VARIATIONAL INEQUALITIES

WITH

MULTIVALUED BIFUNCTIONS

Dissertation

zur Erlangung des

Doktorgrades der Naturwissenschaften
(Dr. rer. nat.)

der

Naturwissenschaftlichen Fakultät II
Chemie, Physik und Mathematik

der Martin-Luther-Universität Halle-Wittenberg

vorgelegt von

Herrn **Christoph Tietz**

geboren am 18. Juli 1990 in Merseburg (Saale)

1. Gutachter: Prof. Dr. Siegfried Carl
2. Gutachter: Prof. Dr. Dumitru Motreanu

Verteidigungsdatum: 17.10.2019

To Luisa, Aya and Zeno.

Contents

Prolog

I am interested in mathematics only as a creative art.
— Godfrey H. Hardy, *A Mathematician's Apology*

[What people] really want is usually not some collection of "answers"—what they want is *understanding*.
— William P. Thurston, *On Proof and Progress in Mathematics*

Once upon a time ...

... there was a real function $f\colon \mathbb{R} \to \mathbb{R}$*, studied by a curios (and mathematically educated) mind which had a basic question:*

Does the equation $f(x) = 0$ *have a* ***solution****?*

The Method of Sub-Supersolutions

There is no answer that fully satisfies a curious mind (which might be a good definition for such a kind of minds), but there are promising candidates. To find one of them, we note that from the solvability of $f(x) = 0$ it follows trivially that there are (not necessarily distinct) real numbers $\underline{x}$ and $\overline{x}$ such that

$$f(\underline{x}) \leqslant 0 \quad \text{and} \quad 0 \leqslant f(\overline{x}),$$

which we will call **subsolution** and **supersolution**, respectively. Now, the task is to find some x^* that is both a sub- and a supersolution, and our mathematical informed guess is that we will find it somewhere between a pair of sub-supersolutions. To simplify notations, let us assume in the following that $\underline{x} \leqslant \overline{x}$ holds for a given subsolution $\underline{x}$ and a given supersolution $\overline{x}$. Then a natural candidate for x^* is the greatest subsolution located between these sub-supersolutions,

$$x^* := \sup \underline{S}, \quad \text{where } \underline{S} := \{x \in [\underline{x}, \overline{x}] : x \text{ is a subsolution, i.e. } f(x) \leqslant 0\}.$$

Since $\underline{S} \subset \mathbb{R}$ is bounded and non-empty, the supremum exists. Thus, in order to check that x^* is indeed the greatest subsolution of f, it suffices to check that x^* is not only a supremum of subsolutions, but a subsolution itself. However, this may not be the case as simple examples demonstrate, and thus we assume that f satisfies the following condition:

(C^*) If $(x_n) \subset \mathbb{R}$ is an increasing sequence converging to x and if $f(x_n) \leqslant 0$ for all n, then $f(x) \leqslant 0$.

It follows readily that x^* is a subsolution. If x^* is in addition a supersolution (and thus the desired solution), then it is obviously the smallest supersolution in the interval $[x^*, \overline{x}]$, and thus it is a good idea to consider the element

$$x_* := \inf \overline{S}, \quad \text{where } \overline{S} := \{x \in [x^*, \overline{x}] : x \text{ is a supersolution, i.e. } 0 \leqslant f(x)\}.$$

Now, x^* is a solution if and only if x_* is a supersolution that equals x^*, and thus we assume the dual condition to (C^*) from above:

(C_*) If $(x_n) \subset \mathbb{R}$ is a decreasing sequence converging to x and if $0 \leqslant f(x_n)$ for all n, then $0 \leqslant f(x)$.

Then, x^* is known to be a supersolution and it follows readily $x^* = x_*$. Indeed, from $x^* < x_*$ it would follow that there is y such that $x^* < y < x_*$ and since y is (as every real number) a sub- or a supersolution, this would contradict the maximality of x^* or the minimality of x_*.

All in all, x^* is the greatest solution of $f(x) = 0$ in the interval $[\underline{x}, \overline{x}]$. Moreover, by dual arguments we find the smallest solution, so that we can state:

Theorem A *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real function satisfying conditions (C^*) and (C_*), and let $(\underline{x}, \overline{x})$ be an ordered pair of sub-supersolutions. Then the equation $f(x) = 0$ has the greatest and the smallest solution in $[\underline{x}, \overline{x}]$.* ○

Entrance of Bifunctions

Of course, every continuous function f satisfies conditions (C^*) and (C_*), thus, we have the following corollary of Theorem A:

Corollary (Intermediate Value Theorem) *Every continuous function $f\colon [\underline{x}, \overline{x}] \to \mathbb{R}$ such that $f(\underline{x}) \leqslant 0 \leqslant f(\overline{x})$ has a zero in $[\underline{u}, \overline{u}]$.* ○

Furthermore, every decreasing function f satisfies both (C^*) and (C_*), but in this case it follows

$$0 \leqslant f(\overline{x}) \leqslant f(\underline{x}) \leqslant 0,$$

so that the sub-supersolutions have to be solutions themselves. Then, of course, our celebrated theorem becomes trivial.

Consequently, a curious mind will ask if there is another example, and fortunately there is a whole class, namely functions $f\colon \mathbb{R} \to \mathbb{R}$ for which there is a further function $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (which we call **bifunction**) such that

(i) $f(x) = g(x, x)$ for all $x \in \mathbb{R}$,

(ii) $s \mapsto g(s, t)$ is continuous in $s = t$ for all $t \in \mathbb{R}$,

(iii) $t \mapsto g(s, t)$ is decreasing for all $s \in \mathbb{R}$.

Indeed, let $(x_n) \subset \mathbb{R}$ be an increasing sequence converging to x such that $f(x_n) \leqslant 0$, then it follows

$$f(x) = g(x, x) \leftarrow g(x_n, x) \leqslant g(x_n, x_n) = f(x_n) \leqslant 0, \quad \text{thus } f(x) \leqslant 0.$$

Similarly, for a decreasing sequence (x_n) converging to x such that $0 \leqslant f(x_n)$, it follows

$$0 \leqslant f(x_n) = g(x_n, x_n) \leqslant g(x_n, x) \to g(x, x) = f(x), \quad \text{thus } 0 \leqslant f(x).$$

Consequently, we have the following corollary:

Corollary B *Let $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a real function being continuous in the first and decreasing in the second argument and let $(\underline{x}, \overline{x})$ be an ordered pair of sub-supersolutions for the function $x \mapsto g(x, x)$. Then the equation $g(x, x) = 0$ has a greatest and a smallest solution in $[\underline{x}, \overline{x}]$.* ○

This exemplifies the fruitful interplay of order theory and topology.

Remark Examples for functions g satisfying (ii) and (iii) are given by defining $g(s, t) := g_1(s) + g_2(t)$ for any continuous $g_1\colon \mathbb{R} \to \mathbb{R}$ and any decreasing $g_2\colon \mathbb{R} \to \mathbb{R}$. Obviously, the function $x \mapsto g(x, x)$ may be neither continuous nor decreasing. ○

Approach via Fixed Points

Speaking of order theory, we want to apply the following famous theorem:

Theorem (Fixed Point Theorem of Tarski) *Every increasing function $f\colon [\underline{x}, \overline{x}] \to [\underline{x}, \overline{x}]$ has the greatest fixed point.* ○

It gives rise to another proof of Corollary B, which main idea is to fix one argument of the bifunction as follows:

Alternative Proof of Corollary B: For any $y \in [\underline{x}, \overline{x}]$, let us define the real function $g_y : x \mapsto g(x, y)$. Then, since g is decreasing in the second argument, we deduce

$$g_y(\underline{x}) = g(\underline{x}, y) \leqslant g(\underline{x}, \underline{x}) \leqslant 0, \quad \text{thus } g_y(\underline{x}) \leqslant 0.$$

Similarly, one deduces $g_y(\overline{x}) \geqslant 0$. Thus, the continuous function $g_y : \mathbb{R} \to \mathbb{R}$ satisfies all conditions of Theorem A, so that we know that the equation $g_y(x) = 0$ has a greatest solution x^*_y in $[\underline{x}, \overline{x}]$. Now, we note that $y \mapsto x^*_y$ is an increasing self-mapping on $[\underline{x}, \overline{x}]$. Indeed, suppose $y_1 \leqslant y_2$, then we have

$$0 = g(x^*_{y_1}, y_1) \geqslant g(x^*_{y_1}, y_2),$$

from which it follows that the equation $g_{y_2}(x) = 0$ has a greatest solution in $[x^*_{y_1}, \overline{x}]$ (which equals $x^*_{y_2}$), and thus $x^*_{y_1} \leqslant x^*_{y_2}$.

Consequently, by the Fixed Point Theorem of Tarski, we know that the equation $y = x^*_y$ has the greatest solution $y^* \in [\underline{x}, \overline{x}]$. Then, by definition of $x^*_{y^*}$, it follows

$$0 = g_{y^*}(y^*) = g(y^*, y^*),$$

and so y^* is a solution of $g(x, x) = 0$. In fact, it readily follows that y^* is even the greatest solution of $g(x, x) = 0$ in $[\underline{x}, \overline{x}]$. Indeed, assume that $\underline{y} \in [\underline{x}, \overline{x}]$ is *any* solution of $g(x, x,) = 0$. Then the same arguments as above, but now with $\underline{y}$ instead of $\underline{x}$, give us some solution y^{**} of $y = x^*_y$ located in $[\underline{y}, \overline{x}]$. Then, by construction of y^*, we have $\underline{y} \leqslant y^{**} \leqslant y^*$, which concludes the proof. ○

At this Point ...

... the curious mind strongly wondered if this alternative approach could really be of any use, maybe for problems that far exceeded all what it had seen in its short life. Exhausted, it fell into a deep sleep full of mathematical dreams.

Introduction

Aim of this Thesis

Let A be a set, and let $\mathsf{S}\colon \mathsf{A} \times \mathsf{A} \to \mathcal{P}_\emptyset(\mathsf{A})$ be a mapping on $\mathsf{A} \times \mathsf{A}$ whose values are subsets of A. Then A is called a **multivalued bifunction**, and the core idea of this thesis can be expressed in form of the following fixed point principle:

Let $\mathsf{F}\colon \mathsf{A} \to \mathcal{P}_\emptyset(\mathsf{A})$ *be a multifunction such that, for all* $\mathsf{a} \in \mathsf{A}$*,*

$$\mathsf{F}(\mathsf{a}) = \{\mathsf{a}^* \in \mathsf{A} : \mathsf{a}^* \in \mathsf{S}(\mathsf{a}^*, \mathsf{a})\}.$$

Then every fixed point of F *is a fixed point of* S*,*
i.e. from $\mathsf{a}^* \in \mathsf{F}(\mathsf{a}^*)$ *it follows* $\mathsf{a}^* \in \mathsf{S}(\mathsf{a}^*, \mathsf{a}^*)$*.*

Let further (P) be a mathematical problem, and suppose that S is defined in such a way that solutions of (P) are fixed points of S and vice versa. Then, if we search for solutions of (P), all we have to do is to search for fixed points of F. However, certainly not every multifunction $\mathsf{F}\colon \mathsf{A} \to \mathcal{P}_\emptyset(\mathsf{A})$ has a fixed point. Thus, we have to investigate in detail under which conditions the set $\operatorname{Fix}\mathsf{F}$ of fixed points of F is non-empty.

The fixed point theory we develop is linked to the special kind of problems we want to solve, namely **variational inequalities** as in [24, 28], in which some parts are given by nonsmooth, multivalued bifunctions. Those problems allow for an application of the powerful concept of sub-supersolutions (see the Introduction to Part II for more information), so our first task reads as follows:

Develop a powerful yet easy to apply mathematical framework
—which bases on order-theoretical fixed point theorems and comparison principles—
in order to solve variational inequalities with multivalued bifunctions.

This framework will give precise conditions on the auxiliary multifunction F, which in turn will give precise conditions on Problem (P) that guarantee the existence of solutions with special properties. Consequently, our second task will be the following one:

Illustrate the applicability of the developed framework
for a wide range of variational problems.

In order to present the main ideas, we will restrict our considerations to more classical function spaces and zero boundary conditions. However, we try our very best to present general ideas which can be extended to more sophisticated problems.

Structure of this Thesis

In Chapter 1, we will present purely order-theoretical results. First, we will devote a section to recall some fundamental notions in partially ordered sets D, such as the set relation $\leqslant^*$ on $\mathcal{P}_\emptyset(D)$, which is defined by

$$A \leqslant^* B \quad :\Longleftrightarrow \quad \text{for every } a \in A \text{ there is } b \in B \text{ such that } a \leqslant b,$$

and the notion of increasing upward multifunctions $F\colon D \to \mathcal{P}_\emptyset(D)$.

After providing a few basic results about multifunctions of isotone type, we then develop in Section 1.2 several purely order-theoretical fixed point theorems. This study starts with the famous theorem of Tarski, whose proof can be generalized so that we obtain more insights in a series of fixed point theorems presented recently in [76, 77, 78, 79]. To obtain an even more general fixed point theorem, we then follow [24, 50] and give a full proof of Theorem 1.59, which guarantees the existence of maximal fixed points under weak conditions. From this theorem, by use of a new order-theoretical property, we then deduce Corollary 1.60, in which conditions involving chains are replaced by conditions involving only sequences. This result serves together with new results about greatest fixed points as a reference for the rest of this thesis.

In Chapter 2, we incorporate topological results in our study. This will lead to fixed point theorems on ordered reflexive Banach spaces which are more easy to apply than Corollary 1.60. Although we roughly follow the path laid out in [50], we develop our own version of the story, whose highlights are the new Theorem 2.31, which ensures that some increasing upward multifunctions have maximal fixed points, and the desired framework of Theorem 2.33, which gives us even a condition under which increasing upward multifunctions have a greatest fixed point. After a quick comparison of our results with other fixed point theorems, we then present some topological results needed in applications.

In Chapter 3, we then consider measurable spaces and its connections with topology and order-theory. To this end, we introduce ordered measurable spaces and investigate conditions under which multifunctions of isotone type are (weakly) measurable. We then extend our study to compact-valued multifunctions $(x, s, t) \mapsto F(x, s, t)$ which depend differently on their arguments: $x \mapsto F(x, s, t)$ is measurable, $s \mapsto F(x, s, t)$ is upper semicontinuous, and $t \mapsto F(x, s, t)$ is increasing upward. In particular, we explore conditions under which F has a measurable single-valued selection. At the end of this chapter, we collect useful facts about spaces of measurable functions.

Starting from Chapter 4, we will apply the theoretical results to various variational problems. Often, such problems are solved with help of topological fixed point theorems, e.g. a suitable version of the Banach or Kakutani fixed point theorem, which require certain continuity properties of the fixed point operator involved. But there are operators of interest that lack continuity, which is why order-theoretical fixed point theorems come into play. In the last years, Carl, Le and Motreanu, among others, have developed an approach that covers a wide range of problems, see, e.g., [20, 22, 23, 24, 27, 28, 67, 68].

They built on theorems presented in [50], see Lemma 1.58, and combined them with the theory of pseudomonotone operators and the concept of sub-supersolutions. Our goal is to extend those results about nonsmooth variational inequalities by use of the general framework developed in Part I.

In Chapter 4, first, for a multivalued bifunction $(x, s, t) \mapsto f(x, s, t)$, we will consider the inclusion

$$u \in K: \quad Au + f(\cdot, u, u) + \partial I_K \ni 0, \tag{P}$$

where A is a Leray-Lions operator on a Sobolev space W, I_K is the indicator function of a given closed and convex set $K \subset W$, and ∂I_K is the subgradient of I_K in the sense of convex analysis. By definition, a solution of (P) is a function $u \in K$ such that

$$\langle Au, w - u\rangle + \langle \eta, w - u\rangle \geqslant 0 \quad \text{for all } w \in K \text{ and some selection } \eta \subset f(\cdot, u, u).$$

Since f is assumed to be upper semicontinuous in the second and increasing in the third argument, we can apply Theorem 2.33 to deduce that there are smallest and greatest solutions between each pair of sub-supersolutions.

Remark The first four chapters present new results and results from the paper *An order theoretic fixed point theorem with application to multivalued variational inequalities* by the author (see [110, 111]). ○

In Chapter 5, we extend our study of Chapter 4 to the following multivalued quasi-variational inequality problem: Find $u \in W$ such that for some $\eta \subset f(\cdot, u, u)$ it holds

$$\langle Au, w - u\rangle + \int_\Omega \eta(w - u) + K(w, u) - K(u, u) \geqslant 0 \quad \text{for all } w \in W,$$

where $A \colon W \to W'$ is, again, a Leray-Lions operator of divergence form, f is a multivalued bifunction being upper semicontinuous in the second and increasing in the third argument, and $K(\cdot, u) \colon W \to \mathbb{R} \cup \{+\infty\}$ is a convex functional for each $u \in W$.

We prove that, under weak assumptions on the data, there are the smallest and the greatest solution between each pair of sub-supersolutions.

Remark Chapter 5 is based on the results of the paper *Multivalued Quasi-Variational Inequalities with Nonsmooth Bifunctions* by the author, which is currently in preparation (see [112]). ○

In Chapter 6, we consider the existence and further qualitative properties of solutions of the Dirichlet problem to quasilinear multivalued elliptic equations with measures of the form

$$Au + G(\cdot, u) \ni f, \tag{P}$$

where A is, again, a second order elliptic operator of Leray-Lions type and $f \in \mathcal{M}_b(\Omega)$ is a given Radon measure on a bounded domain $\Omega \subset \mathbb{R}^N$. The lower order term $s \mapsto G(\cdot, s)$ is assumed to be a multivalued upper semicontinuous function, which includes Clarke's gradient $s \mapsto \partial j(\cdot, s)$ of some locally Lipschitz function $s \mapsto j(\cdot, s)$ as a special case. Our main goals are as follows: First, we develop an existence theory for Problem (P). Second, we propose concepts of sub-supersolutions for this problem and establish an existence and comparison principle. Third, we topologically characterize the solution set enclosed by sub-supersolutions.

Remark Chapter 6 is based on the results of the paper *Quasilinear elliptic equations with measures and multi-valued lower order terms* whose results where developed mainly by the author (see [30]). ○

In Chapter 7, we extend the results of Chapter 6 to the case of bifunctions, that is, we consider multivalued elliptic equations with bifunctions of the form

$$Au + f(\cdot, u, u) \ni \mu, \tag{P}$$

where A is, again, an elliptic Leray-Lions operator, $(x, s) \mapsto f(x, s, t)$ is upper Carathéodory and $t \mapsto f(x, s, t)$ is a decreasing, possibly nonsmooth multifunction, and $\mu \in L^1$.

Since our abstract framework does not apply directly to Problem (P), we introduce the concept of limit-solutions and limit-subsolutions. Then we can prove that there exist smallest and greatest limit-solutions between each ordered pair of sub-supersolutions.

Remark Chapter 7 is based on the results of the paper *Multivalued Elliptic Inclusions with Nonsmooth Bifunctions and* L^1 *Right-Hand Sides* by the author, which is currently in preparation (see [113]). ○

In Chapter 8, we then extend the results of Chapters 4, 5 and 7 to systems

$$A_i(u_i) + F_i(u_i, \mathbf{u}) + \partial K_{i,\mathbf{u}}(u_i) \ni 0 \quad \text{in } W_i', \tag{P_i}$$

and

$$A_i(u_i) + F_i(u_i, \mathbf{u}) \ni \mu_i, \tag{Q_i}$$

which are coupled via the lower-order terms $F_i(u_i, \mathbf{u})$. In order to solve Systems (P) and (Q), we first extend our basic framework to mixed-monotone systems, and then we apply the results of the forgoing chapters.

Remark Chapter 8 is based on the results of the paper *Systems of Quasilinear Elliptic Equations with* L^1*-Functions and of Quasi-Variational Multivalued Variational Inequalities* by the author, which is currently in preparation (see [114]). ○

In the concluding Chapter 9, we finally propose topics for further research.

Acknowledgment

I would like to express my deep gratitude to my supervisor, Prof. Dr. Siegfried Carl. He consistently allowed this thesis to be my own work, but steered me in the right the direction whenever he thought I needed it.

I am grateful to my brother, who read and criticized many drafts of this thesis, and my lovely cohabitant, who provided extensive moral and emotional support.

This study was supported by a doctoral studies grant of Saxony-Anhalt.

Part I

Theory

1 | Order Theory

The common theme of this thesis are order-theoretical considerations. In this first chapter, we are going to developed basic results and powerful fixed point theorems for multifunctions of isotone type that allow for a wide range of applications.

1.1 Basic Concepts

1.1.1 Partially Ordered Sets and Lattices

Let us recall some basic notions. (For more information, see, e.g., [46, 105].)

1.1 Definition Let D be a set and $\mathcal{R} \subset D \times D$. Then the relation $\mathcal{R}$ is called **partial order** on D if the following holds:

(i) $\mathcal{R}$ is **reflexive**, i.e. for all $a \in D$ it holds $(a, a) \in \mathcal{R}$,

(ii) $\mathcal{R}$ is **anti-symmetric**, i.e. from $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$ it follows $a = b$,

(iii) $\mathcal{R}$ is **transitive**, i.e. from $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$ it follows $(a, c) \in \mathcal{R}$.

In this case, we write $a \leqslant_{\mathcal{R}} b$ instead of $(a, b) \in \mathcal{R}$, and we call $(D, \leqslant_{\mathcal{R}})$ a **partially ordered set**, **poset** for short. If no confusion can occur, we only write $a \leqslant b$ instead of $a \leqslant_{\mathcal{R}} b$. Further, we mostly write D instead of $(D, \leqslant)$, and $a < b$ means that both $a \leqslant b$ and $a \neq b$. ○

Note that for any poset $(D, \leqslant_{\mathcal{R}})$ and any $A \subset D$ we easily obtain that $(A, \leqslant_{\mathcal{R} \cap (A \times A)})$ is a poset. Thus, we will consider subsets of a poset naturally as posets.

1.2 Example In this thesis, we use only standard partial orders, such as the usual partial order on $\mathbb{R}$ which makes it to an ordered field, or the componentwise partial order on $\mathbb{R}^N$.

The partial order on a poset D can be extended naturally to the **pointwise partial order** between functions $f, g \colon M \to D$, were M is any set, by setting

$$f \leqslant g \quad :\Longleftrightarrow \quad f(x) \leqslant g(x) \quad \text{for all } x \in M.$$

For (classes) of measurable functions on a measure space, inequality $f(x) \leqslant g(x)$ has to hold for almost every (a.e.) $x \in M$, see Definition 3.62 below. ○

To each poset we can construct another poset, its so called dual, as follows:

1.3 Definition Let $(\mathsf{D}, \leqslant_{\mathcal{R}})$ be a poset and define the set $\mathfrak{R} \subset \mathsf{D} \times \mathsf{D}$ via

$$(a, b) \in \mathfrak{R} \quad :\Longleftrightarrow \quad (b, a) \in \mathcal{R}.$$

Obviously, $\mathfrak{R}$ is a partial order on D, and we call $(\mathsf{D}, \leqslant_{\mathfrak{R}})$ the **dual poset** of $(\mathsf{D}, \leqslant_{\mathcal{R}})$. If no confusion can occur, we write $a \geqslant b$ instead of $a \leqslant_{\mathfrak{R}} b$. ○

In the following, we will often prove assertions about a general poset $(\mathsf{D}, \leqslant)$. Since these assertions hold for its dual poset $(\mathsf{D}, \geqslant)$, too, we obtain effortlessly the so called dual assertion. All we have to do is to replace in the assertion and all involved definitions $\leqslant$ with $\geqslant$ and interpreting thereafter the order-theoretical notions in terms of $\leqslant$. Similarly, all order-theoretical notions have a dual counterpart.

To begin with, let us introduce the following **order-interval** for each element a of a poset $(\mathsf{D}, \leqslant)$ and a subset $\mathsf{C} \subset \mathsf{D}$:

$$a^{\downarrow} := a^{\downarrow}_{\leqslant} := \{b \in \mathsf{D} : b \leqslant a\} \qquad \text{and} \qquad a^{\downarrow}_{\mathsf{C}} := a^{\downarrow} \cap \mathsf{C}.$$

By replacing $\leqslant$ with $\geqslant$, we get the dual counterpart

$$a^{\uparrow} := a^{\downarrow}_{\geqslant} = \{b \in \mathsf{D} : b \geqslant a\} = \{b \in \mathsf{D} : a \leqslant b\}.$$

Those intervals play a central role in defining various order-theoretical concepts, such as the following ones:

1.4 Definition Let D be a poset. An element $a \in \mathsf{D}$ is called

(i) **lower bound** of a subset M of D if $\mathsf{M} \subset a^{\uparrow}$,

(ii) **minimal** element of D if $a^{\downarrow} = \{a\}$, i.e. if, for all $b \in \mathsf{D}$, $b \leqslant a$ implies $b = a$,

(iii) **smallest** element of D if $\mathsf{D} \subset a^{\uparrow}$, i.e. if, for all $b \in \mathsf{D}$, $a \leqslant b$.

By duality, we define upper bounds, maximal and greatest elements, respectively: An element $a \in \mathsf{D}$ is called

$(\text{i})_{\mathrm{d}}$ **upper bound** of a subset M of D if $\mathsf{M} \subset a^{\downarrow}$,

$(\text{ii})_{\mathrm{d}}$ **maximal** element of D if $a^{\uparrow} = \{a\}$, i.e. if, for all $b \in \mathsf{D}$, $b \geqslant a$ implies $b = a$,

$(\text{iii})_{\mathrm{d}}$ **greatest** element of D if $\mathsf{D} \subset a^{\downarrow}$, i.e. if, for all $b \in \mathsf{D}$, $a \geqslant b$. ○

The concepts defined above have the following connection:

1.5 Proposition *Let D be a poset. Then each smallest element is a minimal element but not vice versa, and there is at most one smallest element.*

Proof: Let $\mathsf{D} = \{a, b\}$ (with $a \neq b$) and define $\mathcal{R} = \{(a, a), (b, b)\}$. Then $(\mathsf{D}, \leqslant_{\mathcal{R}})$ is a poset in which all elements are minimal but no element is the smallest. The remaining assertions follow readily by anti-symmetry of the partial order. ○

By duality, we have at once the following corollary:

1.6 Corollary *Let D be a poset. Then each greatest element is a maximal element, and there is at most one greatest element.* ○

Note that it is not guaranteed that there are minimal or even smallest elements in a poset. Therefore, the order-theoretical fixed point theorems to be presented in Section 1.2 below are welcome. They rely on the following notion:

1.7 Definition Let D be a poset.

(i) Let $a, b \in D$. If it exists, we denote by $a \wedge b$ the greatest element of $a^{\downarrow} \cap b^{\downarrow}$ and call it **infimum** of a and b.

(ii) In general, for any $A \subset D$ denote by $\inf A$ the greatest lower bound of A (called the **infimum** of A). Further, we sometimes denote by $y_{\wedge}$ the infimum of any familiy $\{y_i : i \in I\} \subset D$.

(iii) If, for all $a, b \in D$, the infimum $a \wedge b$ exists, we call D an **inf-semilattice**.

By duality, we define the **supremum** of a and b, denoted by $a \vee b$, the **supremum** $\sup A$ of any subset $A \subset D$, the abbreviation $y_{\vee}$ and the notion of **sup-semilattice**.

(iv) If D is both an inf- and a sup-semilattice, we call it a **lattice**.

(v) If D is a lattice, then $A \subset D$ is called a **sub-lattice** of D if A is closed under $\wedge$ and $\vee$ (where infimum and supremum are taken in D). ○

In the sequel, we will often consider poset and lattices which are **order-intervals**

$$[a, b] := \{x \in D : a \leqslant x \leqslant b\} = a^{\uparrow} \cap b^{\downarrow}.$$

If D is a poset or a lattice, then $[a, b]$ is also a poset or a lattice, respectively. If $C \subset D$, we often write $[a, b]_C$ instead of $[a, b] \cap C$.

Finally, let us recall the following properties of subsets of a poset:

1.8 Definition Let D be a poset, and let $A \subset D$.

(i) A is called **increasing** if $a^{\uparrow} \subset A$ for all $a \in A$.

(ii) A is called **order-convex upward** if for all $y_\alpha, y_\beta \in A$ the order-interval $[y_\alpha, y_\vee]$ belongs to A.

By duality, we define what it means to be a **decreasing** or **order-convex downward** set. For instance, A is called order-convex downward if for all $y_\alpha, y_\beta \in A$ the order-interval $[y_\wedge, y_\alpha]$ belongs to A.

(iii) A is called **order-convex** if it is both order-convex upward and order-convex downward. ○

Note, that it is not required in the definition of order-convex sets that $y_\alpha \leqslant y_\beta$, since then one would have $y_\wedge = y_\alpha$ and $y_\vee = y_\beta$ and all three types of convexity would be identical. Further, let y_α and y_β belong to an order-convex set A. Then one has especially $z_1 := y_\wedge \in A$ and $z_2 := y_\vee \in A$ and thus $[y_\wedge, y_\vee] = [z_\wedge, z_2] \subset A$.

1.1.2 Set Relations

1.9 Definition Let X be a set. Then $\mathcal{P}_\emptyset(X)$ denotes the power set of X, and $\mathcal{P}(X)$ denotes the set of all non-empty subsets of X. ○

Let D be a poset, and let $A, B \in \mathcal{P}_\emptyset(D)$ be any subsets. Then it will be convenient to compare A and B with respect to the partial order of D. There are a few possibilities to achieve this, however, the set relations we will use frequently are the following ones:

1.10 Definition Let D be a poset.

(i) The relation $\leqslant^*$ on $\mathcal{P}_\emptyset(D)$ is defined via

$$A \leqslant^* B \quad :\Longleftrightarrow \quad a^\uparrow \cap B \neq \emptyset \quad \text{for all } a \in A,$$

and we read "$A \leqslant^* B$" as "A **looks up to** B".

$(i)_d$ By duality, the relation $\leqslant_*$ on $\mathcal{P}_\emptyset(D)$ is defined via

$$A \leqslant_* B \quad :\Longleftrightarrow \quad B \geqslant^* A \quad \Longleftrightarrow \quad b^\downarrow \cap A \neq \emptyset \quad \text{for all } b \in B,$$

and we read "$A \leqslant_* B$" as "B **looks down on** A". ○

1.11 Remark The notion for those relations is not standardized in the literature; $\leqslant^*$ is denoted by $\leqslant^u$ in [64], by $\leqslant_1$ in [54], by $\preccurlyeq_1$ in [41] and by $\leqslant_*$ in [68]. However, we have chosen the symbol $\leqslant^*$ with $*$ on *top* to indicate that for $A \leqslant^* B$ to be true we have to find, for every $a \in A$, some $b \in B$ which *tops* a. ○

1.12 Remark The set relations $\leqslant^*$ and $\leqslant_*$ were generalized in vector-valued optimization to various kinds of variable set relations, see [39], where $A \preccurlyeq_u^{\mathcal{D}} B \subset Y$ means, for a fixed multifunction $\mathcal{D}\colon Y \to \mathcal{P}(Y)$, that for any $a \in A$ there is some $b \in B$ such that $b \in a + \mathcal{D}(a)$. Those relations can be used with gain in optimization, see, e.g., [48, 83], but are to general for our current purposes. ○

Since those relations are of utmost importance in the sequel, let us reformulate Definition 1.10 in other words. We have

(i) $A \leqslant^* B$ if and only if for every $a \in A$ there is $b \in B$ such that $a \leqslant b$,

(ii) $A \leqslant_* B$ if and only if for every $b \in B$ there is $a \in A$ such that $a \leqslant b$.

Further we have the duality rules $A \leqslant^* B \Longleftrightarrow B \geqslant_* A$ and $A \leqslant_* B \Longleftrightarrow B \geqslant^* A$.

The following property of the so defined set relations is readily seen:

1.13 Proposition *Let D be a poset. Then $\leqslant^*$ is a **pre-order**, i.e. reflexive and transitive. By duality, $\leqslant_*$ is a pre-order, too.* ○

Of course, $\leqslant^*$ is no partial order, since it is not anti-symmetric in general. Consider for a counterexample, e.g., the sets $\{1, 2, 3\}$ and $\{2, 3\}$ of real numbers. However, we can combine those two relations in the following natural way:

1.14 Definition Let D be a poset, and let $A, B \in \mathcal{P}_\emptyset(D)$. Then we write $A \leqslant_*^* B$ if both $A \leqslant_* B$ and $A \leqslant^* B$. ○

It is readily seen that $\leqslant_*^*$ is a pre-order. We further have the following result:

1.15 Proposition *Let D be a poset. Then $\leqslant^*$ and $\leqslant_*^*$ are anti-symmetric in the following sense:*

(i) If $A \leqslant^* B$ and $B \leqslant^* A$ for $A, B \subset D$, then $A = B$ if and only if A and B are decreasing sets in $A \cup B$.

