An identification of groups of Lie type in parabolic characteristic 2

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Abstract

Let G be a K_2-group of parabolic characteristic 2 and H a subgroup of G such that $F^*(H)$ is a simple group of Lie type in characteristic 2. Assume further that for a Sylow 2-subgroup S of H, which is also a Sylow 2-subgroup of G, the following holds: For every non-trivial normal subgroup X of S, it is $\mathrm{N}_G(X)$ contained in H.

Then G equals H or G' is isomorphic to the alternating group A_9. So for $p = 2$ this thesis provides a generalization of Mohammad Reza Salarian's and Gernot Stroth's result "Existence of strongly p-embedded subgroups", see [SaSt], where the property of G being of local characteristic p is assumed.

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Chapter 1

Introduction

In this thesis the result of M. Salarian and G. Stroth from their article "Existence of strongly p-embedded subgroups" [SaSt], which is formulated under the assumption of local characteristic p, is generalized to parabolic characteristic in case of $p = 2$.
Therefore, the present paper is part of the so-called MSS-project. The MSS-project, named after Ulrich Meierfrankenfeld, Bernd Stellmacher and Gernot Stroth, can be seen as a revision project to the classification of the finite simple groups (CFSG).

A finite group is called simple if it has exactly two normal subgroups. Hence the trivial group $\langle 1 \rangle$ is not called simple and the only normal subgroups of a simple group are the trivial subgroup and the whole group itself. Given a finite group $G \neq \langle 1 \rangle$, for a maximal normal subgroup U_1 in G the factor group G/U_1 is simple. If U_1 is not trivial, a maximal normal subgroup U_2 of U_1 can be found and so on. Using this, for every non-trivial finite group G there is a number $n \in \mathbb{N}$ and a series of subnormal groups $G = U_0 \rhd U_1 \rhd \cdots \rhd U_{n-1} \rhd U_n = \langle 1 \rangle$ such that all factors U_{k-1}/U_k with $k \in \{1, \ldots, n\}$ are finite simple groups. These factors are called composition factors of G and the series is called a composition series of G. The Jordan-Hölder Theorem states, that for a given finite group G the length of every composition series is uniquely determined. Furthermore, the composition factors are, up to isomorphisms, uniquely determined, including their multiplicity in the above composition series. The converse of the Jordan-Hölder Theorem is not true, as there are non-isomorphic groups having isomorphic composition factors with the same multiplicities. Nevertheless, the classification of finite simple groups is essential to understand the structure of finite groups.

The task of classifying all finite simple groups (CFSG) is deemed to be completed since the end of the 20th century. The result of this classification says that a finite simple group is either a cyclic group of prime order, an alternating group A_n for $n \geq 5$, a simple group of Lie type or one of 26 so-called sporadic groups. But due to the length and complexity of the original proof, which consists of a large number of different books and articles, there is the ambition to publish the proof in a unified form, thereby closing possibly existing gaps and correcting mistakes. The MSS-project is a revision of parts of the CFSG and this thesis is part of it.

For further argumentation the following definitions are needed: Let p be a prime and G a finite group whose order is divisible by p. Then $O_p(G)$ denotes the largest normal p-subgroup in G. The normalizer $M := \mathrm{N}_G(P)$ of a non-trivial p-subgroup P of G is called a **p-local subgroup**. In particular, P is contained in $O_p(M)$. Furthermore, the elements in the center of a Sylow p-subgroup are called **p-central**.

Definition 1.1: *Let p be a prime and G a finite group whose order is divisible by p.*

(a) If U is isomorphic to a factor group of a subgroup of G, U is called a ***section*** *of G. Hence composition factors are examples of simple sections.*

(b) The group G is of ***characteristic*** *p if $\mathbb{C}_G(O_p(G)) \leq O_p(G)$ or if, equivalently to this, $F^*(G) = O_p(G)$ holds.*

(c) If all p-local subgroups are of characteristic p, G is of ***local characteristic*** *p.*

(d) Let U be a subgroup of G. If U contains a Sylow p-subgroup of G, then U is called a ***parabolic subgroup****.*

(e) Let S be a Sylow p-subgroup of G. The group G is of ***parabolic characteristic*** *p if all p-local subgroups containing S, i.e. all p-local and parabolic subgroups, are of characteristic p.*

(f) The group G is called a ***K_p-group*** *if every simple section of every p-local subgroup of G is a known finite simple group, that means cyclic of prime order, an alternating group, a simple group of Lie type or a sporadic group.*

Every group G of characteristic p is of local characteristic p, which implies G being of parabolic characteristic p. The first implication is quite elementary and can be shown using Thompson's $A \times B$-Lemma, see for example (31.16) in [Asc1]. The second implication is obvious from the above definition. Typical examples for groups of local characteristic p are the groups of Lie type over a field of characteristic p, see 3.1.4 in [GLS3]. But also some of the sporadic groups are of local characteristic p for a certain prime; for example J_4, M_{24} and Th are of local characteristic 2, see for example page 2 in [MSS1].

The prime 2 plays a key role in the proof of the classification of the finite simple groups: By the famous Odd Order Theorem of Walter Feit and John Thompson [FeTh], published in 1963, every non-abelian finite simple group contains an involution, i.e. an element of order

2. Therefore, every non-abelian finite simple group contains a 2-local subgroup, since the centralizer of an involution is a 2-local subgroup. Besides that, a result of Richard Brauer and Kenneth Fowler [BrFo] from 1955 shows that there are only finitely many simple groups such that the centralizer of an involution is isomorphic to a given finite group. This led to the idea of classifying the finite simple groups by the centralizers of their involutions.

To do so, two cases are considered: A simple group G can be of local characteristic 2 or not. In the latter case, the original proof of the CFSG contains an effortful but reliable strategy to classify these groups: Either the 2-rank of the groups is small enough to determine the group effectively or there exists an involution whose centralizer has a known simple group as a component, which also leads to an identification of the simple group.
In the case of G being of local characteristic 2, there is no standard procedure, the situation is more complicated. The original proof of the CFSG deals with these groups by finding an odd prime in order to identify the groups by centralizers of prime elements of the suitably chosen odd prime.

At this point the MSS-project steps in with the aim, to understand the p-local structure of finite simple groups of local characteristic p and, in particular, classifying the simple groups of local characteristic 2. For this purpose the property of local characteristic 2 is used to treat these groups uniformly. To clarify the intention of this thesis, a short overview over the strategy of the MSS-project is given, see [MSS1] for a more detailed description of the MSS-project:

Let p be a prime and let G be a non-abelian finite simple group of local characteristic p. As the generic example of a non-abelian finite simple group of local characteristic p is a group of Lie type over a field of characteristic p, the aim is to show that G is isomorphic to such a simple group of Lie type and to classify the occurring exceptions.
To do so, the structure of some of the maximal p-local subgroups which contain a fixed Sylow p-subgroup is investigated. A subgroup H, which is generated by two of these maximal p-local and parabolic subgroups of G, is constructed. This subgroup itself is not contained in a p-local subgroup. By construction of H, the prime p does not divide the index $|G : H|$, see [PPSS]. It can be deduced, using a result of U. Meierfrankenfeld, G. Stroth and R. Weiss [MSW], that in the typical case H is a group of automorphisms of a group of Lie type over a field of characteristic p. So one can assume that $F^*(H)$ is a group of Lie type over a field of characteristic p.

The aim is to show that H equals G and if not, to determine the exceptions. To do so, the situation is restricted to the cases where the Lie rank of $F^*(H)$ is at least 2. In case of $p = 2$, this provides an identification of the simple groups of local characteristic 2 with a finite group of Lie type over a field of even characteristic, compare [SaSt].

In the MSS-project the concept of a so-called large subgroup is frequently used. So we define:

Definition 1.2: *Let p be a prime and G a finite group. A p-subgroup Q of G is called a* ***large subgroup*** *of G if the following conditions hold:*

- $\mathbb{C}_G(Q) \leq Q$ *and*
- *for all* $1 \neq U \leq \mathbb{C}_G(Q)$ *it is* $\mathrm{N}_G(U) \leq \mathrm{N}_G(Q)$.

A large subgroup Q of G is therefore large in the sense that the normalizer $\mathrm{N}_G(Q)$ contains the normalizer of every non-trivial subgroup of the center of Q. In particular, $\mathrm{N}_G(Q)$ is a parabolic and p-local subgroup of G. Furthermore, by Lemma 1.13 in [MSS2], the existence of a large p-subgroup in G implies G being of parabolic characteristic p.

If Q is a large p-subgroup in G, then $\mathbb{C}_G(O_p(\mathrm{N}_G(Q))) \leq \mathbb{C}_G(Q) \leq Q \leq O_p(\mathrm{N}_G(Q))$ and for each subgroup $1 \neq U \leq \mathbb{C}_G(O_p(\mathrm{N}_G(Q)))$ also $\mathrm{N}_G(U) \leq \mathrm{N}_G(Q) \leq \mathrm{N}_G(O_p(\mathrm{N}_G(Q)))$ holds. Hence $O_p(\mathrm{N}_G(Q))$ is also a large p-subgroup in G.
So, without loss of generality, we may assume in the following for any occurring large p-subgroup Q of G that $Q = O_p(\mathrm{N}_G(Q))$ holds.

The property for a group to be of parabolic characteristic p is weaker than being of local characteristic p and the latter one is often harder to prove. In addition, every simple group, which is not of parabolic characteristic 2, contains a 2-central involution whose centralizer is not of characteristic 2. This is a quite limiting property. These arguments justify the ambition to generalize the assumption of G being of local characteristic p to parabolic characteristic p.

By a result of C. Parker, G. Pientka, A. Seidel and G. Stroth, which is submitted for publication, the following statements hold for $p = 2$, see Main Theorem 1 in [PPSS]:

Theorem 1.3 ([PPSS]): Let G be a finite group, S a Sylow 2-subgroup of G and let H be a subgroup of G with $S \leq H$. Additionally, let the following three conditions hold:

- Let $F^*(H)$ be a group of Lie type in characteristic 2 and with Lie rank at least 2. Further assume $H = \mathrm{N}_G(F^*(H))$;
- it is G a K_2-group;
- it is G of parabolic characteristic 2.

Then either $\mathrm{N}_G(E) \leq H$ holds for all non-trivial, normal subgroups $E \trianglelefteq S$ or $(F^*(H), F^*(G))$ is one of the following pairs:

$$(U_4(2), L_4(3)),\ (L_4(2), A_{10}),\ (Sp_4(2)', Mat(11)),$$

$$(L_3(4), Mat(23)),\ (G_2(2)', G_2(3)) \text{ or } (P\Omega_8^+(2), P\Omega_8^+(3)).$$

Considering this result, in order to prove $G = H$, we work in this thesis under the following hypothesis:

Hypothesis 1.4: (a) Let G be a finite K_2-group and let H be a subgroup of G such that $2 \nmid |G : H|$ holds. Further let $F^*(H)$ be a simple group of Lie type over a field of characteristic 2. So H is a group of automorphisms of a simple group of Lie type.

(b) Let G be a group of parabolic characteristic 2.

(c) Let S be a Sylow 2-subgroup of H. For every subgroup $\langle 1 \rangle \neq Y \trianglelefteq S$, we assume $\mathrm{N}_G(Y) \leq H$.

Under the stronger assumption of G being a group of local characteristic 2, instead of parabolic characteristic 2, the result $G = H$ follows by the article "Existence of strongly p-embedded subgroups" of M. Salarian and G. Stroth [SaSt]. The two authors also show $G = H$ for an arbitrary prime p, whereas for odd primes, G is supposed to be a K_2-group and a K_p-group. Additionally, M. Salarian and G. Stroth require $F^*(H)$ to be of Lie rank at least 2 for $p = 2$ and at least 3 for p being odd.

To generalize their result in case $p = 2$ from G being of local characteristic to parabolic characteristic is the aim of this thesis. Hence we prove the following result:

Theorem 1.5: Suppose that Hypothesis 1.4 holds.

Then $G = H$ or $F^*(G) \cong A_9$ follows.

In the following chapter some preliminary group theoretical results, in particular, some facts considering simple groups of Lie type, are collected.
In Chapter 3 some technical and helpful results are provided which are needed in the following. In particular, we show that for each 2-central involution s, the centralizer $\mathbb{C}_G(s)$ is contained in H and that $F^*(G)$ must be a simple group.
Using this, we deduce in the following chapters that $H \cap F^*(G)$ controls the $F^*(G)$-fusion of such a 2-central involution to obtain with a result due to D. Holt, see Lemma 2.16 in [PaSt2], that either G equals H or that $F^*(H)$ is isomorphic to the alternating group A_8 as a point stabilizer in $F^*(G) \cong A_9$.
In Chapter 4 this is done for $F^*(H)$ being isomorphic to a linear, symplectic or unitary simple group and also for $F^*(H)$ being isomorphic to one of the simple exceptional groups of Lie type $F_4(2^f)$, ${}^2F_4(2^f)'$ and $G_2(2^f)'$.
For $F^*(H)$ being isomorphic either to a simple orthogonal group in even dimension or to $E_6(2^f)$, ${}^2E_6(2^f)$, $E_7(2^f)$, $E_8(2^f)$ or to ${}^3D_4(2^f)$, we consider a large 2-subgroup Q. We prove in Chapter 5 that $\mathbb{C}_G(t)$ is contained in H for every involution $t \in Q$. This can be used

to show again that $H \cap F^*(G)$ controls the $F^*(G)$-fusion of a 2-central involution. Then, again by using Holt's result, we have that G equals H. If $F^*(H)$ is isomorphic to $E_6(2^f)$, ${}^2E_6(2^f)$, $E_7(2^f)$, $E_8(2^f)$ or to ${}^3D_4(2^f)$, this is done in Chapter 6, while Chapter 7 deals with the concluding treatment of $F^*(H) \cong \Omega_{2n}^{\pm}(2^f)$.

Chapter 2

Preliminaries

The notation in this thesis is mostly standard, nevertheless, the definitions of frequently used terms are given in this chapter. Also some basic results, which are needed for the argumentation, are stated here, whereas elementary and well-known results, as for example the Three Subgroups Lemma or Frattini's argument, are not listed, but used without reference. In this chapter a brief introduction to the topic of simple groups of Lie type is given, following [GLS3] in notation and argumentation.

2.1 Notation and general group theoretical results

Notation 2.1: Let G be a finite group. The following terminology is used in this text:

- The notation 1, or more explicit 1_G, is used for the identity element as well as for the trivial subgroup $\langle 1 \rangle$ of G. The set $G \setminus \{1\}$ is denoted by $G^{\#}$.
- The order of G is denoted by $|G|$ and for an element $g \in G$, the order of g is written as $\mathrm{o}(g)$.
- With $\gcd(m, n)$, or in unambiguous cases just (m, n), we denote the greatest common divisor of two integers m and n.
- For $g, h \in G$ and subsets $A, B \subseteq G$, we define $g^h := h^{-1}gh$ and $A^B := \{a^b \mid a \in A, b \in B\}$. If, for elements g, $h \in G$, there is an element $x \in H \leq G$ with $g^x = h$, the elements g and h are called conjugate or fused in H. This is denoted by $g \sim_H h$.
- For $U \leq H \leq G$, it is said that H controls fusion in U with respect to G, or shorter, that H controls the G-fusion in U, whenever two elements g, $h \in U$ which are fused in G are also fused in H. This is when $u^G \cap U = u^H$ holds for all $u \in U$.

- Let $U \leq H \leq G$. Then U is called weakly closed in H with respect to G if $U^g \leq H$ implies $U^g = U$ for all $g \in G$.

- With $\mathrm{N}_G(U)$ we denote the normalizer, with $\mathbb{C}_G(U)$ the centralizer of a subgroup U of G. It is $\mathrm{Z}(G)$ the center of G. As usual, it is $\mathrm{Aut}(G)$ the full automorphism group of G, $\mathrm{Inn}(G)$ the group of inner automorphisms and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ the outer automorphism group of G. For $H \leq G$, the group $\mathrm{Aut}_G(H)$ consists of the automorphisms of H which are induced by elements of G. For a simple group L, we identify L and $\mathrm{Inn}(L)$, so we can write, by abuse of notation, $L \leq \mathrm{Aut}(L)$.

- If A and B are subgroups of G, then $A * B$ is the central product of two subgroups A and B of G, and $A \times B$ is the direct product of these groups. With $A\colon B$ we denote a split extension, which is a semidirect product of a normal subgroup A and a subgroup B, whereas $A^{\bullet}B$ denotes a non-split extension with normal subgroup A and factor group B. If the extension is unspecified, it is merely denoted by AB or $A \cdot B$. If the underlying operations are unambiguous, the wreath product of the group A by the group B is denoted by $A \wr B$.

- Let $\mathbb{P}$ be the set of prime numbers. For $\pi \subseteq \mathbb{P}$ the set π' is defined as $\mathbb{P} \backslash \pi$. Furthermore, $\pi(G)$ is the set of prime numbers which the order of G is divisible by. The group G is a π-group if and only if $\pi(G) \subseteq \pi$. If G is a $\{p\}$-group, we say that G is a p-group.

- With $|G|_p$ we denote the greatest power of a prime p which divides $|G|$. The p-rank of G, which is defined as the logarithm (to base p) of the order of the largest elementary abelian p-subgroup of G, is denoted by $m_p(G)$.

- If p is a prime, $\mathrm{Syl}_p(G)$ is the set of all Sylow p-subgroups of G.

- For $\pi \subseteq \mathbb{P}$ the largest normal π-subgroup contained in G is denoted by $O_\pi(G)$, and $O^\pi(G)$ is the smallest normal subgroup in G for which $G/O^\pi(G)$ is a π-group. It is $O_\pi(G)$ called π-radical and $O^\pi(G)$ is called π-residue.
  In particular, the $2'$-radical $O_{2'}(G)$ is the largest normal subgroup of odd order and it is often referred to as $O(G)$. For a prime number p, $O_p(G)$ is the largest normal p-subgroup of G and $O^p(G)$ is the smallest normal subgroup of G such that $G/O^p(G)$ is a p-group.

- Let p be a prime. The group G is called p-closed if $|\mathrm{Syl}_p(G)| = 1$. In this case, $O_p(G)$ is a Sylow p-subgroup of G.

- The commutator $[a, b]$ of elements a, $b \in G$ is defined as $a^{-1}b^{-1}ab$. The commutator subgroup $\langle [a, b] \mid a, b \in G \rangle$ is denoted by G'. For subsets A, B, $C \subseteq G$, we define $[A, B]$ as $\langle [a, b] \mid a \in A, b \in B \rangle$ and $[A, B, C] := [[A, B], C]$.

- A non-trivial group G is said to be perfect if and only if $G' = G$ holds.

- A perfect group G whose factor group $G/\operatorname{Z}(G)$ is simple is called quasisimple. The quasisimple subnormal subgroups of G are called components and $E(G)$ denotes the central product of the components of G.
- A finite group is called semisimple if and only if it is the central product of quasisimple groups.
- The Frattini subgroup of G is denoted by $\Phi(G)$, $F(G)$ is the Fitting subgroup and $F^*(G) = F(G) * E(G)$ is the generalized Fitting subgroup of G.
- The characteristic subgroup $\operatorname{Z}^*(G)$ of G is defined to be the full preimage of $\operatorname{Z}(G/O(G))$ in G.
- Let P be a p-subgroup of G for a prime number p. Then $J(P)$ denotes the Thompson subgroup of P. It is $J(P)$ generated by all elementary abelian subgroups of P which are of maximal p-rank.
- The subgroup of a p-group P which is generated by the elements of P of order p is denoted by $\Omega_1(P)$, or just by $\Omega(P)$.
- We say a group U is involved in G if U is a section of G, so if U is isomorphic to a factor group of a subgroup of G. And we write $U \lesssim G$ if U is isomorphic to a subgroup of G and say that U can be embedded into G.
- The symmetric group on n letters is denoted by S_n and the corresponding alternating group by A_n. A cyclic group of order n is denoted by Z_n and an elementary abelian p-group of order p^n by E_{p^n}. Additionally, D_{2^n} is a dihedral group of order 2^n and Q_{2^n} is a quaternion group of order 2^n. We also use the ATLAS-notation from [CoCu]; so for example, 2^{k+n} denotes a special group of order 2^{k+n} with elementary abelian center of order 2^k. It is in every case unambiguous, whether a group or an integer is meant by this notation.

Definition 2.2: *Let V be a vector space of finite dimension over a field k. An element $t \in GL(V)$ is called a **transvection** if and only if $[V,t] \subseteq \mathbb{C}_V(t)$ holds and $\mathbb{C}_V(t)$ is a subspace of V of codimension 1.*

The famous Odd Order Theorem of Walter Feit and John G. Thompson is stated here and will be used in the following without being explicitly mentioned.

Theorem 2.3: All finite groups of odd order are solvable.

Proof: See [FeTh]. □

A few more well-known results are stated in the following lemmas. Afterwards some definitions and statements concerning finite groups of Lie type are presented.

Lemma 2.4 (Coprime action): If a finite group A acts coprimely on a finite group G, the following statements hold:

(a) Let N be an A-invariant normal subgroup of G. Then $\mathbb{C}_{G/N}(A) = \mathbb{C}_G(A)N/N$. In particular, if A centralizes N and G/N, then its action on G is also trivial.

(b) $G = \mathbb{C}_G(A)[G, A]$.

(c) $[G, A, A] = [G, A]$.

(d) If G is abelian, then $G = \mathbb{C}_G(A) \times [G, A]$ holds.

(e) If A is abelian but not cyclic, then it is $G = \langle \mathbb{C}_G(a) \mid a \in A^{\#} \rangle$.

Proof: See chapters 8.2–8.4 in [KuSt]. $\square$

Lemma 2.5 ($\boldsymbol{A \times B}$-Lemma): Let $H = A \times B$ be a direct product of a p-group A and a p'-group B. Let G be a p-group such that H acts on G with $\mathbb{C}_G(A) \leq \mathbb{C}_G(B)$. Then B acts trivially on G.

Proof: See 8.2.8 in [KuSt]. $\square$

Lemma 2.6: Let S_n be the symmetric and A_n the alternating group, both acting on the set $\Omega = \{1, 2, \ldots, n\}$. Then the following holds:

(a) For $n \geq 3$, A_n is $(n-2)$-transitive on Ω.

(b) Let $G \leq S_n$ be primitive on Ω and let G contain a transposition, i.e. a 2-cycle. Then $G = S_n$ holds.

(c) Let $G \leq A_n$ be primitive on Ω and let G contain a 3-cycle. Then $G = A_n$ holds.

(d) Let $G \leq A_n$ be primitive on Ω such that $\mathrm{Stab}_G(\Delta)$ is transitive on $\Omega \setminus \Delta$ for a subset Δ of Ω with $\frac{n}{2} < |\Delta| < n$. Then $G = A_n$ follows.

Proof: See 9.7, 13.3 and 13.5 in [Wiel]. $\square$

The following two lemmas provide arguments concerning the fusion of involutions. The first lemma is called Thompson's Transfer Lemma and the second deals with control of fusion in the center of the Thompson subgroup.

Lemma 2.7 (Thompson Transfer): Let G be a finite group, S a Sylow 2-subgroup of G and assume $T \leq S$ with $|S:T| = 2$. If there is an involution $x \in S \setminus T$ with $x^G \cap T = \emptyset$, then G contains a subgroup of index 2. In particular, G has a normal subgroup of index 2.

Proof: See 12.1.1 in [KuSt]. □

Lemma 2.8: Let G be a finite group and S a Sylow 2-subgroup of G. Then $\mathrm{Z}(J(S))$ is a characteristic subgroup of S and $\mathrm{N}_G(J(S))$ controls the G-fusion in $\mathrm{Z}(J(S))$.

Proof: The Thompson subgroup $J(S)$, which by definition is generated by the elementary abelian subgroups of maximal rank in S, is characteristic in S. So $\mathrm{Z}(J(S))$ is a characteristic subgroup of S and $\mathrm{Z}(J(S))$ is the intersection of the elementary abelian subgroups in S, which are of maximal 2-rank.
Let t_1 and t_2 be elements of $\mathrm{Z}(J(S))$, for which there exists an element $g \in G$ with $t_1^g = t_2$. Then $J(S)^g \leq \mathbb{C}_G(t_1)^g = \mathbb{C}_G(t_1^g) = \mathbb{C}_G(t_2)$ and $J(S) \leq \mathbb{C}_G(t_2)$ hold.
Let P_1 and P_2 be Sylow 2-subgroups of $\mathbb{C}_G(t_2)$ such that we have $J(S)^g \trianglelefteq P_1$ and $J(S) \trianglelefteq P_2$. Then there is an element $\tilde{g} \in \mathbb{C}_G(t_2)$ such that $J(S)^{g\tilde{g}} = J(S)$ holds. Hence $g\tilde{g} \in \mathrm{N}_G(J(S))$ and $t_1^{g\tilde{g}} = t_2^{\tilde{g}} = t_2$ follow. So t_1 and t_2 are conjugate in $\mathrm{N}_G(J(S))$. □

The following three results are needed later on, but are presented here, as the statements are well-known.

Lemma 2.9: Let G be a finite group with a subgroup U and let p be a prime.
If $P \in \mathrm{Syl}_p(U)$ and $\mathrm{N}_G(P) \leq U$ hold, then P is a Sylow p-subgroup of G.

Proof: Let Q be a Sylow p-subgroup of $\mathrm{N}_G(P)$. Then PQ is a p-group, which is contained in Q, so $P \leq Q$ holds. Assume now $P \notin \mathrm{Syl}_p(G)$. Then there is a p-subgroup S with $P < \mathrm{N}_S(P) \leq \mathrm{N}_G(P)$. Together with $\mathrm{N}_G(P) \leq U$, this implies $P \notin \mathrm{Syl}_p(U)$, which is a contradiction. So P has to be a Sylow p-subgroup of G. □

Remark 2.10: Let G be a non-abelian quasisimple group. Then $\mathrm{Aut}(G) \lesssim \mathrm{Aut}(G/\mathrm{Z}(G))$ holds by the following: Every element $\alpha \in \mathrm{Aut}(G)$ normalizes $\mathrm{Z}(G)$ and induces an automorphism on $G/\mathrm{Z}(G)$. Assume, that there is an element $\alpha \in \mathrm{Aut}(G)$ such that α acts trivially on $G/\mathrm{Z}(G)$. Then $[G,\alpha] \leq \mathrm{Z}(G)$ implies $[G,\alpha,G] = 1$ and, as G is perfect, the Three Subgroups Lemma implies $[G,\alpha] = 1$. Hence $\mathrm{Aut}(G)$ can be embedded into $\mathrm{Aut}(G/\mathrm{Z}(G))$.

Lemma 2.11: Let $q = 2^f$ for $f \in \mathbb{N}$ and $2 \leq n \in \mathbb{N}$. If $(q,n) \neq (2,6)$, there is a prime number which divides $q^n - 1$ and does not divide $q^k - 1$ for every integer $1 \leq k < n$. A prime number with this property is called a **Zsigmondy prime**.

Proof: See [Zsig]. □

2.2 The simple groups of Lie type

In this section a short introduction to the theory of finite groups of Lie type is given. In particular, simple groups of Lie types over a field of characteristic p are defined and some properties and results are listed which are essential for the proof of the main theorem.

There are 16 families of finite groups of Lie type, each family consisting of infinitely many groups. The so-called classical groups, which are the linear, symplectic, unitary and orthogonal groups, cover 6 of these families. They can be interpreted as a section of the group of isometries of a finitely dimensional vector space preserving a suited bilinear or quadratic form, see for example [Wils].

In the following paragraphs a uniform approach to the simple groups of Lie type is given in terms of linear algebraic groups. This approach covers also the exceptional groups of Lie type and allows a short overview of Dynkin diagrams, root groups, the Lie rank and other terms and connections between them. The argumentation in this section follows mostly [GLS3].

Let p be a prime, $F := GF(p)$ the field of p elements and $\overline{F} := \overline{GF(p)}$ an algebraic closure of F. Furthermore let $\overline{K}$ be a **linear algebraic group** over $\overline{F}$ which is defined to be an affine algebraic variety and additionally a topological group with respect to the Zariski topology. A linear algebraic group $\overline{K}$ can be seen as a, with respect to the Zariski topology, closed subgroup of the linear algebraic group $GL_n(\overline{F})$ for $n \in \mathbb{N}$, where $GL_n(\overline{F})$ denotes the group of invertible $n \times n$ matrices such that all entries are from $\overline{F}$.

A morphism of algebraic groups is a group homomorphism which is additionally a morphism of algebraic varieties. According to this, an isomorphism of algebraic groups is a group isomorphism such that the map itself and its inverse are morphisms of algebraic groups. The algebraic groups and their morphisms build a category.

A connected algebraic group $\overline{K}$ is called **simple** if and only if $[\overline{K}, \overline{K}] \neq 1$ and if all proper, closed normal subgroups are finite and central. These definitions and facts are due to 1.1.6, 1.5.1, 1.7.1 and 1.7.2 in [GLS3]. From now on, $\overline{K}$ is assumed to be a simple algebraic group.

To classify the simple algebraic groups, the terminology of root systems is used. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean vector space equipped with a positive definite scalar product $(\cdot, \cdot)$. For v, $w \in \mathbb{R}^n$ such that $v \neq 0_{\mathbb{R}^n}$, the mapping $r_v \colon \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$r_v(w) = w - 2\frac{(v, w)}{(v, v)}v.$$

Hence r_v describes the reflection that fixes the hyperplane $\langle v \rangle^{\perp}$.

Definition 2.12 ([GLS3], 1.8.1): *A **root system** Σ is a finite set of elements of the Euclidean space $\mathbb{R}^n$ with the following properties:*

- $0_{\mathbb{R}^n} \notin \Sigma$,
- Σ *is a generating set for* $\mathbb{R}^n$,
- $r_v(\Sigma) \subseteq \Sigma$ *for all* $v \in \Sigma$,
- $2 \cdot \frac{(v,w)}{(v,v)} \in \mathbb{Z}$ *for all* $v, w \in \Sigma$ *and*
- *if for* $\lambda \in \mathbb{R}$ *and* $w \in \Sigma$ *also* $\lambda \cdot w \in \Sigma$ *holds, necessarily* $\lambda \in \{1, -1\}$.

The elements of a root system Σ are called roots. Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_l$ be a partition such that $v_i \perp v_j$ for all $v_i \in \Sigma_i$, $v_j \in \Sigma_j$ with $i \neq j$. Such a partition is called an orthogonal decomposition of Σ. If Σ does not have any non-trivial, which means $l \geq 2$, orthogonal decomposition, Σ is called an **irreducible root system**; otherwise it is called reducible, compare 1.8.4 in [GLS3].

For every root system Σ, the group $W(\Sigma) = \langle r_\alpha \mid \alpha \in \Sigma \rangle$, which is called **Weyl group**, is finite. Assume a subset $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ of Σ which is a $\mathbb{R}$-basis of $\mathbb{R}^n$ with the following property: Every element of Σ is either contained in the set $\Sigma^+ = \{\sum\limits_{i=1}^{n} \lambda_i \alpha_i \mid \forall i \in \{1, \ldots, n\} : 0 \leq \lambda_i \in \mathbb{R}\} \cap \Sigma$ or in $\Sigma^- = \{\sum\limits_{i=1}^{n} \lambda_i \alpha_i \mid \forall i \in \{1, \ldots, n\} : 0 \geq \lambda_i \in \mathbb{R}\} \cap \Sigma$. Then the set Σ^+ is called the set of positive roots, and accordingly, Σ^- consists of the negative roots. The basis Π is called a **fundamental system**. For every root system Σ there is a fundamental system Π and $W(\Sigma) = \langle r_\alpha \mid \alpha \in \Pi \rangle$ holds. The dimension of the vector space $\langle \Sigma \rangle_{\mathbb{R}}$, which coincides with the cardinality of the basis Π, is called the rank of the root system Σ. This information about root systems can be found in Section 1.8 in [GLS3].

Definition 2.13 ([GLS3], 1.8.6): *The Dynkin diagram of a root system* Σ *with fundamental system* Π *is defined as a diagram with nodes labeled by the elements of* Π. *Two nodes* v *and* w *are joined by an edge of strength* $4 \cdot \cos^2(\theta) = 2 \cdot \frac{(v,w)}{(v,v)} \cdot 2 \cdot \frac{(w,v)}{(w,w)}$, *where* θ *is the obtuse angle between the vectors* v *and* w. *The strength of such an edge can be* 0, 1, 2 *or* 3, *where an edge of strength* 0 *between two nodes means that these nodes are not directly connected in the diagram. If two nodes are not of equal length as vectors in* $\mathbb{R}^n$ *and not perpendicular, the edge between them is orientated, which is denoted by* $>$ *aiming in direction of the root of smaller length. Two Dynkin diagrams are isomorphic if there is a bijection between the vertices which preserves orientation and strength of all edges.*

Two root systems are isomorphic if and only if their Dynkin diagrams are isomorphic, see 1.8.7 in [GLS3]. Thus the classification of all irreducible root systems corresponds to a list of the corresponding Dynkin diagrams. The Dynkin diagram of an irreducible root system is connected and denoted by A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 for arbitrary $n \in \mathbb{N}$, where the index gives the number of nodes in the Dynkin diagram and therefore the rank of the root system. Considering $A_1 = B_1 = C_1 = D_1$, $B_2 = C_2$, $A_3 = D_3$ and $D_2 = A_1 \times A_1$, the following table gives a complete list of all Dynkin diagrams of irreducible root systems, labeling also the fundamental root system $\Pi = \{\alpha_1, \dots, \alpha_n\}$, compare Table 1.8 in [GLS3].

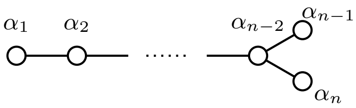
Table 2.1: Dynkin diagrams of irreducible root systems

A_n $(n \geq 1)$

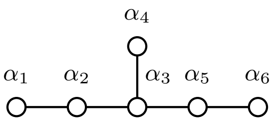
B_n $(n \geq 2)$

C_n $(n \geq 3)$

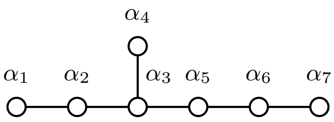
D_n $(n \geq 4)$



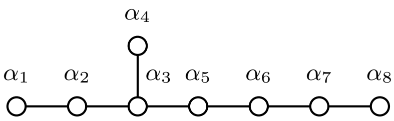
E_6



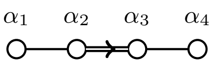
E_7



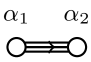
E_8



F_4



G_2



It will be always obvious in this thesis, whether A_n denotes a Dynkin diagram with n nodes or the alternating group on n letters. The same holds for Dynkin diagrams of type D_n and a dihedral group with the same notation.

A **torus** of an algebraic group $\overline{K}$ is a closed subgroup of $\overline{K}$ which is isomorphic as an algebraic group to a direct product of finitely many copies of the algebraic group $GL_1(\overline{F})$. The group $GL_1(\overline{F})$ is identified with the multiplicative group of the field $\overline{F}$. A torus in $\overline{K}$ which is maximal with respect to inclusion is called a maximal torus in $\overline{K}$, see 1.4.1 in [GLS3].
Let $\overline{T}$ be such a maximal torus in $\overline{K}$. A so-called $\overline{T}$**-root subgroup** $\overline{X}$ is a closed and $\overline{T}$-invariant algebraic subgroup of $\overline{K}$ which is isomorphic as an algebraic group to the additive group $\overline{F}_+$ of the field $\overline{F}$. The parametrization map $\overline{F}_+ \to \overline{X}$ with $t \mapsto x(t)$ is an isomorphism of algebraic groups. So by 1.3.4 in [GLS3], one gets

$$\overline{X} = \{x(t) \mid t \in \overline{F}\}\,.$$

A morphism of algebraic groups $\alpha\colon \overline{T} \to GL_1(\overline{F})$ is called a character. For any $\overline{T}$-root subgroup $\overline{X}$ of $\overline{K}$, a character α such that $x(t)^s = x(s^\alpha t)$ holds for any parametrization $x(t)$ of $\overline{X}$ and for all $t \in \overline{F}$ and $s \in \overline{T}$ is called a $\overline{T}$-root in $\overline{K}$. The set of $\overline{T}$-roots in $\overline{K}$ is denoted by $\Sigma_{\overline{K}}(\overline{T})$, following 1.4.4 and 1.9.1–1.9.3 in [GLS3]. The set of $\overline{T}$-roots $\Sigma_{\overline{K}}(\overline{T})$ is a root system whose structure does not depend on the choice of $\overline{T}$. There is a bijection between $\Sigma_{\overline{K}}(\overline{T})$ and the set of $\overline{T}$-root subgroups of $\overline{K}$. Actually, $\Sigma_{\overline{K}}(\overline{T})$ is uniquely determined by $\overline{K}$ alone and therefore $\Sigma_{\overline{K}}(\overline{T})$ is called the **root system of** $\overline{K}$ and it is denoted by just Σ, according to 1.9.5 and 1.9.6 in [GLS3].
Also $\mathbb{C}_{\overline{K}}(\overline{T}) = \overline{T}$ holds and $\mathrm{N}_{\overline{K}}(\overline{T})/\overline{T}$ is finite and is isomorphic to the Weyl group of the root system Σ of $\overline{K}$ by 1.9.5–1.9.6 in [GLS3].

Lemma 2.14: (a) Let $\overline{K}$ be a simple algebraic group. Then the root system Σ of $\overline{K}$ is uniquely determined and it is irreducible.

(b) Let $\overline{K}$ be a simple algebraic group with a maximal torus $\overline{T}$ and root system $\Sigma = \Sigma_{\overline{K}}(\overline{T})$. For each $\overline{T}$-root $\alpha \in \Sigma$ let $\overline{X}_\alpha$ be the uniquely determined $\overline{T}$-root subgroup. Then

$$\overline{K} = \langle \overline{X}_\alpha \mid \alpha \in \Sigma \rangle\,.$$

(c) Let Σ be an irreducible root system. Then there are, up to isomorphisms of algebraic groups, two uniquely determined simple algebraic groups $\overline{K}_u$ and $\overline{K}_a$, both sharing Σ as corresponding root system. And for every simple algebraic group $\overline{K}$ with root system Σ, there are surjective morphisms of algebraic groups $\alpha\colon \overline{K}_u \to \overline{K}$ and $\beta\colon \overline{K} \to \overline{K}_a$ with finite kernels. It is $\overline{K}_u$ called the universal version and $\overline{K}_a$ is called the adjoint version of $\overline{K}$. Furthermore, $\mathrm{Z}(\overline{K}_a) = 1$ holds.

Proof: See 1.10.1–1.10.5 in [GLS3]. □

For a simple algebraic group $\overline{K}$, the corresponding root system is irreducible and so the Dynkin diagram is listed in Table 2.1.

The group structure of $\overline{K}$ can be determined using the structure of the root subgroups and the so-called Chevalley relations, which give relations between root subgroups, compare 1.12.1–1.12.4 in [GLS3]. These relations are not explicitly needed here and hence are omitted.
Using these results and definitions about algebraic groups, it is possible to define finite groups of Lie type as the fixed points of so-called Steinberg endomorphisms:

Definition 2.15 ([GLS3], 1.15.1,2.2.1): *Let $\overline{K}$ be a linear algebraic group over $\overline{GF(p)}$.*

*(a) A **Steinberg endomorphism** $\sigma\colon \overline{K} \to \overline{K}$ is an epimorphism of algebraic groups such that the group of fixed points $\mathrm{C}_{\overline{K}}(\sigma)$ is finite.*

(b) If $\overline{K}$ is simple and $\sigma\colon \overline{K} \to \overline{K}$ is a Steinberg endomorphism, the group

$$K := O^{p'}(\mathrm{C}_{\overline{K}}(\sigma))$$

*is defined to be a **finite group of Lie type** over a field of characteristic p.*
The set of all groups of Lie type over a field of characteristic p is denoted by $\mathrm{Lie}(p)$.

Definition 2.16 ([GLS3],1.15.2,1.15.4): *Let $\overline{K}$ be a simple algebraic group over the field $\overline{F} = \overline{GF(p)}$, $\overline{T}$ a maximal torus and Σ the $\overline{T}$-root system of $\overline{K}$ with fundamental system* Π *and $q = p^a$ for $a \geq 1$.*

(a) Let $a > 1$. Then $\varphi_q\colon \overline{K} \to \overline{K}$ is a so-called field automorphism of $\overline{K}$ which is defined by

$$\overline{X_\alpha}^{\varphi_q} = \{x_\alpha(t) \mid t \in \overline{\mathbb{F}}\}^{\varphi_q} = \{x_\alpha(t^q) \mid t \in \overline{\mathbb{F}}\}$$

for each $\overline{T}$-root subgroup $\overline{X_\alpha}$, using $\overline{K} = \langle \overline{X}_\alpha \mid \alpha \in \Sigma\rangle$ by Lemma 2.14.
For $a > 1$, it is φ_q a Steinberg endomorphism.

(b) If Σ equals A_n for $n \geq 2$ or D_n for $n \geq 4$ or E_6, there is a non-trivial symmetry of the Dynkin diagram which induces an isometry ϱ on $\langle \Sigma \rangle_{\mathbb{R}}$ with $\Pi^\varrho = \Pi$. This isometry induces a so-called graph-automorphism γ_ϱ of the algebraic group $\overline{K}$ which is uniquely determined by

$$x_\alpha(t) \mapsto x_{\alpha^\varrho}(t).$$

The order of ϱ is 2, except for D_4 in which case ϱ can also be of order 3. It is $\mathrm{C}_{\overline{K}}(\gamma_\varrho)$ a simple algebraic group.

(c) For $p = 2$ and $\Sigma \in \{B_2, F_4\}$ or for $p = 3$ in case of $\Sigma = G_2$ there is an angle-preserving bijection $\varrho\colon \Sigma \to \Sigma$ which interchanges short and long roots and with $\Pi^\varrho = \Pi$. The bijection ϱ induces an automorphism ψ of $\overline{K}$ which is also called a graph automorphism. This automorphism ψ is uniquely determined by

$$x_\alpha(t)^\psi := \begin{cases} x_{\alpha^\varrho}(t) \text{ if } \alpha \text{ is a long root,} \\ x_{\alpha^\varrho}(t^p) \text{ if } \alpha \text{ is a short root.} \end{cases}$$

Additionally, ψ is a Steinberg endomorphism of $\overline{K}$ with $\psi^2 = \varphi_p$.

Definition 2.17 ([GLS3], 2.2.1–2.2.8): *Let $\overline{K}$ be a simple algebraic group, σ a Steinberg endomorphism, $\overline{T}$ a σ-invariant maximal torus and Σ the $\overline{T}$-root system of $\overline{K}$. By conjugation with an inner automorphism of $\overline{K}$, which does not effect the isomorphism type of $K := O^{p'}(\mathbb{C}_{\overline{K}}(\sigma))$, it is possible to bring σ in one of the following forms:*

*(a) It is $\sigma = \varphi_q$ with φ_q as defined in Definition 2.16. In this case, $K = O^{p'}(\mathbb{C}_{\overline{K}}(\sigma))$ is an **untwisted group of Lie type** and denoted by $K = \Sigma(q)$, i.e. $A_m(q)$, $B_m(q)$, $C_m(q)$, $D_m(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$ or $G_2(q)$ for $m \geq 2$.*

*(b) It is $\sigma = \gamma_\varrho \circ \varphi_q$ with φ_q and γ_ϱ defined as in Definition 2.16 and it is ϱ of order $d \in \{2, 3\}$. In this case, $K = O^{p'}(\mathbb{C}_{\overline{K}}(\sigma))$ is a so-called **Steinberg group** and denoted by $K = {}^d\Sigma(q)$, i.e. ${}^2A_m(q)$ for $m > 1$, ${}^2D_m(q)$ for $m \geq 3$, ${}^3D_4(q)$ or ${}^2E_6(q)$.*

(c) It is $\sigma = \psi \circ \varphi_q$ with φ_q and ψ defined as in Definition 2.16. Then it is $\sigma = \psi^{2a+1}$ for $q = p^a$.
*In this case, $K = O^{p'}(\mathbb{C}_{\overline{K}}(\sigma))$ is a so-called **Suzuki-Ree group**, leading to groups $K = {}^2B_2(2^{2a+1})$, $K = {}^2F_4(2^{2a+1})$ and $K = {}^2G_2(3^{2a+1})$.*

*Together the Steinberg and Suzuki-Ree groups are the **twisted groups of Lie type**.*

Let $\overline{K}$ be a simple algebraic group. Different versions of $\overline{K}$, compare Lemma 2.14, may lead to non-isomorphic finite groups of Lie type K. If K is an untwisted group of Lie type or a Suzuki-Ree group, the isomorphism type of K is uniquely determined by the version of $\overline{K}$ together with Σ and the integer q. In case of the Steinberg groups, additionally the integer d is needed to determine the isomorphism type of K, see 2.2.5 in [GLS3].

There are, up to isomorphisms, two uniquely determined groups of Lie type K_u and K_a, where K_u comes from the algebraic group $\overline{K}_u$ and K_a comes from the algebraic group $\overline{K}_a$. K_u is called the universal version and K_a is called the adjoint version of K. For every finite group of Lie type K there are surjective homomorphisms $\alpha\colon K_u \to K$ and $\beta\colon K \to K_a$. The kernels of these epimorphisms are central. It is $K/\operatorname{Z}(K) \cong K_u/\operatorname{Z}(K_u) \cong K_a$ by 2.2.6 in [GLS3], and K is a simple group if and only if $K \cong K_a$.

Every version of K is a quasisimple group with the following eight exceptions: $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$, ${}^2B_2(2) = Sz(2)$, $B_2(2)$, $G_2(2)$, ${}^2F_4(2)$ and ${}^2G_2(3)$. The adjoint version K_a is a simple group with the same exceptions. The first four exceptions are solvable groups and the other four are not perfect but have a simple commutator subgroup by 2.2.7 in [GLS3].

