r(\widetilde{a}^i) = \bigcup_{i \in I_q} (\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^i)). \tag{5.26}$$

The next lemma allows us to interpret (5.25) in terms of the sets $\widetilde{T}_1, \dots, \widetilde{T}_4$ introduced in (5.24). In preparation of this result, observe that

$$T_r = \mathrm{MIN}(\mathcal{A}, K_r) = \mathrm{MIN}(\widetilde{\mathcal{A}}, K_r) \quad \text{for all } r \in I_4, \tag{5.27}$$

as a consequence of (1.7). Also, note that all points of $\widetilde{\mathcal{A}}$ are grid points, i.e.,

$$\widetilde{\mathcal{A}} \subseteq \widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2 \tag{5.28}$$

while the particular grid points $e^1, \ldots, e^4$ are the vertices of the rectangular hull $\mathcal{N}(\widetilde{\mathcal{A}})$ and we have

$$\mathcal{N}(\widetilde{\mathcal{A}}) = \bigcap_{r \in I_4} (e^r + K_r) = \operatorname{conv}\{e^1, \ldots, e^4\}. \tag{5.29}$$

**Lemma 5.15** ([3]) For every $r \in I_4$ the following relations hold:

$$\widetilde{S}_r = \mathcal{N}(\widetilde{\mathcal{A}}) \cap (\widetilde{T}_r - \operatorname{int} K_r); \tag{5.30}$$

$$\operatorname{cl} \widetilde{S}_r = \mathcal{N}(\widetilde{\mathcal{A}}) \cap (\widetilde{T}_r - K_r). \tag{5.31}$$

*Proof.* For any $r \in I_4$ define the (possibly empty) set

$$\widetilde{I}^r := \{i \in I_q \mid \widetilde{a}^i \in \widetilde{T}_r\}. \tag{5.32}$$

By definition of $s_r(\cdot)$, we have $\widetilde{T}_r - \operatorname{int} K_r = \bigcup_{i \in \widetilde{I}^r} (\widetilde{a}^i - \operatorname{int} K_r) = \bigcup_{i \in \widetilde{I}^r} s_r(\widetilde{a}^i)$ and $\widetilde{T}_r - K_r = \bigcup_{i \in \widetilde{I}^r} (\widetilde{a}^i - K_r) = \bigcup_{i \in \widetilde{I}^r} \operatorname{cl} s_r(\widetilde{a}^i)$. Therefore, in order to prove (5.30) and (5.31), we just have to show that the following relations hold:

$$\widetilde{S}_r = \bigcup_{i \in \widetilde{I}^r} (\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^i)); \tag{5.33}$$

$$\operatorname{cl} \widetilde{S}_r = \bigcup_{i \in \widetilde{I}^r} (\mathcal{N}(\widetilde{\mathcal{A}}) \cap \operatorname{cl} s_r(\widetilde{a}^i)). \tag{5.34}$$

In view of (5.26), relation (5.33) can be rewritten as

$$\bigcup_{i \in I_q} (\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^i)) = \bigcup_{i \in \widetilde{I}^r} (\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^i)). \tag{5.35}$$

Since $I_q \supseteq \widetilde{I}^r$, the inclusion "$\supseteq$" in (5.35) holds trivially. In order to prove the reverse inclusion "$\subseteq$", consider any point $x \in \bigcup_{i \in I_q} (\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^i))$. Then $x \in \mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^j)$ for some index $j \in I_q$. Since the set $\widetilde{\mathcal{A}} = \{\widetilde{a}^i \mid i \in I_q\}$ is nonempty and compact, it satisfies the domination property with respect to the ordering cone $K_{\psi(r)}$ (see, e.g., Göpfert *et al.* [50]):

$$\widetilde{\mathcal{A}} \subseteq \operatorname{MIN}(\widetilde{\mathcal{A}}, K_{\psi(r)}) + K_{\psi(r)}. \tag{5.36}$$

From (5.27) and (5.36) it follows that there is $k \in I_q$ such that

$$\widetilde{a}^j \in \widetilde{a}^k + K_{\psi(r)} \text{ and } \widetilde{a}^k \in \operatorname{MIN}(\widetilde{\mathcal{A}}, K_{\psi(r)}) = T_{\psi(r)}. \tag{5.37}$$

In view of (5.6), it follows that $s_r(\widetilde{a}^j) = \widetilde{a}^j - \operatorname{int} K_r \subseteq \widetilde{a}^k + K_{\psi(r)} - \operatorname{int} K_r = \widetilde{a}^k - K_r - \operatorname{int} K_r = \widetilde{a}^k - \operatorname{int} K_r = s_r(\widetilde{a}^k)$, hence

$$x \in \mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^j) \subseteq \mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^k). \tag{5.38}$$

By (5.29) and (5.38) we get $x \in \mathcal{N}(\widetilde{\mathcal{A}}) \subseteq e^r + K_r$ and $x \in s_r(\widetilde{a}^k) = \widetilde{a}^k - \operatorname{int} K_r$, hence $\widetilde{a}^k \in x + \operatorname{int} K_r \subseteq e^r + K_r + \operatorname{int} K_r = e^r + \operatorname{int} K_r$. By the second part of (5.37) we infer that $\widetilde{a}^k \in T_{\psi(r)} \cap (e^r + \operatorname{int} K_r) = \widetilde{T}_r$, which means that $k \in \widetilde{I}^r$ according to (5.32). Therefore (5.38) ensures that $x \in \bigcup_{j \in \widetilde{I}^r} (\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^j))$. We conclude that inclusion "$\subseteq$" in (5.35) holds, which ends the proof of (5.33).

Now, let us prove (5.34). The set $\widetilde{I}^r$ being finite, relation (5.33) entails

$$\operatorname{cl}\widetilde{S}_r = \bigcup_{i\in\widetilde{I}^r} \operatorname{cl}(\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^i)). \tag{5.39}$$

Without loss of generality, we can assume that $\widetilde{I}^r \neq \emptyset$. Then, for any $i \in \widetilde{I}^r$ we have $\widetilde{a}^i \in \widetilde{T}^r = T_{\psi(r)} \cap (e^r + \operatorname{int} K_r) = \operatorname{MIN}(\widetilde{\mathcal{A}}, K_{\psi(r)}) \cap (e^r + \operatorname{int} K_r) \subseteq \widetilde{\mathcal{A}} \cap (e^r + \operatorname{int} K_r)$ in view of (5.27). Since $\{e^r, \widetilde{a}^i\} \subseteq \mathcal{N}(\widetilde{\mathcal{A}})$ and $\widetilde{a}^i \in e^r + \operatorname{int} K_r$, it follows that $\frac{1}{2}(e^r + \widetilde{a}^i) \in (e^r + \operatorname{int} K_r) \cap (\widetilde{a}^i - \operatorname{int} K_r) = \operatorname{int}\mathcal{N}(\{e^r, \widetilde{a}^i\}) \subseteq (\operatorname{int}\mathcal{N}(\widetilde{\mathcal{A}})) \cap (\operatorname{int} s_r(\widetilde{a}^i))$, which shows that

$$(\operatorname{int}\mathcal{N}(\widetilde{\mathcal{A}})) \cap (\operatorname{int} s_r(\widetilde{a}^i)) \neq \emptyset.$$

Taking into account that $\mathcal{N}(\widetilde{\mathcal{A}})$ and $s_r(\widetilde{a}^i)$ are convex, we infer by a classical argument in convex analysis (see, e.g., Rockafellar [111, Theorem 6.5]) that

$$\operatorname{cl}(\mathcal{N}(\widetilde{\mathcal{A}}) \cap s_r(\widetilde{a}^i)) = (\operatorname{cl}\mathcal{N}(\widetilde{\mathcal{A}})) \cap (\operatorname{cl} s_r(\widetilde{a}^i)). \tag{5.40}$$

The desired relation (5.34) follows by (5.39) and (5.40), since $\mathcal{N}(\widetilde{\mathcal{A}})$ is closed.

□

We conclude this section by presenting a new characterization of the Pareto efficient solutions to problem ($\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}})$) in terms of the sets $\widetilde{T}_r$, $r \in I_4$.

**Theorem 5.16** ([3]) For any $x \in \mathcal{N}(\widetilde{\mathcal{A}})$, the following assertions are equivalent:

**1°.** $x \in \operatorname{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$.

**2°.** $x$ satisfies both conditions below:

$$x \in (\widetilde{T}_1 - K_1) \cap (\widetilde{T}_2 - K_2) \text{ or } x \notin (\widetilde{T}_1 - \operatorname{int} K_1) \cup (\widetilde{T}_2 - \operatorname{int} K_2); \tag{5.41}$$

$$x \in (\widetilde{T}_3 - K_3) \cap (\widetilde{T}_4 - K_4) \text{ or } x \notin (\widetilde{T}_3 - \operatorname{int} K_3) \cup (\widetilde{T}_4 - \operatorname{int} K_4). \tag{5.42}$$

*Proof.* Directly follows by (5.25) and Lemma 5.15. □

As we shall see in the next section, Theorem 5.16 plays a key role in constructing the set of Pareto efficient solutions (Step 5 of the proposed algorithm).

## 5.3 Rectangular Decomposition Algorithm

In this section, we present a new algorithm for computing the set $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ of all Pareto efficient solutions to problem ($\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})$). It eliminates the nonessential objectives in a first phase and thereafter generates the set $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ as the union of a special family of axis-parallel rectangles and line segments. As far as we know, the first phase of our algorithm is a novel approach in location theory and could be used as a pre-phase for improving other algorithms known in the literature (see, e.g., Chalmet *et al.* [21], Gerth and Pöhler [47] or Wendell *et al.* [128]). On the other hand, the rectangular type decomposition of the set $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ provided by our algorithm differs from other known approaches, since it involves only a reduced number of rectangles (each of them containing in general several "boxes" considered in [21]). Therefore, it can be used as input for other algorithms, as for instance to minimize/maximize an additional cost function over the set of Pareto efficient solutions of the multi-objective location problem ($\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})$) (see Alzorba, Günther and Popovici [2]).

The correctness of this algorithm and detailed explanations of its steps are shown in Subsections 5.3.2 – 5.3.5.

### 5.3.1 Formulation of the Rectangular Decomposition Algorithm

**Input:** The set $\mathcal{A} := \{a^1, \ldots, a^m\}$ (representing the a priori given facilities).

**Step 1.** For every $r \in I_4$ compute the set $T_r$ given by (5.21).

**Step 2.** Construct the reduced set $\widetilde{\mathcal{A}}$ given by (5.22).

**Step 3.** For every $k \in I_2$ compute the set $\widetilde{\mathcal{A}}_k$ given by (5.23) and sort its elements in ascending order deleting the duplicate values, i.e., determine $q_k := \operatorname{card} \widetilde{\mathcal{A}}_k$ and the numbers $u_1 < \cdots < u_{q_1}$ and $v_1 < \cdots < v_{q_2}$ such that $\widetilde{\mathcal{A}}_1 = \{u_1, \ldots, u_{q_1}\}$ and $\widetilde{\mathcal{A}}_2 = \{v_1, \ldots, v_{q_2}\}$.

**Step 4.** For every $r \in I_4$ compute the set $\widetilde{T}_r$ given by (5.24).

**Step 5.** For every $i \in I_{q_1}$ define the set

$$C_i := \bigcup_{j \in I_{q_2}} \{v_j \mid x := (u_i, v_j) \text{ satisfies (5.41) and (5.42)}\}.$$

**Step 6.** For every $i \in I_{q_1}$ define the numbers

$$\underline{c}_i := \min C_i \text{ and } \overline{c}_i := \max C_i.$$

If $q_1 = 1$, then define

$$\mathcal{R}_1^* := \{u_1\} \times [\underline{c}_1, \overline{c}_1];$$
$$\mathcal{R}_2^* := \emptyset$$

and go to Output.

If $q_1 \geq 2$, then define

$$\mathcal{R}_1^* := \bigcup_{i \in I_{q_1}} (\{u_i\} \times [\underline{c}_i, \overline{c}_i]);$$
$$\mathcal{R}_2^* := \bigcup_{i \in I_{q_1 - 1}} \operatorname{conv}\{(u_i, c_i'), (u_i, c_i''), (u_{i+1}, c_i''), (u_{i+1}, c_i')\},$$

where the numbers $c_i'$ and $c_i''$ are defined for all $i \in I_{q_1 - 1}$ by

$$c_i' := \max\{\underline{c}_i, \underline{c}_{i+1}\} \text{ and } c_i'' := \min\{\overline{c}_i, \overline{c}_{i+1}\}.$$

**Output:** The set $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \operatorname{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}) := \mathcal{R}_1^* \cup \mathcal{R}_2^*$ of all Pareto efficient solutions of the initial multi-objective location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$.

In Subsections 5.3.2 – 5.3.5, we will explain each step of this algorithm. We just mention here that, according to Step 6, the set $\mathcal{R}_1^*$ is a union of vertical line segments while the set $\mathcal{R}_2^*$ is a union of axis parallel rectangles (which may degenerate into horizontal line segments) unless it is empty (if $q_1 = 1$). Consequently, the algorithm provides an explicit decomposition of the solution set $\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*$ into a family of (possibly degenerated) rectangles. This feature justifies the name of our algorithm.

### 5.3.2 Analysis and implementation of Steps 1 and 2

Within the first two steps of our algorithm we construct the reduced set $\widetilde{\mathcal{A}}$, which corresponds to the essential objectives of the location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$, as show in Section 5.2.

More precisely, at Step 1 we compute for every $r \in I_4$ the set $T_r$ of all minimal elements of $\mathcal{A} = \{a^1, \ldots, a^m\}$ with respect to $K_r$, by adapting the *Jahn-Graef-Younes Method*. Thereafter, at Step 2 we construct the reduced set $\widetilde{\mathcal{A}}$ according to (5.22).

Next, we present the pseudocode of Steps 1 and 2.

```
/* Step 1                                                    */
for r ← 1 to 4 do
    /* Forward iteration                                     */
    i ← 1;
    b^i ← a^i;
    B ← {b^i};
    for j ← 2 to m do
        if a^j ∉ B + K_r then
            i ← i + 1;
            b^i ← a^j;
            B ← B ∪ {b^i};
        end
    end
    /* Backward iteration                                    */
    T ← {b^i};
    for j ← 1 to i − 1 do
        if b^{i−j} ∉ T + K_r then
            T ← T ∪ {b^{i−j}};
        end
    end
    T_r ← T;
end
/* Step 2                                                    */
Ã ← ∅;
for r ← 1 to 4 do
    Ã ← Ã ∪ T_r;
end
```

### 5.3.3 Analysis and implementation of Steps 3 and 4

At Step 3 we generate the sets $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ according to (5.23). Moreover, we sort the elements of $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$ in ascending order and delete duplicated values. Consequently, we have

$$\widetilde{\mathcal{A}}_1 = \{u_1, \ldots, u_{q_1}\} \quad \text{and} \quad \widetilde{\mathcal{A}}_2 = \{v_1, \ldots, v_{q_2}\} \tag{5.43}$$

for numbers $u_1 < \cdots < u_{q_1}$ and $v_1 < \cdots < v_{q_2}$. Notice that these numbers are not the same as those involved in formula (5.2) although we use the same symbols $u$ and $v$ for simplicity.

At Step 4 we generate the sets $\widetilde{T}_r$, $r \in I_4$, given by (5.24).

The pseudocode for Steps 3 and 4 is given below.

```
/* Step 3                                                                  */
for k ← 1 to 2 do
  | Ã_k ← π_k(Ã) ; // canonical projection of Ã on the k-th coordinate
  | Ã_k ← sort(Ã_k) ; // sorting of Ã_k in ascending order
  | q_k ← card Ã_k;
end
for i ← 1 to q_1 do
  | u_i ← Ã_1(i) ; // Ã_1(i) gives the i-th element of Ã_1
end
for j ← 1 to q_2 do
  | v_j ← Ã_2(j) ; // Ã_2(j) gives the j-th element of Ã_2
end
/* Step 4                                                                  */
T̃_1 ← T_2 ∩ ((u_1, v_1) + int K_1);
T̃_2 ← T_1 ∩ ((u_{q_1}, v_{q_2}) + int K_2);
T̃_3 ← T_4 ∩ ((u_{q_1}, v_1) + int K_3);
T̃_4 ← T_3 ∩ ((u_1, v_{q_2}) + int K_4).
```

### 5.3.4 Analysis and implementation of Step 5

Our next result shows that the sets $C_1, \ldots, C_{q_1}$, generated at Step 5, can be used to determine those efficient solutions of the reduced location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}}))$, which are also grid points (i.e., belong to $\widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2$). To this aim, we introduce the so-called *construction set*

$$C := \bigcup_{(i,j)\in I_{q_1}\times I_{q_2}} \{(u_i, v_j) \mid v_j \in C_i\} = \bigcup_{i\in I_{q_1}} (\{u_i\} \times C_i). \tag{5.44}$$

**Theorem 5.17** ([3]) For every $i \in I_{q_1}$ we have

$$C_i = \bigcup_{j\in I_{q_2}} \{v_j \mid (u_i, v_j) \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})\}. \tag{5.45}$$

Consequently, the construction set contains those Pareto efficient solutions of the reduced location problem, which are also grid points, i.e., we have

$$C = (\widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2) \cap \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}). \tag{5.46}$$

*Proof.* For every $i \in I_{q_1}$, the set $C_i$ has been defined within Step 5 as

$$C_i := \bigcup_{j\in I_{q_2}} \{v_j \mid x := (u_i, v_j) \text{ satisfies (5.41) and (5.42)}\}. \tag{5.47}$$

Since $\{(u_i, v_j) \mid (i,j) \in I_{q_1} \times I_{q_2}\} = \widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2 \subseteq \mathcal{N}(\widetilde{\mathcal{A}})$, relation (5.45) follows from (5.47) in view of Theorem 5.16. Consequently, (5.46) holds. □

**Remark 5.18** By applying Corollary 5.7 (1°) for $\widetilde{\mathcal{A}}$ in the role of $\mathcal{A}$ we infer that $\widetilde{\mathcal{A}} \subseteq \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$. Therefore, (5.28) and (5.46) show that $\widetilde{\mathcal{A}} \subseteq C$. Consequently, for every $i \in I_{q_1}$ the set $C_i$ is nonempty, hence $\underline{c}_i := \min C_i$ and $\overline{c}_i := \max C_i$ are well defined at Step 6.

**Example 5.19** Consider the set $\mathcal{A} = \{a^1, \ldots, a^8\}$ defined in Example 5.10. We have already computed the sets $\mathrm{MIN}(\mathcal{A}, K_r)$ for all $r \in I_4$, so by applying the pseudocode given in Subsection

5.3.2 we recover

$$T_1 = \{a^5, a^7\},\ T_2 = \{a^1, a^3\},\ T_3 = \{a^3, a^4, a^5\} \text{ and } T_4 = \{a^1, a^2, a^6, a^7\}.$$

At Steps 2 and 3 we obtain $\widetilde{\mathcal{A}} = \mathcal{A}\backslash\{a^8\} = \{\widetilde{a}^i \mid i \in I_7\}$ with $\widetilde{a}^i := a^i$ for all $i \in I_7$, then $\widetilde{\mathcal{A}}_1 = \{u_i \mid i \in I_7\}$ and $\widetilde{\mathcal{A}}_2 = \{v_j \mid j \in I_7\}$, where

$$\begin{array}{lllllll} u_1 = 2, & u_2 = 4, & u_3 = 6, & u_4 = 10, & u_5 = 12, & u_6 = 14, & u_7 = 18, \\ v_1 = 2, & v_2 = 4, & v_3 = 6, & v_4 = 8, & v_5 = 9, & v_6 = 10, & v_7 = 12. \end{array}$$

Here we have $e^1 = (2,2)$, $e^2 = (18,12)$, $e^3 = (18,2)$ and $e^4 = (2,12)$. Thus, at Step 4 we find $\widetilde{T}_1 = T_2$, $\widetilde{T}_2 = T_1$, $\widetilde{T}_3 = T_4$ and $\widetilde{T}_4 = T_3$. By means of these sets, at Step 5 we obtain $C_1 = \{v_3\}$, $C_2 = \{v_3, v_4\}$, $C_3 = \{v_1, v_2, v_3, v_4\}$, $C_4 = \{v_2, v_3, v_4, v_5\}$, $C_5 = \{v_2, v_3, v_4, v_5, v_6\}$, $C_6 = \{v_5, v_6, v_7\}$ and $C_7 = \{v_6\}$. By (5.46) we deduce that the set of those Pareto efficient solutions of the reduced location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\widetilde{\mathcal{A}}))$, which are also grid points, is $C = \bigcup_{i \in I_7}(\{u_i\} \times C_i)$ (see Figure 5.3, where $\widetilde{\mathcal{A}} \subseteq C$ in view of Remark 5.18).