(ii) If $A \leqslant_*^* B$ and $B \leqslant_*^* A$ for $A, B \subset D$, then $A = B$ if and only if A and B are order-convex sets in $A \cup B$.

Proof: Suppose $A \leqslant^* B$ and $B \leqslant^* A$. If $A = B$, then of course A is a decreasing set in $A \cup B = A$. The other way around: For any $a \in A$ there is some $b \in B$ such that $a \leqslant b$. If B is a decreasing set in $A \cup B$, then it follows $a \in B$, thus $A \subset B$. Since the problem is symmetric in A and B, (i) holds true.

Now, suppose $A \leqslant_*^* B$ and $B \leqslant_*^* A$. If $A = B$, then A is obviously order convex in $A \cup B = A$. The other way around: For any $a \in A$ there are $b, b' \in B$ such that $b \leqslant a \leqslant b'$. If B is order-convex in $A \cup B$, then it follows $a \in B$, thus $A \subset B$. Since the problem is symmetric in A and B, (ii) holds true. ○

It is illustrating to compare the pre-orders $\leqslant^*$ and $\leqslant_*$, which base directly on the partial order $\leqslant$, with the relations $\preccurlyeq^*$ and $\preccurlyeq_*$, which base on the lattice-operators in the following way:

1.16 Definition Let D be a poset.

(i) The relation $\preccurlyeq^*$ on $\mathcal{P}_\emptyset(D)$ is defined via

$$A \preccurlyeq^* B \quad :\Longleftrightarrow \quad a \vee b \in B \quad \text{for all } a \in A \text{ and all } b \in B,$$

and we read "$A \preccurlyeq^* B$" as "B **is superior to** A".

(i)$_d$ By duality, the relation $\preccurlyeq_*$ on $\mathcal{P}_\emptyset(D)$ is defined via

$$A \preccurlyeq_* B \quad :\Longleftrightarrow \quad a \wedge b \in A \quad \text{for all } a \in A \text{ and all } b \in B,$$

and we read "$A \preccurlyeq_* B$" as "A **is inferior to** B".

(ii) We write $A \preccurlyeq_*^* B$ if both $A \preccurlyeq_* B$ and $A \preccurlyeq^* B$. ○

In general, $\preccurlyeq^*$ is neither reflexive nor transitive and thus no pre-order. Indeed, $A \preccurlyeq^* A$ is the same as to say that A is closed under $\vee$, and on $\mathbb{R}$, equipped with the usual order, $\preccurlyeq^*$ is not transitive, since $\{1\} \preccurlyeq^* \{2\}$ and $\{2\} \preccurlyeq^* \{0, 2\}$, but $\{1\} \preccurlyeq^* \{0, 2\}$ does not hold. However, it is interesting to note that we have the following result:

1.17 Proposition *Let D be a poset. Then $\preccurlyeq_*^*$ is transitive on $\mathcal{P}(D)$.*

Proof: Let $A, C \in \mathcal{P}_{\emptyset}(D)$ and $B \in \mathcal{P}(D)$ be given such that $A \preccurlyeq^* B$ and $B \preccurlyeq^*_* C$. Then, for all $a \in A$, $b \in B$ and $c \in C$ one has $(a \vee (b \wedge c)) \vee c \in C$ and this element clearly is the supremum of a and c. Thus, $A \preccurlyeq^* C$. The remaining relation $A \preccurlyeq_* C$ is proved by duality. ○

The situation is different if D is a sup-semilattice; then $\preccurlyeq^*$ is a pre-order on the non-empty order-convex upward subsets of D, which follows from the next result.

1.18 Proposition *Let D be a poset and $A, B \in \mathcal{P}(D)$. Then $A \preccurlyeq^* B$ implies $A \leqslant^* B$. If D is a sup-semilattice and if B is order-convex upward, the reverse implication holds true, too.*

Proof: It is readily seen that $A \preccurlyeq^* B$ implies $A \leqslant^* B$, since for every $a \in A$ and each $b \in B$ we have $a \leqslant a \vee b \in B$. Now let D be a sup-semilattice and suppose $A \leqslant^* B$ with B being order-convex upward. Then for all $a \in A$ and $b \in B$ there is some $c \in B$ such that $a \leqslant c$ and it follows $b \leqslant a \vee b \leqslant c \vee b \in B$ and thus $a \vee b \in B$. ○

If in Proposition 1.18 the set B is not known to be order-convex upward but only closed under $\vee$, we have the following more technical equivalences:

1.19 Proposition *Let D be a poset and suppose $A, B \in \mathcal{P}(D)$ are closed under $\vee$. Then the following assertions hold true:*

(i) $A \preccurlyeq^* B$ *if and only if there is some $\vee$-closed set $C \subset D$ such that $A, B \subset C$ and $b \in B$ implies $b^{\uparrow}_{C} \subset B$.*

(ii) $A \preccurlyeq^* B$ *if and only if $A \leqslant^* B$ and there is some $\vee$-closed set $C \subset D$ such that $A, B \subset C$ and $b, b'' \in B$ imply $[b, b'']_C \subset B$.*

Especially, if D is a lattice and $B = [B_*, B^*]$, then $A \preccurlyeq^* B$ if and only if $A \leqslant^* B$.

Proof: Ad (i): Suppose $A \preccurlyeq^* B$, then $A \leqslant^* B$, the set $C := A \cup B$ is closed under $\vee$, and for all $b \in B$ we have $b^{\uparrow}_{C} \subset B$. Indeed, suppose $a \in A$ such that $b \leqslant a$, then $A \preccurlyeq^* B$ implies $a = a \vee b \in B$.

Now, let $C \subset D$ be closed under $\vee$ and suppose $A, B \subset C$ and $b^{\uparrow}_{C} \subset B$ for all $b \in B$. Then, for each $a \in A$ and each $b \in B$ we have $b \leqslant a \vee b \in C$ and thus $a \leqslant a \vee b \in B$.

Ad (ii): Suppose $A \preccurlyeq^* B$, then we have $A \leqslant^* B$ and as in (i) we set $C := A \cup B$ and obtain $[b, b'']_{A \cup B} \subset b^{\uparrow}_{A \cup B} \subset B$ for all $b, b'' \in B$.

Now, suppose $A \leqslant^* B$, let $C \subset D$ be closed under $\vee$ and suppose $A, B \subset C$ and $[b, b'']_C \subset B$ for all $b, b'' \in B$. Then, for each $a \in A$ there is $b' \in B$ such that $a \leqslant b'$, and for each $b \in B$ it follows $b \leqslant a \vee b \leqslant b' \vee b$. Since $b' \vee b \in B$ and $a \vee b \in C$, it follows $a \vee b \in B$ and thus $A \preccurlyeq^* B$. ○

1.1.3 Multifunctions of Isotone Type

For functions, we have the following well known notion:

1.20 Definition Let D and D' be posets, and let $f\colon D \to D'$ be a function. Then f is called **increasing** (or **isotone**, or **order-preserving**) if

$$x \leqslant y \quad \text{in } D \quad \Longrightarrow \quad f(x) \leqslant f(y) \quad \text{in } D'.$$

If from $x < y$ it follows $f(x) < f(x)$, f is called **strictly increasing**. By either dualizing the order in D or in D', one obtains the definition of a **(strictly) decreasing** function. ○

In what follows, we generalize Definition 1.20 to multifunctions, defined as follows:

1.21 Definition Let X and Y be sets. Then a **multifunction** is defined to be a mapping $F\colon X \to \mathcal{P}_\emptyset(Y)$. Its **domain** is defined by $\mathcal{D}(F) := \{x \in X : F(x) \neq \emptyset\}$, its **graph** is defined by $\operatorname{gr} F := \{(x, y) \in X \times Y : y \in F(x)\}$, and its **range** is defined by $F(X) := \bigcup\{F(x) : x \in X\}$. If $f\colon X \to Y$ is a function, we may identify f with the single-valued multifunction $F\colon X \to \mathcal{P}(Y)$ defined by $F(x) := \{f(x)\}$. If $G\colon X \to \mathcal{P}_\emptyset(Y)$ is also a multifunction, we write $F \subset G$ if $F(x) \subset G(x)$ for all $x \in X$. If $f \subset G$, i.e. if $f(x) \in G(x)$ for all $x \in X$, then f is called a **selection** of G. ○

By use of the set relations introduced in Definitions 1.10 and 1.14, let us define the following six kinds of monotone multifunctions:

1.22 Definition Let D and D' be posets, and let $F\colon D \to \mathcal{P}_\emptyset(D')$ be a multifunction.

(i) F is called **increasing upward** if $x \leqslant y$ implies $F(x) \leqslant^* F(y)$.

F is called **increasing downward** if $x \leqslant y$ implies $F(x) \leqslant_* F(y)$.

F is called **increasing** if $x \leqslant y$ implies $F(x) \leqslant_*^* F(y)$.

(ii) F is called **decreasing upward** if $x \leqslant y$ implies $F(y) \leqslant_* F(x)$.

F is called **decreasing downward** if $x \leqslant y$ implies $F(y) \leqslant^* F(x)$.

F is called **decreasing** if $x \leqslant y$ implies $F(y) \leqslant_*^* F(x)$. ○

1.23 Remark To apply duality with ease, note that if one replaces the order in D' by its dual, the naming of the properties chances from 'increasing' to 'decreasing' and vice versa. If one wants to swap 'upward' with 'downward', one has to dualize both the order in D and in D' (which might be counterintuitive). Consequently, for a multifunction $F\colon D \to \mathcal{P}_\emptyset(D')$, the following assertions are equivalent:

$$\begin{array}{lll} F\colon (D, \leqslant_D) \to \mathcal{P}_\emptyset(D', \leqslant_{D'}) & \text{is} & \text{increasing upward,} \\ F\colon (D, \geqslant_D) \to \mathcal{P}_\emptyset(D', \geqslant_{D'}) & \text{is} & \text{increasing downward,} \\ F\colon (D, \leqslant_D) \to \mathcal{P}_\emptyset(D', \geqslant_{D'}) & \text{is} & \text{decreasing upward,} \\ F\colon (D, \geqslant_D) \to \mathcal{P}_\emptyset(D', \leqslant_{D'}) & \text{is} & \text{decreasing downward.} \end{array}$$

Thus, it suffices to prove central properties only for increasing upward multifunctions and to rely for the other cases on duality.

Further, if D is an ordered linear space (see Definition 2.12 below), $(-A) \preccurlyeq^* (-B)$ holds if and only if $B \preccurlyeq_* A$. Consequently, the same effects as described above can be achieved by replacing F by $x \mapsto -F(-x)$, $x \mapsto -F(x)$ or $x \mapsto F(-x)$, respectively. ○

1.24 Remark Definition 1.22 generalizes the notion of increasing functions from the single-valued case to multifunctions. This is by no means a new idea, as, in 1971, fixed point theorems for increasing upward multifunctions were derived in [107] (there, those multifunctions were called mappings *satisfying condition I* or *order preserving* multifunctions). Little later, *decreasing* mappings were introduced in [34] and Proposition 1.26 below and further properties were established.

The naming of those multifunctions varies to this day in the vast recent literature. What we call increasing upward is called *upper order-preserving* in, e.g., [80] and *(1)-increasing* in [54]. (The last notion is clearly inspired by the notion $A \leqslant_1 B$ instead of our $A \leqslant^* B$). The naming used in this thesis follows the nomenclature of [24]. Note also that increasing upward multifunctions are used in semantic analysis of logic programs, see, e.g., [52]. There, they are called *Hoare monotonic* after Sir Tony Hoare. ○

1.25 Remark There is another way to generalize monotonicity of real functions to multifunctions, see Section 2.3. In order to distinguish both notions, we will call a multifunction with one of the properties presented in Definition 1.22 a **multifunction of isotone type**. ○

In applications, the values of $F\colon D \to \mathcal{P}_\emptyset(D')$ are often assumed to be order-intervals $[a, b]$, that is, there are single-valued functions $F_*, F^*\colon D \to D'$ called **envelopes** such that $F(x) = [F_*(x), F^*(x)]$. In this setting, the monotonic behavior of F is fully characterized by the monotonic behavior of F_* and F^*, as was noted for the most part in [34]. Let us inspect this connection next in a slightly more abstract setting.

1.26 Proposition *Let D and D' be posets, let $F\colon D \to \mathcal{P}(D')$ be a multifunction, and suppose that all values of F have the greatest element $F^*(x) \in F(x)$. Then F is increasing upward if and only if the function $F^*\colon D \to D'$ is increasing.*

Proof: Let F be increasing upward and $x \leqslant y$ in D. Then $F(x) \leqslant^* F(y)$ and thus there is some $z \in F(y)$ such that $F^*(x) \leqslant z \leqslant F^*(y)$. Consequently, F^* is increasing.

Conversely, if F^* is increasing and $z \in F(x)$, then $z \leqslant F^*(x) \leqslant F^*(y) \in F(y)$, thus F is increasing upward. ○

1.27 Corollary *Let D and D' be posets and let $F\colon D \to \mathcal{P}(D')$ be such that there are functions $F_*, F^*\colon D \to D'$ such that $F(x) = [F_*(x), F^*(x)]$. Then F is decreasing upward if and only if F_* is decreasing, F is increasing downward if and only if F_* is increasing, and F is decreasing downward if and only if F^* is decreasing.* ○

By use of $\preccurlyeq_*$ from Definition 1.16, we can define another six kinds of monotone multifunctions in analogy to Definition 1.22. Let us only define the prototype:

1.28 Definition Let D and D' be posets, and let $\mathsf{F}\colon \mathsf{D} \to \mathcal{P}_\emptyset(\mathsf{D}')$ be a multifunction. Then F is called **lattice-increasing upward** if $x \leqslant y$ implies $\mathsf{F}(x) \preccurlyeq^* \mathsf{F}(y)$. ○

From Proposition 1.18 we obtain at once the following corollary:

1.29 Corollary *Let D be a poset, let D' be a sup-semilattice, and suppose that all values of the multifunction $\mathsf{F}\colon \mathsf{D} \to \mathcal{P}(\mathsf{D}')$ are order-convex upward. Then F is increasing upward if and only if it is lattice-increasing upward.* ○

If the requirements of Corollary 1.29 are not met, one can use the following more general corollary, which can be deduced from Proposition 1.19.

1.30 Corollary *Let D and D' be posets, and let $\mathsf{F}\colon \mathsf{D} \to \mathcal{P}(\mathsf{D}')$ be a multifunction. Then the following assertions hold true:*

(i) If F is lattice-increasing upward, then F is increasing upward and its values are closed under $\vee$.

(ii) If F is increasing upward and its values are $\vee$-closed, then F need not to be lattice-increasing upward; however, if, e.g., there is some $\vee$-closed $\mathsf{C} \subset \mathsf{D}$ such that, for all $x \in \mathsf{D}$, $\mathsf{F}(x) \subset \mathsf{C}$ and $[a, b]_\mathsf{C} \subset \mathsf{F}(x)$ for all $a, b \in \mathsf{F}(x)$, then F is lattice-increasing upward.

Proof: Suppose first that F is lattice-increasing upward. Since $\mathsf{A} \preccurlyeq^* \mathsf{B}$ implies $\mathsf{A} \leqslant^* \mathsf{B}$, F is increasing upward. Furthermore, for all $a, b \in \mathsf{F}(x)$, $\mathsf{F}(x) \preccurlyeq^* \mathsf{F}(x)$ implies $a \vee b \in \mathsf{F}(x)$, thus (i) is proved.

Suppose now that F is increasing upward and that its values are $\vee$-closed. A neat example is the Σ-shaped mapping

$$\mathsf{F}\colon [0, 1] \subset \mathbb{R} \to \mathcal{P}(\mathbb{R}), \quad x \mapsto \mathsf{F}(x) = \{0, 1 - x, 1\}.$$

This map is additionally increasing downward, but neither lattice-increasing upward nor downward. But if there is some $\mathsf{C} \subset \mathsf{D}$ as demanded in (ii), then, by invoking Proposition 1.19, F is readily seen to be lattice-increasing upward. ○

As usual, the dual assertions of Proposition 1.19 and Corollary 1.30 hold true. Especially, if F has order-convex values, then F is increasing if and only if it is lattice-increasing, and if $\mathsf{F}(x) = [\mathsf{F}_*(x), \mathsf{F}^*(x)]$, the former holds if and only if both F^* and F_* are increasing (cf. with [68, Prop. 4.4, 4.5]).

1.31 Example Out of interest, we may add that there are increasing multifunctions whose values are not closed under $\vee$ (and that are thus not order-convex upward) even if they are topological closed. Consider, e.g., the multifunction

$$\mathsf{F}\colon \mathbb{R} \to \mathcal{P}(\mathbb{R}^2), \quad x \mapsto \mathsf{F}(x) = \big((-\infty, x] \times \{1\}\big) \cup \big([x, \infty) \times \{0\}\big).$$

Furthermore, one may consider the (rather exotic) mapping

$$\mathsf{F}\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \mathsf{F}(x) = x + [0, 1]_\mathbb{Q},$$

which is increasing, but not lattice-increasing.

Finally, let us consider the multifunction

$$F\colon (-\infty, 0] \subset \mathbb{R} \to \mathcal{P}([-1,0]), \quad x \mapsto -\mathfrak{C}_{-\lceil x \rceil},$$

where $\lceil x \rceil$ denotes the least integer $z \geqslant x$, and $\mathfrak{C}_n$ denotes the n-th set obtained by constructing the cantor set inductively as usual. This multifunction is obviously increasing and lattice-increasing upward, but not lattice-increasing downward. Since Corollary 1.30 is not applicable, one could generalize it using Proposition 1.19: F is lattice-increasing upward if and only if F is increasing upward, its values are closed under $\vee$ and $x \leqslant y$ in D implies that there is some $\vee$-closed set $C_{x,y}$ such that $F(x), F(y) \subset C_{x,y}$ and $[a,b]_{C_{x,y}} \subset F(y)$ for all $a, b \in F(y)$. ○

Finally, following [67], let us extend $\preccurlyeq^*_*$ to functionals $k\colon D \to \mathbb{R} \cup \{+\infty\}$ over a lattice D. To this end, consider sets $A, B \in \mathcal{P}_\emptyset(D)$ and their **indicator functions** I_A, I_B with

$$I_A\colon D \to \mathbb{R} \cup \{+\infty\}, \quad x \mapsto \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Then we have $A \preccurlyeq^* B$ if and only if $I_A(a) < \infty$ and $I_B(b) < \infty$ imply $I_B(a \vee b) < \infty$. If one developes this idea further with regard to the later applications, on arrives at the following definition:

1.32 Definition Let D be a lattice. Then, for $k, K\colon D \to \mathbb{R} \cup \{+\infty\}$, we define

$$k \preccurlyeq^*_* K \quad :\Longleftrightarrow \quad k(u \wedge v) + K(u \vee v) \leqslant k(u) + K(v) \quad \text{for all } u, v \in D. \qquad ○$$

On readily verifies that $k \preccurlyeq^*_* K$ implies $\mathcal{D}(k) \preccurlyeq^*_* \mathcal{D}(K)$, where $\mathcal{D}(k)$ denotes the **effective domain** $\{u \in D : k(u) \neq \infty\}$ of k. In particular, $A \preccurlyeq^*_* B$ holds if and only if the indicator functions of A and B are related by $I_A \preccurlyeq^*_* I_B$. Further, we have the following result for **distributive lattices** (which are lattices, in which $\wedge$ and $\vee$ distribute over each other):

1.33 Proposition *Let D be a distributive lattice. If functions $j, k, l\colon D \to \mathbb{R} \cup \{+\infty\}$ satisfy $j \preccurlyeq^*_* k \preccurlyeq^*_* k \preccurlyeq^*_* l$ and if $\mathcal{D}(k) \neq \emptyset$, then it holds $j \preccurlyeq^*_* l$.*

Proof: For arbitrary $a, c \in D$ we claim

$$j(a \wedge c) + l(a \vee c) \leqslant j(a) + l(c). \tag{1.1}$$

We can assume $a \in \mathcal{D}(j)$ and $c \in \mathcal{D}(l)$. Further, let $b \in \mathcal{D}(k)$. Then by the relations given and from $a \wedge c = a \wedge (c \wedge (a \vee b))$ it follows

$$j(a \wedge c) + k(a \vee (c \wedge (a \vee b))) \leqslant j(a) + k(c \wedge (a \vee b)). \tag{1.2}$$

By dual reasoning, we further have

$$l(c \vee a) + k(c \wedge (a \vee (c \wedge b))) \leqslant l(c) + k(a \vee (c \wedge b)). \tag{1.3}$$

All terms in (1.2) and (1.3) are finite (recall that $\mathcal{D}(j) \preccurlyeq^*_* \mathcal{D}(k) \preccurlyeq^*_* \mathcal{D}(l)$), and thus (1.1) follows from $k \preccurlyeq^*_* k$ and the identity

$$[a \vee (c \wedge (a \vee b))] \wedge [c \wedge (a \vee (c \wedge b))] = c \wedge (a \vee b)$$

together with its dual counterpart. ○

1.2 Fixed Point Theorems on Posets

Let $f\colon X \to X$ be any function on a set X. Then any point $x \in X$ such that $x = f(x)$ is called **fixed point** of f. This notion generalizes to multifunctions:

1.34 Definition Let $F\colon X \to \mathcal{P}_\emptyset(X)$ be a multifunction on a set X. Then $u \in X$ is called a **fixed point** of F if $u \in F(u)$. The **set of all fixed points** of F is denoted by $\operatorname{Fix} F$. ○

The aim of this section is to investigate the set $\operatorname{Fix} F$ of fixed points of a multifunction $F\colon D \to \mathcal{P}_\emptyset(D)$, defined on a poset D. In doing so, we provide conditions such that

- F has at least one fixed point,
- F has a maximal fixed point u (i.e. for every $v \in \operatorname{Fix} F$, $u \leqslant v$ implies $u = v$),
- F has the greatest fixed point u^* (i.e. for every $v \in \operatorname{Fix} F$ it holds $v \leqslant u^*$).

The order-theoretical conditions we impose on the domain D and the multifunction F will be of different generality, but all fixed point theorems have in common that they assume the existence of a subpoint, which is defined as follows:

1.35 Definition Let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction on a poset D. Then $\underline{u} \in D$ is called a **subpoint** of F if $\underline{u} \leqslant^* F(\underline{u})$. The **set of all subpoints** of F is denoted by $\operatorname{Sub} F$. ○

(In Definition 1.35 and below, $a \leqslant^* A$ is the same as $\{a\} \leqslant^* A$; analogously for other set-relations.)

Obviously, every fixed point of any multifunction $F\colon D \to \mathcal{P}_\emptyset(D)$ is also a subpoint, i.e. $\operatorname{Fix} F \subset \operatorname{Sub} F$. In the following, we will consider cases in which, poetically speaking, each subpoint generates an ascending family of subpoints that can only be stopped if one of these subpoints happens to be a fixed point.

1.2.1 Tarski Fixed Point Theorems

Probably the most widely known order-theoretical fixed point theorem was introduced by Tarski [109] for a single-valued increasing function $f\colon D \to D$. There are numerous versions of this theorem and further generalizations to multifunctions, as we will see below. In order to examine the common core of those theorems, let us first provide two useful results about the connection of fixed points and subpoints of increasing upward multifunctions:

1.36 Proposition *Let D be a poset, let $F\colon D \to \mathcal{P}_\emptyset(D)$ be an increasing upward multifunction, and let u be any subpoint of F. Then there is a subpoint s of F such that $u < s$, or u is a fixed point of F.*

Proof: Let $u \in \operatorname{Sub} F$ be given. Then we have $u \leqslant^* F(u)$ and thus there is $s \in F(u)$ such that $u \leqslant s$. Since F is increasing upward, it follows $F(u) \leqslant^* F(s)$, and, since $s \in F(u)$, we have especially $s \leqslant^* F(s)$, i.e. $s \in \operatorname{Sub} F$. If there is no such s with $u < s$, we have $u = s \in F(u)$, i.e. $u \in \operatorname{Fix} F$. ○

1.37 Proposition *Let D be a poset, and let $\mathsf{F}\colon \mathsf{D} \to \mathcal{P}_\emptyset(\mathsf{D})$ be an increasing upward multifunction. Then the following assertions hold true:*

(i) Every maximal subpoint of F is a maximal fixed point of F.

(ii) If the greatest subpoint of F exists, then it is the greatest fixed point of F.

Proof: Let u^* be a maximal subpoint of F. Then $u^* \in \operatorname{Fix} \mathsf{S}$, since otherwise, by Proposition 1.36, there would be a subpoint s of F such that $u^* < s$, which contradicts the maximality of u^*. Further, from $\operatorname{Fix} \mathsf{F} \subset \operatorname{Sub} \mathsf{F}$ it follows that u^* is a maximal fixed point of F, which proves (i). If u^* is even the greatest subpoint of F, it follows again from $\operatorname{Fix} \mathsf{F} \subset \operatorname{Sub} \mathsf{F}$ that u^* is the greatest fixed point of F. ○

To prove the existence of a maximal (or greatest) fixed point of an increasing upward multifunction F, it thus suffices to prove the existence of a maximal (or greatest) subpoint—which is possible under further order-theoretical conditions on D or F.

First, let us recall a variant of the original theorem of Tarski for complete lattices.

1.38 Definition A poset D is called a **sup-complete lattice** if all $\mathsf{M} \in \mathcal{P}(\mathsf{D})$ have the supremum $\sup \mathsf{M} \in \mathsf{D}$. ○

1.39 Theorem (Tarski) *Let D be a sup-complete lattice, and let $\mathsf{f}\colon \mathsf{D} \to \mathsf{D}$ be an increasing function. If f has a subpoint, then f has the greatest fixed point u^*, which is also the greatest subpoint of f.*

Proof: We are going to prove that $\operatorname{Sub} \mathsf{F}$ has the greatest element. To this end, note that $\operatorname{Sub} \mathsf{f}$ is a non-empty subset of the complete poset D, and define

$$u^* := \sup(\operatorname{Sub} \mathsf{f}).$$

By definition, we have, for all $u \in \operatorname{Sub} \mathsf{f}$, $u \leqslant u^*$, and since f is increasing, it follows $u \leqslant \mathsf{f}(u) \leqslant \mathsf{f}(u^*)$. That means $\operatorname{Sub} \mathsf{f} \leqslant^* \mathsf{f}(u^*)$, so that $\mathsf{f}(u^*)$ is seen to be an upper bound of $\operatorname{Sub} \mathsf{f}$. But u^* is the least upper bound of $\operatorname{Sub} \mathsf{f}$, thus $u^* \leqslant \mathsf{f}(u^*)$, i.e. $u^* \in \operatorname{Sub} \mathsf{f}$. Evidently, u^* is the greatest element of $\operatorname{Sub} \mathsf{f}$. By Proposition 1.37, u^* is the greatest fixed point of f. ○

Now, let us generalize Theorem 1.39 to the case of multifunctions $\mathsf{F}\colon \mathsf{D} \to \mathcal{P}_\emptyset(\mathsf{D})$ with some technical Property (X):

1.40 Corollary *Let D be a sup-complete lattice, and let $\mathsf{F}\colon \mathsf{D} \to \mathcal{P}_\emptyset(\mathsf{D})$ be an increasing upward multifunction. If F has a subpoint and Property (X), given by*

(X) For all $u \in \mathsf{D}$ such that $\operatorname{Sub} \mathsf{F} \leqslant^* \mathsf{F}(u)$ there is $s \in \mathsf{F}(u)$ such that $\operatorname{Sub} \mathsf{F} \leqslant^* s$,

then F has the greatest fixed point u^*, which is also the greatest subpoint of F.

Proof: We set $u^* := \sup(\operatorname{Sub} \mathsf{F})$ and proceed as in the proof of Theorem 1.39 to obtain $\operatorname{Sub} \mathsf{F} \leqslant^* \mathsf{F}(u^*)$. Thus, from (X) we obtain $\operatorname{Sub} \mathsf{F} \leqslant^* s^*$ for some $s^* \in \mathsf{F}(u^*)$. It follows that u^* is the greatest element of $\operatorname{Sub} \mathsf{F}$ and, thanks to Proposition 1.37, also the greatest fixed point of F. ○

Property (X) is formulated in such a way that the idea of the proof of Theorem 1.39 can be used, but it may be a tedious task to check if (X) holds. If one searches for an easy to check Property (X') which implies (X), one could find the following one:

(X') All values of F are *completely directed upward*, i.e. for all $A \subset D$ and $x \in D$ it follows from $A \leqslant^* F(x)$ that there is $b \in F(x)$ such that $A \leqslant^* b$.

Note, however, that (X') is equivalent to the property that all values of F have either the greatest element or are empty (to see this, let $A = F(x)$). In this case, we can give another proof:

1.41 Corollary *Let D be a sup-complete lattice, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be an increasing upward multifunction. If F has a subpoint and if all values of F have either the greatest element $F^*(x) \in F(x)$ or are empty, then F has the greatest fixed point u^*, which is also the greatest subpoint of F.*

Proof: We can assume that all values of F have the greatest element (otherwise, we replace D by the sup-complete lattice $\bigcup\{d^\uparrow : d \in D \text{ and } F(d) \neq \emptyset\}$). Further, it is readily seen that

$$\operatorname{Fix} F^* \subset \operatorname{Fix} F \subset \operatorname{Sub} F = \operatorname{Sub} F^*.$$

Now, due to Proposition 1.26, $F^*\colon D \to D$ is increasing, and since F^* has a subpoint like F, it follows from Theorem 1.39 that F^* has the greatest fixed point, which is also the greatest element of $\operatorname{Sub} F^* = \operatorname{Sub} F$ and thus the greatest fixed point of F. ○

What we have proved so far is summed up in Figure 1.1 (there, a theorem is followed by a ○ if the marked result is shown in the proof of the theorem).

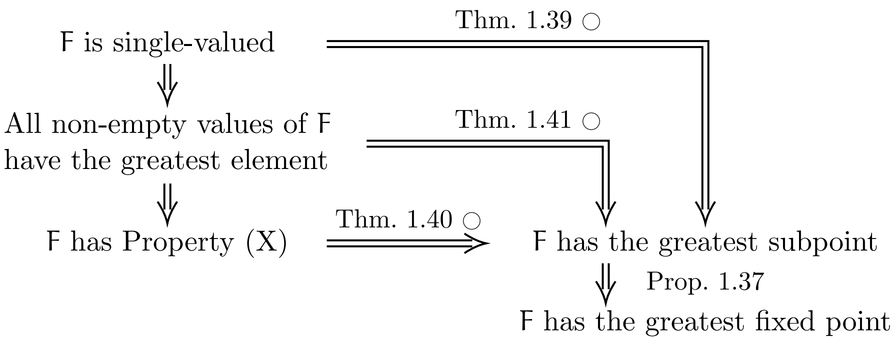

Fig. 1.1: The situation in case D is a sup-complete lattice, $F\colon D \to \mathcal{P}_\emptyset(D)$ is increasing upward, and F has a subpoint.

To be a sup-complete lattice is a rather strong property for a poset, thus there is need for fixed point theorems with weaker assumptions on the underlying poset. To this end, let us recall the results of [76, 77, 78, 79] and restate them with our notation. They are not as general as possible (as we will see below), but the proof is lucid and the theorem would be in fact enough for the applications presented in Part II of this thesis.

Up to now, the main idea was to show that Sub F has a greatest element. Now, we are going to give conditions that guarantee that Sub F has at least a maximal element. To this end, let us recall the famous Lemma of Zorn.

1.42 Definition Let D be a poset.

(i) D is called a **chain** if all elements of D are comparable, i.e. for all $a, b \in D$ it holds either $a < b$, $a = b$ or $b < a$.

(ii) D is called **well-ordered** if each $M \in \mathcal{P}(D)$ has the smallest element.

(iii) D is called **inductive** if each well-ordered subset of D has an upper bound in D. ○

1.43 Lemma (Zorn) *Let D be an inductive poset, and let $\underline{u} \in D$. Then D has a maximal element u^* such that $\underline{u} \leqslant u^*$.* ○

1.44 Remark Zorn's Lemma was introduced as an axiom in [124] (although the ideas circulated even earlier, see [86]), and is equivalent to the Axiom of Choice. A simple proof of Zorn's Lemma from the Axiom of Choice can be found in [75]. See also the Appendix of [18], where a proof via the Bourbaki-Witt fixed point Theorem is presented.

Note, however, that in literature, Zorn's Lemma is usually stated with the presumption that not only well-ordered subsets of D have an upper bound, but all chains. As a consequence thereof, inductive sets are usually defined to be sets in which each chain has an upper bound. However, as noted in [108, Remark 14], Zorn's Lemma in its usual formulation can be strengthened lightly to obtain Lemma 1.43 (to this end, note that for an inductive set D, for each $u \in D$ also the set $u_D^{\uparrow}$ is inductive). We thus followed the spirit of [13] and defined inductive posets in such a way that inductive sets are those sets for which Zorn's Lemma is applicable.