Definition 2.18 ([GLS3], 2.2.8): *The set of **finite simple groups of Lie type** consists of the adjoint versions K_a such that K_a is a simple group, together with the groups $B_2(2)'$, $G_2(2)'$, ${}^2F_4(2)'$ and ${}^2G_2(3)'$.*

The notation $K \cong {}^d\Sigma(q)$ does not include information about the version of K. Unless stated otherwise, in this thesis always the adjoint version is assumed.

The following lemma gives a complete list of isomorphisms between simple groups of Lie type over a field in characteristic p and between a simple group of Lie type and an alternating group A_n.

Lemma 2.19: Let all the listed groups in this Lemma be simple. Then it is

$$\begin{aligned} B_2(q) \cong C_2(q), &\quad D_3(q) \cong A_3(q), \\ {}^2D_3(q) \cong {}^2A_3(q), &\quad {}^2D_2(q) \cong A_1(q^2) \\ A_1(4) \cong A_1(5) \cong {}^2D_2(2) \cong A_5, &\quad A_1(7) \cong A_2(2), \\ A_1(8) \cong {}^2G_2(3)', &\quad A_1(9) \cong B_2(2)' \cong C_2(2)' \cong A_6, \\ {}^2A_2(3) \cong G_2(2)', &\quad A_3(2) \cong A_8, \\ {}^2A_3(2) \cong {}^2D_3(2) \cong B_2(3) \cong C_2(3), &\quad B_m(2^f) \cong C_m(2^f) \text{ for } m, f \in \mathbb{N}, \\ \text{and } {}^{d_1}\Sigma_1(q_1) \cong {}^{d_2}\Sigma_2(q_2) &\text{ if } \Sigma_1 \cong \Sigma_2,\ d_1 = d_2 \text{ and } q_1 = q_2. \end{aligned}$$

Proof: See Theorem 2.2.10 in [GLS3]. □

For the classical groups, i.e. the linear, symplectic and orthogonal groups, the following isomorphisms hold:

Lemma 2.20: Let K be a finite group of Lie type. If ${}^d\Sigma(q)$ is the adjoint version, then it is

$$\begin{aligned} A_m(q) \cong L_{m+1}(q), &\quad {}^2A_m(q) \cong U_{m+1}(q), \\ B_m(q) \cong P\Omega_{2m+1}(q), &\quad C_m(q) \cong PSp_{2m}(q), \\ D_m(q) \cong P\Omega^+_{2m}(q), &\quad {}^2D_m(q) \cong P\Omega^-_{2m}(q). \end{aligned}$$

Proof: See chapter 2.7 in [GLS3]. □

For a group of Lie type which is isomorphic to a classical group the notation of the classical group is used in this text. And instead of ${}^2B_2(2^{2a+1})$ in the adjoint version the notation $Sz(2^{2a+1})$ is used. If q is even, the groups $PSp_{2m}(q)$ and $P\Omega_{2m+1}(q)$ are isomorphic by the above lemmas. Hence they are identified in this thesis to provide a simultaneous treatment. Also the orthogonal groups $P\Omega^{\pm}_{2m}(q)$ for $m \leq 3$ are identified with the isomorphic linear or unitary group.

Definition 2.21 ([Wils], 2.7.1): *A covering group E of a finite group G is a finite group such that $\mathrm{Z}(E) \leq E'$ and $E/\mathrm{Z}(E) \cong G$ hold. A maximal covering group E is maximal in the sense that every covering group is a quotient of E. If G is perfect, then there is, up to isomorphisms, a unique maximal covering group, which is called universal covering group. The center of the universal covering group of G is the so-called* ***Schur multiplier*** *of G.*

The structure of the (uniquely determined) Schur multiplier of the simple groups of Lie type is described in the following lemma:

Lemma 2.22: Let $K \cong {}^d\Sigma(q)'$ be a simple group of Lie type. Then the Schur multiplier is the direct product $M_c(K) \times M_e(K)$, where $M_c(K)$ is the so-called canonical part and $M_e(K)$ is the exceptional part of the Schur multiplier. The canonical part $M_c(K)$ coincides with the center $\mathrm{Z}(K_u)$ of the universal version of K and the exceptional part is mostly trivial except for q being a power of 2 or 3. The structure of $M_c(K)$ and $M_e(K)$, if not trivial, is as follows:

K	$L_n(q)$	$U_n(q)$	$P\Omega_{2n+1}(q)$, $PSp_{2n}(q)$	$P\Omega^+_{2n}(q)$ (n even)	$P\Omega^-_{2n}(q)$ (n even)
$M_c(K)$	$Z_{(n,q-1)}$	$Z_{(n,q+1)}$	$Z_{(2,q-1)}$	$E_{(2,q-1)^2}$	$Z_{(2,q-1)}$

K	$P\Omega^+_{2n}(q)$ (n odd)	$P\Omega^-_{2n}(q)$ (n odd)	$E_6(q)$	${}^2E_6(q)$	$E_7(q)$
$M_c(K)$	$Z_{(4,q-1)}$	$Z_{(4,q+1)}$	$Z_{(3,q-1)}$	$Z_{(3,q+1)}$	$Z_{(2,q-1)}$

The exceptional part is trivial, except for the following groups:

K	$L_2(4)$	$L_3(2)$	$L_3(4)$	$L_4(2)$	$U_4(2)$	$U_6(2)$	$Sp_4(2)'$	$Sp_6(2)$
$M_e(K)$	Z_2	Z_2	$Z_4 \times Z_4$	Z_2	Z_2	$Z_2 \times Z_2$	Z_6	Z_2

K	$Sz(8)$	$P\Omega^+_8(2)$	$G_2(4)$	$F_4(2)$	${}^2E_6(2)$	$L_2(9)$	$U_4(3)$	$\Omega_7(3)$	$G_2(3)$
$M_e(K)$	$Z_2 \times Z_2$	$Z_2 \times Z_2$	Z_2	Z_2	$Z_2 \times Z_2$	Z_3	$Z_3 \times Z_3$	Z_3	Z_3

Proof: See 6.1.1–6.1.4 in [GLS3]. □

Hence for a simple group of Lie type over a field in characteristic 2, which is of main interest in this thesis, the canonical part of the Schur multiplier is trivial, except for $L_n(q)$, $U_n(q)$, $E_6(q)$ and ${}^2E_6(q)$ with $q = 2^a$ and $a \in \mathbb{N}$.
In particular, instead of $PSp_{2n}(q)$ we denote $Sp_{2n}(q)$ and $\Omega^{\pm}_{2n}(q)$ instead of $P\Omega^{\pm}_{2n}(q)$. The universal version of $L_n(q)$ is $SL_n(q)$ and the universal version of $U_n(q)$ is $SU_n(q)$.

By Lemma 2.14, the group theoretical structure of simple algebraic groups is determined by their root subgroups and the Chevalley relations, which are relations between different root groups.
There is an analogy to these root groups for the finite groups of Lie type, which is summarized in the following remarks.

Remark 2.23 ([GLS3], 2.3.1–2.3.3,2.3.6, 2.4.1):

- Let $V := \langle\Sigma\rangle_{\mathbb{R}}$ be the Euclidean space spanned by Σ. Then there is an isometry τ of V as follows: If K is untwisted, then τ is the identity on V. If K is a Steinberg group with $\sigma = \gamma_\varrho \circ \varphi_q$, then τ is the isometry ϱ of V from Definition 2.16. And in case of K being a Suzuki-Ree group, let τ be the isometry of V, which, up to scalars, extends the bijection $\varrho\colon \Sigma \to \Sigma$ from Definition 2.16. The order of τ is 1, 2 or 3 and the image of any positive root $\alpha \in \Sigma^+$ under τ is denoted by $\tilde{\alpha}$ and is a positive multiple of an element of Σ^+. We define

$\tilde{V} := \mathbb{C}_V(\tau)$ and $\tilde{\Sigma}$ as the orthogonal projection of Σ on $\tilde{V}$. It is $\tilde{\Sigma}$ itself a root system with fundamental system $\tilde{\Pi}$ and, in case of an untwisted group K, $\Sigma = \tilde{\Sigma}$ holds. On Σ an equivalence relation $\sim$ can be defined by $\alpha_1 \sim \alpha_2$ if and only if $\tilde{\alpha_1} = c \cdot \tilde{\alpha_2}$ for some $c > 0$. With $\hat{\alpha}$ we denote the equivalence class of α and let $\hat{\Sigma}$ be the set of corresponding equivalence classes. Then $\hat{\Sigma}^+$ and $\hat{\Pi}$ are the images of the set of positive roots Σ^+ and of the fundamental system Π in $\hat{\Sigma}$.

- The cardinality of $\hat{\Sigma}$ is the **Lie rank** of K. In case of an untwisted group K, the Lie rank coincides with the rank of the root system Σ as it is defined above, because all equivalence classes are singletons and $\hat{\Sigma}$ can be identified with Σ in this case. In case of a twisted group of Lie type K, the Lie rank equals the number of orbits of Π under the underlying symmetry of the Dynkin diagram, as the isometry τ is just an extension of this symmetry.
- Further, $\Sigma_{\hat{\alpha}}$ is defined as the subset $\{\beta \in \Sigma \mid \tilde{\alpha} \sim \tilde{\beta}\}$ of Σ. Then

$$X_{\hat{\alpha}} := \prod_{\beta \in \Sigma_{\hat{\alpha}}} \overline{X}_\beta \cap K$$

is called a **root group** of K. If K is an untwisted group of Lie type, $\hat{\alpha} = \alpha$ holds for every root $\alpha \in \Sigma$. Then the root group $X_{\hat{\alpha}}$ is just $\overline{X}_\alpha \cap K = \{x_\alpha(t) \mid t \in GF(q)\}$. So it is an elementary abelian group of order q. The root groups in twisted groups of Lie type need not to be abelian, but there are roots α such that $\hat{\alpha} = \{\alpha\}$, in which case $X_{\hat{\alpha}} = \{x_\alpha(t) \mid t \in GF(q)\}$ holds. Root groups of this type are called **long root groups**, the others are called **short root groups** of K.

An algebraic group is solvable if and only if there is a finite series of closed subnormal subgroups such that all factors are abelian. A closed, connected and solvable subgroup of an algebraic group $\overline{K}$ which is maximal with respect to these three properties is called a **Borel subgroup**, compare 1.5.1, 1.6.1 in [GLS3].
Let $\overline{K}$ be a simple algebraic group, σ a Steinberg endomorphism of $\overline{K}$ and $\overline{T}$ a σ-invariant maximal torus. Then there is a σ-invariant Borel subgroup $\overline{B}$ such that $\overline{B}$ contains $\overline{T}$ and it is $\overline{B} = \overline{U} : \overline{T}$ for a normal subgroup $\overline{U}$ in $\overline{B}$. The group $\overline{U}$ is called the unipotent radical of $\overline{B}$, see 2.1.6 in [GLS3].

For the rest of this section let $\overline{K}$ be a simple algebraic group over a field of characteristic p and let σ be a Steinberg endomorphism of $\overline{K}$, as it is described in Definition 2.17. Further let $\overline{T}$ be a σ-invariant maximal torus which is contained in a σ-invariant Borel subgroup $\overline{B} = \overline{U} : \overline{T}$, and let Σ be the corresponding system of $\overline{T}$-roots with fundamental system Π and positive roots Σ^+. And at last, let $K = O^{p'}(\mathbb{C}_{\overline{K}}(\sigma))$ be the corresponding finite group of Lie type.

For the so-called **Cartan subgroup** $T := \overline{T} \cap K$ and the subgroups $B := \overline{B} \cap K$, $U := \overline{U} \cap K$ and $N := \mathrm{N}_{\overline{K}}(\overline{T}) \cap K$ of K, the following properties hold:

Lemma 2.24: We assume the conditions made in the paragraph above. Then the following statements hold:

(a) $B = U : T$ and $U = O_p(B)$.

(b) $\overline{K} = \overline{B}\,\mathrm{N}_{\overline{K}}(\overline{T})$ with $\overline{B} \cap \mathrm{N}_{\overline{K}}(\overline{T}) = \overline{T}$ and $K = BN$ with $B \cap N = T$. The subgroups B and N build a so-called BN-pair for K.

(c) $W := N/T \cong \mathrm{N}_{\overline{K}}(\overline{T})/\overline{T}$ and W is called the Weyl group of K.

(d) $\overline{U} = \prod_{\alpha \in \Sigma^+} \overline{X}_\alpha$ and $U = \prod_{\hat{\alpha} \in \hat{\Sigma}^+} X_{\hat{\alpha}}$ is a Sylow p-subgroup of K.

(e) It is $\mathrm{Z}(U)$ a long root group of K or K is isomorphic to $Sp_{2n}(2^a)$, $F_4(2^a)$ or $G_2(3^a)$. In these three exceptional cases, $\mathrm{Z}(U)$ is the product of a long and a short root group, which are interchanged by a graph automorphism if there is a symmetry of the corresponding Dynkin diagram.

(f) The Cartan subgroup T normalizes every root group $X_{\hat{\alpha}}$ of K.

Proof: See 1.9.5, 2.3.4, 2.3.7, 2.3.8 and 3.3.1 in [GLS3]. $\square$

Remark 2.25: Let K be a finite group of Lie type over a field in characteristic p and $U \in \mathrm{Syl}_p(K)$. In this thesis a subgroup of K is called parabolic if it contains a Sylow p-subgroup of K, see Definition 1.1.
Usually in the context of groups of Lie type, a subgroup P of K is defined to be parabolic if and only if it contains the K-conjugate of a Borel group B. In this thesis, we call such a group a **Lie-parabolic** subgroup to avoid any misunderstanding.
By a result of Jacques Tits, every parabolic subgroup Q of K is normalized by the Cartan subgroup T and QT is a Lie-parabolic subgroup in K, see 2.6.7 in [GLS3].

In the following three lemmas we collect terminology and some statements to deal with the structure of Lie-parabolic subgroups.

Lemma 2.26: Assume the conditions listed above and let J be a subset of $\hat{\Pi}$ and $J' := \hat{\Pi} \backslash J$. Further define $U_J := \langle X_{\hat{\alpha}} \mid \alpha = \sum_{\alpha_i \in \Pi} \lambda_i \alpha_i \in \Sigma^+ \text{ with at least one } \hat{\alpha_i} \in J' \rangle$,
$M_J := \langle X_{\hat{\alpha}} \mid \pm\hat{\alpha} \in J \rangle$ and $L_J := TM_J$. Then $P_J = U_J L_J$ is a Lie-parabolic subgroup and every Lie-parabolic subgroup of K is K-conjugate to P_J for a uniquely determined set $J \subseteq \Pi$. For $|J| = 1$, the corresponding Lie-parabolic subgroups are called **minimal Lie-parabolic**

subgroups and for $|J| = |\hat{\Pi}| - 1$, the corresponding Lie-parabolic subgroups are called **maximal Lie-parabolic** subgroups.
Furthermore, it is $U_J \cap L_J = 1$, $O_p(P_J) = U_J$ and $\mathbb{C}_K(U_J) = \mathrm{Z}(U_J)\,\mathrm{Z}(K)$. Hence P_J is of characteristic p if $\mathrm{Z}(K) = 1$. It is $\mathbb{C}_{\mathrm{Aut}(K)}(U_J)$ the image of $\mathrm{Z}(U_J)$ in $\mathrm{Aut}(K)$.
The group $O^{p'}(L_J) = M_J$ is a central product of groups of Lie type over a field in characteristic p, where all factors are normalized by T, and with root system $\langle J\rangle_{\mathbb{R}} \cap \Sigma$. This root system is not necessarily irreducible.
In particular, $B = \mathrm{N}_K(U)$ holds.

The decomposition $P_J = U_J L_J$ is called a **Levi decomposition**. The group L_J and its P_J-conjugates are called **Levi complements** and U_J is called **unipotent radical**.

Proof: See 2.6.4-2.6.6 in [GLS3]. □

Lemma 2.27: Let K be a group of Lie type over a field $GF(p^f)$ for a prime p and let U be a Sylow p-subgroup of K.

(a) If L is a proper subgroup of K with $U \leq L$, i.e. L is a parabolic subgroup of K, then L is contained in a maximal Lie-parabolic subgroup of K.

(b) Let X be a non-trivial p-subgroup of K with $X = O_p(\mathrm{N}_K(X))$. Then $P := \mathrm{N}_K(X)$ is a Lie-parabolic subgroup of K and $X = O_p(P)$ is the corresponding unipotent radical.

Proof: The first part is due to Jacques Tits and can be found in Lemma (2.3) in [CKS]. The second statement is Corollary 3.1.5 in [GLS3]. □

Lemma 2.28: Let K be a group of Lie type over a field $GF(p^f)$ for a prime p and V an absolutely irreducible K-module over $GF(p^f)$. If P is a Lie-parabolic subgroup of X, then $\mathbb{C}_V(O_p(P))$ is an irreducible P-module over $GF(p^f)$.
In particular, for $S \in \mathrm{Syl}_p(K)$, the centralizer $\mathbb{C}_V(S)$ is one-dimensional as module over $GF(p^f)$.

Proof: This is the main result in [Smit]. □

To embed a group of Lie type into another group of Lie type, the following lemma gives a necessary condition:

Lemma 2.29: Let K_1 and K_2 be simple groups of Lie type, both over fields of characteristic p and with Lie ranks r_1 and r_2, respectively. If $K_1 \lesssim K_2$, then $r_1 \leq r_2$ holds.

Proof: See 5.2.12 in [KlLi]. □

Also some results concerning the automorphisms of a finite group of Lie type are listed here:

Lemma 2.30 ([GLS3], 2.5.1, 2.5.12): Let K be a finite group of Lie type over a field of characteristic p. Then every automorphism x of K is a product $x = idfg$ with the following properties:

(a) i is an inner automorphism of K;

(b) d is a so-called diagonal automorphism, which means that it is induced by an element of the maximal torus $\overline{T}$ that normalizes K. Then d normalizes every root group of K and p and $\mathrm{o}(d)$ are coprime;

(c) f is called a field automorphism of K, which arises from a field automorphism φ of the underlying field $\overline{GF(p)}$, acting on the root elements by $x_\alpha(t)^\varphi = x_\alpha(t^\varphi)$;

(d) g is a graph automorphism, which arises from a symmetry of the corresponding Dynkin diagram. In particular, if K is twisted, then $g = 1$. If $g \neq 1$, then $\mathrm{o}(g) \in \{2, 3\}$.

It is $\mathrm{Out}(K)$ a split extension of the group of the outer diagonal automorphisms $\mathrm{Outdiag}(K)$ by the group $\Phi_K\Gamma_K$, where Φ_K is the group of field automorphisms and Γ_K is the cyclic group of graph automorphisms. The group $\mathrm{Outdiag}(K)$ is isomorphic to the canonical part of the Schur multiplier, see Lemma 2.22 for its structure, and $\Phi_K\Gamma_K$ is abelian. Hence if K is a finite group of Lie type over a field of even characteristic, the Sylow 2-subgroups of the outer automorphism group are abelian.

Proof: See Theorems 2.5.1 and 2.5.12 in [GLS3]. □

For some simple groups of Lie type $K = {}^d\Sigma(2^f)$ over a field of even characteristic, the conjugacy classes and centralizers of involutions in $\mathrm{Out}(K)$ are described in the following lemma. There we omit the outer automorphisms of orthogonal groups in even dimension. These are needed only in the last chapter and are described there.

Lemma 2.31: Let $K = {}^d\Sigma(2^f)$ be a simple group of Lie type over a field of characteristic 2. Further let K be not isomorphic to $Sp_4(2)'$ or $G_2(2)'$. We suppose x to be an involution in $\mathrm{Aut}(K) \setminus \mathrm{Inn}(K)$. Define $\mathrm{Inndiag}(K)$ as the group generated by the inner and diagonal automorphisms of K. And let S be a Sylow 2-subgroup of K such that S is normalized by x. To simplify the notation, we identify the groups K and $\mathrm{Inn}(K)$. Then the following statements hold:

(a) For $K \cong {}^2F_4(2^f)'$ or $K \cong Sz(2^f)$, there is no involution in $\mathrm{Aut}(K) \setminus \mathrm{Inn}(K)$.

(b) For $K \cong Sp_4(2^f)$, it is $\mathrm{Out}(K)$ cyclic of order $2f$. Then all involutions in $\mathrm{Aut}(K) \setminus \mathrm{Inn}(K)$ are $\mathrm{Inndiag}(K)$-conjugate to x. It is $\mathbb{C}_K(x) \cong Sp_4(2^{\frac{f}{2}})$ for f even and $\mathbb{C}_K(x) \cong Sz(2^f)$ otherwise.

(c) For $K \cong F_4(2^f)$, it is $\mathrm{Out}(K)$ cyclic of order $2f$. Then all involutions in $\mathrm{Aut}(K)\setminus\mathrm{Inn}(K)$ are $\mathrm{Inndiag}(K)$-conjugate to x. It is $\mathbb{C}_K(x) \cong F_4(2^{\frac{f}{2}})$ if f is even and $\mathbb{C}_K(x) \cong {}^2F_4(2^f)$ otherwise.

(d) Assume K to be isomorphic to $G_2(2^f)$, ${}^3D_4(2^f)$, $Sp_{2n}(2^f)$ for $n \geq 3$, $E_7(2^f)$ or $E_8(2^f)$. If f is even, x is $\mathrm{Inndiag}(K)$-conjugate to a field automorphism, for which $O^{2'}(\mathbb{C}_K(x))$ is isomorphic to $G_2(2^{\frac{f}{2}})$, ${}^3D_4(2^{\frac{f}{2}})$, $Sp_{2n}(2^{\frac{f}{2}})$, $E_7(2^{\frac{f}{2}})$ or to $E_8(2^{\frac{f}{2}})$, respectively. If f is odd, there is no involution in $\mathrm{Out}(K)$ for these groups.

(e) Let K be isomorphic to $L_n(2^f)$ or to $E_6(2^f)$ and let x be a field automorphism on K. Then f is even and every involution in the coset Kx is $\mathrm{Inndiag}(K)$-conjugate to x. Additionally, $O^{2'}(\mathbb{C}_K(x))$ is isomorphic to $L_n(2^{\frac{f}{2}})$ or $E_6(2^{\frac{f}{2}})$, respectively.

(f) Let K be isomorphic to $L_n(2^f)$ or to $E_6(2^f)$ and let x induce a graph automorphism. Then the following hold:

   (i) For $K \cong L_n(2^f)$ and n odd, all involutions in Kx are $\mathrm{Inndiag}(K)$-conjugate to x and $\mathbb{C}_K(x) \cong Sp_{n-1}(2^f)$.

   (ii) For $K \cong L_n(2^f)$ and $n = 2m > 2$ even or for $K \cong E_6(2^f)$ each involution $y \in Kx$ is either $\mathrm{Inndiag}(K)$-conjugate to x or to xz for an involution z in a long root group $R \leq \mathrm{Z}(S)$. In this case $\mathbb{C}_K(x) \cong Sp_{2m}(2^f)$ or $\mathbb{C}_K(x) \cong F_4(2^f)$ respectively. And it is $\mathbb{C}_K(xz) = \mathbb{C}_K(x) \cap \mathbb{C}_K(z) = \mathbb{C}_{\mathbb{C}_K(x)}(z)$. Hence for $K \cong L_n(2^f)$, $\mathbb{C}_K(xz)$ is isomorphic to the centralizer of a transvection in $Sp_{2m}(2^f)$.

(g) Let K be isomorphic to $U_n(2^f)$ or ${}^2E_6(2^f)$. If x is the restriction to K of the graph automorphism of the corresponding untwisted group, which is called a field automorphism of K, the following statements hold:

   (i) For $K \cong U_n(2^f)$ and $n \geq 3$ odd, all involutions in Kx are $\mathrm{Inndiag}(K)$-conjugate to x and $\mathbb{C}_K(x) \cong Sp_{n-1}(2^f)$.

   (ii) For $K \cong U_n(2^f)$ and $n = 2m \geq 4$ even or for $K \cong {}^2E_6(2^f)$, it is $\mathbb{C}_K(x) \cong Sp_{2m}(2^f)$ or $\mathbb{C}_K(x) \cong F_4(2^f)$, respectively. Let z be an involution in a long root group $R \leq \mathrm{Z}(S)$. Each involution $y \in Kx$ is either $\mathrm{Inndiag}(K)$-conjugate to x or to xz and $\mathbb{C}_K(xz) = \mathbb{C}_K(x) \cap \mathbb{C}_K(z) = \mathbb{C}_{\mathbb{C}_K(x)}(z)$ holds.

(h) If x is a product of a field and a graph automorphism, a so-called field-graph automorphism, then $K \cong \Sigma(2^f)$ has to be untwisted and $O^{2'}(\mathbb{C}_K(x)) \cong {}^2\Sigma(2^{\frac{f}{2}})$ is the adjoint version. If y is an involutory field-graph automorphism of K, then x and y are conjugate in $\mathrm{Inndiag}(K)$.

Proof: For $K \cong {}^2F_4(2^f)'$ see (9-1) in [GoLy] for $f > 1$ and (18.6) in [AsSe] for $f = 1$. See Section 19 in [AsSe] and 4.9.1–4.9.2 in [GLS3] otherwise. $\square$

In the last lemma of this section, the 2-parts and 2-ranks of some simple groups $K \in \mathrm{Lie}(2)$ are given.

Lemma 2.32 ([GLS3], 3.3.3): Let K be a finite group of Lie type over a field $GF(2^f)$ for $f \in \mathbb{N}$. Let $|K|_2 = (2^f)^N$ for $N \in \mathbb{N}$. Then the integer N and the 2-rank of K are as follows:

K	N	$m_2(K)$
$L_n(2^f)$	$\binom{n}{2}$	$\lfloor (\frac{n}{2})^2 \rfloor f$
$U_n(2^f)$	$\binom{n}{2}$	$\lfloor \frac{n}{2} \rfloor^2 f$
$Sp_{2n}(2^f)$ $(n \geq 2)$	n^2	$\binom{n+1}{2} f$
$Sp_4(2)'$	3	2
$\Omega^{\pm}_{2n}(2^f)$ $(n \geq 2)$	$n(n-1)$	$\binom{n}{2} f$
$G_2(2^f)$ $(f \geq 2)$	6	$3f$
$G_2(2)'$	5	2
$E_6(2^f)$	36	$16f$
${}^2E_6(2^f)$	36	$13f$
$E_7(2^f)$	63	$27f$
$E_8(2^f)$	120	$36f$
$F_4(2^f)$	24	$11f$
${}^2F_4(2^f)$	12	$5f$
${}^3D_4(2^f)$	12	$5f$

Proof: This can be found in Theorems 2.2.9 and 3.3.3 in [GLS3]. $\square$

2.3 Other preliminary group theoretical results

In the following part of this chapter some group theoretical definitions and statements are listed, which are closely related to the simple groups of Lie type.
As semi-extraspecial groups are frequently used in this thesis, a definition and some properties are given in the following lemma. After that, a Levi decomposition of a Lie-parabolic subgroup, which is of importance in the following text, is discussed. Also some results concerning strongly 2-embedded subgroups, a classification result due to D. Holt, which is needed a lot to prove Theorem 1.5, and some facts about so-called minimal parabolic groups are listed. The chapter ends with a result about F-modules, which is needed in Chapter 4. We begin by defining what a semi-extraspecial group is, following Definition 1 in [Beis].

Definition 2.33 ([Beis]): *Let p be a prime.*

(a) A non-abelian p-group G is called special if and only if $\Phi(G) = G' = \mathrm{Z}(G)$ holds.

(b) If additionally $\mathrm{Z}(G)$ is cyclic, G is called extraspecial.

(c) If G is a special group such that for every maximal subgroup M of $\mathrm{Z}(G)$ the factor group G/M is extraspecial, G is called semi-extraspecial.

Lemma 2.34: Let $q = 2^f$ for $f \in \mathbb{N}$.

(a) There are, up to isomorphisms, two semi-extraspecial 2-groups of order $q^{1+2\cdot n}$ for $n \in \mathbb{N}$. They are denoted by $D_n(q) = q_+^{1+2\cdot n}$ and $Q_n(q) = q_-^{1+2\cdot n}$. The group $D_n(q)$ is isomorphic to the central product of n copies of a Sylow 2-subgroup $D_1(q) = q_+^{1+2}$ of $L_3(q)$, while $Q_n(q)$ is isomorphic to the central product of $n-1$ copies of a Sylow 2-subgroup $D_1(q)$ of $L_3(q)$ and one Sylow 2-subgroup $Q_1(q) = q_-^{1+2}$ of $U_3(q)$.

(b) The groups $D_2(q)$ and $Q_2(q)$ are isomorphic.

(c) In case of $q = 2$, the groups $D_n(q)$ and $Q_n(q)$ are extraspecial 2-groups, denoted by $2_+^{1+2\cdot n}$ and $2_-^{1+2\cdot n}$, respectively. In this case, $D_1(2)$ is isomorphic to the dihedral group D_8 and $Q_1(2)$ is isomorphic to the quaternion group Q_8.

(d) Every involution in $Q_1(q)$ is 2-central and $D_1(q)$ is generated by involutions.

Proof: See Lemmas 4 – 7 in [Beis] and for the last part see [Col1] and [Col2]. □

Lemma 2.35: Let Q be an extraspecial 2-group with $R = \mathrm{Z}(Q)$ and $|Q| > 2^3$. If $x \in Q \setminus R$ is an involution, then $\mathbb{C}_Q(x) = \langle x \rangle \times U$ holds for an extraspecial group U such that $\mathrm{Z}(U) = R$.

The group U is of the same type ($+$-type or $-$-type) as Q.
Furthermore, $\mathbb{C}_Q(\mathbb{C}_Q(x)) = \mathrm{Z}(\mathbb{C}_Q(x)) = \langle x\rangle \times R$ holds.[1]

Proof: Let be $R = \langle r\rangle$ and let $x \in Q\backslash R$ be an involution. Then there is an involution $y \in Q$ such that $[x, y] \neq 1$ holds. So it is $[x, y] = r$ and $|Q : \mathbb{C}_Q(x)| = |x^Q| = |\{x, x \cdot r\}| = 2$ follows. For $U := \langle x, y\rangle$, it is $U = \langle x, y, r\rangle \trianglelefteq Q$ an extraspecial group of order 8 and $\mathrm{Z}(U) = \langle r\rangle$. Then U is dihedral of order 8 and, by construction of U, $U \cdot \mathbb{C}_Q(U) \leq Q$ holds. Let g be an arbitrary element in Q. As $U \trianglelefteq Q$ holds, it is $g \in \mathrm{N}_Q(U)$ and it is $x^g = x \cdot [x, g] = x \cdot r^i$ and $y^g = y \cdot [y, g] = y \cdot r^j$ for suitable $i, j \in \{0, 1\}$. Hence there are 4 such automorphisms induced by Q, which are, due to $|\mathrm{Inn}(U)| = |U/\,\mathrm{Z}(U)| = 4$, all inner automorphisms. Without restriction, it is $g \in U \cdot \mathbb{C}_Q(U)$ and so $Q = U \cdot \mathbb{C}_Q(U)$ follows. Furthermore, it is $\mathrm{Z}(\mathbb{C}_Q(U)) = \langle r\rangle$, so $\mathbb{C}_Q(U)$ is extraspecial and, because of $U \cong D_8$, $\mathbb{C}_Q(U)$ is of the same type as Q. Additionally, one gets $\mathbb{C}_Q(x) = (U \cap \mathbb{C}_Q(x)) * \mathbb{C}_Q(U) = \langle x, r\rangle * \mathbb{C}_Q(U) = \langle x\rangle \times \mathbb{C}_Q(U)$. Thus, $\mathbb{C}_Q(\mathbb{C}_Q(x)) = \mathbb{C}_Q(\langle x\rangle \times \mathbb{C}_Q(U)) = \langle x\rangle \times \mathrm{Z}(\mathbb{C}_Q(U)) = \langle x\rangle \times \langle r\rangle$ holds. □

In the following text, it is always unambiguous, whether $D_n(q)$ denotes a semi-extraspecial group or a group of Lie type.

Let now K be a simple group of Lie type over a field of characteristic 2 and with Lie rank at least 2. Let U be a Sylow 2-subgroup of K. We exclude $K \cong Sp_{2n}(2^f)'$ and $K \cong F_4(2^f)$. Then $R := \mathrm{Z}(U)$ is a long root group by Lemma 2.24. In the following the action of $\mathrm{N}_K(R)$ on $Q = O_2(\mathbb{C}_K(R))$ is needed. As the Cartan subgroup normalizes R by Lemma 2.24, $\mathrm{N}_K(R)$ is a Lie-parabolic subgroup of K, which gives rise to a Levi decomposition of $\mathrm{N}_K(R)$. This Levi decomposition is described in the following lemma. We restrict to groups, for which the statement of the lemma is needed in this thesis.

Lemma 2.36: Let K be a simple group of Lie type over $GF(q)$ with $q = 2^f$ and with Lie rank of at least 2 and fundamental root system $\hat{\Pi}$. Additionally, K is assumed to be isomorphic to one of the groups listed below.

Let $R = \mathrm{Z}(U)$ for $U \in \mathrm{Syl}_2(K)$ be a long root group and L a Levi complement in $\mathrm{N}_K(R)$. Then $\overline{Q} = Q/R$ is a faithful L-module, whose structure is given in the following list. In all listed cases, $\overline{Q}$ is a module over the field $GF(q)$, except for $K \cong U_n(q)$ where $\overline{Q}$ is a module over $GF(q^2)$. In each case, Q is a special group with $\mathrm{Z}(Q) = R$.

(a) If $K \cong L_n(q)$ and $n \geq 4$, then it is $O^{2'}(L) \cong SL_{n-2}(q)$ and $\overline{Q} = V_1 \oplus V_2$ such that V_1 is the natural L-module and V_2 the dual module. The order of the module is $|\overline{Q}| = q^{2(n-2)}$.

(b) If $K \cong \Omega^{\pm}_{2n}(q)$ and $n \geq 4$, then it is $O^{2'}(L) \cong \Omega^{\pm}_{2n-4}(q) \times L_2(q)$. Set $L_1 := \Omega^{\pm}_{2n-4}(q)$ and $L_2 := L_2(q) \cong SL_2(q)$. Then it is $\overline{Q} = V_1 \otimes V_2$ such that V_i for $i \in \{1, 2\}$ is the natural L_i-module. Additionally, $|\overline{Q}| = q^{2(2n-4)}$ holds.

[1] Parts of this statement can be found in Excercise 4A.5. in [Isa].

(c) If $K \cong U_n(q)$ and $n \geq 5$, then $O^{2'}(L) \cong SU_{n-2}(q)$ holds. It is $\overline{Q}$ the natural module and $|\overline{Q}| = q^{2(n-2)}$.

(d) If $K \cong E_6(q)$, then $O^{2'}(L) \cong SL_6(q)$ holds. It is $\overline{Q}$ the exterior cube of the natural module with $|\overline{Q}| = q^{20}$.

(e) If $K \cong {}^2E_6(q)$, then it is $O^{2'}(L) \cong SU_6(q)$ and $\overline{Q}$ is the exterior cube of the natural module with $|\overline{Q}| = q^{20}$.

(f) If $K \cong E_7(q)$, then $O^{2'}(L) \cong \Omega_{12}^{+}(q)$ holds. It is $\overline{Q}$ a half-spin module with $|\overline{Q}| = q^{32}$.

(g) If $K \cong E_8(q)$, then it is $O^{2'}(L) \cong E_7(q)$ and $\overline{Q}$ is the natural module of order q^{56}.

(h) If $K \cong {}^3D_4(q)$, then $O^{2'}(L) \cong SL_2(q^3)$ holds. It is $\overline{Q}$ the 8-dimensional module $V \otimes V^{\sigma} \otimes V^{\sigma^2}$ over $GF(q)$, where V is the natural module and σ is the Frobenius automorphism of the field extension $GF(q^3)$ over $GF(q)$. Additionally, $|\overline{Q}| = q^8$ holds.

In case of $K \cong L_n(q)$, it is $\overline{Q}$ the direct sum of two irreducible L-modules over $GF(q)$. In every other case listed in the lemma, $\overline{Q}$ is an irreducible module over $GF(q)$, or over $GF(q^2)$ in case of $K \cong U_n(q)$. Furthermore, the occurring irreducible modules, and for $K \cong L_n(q)$ the two irreducible submodules, are absolutely irreducible.

Proof: This can be found in Lemma 1.8 in [SaSt]. □

The statement of the following lemma describes the structure of the normalizer of a long root group in $G_2(q)$ for $q \geq 2$.

Lemma 2.37: Let $F^*(H) \cong G_2(q)$ for $q = 2^f$ with $f \geq 2$. Let further P be the normalizer of a long root group R in $F^*(H)$ with $Q := O_2(\mathrm{N}_{F^*(H)}(R))$. Then it is $O^{2'}(P) \cong q^{1+4} \colon L_2(q)$, where q^{1+4} denotes a special 2-group of order q^5 with elementary abelian center of order q. For $q > 4$, $O^{2'}(P)/Q$ acts irreducibly on Q/R. For $q = 4$, P acts irreducibly on Q/R, but $O^{2'}(P)/Q \cong L_2(4) \cong A_5$ induces a direct sum of two permutation modules for the alternating group A_5 on Q/R.

Proof: This is 10.10 and page 238 in [DGS]. □

Lemma 2.38: Let K be a simple group of Lie type over $GF(q)$ with $q = 2^f$ and with Lie rank at least 2, excluding $K \cong Sp_{2n}(q)$ and $K \cong F_4(q)$. Further let U be a Sylow 2-subgroup of K. Then $R := \mathrm{Z}(U)$ is a long root group and $Q = O_2(\mathrm{N}_K(R))$ equals $O_2(\mathbb{C}_K(R))$. Additionally, Q is a large subgroup in $\mathrm{Aut}(K)$.

Proof: Due to Lemma 2.24, $R := \mathrm{Z}(U)$ is a long root group of K. And $O_2(\mathrm{N}_K(R))$ equals $O_2(\mathbb{C}_K(R))$, as on the one hand $\mathbb{C}_K(R)$ is a normal subgroup in $\mathrm{N}_K(R)$ and on the other

hand $O_2(\mathrm{N}_K(R)) \leq U \leq \mathbb{C}_K(R)$ holds.
And due to Lemma 2.11 in [Pie], Q is a large subgroup in $\mathrm{Aut}(K)$. □

Lemma 2.39: Let K be a simple group of Lie type over $GF(q)$ for $q = 2^f$ and with Lie rank at least 2. Let K be isomorphic to one of the groups listed in Lemma 2.36. For $U \in \mathrm{Syl}_2(K)$, $R = \mathrm{Z}(U)$ and $Q = O_2(\mathrm{N}_K(R))$, it is $\mathrm{Z}_2(U) \leq Q$ of order q^2 or K is isomorphic to $U_n(q)$ or to $L_n(q)$. In the last two cases, $|\mathrm{Z}_2(U)| = q^3$ holds.

Proof: Due to Lemma 2.24, $R := \mathrm{Z}(U)$ is a long root group of K. By Lemma 2.38, $Q = O_2(\mathrm{N}_K(R))$ is a large subgroup in K with $\mathbb{C}_K(Q) = \mathrm{Z}(Q) = R$ and with Levi complement L in $\mathrm{N}_K(R)$ acting faithfully on $\overline{Q} = Q/R$, as it is described in Lemma 2.36. Then $\mathrm{Z}_2(U) \leq \mathrm{N}_K(Q)$. Together with $[\mathrm{Z}_2(U), Q] \leq R$, the faithful action of L on Q/R implies $\mathrm{Z}_2(U) \leq Q$. If L acts absolutely irreducible on $\overline{Q}$ over $GF(q)$, then $\mathbb{C}_{\overline{Q}}(U)$ is one-dimensional as a $GF(q)$-module by Lemma 2.28. Hence $|\mathrm{Z}_2(U)| = q^2$ follows. If L does not act absolutely irreducible on $\overline{Q}$ over $GF(q)$, then $K \cong U_n(q)$ or $K \cong L_n(q)$. For $K \cong L_n(q)$, $\overline{Q}$ is a direct sum of absolutely irreducible L-modules over $GF(q)$. Then by Lemma 2.28, $|\mathbb{C}_{\overline{Q}}(U)| = q^2$ holds, implying $|\mathrm{Z}_2(U)| = q^3$. If $K \cong U_n(q)$, $\overline{Q}$ is an absolutely irreducible L-module over $GF(q^2)$. Hence it is $|\mathbb{C}_{\overline{Q}}(U)| = q^2$, which implies $|\mathrm{Z}_2(U)| = q^3$. □

Definition 2.40: *Let p be a prime and G a finite group with a proper subgroup H such that $|H|$ is divisible by p. It is H called **strongly p-embedded** if and only if for all $g \in G \setminus H$, $|H \cap H^g|$ is not divisible by p.*

Remark 2.41: Assume that a proper subgroup $H < G$ is strongly p-embedded. This is equivalent to $\mathrm{N}_G(X) \leq H$ for every p-subgroup $1 \neq X \leq H$.

Proof: Let H be a strongly p-embedded subgroup of G. Assume $\mathrm{N}_G(X) \not\leq H$ for a non-trivial p-subgroup X of H. Then there is an element $g \in \mathrm{N}_G(X) \setminus H$ and $X^g = X \leq H$, implying $1 \neq X = X^g \cap H \leq H^g \cap H$. So p divides the order of $H \cap H^g$. But this contradicts H being strongly p-embedded in G.
Let now $\mathrm{N}_G(X)$ be contained in H for every non-trivial p-subgroup $X \leq H$. Let g be an element in G. Then it is $\mathrm{N}_G(Y) \leq H^g$ for every non-trivial p-subgroup $Y \leq H^g$.
If there is a non-trivial p-subgroup $U \leq H \cap H^g$, then there is $P \in \mathrm{Syl}_p(H \cap H^g)$ such that $U \leq P$ holds. With $P \leq H^g$ also $P^{g^{-1}} \leq H$ holds. There is a Sylow p-subgroup S of H with $P \leq S$. Then it is $\mathrm{N}_S(P) \leq H \cap H^g$, according to the previous paragraph. In particular, it is $P = S$ a Sylow 2-subgroup of H. By Sylow's Theorem there is an element $h \in H$ such that $P^{g^{-1}h} = P$ holds. Hence it is $g^{-1}h \in \mathrm{N}_G(P) \leq H$ and so $g \in H$ follows, which is a contradiction. □

In case of $p = 2$, there are well-known results to classify strongly p-embedded subgroups. Two of them are given in the following lemmas.

Lemma 2.42: Let G be a finite group with a proper subgroup H of even order and let S be a Sylow 2-subgroup of H. Then H is strongly 2-embedded in G if and only if $\mathbb{C}_G(t) \leq H$ holds for every involution $t \in S$ and if additionally $\mathrm{N}_G(S)$ is contained in H.

Proof: See Proposition 17.11 in [GLS2]. □

Lemma 2.43 ([Bend]): Let G be a finite group with a strongly 2-embedded subgroup. Then the following holds:

- A Sylow 2-subgroup of G is cyclic or a quaternion group or
- there is a series of subgroups $1 \leq M \leq L \leq G$, all normal in G, such that M and G/L are of odd order and L/M is isomorphic to one of the simple groups $L_2(2^f)$, $Sz(2^f)$ or $U_3(2^f)$ for suitable $f \geq 2$.

Proof: This is the main result in [Bend]. □

The classification of groups with a strongly 2-embedded subgroup in the previous lemma was published by Helmut Bender in 1971. The simple groups listed in this lemma, $L_2(2^f)$, $Sz(2^f)$ and $U_3(2^f)$, are of Lie rank 1. A Sylow 2-subgroup of a finite non-abelian simple group G can neither be a cyclic nor a quaternion group. The former is due to Burnside's theorem about p-complements, see 7.2.2 in [KuSt], the latter follows from a result of Brauer-Suzuki, see Theorem 15.2 in [GLS2], stating that the 2-central involution would be contained in $Z^*(G)$, which contradicts the simplicity of G.

The following lemma, based on a result of Derek Holt, see [Holt], is used in this thesis in the following form to characterize simple groups by the fusion of a 2-central element. It allows, roughly spoken, to restrict the consideration of centralizers of arbitrary involutions to involutions which are conjugate to a 2-central involution. This can be seen as a key result in the following in order to prove the main theorem of this thesis.

Lemma 2.44: Let G be a simple group and H a proper subgroup of G. Further let r be a 2-central element in G such that $\mathbb{C}_G(r) \leq H$ and $r^G \cap H = r^H$ hold. Then G is isomorphic to $L_2(2^f)$ for $f \geq 2$, $U_3(2^f)$ for $f \geq 2$, $Sz(2^f)$ for $f \geq 3$ odd, or to an alternating group A_n. In the first three cases, H is a Borel subgroup of G. In particular, H is solvable. And in the last case, one gets $H \cong A_{n-1}$.

Proof: This is Lemma 2.16 in [PaSt2]. □

Additionally, we denote a result, which is due to Bernd Baumann. The result is needed in Chapter 5 to provide, under certain conditions, a restriction of the set of possible components to some groups of Lie type of Lie rank 1 or 2.

Lemma 2.45 ([Baum]): Let G be a finite group without any non-trivial solvable normal subgroups. Further we assume that G contains an involution, in whose centralizer a Sylow 2-subgroup of G is normal. Then the smallest normal subgroup N of G with 2-closed factor group is a direct product of simple groups. Every simple, direct factor of N is isomorphic to one of the following groups:

(a) $L_2(q)$, $Sz(q)$, $U_3(q)$, $L_3(q)$, $Sp_4(q)$ for $q = 2^f > 2$;

(b) $L_2(q)$ for $q = 2^f \pm 1 > 3$.