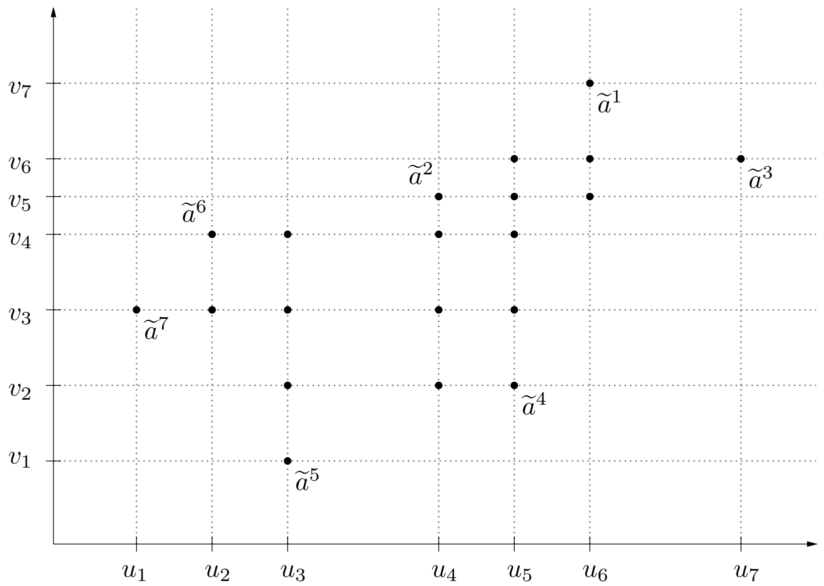


Figure 5.3: The construction set for the set $\mathcal{A}$ in Example 5.19.

Next, we present the pseudocode of Step 5.

```
/* Step 5                                                        */
for i ← 1 to q_1 do
 | C_i ← ∅;
end
for i ← 1 to q_1 do
  for j ← 1 to q_2 do
    if (u_i, v_j) ∈ (T̃_1 − K_1) ∩ (T̃_2 − K_2) ∨ (u_i, v_j) ∉ (T̃_1 − int K_1) ∪ (T̃_2 − int K_2) then
      if (u_i, v_j) ∈ (T̃_3 − K_3) ∩ (T̃_4 − K_4) ∨ (u_i, v_j) ∉ (T̃_3 − int K_3) ∪ (T̃_4 − int K_4) then
        C_i ← C_i ∪ {v_j};
      end
    end
  end
end
```

### 5.3.5 Analysis and implementation of Step 6

In order to explain the procedure used in Step 6 for generating the whole set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$, we need four preliminary results.

**Lemma 5.20** ([3]) Let $R$ be an axis-parallel (possibly degenerated) rectangle. If all vertices (extreme points) of $R$ belong to $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$, then $R \subseteq \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$.

*Proof.* Without loss of generality, we can assume that $R$ is not a singleton. When $R$ is degenerated into a (horizontal or vertical) line segment, the conclusion follows from a known result by Wendell *et al.* [128, Corollary 1]. Consequently, when $R$ is a non-degenerated rectangle, then its horizontal edges are subsets of $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$. Moreover, since any point $x$ of $R$ belongs to a vertical line segment whose extreme points are located on the horizontal edges of $R$, it follows that $x \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$. □

**Lemma 5.21** ([3]) Assume that $q_1, q_2 > 1$. If $(u_i, v_j), (u_{i+1}, v_k) \in C$ for some $i \in I_{q_1-1}$, $j \in I_{q_2}$ and $k \in I_{q_2}\backslash\{j\}$ (i.e., $v_k \neq v_j$), then we have $(u_{i+1}, v_j) \in C$ or $(u_i, v_l) \in C$ for some $l \in I_{q_2}\backslash\{j\}$ such that $\min\{j,k\} \leq l \leq \max\{j,k\}$ (i.e., $v_j < v_l \leq v_k$ or $v_k \leq v_l < v_j$, depending on the values of the given indices, $j < k$ or $k < j$, respectively).

*Proof.* Follows by a known result by Wendell *et al.* [128, Corollary 3]. □

If $q_1 > 1$, then we consider the following elements for all $i \in I_{q_1-1}$:

$$c'_i := \max\{\underline{c}_i, \underline{c}_{i+1}\} \text{ and } c''_i := \min\{\overline{c}_i, \overline{c}_{i+1}\}.$$

**Lemma 5.22** ([3]) Assume that $q_1 > 1$. For every $i \in I_{q_1-1}$, we have

$$(u_i, c'_i), (u_i, c''_i), (u_{i+1}, c''_i), (u_{i+1}, c'_i) \in C. \tag{5.48}$$

*Proof.* We first note that (5.44) and (5.46) yield

$$(u_i, \underline{c}_i), (u_i, \overline{c}_i) \in \{u_i\} \times C_i \subseteq C \subseteq \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}); \tag{5.49}$$

$$(u_{i+1}, \underline{c}_{i+1}), (u_{i+1}, \overline{c}_{i+1}) \in \{u_{i+1}\} \times C_{i+1} \subseteq C \subseteq \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}). \tag{5.50}$$

Without any loss of generality, we suppose in what follows that

$$\max\{\underline{c}_i, \underline{c}_{i+1}\} = \underline{c}_i, \text{ i.e., } c'_i = \underline{c}_i \geq \underline{c}_{i+1};$$
$$\min\{\overline{c}_i, \overline{c}_{i+1}\} = \overline{c}_i, \text{ i.e., } c''_i = \overline{c}_i \leq \overline{c}_{i+1}.$$

Under these assumptions, we can easily deduce by (5.49) that

$$(u_i, c'_i) = (u_i, \underline{c}_i) \in C \text{ and } (u_i, c''_i) = (u_i, \overline{c}_i) \in C. \tag{5.51}$$

In order to show that $(u_{i+1}, c''_i) \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$, we distinguish two situations. If $\overline{c}_i = \overline{c}_{i+1}$, then we have $(u_{i+1}, c''_i) = (u_{i+1}, \overline{c}_{i+1}) \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$ by (5.50). Otherwise, if $\overline{c}_i < \overline{c}_{i+1}$, then $q_2 > 1$ and there exist $j, k \in I_{q_2}$, $j < k$, such that $\overline{c}_i = v_j$ and $\overline{c}_{i+1} = v_k$. Thus we have $(u_i, v_j) = (u_i, \overline{c}_i) \in C$ by (5.49) and $(u_{i+1}, v_k) = (u_{i+1}, \overline{c}_{i+1}) \in C$ by (5.50). Moreover, since $\overline{c}_i = \max C_i$, there is no $l \in I_{q_2}$ such that $v_l > \overline{c}_i = v_j$ and $(u_i, v_l) \in C$. By Lemma 5.21 we infer

$$(u_{i+1}, c''_i) = (u_{i+1}, \overline{c}_i) = (u_{i+1}, v_j) \in C. \tag{5.52}$$

Similarly, for proving that $(u_{i+1}, c'_i) \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$, we should analyze two possible situations. If $\underline{c}_i = \underline{c}_{i+1}$, then $(u_{i+1}, c'_i) = (u_{i+1}, \underline{c}_{i+1}) \in \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$ by (5.50). If $\underline{c}_i > \underline{c}_{i+1}$, then $\underline{c}_i = v_j$ and $\underline{c}_{i+1} = v_k$ for some $j, k \in I_{q_2}$ such that $j > k$. By (5.49) and (5.50) it follows that $(u_i, v_j) = (u_i, \underline{c}_i) \in C$ and $(u_{i+1}, v_k) = (u_{i+1}, \underline{c}_{i+1}) \in C$. On the other hand, since $\underline{c}_i = \min C_i$, there is no $l \in I_{q_2}$ such that $v_l < \underline{c}_i = v_j$ and $(u_i, v_l) \in C$. Lemma 5.21 yields

$$(u_{i+1}, c'_i) = (u_{i+1}, \underline{c}_i) = (u_{i+1}, v_j) \in C. \tag{5.53}$$

By (5.51), (5.52) and (5.53) we infer (5.48).

□

**Lemma 5.23** ([3]) Assume that $q_1 > 1$. For every $i \in I_{q_1-1}$, we have

$$\max\{\underline{c}_i, \underline{c}_{i+1}\} = c_i' \leq c_i'' = \min\{\overline{c}_i, \overline{c}_{i+1}\}.$$

*Proof.* For any $i \in I_{q_1-1}$ it holds $(u_i, c_i'), (u_{i+1}, c_i') \in C$ by Lemma 5.22, hence $c_i' \leq \overline{c}_i$ and $c_i' \leq \overline{c}_{i+1}$. Thus we have $c_i' \leq \min\{\overline{c}_i, \overline{c}_{i+1}\} = c_i''$.

□

**Remark 5.24** According to Lemma 5.23, if $q_1 > 1$, then at Step 6 we have

$$\mathcal{R}_2^* = \bigcup_{i \in I_{q_1-1}} ([u_i, u_{i+1}] \times [c_i', c_i'']).$$

In preparation for the next results, we introduce the set

$$\mathcal{R}_1^* := \bigcup\nolimits_{i \in I_{q_1}} (\{u_i\} \times [\underline{c}_i, \overline{c}_i]). \tag{5.54}$$

We are now ready to show that the whole set of Pareto efficient solutions of problem $(\text{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ can be recovered by means of the sets $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ generated at Step 6 of the algorithm.

**Theorem 5.25** ([3]) The following representation holds:

$$\text{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*.$$

*Proof.* In view of Corollary 5.13, it suffices to prove that

$$\text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*. \tag{5.55}$$

For proving inclusion "$\supseteq$" in (5.55), let $x^* \in \mathcal{R}_1^* \cup \mathcal{R}_2^*$ be arbitrarily chosen.
*Case 1*: If $x^* \in \mathcal{R}_1^*$, then there exists $i \in I_{q_1}$ such that

$$x^* \in \{u_i\} \times [\underline{c}_i, \overline{c}_i]. \tag{5.56}$$

Observe that $(u_i, \underline{c}_i), (u_i, \overline{c}_i) \in \{u_i\} \times C_i \subseteq C \subseteq \text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$, according to (5.44) and (5.46). Therefore, by Lemma 5.20 we have $\{u_i\} \times [\underline{c}_i, \overline{c}_i] \subseteq \text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$, hence $x^* \in \text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$ by (5.56).
*Case 2*: If $x^* \in \mathcal{R}_2^*$, then $q_1 > 1$ and we can find $i \in I_{q_1-1}$ such that

$$x^* \in [u_i, u_{i+1}] \times [c_i', c_i''] \tag{5.57}$$

in view of Remark 5.24. Due to Lemma 5.22 and (5.46), we know that

$$(u_i, c_i'), (u_i, c_i''), (u_{i+1}, c_i''), (u_{i+1}, c_i') \in \text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}). \tag{5.58}$$

Hence, by (5.57), (5.58) and Lemma 5.20 we deduce that $x^* \in \text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$.
Consequently inclusion "$\supseteq$" in (5.55) holds.
In order to prove the reverse inclusion "$\subseteq$" in (5.55), we observe first that, according to Corollary 5.7 (2°) applied for $\widetilde{\mathcal{A}}$ in the role of $\mathcal{A}$, we have $\text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}) = \bigcup_{\lambda \in \text{int}\, \mathbb{R}^q_+} \text{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\widetilde{\mathcal{A}}}(\cdot) \rangle)$. Therefore, it suffices to prove that for any $\lambda \in \text{int}\, \mathbb{R}^q_+$ the following inclusion holds:

$$\text{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\widetilde{\mathcal{A}}}(\cdot) \rangle) \subseteq \mathcal{R}_1^* \cup \mathcal{R}_2^*. \tag{5.59}$$

To this aim we distinguish the following four cases, according to Lemma 5.2 (1° – 4°) applied for $\widetilde{\mathcal{A}}$ in the role of $\mathcal{A}$:
*Case 1*: If $\text{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\widetilde{\mathcal{A}}}(\cdot) \rangle)$ is a singleton, then there is $(i, j) \in I_{q_1} \times I_{q_2}$ with $\text{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\widetilde{\mathcal{A}}}(\cdot) \rangle) = \{(u_i, v_j)\} \subseteq \widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2$. Since we have also $\text{Sol}(\mathbb{R}^2 \mid \langle \lambda, g_{\widetilde{\mathcal{A}}}(\cdot) \rangle) \subseteq \text{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}})$, we infer

by (5.46) and (5.54) that $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle) \subseteq C \subseteq \mathcal{R}_1^*$.

*Case 2*: If $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle)$ is a horizontal line segment, then $q_1 > 1$ and there is $(i,j) \in I_{q_1-1} \times I_{q_2}$ such that $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle) = [u_i, u_{i+1}] \times \{v_j\}$. Since $c_i' \leq v_j \leq c_i''$, we conclude that $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle) \subseteq [u_i, u_{i+1}] \times [c_i', c_i''] \subseteq \mathcal{R}_2^*$.

*Case 3*: If $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle)$ is a vertical line segment, then $q_2 > 1$ and there is $(i,j) \in I_{q_1} \times I_{q_2-1}$ with $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle) = \{u_i\} \times [v_j, v_{j+1}] \subseteq \{u_i\} \times [\underline{c}_i, \overline{c}_i] \subseteq \mathcal{R}_1^*$.

*Case 4*: If $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle)$ is a non-degenerated rectangle, then $q_1, q_2 > 1$ and there is $(i,j) \in I_{q_1-1} \times I_{q_2-1}$ such that $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle) = [u_i, u_{i+1}] \times [v_j, v_{j+1}]$. Since $c_i' \leq v_j < v_{j+1} \leq c_i''$, we infer that $\mathrm{Sol}(\mathbb{R}^2 \mid \langle\lambda, g_{\widetilde{\mathcal{A}}}(\cdot)\rangle) \subseteq [u_i, u_{i+1}] \times [c_i', c_i''] \subseteq \mathcal{R}_2^*$.

Thus (5.59) holds in all cases. □

The pseudocode of Step 6 is given below.

```
/* Step 6                                                    */
for i ← 1 to q_1 do
    c̲_i ← min C_i;
    c̄_i ← max C_i;
end
if q_1 = 1 then
    R*_1 ← {u_1} × [c̲_1, c̄_1];
    R*_2 ← ∅;
end
if q_1 ≥ 2 then
    for i ← 1 to q_1 − 1 do
        c'_i ← max{c̲_i, c̲_{i+1}};
        c''_i ← min{c̄_i, c̄_{i+1}};
    end
    R*_1 ← ∅;
    R*_2 ← ∅;
    for i ← 1 to q_1 do
        R*_1 ← {u_i} × [c̲_i, c̄_i];
    end
    for i ← 1 to q_1 − 1 do
        R*_2 ← R*_2 ∪ conv{(u_i, c'_i), (u_i, c''_i), (u_{i+1}, c''_i), (u_{i+1}, c'_i)};
    end
end
```

**Example 5.26** Consider once again the set $\mathcal{A} = \{a^1, \dots, a^8\}$ introduced in Example 5.10. By means of the sets $C_1, \dots, C_7$ constructed within Example 5.19, we apply the procedure described at Step 6 to generate the sets

$$
\begin{aligned}
\mathcal{R}_1^* =& \{(u_1, v_3)\} \cup \big(\{u_2\} \times [v_3, v_4]\big) \cup \big(\{u_3\} \times [v_1, v_4]\big) \cup \big(\{u_4\} \times [v_2, v_5]\big) \\
& \cup \big(\{u_5\} \times [v_2, v_6]\big) \cup \big(\{u_6\} \times [v_5, v_7]\big) \cup \{(u_7, v_6)\}; \\
\mathcal{R}_2^* =& \big([u_1, u_2] \times \{v_3\}\big) \cup \big([u_2, u_3] \times [v_3, v_4]\big) \cup \big([u_3, u_4] \times [v_2, v_4]\big) \\
& \cup \big([u_4, u_5] \times [v_2, v_5]\big) \cup \big([u_5, u_6] \times [v_5, v_6]\big) \cup \big([u_6, u_7] \times \{v_6\}\big).
\end{aligned}
$$

The sets $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ can be visualized in Figures 5.4 and 5.5, respectively. Finally, we obtain the representation of the set of all Pareto efficient solutions

$$\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\widetilde{\mathcal{A}}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*$$

as the union of eight rectangles, among which four are degenerated into line segments (see Figure 5.6).

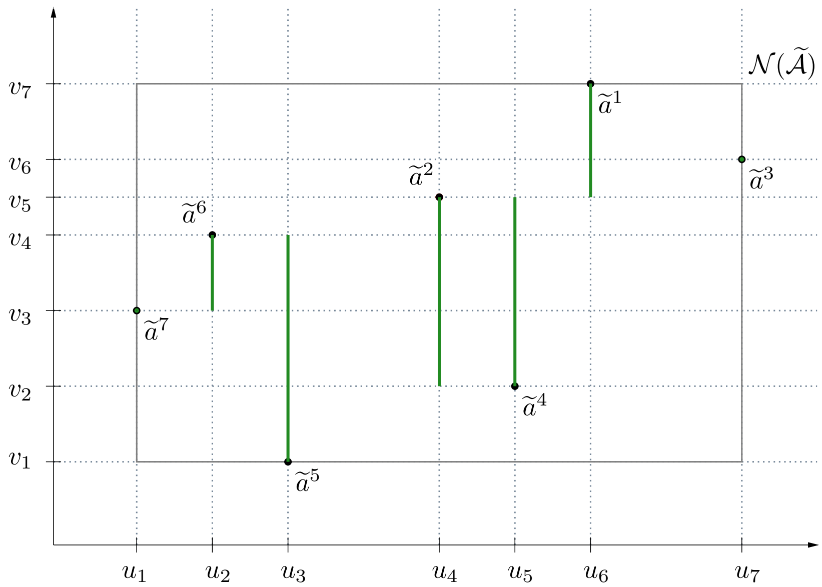


Figure 5.4: The set $\mathcal{R}_1^*$ of Example 5.26.

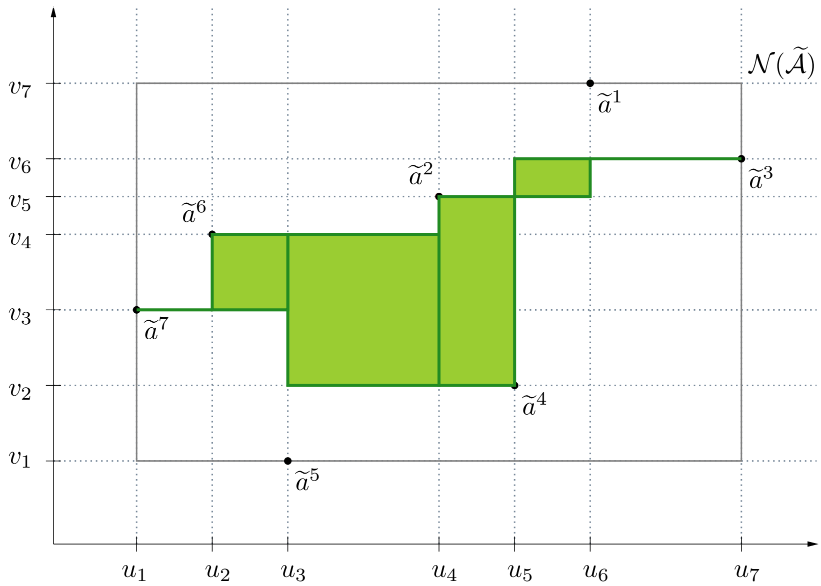


Figure 5.5: The set $\mathcal{R}_2^*$ of Example 5.26.

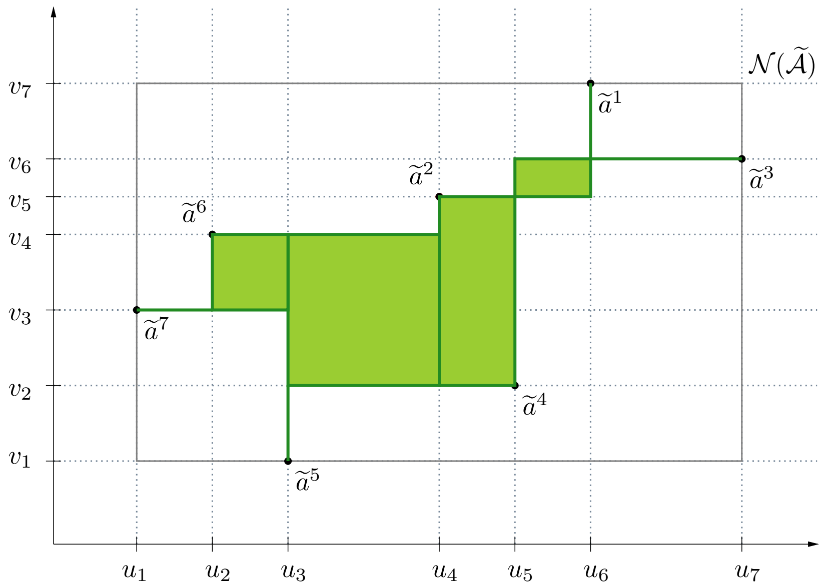


Figure 5.6: Output of the algorithm for the set $\mathcal{A}$ considered in Example 5.26.