Note moreover that an inductive poset D is always non-empty, since $\emptyset$ is a well-ordered subset of D and thus there has to be an upper bound of $\emptyset$ in D (and, in fact, every element of D is an upper bound of $\emptyset$). ○

From Lemma 1.43 and Proposition 1.37 we have at once the following theorem:

1.45 Theorem *Let D be a poset, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be an increasing upward multifunction. If* Sub F *is inductive, then for each subpoint $\underline{u}$ of F there is a maximal fixed point u^* of F such that $\underline{u} \leqslant u^*$.* ○

However, it is not easy to check if Sub F is inductive. Thus, we once again introduce a handy property, which is the property given in [77] with the difference that we consider not all chains, but only well-ordered ones.

1.46 Definition Let D be a poset. A subset $B \in \mathcal{P}_\emptyset(D)$ is called **universally inductive** in D if for any well-ordered set $A \in \mathcal{P}_\emptyset(D)$ such that $A \leqslant^* B$ there is $b \in B$ such that $A \leqslant^* b$. ○

Note that the definition of universally inductive sets closely resembles the definition of completely directed upward sets we introduced in Property (X') above. The important difference is that A is now assumed to be well-ordered, so that a universally inductive set B is only forced to have a greatest element if B is well-ordered, too.

The following theorem slightly generalizes [77, Theorem 3.4] which presumes that all non-empty chains have a supremum. However, the proof follows the same lines, and the calculations are very similar to the ones presented in the proof of Theorem 1.39.

1.47 Theorem *Let D be a poset in which each non-empty well-ordered set has the supremum, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be an increasing upward multifunction. If F has a subpoint $\underline{u}$ and if all values of F are universally inductive in D, then F has a maximal fixed point u^* such that $\underline{u} \leqslant u^*$.*

Proof: In view of Theorem 1.45 we have only to show that $\operatorname{Sub} F$ is inductive. To this end, let $A \subset \operatorname{Sub} F$ be any well-ordered set. If $A = \emptyset$, then $\underline{u} \in \operatorname{Sub} F$ is an upper bound of A. Otherwise, $\alpha := \sup A$ exists. Then, for all $a \in A$, we have $a \leqslant \alpha$, and since F is increasing upward, it follows $a \leqslant^* F(a) \leqslant^* F(\alpha)$. Since $F(\alpha)$ is universally inductive, there is $\beta \in F(\alpha)$ such that $A \leqslant^* \beta$, which implies $\alpha \leqslant \beta \in F(\alpha)$, i.e. $\alpha \in \operatorname{Sub} F$ is an upper bound of A. Therefore, $\operatorname{Sub} F$ is inductive. ○

The new results are summed up in Figure 1.2:

Each well-ordered set in D has the supremum
and all values of F are universally inductive

$\Downarrow$ Thm. 1.47 ○

$\operatorname{Sub} F$ is inductive

$\Downarrow$ Thm. 1.43

F has a maximal subpoint above $\underline{u}$

$\Downarrow$ Prop. 1.37

F has a maximal fixed point above $\underline{u}$

Fig. 1.2: The situation in case D is a poset, $F\colon D \to \mathcal{P}_\emptyset(D)$ is increasing upward, and F has a subpoint $\underline{u}$.

For applications it is convenient to have a topological criterion that guarantees that a set is universally inductive, such as [77, Lemma 3.6], which states that every nonempty compact subset of an *ordered Hausdorff space* is universally inductive. Thus, in Section 2.1 we will examine our fixed point theorems in the context of ordered topological spaces. But next let us bring purely order-theoretical fixed point theorems to the next level.

1.2.2 Maximal Fixed Points

Let $F, S\colon D \to \mathcal{P}_\emptyset(D)$ be multifunctions such that $S \subset F$. Then we have $\operatorname{Fix} S \subset \operatorname{Fix} F$, so that F has a fixed point if S satisfies the requirements of any fixed point theorem, say

Theorem 1.45. In the following, we are going to extend this principle in such a way, that a multifunction F is known to have a maximal fixed point if only some increasing parts of F behave well. To make this more clear, let us first provide the following proposition about selections:

1.48 Proposition *Let D be a poset, let $F\colon D \to \mathcal{P}(D)$ be an increasing upward multifunction, and let $s\colon D \to D$ be a single-valued increasing function such that $s \subset F$ (i.e. $s(d) \in F(d)$ for all $d \in D$). Then the following assertions hold true:*

(i) If $U \subset D$ is well-ordered, then $s(U)$ is well-ordered.

(ii) If $\operatorname{Sub} F$ is inductive, then for any well-ordered set $U \subset \operatorname{Sub} s$ the set $s(U)$ has an upper bound in $\operatorname{Sub} F$.

Proof: To proof (i), let $U \subset D$ be well-ordered and let $M \subset s(U)$ be non-empty. Then $s^{-1}(M)$ is non-empty as well and has the smallest element a. Then it follows $s(a) \leqslant s(b)$ for all $b \in s^{-1}(M)$, i.e. $s(a)$ is the smallest element of M.

To prove (ii), note first that $u \in \operatorname{Sub} s$ is the same as to say $u \leqslant s(u)$, from which it follows $s(u) \in F(u) \leqslant^* F(s(u))$, i.e. $s(u) \in \operatorname{Sub} F$. By (i) and the inductivity of $\operatorname{Sub} F$ the assertion follows. ○

Thus, we have proved that for any multifunction F satisfying the conditions of Theorem 1.45 we have that $\operatorname{Sub} F$ is F-inductive in the following sense:

1.49 Definition Let D be a poset, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction.

(i) A subset $V \subset D$ is called F-**selected** if there is a well-ordered set $U \subset \operatorname{Sub} F$ and a strictly increasing, bijective function $s\colon U \to V$ such that

$$u \leqslant s(u) \in F(u) \quad \text{for all } u \in U.$$

(ii) $\operatorname{Sub} F$ is called F-**inductive** if each F-selected set $V \subset D$ has an upper bound $s^* \in \operatorname{Sub} F$. ○

We have the following elementary result about multifunctions F whose set of subpoints is F-inductive:

1.50 Proposition *Let D be a poset, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction such that $\operatorname{Sub} F$ is F-inductive. Then the following holds true:*

(i) F has a subpoint.

(ii) Every maximal subpoint of F is a maximal fixed point of F (cf. Proposition 1.37).

Proof: Ad (i): The empty function $s\colon \emptyset \to \emptyset$ is strictly increasing, surjective, and it holds $u \leqslant s(u) \in F(u)$ for all $u \in \emptyset$. Thus, $\emptyset$ is an F-selected subset of $\operatorname{Sub} F$, and since $\operatorname{Sub} F$ is F-inductive, it follows that $\emptyset$ has an upper bound in $\operatorname{Sub} F$, i.e. $\operatorname{Sub} F$ is non-empty.

Ad (ii): Let $s^* \in \operatorname{Sub} F$ be maximal and let $a \in D$ be such that $s^* \leqslant a \in F(s^*)$. By setting $U := \{s^*\}$ and defining $s\colon U \to \{a\}$ trivially by $s(s^*) := a$, we see that $\{a\}$

is an F-selected subset of the F-inductive set $\operatorname{Sub} F$. Thus, there is $\bar{s} \in \operatorname{Sub} F$ such that $s^* \leqslant a \leqslant \bar{s}$. From the maximality of s^* it then follows $s^* = a = \bar{s}$, and so $s^* \in F(s^*)$, meaning that s^* is a fixed point of F. Moreover, from $\operatorname{Fix} F \subset \operatorname{Sub} F$ it follows that s^* is also maximal in $\operatorname{Fix} F$. ○

The aim of this subsection is to prove the following fixed point theorem, that generalizes Theorem 1.45:

1.51 Theorem *Let D be a poset, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction. If $\operatorname{Sub} F$ is F-inductive, then for each subpoint $\underline{u}$ of F there is a maximal fixed point u^* of F such that $\underline{u} \leqslant u^*$.*

To prove this central theorem (on page 40 below), we proceed in three steps:

1. We recall the comparison principle for well-ordered sets, Proposition 1.55.
2. We recall the chain generating principle, Lemma 1.58, introduced in [50].
3. We deduce from Lemma 1.58 the very general fixed point Theorem 1.59, from which Theorem 1.51 then follows easily like in [24].

Although those results are already known in literature, we decided to give full-length proofs for good reasons: The proofs nicely illustrate the concepts of order theory, our notations differ in some aspects from the notations in literature, and—most important—Theorem 1.51 is the theoretical core of this thesis and should thus be proved as comprehensible as possible.

1.52 Remark The subsequent proofs make no use of transfinite induction or Zorn's Lemma, but need only elementary tools of set theory. In order to prove Proposition 1.55 and Lemma 1.58, we even do not need the Axiom of Choice. However, we have to use the Axiom of Choice in order to prove Theorem 1.59. ○

First, let us recall that well-ordered posets have the very useful property that, given two of them, one can mapped via an increasing bijection (which is readily seen to be strictly increasing) to the first elements of the other. To make this more clear, let us use the following order-theoretical notations:

1.53 Definition For a poset D and $a \in D$ we call

$$a^{\downarrow\downarrow} := \{x \in D : x \leqslant a \text{ and } x \neq a\} = \{x \in D : x < a\}$$

the **initial segment** generated by a. If $M \subset D$, we set $a_M^{\downarrow\downarrow} := a^{\downarrow\downarrow} \cap M$. ○

1.54 Definition Two posets D and D' are called **isomorphic** (as posets) if there is an increasing bijection $\varphi\colon D \to D'$ whose inverse φ^{-1} is also increasing. In this case, we write $D \sim D'$. ○

With these notations, the basic comparison principle for well-ordered sets reads as follows:

1.55 Proposition *Let* A *and* B *be well-ordered posets. Then it holds* $A \sim B$ *or there is* $\beta \in B$ *such that* $A \sim \beta^{\downarrow\downarrow}$ *or there is* $\alpha \in A$ *such that* $\alpha^{\downarrow\downarrow} \sim B$.

The proof of Proposition 1.55 is well-known and roughly sketched in, e.g., [47]. However, we will give it here in full length, preceded by two basic propositions:

1.56 Proposition *Let* D *be a chain, let* D' *be a poset and let* $\varphi\colon D \to D'$ *be an increasing bijection. Then* $\varphi^{-1}\colon D' \to D$ *is an increasing bijection, too, and for all* $a \in D$ *it holds* $\varphi(a^{\downarrow\downarrow}) = \varphi(a)^{\downarrow\downarrow}$.

Proof: To prove that φ^{-1} is increasing, let $a' < b'$ in D'. Then it does not hold $\varphi^{-1}(a') > \varphi^{-1}(b')$, since otherwise monotonicity of φ would imply $a' > b'$. Thus, since all elements in D are comparable, we have $\varphi^{-1}(a') < \varphi^{-1}(b')$. Consequently, φ^{-1} is increasing.

The second assertion is an easy consequence of the monotonicity of φ and φ^{-1}: For each $b' \in \varphi(a^{\downarrow\downarrow})$ there is $a' < a$ such that $b' = \varphi(a') < \varphi(a)$, thus $b' \in \varphi(a)^{\downarrow\downarrow}$ and, consequently, $\varphi(a^{\downarrow\downarrow}) \subset \varphi(a)^{\downarrow\downarrow}$. The reversed inclusion is obtained by taking any $b' \in \varphi(a)^{\downarrow\downarrow}$ and applying φ^{-1} to $b' < \varphi(a)$. Since φ^{-1} is strictly increasing, we obtain $a' := \varphi^{-1}(b') < a$ and thus $b' = \varphi(a') \in \varphi(a^{\downarrow\downarrow})$. ○

1.57 Proposition *Let* D *be a well-ordered set. Then each decreasing subset* A *of* D *either equals* D *or is an initial segment of* D.

Proof: Let $A \subset D$ be decreasing and assume that $D \setminus A$ is non-empty. Then the smallest element β of $D \setminus A$ exists and we claim $A = \beta^{\downarrow\downarrow}$. Indeed, for all $a < \beta$ we have $a \in A$ per definition of β, and from $a \in A$ it follows $a < \beta$, since otherwise we would have $\beta \leqslant a$ and thus, since A is decreasing, $\beta \in A$, which contradicts the choice of β. ○

Now, let us provide the promised proof of Proposition 1.55:

Proof of Proposition 1.55: Let A and B be well-ordered posets, and define the multi-function

$$\psi\colon A \to \mathcal{P}_{\emptyset}(B), \quad a \mapsto \{b \in B : a^{\downarrow\downarrow} \sim b^{\downarrow\downarrow}\}.$$

We claim that ψ has the following properties:

(i) ψ is increasing downward,

(ii) $\psi(A)$ is a decreasing set,

(iii) the values of ψ have at most one element.

To prove (i), suppose $a_1 < a_2$ in A and let $b_2 \in \psi(a_2)$ be given. Then there is an increasing bijection $\varphi_2\colon a_2^{\downarrow\downarrow} \to b_2^{\downarrow\downarrow}$. Since $a_1 < a_2$, $b_1 := \varphi_2(a_1) < b_2$ is well-defined, and we conclude that the function

$$\varphi_1\colon a_1^{\downarrow\downarrow} \to b_1^{\downarrow\downarrow}, \quad x \mapsto \varphi_2(x)$$

is well-defined, increasing and, thanks to Proposition 1.56, bijective. Thus, we have shown $a_1^{\downarrow\downarrow} \sim b_1^{\downarrow\downarrow}$, which means $b_1 \in \psi(a_1)$. Consequently, ψ is increasing downward.

To prove (ii), let $b_2 \in \psi(A)$ and $b_1 \in b_2^{\downarrow\downarrow}$ be given. Then there is $a_2 \in A$ such that $b_2 \in \psi(a_2)$, which implies that there is an increasing bijection $\varphi_2 \colon b_2^{\downarrow\downarrow} \to a_2^{\downarrow\downarrow}$. Then, as above, there is $a_1 \in A$ such that $b_1^{\downarrow\downarrow} \sim a_1^{\downarrow\downarrow}$, implying $b_1 \in \psi(A)$. Thus, $\psi(A)$ is decreasing.

Finally, to prove (iii), suppose $b_1, b_2 \in \psi(a)$ for some $a \in A$. Then there are increasing bijections $\varphi_1 \colon a^{\downarrow\downarrow} \to b_1^{\downarrow\downarrow}$ and $\varphi_2 \colon a^{\downarrow\downarrow} \to b_2^{\downarrow\downarrow}$ and we infer that

$$\varphi := \varphi_2 \circ \varphi_1^{-1} \colon b_1^{\downarrow\downarrow} \to b_2^{\downarrow\downarrow}$$

is an increasing bijection. If $b_1 \neq b_2$, we may assume $b_2 < b_1$, i.e. $b_2 \in b_1^{\downarrow\downarrow}$. Since $b_2 \notin b_2^{\downarrow\downarrow}$, we have $b_2 \neq \varphi(b_2)$. Now, let β be the smallest element of $b_1^{\downarrow\downarrow}$ such that $\beta \neq \varphi(\beta)$. Then we have $\beta < \varphi(\beta)$, as otherwise we would have $\varphi(\beta) < \beta$ and thus, due to the definition of β and the monotonicity of φ, $\varphi(\beta) = \varphi(\varphi(\beta)) < \varphi(\beta)$, which is ridiculous. But from $\beta < \varphi(\beta) \in b_2^{\downarrow\downarrow}$ it follows $\beta < b_2$ and thus there has to be $\gamma \in b_1^{\downarrow\downarrow}$ such that $\beta = \varphi(\gamma)$. But this is not possible, neither for $\gamma < \beta$ (for which $\varphi(\gamma) = \gamma < \beta$) nor $\gamma = \beta$ (since $\beta \neq \varphi(\beta)$) nor $\gamma > \beta$ (for which, by monotonicity of φ, $\varphi(\gamma) > \varphi(\beta)$ in contrast to $\beta < \varphi(\beta)$). Thus, our assumption $b_1 \neq b_2$ does not hold and we have proved that the values of ψ have at most one element.

Now, let us define

$$A_0 := \{a \in A : \psi(a) \neq \emptyset\}.$$

In consequence of (i) and (iii), we can interpret $\psi \colon A_0 \to \mathcal{P}(B)$ as an increasing, single-valued function. Furthermore, the proof of (i) reveals that ψ is even strictly increasing, so that $\psi \colon A_0 \to \psi(A_0)$ is an increasing bijection, i.e. $A_0 \sim \psi(A_0)$.

Now note that A_0 is a decreasing subset of A (which follows readily from (i)) and that $\psi(A_0)$ is a decreasing subset or B (which follows readily from (i) and (ii)). Thus, in view of Proposition 1.57, we have one of the following three cases:

(i) $A_0 = A$ and $\psi(A_0) = B$, thus $A \sim B$.

(ii) $A_0 = A$ and $\psi(A_0) \neq B$, thus $A \sim \beta^{\downarrow\downarrow}$ for some $\beta \in B$.

(iii) $A_0 \neq A$ and $\psi(A_0) = B$, thus $\alpha^{\downarrow\downarrow} \sim B$ for some $\alpha \in A$.

The forth conceivable case $A_0 \neq A$ and $\psi(A_0) \neq B$ is not possible, since in this case we would have $\psi(\alpha^{\downarrow\downarrow}) = \psi(A_0) = \beta^{\downarrow\downarrow}$ for some $\alpha \in A \setminus A_0$ and $\beta \in B$, thus $\beta \in \psi(\alpha)$ per definition of ψ, which contradicts $\alpha \notin A_0$. ○

With help of this comparison principle, we can prove the following chain generating principle like in [24]. Its merit is that it provides a tool to generate a unique chain with a nice order-theoretical property (which will be used to prove Theorem 1.59).

1.58 Lemma *Let* D *be a poset, let* $\mathcal{D} \subset \mathcal{P}_{\emptyset}(D)$ *be a family of subsets of* D *with* $\emptyset \in \mathcal{D}$, *and let* $d\colon \mathcal{D} \to D$ *be a single-valued function. Then there is a unique well-ordered set* $C \in \mathcal{P}(D)$ *such that*

$$c \in C \quad \textit{if and only if} \quad c = d(c^{\downarrow\downarrow}_{C}). \tag{1.4}$$

Furthermore, if $C \in \mathcal{D}$, *then* $d(C)$ *is not a strict upper bound of* C.

Proof: Let us call $A \in \mathcal{P}(D)$ a **d-set** if A is well-ordered and if

$$\alpha \in A \quad \text{implies} \quad \alpha = d(\alpha^{\downarrow\downarrow}_{A}). \tag{1.5}$$

Note that there are d-sets, e.g. $\{d(\emptyset)\}$. Our aim is to show that the union of all d-sets is the well-ordered set $C \in \mathcal{P}(D)$ we are searching for. To this end, let us first prove that by use of Property (1.5) we can strengthen Proposition 1.55 in the following way:

(D) For all d-sets A and B it either holds $A = B$ or there is $\beta \in B$ such that $A = \beta^{\downarrow\downarrow}_{B}$ or there is $\alpha \in A$ such that $\alpha^{\downarrow\downarrow}_{A} = B$.

Indeed, for d-sets A and B we have, due to Proposition 1.55, $\varphi\colon A \to B$ or $\varphi\colon A \to \beta^{\downarrow\downarrow}_{B}$ or $\varphi\colon \alpha^{\downarrow\downarrow}_{A} \to B$ for suitable elements $\beta \in B$ or $\alpha \in A$ and an increasing bijection φ. In each of these cases, we are done if $\varphi(a) = a$ for all $a \in A_0$, A_0 being the domain of φ. So let us assume that $M := \{a \in A_0 : a \neq \varphi(a)\} \subset A$ is non-empty and let γ be the smallest element of M. Then we have, due to Proposition 1.56,

$$\gamma^{\downarrow\downarrow}_{A_0} = \varphi(\gamma^{\downarrow\downarrow}_{A_0}) = \varphi(\gamma)^{\downarrow\downarrow}_{\varphi(A_0)}. \tag{1.6}$$

But A_0 and $\varphi(A_0)$ are readily seen to be d-sets like A and B (in fact, each descreasing subset of a d-set is a d-set, too), so that from (1.6) we have

$$\gamma = d(\gamma^{\downarrow\downarrow}_{A_0}) = d(\varphi(\gamma)^{\downarrow\downarrow}_{\varphi(A_0)}) = \varphi(\gamma),$$

which contradicts $\gamma \in M$. Thus, M is empty and Property (D) of d-sets holds true.

Now, let us prove that the union of all d-sets

$$C := \bigcup \{A \in \mathcal{P}(D) : A \text{ is a } d\text{-set}\}$$

has all demanded properties: First, to prove that C is well-ordered, let $N \subset C$ be non-empty. Then let A be a d-set such that $N \cap A$ is non-empty and let α be the smallest element of $N \cap A$. We claim that α is the smallest element of N. To this end, let $\beta \in N \setminus A$ be arbitrary and let us show that $\alpha \leqslant \beta$: From $\beta \in C \setminus A$ it follows that there is a d-set B such that $\beta \in B$ and $B \not\subset A$. From Property (D) we deduce that there is $\gamma \in B$ such that $A = \gamma^{\downarrow\downarrow}_{B}$. In particular, both α and β belong to the chain B, so they are comparable, and the only possibility is $\alpha < \beta$. Indeed, from $\beta \leqslant \alpha$ it would follow $\beta < \gamma$ and thus $\beta \in A$, which contradicts the choice of β. Thus, α is indeed the smallest element of N, and C is seen to be well-ordered.

Second, let us prove that C is not only well-ordered, but even a d-set: Let $\alpha \in C$, then we have $\alpha \in A$ for some d-set A and, since $A \subset C$, we have $\alpha_A^{\downarrow\downarrow} \subset \alpha_C^{\downarrow\downarrow}$. To show the reversed inclusion, let $\beta \in C$ be given such that $\beta < \alpha$. Then, like above, from $\beta \notin A$ it would follow that A is the initial segment of some other d-set B containing β, and thus $\beta \in A$ (implying that $\beta \notin A$ is not possible), and so we have $\beta \in \alpha_A^{\downarrow\downarrow}$. Since A is a d-set, we obtain finally $\alpha = d(\alpha_A^{\downarrow\downarrow}) = d(\alpha_C^{\downarrow\downarrow})$, thus C is a d-set, too.

Third, to prove that not only (1.5), but even (1.4) holds, suppose $c = d(c_C^{\downarrow\downarrow})$. Then, since C is a d-set, $c_C^{\downarrow\downarrow} \cup \{c\}$ is a d-set, too, and thus c belongs to the union C of all d-sets.

Fourth, to prove that C is the unique well-ordered set satisfying (1.4), let B be a d-set such that $b = d(b_B^{\downarrow\downarrow})$ implies $b \in B$. Then we have $B \subset C$ and due to Property (D) it follows $B = C$, since otherwise there would be $\gamma \in C$ such that $B = \gamma_C^{\downarrow\downarrow}$, implying $d(\gamma_B^{\downarrow\downarrow}) = d(\gamma_C^{\downarrow\downarrow}) = \gamma$ and thus $\gamma \in B$, which is a contradiction.

Finally, suppose that $d(C)$ is defined, then $d(C)$ is no strict upper bound of C. Otherwise, we have $c < d(C)$ for all $c \in C$ which is the same as to say $C = d(C)_C^{\downarrow\downarrow}$, from which it follows that $C \cup \{d(C)\}$ is a d-set not contained in C, which contradicts the definition of C. ○

Consequences of Lemma 1.58 and the Axiom of Choice are a generalized version of Zorn's Lemma (for details we refer to [50]), and Theorem 1.51. To prove the latter, we proceed by deducing from Lemma 1.58 the following basic fixed point theorem like in [24, Lemma 2.7]).

1.59 Theorem *Let D be a poset, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction. If* Sub F *is F-inductive, i.e. if F has Property (Y), given by*

(Y) If $U \subset \operatorname{Sub} F$ is a well-ordered set, $s\colon U \to D$ a strictly increasing mapping and $u \leqslant s(u) \in F(u)$ for all $u \in U$, then $s(U)$ has an upper bound $s^* \in \operatorname{Sub} F$,

then F has a maximal fixed point which is also a maximal subpoint of F.

Proof: We are going to apply Lemma 1.58. To this end, set

$$\mathcal{D} := \{W \subset D : W \text{ is well-ordered and has a strict upper bound in } \operatorname{Sub} F\}.$$

Then, we note first that $\emptyset$ belongs to $\mathcal{D}$, since F has at least one subpoint due to Propsition 1.50, which is a strict upper bound of $\emptyset$. Further, let us define a function $d\colon \mathcal{D} \to D$ as follows: Let $\nu\colon \mathcal{D} \to \operatorname{Sub} F$ be a function such that $\nu(W)$ is a strict upper bound of W and choose, for any $W \in \mathcal{D}$, $d(W) \in D$ such that

$$\nu(W) \leqslant d(W) \in F(\nu(W))$$

(where we have used the Axiom of Choice twice). Then, thanks to Lemma 1.58, there is a unique well-ordered set $C \in \mathcal{P}(D)$ such that

$$c \in C \quad \text{if and only if} \quad c = d(c_C^{\downarrow\downarrow}).$$

In particular, for each $c \in C$ we have $c_C^{\downarrow\downarrow} \in \mathcal{D}$, so that the function

$$u\colon C \to \operatorname{Sub} F, \quad c \mapsto u(c) = \nu(c_C^{\downarrow\downarrow})$$

is well-defined. By definitions of ν and d it follows

$$u(c) = \nu(c_C^{\downarrow\downarrow}) \leqslant d(c_C^{\downarrow\downarrow}) \in F(\nu(c_C^{\downarrow\downarrow})) = F(u(c)) \quad \text{for all } c \in C. \tag{1.7}$$

Now, let $U := u(C)$ and note the following:

(a) From (1.7) it follows $U \subset \operatorname{Sub} F$ and, since C is a d-set, $u(c_1) \leqslant c_1$ for any $c_1 \in C$. If $c_1 < c_2$ for some $c_2 \in C$, we deduce, by definition of u and ν, $u(c_1) \leqslant c_1 < u(c_2)$. Since C is a chain, it follows that $u\colon C \to U$ is an increasing bijection.,

(b) From (a) and Proposition 1.56 we have that $s := u^{-1}\colon U \to D$ is strictly increasing. Since C is well-ordered and non-empty, we further readily obtain that $U \subset \operatorname{Sub} F$ is well-ordered and non-empty, too. Lastly, from (1.7) and since C is a d-set, we have $u \leqslant s(u) \in F(u)$ for all $u \in U$.

From (b) and (Y) it now follows that $C = s(U)$ has an upper bound $s^* \in \operatorname{Sub} F$, which is maximal in $\operatorname{Sub} F$. Indeed, for any $\bar{s} \in \operatorname{Sub} F$ we cannot have $s^* < \bar{s}$, since otherwise $C \in \mathcal{D}$ and $d(C)$ is a strict upper bound of C which contradicts Lemma 1.58. Finally, Proposition 1.50 ensures that s^* is a maximal fixed point of F. ○

Now, our central fixed point theorem follows readily:

Proof of Theorem 1.51: Let D and F be such that the assumptions of Theorem 1.51 are satisfied, i.e. the assumptions of Theorem 1.59 hold true and a subpoint $\underline{u}$ of F is fixed. Then let F' be the restriction of F to the poset $D' := \underline{u}^{\uparrow}$, that is

$$F'\colon D' \to \mathcal{P}_{\emptyset}(D'), \quad a \mapsto F(a) \cap D'.$$

We claim that $\operatorname{Sub} F'$ is F'-inductive. Indeed, let $U \subset \operatorname{Sub} F' \subset \operatorname{Sub} F$ be a well-ordered set in D', and let $s\colon U \to D' \subset D$ be a strictly increasing function such that

$$u \leqslant s(u) \in F'(u) \subset F(u) \quad \text{for all } u \in U.$$

If $s(U) = \emptyset$, then $\underline{u} \in \operatorname{Sub} F'$ is an upper bound of $s(U)$. If $s(U) \neq \emptyset$, let $s^* \in \operatorname{Sub} F$ be any upper of $s(U)$ (which exists since $\operatorname{Sub} F$ is F-inductive). Then, for any $v \in s(U)$, it holds $\underline{u} \leqslant v \leqslant s^* \leqslant^* F(s^*)$, from which it follows $s^* \in D'$. Thus, $s^* \in \operatorname{Sub} F'$ is an upper bound of $s(U)$.

Since $\operatorname{Sub} F'$ is F'-inductive, F' has, thanks to Theorem 1.59, a maximal fixed point u^*, which is, in view of $F' \subset F$, also a fixed point of F. Moreover, if $v^* \in D$ is any fixed point of F such that $u^* \leqslant v^*$, we have $\underline{u} \leqslant u^* \leqslant v^*$, thus $v^* \in D'$ is a fixed point of F'. Since u^* is maximal in $\operatorname{Fix} F'$, it follows $u^* = v^*$, and so u^* is a maximal fixed point of F such that $\underline{u} \leqslant u^*$. ○

Comparing the structure of Theorems 1.47 and 1.51, we see that the technical condition (Y) (meaning that $\operatorname{Sub} F$ is F-inductive) comprises both conditions on D and F and the existence of at least on subpoint. However, in analytic applications dealing with well-ordered sets is not standard, so it is convenient to introduce conditions on D such that (Y) can be formulated in terms of increasing sequences.

The naive approach would be to consider only posets D in which all well-ordered chains are increasing sequences. However, this is a very strong property which not even holds for the basic set of analysis, the compact interval $[0,1] \subset \mathbb{R}$. Indeed, consider the sets

$$C_0 := \{1 - 1/n : n \in \mathbb{N}\} \quad \text{and} \quad C_1 := C_0 \cup \{1\}.$$

If $M \subset C_0$ is non-empty and $N \subset \mathbb{N}$ such that $M = \{1 - 1/n : n \in N\}$, then $1 - 1/\min N$ is the smallest element of M. Thus, C_1 is seen to be well-ordered. However, if $C_1 = (c_n)$ for some sequence (c_n) (i.e. $C_1 = \{c_n : n \in \mathbb{N}\}$), then there is n_1 such that $1 = c_{n_1}$, and we have $c_{n_2} < c_{n_1}$ for every (of infinitely many) n_2 such that $c_{n_2} \in C_0$, so that (c_n) is seen to be not increasing.

Fortunately, there is a much more weaker assumption on a poset D which allows us to consider only (possibly finite) strictly increasing sequences. It reads as follows:

(B) For each well-ordered subset C of D such that each possibly finite strictly increasing sequence in C has an upper bound in D, there is a possibly finite strictly increasing sequence in C that has the same upper bounds as C.

(To avoid confusion, it should be noted that we allow also for the empty sequence when speaking of possibly finite sequences.)

1.60 Corollary *Let D be a poset satisfying Property (B), and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction. If F has Property (Z), given by*

(Z) For all possibly finite strictly increasing sequences $(u_n) \subset \operatorname{Sub} F$ and $(s_n) \subset D$ such that $u_n \leqslant s_n \in F(u_n)$ for all n, there is an upper bound $s^* \in \operatorname{Sub} F$ of (s_n),

then F has a maximal fixed point u^*. If $\underline{u}$ is any subpoint of F, then u^* can be chosen such that $\underline{u} \leqslant u^*$.

Proof: Let $U \subset \operatorname{Sub} F$ be a well-ordered set and let $s\colon U \to D$ be a strictly increasing mapping such that $u \leqslant s(u) \in F(u)$ for all $u \in U$. Then for every possibly finite strictly increasing sequence $(s_n) \subset s(U) \subset D$ let $u_n := s^{-1}(s_n)$, such that $(u_n) \subset U \subset \operatorname{Sub} F$ is a possibly finite strictly increasing sequence with $u_n \leqslant s_n \in F(u_n)$. From the assumptions, (s_n) has an upper bound $s^* \in \operatorname{Sub} F \subset D$. Thus, due to (B) and since $s(U)$ is well-ordered, there is a possibly finite strictly increasing sequence in $s(U)$ which has the same upper bounds as $s(U)$, and we can assume that this sequence is given by (s_n). Consequently, $s^* \in \operatorname{Sub} F$ is an upper bound of $s(U)$, and $\operatorname{Sub} F$ is seen to be F-inductive. This shows that all assumption of Theorem 1.51 are satisfied, thus F has a maximal fixed point u^* with the required properties. ○

It may seem that (B) is a rather general property for a poset D, but in fact it implies that every well-ordered subset of D is countable, which gives a nice necessary condition for (B).

1.61 Lemma *Let D be a poset. Then Property (B) implies that each well-ordered subset C of D is countable, which in turn is equivalent to the property*

(C) Each well-ordered subset C of D possesses a countable subset B such that $C \leqslant^* B$ (such a set B is called ***cofinal chain*** *of C).*

Proof: Let us first prove the claimed equivalence. To this end, it suffices to prove that (C) implies that all well-ordered subsets of D are countable. In order to do so, we follow the proof of [50, Lemma 1.1.4]):

Suppose that (C) holds true and let $C \subset D$ be well-ordered. Then we distinguish two cases:

(i) If C has the greatest element α, then it follows $C = \alpha^{\downarrow\downarrow} \cup \{\alpha\}$.