Proof: This is the main result in [Baum]. □

In the following, the characterization of so-called minimal parabolic groups is needed. Despite this term is closely related to minimal Lie-parabolic groups, the meaning is different and the two terms should be distinguished carefully.

Definition 2.46: *Let T be a Sylow p-subgroup of a finite group G. The group G is called **minimal parabolic** (with respect to T) if and only if T is not normal in G and there is a unique maximal subgroup in G which contains T.*

The characterization of minimal parabolic groups with respect to a Sylow 2-subgroup, which arise from simple groups of Lie type in even characteristic, from sporadic or from alternating groups, is well-known, as well as the arguments to prove this result. This characterization is stated in the following three lemmas. We omit a corresponding statement concerning simple groups of Lie type over a field in odd characteristic, as it is not needed in the proof of the main theorem.

Lemma 2.47: Let $K = LT$ be a finite group such that $F^*(K) = L$ is a non-abelian simple group of Lie type over a field of characteristic 2. Additionally, let K be minimal parabolic with respect to $T \in \mathrm{Syl}_2(K)$.
Then L is isomorphic to $L_2(2^f)$, $Sz(2^f)$, $U_3(2^f)$, $L_3(2^f)$ or $Sp_4(2^f)$ for $f \in \mathbb{N}$ such that L is simple. In the last two cases, hence for $L \cong L_3(2^f)$ or $L \cong Sp_4(2^f)$, T acts non-trivially on the Dynkin diagram of L.
Furthermore, it is $\mathrm{N}_K(T \cap L)$ the uniquely determined maximal subgroup of K which contains the Sylow 2-subgroup T.

Proof: It is $K = LT$ an automorphism group of a simple group of Lie type. Let M be the uniquely determined maximal subgroup of K which contains T. Let further Ω be the set of minimal Lie-parabolic subgroups of L which contain $T \cap L$.

To show that T acts transitively on Ω, let k be the number of T-orbits Ω_i of Ω and define $U_i := \langle \Omega_i, T\rangle$. Then K is generated by the groups $U_i := \langle \Omega_i, T\rangle$. This is a consequence of Lemma 2.14.
For $k > 1$, each group U_i is properly contained in K and contains T, so $U_i = \langle \Omega_i, T\rangle \leq M$ follows for each i. But this contradicts $K = \langle U_i \mid 1 \leq i \leq k\rangle$. So T acts transitively on Ω.

The elements of Ω are minimal Lie-parabolic subgroups of L. As the action of T on Ω corresponds to graph automorphisms of even order, the size of Ω is at most 2, by Lemma 2.30. Hence L has to be of Lie rank 1 or of Lie rank 2, where in the latter case T induces a graph automorphism on the Dynkin diagram of L. This implies the list of simple groups, given in the lemma.
It is $\mathrm{N}_L(T \cap L) \leq M \cap L$ a Lie-parabolic subgroup of L, so $M \cap L = \mathrm{N}_L(T \cap L)$ holds. This implies the statement of the lemma. □

Lemma 2.48: Let $K = LT$ be a finite group such that $F^*(K) = L$ is non-abelian simple. Let further K be minimal parabolic with respect to $T \in \mathrm{Syl}_2(K)$. Then L cannot be a sporadic group.

Proof: We assume that $K = LT$ is an automorphism group of a sporadic simple group L. By Table 1 in [RoSt], we are left with L being isomorphic to the Mathieu group M_{11} or to the Janko group J_1. As $\mathrm{Out}(L)$ is trivial for these groups, compare [CoCu], we may assume $K = J_1$ or $K = M_{11}$. But, according to [CoCu], J_1 contains two maximal subgroups which contain a Sylow 2-subgroup, and in M_{11} there are three maximal subgroups with this property. Hence L is not a sporadic group. □

Lemma 2.49: Let $K = LT$ be a finite group such that $F^*(K) = L$ is isomorphic to an alternating group A_n for $n \geq 5$ and let K be minimal parabolic with respect to $T \in \mathrm{Syl}_2(K)$. Let M be the unique maximal subgroup of K which contains T. Then either it is

- $n = 2^m + 1$ and M is the stabilizer of a point in the permutation representation of degree $n = 2^m + 1$ in $K \cong A_{2^m+1}$ or $K \cong S_{2^m+1}$, or
- $L \cong A_6$ with $M = T$ holds.

Proof: The argumentation in this proof follows mostly Lemma 2.2 in [LPR].
Except for $n = 6$, $K \cong A_n$ or $K \cong S_n$ holds. For $n = 5$ and $n = 6$, the lemma holds by using [CoCu] to look up the maximal subgroups of A_n.
So from now on let be $n \geq 7$. It is L the alternating group on $\Omega = \{1, 2, \ldots, n\}$ for $n \geq 7$. Let $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k}$ with $n_1 > n_2 > \ldots > n_k \geq 0$ be the 2-adic representation of n. To consider the structure of a Sylow 2-subgroup of L, let S be a Sylow 2-subgroup of

the symmetric group S_n. Then, for example by 15.3 in [Hupp], S is isomorphic to a direct product $P_{n_1} \times \ldots \times P_{n_k}$, where each direct factor P_{n_i} is a Sylow 2-subgroup of $S_{2^{n_i}}$ and is isomorphic to an iterated wreath product of n_i many cyclic groups of order 2. So it is $P_{n_i} \cong \underbrace{(Z_2 \wr Z_2) \wr \ldots \wr Z_2}_{n_i} \cong D_8 \wr P_{n_i-2}$, using $Z_2 \wr Z_2 \cong D_8$.

We assume first $K \cong A_n$, which implies $K = L$.

For $n = 2^{n_1} \geq 8$ and $T \in \mathrm{Syl}_2(L)$, T equals $S \cap L$ for S being isomorphic to a Sylow 2-subgroup of the symmetric group S_n. Hence T stabilizes a system Γ_1 of blocks of size 2 and a system Γ_2 of blocks of size 4. Without restriction, we assume $\Gamma_1 = \{\{1,2\},\{3,4\},\ldots,\{2^{n_1}-1,2^{n_1}\}\}$ and $\Gamma_2 = \{\{1,2,3,4\},\ldots,\{2^{n_1}-3,2^{n_1}-2,2^{n_1}-1,2^{n_1}\}\}$. Hence $U_1 := \mathrm{Stab}_L(\Gamma_1)$ and $U_2 := \mathrm{Stab}_L(\Gamma_1)$ are subgroups of L which contain T. As L acts $(n-2)$-transitively on Ω, U_1 and U_2 are proper subgroups of L. Additionally, $\langle U_1, U_2\rangle$ acts 2-transitively on Ω and $\langle U_1, U_2\rangle$ contains the 3-cycle $(1,2,3)$. Hence by Lemma 2.6, $\langle U_1, U_2\rangle = L$ holds and so K is not minimal parabolic with respect to T.

Let now be $n = 2^{n_1} + 1 \geq 9$. Then $T \in \mathrm{Syl}_2(L)$ may be chosen to be contained in $\mathrm{Stab}_L(\{n\}) \cong A_{n-1}$. As L acts primitively on Ω, it is $\mathrm{Stab}_L(\{n\})$ a maximal subgroup of L. To show that $\mathrm{Stab}_L(\{n\})$ is the only maximal subgroup which contains T, we assume the existence of a maximal subgroup $\tilde{M}$ in L such that $T \leq \tilde{M} \not\leq \mathrm{Stab}_L(\{n\})$. As L acts $(n-2)$-transitively on Ω, $\tilde{M}$ acts 2-transitively and thus primitively on Ω. Additionally, $T \leq \tilde{M}$ contains the subgroup $\langle (1,2)(3,4),(1,3)(2,4)\rangle$, which acts transitively on $\{1,2,3,4\}$. As it is $1 < |\{1,2,3,4\}| < \frac{n}{2}$, Lemma 2.6 implies $\tilde{M} = L$, in contradiction to $\tilde{M}$ being a maximal subgroup. Hence $\mathrm{Stab}_L(\{n\})$ is the only maximal subgroup of L which contains T. Therefore, K is minimal parabolic with respect to T.

For $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_k} \geq 7$, we consider the remaining cases. Hence $T \in \mathrm{Syl}_2(L)$ acts intransitively on Ω. Without restriction, let $\Omega_1 = \{1,2,\ldots,2^{n_1}\}$ be one of the T-orbits. Additionally, as described in the first case, T stabilizes a system Γ_1 of blocks of size 2. Then $U_1 := \mathrm{Stab}_L(\Omega_1)$ and $U_2 := \mathrm{Stab}_L(\Gamma_1)$ are proper subgroups of L, both containing T. And as before, $\langle U_1, U_2\rangle$ acts primitively on Ω and contains the 3-cycle $(1,2,3)$. Hence by Lemma 2.6, $\langle U_1, U_2\rangle = L$ holds. Hence K is not minimal parabolic with respect to T. Altogether, for $K \cong A_n$, the lemma holds.

Now let be $K \cong S_n$:

Without loss of generality, T contains the transposition $(1,2)$ and the rest of the proof is analogously to the arguments above, using again Lemma 2.6. □

The statements of the following three lemmas allow to generate some quasisimple groups by centralizers of certain involutions which act on the quasisimple group. In particular, the minimal parabolic groups, which are listed in the previous lemma, have this property. And also most quasisimple groups of Lie type over a field in odd characteristic, for which we have not given a classification of minimal parabolic groups, can be generated by centralizers of suitable involutions.

Lemma 2.50: Let K be a group such that $K/\mathrm{Z}(K)$ is an alternating group on a set Ω with $|\Omega| = n \geq 5$ and n an odd integer. Let E be an elementary abelian 2-group of 2-rank $m \geq 2$ such that E acts on K. So the isomorphic image E^* of E in $\mathrm{Aut}(K)$ is contained in the symmetric group on the set Ω. The partition of Ω into E^*-orbits leads to $n = a_0 + a_1 \cdot 2 + \ldots + a_r \cdot 2^r$, where a_i is the number of orbits of size 2^i. Additionally, we suppose $a_0 \neq 0 \neq a_r$, where 2^r is the size of the longest orbit.
Then $K = \langle \mathbb{C}_K(F) \mid F \leq E,\, m_2(F) \geq m - r \rangle$ holds.

Proof: See 7.5.1 in [GLS3]. □

And also the alternating group $A_6 \cong Sp_4(2)'$ can be generated by centralizers of involutions.

Lemma 2.51: Let E be an elementary abelian subgroup of order 8 which acts faithfully on the alternating group A_6. Then $A_6 = \langle \mathbb{C}_{A_6}(t) \mid t \in E^{\#} \rangle$.

Proof: As E acts faithfully on A_6, we may assume that E is a subgroup of $\mathrm{Aut}(A_6)$. We identify A_6 and $\mathrm{Inn}(A_6)$ to simplify notation. It is $m_2(A_6) = 2$ and $m_2(\mathrm{Aut}(A_6)) = 3$. As $|E| = 8$, E is a maximal elementary abelian subgroup of $\mathrm{Aut}(A_6)$ which is not contained in A_6. If τ is an involution in $\mathrm{Aut}(A_6) \setminus A_6$, then by [CoCu] there are two possibilities: Either $\mathbb{C}_{A_6}(\tau)$ is of order 10 and, by examining the structure of maximal subgroups of A_6, τ is not contained in a maximal elementary abelian subgroup of $\mathrm{Aut}(A_6)$ or $\tau \in S_6 \setminus A_6$ and $\mathbb{C}_{A_6}(\tau) \cong S_4$ is a maximal subgroup in A_6. Let T be a Sylow 2-subgroup of A_6. We may assume $T = \langle (1,2)(3,4), (1,3)(2,4), (1,2)(5,6) \rangle$. Using the information about centralizers of outer involutions in $\mathrm{Aut}(A_6)$, we may further assume that the elementary abelian subgroup V equals $\langle (1,2)(3,4), (1,3)(2,4), (5,6) \rangle = \langle (1,2)(3,4), (1,3)(2,4), (1,2)(3,4)(5,6) \rangle$. We set $t_1 := (1,3)(2,4)$ and $t_2 := (1,2)(3,4)(5,6)$. It is $\mathbb{C}_{A_6}(t_2) \cong S_4$ a maximal subgroup of A_6 which does not contain the element $(2,4)(5,6) \in \mathbb{C}_{A_6}(t_1)$, see [CoCu]. Hence A_6 is generated by the centralizers of the involutions $t_1 \in V$ and $t_2 \in V$. □

Lemma 2.52: Let K be a quasisimple group of Lie type over a field of odd characteristic. Further let $E \neq 1$ be a non-cyclic, elementary abelian 2-group which acts faithfully on K. Then either $K = \langle \mathbb{C}_K(F) \mid 1 \neq F \leq E \rangle$ holds or $K/\mathrm{Z}(K)$ is isomorphic to one of the following groups:
$L_2(5)$, $L_2(7)$, $L_2(9)$, ${}^2G_2(3)'$ or $PSp_4(3)$.

Proof: See 7.3.4 in [GLS3]. □

At the end of this chapter, we introduce the terminology of so-called F-modules and the corresponding offenders.

Definition 2.53: *Let G be a finite group and V a G-module which is of finite dimension over $GF(2)$. Then $V \neq 1$ is called* ***F-module*** *for G if G contains a subgroup A such that $A/\mathbb{C}_A(V)$ is an elementary abelian 2-group and*

$$|V| \cdot |\mathbb{C}_A(V)| \leq |A| \cdot |\mathbb{C}_V(A)|$$

holds.

Such a subgroup $A \leq G$ is called a non-trivial ***offender*** *for V if additionally $[V, A] \neq 1$ holds. The subgroup A is called a* ***quadratic offender*** *if the action of A on V is quadratic, hence if $[V, A, A]$ is trivial.*

A subgroup $A \leq G$ is called a ***best offender*** *for V if $A/\mathbb{C}_A(V)$ is an elementary abelian 2-group and if for every subgroup $B \leq A$ the inequality $|B| \cdot |\mathbb{C}_V(B)| \leq |A| \cdot |\mathbb{C}_V(A)|$ holds. Using $B = \mathbb{C}_A(V)$, every best offender for V is an offender for V.*

And A is called an ***over-offender*** *for V if A is an offender for V and*

$$|V| \cdot |\mathbb{C}_A(V)| < |A| \cdot |\mathbb{C}_V(A)|$$

holds.

Timmesfeld's Replacement Theorem, see for example 9.2.3 in [KuSt], implies that if $A \leq G$ is a best offender for V, then $\mathbb{C}_A([V, A])$ is a quadratic best offender. As a consequence, if there is an offender $A \leq G$ for V, then one can choose a group $\tilde{A}$ among the subgroups of A such that $|\tilde{A}| \cdot |\mathbb{C}_V(\tilde{A})|$ is maximal to get a best offender. By Timmesfeld's Replacement Theorem, then there exists also a quadratic best offender for V.
The following result on offenders and F-modules can be found in [MeSt]:

Lemma 2.54: Let G be isomorphic to $SL_n(q)$ with $n \geq 2$ and with q a power of 2. Let further $V = N^n$ be a direct product of n natural $SL_n(q)$-modules N. Then V is no F-module for G.

Proof: Let be $G \cong SL_n(q)$ with $n \geq 2$ and with q a power of 2. Then it is $O_2(G) = 1$. We suppose that $V = N^n$ is a F-module for G. Then the normal subgroup J of G, which is generated by the best offenders in G for V, is not trivial. A non-trivial subgroup K of J is called a J-component if K is minimal with respect to $K = [K, J]$. Let K be such a J-component.
Then by Lemma 2.2 and Theorem 8.2 in [MeSt], which are included in Theorem 1 of the same paper, the following statements hold:

- For $n = 2$, it is $K \cong SL_2(2)' \cong Z_3$ and $[V, K]$ is a natural $SL_2(2)$-module, contradicting $V = N^2$.

- For $n \geq 3$, it is $K \cong SL_n(q)$ and $V = N^r \oplus N^{*\,s}$, where N is a natural $SL_n(q)$-module, N^* its dual, and r, s are integers with $0 \leq r, s < n$. Hence this also contradicts $V = N^n$ being a direct product of n natural $SL_n(q)$-modules.

Hence $V = N^n$ is no F-module for G. $\square$

Chapter 3

Preparatory results and centralizers of 2-central involutions

In order to outline the strategy of the proof of Theorem 1.5, some definitions are introduced and first conclusions from the assumptions in Hypothesis 1.4 are made in this chapter. We show that for $F^*(H) \cong L_4(2)$ there exists an example with $G \neq H$. Also it is shown that the centralizer $\mathbb{C}_G(t)$ is contained in H for every 2-central involution $t \in H$. We start with a short remark.

Remark 3.1: Suppose that Hypothesis 1.4 holds.

As the index $|G : H|$ is odd, $S \in \mathrm{Syl}_2(H)$ implies $S \in \mathrm{Syl}_2(G)$. According to Hypothesis 1.4 (c), it is $\mathrm{N}_G(S) \leq H$ and, using Frattini's argument, $\mathrm{N}_G(H) = H$ follows.

Let $\tilde{S}$ be an arbitrary Sylow 2-subgroup of H. Then it is $\tilde{S}^h = S$ for an element $h \in H$ by Sylow's Theorem. If X is a non-trivial and normal subgroup of $\tilde{S}$, then $1 \neq X^h \trianglelefteq S$ holds. So it is $\mathrm{N}_G(X) = \mathrm{N}_G(X^{hh^{-1}}) = \mathrm{N}_G(X^h)^{h^{-1}} \leq H^{h^{-1}} = H$. In particular, the statements of Hypothesis 1.4 hold for every Sylow 2-subgroup of H.

Lemma 3.2: Let Hypothesis 1.4 hold. Without loss of generality, $F^*(G)$ may supposed to be a non-abelian simple group. Additionally, $F^*(H)$ is a subgroup of $F^*(G)$.

Proof: If $G = H$ holds, then Theorem 1.5 is proved. So let H be a proper subgroup of G. Let further S be a Sylow 2-subgroup of H and $S_1 := S \cap F^*(H)$. Then $S_1 \in \mathrm{Syl}_2(F^*(H))$ and, by the remark above, $S \in \mathrm{Syl}_2(G)$ follows.

We assume first $O_2(G) \neq 1$. Then Hypothesis 1.4 (c) implies $G = \mathrm{N}_G(O_2(G)) \leq H$, contradicting $H \neq G$. So without loss of generality, $O_2(G)$ is trivial.
Now we assume $O_p(G) \neq 1$ for an odd prime p. As $F^*(H)$ is a simple group, $O_p(G) \cap H$ is

trivial. As $O_p(G)$ by assumption is not trivial, $V := \Omega_1(\mathrm{Z}(O_p(G)))$ is a non-trivial elementary abelian p-group. The simple group $F^*(H)$ acts faithfully on V, as otherwise $S_1 \leq F^*(H)$ centralizes V; but $V \leq \mathbb{C}_G(S_1) \leq H$ contradicts $O_p(G) \cap H = 1$. So $F^*(H)$ is isomorphic to a subgroup of $GL(V)$. Since p is odd, the minimal polynomial of every involution $t \in F^*(H)$ divides $x^2 - 1 = (x-1)(x+1)$. Either such an involution $t \in F^*(H)$ inverts every element in V or it fixes a non-trivial element in V. If $t \in S_1$ acts by inverting all elements of V, then $t^{F^*(H)} \cap \mathbb{C}_{S_1}(t) = \{t\}$, so by Glauberman's Z^*-Theorem, see [Glau], $t \in Z^*(F^*(H)) = 1$ holds. Hence every involution $t \in \mathrm{Z}(S_1)$ fixes an element $1 \neq v \in V$. Again by Hypothesis 1.4 (c), $v \in \mathrm{N}_G(\langle t \rangle)$ is contained in $H \cap O_p(G) = 1$. This implies $O_p(G) = 1$ for every prime number p. Thus G is a group of automorphisms of a semisimple group.

As $F^*(G) \cap F^*(H) \trianglelefteq F^*(H)$ and $F^*(H)$ is simple, either $F^*(H) \leq F^*(G)$ or $F^*(G) \cap F^*(H) = 1$ holds.

Assume first that $F^*(H) \cap F^*(G) = 1$ holds. It is $T := S \cap F^*(G)$ not trivial and, using $F^*(H) \geq [F^*(H), T] \leq [F^*(H), F^*(G)] \leq F^*(G)$, one gets $[F^*(H), T] = 1$. So $T \leq \mathbb{C}_H(F^*(H)) \leq F^*(H)$ follows, which contradicts $F^*(H) \cap F^*(G) = 1$. Hence it is $F^*(H) \leq F^*(G)$.

Let $F^*(G) = L_1 \times \cdots \times L_k$ for $k > 1$ with non-abelian simple groups $L_1, \ldots, L_k$. For every $i \in \{1, \ldots, k\}$, the group $T_i := S \cap L_i \neq 1$ is a Sylow 2-subgroup of L_i. As before, one gets $F^*(H) \geq [F^*(H), T_i] \leq [F^*(H), L_i] \leq L_i$.

This implies $[F^*(H), T_i] \leq F^*(H) \cap L_i \trianglelefteq F^*(H)$. If there is an $i \in \{1, \ldots, k\}$ such that $F^*(H) \cap L_i = 1$ holds, then $T_i \leq \mathbb{C}_H(F^*(H)) \leq F^*(H)$ follows, which is an immediate contradiction.

So $F^*(H) \cap L_i = F^*(H)$ and therefore $F^*(H) \leq L_i$ holds for every $i \in \{1, \ldots, k\}$. As $F^*(H)$ is simple, this implies $k = 1$. Hence $F^*(G)$ is a non-abelian simple group.

The aim of this thesis is to show that, up to the exceptions listed in Theorem 1.5, always $F^*(H) = F^*(G)$ holds. Using Hypothesis 1.4 and Frattini's argument, this implies $G = F^*(H) \cdot \mathrm{N}_G(S_1) \leq H$ and therefore $G = H$. □

Hence we assume in the following that $F^*(G)$ is a non-abelian simple group which contains the also simple group $F^*(H)$. This assumption is made throughout the rest of this text without being explicitly mentioned in every case.

In M. Salarian's and G. Stroth's article [SaSt], a certain set of subgroups of G is defined, to show that $\mathbb{C}_G(t)$ is contained in H for every involution $t \in S \in \mathrm{Syl}_2(H)$. This implies $\mathbb{C}_{F^*(G)}(t) \leq H \cap F^*(G)$ for every involution $t \in S$. And $\mathrm{N}_{F^*(G)}(S) \leq H \cap F^*(G)$ holds, compare for example Remark 3.1.

Then it is $H \cap F^*(G)$ a strongly 2-embedded subgroup in $F^*(G)$, by Lemma 2.42.

It is $F^*(H)$ a simple group of Lie type which is additionally assumed to be of Lie rank at least 2 in [SaSt]. As also $F^*(G)$ is simple, the simple sections $L_2(2^f)$, $Sz(2^f)$ and $U_3(2^f)$

listed in Lemma 2.43 can be excluded as well as $S \cap F^*(G) \in \mathrm{Syl}_2(F^*(G))$ being a cyclic or quaternion group. This is due to the comment following Lemma 2.43.
But this is a contradiction to $F^*(H)$ being strongly 2-embedded in $F^*(G)$, so the assumption $G \neq H$ is disproved.

The strategy in this thesis is similar and in order to prove Theorem 1.5, the following definitions from [SaSt] are required.

Definition 3.3: *Let X be a non-trivial 2-subgroup of H.*

(a) The set $M(X)$ consists of subgroups of G which are not contained in H and which contain a non-trivial normal 2-subgroup:

$$M(X) := \{K \mid X \leq K \leq G, K \not\leq H, O_2(K) \neq 1\}.$$

(b) With $\sqsubset$ we denote a relation on the elements of $M(X)$ which is defined as follows. Let K_1, K_2 be elements of $M(X)$.

$$K_1 \sqsubset K_2 \;:\Longleftrightarrow \exists T \in \mathrm{Syl}_2(K_2 \cap H) : (T \cap K_1 \in \mathrm{Syl}_2(K_1 \cap H) \wedge T \neq T \cap K_1).$$

The set $M(X)$ is finite. Thus, despite $\sqsubset$ does not give a partial order on $M(X)$, maximal elements with respect to $\sqsubset$ can be defined as follows:

$$M_{max}(X) := \{K \mid K \in M(X), K \text{ is maximal with respect to } \sqsubset\}.$$

(c) Among these maximal elements of $M(X)$ we are interested in the ones which are minimal by inclusion. So we define

$$P(X) := \{K \mid K \in M_{max}(X), K \text{ is minimal with respect to inclusion}\}.$$

(d) Eventually, the set $P^*(X)$ consists of the 2-constrained elements in $P(X)$ for which $O_{2'}(K) \leq H$ holds, so

$$P^*(X) := \{K \mid K \in P(X), O_{2'}(K) \leq H, F^*(K/O_{2'}(K)) = O_2(K/O_{2'}(K))\}.$$

In the following remark some direct consequences from these definitions are stated.

Remark 3.4: Suppose that Hypothesis 1.4 holds.

- For $S \in \mathrm{Syl}_2(H)$, the set $M(S)$ is empty:
  Assume $K \in M(S)$. Then $K = \mathrm{N}_K(O_2(K))$ and $S \leq K$. So one gets $1 \neq O_2(K) \trianglelefteq S$. Due to Hypothesis 1.4 (c), $\mathrm{N}_G(O_2(K)) \leq H$ follows. Altogether, $K = \mathrm{N}_K(O_2(K)) \leq \mathrm{N}_G(O_2(K)) \leq H$, which contradicts $K \in M(S)$.

- For $1 \neq X \leq S$ with $\mathrm{N}_G(X) \nleq H$, the set $M(X)$ is not empty:
  For $K = \mathrm{N}_G(X)$ it is $X \trianglelefteq \mathrm{N}_G(X) = K$ and, as $X \leq S$ is a 2-group, $O_2(K) = O_2(\mathrm{N}_G(X)) \neq 1$ and $K \nleq H$. Hence $K \in M(X)$.

- If $M(X) \neq \emptyset$, then also $P(X) \neq \emptyset$. So $P(X) = \emptyset$ implies $M(X) = \emptyset$, from which $\mathrm{N}_G(X) \leq H$ follows.

Whereas in [SaSt], due to the assumption of G being of local characteristic 2, the set $P^*(X)$ of 2-constrained groups is in the spotlight, in this paper the main focus is on the set $P(X)$. Under the assumption $G \neq H$, the existence of an involution $t \in H$ is assumed such that $\mathbb{C}_G(t) \nleq H$ holds. Then for $1 \neq X \leq S$, the structure of possible groups $K \in P(X)$ is investigated, in order to get a contradiction. Therefore, $P(X)$ is the empty set and so $\mathbb{C}_G(t) \leq H$ holds.
The main difference to the strategy pursued in [SaSt] is the usage of D. Holt's result, given in Lemma 2.44. This allows to eliminate a lot of possible groups for $F^*(H)$ nearly from the beginning and quite simplifies the proof. If $F^*(H)$ is isomorphic to a linear, symplectic, unitary or to one of the following groups of Lie type $G_2(2^f)'$, $F_4(2^f)$, ${}^2F_4(2^f)'$, $Sz(2^f)$, this is done in Chapter 4.
The strategy to prove Theorem 1.5 for the remaining possibilities for $F^*(H)$ is similar to the approach of M. Salarian and G. Stroth in [SaSt].

The following lemma illustrates that $F^*(H) \cong A_8$ as proper subgroup of $F^*(G) \cong A_9$ satisfies the assumptions in Hypothesis 1.4, so it is really an exception to $G = H$ in Theorem 1.5.

Lemma 3.5: Let $F^*(G) = A_9$ be the alternating group acting on the set $\{1, \dots, 9\}$ and let $F^*(H)$ be a point stabilizer in $F^*(G)$. This situation satisfies the conditions of Hypothesis 1.4 and $G \neq H$ holds.

Proof: We identify $F^*(H)$ with the point stabilizer of the letter 9 in $F^*(G) = A_9$. Then it is $F^*(H) \cong A_8$ a maximal subgroup in $F^*(G)$ and $F^*(H) \cong A_8 \cong L_4(2)$ is a simple group of Lie type over a field of characteristic 2.
As $\mathrm{Aut}(A_8) \cong S_8$ and $\mathrm{Aut}(A_9) \cong S_9$, there are two possibilities to consider: Either it is $G = F^*(G) = A_9$ or it is $G = S_9$. In the first case, it is $H = F^*(H)$, as A_8 is a maximal subgroup in A_9. In the second case, it is $H \cong S_8$, as otherwise the index $|G : H|$ is even.
In both cases, G is of parabolic characteristic 2, as the centralizer of a 2-central involution in A_9 and also in S_9 is of characteristic 2. Also in both cases, H is a point stabilizer and hence a maximal subgroup in G. Let S be a Sylow 2-subgroup of H. As G is minimal parabolic with respect to S, see Lemma 2.49, H is the only maximal subgroup of G that contains S.
For every non-trivial subgroup $Y \trianglelefteq S$, one gets $1 \neq \mathrm{N}_G(Y) \neq G$, as $O_2(G)$ is trivial. So $\mathrm{N}_G(Y)$ is contained in a maximal subgroup that contains S. Hence $\mathrm{N}_G(Y) \leq H$ holds for

every non-trivial normal subgroup Y of S. Thus G and H satisfy all properties listed in Hypothesis 1.4.

This exception to $G = H$ does not occur, if we require G to be of local characteristic 2 instead of parabolic characteristic 2: For the non-trivial 2-subgroup $X = \langle (1,2)(3,4), (1,3)(2,4) \rangle$, the normalizer $\mathrm{N}_G(X)$ contains a subgroup $U \cong A_5$ on the letters 5 to 9, so $\mathrm{N}_G(X)$ is not contained in the point stabilizer H and therefore $M(X) \neq \emptyset$ holds. In particular, the groups A_9 and S_9 are of parabolic characteristic 2 but not of local characteristic 2. □

In the following three lemmas some basic properties of the groups in $P(X)$ are collected. Most of the results could just have been quoted from [SaSt]. But as the results are heavily used in the following and to make this thesis more self-contained, they are given with a proof. The first four statements of the first lemma are citations from Lemma 1.2 in [SaSt].

Lemma 3.6 ([SaSt], Lemma 1.2): Suppose Hypothesis 1.4. Further let S be a Sylow 2-subgroup of H,
$1 \neq X \leq S$, $K \in M_{max}(X)$ and $T \in \mathrm{Syl}_2(K \cap H)$ such that $X \leq T$. Then the following statements hold:

(a) It is $T \in \mathrm{Syl}_2(K)$ and $\mathrm{N}_K(T) \leq H$.

(b) If Y is a non-trivial 2-group which is normal in K, then K contains a Sylow 2-subgroup of $\mathrm{N}_G(Y)$.

(c) Let C be non-trivial characteristic subgroup of T. Then $\mathrm{N}_G(C) \leq H$ holds.

(d) If $K \in P(X)$, then K is minimal parabolic with respect to T.

(e) Let $L \in M(X)$ such that $T \in \mathrm{Syl}_2(L)$. Then $L \in M_{max}(X)$.

(f) It is $\mathrm{N}_G(O_2(K)) \in M_{max}(X)$.

Proof:

(a) By definition of $K \in M(X)$, it is $K \not\leq H$, $X \leq K$ and $O_2(K) \neq 1$. Using $T \leq K \cap H \leq K$, then $K \in M(T)$. If $T \in \mathrm{Syl}_2(H)$, by Remark 3.4, $M(T) = \emptyset$. Thus $T \notin \mathrm{Syl}_2(H)$; so T is a proper subgroup of S. Hence T is a proper subgroup of $\mathrm{N}_S(T)$.
We assume now $\mathrm{N}_K(T) \not\leq H$. Then also $\mathrm{N}_G(T) \not\leq H$ holds and so, using $X \leq T \leq \mathrm{N}_G(T)$ and $O_2(\mathrm{N}_G(T)) \geq T \neq 1$, this implies $\mathrm{N}_G(T) \in M(X)$.
As T is a proper subgroup of $\mathrm{N}_S(T)$, $T \notin \mathrm{Syl}_2(\mathrm{N}_G(T) \cap H)$. So there is a group $\tilde{T} \in \mathrm{Syl}_2(\mathrm{N}_G(T) \cap H)$ with $T \lneq \tilde{T}$. Because of that, $T \in \mathrm{Syl}_2(K \cap H)$ is contained in

$\tilde{T} \cap K$. Hence $T = \tilde{T} \cap K \in \mathrm{Syl}_2(K \cap H)$ and, using $T \lneq \tilde{T}$, one gets $K \sqsubset \mathrm{N}_G(T)$. But this contradicts $K \in M_{max}(X)$. Therefore, $\mathrm{N}_K(T) \leq K \cap H$ holds.
Applying Lemma 2.9, T is a Sylow 2-subgroup of K.

(b) Let $1 \neq Y \trianglelefteq K$. Then $K = \mathrm{N}_K(Y) \leq \mathrm{N}_G(Y)$, so $\mathrm{N}_G(Y) \nleq H$. Therefore, $\mathrm{N}_G(Y) \in M(X)$. Due to part (a), also $T \in \mathrm{Syl}_2(K)$ holds. Hence it is $K \leq \mathrm{N}_G(Y)$, $\mathrm{N}_G(Y) \in M(X)$ and $K \in M_{max}(X)$. Then one gets $T \in \mathrm{Syl}_2(\mathrm{N}_H(Y))$. Hence $\mathrm{N}_G(Y) \in M_{max}(X)$. By using part (a), T is a Sylow 2-subgroup of $\mathrm{N}_G(Y)$.
In particular, $T = \mathrm{N}_S(O_2(K))$.

(c) According to part (a), $T \in \mathrm{Syl}_2(K)$. Assuming $\mathrm{N}_G(C) \nleq H$, it is $\mathrm{N}_G(C) \in M(X)$. Furthermore, $T \leq \mathrm{N}_G(C)$ and $K \in M_{max}(X)$, so exactly as in the arguments above $T \in \mathrm{Syl}_2(\mathrm{N}_H(C))$. So $\mathrm{N}_G(C) \in M_{max}(X)$ holds, and again with part (a) of this lemma, $T \in \mathrm{Syl}_2(\mathrm{N}_G(C))$ follows.
But exactly as in the proof of part (a), $T \neq \mathrm{N}_S(T) \leq \mathrm{N}_G(C)$, which is a contradiction. So $\mathrm{N}_G(C) \leq H$ holds.

(d) Let S, T, X and K as before and additionally let $K \in P(X)$. Using part (a) of this lemma gives $\mathrm{N}_K(T) \leq H$, so T is not normal in K.
Assume X_1 to be a proper subgroup of K such that $T \leq X_1$ and $X_1 \nleq H$. Then $X_1 \in M(X)$ and, due to (a), $T \in \mathrm{Syl}_2(K)$. So also $T \in \mathrm{Syl}_2(X_1)$ holds.
We assume that there is a group $X_2 \in M(X)$ such that $X_1 \sqsubset X_2$. Then there is a Sylow 2-subgroup T_2 of $X_2 \cap H$ such that $T_2 \cap X_1 \in \mathrm{Syl}_2(X_1 \cap H)$ and $T_2 \neq T_2 \cap X_1$ holds. Using $T \in \mathrm{Syl}_2(X_1)$, without loss of generality $T_2 > T$ follows, which implies $K \sqsubset X_2$. This contradicts $K \in M_{max}(X)$. So $X_1 \in M_{max}(X)$ holds. But it is $K \in P(X)$, which is a contradiction to $X_1 < K$. Therefore, $H \cap K$ is the unique maximal subgroup of K containing T.

(e) We assume $L \notin M_{max}(X)$. Then there is $L_1 \in M(X)$ with a Sylow 2-subgroup $T_1 \in \mathrm{Syl}_2(L_1 \cap H)$ such that $T_1 \cap L \in \mathrm{Syl}_2(L \cap H)$ and $T_1 \neq T_1 \cap L$. Then $T_1^h \cap L = T$ for an element $h \in L \cap H$ and $T_1^h \neq T \cap L$. So T_1 can be chosen such that $T_1 \cap L = T$ and T is a proper subgroup of T_1. But then also $K \cap T_1 = T$ holds, contradicting $K \in M_{max}(X)$. So $L \in M_{max}(X)$ follows.

(f) We denote $L := \mathrm{N}_G(O_2(K))$. According to part (b), it is $T = \mathrm{N}_S(O_2(K)) \leq L$. So $T \in \mathrm{Syl}_2(L)$ holds. Part (e) implies $L \in M_{max}(X)$.

□

Lemma 3.7 ([SaSt], Lemma 1.4): Let $K \in P(X)$ and $T \in \mathrm{Syl}_2(K \cap H)$ such that $X \leq T$. Then $Y \leq T$ implies $K \in P(Y)$.

Proof: As $K \in M(X)$ and $Y \leq T \leq K$, it is $K \in M(Y)$.
Assume $K \notin M_{max}(Y)$. Then there is a group $K_1 \in M(Y)$ and a Sylow 2-subgroup T_1 of $H \cap K_1$ such that $T_1 \cap K \in \mathrm{Syl}_2(K \cap H)$ and $T_1 \neq T_1 \cap K$ hold. Then $(T_1 \cap K)^h = T$ for an element $h \in H \cap K$. Hence $Y \leq T \leq K_1^h$ and, using $O_2(K_1^h) \neq 1$ and $K_1^h \not\leq H$, it implies $K_1^h \in M(Y)$. Because of $X \leq T \leq K_1^h$, also $K_1^h \in M(X)$. But $T_1^h \in \mathrm{Syl}_2(H \cap K_1^h)$, $T_1^h \cap K = (T_1 \cap K)^h = T$ and $T_1^h > T$. So it is $K \sqsubset K_1^h$, contradicting $K \in M_{max}(X)$. Therefore, it is $K \in M_{max}(Y)$ and so $M_{max}(X) \subseteq M_{max}(Y)$. Then $K \in P(Y)$ follows. □

Lemma 3.8 ([SaSt], Lemma 1.3): Suppose Hypothesis 1.4.

(a) Let $K \in P^*(X)$, then $O_{2',2}(K) = O_2(K) \times O(K)$.

(b) Let be $K \in P(X) \setminus P^*(X)$. Then $\mathrm{Z}(K)$ is of even order. If $O(K) \not\leq H$, then $K = O(K)T$.

Proof: Let $T \in \mathrm{Syl}_2(K \cap H)$ such that $X \leq T$. According to Lemma 3.6, it is $T \in \mathrm{Syl}_2(K)$.

(a) Denote $U := T \cap O_{2',2}(K)$. Then $U \in \mathrm{Syl}_2(O_{2',2}(K))$ and Frattini's argument implies $K = O_{2',2}(K)\,\mathrm{N}_K(U)$. Hence $O_{2',2}(K)/O(K) = O_2(K/O(K))$ is a normal 2-subgroup of $TO(K)/O(K)$, so $O_{2',2}(K)$ is a normal subgroup of $TO(K)$. By definition of $K \in P^*(X)$, it is $O(K) \leq H$. Hence $TO(K) \leq H$ and therefore it is $O_{2',2}(K) \leq H$.
Because of $K \not\leq H$, then $\mathrm{N}_K(U) \not\leq H$ holds. According to this, $\mathrm{N}_K(U) \in M(X)$ and, as $T \leq \mathrm{N}_K(U)$, also $\mathrm{N}_K(U) \in M_{max}(X)$ follows. It is $K \in P(X)$ minimal in $M_{max}(X)$, hence $K = \mathrm{N}_K(U)$ and so $U = O_2(K)$. This implies the assertion.

(b) For $K \in P(X) \setminus P^*(X)$ with $O(K) \leq H$, it is $E(K/O(K)) \neq 1$. Let U be the full preimage of $E(K/O(K))$ in K. Then $U \trianglelefteq K$ follows and, because of $T \cap U \in \mathrm{Syl}_2(U)$ and Frattini's argument, also $K = UN_K(T \cap U)$. In particular, it is $U \not\leq H$ and, by the minimality of $K \in P(X)$, it is $K = UT$. Furthermore, $1 \neq \mathrm{Z}(T) \cap O_2(K) \leq \mathrm{Z}(K)$ holds.

If $O(K) \not\leq H$, because of the minimality of $K \in P(X)$, it is $K = O(K)T$. So again $1 \neq \mathrm{Z}(T) \cap O_2(K) \leq \mathrm{Z}(K)$ follows. Altogether, $\mathrm{Z}(K)$ is of even order.

□

Throughout the following, we assume that Hypothesis 1.4 holds and, using Lemma 3.2, that $F^*(G)$ is a non-abelian simple group which contains $F^*(H)$.

So $F^*(H)$ is a simple group of Lie type over a field $GF(q)$ for $q = 2^f$. We fix $S \in \mathrm{Syl}_2(H)$ and $S_1 := S \cap F^*(H)$. We further choose a root system Σ for K such that $S_1 = \prod_{\alpha \in \hat{\Sigma}^+} X_{\hat{\alpha}}$ holds. Let R be a long root subgroup in $\mathrm{Z}(S_1)$. In particular, R is an elementary abelian group of order q. We fix this notation for the rest of this thesis.
We prove $\mathbb{C}_G(s) \leq H$ for every involution s in R if $F^*(H)$ is neither isomorphic to $Sp_{2n}(2^f)'$, $F_4(2^f)$ nor to a Suzuki-Ree group. In general we show that the centralizer of every 2-central involution in G coincides with the centralizer in H.

Lemma 3.9: Let the notation and properties listed in Hypothesis 1.4 hold.

(a) For every element $s \in \mathrm{Z}(S)^\#$, the centralizer $\mathbb{C}_G(s)$ is contained in H.

(b) If $F^*(H)$ is not isomorphic to $Sp_{2n}(2^f)'$, $F_4(2^f)$, ${}^2F_4(2^f)'$ or to $Sz(2^f)$, then $\mathbb{C}_G(r) \leq H$ holds for every $r \in R^\#$.

Proof: For $s \in \mathrm{Z}(S)^\#$ one gets $1 \neq \langle s \rangle \trianglelefteq S$ and therefore, by Hypothesis 1.4, $\mathbb{C}_G(s) = \mathrm{N}_G(\langle s \rangle) \leq H$. So part (a) holds.
Set $|R| = q = 2^f$ for $f \in \mathbb{N}$ and $S_1 = S \cap F^*(H)$. If $F^*(H)$ is not isomorphic to $Sp_{2n}(2^f)'$ or to $F_4(2^f)$, it is $1 \neq R = \mathrm{Z}(S_1) \trianglelefteq S$. So, again by 1.4, $\mathrm{N}_G(R) = \mathrm{N}_H(R)$ holds. Furthermore, there is an involution $s \in \mathrm{Z}(S) \cap R$, implying $\mathbb{C}_G(s) \leq H$.
It is $F^*(H)$ not a Suzuki-Ree group, so by Example 3.2.6 and Theorem 2.4.8 in [GLS3], all elements of $R^\#$ are conjugate in H, as the Lie-parabolic subgroup $\mathrm{N}_H(R)$ acts transitively on the elements of $R^\# = \mathrm{Z}(S_1)^\#$. Hence part (b) follows. □

The previous lemma provides $\mathbb{C}_G(r) \leq H$ for a 2-central involution r. Hence it is $\mathbb{C}_{F^*(G)}(r) \leq H \cap F^*(G)$. We need to show in the following $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ to apply D. Holt's result from Lemma 2.44 to the simple group $F^*(G)$.
In the following chapter the previous lemma is heavily used to prove Theorem 1.5 for $F^*(H)$ being isomorphic to a linear, unitary or symplectic group or to an exceptional group of type $Sz(2^f)$, $G_2(2^f)'$, ${}^2F_4(2^f)'$ or $F_4(2^f)$.

Chapter 4

Some families of classical and exceptional groups

In this chapter the statement of Theorem 1.5 is proved for $F^*(H)$ being isomorphic to a linear, unitary or symplectic group. The main result is also shown if $F^*(H)$ is isomorphic to $Sz(2^f)$, $F_4(2^f)$, ${}^2F_4(2^f)'$ or $G_2(2^f)'$ for some $f \in \mathbb{N}$. The strategy of the proof is to show first that $H \cap F^*(G)$ controls the $F^*(G)$-fusion of a 2-central involution. This, together with Lemma 3.9 and the result of Holt in Lemma 2.44, implies that either $F^*(G)$ is an alternating group or that $F^*(G) = H \cap F^*(G)$ holds. We show that the first case may only occur for $F^*(H) \cong L_4(2) \cong A_8$. In every other case, as $F^*(H)$ is a subgroup of $F^*(G)$, it is $F^*(H) = F^*(G)$. By Frattini's argument, this implies $G = F^*(H)\,\mathrm{N}_G(S_1) \leq H$.

In the following lemma, Theorem 1.5 is proved for $F^*(H)$ being a group of Lie rank 1.

Lemma 4.1: Assume Hypothesis 1.4. If $F^*(H)$ is isomorphic to a simple group $Sz(q)$, $L_2(q)$ or $U_3(q)$ for $q = 2^f$, then $G = H$ holds.

Proof: We assume Hypothesis 1.4. So let be $S \in \mathrm{Syl}_2(H)$ and let be $S_1 := S \cap F^*(H)$.

We first consider $F^*(H)$ being isomorphic to $Sz(q)$ for $q = 2^f \geq 8$. Then all involutions in $F^*(H)$ are conjugate in $F^*(H)$ to an involution $s \in \mathrm{Z}(S_1)$. Additionally, $|\mathrm{Out}(F^*(H)|$ is odd by Lemma 2.31, so it is $S = S_1$. Using Lemma 3.9(a), $\mathbb{C}_{F^*(G)}(s) \leq H \cap F^*(G)$ holds. As there is only one conjugacy class of involutions in S, using $2 \nmid |G : H|$, $s^{F^*(G)} \cap (H \cap F^*(G)) = s^{H \cap F^*(G)}$ follows.