## 5.4 Computational analysis of the algorithm

### 5.4.1 Complexity analysis

Based on the pseudocodes given in Subsections 5.3.2 – 5.3.5 one can see that the *Rectangular Decomposition Algorithm* constructs the set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ with a worst case complexity of

$$\mathcal{O}(\underbrace{m^2}_{\text{Step 1}} + \underbrace{q \cdot \log q}_{\text{Steps 2-4}} + \underbrace{q_1 \cdot q_2 \cdot q}_{\text{Step 5}} + \underbrace{q_1 \cdot q_2}_{\text{Step 6}})$$

where, according to (5.22) and (5.23), we have

$$m = \operatorname{card} \mathcal{A} \geq q = \operatorname{card} \widetilde{\mathcal{A}} \geq q_k = \operatorname{card} \widetilde{\mathcal{A}}_k \quad \text{for all } k \in I_2.$$

Notice that, among other algorithms for solving location problems, the classical *Row Algorithm* proposed by Chalmet, Francis and Kolen computes the set of Pareto efficient solutions with complexity $\mathcal{O}(m \cdot \log m)$ and indeed there is no algorithm with a smaller complexity (see [21, Theorem 1]). However, our algorithm is more appropriate for decomposing the set of Pareto efficient solutions in a small number of rectangles and line segments, as we will see in the next subsection. On the other hand, the early algorithm proposed by Wendell *et al.* [128] computes the *periphery* of the set of Pareto efficient solutions with complexity $\mathcal{O}(m^2)$, but does not contain an explicit procedure for decomposing the set of Pareto efficient solutions in rectangles and line segments.

### 5.4.2 Numerical tests

The *Rectangular Decomposition Algorithm* is conceived to solve general multi-objective location problems of type $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ in absence of any (a priori given) information about the existence/number of nonessential objectives. Therefore we have implemented two variants of this algorithm, namely:

**Variant I**, obtained from the implicit form of the algorithm (formulated in Subsection 5.3.1), and

**Variant II**, obtained from the Variant I by bypassing the procedure described at Steps 1 and 2 (identification of all essential objectives and elimination of the nonessential ones) and letting $\widetilde{\mathcal{A}} := \mathcal{A}$ in Step 3.

Both variants have been implemented in MATLAB and tested on various location problems of type (POLP$^1_{\mathbb{R}^2}(\mathcal{A})$). Our computational experiments show that Variant I is highly effective when the number of nonessential objectives, i.e., $m - q$, is large. In what follows, we present some numerical results obtained by solving several location problems with $m \in \{2^4, 2^6, 2^8, 2^{10}, 2^{12}, 2^{14}\}$ on a Core i5 4570 4x 3.20GHz CPU computer. For every $m$ we have considered 15 test problems of type (POLP$^1_{\mathbb{R}^2}(\mathcal{A})$) where the coordinates of the location points of $\mathcal{A}$ were generated as uniformly distributed random numbers in the interval $[0, 1]$ (using the MATLAB function "rand"). The averaged running times (in seconds) for both variants I and II are listed in Table 5.1. Moreover, by Variant I we have also determined the number of essential objectives for each of these test problems, i.e., the cardinality of the reduced set $\widetilde{\mathcal{A}}$ generated at Step 2. The averaged values of $q$ are also listed in Table 5.1.

| $m$ | 16 | 64 | 256 | 1024 | 4096 | 16384 |
|---|---|---|---|---|---|---|
| $q$ | 8 | 14 | 22 | 26 | 33 | 37 |
| Variant I | 0.005 | 0.018 | 0.049 | 0.098 | 0.226 | 0.505 |
| Variant II | 0.017 | 0.292 | 5.554 | 99.951 | 1415.953 | 35379.250 |

Table 5.1: Running times in seconds.

Figures 5.7 and 5.8 contain screen captures obtained in MATLAB for a test problem (POLP$^1_{\mathbb{R}^2}(\mathcal{A})$) with $m = 64$, illustrating the rectangular decompositions of Eff($\mathbb{R}^2 \mid g_{\mathcal{A}}$) obtained by Variant I (14 rectangles and 2 segments) and by Variant II (40 rectangles and 2 segments). In contrast to our algorithm, the number of "boxes" involved in the approach proposed by Chalmet, Francis and Kolen [21] (obtained by splitting the 40 rectangles in Figure 5.8 by horizontal lines through the location points) is very large, namely 2858.

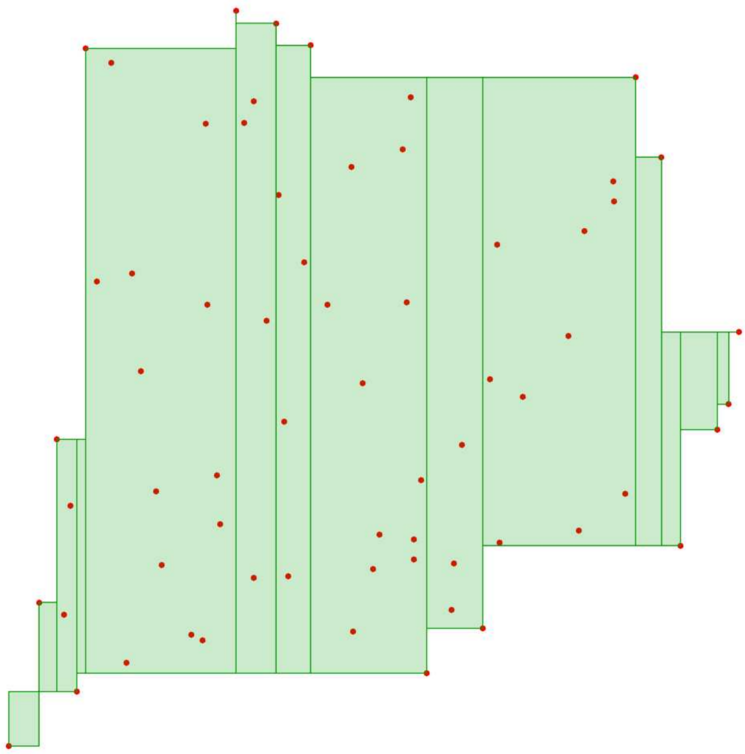

Figure 5.7: Rectangular decomposition of Eff($\mathbb{R}^2 \mid g_{\mathcal{A}}$) by Variant I.

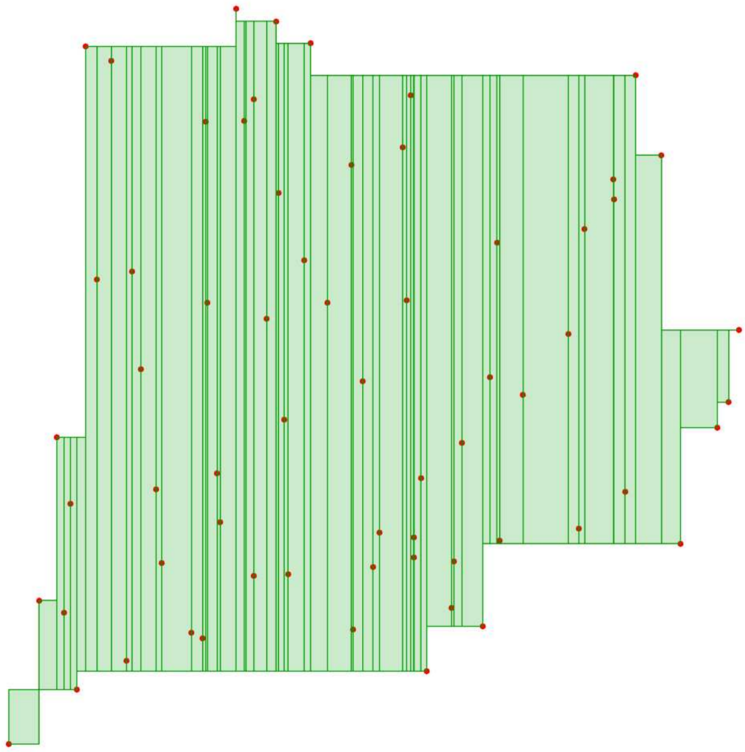

Figure 5.8: Rectangular decomposition of $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ by Variant II.

## 5.5 Extension to constrained problems

Let us consider a constrained version of our initial point-objective location problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$:

$$\begin{cases} g_{\mathcal{A}}(x) = \big(\|x-a^1\|_1, \cdots, \|x-a^m\|_1\big) \to \min \\ x \in X, \end{cases} \qquad (\mathrm{POLP}^1_X(\mathcal{A}))$$

where $X$ is a nonempty closed, convex set in $\mathbb{R}^2$. We are going to emphasize how to generate the set of Pareto efficient solutions of problem $(\mathrm{POLP}^1_X(\mathcal{A}))$ by using our penalization approach derived in Chapter 2 and by applying the *Rectangular Decomposition Algorithm* proposed in the preceding sections. Notice that the *projection property* fails for the set of Pareto efficient solutions, i.e., the projection of $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ onto $X$ does not coincide with $\mathrm{Eff}(X \mid g_{\mathcal{A}})$ in general, since the Manhattan norm $||\cdot||_1$ is not strictly convex (see Remark 4.7). However, the *projection property* holds for the set of weakly Pareto efficient solutions, as stated in the next theorem.

**Theorem 5.27** We have

$$\mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{Proj}_X^{||\cdot||_1}(\mathcal{N}(\mathcal{A})). \tag{5.60}$$

*Proof.* Since in the space $(\mathbb{R}^2, ||\cdot||_1)$ the *projection property* concerning the set of weakly Pareto efficient solutions holds (see Remark 4.7), we get (5.60) by the fact that $\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathcal{N}(\mathcal{A})$ in view of Theorem 5.5. □

Now, according to our penalization approach in Chapter 2, let us introduce the penalized problem

$$\begin{cases} g^{\oplus}_{\mathcal{A}}(x) = \big(\|x-a^1\|_1, \cdots, \|x-a^m\|_1, \phi(x)\big) \to \min \\ x \in \mathbb{R}^2, \end{cases} \qquad (\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})^{\oplus})$$

where $\phi : \mathbb{R}^2 \to \mathbb{R}$ represents the penalization function.

**Example 5.28** We consider the problem $(\mathrm{POLP}^1_X(\mathcal{A}))$ with three given points $a^1, a^2, a^3 \in \mathbb{R}^2$ (i.e., $m = 3$) and a feasible set $X$ represented by a closed Manhattan ball centered at $x' \in \mathbb{R}^2$ with positive radius, as shown in Figure 5.9. Due to the structure of the set $X$, it is possible to compute the sets $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ and $\mathrm{Eff}(\mathbb{R}^2 \mid g^{\oplus}_{\mathcal{A}})$ by using the *Rectangular Decomposition Algorithm*, where $g^{\oplus}_{\mathcal{A}}$ is defined as in problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})^{\oplus})$ with $\phi(\cdot) := ||\cdot - x'||_1$. Figure 5.9 shows the procedure for computing the set

$$\mathrm{Eff}(X \mid g_{\mathcal{A}}) = \left[X \cap \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})\right] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(\mathbb{R}^2 \mid g^{\oplus}_{\mathcal{A}})\right].$$

In the left part of Figure 5.10, the sets of (strictly, weakly) Pareto efficient solutions for the unconstrained case ($X = \mathbf{E} = \mathbb{R}^2$) are shown, while in the right part, the sets of (strictly, weakly) Pareto efficient solutions for the constrained case are illustrated.

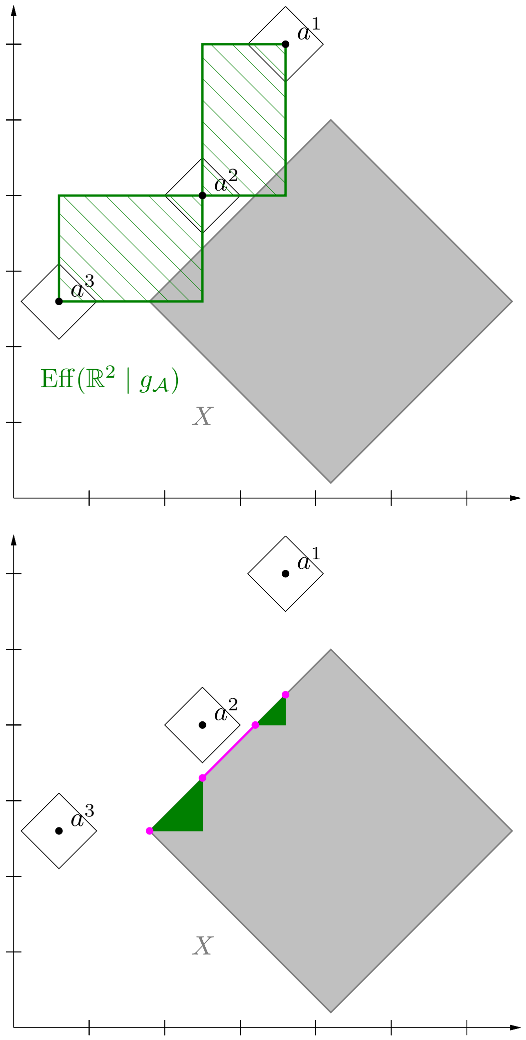


Figure 5.9: Construction of the set of Pareto efficient solutions of the problem $(\mathrm{POLP}^1_X(\mathcal{A}))$, where $X$ is given by a closed Manhattan ball in $\mathbf{E} = \mathbb{R}^2$ with center point $x' \in \mathbb{R}^2$.

Due to Theorem 5.5, Theorem 5.60 and Proposition 4.5, we have

$$\begin{aligned}\mathrm{WEff}(X \mid g_{\mathcal{A}}) &= \left[X \cap \mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})\right] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{WEff}(\mathbb{R}^2 \mid g^{\oplus}_{\mathcal{A}})\right] \\ &= \mathrm{Proj}^{||\cdot||_1}_X(\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})) \\ &= \mathrm{Proj}^{||\cdot||_1}_X(\mathcal{N}(\mathcal{A})).\end{aligned}$$

According to Ndiaye and Michelot [88], the *projection property* does not hold for the sets of (strictly) Pareto efficient solutions of problem $(\mathrm{POLP}^1_X(\mathcal{A}))$ involving the Manhattan norm $||\cdot||_1$ (that is not strictly convex) in general. However, due to our vectorial penalization approach derived in Chapter 2, we can construct these sets of (strictly) Pareto efficient solutions. Figure 5.10 shows the sets of (strictly, weakly) Pareto effient solutions for the unconstrained case (left part) as well as for the constrained case (right part). Notice, in view of Durier and Michelot [27, Rem. 3.1], $\mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ coincides with the rectangular hull (in the sense of Love and Morris [78]) of $\mathcal{A} \subseteq \mathbb{R}^2$

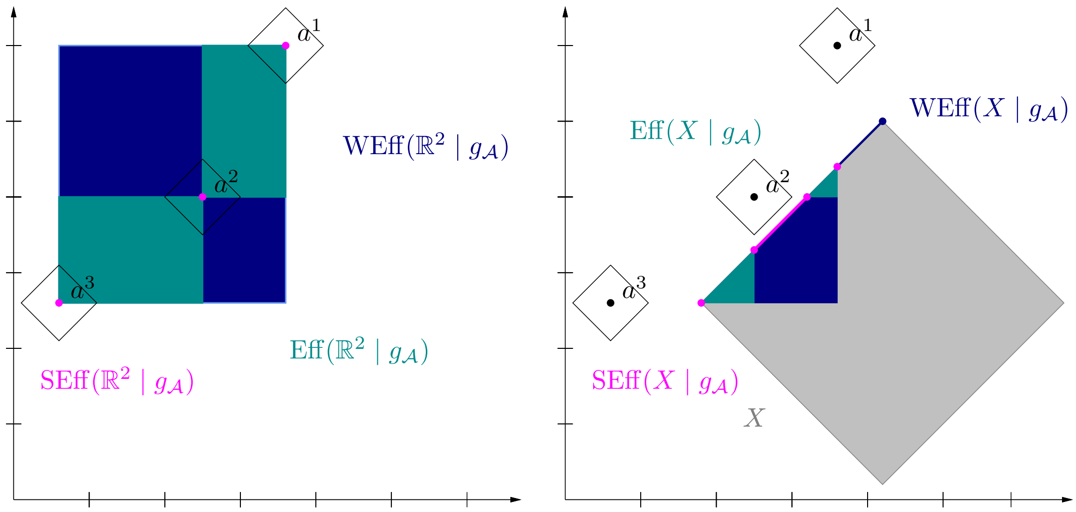


Figure 5.10: The figure shows the sets of (strictly, weakly) Pareto efficient solutions of the problems $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ and $(\mathrm{POLP}^1_X(\mathcal{A}))$. In the left part of the figure, the unconstrained case (i.e, $X = \mathbb{R}^2$) is illustrated, while in the right part, the feasible set $X \subsetneq \mathbb{R}^2$ is given by a certain closed Manhattan ball (grey colored region).

The reader should pay attention to the fact that

$$\begin{aligned}
\mathrm{SEff}(X \mid g_{\mathcal{A}}) &\subsetneq \mathrm{Proj}_X^{||\cdot||_1}(\mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}));\\
\mathrm{Eff}(X \mid g_{\mathcal{A}}) &\subsetneq \mathrm{Proj}_X^{||\cdot||_1}(\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}));\\
\mathrm{WEff}(X \mid g_{\mathcal{A}}) &= \mathrm{Proj}_X^{||\cdot||_1}(\mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}})).
\end{aligned}$$

In the concluding remarks given by Ndiaye and Michelot [88], the question about the validity of the above first two (not necessarily strict) inclusions for point-objective location problems involving a polyhedral norm arises. Since

$$\begin{aligned}
X \cap \mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) &\subseteq \mathrm{Proj}_X^{||\cdot||_1}(\mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}));\\
X \cap \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) &\subseteq \mathrm{Proj}_X^{||\cdot||_1}(\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})),
\end{aligned}$$

it remains to check whether the inclusions

$$\begin{aligned}
(\mathrm{bd}\, X) \cap \mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus}) &\subseteq \mathrm{Proj}_X^{||\cdot||_1}(\mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}));\\
(\mathrm{bd}\, X) \cap \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus}) &\subseteq \mathrm{Proj}_X^{||\cdot||_1}(\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}))
\end{aligned}$$

holds true in general. In this thesis, we do not give an answer to this question, however it is worth enough to investigate it, since these projections help to localize (strictly) Pareto efficient solutions.

## 5.6 Concluding remarks

The *Rectangular Decomposition Algorithm* is a new effective numerical method for solving multi-objective location problems of type $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$, where $\mathcal{A}$ is a finite set of points in $\mathbb{R}^2$ representing a priori given facilities, while the distance-type objective functions are defined by means of the Manhattan norm. Its mathematical background mainly relies on a dual characterization of Pareto efficient solutions of $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ given by Gerth and Pöhler [47], several structural properties of the set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ established by Wendell, Hurter and Lowe [128], and a new characterization of this set in terms of the minimal elements of $\mathcal{A}$ with respect to four ordering cones of $\mathbb{R}^2$. The latter characterization leads to an implementable procedure, embedded in the first phase of the *Rectangular Decomposition Algorithm*, based on the *Jahn-Graef-Younes Method* for solving discrete vector optimization problems.

In comparison with other known algorithms for solving problems of type $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$, our algorithm has several special features. First of all it identifies all nonessential objectives, hence the running time needed to generate $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ can be reduced drastically, by eliminating these objectives. Actually, our reduction procedure can be used as a pre-phase for other algorithms known in the literature. On the other hand, in contrast to certain algorithms which produce one Pareto efficient solution or a part of the set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ corresponding to a particular choice of weights or other scalarization parameters, our algorithm provides a well structured representation of the whole set $\mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}})$ as a finite union of line segments and rectangles. This representation can be used as input for further applications, as for instance to minimize/maximize an additional cost function over the set of Pareto efficient solutions (see Alzorba, Günther and Popovici [2]).