(ii) If C has no greatest element, let $B \subset C$ be a countable cofinal chain of C. Then, for all $c \in C$ there is $c' \in C$ and some $b \in B$ such that $c < c' \leqslant b$, which implies that C is the union of all (countably many) $b^{\downarrow\downarrow}_C$, $b \in B$.

Now, if $c^{\downarrow\downarrow}$ is countable for all $c \in C$, it follows in both cases (i) and (ii) that C is countable. So let us assume that there is $z \in C$ such that $z^{\downarrow\downarrow}_C$ is not countable, and let z be the smallest such element. Then, by construction, $Z := z^{\downarrow\downarrow}_C$ is a well-ordered subset of D such that $c^{\downarrow\downarrow}_Z = c^{\downarrow\downarrow}_C$ is countable for all $c \in Z$, from which it follows like above that $Z = z^{\downarrow\downarrow}_C$ is countable—which contradicts our assumption. Thus, such z does not exist and C is seen to be countable.

Now, let us prove that also under condition (B) all well-ordered subsets of D are countable. To this end, we could use some knowledge about infinite ordinals and the least uncountable ordinal ω_1. However, since uncountable ordinals play no further role in this thesis, let us formulate the proof more elementary:

Let us by way of contraposition assume that there is a well-ordered uncountable chain C in D. Then we can assume that $c^{\downarrow\downarrow}_C$ is countable for each $c \in C$ (as otherwise there is a smallest $z \in C$ such that $z^{\downarrow\downarrow}_C$ is uncountable, in which case we replace C by $z^{\downarrow\downarrow}_C$). Now take any possibly finite strictly increasing sequence (c_n) in C and let B be the union of all ${c_n}^{\downarrow\downarrow}_C$. Since $B \subset C$, B is well-ordered like C, and from the countability of each ${c_n}^{\downarrow\downarrow}_C$ it follows that B is countable. Further, B is a decreasing set, from which it follows readily $C' = B$ whenever B is a cofinal chain of some set $C' \subset C$. Especially, such sets C' are countable, whence B is not a cofinal chain of C. Thus in C there is a strict upper bound of B, and we take c to be the smallest one. This element c cannot be an upper bound (and thus a maximal element) of C, since otherwise we would have $C = c^{\downarrow} = c^{\downarrow\downarrow}_C \cup \{c\} = B \cup \{c\}$ and thus C would be countable. Thus, all possibly finite strictly increasing sequences in C have an upper bound c which is no upper bound of C, whence Property (B) does not hold. ○

The core results of this subsection are summed up in Figure 1.3.

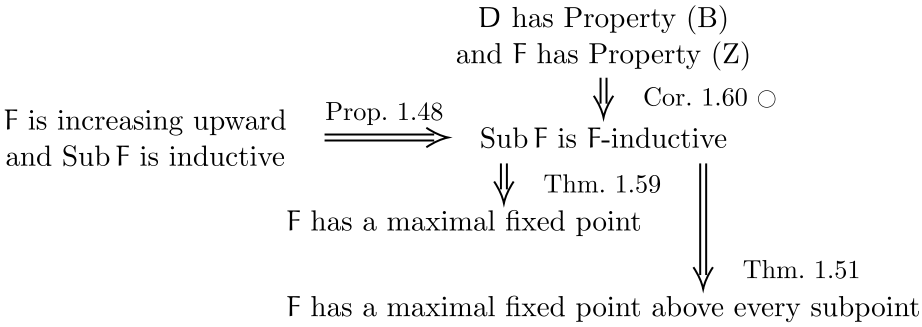

Fig. 1.3: The situation in case D is a poset and $F\colon D \to \mathcal{P}_\emptyset(D)$ is a multifunction (which has, implicitly, a subpoint)

In Section 2.1 we will give some mild topological conditions which guarantee that Properties (B) and (Z) hold, see Propositions 2.6 and 2.7, and Theorems 2.10 and 2.30.

1.2.3 Greatest Fixed Points

The fixed point theorems presented in Subsection 1.2.2 do not ensure that there are greatest fixed points. However, there is no need to use a theorem for complete posets, since there is a simple result establishing a connection with directed upward sets:

1.62 Definition Let D be a poset. A set $A \subset D$ is called **directed upward** if for all $a, b \in A$ there is $c \in A$ such that $\{a, b\} \leqslant^* c$. By duality, A is called **directed downward** if for all $a, b \in A$ there is $c \in A$ such that $c \leqslant_* \{a, b\}$. ○

1.63 Lemma *Let D be a poset and suppose that $F\colon D \to \mathcal{P}_\emptyset(D)$ has a maximal fixed point. Then F has a greatest fixed point if and only if $\operatorname{Fix} F$ is directed upward.*

Proof: If a is the greatest fixed point of F, than $\operatorname{Fix} F$ is obviously directed upward. If $\operatorname{Fix} F$ is directed upward, then for a maximal fixed point $a \in \operatorname{Fix} F$ and any $b \in \operatorname{Fix} F$ there is some $c \in \operatorname{Fix} F$ such that $a \leqslant c$ and $b \leqslant c$. Since a is maximal, it follows $a = c$ and thus $b \leqslant a$, proving that a is the greatest fixed point. ○

To apply Lemma 1.63, we need conditions that guarantee that the set of fixed points is directed upward. For this purpose, we introduce the following properties:

1.64 Definition Let D be a poset and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction.

(i) F is called **permanent upward** if $x \leqslant y$ in D and $z \in F(x)$ imply $z \in F(y)$.

(ii) F is called **fixed upward** if $a \leqslant b \in F(a)$ imply $b \in F(b)$.

Now let D be a sup-semilattice.

(iii) F is said to be of **type (U)** if

$$a, b \in F(a \vee b) \quad \text{imply} \quad a \vee b \leqslant^* F(a \vee b).$$

(iv) F is said to be of **type (U+)** if

$$c \leqslant a \vee b \quad \text{and} \quad a, b \in F(c) \quad \text{imply} \quad a \vee b \leqslant^* F(c). \qquad \bigcirc$$

1.65 Remark Clearly, any operator that is permanent upward is both increasing upward and fixed upward, whereas the reversed implication does not hold in general. Furthermore, if F has directed upward values, it is of type (U+), and if it is of type (U+), it is of type (U), whereas the reversed implications do not hold in general. ○

1.66 Lemma *Let D be a sup-semilattice and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a multifunction. Suppose further that one of the following conditions holds true:*

(i) F is permanent upward and of type (U).

(ii) F is increasing upward, fixed upward and of type (U+).

Then $\operatorname{Fix} F$ *is directed upward.*

Proof: First, let F satisfy the conditions in (i), let $a_i \in \operatorname{Fix} F$, $i = 1, 2$, be arbitrary fixed points and set $a_3 := a_1 \vee a_2$. F is permanent upward, so from $a_i \leqslant a_3$ it follows $a_i \in F(a_i) \subset F(a_3)$, $i = 1, 2$. Since F if of type (U), we deduce $a_3 \leqslant b$ for some $b \in F(a_3)$ and it follows, again since F is permanent upward, $b \in F(b)$, that is, $b \in \operatorname{Fix} F$. Thus, $\operatorname{Fix} F$ is directed upward.

Now, let F satisfy the conditions in (ii) and let a_i, $i = 1, 2, 3$, be given as above. Since F is increasing upward, there are $a_i' \in F(a_3)$ such that $a_i \leqslant a_i'$, $i = 1, 2$. Since $a_3 \leqslant a_1' \vee a_2'$ and since F is of type (U+), there is some $b \in F(a_3)$ such that it holds $a_3 \leqslant a_1' \vee a_2' \leqslant b$. Since F is fixed upward, we deduce $b \in F(b)$, that is, $b \in \operatorname{Fix} F$. Consequently, $\operatorname{Fix} F$ is directed upward. ○

For applications, usually the following corollary of Lemma 1.66 is enough:

1.67 Corollary *Let D be a sup-semilattice, and let $F\colon D \to \mathcal{P}_\emptyset(D)$ be a permanent upward multifunction with directed upward values. Then* $\operatorname{Fix} F$ *is directed upward.* ○

1.68 Remark Condition (ii) of Lemma 1.66 can be weakened in such a way, that F is not supposed to be fixed upward, but only *weakly fixed upward* in the sense that $u \leqslant v \in F(u)$ implies $w \in F(w)$ for some $w \geqslant v$. ○

1.69 Remark Simple examples of multifunctions can be visualized with graphs in the following way: The sup-semilattice $D \subset \mathbb{R}^2$ is taken as the set of nodes of a directed graph, and there is an edge (a, b) that points to b if and only if $b \in F(a)$. Thus, F is, e.g., fixed upward if and only if any upward pointing edge (where $\mathbb{R}^2$ is equipped with the usual componentwise partial order) points to a node with a loop. The examples in Figures 1.4 and 1.5 show that Lemma 1.66 is optimal in the following two senses:

(i) No weaker combination of the considered properties is enough to guarantee that a multivalued operator over a sup-semilattice has a directed upward set of fixed points.

(ii) No logically permitted combination of these properties in combination with $\operatorname{Fix} F$ having a maximal and a minimal fixed point (and thus being directed upward) implies another of these properties.

To shorten notations, we have captioned a diagram with the letters p, i, f, U and U+ if and only if the depicted operator is permanent upward, increasing upward, fixed upward, or is of type (U) or type (U+), respectively.

For example, the first graph of Figure 1.4 depicts a multifunction F on the lattice $\{a, b, c, d\}$. Its set of fixed points $\operatorname{Fix} F = \{a, b, d\}$ is not directed upward since the only upper bound of $\{b, d\}$ is c. Further, F is permanent upward, since for every edge (b, x) and (d, y) there is an edge (c, x) and (c, y), respectively, and for every edge (a, x) (which is only (a, a)) there are edges (b, x), (c, x) and (d, x). Consequently, F is also increasing upward and fixed upward. Further, F is not of type (U), since $c = b \vee d$, $b, d \in F(c)$, but there is no $x \in F(c)$ such that $c \leqslant x$. Consequently, F has neither Property $(U+)$ nor are all its values directed upward. ○

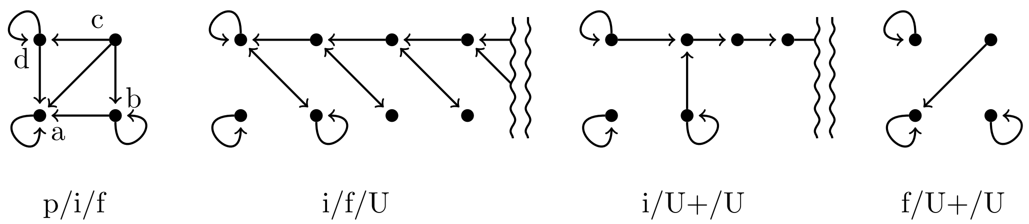

Fig. 1.4: Multifunctions whose set of fixed points is not directed upward.

In the analytic application of Part II, we will work not only with one, but with two multifunctions. The main operator S maps functions to solutions of the considered problem, while the suboperator $\underline{S}$ maps functions to so called subsolutions. As a matter of fact, solutions and subsolution are defined carefully in such a way that the suboperator $\underline{S}$ has good order-theoretical properties while the main operator S has good analytic properties. The following proposition demonstrates the order-theoretical interplay of S and $\underline{S}$ in an abstract setting.

1.70 Proposition *Let D and D' be posets, and let $F, G\colon D \to \mathcal{P}_{\emptyset}(D')$ be two multifunctions.*

(i) Suppose that F and G are equivalent in the sense that $F(v) \leqslant^* G(v) \leqslant^* F(v)$ for all $v \in D$. Then F is increasing upward if and only if G is increasing upward.

(ii) Suppose that F and G are equivalent in the sense that $\operatorname{Fix} F \leqslant^* \operatorname{Fix} G \leqslant^* \operatorname{Fix} F$. Then F and G have the same maximal and greatest fixed points.

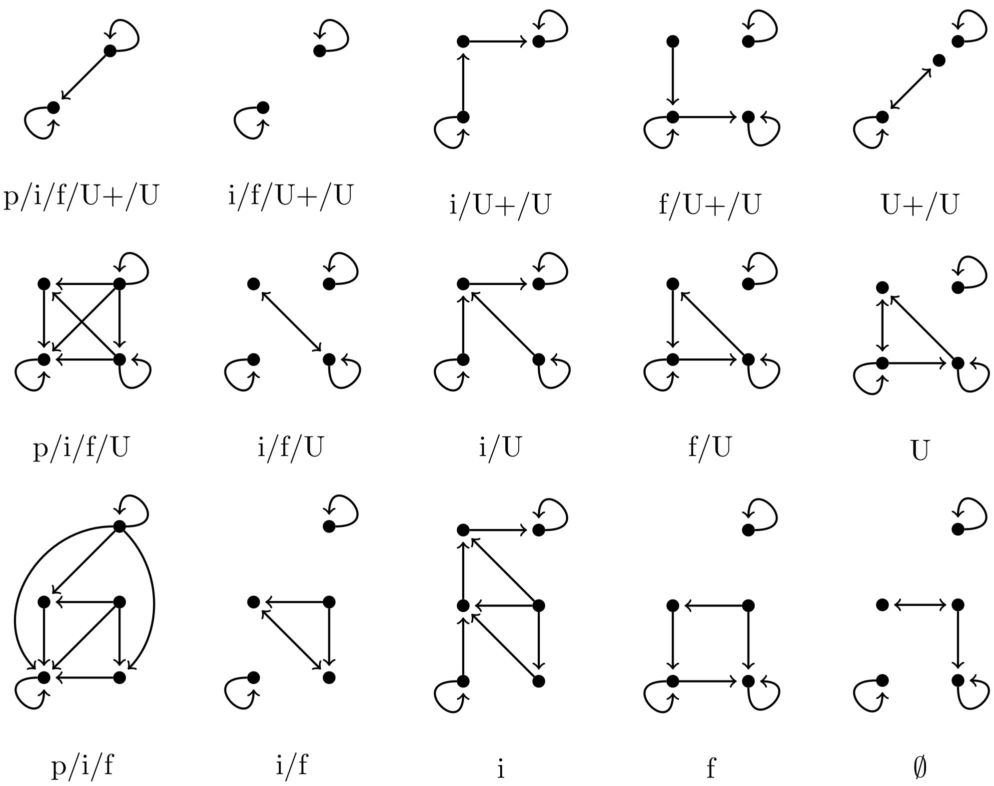

Fig. 1.5: Multifunctions with minimal and maximal fixed points.

Proof: Suppose first that $F(v) \leqslant^* G(v) \leqslant^* F(v)$ for all $v \in D$ and assume that G is increasing upward. Then, for all $a, b \in D$ such that $a \leqslant b$ it follows

$$F(a) \leqslant^* G(a) \leqslant^* G(b) \leqslant^* F(b),$$

which means that F is increasing upward. By symmetry, the claim in (i) is proved.

Suppose now that $\operatorname{Fix} F \leqslant^* \operatorname{Fix} G \leqslant^* \operatorname{Fix} F$ and let a be a maximal fixed point of F. Then there are $b \in \operatorname{Fix} G$ and $c \in \operatorname{Fix} F$ such that $a \leqslant b \leqslant c$ and since a is maximal, it follows $a = c$ and thus $a = b \in \operatorname{Fix} G$. Further, let $d \in \operatorname{Fix} G$ be such that $a \leqslant d$, then there is $e \in \operatorname{Fix} F$ such that $a \leqslant d \leqslant e$, implying $a = e$ and thus $a = d$, so that a is maximal in $\operatorname{Fix} G$. By symmetry, we deduce that F and G have the same maximal fixed points.

If the greatest fixed point a of F exists, it is the only maximal fixed point of F and thus the only maximal fixed point of G. Since for any $b \in \operatorname{Fix} G$ there is some $c \in \operatorname{Fix} F$ such that $b \leqslant c \leqslant a$, we deduce that a is in fact the greatest element of $\operatorname{Fix} G$. By symmetry, the greatest fixed points of F and G coincide. ○

Finally, let us combine the results so far to obtain an abstract purely order-theoretical framework combining the concepts of subsolutions and subpoints.

1.71 Theorem *Let* D *be a poset, and let* $\mathsf{S}\colon \mathsf{D} \to \mathcal{P}_\emptyset(\mathsf{D})$ *and* $\underline{\mathsf{S}}\colon \mathsf{D} \to \mathcal{P}_\emptyset(\mathsf{D})$ *be multifunctions such that the following conditions are satisfied:*

(i) $\underline{\mathsf{S}}$ *has a subpoint* $\underline{u}$*, i.e.* $\underline{u} \leqslant^* \underline{\mathsf{S}}(\underline{u})$.

(ii) $\underline{\mathsf{S}}$ *is permanent upward and of type (U), or* $\underline{\mathsf{S}}$ *is increasing upward, fixed upward and of type (U+)*

(which holds especially if $\underline{\mathsf{S}}$ *is permanent upward and has directed upward values).*

(iii) $\operatorname{Sub}\mathsf{S}$ *is* S*-inductive*

(which holds especially if D *has property (B) and if* S *has property (Z)).*

(iv) For all $v \in \mathsf{D}$ *it holds* $\mathsf{S}(v) \leqslant^* \underline{\mathsf{S}}(v) \leqslant^* \mathsf{S}(v)$.

(v) It holds $\operatorname{Fix}\mathsf{S} \leqslant^* \operatorname{Fix}\underline{\mathsf{S}}$.

Then $\operatorname{Fix}\mathsf{S}$ *has the greatest element* u^* *and it holds* $\underline{u} \leqslant u^*$.

Proof: Due to the first part of Proposition 1.70, $\mathsf{S}\colon \mathsf{D} \to \mathcal{P}_\emptyset(\mathsf{D})$ is increasing upward like $\underline{\mathsf{S}}$, and we have, since $\leqslant^*$ is transitive, $\underline{u} \leqslant^* \mathsf{S}(\underline{u})$. Thus, Theorem 1.51 ensures the existence of some maximal element $u^* \in \operatorname{Fix}\mathsf{S}$ such that $\underline{u} \leqslant u^*$. Since $\underline{u} \in \operatorname{Sub}\underline{\mathsf{S}}$ can be an arbitrary subpoint of $\underline{\mathsf{S}}$, in particular we infer $\operatorname{Fix}\underline{\mathsf{S}} \leqslant^* \operatorname{Fix}\mathsf{S}$. Thus, by the second part of Proposition 1.70, it follows that S and $\underline{\mathsf{S}}$ have the same maximal and greatest fixed points. In particular, u^* is a maximal fixed point of $\underline{\mathsf{S}}$. From Lemmata 1.63 and 1.66 we thus deduce that $\underline{\mathsf{S}}$ has the greatest fixed point u^{**}, which is also the greatest fixed point of S. Since $\underline{u} \leqslant u^* \leqslant u^{**}$, the proof is complete. ○

In the next chapter, we will extend this framework by topological methods in order to find, in Part II, smallest and greatest solutions of multivalued variational inequalities.

2 | Topology

In this chapter, let us strengthen our purely order-theoretical insights by use of topological results on linear spaces, in order to obtain a general framework for solving variational inequalities with multivalued bifunctions. Further, let us collect some well-known results from Functional Analysis.

The notions of posets with special structure, such as ordered topological spaces or ordered Banach spaces, and their basic properties are standard and only presented to the extent required in this thesis. For more information and deeper results, we refer to standard textbooks, e.g. [37, 56, 100, 103, 120, 121, 122, 123], to the monographs [28, 50], and—for a wealth of information about multifunctions—to [53].

2.1 Order-Topological Fixed Point Theorems

In this first section, we are going to continue the study of fixed point theorems started in Section 1.2. The order-theoretical fixed point theorem that is most suitable for applications is Corollary 1.60. There, the poset D is assumed to satisfy Property (B), which reads as follows:

(B) For each well-ordered subset C of D such that each possibly finite strictly increasing sequence in C has an upper bound in D, there is a possibly finite strictly increasing sequence in C that has the same upper bounds as C.

To ensure the existence of such a sequence, we will use a central tool of topology: compactness. Especially, this will be useful if D is not only an ordered topological space, but even a reflexive ordered Banach space, because then we have that bounded sets are sequentially weakly compact. Further, we will find simple topological conditions such that a multifunction $F\colon D \to \mathcal{P}_\emptyset(D)$ has Property (Z) presented above, which reads as follows:

(Z) For all possibly finite strictly increasing sequences $(u_n) \subset \operatorname{Sub} F$ and $(s_n) \subset D$ such that $u_n \leqslant s_n \in F(u_n)$ for all n, there is an upper bound $s^* \in \operatorname{Sub} F$ of (s_n).

To this end, we will demand that the values of F are weakly closed subsets of a reflexive Banach space. These results then will constitute our central framework for the applications in Part II.

In the following, we will work only with posets D that have more structure. For a steady reminder, such spaces will by denoted by capitals W, X, Y or Z, or sometimes R.

2.1.1 Ordered Topological Spaces

Recall that a **topological space** is a set X together with a **topology** τ, which is a family $\tau \subset \mathcal{P}_\emptyset(X)$ of subsets of X such that $\emptyset \in \tau$ and $X \in \tau$, and such that τ is closed under unions and finite intersections. The elements of τ are called **open** and their complements are called **closed**. For a sequence $(x_n) \subset X$, the topology τ defines a **convergence** $x_n \to x$ in the usual way, in which case we also write $\lim_n x_n = x$. If $C \subset X$ is closed and $(x_n) \subset C$ converges, then $\lim_n x_n \in C$. If for all closed $C \subset X$ and each element $x \in C$ there is a sequence $(x_n) \subset C$ such that $x_n \to x$, then $X = (X, \tau)$ is called **Fréchet-Uryson**. If Y is also a topological space, then a function $f \colon X \to Y$ is called **continuous** if $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X for every open subset U of Y.

To bring topology and order theory together, we will use the following compatibility condition:

2.1 Definition Let $(X, \leqslant)$ be a poset, and let (X, τ) be a topological space. Then the triple $X = (X, \leqslant, \tau)$ is called **ordered topological space** if for all $a \in X$ the sets $a^{\uparrow}$ and $a^{\downarrow}$ are closed. ○

Of course, $\mathbb{R}$ is an ordered topological space. This generalizes to chains in the following way:

2.2 Definition Let X be a chain. Then a topology τ on X is called **interval topology** on X if the sets $a^{\downarrow\downarrow}$ and $a^{\uparrow\uparrow}$ are a **subbase** of τ (i.e. all sets in τ are a union of (possibly infinitely many) finite intersections of sets $a^{\downarrow\downarrow}$ and $b^{\uparrow\uparrow}$). ○

2.3 Proposition *Let $(X, \leqslant)$ be a chain, and let τ be its interval topology. Then the triple $(X, \leqslant, \tau)$ is an ordered topological space.*

Proof: Let $a \in X$, then $a^{\uparrow} = X \setminus a^{\downarrow\downarrow}$ and $a^{\downarrow} = X \setminus a^{\uparrow\uparrow}$ are complements of open sets and thus closed. ○

2.4 Remark If $(X, \leqslant, \tau)$ is an ordered topological space, then $(X, \geqslant, \tau)$ is an ordered topological space, so that order-theoretical duality methods apply. ○

The definition of ordered topological spaces is chosen such that limits preserve the order-structure of X and such that we have a way of computing suprema of increasing sequences via subsequences:

2.5 Proposition *Let X be an ordered topological space, and let $(x_n) \subset X$ be a sequence.*

(i) If (x_n) converges and if $x \in X$ is such that $x_n \leqslant x$ for all n, then $\lim_n x_n \leqslant x$.

(ii) If (x_n) is increasing, then each convergent subsequence (y_n) of (x_n) converges to the supremum $x_\vee$ of (x_n).

Proof: Assertion (i) follows readily since $x^{\downarrow}$ is closed.

To prove (ii), let $y \in D$ be such that $\lim_n y_n = y$. Then we have for every m, up to a subsequence, $(y_n) \subset {x_m}^{\uparrow}$, thus $x_m \leqslant y$ by the dual of (i), from which it is seen that y is an upper bound of (x_n). Further, for all upper bounds x of (x_n) it holds $y_n \in x^{\downarrow}$ and thus $y \leqslant x$ by (i), implying that y is the smallest upper bound of (x_n), i.e. $y = x_\vee$. ○

Proposition 2.5 helps us to find a topological criterion that guarantees that a poset has Property (B). Let us start with the following intermediate step:

2.6 Proposition *Let* X *be an ordered topological space with Property (B'), given by*

(B') Each non-empty well-ordered subset C *of* X *contains an increasing sequence which converges to* $\sup C$*.*

Then X *has Property (B).*

Proof: Let C be a well-ordered subset of X. The case of $C = \emptyset$ makes no trouble, so let us assume that C is non-empty. Then, by Property (B'), there is an increasing sequence $(x_n) \subset C$ such that $\lim_n x_n = \sup C$. If c is any upper bound of (x_n), it follows by Proposition 2.5 that $\sup C \leqslant c$, implying that c is an upper bound of C, too. Thus, C and (x_n) have the same upper bounds. Since (x_n) is maybe not strictly increasing, define (y_n) recursively via

$$y_1 := x_1, \quad y_{n+1} := \min\{x_k : x_k > y_n\}$$

for all n for which y_n is well-defined. Then, (y_n) is a possibly finite strictly increasing sequence with the same upper bounds as (x_n) and thus with the same upper bounds as C, whence X has Property (B). ○

Now, we have two tasks: We have to find a topological criterion which guarantees that a well-ordered set C has the supremum, and we have to make sure that this supremum is attained as the limit of some subsequence of C. Both tasks can be tackled by incorporating compactness. To this end, recall that a subset M of a topological space X is called **compact** if every open cover of M has a finite subcover, and that M is called **relatively compact** if the closure of M is compact. Then, with the ideas of the proof of [50, Proposition 1.1.4], we can prove the following proposition:

2.7 Proposition *Let* X *be an ordered topological space, and let* $C \subset X$ *be a relatively compact chain. Then* $\sup C$ *exists. If* X *is Fréchet-Urysohn, then* $\lim_n x_n = \sup C$ *for some increasing sequence* $(x_n) \subset C$*.*

Proof: Let $\overline{C}$ be the closure of C. Then the family $\mathcal{C} := \{c^\uparrow \cap \overline{C} : c \in C\}$ consists of closed subsets of $\overline{C}$ such that the intersection of finitely many members of $\mathcal{C}$ is non-empty. Since each $c^\uparrow \cap \overline{C}$ is compact, it follows that there is $\alpha \in \bigcap \mathcal{C}$, i.e. it holds $\alpha \in \overline{C}$ and $c \leqslant \alpha$ for all $c \in C$. It follows that $\alpha = \sup C$. Indeed, α is an upper bound of C, and for all upper bounds β of C we have $\overline{C} \subset \beta^\downarrow$, since $\beta^\downarrow$ is closed, and so $\alpha \leqslant \beta$.

Next, let X be Fréchet-Urysohn. Then from $\sup C \in \overline{C}$ it follows that there is a sequence $(x_n) \subset C$ which converges to $\sup C$. However, (x_n) may be not increasing. In order to fix this, let us first assume that (x_n) has the greatest element x_{n_0}. Then, by Proposition 2.5, it follows $\sup C = \lim_n x_n \leqslant x_{n_0} \leqslant \sup C$, such that $x_{n_0} \in C$ is seen to be the greatest element of C. Then, (x_n) may not be increasing, but the constant sequence $(\sup C)$ is increasing and converges to $\sup C$.

Now, let us assume that (x_n) has no greatest element. Then to each x_n there is x_m, $m > n$, such that $x_n < x_m$ (as otherwise $x_m \leqslant x_n$ for all $m > n$ and the greatest element of $\{x_1, x_2, \ldots, x_n\}$ would be the greatest element of (x_n)). Thus, we can recursively define an increasing subsequence (y_n) of (x_n) by setting

$$y_1 := x_1 \quad \text{and} \quad y_{n+1} := \min\{x_m : x_n < x_m \text{ and } n < m\}.$$

It is readily seen that $\lim_n y_n = \lim_n x_n$, i.e. $(y_n) \subset C$ converges to $\sup C$. ○

From Propositions 2.6 and 2.7 and the fact that each subset of a compact set is relatively compact, we have at once the following corollary of Corollary 1.60, which is our first fixed point result on ordered topological spaces:

2.8 Corollary *Let* X *be an ordered topological space which is compact and Fréchet-Urysohn, and let* $F\colon X \to \mathcal{P}_\emptyset(X)$ *be a multifunction. If* F *has a subpoint* $\underline{u}$, *and if* F *has Property (Z), then* F *has a maximal fixed point* u^* *which is also a maximal element of* Sub F *and which satisfies* $\underline{u} \leqslant u^*$. ○

Now, it is the right time to consider again increasing upward multifunctions F to obtain the analogue of Theorem 1.47 for compact ordered topological spaces. To this end, let us introduce the following analogue of universally inductive sets:

2.9 Definition Let D be a poset. A subset $B \in \mathcal{P}_\emptyset(D)$ is called **countably universally inductive** in D if for any increasing sequence $(a_n) \subset D$ such that $(a_n) \leqslant^* B$ there is $b \in B$ such that $(a_n) \leqslant^* b$. ○

Note that each universally inductive set is countably universally inductive, since an increasing sequence (a_n) is a well-ordered chain. Indeed, if $M \subset (a_n)$ is non-empty, set $N := \{n \in \mathbb{N} : a_n \in M\}$, then $\min M = a_{\min N}$.

2.10 Theorem *Let* X *be an ordered topological space which is compact and Fréchet-Urysohn, and let* $F\colon X \to \mathcal{P}_\emptyset(X)$ *be an increasing upward multifunction. If* F *has a subpoint* $\underline{u}$ *and if all values of* F *are countably universally inductive, then* F *has a maximal fixed point* u^* *which is also a maximal element of* Sub F *and which satisfies* $\underline{u} \leqslant u^*$.

Proof: In view of Corollary 2.8, we have only to prove that F has Property (Z), so let (u_n) and (s_n) be increasing sequences in X such that $u_n \leqslant s_n \in F(u_n)$ for all n. Then, by Proposition 2.7, $s := \sup(s_n)$ exists, and we have $u_n \leqslant s_n \leqslant s$. Since F is increasing upward, it follows $s_n \in F(u_n) \leqslant^* F(s)$, i.e. $(s_n) \leqslant^* F(s)$. By assumption, $F(s)$ is countably universally inductive, so that $(s_n) \leqslant s^*$ for some $s^* \in F(s)$. By definition of s, we have $s \leqslant s^*$, from which we have $s^* \in F(s) \leqslant^* F(s^*)$. Thus, $s^* \in \text{Sub}\, F$ is an upper bound of (s_n), and F has Property (Z). ○

In applications, compactness of X is a rather strong property, especially in function spaces, and it is rather technical to check if the values of F are countably universally inductive. We will overcome these problems in the next subsection by considering ordered linear spaces, especially reflexive ordered Banach spaces.

2.1.2 Reflexive Ordered Banach Spaces

Recall that a **linear space** over a field $\mathbb{K}$ is a set X together with an addition $+$ and a scalar multiplication $\cdot$ that satisfy the usual axioms. Especially, there exists some element $0 \in X$ such that $u + 0 = u$ for all $u \in X$. In this thesis, we will always set $\mathbb{K} := \mathbb{R}$.

On linear spaces we have the following two central notions:

2.11 Definition Let X and Y be linear spaces.

(i) A set $K \subset X$ is called **convex** if $(1-\lambda)u + \lambda v \in K$ for all $u, v \in K$ and all $\lambda \in [0, 1]$.