Let be $F^*(H) \cong L_2(q)$ with $q = 2^f \geq 4$. There is only one conjugacy class of involutions in $F^*(H)$, so we consider an involution $r \in \mathrm{Z}(S_1)$. Then by Lemma 3.9 it is $\mathbb{C}_G(r) \leq H$. It is $\mathbb{C}_{F^*(H)}(r) = S_1$, so $\mathbb{C}_G(r) = \mathbb{C}_H(r)$ is solvable.
For $F^*(H) \neq H$, let t be an involution in $S \setminus S_1$. Then t induces a field automorphism

on $F^*(H)$ and $\mathbb{C}_H(t)$ involves a group $L_2(q_0)$ for $q_0^2 = q$. If $F^*(H)$ is not isomorphic to $L_2(4)$, it is $L_2(q_0)$ a non-abelian simple group; then $\mathbb{C}_H(t)$ cannot be embedded into $\mathbb{C}_H(r)$. For $F^*(H) \cong L_2(4)$, it is $\mathrm{Out}(F^*(H)) \cong Z_2$. In this case $\mathbb{C}_H(r)$ does not involve a group $L_2(q_0) \cong S_3$. Altogether, it is $\mathbb{C}_{F^*(G)}(r) \leq H \cap F^*(G)$ and $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$.

Let be $F^*(H) \cong U_3(q)$ with $q = 2^f \geq 4$. Also in this case, there is only one conjugacy class of involutions in $F^*(H)$. Let r be an involution in $\mathrm{Z}(S_1)$. Again by Lemma 3.9, it is $\mathbb{C}_G(r) \leq H$. It is $\mathbb{C}_{F^*(H)}(r) \cong Q_1(q){:}\, Z_{\frac{q+1}{(3,q+1)}}$, where $Q_1(q) = q_-^{1+2}$ denotes a semi-extraspecial group of $-$-type, see Lemma 2.34. So $\mathbb{C}_G(r) = \mathbb{C}_H(r)$ is solvable.
In case of $F^*(H) \neq H$, we consider an involution $t \in S \setminus S_1$. Then $\mathbb{C}_H(t)$ involves a group $L_2(q)$. As $q \geq 4$ holds, it is $L_2(q)$ a non-abelian simple group. Hence $\mathbb{C}_H(t)$ cannot be embedded into $\mathbb{C}_H(r)$. So also for $F^*(H) \cong U_3(q)$ it is $\mathbb{C}_{F^*(G)}(r) \leq H \cap F^*(G)$ and $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$.

So in all considered cases, we can apply Holt's result from Lemma 2.44. Then either $F^*(G)$ is an alternating group or it is $F^*(G) = H \cap F^*(G)$. The first case could possibly occur for $F^*(H) \cong A_5$ and $F^*(G) \cong A_6$. But this contradicts $2 \nmid |G : H|$ from Hypothesis 1.4. Hence $F^*(G) = F^*(H)$ holds. By Frattini's argument, this implies $G = F^*(H)\,\mathrm{N}_G(S) \leq H$.

□

Lemma 4.2: Assume Hypothesis 1.4. If $F^*(H)$ is isomorphic to $L_n(q)$ or $U_n(q)$ with $q = 2^f$ and $n \leq 4$, then Theorem 1.5 holds.

Proof: Let $F^*(H)$ be isomorphic to $L_n(q)$ or $U_n(q)$ with $q = 2^f$ and $n \leq 4$. By Lemma 3.9, $\mathbb{C}_G(r) = \mathbb{C}_H(r)$ holds for all elements $r \in R^\# = \mathrm{Z}(S_1)$. Then also $\mathbb{C}_{F^*(G)}(r) \leq H \cap F^*(G)$ follows. By the previous lemma, we may assume that $F^*(H)$ is isomorphic to $L_3(q)$, $L_4(q)$ or $U_4(q)$ for $q = 2^f$.
By (4.2) and (6.1) in [AsSe], there is only one conjugacy class of involutions in $L_3(q)$ and exactly two conjugacy classes in $L_4(q)$ and $U_4(q)$. The information about centralizers of involutions in these groups can be found using (4.1)–(4.6) and (6.1)–(6.2) in [AsSe] and by using the information about outer automorphisms collected in Lemma 2.31.

At first, let be $F^*(H) \cong L_3(q)$. Then all involutions in $F^*(H)$ are conjugate in H to an involution $r \in \mathrm{Z}(S_1)$. If $F^*(H) = H$, then $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ holds. If $F^*(H)$ is a proper subgroup of H, let x be an involution in $S \setminus S_1$. It is $\mathbb{C}_{F^*(H)}(r) \cong D_1(q){:}\, Z_{\frac{q-1}{(3,q-1)}}$, where $D_1(q) = q_+^{1+2}$ denotes a semi-extraspecial group of $+$-type, see Lemma 2.34. In particular, $\mathbb{C}_H(r)$ is solvable. Now x can be chosen to induce a field automorphism, a graph automorphism or a field-graph automorphism of $F^*(H) \cong L_3(q)$. In the field automorphism case, $\mathbb{C}_{F^*(H)}(x)$ involves $L_3(q_0)$ with $q_0^2 = q$, which is not solvable. If x induces a graph automorphism, then we have $\mathbb{C}_{F^*(H)}(x) \cong L_2(q)$. And if x induces a field-graph automorphism, then $\mathbb{C}_{F^*(H)}(x)$ involves $U_3(q_0)$ with $q_0^2 = q$. In all three cases $\mathbb{C}_G(x)$ is not solvable, except for $\mathbb{C}_{L_3(2)}(x) \cong L_2(2) \cong S_3$ or $\mathbb{C}_{L_3(4)}(x) \cong U_3(2) \cong E_{3^2} : Q_8$. For $F^*(H) \cong L_3(2)$, it is $\mathbb{C}_{F^*(H)}(r)$ a 2-group and $\mathrm{Out}(F^*(H)) \cong Z_2$, so $\mathbb{C}_H(r)$ does not involve a symmetric group S_3. And for $F^*(H) \cong L_3(4)$, $\mathbb{C}_{F^*(H)}(r)$ is also a 2-group. Hence $\mathbb{C}_H(r)$ does not involve $E_{3^2} : Q_8$, as it is $\mathrm{Out}(L_3(4)) \cong Z_2 \times S_3$ by [CoCu]. Therefore, $\mathbb{C}_{F^*(G)}(x)$ cannot be embedded into $\mathbb{C}_H(r) = \mathbb{C}_G(r)$. Hence $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ holds for $F^*(H) \cong L_3(q)$. This, together with $\mathbb{C}_{F^*(G)}(r) \leq H \cap F^*(G)$, see Lemma 3.9, implies $F^*(H) = F^*(G)$, by using Holt's result from Lemma 2.44. By Frattini's argument and Hypothesis 1.4, $G = F^*(H)\,\mathrm{N}_G(S) \leq H$ follows.

For $F^*(H) \cong L_4(q)$ or $F^*(H) \cong U_4(q)$ let $r \in \mathrm{Z}(S_1)$ be a 2-central involution and $t \in S_1$ a representative of the conjugacy class of involutions, which are not 2-central in $F^*(H)$. We identify in this proof $F^*(H)$ with $L_4(q)$ or $U_4(q)$, respectively. Then it is $\mathbb{C}_{L_4(q)}(r) \cong q^{1+4} : GL_2(q)$, $\mathbb{C}_{U_4(q)}(r) \cong q^{1+4} : GU_2(q)$ and $O^{2'}(\mathbb{C}_{F^*(H)}(t)) \cong E_{q^4} : L_2(q)$. As $O_2(\mathbb{C}_{F^*(H)}(r))$ does not contain an elementary abelian normal subgroup of order q^4, a group $L_2(q)$ acts on, $O^{2'}(\mathbb{C}_{F^*(H)}(t))$ cannot be embedded into $\mathbb{C}_{F^*(H)}(r)$. So $r^{F^*(G)} \cap F^*(H) = r^{H \cap F^*(G)}$ follows. In case of $S_1 = S$, $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ holds.

For $S_1 \neq S$, let x be an involution in $S \setminus S_1$. Then in case of $F^*(H) \cong L_4(q)$, the following holds: The involution x can be chosen to induce a field, a field-graph or a graph automorphism of $L_4(q)$. If x induces a field or a field-graph automorphism, $\mathbb{C}_{L_4(q)}(x)$ involves a simple group $L_4(q_0)$ or $U_4(q_0)$ for $q_0^2 = q$. So both centralizers are not isomorphic to a subgroup of $\mathbb{C}_H(r)$, see Lemma 2.29. If x induces a graph automorphism, there are two conjugacy classes of involutions: Either it is $\mathbb{C}_{L_4(q)}(x) \cong Sp_4(q)$, which is not involved in $\mathbb{C}_H(r)$, or $\mathbb{C}_{L_4(q)}(x) \cong Sp_4(q) \cap \mathbb{C}_{Sp_4(q)}(r) \cong O_2(\mathbb{C}_{L_4(q)}(x))L_2(q)$ holds, where $O_2(\mathbb{C}_{L_4(q)}(x))$ is elementary abelian of order q^3 with $R \leq O_2(\mathbb{C}_{L_4(q)}(x))$. It is $O_2(\mathbb{C}_{L_4(q)}(x))/R$ the natural $L_2(q)$-module.
In the latter case, we distinguish between the cases $q > 2$ and $q = 2$: For $q > 2$, $\mathbb{C}_{L_4(q)}(x)$ involves a simple group $L_2(q)$ and an embedding of $\mathbb{C}_{L_4(q)}(x)$ into $\mathbb{C}_H(r)$ embeds $O_2(\mathbb{C}_{L_4(q)}(x))$ into $O_2(\mathbb{C}_H(r))$ and maps R onto R, compare Proposition 3.2 in [CKS]. Hence if there is an element $g \in G$ such that $(\mathbb{C}_{L_4(q)}(x))^g \leq \mathbb{C}_H(r)$ holds, g normalizes $R \trianglelefteq S$, implying $g \in H$. The case $q = 2$ is treated at the end of this proof.

In case $F^*(H) \cong U_4(q)$ with $q > 2$, the arguments are the same as in the $L_4(q)$-case: There are also two conjugacy classes of outer involutions such that for an involution $x \in S \setminus S_1$, either $\mathbb{C}_{U_4(q)}(x) \cong Sp_4(q)$ or $\mathbb{C}_{U_4(q)}(x) \cong O_2(\mathbb{C}_{U_4(q)}(x))L_2(q)$ holds, where $O_2(\mathbb{C}_{U_4(q)}(x))$ is elementary abelian of order q^3 with $R \leq O_2(\mathbb{C}_{U_4(q)}(x))$. Also here it is $O_2(\mathbb{C}_{U_4(q)}(x))/R$ the natural $L_2(q)$-module; this is exactly as before in the case $F^*(H) \cong L_4(q)$. As $Sp_4(q)$ cannot be embedded into $\mathbb{C}_H(r)$, we are left with $\mathbb{C}_{U_4(q)}(x) \cong q^3 L_2(q)$, which can be embedded into $\mathbb{C}_H(r)$. As above for $q > 2$, an embedding of $\mathbb{C}_{U_4(q)}(x)$ into $\mathbb{C}_{U_4(q)}(r)$ via conjugation with an element $g \in G$ maps R onto R. Hence $g \in \mathrm{N}_G(R) \leq H$ follows.

Altogether, if it is $q > 2$, hence if $F^*(H)$ is not isomorphic to $L_4(2)$ or $U_4(2)$, $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ holds. For $H \cong L_4(2)$ or $H \cong U_4(2)$, also $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ follows. Additionally, $\mathbb{C}_{F^*(G)}(r)$ is contained in $H \cap F^*(G)$ by Lemma 3.9. Then Holt's result, compare Lemma 2.44, implies that either $F^*(H)$ equals $F^*(G)$ or it is $F^*(H) \cong A_{n-1}$ and $F^*(G) \cong A_n$. Hence either $G = H$ follows by Frattini's argument or $F^*(H) \cong L_4(2) \cong A_8$ together with $F^*(G) \cong A_9$ arises. This exceptional case is explicitly described in Lemma 3.5.

We are left with $F^*(H) \cong L_4(2)$ or $F^*(H) \cong U_4(2)$ and $F^*(H)$ being a proper subgroup of H. For $F^*(H) \cong L_4(2) \cong A_8$, it is $H \cong L_4(2){:}\, Z_2 \cong S_8$ and for $F^*(H) \cong U_4(2) \cong PSp_4(3)$, it is $H \cong U_4(2){:}\, Z_2$ by [CoCu]. In particular, it is $H = H \cap F^*(G)$ if we suppose that $F^*(H)$ is a proper subgroup of $F^*(G)$. Let be $S \in \mathrm{Syl}_2(H)$. Then $S \cong D_8 \wr Z_2$ holds. By Lemma 3.2, $F^*(G)$ is simple. By Theorem 3.15 in [Mas], $F^*(G)$ must be isomorphic to one of the following groups: A_{10}, A_{11}, $PSL_4(m)$ for $m \equiv 3 \mod 4$ or $PSU_4(m)$ for $m \equiv 1 \mod 4$. To restrict these possibilities to $F^*(G) \cong PSL_4(3)$ and $H = H \cap F^*(G) \cong U_4(2){:}\, Z_2$, we use arguments from Proposition 14.3 in [PPSS]. Calculations, excluding the trivial ones, were performed using [GAP].

Let first $F^*(G)$ being isomorphic to A_{10}. As by Table 5.2.A in [KlLi], a minimal permutation representation of $U_4(2)$ is of degree 27, we may identify H with the symmetric group S_8 on the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ where each element in $S_8 \setminus A_8$ is multiplied with the additional transposition $(9, 10)$. Let S be a Sylow 2-subgroup of H with $\mathrm{Z}(S) = \langle (1, 2)(3, 4)(5, 6)(7, 8) \rangle$. It is $E := \langle (1, 2)(9, 10), (3, 4)(9, 10), (5, 6)(9, 10), (7, 8)(9, 10) \rangle \leq A_{10}$ a normal subgroup of S. Then $\mathrm{N}_{A_{10}}(E)/E \cong S_5$, while $\mathrm{N}_H(E)/E \cong S_4$ holds. This contradicts Hypothesis 1.4, as $\mathrm{N}_G(E) \neq \mathrm{N}_H(E)$. So $F^*(G)$ cannot be isomorphic to A_{10}.

We assume now that $F^*(G)$ is isomorphic to A_{11} and identify these groups to simplify notation. It is $r = (1, 2)(3, 4)(5, 6)(6, 7)$ a 2-central involution in A_{11}. Then it is $O_3(\mathbb{C}_{A_{11}}(r)) = \langle (9, 10, 11) \rangle$. But by Hypothesis 1.4, G is of parabolic characteristic 2, so $O_3(\mathbb{C}_{F^*(G)}(r))$ must be trivial. Hence $F^*(G)$ cannot be isomorphic to A_{11}, either.

Let now $F^*(G)$ be isomorphic to $PSL_4(m)$ for $m \equiv 3 \mod 4$ or to $PSU_4(m)$ for $m \equiv 1 \mod 4$. Let $r \in S$ be a 2-central involution. For $m > 3$, $\mathbb{C}_{F^*(G)}(r)$, and then $\mathbb{C}_G(r)$ has components $SL_2(m)$, see Table 4.5.1 in [GLS3]. As G is of parabolic characteristic 2, this implies $m = 3$. So $F^*(G) \cong PSL_4(3)$ follows. As $|PSL_4(3)|$ is not divisible by 7 by [CoCu], it is $F^*(H) \cong U_4(2)$. Hence it is $H \cong U_4(2){:}\, Z_2$ and $F^*(G) \cong PSL_4(3)$. As a dihedral group of order 8 contains exactly two elementary abelian subgroups of order 4, $S \cong D_8 \wr Z_2$ has exactly two normal elementary abelian subgroups of order 16. Let E_1 and E_2 be these normal elementary abelian subgroups. Let E_1 consist of 5 involutions out of the conjugacy class of the 2-central involutions and 10 involutions from the other class. Then the 2-local parabolic group $\mathrm{N}_H(E_1)$ is isomorphic to $E_1{:}\, S_5$ and is contained in H by page 26 in [CoCu]. For the other group E_2, in analogy to the A_{10}-case above, $\mathrm{N}_H(E_2)/E_2 \cong S_4$ holds. But $\mathrm{N}_{F^*(G)}(E_2)/E_2$ is isomorphic to S_5. So $\mathrm{N}_G(E_2) \neq \mathrm{N}_H(E_2)$ follows. Therefore, $H \cap F^*(G) \cong U_5(2)\colon Z_2$ in $F^*(G) \cong PSL_4(3)$ contradicts Hypothesis 1.4.

Hence all cases with $F^*(H) \cong L_4(2)$ or $F^*(H) \cong U_4(2)$ and $F^*(H)$ being a proper subgroup of H cannot occur and the statement of the lemma holds. $\square$

In the following four lemmas the statement of Theorem 1.5 is shown for $F^*(H)$ being a linear or unitary group over a vector space in dimension at least 5.

Lemma 4.3: Assume Hypothesis 1.4.

(a) Let $F^*(H)$ be isomorphic to $L_n(q)$ with $n = 2k+1$ and $q = 2^f$ for $f, k \in \mathbb{N}$ and $k \geq 2$. Then $H \cap F^*(G)$ controls the $F^*(G)$-fusion of involutions in $F^*(H)$.

(b) Let $F^*(H)$ be isomorphic to $L_{2k}(q)$ with $q = 2^f$ for $k, f \in \mathbb{N}$ and $k \geq 3$. For a Sylow 2-subgroup S of H, the Thompson subgroup $J(S)$ is a maximal elementary abelian subgroup of H. And $H \cap F^*(G)$ controls the $F^*(G)$-fusion of involutions in $F^*(H)$.

Proof:

(a) Suppose that $F^*(H)$ is isomorphic to $L_n(q)$ with $n = 2k+1$ and $q = 2^f$ for f, $k \in \mathbb{N}$ and $k \geq 2$. Let V be the natural module for $F^*(H)$, $S \in \mathrm{Syl}_2(H)$ and $S_1 = S \cap F^*(H)$. By Lemma 2.32, the 2-rank of $F^*(H)$ equals $k(k+1)f$.
Without restriction, we identify S_1 with the set of upper right $n \times n$-matrices where all diagonal entries are equal to 1. We denote the $m \times m$-identity matrix with I_m. Then we choose elementary abelian 2-groups E_1 and E_2 of maximal rank in S_1 in the following way: Let E_1 be the set of block matrices of shape $\begin{pmatrix} I_k & X \\ 0 & I_{k+1} \end{pmatrix}$, where X is a $k \times (k+1)$-block, and let E_2 be the set of matrices of shape $\begin{pmatrix} I_{k+1} & Y \\ 0 & I_k \end{pmatrix}$, where Y is a $(k+1) \times k$-block.
Then $[V, E_1] \leq \mathbb{C}_V(E_1)$ holds, where $\mathbb{C}_V(E_1)$ is of $GF(q)$-dimension $k+1$. Also $[V, E_2] \leq \mathbb{C}_V(E_2)$ holds, where $[V, E_2]$ is a hyperplane in $\mathbb{C}_V(E_1)$. So for every element $x \in E_1 \cap E_2$, $[V, x] \leq [V, E_2] \leq \mathbb{C}_V(E_1)$ holds. It is $\langle E_1, E_2\rangle = E_1E_2$, as E_1 and E_2 are normal in S_1. Additionally, $E_1 \cap E_2 = \mathrm{Z}(E_1E_2)$ is elementary abelian of order $q^{(k^2)}$.
Let $t \in S_1$ be an involution. As t centralizes a subspace of dimension $k+1$, t is conjugate in $L_n(q)$ to an involution in E_1, so we may assume $t \in E_1$. Then $[V, t]$ is contained in a hyperplane of $\mathbb{C}_V(E_1)$. As E_1 can be seen as a vector space over $GF(q)$, $\mathrm{Aut}(E_1)$

induces a group $L_{k+1}(q)$ on $\mathbb{C}_V(E_1)$; thus all hyperplanes in $\mathbb{C}_V(E_1)$ may supposed to be conjugate. In particular, t is $F^*(H)$-conjugate to an involution in $E_1 \cap E_2$.

It is $M := \mathrm{N}_{F^*(H)}(E_1E_2)/(E_1E_2) \cong L_k(q) \times L_k(q)$ and $E := E_1 \cap E_2$ is the corresponding tensor product module for M. So for both factors of M, E is a direct product of k natural $L_k(q)$-modules. Then by Lemma 2.54, E is no F-module for $L_k(q)$.
To show that E is no F-module for $M = \mathrm{N}_{F^*(H)}(E_1E_2)/(E_1E_2)$, we assume the existence of a quadratic offender A in M.
If $(k, q) \neq (2, 2)$ holds, M is a direct product of two non-abelian simple groups, hence $A \leq M$ has to normalize both simple components of M, which are isomorphic to $L_k(q)$. We denote these components by L_1 and L_2. If (k, q) equals $(2, 2)$, in which case $F^*(H) \cong L_5(2)$ holds, M is isomorphic to $S_3 \times S_3$. And also here A normalizes the factors $L_i \cong S_3$ for $i \in \{1, 2\}$.
We assume first that A is a direct product $A_1 \times A_2$ such that A_1 centralizes L_1 and acts faithfully on L_2, whereas A_2 acts trivially on L_2 and faithfully on L_1. As A acts quadratically on E, $[E, A_2] \leq \mathbb{C}_E(A_1)$ holds and L_2 acts on $[E, A_2]$. Then $[E, A_2, A_1] = 1$, so also $[E, A_2, L_2]$ is trivial. But E is a direct product of natural L_2-modules, so $\mathbb{C}_E(L_2) = 1$ holds. Hence, it is $[E, A_2] = 1$ and $A_2 \leq E_1E_2$. So A_2 is trivial. Analogously, $A_1 \leq E_1E_2$ follows, so A_1 is trivial. Hence A acts faithfully on L_1, but E is no F-module for $L_1 \cong L_k(q)$, implying that $A \leq E_1E_2$ holds. But then A centralizes $\mathrm{Z}(E_1E_2) = E$. So E is no F-module for M. This implies $J(S_1) = E_1E_2$, as otherwise an elementary abelian 2-subgroup of maximal rank, which is not contained in E_1E_2, is an offender with E an F-module for M.

To show that E_1E_2 coincides with the Thompson subgroup $J(S)$, we assume that an involution $x \in S \setminus S_1$ is contained in a maximal elementary abelian subgroup A. Then, using Lemmas 2.31 and 2.32 for information about automorphisms and 2-ranks, the following cases are possible:
The involution x admits a field automorphism of $F^*(H) \cong L_{2k+1}(q)$, in which case $O^{2'}(\mathbb{C}_{F^*(H)}(x)) \cong L_{2k+1}(q_0)$ for $q_0^2 = q$ holds. Then $m_2(\mathbb{C}_H(x)) = \frac{m_2(F^*(H))}{2} + 1$ holds. Or x admits a graph automorphism, in which case $\mathbb{C}_{F^*(H)}(x)$ is isomorphic to $Sp_{2k}(q)$. So in both cases, it is $m_2(\mathbb{C}_H(x)) = \frac{m_2(F^*(H))}{2} + 1 = \frac{k \cdot (k+1) \cdot f}{2} + 1 < k \cdot (k+1) \cdot f = m_2(F^*(H))$, using $k \geq 2$. Thus, there cannot be an elementary abelian 2-group A of maximal rank in S with $x \in A$. Obviously the same holds if x admits a field-graph automorphism. Hence $E_1E_2 = J(S)$ holds.

As every involution $t \in F^*(H)$ is $F^*(H)$-conjugate to an involution in $E_1 \cap E_2 = \mathrm{Z}(E_1E_2) = \mathrm{Z}(J(S))$, Lemma 2.8 implies that $\mathrm{N}_G(J(S))$ controls the G-fusion of involutions in $F^*(H)$. It is $1 \neq J(S) \trianglelefteq S$, so by Hypothesis 1.4, it is $\mathrm{N}_G(J(S)) \leq H$. In particular, also $\mathrm{N}_{F^*(G)}(J(S)) \leq H \cap F^*(G)$ holds and $H \cap F^*(G)$ controls the $F^*(G)$-fusion of involutions in $F^*(H)$.

(b) Suppose now that $F^*(H)$ is isomorphic to $L_n(q)$ with $n = 2k$ and $q = 2^f$ for $f, k \in \mathbb{N}$ and $k \geq 3$. Let V be the natural module for $F^*(H)$, $S \in \mathrm{Syl}_2(H)$ and $S_1 = S \cap F^*(H)$. By Lemma 2.32, the 2-rank of $F^*(H)$ equals $k^2 \cdot f$.
Without restriction, we identify S_1 with the set of upper right $n \times n$-matrices where all diagonal entries are equal to 1. We denote the $m \times m$-identity matrix with I_m. Then we choose an elementary abelian 2-group E of maximal rank in S_1 in the following way:
Let E be the set of block matrices of shape $\begin{pmatrix} I_k & X \\ 0 & I_k \end{pmatrix}$, where X is a $k \times k$-block.
Then $[V, E] \leq \mathbb{C}_V(E)$ holds, where $\mathbb{C}_V(E)$ is of $GF(q)$-dimension k. It is E normal in S_1 and E is elementary abelian of order $q^{(k^2)}$.
Let $t \in S_1$ be an involution. As t centralizes a subspace of dimension k, t is conjugate in $L_n(q)$ to an involution in E, so we may assume $t \in E$.

It is $M := \mathrm{N}_{F^*(H)}(E)/E \cong L_k(q) \times L_k(q)$ and E is the corresponding tensor product module for M. So for each factor $L_k(q)$, E is a direct product of k natural modules. Then by Lemma 2.54, E is no F-module for the simple components $L_k(q)$. To show that E is no F-module for $M = \mathrm{N}_{F^*(H)}(E)/E$, one can simply copy the above arguments. Then $E = J(S_1)$ follows.

To show that E coincides with the Thompson subgroup $J(S)$, we assume that an involution $x \in S \setminus S_1$ is contained in a maximal elementary abelian subgroup A. Then, using Lemmas 2.31 and 2.32 for information about automorphisms and 2-ranks, the following cases are possible:
If x admits a field automorphism of $F^*(H) \cong L_{2k}(q)$, $O^{2'}(\mathbb{C}_{F^*(H)}(x)) \cong L_{2k}(q_0)$ with $q_0^2 = q$ holds. In this case, it is $m_2(\mathbb{C}_{F^*(H)}(x)) = \frac{m_2(F^*(H))}{2}$. But as $m_2(\mathbb{C}_H(x)) = \frac{m_2(F^*(H))}{2} + 1 = \frac{k^2 \cdot f}{2} + 1 < k^2 \cdot f = m_2(F^*(H))$ holds, there cannot exist an elementary abelian 2-group of maximal rank in S with $x \in A$.
So we assume now that x admits a graph automorphism of $F^*(H)$. Then one gets either $x = y$ or $x = ys$ for $s \in \mathrm{Z}(S_1)$ and y such that $\mathbb{C}_{F^*(H)}(y) \cong Sp_{2k}(q)$. It is $\mathbb{C}_{F^*(H)}(ys) = \mathbb{C}_{F^*(H)}(y) \cap \mathbb{C}_{F^*(H)}(s)$. Hence, it is $m_2(Sp_{2k}(2^f)) + 1 = \binom{k+1}{2} \cdot f + 1 < k^2 \cdot f = m_2(F^*(H))$, using $k \geq 3$. So also in this case, there cannot exist an elementary abelian 2-group of maximal rank in S with $x \in A$. Obviously for the same reason, x cannot admit a field-graph automorphism either. Hence $E = J(S)$ holds and $J(S)$ is a maximal elementary abelian subgroup in H.

As every involution $t \in F^*(H)$ is $F^*(H)$-conjugate to an involution in $E = J(S) = \mathrm{Z}(J(S))$, Lemma 2.8 implies that $\mathrm{N}_G(J(S))$ controls the G-fusion of involutions in $F^*(H)$ and by Hypothesis 1.4, it is $\mathrm{N}_G(J(S)) \leq H$. Then also $H \cap F^*(G)$ controls the $F^*(G)$-fusion of involutions in $F^*(H)$.

$\square$

Lemma 4.4: Assume Hypothesis 1.4. If $F^*(H)$ is isomorphic to $L_n(q)$ with $n \geq 5$ and $q = 2^f$ for $f \in \mathbb{N}$, then $G = H$ holds.

Proof: As before, we fix $S \in \mathrm{Syl}_2(H)$ and $S_1 = S \cap F^*(H)$. Let $r \in R = \mathrm{Z}(S_1)$ be an involution. Lemma 3.9 implies $\mathbb{C}_{F^*(G)}(r) = \mathbb{C}_{H \cap F^*(G)}(r)$. And by Lemma 4.3, $r^{F^*(G)} \cap F^*(H) = r^{H \cap F^*(G)}$ holds.
Let x be an involution in $S \setminus S_1$. To show that x cannot be $F^*(G)$-conjugate to r, we assume the opposite. Then $\mathbb{C}_{F^*(G)}(x) \cong \mathbb{C}_{H \cap F^*(G)}(r)$ holds. The simple group $L_{n-2}(q)$ is involved in the Levi complement of $\mathbb{C}_{H \cap F^*(G)}(r)$, see Lemma 2.36.
Using Lemma 2.30, x induces a field or a graph or a field-graph automorphism. If x induces a field automorphism, $\mathbb{C}_{F^*(H)}(x)$ involves the simple group $L_n(q_0)$ with $q_0^2 = q$, see Lemma 2.31. To embed $L_n(q_0)$ into $\mathbb{C}_{H \cap F^*(G)}(r)$, $L_n(q_0)$ must be embedded into $L_{n-2}(q)$, which contradicts Lemma 2.29.
If x induces a field-graph automorphism, then $\mathbb{C}_{F^*(H)}(x)$ involves a simple group $U_n(q_0)$ with $q_0^2 = q$, see Lemma 2.31. So $\mathbb{C}_{H \cap F^*(G)}(x)$ cannot be embedded into $\mathbb{C}_{H \cap F^*(G)}(r) = \mathbb{C}_{F^*(G)}(r)$ by construction of $U_n(q_0)$ and Lemma 2.29. Hence we may assume that x induces a graph automorphism.
In case of n being odd, $\mathbb{C}_{F^*(H)}(x)$ involves a simple group $Sp_{n-1}(q)$. Then $q^{n-1} - 1$ divides $|Sp_{n-1}(q)|$, and, except for $(q, n-1) = (2, 6)$, there is a Zsigmondy prime that does not divide $|L_{n-2}(q)| = q^{\binom{n-2}{2}} \prod_{i=2}^{n-2} (q^i - 1)$, see Lemma 2.11. For $n = 7$, the order of $Sp_6(2)$ equals $2^9 \cdot 3^4 \cdot 5 \cdot 7$, which does not divide $|L_5(2)| = 2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$ either. Hence $Sp_{n-1}(q)$ cannot be embedded into $L_{n-2}(q)$. In particular, if n is odd, $x \in S \setminus S_1$ cannot be $F^*(G)$-conjugate to r.
If n is even, there are two possibilities for $\mathbb{C}_{F^*(H)}(x)$, see Lemma 2.31: Either it is $\mathbb{C}_{F^*(H)}(x) \cong Sp_n(q)$ or $\mathbb{C}_{F^*(H)}(x) \cong \mathbb{C}_{Sp_n(q)}(s)$ for $s \in R$.
It is $|Sp_n(q)|$ divisible by $q^n - 1$, so again by Lemma 2.11 there is a Zsigmondy prime that divides the order of $Sp_n(q)$ but not the order of $L_{n-2}(q)$, except for $n = 6$. But also $|Sp_6(2)|$ does not divide $|L_4(2)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$. Hence in the first case, $\mathbb{C}_{H \cap F^*(G)}(x)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(r)$.
In the second case, using Proposition 3.2 in [CKS], $\mathbb{C}_{F^*(H)}(x)$ involves $L \cong Sp_{n-2}(q)$, which acts indecomposably on $O_2(\mathbb{C}_{F^*(H)}(x))/R$. In particular, an embedding of $\mathbb{C}_{H \cap F^*(G)}(x)$ into $\mathbb{C}_{F^*(G)}(r)$ normalizes R. Hence, if there exists an element $g \in F^*(G)$ such that $x^g = r$ holds, then it is $g \in \mathrm{N}_{F^*(G)}(R) \leq H \cap F^*(G)$ by Hypothesis 1.4.
Altogether, $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ follows. Together with $\mathbb{C}_{F^*(G)}(r) \leq H \cap F^*(G)$, Holt's result, compare Lemma 2.44, implies $F^*(G) = F^*(H)$ and therefore $G = H$. □

This completes the proof of Theorem 1.5 for $F^*(H)$ being isomorphic to a simple group $L_n(2^f)$ for n, $f \in \mathbb{N}$. In the following two lemmas, we use similar methods to prove Theorem 1.5

for $F^*(H) \cong U_n(2^f)$ and $n \geq 4$. The statements concerning the Thompson subgroup in Lemma 4.3 and Lemma 4.5 are used later in the proof of Lemma 6.3.

Lemma 4.5: Assume Hypothesis 1.4. Let S be a Sylow 2-subgroup of H and $S_1 = S \cap F^*(H)$. Let $F^*(H)$ be isomorphic to $U_n(q)$ with $n \geq 5$ and $q = 2^f$ for $f \in \mathbb{N}$. Then the following statements hold:

(a) If $(n, q) \neq (5, 2)$, then $H \cap F^*(G)$ controls the $F^*(G)$-fusion of involutions in $F^*(H)$.

(b) If $(n, q) = (5, 2)$, then $H \cap F^*(G)$ controls the $F^*(G)$-fusion of 2-central involutions in H.

(c) If n is even, the Thompson subgroup $J(S)$ is a maximal elementary abelian subgroup of H.

Proof: Suppose that $F^*(H)$ is isomorphic to $U_n(q)$ with $n = 2k+1$ for $k \geq 2$ or $n = 2k$ for $k \geq 3$ and $q = 2^f$ for $f \in \mathbb{N}$. Let V be the natural unitary module for $F^*(H)$, $S \in \mathrm{Syl}_2(H)$ and $S_1 = S \cap F^*(H)$. By Lemma 2.32, the 2-rank of $F^*(H)$ is $k^2 \cdot f$.
We consider an arbitrary involution $t \in S_1$. As t acts quadratically on V and centralizes a subspace of V of at least $GF(q)$-dimension k, we can choose a k-dimensional subspace U of V such that $[V, t] \leq U \leq \mathbb{C}_V(t) = [V, t]^\perp$ holds. Hence U is a maximal totally isotropic subspace.
The stabilizer of U in $F^*(H)$ is a maximal Lie-parabolic subgroup P of shape

$$q^{k\cdot(2n-3k)} \colon (GL_k(q^2) \times SU_{n-2k}(q)),$$

where $O_2(P)$, denoted by $q^{k\cdot(2n-3k)}$, is a special 2-group, where $E := \mathrm{Z}(O_2(P))$ is elementary abelian of order $q^{(k^2)}$ and $O_2(P) \trianglelefteq S_1$, see 2.6.2 in [Wils]. Additionally, E is the tensor product module for $L_k(q^2)$.
As a consequence of Witt's Lemma, see (20.8) in [Asc1], all the maximal totally isotropic subspaces are conjugate in $\mathrm{Aut}(F^*(H))$. Hence without loss of generality, every involution in $F^*(H)$ is H-conjugate to an involution in E.
By Theorem B in [GuMa], E is no F-module for $L_k(q^2)$. Let $A \leq S_1$ be an arbitrary maximal elementary abelian subgroup. Then A cannot be a quadratic offender in $P/O_2(P)$ for E; hence $A \leq O_2(P)$ follows. So $A \leq \mathbb{C}_{S_1}(E)$ and $J(S_1) \leq O_2(P)$ follow and $E = \mathrm{Z}(O_2(P)) = \mathrm{Z}(J(S_1))$ holds.
In case $n = 2 \cdot k$, we additionally have that $O_2(P)$ is of order $q^{(k^2)}$; thus $E = O_2(P) = J(S_1)$ follows.
Let $(n, q) \neq (5, 2)$, so we have $k \geq 3$. Then we can show that $J(S_1)$ coincides with $J(S)$: We assume first that an involution $x \in S \setminus S_1$ is contained in a maximal elementary abelian subgroup A. By Lemma 2.31, for $n = 2k+1$, it is $\mathbb{C}_{F^*(H)}(x) \cong Sp_{2k}(2^f)$ and for $n = 2k$, it

is either $\mathbb{C}_{F^*(H)}(x) \cong Sp_{2k}(2^f)$ or $\mathbb{C}_{F^*(H)}(x) \cong \mathbb{C}_{Sp_{2k}(2^f)}(s)$ for an element $s \in \mathrm{Z}(S_1)$. By Lemma 2.32 for $k \geq 3$, it is $m_2(Sp_{2k}(2^f)) + 1 = \binom{k+1}{2} \cdot f + 1 < k^2 \cdot f = m_2(F^*(H))$. So an elementary abelian 2-group A of maximal rank in S which contains x cannot exist.

Hence $E = \mathrm{Z}(J(S))$ holds. If additionally n is even, $E = J(S_1) = J(S)$ holds.

As every involution in $F^*(H)$ is $\mathrm{Aut}(F^*(H))$-conjugate to an involution in E, $\mathrm{N}_{F^*(G)}(J(S)) \leq H \cap F^*(G)$ controls the $F^*(G)$-fusion of involutions in $F^*(H)$, using Hypothesis 1.4 and Lemma 2.8.

Let now $F^*(H)$ be isomorphic to $U_5(2)$. There are exactly two conjugacy classes of involutions in $F^*(H)$, see [CoCu]. Let r be a representative of the conjugacy class of 2-central involutions and let t be a representative of the conjugacy class of involutions, which are not 2-central in $F^*(H)$. It is $\mathbb{C}_{F^*(G)}(r) = \mathbb{C}_{H \cap F^*(G)}(r)$ by Lemma 3.9.

If H equals $F^*(H)$, then also $E = \mathrm{Z}(J(S))$ holds and we are done, using Lemma 2.8 and Hypothesis 1.4.

So we assume that $F^*(H)$ is a proper subgroup of $H \cap F^*(G)$. Then $H = H \cap F^*(G)$ is isomorphic to $\mathrm{Aut}(U_5(2))$ by [CoCu]. Let $x \in S \setminus S_1$ be an involution. Then $5 \mid |\mathbb{C}_{H \cap F^*(G)}(x)|$ as $\mathbb{C}_{F^*(H)}(x) \cong Sp_4(2) \cong S_6$ by Lemma 2.31. But $5 \nmid |\mathbb{C}_{H \cap F^*(G)}(r)|$, as $\mathbb{C}_{F^*(G)}(r) = \mathbb{C}_{H \cap F^*(G)}(r) = \mathbb{C}_H(r)$ is of order 156888 by [CoCu], so $\mathbb{C}_{F^*(G)}(x)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(r)$. Hence x cannot be $F^*(G)$-conjugate to r.

Assume now that t is $F^*(G)$-conjugate to r. Then it is $|S : S_1| = 2$ and every involution in S_1 is $F^*(G)$-conjugate to r. Hence $x^{F^*(G)} \cap S_1 = \emptyset$ holds. But by Thompson Transfer, see Lemma 2.7, then $F^*(G)$ cannot be simple which contradicts Lemma 3.2. Hence for $F^*(H) \cong U_5(2)$, $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ holds. □

Lemma 4.6: Assume Hypothesis 1.4. If $F^*(H)$ is isomorphic to $U_n(q)$ with $n \geq 5$ and $q = 2^f$ for $f \in \mathbb{N}$, then $G = H$ holds.

Proof: As before, we fix $S \in \mathrm{Syl}_2(H)$ and $S_1 = S \cap F^*(H)$. Let $r \in R = \mathrm{Z}(S_1)$ be an involution. Lemma 3.9 implies $\mathbb{C}_{F^*(G)}(r) = \mathbb{C}_{H \cap F^*(G)}(r)$. And by Lemma 4.5, $r^{F^*(G)} \cap F^*(H) = r^{H \cap F^*(G)}$ holds.

Let x be an involution in $S \setminus S_1$. To show that x cannot be $F^*(G)$-conjugate to r, we assume the opposite. Then $\mathbb{C}_{F^*(G)}(x) \cong \mathbb{C}_{H \cap F^*(G)}(r)$ holds. The group $U_{n-2}(q)$ is involved in the Levi complement of $\mathbb{C}_{F^*(G)}(r)$, see Lemma 2.36. The group $U_{n-2}(q)$ is simple, except for $F^*(H) \cong U_5(2)$.

At first let n be odd with $(n, q) \neq (5, 2)$. Using Lemma 2.31, it is $\mathbb{C}_{F^*(H)}(x) \cong Sp_{n-1}(q)$. As the Lie rank of $Sp_{n-1}(q)$ is $\frac{n-1}{2}$ and the Lie-rank of $U_{n-2}(q)$ is $\frac{n-3}{2}$, Lemma 2.29 implies that $\mathbb{C}_{H \cap F^*(G)}(x)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(r)$. Hence if n is odd, $x \in S \setminus S_1$ cannot be $F^*(G)$-conjugate to r.

If n is even, there are two possibilities for $\mathbb{C}_{F^*(H)}(x)$, see Lemma 2.31: Either it is $\mathbb{C}_{F^*(H)}(x) \cong Sp_n(q)$ or $\mathbb{C}_{F^*(H)}(x) \cong \mathbb{C}_{Sp_n(q)}(s)$ for an involution $s \in R$. The Lie-rank

of $Sp_n(q)$ is $\frac{n}{2}$. This exceeds $\frac{n-2}{2}$, which is the Lie rank of $U_{n-2}(q)$. Hence in the first case, again with Lemma 2.29, $\mathbb{C}_{H\cap F^*(G)}(x)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(r)$. In the second case, using Proposition 3.2 in [CKS], $\mathbb{C}_{F^*(H)}(x)$ involves $L \cong Sp_{n-2}(q)$, which acts indecomposably on $O_2(\mathbb{C}_{F^*(H)}(x))/R$. In particular, an embedding of $\mathbb{C}_{H\cap F^*(G)}(x)$ into $\mathbb{C}_{F^*(G)}(r)$ normalizes R. Hence, if there exists an element $g \in G$ such that $x^g = r$, then $g \in \mathrm{N}_G(R) \le H$ by Hypothesis 1.4.
We consider now $F^*(H) \cong U_5(2)$. Then by Lemma 4.5, $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H\cap F^*(G)}$ holds.
Altogether, $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H\cap F^*(G)}$ follows. Together with $\mathbb{C}_{F^*(G)}(r) \le H \cap F^*(G)$ this implies $F^*(G) = F^*(H)$, using Holt's result from Lemma 2.44. Using Frattini's argument, $G = H$ follows. □

Next we show for $F^*(H)$ being isomorphic to $G_2(2^f)'$, ${}^2F_4(q)'$, $F_4(q)$ or a symplectic group that $G = H$ holds. In all these cases, we use Lemma 3.9 and Holt's result from Lemma 2.44. To do so, we have to show that $H \cap F^*(G)$ controls the $F^*(G)$-fusion of a 2-central involution.

Lemma 4.7: Assume Hypothesis 1.4. If it is $F^*(H) \cong G_2(q)'$ for $q = 2^f$, then $G = H$ holds.

Proof: Suppose $F^*(H) \cong G_2(q)'$ for $q = 2^f$. We identify $F^*(H)$ and $G_2(q)'$ to simplify notation. Due to Lemma 3.9, it is $\mathbb{C}_{F^*(G)}(r) \le H \cap F^*(G)$ for all elements $r \in R^{\#}$.
Assume first $q \ge 4$. Then there are, according to (18.2) in [AsSe], exactly two conjugacy classes of involutions in $G_2(q)$, one of which is 2-central. Let $r \in R^{\#}$ be a representative of the class of 2-central involutions and let t be a representative of the other class. By (18.4) in [AsSe], it is $O^{2'}(\mathbb{C}_{G_2(q)}(r)) \cong q^{1+4} \colon L_2(q)$ and $O^{2'}(\mathbb{C}_{G_2(q)}(t)) \cong q^3 \colon L_2(q)$. Assuming r and t being $F^*(G)$-conjugate, implies the existence of an embedding of $\mathbb{C}_{F^*(G)}(t)$ into $\mathbb{C}_{F^*(G)}(r) = \mathbb{C}_{H\cap F^*(G)}(r)$ which maps $O^{2'}(\mathbb{C}_{G_2(q)}(t))/O_2(\mathbb{C}_{G_2(q)}(t)) \cong L_2(q)$ onto $O^{2'}(\mathbb{C}_{G_2(q)}(r))/O_2(\mathbb{C}_{G_2(q)}(r)) \cong L_2(q)$. Hence an embedding of $\mathbb{C}_{F^*(G)}(t)$ into $\mathbb{C}_{F^*(G)}(r)$ embeds $O_2(\mathbb{C}_{G_2(q)}(t))$ into $O_2(\mathbb{C}_H(r))$.
But for $q > 4$, $L_2(q)$ acts irreducibly on $O_2(\mathbb{C}_{G_2(q)}(r))$ modulo its center and there is no elementary abelian normal subgroup of order q^3 in $O_2(\mathbb{C}_{G_2(q)}(r)) \cong q^{1+4}$. Hence $O^{2'}(\mathbb{C}_{G_2(q)}(t))$ cannot be embedded into $\mathbb{C}_{F^*(G)}(r)$. So r and t are not conjugate in $F^*(G)$ for $q > 4$.
In case $q = 4$, by Lemma 2.37, $L_2(4) \cong A_5$ induces a direct sum of two permutation modules for the alternating group A_5 on $O_2(\mathbb{C}_{G_2(4)}(r))$ modulo its center, in which case it is not possible to embed an elementary abelian group of order 4^3 into $O_2(\mathbb{C}_{G_2(4)}(r))$. So for $q = 4$, also $r^{F^*(G)} \cap F^*(H) = r^{H\cap F^*(G)}$ holds.
Additionally, for $q \ge 4$, every involution $x \in S$ which induces an outer automorphism on $G_2(q)$, is, due to Lemma 2.31, conjugate to a field automorphism. Therefore, the centralizer of x in $G_2(q)$ involves a group $G_2(q_0)$ for $q_0^2 = q$. As Sylow 2-subgroups of $G_2(q_0)$ are not abelian, $G_2(q_0)$ does not occur as a subgroup of $L_2(q)$, because the Sylow 2-subgroups of $L_2(q)$

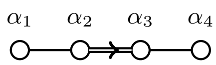
are abelian. Hence $\mathbb{C}_{F^*(G)}(x)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(r)$ and $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ holds for $q \geq 4$.
So we are left with $q = 2$. According to [CoCu], there is exactly one conjugacy class of involutions in $G_2(2)'$. So every involution $r \in G_2(2)'$ is 2-central and by Lemma 3.9, $\mathbb{C}_{F^*(G)}(r) = \mathbb{C}_{H \cap F^*(G)}(r)$ holds. The outer automorphism group of $G_2(2)'$ is of order 2. So let x be an involution in $S \setminus S_1$, inducing an outer automorphism. Then, by [CoCu], it is $\mathbb{C}_G(r) = \mathbb{C}_H(r) \cong 2_+^{1+4} S_3$, in which case no alternating group A_4 is involved in $\mathbb{C}_{F^*(G)}(r)$. But, due to [CoCu], $\mathbb{C}_{F^*(G)}(x)$ involves a group S_4, so also an alternating group A_4. So again $r^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ follows.
Altogether, Lemma 2.44 is applicable and $F^*(G) = F^*(H)$ follows which implies $G = H$ as before. □

Lemma 4.8: Assume Hypothesis 1.4. If it is $F^*(H) \cong {}^2F_4(q)'$, then $G = H$ holds.