The construction procedure described at Step 5 of our algorithm is based on the characterization of efficient grid points. However, it can be easily adapted to other known algorithms that compute the periphery/grid efficient points (e.g, the algorithms by Chalmet, Francis and Kolen [21], Pelegrin and Fernández [97], or Wendell, Hurter and Lowe [128]).

In order to develop effective algorithms for computing the sets of (strictly, weakly) Pareto efficient solutions to problem $(\mathrm{POLP}^1_{X}(\mathcal{A}))$ involving a nonempty, closed, convex set $X \subsetneq \mathbb{R}^2$, it is important to analyze the penalized problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})^{\oplus})$ in detail. In the case that $X$ is given by a polytope with nonempty interior (i.e, the penalization function $\phi$ can be chosen as a polyhedral Minkowski gauge), one could use ideas that are known for point-objective location problems involving mixed polyhedral Minkowski gauges (see, e.g., Durier [26] and Kaiser [69]) and corresponding generalizations of these problems (see, e.g., Nickel *et al.* [90]). So, we are able to compute the sets of (strictly, weakly) Pareto efficient of the unconstrained problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A})^{\oplus})$, hence we can also generate the sets of (strictly, weakly) Pareto efficient solutions of the initial constrained problem $(\mathrm{POLP}^1_{X}(\mathcal{A}))$ taking into account results that are known for the unconstrained problem $(\mathrm{POLP}^1_{\mathbb{R}^2}(\mathcal{A}))$ (see the preceding sections of this chapter) as well as our new penalization approach (see Chapter 2).

# Chapter 6

## Point-objective location problems in a finite-dimensional Hilbert space

Throughout this chapter, we assume that $(\mathbf{E}, ||\cdot||)$ is a real finite-dimensional Hilbert space. Consider $m$ a priori given facilities located at the points $a^1, \cdots, a^m \in \mathbf{E}$. Again, for notational convenience, we consider the set of all existing facilities by

$$\mathcal{A} = \{a^1, \cdots, a^m\}.$$

Let $X \subseteq \mathbf{E}$ be a nonempty, closed set. Our aim is to find a point $x \in X$ for a new facility such that the distances (associated with the norm $||\cdot||$) between $x$ and the given points $a^1, \cdots, a^m$ are to be simultaneously minimized. Such a problem can be modeled as follows:

$$\begin{cases} g_{\mathcal{A}}(x) := \big(||x - a^1||, \cdots, ||x - a^m||\big) \to \min \\ x \in X. \end{cases} \tag{$\mathrm{POLP}^2_X(\mathcal{A})$}$$

Now, let the feasible set $X$ of problem $(\mathrm{POLP}^2_X(\mathcal{A}))$ be given by the whole pre-image space $\mathbf{E}$ excepting some forbidden regions that are defined by open balls with respect to the norm $||\cdot||$. More precisely, throughout this chapter, we assume that the following assumptions are fulfilled:

$$\begin{cases} \text{Let } (\mathbf{E}, ||\cdot||) \text{ be a real finite-dimensional Hilbert space;} \\ \text{let } D_i := \overline{B}_{||\cdot||}(d^i, r_i) \text{ with } d^i \in \mathbf{E},\ r_i \in \mathbb{R}_{++},\ i \in I_l,\ l \in \mathbb{N}; \\ \text{let } X := \bigcap_{i \in I_l} X_i \text{ with } X_i := \mathbf{E} \setminus \operatorname{int} D_i,\ i \in I_l. \end{cases} \tag{6.1}$$

As one can see in (6.1), the feasible set $X$ is given by an intersection of reverse convex sets $X_1, \cdots, X_l$. For convenience, the reader may assume that $\mathbf{E} = \mathbb{R}^n$ and that $||\cdot||$ is given by the Euclidean norm.

Notice that the Hilbert space $(\mathbf{E}, ||\cdot||)$ is strictly convex. Hence, for any $i \in I_l$, we have

$$]x^1, x^2[ \subseteq \operatorname{int} D_i \quad \text{for all } x^1, x^2 \in \operatorname{bd} D_i, x^1 \neq x^2.$$

Moreover, we have

$$||d^i - d^j|| > r_i + r_j \quad \text{for all } i, j \in I_l, i \neq j, \tag{6.2}$$

if and only if the balls $D_1, \cdots, D_l$ are pairwise disjoint. Furthermore, we have

$$||d^i - d^j|| \geq r_i + r_j \quad \text{for all } i, j \in I_l, i \neq j, \tag{6.3}$$

if and only if the interiors $\operatorname{int} D_1, \cdots, \operatorname{int} D_l$ of the balls $D_1, \cdots, D_l$ are pairwise disjoint. Obviously, (6.3) follows by (6.2).

In general, we have $\operatorname{bd} X \subseteq \bigcup_{i \in I_l} \operatorname{bd} D_i$. Under the assumption (6.3), in view of Lemma 3.12, we actually have

$$\operatorname{bd} X = \bigcup_{i \in I_l} \operatorname{bd} D_i$$

For every $i \in I_l$, we consider a penalized point-objective location problem by

$$\begin{cases} g_{\mathcal{A}}^{\oplus_i}(x) := (g_{\mathcal{A}}, \phi_i) = \left(||x - a^1||, \cdots, ||x - a^m||, -||x - d^i||\right) \to \min \\ x \in \mathbf{E}, \end{cases} \qquad (\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A})^{\oplus_i})$$

where we define the penalization function $\phi_i : \mathbf{E} \to \mathbb{R}$ by

$$\phi_i(x) := -||x - d^i|| \quad \text{ for all } x \in \mathbf{E}.$$

Notice that $(\mathrm{POLP}^2_X(\mathcal{A}))$ involves a convex objective function $g_{\mathcal{A}}$ and a nonconvex feasible set $X$. In contrast to that, $(\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A})^{\oplus_i})$ involves a nonconvex objective function $g_{\mathcal{A}}^{\oplus_i}(x)$ and a convex feasible set $\mathbf{E}$ for every $i \in I_l$. According to Jourani, Michelot and Ndiaye [67], the problem $(\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A})^{\oplus_i})$ can be seen as the problem of locating a new facility $x \in \mathbf{E}$ in presence of attracting points $a^1, \cdots, a^m$ and a repulsive demand point $d^i$ in a continuous location space $\mathbf{E}$.

**Remark 6.1** By Example 3.15 and Lemma 3.16, we know that the function $\widehat{\phi}_i : \mathbf{E} \to \mathbb{R}$, defined for every $x \in \mathbf{E}$ by

$$\widehat{\phi}_i(x) := -\frac{1}{r_i}||x - d^i|| = -\inf\{\lambda \in \mathbb{R}_+ \mid x - d^i \in \lambda \cdot (-d^i + D_i)\},$$

fulfils Assumptions (A1), (A2) and (A6) (with $\widehat{\phi}_i$ in the role of $\phi$ and $X_i$ in the role of $X$) for every $i \in I_l$. In view of Lemma 2.10 (5°, 6°), we actually get that $\phi_i(\,\cdot\,) = -||\cdot - d^i||$ fulfils Assumptions (A1), (A2) and (A6) (with $\phi_i$ in the role of $\phi$ and $X_i$ in the role of $X$) for every $i \in I_l$.

## 6.1 Structure of the sets of (strictly, weakly) Pareto efficient solutions

By Durier and Michelot [27, Prop. 1.3], we know that one can determine completely the set of (strictly, weakly) Pareto efficient solutions for the problem $\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A})$ (defined as $(\mathrm{POLP}^2_X(\mathcal{A}))$ with $\mathbf{E}$ in the role of $X$) as stated in the next lemma.

**Lemma 6.2** Assume that (6.1) holds. Then, we have

$$\mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}) = \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}) = \mathrm{conv}\mathcal{A}.$$

By Jourani, Michelot and Ndiaye [67] we get the following characterizations for the sets of (strictly, weakly) Pareto efficient solutions of the nonconvex location problem $(\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A})^{\oplus_i})$.

**Lemma 6.3** Assume that (6.1) holds. For every $i \in I_l$, the following assertions hold:

**1°.** $\mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \mathrm{conv}\,\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)$.

**2°.** $d^i \in \mathrm{int}(\mathrm{conv}\,\mathcal{A})$ if and only if $\mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \mathbf{E}$.

**3°.** If $d^i \notin \mathrm{conv}\,\mathcal{A}$, then $\mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) \neq \mathbf{E}$.

**4°.** $d^i \in \mathrm{conv}\,\mathcal{A}$ if and only if $\mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \mathbf{E}$.

**5°.** $d^i \notin \mathrm{rint}(\mathrm{conv}\,\mathcal{A})$ if and only if $\mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) \neq \mathbf{E}$.

**6°.** $d^i \in \mathrm{rint}(\mathrm{conv}\,\mathcal{A})$ if and only if $\mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \mathbf{E}$.

**7°.** $\mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i}) = \{x \in \mathbf{E} \mid (\mathrm{conv}\,\mathcal{A}) \cap \mathrm{conv}\,\{x, d^i\} \neq \emptyset\}$.

**8°.** $\mathrm{rint}(\mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i})) = \{x \in \mathbf{E} \mid \mathrm{rint}(\mathrm{conv}\,\mathcal{A}) \cap \mathrm{rint}(\mathrm{conv}\,\{x, d^i\}) \neq \emptyset\}$.

*Proof.* First, recalling Lemma 1.29 (2°), we have $\mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right) = \mathbf{E}$ if and only if $d^i \in \mathrm{int}(\mathrm{conv}\mathcal{A})$. Now, 1° follows by [67, Cor. 4.1]; 2° follows by 1°; 3° follows by [67, Th. 4.5] and by 2°; 4° follows by [67, Prop. 4.2]; 5° follows by [67, Th. 4.3]; 6° follows by [67, Prop. 4.1]; 7° follows by [67, Th. 4.4]; 8° follows by [67, Th. 4.2]. □

**Remark 6.4** Notice that Lemma 6.2 is actually true for infinite-dimensional Hilbert spaces (see Durier and Michelot [27, Prop. 1.3]) taking into account that $\mathrm{conv}\mathcal{A}$ is compact for the finite set $\mathcal{A}$ (see Aliprantis and Border [1, Cor. 5.30]). According to Jourani, Michelot and Ndiaye [67], the results given in Lemma 6.3 are valid for finite-dimensional inner product spaces (hence finite-dimensional Hilbert spaces). For that reason, we assume in our main assumption (6.1) that $\mathbf{E}$ is a finite-dimensional Hilbert space.

Since $\mathcal{A}$ is finite, the set $\mathrm{conv}\mathcal{A}$ is a polytope. In the case $d^i \notin \mathrm{int}(\mathrm{conv}\,\mathcal{A})$, for any $i \in I_l$, the set $\mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)$ is a (closed and convex) polyhedral cone and $\mathrm{conv}\,\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)$ is a polyhedral set. Otherwise, if $d^i \in \mathrm{int}(\mathrm{conv}\,\mathcal{A})$, then both sets are equal to $\mathbf{E}$. In addition, for any $d^i \in \mathrm{conv}\,\mathcal{A}$, we have

$$T(\mathrm{conv}\mathcal{A}, d^i) = \mathrm{cl}\left(\mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)\right) = \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right),$$

where $T(\mathrm{conv}\mathcal{A}, d^i)$ stands for the contingent cone of $\mathrm{conv}\mathcal{A}$ at the point $d^i$ (see Remark 1.28).

As mentioned by Jourani, Michelot and Ndiaye [67], these complete geometrical descriptions of the sets of (strictly, weakly) Pareto efficient solutions given in Lemma 6.3 are surprising due to the nonconvexity of the objective function $g_{\mathcal{A}}^{\oplus_i}$, $i \in I_l$.

In the next lemma, we will see that Lemmata 6.2 and 6.3 are very important results in order to obtain complete geometrical descriptions for the sets of (strictly, weakly) Pareto efficient solutions (under the validity of (6.2) and (6.3), respectively) of the nonconvex problem $(\mathrm{POLP}^2_X(\mathcal{A}))$.

**Lemma 6.5** ([53]) Let (6.1) be fulfilled. Then, the following assertions are satisfied:

**1°.** We have

$$\mathrm{SEff}(X \mid g_{\mathcal{A}}) \supseteq [X \cap \mathrm{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i})\right];$$

$$\mathrm{Eff}(X \mid g_{\mathcal{A}}) \supseteq [X \cap \mathrm{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i})\right];$$

$$\mathrm{WEff}(X \mid g_{\mathcal{A}}) \supseteq [X \cap \mathrm{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i})\right]$$

and

$$\mathrm{SEff}(X \mid g_{\mathcal{A}}) \subseteq \mathrm{Eff}(X \mid g_{\mathcal{A}}) \subseteq \mathrm{WEff}(X \mid g_{\mathcal{A}}) \subseteq [X \cap \mathrm{conv}\mathcal{A}] \cup \mathrm{bd}\, X.$$

**2°.** Assume that (6.2) holds. Then, we have

$$\mathrm{SEff}(X \mid g_{\mathcal{A}}) = [X \cap \mathrm{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{SEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i})\right].$$

**3°.** Assume that (6.3) holds. Then, we have

$$\mathrm{Eff}(X \mid g_{\mathcal{A}}) = [X \cap \mathrm{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{Eff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i})\right];$$

$$\mathrm{WEff}(X \mid g_{\mathcal{A}}) = [X \cap \mathrm{conv}\mathcal{A}] \cup \left[\bigcup_{i \in I_l} (\mathrm{bd}\, D_i) \cap \mathrm{WEff}(\mathbf{E} \mid g_{\mathcal{A}}^{\oplus_i})\right].$$

*Proof.* Follows by Corollary 3.22 and Lemma 6.2. ☐

The reverse inclusions in 1° of Lemma 6.5 do not hold in general, as shown in the next example.

**Example 6.6** Consider the space $\mathbf{E} = \mathbb{R}^2$, the set $\mathcal{A} = \{a^1\} = \{(0,0)\}$, and three Euclidean balls in $\mathbb{R}^2$, namely

$$\begin{aligned}
&D_1 \text{ with center } d^1 = (-2,0) \text{ and radius } r_1 = 3,\\
&D_2 \text{ with center } d^2 = (2,0) \text{ and radius } r_2 = 3,\\
&D_3 \text{ with center } d^3 = (0,2) \text{ and radius } r_3 = 3.
\end{aligned}$$

For the problem $(\text{POLP}^2_X(\mathcal{A}))$ (with $m = 1$), we suppose that $X = X_1 \cap X_2 \cap X_3$ with $X_i = \mathbb{R}^2 \setminus \operatorname{int} D_i$ for every $i \in I_3$. Then, we have $\operatorname{conv}\mathcal{A} = \{(0,0)\}$, hence

$$X \cap \operatorname{conv}\mathcal{A} = \emptyset.$$

Moreover, we get for $d^1, d^2, d^3 \notin \operatorname{conv}\mathcal{A}$,

$$\begin{aligned}
\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_1}) &= -\operatorname{cone}\{d^1\} = [0,\infty) \times \{0\},\\
\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_2}) &= -\operatorname{cone}\{d^2\} = (-\infty,0] \times \{0\},\\
\operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_3}) &= -\operatorname{cone}\{d^3\} = \{0\} \times (-\infty,0]
\end{aligned}$$

by Lemma 6.3. We thus infer

$$\begin{aligned}
X \cap (\operatorname{bd} D_1) \cap \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_1}) &= X \cap \{(1,0)\} = \emptyset,\\
X \cap (\operatorname{bd} D_2) \cap \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_2}) &= X \cap \{(-1,0)\} = \emptyset,\\
X \cap (\operatorname{bd} D_3) \cap \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_3}) &= X \cap \{(0,-1)\} = \emptyset.
\end{aligned}$$

Notice, in view of Lemma 6.3, we have

$$\operatorname{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_i}) = \operatorname{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_i}) = \operatorname{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}^{\oplus_i}) \quad \text{for all } i \in I_3.$$

However, it can easily be checked that

$$\emptyset \neq \{(0,-\sqrt{5})\} = \operatorname{argmin}_{x \in X} ||x||_2 = \operatorname{SEff}(X \mid g_{\mathcal{A}}) = \operatorname{Eff}(X \mid g_{\mathcal{A}}) = \operatorname{WEff}(X \mid g_{\mathcal{A}}).$$

This means that the reverse inclusions in 1° of Lemma 6.5 do not hold for this example problem. Notice that neither (6.2) nor (6.3) is fulfilled in this example.

In preparation of the next theorem, we define the following three sets of indices:

$$\begin{aligned}
I^{conv} &:= \{i \in I_l \mid d^i \in \operatorname{conv}\mathcal{A}\};\\
I^{i-conv} &:= \{i \in I_l \mid d^i \in \operatorname{int}(\operatorname{conv}\mathcal{A})\};\\
I^{ri-conv} &:= \{i \in I_l \mid d^i \in \operatorname{rint}(\operatorname{conv}\mathcal{A})\}.
\end{aligned}$$

We now present the main theorem of this section where we give complete geometrical descriptions for the sets of (strictly, weakly) Pareto efficient solutions of $(\text{POLP}^2_X(\mathcal{A}))$ that are valid under the assumptions (6.1) and (6.2) (respectively, (6.3)).

**Theorem 6.7** ([53]) Let (6.1) be fulfilled. Then, the following assertions hold:

**1°.** Assume that (6.2) holds. Then, we have

$$\begin{aligned}\mathrm{SEff}(X \mid g_{\mathcal{A}}) &= X \cap \mathrm{conv}\mathcal{A}\\ &\cup \left[\bigcup_{i \in I_l \setminus I^{i-conv}} (\mathrm{bd}\, D_i) \cap \left(\mathrm{conv}\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)\right)\right]\\ &\cup \left[\bigcup_{i \in I^{i-conv}} \mathrm{bd}\, D_i\right]\\ &\supseteq X \cap \mathrm{conv}\mathcal{A}\\ &\cup \left[\bigcup_{i \in I_l \setminus I^{i-conv}} \left\{x \in \mathrm{bd}\, D_i \mid \mathrm{rint}(\mathrm{conv}\, \mathcal{A}) \cap \mathrm{rint}(\mathrm{conv}\, \{x, d^i\}) \neq \emptyset\right\}\right]\\ &\cup \left[\bigcup_{i \in I^{i-conv}} \mathrm{bd}\, D_i\right].\end{aligned}$$

**2°.** Assume that (6.3) holds. Then, we have

$$\begin{aligned}\mathrm{Eff}(X \mid g_{\mathcal{A}}) &= X \cap \mathrm{conv}\mathcal{A}\\ &\cup \left[\bigcup_{i \in I_l \setminus I^{ri-conv}} (\mathrm{bd}\, D_i) \cap \left(\mathrm{conv}\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)\right)\right]\\ &\cup \left[\bigcup_{i \in I^{ri-conv}} \mathrm{bd}\, D_i\right];\\ \mathrm{WEff}(X \mid g_{\mathcal{A}}) &= X \cap \mathrm{conv}\mathcal{A}\\ &\cup \left[\bigcup_{i \in I_l \setminus I^{conv}} (\mathrm{bd}\, D_i) \cap \left(\mathrm{conv}\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)\right)\right]\\ &\cup \left[\bigcup_{i \in I^{conv}} \mathrm{bd}\, D_i\right]\\ &= X \cap \mathrm{conv}\mathcal{A}\\ &\cup \left[\bigcup_{i \in I_l \setminus I^{conv}} \left\{x \in \mathrm{bd}\, D_i \mid (\mathrm{conv}\mathcal{A}) \cap \mathrm{conv}\, \{x, d^i\} \neq \emptyset\right\}\right]\\ &\cup \left[\bigcup_{i \in I^{conv}} \mathrm{bd}\, D_i\right].\end{aligned}$$

*Proof.* Follows by Lemmata 6.3 and 6.5. □

**Corollary 6.8** ([53]) Let (6.1) be fulfilled. Then, the following assertions hold:

**1°.** Assume that (6.2) holds. Then, $\mathrm{SEff}(X \mid g_{\mathcal{A}})$ is a compact set.

**2°.** Assume that (6.3) holds. Then, $\mathrm{Eff}(X \mid g_{\mathcal{A}})$ and $\mathrm{WEff}(X \mid g_{\mathcal{A}})$ are compact sets.