(ii) A function $f\colon X \to Y$ is called **linear** if $f(\lambda u + v) = \lambda f(u) + v$ for all $u, v \in X$ and all $\lambda \in \mathbb{R}$. ○

To bring linearity and order theory together, we will use the following compatibility condition:

2.12 Definition Let $(X, \leqslant)$ be a poset, and let $(X, +, \cdot)$ be a linear space over $\mathbb{R}$. Then $X = (X, \leqslant, +, \cdot)$ is called **ordered linear space** if $a \leqslant b$ is equivalent to $0 \leqslant b - a$ and if $0 \leqslant \lambda$ in $\mathbb{R}$ and $0 \leqslant a$ in X imply $0 \leqslant \lambda a$. ○

2.13 Remark Let $(X, \leqslant, +, \cdot)$ be an ordered linear space. If we consider the dual order $\geqslant$, then it follows

$$u \geqslant v \quad \Leftrightarrow \quad v \leqslant u \quad \Leftrightarrow \quad 0 \leqslant u - v = 0 - (v - u) \quad \Leftrightarrow \quad v - u \leqslant 0 \quad \Leftrightarrow \quad 0 \geqslant v - u,$$

and from $0 \leqslant \lambda$ in $\mathbb{R}$ and $0 \geqslant u$ in X it follows $0 \leqslant -u$, thus

$$0 \leqslant \lambda(-u) = -(\lambda u) \quad \Leftrightarrow \quad \lambda u \leqslant 0 \quad \Leftrightarrow \quad 0 \geqslant \lambda u.$$

Thus, $(X, \geqslant, +, \cdot)$ is also an ordered linear space and duality applies. (Note that the partial order in $\mathbb{R}$ was not changed.) ○

2.14 Definition Let K be a convex subset of a linear space X, and let Y be an ordered linear space. A function $f\colon K \to Y$ is called **convex** if $f((1-\lambda)u + \lambda v) \leqslant (1-\lambda)f(u) + \lambda f(v)$ for all $u, v \in X$ and all $\lambda \in [0, 1]$. ○

Ordered linear spaces without any further structural property will not play a role in this thesis. However, we have at least a useful connection to convex sets:

2.15 Proposition *Let X be an ordered linear space, and let $u, v \in X$. Then $u^{\uparrow}$, $v^{\downarrow}$ and $[u, v]$ are convex.*

Proof: Let $u \in X$, $x, y \in u^{\uparrow}$ and $\lambda \in [0, 1]$ be given. Then we have $0 \leqslant (1 - \lambda)$ and $0 \leqslant x - u$, from which it follows $0 \leqslant (1 - \lambda)(x - u)$. Furthermore, we have $0 \leqslant \lambda$ and $0 \leqslant y - u$, from which it follows $0 \leqslant \lambda(y - u) \Leftrightarrow \lambda(u - y) \leqslant 0$. By transitivity of $\leqslant$ it follows $\lambda(u - y) \leqslant (1 - \lambda)(x - u)$, which is equivalent to $u \leqslant (1 - \lambda)x + \lambda y$. Thus, $u^{\uparrow}$ is convex.

Let furthermore $v \in X$ be given. Then $v^{\downarrow}$ is convex by duality, and $[u, v]$ is convex, since it is the intersection of the convex sets $u^{\uparrow}$ and $v^{\downarrow}$. ○

Next, let us combine order theory, topology and linearity:

2.16 Definition Let $(X, \leqslant, \tau)$ be an ordered topological space, and let $(X, \leqslant, +, \cdot)$ be an ordered linear space. Then $X = (X, \leqslant, \tau, +, \cdot)$ is called **ordered topological vector space** if $+$ and $\cdot$ are continuous. If furthermore τ stems from a norm $\|\cdot\|$ under which X is complete (i.e. all Cauchy sequences are convergent), then $X = (X, \leqslant, \|\cdot\|)$ is called an **ordered Banach space**. ○

2.17 Remark The order cone $K = 0^{\uparrow}$ of an ordered Banach space is closed, satisfies $x \leqslant y$ if and only if $y - x \in K$, and $K + K \subset K$, $K \cap (-K) = \{0\}$ and $\alpha K \subset K$ for each $\alpha \geqslant 0$. Further, each ordered Banach space is Fréchet-Urysohn. ○

On ordered Banach spaces we can use all tools of analysis. To this end, let us recall some standard definitions and theorems. (Recall that in this thesis all Banach spaces are real Banach spaces.)

2.18 Definition Let $(X, \|\cdot\|)$ and Y be normed linear spaces, and let $A\colon X \to Y$ be a function. We call A an **operator** and write Au instead of $A(u)$ for the image of $u \in X$.

(i) A subset M of X is called **bounded** if there is $r \geqslant 0$ such that $\|u\| \leqslant r$ for all $u \in M$.

(ii) A is called **bounded** if it maps bounded sets to bounded ones. ○

2.19 Remark Note that also a subset M of an *ordered* Banach space X is called bounded if it is bounded with respect to the norm of X. If there are $u, v \in X$ such that $u \leqslant^* M \leqslant^* v$, we call M **order-bounded**. If $M = [u, v]$, then M is obviously order-bounded. An order-bounded set M is easily seen to be bounded provided there is a constant c such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant c\|y\|$ for all $x, y \in X$. In this case, the norm is called **normal**. ○

It is well-known that the set $L(X, Y)$ of all linear and bounded operators $A\colon X \to Y$ between normed spaces is a normed space with respect to the operator norm given by

$$\|A\| := \sup\{Au : u \in X, \|u\| \leqslant 1\}.$$

Further, any $A \in L(X, Y)$ is continuous with respect to the norms on X and Y. An important special case is given by $Y = \mathbb{R}$:

2.20 Definition Let X be a normed space.

(i) We call $X' := L(X, \mathbb{R})$ the **dual space** of X and its elements $\varphi\colon X \to \mathbb{R}$ **linear continuous functionals**.

(ii) Let $u \in X$ and $\varphi \in X'$. Then we define $\langle \varphi, u \rangle := \langle \varphi, u \rangle_X := \varphi(u)$, the **duality pairing**. ○

If X is an ordered normed space, then functionals $\varphi \in X'$ such that $\langle \varphi, u \rangle \geqslant 0$ for all $u \geqslant 0$, so called *positive functionals*, may be useful. However, in this thesis we will work only with the whole space X', which is large enough to give us the following result:

2.21 Proposition *Let* X *be a normed space, and let* $u \in X$ *be such that* $\langle \varphi, u \rangle = 0$ *for all* $\varphi \in X'$. *Then* $u = 0$. ○

On a Banach space X we have not only the topology induced by the norm, but the weak topology, which has fewer open sets and allows thus for more convergent sequences and more compact sets. Let us recall the basics:

2.22 Definition Let X be a Banach space, and let $u_n, u \in X$ for $n \in \mathbb{N}$.

(i) The norm on X induces a topology which we call the **strong topology**. Convergence of (u_n) to u with respect to this topology is denoted by $u_n \to u$.

(ii) The coarsest topology on X such that each element of X' remains continuous is called the **weak topology**. Convergence of (u_n) to u with respect to this topology is called **weak convergence** and is denoted by $u_n \rightharpoonup u$.

(iii) A subset M of X is called **weakly closed** or **weakly compact** if it is closed or compact with respect to the weak topology, respectively. M is called **weakly sequentially compact** if each sequence in M contains a subsequence that converges weakly to some element of M. ○

2.23 Proposition *Let* X *be a Banach space, and let* $u_n, u \in X$ *for* $n \in \mathbb{N}$.

(i) The weak convergence $u_n \rightharpoonup u$ *holds if and only if* $\langle \varphi, u_n \rangle \to \langle \varphi, u \rangle$ *for all* $\varphi \in X'$.

(ii) If $u_n \to u$, *then* $u_n \rightharpoonup u$.

(iii) If $u_n \rightharpoonup u$, *then* (u_n) *is bounded and* $\|u\| \leqslant \liminf_n \|u_n\|$.

(iv) If $u_n \rightharpoonup u$ *in* X *and* $\varphi_n \to \varphi$ *in* X', *or if* $u_n \to u$ *in* X *and* $\varphi_n \rightharpoonup \varphi$ *in* X', *then* $\langle \varphi_n, u_n \rangle \to \langle \varphi, u \rangle$.

(v) Let K *be a convex subset of* X. *Then,* K *is closed if and only if* K *is weakly closed.* ○

2.24 Definition Let X be a normed space.

(i) We define the **canonical embedding** $j \colon X \to (X')'$ via $\langle j(u), \varphi \rangle := \langle \varphi, u \rangle$ for all $u \in X$ and $\varphi \in X'$.

(ii) X is called **reflexive** if the canonical embedding $j \colon X \to (X')'$ is surjective. ○

2.25 Proposition *Let* X *be a reflexive normed space. Then the following assertions hold true:*

(i) X *is a Banach space.*

(ii) Every bounded subset of X *is weakly relatively compact.*

(iii) A subset of X *is weakly compact if and only if it is weakly sequentially compact. If* $M \subset X$ *is bounded, then each point in the weak closure of* M *is the weak limit of some sequence* $(u_n) \subset M$. [Eberlein-Šmulian] ○

2.26 Remark The last assertion in Proposition 2.25 follows from the proof given first in [118]. ○

As an application, let us provide a standard result about the existence of weakly convergent subsequences in intersections of continuously descending reflexive Banach spaces. To this end, recall the following definition:

2.27 Definition Let X and Y be normed spaces.

(i) We say that X is **continuously embedded** in Y (and write $X \hookrightarrow Y$) if the embedding operator $i\colon X \to Y$, defined via $i(u) = u$, is continuous.

(ii) We say that X is even **compactly embedded** in Y if the embedding operator $i\colon X \to Y$ is **compact**, i.e. if i is continuous and maps bounded sets to relatively compact ones. ○

2.28 Proposition *Let $I \subset \mathbb{R}$, let X_i be reflexive Banach spaces, $i \in I$, such that $X_j \hookrightarrow X_i$ is a continuous embedding for $i < j$, and let $(u_n) \subset X := \bigcap_i X_i$. If (u_n) is bounded in every X_i, then there is $u \in X$ and a subsequence of (u_n) converging weakly in every X_i to u.*

Proof: Fix some $i_0 \in I$. Then, since (u_n) is a bounded sequence in the reflexive Banach space X_{i_0}, there is $u \in X_{i_0}$ such that a subsequence of (u_n) converges weakly to u in X_{i_0}. W.l.o.g. we can assume $u_n \rightharpoonup u$ in X_{i_0}.

We claim that even $u_n \rightharpoonup u$ in every X_i. To this end, let $i_1 \in I$ be arbitrary. We consider two cases:

(i) Let $i_0 > i_1$. Then we have the continuous embedding $i\colon X_{i_0} \hookrightarrow X_{i_1}$ with the continuous dual embedding $i^*\colon X'_{i_1} \hookrightarrow X'_{i_0}$ (cf. Remark 3.77). Then we have, for every $\varphi \in X'_{i_1}$,

$$\langle \varphi, iu_n \rangle = \langle i^*\varphi, u_n \rangle \to \langle i^*\varphi, u \rangle = \langle \varphi, iu \rangle,$$

implying $iu_n \rightharpoonup iu$ in X_{i_1}. Since i is the embedding operator, we conclude $u_n \rightharpoonup u$ in X_{i_1}.

(ii) Let $i_1 > i_0$. Then (u_n) is bounded in X_{i_1} and thus there is a subsequence u_{n_k} that converges weakly in X_{i_1} to some $v \in X_{i_1}$. As in case (i), we conclude $u_{n_k} \rightharpoonup v$ in X_{i_0} and thus, since weak limits are unique, $v = u$.

Now let us assume that (u_n) does not converge weakly in X_{i_1} to u. Then there are a subsequence (u_{n_k}), a functional $\varphi \in X'_{i_1}$ and $\varepsilon > 0$ such that $|\langle \varphi, u_{n_k} - u \rangle| \geqslant \varepsilon$. But, according to the same arguments as above, (u_{n_k}) has a weakly convergent subsequence with weak limit u, contradicting that $\langle \varphi, u_{n_k} - u \rangle$ is bounded away from 0. Thus the assumption is false and we have $u_n \rightharpoonup u$ in X_{i_1}.

In both cases, we have $u_n \rightharpoonup u$ in X_{i_1}, which concludes the proof. ○

Finally, we have the following order-theoretical result:

2.29 Proposition *Let* $X = (X, \leqslant, \|\cdot\|, +, \cdot)$ *be an ordered Banach space, and let* σ *be the weak topology of* X*. Then* $Y = (X, \leqslant, \sigma, +, \cdot)$ *is an ordered topological vector space.*

Proof: Let $u \in X$. Since X is especially an ordered topological space, the order-intervals $u^{\uparrow}$ and $u^{\downarrow}$ are closed. Further, X is an ordered linear space, from which it follows (see Proposition 2.15) that $u^{\uparrow}$ and $u^{\downarrow}$ are convex. Thus, by Proposition 2.23, $u^{\uparrow}$ and $u^{\downarrow}$, are weakly closed, so that $(X, \leqslant, \sigma)$ is an ordered topological space. Further, it is readily seen that $+$ and $\cdot$ are continuous with respect to σ, thus $Y = (X, \leqslant, \sigma, +, \cdot)$ is an ordered topological vector space. ○

Now, from Theorem 2.10 we deduce the following basic fixed point theorem on reflexive Banach spaces:

2.30 Theorem *Let* D *be a bounded and weakly sequentially closed subset of a reflexive ordered Banach space* X*, and let* $F\colon D \to \mathcal{P}_{\emptyset}(D)$ *be an increasing upward multifunction. If* F *has a subpoint* $\underline{u}$ *and if all values of* F *are weakly sequentially closed, then* F *has a maximal fixed point* u^* *which satisfies* $\underline{u} \leqslant u^*$*.*

Proof: We are going to apply Theorem 2.10. To this end, note first that, by Proposition 2.25, D is compact and Fréchet-Urysohn with respect to the weak topology of X. Second, all values of F are countably universally inductive. Indeed, let $u \in D$ and any increasing sequence $(a_n) \subset D$ be given such that $(a_n) \leqslant^* F(u)$. Then there is a sequence (b_n) such that $a_n \leqslant b_n \in F(u)$ for all n. Since $(b_n) \subset D$, there is a subsequence (b_{k_n}) of (b_n) and $b \in D$ such that $b_{k_n} \rightharpoonup b$ (where $k\colon \mathbb{N} \to \mathbb{N}$ is strictly increasing). Now, for each n and all $m \geqslant n$ we have $a_n \leqslant a_{k_n} \leqslant a_{k_m} \leqslant b_{k_m}$, so that, by the dual of Proposition 2.5(i), $a_n \leqslant b$. Furthermore, we obtain $b \in F(u)$, since $F(u)$ is weakly sequentially closed. Thus, we have $(a_n) \leqslant^* b \in F(u)$, verifying that $F(u)$ is countable universally inductive. Thus, by Theorem 2.10, all claims follow. ○

Next, let us slightly generalize Theorem 2.30 to the following case, in which two reflexive ordered Banach spaces V and W are considered such that $W \subset V$ as posets, which means that we have $W \subset V$ and $u \leqslant v$ in W only if $u \leqslant v$ in V. A further compatibility condition between V and W is not needed. (If W is continuously embedded in V, then the proof simplifies and can be done similar as the proof of Theorem 2.30.)

2.31 Theorem *Let* V *and* W *be reflexive ordered Banach spaces such that* $W \subset V$ *as posets, and let* $F\colon D \subset V \to \mathcal{P}_{\emptyset}(D \cap W)$ *be a multifunction such that the following hypotheses are satisfied:*

(i) D *is bounded and weakly sequentially closed in* V*, and* F *has a subpoint* $\underline{u}$*.*

(ii) F *is increasing upward and has weakly sequentially closed values in* W*.*

(iii) The values of F *are uniformly bounded in* W*.*

Then F *has a maximal fixed point* u^* *such that* $\underline{u} \leqslant u^*$*.*

Proof: From $W \subset V$ and Proposition 2.25 we obtain that $F\colon D \to \mathcal{P}_\emptyset(D)$ is an increasing upward multifunction on a weakly compact and Fréchet-Urysohn subset D of V. Thus, by Corollary 2.8, it suffices to show that F has Property (Z). To this end, let increasing sequences $(u_n) \subset D \subset V$ and $(s_n) \subset D \cap W \subset V$ be given such that $u_n \leqslant s_n \in F(u_n)$. We have to prove that (s_n) has an upper bound $s^* \in \operatorname{Sub} F$.

Since (u_n) belongs to D, which is bounded and weakly sequentially closed in V, and since (s_n) belongs to $\bigcup_n F(u_n)$, which is bounded in W, there are subsequences (u_{k_n}) of (u_n) and (s_{k_n}) of (s_n) and $u \in D$ and $s \in W$ such that $u_{k_n} \rightharpoonup u$ in V and $s_{k_n} \rightharpoonup s$ in W. Since both (u_n) and (s_n) are increasing, we have furthermore from Proposition 2.5 that $u = u_\vee$ (the supremum in U) and $s = s_\vee$ (the supremum in W), and since s is an upper bound of (s_n) in W (and thus in V) and so of (u_n), we infer $u \leqslant s$.

Further, since F is increasing upward, it follows $s_{k_n} \leqslant^* F(u)$ for all n. Let $(s^*_{k_n}) \subset F(u)$ be a sequence such that $s_{k_n} \leqslant s^*_{k_n}$. Since $F(u)$ is bounded in W, there is $s^* \in W$ and a subsequence $(s^*_{l_n})$ of $(s^*_{k_n})$ such that $s^*_{l_n} \rightharpoonup s^*$ in W, and since $F(u)$ is weakly sequentially closed in W, we have $s^* \in F(u) \subset D$.

Now, from $s_{l_n} \leqslant s^*_{l_n}$ we infer, since W together with its weak topology is an ordered topological vector space, $s \leqslant s^*$. Further, from $u \leqslant s$ it follows $u \leqslant s^*$ and thus, again since F is increasing upward, $s^* \in F(u) \leqslant^* F(s^*)$, from which we have at once $s^* \in \operatorname{Sub} F$. Thus, s^* is the upper bound of (s_n) we searched for, and thus, by Corollary 2.8, F has a maximal fixed point u^* such that $\underline{u} \leqslant u^*$. ○

Of course, in Theorem 2.31 one can choose $V = W$ to obtain as a special case again Theorem 2.30.

2.32 Remark By inspecting the proof of Theorem 2.31, we see that we can replace Condition (iii) by the following more general condition:

(iii') Each value of F is bounded in W and if $(u_n) \subset D$ and $(s_n) \subset W$ are increasing sequences such that $u_n \leqslant s_n \in F(u_n)$, then (s_n) is bounded in W. ○

Finally, let us combine all results in the spirit of Theorem 1.71 to obtain an abstract framework that combines order-theoretical and topological properties with the concept of sub-supersolutions. This framework will be applied later to guarantee the existence of smallest and greatest solutions of multivalued variational inequalities. Since we will not need the framework in its most general form, let us formulate only a special version:

2.33 Theorem *Let V and W be reflexive ordered Banach spaces such that $W \subset V$ as ordered sets, and let $S\colon D \subset V \to \mathcal{P}_\emptyset(D \cap W)$ and $\underline{S}\colon D \to \mathcal{P}_\emptyset(V)$ be multifunctions such that the following conditions are satisfied:*

(i) D is a sup-semilattice, bounded and weakly sequentially closed in V, and there is $\underline{u} \in D$ such that $\underline{u} \leqslant^* \underline{S}(\underline{u})$.

(ii) In W, $S(D)$ is bounded and S has weakly sequentially closed values.

(iii) $\underline{S}$ is permanent upward, its values are directed upward and for all $v \in D$ it holds $S(v) \subset \underline{S}(v) \leqslant^* S(v)$.

Then $\operatorname{Fix} S$ has the greatest element u^* and it holds $\underline{u} \leqslant u^*$.

Proof: We proceed along the same lines as in the proof of Theorem 1.71, but of course the existence of some maximal element $u^* \in \operatorname{Fix} S$ such that $\underline{u} \leqslant u^*$ now follows from Theorem 2.31. To conclude the proof, note that from $S(v) \subset \underline{S}(v)$ it readily follows $\operatorname{Fix} S \subset \operatorname{Fix} \underline{S}$ (which is a special case of $\operatorname{Fix} S \leqslant^* \operatorname{Fix} \underline{S}$). ○

2.34 Remark In the analytic application in Part II, the properties of S and $\underline{S}$ which are hard to show are the relation $\underline{S}(v) \leqslant^* S(v)$ and that the values of $\underline{S}$ are directed upward. Those properties can be derived by use of involved analytic tools if one has not only $\underline{u} \in \operatorname{Sub} \underline{S}$ but even $\underline{u} \in \operatorname{Fix} \underline{S}$. Furthermore, we usually will introduce the superoperator $\overline{S}\colon D \to \mathcal{P}(D)$ as the dual counterpart to $\underline{S}$. Then, the existence of some $\overline{u} \in \operatorname{Fix} \overline{S}$ such that $\underline{u} \leqslant \overline{u}$ can be used to find both smallest and greatest fixed points of S between $\underline{u}$ and $\overline{u}$. This can be seen as a special case of Theorem 2.33 if one chooses $D = [\underline{u}, \overline{u}]$.

However, for our aims it is not enough to formulate only an even more special case of Theorem 2.33, as we will see in Chapter 7. There, we do not need a superoperator $\overline{S}$, and $\underline{u}$ can only be guaranteed to be a subpoint of the suboperator $\underline{S}$. ○

2.1.3 Various Fixed Point Theorems

It is illustrating to compare Theorem 2.30 with the following fixed point theorem, which is [24, Theorem 4.36].

2.35 Theorem *Let X be a lattice-ordered reflexive Banach space, let $D \subset X$, and let $S\colon D \to \mathcal{P}(D)$ be a multifunction such that the following hypotheses are satisfied:*

(i) *D is bounded, weakly sequentially closed and has an* inf-center $\underline{u}$, *i.e.* $\underline{u} \in X$ *and* $\underline{u} \wedge v \in D$ *for all* $v \in D$;

(ii) *S is increasing and has weakly sequentially closed values.*

Then S has a maximal fixed point. ○

The differences between our Theorem 2.30 and Theorem 2.35 are the following:

(i) We ask X only to be an ordered reflexive Banach space and F only to be increasing upward. It should be possible to prove Theorem 2.35 also under these more restrictive conditions.

(ii) We ask for an element $\underline{u} \in D$ such that $\underline{u} \leqslant^* S(\underline{u})$, whereas Theorem 2.35 asks for an element $\underline{u} \in X$ such that $\underline{u} \wedge v \in D$ for all $v \in D$. Both conditions are fulfilled if D is an order-interval $[\underline{u}, \overline{u}]$ or the increasing set $\underline{u}^{\uparrow}$.

Since we can reduce our considerations to the poset $D \cap \underline{u}^{\uparrow}$ (cf. with the deduction of Theorem 1.51 from Theorem 1.59), we can deduce Theorem 2.30 from Theorem 2.35 (if we ignore the differences listed in (i)) or from a weaker form of this theorem in which the set D is of the form $D \cap u^{\uparrow}$ for some $u \in D$. However, the proof of Theorem 2.35 bases also on Lemma 1.58 and is more involved.

Let us also compare our derived fixed point theorem with other widely used fix point theorems for multifunctions on a Banach space (where the following list is by no means conclusive; for more fixed point results (as well as related topics), we refer to, e.g., [24, 44, 45, 58, 63]).

As a rule of thumb, if one searches for fixed points of some multifunction $F\colon X \to \mathcal{P}(X)$, one needs two ingredients: the existence of elements with some special property, and the right compatibility of F with this property. In order-theory, usually there exist suprema of special sets, and F is of isotone type. If one has no partial order (or chooses to ignore it), usually topological fixed point theorems are considered. There, the special elements are convergent sequences which exist due to some compactness property of X, and the graph of F is closed in some sense, which results in some kind of continuity.

First, let us inspect two fixed point theorems for single-valued functions, the classical Theorem of Schauder and its generalization, a strong version of the Tychonoff fixed point theorem for *Hausdorff locally convex topological vector spaces* (see [88]):

2.36 Theorem (Schauder) *Let X be a Banach space, let $M \subset X$, and let $f\colon M \to M$ be a function such that the following hypotheses are satisfied:*

(i) M *is non-empty, compact and convex.*

(ii) f *is continuous.*

Then $\operatorname{Fix} f$ *is non-empty and compact.* ○

2.37 Theorem (Tychonoff) *Let X be a Hausdorff locally convex topological vector space, let $M \subset X$, and let $f\colon M \to M$ be a function such that the following hypotheses are satisfied:*

(i) M *is non-empty and convex.*

(ii) f *is continuous and* $f(M)$ *is contained in a compact subset of* M.

Then $\operatorname{Fix} f$ *is non-empty and compact.* ○

Those theorems rely on compactness—but we have used at least weak compactness in the deduction of Theorem 2.30, too. So the core difference between those topological fixed point theorems and ours is the continuity of the considered function f and the convexity assumption. If f is not continuous, purely topological fixed point theorems fail, while one has the chance of f being monotonous. But of course, if f is not monotonous, order-theoretical fixed point theorems fail and one might apply a topological result. Thus, these various fixed point results complement each other.

Interestingly enough, the fixed point theorem of Banach (for multifunctions on *complete metric spaces*) is proved in [52] in the context of *quasi-metric spaces* which are also used to provide a multivalued version of Kleene's order-theoretical fixed point theorem. To this end, a quasi-metric d on a poset D with values 0 and 1 is defined via $d(x, y) = 0$ if and only if $x \leqslant y$. Thus, there is a connection between the following two rather special fixed point theorems:

2.38 Theorem (Banach) *Let* X *be a complete metric space with metric* d*, and let* $F\colon X \to \mathcal{P}(X)$ *be a multifunction such that the following hypotheses are satisfied:*

(i) F *is a* contraction*, i.e. there is* $L \in [0, 1)$ *such that for all* $x, y \in X$ *and all* $a \in F(x)$ *there is* $b \in F(y)$ *such that* $d(a, b) \leqslant Ld(x, y)$.

(ii) F *has closed values.*

Then F *has a unique fixed point.* ○

2.39 Theorem (Kleene) *Let* D *be a poset in which each increasing sequence has a supremum, and let* $F\colon D \to \mathcal{P}(D)$ *be a multifunction such that the following hypotheses are satisfied:*

(i) F *has a subpoint.*

(ii) F *is increasing upward, and further, for each increasing sequence* $(x_n) \subset D$ *such that* $x_{n+1} \in F(x_n)$*, it holds* $x_\vee \in F(x_\vee)$ *(such a multifunction* F *is sometimes called* order-continuous*).*

Then F *has a fixed point. If* F *is furthermore increasing downward and if all values of* F *have the smallest element, then* F *has the smallest fixed point.* ○

Although Kleene's fixed point theorem is similar to Tarski's one, it is not useful for our targeted application, because it imposes a rather strong order-theoretical condition on F. More useful in the treatment of variational problems are the following generalized version of the Kakutani fixed point theorem (see [2]) and Kluge's fixed point theorem, which is applied in, e.g., [60, 61].

2.40 Theorem (Kakutani-Fan-Glicksberg) *Let* X *be a locally convex Hausdorff space, let* $M \subset X$*, and let* $F\colon M \to \mathcal{P}(M)$ *be a multifunction such that the following hypotheses are satisfied:*

(i) M *is non-empty, compact and convex.*

(ii) F *has a closed graph and its values are convex.*

Then $\operatorname{Fix} F$ *is non-empty and compact.* ○

2.41 Theorem (Kluge) *Let* X *be a reflexive ordered Banach space, let* $M \subset X$*, and let* $F\colon M \to \mathcal{P}(M)$ *be a multifunction such that the following hypotheses are satisfied:*

(i) M *is non-empty, weakly closed and convex.*

(ii) F *has weakly closed graph and closed and convex values.*

(iii) Either M *is bounded or* $F(M)$ *is bounded.*

Then F *has a fixed point.* ○

It could be an interesting endeavor to combine those purely topological fixed point theorems with our order-theoretical ones or to find a common generalization.

2.2 Continuous Operators

The concept of continuity generalizes with ease to multifunctions $F\colon X \to \mathcal{P}_\emptyset(Y)$ between topological spaces if one defines pre-images. To this end, one has to generalize the condition $f(x) \in U$ in the definition of pre-images of single-valued functions to the set-valued case, and as a matter of fact, there are two different ways of doing so:

2.42 Definition Let X and Y be sets, let $F\colon X \to \mathcal{P}_\emptyset(Y)$ be a multifunction, and let M be a subset of Y.

(i) The **(large) pre-image** of F is defined by $F_-^{-1}(M) := \{x \in X : F(x) \cap M \neq \emptyset\}$.

(ii) The **small pre-image** of F is defined by $F_+^{-1}(M) := \{x \in X : F(x) \subset M\}$. ○

Thus, we have two different notions of continuity for multifunctions (however, in this thesis, we will deal only with upper semicontinuous multifunctions, as they allow for measurable selections in bifunctions, see Theorem 3.47 below):

2.43 Definition Let X and Y be topological spaces, and let $F\colon X \to \mathcal{P}_\emptyset(Y)$ be a multifunction.

(i) F is called **upper semicontinuous** if $F_+^{-1}(U)$ is open for all open $U \subset Y$.

(ii) F is called **lower semicontinuous** if $F_-^{-1}(U)$ is open for all open $U \subset Y$. ○

2.44 Remark It is readily seen that $X \setminus F_+^{-1}(M) = F_-^{-1}(Y \setminus M)$. By this knowledge, we can apply the so called topological duality. For instance, $F\colon X \to \mathcal{P}_\emptyset(Y)$ is upper semicontinuous if and only if $F_-^{-1}(C)$ is closed for any closed subset C of Y. ○

In Proposition 1.26 we have seen that some properties of a multifunction F of isotone type can be derived by knowledge of the properties of F^*. Similar results hold for semicontinuous multifunctions from X to Y if Y is an ordered topological space. To this end, let us define first what it means for single-valued functions to be semicontinuous:

2.45 Definition Let X be a topological space, let R be an ordered topological space, and let $f\colon X \to R$ be a function.

(i) f is called **upper semicontinuous** if $f^{-1}(\alpha^\uparrow)$ is closed for all $\alpha \in R$.

(i)$_d$ f is called **lower semicontinuous** if $f^{-1}(\alpha^\downarrow)$ is closed for all $\alpha \in R$. ○

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ be partially ordered by extending the natural order on $\mathbb{R}$ by setting $-\infty < x < +\infty$ for all $x \in \mathbb{R}$, and $-\infty < +\infty$. If the topology on $\overline{\mathbb{R}}$ is chosen to be its interval topology, $\overline{\mathbb{R}}$ is an ordered topological space, and if a functional $f\colon X \to \overline{\mathbb{R}}$ is semicontinuous in the sense of Definition 2.45 then it is semicontinuous in the usual sense. This holds also if X is a normed linear space with its weak topology, in which case an upper semicontinuous function is called **weakly upper semicontinuous**. Indeed, we have the following well-known result:

2.46 Proposition *Let* X *be a topological space and let* $f\colon X \to \overline{\mathbb{R}}$ *be a function. If* f *is upper semicontinuous with respect to Definition 2.45, then for all* $x \in X$ *and all sequences* $(x_n) \subset X$ *with* $x_n \to x$ *it holds* $\limsup_n f(x_n) \leqslant f(x)$*. The converse holds true if* X *is* ***sequential****, i.e. if a set* C *is closed if and only if for each convergent sequence* $(x_n) \subset C$ *one has* $\lim_n x_n \in C$*.*

Proof: Let f be upper semicontinuous, let $x_n \to x$ in X and assume $\limsup_n f(x_n) > f(x)$. Then there is a subsequence (y_n) of (x_n) and $y > f(x)$ such that $f(y_n) \geqslant y$. Since $f^{-1}(y^{\uparrow})$ is closed (and thus sequentially closed), we deduce $f(x) \geqslant y > f(x)$, which is a clear contradiction. Thus, $\limsup_n f(x_n) \leqslant f(x)$.

The other way around, assume that $f^{-1}(y^{\uparrow})$ is not closed for some $y \in \overline{\mathbb{R}}$. Then, since X is sequential, we have $x_n \to x$ for some sequence $(x_n) \subset f^{-1}(y^{\uparrow})$ and some $x \notin f^{-1}(y^{\uparrow})$. Then, there is $\varepsilon > 0$ such that $f(x) + \varepsilon < y$ and some element x_0 of (x_n) such that

$$y \leqslant f(x_0) \leqslant \limsup_n f(x_n) + \varepsilon \leqslant f(x) + \varepsilon < y,$$

which is a contradiction. Thus, f is upper semicontinuous. ○

We have chosen our general definition of upper semicontinuous functions such that it fits nicely to multivalued upper semicontinuity, as seen in the next result:

2.47 Proposition *Let* X *be a topological space, let* R *be an ordered topological space, and let* $F\colon X \to \mathcal{P}(R)$ *be a multifunction.*

(i) *Let* F *be upper semicontinuous, and suppose that the values of* F *have the greatest element* $F^*(x) \in F(x)$*. Then* $F^*\colon X \to R$ *is upper semicontinuous.*

(i)$_d$ *Let* F *be upper semicontinuous, and suppose that the values of* F *have the smallest element* $F_*(x) \in F(x)$*. Then* $F_*\colon X \to R$ *is lower semicontinuous.*

(ii) *Suppose that* $F(x) = [F_*(x), F^*(x)]$ *for all* $x \in X$*, and let* either X *be sequential and* $R = \mathbb{R}$*,* or*, more generally, let* R *be a chain equipped with its interval topology. If* F_* *and* F^* *are lower semicontinuous and upper semicontinuous, respectively, then* F *is upper semicontinuous.*

Proof: Concerning (i), let us note that for all $\alpha, \beta \in R$ it holds

$$(F^*)^{-1}(\alpha^{\uparrow}) = F_-^{-1}(\alpha^{\uparrow}) \quad \text{and} \quad (F^*)^{-1}(\beta^{\downarrow}) = F_+^{-1}(\beta^{\downarrow}). \tag{2.1}$$

From the first equality in (2.1), assertion (i) follows, and (i)$_d$ follows by duality (i.e. by application of the second equality in (2.1)).