Proof: We identify the groups $F^*(H)$ and ${}^2F_4(q)'$. By Lemma 2.31, there is no involution in $\mathrm{Aut}(F^*(H)) \setminus \mathrm{Inn}(F^*(H))$. So every involution in H is contained in $F^*(H)$. Furthermore, in ${}^2F_4(q)'$ there are, due to (18.2) in [AsSe], exactly two classes of involutions with representatives s and t, one of which is 2-central. Let the involution s be 2-central in H, so $\mathbb{C}_{F^*(G)}(s) = \mathbb{C}_{H \cap F^*(G)}(s)$ holds by the first part of Lemma 3.9. By (18.6) in [AsSe], $|\mathbb{C}_{F^*(G)}(s)|$ is not divisible by 3, whereas the order of $\mathbb{C}_{F^*(H)}(t)$, and therefore $\mathbb{C}_{F^*(G)}(t)$, is divisible by 3. So $s^{F^*(G)} \cap (H \cap F^*(G)) = s^{H \cap F^*(G)}$ follows. Again Lemma 2.44 implies $F^*(G) = F^*(H)$ and therefore $G = H$. □

Lemma 4.9: Assume Hypothesis 1.4. For $F^*(H) \cong F_4(q)$ with $q = 2^f$ and $f \in \mathbb{N}$, it is $G = H$.

Proof: Let $F^*(H) \cong F_4(q)$ with $q = 2^f$ and $f \in \mathbb{N}$. We identify the groups $F^*(H)$ and $F_4(q)$ to simplify notation and fix $S \in \mathrm{Syl}_2(H)$ with $S_1 = S \cap F^*(H) \in \mathrm{Syl}_2(F^*(H))$. It is $\mathrm{Z}(S_1) = R_1 R_2$, where R_1 is a long root subgroup of $F^*(H)$ and R_2 a short root subgroup of $F^*(H)$, compare Lemma 2.24.
By (13.1) in [AsSe] there are exactly four conjugacy classes of involutions in $F^*(H)$. Let t_1, t_4, $s := t_1 \cdot t_4$ and v be representatives of these four conjugacy classes in S_1. Thereby t_1, t_4 and s are 2-central involutions in $F^*(H)$, while v is not 2-central. We can choose $s \in \mathrm{Z}(S)$ such that s is 2-central in H, as a graph automorphism of $F_4(q)$ fuses t_1 and t_4, see (13.1) in [AsSe]. Hence $\mathbb{C}_G(s) = \mathbb{C}_H(s)$ follows from the first part of Lemma 3.9. This implies $\mathbb{C}_{F^*(G)}(s) = \mathbb{C}_{H \cap F^*(G)}(s)$.

By (13.1)–(13.3) in [AsSe], it is $\mathbb{C}_{F^*(H)}(t_1) = O^{2'}(P_{\{1\}'})$ and $\mathbb{C}_{F^*(H)}(t_4) = O^{2'}(P_{\{4\}'})$ and $\mathbb{C}_{F^*(H)}(s) = O^{2'}(P_{\{1,4\}'})$, where P_J is a Lie-parabolic subgroup of $F^*(H)$, which corresponds to the Dynkin diagram of type F_4  with the roots labelled by elements in

$J = \hat{\Pi} \setminus J'$. Additionally, a graph automorphism interchanges $P_{\{1\}'}$ and $P_{\{4\}'}$.
Section 13 in [AsSe] provides the following information:
It is $\mathbb{C}_{F^*(H)}(t_1) = Q_1\colon L_1$, $\mathbb{C}_{F^*(H)}(t_4) = Q_4\colon L_4$, $\mathbb{C}_{F^*(H)}(s) = Q_{1,4}\colon L_{1,4}$ and $\mathbb{C}_{F_4(q)}(v) = Q_v\colon L_v$ with $L_1 \cong L_4 \cong Sp_6(q)$, $L_{1,4} \cong Sp_4(q)$ and $L_v \cong L_2(q)\times L_2(q) \cong \Omega_4^+(q)$. Additionally, $Q_1 = O_2(\mathbb{C}_{F^*(H)}(t_1)) \cong O_2(\mathbb{C}_{F^*(H)}(t_4)) = Q_4$ is a group of order q^{15} with $\Phi(Q_1) = Q_1' = R_1$ and $\Phi(Q_4) = Q_4' = R_2$. By Example 3.2.4 in [GLS3], for both $i \in \{1,4\}$, $Q_i/\,\mathrm{Z}(Q_i)$ is a spin module, $\mathrm{Z}(Q_i)/R_i$ is a natural module and R_i is a trivial module of $GF(q)$-dimensions 8, 6 and 1, respectively. Additionally, by the same source, $\mathrm{Z}(Q_i)$ and Q_i/R_i are both indecomposable as L_i-modules over $GF(q)$. Furthermore, it is $|\mathrm{Z}(\mathbb{C}_{F^*(H)}(v))| = q^2$ and $\mathrm{Z}(\mathbb{C}_{F^*(H)}(s)) = R_1R_2$ is of order q^2 by (13.3) in [AsSe].

The simple group $Sp_6(q)$ is involved in $\mathbb{C}_{F^*(H)}(t_1)$ and in $\mathbb{C}_{F^*(H)}(t_4)$. So these centralizers cannot be embedded into $\mathbb{C}_{H\cap F^*(G)}(s) = \mathbb{C}_{F^*(G)}(s)$, as $Sp_6(q)$ cannot be embedded into $Sp_4(q)$, due to Lemma 2.29. Hence both involutions, t_1 and t_4, cannot be $F^*(G)$-conjugate to s.

To show that the remaining involution v cannot be conjugate to s in $F^*(G)$, the structure of the centralizers of these involutions needs to be considered in more detail:
By (13.3) in [AsSe], $Q_{1,4} = O_2(\mathbb{C}_{F^*(H)}(s))$ equals Q_1Q_4, which is of order q^{20}. Additionally, by the same source, $Q_{1,4} = [Q_{1,4}, L_{1,4}]$ holds. It is $\Phi(Q_1 \cap Q_4) \leq \Phi(Q_1) \cap \Phi(Q_4) = R_1 \cap R_2$, which is trivial. Hence $Q_1 \cap Q_4$ is elementary abelian and $|Q_i/(Q_1 \cap Q_2)| = |Q_{1,4}/Q_i| = q^{20-15} = q^5$ holds for both $i \in \{1,4\}$. Hence $Q_1 \cap Q_4$ is of order q^{10}. As $\mathrm{N}_{L_1}(\mathbb{C}_{F^*(H)}(s)/Q_1)$ is a Lie-parabolic subgroup in $L_1 \cong Sp_6(q)$, Proposition (3.2) in [CKS] provides that $Q_{1,4}/Q_1 \cong Q_4/(Q_1 \cap Q_4)$ is an elementary abelian and indecomposable $L_{1,4}$-module of $GF(q)$-dimension 5. Additionally, it is $Q_4/(Q_1 \cap Q_4)$ a non-split extension of a 1-dimensional module, $L_{1,4}$ acts trivially on, and an irreducible $L_{1,4}$-module of $GF(q)$-dimension 4. As $P_{\{1\}'}$ and $P_{\{4\}'}$ are interchanged by a graph automorphism, the same holds for $Q_1/(Q_1 \cap Q_4)$.
Hence $Q_{1,4}/(Q_1 \cap Q_4)$ is a direct sum of two indecomposable $L_{1,4}$-modules of dimension 5 and involves a direct sum of two irreducible $L_{1,4}$-modules of $GF(q)$-dimension 4.
Example 3.2.4 in [GLS3] states that $\mathrm{Z}(Q_1)/R_1$ is a natural L_1-module of $GF(q)$-dimension 6. Hence, using $|Q_1 \cap Q_4| = q^{10}$ and the symmetry of Q_1 and Q_4, $(Q_1 \cap Q_4)/(R_1R_2)$ is a direct sum of two irreducible $L_{1,4}$-modules of $GF(q)$-dimension 4.
As $L_v \cong \Omega_4^+(q)$ can be embedded into $L_{1,4} \cong Sp_4(q)$, we assume the existence of an embedding of $\mathbb{C}_{F^*(H)}(v)$ into $\mathbb{C}_{F^*(G)}(s) = \mathbb{C}_{H\cap F^*(G)}(s)$. Let X be the corresponding image of $\mathbb{C}_{F^*(H)}(v)$ in $\mathbb{C}_{F^*(G)}(s)$. Then, as $\Omega_4^+(q) \lesssim L_{1,4}$ cannot normalize a non-trivial 2-subgroup in $L_{1,4} \cong Sp_4(q)$, there has to exist an isomorphic image of $Q_v = O_2(\mathbb{C}_{F^*(H)}(v))$ in $Q_{1,4}$. It is Q_v a group of order q^{18} by (13.1) in [AsSe] and $Q_{1,4}$ is of order q^{20}. The existence of a subgroup $Y \leq X$ of order q^{18} in $Q_{1,4}$, which is normalized by an orthogonal group $\Omega_4^+(q)$, implies that Y involves the four irreducible $L_{1,4}$-modules of $GF(q)$-dimension 4 and $R_1R_2 = \mathrm{Z}(\mathbb{C}_{F^*(H)}(s))$. In particular, $\mathrm{Z}(\mathbb{C}_{F^*(H)}(s)) = R_1R_2 \leq \mathrm{Z}(X)$. By (13.3) in [AsSe], it is $|\mathrm{Z}(X)| = q^2$, hence

$\mathrm{Z}(X) = R_1R_2$. But this is a contradiction to $|\mathrm{Z}(X)| \geq |\langle R_1, R_2, v\rangle| = 2q^2$. Hence there is no such embedding of $\mathbb{C}_{F^*(H)}(v)$ into $\mathbb{C}_{F^*(G)}(s) = \mathbb{C}_{H\cap F^*(G)}(s)$ and v cannot be $F^*(G)$-conjugate to s.

So it is $s^{F^*(G)} \cap F^*(H) = s^{H\cap F^*(G)}$. If there is an involution $x \in S \setminus S_1$, the following holds: If f is even, every involution $x \in S \setminus S_1$ which induces an outer automorphism on $F^*(H)$ induces a field automorphism, see Lemma 2.31. Then it is $O^{2'}(\mathbb{C}_{F^*(H)}(x)) \cong F_4(q_0)$ for $q_0^2 = q$, again by Lemma 2.31. The group $F_4(q_0)$ is simple and of Lie rank 4, whereas $Sp_4(q)$ is of Lie rank 2. So by Lemma 2.29, $\mathbb{C}_{F^*(H)}(x)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(s)$. Hence x is not $F^*(G)$-conjugate to s if f is even.
If f is odd, every involution $x \in S \setminus S_1$ induces a graph automorphism with $\mathbb{C}_{F^*(H)}(x) \cong {}^2F_4(q)$, see Lemma 2.31. Here we use Lemma 2.32 to see that $|{}^2F_4(q)|_2 > |Sp_4(q)|_2$ holds. Hence again, $\mathbb{C}_{F^*(H)}(x)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(s)$. So x cannot be $F^*(G)$-conjugate to s if f is odd.

Altogether, $s^{F^*(G)} \cap (H \cap F^*(G)) = s^{H\cap F^*(G)}$ and $\mathbb{C}_{F^*(G)}(s) \leq H \cap F^*(G)$ hold. So Lemma 2.44 implies $F^*(G) = F^*(H)$. By Frattini's argument $G = H$ follows. □

If $F^*(H)$ is isomorphic to a symplectic group $F^*(H) \cong Sp_{2n}(2^f)'$ for $f \in \mathbb{N}$, then $G = H$ holds, which is shown in the following three lemmas.
First we deal with $F^*(H) \cong Sp_4(2)' \cong A_6$.

Lemma 4.10: Assume Hypothesis 1.4. If it is $F^*(H) \cong Sp_4(2)'$, then $G = H$ holds.

Proof: There is only one conjugacy class of involutions in $Sp_4(2)' \cong A_6$. Let S be a Sylow 2-subgroup of $H \cap F^*(G)$, so $S_1 = S \cap F^*(H)$ is a Sylow 2-subgroup of $F^*(H)$. It is S_1 a dihedral group of order 8 and every involution $z \in F^*(H)$ is conjugate in $F^*(H)$ to a 2-central involution. Let $z \in S_1$ be an involution. Then we may assume that $\langle z\rangle = \mathrm{Z}(S_1)$ holds. Hence z is normalized by every element in $S \in \mathrm{Syl}_2(H)$. So $1 \neq \langle z\rangle \trianglelefteq S$ holds, which implies $\mathbb{C}_G(z) \leq H$, by Hypothesis 1.4. Then also $\mathbb{C}_{F^*(G)}(z) \leq H \cap F^*(G)$ holds.
The outer automorphism group of $F^*(H) \cong Sp_4(2)'$ is elementary abelian of order 4, so it is $\mathrm{Out}(F^*(H)) = \langle \alpha_1, \alpha_2\rangle$ for involutions α_1 and α_2. If there are involutions in $S \setminus S_1$, then four possibilities for H need to be investigated. The used information about involutions in $S \setminus S_1$ can be found in [CoCu].
For $H\cap F^*(G) \cong Sp_4(2)'\langle\alpha_1\rangle \cong S_6$, one gets $\mathbb{C}_{F^*(G)}(z) = \mathbb{C}_{H\cap F^*(G)}(z) \cong D_8 \times Z_2$. There are two conjugacy classes of outer involutions; for every involution $t \in S \setminus S_1$, it is $\mathbb{C}_{H\cap F^*(G)}(t) \cong S_4 \times Z_2$. So the order of $\mathbb{C}_{F^*(G)}(t)$ is divisible by 3, while $|\mathbb{C}_{F^*(G)}(z)|$ is not.
For $H \cap F^*(G) \cong Sp_4(2)'\langle\alpha_2\rangle \cong PGL_2(9)$, $\mathbb{C}_{F^*(G)}(z) = \mathbb{C}_{H\cap F^*(G)}(z)$ is of order 16 and there is only one class of outer involutions. For every involution $t \in S \setminus S_1$, the order of $\mathbb{C}_{H\cap F^*(G)}(t)$ equals 20. So the order of $\mathbb{C}_{F^*(G)}(t)$ is divisible by 5, while $|\mathbb{C}_{F^*(G)}(z)|$ is not.

For $H \cap F^*(G) \cong Sp_4(2)'\langle \alpha_1 \cdot \alpha_2 \rangle \cong M_{10}$, there are no involutions in $S \setminus S_1$.
And for $H \cap F^*(G) \cong \mathrm{Aut}(Sp_4(2)')$, it is $\mathbb{C}_{F^*(G)}(z) = \mathbb{C}_{H \cap F^*(G)}(z)$ of order 32 and there are exactly two conjugacy classes of outer involutions with representatives t_1 and t_2, such that $\mathbb{C}_{H \cap F^*(G)}(t_1)$ is of order 48 and $\mathbb{C}_{H \cap F^*(G)}(t_2)$ is of order 40. Then $|\mathbb{C}_{F^*(G)}(t_1)|$ is divisible by 3 and $|\mathbb{C}_{F^*(G)}(t_2)|$ is divisible by 5, while $|\mathbb{C}_{F^*(G)}(z)|$ is not.
Altogether, $z^{F^*(G)} \cap (H \cap F^*(G)) = z^{H \cap F^*(G)}$ holds. Together with $\mathbb{C}_{F^*(G)}(z) \leq H \cap F^*(G)$ we can apply Holt's result from Lemma 2.44. Then either $F^*(G) = F^*(H)$ holds or it is $F^*(G) \cong A_7$. For $F^*(G) \cong A_7$ it is $G \cong A_7$ or $G \cong S_7$. Considering $N_{A_7}(\langle (1,2)(3,4), (1,3)(2,4) \rangle)$ and $N_{S_7}(\langle (1,2)(3,4), (1,3)(2,4) \rangle)$, which both contain a full Sylow 2-subgroup and a non-trivial normal 3-subgroup, neither A_7 nor S_7 are of parabolic characteristic 2. Hence we get $F^*(G) = F^*(H)$ and as before $G = H$. □

Lemma 4.11: Assume Hypothesis 1.4. If it is $F^*(H) \cong Sp_4(q)$ for $q = 2^f > 2$, then $G = H$ holds.

Proof: Let $F^*(H) \cong Sp_4(q)$ for $q = 2^f > 2$ and let S_1 be a Sylow 2-subgroup of $F^*(H) \cong Sp_4(q)$. Due to Theorem 6 and Theorem 10 in [Dye1], there are exactly three conjugacy classes of involutions, which are all 2-central in $F^*(H)$. It is $\mathrm{Z}(S_1)$ the product of two root groups R_1 and R_2, which are interchanged in case of a graph automorphism, see Lemma 2.24. The information about centralizers of involutions in $Sp_4(q)$ are taken from (7.9)–(7.11) in [AsSe]. Let $r_1 \in R_1$ and $r_2 \in R_2$ be involutions. Then their centralizers in $F^*(H)$ involve a simple group $L_2(q)$, hence they are not solvable, as $q \neq 2$. We choose r_1 and r_2 such that $s := r_1 \cdot r_2$ is an involution which is diagonal in $\mathrm{Z}(S_1)$ and for which $\mathbb{C}_H(s) = S_1 \cdot A$ holds, where $A \leq H$ contains the elements of H which induce outer automorphisms on $F^*(H)$. Hence s is 2-central in H and A is isomorphic to a subgroup of $\mathrm{Out}(Sp_4(q))$. In particular, $\mathbb{C}_H(s)$, and therefore $\mathbb{C}_{H \cap F^*(G)}(s)$ is solvable by Lemma 2.31. As r_1, r_2 and s represent all three conjugacy classes of involutions in $F^*(H)$, we have shown $s^{F^*(G)} \cap F^*(H) = r^{H \cap F^*(G)}$. As s is 2-central in H, by applying Lemma 3.9 (a), it is $\mathbb{C}_{F^*(G)}(s) = \mathbb{C}_{H \cap F^*(G)}(s)$.
It is $\mathrm{Out}(F^*(H))$ a cyclic group of order $2 \cdot f$ and all involutions in $S \setminus S_1$ are conjugate in H. If f is even, every involution $x \in S \setminus S_1$ can be assumed to be a field automorphism with $\mathbb{C}_{F^*(H)}(x) \cong Sp_4(q_0)$ for $q_0^2 = q$, which is not solvable. If f is odd, the centralizer $\mathbb{C}_{Sp_4(q)}(x)$ of any involution $x \in H \setminus F^*(H)$ is isomorphic to $Sz(q)$, which is also not solvable. Hence $Sp_4(q_0)$ with $q_0^2 = q$ or $Sz(q)$ cannot be embedded into $\mathbb{C}_{F^*(G)}(s)$.
Altogether, $\mathbb{C}_{F^*(G)}(s) \leq H \cap F^*(G)$ and $s^{F^*(G)} \cap (H \cap F^*(G)) = s^{H \cap F^*(G)}$ hold. The result follows as before with Lemma 2.44. □

Lemma 4.12: Assume Hypothesis 1.4. If $F^*(H) \cong Sp_{2n}(q)$ holds for $q = 2^f$ and $n \geq 3$, then $G = H$ follows.

Proof: Let $F^*(H) \cong Sp_{2n}(q)$ for $q = 2^f$ and $n \geq 3$ and let t be an involution in $S_1 =$

$S \cap F^*(H) \in \mathrm{Syl}_2(F^*(H))$. Let V be the natural symplectic module. Then t normalizes the natural module V for $Sp_{2n}(q)$ and, as t is an involution, $[V,t] \leq \mathbb{C}_V(t) = [V,t]^\perp$ holds. Hence $[V,t]$ is a totally isotropic subspace. Then there is a maximal totally isotropic subspace U such that $[V,t] \leq U \leq [V,t]^\perp = \mathbb{C}_V(t)$ holds. In particular, $[U,t] = 1$ follows.
Let P be the stabilizer of U in $F^*(H)$. Then P is a Lie-parabolic subgroup of $F^*(H)$ with unipotent radical $O_2(P)$. Hence $O_2(P)$ is elementary abelian of maximal 2-rank, so by 3.12 in [GuMa] the rank of $O_2(P)$ equals $f \cdot \binom{n+1}{2}$. By 3.13 in [GuMa], every unipotent elementary abelian subgroup of $F^*(H)$ which is of maximal 2-rank is $F^*(H)$-conjugate to $O_2(P)$. As $O_2(P)$ has to be a weakly closed subgroup in S_1 relative to $Sp_{2n}(q)$ by 4.2 in [Gro], $O_2(P)$ equals the Thompson subgroup $J(S_1)$.
As a consequence of Witt's Lemma, see (20.8) in [Asc1], all the maximal totally singular subspaces are conjugate in $F^*(H)$, so every involution in $F^*(H)$ is $F^*(H)$-conjugate to an involution in $J(S_1) \trianglelefteq S$ for $S_1 \leq S \in \mathrm{Syl}_2(H)$.

To show that $J(S_1) = J(S)$, we assume the existence of an elementary abelian 2-group A of maximal rank which contains an involution $x \in S \setminus S_1$. Lemma 2.31 implies that every involution $x \in S \setminus S_1$ can be assumed to induce a field automorphism, for which $O^{2'}(\mathbb{C}_{F^*(H)}(x)) \cong Sp_{2n}(q_0)$ with $q_0^2 = q$ holds. But it is $m_2(Sp_{2n}(q_0)) + 1 = \frac{n\cdot(n+1)\cdot f}{4} + 1 < \frac{n\cdot(n+1)\cdot f}{2} = m_2(Sp_{2n}(q))$, hence such an elementary abelian group A cannot exist. Hence $J(S) = J(S_1)$ and so Hypothesis 1.4 and Lemma 2.8 imply that $\mathrm{N}_{F^*(G)}(J(S)) \leq H \cap F^*(G)$ controls the $F^*(G)$-fusion of involutions in $J(S)$.

Let $s \in \mathrm{Z}(S)$ be an involution. Then by Lemma 3.9 (a), it is $\mathbb{C}_{F^*(G)}(s) = \mathbb{C}_{H\cap F^*(G)}(s)$. Additionally, for an involution $x \in S \setminus S_1$ such that $x \sim_{F^*(G)} s$ holds, $Sp_{2n}(q_0)$ has to be embedded into $Sp_{2n-4}(q)$, see Lemma 2.31 and (7.9) in [AsSe]. But this contradicts Lemma 2.29. In conclusion, one gets $\mathbb{C}_{F^*(G)}(s) \leq H \cap F^*(G)$ and $s^{F^*(G)} \cap (H \cap F^*(G)) = s^{H\cap F^*(G)}$. Thus Lemma 2.44 implies $F^*(G) = F^*(H)$. Hence $G \leq F^*(H)\,\mathrm{N}_G(S_1) \leq H$ follows. □

The following lemma from [SaSt] states that $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ can be assumed to be not contained in any group $K \in P(X)$.

Lemma 4.13 ([SaSt], Lemma 4.9): Assume Hypothesis 1.4. If $F^*(H)$ is not isomorphic to ${}^2F_4(2^f)'$, $L_3(2^f)$, $Sp_4(2^f)'$, $G_2(2)'$ or to a simple group of Lie rank 1, then $M(Q) = \emptyset$ holds. For $K \in P(X)$ with $1 \neq X \leq S$, this implies $Q \not\leq K$.
For $F^*(H)$ being isomorphic to ${}^2F_4(2^f)'$, $L_3(2^f)$, $Sp_4(2^f)'$, $G_2(2)'$ or a simple group of Lie rank 1, the result $G = H$ follows by the previous results; so we may assume in the following that $Q \not\leq K$ for every $K \in P(X)$ with $1 \neq X \leq S$.

Proof: This is Lemma 4.9 in [SaSt]. □

We end this chapter with a concluding remark, assembling some facts, which have been proved so far.

Remark 4.14: Let the notation and properties listed in Hypothesis 1.4 hold. Then the following statements hold:

(a) Lemma 4.1, Lemma 4.2, 4.4 and 4.6 imply that Theorem 1.5 holds for $F^*(H) \cong L_n(2^f)$ or $F^*(H) \cong U_n(2^f)$ and $F^*(H)$ being isomorphic to a simple group of Lie rank 1 over a field of characteristic 2. Lemma 4.10, 4.11 and 4.12 show that $G = H$ also holds for $F^*(H)$ being isomorphic to a simple symplectic group. And by Lemma 4.9, Lemma 4.8 and Lemma 4.7, the result $G = H$ also holds for $F^*(H)$ being isomorphic to any of the simple exceptional groups $F_4(2^f)$, ${}^2F_4(2^f)'$ and $G_2(2^f)'$.

(b) Lemma 2.19 and Lemma 2.20 provide $\Omega_4^+(2^f) \cong L_2(2^f) \times L_2(2^f)$, $\Omega_4^-(2^f) \cong L_2(2^{2f})$, $\Omega_6^+(2^f) \cong L_4(2^f)$ and $\Omega_6^-(2^f) \cong U_4(2^f)$. Hence Theorem 1.5 still has to be shown for $F^*(H)$ being isomorphic to one of the following groups: $\Omega_{2m}^{\pm}(2^f)$ with $m \geq 4$, ${}^3D_4(2^f)$, $E_6(2^f)$, ${}^2E_6(2^f)$, $E_7(2^f)$ and $E_8(2^f)$ for $f \in \mathbb{N}$.

(c) For $F^*(H)$ being isomorphic to $\Omega_{2m}^{\pm}(q)$ with $m \geq 4$, ${}^3D_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$, each with $q = 2^f$, it is $|Q| \geq q^9$ for $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ by Lemma 2.36. Hence we assume $|Q| \geq q^9$ in the following.

Chapter 5

Centralizers of involutions in Q

Throughout the following section we assume that Hypothesis 1.4 holds and, by Lemma 3.2, that $F^*(G)$ is a non-abelian simple group. We fix a Sylow 2-subgroup $S \in \mathrm{Syl}_2(H)$, hence $S_1 = S \cap F^*(H)$ is a Sylow 2-subgroup of the simple group of Lie type $F^*(H)$.
Using the results of the previous chapter, collected in Remark 4.14, we may assume that $F^*(H)$ is isomorphic to one of the following groups: $\Omega^{\pm}_{2m}(2^f)$ with $m \geq 4$, ${}^3D_4(2^f)$, $E_6(2^f)$, ${}^2E_6(2^f)$, $E_7(2^f)$ or $E_8(2^f)$.

It is $R = \mathrm{Z}(S_1)$ a long root group, hence elementary abelian of order $q = 2^f$, see Remark 2.23. And it is $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ a large subgroup in $F^*(H)$ and also in H by Lemma 2.38. Particularly, as $S \in \mathrm{Syl}_2(H)$ centralizes a non-trivial element in $R = \mathrm{Z}(Q)$, $S \leq \mathrm{N}_H(Q)$ holds. Hence, using Hypothesis 1.4, it is $\mathrm{N}_G(Q) = \mathrm{N}_H(Q)$. Additionally, Q is a special 2-group with $R = Q' = \Phi(Q) = \mathrm{Z}(Q)$, compare Lemma 2.36. By Remark 4.14, we may assume $|Q| \geq q^9$.

The aim of this chapter is to show $\mathbb{C}_G(z) \leq H$ for every involution $z \in Q$. As z is an element of $O_2(\mathbb{C}_G(z))$, it is sufficient to show that the set $\{K \mid K \in P(\mathbb{C}_Q(z)),\, z \in O_2(K)\}$ is empty, as then also $\{K \mid K \in M(\mathbb{C}_Q(z)),\, z \in O_2(K)\} = \emptyset$ holds. This implies $\mathbb{C}_G(z) \leq H$.
Hence we assume the existence of an involution $z \in Q$ such that $\mathbb{C}_G(z) \not\leq H$ holds. Then it is $\mathbb{C}_G(z) \in M(\mathbb{C}_Q(z))$ and so $P(\mathbb{C}_Q(z)) \neq \emptyset$. We suppose $K \in P(\mathbb{C}_Q(z))$ and additionally let z be an element of $O_2(K)$.

The structure of the centralizer of an involution in Q is described in the following lemma. In its proof we cite Lemma 2.36, so we restrict $F^*(H)$ to be isomorphic to the groups listed above, although the lemma holds in more generality.

Lemma 5.1: Let Hypothesis 1.4 hold and let $F^*(H)$, R and $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ be as described in the paragraphs at the beginning of this chapter. In particular, we assume $F^*(H)$ being isomorphic to $\Omega^{\pm}_{2m}(q)$ with $m \geq 4$, ${}^3D_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ for $q = 2^f \in \mathbb{N}$. Then the following statements hold:

(a) The group Q ist semi-extraspecial. Hence for every maximal subgroup M in $R = \mathrm{Z}(Q)$, the factor group Q/M is extraspecial.

(b) For $|R| = q > 2$, there is an integer l such that $Q \cong D_l(q)$ holds. By Lemma 2.34, Q is a central product of groups $D_1(q) \in \mathrm{Syl}_2(L_3(q))$.
In case of $|Q| > q^3$, for every involution $x \in Q \setminus R$, the centralizer $\mathbb{C}_Q(x)$ is isomorphic to $\tilde{Z} \times U$ for an elementary abelian subgroup $\tilde{Z}$ of order q and a semi-extraspecial 2-group U with $\mathrm{Z}(U) = R$. Additionally, U is of the same type ($+$ or $-$) as Q.

Proof:

(a) Due to Remark 4.14, all possibilities for $F^*(H)$ and Q are listed in Lemma 2.36. In particular, Q is special. Let M be a maximal subgroup of $R = \mathrm{Z}(Q)$. As R is elementary abelian, R/M is a group of order 2. It is $(Q/M)' = Q'/M = R/M$, $\Phi(Q/M) = \Phi(Q)/M = R/M$ and $R/M \leq \mathrm{Z}(Q/M)$. Due to Lemma 7.1 in [Pie], $[Q, y] = R$ holds for every element $y \in Q \setminus R$. For $yM \notin R/M$ then $yM \notin \mathrm{Z}(Q/M)$ follows, as it is $y \in Q \setminus R$. So $\mathrm{Z}(Q/M) = R/M$ holds. Hence Q is semi-extraspecial.

(b) The structure of Q follows from [Beis] and [Timm]. Let $x \in Q \setminus R$ be an involution. If $x \in D_1(q)$, then by (4.2) in [Col2], $\mathbb{C}_{D_1(q)}(x)$ is elementary abelian of order q^2; for $x \in Q_1(q)$, one gets that $\mathbb{C}_{Q_1(q)}(x)$ is a homocyclic group of order q^2, see [Col1]. For $F^*(H) \cong U_n(q)$, the structure of $\mathbb{C}_Q(x)$ follows from [Timm] and in this case, Q/R is the natural unitary module over $GF(q^2)$ for $U_{n-2}(q)$. Hereby $U_{n-2}(q)$ acts transitively on the one-dimensional subspaces of the module, see 10.12 in [Tayl]. As every involution generates such a subspace, we may assume $x \in D_1(q)$ and the claim, concerning the structure of $\mathbb{C}_Q(x)$, follows.
As by assumption $F^*(H)$ is not isomorphic to $U_n(q)$, it is $Q \cong D_l(q)$ with $l \in \mathbb{N}$.

□

For the remainder of this chapter we work under the following hypothesis:

Hypothesis 5.2: Let the notation and properties listed in Hypothesis 1.4 hold. Using Remark 4.14, we assume $F^*(H)$ to be isomorphic to $\Omega^{\pm}_{2m}(q)$ with $m \geq 4$, ${}^3D_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$ or $E_8(q)$ for $q = 2^f \in \mathbb{N}$. We further let be $S \in \mathrm{Syl}_2(H)$, $S_1 = S \cap F^*(H)$, $R = \mathrm{Z}(S_1)$ and $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ and we assume the existence of an involution $z \in Q \setminus R$ such that $\mathbb{C}_G(z) \not\leq H$ holds.
This implies $\mathbb{C}_G(z) \in M(\mathbb{C}_Q(z))$ with $z \in O_2(\mathbb{C}_G(z))$, hence $P(\mathbb{C}_Q(z)) \neq \emptyset$. Suppose $K \in P(\mathbb{C}_Q(z))$ with additional condition $z \in O_2(K)$.
Remark 4.14 allows to assume $|Q| \geq q^9$. Due to Lemma 5.1, $\mathbb{C}_Q(z) = \tilde{Z} \times U$ holds such that

$\tilde{Z}$ is elementary abelian of order $q = |R|$ and U is special. And we may assume that Q is of +-type, as $F^*(H)$ is not isomorphic to an unitary group. In particular, using Lemma 2.34 and Lemma 5.1, Q, and hence also U, is generated by involutions.

Lemma 5.3: Suppose Hypothesis 5.2. Then $O(K) \leq H$ holds for every $K \in P(\mathbb{C}_Q(z))$.

Proof: Let $z \in Q$ be an involution such that $\mathbb{C}_G(z) \not\leq H$ holds. This implies $\mathbb{C}_G(z) \in M(\mathbb{C}_Q(z))$. So it is $P(\mathbb{C}_Q(z)) \neq \emptyset$. Suppose $K \in P(\mathbb{C}_Q(z))$, thus $\mathbb{C}_Q(z) \leq K \not\leq H$ and $O_2(K) \neq 1$ holds. Because of $\mathbb{C}_Q(z) \leq S \cap F^*(H) = S_1$, we can choose $T \in \mathrm{Syl}_2(K \cap H)$ such that $\mathbb{C}_Q(z) \leq T \leq S$ holds. Particularly, it is $R = \mathrm{Z}(Q) \leq T$ and we may assume $K \in P(\mathbb{C}_Q(z)) \setminus P^*(\mathbb{C}_Q(z))$, as otherwise $O(K) \leq H$ holds by definition of $P^*(\mathbb{C}_Q(z))$.
It is $O_2(K)$ a non-trivial normal 2-subgroup in K, so, using Lemma 3.6, K contains a Sylow 2-subgroup of $\mathrm{N}_G(O_2(K))$. Then $T \leq \mathrm{N}_G(O_2(K))$ and $T \in \mathrm{Syl}_2(K)$ together imply $T \in \mathrm{Syl}_2(\mathrm{N}_G(O_2(K)))$ and, because of $T \leq S$, one gets $T = \mathrm{N}_S(O_2(K))$.

We assume now that $O(K)$ is not a subgroup of H. Because of the minimality of elements in $P(\mathbb{C}_Q(z))$, then $K = O(K)\colon T$ follows. Hence K is solvable, using the Odd Order Theorem 2.3. If R is not cyclic, it is $O(K) = \langle \mathbb{C}_{O(K)}(r) \mid r \in R^\# \rangle \leq H$ by Coprime action, see Lemma 2.4. Hence R has to be cyclic and, as it is elementary abelian, $|R| = 2$ follows. In particular, Q is an extraspecial 2-group. We set $R = \langle r \rangle$. Lemma 2.35 implies $|Q : \mathbb{C}_Q(z)| = 2$ and $\mathbb{C}_Q(z) = \langle z \rangle \times U$ for an extraspecial 2-group U with $\mathrm{Z}(U) = \langle r \rangle$.
Additionally, by Lemma 4.13, it is $Q \not\leq K$, so $Q \cap K = \mathbb{C}_Q(z)$ follows.

By Hypothesis 5.2, $F^*(H)$ is not isomorphic to $L_n(q)$ or $U_n(q)$ for $n \in \mathbb{N}$. So by Lemma 2.39, $\mathrm{Z}_2(S_1)$ is an elementary abelian subgroup of Q of order 4, and $x \sim_H r$ holds for any $x \in \mathrm{Z}_2(S_1)^\#$. As $\mathrm{Z}_2(S_1)$ is elementary abelian of order 4, $|S_1 : \mathbb{C}_{S_1}(\mathrm{Z}_2(S_1))| = 2$ holds. As $F^*(H)$ is a simple group, Thompson's Transfer Lemma, see Lemma 2.7, implies that every involution in S_1 is H-conjugate to an involution in $\mathbb{C}_{S_1}(\mathrm{Z}_2(S_1))$. So without loss of generality, $[\mathrm{Z}_2(S_1), z] = 1$ holds. Hence, it is $\mathrm{Z}_2(S_1) \leq \mathbb{C}_Q(z) \leq T$. So $\mathrm{Z}_2(S_1)$ is a non-cyclic, elementary abelian group whose involutions all are H-conjugate to r. Lemma 2.4 (Coprime action) implies $O(K) = \langle \mathbb{C}_{O(K)}(x) \mid x \in \mathrm{Z}_2(S_1)^\# \rangle \leq H$.

□

Lemma 5.4: Suppose Hypothesis 5.2. Then $O(K) = 1$ holds for every $K \in P(\mathbb{C}_Q(z))$.

Proof: Suppose $K \in P(\mathbb{C}_Q(z))$. Due to Lemma 5.3, $O(K)$ is contained in H. We assume first $|R| = q > 2$. For every involution $r \in R$, the centralizer $\mathbb{C}_{O(K)}(r)$ is contained in $\mathrm{N}_H(Q)$, as any group which centralizes an element in $\mathrm{Z}(Q)^\#$ normalizes the large subgroup Q. Hence it is $[\mathbb{C}_Q(z), \mathbb{C}_{O(K)}(r)] \leq Q \cap O(K) = 1$. Let $\omega \in \mathbb{C}_{O(K)}(r)$ and $u \in Q \setminus \mathbb{C}_Q(z)$ be arbitrary elements. Then $R \ni [z, u] = [z, u]^\omega = [z, u^\omega]$ holds, so it is $u^{-1}u^\omega \in \mathbb{C}_Q(z)$. Hence

$[Q,\omega] \le \mathbb{C}_Q(z)$ follows, and every element in $\mathbb{C}_{O(K)}(r)$ centralizes $Q/\mathbb{C}_Q(z)$. In conclusion, using Coprime action, $[Q,\omega] = 1$ holds for every element $\omega \in \mathbb{C}_{O(K)}(r)$. But as Q is a large subgroup in H, it is $\mathbb{C}_H(Q) = \mathrm{Z}(Q)$. Hence $\mathbb{C}_{O(K)}(r) = 1$ follows. Again by Coprime action, $O(K)$ equals $\langle \mathbb{C}_{O(K)}(r) \mid r \in R^{\#} \rangle = 1$.
So we may assume that $R = \langle r \rangle$ is cyclic. We use the same arguments as in Lemma 5.3 to show that $\mathrm{Z}_2(S_1) \le Q \cap K = \mathbb{C}_Q(z)$ holds: It is $\mathrm{Z}_2(S_1)$ an elementary abelian subgroup of Q of order 4, and $x \sim_H r$ holds for any $x \in \mathrm{Z}_2(S_1)^{\#}$. So $|S_1 : \mathbb{C}_{S_1}(\mathrm{Z}_2(S_1))|$ equals 2 and by Lemma 2.7, every involution in S_1 is H-conjugate to an involution in $\mathbb{C}_{S_1}(\mathrm{Z}_2(S_1))$. Thus, without loss of generality, $[\mathrm{Z}_2(S_1), z] = 1$ holds. Hence it is $\mathrm{Z}_2(S_1) \le \mathbb{C}_Q(z) \le T$.
So we may assume $\mathrm{Z}_2(S_1) = \langle r, r_1 \rangle$ with $r \in R$ and $r_1 = r^h$ for a suitable element $h \in H$. With the same arguments as above $\mathbb{C}_{O(K)}(r) = 1$ holds. If $\mathbb{C}_{O(K)}(r_1)$ is trivial, we are done by Coprime action as above. Hence we assume that $\mathbb{C}_{O(K)}(r_1)$ is not trivial. So it is $\mathbb{C}_{O(K)}(r_1) = \mathbb{C}_{O(K)}(r^h) \le \mathrm{N}_H(Q^h)$ and r normalizes $\mathbb{C}_{O(K)}(r_1)$. As $h \in H$ acts like an element in $\mathrm{N}_H(\mathrm{Z}_2(S_1))$ on r and $r_1 \in Q \cap Q^h$, also $r \in Q^h$ holds. Together with Coprime action, this implies $\mathbb{C}_{O(K)}(r_1) \le [r, \mathbb{C}_{O(K)}(r_1)] \cdot \mathbb{C}_{O(K)}(r) = [r, \mathbb{C}_{O(K)}(r_1)] \le O(K) \cap Q^h = 1$. And again by using Coprime action, $O(K)$ is generated by trivial centralizers, hence $O(K) = 1$ follows.

□

In the remainder of this chapter $K \in P(\mathbb{C}_Q(z))$ is further investigated. By Proposition 5.11 in [SaSt], it is $P^*(\mathbb{C}_Q(z)) = \emptyset$.
So from now on we assume that K is contained in $P(\mathbb{C}_Q(z)) \setminus P^*(\mathbb{C}_Q(z))$ with $O(K) = 1$. Then there are components $L_1, \dots, L_m$ of K such that $K = (L_1 * \cdots * L_m) \cdot T$ with $R \le T \le S$, where T is a Sylow 2-subgroup of $K \cap H$ and, using Lemma 3.6, also from K.

In the following lemma, the possible components $L_1, \dots, L_m$ of K are determined under the assumption that there is more than one component.

Lemma 5.5: Suppose Hypothesis 5.2. Then it is $K \in P(\mathbb{C}_Q(z)) \setminus P^*(\mathbb{C}_Q(z))$ with $z \in O_2(K)$. Additionally, let be $O(K) = 1$ and K having $m > 1$ many components $L_1, \dots, L_m$. As before, let be $T \in \mathrm{Syl}_2(K \cap H)$ and $R \le \mathbb{C}_Q(z) \le T \le S$. Then the following statements hold:

(a) All components are isomorphic and T acts transitively on the set $\{L_1, \dots, L_m\}$.

(b) Every component L_i of K is normalized by R.

(c) For every $1 \le i \le m$, $L_i/\mathrm{Z}(L_i)$ is isomorphic to one of the following simple groups:

$$L_2(2^n),\ Sz(2^n),\ Sp_4(2^n) \text{ for } n \ge 2,\ L_3(4) \text{ or } L_2(q) \text{ for } q = 2^n \pm 1 > 3.$$

Proof: We assume $K \in P(\mathbb{C}_Q(z)) \setminus P^*(\mathbb{C}_Q(z))$ with $O(K) = 1$ and $K = (L_1 * \cdots * L_m) \cdot T$. As before, let $L_1, \ldots, L_m$ with $m > 1$ be the components of K and $T \in \mathrm{Syl}_2(K \cap H)$. In consequence of the minimality of $K \in P(\mathbb{C}_Q(z))$, the Sylow 2-subgroup T acts transitively on $\{L_1, \ldots, L_m\}$ and none of the components is contained in H. In particular, the number of components m divides the order of T and all the components are isomorphic.
Also $[R, O_2(K)] \leq R \cap O_2(K) = 1$ holds, as otherwise there would exist a root element $1 \neq r \in O_2(K) \cap R$ such that $E(K) \leq \mathbb{C}_G(r) \leq H$ holds by Lemma 3.9. But this contradicts the fact that no component is contained in H. With the same argumentation also $R \cap L_i = 1$ holds for all $1 \leq i \leq m$, using $m > 1$ and the well-known fact that different components centralize each other.
Now we show that every component L_i for $1 \leq i \leq m$ is normalized by R: We assume that there is an element $r \in R$ such that $L_i^r = L_j$ holds for some $i \neq j$. By Lemma 3.6, K is minimal parabolic and $H \cap K$ is the unique maximal subgroup of K which contains T. It is $H \cap E(K) \leq H \cap K$ and $E(K) \not\leq H$. Since it is $L_i \not\leq H$, there is an element $u \in L_i \setminus H$ and $u \cdot u^r \in \mathbb{C}_{L_i * L_j}(r) \leq H$ and $E(K) \leq \langle H \cap K, u \cdot u^r \rangle \leq H$, by Lemma 3.9. This is a contradiction, so (b) holds.