*Proof.* The sets $D_i$, $i \in I_l$, and $\mathrm{conv}\mathcal{A}$ are compact sets. Moreover, notice that the sets $X$ and $\mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)$, $i \in I_l$, are closed. Hence, we easily obtain that $\mathrm{SEff}(X \mid g_{\mathcal{A}})$, $\mathrm{Eff}(X \mid g_{\mathcal{A}})$ and $\mathrm{WEff}(X \mid g_{\mathcal{A}})$ are closed and bounded sets by Theorem 6.7. Notice that the sum of a compact set and a closed set in $\mathbf{E}$ is closed (see, e.g., Jahn [64, Lem. 1.34]). Since $\mathbf{E}$ is a finite-dimensional normed space, both assertions of this corollary follow immediately. □

Next, we present two examples in order to illustrate (for the case $l = 1$ as well as for the case $l = 2$) the geometrical descriptions given for the sets of (strictly, weakly) Pareto efficient solutions of the problem $(\mathrm{POLP}_X^2(\mathcal{A}))$ in Theorem 6.7.

**Example 6.9** We consider a planar point-objective location problem $(\mathrm{POLP}_X^2(\mathcal{A}))$ involving the Euclidean norm $||\cdot||_2$ where the set of existing facilities is given by

$$\mathcal{A} = \{a^1, a^2, a^3\} \subseteq \mathbb{R}^2 = \mathbf{E}$$

and the feasible set is given by $X = X_1 = \mathbb{R}^2 \setminus \operatorname{int} D_1$. Figure 6.1 shows the location problem as well as the procedure for computing the set $\mathrm{Eff}(X \mid g_{\mathcal{A}})$. Notice that $d^1 \in (\operatorname{conv}\mathcal{A}) \setminus \operatorname{int}(\operatorname{conv}\mathcal{A})$. Due to Lemma 6.2 and Theorem 6.7, we have

$$\mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \operatorname{conv}\mathcal{A}$$

and

$$\begin{aligned}
\mathrm{SEff}(X \mid g_{\mathcal{A}}) &= [X \cap \operatorname{conv}\mathcal{A}] \cup \left[(\operatorname{bd} D_1) \cap \left(\operatorname{conv}\mathcal{A} + \operatorname{cone}\left(\operatorname{conv}\mathcal{A} - d^1\right)\right)\right];\\
\mathrm{Eff}(X \mid g_{\mathcal{A}}) &= \mathrm{SEff}(X \mid g_{\mathcal{A}});\\
\mathrm{WEff}(X \mid g_{\mathcal{A}}) &= [X \cap \operatorname{conv}\mathcal{A}] \cup \operatorname{bd} D_1.
\end{aligned}$$

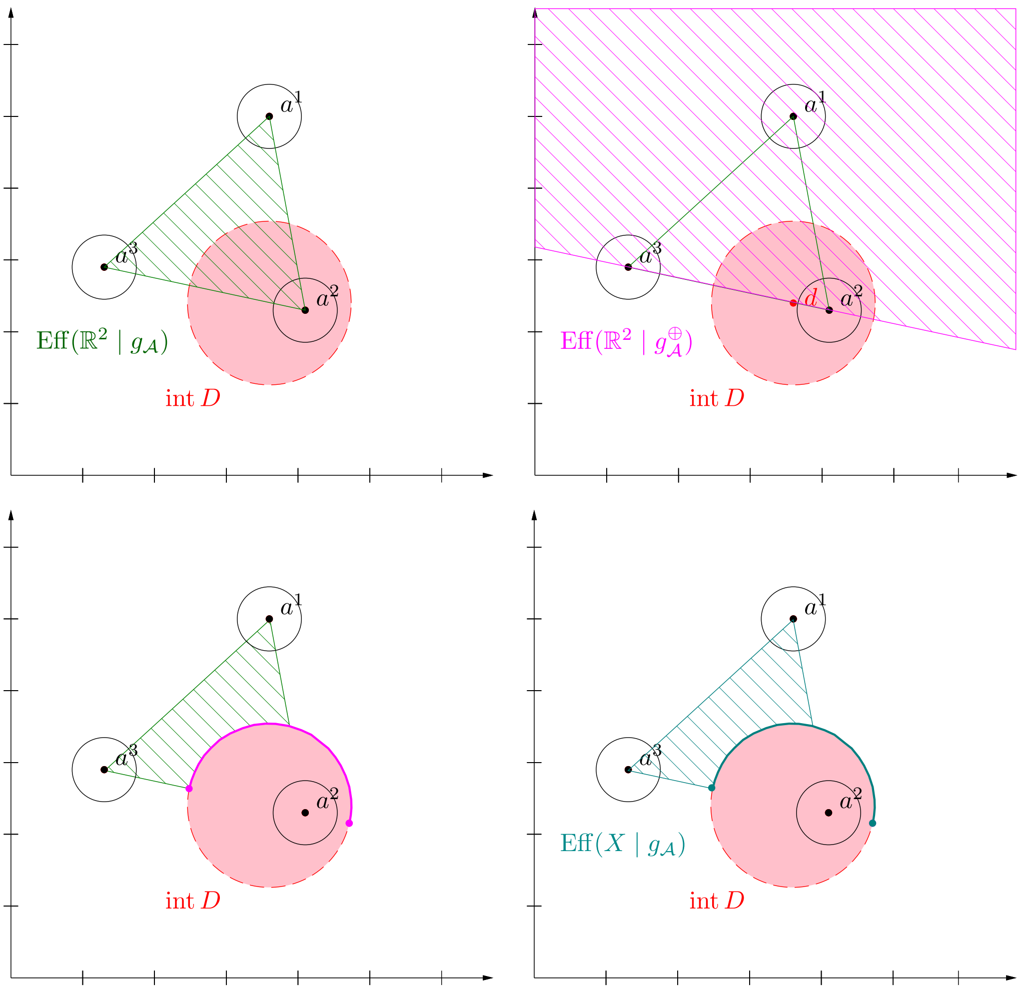


Figure 6.1: Construction of the set of Pareto efficient solutions of the problem $(\mathrm{POLP}_X^2(\mathcal{A}))$ with $m = 3$ and $l = 1$.

**Example 6.10** Again, let us consider a planar point-objective location problem $(\mathrm{POLP}^2_X(\mathcal{A}))$ involving the Euclidean norm $||\cdot||_2$ where the set of existing facilities is given by

$$\mathcal{A} = \{a^1, a^2, a^3\} \subseteq \mathbb{R}^2 = \mathbf{E}.$$

We assume that $X$ is an intersection of two reverse convex sets, i.e., we have

$$X = X_1 \cap X_2 = (\mathbb{R}^2 \setminus \operatorname{int} D_1) \cap (\mathbb{R}^2 \setminus \operatorname{int} D_2).$$

Figure 6.2 illustrates this problem and shows how the set $\mathrm{Eff}(X \mid g_{\mathcal{A}})$ can be computed. Notice that $d^1 \in \operatorname{int}(\operatorname{conv}\mathcal{A})$ and $d^2 \notin \operatorname{conv}\mathcal{A}$. In view of Lemma 6.2 and Theorem 6.7, we infer

$$\mathrm{SEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{Eff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \mathrm{WEff}(\mathbb{R}^2 \mid g_{\mathcal{A}}) = \operatorname{conv}\mathcal{A}$$

and

$$\begin{aligned}\mathrm{SEff}(X \mid g_{\mathcal{A}}) &= [X \cap \operatorname{conv}\mathcal{A}] \cup \operatorname{bd} D_1 \\ &\quad \cup \left[(\operatorname{bd} D_2) \cap \left(\operatorname{conv}\mathcal{A} + \operatorname{cone}\left(\operatorname{conv}\mathcal{A} - d^2\right)\right)\right]; \\ \mathrm{Eff}(X \mid g_{\mathcal{A}}) &= \mathrm{WEff}(X \mid g_{\mathcal{A}}) = \mathrm{SEff}(X \mid g_{\mathcal{A}}).\end{aligned}$$

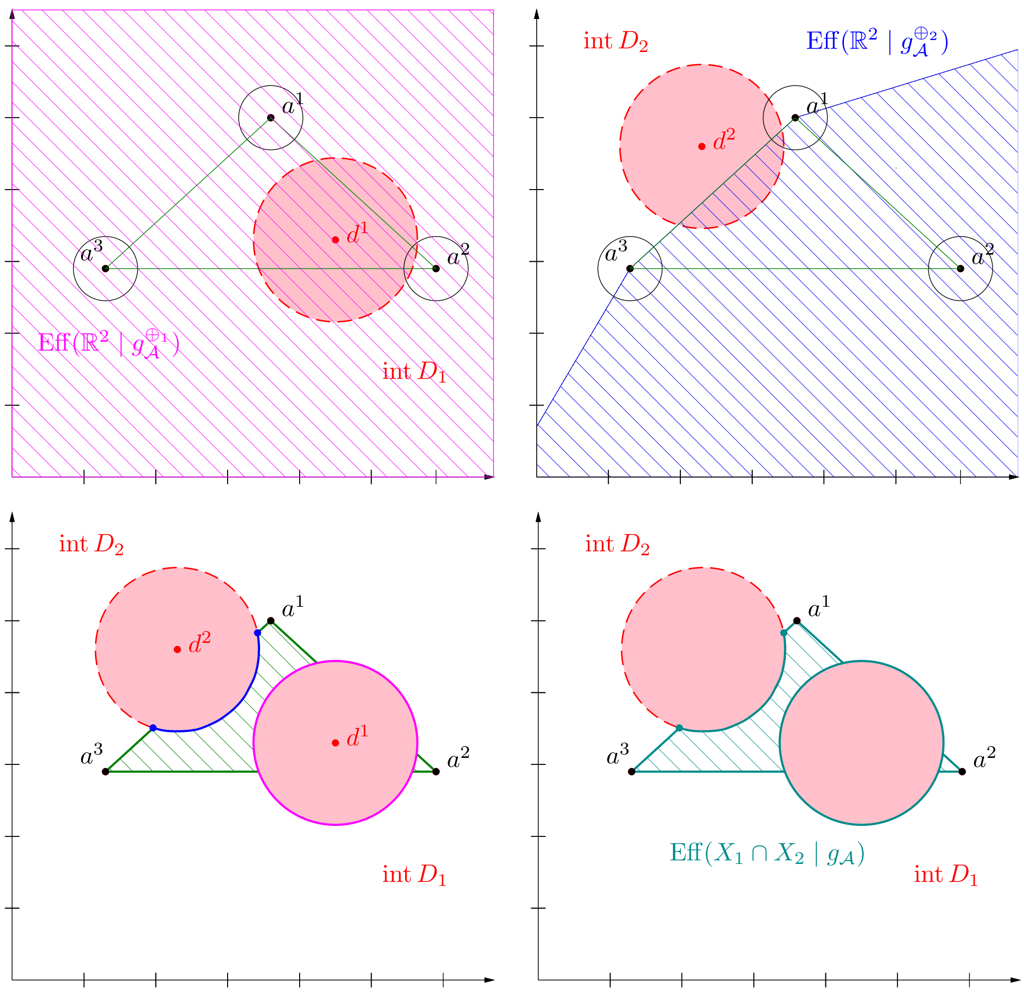


Figure 6.2: Construction of the set of Pareto efficient solutions of the problem $(\mathrm{POLP}^2_X(\mathcal{A}))$ with $m = 3$ and $l = 2$.

In Proposition 6.11, we present some characterizations related to the sets of (strictly, weakly) Pareto efficient solutions.

**Proposition 6.11** ([53]) Let (6.1) and (6.3) be fulfilled. Then, the following assertions are true:

1°. Assume that (6.2) holds. Then, we have

$$I^{i-conv} = I_l \iff \mathrm{SEff}(X \mid g_{\mathcal{A}}) = [X \cap \mathrm{conv}\mathcal{A}] \cup \mathrm{bd}\, X.$$

2°. Assume that (6.2) or $\dim \mathbf{E} \geq 2$ holds. Then, we have

$$I^{ri-conv} = I_l \iff \mathrm{Eff}(X \mid g_{\mathcal{A}}) = [X \cap \mathrm{conv}\mathcal{A}] \cup \mathrm{bd}\, X.$$

3°. Assume that (6.2) or $\dim \mathbf{E} \geq 2$ holds. Then, we have

$$I^{conv} = I_l \iff \mathrm{WEff}(X \mid g_{\mathcal{A}}) = [X \cap \mathrm{conv}\mathcal{A}] \cup \mathrm{bd}\, X.$$

4°. Assume that (6.2) holds. Then, we have

$$I^{ri-conv} = I^{i-conv} \iff \mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}}),$$

or, equivalently, we have

$$\mathrm{int}(\mathrm{conv}\mathcal{A}) \neq \emptyset \vee I^{ri-conv} = \emptyset \iff \mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}}).$$

5°. Assume that (6.2) or $\dim \mathbf{E} \geq 2$ holds. Then, we have

$$I^{conv} = I^{ri-conv} \iff \mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}}).$$

6°. Assume that (6.2) holds. Then, we have

$$I^{conv} = I^{i-conv} \iff \mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}}).$$

7°. Assume that (6.2) holds. Then, we have

$$\emptyset = I^{i-conv} \subsetneq I^{ri-conv} \subsetneq I^{conv} \iff \mathrm{SEff}(X \mid g_{\mathcal{A}}) \subsetneq \mathrm{Eff}(X \mid g_{\mathcal{A}}) \subsetneq \mathrm{WEff}(X \mid g_{\mathcal{A}}).$$

To prove Proposition 6.11, we need the following key lemma.

**Lemma 6.12** ([53]) Let (6.1) be fulfilled. The following assertions hold:

1°. For any $j \in I_l$, we have

$$j \in I^{i-conv} \iff (\mathrm{bd}\, D_j) \cap \big(\mathrm{conv}\mathcal{A} + \mathrm{cone}\,\big(\mathrm{conv}\mathcal{A} - d^j\big)\big) = \mathrm{bd}\, D_j.$$

2°. Let $j \in I_l \setminus I^{i-conv}$. Then, we have

$$(\mathrm{bd}\, D_j) \cap (\mathrm{conv}\mathcal{A}) \subseteq (\mathrm{bd}\, D_j) \cap \big(\mathrm{conv}\mathcal{A} + \mathrm{cone}\,\big(\mathrm{conv}\mathcal{A} - d^j\big)\big) \subsetneq \mathrm{bd}\, D_j.$$

Hence, the set

$$(\mathrm{bd}\, D_j) \setminus \big(\mathrm{conv}\mathcal{A} + \mathrm{cone}\,\big(\mathrm{conv}\mathcal{A} - d^j\big)\big)$$

is nonempty, and if $\dim \mathbf{E} \geq 2$, has an infinite number of elements.

3°. Assume that (6.3) holds. For any $i, j \in I_l$, $i \neq j$, the set $(\mathrm{bd}\, D_i) \cap (\mathrm{bd}\, D_j)$ is a singleton set or the empty set. Hence, for any $j \in I_l$, the set $(\mathrm{bd}\, D_j) \cap \bigcup_{i \in I_l \setminus \{j\}} \mathrm{bd}\, D_i$ has at most $l - 1$ elements.

4°. Assume that (6.2) holds. For any $i, j \in I_l$, $i \neq j$, the set $(\mathrm{bd}\, D_i) \cap (\mathrm{bd}\, D_j)$ is the empty set.

*Proof.* For notational convenience, we define $C_j := \mathrm{cone}\,\big(\mathrm{conv}\mathcal{A} - d^j\big)$ for all $j \in I_l$.

**1°.** Since for $j \in I^{i-conv}$ we have $C_j = \mathbf{E}$ by Lemma 1.29, the implication "$\Longrightarrow$" follows immediately. Now, let us establish the reverse implication "$\Longleftarrow$".

Let $j \in I_l$. Since $\operatorname{bd} D_j \subseteq \operatorname{conv}\mathcal{A} + C_j$ and $\operatorname{conv}\mathcal{A} + C_j$ is a convex set, we get

$$d^j \in \operatorname{int} D_j \subseteq D_j \subseteq \operatorname{conv}\mathcal{A} + C_j,$$

hence

$$d^j \in \operatorname{int}(\operatorname{conv}\mathcal{A} + C_j). \tag{6.4}$$

Assume that the contrary holds, i.e., $j \in I_l \setminus I^{i-conv}$, hence $d^j \notin \operatorname{int}(\operatorname{conv}\mathcal{A})$. First, we show that

$$\exists\, v \in \mathbf{E} \setminus \{0\}\, \forall\, \delta \in \mathbb{R}_{++} :\; d^j + \delta v \notin \operatorname{conv}\mathcal{A} \tag{6.5}$$

by considering two cases:

Case 1: Assume that $d^j \notin \operatorname{conv}\mathcal{A}$. By the *Seperation Theorem for Convex Sets* in Corollary 1.16, we infer that there exists a linear functional $\psi : \mathbf{E} \to \mathbb{R}$ such that

$$\sup_{a \in \operatorname{conv}\mathcal{A}} \psi(a) < \psi(d^j). \tag{6.6}$$

Assume that the contrary of (6.5) holds. Then, for $v := d^j - a$ with $a \in \operatorname{conv}\mathcal{A}$, there exists $\delta \in \mathbb{R}_{++}$ such that $d^j + \delta v \in \operatorname{conv}\mathcal{A}$. So, we have

$$\begin{aligned}\psi(d^j) &\overset{(6.6)}{>} \psi(d^j + \delta v)\\ &= \psi(d^j) + \delta\psi(d^j - a)\\ &= \psi(d^j) + \delta\psi(d^j) - \delta\psi(a),\end{aligned}$$

which implies $\psi(a) > \psi(d^j)$, a contradiction to (6.6). Thus, (6.5) holds.

Case 2: Assume that $d^j \in \operatorname{bd}(\operatorname{conv}\mathcal{A})$. Since $d^j \in \operatorname{conv}\mathcal{A}$ is not an interior point of $\operatorname{conv}\mathcal{A}$, it follows

$$\exists\, \overline{v} \in \mathbf{E} \setminus \{0\}\, \forall\, \delta \in \mathbb{R}_{++}\, \exists\, \theta \in ]0, \delta] :\; d^j + \theta\overline{v} \notin \operatorname{conv}\mathcal{A} \tag{6.7}$$

in the finite-dimensional normed space $(\mathbf{E}, ||\cdot||)$ (see Lemma 1.20). If we suppose that $d^j + \overline{\delta}\overline{v} \in \operatorname{conv}\mathcal{A}$ for some $\overline{\delta} \in \mathbb{R}_{++}$, then

$$d^j + [0, \overline{\delta}] \cdot \overline{v} \subseteq \operatorname{conv}\mathcal{A}$$

by the convexity of $\operatorname{conv}\mathcal{A}$, a contradiction to (6.7). This shows (6.5) with $v := \overline{v}$.

In both cases (6.5) holds.

In view of (6.4), for $v \in \mathbf{E} \setminus \{0\}$ given in (6.5), we get that there exists $\widehat{\delta} \in \mathbb{R}_{++}$ such that $d^j + \widehat{\delta}v \in \operatorname{conv}\mathcal{A} + C_j$. So, there exist $k \in \mathbb{R}_+$, $a', a'' \in \operatorname{conv}\mathcal{A}$, such that $d^j + \widehat{\delta}v = a' + k(a'' - d^j)$. This means that

$$d^j + \frac{\widehat{\delta}}{1+k}v = \frac{1}{1+k}a' + \frac{k}{1+k}a'' = \left(1 - \frac{k}{1+k}\right)a' + \frac{k}{1+k}a'' \in \operatorname{conv}\mathcal{A},$$

a contradiction to (6.5).

The proof of assertion 1° is complete.

**2°.** We have $0 \in C_j$, hence $\operatorname{conv}\mathcal{A} \subseteq \operatorname{conv}\mathcal{A}+C_j$, which shows the first inclusion in assertion 2°. By 1° of this lemma, we get the second strict inclusion. Hence, we infer that $(\operatorname{bd} D_j)\setminus(\operatorname{conv}\mathcal{A}+C_j) \neq \emptyset$.

Now, we show that $(\operatorname{bd} D_j) \setminus (\operatorname{conv}\mathcal{A} + C_j)$ has an infinite number of elements. Let us consider two cases:

Case 1: Assume that $d^j \notin \operatorname{conv}\mathcal{A}$. Then, we get $d^j \notin \operatorname{conv}\mathcal{A} + C_j$. Indeed, if there exist $k \in \mathbb{R}_+$, $a', a'' \in \operatorname{conv}\mathcal{A}$, such that $d^j = a' + k(a'' - d^j)$, then

$$d^j = \frac{1}{1+k}a' + \frac{k}{1+k}a'' = \left(1 - \frac{k}{1+k}\right)a' + \frac{k}{1+k}a'' \in \operatorname{conv}\mathcal{A},$$

a contradiction.