To prove (ii), suppose first that X is sequential and that $R = \mathbb{R}$, let $C \subset \mathbb{R}$ be closed and let $(x_n) \subset F_-^{-1}(C)$ be a sequence converging to x. Then there are $y_n \in F(x_n) \cap C$ and it follows

$$F_*(x) \leqslant \liminf_n F_*(x_n) \leqslant \liminf_n y_n \leqslant \limsup_n y_n \leqslant \limsup_n F^*(x_n) \leqslant F^*(x),$$

implying, e.g., $\liminf_n y_n \in F(x) \cap C$ and thus $x \in F_-^{-1}(C)$. Since X is sequential, $F_-^{-1}(C)$ is closed. Consequently, F is upper semicontinuous.

If R is a general chain equipped with its interval topology, let any open set $U \subset R$ be given and let $x \in X$ be such that $x \in F_+^{-1}(U)$, implying $[F_*(x), F^*(x)] \subset U$. Since $F^*(x) \in U$, either $F^*(x)$ is the greatest element of R, or there is $y^* > F^*(x)$ such that $[F^*(x), y^*) \subset U$ (where $[a, b) = a^{\uparrow} \cap b^{\downarrow\downarrow}$). In the latter case, $\{x' \in X : F^*(x') < y^*\}$ is open due to the upper semicontinuity of F^*. By dual reasoning, either $F_*(x)$ is the smallest element of R, or there is $y_* < F_*(x)$ such that $(y_*, F_*(x)] \subset U$ and the set $\{x' \in X : F_*(x') > y^*\}$ is open. Consequently, there is an open neighborhood N of x such that for all $x' \in N$ one has $[F_*(x'), F^*(x')] \subset U$, meaning $x' \in F_+^{-1}(U)$, which implies that F is upper semicontinuous. ○

2.48 Remark A single-valued function $f\colon \mathbb{R} \to \mathbb{R}$ which is upper semicontinuous as a single-valued function is in general not upper semicontinuous as a multivalued function. For instance, consider the function $f\colon \mathbb{R} \to \{0, 2\}$ with $f(x) = 0$ if and only if $x < 0$. Then f is obviously upper semicontinuous as a single-valued function, but not upper semicontinuous as a multifunction, since we have, e.g., $0 \in f_+^{-1}((1, 3))$, but, for every $x < 0$, $x \notin f_+^{-1}((1, 3))$, so that $f_+^{-1}((1, 3))$ is not open. ○

Finally, let us state two well known results about functions of continuous type:

2.49 Proposition *Let X and Y be Banach spaces.*

(i) If $K \subset X$ is convex and if $f\colon K \to [-\infty, +\infty]$ is a convex functional, then f is lower semicontinuous if and only if it is weakly lower semicontinuous.

*(ii) If $A\colon X \to Y$ is linear and continuous, then $u_n \rightharpoonup u$ in X implies $Au_n \rightharpoonup Au$ in Y (which means that A is **weakly continuous**).* ○

2.3 Operators of Monotone Type

Until now, we have only considered multifunctions of isotone type as introduced in Subsection 1.1.3. The basic idea was to generalize the notion of increasing functions $f\colon \mathbb{R} \to \mathbb{R}$ to multifunctions $F\colon D \to \mathcal{P}_\emptyset(D')$ on arbitrary posets D and D'.

However, there is another way of generalization. To this end, note that $f\colon \mathbb{R} \to \mathbb{R}$ is increasing if and only if for all $x, y \in \mathbb{R}$ it holds $(f(x) - f(y))(x - y) \geqslant 0$. This characterization can be generalized to multifunctions $F\colon X \to \mathcal{P}_\emptyset(X')$ from a normed space to its dual (as seen below). This concept can be generalized even further to so called generalized pseudomonotone operators. This theory was developed in order to obtain very general existence results for variational inequalities, and thus the results presented in this section will be of equal importance as the fixed point results presented above.

2.3.1 Basic Concepts

Let us first recall the basic definitions for single-valued operators $A\colon X \to X'$:

2.50 Definition Let X be a Banach space, and let $A\colon X \to X'$ be an operator.

(i) A is called **coercive** if

$$\frac{\langle Au_n, u_n\rangle}{\|u_n\|} \to \infty \quad \text{for all sequences } (u_n) \subset X \text{ such that } \|u_n\| \to \infty.$$

(ii) A is called **monotone** if $\langle Au - Av, u - v\rangle \geqslant 0$ for all $u, v \in X$.

(iii) A is called **pseudomonotone** if from

$$u_n \rightharpoonup u \text{ in } X \quad \text{and} \quad \limsup_n \langle Au_n, u_n - u\rangle \leqslant 0 \tag{2.2}$$

it follows

$$\langle Au, u - v\rangle \leqslant \liminf_n \langle Au_n, u_n - v\rangle \quad \text{for all } v \in X. \qquad \bigcirc$$

We have the following connection to monotone operators and a useful property of pseudomonotone operators:

2.51 Proposition *Let X be a Banach space, and let $A\colon X \to X'$ be an operator.*

*(i) If A is **completely continuous** (i.e. $u_n \rightharpoonup u$ in X implies $Au_n \to Au$ in X'), then A is pseudomontone.*

(ii) A is pseudomonotone if and only if (2.2) implies

$$Au_n \rightharpoonup Au \quad \textit{and} \quad \langle Au_n, u_n\rangle \to \langle Au, u\rangle. \qquad \bigcirc$$

In [81] (see also, e.g., [115]) the so called **Leray-Lions operators** $A\colon X \to X'$ are defined, which are operators such that $Au = a(u, u)$ for some operator $a\colon X \times X \to X'$ satisfying certain topological conditions. As seen in [115], each Leray-Lions operator is pseudomonotone. We will give an example of outmost importance for our study of variational inequalities in Subsection 3.3.3 below, where Sobolev spaces are studied.

Next, let us consider multivalued operators $A\colon X \to \mathcal{P}_\emptyset(X')$.

2.52 Definition Let X be a Banach space, and let $A\colon X \to \mathcal{P}_\emptyset(X')$ be a multivalued operator.

(i) A is called **bounded** if it maps bounded sets to bounded ones.

(ii) A is called **coercive** with respect to $u_0 \in X$ if there exists a function $c\colon \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{s\to\infty} c(s) = \infty$ such that

$$\langle u', u - u_0\rangle \geqslant c(\|u\|)\|u\| \quad \text{for all } u \in X \text{ and all } u' \in Au.$$

(iii) A is called **monotone** if $\langle u' - v', u - v\rangle \geqslant 0$ for all $(u, u'), (v, v') \in \operatorname{gr} A$.

(iv) A is called **maximal monotone** if A is maximal in the set of all monotone operators on X, ordered by inclusion.

(v) A is called **pseudomonotone** if

(α) A has nonempty, bounded, closed and convex values,

(β) A is upper semicontinuous from each finite-dimensional subspace of X to X' with its weak topology,

(γ) if $u_n \rightharpoonup u$ in X and if $u'_n \in Au_n$ is such that $\limsup_n \langle u'_n, u_n - u\rangle \leqslant 0$, then for each $v \in X$ there is $u' \in Au$ such that $\langle u', u - v\rangle \leqslant \liminf_n \langle u'_n, u_n - v\rangle$. ○

For pseudomonotone multifunctions we have the following results, which simplify some calculations:

2.53 Proposition *Let X be a Banach space, and let $A, B\colon X \to \mathcal{P}_\emptyset(X')$ be multivalued operators.*

(i) Let A be single-valued. Then A is pseudomonotone with respect to Definition 2.50 if and only if A is pseudomonotone with respect to Definition 2.52.

(ii) Let A and B be pseudomonotone. Then $A + B$ is pseudomonotone. ○

From [92, Prop. 2.2] we deduce the following result (which can be stated in more generality):

2.54 Proposition *Let X be a reflexive Banach space and let $A\colon X \to \mathcal{P}(X')$ be a multivalued operator having the following properties:*

(i) A has non-empty, closed and convex values,

(ii) A is bounded,

*(iii) the graph of A is **sequentially weakly closed**, i.e. for all weakly convergent sequences $u_n \rightharpoonup u$ in X and $u^*_n \rightharpoonup u^*$ in X' with $u^*_n \in Au_n$ one has $u^* \in Au$,*

*(iiv) the duality pairing is **(w×w)-continuous** on $\operatorname{gr} A$, i.e. $u_n \rightharpoonup u$ in X and $u^*_n \rightharpoonup u^*$ in X' with $u^*_n \in Au_n$ imply $\langle u^*_n, u_n\rangle \to \langle u^*, u\rangle$.*

Then A is pseudomonotone. ○

2.3.2 Surjectivity Results

For single-valued pseudomonotone operators, we have the following result:

2.55 Theorem (Main Theorem on Single-Valued Pseudomonotone Operators) *Let X be a reflexive Banach space, and let $A\colon X \to X'$ be a pseudomonotone, bounded and coercitive operator. Then A is surjective.* ○

It should be noted that the proof of Theorem 2.55, as presented, e.g., in [98], allows for a slightly more general theorem. Moreover, we refer the interested reader to the original papers [14], [17] and [85].

A consequence of Theorem 2.55 is the well-known Lax-Milgram theorem about bilinear forms. (If in doubt about the validity of this corollary, please see [33] for a very detailed and elementary proof.)

2.56 Corollary (Lax-Milgram) *Let* H *be a real* Hilbert space, *and let* $A\colon H \to H'$ *be a linear bounded operator (which induces a bounded* bilinear *function* $a\colon X \times X \to \mathbb{R}$ *via* $a(u,v) = \langle Au, v\rangle$, *and vice versa) such that there is* $c > 0$ *such that* $\langle Au, u\rangle \geqslant c\|u\|^2$ *for all* $u \in X$. *Then* A *is surjective (and has, in fact, a bounded inverse).* ○

In the sequel, the following existence theorem for perturbed coercive pseudomonotone operators will be a key element:

2.57 Theorem (Main Theorem on Multivalued Pseudomonotone Operators) *Let* X *be a reflexive Banach space, let* $A\colon X \to \mathcal{P}(X')$ *be a bounded, pseudomonotone multifunction with closed and convex values, let* $M\colon X \to \mathcal{P}_{\emptyset}(X)$ *be a maximal monotone multifunction, and suppose* $M(u_0) \neq \emptyset$ *for some* $u_0 \in X$. *If* A *is coercive with respect to* u_0, *then* $A+M$ *is surjective, i.e. for all* $u' \in X'$ *there is* $u \in X$ *such that* $u' \in (A+M)(u)$. ○

The proof of Theorem 2.57 along with other information is contained in, e.g., [92]. There, also generalized pseudomonotone operators and more general surjectivity results are presented. We also refer to [65, Theorem 2.2] for the following result (among others) in which the whole operator $A + M$ is assumed to be generalized coercive:

2.58 Theorem *Let* X *be a reflexive Banach space, let* $A\colon X \to \mathcal{P}(X')$ *be a bounded and pseudomonotone multifunction, let* $M\colon \mathcal{D}(B) \subset X \to \mathcal{P}(X')$ *be a maximal monotone multifunction, and let* $u' \in X'$. *Assume there exist* $u_0 \in X$ *and* $R \geqslant \|u_0\|$ *such that*

$$\mathcal{D}(M) \cap \{x \in X : \|x\| < R\} \neq \emptyset \qquad \text{and} \qquad \langle \eta + \xi - u', u - u_0\rangle > 0$$

for all $u \in \mathcal{D}(M)$ *with* $\|u\| = R$, *all* $\eta \in A(u)$, *and all* $\xi \in M(u)$. *Then the inclusion* $A(u) + M(u) \ni u'$ *has a solution.* ○

2.3.3 Subdifferentials

Let us start with the standard subdifferential of Convex Optimization:

2.59 Definition Let X be a Banach space, and let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a functional.

(i) f is called **proper** if its effective domain $\mathcal{D}(f)$ is non-empty.

(ii) Let f be convex and proper. An element $u' \in X'$ is called a **subgradient** of f at $u \in \mathcal{D}(f)$ if
$$f(v) \geqslant f(u) + \langle u', v - u\rangle \quad \text{for all } v \in X.$$

(iii) The set of all subgradients of f at u is called **subdifferential** of f at u, and is denoted by $\partial f(u)$. ○

2.60 Example Let K be a non-empty, closed and convex subset of a Banach space X. Then the **indicator function** $I_K\colon X \to \mathbb{R} \cup \{+\infty\}$ of K, defined by

$$I_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{otherwise,} \end{cases}$$

is a convex and proper functional with effective domain $\mathcal{D}(I_K) = K$. Moreover, we have the following variational characterization of ∂I_K, which follows readily from its definition: For any $u \in X$, we have $u' \in \partial I_K(u)$ if and only if

$$u \in K \quad \text{and} \quad \langle u', v - u \rangle \leqslant 0 \quad \text{for all } v \in K.$$

We will use this special functional (and some generalization of it) in the applications below in order to transform a variational inequality into a multivalued inclusion. ○

To apply the surjectivity result Theorem 2.57, we need further the following classical result, proved in [97]:

2.61 Theorem *Let X be a reflexive Banach space, and let $f\colon X \to \mathbb{R}\cup\{+\infty\}$ be proper, convex and lower semicontinuous. Then $\partial f\colon X \to \mathcal{P}_\emptyset(X')$ is maximal monotone.* ○

Now, let us generalize the subgradient to the famous generalized gradient of Clarke (for the definitions and basic results see, e.g., [32]), which can be formed for all locally Lipschitz functions and has a wide range of applications in Variational Analysis.

2.62 Definition Let X be a Banach space, and let $f\colon X \to \mathbb{R}$ be a function.

(i) f is said to be **locally Lipschitz** if for every $x \in X$ there are an open neighborhood U of x and a constant $L \geqslant 0$ such that $|f(x) - f(y)| \leqslant L\|x - y\|$ for all $y \in U$.

(ii) Let f be locally Lipschitz. Then the **generalized directional derivative of f at x in the direction v** is defined as

$$f^\circ(x, v) := \limsup_{y \to x, t \downarrow 0} \frac{f(y + tv) - f(y)}{t}$$

and the **generalized gradient of f at x** is defined as

$$\partial f(x) := \{\varphi \in X' : \langle \varphi, v \rangle \leqslant f^\circ(x, v) \text{ for all } v \in X\}.$$ ○

Let us recall some basic properties of the generalized gradient:

2.63 Lemma *Let X be a Banach space, and let $f\colon X \to \mathbb{R}$ be locally Lipschitz. Then the following holds true:*

(i) For all $x, v \in X$, $f^\circ(x, v)$ is well-defined and finite.

(ii) For all $x \in X$, the mapping $v \mapsto f^\circ(x, v)$ is positively homogeneous and subadditive.

(iii) For all $x \in X$ one has $f^\circ(x, -v) = (-f)^\circ(x, v)$.

(iv) The function $-f$ is locally Lipschitz, too, and one has $\partial(-f) = -\partial f$.

(v) For all $v \in X$, the mapping $x \mapsto f^\circ(x, v)$ is upper semicontinuous. ○

There is a rich theory about Clarke's generalized gradient. However, in the sequel, we need only the following simple result:

2.64 Proposition *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be locally Lipschitz. Then for all* $x \in \mathbb{R}$ *we have*

$$\partial f(x) = [-f^{\circ}(x,-1), f^{\circ}(x,1)] \neq \emptyset. \tag{2.3}$$

Proof: In the case $X = \mathbb{R}$, Clarke's generalized gradient is given by

$$\partial f(x) = \{\varphi \in \mathbb{R} : \varphi v \leqslant f^{\circ}(x,v) \text{ for all } v \in \mathbb{R}\}. \tag{2.4}$$

As $v \mapsto f^{\circ}(x,v)$ is positively homogeneous, (2.4) implies that $\varphi \in \partial f(x)$ if and only if $\varphi \leqslant f^{\circ}(x,1)$ for all $v > 0$ and $\varphi \geqslant -f^{\circ}(x,-1)$ for all $v < 0$. This proves (2.3). Furthermore, from

$$0 = f^{\circ}(x,0) \leqslant f^{\circ}(x,-1) + f^{\circ}(x,1).$$

it follows that $\partial f\colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is well-defined. ○

At the end of this subsection, it should be noted that there is a more general subdifferential, the so called approximate subdifferential introduced by Mordukhovich, which is widely used in contemporary Optimization, but has, in general, not so good analytic properties (see, e.g., [87]). It would be interesting to investigate in what extend the results of this thesis can be generalized in this direction.

3 | Measure Theory

In this last theoretical chapter, we investigate the connection between order-theoretical, topological and measure-theoretical concepts for multifunctions. In particular, we explore conditions under which multifunctions have measurable single-valued selections.

For basic notations and results we refer, e.g., to [11, 12, 53, 96, 119].

3.1 Measurable Multifunctions

3.1.1 Ordered Measurable Spaces

Recall that a **measurable space** is a set X together with a **σ-algebra** $\mathcal{A}$, which is a family $\mathcal{A} \subset \mathcal{P}_\emptyset(X)$ of subsets of X such that $X \in \mathcal{A}$ and such that $\mathcal{A}$ is closed under complementation and countable unions. The sets in $\mathcal{A}$ are called **measurable**. To every family $\mathcal{M} \subset \mathcal{P}_\emptyset(X)$ let $\sigma(\mathcal{M})$ be the smallest σ-algebra on X (with respect to set-inclusion) which contains $\mathcal{M}$, and $\mathcal{M}$ is called **base** of a σ-algebra $\mathcal{A}$ if $\sigma(\mathcal{M}) = \mathcal{A}$. If (X, τ) is a topological space, then, unless otherwise stated, we will equip X with its **Borel σ-algebra** $\mathcal{B}(X) := \sigma(\tau)$. For $X = \mathbb{R}^n$, the **Lebesgue σ-algebra** $\mathcal{L}(X)$ is the *completion* of $\mathcal{B}(X)$ with respect to the *Lebesgue measure* λ. Measurable functions into a topological space are defined as follows:

3.1 Definition Let X be a measurable space, and let R be a topological space. Then a function $f\colon X \to R$ is called **measurable** if, for any open set $A \subset R$, the pre-image $f^{-1}(A)$ is measurable in X. ○

Evidently, if X is a topological space and if $(X, \mathcal{A})$ is a measurable space such that $\mathcal{B}(X) \subset \mathcal{A}$, and if R is any topological space, then any continuous function $f\colon X \to R$ is measurable. To have an analogous result for increasing functions, we have to bring order and measure together. To this end, we recall from Definition 1.8 that a subset A of a poset D is called increasing if $a^\uparrow \subset A$ for all $a \in A$, and introduce the following compatibility condition:

3.2 Definition Let $(X, \leqslant)$ be a poset, and let $(X, \mathcal{A})$ be a measurable space. Then $X = (X, \leqslant, \mathcal{A})$ is called **ordered measurable space** if all increasing subsets of X are measurable. ○

3.3 Remark Let M be a decreasing subset of a poset D. Then the complement $D \setminus M$ is increasing. Indeed, let $a \in D \setminus M$ and $b \in D$ with $a \leqslant b$ be given, and assume that $b \notin D \setminus M$. Then $b \in M$ and thus $a \in M$, which is a contradiction. Thus, $D \setminus M$ is increasing. It follows that in any ordered measurable space decreasing sets are measurable, too, so that we can apply order-duality. ○

The notion of ordered measurable spaces is a natural combination of order and measurability that mimics the way how ordered topological spaces are defined: As well as it is favored that order-intervals $a^{\uparrow}$ and $b^{\downarrow}$ are closed, it is favored that increasing and decreasing sets are measurable, which implies that multifunctions of isotone type with compact values in $\mathbb{R}$ are measurable, see Corollary 3.21 below. The same holds, of course, for single-valued functions, along with its converse:

3.4 Proposition *Let $(X, \leqslant)$ be a poset, and let $(X, \mathcal{A})$ be a measurable space. Then the following assertions are equivalent:*

(i) $X = (X, \leqslant, \mathcal{A})$ is an ordered measurable space.

(ii) Every increasing function $f \colon X \to \mathbb{R}$ is measurable.

Proof: First, let $(X, \leqslant, \mathcal{A})$ be an ordered measurable space and let $f \colon X \to \mathbb{R}$ be increasing. To show that f is measurable, let $a \in \mathbb{R}$, and $x \geqslant y$ for some $y \in f^{-1}(a^{\uparrow})$. Then it follows $f(x) \geqslant f(y) \geqslant a$, which implies $x \in f^{-1}(a^{\uparrow})$. Thus, the set $f^{-1}(a^{\uparrow})$ is increasing and thus measurable. Since the sets $a^{\uparrow}$ form a base of $\mathcal{B}(\mathbb{R})$, f is measurable.

Now, suppose that (ii) holds true, and let $A \subset X$ be any increasing set. Then the **characteristic function**

$$\chi_A \colon X \to \mathbb{R}, \quad x \mapsto \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

is increasing. Indeed, for $x \leqslant y$ we have either $f(x) = 0$ and thus $f(x) \leqslant f(y)$, or $f(x) = 1$ and thus $x \in A$, thus $y \in A$, thus $f(y) = 1$, and thus $f(x) \leqslant f(y)$. So, f is measurable, which implies that $A = f^{-1}(1)$ is measurable, too. Consequently, $(X, \leqslant, \mathcal{A})$ is an ordered measurable space. ○

Our notion of ordered measurable spaces is very natural, but it seems to be no standard in literature. However, there are other notions of ordered measurable spaces, which serve other purposes.

First, we may mention the extensively studied so called **totally ordered topological spaces**, which are chains X equipped with their interval topology and Baire or Borel measures, see, e.g., [102]. We will not deal with them, since the presumption that X is a chain is to strong for our aim. Moreover, it is no option to drop the requirement that X is a chain, since then increasing sets may not be measurable, as the following example shows.

3.5 Example Consider $D = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$, equipped with component-wise ordering. Then, all subsets of D are increasing, but $a^{\uparrow\uparrow} = a^{\downarrow\downarrow} = \emptyset$ for all $a \in D$, so that both the interval topology of D and the induced Borel algebra equal $\{\emptyset, D\}$. Thus, there are many increasing sets which a not measurable. ○

Second, we mention so called proper ordered spaces as defined in [117], which allow to extend orders to spaces of probabilities, which leads to economic applications.

3.6 Definition Let $(X, \leqslant)$ be a poset, and let $(X, \mathcal{A})$ be a measurable space. Then $X = (X, \leqslant, \mathcal{A})$ is called **proper ordered space** if $\mathcal{A}$ has a base $\mathcal{B}$ such that $\mathcal{B}$ generates $\leqslant$ in the sense that $a \leqslant b$ holds if and only if, for all $B \in \mathcal{B}$, $a \in B$ implies $b \in B$. ○

Evidently, ordered measurable spaces and proper ordered spaces share a common core. To make the connection clear, we denote by $\mathcal{I}$ the **set of all increasing subsets** of a poset D. Then we have the following results:

(i) Let $\mathcal{B}$ be a base of a proper ordered space $(X, \leqslant, \mathcal{A})$, then clearly every $B \in \mathcal{B}$ is an increasing set. Thus, we have

$$\mathcal{A} = \sigma(\mathcal{B}) \subset \sigma(\mathcal{I}),$$

and since $\mathcal{B}$ generates $\leqslant$, $\mathcal{I}$ generates $\leqslant$, too.

(ii) Let $(X, \leqslant, \mathcal{A})$ be an ordered measurable space, then the set $\mathcal{B}$ consisting of all intervals $x^{\uparrow}$ generates $\leqslant$ (and thus $\mathcal{I}$ generates $\leqslant$, too) and we have

$$\sigma(\mathcal{B}) \subset \sigma(\mathcal{I}) \subset \mathcal{A}.$$

Thus, every proper ordered space can be made to be an ordered measurable space if one extends $\mathcal{A}$ such that $\mathcal{A} = \sigma(\mathcal{I})$, and every ordered measurable space can be made to be a proper ordered space if one restricts $\mathcal{A}$ such that $\mathcal{A} = \sigma(\mathcal{I})$, in which case $\mathcal{I}$ is the base. However, neither of those two kinds of measurable spaces equipped with a partial order is a special case of the other, as the following simple examples show:

3.7 Example The set $\mathbb{R}$, ordered canonically and equipped with the Lebesgue σ-algebra $\mathcal{L}(\mathbb{R})$, is an ordered measurable space, since each increasing, non-trivial set M is of the form $\alpha^{\uparrow}$ or $\alpha^{\uparrow\uparrow}$ with $\alpha = \inf M$, such that M is even contained in $\mathcal{B}(\mathbb{R})$. But $\mathbb{R}$ is not a proper measurable space, since $\sigma(\mathcal{I}) = \mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$. ○

For the next example, we need the following projection theorem:

3.8 Theorem *Let $\Omega \subset \mathbb{R}^n$ be open, and let $M \subset \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$ be given (where $\mathcal{L} \otimes \mathcal{B}$ denotes the* product σ-algebra *of two σ-algebras $\mathcal{L}$ and $\mathcal{B}$). Then the projection* $\mathrm{proj}_{\Omega}(M)$ *of M to Ω belongs to $\mathcal{L}(\Omega)$.* ○

For a proof, see, e.g., [31, Theorem III.23], or [53, Theorem II.1.33], where a far more general version is presented.

3.9 Example The set $\mathbb{R}^2$, equipped with coordinate-wise ordering and the σ-algebra $\mathcal{A}$ generated by all sets of the form $x^{\uparrow}$ is a proper measurable space, since the sets $x^{\uparrow}$ generate $\leqslant$. But the considered space is not an ordered measurable space. Indeed, since $x^{\uparrow}$ is closed with respect to the Euclidean topology, all measurable sets are Borel measurable, i.e. $\mathcal{A} \subset \mathcal{B}(\mathbb{R}^2)$. However, let $A \subset \mathbb{R}$ be *not* Lebesgue measurable, then $M := \bigcup_{a \in A} (a, -a)^{\uparrow}$ is increasing, but not Borel measurable (which is a consequence of Theorem 3.8, since $A = \mathrm{proj}_{\mathbb{R}}(M \setminus \{(x, y) \in \mathbb{R}^2 : x + y > 0\})$.) Thus, there are non-measurable increasing sets. (Cf. Example 3.5.) ○

Third, let us shortly consider so called **natural proper ordered spaces** (see [117]), which are proper ordered spaces whose σ-algebra is generated by the sets $x^{\uparrow}$ and $x^{\downarrow}$. Those natural proper ordered spaces have, despite their name, the unnatural defect that a subset of them may not be again a natural proper ordered space: Consider, e.g., the set $M := \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ as a subset of $\mathbb{R}^2$. All sets $x^{\uparrow}$ and $x^{\downarrow}$ in M are singletons. Thus, the generated σ-algebra is only the algebra of all countable and co-countable subsets of M, which is smaller than the induced σ-algebra on M.

Fortunately, this unpleasant property does not hold neither for ordered measurable spaces nor for proper ordered spaces:

3.10 Proposition *Let $(X, \leqslant, \mathcal{A})$ be an ordered measurable space or a proper ordered space, and let $M \subset X$ be any subset. Then $(M, \leqslant_{|M}, \mathcal{A}_{|M})$ is an ordered space or an proper ordered space, respectively, where $\leqslant_M$ and $\mathcal{A}_{|M}$ denote the canonical restriction of $\leqslant$ and $\mathcal{A}$ to M, i.e. for $a, b \in M$ it holds $a \leqslant_{|M} b$ if and only if $a \leqslant b$, and $A \in \mathcal{A}_{|M}$ if and only if $A = B \cap M$ for some $B \in \mathcal{A}$.*

Proof: First, let $(X, \leqslant, \mathcal{A})$ be an ordered measurable space and $M \subset X$. Then, for every $A \subset M$, we have in $(X, \leqslant)$ that the set $A^{\uparrow} := \bigcup_{a \in A} a^{\uparrow}$ is increasing and thus $A^{\uparrow} \in \mathcal{A}$. Further, if A is increasing in $(M, \leqslant_{|M})$, it holds $A = A^{\uparrow} \cap M$, as, obviously, $A \subset A^{\uparrow} \cap M$, and $b \in A$ for all $b \in M$ such that there is $a \in A$ with $a \leqslant b$. Thus, $A \in \mathcal{A}_{|M}$ and $(M, \leqslant_{|M}, \mathcal{A}_{|M})$ is seen to be an ordered measurable space.

Second, let $(X, \leqslant, \mathcal{A})$ be a proper ordered space with base $\mathcal{B}$ and let $M \subset X$. Then $\mathcal{B}_M := \{B \cap M : B \in \mathcal{B}\}$ is obviously a base of $\mathcal{A}_{|M}$. Further, we have $a \leqslant_{|M} b$ if and only if $a \in B \cap M$ implies $b \in B \cap M$ for all $B \in \mathcal{B}$. Indeed, since any $B \in \mathcal{B}$ is increasing with respect to $\leqslant$, $B \cap M$ is increasing with respect to $\leqslant_{|M}$; and if $a \leqslant_{|M} b$ does not hold, then $a \leqslant b$ does not hold, and thus there is some $B \in \mathcal{B}$ such that $a \in B$ and $b \notin B$, wherefore $a \in B \cap M$ and $b \notin B \cap M$. Thus, $(M, \leqslant_{|M}, \mathcal{A}_{|M})$ is seen to be a proper ordered measurable space. ○

It follows that $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ is not an ordered measurable space (since all subsets of its subset $\{(x, y) \in \mathbb{R}^2 : x + y = 0\}$ are increasing, but in general not Borel measurable). However, we already know that $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is an ordered measurable space (cf. Example 3.7). Thus, if $(X, \leqslant)$ is not a chain, the measurability of null-sets is of interest, which leads us to the following examples of ordered measurable spaces.

3.11 Proposition *For any $n \in \mathbb{N}$, $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$ is an ordered measurable space.*

Proof: All subsets of $\mathbb{R}^0 = \{0\}$ are Borel measurable. Now let us proceed by induction and assume that $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$ is an ordered measurable space. Let $M \subset \mathbb{R}^{n+1}$ be increasing and define $f \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ by setting pointwise

$$f(x) := \inf M_x, \quad \text{where} \quad M_x := \{y \in \mathbb{R} : (x, y) \in M\}.$$

Since M is increasing, f is decreasing. To see this, let $x \leqslant x'$ in $\mathbb{R}^n$ and let $y \in M_x$ be arbitrary. It follows $(x, y) \in M$ and $(x, y) \leqslant (x', y)$, thus $(x', y) \in M$ and $y \in M_{x'}$. Consequently, $M_x \subset M_{x'}$ and thus $f(x) = \inf M_x \geqslant \inf M_{x'} = f(x')$.

Thanks to the dual of Proposition 3.4 and due to the induction hypothesis, f is measurable, and thus its epigraph $\operatorname{epi} f = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leqslant y\}$ is measurable. Now, let $g\colon \mathbb{R}^n \to \mathbb{R}_{>0}$ be integrable. Then, for each $\varepsilon > 0$, we have

$$\operatorname{epi}(f + \varepsilon g) \subset M \subset \operatorname{epi} f,$$

where the first inclusion follows readily since M is increasing (such that $(x, y') \in M$ implies $(x, y) \in M$ if $y' \leqslant y$). Because $\epsilon > 0$ can be made arbitrary small, we conclude that M and $\operatorname{epi} f$ differ only be a null-set and thus $M \in \mathcal{L}(\mathbb{R}^{n+1})$. ○

3.1.2 Weakly Measurable Multifunctions of Isotone Type

Now, let us extend Proposition 3.4 to multifunctions. To this end, let us first generalize the concept of measurability to multifunctions. Since there are two pre-image, we have two different notions:

3.12 Definition Let $(X, \mathcal{A})$ be a measurable space, let R be a topological space, and let $F\colon X \to \mathcal{P}_\emptyset(R)$ be a multifunction.

(i) F is called **(strongly) measurable** if, for every open $U \subset R$, the small pre-image $F_+^{-1}(U) = \{x \in X : F(x) \subset U\}$ is measurable.

(ii) F is called **weakly measurable** if, for every open $U \subset R$, the (large) pre-image $F_-^{-1}(U) = \{x \in X : F(x) \cap U \neq \emptyset\}$ is measurable. ○

3.13 Remark Unlike in the single-valued case, the pre-image $F_-^{-1}(B)$ of some Borel set B may not be measurable, even if F is measurable. If one tries to prove otherwise, one is confronted with the fact, that, in general, the large pre-image commutes not with intersections.

In contrast, for any multifunction $F\colon X \to \mathcal{P}_\emptyset(R)$, where X and R are only supposed to be sets, it holds $F_-^{-1}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} F_-^{-1}(A_i)$ for any family of sets $A_i \subset R$. ○

In general, neither of the two measurability notions implies the other (like it is with lower and upper semicontinuity). However, the situation is different if the topological space R has better topological properties. To this end, recall the following notions:

(i) R is called **separable** if it contains a countable, dense subset.

(ii) R is called **σ-compact** if it is the countable union of compact sets.