The strategy for the third statement is to apply a result of Bernd Baumann, see Lemma 2.45, to restrict the set of possible components. We set $\overline{K} = K/O_2(K)$ and $\overline{U} = UO_2(K)/O_2(K)$ for subgroups U of K. It is $K = E(K) \cdot T$, with $T \in \mathrm{Syl}_2(K)$ and we know that all components of K are isomorphic. Additionally, it is $O(\overline{K})$ normal in $\overline{L_1} * \cdots * \overline{L_m}$ and, as the groups $\overline{L_i}$ are all isomorphic, $O(K) = 1$ implies also $O(\overline{K}) = 1$.
Hence $\overline{K} = (\overline{L_1} \times \cdots \times \overline{L_m})\overline{T}$ follows, where $\overline{T}$ is a Sylow 2-subgroup of $\overline{K}$.

Choose $1 \neq s \in R \cap \mathrm{Z}(S)$. We show now that $\mathbb{C}_{L_i}(s)$ is a 2-group. Without loss of generality, we fix $i = 1$. It is Q a large subgroup in H, see Lemma 2.38, and it is $[T \cap L_1, s] = 1$. So $T \cap L_1 \leq \mathrm{N}_H(Q)$ holds. We assume now the existence of an element of odd order $\omega \in \mathbb{C}_{L_1}(s)$. By Lemma 3.9, it is $\omega \in H$. So ω centralizes $1 \neq \langle s \rangle \leq R = \mathrm{Z}(Q)$, hence ω normalizes the large subgroup Q. Therefore, ω also normalizes $\mathrm{Z}(Q) = R$. Then $[R, \omega] \leq R \cap L_1 = 1$ follows, as R normalizes L_1, using $m > 1$.
We now suppose the existence of an element $x \in Q \cap K$ such that $L_1^x = L_j$ holds for $j \neq 1$. Then, using $\omega \in \mathrm{N}_H(Q)$, $[\omega, x] \in \langle \omega \rangle * \langle \omega^x \rangle \cap Q = 1$ follows. So it is $\omega^x = \omega \in L_1 \cap L_j$, but, using $O(K) = 1$ this implies $\omega = 1$, as ω is of odd order.
Therefore, we may assume that $Q \cap K$ normalizes L_1. Hence it is $[L_1, K \cap Q] \leq L_1$. In particular, $[L_1, \mathbb{C}_Q(z)] \leq L_1 \cap Q$ holds.
This implies $[\omega, \mathbb{C}_Q(z), \mathbb{C}_Q(z)] \leq L_1 \cap Q' = L_1 \cap R = 1$. Then $[\omega, \mathbb{C}_Q(z)] \leq L_1 \cap \mathbb{C}_Q(\mathbb{C}_Q(z)) = \mathrm{Z}(\mathbb{C}_Q(z)) \cap L_1$ follows. As $R \cap L_1$ is trivial, we may assume that $\mathbb{C}_Q(z)$ equals $Z \times Q_1$, where Z is an elementary abelian group of order q, Q_1 is special and $\mathrm{Z}(\mathbb{C}_Q(z)) \cap L_1 \leq Z$. Hence ω centralizes $\mathbb{C}_Q(z)/Z$.
By Hypothesis 5.2, it is $z \in Q \cap O_2(K)$ and, as ω is an element in the component L_1, it is

$[Q \cap O_2(K), \omega] = 1$. This implies $[z, \omega] = 1$. And as before, $[R, \omega] \leq R \cap L_1 = 1$ holds.
Let u be an element in $Q \setminus \mathbb{C}_Q(z)$. Then $R \ni [u, z] = [u, z]^\omega = [u^\omega, z]$ holds. This implies $u^{-1} \cdot u^\omega \in \mathbb{C}_Q(z)$. Therefore, ω centralizes $Q/\mathbb{C}_Q(z)$ and, as proved before, also $\mathbb{C}_Q(z)/Z$. So by Coprime action, $[Q/Z, \omega] = 1$ follows.
Hence there is an element $u \in Q \setminus \mathbb{C}_Q(z)$ with $u^\omega = u$. Let x be an arbitrary element in $Z^\#$. Then $R \ni [x, u] = [x, u]^\omega = [x^\omega, u]$ holds. So it is $[x, \omega] \in \mathbb{C}_Q(u)$ and, as ω normalizes Z, $[x, \omega] \in \mathbb{C}_Z(u)$ holds. But $\mathbb{C}_Z(u)$ is trivial, as otherwise there is a non-trivial element $\tilde{z} \in Z$ which centralizes $u \in Q \setminus \mathbb{C}_Q(z)$. In this case, $\tilde{z}$ is centralized by u, so $u \in \mathbb{C}_Q(\tilde{z}) = \mathbb{C}_Q(z)$, which contradicts the choice of u. Altogether, ω centralizes Z and Q/Z, so by Coprime action $[Q, \omega] = 1$ holds. As ω is an element of H and Q is a large subgroup in H, this implies $\omega \in \mathbb{C}_H(Q) \leq Q$. Thus, ω must be trivial.
Therefore, $\mathbb{C}_{L_i}(s)$ is a 2-group for all $i \in \{1, \ldots, m\}$.

Next we show that also $\mathbb{C}_{\overline{L_i}}(\overline{s})$ is a 2-group. To do so, let $\overline{\omega} \in \mathbb{C}_{\overline{L_i}}(\overline{s})$ be an element of odd order. Then it is $[sO_2(K), \omega O_2(K)] \leq O_2(K)$ and $\omega \in O^2(L_i O_2(K)) = L_i$. Hence we have $[\omega, O_2(K)] = 1$ and $[\langle s \rangle O_2(K), \langle \omega \rangle, \langle \omega \rangle] = 1$. By Coprime action, $[\langle s \rangle O_2(K), \langle \omega \rangle]$ must be trivial. And, again by Coprime action, $[\langle s \rangle, \langle \omega \rangle]$ is trivial. So we have $\omega \in \mathbb{C}_{L_i}(s)$ and, as $\mathbb{C}_{L_i}(s)$ is a 2-group by the arguments above, $\mathbb{C}_{\overline{L_i}}(\overline{s})$ is a 2-group.

Let $\overline{x}$ be an element in $\mathbb{C}_{\overline{K}}(\overline{s})$. Then it is $\overline{x} = \overline{x_1} \ldots \overline{x_m} \cdot \overline{t}$ for elements $x_i \in L_i$ and $t \in T$. It is $\overline{t} \in \mathbb{C}_{\overline{K}}(\overline{s})$, so $\overline{x_1} \ldots \overline{x_m} \in \mathbb{C}_{\overline{K}}(\overline{s}) \cap \overline{L_1} \times \cdots \times \overline{L_m}$ follows. As the components L_i are normalized by $s \in R^\#$, we have $\overline{x_1} \ldots \overline{x_m} = (\overline{x_1} \ldots \overline{x_m})^{\overline{s}} = \overline{x_1}^{\overline{s}} \ldots \overline{x_m}^{\overline{s}} \in \overline{L_1} \times \cdots \times \overline{L_m}$. For all $i \in \{1, \ldots, m\}$, then $[\overline{x_i}, \overline{s}] \in \overline{L_i} \cap \bigtimes_{j \neq i} \overline{L_j}$ follows by induction, which leads to $\overline{x_i} \in \mathbb{C}_{\overline{L_i}}(\overline{s})$.
Altogether, $\overline{x}$ is a 2-element, so $\mathbb{C}_{\overline{K}}(\overline{s})$ is a 2-group.
Now we can apply Lemma 2.45, to get that the simple groups $\overline{L_i}$ must be isomorphic to one of the following groups: $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, $L_3(2^n)$, $Sp_4(2^n)$ for $n \geq 2$ or $L_2(q)$ for $q = 2^n \pm 1 > 3$.

In case of $\overline{L_i} \cong L_3(r)$ with $r = 2^n > 2$, we consider the subgroup

$$U := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in GF(r)^* \right\}$$

in $SL_3(r)$. It is U of order $r-1$. The projection of $\overline{s}$ into $\overline{L_i}$ can be identified with the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is centralized by U. The group $U/(U \cap \mathrm{Z}(SL_3(r)))$ can be identified with a subgroup of $L_3(r)$. As $\mathrm{Z}(SL_3(r))$ is of order $\gcd(3, r-1)$, $U/(U \cap \mathrm{Z}(SL_3(r)))$ can be a 2-group only if $r-1=3$ holds; hence for $r=4$. As $\mathbb{C}_{\overline{L_i}}(\overline{s})$ is a 2-group, the only remaining possibility is $\overline{L_i} \cong L_3(4)$.

For $\overline{L_i} \cong U_3(r)$ with $r=2^n>2$, the group

$$\tilde{U} := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & a \end{pmatrix} \mid a \in GF(r^2)^*,\ a^{r+1}=1 \right\}$$

is a subgroup of $SU_3(r)$, as it is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-2} & 0 \\ 0 & 0 & a \end{pmatrix} \cdot \begin{pmatrix} a^r & 0 & 0 \\ 0 & a^{-2r} & 0 \\ 0 & 0 & a^r \end{pmatrix} = \begin{pmatrix} a \cdot a^r & 0 & 0 \\ 0 & (a \cdot a^r)^{-2} & 0 \\ 0 & 0 & a \cdot a^r \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is $GF(r^2)^*$ cyclic of order $r^2-1=(r-1)(r+1)$, so the order of $\tilde{U}$ equals $r+1$. Again, the projection of $\overline{s}$ into $\overline{L_i}$ can be identified with

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is centralized by $\tilde{U}$. The group $\tilde{U}/(\tilde{U} \cap \mathrm{Z}(SU_3(r)))$ can be identified with a subgroup of $U_3(r)$. As $\mathrm{Z}(SU_3(r))$ is of order $\gcd(3, r+1)$, $\tilde{U}/(\tilde{U} \cap \mathrm{Z}(SU_3(r)))$ can only be a 2-group if $r+1=3$ holds; hence for $r=2$. As $\mathbb{C}_{\overline{L_i}}(\overline{s})$ is a 2-group and $r>2$ holds, the claim of the lemma follows. □

In the following remark some facts about K and its components are collected, which are frequently needed in the following. The terminology is the one from the previous results.

Remark 5.6: (a) The structure of the Sylow 2-subgroups of the simple groups, listed in the previous lemma, is well-known:

L	$L_2(2^n)$	$Sp_4(2^n)$	$L_2(2^n \pm 1)$	$L_3(4)$	$Sz(2^n)$
$P \in \mathrm{Syl}_2(L)$	E_{2^n}	$E_{2^{3n}} : E_{2^n}$	D_{2^n}	$D_1(4) \cong 4^{1+2}$	2^{n+n}

(b) It is $T \leq \mathbb{C}_H(s)$ for an element $1 \neq s \in \mathrm{Z}(S) \cap R$; hence T normalizes the large subgroup Q and its center R.

(c) The components $L_1, \ldots, L_m$ are normalized by R, see Lemma 5.5. And for all $i \in \{1, \ldots, m\}$, $L_i \cap R = 1$ as both as $[O_2(K), R] \leq O_2(K) \cap R = 1$ hold. Otherwise, there would be at least one component in the centralizer of a non-trivial element in R and therefore contained in H, by Lemma 3.9.

In the following two lemmas, we show that K can only have one component. To do so, we first show that, in case of more than one component, each component is normalized by $\mathbb{C}_Q(z)$.

Lemma 5.7: Let Hypothesis 5.2 hold and K be an element of $P(\mathbb{C}_Q(z))$ with $O(K) = 1$ and components $L_1, \ldots, L_m$ and $m > 1$. Further let be $T \in \mathrm{Syl}_2(K \cap H)$ and $R \leq \mathbb{C}_Q(z) \leq T \leq S$. Then L_1 is normalized by $\mathbb{C}_Q(z)$.

Proof: To show that L_1 is normalized by $\mathbb{C}_Q(z)$, we assume the opposite. By Lemma 5.4, we may assume $O(K) = 1$ and we define $\overline{K} := K/O_2(K) = (\overline{L_1} \times \cdots \times \overline{L_m}) \cdot \overline{T}$, where the components $\overline{L_i}$ of $\overline{K}$ are isomorphic to exactly one of the following groups:
$L_2(2^n)$, $Sz(2^n)$, $Sp_4(2^n)$ for $2^n > 2$, $L_3(4)$ or $L_2(2^n \pm 1)$ for $2^n \pm 1 > 3$.
The group R normalizes each component, see Lemma 5.5, and for all $i \in \{1, \ldots, m\}$, it is $L_i \cap R = 1$ and $O_2(K) \cap R = 1$, as otherwise at least one component would be in the centralizer of an involution in R and hence in H. Also $T_1 := T \cap L_1$ centralizes a non-trivial element $s \in \mathrm{Z}(S) \cap R$ and hence, normalizes the large subgroup Q and its center $\mathrm{Z}(Q) = R$. Additionally, it is $[T_1, R] \leq L_1 \cap R = 1$. The same holds for $\overline{T_1} = T_1 O_2(K)/O_2(K)$, instead of T_1.
Due to Lemma 5.1, it is $\mathbb{C}_Q(z) = \tilde{Z} \times Q_1$, where $\tilde{Z}$ is elementary abelian of order $q = |R|$ and Q_1 is a semi-extraspecial 2-group with $\mathrm{Z}(Q_1) = R$.
The component L_1 is not normalized by $\mathbb{C}_Q(z)$ by supposition. We define $U := \mathrm{N}_{K \cap Q}(L_1)$ and $Q_L := U \cap \mathbb{C}_Q(z)$.
There is no element x of order 4 in $\mathbb{C}_Q(z)$ which permutes 4 different components, as $x^2 \in \Phi(Q) = R$ normalizes every component by Lemma 5.5.
Hence for $1 \neq t \in \mathbb{C}_Q(z) \setminus Q_L$, $[T_1, t]$ is a diagonal subgroup in $L_1 * L_1^t$ and it is $[T_1, t] \leq Q$, as T_1 normalizes Q. If there is an element $\tilde{t} \in \mathbb{C}_Q(z) \setminus Q_L$ for which $L_1^{\tilde{t}} \neq L_1^t$ holds, then $\left[T_1, t, \tilde{t}\right] \leq Q' = R$ is a diagonal subgroup in the central product of the corresponding four components.
So every orbit of the component L_1 under the action of $\mathbb{C}_Q(z)$ on $E(K)$ contains at most 4 components, as there are non-trivial elements of R contained in this orbit, which cannot be centralized by any component, due to Lemma 3.9.
Let m_1 be the length of the orbit of L_1 under $\mathbb{C}_Q(z)$. So, as $\mathbb{C}_Q(z)$ is a 2-group, which by assumption does not normalize L_1, we are left with orbits of length $m_1 = 2$ or $m_1 = 4$ within this lemma.
Furthermore, $U = \mathrm{N}_{K \cap Q}(L_1)$ equals $\mathrm{N}_{K \cap Q}(L_i)$ for every component L_i in the orbit of L_1 under Q: We suppose an element $u \in U$ and $L_i = L_1^y$ for some $y \in Q$. Then it is $L_i^u = L_1^{yu} = L_1^{uys} = L_1^{ys} = L_i^s = L_i$, where s is an element in R that therefore normalizes every component. So U normalizes every component L_i and, analogously, we deduce $\mathrm{N}_{K \cap Q}(L_i) \leq U$. The same arguments imply $Q_L = \mathrm{N}_{\mathbb{C}_Q(z)}(L_i)$ for all $i \in \{1, \ldots, m\}$.
In the following, we distinguish between $\mathbb{C}_Q(z)$-orbits of length $m_1 = 4$ and $m_1 = 2$.

- At first, we consider the case $m_1 = 4$, where $\mathbb{C}_Q(z)$ acts transitively on the set of 4 components which build the orbit of L_1 under $\mathbb{C}_Q(z)$. Let $B := L_1 * L_1^t * L_1^{\tilde{t}} * L_1^{t\tilde{t}}$ be the $\mathbb{C}_Q(z)$-orbit of L_1 for suitable elements $t, \tilde{t} \in \mathbb{C}_Q(z)$. It is $\left[T_1, t, \tilde{t}\right] \leq R$, hence $\left[\overline{T_1}, t, \tilde{t}\right]$ is a diagonal subgroup of $\overline{B}$, and $\left[\overline{T_1}, t, \tilde{t}\right]$ is an isomorphic copy of $\overline{T_1}$. As R is elementary abelian, the same holds for $\overline{T_1}$. This implies $\overline{L_1} \cong L_2(r)$ for $r = 2^n \geq 4$, by Remark 5.6. The Schur multiplier of $L_2(2^n)$ is trivial, except for $L_2(4)$ by Lemma 2.22. Hence, up to $L_2(4)$, it is $L_1 \cong \overline{L_1}$ and $\overline{T_1} = T_1$.
  Additionally, it is $4 \leq r = |\overline{T_1}| \leq |R|$. Due to the orbit length, $|\mathbb{C}_Q(z) : Q_L| = 4$ holds and the elements in Q_L induce automorphisms on $\overline{L_1}$ which normalize the Sylow 2-subgroup $\overline{T_1}$. Looking at the Dynkin diagram of $L_2(r)$, these automorphisms have to be inner or field automorphisms, where $\overline{T_1}$ cannot be centralized by a field automorphisms. As $[\overline{T_1}, R] \leq R \cap \overline{L_1} = 1$ holds, the automorphisms on $\overline{L_1}$, which are induced by elements of R, have to be inner. We assume that an element $x \in Q_L$ induces a field automorphism. Then, using $R \leq \mathrm{Z}(\mathbb{C}_Q(z))$, it is $|\mathbb{C}_{\overline{T_1}}(x)| \geq |R| \geq |\overline{T_1}|$; so $x \in Q_L$ acts trivially on $\overline{T_1}$. Therefore, x cannot induce a field automorphism. Hence Q_L induces inner automorphisms on $\overline{L_1}$ only, which all centralize $\overline{T_1}$. Therefore, Q_L is an abelian subgroup of $\mathbb{C}_Q(z)$ of index 4. This implies $|R| \leq 4$, so in total, it is $r = |R| = 4$ and $\overline{L_1} \cong L_2(4)$.
  In particular, $|\mathbb{C}_Q(z)| \leq |R|^4 = 4^4 = q^4$ holds. Hence it is $|Q| \leq q^5$ for $q = 4$.
  This contradicts $|Q| \geq q^9$ in Hypothesis 5.2. As these arguments hold for all $i \in \{1, \dots, m\}$ and not only for $i = 1$, $\mathbb{C}_Q(z)$-orbits of length 4 in the set of components are not possible.

- Let the orbit of L_1 under $\mathbb{C}_Q(z)$ be of length 2. Hence $|\mathbb{C}_Q(z) : Q_L| = 2$ holds. Then there is an element $t \in \mathbb{C}_Q(z) \setminus Q_L$ such that $[T_1, t] \leq Q$ and Q_L normalizes both components L_1 and $L_2 := L_1^t$. At this, $t \in \mathbb{C}_Q(z)$ is either an involution or an element of order 4 with $t^2 \in R$.
  There is no element of order at least 8 in $\overline{T_1}$: If there was an element $x \in \overline{T_1}$ with $\mathrm{o}(x) \geq 8$, it would be $[x, t] \in Q$ and $\mathrm{o}([x, t]) \geq 8$. But there is no element of such order in the special group Q.
  For $\overline{T_1} \in \mathrm{Syl}_2(L_2(2^n \pm 1))$ with $2^n \pm 1 > 3$ this implies $\overline{T_1} \cong D_8$ and hence $\overline{L_1} \cong L_2(7)$ or $\overline{L_1} \cong L_2(9)$, using Remark 5.6. So, if $\overline{L_1}$ is not isomorphic to one of the groups $L_2(4)$, $L_3(4)$, $Sz(8)$, $L_2(7)$ or $L_2(9)$, then L_1 is a simple group, using the information about Schur multipliers provided in Lemma 2.22.

  We now consider the case that L_1 and L_2 are simple components. In this case $L_1 \cap L_2 = 1$ holds. So we may assume for now that $L_1 \times L_2$ is the orbit of L_1 under $\mathbb{C}_Q(z)$. We remember that $Q_L = \mathrm{N}_{\mathbb{C}_Q(z)}(L_1)$ is a subgroup of index 2 in $\mathbb{C}_Q(z)$.
  We want to show $q = 2$. Therefore, we assume $q > 2$. As before, it is $[T_1, Q] \leq Q$ and $[T_1, R] = 1$. Further, $[T_1 \cap Q, Q_L] \leq Q' \cap T_1 = R \cap T_1 = 1$ holds and so

$T_1 \cap Q \leq \mathbb{C}_Q(Q_L) \cap T_1$ follows. As $\mathbb{C}_{T_1}(Q_L)$ equals $\mathbb{C}_{T_1}(\mathbb{C}_Q(z))$, this implies $T_1 \cap Q \leq \mathbb{C}_Q(\mathbb{C}_Q(z)) \cap T_1 = \mathrm{Z}(\mathbb{C}_Q(z)) \cap T_1$. Particularly, $T_1 \cap Q \leq T_1 \cap T_1^t \leq L_1 \cap L_2 = 1$ holds and so $[T_1, Q_L] \leq T_1 \cap Q$ is trivial. So the elements in T_1 act on Q/R and centralize a subgroup of index at most $2 \cdot q$ in Q.
As L_1 is a component, $\mathrm{N}_{L_1}(T_1)$ is a proper subgroup in L_1 and, using the minimality of $K \in P(\mathbb{C}_Q(z))$, $\mathrm{N}_{L_1}(T_1) \leq H$ holds. If T_1 is abelian, then it is $T_1 = [T_1, \mathrm{N}_{L_1}(T_1)] \leq H'$ and, as the Sylow 2-subgroups of $\mathrm{Out}(L_1)$ are abelian by Lemma 2.30, we may assume the existence of an involution $x \in T_1 \cap F^*(H)$. If T_1' is not trivial, for the same reason, we can find an involution $x \in T_1' \cap F^*(H)$. Altogether, we choose an involution $x \in T_1 \cap F^*(H)$ which induces an inner automorphism on L_1 and hence induces a $GF(q)$-linear action on Q/R. Then $|Q : \mathbb{C}_Q(x)|$ is a power of q.
And, as by assumption $q > 2$ holds, x centralizes a subgroup of index at most q in Q. So either $x \in T_1$ centralizes Q or x induces a $GF(q)$-transvection on Q/R. If x induces a transvection, the Levi complement in $\mathrm{N}_{F^*(H)}(Q)$ is a classical group of Lie type and Q/R the corresponding natural module. But unitary transvections are defined over $GF(q^2)$ and not over $GF(q)$, and linear and symplectic transvections act transitively on the one-dimensional subspaces of Q/R, hence they induce a transitive action on $Q^{\#}$, which consists of elements of order 2 and 4, a contradiction. Due to Lemma 2.36, there is no orthogonal group acting as Levi complement in the described way on Q/R, so x induces no orthogonal transvection. Hence it is $x \in \mathbb{C}_H(Q) \cap T_1$ and, as $\mathbb{C}_H(Q) \leq Q$ holds, we get $x \in Q \cap T_1 = 1$, which is a contradiction. So without restriction, we may assume $q = 2$.
Therefore, Q is extraspecial and it is $\mathbb{C}_Q(z) = \langle z \rangle \times Q_1$, where Q_1 is an extraspecial 2-group with $\mathrm{Z}(Q_1) = R = \langle s \rangle$, see Lemma 2.35. It is $T_1 \cong [T_1, t]$ and $[T_1, t]' \leq Q' = R$. This and $q = |R| = 2$ imply $|T_1'| \leq 2$. So either it is T_1 abelian or it is $|T_1'| = 2$ by Remark 5.6. If T_1 is abelian, it is $L_1 \cong L_2(2^n)$ for $n \geq 2$. If T_1 is not abelian, $L_1 \cong L_2(7)$ or $L_1 \cong L_2(9)$ and $T_1 \cong D_8$ are the only possibilities for the component L_1, due to Remark 5.6.

It is $Q_L = \mathrm{N}_{\mathbb{C}_Q(z)}(L_1)$ a subgroup of $\mathbb{C}_Q(z)$ of index 2. Hence Q_L is isomorphic to $\tilde{Z}_2 \times Q_2$, where $\tilde{Z}_2$ is an abelian group and Q_2 is an extraspecial group with center $R = \langle s \rangle$. Hence Q_2 acts faithfully on L_1 and normalizes $T_1 = T \cap L_1$ by definition. By the paragraph above, we only have to deal with L_1 being isomorphic to $L_2(2^n)$ for $n \geq 2$, $L_2(7)$ or $L_2(9)$, where Q_2 acts faithfully on L_1 and normalizes T_1.
For $L_1 \cong L_2(2^n)$ and $n \geq 2$, $\mathrm{Out}(L_1)$ is cyclic of order n, so Q_2 is of order 8 and $L_1 \cong L_2(4)$. It is $\mathrm{Out}(L_2(7)) \cong Z_2$ and $\mathrm{Out}(L_2(9)) \cong E_4$. So in all three cases, Q_2 has to be a group of order 8. Then $|Q| = 2^7$ follows, which contradicts $|Q| \geq q^9$ in Hypothesis 5.2.

Until now, we dealt with the case that the Schur multiplier of L_1 is trivial. So we now

have to consider a $\mathbb{C}_Q(z)$-orbit $L_1 * L_2$, where $L_1 \cong L_2$ is a quasisimple covering group of one of the following groups: $L_2(4) \cong L_2(5)$, $L_2(7)$, $L_2(9)$, $Sz(8)$ or $L_3(4)$.
As $O(K) = 1$ holds, the Schur multiplier is a 2-group, hence L_1 is isomorphic to $SL_2(5)$, $SL_2(7)$, $SL_2(9)$ or to a Schur cover of $Sz(8)$ or of $L_3(4)$.
We again consider $Q_L = \tilde{Z}_2 \times Q_2$, which is a subgroup of index 2 in $\mathbb{C}_Q(z)$. And again $\tilde{Z}_2$ is abelian and Q_2 is a semi-extraspecial 2-group with $\mathrm{Z}(Q_2) = R$ by Lemma 5.1. Then Q_2 acts faithfully on L_1, as otherwise there would exist an element $r \in R^{\#}$ in the kernel of the operation and $L_1 \leq \mathbb{C}_G(r) \leq H$ would follow by Lemma 3.9. Additionally, it is $\mathrm{Aut}(L_1) \lesssim \mathrm{Aut}\,\overline{L_1}$ by Remark 2.10. Because of that, if Q_2 cannot be embedded into $\mathrm{Aut}(\overline{L_1})$, it cannot be embedded into $\mathrm{Aut}(L_1)$.
We first consider $\overline{L_1}$ being isomorphic to $L_2(4)$, $L_2(7)$ or to $L_2(9)$. The same arguments as before imply that Q_2 can only be embedded into $\mathrm{Aut}(\overline{L_1})$, and therefore in $\mathrm{Aut}(L_1)$ if $|Q_2| = 8$ holds. So we are left with L_1 being isomorphic to a Schur cover of $Sz(8)$ or of $L_3(4)$, as $|Q_2| = 8$ implies $|Q| = 2^7$, which contradicts $|Q| \geq q^9$, compare Hypothesis 5.2.
For $\overline{L_1} \cong Sz(8)$, there is no outer automorphism of even order, see Lemma 2.31. So Q_2 is isomorphic to a subgroup of $\overline{T_1} \cong 2^{3+3}$, using Remark 5.6. This only works out for $q = 2$ and $|Q_2| = 8$. So again $|Q| = 2^7$ holds, which contradicts $|Q| \geq q^9$ in Hypothesis 5.2.
For $\overline{L_1} \cong L_3(4)$, the diagonal subgroup $[\overline{T_1}, t]$ is contained in Q and is isomorphic to $\overline{T_1} \cong D_1(4)$. This implies $q \geq 4$, and because of $|\mathrm{Aut}(L_3(4))|_2 = 2^8$ and $|Q_2| \geq q^3$, necessarily $q = 4$ follows. But for $q = 4$, it is $|Q_2| = q^3 = 4^3$. This implies $|Q| = q^7$. Hence it contradicts $|Q| \geq q^9$, compare Hypothesis 5.2.

Altogether, we can conclude that $\mathbb{C}_Q(z)$ normalizes L_1 and therefore each component of K. □

Lemma 5.8: Suppose Hypothesis 5.2, $K \in P(\mathbb{C}_Q(z))$ with $O(K) = 1$ and components $L_1, \ldots, L_m$, $T \in \mathrm{Syl}_2(K \cap H)$ and $R \leq \mathbb{C}_Q(z) \leq T \leq S$. If L_1 is normalized by $\mathbb{C}_Q(z)$, then $m = 1$ follows.

Proof: To show $m = 1$, we assume $m > 1$. For $T_1 := T \cap L_1 \in \mathrm{Syl}_2(L_1)$, it is $[T_1, R] \leq L_1 \cap R = 1$, as T normalizes $R = \mathrm{Z}(S_1)$. By supposition, $\mathbb{C}_Q(z) \leq K$ normalizes L_1 and, it is $\mathbb{C}_Q(z) = \tilde{Z} \times Q_1$, where $\tilde{Z}$ is elementary abelian of order $|R| = q$ and Q_1 is semi-extraspecial with $\mathrm{Z}(Q_1) = \mathrm{Z}(Q) = R$, by Lemma 5.1. As before, $[R, O_2(K)] \leq O_2(K) \cap R = 1$ holds. The group Q_1 acts faithfully on L_1, as otherwise there would exist an element $r \in R^{\#}$ in the kernel of the operation, so $L_1 \leq \mathbb{C}_G(r) \leq H$ would hold, by Lemma 3.9. So the elements in Q_1 induce non-trivial automorphisms on L_1 and, in case of a non-trivial Schur multiplier, also on $\overline{L_1} = L_1 O_2(K)/O_2(K)$.

The strategy of this proof is to show that, if Q_1 can be embedded into the automorphism group of L_1, the order of Q_1 is at most q^5 for $|R| = q$, which implies $|Q| \leq q^7$.
In case of a non-trivial Schur multiplier, which by $O(K) = 1$ can only be of even order, Q_1 can only be embedded into $\mathrm{Aut}(L_1)$ if this also holds for $\mathrm{Aut}(\overline{L_1})$, see Remark 2.10. Hence without loss of generality, we may assume that L_1 is simple.
Using Lemma 5.5, L_1 has to be isomorphic to $L_2(2^n)$, $Sz(2^n)$, $Sp_4(2^n)$, all for $n \geq 2$, $L_3(4)$ or $L_2(2^n \pm 1)$ for $2^n \pm 1 > 3$. The Sylow 2-subgroups of $\mathrm{Out}(L_1)$ are abelian by Lemma 2.30, as either $\mathrm{Outdiag}(L_1)$ is trivial or of order 2 in case of $L_2(2^n \pm 1)$. As Q_1 is not abelian, it cannot be embedded into $\mathrm{Out}(L_1)$.

- Let $L_1 \cong L_2(2^n)$ for $n \geq 2$. Then T_1 is elementary abelian of order 2^n and $\mathrm{Out}(L_2(2^n))$ is cyclic of order n. Hence the elements of Q_1' induce inner automorphisms. As Q_1/Q_1' is elementary abelian, Q_1 modulo inner automorphisms has to be cyclic of order 2. This implies $q = 2$. Hence we have $L_1 \cong L_2(4)$ and $|Q_1| = 2^3$ follows.

- Let $L_1 \cong Sz(2^n)$ for $n \geq 2$. Then $T_1 \cong 2^{n+n}$ by Remark 5.6, so T_1 is a special group with $\mathrm{Z}(T_1) \cong E_{2^n}$ and every involution in T_1 is 2-central, see [Hig]. By Lemma 2.31, there is no outer automorphism of even order in $\mathrm{Aut}(L_1)$, so Q_1 has to be isomorphic to a subgroup of T_1. As in $Sz(2^n)$ every involution is 2-central, $\Omega_1(Q_1) = \mathrm{Z}(Q_1)$ follows, which implies $Q_1 \cong Q_1(q)$, using Lemma 2.34 and the structure of T_1. Hence Q_1 is of $-$-type and of order q^3, which both is a contradiction to Hypothesis 5.2.

- Let $L_1 \cong Sp_4(2^n)$ for $n \geq 2$. It is T_1 generated by two elementary abelian groups E_1 and E_2, which intersect in $\mathrm{Z}(T_1)$ and for every involution $i \in T_1 \setminus \mathrm{Z}(T_1)$, $\mathbb{C}_{T_1}(i)$ is abelian. Additionally, $\mathrm{Out}(L_1)$ is cyclic of order $2n$, see Lemma 2.31, and, as Q_1 is a special 2-group, every abelian factor group of Q_1 is elementary abelian. Hence $|Q_1 : Q_1 \cap T_1| \leq 2$ holds. To show $|Q_1| \leq q^3$, we assume $|Q_1| > q^3$.
  It is Q_1 generated by involutions, using Lemma 2.34 and Lemma 5.1. And for all involutions $i \in Q_1$, it is $\mathbb{C}_{Q_1}(i)$ not abelian, using $|Q_1| \geq q^5$. For $Q_1 \leq T_1$, every involution in Q_1 must be contained in $\mathrm{Z}(T_1)$. But Q_1 is special, so it cannot be embedded into the abelian group $\mathrm{Z}(T_1)$. Thus $|Q_1 : Q_1 \cap T_1| = 2$ follows. Let i be an involution in $Q_1 \cap T_1$ which is not contained in $\mathrm{Z}(T_1)$. Then $2q \geq |Q_1 : \mathbb{C}_{Q_1 \cap T_1}(i)|$ holds and this index is of at least q^2, as $|Q_1| \geq q^5$ and the centralizer $\mathbb{C}_{Q_1 \cap T_1}(i)$ is elementary abelian. This inequality only holds for $q = 2$ and $|Q_1| \leq 2^5$. Hence for $q > 2$ or $|Q_1| > 2^5$, every involution in Q_1 which not embeds into $\mathrm{Z}(T_1)$, embeds into $\mathrm{Out}(L_1)$. Thus there is an elementary abelian subgroup of index 2 in Q_1, which implies $Q_1 \cong D_8$. Altogether, $|Q_1| \leq q^5$ holds.

- Let $L_1 \cong L_3(4)$. Then it is $\mathrm{Aut}(L_1) \cong L_3(4){:}\, D_{12}$ by [CoCu], so $|\mathrm{Aut}(L_1)|_2 = 2^8$. Comparing orders, for $q > 4$ there is no semi-extraspecial subgroup in $\mathrm{Aut}(L_1)$. As

$T_1 \cong D_1(4)$ holds, one gets $|Q_1| = q^3$ for $q = 4$. A Sylow 2-subgroup of $\mathrm{Aut}(L_1)$ is, by [CoCu], of shape $2^{1+4} \cdot D_8$, where the product is not central, so $\mathrm{Aut}(L_1)$ does not contain an extraspecial subgroup of order 2^7. Hence for $q = 2$, there is no extraspecial subgroup in $\mathrm{Aut}(L_1)$ of order greater than 2^5. So, also in this case, $|Q_1| \leq q^5$ holds.

- Let $L_1 \cong L_2(2^n \pm 1)$ for $2^n \pm 1 > 3$. Then $(2^n \pm 1) \equiv \pm 1 \mod 8$ holds, so T_1 is a dihedral group of order 2^n. With $m_2(T_1) = 2$, there is an elementary abelian 2-group of order at most 2^4 in $\mathrm{Aut}(L_1)$, see Lemma 2.30. As Q_1 is special and Sylow 2-subgroups of $\mathrm{Out}(L_1)$ are abelian, Q_1' induces inner automorphisms. For $q > 2$, this implies $q = 4$, but then there has to be an elementary abelian 2-group of order $q^2 > 4$ in $\mathrm{Out}(L_1)$, which is impossible. For $q = 2$, only an extraspecial group of order 8 can be embedded into T_1. And using $|Q_1 : T_1 \cap Q_1| \leq 4$, the order of Q_1 is at most 2^5.

So it is $|Q_1| \leq q^5$ and therefore, $|Q| \leq q^7$ in all cases. Again, this contradicts $|Q| \geq q^9$ in Hypothesis 5.2. Hence $m = 1$ holds. □

Remark 5.9: Suppose Hypothesis 5.2. Using the previous results for $K \in P(\mathbb{C}_Q(z))$, we may assume $K = L \cdot T$ with $O(K) = 1$ and $O_2(K) \neq 1$, where L is a component of K and T a Sylow 2-subgroup of $K \cap H$ and also of K, see Lemma 3.6 (a). Additionally, we assume $R \leq T \leq S$ and $S_1 = S \cap F^*(H)$. It is K minimal parabolic with respect to T, see Lemma 3.6(d). By defining $\overline{K} = K/O_2(K)$, one gets $\overline{K} = \overline{L}\,\overline{T}$ for $\overline{L} = L\,O_2(K)/O_2(K)$ and $\overline{T} = T/O_2(K) \in \mathrm{Syl}_2(\overline{K})$. Then $\overline{M} = (K \cap H)/O_2(K)$ is the only maximal subgroup of $\overline{K}$ which contains $\overline{T}$. Additionally, $F^*(\overline{K}) = \overline{L}$ is a non-abelian simple group and, because of $\mathbb{C}_K(L) = O_2(K)$, $\overline{K}$ is a group of automorphisms of the simple group $\overline{L}$.
As K is a subgroup of the 2-local subgroup $\mathrm{N}_G(O_2(K))$, by the K_2-group assumption in Hypothesis 1.4, $\overline{L}$ is a known finite simple group. As Lemma 5.5 is formulated under the assumption of K having more than one component, the possible component is not (only) one considered in the previous lemmas.

Lemma 5.10: Let Hypothesis 5.2 hold. Then it is $\mathbb{C}_G(z) \leq H$ for all $z \in Q$.

Proof: Using the results of the previous lemmas, collected in Remark 5.9, we have to deal with $\overline{K} = \overline{L} \cdot \overline{T}$, with properties and notation as above.
Also $\mathbb{C}_Q(z) \leq K$ normalizes $E(K) = L$ and $\mathbb{C}_Q(z) = \tilde{Z} \times Q_1$, where $\tilde{Z}$ is elementary abelian of order $|R| = q$ and Q_1 is semi-extraspecial with $\mathrm{Z}(Q_1) = \mathrm{Z}(Q) = R$, see Lemma 5.1. For $q = 2$, Q_1 and Q are extraspecial. As before, $O_2(K) \cap R = 1$ holds and it is $T_1 := T \cap L \in \mathrm{Syl}_2(L)$. Then $\mathbb{C}_Q(z)$, and in particular Q_1, normalize T_1.
Additionally, Q_1 acts faithfully on L, as otherwise there would be a non-trivial element $s \in R$ in the kernel of the operation, so $L \leq \mathbb{C}_G(s) \leq H$ would hold by Lemma 3.9. Hence Q_1 is isomorphic to a subgroup of $\mathrm{Aut}(L)$. We now consider the possibilities for $\overline{L}$:

If $\overline{L}$ is a simple group of Lie type over a field of characteristic 2, then Lemma 2.47 states that $\overline{L}$ is isomorphic to one of the simple groups: $L_2(2^n)$, $Sz(2^n)$, $U_3(2^n)$, $L_3(2^n)$ or $Sp_4(2^n)$, where in the last two cases, $\overline{T}$ acts non-trivially on the Dynkin diagram of $\overline{L}$.
If $\overline{L}$ is an alternating group, then Lemma 2.49 states that either

- $\overline{L} \cong A_{2^m+1}$ for $m \in \mathbb{N}$ holds, where $\overline{H \cap K}$ is the stabilizer of a point in the permutation representation of degree $2^m + 1$ in $\overline{K} \cong A_{2^m+1}$ or $\overline{K} \cong S_{2^m+1}$, or
- $\overline{L} \cong A_6$ holds.

By Lemma 2.48, $\overline{L}$ cannot be isomorphic to one of the 26 sporadic simple groups.
If $\overline{L}$ is a simple group of Lie type over a field of odd characteristic, we distinguish between the cases that R is cyclic and that R is not cyclic. The group R acts on L and $\mathbb{C}_L(s) \leq H$ holds for all elements $s \in R^{\#}$ by Lemma 3.9. Then also $\mathbb{C}_R(L)$ is trivial, as otherwise L centralizes an element in $s \in R^{\#}$, implying $L \leq H$.

If R is not cyclic, we can apply Lemma 2.52.
If $R = \langle s \rangle$ is cyclic, we consider $\mathrm{Z}_2(S_1)$. As $F^*(H)$ is not isomorphic to $L_n(q)$ or $U_n(q)$, Lemma 2.39 states that $\mathrm{Z}_2(S_1)$ is an elementary abelian subgroup of Q of order 4, and $x \sim_H s$ holds for any $x \in \mathrm{Z}_2(S_1)^{\#}$. As $\mathrm{Z}_2(S_1)$ is elementary abelian of order 4, $|S_1 : \mathbb{C}_{S_1}(\mathrm{Z}_2(S_1))| = 2$ holds. As $F^*(H)$ is non-abelian simple, Thompson's Transfer, see Lemma 2.7, implies that every involution in S_1 is H-conjugate to an involution in $\mathbb{C}_{S_1}(\mathrm{Z}_2(S_1))$. So without loss of generality, it is $\mathrm{Z}_2(S_1) \leq \mathbb{C}_Q(z) \leq K$. So $\mathrm{Z}_2(S_1)$ is a non-cyclic abelian group of H-conjugates of s. Therefore, $\mathbb{C}_L(x) \leq H$ holds for every element $x \in \mathrm{Z}_2(S_1)^{\#}$ and $\mathrm{Z}_2(S_1)$ acts on L with $\mathbb{C}_{\mathrm{Z}_2(S_1)}(L) = 1$.
So in both cases, Lemma 2.52 is applicable. Thus $\overline{L}$ is isomorphic to $L_2(5) \cong L_2(4)$, $L_2(7) \cong L_3(2)$, ${}^2G_2(3)' \cong L_2(8)$, $L_2(9) \cong A_6$ or to $PSp_4(3) \cong U_4(2)$, where $PSp_4(3)$ is not minimal parabolic, due to [CoCu].
We treat these possibilities case by case. The group $L_2(5)$ is treated as Lie type group $L_2(4)$, A_6 is treated as Lie type group $L_2(9)$ and ${}^2G_2(3)'$ is treated as $L_2(8)$.

- If $\overline{L}$ is isomorphic to $L_2(2^n)$, $Sz(2^n)$ or $Sp_4(2^n)$ for $n \geq 2$, the argumentation is exactly the same as in Lemma 5.8. So in all these cases, one ends up with $|Q| < q^9$, which contradicts Hypothesis 5.2.
- Let $\overline{L} \cong U_3(2^n)$ with $n \geq 2$. The strategy is again to embed Q_1 into $\mathrm{Aut}(L)$ to see that this only works out for small cases, which have been already considered. It is $L \cong U_3(2^n)$, as there is no non-trivial Schur multiplier of even order, see Lemma 2.22. By Lemma 2.34, it is $T_1 \cong Q_1(2^n)$, hence every involution in $U_3(2^n)$ is 2-central and, using Lemma 2.30, $\mathrm{Out}(L)$ is cyclic. So in particular, Q' embeds into T_1 and $|Q : Q \cap T_1| \leq 2$.

If Q_1 completely embeds into T_1, then $\Omega(Q_1) = \mathrm{Z}(Q_1)$ follows, implying $Q_1 \cong Q_1(q)$, hence $|Q_1| = q^3$. If Q_1 does not embed into T_1, assuming $|Q_1| > q^3$, then Q_1 contains an elementary abelian subgroup of index 2, which implies $Q_1 \cong D_8$. Altogether, $|Q_1| = q^3$ holds. This contradicts $|Q| \geq q^9$ in Hypothesis 5.2.