Since $\operatorname{conv}\mathcal{A} + C_j$ is closed and convex, we infer that there exists a linear functional $\psi : \mathbf{E} \to \mathbb{R}$ such that

$$\sup_{c \in \operatorname{conv}\mathcal{A}+C_j} \psi(c) < \psi(d^j) \tag{6.8}$$

by the *Seperation Theorem for Convex Sets* in Corollary 1.16. Without loss of generality, assume that $\mathbf{E}$ has dimension $n \geq 2$. By the well-known *Dimension Theorem*, we know that the sum of the dimensions of the *kernel of* $\psi$ (denoted by $\ker\psi := \{x \in \mathbf{E} \mid \psi(x) = 0\}$) and the *image of* $\psi$ (denoted by $\operatorname{img}\psi := \{\psi(x) \mid x \in \mathbf{E}\}$) is equal to $n$. More precisely, we have $\dim\ker\psi = n-1$ and $\dim\operatorname{img}\psi = 1$. Consider $\overline{x} \in (\ker\psi) \setminus \{0\}$. Since $d^j \in \operatorname{int} D_j$ and $D_j$ is convex, it exists $\overline{\delta} \in \mathbb{R}_{++}$ such that $S := d^j + [0, \overline{\delta}] \cdot \overline{x} \subseteq \operatorname{int} D_j$ by Lemmata 1.13 and 1.20. Notice that $S$ has an infinite number of elements. Define $v := d^j - c$ for some $c \in \operatorname{conv}\mathcal{A} + C_j$. For any $y \in S$, we define a function $h_y : \mathbb{R} \to \mathbb{R}$ by

$$h_y(t) := ||y + tv - d^j|| \quad \text{for all } t \in \mathbb{R}. \tag{6.9}$$

Consider $y \in S$. Since $D_j$ is bounded and $v \neq 0$, there exists $t_y \in \mathbb{R}_{++}$ such that $y + t_y v \notin D_j$. By the continuity of $h_y$ and by $h_y(0) < r_j < h_y(t_y)$, we get some $t_y^* \in ]0, t_y[ \subseteq \mathbb{R}_{++}$ such that $h_y(t_y^*) = r_j$ by the well-known *Intermediate Value Theorem*, hence $x_y := y + t_y^* v \in \operatorname{bd} D_j$. Since $y \in S$ we know that $y = d^j + \delta\overline{x}$ for some $\delta \in [0, \overline{\delta}]$. Then, due to $\overline{x} \in \ker\psi$ and (6.8), we infer

$$\psi(x_y) = \psi(d^j) + \delta\psi(\overline{x}) + t_y^*(\psi(d^j) - \psi(c)) > \psi(d^j),$$

which implies $x_y \notin \operatorname{conv}\mathcal{A} + C_j$ in view of (6.8). We conclude that

$$x_y \in (\operatorname{bd} D_j) \setminus (\operatorname{conv}\mathcal{A} + C_j).$$

Moreover, the map $y \mapsto x_y$ is injective. Assume the contrary holds, i.e., there exist $y', y'' \in S$, $y' \neq y''$, such that

$$x_{y'} = y' + t_{y'}^* v = y'' + t_{y''}^* v = x_{y''}.$$

Of course, $t_{y'}^* = t_{y''}^*$ implies $y' = y''$, a contradiction. Without loss of generality, assume that $t_{y'}^* > t_{y''}^*$. We get $y' - y'' = (t_{y'}^* - t_{y''}^*)v$, hence

$$0 = \psi(y' - y'') = (t_{y'}^* - t_{y''}^*)\psi(v) = (t_{y'}^* - t_{y''}^*)(\psi(d^j) - \psi(c)) > 0$$

taking into account the definition of $S$ and formula (6.8), a contradiction.

This completes the proof in the first case.

Case 2: Assume that $d^j \in \operatorname{bd}(\operatorname{conv}\mathcal{A})$. We must have $d^j \notin \operatorname{int}(\operatorname{conv}\mathcal{A} + C_j)$, otherwise $d^j \in \operatorname{int}(\operatorname{conv}\mathcal{A})$ by the ideas given in the proof of assertion 1° in this lemma. Notice that the case $d^j \notin \operatorname{conv}\mathcal{A} + C_j$ is considered in Case 1 (assertion 2°). Now, assume that $d^j \in \operatorname{bd}(\operatorname{conv}\mathcal{A} + C_j)$. Then, similar to the proof given in 1° of this lemma, there exists $v \in \mathbf{E} \setminus \{0\}$ such that $d^j + \delta v \notin \operatorname{conv}\mathcal{A} + C_j$ for all $\delta \in \mathbb{R}_{++}$. Since $d^j \in \operatorname{int} D_j$ and $D_j$ is a convex set, there is $\overline{\delta} \in \mathbb{R}_{++}$ such that $x^0 := d^j + \overline{\delta} v \in \operatorname{int} D_j$ by Lemmata 1.13 and 1.20. So, we get $x^0 \in (\operatorname{int} D_j) \setminus (\operatorname{conv}\mathcal{A} + C_j)$. Now, the proof is analogous to the proof given in Case 1 (assertion 2°) where $x^0$ is in the role of $d^j$ (except in the definition of the function $h_y$ given in (6.9)).

The proof of assertion 2° is complete.

**3°, 4°.** Directly follow by the assumptions (6.3) and (6.2), respectively.

□

Now, we are going to show Proposition 6.11.

*Proof.* **1°.** In view of assertion 1° in Theorem 6.7, the implication "$\Longrightarrow$" is obvious. Let us prove the reverse implication "$\Longleftarrow$". Let

$$\mathrm{SEff}(X \mid g_{\mathcal{A}}) = [X \cap \mathrm{conv}\mathcal{A}] \cup \mathrm{bd}\, X. \tag{6.10}$$

Assume that the contrary holds, i.e, there exists $j \in I_l \setminus I^{\mathrm{i-conv}}$. Then, due to Theorem 6.7 (1°) and formula (6.10), we must have

$$\begin{aligned} \mathrm{bd}\, D_j \subseteq & \bigcup_{i \in I_l \setminus \{j\}} \mathrm{bd}\, D_i \\ & \cup (\mathrm{bd}\, D_j) \cap \mathrm{conv}\, \mathcal{A} \\ & \cup \big((\mathrm{bd}\, D_j) \cap (\mathrm{conv}\, \mathcal{A} + \mathrm{cone}(\mathrm{conv}\, \mathcal{A} - d^j))\big) \\ = & \bigcup_{i \in I_l \setminus \{j\}} \mathrm{bd}\, D_i \\ & \cup \big((\mathrm{bd}\, D_j) \cap (\mathrm{conv}\, \mathcal{A} + \mathrm{cone}(\mathrm{conv}\, \mathcal{A} - d^j))\big)\,. \end{aligned}$$

Then, it can easily be seen that we get a contradiction by Lemma 6.12 (2°, 4°). So, 1° holds.

**2°, 3°.** Analogous to the proof of 1° in this proposition by using Theorem 6.7 (2°) and Lemma 6.12 (2°, 3°, 4°).

**4°.** By Theorem 6.7 (1°, 2°), the implication "$\Longrightarrow$" holds. Now, we prove the reverse implication "$\Longleftarrow$". Assume that the contrary holds, i.e, there exists $j \in I^{\mathrm{ri-conv}} \setminus I^{\mathrm{i-conv}}$. Then, in view of Theorem 6.7 (1°, 2°) and because of the assumption $\mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}})$, we obtain a contradiction by Lemma 6.12 (2°, 4°) and by the ideas given in the proof of 1° of this proposition.

**5°.** Analogous to the proof of 4° in this proposition by using Theorem 6.7 (2°) and Lemma 6.12 (2°, 3°, 4°).

**6°, 7°.** Follow by assertions 4° and 5° of this proposition.

□

## 6.2 The special case $d^i = a^i$, $i \in I_m$

Usually, the new facility $x \in \mathbf{E}$ should be located as close a possible to the existing facilities $a^i$, $i \in I_m$. In our model, each existing facility is located at exactly one point $a^i$ in $\mathbf{E}$ and has no expansion around this point. In particular, in the field of town planning, a given facility has a certain expansion. Hence, it is convenient to consider a forbidden region around $a^i$ defined by a certain open ball centered at $a^i$ with positive radius. So, it is possible to include information about the sizes of the existing facilities in the model. This means we are going to study the special case

$$l = m \;\text{ and } d^i = a^i \text{ for all } i \in I_l = I_m. \tag{6.11}$$

**Corollary 6.13** ([53]) Let (6.1), (6.3) and (6.11) be fulfilled. Then, the following assertions hold:

**1°.** Assume that (6.2) holds. Then, we have

$$\begin{aligned} \mathrm{SEff}(X \mid g_{\mathcal{A}}) = & \; X \cap \mathrm{conv}\mathcal{A} \\ & \cup \left[ \bigcup_{i \in I_m \setminus I^{i-conv}} (\mathrm{bd}\, D_i) \cap \left(\mathrm{conv}\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)\right) \right] \\ & \cup \left[ \bigcup_{i \in I^{i-conv}} \mathrm{bd}\, D_i \right]. \end{aligned}$$

**2°.** Assume that (6.2) or $\dim \mathbf{E} \geq 2$ holds. Then, we have

$$\begin{aligned}
\mathrm{Eff}(X \mid g_{\mathcal{A}}) = {} & X \cap \mathrm{conv}\mathcal{A} \\
& \cup \left[ \bigcup_{i \in I_m \setminus I^{ri-conv}} (\mathrm{bd}\, D_i) \cap \left(\mathrm{conv}\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - d^i\right)\right) \right] \\
& \cup \left[ \bigcup_{i \in I^{ri-conv}} \mathrm{bd}\, D_i \right]; \\
\mathrm{WEff}(X \mid g_{\mathcal{A}}) = {} & [X \cap \mathrm{conv}A] \cup \mathrm{bd}\, X.
\end{aligned}$$

*Proof.* Follows by Theorem 6.7. □

**Corollary 6.14** ([53]) Let (6.1), (6.3) and (6.11) be fulfilled. Then, the following assertions hold:

**1°.** Assume that (6.2) holds. Then, we have

$$\mathrm{int}(\mathrm{conv}\mathcal{A}) \neq \emptyset \vee I^{ri-conv} = \emptyset \iff \mathrm{SEff}(X \mid g_{\mathcal{A}}) = \mathrm{Eff}(X \mid g_{\mathcal{A}}).$$

**2°.** Assume that (6.2) or $\dim \mathbf{E} \geq 2$ holds. Then, we have

$$\mathrm{card}\ \mathcal{A} = 1 \iff \mathrm{Eff}(X \mid g_{\mathcal{A}}) = \mathrm{WEff}(X \mid g_{\mathcal{A}}).$$

*Proof.* Directly follows by 4° and 5° in Proposition 6.11. □

Next, we present an applied example of a location problem of type ($\mathrm{POLP}^2_X(\mathcal{A})$) in which the conditions (6.2), (6.3) and (6.11) are fulfilled.

**Example 6.15** A new central taxi station should be located in the district around *La Habana* on *Cuba*. We assume that the new location will be located as close as possible to each center of the cities L*a Habana*, *Guanabo*, *San José de las Lajas*, *Santiago de las Vegas*, and *Playa Baracoa*. Due to the high car traffic in the centers of the cities we want to avoid to place the new facility in the near of the city centers. This means that we consider some forbidden regions around the given city centers. Figure 6.3 illustrates this example and shows the whole set of Pareto efficient solutions of the nonconvex location problem ($\mathrm{POLP}^2_X(\mathcal{A})$).

## 6.3 Extension to problems with attraction and repulsion

Under the assumptions given in (6.1), we consider the problem of locating a new facility in presence of *attraction points* $a^1, \cdots, a^m \in \mathbf{E}$, $m \in \mathbb{N}$, and *repulsion points* (i.e. *undesirable facilities*, such as *polluting factories* or *nuclear plants*) $b^1, \cdots, b^k \in \mathbf{E}$, $k \in \mathbb{N}$. For notational convenience, we put

$$\mathcal{A} := \{a^1, \cdots, a^m\} \quad \text{and} \quad \mathcal{B} := \{b^1, \cdots, b^k\}.$$

This nonconvex class of location problems is discussed by Jourani, Michelot and Ndiaye [67] and can be modeled as follows:

$$\begin{cases} \widehat{g}_{\mathcal{A},\mathcal{B}}(x) := \left(||x - a^1||, \cdots, ||x - a^m||, -||x - b^1||, \cdots, -||x - b^k||\right) \to \min \\ x \in X \end{cases} \qquad (\mathrm{POLP}^2_X(\mathcal{A}, \mathcal{B}))$$

Then, for any $i \in I_l$, the penalized multi-objective location problem is given by

$$\begin{cases} \widehat{g}^{\oplus_i}_{\mathcal{A},\mathcal{B}}(x) := \left(\widehat{g}_{\mathcal{A},\mathcal{B}}(x), -||x - d^i||\right) \to \min \\ x \in \mathbf{E}, \end{cases} \qquad (\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A}, \mathcal{B})^{\oplus_i})$$

where $\mathcal{B}^i := \mathcal{B} \cup \{d^i\}$ consists of all repulsion points belonging to $\mathcal{B}$ and one additional point $d^i \in \mathrm{int} D_i$. In fact, ($\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A}, \mathcal{B})^{\oplus_i}$) can be seen as the problem ($\mathrm{POLP}^2_{\mathbf{E}}(\mathcal{A}, \mathcal{B}^i)$).

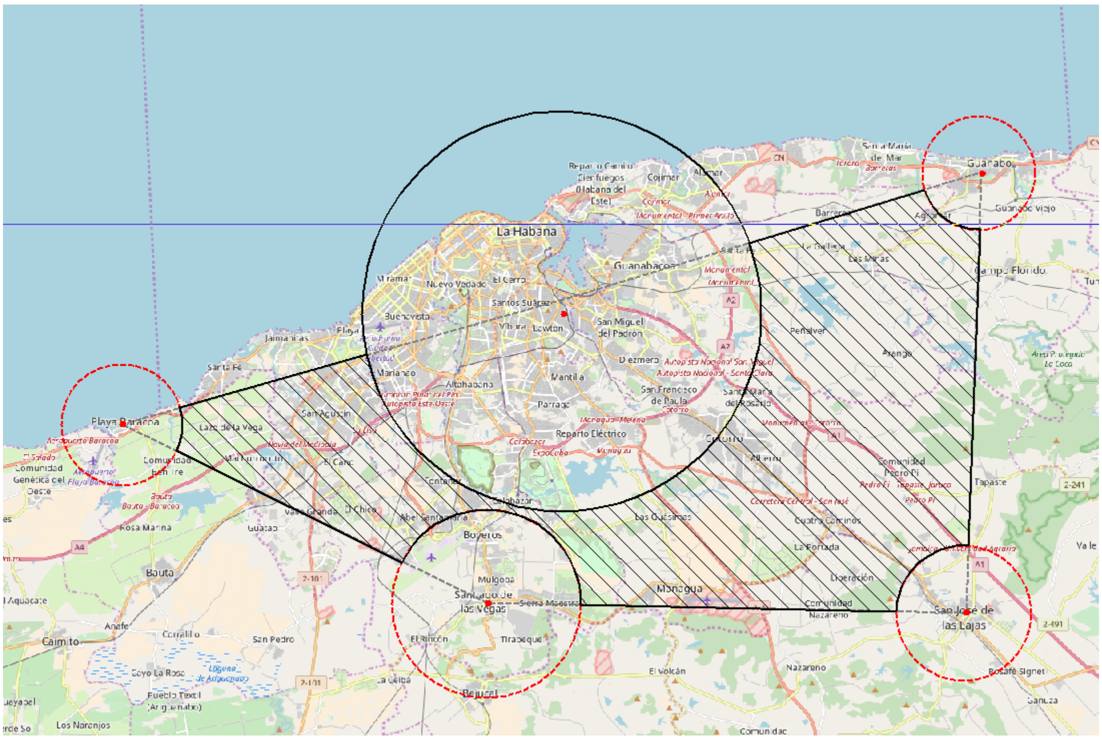


Figure 6.3: The set of Pareto efficient solutions (black colored region) of the problem $(\text{POLP}^2_X(\mathcal{A}))$ for Example 6.15 illustrated on a map (from OpenStreetMap) of La Habana in Cuba.

It is important to mention that $\widehat{g}_{\mathcal{A},\mathcal{B}}$ is neither componentwise semi-strictly quasi-convex nor componentwise quasi-convex. The sets of (strictly, weakly) Pareto efficient solutions of the unconstrained problems $(\text{POLP}^2_{\mathbf{E}}(\mathcal{A},\mathcal{B}))$ (defined as $(\text{POLP}^2_X(\mathcal{A},\mathcal{B}))$ with $\mathbf{E}$ in the role of $X$) and $(\text{POLP}^2_{\mathbf{E}}(\mathcal{A},\mathcal{B})^{\oplus_i})$ can be completely characterized by using results in Jourani, Michelot and Ndiaye [67], as to see in the next lemma.

**Lemma 6.16** ([67]) Assume that (6.1) holds. Let $B \subseteq \mathbf{E}$ be a nonempty finite set. Then, the following assertions hold for the problem $\text{POLP}^2_{\mathbf{E}}(\mathcal{A},B)$ (defined as $(\text{POLP}^2_X(\mathcal{A},\mathcal{B}))$ with $B$ in the role of $\mathcal{B}$ and $\mathbf{E}$ in the role of $X$):

**1°.** $\text{SEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) = \text{conv}\mathcal{A} + \text{cone}\,(\text{conv}\mathcal{A} - \text{conv}B)\,.$

**2°.** If $(\text{conv}\mathcal{A}) \cap (\text{conv}B) = \emptyset$, then $\text{SEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) = \text{Eff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) = \text{WEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) \neq \mathbf{E}$.

**3°.** $(\text{conv}\mathcal{A}) \cap (\text{conv}B) \neq \emptyset$ if and only if $\text{WEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) = \mathbf{E}$.

**4°.** $\text{rint}(\text{conv}\mathcal{A}) \cap \text{rint}(\text{conv}B) = \emptyset$ if and only if $\text{Eff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) = \text{SEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) \neq \mathbf{E}$.

**5°.** $\text{rint}(\text{conv}\mathcal{A}) \cap \text{rint}(\text{conv}B) \neq \emptyset$ if and only if $\text{Eff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) = \mathbf{E}$.

**6°.** $\text{WEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B}) = \{x \in \mathbf{E} \mid (\text{conv}\,\mathcal{A}) \cap \text{conv}(\{x\} \cup B) \neq \emptyset\}$.

**7°.** $\text{rint}(\text{SEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B})) = \{x \in \mathbf{E} \mid \text{rint}(\text{conv}\,\mathcal{A}) \cap \text{rint}(\text{conv}(\{x\} \cup B)) \neq \emptyset\}$.

**Remark 6.17** Notice that $\text{SEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B})$, $\text{Eff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B})$ and $\text{Eff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},B})$ are closed, convex sets.

By the next Corollary 6.18 and by Lemma 6.16 (applied for the set $B := \mathcal{B}$ as well as for each of the sets $B := \mathcal{B}^i$, $i \in I_l$), we obtain useful inner approximations (lower bounds) for sets of (strictly, weakly) Pareto efficient solutions of $(\text{POLP}^2_X(\mathcal{A},\mathcal{B}))$.

**Corollary 6.18** ([53]) The following assertions hold:

$$\mathrm{SEff}(X \mid \widehat{g}_{\mathcal{A},\mathcal{B}}) \supseteq [X \cap \mathrm{SEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},\mathcal{B}})] \cup \left[\bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{SEff}\left(\mathbf{E} \mid \widehat{g}^{\oplus_i}_{\mathcal{A},\mathcal{B}}\right)\right];$$

$$\mathrm{Eff}(X \mid \widehat{g}_{\mathcal{A},\mathcal{B}}) \supseteq [X \cap \mathrm{Eff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},\mathcal{B}})] \cup \left[\bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{Eff}\left(\mathbf{E} \mid \widehat{g}^{\oplus_i}_{\mathcal{A},\mathcal{B}}\right)\right];$$

$$\mathrm{WEff}(X \mid \widehat{g}_{\mathcal{A},\mathcal{B}}) \supseteq [X \cap \mathrm{WEff}(\mathbf{E} \mid \widehat{g}_{\mathcal{A},\mathcal{B}})] \cup \left[\bigcup_{i \in I_l} X \cap (\mathrm{bd}\, D_i) \cap \mathrm{WEff}\left(\mathbf{E} \mid \widehat{g}^{\oplus_i}_{\mathcal{A},\mathcal{B}}\right)\right].$$

Under the assumption (6.3), we actually have $X \cap (\mathrm{bd}\, D_i) = \mathrm{bd}\, D_i$ for every $i \in I_l$.