(iii) A **metric space** is a pair $R = (R, d)$ of a set R and a mapping $d\colon R \times R \to \mathbb{R}_{\geqslant 0}$ which is positive definite, symmetric, and subadditive. Each metric space (R, d) is also a topological space; its topology is induced by d by taking all **open balls** $B_\varepsilon := \{y \in R : d(x, y) < \varepsilon\}$ as a base.

Then, we have, among others, the following results from [51]:

3.14 Proposition *Let X be a measurable space, let R be a topological space, and let $F\colon X \to \mathcal{P}_\emptyset(R)$ be a multifunction.*

(i) *Let* F *be measurable and let* R *be such that every open set is the union of countably many closed sets (which is the case if, e.g., the topology of* R *is generated by a metric). Then* F *is weakly measurable.*

(ii) *Let* F *be weakly measurable, let the values of* F *be closed, and let* R *be a separable metric space. Then* F *is measurable.* ○

In order to generalize Proposition 3.4, we now have to provide a few results that investigate the connections between order and measurability. Let us start with the following result:

3.15 Proposition *Let* D *and* D' *be posets, and let* $F\colon D \to \mathcal{P}_\emptyset(D')$ *be increasing upward. Then, for any increasing set* $B \subset D'$*, the set* $F_-^{-1}(B) \subset D$ *is increasing.*

Proof: Let $B \subset D'$ be increasing and let $a \in F_-^{-1}(B)$ be arbitrary. Then, by definition, there is $b \in F(a) \cap B$. Since F is increasing upward, for all $a' \geqslant a$ there is $b' \in F(a')$ such that $b' \geqslant b$, and since B is increasing, it follows $b' \in B$ and thus $a' \in F_-^{-1}(B)$. Consequently, $F_-^{-1}(B)$ is increasing. ○

As stated in Remark 3.13, one major problem in dealing with multifunctions is the defect that pre-images do not commute with intersections. Thus, to give an useful measurability result for increasing upward multifunctions $F\colon X \to \mathcal{P}_\emptyset(R)$, we furthermore have to develop a condition on F and subsets $A, B \subset R$ under which

$$F_-^{-1}(A \cap B) = F_-^{-1}(A) \cap F_-^{-1}(B) \tag{3.1}$$

holds. In order to approach this question, note that

- it is well-known that $F_-^{-1}(A \cap B) \subset F_-^{-1}(A) \cap F_-^{-1}(B)$ holds, as obviously for all $x \in X$ such that $f(x) \cap A \cap B \neq \emptyset$ one has $F(x) \cap A \neq \emptyset$ and $F(x) \cap B \neq \emptyset$;
- (3.1) holds if F is single-valued; $F(x) \cap A \neq \emptyset$ and $F(x) \cap B \neq \emptyset$ imply $F(x) \cap A \cap B \neq \emptyset$;
- if F is not single-valued, i.e. if there is some $x \in X$ such that $F(x)$ contains at least two distinct values y_1 and y_2, one has $F(x) \cap \{y_1\} \neq \emptyset$ and $F(x) \cap \{y_2\} \neq \emptyset$, but $F(x) \cap \{y_1\} \cap \{y_2\} = \emptyset$.

Thus, a reasonable (but neither sufficient nor necessary) condition for Equation (3.1) to hold is $A \cap B \neq \emptyset$. If the underlying space is a poset, a large class of such pairs is given by

$$A = \alpha^{\uparrow} \quad \text{and} \quad B = \beta^{\downarrow}, \quad \text{provided} \quad \alpha \leqslant \beta.$$

That given, we have $A \cap B = [\alpha, \beta] \neq \emptyset$ and (3.1) reduces to the question if

$$F_-^{-1}([\alpha, \beta]) \supset F_-^{-1}(\alpha^{\uparrow}) \cap F_-^{-1}(\beta^{\downarrow}). \tag{3.2}$$

To tackle this inclusion, let us first notate the following auxiliary result:

3.16 Proposition *Let* R *be a lattice, and let* $\alpha, y_\alpha, \beta, y_\beta \in R$ *be given such that* $\alpha \leqslant \beta$*,* $\alpha \leqslant y_\alpha$ *and* $y_\beta \leqslant \beta$*. Then*

$$[\alpha, \beta] \cap [y_\wedge, y_\alpha] \neq \emptyset, \quad [\alpha, \beta] \cap [y_\beta, y_\vee] \neq \emptyset, \quad [\alpha, \beta] \cap [y_\wedge, y_\vee] \neq \emptyset.$$

Proof: By assumption, we have $y_\wedge \leqslant y_\beta \leqslant \beta$ and $\alpha \leqslant y_\alpha \leqslant y_\vee$, which together with $\alpha \leqslant \beta$ and $y_\wedge \leqslant y_\vee$ imply $\gamma := \alpha \vee y_\wedge \leqslant \beta \wedge y_\vee =: \delta$. This implies

$$\gamma, \delta \in [\gamma, \delta] \subset [\alpha, \beta] \cap [y_\wedge, y_\vee].$$

Furthermore, it holds $y_\wedge \leqslant \gamma \leqslant y_\alpha$ and $y_\beta \leqslant \delta \leqslant y_\vee$, thus

$$\gamma \in [\gamma, y_\alpha \wedge \beta] \subset [\alpha, \beta] \cap [y_\wedge, y_\alpha], \qquad \delta \in [y_\beta \vee \alpha, \delta] \subset [\alpha, \beta] \cap [y_\beta, y_\vee]. \qquad \bigcirc$$

In view of Proposition 3.16, the appropriate condition for Equation (3.2) to hold is, interestingly enough, that the values of F are of order-convex type (see Definition 1.8):

3.17 Lemma *Let* X *be a set, let* R *be a sup-semilattice with elements* $\alpha \leqslant \beta$*, and let* $F\colon X \to \mathcal{P}_\emptyset(R)$ *be a multifunction whose values are order-convex upward. Then it holds*

$$F_{-}^{-1}([\alpha, \beta]) = F_{-}^{-1}(\alpha^{\uparrow}) \cap F_{-}^{-1}(\beta^{\downarrow}).$$

Proof: Suppose $x \in F_{-}^{-1}(\alpha^{\uparrow}) \cap F_{-}^{-1}(\beta^{\downarrow})$, then there exist $y_\alpha, y_\beta \in F(x)$ such that $\alpha \leqslant y_\alpha$ and $y_\beta \leqslant \beta$. Obviously, this implies

$$y_\beta \leqslant \alpha \vee y_\beta \leqslant y_\vee \quad \text{and} \quad \alpha \leqslant \alpha \vee y_\beta \leqslant \beta,$$

such that $\alpha \vee y_\beta \in F(x) \cap [\alpha, \beta]$, since $F(x)$ is order-convex upward. Thus, $x \in F_{-}^{-1}([\alpha, \beta])$ and by the preceding notes, the proof is complete. ○

To bring order, topology and measurability together, we formulate the following property of an ordered topological space:

(Q) Every open set is a countable union of order-intervals $[\alpha, \beta]$.

An example for an ordered topological space having property (Q) is $\mathbb{R}$ (which is, in fact, the only case we consider in the applications in Part II). For $\mathbb{R}$, we have even the following folkloric lemma:

3.18 Lemma *Every open set* $U \subset \mathbb{R}$ *is the union of countably many disjoint open intervals.*

Proof: Let U be an open subset of $\mathbb{R}$ and consider the equivalence relation $\sim$ on U, defined by $a_1 \sim a_2$ if and only if $[a_\wedge, a_\vee] \subset U$. Since U is the disjoint union of all equivalence classes C of $\sim$, which are open intervals $(\inf C)^{\uparrow\uparrow} \cap (\sup C)^{\downarrow\downarrow}$ containing at least one rational number, the assertion follows. ○

The merit of property (Q) is obvious: If $F_{-}^{-1}(\alpha^{\uparrow})$ and $F_{-}^{-1}(\alpha^{\downarrow})$ are measurable for all $\alpha \in R$, property (Q) and Lemma 3.17 imply that F is measurable, as the large pre-image commutes with unions. This idea gives us the desired generalization of Proposition 3.4:

3.19 Theorem *Let X be an ordered measurable space, let R be an ordered topological space with sup-semilattice-structure satisfying property (Q), and let $F\colon X \to \mathcal{P}_\emptyset(R)$ be an increasing multifunction with order-convex upward values. Then F is weakly measurable.*

Proof: Let $U \subset R$ be any open set. Then, thanks to property (Q), there are countably many $\alpha_i, \beta_i \in R$ such that

$$U = \bigcup_{i\in\mathbb{N}} [\alpha_i, \beta_i] = \bigcup_{i\in\mathbb{N}} \alpha_i^{\uparrow} \cap \beta_i^{\downarrow}.$$

Due to Lemma 3.17 and Remark 3.13 it follows

$$F_-^{-1}(U) = F_-^{-1}(\bigcup_{i\in\mathbb{N}} \alpha_i^{\uparrow} \cap \beta_i^{\downarrow}) = \bigcup_{i\in\mathbb{N}} F_-^{-1}(\alpha_i^{\uparrow} \cap \beta_i^{\downarrow}) = \bigcup_{i\in\mathbb{N}} F_-^{-1}(\alpha_i^{\uparrow}) \cap F_-^{-1}(\beta_i^{\downarrow}).$$

By Proposition 3.15, $F_-^{-1}(\alpha_i^{\uparrow})$ is increasing and thus measurable, and by the dual of Proposition 3.15 for an increasing downward multifunction, $F_-^{-1}(\beta_i^{\downarrow})$ is decreasing and thus measurable, too. It follows that $F_-^{-1}(U)$ is measurable and thus that F is a weakly measurable multifunction, which concludes the proof. ○

3.20 Remark Let X be a topological space equipped with the Borel σ-algebra or some finer σ-algebra, and let $F\colon X \to \mathcal{P}(\mathbb{R})$ be a multifunction with non-empty and compact values. Then the assertion of Theorem 3.19 can be deduced as follows: Since F is increasing if and only if its envelopes $F_*\colon X \to \mathbb{R}$ and $F^*\colon X \to \mathbb{R}$ are increasing (see Proposition 1.26), and since increasing real functions are Borel measurable, Proposition 3.31 below implies that F is weakly measurable.

(The same holds if the values of F are such that $\inf F(x)$ and $\sup F(x)$ exist in $\mathbb{R}$, since F is measurable if and only if the multifunction $x \mapsto \overline{F(x)}$ is measurable, where $\overline{F(x)}$ denotes the closure of $F(x)$.) ○

3.21 Corollary *Let $\mathbb{R}$ be equipped with the usual order, the Euclidean metric and its Borel σ-algebra, and let $F\colon \mathbb{R} \to \mathcal{P}_\emptyset(\mathbb{R})$ be an increasing multifunction whose values are closed intervals. Then F is measurable.* ○

The space $\mathbb{R}$ has property (Q). To illustrate the applicability of Theorem 3.19, we are going to provide further examples for ordered topological spaces having property (Q). To this end, we need the following definitions and well-known insights.

3.22 Definition Let R be an ordered normed space with order cone K. We say that K is **proper** if $K \neq R$ and if K has non-empty interior $\operatorname{int} K$. In this case, for $s, t \in R$ we write $s \ll t$ and $t \gg s$ if $t - s \in \operatorname{int} K$, i.e. if there is some $\varepsilon > 0$ such that the ball $B_\varepsilon(t-s) := \{r \in R : \|r - (t-s)\| < \varepsilon\}$ is a subset of K. ○

For example, consider $R = \mathbb{R}$ with its usual order cone K, then we have $\operatorname{int} K = (0, \infty)$ and thus $s \ll t$ if and only if $s < t$. In case $R = \mathbb{R}^N$, we have $\operatorname{int} K = \{s : s_i > 0\}$ and thus $s \ll t$ implies $s < t$, but not vice versa.

3.23 Proposition *Let* R *be an ordered normed space with proper order cone* K. *Then* $\ll$ *is a* ***strict partial order*** *(i.e.* $\ll$ *is irreflexive and transitive) and the following assertions hold true:*

(i) If $s \ll s'$ *and* $t \leqslant t'$, *then* $s + t \ll s' + t'$.

(ii) If $s \ll t$ *and* $\alpha > 0$, *then* $\alpha s \ll \alpha t$.

(iii) If $s \leqslant t \ll r$ *or* $s \ll t \leqslant r$, *then* $s \ll r$.

Proof: We first prove that $\ll$ is irreflexive. To this end, suppose $r \ll r$ for some $r \in \mathsf{R}$. Then $0 \in \operatorname{int} \mathsf{K}$, that is, $B_\varepsilon(0) \subset \mathsf{K}$ for some $\varepsilon > 0$, and thus $\mathsf{R} \subset \mathsf{K}$, since every $s \in \mathsf{R}$ is of the form $s = \alpha s'$ for some $\alpha \geqslant 0$ and some $s' \in B_\varepsilon(0)$. This contradicts $\mathsf{K} \neq \mathsf{R}$, thus $r \ll r$ does not hold.

Furthermore, $\ll$ is transitive (see (iii)), and thus a strict partial order.

Now, assume $s \ll s'$, then $B_\varepsilon(s' - s) \subset \mathsf{K}$ for some $\varepsilon > 0$, and thus for every $t \in \mathsf{R}$ it follows $B_\varepsilon(s' + t - (s + t)) \subset \mathsf{K}$, that is, $s + t \ll s' + t$. Assertion (i) follows by (iii).

To prove (ii), suppose $B_\varepsilon(t - s) \subset \mathsf{K}$ and $\alpha > 0$. Since $y \in B_\varepsilon(t - s)$ if and only if $\alpha y \in B_{\alpha\varepsilon}(\alpha(t - s))$, it follows $B_{\alpha\varepsilon}(\alpha(t - s)) \subset \mathsf{K}$, and thus $\alpha s \ll \alpha t$.

At last, assume $s \leqslant t$ and $B_\varepsilon(r - t) \subset \mathsf{K}$ for some $\varepsilon > 0$. Then

$$B_\varepsilon(r - s) = B_\varepsilon(r - t + t - s) = B_\varepsilon(r - t) + t - s \subset \mathsf{K},$$

since $t - s \in \mathsf{K}$ and K is closed under addition. Thus, $s \ll r$. If $s \ll t \leqslant r$, then $s \ll r$ follows similarly and (iii) is proved. ○

3.24 Definition Let R be an ordered normed space with proper order cone K. Then R is said to be **order-separable** if there is some countable set $\mathsf{Q} \subset \mathsf{R}$ such that for all $x, y \in \mathsf{R}$ such that $x \ll y$ there is some $q \in \mathsf{Q}$ such that $x \leqslant q \leqslant y$. ○

3.25 Remark There are various variants of defining order-separable posets, see, e.g., [16] for the definitions of order-separable posets in the sense of Cantor, Debreu, Jaffray and Birkhoff, respectively. However, our natural notion seems not to be standard. ○

3.26 Proposition *Let* R *be an order-separable ordered normed space with proper order cone, and suppose that the norm is normal, i.e. there is some* $c_\mathsf{K} > 0$ *such that* $0 \leqslant x \leqslant y$ *in* R *implies* $\|x\| \leqslant c_\mathsf{K}\|y\|$. *Then* R *has property (Q).*

Proof: Let $e \in \operatorname{int} \mathsf{K}$, let $\mathsf{U} \subset \mathsf{R}$ be open and consider some $x \in \mathsf{U}$ and some $r > 0$ such that $B_r(x) \subset \mathsf{U}$. Then, for every $\varepsilon > 0$, we have $\varepsilon e \gg 0$ and $x - \varepsilon e \ll x \ll x + \varepsilon e$, and therefore there are $\alpha, \beta \in \mathsf{Q}$ such that $x - \varepsilon e \leqslant \alpha \leqslant x \leqslant \beta \leqslant x + \varepsilon e$. For every $z \in [\alpha, \beta]$ we conclude $0 \leqslant z - x + \varepsilon e \leqslant 2\varepsilon e$ and thus $\|z - x\| \leqslant (2c_\mathsf{K} + 1)\varepsilon\|e\|$. Thus, we have $x \in [\alpha, \beta] \subset B_r(x) \subset \mathsf{U}$ if ε is sufficiently small. Consequently, U is the union of countably many such order-intervals $[\alpha, \beta]$. ○

3.27 Corollary *Let* X *be a measurable space, let* R *be an order-separable ordered normed space with proper order cone and normal norm, and let* $\mathsf{F}\colon \mathsf{X} \to \mathcal{P}_\emptyset(\mathsf{R})$ *be a multifunction such that* $\mathsf{F}_-^{-1}([\alpha, \beta])$ *is measurable for all* $\alpha, \beta \in \mathsf{R}$. *Then* F *is measurable.* ○

3.28 Example Besides $\mathbb{R}^n$ there are other order-separable ordered normed spaces with proper order cone and normal norm. Consider, e.g., the set $R = C(I)$ of continuous real functions over a compact interval $I \subset \mathbb{R}$. Equipped with the usual sup-norm $\|\cdot\|_\infty$ and ordered point-wise, R is clearly an ordered normed space with proper order cone and normal norm. Further, it is order-separable: Assume $f \ll g$, then $(g-f)(x) \geqslant 2\varepsilon > 0$ for all $x \in I$ and some rational ε. As there is a polynomial p with rational coefficients such that $\|f-p\|_\infty < \varepsilon$, we have $f \leqslant p + \varepsilon \leqslant g$. ○

3.1.3 Measurable Selections

In applications, often the question arises if a given multifunction has a measurable selection. Depending on what we know about the underlying space, we have two slightly different definitions:

3.29 Definition Let X be a measurable space, let R be a topological space, let $F\colon X \to \mathcal{P}(R)$ be a multifunction, and let $f\colon X \to R$ be a single-valued measurable function.

(i) f is called a **measurable selection** of F if $f \subset F$, i.e. $f(x) \in F(x)$ for all $x \in X$.

Now let X be a **complete measure space**, i.e. a measurable space $(X, \mathcal{A})$ endowed with a measure $\mu\colon \mathcal{A} \to [0, +\infty]$ such that each subset of a null-set is measurable.

(ii) f is called a **measurable selection** of F if $f(x) \in F(x)$ for a.e. $x \in X$ (where, as usual, 'a.e.' stands for 'almost every', meaning that the set of exception is a null-set). In this case, we also write $f \subset F$. ○

3.30 Remark This abuse of notation makes no trouble. Indeed, if f is a measurable selection of F in the sense of (ii), there is a measurable selection g of F in the sense of (i) such that $f(x) = g(x)$ for a.e. $x \in X$, since we can redefine f on the null-set $\{x \in X : f(x) \notin F(x)\}$ without losing measurability. Thus, when identifying functions which differ only on a null-set, f is a measurable selection of F in the sense of (i). This result simplifies some arguments for complete measure spaces. ○

In the following, let us present a few results guaranteeing the existence of measurable selections. To this end, first let us connect measurability of multifunctions with measurability of its envelopes, in the spirit of Propositions 1.26 and 2.47.

3.31 Proposition *Let X be a measurable space, let R be an ordered topological space with property (Q), and suppose that the values $F(x)$ of the multifunction $F\colon X \to \mathcal{P}(R)$ have greatest elements $F^*(x) \in F(x)$.*

(i) If F is weakly measurable and measurable, then F^* is measurable.

Additionally, suppose that $F(x) = [F_*(x), F^*(x)]$ for all $x \in X$, and suppose that R is a sup-semilattice or an inf-semilattice.

(ii) If F_* and F^* are measurable, then F is weakly measurable.

Proof: Recall that for all $\alpha, \beta \in R$ it holds

$$(F^*)^{-1}(\alpha^{\uparrow}) = F_-^{-1}(\alpha^{\uparrow}) \quad \text{and} \quad (F^*)^{-1}(\beta^{\downarrow}) = F_+^{-1}(\beta^{\downarrow}).$$

Thus, measurability of F implies that $(F^*)^{-1}(\alpha^{\uparrow})$ is measurable, and weak measurability of F implies that $(F^*)^{-1}(\beta^{\downarrow})$ is measurable. Since the order-intervals $\alpha^{\uparrow}$ and $\beta^{\downarrow}$ generate $\mathcal{B}(R)$, F^* is measurable. In view of the proof of Theorem 3.19 and the dual equations for F_*, we easily deduce assertion (ii). ○

From Proposition 3.31 we have especially the result that F has the measurable selection F^*. If we have no partial order, we have the following result which shows that it is not hard to find measurable selections of a measurable multifunction F with closed values. To this end, recall that a **Polish space** is a topological space whose topology is generated by a metric d such that (R, d) is a complete separable metric space.

3.32 Theorem *Let* X *be a measurable space, and let* R *be a Polish space, and let* $F\colon X \to \mathcal{P}(R)$ *be a multifunction with closed values. Then* F *is measurable if and only if* F *admits a* measurable exhaustion (f_n), *that is, a sequence of measurable functions* $f_n\colon X \to R$ *such that, for all* $x \in X$, $F(x)$ *equals the closure of* $\{f_n(x) : n \in \mathbb{N}\}$.

Proof: See [53, Prop. II.2.3] or [51, Theorem 5.6]. ○

Theorem 3.32 is a very useful result. However, in the applications in Part II we do not seek simply for measurable selections of a mere multifunction, but for measurable selections f of a multifunction $x \mapsto F(x, u(x), v(x))$, where $u\colon X \to R_1$ and $v\colon X \to R_2$ are measurable functions and $F\colon X \times R_1 \times R_2 \to \mathcal{P}(R_3)$ is a bifunction defined on the product space $X \times R_1 \times R_2$, equipped with the product σ-algebra $\mathcal{A} \otimes \mathcal{B}(R_1) \otimes \mathcal{B}(R_2)$.

In order to apply Theorem 3.32 to this case, we would have to make sure that the multifunction $x \mapsto F(x, u(x), v(x))$ is measurable—but we will see in Remark 3.36 and Example 3.46 that this is not always the case in our targeted application. Thus, we are going to introduce more notions of measurability, which extend those introduced in Definition 3.12. To emphasize the main ideas and in order to avoid too much technicalities, let us restrict our considerations to the case of $R_1 \times R_2 = \mathbb{R}^n$ and $R_3 = \mathbb{R}$, and let, in view of the following applications, $\Omega \subset \mathbb{R}^N$ be an open set equipped with its Lebesgue σ-algebra $\mathcal{L}(\Omega)$ and its Lebesgue measure μ, so that $(\Omega, \mathcal{L}(\Omega), \mu)$ is a complete measure space.

3.33 Definition Let $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}_\emptyset(\mathbb{R})$ be a multifunction.

(i) F is called **(weakly) product-measurable** if it is (weakly) measurable with respect to the product σ-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$.

(ii) F is called **graph-measurable** if its graph gr F is measurable with respect to the product σ-algebra $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1})$.

(iii) F is called **superpositionally measurable** if, for any measurable single-valued function $u\colon \Omega \to \mathbb{R}^n$, the multifunction $x \mapsto F(x, u(x))$ is measurable.

(iv) F is called **weakly superpositionally measurable** if, for any measurable function $u\colon \Omega \to \mathbb{R}^n$, the multifunction $x \mapsto F(x, u(x))$ has a measurable selection. ○

There are various connections between the several types of measurability. In our setting and if $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ has closed values, we have the following implications:

$$\begin{aligned}
&\text{product-measurable} \\
&\iff \text{weakly product-measurable} \\
&\qquad \implies \text{graph-measurable} \\
&\qquad\qquad \implies \text{superpositionally measurable} \\
&\qquad\qquad\qquad \implies \text{weakly superpositionally measurable.}
\end{aligned}$$

The equivalence was provided in Proposition 3.14 (which is why we do not need to differentiate between weak and strong measurability in the following), and the last implication follows from Theorem 3.32. Next, let us provide results for the other implications. We start with the following result from [3, Lemma 7.2] (see also [125]):

3.34 Lemma *Let $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}_\emptyset(\mathbb{R})$ be a multifunction. If F is (weakly) product-measurable, then F is superpositionally measurable.*

Proof: Let $u\colon \Omega \to \mathbb{R}^n$ be any measurable function and define $\hat{u}\colon \Omega \to \Omega \times \mathbb{R}^n$ by $\hat{u}(x) = (x, u(x))$. Then, for any $A \in \mathcal{L}(\Omega)$ and any $B \in \mathcal{B}(\mathbb{R}^n)$ one has

$$\hat{u}^{-1}(A \times B) = \{x : x \in A, u(x) \in B\} = A \cap u^{-1}(B) \in \mathcal{L}(\Omega).$$

Since the set $\{C \subset \Omega \times \mathbb{R}^n : \hat{u}^{-1}(C) \in \mathcal{L}(\Omega)\}$ is a *Dynkin system* (even a σ-algebra) that contains $\mathcal{L}(\Omega) \times \mathcal{B}(\mathbb{R}^n)$, it contains also $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$, and thus $\hat{u}^{-1}(M) \in \mathcal{L}(\Omega)$ for all $M \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$. So, if $U \subset \mathbb{R}$ is open, we have for the multifunction $F(\cdot, u(\cdot))$

$$F(\cdot, u(\cdot))_-^{-1}(U) = \{x \in \Omega : F(x, u(x)) \cap U \neq \emptyset\} = \hat{u}^{-1}(F_-^{-1}(U)) \in \mathcal{L}(\Omega),$$

since $F_-^{-1}(U) \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$. Consequently, F is superpositionally measurable. ○

3.35 Remark There is also a converse of Lemma 3.34: If F is compact-valued, superpositionally measurable and upper Carathéodory (see Definition 3.45 below), then F is product-measurable. As for the proof, we note that Ω has the projection property, i.e. the projection $\mathrm{proj}_\Omega(M)$ of every $M \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$ to Ω belongs to $\mathcal{L}(\Omega)$. Using this, the proof can be done as in [3, Lemma 7.3]. ○

3.36 Remark It may be a difficult task to prove the product-measurability of some multifunction $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}_\emptyset(\mathbb{R})$. Unfortunately, it is not enough that $x \mapsto F(x, t)$ is measurable and that $t \mapsto F(x, t)$ is increasing. To see this, consider the following example from [4]: Let $N \subset [0, 1]$ be *not* Lebesgue measurable and define the function

$$f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}, \qquad (x, t) \mapsto f(x, t) = \begin{cases} 1 & \text{if } x < t, \text{ or } x = t \text{ and } x \in N, \\ 0 & \text{if } x > t, \text{ or } x = t \text{ and } x \notin N. \end{cases}$$

Then, $x \mapsto f(x, x)$ equals the non-measurable characteristic function of N. Corollary 3.21 shows that no such problem occurs if f does not depend on $x \in \Omega$. ○

Next, let us give two conditions which imply that a given multifunction with closed values is graph-measurable:

3.37 Proposition *Let* $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ *be a multifunction. If* F *has closed values and is product-measurable, or if* F *has compact values and both* F^* *and* F_* *are product-measurable, then* F *is graph-measurable.*

Proof: Let us set $X = \Omega \times \mathbb{R}^n$ and $\mathcal{A} = \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n)$. Then, the first part of this proposition is proven in, e.g., [51].

Suppose now that both $F^*\colon X \to \mathbb{R}$ and $F_*\colon X \to \mathbb{R}$ exist and are measurable with respect to $\mathcal{A}$. Due to Proposition 3.31 and the first part of this proposition, F is graph-measurable. But let us give another proof: We have

$$\operatorname{gr} F = \{(x,t) \in X \times \mathbb{R} : F_*(x) \leqslant t \leqslant F^*(x)\} = \operatorname{epi} F_* \cap \operatorname{hypo} F^*,$$

where $\operatorname{epi} F^* := \{(x,t) : F_*(x) \leqslant t\}$ is the *epigraph* of F_* and $\operatorname{hypo} F^* := \{(x,t) : F^*(x) \geqslant t\}$ its *hypograph.* Since both the epigraph and the hypograph of a measurable function are measurable, we are done. ○

Finally, following [68, Prop. 2.3], let us show that graph-measurable multifunctions are indeed superpositionally measurable.

3.38 Proposition *Let* $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}_\emptyset(\mathbb{R})$ *be a graph-measurable multifunction. Then* F *is superpositionally measurable.*

Proof: Let $u\colon \Omega \to \mathbb{R}^n$ be any measurable function, and let $U \subset \mathbb{R}$ be any open set. Then we have

$$F(\cdot, u(\cdot))_-^{-1}(U) = \{x \in \Omega : F(x, u(x)) \cap U \neq \emptyset\} = \operatorname{proj}_\Omega(M(F,u,U)),$$

where

$$\begin{aligned} M(F,u,U) &= \{(x,s,r) \in \Omega \times \mathbb{R}^n \times \mathbb{R} : s = u(x),\ r \in F(x,s) \cap U\} \\ &= (\operatorname{gr}(u) \times \mathbb{R}) \cap \operatorname{gr}(F) \cap (\Omega \times \mathbb{R}^n \times U)\,. \end{aligned}$$

Since u is measurable, it is graph-measurable, and thus $M(F,u,U) \in \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1})$. By Theorem 3.8, $\operatorname{proj}_\Omega(M(F,u,U))$ belongs to $\mathcal{L}(\Omega)$ and the assertion follows. ○

At the end of this subsection, let us illuminate the gap between product-measurable and graph-measurable multifunctions by inspecting some more technical results:

Multifunctions with non-closed values If F is not closed-valued, measurability needs not to imply graph-measurability, as the following simple example (which is [53, Example II.1.38]) illustrates.

3.39 Example Let $\Omega = R = [0,1] \subset \mathbb{R}$ be equipped with the Lebesgue σ-algebra $\mathcal{L}(\Omega)$. Let $A \subset [0,1]$ be a non-measurable set, and define $F\colon \Omega \to \mathcal{P}(R)$ by

$$F(x) = \begin{cases} R & \text{if } x \in A, \\ R \setminus \{x\} & \text{if } x \notin A. \end{cases}$$

Then F is measurable, as large pre-images are of the form Ω or $\Omega \setminus \{x\}$, but not graph-measurable. Indeed, if we suppose otherwise, then we have $\operatorname{gr} F \in \mathcal{L}(\Omega) \times \mathcal{B}(R)$ and thus $\operatorname{gr}(F) \cap \Delta \in \mathcal{L}(\Omega) \times \mathcal{B}(R)$ with $\Delta = \{(x,x) : x \in [0,1]\}$, so $A = \operatorname{proj}_\Omega(\operatorname{gr}(F) \cap \Delta) \in \mathcal{L}(\Omega)$ by Theorem 3.8, contradicting the choice of A. ○

Complete Measure Spaces We know that a measurable multifunction F with closed values is graph-measurable, see Proposition 3.37. If the domain of F is a complete measure space, the converse holds true, which is a consequence of Theorem 3.8. To be more precise, we have the following theorem (see [53, Theorem II.1.35]):

3.40 Theorem *Let $(X, \mathcal{A}, \mu)$ be a complete measure space, let R be a Polish space, and let $F\colon X \to \mathcal{P}(R)$ be a multifunction with closed values. Then F is graph-measurable if and only if $F_-^{-1}(D) \in \mathcal{A}$ for all $D \in \mathcal{B}(R)$.* ○

However, this result can not be applied to a multifunction $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$, since $\mathcal{B}(\mathbb{R}^n)$ is not complete with respect to the Borel measure.

Souslin Sets and Standard Borel Spaces To present deeper results about measurable functions, we have to recall a few more definitions:

(i) A topological space R is called **Hausdorff** if for any two distinct points $x, y \in R$ there are distinct open sets $U, V \subset R$ such that $x \in U$ and $y \in V$.

(ii) A Hausdorff space is called **Souslin** if it is the image of a Polish space under a continuous mapping. A subset M of a topological space is called **Souslin** if M is a Souslin space with respect to its induced topology.

(iii) A measurable space $(X, \mathcal{A})$ is called **standard Borel space** if it is isomorphic to $(Y, \mathcal{B}(Y))$ for some Polish space Y, i.e. if there is a bijection $f\colon X \to Y$ such that both f and f^{-1} are measurable.