- Let $\overline{L} \cong L_3(2^n)$ and $\overline{M} = \mathrm{N}_{\overline{K}}(\overline{T} \cap \overline{L})$. For $\overline{L_3(2)}$, the 2-part of $\mathrm{Aut}(L)$ is 2^4, which immediately implies $|Q_1| = 2^3$. For $\overline{L} \cong L_3(4)$ the argumentation is identical to the one in Lemma 5.8, leading to $|Q_1| \leq q^5$. So one may assume $n \geq 3$. The idea is again to embed Q_1 into $\mathrm{Aut}(L)$, to see that this only works out for small cases of Q, leading to $F^*(H)$ being isomorphic to groups which have been excluded in Hypothesis 5.2. It is $L \cong L_3(2^n)$, as there is no non-trivial Schur multiplier of even order in these groups for $n > 2$ by Lemma 2.22.
The argumentation of the following is similar to the one in Lemma 5.8 for $L \cong Sp_4(2^n)$. To show $|Q_1| \leq q^5$, we assume $|Q_1| \geq q^7$. Then Q_1 is generated by involutions and for every involution $i \in Q_1$, $\mathbb{C}_{Q_1}(i)$ is not abelian. It is $T_1 \in \mathrm{Syl}_2(L)$ isomorphic to $D_1(q)$, see Lemma 2.34, so there are two elementary abelian subgroups E_1 and E_2 in T_1, each of order 2^{2n}. Every involution in T_1 is contained in $E_1 \cup E_2$ and it is $E_1 \cap E_2 = \mathrm{Z}(T_1) \cong E_{2^n}$. The centralizer $\mathbb{C}_{T_1}(t)$ of each involution $t \in T_1$ is elementary abelian, except for $t \in \mathrm{Z}(T_1)$ by [Col2]. So if Q_1 embeds into T_1, it embeds into $\mathrm{Z}(T_1)$, which is not possible, as Q_1 is special. Hence Q_1 has to induce outer automorphisms. The Sylow 2-subgroups of $\mathrm{Out}(L)$ are abelian and of 2-rank at most 2 by Lemma 2.30, while Q_1 is not abelian; so Q_1 cannot be embedded into $\mathrm{Out}(L_1)$ and $Q_1' = R$ embeds into T_1. As Q_1/Q_1' is elementary abelian, $1 < |Q_1 : T_1 \cap Q_1| \leq 4$ holds. Let i be an involution in $(T_1 \cap Q) \setminus \mathrm{Z}(T_1)$. Then $\mathbb{C}_{T_1}(i)$ is elementary abelian and $q^3 \leq |Q_1 : \mathbb{C}_{Q_1 \cap T_1}(i)| \leq 4q$ holds, where the first inequality follows from the assumption $|Q_1| \geq q^7$. This inequality works out only for $q = 2$ and $Q_1 \cong 2^{1+6}_+$. Hence for $q > 2$, $|Q_1| \leq q^5$ holds or every involution embeds either in $\mathrm{Z}(T_1)$ or in $\mathrm{Out}(L)$, which implies that there is an elementary abelian subgroup of index at most 4 in Q_1, which is impossible for $|Q_1| \geq q^5$.
So assume $q = 2$ and $Q_1 \cong 2^{1+6}_+$. By the argumentation above, Q_1 embeds into $\mathrm{Aut}(L)$ only in case $|Q_1 : (T_1 \cap Q_1)| = 4$. Hence Q_1 induces a graph and also a field automorphism on L. Now we show that $|Q_1 : (T_1 \cap Q_1)| = 2$ holds, to get the modified inequality $q^3 \leq |Q_1 : \mathbb{C}_{Q_1 \cap T_1}(i)| \leq 2q$, which is wrong even for $q = 2$. So, even if $|Q_1 : T_1 \cap Q_1| = 2$ holds, every involution in Q_1 is either embedded into $\mathrm{Z}(T_1)$ or into $\mathrm{Out}(L)$. Then there is an elementary abelian subgroup of index 2 in Q_1, which is impossible for $|Q_1| \geq 2^5$. Then also in the extraspecial case the order of Q_1 is at most 2^5.
It remains to show $|Q_1 : T_1 \cap Q_1| = 2$. We assume x_1 and x_2 to be involutions in Q_1 such that x_1 induces a field automorphism and x_2 a graph automorphism on L. It is

$R = \langle s \rangle$ with $s \in \mathrm{Z}(S)$. Hence R embeds into $\mathrm{Z}(T_1)$.
The involution x_1 normalizes the elementary abelian subgroup E_1, so $[E_1, x_1] \leq E_1$ holds. But as E_1 centralizes s, it normalizes Q, so it is $[E_1, x_1] \leq E_1 \cap Q$. The graph automorphism x_2 interchanges E_1 and E_2, see [Col2], so $[[E_1, x_1], x_2] \leq [E_1 \cap Q, x_2] \leq Q' = R \leq \mathrm{Z}(T_1)$ holds. But it is $[E_1, x_1] \not\leq \mathrm{Z}(T_1)$ and therefore also $[[E_1, x_1], x_2] \not\leq \mathrm{Z}(T_1)$. This is a contradiction, so $|Q_1| \leq q^5$ follows.

- Let $\overline{L} \cong A_{2^n+1}$, where $\overline{M} = \overline{K \cap H}$ is the stabilizer of a point in the permutation representation of degree $2^n + 1$ in $\overline{K} \cong A_{2^n+1}$ or in $\overline{K} \cong S_{2^n+1}$. As $\overline{L} \cong A_5 \cong L_2(4)$ has been already treated above, without loss of generality $n \geq 3$ holds. Without restriction, let $\overline{M}$ be the stabilizer of the point $2^n + 1$. It is $R \cong \overline{R} \leq \overline{T} \cap \overline{M}$. Fix $\overline{s} = (1, 2)(3, 4) \cdots (2^n - 1, 2^n)$, which can be chosen to be in R. This is a product of 2^{n-1} transpositions. As $\overline{Q \cap L} \leq \overline{K \cap H} = \overline{M}$ centralizes $\overline{s}$, $\overline{Q \cap L}$ is contained in $\mathbb{C}_{\overline{M}}(\overline{s})$. As Q is a large subgroup in H and $L = E(K)$ holds, $\mathbb{C}_M(s)$ normalizes $Q \cap L$. Hence $\overline{Q \cap L}$ is a normal 2-subgroup in $\mathbb{C}_{\overline{M}}(\overline{s})$.
  It is $\mathbb{C}_{\overline{M}}(\overline{s}) = \langle (1, 2), (3, 4), \ldots, (2^n - 1, 2^n) \rangle \colon S_{2^{n-1}} \cong E_{2^{(2^{n-1})}} \colon S_{2^{n-1}}$ for $\overline{K} \cong S_{2^n+1}$ and $\mathbb{C}_{\overline{M}}(\overline{s}) = (\langle (1, 2), (3, 4), \ldots, (2^n - 1, 2^n) \rangle \colon S_{2^{n-1}}) \cap A_{2^n+1} \cong E_{2^{(2^{n-1})}} \colon A_{2^{n-1}}$ for $\overline{K} \cong A_{2^n+1}$.
  In both cases, for $2^{n-1} > 4$, every normal 2-subgroup of $\mathbb{C}_{\overline{M}}(\overline{s})$ is contained in $\langle (1, 2), (3, 4), \ldots, (2^n - 1, 2^n) \rangle$, as $O_2(S_{2^{n-1}}) = O_2(A_{2^{n-1}}) = 1$. Hence the non-abelian group $\overline{Q \cap L}$ has to be contained in an elementary abelian group, which is not possible. So we may assume $2^{n-1} \leq 4$, hence $n = 3$. Then $\overline{L} \cong A_9$ and $\mathbb{C}_{\overline{M}}(\overline{s})$ is isomorphic to $E_{16} \colon S_4$ or to $E_{16} \colon A_4$. These groups contain an extraspecial 2-group of order at most 2^5, so it is $|Q_1| \leq 2^5$, implying $|Q| \leq 2^7$. This contradicts the assumption of $|Q| \geq q^9$ in Hypothesis 5.2.

- Let $\overline{L} \cong L_2(7)$ or $L_2(9)$. Again the idea is to show that an embedding of Q_1 into a Sylow 2-subgroup of $\mathrm{Aut}(L)$ only works out for small cases, which have already been treated. As $\mathrm{Aut}(L) \lesssim \mathrm{Aut}(\overline{L})$ holds by Remark 2.10, we may assume that L is simple, as an embedding of Q_1 into a Sylow 2-subgroup of $\mathrm{Aut}(L)$ implies an embedding into $\mathrm{Aut}(\overline{L})$. For $L \cong L_2(7)$ or $L \cong L_2(9)$, it is T_1 a dihedral group of order 8 and $|\mathrm{Aut}(L)|_2 \leq 2^5$ holds. So the order of Q_1 is at most 2^5 and so $|Q| \leq 2^7$ follows. Again, this situation arises only in cases which have been excluded in Hypothesis 5.2.

$\square$

Chapter 6

The remaining families of exceptional groups of Lie type

In this chapter we show Theorem 1.5 for $F^*(H)$ being isomorphic to one of the exceptional groups of Lie type which have not been treated before. These are the groups $E_6(2^f)$, $E_7(2^f)$, $E_8(2^f)$, ${}^2E_6(2^f)$ and ${}^3D_4(2^f)$ for $f \in \mathbb{N}$.

The strategy is again to show that Holt's result in Lemma 2.44 can be applied to prove $G = H$. Using the results of the previous chapters, we can work under the following hypothesis:

Hypothesis 6.1: Assume Hypothesis 1.4 and let be $S_1 = S \cap F^*(H)$, $R = \mathrm{Z}(S_1)$ and $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ as before. Additionally, we assume $F^*(H)$ to be isomorphic to one of the following groups: $E_6(q)$, $E_7(q)$, $E_8(q)$, ${}^2E_6(q)$ or ${}^3D_4(q)$ for $q = 2^f$ and $f \in \mathbb{N}$, using Remark 4.14. By the same remark, we may assume $|Q| \geq q^9$.
Lemma 5.10 implies $\mathbb{C}_G(t) \leq H$ for every involution $t \in Q$.

The following three lemmas are needed in the proof of Lemma 6.5, where we show that $H \cap F^*(G)$ controls the $F^*(G)$-fusion of 2-central involutions.

Lemma 6.2: Let Hypothesis 6.1 hold. Then $|H : \mathrm{N}_H(Q)|$ is odd and Q is weakly closed in S with respect to H.

Proof: By Lemma 2.38, $Q \leq S$ is a large 2-subgroup in H, hence $Q = O_2(\mathrm{N}_H(Q))$ holds. Suppose a conjugate $Q^h \leq S$ with $h \in H$. Then Q and Q^h are centralized by $\mathrm{Z}(S)$. Hence, by definition of a large subgroup, $\mathrm{N}_H(S) \leq \mathrm{N}_H(\mathrm{Z}(S)) \leq \mathrm{N}_H(Q)$ and $\mathrm{N}_H(S) \leq \mathrm{N}_H(\mathrm{Z}(S)) \leq \mathrm{N}_H(Q^h)$ hold. In particular, $|H : \mathrm{N}_H(Q)|$ is odd.
Sylow's Theorem implies $\mathrm{N}_H(Q) = \mathrm{N}_H(Q^h)$. So $Q^h = O_2(\mathrm{N}_H(Q^h)) = O_2(\mathrm{N}_H(Q)) = Q$ follows. □

Lemma 6.3: Assume Hypothesis 6.1 with $S \in \mathrm{Syl}_2(H)$ and $S_1 = S \cap F^*(H)$. Let L be the Levi complement in the Lie-parabolic group $\mathrm{N}_{F^*(H)}(Q)$. We denote the subgroup of H whose elements induce automorphisms on L by $\mathrm{Aut}_H(L)$.
Then for every maximal elementary abelian 2-subgroup $V \leq \mathrm{Aut}_H(L) \cap S$, it is $V \leq S_1$. In particular, $m_2(\mathrm{Aut}_H(L)) = m_2(L)$ holds.

Proof: As the corresponding Cartan subgroup is a $2'$-group, we may identify $O^{2'}(L)$ with the group of inner automorphisms in $O^{2'}(\mathrm{Aut}_H(L))$. Lemmas 2.36 and 2.32 imply that $O^{2'}(L)$ and $m_2(L)$ are as follows: For $F^*(H) \cong E_6(2^f)$, it is $O^{2'}(L) \cong SL_6(2^f)$ and $m_2(L) = 9f$; for $F^*(H) \cong {}^2E_6(2^f)$, it is $O^{2'}(L) \cong SU_6(2^f)$ and $m_2(L) = 9f$; for $F^*(H) \cong E_7(2^f)$, it is $O^{2'}(L) \cong \Omega_{12}^+(2^f)$ and $m_2(L) = 15f$; for $F^*(H) \cong E_8(2^f)$, it is $O^{2'}(L) \cong E_7(2^f)$ and $m_2(L) = 27f$; and for $F^*(H) \cong {}^3D_4(2^f)$, it is $O^{2'}(L) \cong SL_2(2^{3f})$ and $m_2(L) = 3f$, compare also Table 6.1 on the page after next.

Let V be a maximal elementary abelian 2-subgroup in $\mathrm{Aut}_H(L) \cap S$. By Lemma 4.3 for $F^*(H) \cong E_6(2^f)$ and by Lemma 4.5 for $F^*(H) \cong {}^2E_6(2^f)$, it is $V = J(S) = J(S_1)$. In particular, $m_2(\mathrm{Aut}_H(L)) = m_2(L)$ holds.
For $F^*(H) \cong E_7(2^f)$, $F^*(H) \cong E_8(2^f)$ or $F^*(H) \cong {}^3D_4(2^f)$, we assume the existence of an involution $x \in S \setminus S_1$ with $x \in V$. So x induces a field automorphism on $F^*(H)$, using Lemma 2.31. Then x can only induce a field automorphism on L. By Lemma 2.31, f must be even and it is $m_2(\mathbb{C}_{\mathrm{Aut}_H(L)}(x)) = \frac{m_2(L)}{2} + 1 < m_2(L)$. So in all considered cases we have $V \leq S_1$. Therefore, $m_2(L) = m_2(\mathrm{Aut}_H(L))$ holds. □

Lemma 6.4: Suppose Hypothesis 6.1 and additionally we allow $F^*(H)$ to be isomorphic to $\Omega_{2n}^{\pm}(2^f)$ with $n \geq 4$ and $f \in \mathbb{N}$. Further let $t \in S \setminus Q$ be an involution and let Q be of order q^{1+2m} for $q = 2^f$. Then $\mathbb{C}_Q(t)$ has an abelian 2-subgroup of order q^m which contains an elementary abelian 2-subgroup of order q^{m-1}.

Proof: It is $|Q| = q^{1+2\cdot m}$ and Q/R a vector space of dimension $2m$ over $GF(q)$. The space Q/R is equipped with a quadratic form, see Satz 4 in [Beis]. The involution $t \in \mathbb{C}_S(R)$ normalizes the large 2-subgroup Q, so $[Q,t] \leq Q$ holds and t acts as a $GF(q)$-linear map on Q/R. By Satz 4 in [Beis], t can be seen as an element of $O_{2m}^+(q)$, which therefore leaves the quadratic form, defined on Q/R, invariant. By Hypothesis 6.1, it is $m \geq 4$, so t is a product of reflections. As t is an involution, it centralizes a subspace of at least dimension m in Q/R and $[Q,t]/R = [Q/R,t] \leq \mathbb{C}_{Q/R}(t)$ holds. Hence, using Lemma 11.10 in [Tayl], it is $\mathbb{C}_{Q/R}(t) = [Q/R,t]^{\perp} = [Q,t]/R \times \tilde{Q}/R$, where $\tilde{Q}/R$ is the complement of $[Q,t]/R$ in $\mathbb{C}_{Q/R}(t)$. Let $|\tilde{Q}/R| = q^{2r}$ for $r \in \mathbb{N}_0$. It is $\mathbb{C}_Q(t) = \mathbb{C}_{[Q,t]}(t) * U$ with amalgamated center R, where U is the full preimage of $\tilde{Q}/R$ in $\mathbb{C}_Q(t)$. Then $\mathbb{C}_{[Q,t]}(t)$ contains an elementary abelian subgroup of order $|[Q,t]/R|$, as the order probably loses a factor $q = |R|$ by going back to the preimage which is compensated by the amalgamated center R. As t is a product of reflections, the proof of Satz 4 in [Beis] implies that U is isomorphic to either $D_r(q)$ or $Q_r(q)$; in case $r = 0$, it is U trivial. Hence U contains an elementary abelian 2-subgroup of order at least q^r and an abelian subgroup of order at least q^{r+1} if $r \geq 1$. In any case, $\mathbb{C}_Q(t)$ contains an abelian 2-subgroup of order q^m which itself contains an elementary abelian 2-subgroup of order q^{m-1}. □

Lemma 6.5: Assume Hypothesis 6.1. Let $t \in S_1 \setminus Q$ be an involution which is G-conjugate to an involution $s \in R$. Then $\mathbb{C}_G(t) \leq H$ follows and t must be H-conjugate to s.
In particular, it is $s^{F^*(G)} \cap F^*(H) = s^{H \cap F^*(G)}$.

Proof: Let $t \in S_1 \setminus Q$ be an involution which is G-conjugate to an involution $s \in R \cap \mathrm{Z}(S)$. In order to show $\mathbb{C}_G(t) \leq H$, we assume $\mathbb{C}_G(t) \nleq H$.
Because of $t \sim_G s$, there is an element $g \in G$ such that $t = s^g$ and $\mathbb{C}_G(t) \cong \mathbb{C}_G(s) = \mathbb{C}_H(s)$ holds, using Lemma 3.9. It is $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ a normal subgroup in $\mathrm{N}_H(Q)$ and in $\mathbb{C}_G(s)$. Define Q_t to be the isomorphic copy of Q in $\mathbb{C}_G(t)$. This implies $Q_t \trianglelefteq \mathbb{C}_G(t)$ and therefore $Q_t\,\mathbb{C}_Q(t)$ is a 2-subgroup of $\mathbb{C}_G(t)$.
To show that $Q_t \cap Q$ is trivial, we assume the existence of an involution $x \in Q_t \cap Q$. As x is an element in Q, Hypothesis 6.1 implies $\mathbb{C}_G(x) \leq H$. In particular, $\mathbb{C}_{Q_t}(x) \leq H$ holds.
It is $t \in \mathrm{Z}(Q_t)$ by construction. To show that without loss of generality $x \notin \mathrm{Z}(Q_t)$ holds, we assume $x \in \mathrm{Z}(Q_t)$. This implies $Q_t \leq H$. As Q is a large subgroup, Q_t and Q are H-conjugate. Hence we may assume $x \notin \mathrm{Z}(Q_t)$ in the following.
It is $|Q_t : \mathbb{C}_{Q_t}(x)| = |R| = q$ with $\mathbb{C}_{Q_t}(x) \leq H$, using Lemma 5.1. Additionally, this lemma implies $\mathbb{C}_{Q_t}(x) = Z \times U$, for an elementary abelian group Z of order q and a special group U of index q^2 in Q_t. Using $\mathbb{C}_{Q_t}(x) \leq H$, in particular, $U \leq H$ holds and one gets $\mathrm{Z}(Q_t) = \mathrm{Z}(U)$ by definition of Q_t. It is $U^h \leq S$ for an element $h \in H$, so without loss of generality we may assume that $U \leq S$ holds, implying $U \leq \mathrm{N}_H(Q)$. Additionally, $\mathrm{Z}(U)$ is not contained in Q, as $\mathrm{Z}(Q_t) \cap Q$ is trivial.
As U normalizes Q, $U \cap Q$ is a normal subgroup in U. If $U \cap Q$ is not trivial, also $(U \cap Q) \cap \mathrm{Z}(U)$ is not trivial. But as all elements in $\mathrm{Z}(U)$ are H-conjugates of t, this contradicts $\mathbb{C}_G(t) \nleq H$. Hence $U \cap Q = 1$ holds. We set $|Q| = q^{1+2m}$ for $m \in \mathbb{N}$. Then, using $U \cap Q = 1$, $\mathrm{N}_H(Q)/Q$ contains a special group, isomorphic to U, of order q^{2m-1}.
Lemma 2.36 and Lemma 2.32 provide the information in Table 6.1 on the following page. In this table we set $q = 2^f$ for $f \in \mathbb{N}$ and let L be the Levi complement in the Lie-parabolic group $\mathrm{N}_{F^*(H)}(Q)$.

If there is an involution $x \in S \setminus S_1$, then x induces an automorphism on the Levi complement L. To see that $\mathrm{N}_H(Q)/Q$ cannot contain a special group of order q^{2m-1}, the 2-rank of $\mathrm{Aut}_H(L)$ is considered. By Lemma 6.3, it is $m_2(L) = m_2(\mathrm{Aut}_H(L))$.
A comparison of $m_2(L)$ and the 2-rank of U gives $m_2(L) < m_2(U)$ in all considered cases. So U cannot be embedded into L for $F^*(H)$ being isomorphic to $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$

Table 6.1: Orders and ranks of certain subgroups

$F^*(H)$	$O^{2'}(L)$	$\|L\|_2$	$m_2(L)$	Q	U	$m_2(U)$
$E_6(q)$	$SL_6(q)$	q^{15}	$9f$	q^{1+20}	q^{1+18}	$10f$
${}^2E_6(q)$	$SU_6(q)$	q^{15}	$9f$	q^{1+20}	q^{1+18}	$10f$
$E_7(q)$	$\Omega_{12}^+(q)$	q^{30}	$15f$	q^{1+32}	q^{1+30}	$16f$
$E_8(q)$	$E_7(q)$	q^{63}	$27f$	q^{1+56}	q^{1+54}	$28f$
${}^3D_4(q)$	$SL_2(q^3)$	q^3	$3f$	q^{1+8}	q^{1+6}	$4f$

or ${}^3D_4(q)$.
This implies $Q_t \cap Q = 1$ and in particular $\mathbb{C}_Q(t) \cap Q_t = 1$. So $\mathbb{C}_G(t)$ contains the semidirect product $Q_t\colon \mathbb{C}_Q(t)$. As t is a 2-central involution in H^g and Q_t is isomorphic to Q, an isomorphic copy of $\mathbb{C}_Q(t)$ is involved in $\mathrm{N}_H(Q)/Q$ and thus also in $\mathrm{Aut}_H(L)$.
By Lemma 6.4, t centralizes an abelian 2-subgroup of order q^m in Q, which contains an elementary abelian subgroup of order q^{m-1}.
We denote an isomorphic copy of this abelian group of order q^m in $\mathrm{N}_H(Q)/Q$ by A and the elementary abelian subgroup of order q^{m-1} by V. Let further T be a Sylow 2-subgroup of $\mathrm{Aut}_H(L)$ which contains A. Thus it is $V \leq A \leq T$.
A comparison of $m_2(Q)$ and $m_2(L)$, using Table 6.1, gives that V is a maximal elementary abelian 2-subgroup in L. By Lemma 6.3, additionally $V \leq L$ holds.

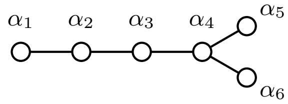
If it is $F^*(H) \cong {}^3D_4(q)$, hence in case $O^{2'}(L) \cong SL_2(q^3)$, V equals a Sylow 2-subgroup of $O^{2'}(L)$. So $V = T \cap L$ follows. But in the outer automorphism group of $SL_2(q^3)$ there is no involution which centralizes a full Sylow 2-subgroup of $SL_2(q^3)$, as every involution which induces an outer automorphism on $SL_2(q^3)$ is a field automorphism. This contradicts the existence of an abelian group A which properly contains V. Hence for $F^*(H) \cong {}^3D_4(q)$, the assumption from the beginning of this proof is refuted. And therefore, $\mathbb{C}_G(t) \leq H$ follows.

Let now $F^*(H)$ be isomorphic to $E_6(q)$ or to ${}^2E_6(q)$. Then it is $O^{2'}(L) \cong SL_6(q)$ or $O^{2'}(L) \cong SU_6(q)$. Additionally, it is V of order q^9. According to Lemma 4.3 and 4.5, V coincides with $J(T \cap L) = J(T)$. Then it is V the unipotent radical of a maximal Lie-parabolic subgroup in $O^{2'}(L)$: For $O^{2'}(L) \cong SL_6(q)$ the Dynkin diagram is of type A_5 and the Lie-parabolic subgroup is $P_{\{\alpha_1,\alpha_2,\alpha_4,\alpha_5\}}$ by Example 3.2.3 on page 97 in [GLS3]. For $O^{2'}(L) \cong SU_6(q)$, the Lie-parabolic subgroup is $P_{\{\tilde{\alpha_1},\tilde{\alpha_2}\}}$ by Example 3.2.5 on page 101 in [GLS3]. Then by Lemma 2.26, it is $\mathbb{C}_{\mathrm{Aut}_H(L)}(V) \leq V$. In particular, it is $\mathbb{C}_T(V) = V$, hence V cannot be properly contained in an abelian 2-subgroup A. So the assumption $\mathbb{C}_G(t) \not\leq H$ is disproved for $F^*(H) \cong E_6(q)$ and for $F^*(H) \cong {}^2E_6(q)$.

For $F^*(H) \cong E_8(q)$, we consider a maximal elementary abelian 2-subgroup V in $O^{2'}(L) \cong$

$E_7(q)$. By Theorem B in [GuMa], V is no F-module for $O^{2'}(L)$. This implies $J(T) = V$, as otherwise an elementary abelian 2-subgroup of maximal rank, which is not contained in V, is an offender, and V would be an F-module for $O^{2'}(L)$. Then by Example 3.2.4 on page 100 in [GLS3], it is V the unipotent radical of the maximal Lie-parabolic subgroup $P_{\{\alpha_6\}'}$ in $O^{2'}(L) \cong E_7(q)$. Using Lemma 2.26, then also $\mathbb{C}_T(V) = V$ follows. So V cannot be properly contained in an abelian 2-subgroup A. This disproves the assumption $\mathbb{C}_G(t) \nleq H$ for $F^*(H) \cong E_8(q)$.

For $F^*(H) \cong E_7(q)$, the situation is more complicated, as there is more than one maximal elementary abelian subgroup in T. We consider the maximal Lie-parabolic subgroup $P_{\{\alpha_6\}'}$ (or $P_{\{\alpha_5\}'}$, alternatively) in $L \cong \Omega^+_{12}(q)$. Using the structure of the Dynkin diagram of $\Omega^+_{12}(q)$ and Lemma 2.26, $L_6(q)$ is the corresponding simple part of the Levi complement.



Additionally, we consider the unipotent radical E of this Lie-parabolic subgroup, which is maximal elementary abelian of order q^{15}. It is E the alternating square $\Lambda^2(Y_6)$, where Y_6 denotes the natural $GF(q)$-module for $L_6(q)$, see Theorem B in [GuMa]. In particular, E is a faithful $L_6(q)$-module. If V equals E, then by Lemma 2.26, V is self-centralizing in T, and we are done. So let V be an arbitrary maximal elementary abelian 2-subgroup in T with $E \neq V$. Then V is an offender for E and by Lemma 6.3, it is $V \leq T \cap L$. Theorem 3 (Best offender theorem) in [MeSt] states that a non-trivial offender V for E is uniquely determined in its Sylow 2-subgroup of $L_6(q)$. The source states further that $|E/\mathbb{C}_E(V)| = |V/\mathbb{C}_V(E)| = q^5$ holds. In particular, there are no over-offenders for E. It is $|V/(E \cap V)| = |EV/E| = |E/\mathbb{C}_E(V)| = q^5$. Hence $|E \cap V| = q^{10}$ follows. As $EV/E \cong V/(E \cap V)$ is a natural module for $L_5(q) \leq L_6(q)$, the simple group $L_5(q)$ acts faithfully on $V/(E \cap V)$. Additionally, elements in $E \setminus V$ cannot centralize V, as there are no over-offenders for E. Therefore, it is V self-centralizing in T. In particular, V cannot be properly contained in an abelian 2-subgroup A.

Altogether, $\mathbb{C}_G(t) \leq H$ follows. Thus, using the notation of the beginning of this proof, it is $\mathbb{C}_G(s) = \mathbb{C}_G(t)^{g^{-1}} = \mathbb{C}_H(t)^{g^{-1}} = \mathbb{C}_{H^{g^{-1}}}(s) \leq H$. Hence $g \in \mathrm{N}_G(\mathbb{C}_H(s)) = \mathbb{C}_H(s)\,\mathrm{N}_G(S) \leq H$ holds by Frattini's argument. In particular, for $g \in F^*(G)$ it is $g \in H \cap F^*(G)$. Hence $s^{F^*(G)} \cap F^*(H) = s^{H \cap F^*(G)}$ holds. □

For $F^*(H) = H$, we now can apply Holt's result, Lemma 2.44. Hence in the following lemma, we assume the existence of an involution in $S \setminus S_1$, to show $s^{F^*(G)} \cap (H \cap F^*(G)) = r^{H \cap F^*(G)}$ for a 2-central involution $s \in S$.

Lemma 6.6: Assume Hypothesis 6.1. Let $\tau \in S \setminus S_1$ be an involution which is not H-conjugate to an involution $s \in \mathrm{Z}(S)$. Then τ is not G-conjugate to s either.

Proof: Let $\tau \in S\backslash S_1$ be an involution such that τ is G-conjugate to an involution $s \in \mathrm{Z}(S)$. By Lemma 2.30, τ induces a field automorphism, a graph automorphism or a field-graph automorphism. The information on centralizers of τ can be found in Lemma 2.31.
If $F^*(H)$ is isomorphic to one of the groups $E_7(q)$, $E_8(q)$ or ${}^3D_4(q)$, τ induces a field automorphism and it is $O^{2'}(\mathbb{C}_{E_7(q)}(\tau)) \cong E_7(q_0)$, $O^{2'}(\mathbb{C}_{E_8(q)}(\tau)) \cong E_8(q_0)$ and $O^{2'}(\mathbb{C}_{{}^3D_4(q)}(\tau)) \cong {}^3D_4(q_0)$ with $q_0^2 = q$. Hence the Lie rank of $\mathbb{C}_{F^*(H)}(\tau)$ equals the Lie rank of $F^*(H)$.
The Lie rank of the Levi complement L of $\mathbb{C}_H(s)$, as it can be seen in Table 6.1 above, is 6 for $F^*(H) \cong E_7(q)$, it is 7 for $F^*(H) \cong E_8(q)$ and it equals 1 for $F^*(H) \cong {}^3D_4(q)$. Hence in all that cases, $O^{2'}(\mathbb{C}_{F^*(H)}(\tau))$ cannot be embedded into $\mathbb{C}_G(s) = \mathbb{C}_H(s)$, using Lemma 2.29. Thus, we are left with $F^*(H) \cong E_6(q)$ and $F^*(H) \cong {}^2E_6(q)$. If τ induces a field automorphism, then it is $O^{2'}(\mathbb{C}_{E_6(q)}(\tau)) \cong E_6(q_0)$ with $q_0^2 = q$ and $O^{2'}(\mathbb{C}_{{}^2E_6(q)}(\tau)) \cong F_4(q)$, see Lemma 2.31. The Lie rank of $E_6(q_0)$ is 6. For $F^*(H) \cong E_6(q)$, $\mathbb{C}_{F^*(H)}(\tau)$ cannot be embedded into $\mathbb{C}_G(s)$, again by Lemma 2.29, as the corresponding Levi complement is isomorphic to $SL_6(q)$, which is of Lie rank 5. And for $F^*(H) \cong {}^2E_6(q)$ it is $\mathbb{C}_{F^*(H)}(\tau)$ of Lie-rank 4. Hence $\mathbb{C}_{F^*(H)}(\tau)$ cannot be embedded into $\mathbb{C}_G(s)$, as the Levi complement of $\mathbb{C}_G(s)$ is isomorphic to $SU_6(q)$, which is of Lie rank 3.
If τ induces a field-graph automorphism, which is only possible for $F^*(H) \cong E_6(q)$, it is $\mathbb{C}_{F^*(H)}(\tau)' \cong {}^2E_6(q_0)$ with $q_0^2 = q$. An embedding of $\mathbb{C}_{F^*(H)}(\tau)$ into $\mathbb{C}_G(s) = \mathbb{C}_H(s)$ implies an embedding of ${}^2E_6(q_0)$ into $SL_6(q) = SL_6(q_0^2)$. But this is impossible, because of $|{}^2E_6(q_0)|_2 = q_0^{36} > q_0^{30} = q^{15} = |SL_6(q)|_2$.
If τ induces a graph automorphism, then there are precisely two possibilities for $\mathbb{C}_{F^*(H)}(\tau)$: Either it is $\mathbb{C}_{F^*(H)}(\tau) \cong F_4(q)$ or $\mathbb{C}_{F^*(H)}(\tau)$ is isomorphic to the centralizer of a 2-central involution in $F_4(q)$. In any case, s is contained in $\mathbb{C}_S(\tau)' \leq F^*(H)$. Without loss of generality, we may assume that $\mathbb{C}_S(\tau)$ is a Sylow 2-subgroup of $\mathbb{C}_H(\tau)$. If $\mathbb{C}_S(\tau)$ is not a Sylow 2-subgroup of $\mathbb{C}_G(\tau)$, let T be a subgroup of $\mathbb{C}_G(\tau)$ such that $|T : \mathbb{C}_S(\tau)| = 2$ holds. Then there is an element $t \in T \setminus \mathbb{C}_S(\tau)$. Hence it is $s^t \in \mathbb{C}_{S_1}(\tau)$. But by Lemma 6.5, H controls the fusion of s in S_1. Hence there is an element $h \in H$ such that $s^t = s^h$ holds. Hence it is $h^{-1} \cdot t \in \mathbb{C}_G(s) \leq H$, which contradicts $\mathbb{C}_S(\tau) \in \mathrm{Syl}_2(\mathbb{C}_H(\tau))$. Thus, $\mathbb{C}_S(\tau)$ is a Sylow 2-subgroup of $\mathbb{C}_G(\tau)$. By assumption, it is $\tau^g = s$ for an element $g \in G$, so $\mathbb{C}_G(\tau)^g = \mathbb{C}_G(s)$ follows. In particular, $\mathbb{C}_S(\tau) \in \mathrm{Syl}_2(\mathbb{C}_G(\tau))$ and $S \in \mathrm{Syl}_2(\mathbb{C}_G(s))$ are isomorphic and therefore equal. But $\mathbb{C}_S(\tau) = S$ is wrong, as τ is not a 2-central involution in H. Hence τ cannot be G-conjugate to s. $\square$

Corollary 6.7: Assume Hypothesis 6.1. Then $G = H$ holds.

Proof: It is $F^*(G)$ a simple group by Lemma 3.2. Let $s \in R \cap \mathrm{Z}(S)$ be an involution. Then it is $\mathbb{C}_{F^*(G)}(s) \leq H \cap F^*(G)$, see Lemma 3.9. Additionally, the Lemmas 6.5 and 6.6 give $s^{F^*(G)} \cap (H \cap F^*(G)) = s^{H \cap F^*(G)}$. So by Holt's result, see Lemma 2.44, $F^*(H) = F^*(G)$ follows. And therefore, $G = F^*(H)\,\mathrm{N}_G(S_1) \leq H$ holds. $\square$

Chapter 7

The orthogonal groups in even dimension

In this chapter we discuss the main theorem of this thesis for $F^*(H)$ being isomorphic to an orthogonal group $\Omega_{2n}^{\pm}(q)$ for $q = 2^f$ with $f \in \mathbb{N}$. Notation and properties of Hypothesis 1.4 are assumed. So let S be a Sylow 2-subgroup of H, $S_1 = S \cap F^*(H)$ a corresponding Sylow 2-subgroup of $F^*(H)$ and $R = \mathrm{Z}(S_1)$ an elementary abelian (long) root subgroup of order q.

Using Remark 4.14, we may assume $F^*(H) \cong \Omega_{2n}^{\pm}(q)$ with $n \geq 4$. As before, it is $Q = O_2(\mathbb{C}_{F^*(H)}(R))$ a large subgroup in $F^*(H)$ and also in H, compare Lemma 2.38. Additionally, by using Lemma 2.36, $F^*(H) \cong \Omega_{2n}^{\pm}(q)$ with $n \geq 4$ implies $|Q| \geq q^9$.
It is well-known that $\Omega_{2n}^{\pm}(q)$ is the commutator subgroup of the full orthogonal group $O_{2n}^{\pm}(q)$ and that $|O_{2n}^{\pm}(q) : \Omega_{2n}^{\pm}(q)| = 2$ holds.

In the following remark some properties of involutions and their centralizers in the orthogonal group are listed. For involutions in $O_{2n}^{\pm}(q)$ we follow the notation as it is used in [AsSe].

Remark 7.1 ([AsSe], Section 8): Let be $F^*(H) \cong \Omega_{2n}^{\pm}(q)$ for $n \geq 4$ with $q = 2^f$. Let further be t an involution in the orthogonal group $O_{2n}^{\pm}(q)$ and V the corresponding orthogonal module. Then the following statements hold:

- The involution t is either of type a_l, b_l or c_l, where the rank $l \in \mathbb{N}$ is the dimension of the commutator space $[V, t]$.
- It is $t \in \Omega_{2n}^{\pm}(q)$ if and only if t is of type a_l or c_l, while involutions of type b_l are in $O_{2n}^{\pm}(q) \setminus \Omega_{2n}^{\pm}(q)$.
- For t of type a_l or c_l, l must be even. For involutions of type b_l, l is odd.
  In particular, elements of type a_n or c_n in $O_{2n}^{\pm}(q)$ only occur for n even, while involutions of type b_n require n to be odd.

- If t is of type a_l, it is $\mathbb{C}_{O^{\pm}_{2n}(q)}(t)/O_2(\mathbb{C}_{O^{\pm}_{2n}(q)}(t)) \cong Sp_l(q) \times O^{\pm}_{2n-2l}(q)$. If t is of type b_l, then $\mathbb{C}_{O^{\pm}_{2n}(q)}(t)/O_2(\mathbb{C}_{O^{\pm}_{2n}(q)}(t)) \cong Sp_{l-1}(q) \times Sp_{2n-2l}(q)$. And for t of type c_l, it is $\mathbb{C}_{O^{\pm}_{2n}(q)}(t)/O_2(\mathbb{C}_{O^{\pm}_{2n}(q)}(t)) \cong Sp_{l-2}(q) \times Sp_{2n-2l}(q)$.

- Let be $r \in R^{\#} = \mathrm{Z}(S_1)^{\#}$ with $S_1 \in \mathrm{Syl}_2(F^*(H))$. Then it is r of type a_2 and $\mathbb{C}_{O^{\pm}_{2n}(q)}(r) = Q\colon (O^{\pm}_{2n-4}(q) \times L_2(q))$ holds with $Q \cong D^{+}_{2n-4}(q)$.

- The conjugacy class $t^{O^{\pm}_{2n}(q)}$ contains the involutions of the same type as t. The same holds for $t^{\Omega^{\pm}_{2n}(q)}$ with exception of an involution t of type a_n in $\Omega^{+}_{2n}(q)$: Then $t^{O^{+}_{2n}(q)}$ splits into two conjugacy classes in $\Omega^{+}_{2n}(q)$.

Proof: The statements above can be found in Section 8 in [AsSe]. □

Lemma 7.2: Assume Hypothesis 1.4. Let $F^*(H)$ be isomorphic to $\Omega^{\pm}_8(q)$ and t an involution in H. Then $G = H$ holds.

Proof: To simplify notation we identify $F^*(H)$ and $\Omega^{\pm}_8(q)$. As before, we fix an involution $r \in \mathrm{Z}(S \cap F^*(H)) = R$. By Remark 7.1, it is $r \in R$ of type a_2. Additionally, from Lemmas 3.9 and the above remark, we deduce $\mathbb{C}_G(r) = \mathbb{C}_H(r)$ and $\mathbb{C}_{O^{\pm}_8(q)}(r)/Q \cong O^{\pm}_4(q) \times L_2(q)$ with $Q = O_2(\mathbb{C}_{F^*(H)}(R))$.

At first, we show that every involution in $F^*(H)$ is H-conjugate to an involution in Q. To see that, we consider the conjugacy classes of involutions in $F^*(H)$ and list a representative in Q for each conjugacy class. The calculations were performed in GAP, compare [GAP].
For $F^*(H) \cong \Omega^{+}_8(q)$, there are five conjugacy classes of involutions in $F^*(H)$: One of type a_2, two of type a_4 (which are fused in $O^{\pm}_8(q)$), one of type c_2 and one of type c_4. Let r be the orthogonal matrix

$$\begin{pmatrix} 0&0&1&1&0&1&1&1\\ 0&0&1&1&1&0&1&1\\ 1&1&1&0&1&1&0&0\\ 1&1&0&1&1&1&0&0\\ 0&1&1&1&0&0&1&1\\ 1&0&1&1&0&0&1&1\\ 1&1&0&0&1&1&1&0\\ 1&1&0&0&1&1&0&1 \end{pmatrix}.$$

It is r of type a_2 and by definition contained in Q. In the following, a representative for each remaining conjugacy class of involutions is given such that all the given representatives are contained in Q. The involutions are listed in order c_2, c_4, a_4, a_4:

$$\begin{pmatrix}0&1&1&1&1&1&0&0\\1&0&1&1&1&1&0&0\\1&1&1&0&1&1&0&0\\1&1&0&1&1&1&0&0\\1&1&1&1&0&1&0&0\\1&1&1&1&1&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0&0&1&1&1&1\\1&0&1&0&0&0&0&0\\0&0&0&0&1&1&1&1\\1&0&0&0&1&0&0&0\\0&1&0&1&1&1&1&1\\0&1&1&1&1&1&0&0\\0&1&0&1&0&1&1&0\\0&1&0&1&0&1&0&1\end{pmatrix}, \begin{pmatrix}0&0&1&0&0&1&1&0\\0&0&0&0&0&1&0&1\\0&0&0&0&1&1&1&1\\0&1&0&0&0&1&0&0\\1&1&1&1&1&0&1&0\\0&0&0&1&0&1&0&1\\1&0&0&1&1&0&1&0\\0&1&0&1&0&1&0&1\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&1&0&1&0\\0&0&1&0&1&0&0&1\\0&0&0&0&1&1&1&1\\1&0&0&0&1&0&0&0\\0&0&0&1&1&0&1&0\\1&1&1&1&0&1&0&1\\1&0&0&1&1&0&1&0\\0&1&0&1&0&1&0&1\end{pmatrix}.$$

For $F^*(H) \cong \Omega_8^-(q)$, there are three conjugacy classes of involutions in $F^*(H)$: One of type a_2, one of type c_2 and one of type c_4. Let r be the orthogonal matrix

$$\begin{pmatrix}0&0&0&0&1&0&0&0\\0&0&1&1&0&1&1&1\\1&0&1&0&1&0&0&0\\1&0&0&1&1&0&0&0\\1&0&0&0&0&0&0&0\\0&1&1&1&0&0&1&1\\1&0&0&0&1&0&1&0\\1&0&0&0&1&0&0&1\end{pmatrix}.$$

As before, we list a representative for each conjugacy class of involutions, such that the given representatives are contained in Q. The involutions are listed in order c_2, c_4:

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\1&1&1&1&1&0&0&0\\1&0&1&0&1&0&0&0\\1&0&0&1&1&0&0&0\\0&0&0&0&1&0&0&0\\1&0&1&1&1&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0&0&1&0&0&1\\1&1&1&0&1&0&0&1\\0&0&1&0&1&0&0&1\\1&0&0&1&0&0&0&1\\0&0&0&0&0&0&0&1\\1&1&0&1&1&0&1&0\\1&1&1&1&0&1&0&0\\0&0&0&0&1&0&0&0\end{pmatrix}.$$

The statement of the lemma for involutions in $F^*(H)$ follows from Lemma 5.10.

Involutions in $O_8^{\pm}(q) \setminus \Omega_8^{\pm}(q)$ can be of type b_1 or b_3. If t is an involution of type b_1, then

$\mathbb{C}_G(t)$ involves a simple group $Sp_6(q)$, which cannot be embedded into $\mathbb{C}_G(r) = \mathbb{C}_H(r)$. If t is of type b_3, it is $\mathbb{C}_{O_8^{\pm}(q)}(t)/O_2(\mathbb{C}_{O_8^{\pm}(q)}(t)) \cong L_2(q) \times L_2(q)$ and $O_2(\mathbb{C}_{O_8^{\pm}(q)}(t))$ is of order $2 \cdot q^7$ where $O_2(\mathbb{C}_{O_8^{\pm}(q)}(t))' = O_2(\mathbb{C}_{\Omega_8^{\pm}(q)}(t))'$ is of order q and contains only involutions of type a_2. Hence conjugation with an element $g \in G$, which embeds $\mathbb{C}_H(t)$ into $\mathbb{C}_H(r)$ maps $O_2(\mathbb{C}_{\Omega_8^{\pm}(q)}(t))'$ on $O_2(\mathbb{C}_{\Omega_8^{\pm}(q)}(r))' = R$; therefore g maps a H-conjugate of R onto R. In particular, $g \in H$ follows.
Hence we may assume $t \notin \mathbb{C}_S(R)$.

Using Section 19 in [AsSe], this can only occur for $F^*(H) \cong \Omega_8^+(q)$. For $F^*(H) \cong \Omega_8^+(q)$, t can only induce a field or a graph-field automorphism. If t induces a field automorphism, then by (19.1) in [AsSe], it is $O^{2'}(\mathbb{C}_{F^*(H)}(t)) \cong \Omega_8^+(q_0)$ with $q_0^2 = q$. Hence $\mathbb{C}_G(t)$ involves a simple group of Lie rank n. By Lemma 2.29, $\Omega_8^+(q_0)$ cannot be embedded into $\Omega_4^{\pm}(q) \times L_2(q)$. So it is impossible to embed $\mathbb{C}_G(t)$ into $\mathbb{C}_G(r) = \mathbb{C}_H(r)$. If t induces a field-graph automorphism, it is $\mathbb{C}_{F^*(H)}(t) \cong \Omega_8^-(q_0)$ with $q_0^2 = q$. Hence $\mathbb{C}_{F^*(H)}(t)$ is a simple group of Lie rank $n-1$. Again by Lemma 2.29, it is impossible to embed $\mathbb{C}_G(t)$ into $\mathbb{C}_G(r) = \mathbb{C}_H(r)$.
Altogether, $r^G \cap H = r^H$ holds. Then Holt's result from Lemma 2.44 implies $G = H$.

□

Using the result of the previous lemma, we assume the following hypothesis in this chapter:

Hypothesis 7.3: Let the notation and properties listed in Hypothesis 1.4 hold. Additionally, let be $F^*(H) \cong \Omega_{2n}^{\pm}(q)$ with $n \geq 5$ and $q = 2^f$ for $f \in \mathbb{N}$. To simplify notation, we identify $F^*(H)$ and $\Omega_{2n}^{\pm}(q)$. It is $\mathbb{C}_{\Omega_{2n}^{\pm}(q)}(R)/Q \cong \Omega_{2n-4}^{\pm}(q) \times L_2(q)$ and it is $\mathbb{C}_{O_{2n}^{\pm}(q)}(R)/Q \cong O_{2n-4}^{\pm}(q) \times L_2(q)$, using Lemma 2.36. Additionally, it is $|Q| \geq q^{13}$, using Lemma 2.36.
We assume the existence of an involution $z \in \mathbb{C}_H(R)$ such that $\mathbb{C}_G(z) \not\leq H$. We may choose $z \in \mathbb{C}_H(R)$ such that $\mathbb{C}_S(z)$ is a Sylow 2-subgroup of $\mathbb{C}_H(z)$. Then it is $P(\mathbb{C}_S(z)) \neq \emptyset$. So let K be a group in $P(\mathbb{C}_S(z))$. By Lemma 6.12 and Proposition 8.8 in [SaSt], it is $P^*(\mathbb{C}_S(z)) = \emptyset$. Hence we may assume $K \in P(\mathbb{C}_S(z)) \setminus P^*(\mathbb{C}_S(z))$.
As a direct consequence of Lemma 5.10, z cannot be H-conjugate to an involution in Q.