*Proof.* Directly follows by Corollary 3.22. □

**Example 6.19** In Example 6.10, we studied a planar constrained point-objective location problem (POLP$^2_X(\mathcal{A})$) involving the Euclidean norm $||\cdot||_2$, three given facilities

$$a^1, a^2, a^3 \in \mathbb{R}^2 = \mathbf{E},$$

and a feasible set

$$X = X_1 \cap X_2 = (\mathbb{R}^2 \setminus \mathrm{int}\, D_1) \cap (\mathbb{R}^2 \setminus \mathrm{int}\, D_2).$$

Now, we extend Example 6.10. We consider one additional attraction demand point $a^4 := d^1$ (i.e., $\mathcal{A} = \{a^1, \cdots, a^4\}$) and we introduce one repulsion point $b^1 := d^2$ (i.e., $\mathcal{B} = \{b^1\}$). So, our location problem is of type (POLP$^2_X(\mathcal{A}, \mathcal{B})$) with $m = 4$, $k = 1$, and $l = 2$.

In our model, we assume that a certain *minimum safe distance* between the new facility $x \in \mathbb{R}^2$ and the *undesirable facility* $b^1$ (e.g., *polluting factory*) is maintained. This is ensured in our model by considering a forbidden region int $D_2$ around the point $b^1$. Moreover, in our example, the new facility $x \in \mathbb{R}^2$ can not be located in a region around the *attraction facility* $a^4$ (e.g., due to certain territorial circumstances), again modeled by a certain forbidden region int $D_1$ around the point $a^4$.

Now, we are interested in computing Pareto efficient solutions for the problem (POLP$^2_X(\mathcal{A}, \mathcal{B})$). The objective functions considered in our approach are given by

$$\begin{aligned}
\widehat{g}_{\mathcal{A},\mathcal{B}}(x) &= \left(||x - a^1||_2, \cdots, ||x - a^4||_2, -||x - b^1||\right);\\
\widehat{g}^{\oplus_1}_{\mathcal{A},\mathcal{B}}(x) &= \left(||x - a^1||_2, \cdots, ||x - a^4||_2, -||x - b^1||, -||x - a^4||\right);\\
\widehat{g}^{\oplus_2}_{\mathcal{A},\mathcal{B}}(x) &= \left(||x - a^1||_2, \cdots, ||x - a^4||_2, -||x - b^1||, -||x - b^1||\right).
\end{aligned}$$

Due to $b^1 \notin \mathrm{conv}\mathcal{A}$, in view of Lemma 6.16 (1°, 2°), we get

$$\mathrm{Eff}(\mathbb{R}^2 \mid \widehat{g}_{\mathcal{A},\mathcal{B}}) = \mathrm{conv}\mathcal{A} + \mathrm{cone}\left(\mathrm{conv}\mathcal{A} - b^1\right).$$

Moreover, since $\mathrm{int}(\mathrm{conv}\mathcal{A}) \cap \mathrm{rint}(\mathrm{conv}\{b^1, a^4\}) \neq \emptyset$, it follows

$$\mathrm{Eff}\left(\mathbb{R}^2 \mid \widehat{g}^{\oplus_1}_{\mathcal{A},\mathcal{B}}\right) = \mathbb{R}^2$$

by Lemma 6.16 (4°). It is easily seen that

$$\mathrm{Eff}\left(\mathbb{R}^2 \mid \widehat{g}^{\oplus_2}_{\mathcal{A},\mathcal{B}}\right) = \mathrm{Eff}(\mathbb{R}^2 \mid \widehat{g}_{\mathcal{A},\mathcal{B}}).$$

We conclude that

$$\mathrm{Eff}_{\mathrm{iapp}} := \left[X \cap \mathrm{Eff}(\mathbb{R}^2 \mid \widehat{g}_{\mathcal{A},\mathcal{B}})\right] \cup [\mathrm{bd}\, D_1] \cup \left[(\mathrm{bd}\, D_2) \cap \mathrm{Eff}\left(\mathbb{R}^2 \mid \widehat{g}^{\oplus_2}_{\mathcal{A},\mathcal{B}}\right)\right]$$

is an inner approximation for the set $\mathrm{Eff}(X \mid \widehat{g}_{\mathcal{A},\mathcal{B}})$, taking into account Corollary 6.18.

Figure 6.4 shows the sets $\mathrm{Eff}\left(\mathbb{R}^2 \mid \widehat{g}^{\oplus_1}_{\mathcal{A},\mathcal{B}}\right)$ (upper left image), $\mathrm{Eff}\left(\mathbb{R}^2 \mid \widehat{g}^{\oplus_2}_{\mathcal{A},\mathcal{B}}\right)$ (upper right image), $\mathrm{Eff}(\mathbb{R}^2 \mid \widehat{g}_{\mathcal{A},\mathcal{B}})$ (lower left image), and $\mathrm{Eff}_{\mathrm{iapp}}$ (lower right image).

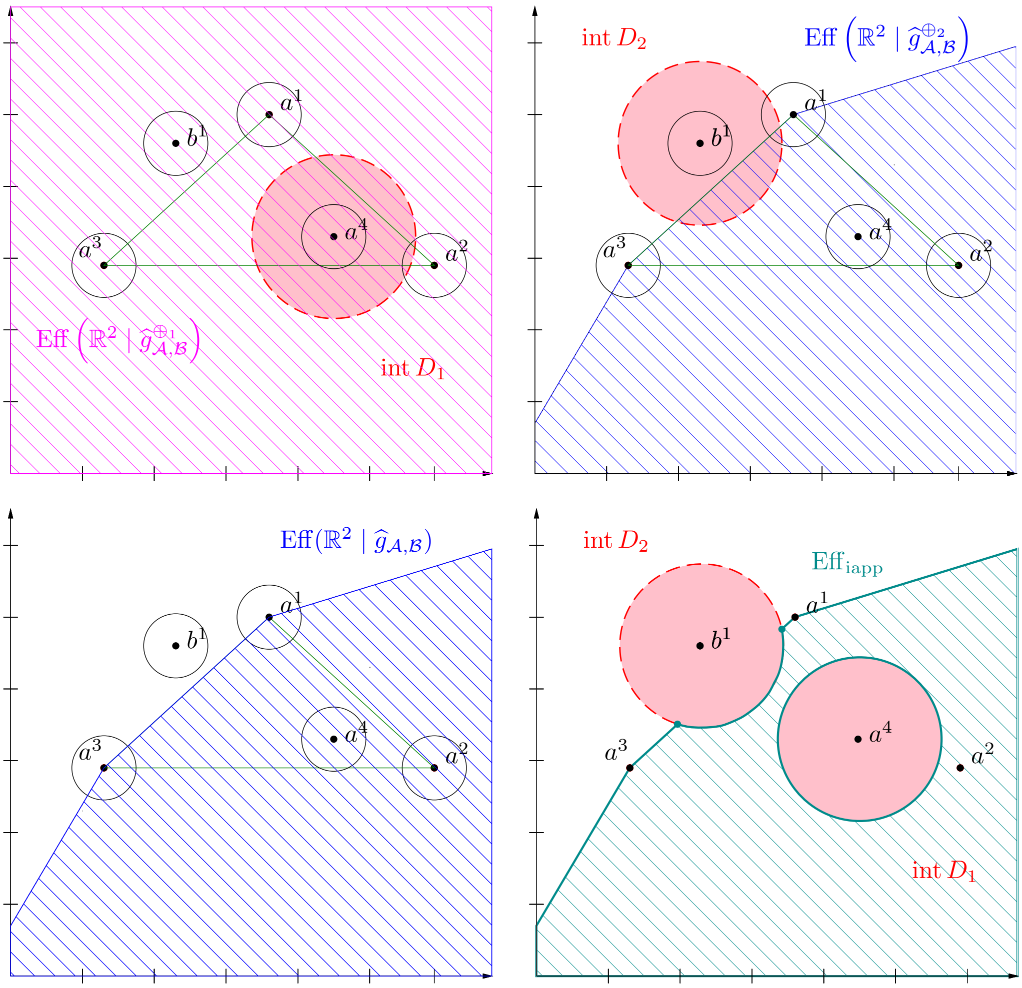


Figure 6.4: Computation of Pareto efficient solutions for the nonconvex location problem $(\mathrm{POLP}^2_X(\mathcal{A},\mathcal{B}))$ with $m = 4$, $k = 1$ and $l = 2$.

## 6.4 Concluding remarks

In this chapter, we applied our results to a special multi-objective location problem (known as *point-objective location problem*) that consists of locating a new facility in a continuous location space (a finite-dimensional Hilbert space) in the presence of a finite number of demand points. For the choice of the new location point, we took into consideration some forbidden regions that are given by open balls (defined with respect to the underlying norm). For such a nonconvex location problem, under the assumption that the forbidden regions are pairwise disjoint, we characterized completely the set of (strictly, weakly) Pareto efficient solutions by using our penalization approach from Chapters 2 and 3, and results obtained by Jourani, Michelot and Ndiaye [67].

It is important to mention that our approach relies essentially on the fact that the objective function in $(\mathrm{POLP}^2_X(\mathcal{A}))$ as well as the unit balls $D_1, \cdots, D_l$ (see the assumptions given in (6.1)) are defined with respect to a norm induced by a scalar product. This ensures that we can apply the results derived by Jourani, Michelot and Ndiaye [67].

It would be interesting to study other types of balls $D_1, \cdots, D_l$, for instance balls defined with respect to a polyhedral norm $\mu : \mathbf{E} \to \mathbb{R}$. Therefore, in a forthcoming work, we will analyze a *planar point-objective location problem involving a polyhedral norm* $\eta : \mathbb{R}^2 \to \mathbb{R}$. It is known that such a problem without considering constraints can be solved completely. In order to solve a corresponding constrained problem with a feasible set that is given by the complement of a finite union of open balls with respect to a polyhedral norm $\mu : \mathbb{R}^2 \to \mathbb{R}$, we have to compute the set of (strictly, weakly) Pareto efficient solutions of the problem

$$\begin{cases} \big(\eta(x-a^1), \cdots, \eta(x-a^m), -\mu(x-d^i)\big) \to \min \\ x \in \mathbf{E} = \mathbb{R}^2. \end{cases}$$

Notice that this problem includes only one repulsive demand point, namely the point $d^i$. Hence, as mentioned by Jourani, Michelot and Ndiaye [67] in their conclusion, some results for the polyhedral case in presence of only one repulsive demand point could be expected.

Since *constrained point-objective location problems involving attraction points and repulsion points* are of great practical relevance, the analysis given in Section 6.4 could be extended with the aim to find representations (not only inner or outer approximations) for the sets of (strictly, weakly) Pareto efficient solutions of the nonconvex location problem $(\mathrm{POLP}^2_X(\mathcal{A}, \mathcal{B}))$.

# Conclusions

This thesis studies constrained multi-objective optimization problems involving componentwise generalized-convex (semi-strictly quasi-convex, quasi-convex, or explicitly quasi-convex) vector-valued objective functions that are acting between a real linear topological pre-image space and a finite-dimensional image space. In particular, the case of nonconvex constraints is analyzed in detail. Solutions of the multi-objective optimization problems are defined with respect to the well-known concepts of (strict, weak, proper) Pareto efficiency.

In the following, we highlight some of our new results:

- We derived a new vectorial penalization approach for multi-objective optimization problems with generalized-convexity assumptions on the objective functions. We showed that the set of (strictly, weakly) Pareto efficient solutions of such a problem with a not necessarily convex feasible set can be computed completely by using the sets of (strictly, weakly) Pareto efficient solutions of at most two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set (see Chapter 2). Our approach relies on the fact that the original feasible set can be described by using level sets of a certain scalar penalization function.

- In particular, we obtained useful relationships between constrained and unconstrained multi-objective optimization. Our results show that constrained generalized-convex multi-objective optimization (involving convex constraints) is in a certain sense equivalent to unconstrained generalized-convex multi-objective optimization.

- By the application of our penalization approach, we succeeded to characterize solutions sets for two classes of nonconvex constrained multi-objective optimization problems (the feasible set is given by a finite union of convex sets or by the whole pre-image space excepting some forbidden regions that are given by convex sets; see Chapter 3) in terms of the solutions sets of corresponding unconstrained multi-objective optimzation problems.

- For special classes of multi-objective location problems, it is known that the so-called *projection property* holds, i.e., the set of Pareto efficient solutions of the constrained problem with a closed, convex feasible set $X$ can be obtained by projecting the set of Pareto efficient solutions of the corresponding unconstrained problem onto $X$ (see Chapter 4). We pointed out that our penalization approach is in particular very useful in the case that the projection property does not hold (e.g., for the class of point-objective location problems if the distances are induced by a not strictly convex norm; see Remark 4.7). Of course, our approach can also be used for problems involving nonconvex constraints, hence it opens the way for developing algorithms for such problems (as illustrated for a particular nonconvex multi-objective location problem in Chapter 6).

- We showed that the set of (strictly, weakly) Pareto efficient solutions of a problem with closed, convex constraints that belongs to a well-known class of multi-objective location problems (e.g., *point-objective location problems*, *multi-objective min-sum location problems*, *multi-objective min-max location problems*, *convex multi-objective ordered median problems*) can be completely characterized in terms of the sets of (strictly, weakly) Pareto efficient solutions of at most two corresponding unconstrained multi-objective location problems of the same class (see Chapter 4).

- In Chapter 5, we studied *unconstrained planar point-objective location problems* where the distances between points are defined by means of the Manhattan norm. By characterizing the nonessential objectives of such location problems and, by eliminating them, we developed an effective algorithm (the *Rectangular Decomposition Algorithm*) for generating the whole set of Pareto efficient solutions as the union of a special family of rectangles and line segments.

- We analyzed *point-objective location problems* in finite-dimensional Hilbert spaces involving multiple forbidden regions (see Chapter 6). For the choice of the new location point, we are taking into consideration a finite number of forbidden regions that are given by open balls (defined with respect to the underlying norm). For such a nonconvex multi-objective location problem, under the assumption that the forbidden regions are pairwise disjoint, we succeeded to give complete geometrical descriptions for the sets of (strictly, weakly) Pareto efficient solutions.

The results of this thesis are useful for deriving solution procedures for different classes of nonconvex multi-objective optimization problems. This will be the main topic for future research. Some further ideas for extending the results of this thesis are given in the concluding remarks of the preceding chapters.

## Summary of Contributions

This thesis is mainly based on four articles that are published in peer-reviewed international journals:

- [3] Alzorba S, Günther C, Popovici N, Tammer C (2017) A new algorithm for solving planar multi-objective location problems involving the Manhattan norm. European Journal of Operational Research 258(1):35–46;
- [53] Günther C (2018) Pareto efficient solutions in multi-objective optimization involving forbidden regions. Revista de Investigacion Operacional (to appear);
- [55] Günther C, Tammer C (2016) Relationships between constrained and unconstrained multi-objective optimization and application in location theory. Mathematical Methods of Operations Research 84(2):359–387;
- [56] Günther C, Tammer C (2018) On generalized-convex constrained multi-objective optimization. Pure and Applied Functional Analysis (accepted).

Let us outline the author's contributions to each chapter:

- A part of the facts that are given in the Introduction were already mentioned in [3, 55, 56, 53].
- Chapter 1 contains some basic facts that were also given in [3, 55, 56, 53]. However, in this thesis, we take an extended view on the theories of convex analysis, generalized-convex analysis, and multi-objective optimization. Section 1.3 is based on a joint work [56] with Christiane Tammer.
- Chapter 2 is based on joint works [55, 56] with Christiane Tammer. However, in this thesis, we consider functions that are defined on a nonempty subset $\mathcal{D}$ of the linear topological space $\mathbf{E}$. In [56], the case $\mathcal{D} = \mathbf{E}$ is considered, while in [55] it is assumed that $\mathcal{D} = \mathbf{E} = \mathbb{R}^n$. Section 2.7 contains some new observations.
- In Chapter 3, Section 3.1 is new, while Section 3.2 is part of [53] which is the sole work of the author.
- Chapter 4 is new, however within this chapter we use some ideas that were presented in the articles [3, 55, 56, 53].
- Chapter 5 (except of Section 5.5) is based on the joint work [3] with Shaghaf Alzorba, Nicolae Popovici and Christiane Tammer. Section 5.5 contains an example from [55].
- Chapter 6 is based on [53] which is the sole work of the author. Example 6.19 is new.

# List of References


[1] Aliprantis CD, Border KC (2006) Infinite dimensional analysis, 2nd edn. Springer, Berlin, Heidelberg

[2] Alzorba S, Günther C, Popovici N (2015) A special class of extended multicriteria location problems. Optimization 64(5):1305–1320

[3] Alzorba S, Günther C, Popovici N, Tammer C (2017) A new algorithm for solving planar multiobjective location problems involving the Manhattan norm. European Journal of Operational Research 258(1):35–46

[4] Apetrii M, Durea M, Strugariu R (2014) A new penalization tool in scalar and vector optimization. Nonlinear Analysis 107:22–33

[5] Arrow KJ, Barankin EW, Blackwell D (1953) Admissible points of convex sets. In: Kuhn HW, Tucker AW (Eds.), Contribution to the Theory of Games (pp. 87–91). Princeton University Press, Princeton, NJ

[6] Arrow KJ, Enthoven AC (1961) Quasi-concave programming. Econometrica 29:779–800

[7] Avriel M, Diewert WE, Schaible S, Zang I (1988) Generalized Concavity. Plenum Press, New York

[8] Bagdasar O, Popovici N (2015) Local maximum points of explicitly quasiconvex functions. Optimization Letters 9(4):769–777

[9] Bagdasar O, Popovici N (2017) Local maximizers of generalized convex vector-valued functions. Journal of Nonlinear and Convex Analysis 18(12):2229–2250

[10] Bagdasar O, Popovici N (2018) Unifying local-global type properties in vector optimization. Journal of Global Optimization (DOI: 10.1007/s10898-018-0656-8)

[11] Barbu V, Precupanu T (2012) Convexity and Optimization in Banach Spaces. Springer Netherlands

[12] Benker H, Hamel A, Tammer C (1997) An algorithm for vectorial control approximation problems. Multiple Criteria Decision Making 3–12

[13] Boţ RI (2003) Duality and optimality in multiobjective optimization. Dissertation, Technical University of Chemnitz

[14] Boţ RI, Vargyas E, Wanka G (2007) Conjugate duality for multiobjective composed optimization problems. Acta Mathematica Hungarica 116(3):177–196

[15] Bozau M (2017) Ein Lösungsverfahren für Centerprobleme mit anziehenden und abstoßenden Standorten. Master thesis, Martin Luther University Halle-Wittenberg

[16] Brimberg J, Juel H (1998) A minisum model with forbidden regions for locating a semi-desirable facility in the plane. Location Science 6(1-4):109–120

[17] Cambini A, Martein L (2009) Generalized Convexity and Optimization: Theory and Applications. Springer, Berlin, Heidelberg

[18] Carrizosa E, Conde E, Fernandez FR, Puerto J (1993) Efficiency in Euclidean constrained location problems. Operations Research Letters 14(5):291–295

[19] Carrizosa E, Conde E, Fernandez FR, Puerto J (1995) Planar point-objective location problems with nonconvex constraints: A geometrical construction. Journal of Global Optimization 6(1):77–86

[20] Carrizosa E, Plastria F (1996) A characterization of efficient points in constrained location problems with regional demand. Operations Research Letters 19(3):129–134

[21] Chalmet L, Francis RL, Kolen A (1981) Finding efficient solutions for rectilinear distance location problems efficiently. European Journal of Operational Research 6:117–124

[22] Clarke FH (1983) Optimization and Nonsmooth Analysis. Wiley-Interscience, New York

[23] De Finetti B (1949) Sulle stratificazioni convesse. Annali di Matematica Pura ed Applicata 30:173–183

[24] Dempe S, Eichfelder G, Fliege J (2015). On the effects of combining objectives in multi-objective optimization. Mathematical Methods of Operations Research 82(1):1–18

[25] Durea M, Strugariu R, Tammer C (2017) On some methods to derive necessary and sufficient optimality conditions in vector optimization. Journal of Optimization Theory and Applications 175(3):738–763

[26] Durier R (1990) On Pareto optima and the Fermat-Weber problem. Mathematical Programming 47(1-3):65–79

[27] Durier R, Michelot C (1986) Sets of efficient points in a normed space. Journal of Mathematical Analysis and Applications 117(2):506–528

[28] Edgeworth FY (1987) Mathematical psychics. University Microfilms International (the original edition in 1881)

[29] Ehrgott M (2005) Multicriteria Optimization, 2nd edn. Springer, Berlin, Heidelberg