For $H = F^*(H)$, it is z an involution in $\Omega_{2n}^{\pm}(q)$. For $H \neq F^*(H)$, $z \in \mathbb{C}_H(R)$ could also be an involution of b-type in $O_{2n}^{\pm}(q)$. Hence in the rest of this chapter, we assume $z \in O_{2n}^{\pm}(q)$; nevertheless in case of $H = F^*(H)$ we can always replace $O^{\pm}(q)$ by $\Omega^{\pm}(q)$.
The following statement can be found as Lemma 6.3 in [SaSt]. For $K \in P(\mathbb{C}_S(z))$, it allows us to choose $T \in \mathrm{Syl}_2(K \cap H)$ such that $T \leq S \cap K$ holds.

Lemma 7.4: Assume Hypothesis 7.3. Without loss of generality $T \leq S \cap K$ is a Sylow 2-subgroup of $K \cap H$.

Proof: Let K be an element of the set $P(\mathbb{C}_S(z))$ and let T be a Sylow 2-subgroup of $K \cap H$ with $\mathbb{C}_S(z) \leq T$. Using Lemma 3.6, $T \in \mathrm{Syl}_2(K \cap H)$ is also a Sylow 2-subgroup of K. By choice of $z \in \mathbb{C}_S(R)$, $R \leq K$ holds. As $\mathbb{C}_S(z)$ is a Sylow 2-subgroup of $\mathbb{C}_H(z)$, $\mathbb{C}_S(z) \leq T$ implies $\mathbb{C}_S(z) = \mathbb{C}_T(z)$. So also $\mathbb{C}_T(z) \cap H$ is a Sylow 2-subgroup of $\mathbb{C}_H(z)$. By Sylow's Theorem, there is an element $h \in H$ such that $T^h \leq S$ holds. Then $\mathbb{C}_{T^h}(z^h) \cap H = \mathbb{C}_T(z)^h \cap H$ is a Sylow 2-subgroup of $\mathbb{C}_H(z^h)$. Hence $\mathbb{C}_{T^h}(z^h) \leq \mathbb{C}_S(z^h) \cap K^h$ follows. It is $K^h \in P(\mathbb{C}_S(z^h))$ and $z^h \in S$ centralizes R. Hence $R \leq K^h$ holds. Replacing z by z^h and K by K^h gives the statement of this lemma. □

Lemma 7.5: Assume Hypothesis 7.3. Then $O(K) \leq H$ holds.

Proof: Let $z \in \mathbb{C}_S(R)$ be an involution such that $\mathbb{C}_G(z) \not\leq H$ holds and let $K \in P(\mathbb{C}_S(z)) \setminus P^*(\mathbb{C}_S(z))$. By Hypothesis 7.3, it is $z \in S \setminus Q$. Because of $[\langle z \rangle, R] = 1$ and as Q is a large subgroup in H, $[\langle z \rangle, Q] \leq Q$ holds. Then the action of z on Q/R is linear. Hypothesis 7.3 implies $|Q/R| \geq q^{12}$. Hence, as z is an involution, $\mathbb{C}_Q(z) \leq K$ contains an elementary abelian subgroup V of order q^5 by Lemma 6.4. Coprime action and Lemma 5.10 give $O(K) = \langle \mathbb{C}_{O(K)}(v) \mid v \in V^\# \rangle \leq H$. □

So we have $K/O(K) = E(K/O(K)) \cdot (TO(K)/O(K))$ with $E(K/O(K)) = (L_1 * \cdots * L_m)O(K)/O(K)$ for an integer $m \in \mathbb{N}$. It is $L_1 * \cdots * L_m$ a central product of components of K and $T \in \mathrm{Syl}_2(K \cap H)$. By definition of $P(\mathbb{C}_S(z))$, it is $O_2(K) \neq 1$ and by Lemma 7.4, we may assume $R \leq T \leq S$. By minimality of $K \in P(\mathbb{C}_S(z))$, T acts transitively on the set $\{L_1, \ldots, L_m\}$. Hence all the components of K are isomorphic and none of the components is contained in H. We assume this setting for the rest of this chapter.

By construction, it is K a subgroup of the 2-local subgroup $\mathrm{N}_G(O_2(K))$ with $O_2(K) \neq 1$. Hence the K_2-group assumption in Hypothesis 1.4 implies that for every component L of K, $\overline{L} = L/\mathrm{Z}(L)$ is a known finite simple group.

This is used in the following lemma to collect some statements, which in particular restrict the set of possible components of K.

Lemma 7.6: Assume Hypothesis 7.3. Let L be an arbitrary component of K and $T \in \mathrm{Syl}_2(K \cap H)$ with $R \leq T \leq S$. Then the following statements hold:

(a) It is $[Q \cap K, O_2(K)] = 1$ and $Q \cap O_2(K) = 1$.

(b) The component L is normalized by every involution x in K for which $\mathbb{C}_G(x) \leq H$ holds. In particular, this holds for every involution in $Q \cap K$.

(c) No involution x in K for which $\mathbb{C}_G(x) \leq H$ holds centralizes L.
In particular, the action of $\Omega_1(Q \cap K)$ on L is faithful.

(d) If there is more than one component in K, then $Q \cap L$ is trivial.

(e) It is $L/\mathrm{Z}(L)$ isomorphic to one of the following groups: $L_2(2^j)$ with $j \geq 4$, $Sz(2^j)$ with $j \geq 4$, $U_3(2^j)$ with $j \geq 3$, $L_3(2^j)$ with $j \geq 3$ or $Sp_4(2^j)$ with $j \geq 3$.

Proof: By minimality of $K \in P(\mathbb{C}_S(z))$, it is L, and therefore $E(K)$, not contained in H. As before, $O_2(K) \leq T \leq S$ centralizes some element $1 \neq s \in \mathrm{Z}(S) \cap R$. So $O_2(K)$ normalizes the large subgroup Q. Hence it is $[Q \cap K, O_2(K)] \leq Q \cap O_2(K) = 1$, as otherwise there is an involution $x \in O_2(K) \cap Q$ and, using $[E(K), O_2(K)] = 1$, $E(K) \leq \mathbb{C}_G(x) \leq H$ follows, applying Lemma 5.10. This proves (a).

Now let $x \in K$ be an involution such that $\mathbb{C}_G(x) \leq H$ holds. We assume that $L_i^x = L_j$ holds for different components L_i and L_j of K. By Lemma 3.6, it is $T \in \mathrm{Syl}_2(K)$ and K minimal parabolic such that $H \cap K$ is the unique maximal subgroup of K which contains T. By construction of K, it is $H \cap E(K) \leq H \cap K$ and $E(K) \nleq H$. As $L_i \nleq H$, there is an element $u \in L_i \setminus H$. Then it is $u \cdot u^x \in \mathbb{C}_{L_i * L_j}(x) \leq H$. So $E(K) \leq \langle H \cap K, u \cdot u^x \rangle \leq H$ follows, which is a contradiction. Hence every involution x in K with $\mathbb{C}_G(x) \leq H$ normalizes every component of K and part (b) holds.
Additionally, no such involution centralizes L, as otherwise L would be contained in H. This is part (c). By Lemma 5.10, parts (b) and (c) hold for every involution in $Q \cap K$.

If K has $m > 1$ many components $L_1, \ldots, L_m$, then $Q \cap L_i$ is trivial for each $i \in \{1, \ldots, m\}$. This is, because otherwise every other component would be contained in the centralizer of an involution of Q, which contradicts Lemma 5.10. Hence part (d) holds.

It is $K \in P(\mathbb{C}_S(z))$ minimal parabolic with respect to T. Next we show that $K_L := L\,\mathrm{N}_T(L)$ is minimal parabolic with respect to $\mathrm{N}_T(L)$. As $L \leq K_L$ is quasisimple, $\mathrm{N}_T(L)$ is not normal in K_L. Assume now the existence of a proper subgroup X of K_L which contains $\mathrm{N}_T(L)$ and is not contained in H. Then $\langle X, T \rangle$ is a proper subgroup of K which is not contained in $K \cap H$. But this contradicts K being a minimal parabolic subgroup with respect to T. So K_L is minimal parabolic with respect to $\mathrm{N}_T(L)$.
It is $K \leq \mathrm{N}_G(O_2(K))$ and $O_2(K) \neq 1$. By Hypothesis 1.4, $\overline{L} = L/\mathrm{Z}(L)$ is a known non-abelian, finite simple group.
As K_L is minimal parabolic with respect to $\mathrm{N}_T(L)$, $\overline{L}$ is not isomorphic to a sporadic group by Lemma 2.48.

As the involution z centralizes $R = \mathrm{Z}(Q)$, it induces a $GF(q)$-linear action on Q/R. Hence z centralizes a subspace of dimension at least $\frac{\dim_{GF(q)}(Q/R)}{2}$ in Q/R. As for $F^*(H) \cong \Omega_{2n}^{\pm}(q)$ with $n \geq 5$ the order of Q/R is at least q^{12}, Lemma 6.4 implies that z centralizes an elementary abelian subgroup V of order at least q^5 in Q. So it is $V \leq \mathbb{C}_Q(z) \leq \mathbb{C}_S(z) \leq K$ and V acts on each component L of K. Additionally, the action of V on L is faithful by part (c) of this lemma. Using Remark 2.10, V is involved in $\mathrm{Aut}(\overline{L})$.

Suppose now that $\overline{L} = L/\operatorname{Z}(L)$ is isomorphic to an alternating group.
By Lemma 2.49, it is $\overline{L}$ either isomorphic to A_6 or it is $\overline{L} \cong A_{2^k+1}$ for $k \in \mathbb{N}$. In case of $\overline{L} \cong A_6$, it is $\overline{L}$ generated by centralizers of involutions in $V \leq Q$, using Lemma 2.51. If $\overline{L}$ is generated by centralizers of involutions in $V \leq Q$, then $L \leq H$ follows, as the centralizer of every involution $v \in V$ is contained in H by Lemma 5.10. This is a contradiction to $L \nleq H$. For $\overline{L} \cong A_5$ it is $m_2(\operatorname{Aut}(\overline{L})) \leq m_2(S_5) = 2$ and for $\overline{L} \cong A_9$ it is $m_2(\operatorname{Aut}(A_9)) = m_2(S_9) = 4$. Hence there is no elementary abelian subgroup of order q^5 in $\operatorname{Aut}(\overline{L})$ for these groups.
Hence we can identify $\overline{L}$ with A_{2^k+1} for $4 \leq k \in \mathbb{N}$. We consider the stabilizer $M = K_L \cap H$ of a point in the permutation representation of degree $2^k{+}1$ in A_{2^k+1} or S_{2^k+1}. Without restriction let M be the stabilizer of the point 2^k+1. It is $R \leq T$, hence $R \leq T \cap M$. We may assume that $\hat{s} = (1,2)(3,4)\cdots(2^k-1, 2^k)$ is contained in the projection $\hat{R}$ of R in L. By part (b) of this lemma L is normalized by $\Omega_1(Q \cap K)$. In particular, it is $\Omega_1(Q \cap K) \leq K_L \cap H = M$. It is $V \leq \Omega_1(Q \cap K)$ and the isomorphic projection $\Omega_1\widehat{(Q \cap K)}$ of $\Omega_1(Q \cap K)$ in L centralizes $\hat{s}$. As Q is a large subgroup in H, $\mathbb{C}_M(\hat{s})$ normalizes $\Omega_1\widehat{(Q \cap K)}$. Hence $\Omega_1\widehat{(Q \cap K)}$ is a normal 2-subgroup in $\mathbb{C}_M(\hat{s})$. Let $\hat{V}$ be the projection of $V \leq \Omega_1(Q \cap K)$ in L. In analogy to Lemma 5.10, it is $\hat{V}$ contained in $\langle (1,2), (3,4), \ldots, (2^k-1, 2^k)\rangle \cap K_L$ for $2^k+1 \geq 17$.
To apply Lemma 2.50, we have to make sure that there is no elementary abelian 2-group of rank at most k which acts on the set $\{1, 2, \ldots, 2^k+1\}$ by fixing the point 2^k+1 and acting regularly on the remaining set $\{1, 2, \ldots, 2^k\}$. It is easy to see that the described situation can only arise for $k = 3$, so for $\overline{L} \cong A_9$. As the group A_9 has already been excluded from the set of possible components, Lemma 2.50 implies that $\overline{L} \cong A_{2^k+1}$ for $k \geq 4$ is generated by centralizers of involutions in $V \leq Q$, which contradicts Lemma 5.10. So $\overline{L}$ is not isomorphic to an alternating group.

If $\overline{L} = L/\operatorname{Z}(L)$ is isomorphic to a group of Lie type over a field in odd characteristic, Lemma 2.52 implies that either $\overline{L}$ is generated by centralizers of non-trivial elements in V or that $\overline{L}$ is isomorphic to one of the following groups: $L_2(5)$, $L_2(7)$, $L_2(9)$, ${}^2G_2(3)'$ or $PSp_4(3)$. As it is $L_2(5) \cong A_5$ and $L_2(9) \cong A_6$, these possibilities have been excluded in the previous paragraphs.
Using [CoCu], it is $m_2(\operatorname{Aut}(L_2(7))) = 2$. Hence there is no elementary abelian 2-subgroup of order q^5 in the automorphism group of $L_2(7)$. And it is $PSp_4(3)$ not minimal parabolic, see [CoCu]. Hence $\overline{L}$ is not isomorphic to $PSp_4(3)$. It is ${}^2G_2(3)' \cong L_2(8)$ and it is $m_2(\operatorname{Aut}(L_2(8))) = 3$, which again contradicts V being involved in $\operatorname{Aut}({}^2G_2(3)')$.

Altogether, we have that $\overline{L}$ is a simple group of Lie type over a field of even characteristic. Hence Lemma 2.47 gives that $\overline{L}$ is isomorphic to one of the following groups: $L_2(2^j)$, $Sz(2^j)$, $U_3(2^j)$, $L_3(2^j)$ or $Sp_4(2^j)'$ with $j \in \mathbb{N}$, where in the last two cases $\operatorname{N}_T(L)$ involves a graph automorphism of $\overline{L}$. Additionally, the groups $L_2(2)$, $Sz(2)$ and $U_3(2)$ are solvable and it is $L_2(4) \cong L_2(5)$, $L_3(2) \cong L_2(7)$ and $Sp_4(2)' \cong A_6$. Components are never solvable and the latter three groups have been excluded above, which also holds for the group $L_2(8)$. And

$m_2(\mathrm{Aut}(Sz(8))) = 3$, $m_2(\mathrm{Aut}(L_3(4))) = 3$, $m_2(\mathrm{Aut}(U_3(4))) = 2$ and $m_2(\mathrm{Aut}(Sp_4(4))) = 2$ are contradictions to V being involved in $\mathrm{Aut}(\overline{L})$ for $\overline{L}$ being isomorphic to $L_2(2^j)$ or $Sz(2^j)$ for $j \leq 4$ or $U_3(2^j)$, $L_3(2^j)$ or $Sp_4(2^j)$ for $j \leq 3$. □

Lemma 7.7: Assume Hypothesis 7.3. Let L be a component of K. Then every involution $\tau \in K$ with $\mathbb{C}_G(\tau) \leq H$ induces an inner automorphism on L, which projects into $\mathrm{Z}(T \cap L)$. So τ acts like a 2-central involution on L.
Additionally, $\mathbb{C}_Q(z)$ is elementary abelian and it is $2 \cdot \dim(\mathbb{C}_{Q/R}(z)) = \dim(Q/R)$ over $GF(q)$ with $\mathbb{C}_{Q/R}(z) = [Q/R, z]$.

Proof: Let L be a component of $K \in P(\mathbb{C}_S(z))$ and $T \in \mathrm{Syl}_2(K \cap H)$ with $\mathbb{C}_S(z) \leq T$. Then $K_L := L\,\mathrm{N}_T(L)$ is minimal parabolic with respect to $\mathrm{N}_T(L)$ and $L/\mathrm{Z}(L)$ is a simple group of Lie type over a field of characteristic 2 listed in Lemma 7.6. By Lemma 2.47, $K_L \cap H = \mathrm{N}_{K_L}(T \cap L)$ is the only maximal subgroup of K_L which contains $\mathrm{N}_T(L)$.
We denote by $\mathrm{Aut}_K(L)$ the group of automorphisms of L which are induced by elements of K. By Lemma 7.6, $\Omega_1(Q \cap K)$ embeds into $\mathrm{Aut}_K(L)$ and the involutions in $Q \cap K$ induce non-trivial automorphisms on $L/\mathrm{Z}(L)$, see Remark 2.10. The possible components have no center of even order, compare Lemma 2.22. Additionally, we only use the 2-structure of these groups. Hence, we may assume that L is simple.

We first show that every involution $\tau \in T$ with $\mathbb{C}_G(z) \leq H$ induces an inner automorphism on L. Therefore, we assume the existence of such an involution τ which induces an outer automorphism on L. For $L \cong Sz(2^j)$ there is no involution in $\mathrm{Out}(L)$ by Lemma 2.31. If τ induces a field automorphism on L, then it is $O^{2'}(\mathbb{C}_L(\tau))$ isomorphic to a group of Lie type of the same type but over $GF(2^{\frac{j}{2}})$, except for $L \cong U_3(2^j)$, by Lemma 2.31. In particular, τ centralizes an element in L which does not normalize $T \cap L$. Hence τ centralizes an element which is not contained in the maximal subgroup $\mathrm{N}_{K_L}(T \cap L)$ of K_L. Thus τ centralizes an element not contained in H. This is a contradiction. If τ induces a field automorphism on $L \cong U_3(2^j)$ or a graph or a field-graph automorphism on L, $\mathbb{C}_L(\tau)$ can be isomorphic to $L_2(2^j)$ for $L \cong U_3(2^j)$ or for $L \cong L_3(2^j)$, in which case also $\mathbb{C}_L(\tau) \cong L_2(2^{\frac{j}{2}})$ is possible. Also $\mathbb{C}_L(\tau)$ can be isomorphic to $Sz(2^j)$ for $L \cong Sp_4(2^j)$, see Lemma 2.31. But in all these cases, $\mathbb{C}_L(\tau)$ is not contained in the normalizer of $T \cap L$ and therefore not in H, which is impossible. So every involution $\tau \in K$ with $\mathbb{C}_G(z) \leq H$ induces an inner automorphism on L.
Next we show that such an involution $\tau \in T$ acts like a 2-central involution on L. Let $\hat{\tau}$ denote the projection of τ in $T \cap L$. For $L \cong L_2(2^j)$, $L \cong Sz(2^j)$ or $L \cong U_3(2^j)$, every involution in $T \cap L$ is 2-central, see [Hig] for $L \cong Sz(2^j)$ and Lemma 2.34 for $L \cong U_3(2^j)$. Since it is $z \in \mathbb{C}_S(R)$, we have $R \leq \mathrm{Z}(\mathbb{C}_Q(z))$. Hence $R \leq S \cap K = T$ holds, using Lemma 7.4. In particular, there is an involution in R which is projected into $\mathrm{Z}(T \cap L)$.
For $L \cong L_3(2^j)$ or $L \cong Sp_4(2^j)$, we assume that there are involutions τ_1 and τ_2 in T

with $\mathbb{C}_G(\tau_1) \leq H$ and $\mathbb{C}_G(\tau_2) \leq H$ such that the projection $\widehat{\tau_1}$ is contained in $\mathrm{Z}(T \cap L)$ and $\widehat{\tau_2} \in T \cap L$ is not 2-central. But then these centralizers generate $L \nleq H$, which is a contradiction. Therefore, $\hat{\tau}$ is contained in $\mathrm{Z}(T \cap L)$ and τ acts like a 2-central involution on L.

In particular, using Lemma 5.10, this implies that $\Omega_1(Q \cap K)$ embeds into $\mathrm{Z}(T \cap L)$. Hence $\Omega_1(Q \cap K)$ is elementary abelian.

Now we can show that $\mathbb{C}_Q(z)$ is elementary abelian. The involution z cannot centralize a semi-extraspecial group of type $D_1(q)$ in Q, as by Lemma 2.34 $D_1(q)$ is generated by involutions and $\Omega_1(\mathbb{C}_Q(z))$ is abelian. We assume now that $\mathbb{C}_Q(z)$ is not abelian. Then, using $D_2(q) \cong Q_2(q)$, it is $\mathbb{C}_Q(z) = \mathbb{C}_{[Q,z]}(z) * X$ with $X \cong Q_1(q)$. It is $\mathbb{C}_{[Q,z]}(z)$ elementary abelian. For $\mathbb{C}_Q(z) \cong Q_1(q)$, it is $[Q, z]$ a homocyclic group and therefore, $|Q| = q^5$ holds. But by assumption, it is $F^*(H) \cong \Omega^{\pm}_{2n}(q)$ for $n \geq 5$ in this chapter, so it is $|Q| \geq q^{13}$. Hence it is $\mathbb{C}_Q(z) \cong E \times Q_1(q)$ where $E \neq 1$ is elementary abelian.

By Hypothesis 7.3, it is z an involution in $\mathbb{C}_{O^{\pm}_{2n}(q)}(r) \cong Q\colon (O^{\pm}_{2n-4}(q) \times L_2(q))$ such that z is not H-conjugate to an involution in Q. We fix a Levi complement $K_1 \times K_2$ with $K_1 \cong O^{\pm}_{2n-4}(q)$ and $K_2 \cong L_2(q)$. The module Q/R consists of $q+1$ many orthogonal K_1-modules, and K_2 acts transitively on this set of orthogonal modules. Every involution in K_1 acts in the same way on each of these modules. As $q+1$ is odd, every involution in K_2 fixes at least one orthogonal K_1-module. We distinguish the two cases $z \in Q\colon K_1$ and $z \notin Q\colon K_1$. For the rest of this lemma, we denote the coset of z modulo Q by $\overline{z}$.

For $z \in Q\colon K_1$, it is $\mathbb{C}_Q(z)$ generated by the centralizers of z in the orthogonal K_1-modules. Hence $\mathbb{C}_Q(z)$ is generated by involutions, while $Q_1(q)$ is not. As seen above, it is $\Omega_1(Q \cap K)$ elementary abelian, so also $\mathbb{C}_Q(z) \leq K$ is elementary abelian.

Now we show that $2 \cdot \dim_{GF(q)}(\mathbb{C}_{Q/R}(z)) = \dim_{GF(q)}(Q/R)$ holds in this case. We have $|Q/R| = q^{4(n-2)}$ for $q = 2^f$, compare Lemma 2.36. Then the involution $\overline{z}$ centralizes a subspace in Q/R of $GF(q)$-dimension at least $2(n-2)$. We assume that $|\mathbb{C}_{Q/R}(\overline{z})| > q^{2(n-2)}$. As we have $[\overline{z}, K_2] = 1$ in this case, $\mathbb{C}_{Q/R}(\overline{z})$ is a K_2-module. So there is an integer $m \in \mathbb{N}$ such that $|\mathbb{C}_{Q/R}(\overline{z})| = q^{2m}$ holds. In particular, $\dim_{GF(q)}(\mathbb{C}_{Q/R}(\overline{z}))$ is even. Hence it is $|\mathbb{C}_{Q/R}(\overline{z})| \geq q^{2(n-2)+2}$. By the assumption, it is $|\mathbb{C}_{Q/R}(\overline{z}) : \mathbb{C}_Q(\overline{z})/R| = q$. So $|\mathbb{C}_Q(\overline{z})/R| \geq q^{2(n-2)+1}$ follows. Hence it is $|\mathbb{C}_Q(\overline{z})| \geq q^{2(n-2)+2}$. As we have already shown that $\mathbb{C}_Q(z)$ is elementary abelian in this case, there is an elementary abelian subgroup of order at least $q^{2(n-2)+2}$ in Q. But this contradicts $m_2(Q) = (2(n-2)+1)f$, which follows from the structure of Q, compare Lemma 2.34. Therefore, $2 \cdot \dim_{GF(q)}(\mathbb{C}_{Q/R}(z)) = \dim_{GF(q)}(Q/R)$ and $\mathbb{C}_{Q/R}(z) = [Q/R, z]$ hold in this case.

For $z \notin Q\colon K_1$, z interchanges at least two orthogonal modules and the projection of $\overline{z}$ in K_1 acts equally on every K_1-module. As it is $\overline{z} \notin K_1$, there is an element $\omega \in K_2$ of order $q+1$ such that ω is inverted by $\overline{z}$. The action of $\omega \in K_2$ on Q/R is fixed point free. Then $\mathbb{C}_{Q/R}(\overline{z}) = [Q/R, \overline{z}]$ holds. Therefore, we have $\mathbb{C}_Q(z)/R \leq \mathbb{C}_{Q/R}(z) = [Q/R, z]$, $R \leq [Q, z] \trianglelefteq Q$ and $2 \cdot \dim_{GF(q)}(\mathbb{C}_{Q/R}(z)) = \dim_{GF(q)}(Q/R)$. It is $\mathbb{C}_Q(z) \leq [Q, z]$ and, as z inverts every element in $[Q, z]$, $[Q, z]$ is abelian. In particular, $\mathbb{C}_Q(z)$ is abelian. As z acts on $\mathbb{C}_Q(z)$ by inverting the elements, there cannot be an element of order 4 in $\mathbb{C}_Q(z)$. So $\mathbb{C}_Q(z)$ is elementary abelian.

Hence in every case, the statement of the lemma holds. □

In the following lemma, we show that H controls the G-fusion of 2-central elements in $O^{\pm}_{2n}(q)$. In particular, the lemma provides $r^G \cap F^*(H) = r^H$ for $r \in R$. In case of $H = F^*(H)$, it implies $r^G \cap H = r^H$, by replacing each group $O^{\pm}(q)$ by $\Omega^{\pm}(q)$ and omitting all cases where z is an involution of b-type.

Lemma 7.8: Assume Hypothesis 7.3. Then $r^G \cap O^{\pm}_{2n}(q) = r^H$ holds for $r \in R^{\#}$.

Proof: As before, we identify $F^*(H)$ with $\Omega^{\pm}_{2n}(q)$. Let V be the orthogonal module for $O^{\pm}_{2n}(q)$. We consider an involution $r \in R = \mathrm{Z}(S \cap O^{\pm}_{2n}(q))$ and an involution $z \in S \cap O^{\pm}_{2n}(q)$ such that $z \sim_G r$ holds. We show that this implies $z \sim_H r$.

It is z contained in $O^{\pm}_{2n}(q)$, so z is an involution of type a_l, b_l or c_l for $l \in \mathbb{N}$. Let K be a group in $P(\mathbb{C}_S(z)) \setminus P^*(\mathbb{C}_S(z))$, compare Hypothesis 7.3.
By Lemma 2.36, we have $\mathbb{C}_{O^{\pm}_{2n}(q)}(r) = Q\colon (K_1 \times K_2)$ for $K_1 \cong O^{\pm}_{2n-4}(q)$ and $K_2 \cong L_2(q)$, and additionally, $Q/R = V_1 \otimes V_2$, where V_1 is a natural K_1-module and V_2 a natural K_2-module. Hence there are $q+1$ many orthogonal K_1-modules and K_2 acts transitively on this set of orthogonal K_1-modules. For all involutions $t_1 \in K_1$ and $t_2 \in K_2$, the action of $\langle t_1 \rangle$ on every K_1-module is equal and t_2 fixes one of the K_1-modules and interchanges the other ones pairwise.
By Hypothesis 7.3, it is z not H-conjugate to an involution in Q. The coset of z modulo Q is denoted by $\overline{z}$. We distinguish the following cases: $z \in Q\colon K_2$, $z \in Q\colon K_1$ and $z \in Q\colon (K_1 \times K_2)$, where in the third case $\overline{z}$ is a diagonal involution in $(K_1 \times K_2)Q/Q$.

We begin with an observation about the centralizer of a singular vector v in the orthogonal $O^{\pm}_{2n}(q)$-module V. It is $U := \mathbb{C}_{O^{\pm}_{2n}(q)}(v) = E\colon A$ with $A \cong O^{\pm}_{2n-2}(q)$ and E the natural A-module. It is U a parabolic subgroup in $O^{\pm}_{2n}(q)$. Hence we can choose v such that $S \cap O^{\pm}_{2n}(q) \leq U$ holds. Hence it is E a normal subgroup in $S \cap O^{\pm}_{2n}(q)$ and it is $R = \mathrm{Z}(S \cap O^{\pm}_{2n}(q)) \leq E$ a singular subspace of $GF(q)$-dimension 1 in the A-module E. So, it is E normalized by $z \in S$ and z centralizes R. In particular, z acts on $Q \cap E$.
We now consider $r \in R$. As r is a singular element in E, it is $\mathbb{C}_U(r) \leq \mathbb{C}_{O^{\pm}_{2n}(q)}(r)$. And without loss of generality, it is $\mathbb{C}_U(r) = E\colon (W\colon K_1)$, where W is a natural K_1-module. As $E\colon W$ splits, we may assume that W is a subgroup of $O_2(\mathbb{C}_U(r))$. It is $\mathbb{C}_U(r)$ a parabolic subgroup in $O^{\pm}_{2n}(q)$.

Hence Q is a normal subgroup in $\mathbb{C}_U(r)$. This implies $Q \leq O_2(\mathbb{C}_U(r))$. Since W is a natural K_1-module, it is W irreducible. Additionally, $Q \cap E$ is K_1-invariant. This and a comparison of $|Q| = q^{4n-7}$ and $|O_2(\mathbb{C}_U(r))| = q^{4n-6}$ provide $W \leq Q$ and $|E : Q \cap E| = q$. Hence it is $Q \cap E$ a submodule of codimension 1 in the natural A-module E. By Witt's Theorem, see Theorem 7.4 in [Tayl], A acts transitively on the set of hyperplanes in E, so every element in E is H-conjugate to an element in $Q \cap E$.
Additionally, it is $R^\perp = Q \cap E$, as otherwise it is $R^\perp \cap (Q \cap E)$ of codimension 2 in E and an A-invariant complement of R in $R^\perp$, which is impossible.

We now consider the case $z \in Q\colon K_2$:
Using the paragraph above, it is QE a Sylow 2-subgroup of $Q\colon K_2$. We may assume that $z \in QE$ holds. For $z \in E$, it is z conjugate in H to an involution in Q, which contradicts Hypothesis 7.3. Hence we may assume $z = z_1 \cdot z_2$ for $z_1 \in Q \setminus E$ and an involution $z_2 \in E \setminus Q$. Then it is $1 = z^2 = z_1^2 \cdot [z_1, z_2]$. From this $z_1^2 = [z_1, z_2]^{-1}$ follows. As Q is semi-extraspecial, it is $z_1^2 \in R$. Hence $[z_1, z_2] \in R$ holds. As E is abelian, $z_1 \in Q \setminus E$ acts as an involution in W on E. Applying Lemma 3.1(b) in [MSS2], we get $[z_1, z_2] \notin R$, a contradiction.

For the remaining two cases, we try to find an involution $y \in K$ such that y is H-conjugate to an involution in Q and such that y acts non-trivially on $\mathbb{C}_Q(z)$.
Such an involution y, as it is H-conjugate to an involution in Q, induces a non-trivial automorphism on each component L of K and also on $L/Z(L)$ by Lemma 7.6. Then Lemma 7.7 provides that y induces an inner automorphism, which acts like a 2-central involution in L. Additionally, by Lemma 7.7, it is $\mathbb{C}_Q(z) \leq K$ elementary abelian and, applying Lemma 7.6, $\mathbb{C}_Q(z)$ is contained in $E(K)$ by the minimality of K. But this contradicts the assumption that y acts non-trivially on $\mathbb{C}_Q(z)$. Hence K cannot have any components. But then K is not an element of $P(\mathbb{C}_S(z)) \setminus P^*(\mathbb{C}_S(z))$, as by Lemma 7.5, it is $O(K) \leq H$, so $E(K)$ cannot be trivial. Using M. Salarian's and G. Stroth's result in [SaSt], it is $P^*(\mathbb{C}_S(z)) = \emptyset$. This implies $P(\mathbb{C}_S(z)) = \emptyset$ and therefore, $\mathbb{C}_G(z) \leq H$ holds. Hence $z \sim_H r$ follows.
So we have to find an involution y with the described properties.

Let now $z \in S$ be an involution in $Q\colon K_1$. As before, it is z not H-conjugate to an element in Q and we use the notation from the paragraphs above. We prove the existence of an involution $y \in E \setminus (Q \cap E)$ with the following properties: It is $[y, z] = 1$ and y acts non-trivially on $\mathbb{C}_Q(z)$. Then, as y centralizes z, it is $y \in \mathbb{C}_S(z) \leq K$. And $y \in Q\colon K_2$ implies, as described in the first case above, that y is H-conjugate to an involution in Q. Therefore, y normalizes every component of $K \in P(\mathbb{C}_S(z))$, compare Lemma 7.6, and we are done.

The previous Lemma 7.7 implies $2 \cdot \dim_{GF(q)}(\mathbb{C}_{Q/R}(z)) = \dim_{GF(q)}(Q/R)$. In particular, it is $2 \cdot \dim_{GF(q)}([Q/R, z]) = \dim_{GF(q)}(Q/R)$.
As before, it is $(E \cap Q)/R = R^\perp/R$ a z-invariant subspace of Q/R. Now we construct a z-invariant complement of $(E \cap Q)/R$ in Q/R. As we have seen before, it is $W \leq Q$

and K_1 normalizes W. We now denote $z = t_1 \cdot t_2$ with $t_1 \in Q$ and $t_2 \in K_1$. As Q/R is elementary abelian, $z \in Q\colon K_1$ and $t_2 \in K_1$ act in the same way on $WR/R \leq Q/R$. Hence WR/R is a z-invariant subspace of Q/R. And it is $Q/R = (E \cap Q)/R \oplus WR/R$ by comparing orders. Hence it is $[Q/R, z] = [(E \cap Q)/R, z] \oplus [WR/R, z]$. So we have $2 \cdot \dim_{GF(q)}(\mathbb{C}_{(E \cap Q)/R}(z)) = \dim_{GF(q)}((E \cap Q)/R)$. As z normalizes E/R and $\dim_{GF(q)}(E/R)$ is odd, it is $2 \cdot \dim_{GF(q)}(\mathbb{C}_{E/R}(z)) > \dim_{GF(q)}(E/R)$. Therefore, we can find an involution $y \in E \setminus (E \cap Q)$ such that $y^z = y \cdot t$ holds for a suitable element $t \in R$. We want to show that we can choose y such that $[y, z] = 1$ holds. For $t \neq 1$, the involution $t \in R$ is singular and we have $y \notin (Q \cap R) = R^{\perp}$. Then the quadratic form of y^z differs from the quadratic form of $y \cdot t$. This implies $t = 1$. So we can find an involution $y \in E \setminus (Q \cap E)$ such that $[y, z] = 1$ holds. And every such involution is H-conjugate to an involution in Q, by the argumentation above.

Additionally, z is non-trivial on $R^{\perp} = Q \cap E$ by Theorem 11.11 in [Tayl]. As E is abelian, it is $Q \cap E \leq \mathbb{C}_Q(y)$. And, by Lemma 3.1(b) in [MSS2], $\mathbb{C}_Q(y) \leq E$ holds. So it is $Q \cap E = \mathbb{C}_Q(y)$. As $\mathbb{C}_Q(z) \setminus (E \cap Q)$ is not empty, it is $\mathbb{C}_Q(z) \not\leq \mathbb{C}_Q(y)$. Hence y acts non-trivially on $\mathbb{C}_Q(z)$.

In the third case we consider $z \in Q\colon (K_1 \times K_2)$ such that the projections of $\overline{z}$ in $(Q\colon K_1)/Q$ and also in $(Q\colon K_2)/Q$ are not trivial.

There is an element $\omega \in K_2 \cong L_2(q)$ of order $q+1$ such that ω is, modulo Q, inverted by conjugation with z. Applying the Baer Theorem, compare 6.7.7 in [KuSt], there is an element ω_1 of order $q+1$ such that $\omega_1^z = \omega_1^{-1}$ holds. It is ω_1 not necessarily contained in K_2, but there is a conjugate $\tilde{\omega}$ of ω_1 in K_2, such that $\tilde{\omega}^{\tilde{z}} = \tilde{\omega}^{-1}$ holds for a Q-conjugate $\tilde{z}$ of z. In particular, it is $\tilde{z} \in S$, as Q is contained in S.

As $\tilde{\omega} \in K_2$ acts by transitively permuting the $q+1$ many orthogonal K_1-modules, it is $\mathbb{C}_Q(\tilde{\omega}) = R$. Then it is $\mathbb{C}_{QK_2}(\tilde{\omega}) = R \times \langle\tilde{\omega}\rangle$. So we have $\mathbb{C}_{\mathbb{C}_{O^{\pm}_{2n}(q)}(r)}(\tilde{\omega}) = R \times K_1 \times \langle\tilde{\omega}\rangle$. The normalizer $\mathrm{N}_{\mathbb{C}_{O^{\pm}_{2n}(q)}(r)}(\tilde{\omega})$ equals $R \times K_1 \times (\langle\tilde{\omega}\rangle\colon Z)$ with $Z \leq K_2$ a group of order 2 that inverts $\tilde{\omega}$. As $\tilde{z}$ normalizes $\langle\tilde{\omega}\rangle$, there is an element $t \in R$ such that $\tilde{z} \cdot t \in K_1 \times (\langle\tilde{\omega}\rangle\colon Z)$ holds. It is $\langle\tilde{\omega}\rangle\colon Z \cong D_{2(q+1)}$ a dihedral group of order $2(q+1)$. Then $\tilde{z} \in S$ centralizes a 2-central involution $\tilde{y}$ in $S \cap K_1$. It is $\tilde{y} \in K_1$ not contained in Q and in particular, not contained in $\mathrm{Z}(S \cap O^{\pm}_{2n}(q))$.

To show that $\tilde{y}$ is H-conjugate to an involution in Q, we consider the centralizer in $O^{\pm}_{2n}(q)$ of a hyperbolic pair $\{e_1, f_1\}$ in the natural $O^{\pm}_{2n}(q)$-module V. This centralizer induces an orthogonal group $O^{\pm}_{2n-2}(q)$, which acts naturally on $\langle e_1, f_1\rangle^{\perp}$ and centralizes $\langle e_1, f_1\rangle$. An involution in $O^{\pm}_{2n-2}(q)$, which acts as an involution of type a_2 or c_2 on $\langle e_1, f_1\rangle^{\perp}$ and centralizes $\langle e_1, f_1\rangle$, must be of type a_2 or c_2 in $O^{\pm}_{2n}(q)$. We can repeat this consideration for the centralizer of a hyperbolic pair in $O^{\pm}_{2n-2}(q)$ to get a group $O^{\pm}_{2n-4}(q)$, such that an involution of type a_2 in $O^{\pm}_{2n-4}(q)$ acts as an involution of type a_2 or c_2 on $\langle e_1, f_1\rangle^{\perp}$ and then also on V.

It is $\tilde{y}$ a 2-central involution in $K_1 \cong O^{\pm}_{2n}(q)$. By a suitable choice of a basis for V, we can

apply the previous argument to $K_1 \cong O^{\pm}_{2n-4}(q)$ and $O^{\pm}_{2n}(q)$. Hence we may assume, that $\tilde{y}$ acts as an involution of type a_2 or c_2 on V. So $\tilde{y}$ is H-conjugate to an involution in Q.
Going back from $\tilde{z}$ to z by conjugation with an element in $Q \leq S$, we have an involution $y \in \mathbb{C}_S(z) \leq K$ which is H-conjugate to an involution in Q. By Lemma 7.6, y induces a non-trivial automorphism on each component L of K. As $y \in \mathbb{C}_S(z)$ centralizes R, it normalizes Q and therefore acts on $\mathbb{C}_Q(z)$.
It remains to show that the action of y on $\mathbb{C}_Q(z)$ is not trivial. As it is $\tilde{y} \notin Q$ and $\tilde{y} \sim_Q y$, also $y \notin Q$ holds. So it is $y \in S$ a 2-central involution in H, but not contained in $\mathrm{Z}(S \cap O^{\pm}_{2n}(q)) = R$.
We now consider the action of $\langle z \rangle$ on the abelian group Q/R in order to construct a z-invariant complement of $\mathbb{C}_{Q/R}(y)$ of $GF(q)$-dimension 4. As Q/R is abelian, we may assume $z = z_1 \cdot z_2$ for $z_1 \in K_1$ and $z_2 \in K_2$. We choose a K_1-module M in $Q/R = V_1 \otimes V_2$ which is not fixed by z_2. Hence we can write $M = V_1 \otimes \langle a \rangle$ for an element $a \in V_2$. Then it is $a^z = a^{z_1 \cdot z_2} = a^{z_2} =: b$ for $a \neq b \in V_2$. It is $(V_1 \otimes \langle a \rangle) \cap (V_1 \otimes \langle b \rangle)$ trivial, so $Q/R = (V_1 \otimes \langle a \rangle) \oplus (V_1 \otimes \langle b \rangle) = M \oplus M^z$ holds. As $y \in K_1$ acts on M and M^z in the same way, $1 = [y, z] = [y, z_1]$ implies, that M and also $\mathbb{C}_M(y)$ are z_1-invariant and $\mathbb{C}_M(y)^z = \mathbb{C}_{M^z}(y)$ holds. Hence it is $\mathbb{C}_{Q/R}(y) = \mathbb{C}_M(y) \oplus \mathbb{C}_{M^z}(y)$. Additionaly, $\langle a \rangle$ is $\langle z_1 \rangle$-invariant and $\mathbb{C}_{V_1}(y)$ is z-invariant. As y acts as an involution of type a_2 on M, we can find a complement N of $GF(q)$-dimension 2 of $\mathbb{C}_M(y)$ in M such that N^z is a complement of $\mathbb{C}_M(y)^z$ in M^z. Then $N \oplus N^z$ is a $\langle z \rangle$-invariant complement of $\mathbb{C}_{Q/R}(y)$ in Q/R with $\dim_{GF(q)}(N \oplus N^z) = 4$.
Applying Lemma 7.7, it is $\dim_{GF(q)}(\mathbb{C}_{N \oplus N^z}(z)) = 2$.
Let the preimage of $\mathbb{C}_{N \oplus N^z}(z)$ in Q be denoted by Y. It is $|Y| = q^3$. Additionally, it is $[Y, z] \leq \mathrm{Z}(Y) \leq R$. Then 1.5.4 in [KuSt] implies that the map $Y \to R$ with $x \mapsto [x, z]$ is a homomorphism. As the kernel of this map is of order at least q^2, we have an element $x \in W \setminus R$ with $[x, z] = 1$. Altogether, it is $x \in \mathbb{C}_Q(z) \setminus \mathbb{C}_Q(y)$. Hence y is not trivial on $\mathbb{C}_Q(z)$ and we are done. □

Now we can prove Theorem 1.5 for $F^*(H) \cong \Omega^{\pm}_{2n}(q)$ with $n \geq 4$. For information about centralizers of involutions which are not contained in $\mathbb{C}_S(R)$, we use Section 19 in [AsSe].

Lemma 7.9: We assume Hypothesis 7.3 and let be $F^*(H) \cong \Omega^{\pm}_{2n}(q)$ for $n \geq 4$ and $q = 2^f$ with $f \in \mathbb{N}$. Then $G = H$ holds.

Proof: Let $r \in R$ be an involution. By Lemma 3.9, it is $\mathbb{C}_G(r) \leq H$.
Now let t be an arbitrary involution in $S \in \mathrm{Syl}_2(H)$ which is G-conjugate to r. For $t \in \mathbb{C}_S(R)$, it is $t \sim_H r$ by Lemma 7.8. Hence we may assume $t \notin \mathbb{C}_S(R)$. Using Section 19 in [AsSe], this can only occur for $F^*(H) \cong \Omega^{+}_{2n}(q)$, as the only involutions in $\mathrm{Aut}(\Omega^{-}_{2n}(q)) \setminus \Omega^{-}_{2n}(q)$ are involutions of type b_l for suitable $l \in \mathbb{N}$. And such involutions of b-type are contained in $\mathbb{C}_H(R)$.
For $F^*(H) \cong \Omega^{+}_{2n}(q)$, t can only induce a field or a graph-field automorphism. If t induces

a field automorphism, then by (19.1) in [AsSe], it is $O^{2'}(\mathbb{C}_{F^*(H)}(t)) \cong \Omega_{2n}^{+}(q_0)$ with $q_0^2 = q$. Hence $\mathbb{C}_G(t)$ involves a simple group of Lie rank n. By Lemma 2.29, $\Omega_{2n}^{+}(q_0)$ cannot be embedded into $\Omega_{2n-4}^{\pm}(q) \times L_2(q)$. So it is impossible to embed $\mathbb{C}_G(t)$ into $\mathbb{C}_G(r) = \mathbb{C}_H(r)$. If t induces a field-graph automorphism, then by (19.6) in [AsSe], it is $\mathbb{C}_{F^*(H)}(t) \cong \Omega_{2n}^{-}(q_0)$ with $q_0^2 = q$. Hence $\mathbb{C}_{F^*(H)}(t)$ is a simple group of Lie rank $n-1$. Again by Lemma 2.29, it is impossible to embed $\mathbb{C}_G(t)$ into $\mathbb{C}_G(r) = \mathbb{C}_H(r)$.
Altogether, $r^G \cap H = r^H$ holds. Then Holt's result from Lemma 2.44 implies $G = H$. □

By Lemma 7.9, Theorem 1.5 holds for $F^*(H) \cong \Omega_{2n}^{\pm}(q)$. Therefore, by Remark 4.14 and Corollary 6.7, the proof of Theorem 1.5 is complete.

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Appendix

Eigenständigkeitserklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit eigenständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe. Die Stellen, die anderen Werken wörtlich oder sinngemäß entnommen sind, wurden unter Angabe der Quelle kenntlich gemacht.

Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder in ähnlicher Form in einem anderen Prüfungsverfahren vorgelegt.

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