[30] Ehrgott M, Nickel S (2002) On the number of criteria needed to decide Pareto optimality. Mathematical Methods of Operations Research 55(3):329–345

[31] Eichfelder G (2008) Adaptive Scalarization Methods in Multiobjective Optimization. Springer, Berlin, Heidelberg

[32] Eichfelder G (2014) Variable Ordering Structures in Vector Optimization. Springer, Berlin, Heidelberg

[33] Eichfelder G, Jahn J (2012) Vector optimization problems and their solution concepts. In: Ansari QH, Yao J-C (Eds.), Recent Developments in Vector Optimization (pp. 1–275). Springer, Berlin, Heidelberg

[34] Engau A, Wiecek M (2008) Interactive coordination of objective decompositions in multiobjective programming. Management Science 54(7):1350–1363

[35] Eremin II (1966) Penalty method in convex programming. Soviet Math Dolk 8:459–462

[36] Farahani RZ, Hekmatfar M (2009) Facility Location: Concepts, Models, Algorithms and Case Studies. Physica, Heidelberg

[37] Farahani RZ, SteadieSeifi M, Asgari N (2010) Multiple criteria facility location problems: A survey. Applied Mathematical Modelling 34(7):1689–1709

[38] Fejer L (1922) Über die Lage der Nullstellen von Polynomen, die aus Minimumforderungen gewisser Art entspringen. Mathematische Annalen 85:41–48

[39] Fenchel W (1953) Convex Cones, Sets and Functions. Princeton Univ. Press, Princeton, NJ

[40] Fliege J (2007) The effects of adding objectives to an optimisation problem on the solution set. Operations Research Letters 35(6):782–790

[41] Flores-Bazán F (2004) Semistrictly quasiconvex mappings and non-convex vector optimization. Mathematical Methods of Operations Research 59(1):129–145

[42] Gal T, Hanne T (2006) Nonessential objectives within network approaches for MCDM. European Journal of Operational Research 168(2):584–592

[43] Gal T, Leberling H (1977) Redundant objective functions in linear vector maximum problems and their determination. European Journal of Operational Research 1(3):176-184

[44] Geoffrion AM (1968) Proper efficiency and the theory of vector maximization. Journal of Mathematical Analysis and Applications 22(3):618–630

[45] Gerstewitz (Tammer) C (1983) Nichtkonvexe Dualität in der Vektoroptimierung. Wiss. Zeitschr. TH Leuna-Merseburg, 25:357–364

[46] Gerstewitz C, Iwanow E (1985) Dualität für nichtkonvexe Vektoroptimierungsprobleme. Wiss. Z. Tech. Hochsch. Ilmenau 2:61–81

[47] Gerth (Tammer) C, Pöhler K (1988) Dualität und algorithmische Anwendung beim vektoriellen Standortproblem. Optimization 19:491–512

[48] Gerth (Tammer) C, Weidner P (1990) Nonconvex separation theorems and some applications in vector optimization. Journal of Optimization Theory and Applications 67(2):297–320

[49] Giorgi G, Guerraggio A, Thierfelder J (2004) Mathematics of Optimization - Smooth and Nonsmooth Case. Elsevier, Amsterdam

[50] Göpfert A, Riahi H, Tammer C, Zălinescu C (2003) Variational Methods in Partially Ordered Spaces. Springer, New York

[51] Gromicho J (1998) Quasiconvex Optimization and Location Theory. Springer US

[52] Günther C (2013) Dekomposition mehrkriterieller Optimierungsprobleme und Anwendung bei nicht konvexen Standortproblemen. Master thesis, Martin Luther University Halle-Wittenberg

[53] Günther C (2018) Pareto efficient solutions in multi-objective optimization involving forbidden regions. Revista de Investigacion Operacional (to appear)

[54] Günther C, Hillmann M, Tammer C, Winkler B (2015) Facility Location Optimizer (FLO) - A tool for solving location problems. http://www.project-flo.de

[55] Günther C, Tammer C (2016) Relationships between constrained and unconstrained multi-objective optimization and application in location theory. Mathematical Methods of Operations Research 84(2):359–387

[56] Günther C, Tammer C (2018) On generalized-convex constrained multi-objective optimization. Pure and Applied Functional Analysis (accepted)

[57] Hamacher HW (1995) Mathematische Lösungsverfahren für planare Standortprobleme. Vieweg, Braunschweig/Wiesbaden

[58] Hamacher HW, Nickel S (1995) Restricted planar location problems and applications. Naval Research Logistics 42:967-992

[59] Hamacher HW, Nickel S (1996) Multicriteria planar location problems. European Journal of Operational Research 94(1):66–86

[60] Hansen P, Peeters D, Thisse JF (1982) An algorithm for a constrained Weber problem. Management Science 28(11):1285-1295

[61] Hanson MA (1964) Bounds for functionally convex optimal control problems, Journal of Mathematical Analysis and Applications 8(1):84–89

[62] Hiriart-Urruty JB (1979) New concepts in nondifferentiable programming. Mémoires de la Société Mathématique de France 60:57–85

[63] Hiriart-Urruty JB, Lemaréchal C (1993) Convex Analysis and Minimization Algorithms I & II. Springer, Berlin, Heidelberg

[64] Jahn J (2011) Vector Optimization - Theory, Applications, and Extensions, 2nd edn. Springer, Berlin, Heidelberg

[65] Jahn J, Sachs E (1986) Generalized quasiconvex mappings and vector optimization. SIAM Journal on Control and Optimization 24:306–322

[66] Jeyakumar V, Yang XQ (1993) Convex composite multi-objective nonsmooth programming. Mathematical Programming 59(3):325–343

[67] Jourani A, Michelot C, Ndiaye M (2009) Efficiency for continuous facility location problems with attraction and repulsion. Annals of Operations Research 167:43–60

[68] Khan AA, Tammer C, Zălinescu C (2015) Set-valued Optimization: An Introduction with Applications. Springer, Berlin, Heidelberg

[69] Kaiser M (2015) Spatial uncertainties in continuous location problems. Dissertation, Bergische Universität Wuppertal

[70] Karamardian S (1967) Strictly quasi-convex (concave) functions and duality in mathematical programming. Journal of Mathematical Analysis and Applications 20(2):344–358

[71] Klamroth K (2002) Single-Facility Location Problems with Barriers. Springer, Berlin, Heidelberg

[72] Klamroth K, Tind J (2007) Constrained optimization using multiple objective programming. Journal of Global Optimization 37(3):325–355

[73] Klamroth K, Wiecek MM (2002) A bi-objective median location problem with a line barrier. Operations Research 50(4):670–679

[74] Kuhn HW (1967) On a pair of dual non linear programs. In: Abadie J (Ed.), Nonlinear Programming (pp. 38–54). North-Holland, Amsterdam

[75] Liu CG, Ng KF, Yang WH (2009) Merit functions in vector optimization. Mathematical Programming 119(2):215–237

[76] Löhne A (2011) Vector Optimization with Infimum and Supremum. Springer, Berlin, Heidelberg

[77] Löhne A, Wagner A (2017) Solving DC programs with a polyhedral component utilizing a multiple objective linear programming solver. Journal of Global Optimization 69(2):369–385

[78] Love RF, Morris JG (1975) A computation procedure for the exact solution of location-allocation problems with rectangular distances. Naval Research Logistics 22(3):441–453

[79] Lowe TJ, Thisse JF, Ward JE, Wendell RE (1984) On efficient solutions to multiple objective mathematical programs. Management Science 30(11):1346–1349

[80] Luc DT (1989) Theory of Vector Optimization. Springer, Berlin, Heidelberg

[81] Luc DT (1998) Generalized Convexity and Some Applications to Vector Optimization. Vietnam Journal of Mathematics 26(2):95–110

[82] Mäkelä MM, Nikulin Y (2008) Properties of efficient solution sets under addition of objectives. Operations Research Letters 36(6):718–721

[83] Mäkelä MM, Eronen V-P, Karmitsa N (2010) On generalized pseudo- and quasiconvexities for nonsmooth functions. TUCS Technical Report 989, Turku Centre for Computer Science, Turku

[84] Mäkelä MM, Eronen V-P, Karmitsa N (2014) On nonsmooth optimality conditions with generalized convexities. Optimization in Science and Engineering 333-357

[85] Malivert C, Boissard N (1994) Structure of efficient sets for strictly quasi-convex objectives. Journal of Convex Analysis 1(2):143–150

[86] Mangasarian OL (1965) Pseudo-convex functions. Journal of the Society for Industrial and Applied Mathematics Series A Control 3(2):281–290

[87] Mordukhovich SB, Nguyen Mau Nam (2006) Subgradients of distance functions at out-of-set points. Taiwanese Journal of Mathematics 10(2):299–326

[88] Ndiaye M, Michelot C (1998) Efficiency in constrained continuous location. European Journal of Operational Research 104(2):288–298

[89] Nickel S (1995) Discretization of Planar Location Problems. Shaker, Aachen

[90] Nickel S, Puerto J (2005) Location Theory: A Unified Approach. Springer, Berlin Heidelberg

[91] Nickel S, Puerto J, Rodríguez-Chía AM (2003) An approach to location models involving sets as existing facilities. Mathematics of Operations Research 28(4):693–715

[92] Nickel S, Puerto J, Rodríguez-Chía AM (2005) MCDM location problems. In: Figueira J, Greco S, Ehrgott M (Eds.), Multiple criteria decision analysis: State of the art surveys (pp. 761–795). Springer, New York

[93] Nickel S, Puerto J, Rodríguez-Chía AM (2015) Location problems with multiple criteria. In: Laporte G, Nickel S, Saldanha da Gama F (Eds.), Location Science (pp. 205-247). Springer International Publishing, Berlin

[94] Nickel S, Puerto J, Rodríguez-Chía AM, Weissler A (2005) Multicriteria planar ordered median problems. Journal of Optimization Theory and Applications 126(3):657–683

[95] Ohsawa Y, Ozaki N, Plastria F, Tamura K (2007) Quadratic ordered median location problems. Journal of the Operations Research Society of Japan 50(4):540–562

[96] Pareto V (1964) Cours d'Économie Politique. Libraire Droz, Genève (the first edition in 1896)

[97] Pelegrín B, Fernández FR (1988) Determination of efficient points in multiple-Objective location problems. Naval Research Logistics 35(6):697–705

[98] Plastria F (1983) Continuous location problems and cutting plane algorithms. Dissertation, Vrije Universiteit Brussel

[99] Plastria F (1984) Localization in single facility location. European Journal of Operational Research 18(2):215–219

[100] Plastria F, Carrizosa E (1996) Geometrical characterization of weakly efficient points. Journal of Optimization Theory and Applications 90(1): 217–223

[101] Ponstein J (1967) Seven kinds of convexity. SIAM Review 9(1):115–119

[102] Popovici N (2005) Pareto reducible multicriteria optimization problems. Optimization 54(3):253–263

[103] Popovici N (2006) Structure of efficient sets in lexicographic quasiconvex multicriteria optimization. Operations Research Letters 34(2):142–148

[104] Popovici N (2008) Involving the Helly number in Pareto reducibility. Operations Research Letters 36(2):173–176

[105] Popovici (2016) The role of generalized convexity in vector optimization and related variational problems. Habilitation thesis, Babeş-Bolyai University Cluj-Napoca

[106] Puerto J, Fernández FR (1998) A convergent approximation scheme for efficient sets of the multi-criteria weber location problem. Top 6(2):195–204

[107] Puerto J, Fernández FR (1999) Multi-criteria minisum facility location problems. Journal of Multicriteria Decision Analysis 18:268-280

[108] Puerto J, Fernández FR (2000) Geometrical properties of the symmetrical single facility location problem. Journal of Nonlinear and Convex Analysis 1(3):321–342

[109] Puerto J, Rodríguez-Chía AM (2002) Geometrical description of the weakly efficient solution set for multicriteria location problems. Annals of Operations Research 111(1-4):181-196

[110] Puerto J, Rodríguez-Chía AM (2008) Quasiconvex constrained multicriteria continuous location problems: Structure of nondominated solution sets. Computers & Operations Research 35(3):750–765

[111] Rockafellar RT (1970) Convex Analysis. Princeton University Press, Princeton

[112] Rudin W (1973) Functional analysis. McGraw-Hill Inc, New York

[113] Schöbel A (1999) Locating Lines and Hyperplanes: Theory and Algorithms. Springer US

[114] Schöbel A (2015) Location of dimensional facilities in a Continuous Space. In: Laporte G, Nickel S, Saldanha da Gama F (Eds.), Location Science (pp. 135-175). Springer International Publishing, Berlin

[115] Singer I (1979) A Fenchel-Rockafellar type duality theorem for maximization. Bulletin of the Australian Mathematical Society 20(2):193–198

[116] Tammer C, Tammer K (1991) Duality results for convex vector optimization problems with linear restrictions. In: Davisson LD *et al.* (Eds.), System Modelling and Optimization. Lecture Notes in Control and Information Sciences, vol 180 (pp. 55–64). Springer, Berlin, Heidelberg

[117] Thisse JF, Ward JE, Wendell RE (1984) Some properties of location problems with block and round norms. Operations Research 32(6):1309–1327

[118] Toland JF (1978) Duality in nonconvex optimization. Journal of Mathematical Analysis and Applications 66(2):399–415

[119] Vargyas E (2004) Duality for convex composed programming problems. Dissertation, Technical University of Chemnitz

[120] Wagner A (2015) A new duality based approach for the problem of locating a semi-obnoxious facility. Dissertation, Martin Luther University Halle-Wittenberg

[121] Wagner A, Martínez-Legaz JE, Tammer C (2016) Locating a semi-obnoxious facility - a Toland-Singer duality based approach. Journal of Convex Analysis 23(4):1185–1204

[122] Wanka G (2000) Multiobjective control approximation problems: duality and optimality. Journal of Optimization Theory and Applications 105(2):457–475

[123] Wanka G, Boţ RI, Vargyas E (2003) Duality for the multiobjective location model involving sets as existing facilities. In: Pardalos PM, Tseveendorj I, Enkhbat R (Eds.), Optimization and Optimal Control (pp. 307–333). World Scientific, New Jersey

[124] Wanka G, Boţ RI, Vargyas E (2003) Duality for location problems with unbounded unit balls. European Journal of Operational Research 179(3):1252–1265

[125] Ward J (1989) Structure of efficient sets for convex objectives. Mathematics of Operations Research 14(2):249–257

[126] Weber A (1909) Über den Standort der Industrien, 1. Teil: Reine Theorie des Standortes. J. C. B. Mohr, Tübingen

[127] Wendell RE, Hurter AP (1973) Location theory, dominance and convexity. Operations Research 21(1):314–320

[128] Wendell RE, Hurter AP, Lowe TJ (1977) Efficient points in location problems. AIIE Transactions 9(3):238–246

[129] Ye JJ (2012) The exact penalty principle. Nonlinear Analysis 75(3):1642–1654

[130] Zaffaroni A (2003) Degrees of efficiency and degrees of minimality. SIAM Journal on Control and Optimization 42(3):1071–1086

[131] Zălinescu C (2002) Convex Analysis in General Vector Spaces. World Scientific Publishing, River Edge, NJ

[132] Zangwill WI (1967) Nonlinear programming via penalty functions. Management Science 13(5):344–358

# List of Figures

# Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit

**On Generalized-Convex Constrained Multi-Objective Optimization and Application in Location Theory**

selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

Halle (Saale), 24. Mai 2018

(Christian Günther)

# Lebenslauf

## Persönliche Angaben

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| Name: | Günther |
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## Beruf

seit 05/2018 — Wissenschaftlicher Mitarbeiter am Institut für Mathematik der Martin-Luther-Universität Halle-Wittenberg

- Mitarbeiter in der Arbeitsgruppe Optimierung von Prof. Dr. Christiane Tammer
- Arbeitsthema: “Vektor- und Mengenoptimierung (Vector and set optimization)”

11/2017 - 04/2018 — Wissenschaftlicher Mitarbeiter am Institut für Informatik der Martin-Luther-Universität Halle-Wittenberg

- Mitarbeiter in der Arbeitsgruppe Bioinformatik von Prof. Dr. Ivo Große
- Arbeitsthema: “Verallgemeinerte Entropie-Modelle und deren Anwendung in der DNA-Sequenzanalyse (Generalized entropy models in DNA sequence analysis)”

## Ausbildung

### Stipendien

| | |
|---|---|
| 06/2014 - 06/2017 | Förderung durch das Land Sachsen-Anhalt (Graduiertenstipendium) |
| seit 07/2014 | Reisestipendium des Allgemeinen Stiftungsfonds Theoretische Physik und Mathematik, Martin-Luther-Universität Halle-Wittenberg |

### Universität

seit 01/2014 — Promotionsstudium Mathematik an der Martin-Luther-Universität Halle-Wittenberg

- Thema der Dissertation: ”Verallgemeinert konvexe mehrkriterielle Optimierung und deren Anwendung in der Standorttheorie (On generalized-convex constrained multi-objective optimization and application in location theory)”
- Betreuung: Prof. Dr. Chr. Tammer

| | |
|---|---|
| 10/2011 - 09/2013 | Masterstudiengang Wirtschaftsmathematik<br>an der Martin-Luther-Universität Halle-Wittenberg<br>• Abschluss Master of Science (M.Sc.)<br>• Thema der Masterarbeit: “Dekomposition mehrkriterieller Optimierungsprobleme und Anwendung bei nicht konvexen Standortproblemen”<br>• Betreuung: Prof. Dr. Chr. Tammer |
| 10/2008 - 09/2011 | Bachelorstudiengang Wirtschaftsmathematik<br>an der Martin-Luther-Universität Halle-Wittenberg<br>• Abschluss Bachelor of Science (B.Sc.)<br>• Thema der Bachelorarbeit: “Standort-Medianprobleme mit variablen Anlagen”<br>• Betreuung: Prof. Dr. Chr. Tammer |

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|---|---|
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| 09/1998 - 06/2000 | Sekundarschule Heide-Nord 1, Halle (Saale) |
| 09/1994 - 06/1998 | Grundschule Lettin, Halle (Saale) |

## Software

| | |
|---|---|
| 2011 - heute | Veröffentlichung der Software “Facility Location Optimizer”:<br>Facility Location Optimizer (FLO) ist eine auf MATLAB basierende Software zur Lösung einer Vielzahl von skalaren und mehrkriteriellen Standortproblemen. Die Entwicklung der Software wurde 2011 im Rahmen meiner Bachelorarbeit begonnen. Über das Masterstudium einschließlich der Masterarbeit und über meine Promotionszeit wurde die Software von mir weiterentwickelt. Am 22. April 2015 wurde die Software veröffentlicht und ist nun unter<br>`http://www.project-flo.de`.<br>frei verfügbar. Am 12. Mai 2015 wurde die Software auf der MATLAB Expo in München vorgestellt. |

## Publikationen

| | |
|---|---|
| 2018 | C. Günther and N. Popovici, *New algorithms for discrete vector optimization based on the Graef-Younes method and cone-monotone sorting functions*, Optimization<br>DOI: 10.1080/02331934.2018.1474469 |
| 2018 | C. Günther, *Pareto efficient solutions in multi-objective optimization involving forbidden regions*, Revista de Investigacion Operacional (to appear)<br>`http://www.optimization-online.org/DB_HTML/2017/10/6306.html` |
| 2018 | C. Günther and C. Tammer, *On generalized-convex constrained multi-objective optimization*, Pure and Applied Functional Analysis (accepted)<br>`http://www.optimization-online.org/DB_HTML/2017/04/5941.html` |
| 2017 | S. Alzorba, C. Günther, N. Popovici and C. Tammer, *A new algorithm for solving planar multiobjective location problems involving the Manhattan norm*, European Journal of Operational Research, Volume 258, Issue 1, pp 35-46<br>DOI: 10.1016/j.ejor.2016.10.045 |
| 2016 | C. Günther and C. Tammer, *Relationships between constrained and unconstrained multi-objective optimization and application in location theory*, Mathematical Methods of Operations Research, Volume 84, Issue 2, pp 359-387<br>DOI: 10.1007/s00186-016-0547-z |
| 2015 | S. Alzorba, C. Günther and N. Popovici, *A special class of extended multicriteria location problems*, Optimization, Volume 64, Issue 5, pp 1305-1320<br>DOI: 10.1080/02331934.2013.869810 |
| 2012 | S. Alzorba and C. Günther, *Algorithms for multicriteria location problems*, Numerical Analysis and Applied Mathematics ICNAAM, AIP Conference Proceedings, Volume 1479, pp 2286-2289<br>DOI: 10.1063/1.4756650 |

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