Souslin sets, which are more general than Borel sets, relate measurable functions with measurable graphs. The first result in this direction can be found in [12, Lemma 6.7.1] and reads as follows:

3.41 Lemma *Let R and S be Souslin spaces. Then the graph of any Borel measurable function $f\colon R \to S$ (where R and S are equipped with their Borel σ-algeba) is a Borel, hence Souslin, subset of the Souslin space $R \times S$. Conversely, if $f\colon R \to S$ has a Souslin graph, then f is Borel measurable.* ○

A similar result about standard Borel spaces is [59, Theorem 14.12], which reads a follows:

3.42 Lemma *Let* X *and* Y *be standard Borel spaces, and let* $f\colon X \to Y$ *be a function. Then* f *is measurable if and only if* $\operatorname{gr} f$ *is measurable.* ○

In order to apply this result, we need to know which sets are standard Borel spaces. The following proposition (which is [59, Corollary 13.4]) gives a result in this direction:

3.43 Proposition *Let* $(X, \mathcal{A})$ *be a standard Borel space and* $Y \in \mathcal{A}$*. Then* $(Y, \mathcal{A}_{|Y})$ *is also a standard Borel space (where* $\mathcal{A}_{|Y} = \{A \subset Y : A \in \mathcal{A}\}$*, since* $Y \in \mathcal{A}$*).*

Proof: We can assume that X is a Polish space with topology $\mathcal{T}$ and $\mathcal{A} = \mathcal{B}(\mathcal{T})$. Then, Y is a Borel set, and thus there is a topology $\mathcal{S}$ on X such that $(X, \mathcal{S})$ is a Polish space, $\mathcal{T} \subset \mathcal{S}$, $\sigma(\mathcal{S}) = \sigma(\mathcal{T})$, and such that Y is clopen (i.e. both closed and open) with respect to $\mathcal{S}$ (see [59, Theorem 13.1]). Thus, $(Y, \mathcal{S}_{|Y})$ is a Polish space, and since $\mathcal{B}(X)_{|Y} = \mathcal{B}(Y)$, $(Y, \mathcal{B}(X)_{|Y})$ is a standard Borel space. ○

Especially, $(\Omega, \mathcal{B}(\Omega))$ is a standard Borel space, and so is $(\Omega \times \mathbb{R}^n, \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R}^n))$.

For multifunctions, we have a similar result. To this end, recall that the **Vietoris topology** $\mathcal{V}$ on the set $\mathrm{Cp}(R)$ of all non-empty, compact subsets of a topological space R is generated by the sets

$$U^- = \{C \in \mathrm{Cp}(R) : C \cap U \neq \emptyset\}, \qquad U^+ = \{C \in \mathrm{Cp}(R) : C \subset U\}$$

for U open in R. Thus, for a mapping $F\colon X \to \mathrm{Cp}(R)$ we have, for any open subset U of R,

$$F_-^{-1}(U) = \{x \in X : F(x) \cap U \neq \emptyset\} = \{x \in X : F(x) \in U^-\} = F^{-1}(U^-),$$

where on the left F is treated as a multifunction whose values are subsets of R, and on the right F is treated as a single-valued function whose values are elements of the codomain $\mathrm{Cp}(R)$. Consequentely, if F is Borel measurable with respect to the Vietoris topology, then F is weakly measurable in our usual sense. With this in mind, we can appreciate the next result, which is [59, Theorem 28.8]:

3.44 Theorem *Let* X *be a standard Borel space, let* R *be a Polish space, and let* $F\colon X \to \mathrm{Cp}(R)$*. Then* F *is measurable (with respect to the Vietoris topology) if and only if* $\operatorname{gr} F \subset X \times R$ *is measurable.* ○

Finally, let us apply Theorem 3.44 to our setting: If $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ is product-measurable and compact-valued, its graph belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1})$ (due to Proposition 3.37). If even $\operatorname{gr} F \in \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1})$, then F is measurable with respect to the product σ-algebra $\mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1}) \subset \mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1})$ and thus product-measurable. Thus, when restricting our considerations to product-measurable multifunctions instead of graph-measurable ones, we exclude only such multifunctions whose graph is contained in $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1}) \setminus \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{R}^{n+1})$ and which have at least one non-product-measurable envelop. It would be interesting to know such a multifunction.

3.2 Multivalued Bifunctions

In the applications in Part II, a **bifunction** is a multifunction $F\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, where, as above, $\Omega \subset \mathbb{R}^N$ is an open set equipped with its Lebesgue algebra $\mathcal{L}(\Omega)$ and its Lebesgue measure, $\mathbb{R}$ is equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{R})$, and all sets are equipped with the usual componentwise ordering and the Euclidean metric.

The interesting part is that F depends on various arguments, but differently. To be more clear, we impose the following hypotheses on F:

(i) For fixed x and s, the mapping $t \mapsto F(x, s, t)$ is of isotone type as defined in Definition 1.22, such that the developed order-theoretical methods apply.

(ii) For fixed s, the mapping $(x, t) \mapsto F(x, s, t)$ is product-measurable, and for fixed x and t, the mapping $s \mapsto F(x, s, t)$ is upper semicontinuous, such that measure-theoretical and topological methods apply.

This section provides the theoretical background which is needed to treat multivalued variational inequality problems with nonsmooth bifunctions. To this end, we combine some known results concerning various kinds of measurability, and then we present examples of bifunctions that satisfy requirements (i) and (ii).

3.2.1 Upper Carathéodory Multifunctions

Recall that a single-valued function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is called **Carathéodory** if the mapping $x \mapsto f(x, s)$ is measurable for all $s \in \mathbb{R}$, and if the mapping $s \mapsto f(x, s)$ is continuous for a.e. $x \in \Omega$. This notion generalizes to multifunctions as follows:

3.45 Definition A multifunction $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ is called **upper Carathéodory** if, for all $s \in \mathbb{R}^n$, $x \mapsto F(x, s)$ is measurable and if, for a.e. $x \in \Omega$, $s \mapsto F(x, s)$ is upper semicontinuous. ○

Upper Carathéodory multifunctions may not be superpositionally measurable, as the following example ([3, Example 7.1]) illustrates:

3.46 Example Let $\Omega = [0, 1]$, let $D \subset \Omega$ be a non-measurable subset, and define $F\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$F(x, s) := \begin{cases} [0, 1] & \text{if } s = x \text{ and } x \in D, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then F is obviously upper Carathéodory and compact-valued, but not superpositionally measurable. Indeed, F maps the function $u\colon x \mapsto x$ into the multifunction $G\colon \Omega \to \mathcal{P}(\mathbb{R})$ defined by

$$G(x) := \begin{cases} [0, 1] & \text{if } x \in D, \\ \{1\} & \text{otherwise,} \end{cases}$$

which is not measurable. ○

However, note that G in Example 3.46 has a measurable selection, therefore F is weakly superpositionally measurable. The forthcoming Theorem (which is [3, Lemma 7.1]) shows that this is true for all upper Carathéodory multifunctions with compact values (which is a result we will apply frequently in the applications).

3.47 Theorem *Let $F\colon \Omega \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ be upper Carathéodory and compact-valued. Then F is weakly superpositionally measurable.*

Proof: Let $u\colon \Omega \to \mathbb{R}^n$ be any measurable function. Then there is a sequence (u_n) of simple functions (i.e. finite linear combinations of measurable characteristic functions) which converges a.e. to u. Then, all multifunctions $F_n\colon x \mapsto F(x, u_n(x))$ are measurable, as we have, per definition of a simple function, $u_n = \sum_j a_j \chi_{A_j}$, where each A_j is measurable and $\Omega = \bigcup_j A_j$ is a disjoint union, and thus, for each open set U,

$$(F_n)_-^{-1}(U) = \{x \in \Omega : F(x, \textstyle\sum_j a_j \chi_{A_j}(x)) \cap U \neq \emptyset\} = \bigcup_j F(\cdot, a_j)_-^{-1}(U) \cap A_j \in \mathcal{L}(\Omega).$$

Hence, also the multifunction

$$G\colon \Omega \to \mathcal{P}_\emptyset(\mathbb{R}), \quad x \mapsto \bigcap_{k\geqslant 1} \operatorname{cl} \bigcup_{n \geqslant k} F_n(x)$$

(where cl denotes the closure) is measurable (since the intersection of measurable closed-valued multifunctions is measurable, see [51, Corollary 4.2]).

Further, we claim $G \subset F(\cdot, u)$. To see this, suppose $y \notin F(x, u(x))$ for some $x \in \Omega$ for which $u_n(x) \to u(x)$ and for which $s \mapsto F(x, s)$ is upper semicontinuous. Then, since $F(x, u(x))$ is closed, we have $y \notin F(x, u(x)) + B_\varepsilon$ for $\varepsilon := \operatorname{dist}(y, F(x, u(x)))/2 > 0$. Because $s \mapsto F(x, s)$ is upper semicontinuous, we have $F(x, u_n(x)) \subset F(x, u(x)) + B_\varepsilon$ for all $n \geqslant k_0$, where k_0 is chosen appropriately. Hence, $y \notin \operatorname{cl} \bigcup_{n \geqslant k_0} F_n(x)$, and hence $y \notin G(x)$. This implies $G(x) \in F(x, u)$ for a.e. $x \in \Omega$.

Consequently, if G has a measurable selection, also $F(\cdot, u)$ has. In view of Theorem 3.32 we only have to show that $G(x) \neq \emptyset$ for a.e. $x \in \Omega$ (note that G is also measurable on the measurable space $\Omega \setminus N$ for any null-set $N \subset \Omega$). To this end, we recall that F has bounded values and deduce similar as above that $\bigcup_{n \geqslant k} F_n(x)$ is bounded for a.e. $x \in \Omega$. Thus, for a.e. $x \in \Omega$, $(\operatorname{cl} \bigcup_{n \geqslant k} F_n(x))$ is a decreasing sequence of non-empty compact sets, which implies $G(x) \neq \emptyset$. ○

3.48 Remark In the last two lines of the above proof we used that bounded and closed sets in $\mathbb{R}$ are compact. We currently don't know if Theorem 3.47 can be generalized to topological spaces without this property. ○

3.49 Remark An analogous result for a lower Carathéodory multifunction (defined analogous to upper Carathéodory multifunctions) does *not* hold. To see this, consider the following example, which is [3, Example 7.2]:

Let $\Omega = [0, 1]$, let $D \subset \Omega$ be a non-measurable subset, and let $F\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be defined by

$$F(x, s) := \begin{cases} \{0\} & \text{if } s = x \text{ and } x \in D, \\ \{1\} & \text{if } s = x \text{ and } x \in \Omega \setminus D, \\ [0, 1] & \text{otherwise.} \end{cases}$$

Then F is lower Carathéodory, but not weakly superpositionally measurable, since F maps the function $u\colon x \to x$ into the multifunction $G\colon \Omega \to \mathcal{P}(\mathbb{R})$ defined by

$$G(t) := \begin{cases} \{0\} & \text{if } x \in D, \\ \{1\} & \text{if } x \in \Omega \setminus D, \end{cases}$$

which has no measurable selection. ○

3.2.2 General Bifunctions

Now, let us consider bifunctions $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ with compact values. From what we have proved so far, we have at once the following result about measurable selections:

3.50 Corollary *Let $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ be a multifunction with compact values such that $(x, t) \mapsto F(x, s, t)$ is superpositionally measurable or product-measurable for all $s \in \mathbb{R}$, and such that $s \mapsto F(x, s, t)$ is upper semicontinuous for a.e. $x \in \Omega$ and all $t \in \mathbb{R}^n$. Then F is weakly superpositionally measurable.*

Proof: Let $u\colon \Omega \to \mathbb{R}$ and $v\colon \Omega \to \mathbb{R}^n$ be measurable functions. Then, for all $s \in \mathbb{R}$, the multifunction $x \mapsto F(x, s, v(x))$ is measurable due to Lemma 3.34. Thus, by definition, the multifunction $(x, s) \mapsto F(x, s, v(x))$ is upper Carathéodory. Due to Theorem 3.47, $x \mapsto F(x, u(x), v(x))$ has a measurable selection, that is, F is weakly superpositionally measurable. ○

Corollary 3.50 gives an applicable criterion for a bifunction to be weakly superpositionally measurable. However, it does not make use of order-theoretical considerations (except the elementary result that bounded and closed subsets of $\mathbb{R}$ are compact). It would be convenient to use that $t \mapsto F(x, s, t)$ is of isotone type in order to provide the product-measurability of $(x, t) \mapsto F(x, s, t)$. But there is no simple answer, as we have seen in Remark 3.36. Thus, we are going to provide a criterion (M) which guarantees that $(x, t) \mapsto F(x, s, t)$ is product-measurable if it is measurable in x and increasing in t. This generalizes a finding for single-valued functions (see [55]).

To make the proof more lucid, we use the notion of permanent multifunctions and a technical proposition. To this end, let us slightly extend Definition 1.64:

3.51 Definition Let D be a poset, let Y be a set, and let $F\colon D \to \mathcal{P}_\emptyset(Y)$ be a multifunction.

(i) F is called **permanent upward** if $x \leqslant y$ implies $F(x) \subset F(y)$.

(i)$_d$ F is called **permanent downward** if $x \leqslant y$ implies $F(y) \subset F(x)$. ○

3.52 Proposition *Let $(X, \mathcal{A})$ be a measurable space, and let $m\colon \mathbb{R} \to \mathcal{A}$ be a permanent upward multifunction. Then the function $g\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$, defined by*

$$g(x) := \inf\{t : x \in m(t)\},$$

is measurable.

Proof: Let (ε_n) be a sequence of real numbers $\varepsilon_n > 0$ converging to 0. Then for each $s \in \mathbb{R}$, $g(x) \leqslant s$ is equivalent to $x \in m(s + \varepsilon_n)$ for all $n \in \mathbb{N}$, since m is permanent upward. Thus, we have

$$g^{-1}(s^{\downarrow}) = g^{-1}([-\infty, s]) = \bigcap\nolimits_n m(s + \varepsilon_n) \in \mathcal{A}.$$

Since the sets $s^{\downarrow}$ generate the Borel algebra on $\mathbb{R} \cup \{-\infty, +\infty\}$, g is measurable. ○

After this preparation, let us prove the main result of this subsection:

3.53 Theorem *Let $F\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multifunction whose values $F(x,t)$ have the greatest element $F^*(x,t)$, and assume that*

(i) for all $t \in \mathbb{R}$ the multifunction $x \mapsto F(x,t)$ is measurable on Ω,

(ii) for a.e. $x \in \Omega$ the multifunction $t \mapsto F(x,t)$ is increasing upward.

Then, for $\alpha \in \mathbb{R}$, $F_{-}^{-1}(\alpha^{\uparrow})$ is product-measurable if and only if the set

$$M^*(\alpha) = \{(x,t) \in \Omega \times \mathbb{R} : \min F^*(x,\cdot)^{-1}(\alpha^{\uparrow}) = t\}$$

is product-measurable.

Proof: Let $\alpha \in \mathbb{R}$ be arbitrary and consider the pre-image

$$M := F_{-}^{-1}(\alpha^{\uparrow}) = \{(x,t) \in \Omega \times \mathbb{R} : F(x,t) \cap \alpha^{\uparrow} \neq \emptyset\} = \{(x,t) \in \Omega \times \mathbb{R} : F^*(x,t) \geqslant \alpha\}.$$

We are going to reveal a connection between M and $M^*(\alpha)$. To this end, define the multifunction $m\colon \mathbb{R} \to \mathcal{P}_{\emptyset}(\Omega)$ by

$$m(t) := F(\cdot,t)_{-}^{-1}(\alpha^{\uparrow}),$$

such that $(x,t) \in M$ if and only if $x \in m(t)$. Due to (i), $m\colon \mathbb{R} \to \mathcal{P}_{\emptyset}(\Omega)$ has measurable values, and due to (ii), m is permanent upward. To show the latter, assume $x \in m(t)$ and $t \leqslant t'$. Then $\alpha \leqslant F^*(x,t) \leqslant F^*(x,t')$, which implies $x \in m(t')$. Thus, from Proposition 3.52 we know that the function $g\colon \Omega \to \mathbb{R} \cup \{-\infty, +\infty\}$, defined by

$$g(x) := \inf\{t \in \mathbb{R} : (x,t) \in M\} = \inf\{t \in \mathbb{R} : x \in m(t)\},$$

is measurable. Furthermore, consider the set

$$M' := \{(x,t) \in \Omega \times \mathbb{R} : g(x) < t\},$$

which is a subset of M. Indeed, for all $(x,t') \in M'$ there is some t such that $t \leqslant t'$ and $(x,t) \in M$; since m is permanent upward, we deduce $(x,t') \in M$. Therefore, we have the representation $M = M' \cup (M \setminus M')$. Further, we have

$$\begin{aligned} M \setminus M' &= \{(x,t) \in \Omega \times \mathbb{R} : (x,t) \in M \text{ and } g(x) \geqslant t\} \\ &= \{(x,t) \in \Omega \times \mathbb{R} : F^*(x,t) \geqslant \alpha \text{ and } \inf\{s : F^*(x,s) \geqslant \alpha\} \geqslant t\} \\ &= \{(x,t) \in \Omega \times \mathbb{R} : \min F^*(x,\cdot)^{-1}(\alpha^{\uparrow}) = t\}, \end{aligned}$$

that is, $M \setminus M' = M^*(\alpha)$ and $M = M' \cup M^*(\alpha)$. Since the functions $(x,t) \mapsto g(x)$ and $(x,t) \mapsto t$ are $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R})$-measurable, the set M' is product-measurable, and thus $M = F_{-}^{-1}(\alpha^{\uparrow})$ is product-measurable if and only if $M^*(\alpha)$ is product-measurable. ○

3.54 Corollary *Let $F\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multifunction with compact and convex values such that*

(i) for all $t \in \mathbb{R}$ the multifunction $x \mapsto F(x,t)$ is measurable on Ω,

(ii) for a.e. $x \in \Omega$ the multifunction $t \mapsto F(x,t)$ is increasing.

Then, F is product-measurable if and only if for every $\alpha \in \mathbb{R}$ the sets

$$M^*(\alpha) := \{(x,t) \in \Omega \times \mathbb{R} : \min F^*(x,\cdot)^{-1}(\alpha^\uparrow) = t\},$$
$$M_*(\alpha) := \{(x,t) \in \Omega \times \mathbb{R} : \max F_*(x,\cdot)^{-1}(\alpha^\downarrow) = t\}$$

are product-measurable.

Proof: Suppose first that for every $\alpha \in \mathbb{R}$ both $M^*(\alpha)$ and $M_*(\alpha)$ are measurable. Then, by Theorem 3.53, $F_-^{-1}(\alpha^\uparrow)$ is product-measurable, and the some holds by duality for $F_-^{-1}(\alpha^\downarrow)$. By Lemma 3.17 and Corollary 3.27, F is product-measurable.

Conversely, if F is product-measurable, for all $\alpha \in \mathbb{R}$ the pre-images $F_-^{-1}(\alpha^\uparrow)$ and $F_-^{-1}(\alpha^\downarrow)$ are product-measurable and thus, again by Theorem 3.53 and its dual, $M^*(\alpha)$ and $M_*(\alpha)$ are product-measurable, too. ○

3.55 Remark The proof of Corollary 3.54 simplifies slightly if one incorporates Proposition 3.31 and proves Theorem 3.53 only for single-valued functions. However, the conditions remain the same. ○

3.56 Remark In the setting of Corollary 3.54, one has that $M := F^*(x,\cdot)^{-1}(\alpha^\uparrow)$ is an increasing set in $\mathbb{R}$ and thus $M = \mathbb{R}$, $M = t^\uparrow$, $M = t^{\uparrow\uparrow}$ or $M = \emptyset$.

To simplify the condition that $M^*(\alpha)$ is product-measurable, let us define the set Ω^* and the function $t^*\colon \Omega^* \to \mathbb{R}$ via

$$\Omega^* := \{x \in \Omega : F^*(x,\cdot)^{-1}(\alpha^\uparrow) \text{ has a minimum}\}, \quad t^*(x) := \min F^*(x,\cdot)^{-1}(\alpha^\uparrow).$$

Then $M^*(\alpha)$ is measurable if and only if the set $\{(x,t) \in \Omega \times \mathbb{R} : x \in \Omega^*, t = t^*(x)\}$ is measurable, that is if the graph of t^* is measurable. This is the case if Ω^* and t^* are both measurable, which is a fairly simple criterion. ○

3.57 Remark In the setting of Corollary 3.54, if F is constant with respect to its first argument, then we have for every $\alpha \in \mathbb{R}$

$$M^*(\alpha) = \{(x,t) \in \Omega \times \mathbb{R} : \min F^*(x,\cdot)^{-1}(\alpha^\uparrow) = t\} = \Omega \times A(\alpha),$$

where either $A(\alpha)$ is the singleton $\{\min F^*(x_0,\cdot)^{-1}(\alpha^\uparrow)\}$ for a fixed $x_0 \in \Omega$ (if there is such a minimum), or $A(\alpha) = \emptyset$. In both cases, $M^*(\alpha)$ is product-measurable. With dual reasoning, $M_*(\alpha)$ is product-measurable, too. Thus, we obtain as a special case again Corollary 3.21. ○

3.58 Remark Finally, let us also mention a related result from [4, Th. 1.9]:

Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $x \mapsto f(x,s)$ is measurable for all $s \in \mathbb{R}$ and such that $s \mapsto f(x,s)$ is increasing for a.e. $x \in \Omega$. Then the following assertions are equivalent:

(i) f is superpositionally measurable;

(ii) f is a *Shragin function*, i.e. there exists a null-set $N \in \mathcal{L}(\Omega)$ such that, for any $B \in \mathcal{B}(\mathbb{R})$, the set $f^{-1}(B) \setminus (N \times \mathbb{R})$ belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R})$.

Note also that all Shragin functions are superpositionally measurable ([4, Th. 1.1]). ○

3.2.3 Construction of Bifunctions

In [22, 27], bifunctions $f\colon (s,t) \mapsto \partial_1 g(s,t)$, were considered, where $\partial_1 g(s,t)$ denotes Clarke's generalized gradient (see Definition 2.62) with respect to the first argument of a function $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The following corollary provides the properties of f under the assumptions used in [22, 27].

3.59 Corollary *Let $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be locally Lipschitz in the first argument and assume that $t \mapsto g^{\circ}(s,t,1)$ is decreasing, and that $t \mapsto g^{\circ}(s,t,-1)$ is increasing for all $s \in \mathbb{R}$. Then $f = \partial_1 g$ is upper semicontinuous in the first and decreasing in the second argument.*

Proof: This follows readily from Propositions 2.64 and 3.31 and Corollary 1.27. ○

In the following, we present two ways to construct various examples of multifunctions $F\colon \mathbb{R}\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ that are upper semicontinuous in the first and increasing upward in the second argument. This extends the examples given in [22], where Clarke's generalized gradient ∂g of a locally Lipschitz function $g\colon \mathbb{R} \to \mathbb{R}$ and an increasing function $h\colon \mathbb{R} \to \mathbb{R}$ were combined to generate bifunctions as follows:

$$(s,t) \mapsto \partial g(s) + h(t) \quad \text{and} \quad (s,t) \mapsto \partial g(s) h(t)$$

(with the requirement $\partial g \subset [0,\infty)$ in the second case).

3.60 Proposition *Let $F\colon \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and $G, H\colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be multifunctions and suppose one of the following conditions:*

(i) G is single-valued and continuous, H is single-valued and increasing upward, and F is upper semicontinuous in the first and increasing upward in the second argument.

(ii) G is compact-valued and upper semicontinuous, the values of H and F are compact intervals, H is increasing upward, and F is continuous in the first and increasing upward in the second argument.

Then the composition

$$F \circ (G,H)\colon \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R}), \quad (s,t) \mapsto H(G(s),H(t)) = \{H(x,y) : x \in G(s), y \in H(t)\}$$

is upper semicontinuous in the first and increasing upward in the second argument.

Proof: Let us first assume that (i) holds and set $g := G$ and $h := H$. Then, for all $s_0, t_0 \in \mathbb{R}$ and any open set $U \subset \mathbb{R}$ such that $F(g(s_0), h(t_0)) \subset U$ it follows from the upper semicontinuity of $F(\cdot, h(t_0))$ that $F(g(s), h(t_0)) \subset U$ if only $g(s)$ is near $g(s_0)$. Since g is continuous, this holds for all s near s_0 and thus $F \circ (g, h)$ is upper semicontinuous in the first argument.

Further, let $t_1 \in \mathbb{R}$ be such that $t_0 \leqslant t_1$. Then, for any $b_0 \in h(t_0)$ there is some $b_1 \in h(t_1)$ such that $b_0 \leqslant b_1$, and thus for all $a \in g(s_0)$ and $c_0 \in F(a, b_0)$ there is some $c_1 \in F(a, b_1)$ such that $c_0 \leqslant c_1$. This implies that $F \circ (g, h)$ is increasing upward in the second argument.

Now, let us assume that conditions (ii) holds and let again $s_0, t_0 \in \mathbb{R}$ be fixed. The proof of $F \circ (G, H)$ being increasing upward in the second argument goes exactly as above, so that we only have to prove that $F \circ (G, H)$ is upper semicontinuous in the first argument. To this end, note that for $a \in \mathbb{R}$ we have

$$F(a, H(t_0)) = [F_*(a, H_*(t_0)), F^*(a, H^*(t_0))]$$

(where as usual F_* and F^* are the envelopes of F, and H_* and H^* those of H). Further, the continuity of F implies (thanks to Proposition 2.47) the continuity of F_* and F^* in the first argument and thus, for any $s_0 \in \mathbb{R}$, there are a_* and a^* in the compact set $G(s_0)$ such that

$$F(G(s_0), H(t_0)) = [F_*(a_*, H_*(t_0)), F^*(a^*, H^*(t_0))]. \tag{3.3}$$

Now let $U \subset \mathbb{R}$ be an open set such that $F(G(s_0), H(t_0)) \subset U$. In view of (3.3) and the continuity of F_* and F^* we conclude $F(G(s), H(t_0)) \subset U$ if $G(s) \subset G(s_0) + [0, \varepsilon)$ for some $\varepsilon > 0$. By means of the upper semicontinuity of G, this holds for all s near s_0, thus $F(G(\cdot), h(t_0))$ is upper semicontinuous. ○

3.61 Example There are elementary functions $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfying the conditions in (ii) above. One can define $f(s, t)$ to be equal to, e.g,

$$s + t, \qquad |s| \cdot t, \qquad s \wedge t, \qquad s \vee t, \qquad (|s| + 1)^t, \qquad \operatorname{sgn}(t)|t|^{|s|}.$$

Further, in the terms above any s can be replaced by $f_1(s)$, where f_1 is a continuous real function, and any t can be replaced by $f_2(t)$, where f_2 is an increasing real function. Let us give the following illuminating example (that closely resembles the one given in [22]):

$$g(x) := \begin{cases} -x + 1 & \text{if } x < 0, \\ [0, 1] & \text{if } x = 0, \\ 0 & \text{if } x > 0, \end{cases} \qquad h(x) := \begin{cases} -2 & \text{if } x \leqslant 0, \\ x & \text{if } x > 0. \end{cases}$$

As one sees, $g + h$ is neither upper semicontinuous (nor lower semicontinuous) nor in some way increasing or decreasing. ○

3.3 Spaces of Measurable Functions

In applications, the spaces we will work with are mostly the well-known Sobolev spaces $W^{1,p}(\Omega)$ over a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary. In this section, we collect some basic facts. The stated notations and results are standard. For more information, we refer, e.g., to [11, 12, 40, 96, 119].

To avoid technicalities, we confine our considerations mostly to the Sobolev space $W_0^{1,p}(\Omega)$ whose elements have zero boundary in the sense of traces. Note, however, that most of the results can be generalized to more general function spaces, e.g. spaces of Sobolev functions with non-trivial boundary values, Sobolev spaces with respect to variable exponents, or Orlicz-Sobolev spaces. Consequently, most of the methods developed in this thesis can be applied also to the more general case. For further information, we refer, e.g., to [36, 49, 101].

3.3.1 Spaces of Merely Measurable Functions

Let $\mathbb{R}^N$, $N \geqslant 1$, and let $\Omega \subset \mathbb{R}^N$ be a **domain**, i.e. an open and connected subset of $\mathbb{R}^N$. We equip Ω with its Lebesgue σ-algebra and its Lebesgue measure λ, such that $\Omega = (\Omega, \mathcal{L}(\Omega), \lambda)$ becomes a complete measure space. For a set $M \in \mathcal{L}(\Omega)$, we denote by $|M| := \lambda(M)$ its measure. As usual, we identify functions $u, v\colon \Omega \to \mathbb{R}$ with each other if $u(x) = v(x)$ for a.e. $x \in \Omega$. If we refer to some value $u(x)$ of a class u, we mean the value of an arbitrary representative of the class u.

3.62 Definition Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We denote by $L^0 = L^0(\Omega)$ the space of (equivalence classes of) measurable functions $u\colon \Omega \to \mathbb{R}$ (where $\mathbb{R}$ is equipped with its Borel σ-algebra). We equip L^0 (and all its subspaces) with the partial order $\leqslant$ defined via

$$u \leqslant v \quad :\Longleftrightarrow \quad u(x) \leqslant v(x) \quad \text{for a.e. } x \in \Omega.$$

Addition and scalar multiplication are defined analogously. Further, for $M \subset L^0$ we set

$$M_+ := 0_M^{\uparrow} = \{u \in M : 0 \leqslant u\},$$

and for $u \in L^0$ we define functions $|u|\colon \Omega \to \mathbb{R}$, $u^+\colon \Omega \to \mathbb{R}$ and $u^-\colon \Omega \to \mathbb{R}$ via

$$|u|\colon x \mapsto |u(x)|, \quad u^+\colon x \mapsto u(x) \vee 0, \quad \text{and} \quad u^-\colon x \mapsto (-u(x)) \vee 0. \qquad \circ$$

3.63 Proposition *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Then $L^0 = (L^0, \leqslant, +, \cdot)$ is an ordered linear space with lattice structure (and thus a distributive lattice). For $u, v \in L^0$ the function $u \vee v$ is given by $u \vee v\colon x \mapsto u(x) \vee v(x)$, $u \wedge v$ is given by duality. Further, $|u| \in L^0$, $u^+ \in L^0$ and $u^- \in L^0$.* ○

If $u_1, \dots, u_n \in L^0$, then

$$u\colon \Omega \to \mathbb{R}^n, \quad x \mapsto (u_1(x), \dots, u_n(x))$$

is a measurable vector-valued function (where $\mathbb{R}^n$ is equipped with its Borel σ-algebra). By abuse of notation, we write $u \in L^0$ instead of $u \in (L^0)^n$. We will use the same convention for all subspaces of L^0.

3.3.2 Lebesgue Spaces

Although L^0 has good order-theoretical properties, its topological properties are not good enough. Thus, the following normed subspaces of L^0 are welcome. We construct them by use of the usual Lebesgue integration. Note, however, that we usually write $\int_\Omega u$ instead of $\int_\Omega u \, d\lambda$ or $\int_\Omega u(x) \, d\lambda(x)$ or $\int_\Omega u(x) \, dx$.

3.64 Definition Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $1 \leqslant p < \infty$.

(i) For a measurable function $u \in L^0$ we define $\|u\|_p$ via

$$\|u\|_p^p := \int_\Omega |u|^p.$$

We denote by $L^p = L^p(\Omega)$ the space of functions $u \in L^0$ satisfying $\|u\|_p < \infty$.

(ii) Let $u_1, \ldots, u_n \in L^p$ be given. Then, for the measurable vector-valued function

$$u \colon \Omega \to \mathbb{R}^n, \quad x \mapsto (u_1(x), \ldots, u_n(x)),$$

we define $\|u\|_p$ via $\|u\|_p^p := \sum_{i=1}^n \|u_i\|_p^p$.

(iii) We denote by $L^p_{loc} = L^p_{loc}(\Omega)$ the space of functions $u \in L^0$ such that $\int_K |u|^p < \infty$ for every compact subset K of Ω.

(iv) For a measurable function $u \in L^0$, we put

$$\|u\|_\infty := \inf\{\alpha \in \mathbb{R} : |\{x \in \Omega : |u(x)| > \alpha\}| = 0\}.$$

We denote by $L^\infty = L^\infty(\Omega)$ the space of functions $u \in L^0$ satisfying $\|u\|_\infty < \infty$. L^∞_{loc} is defined analogously to L^p_{loc}. ○

We have the following structural properties of the Lebesgue spaces L^p and L^∞:

3.65 Proposition *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $1 \leqslant p \leqslant \infty$.*

(i) L^p is a sublattice and a linear subspace of L^0 and $L^p = (L^p, \|\cdot\|_p)$ is an ordered Banach space with normal order cone.

(ii) Denote by $p' \in [1, \infty]$ the ***Hölder conjugate*** *of p defined via $1/p + 1/p' = 1$. Then for $u \in L^p$ and $v \in L^{p'}$ it holds $\|uv\|_1 \leqslant \|u\|_p \|v\|_{p'}$.* [Hölder's Inequality]

(iii) Let $1 < p < \infty$, $u \in L^p$ and $v \in L^{p'}$, and $\varepsilon > 0$. Then $\|uv\|_1 \leqslant \varepsilon \|u\|_p^p + c_\varepsilon \|v\|_{p'}^{p'}$ with $c_\varepsilon = (\varepsilon p)^{p'/p}/p'$. [Young's Inequality with Epsilon]

(iv) If $1 \leqslant p < \infty$, then there is an isometric bijection $i \colon (L^p)' \to L^{p'}$ such that $\langle \varphi, u \rangle = \int_\Omega i(\varphi) u$ for all $\varphi \in (L^p)'$ and $u \in L^p$. We thus identify $(L^p)'$ with $L^{p'}$.

(v) If $1 < p < \infty$, then L^p is reflexive. ○

Concerning different types of convergence, we have the following results:

3.66 Theorem *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $(u_n) \subset L^1$, $u \in L^0$ and $g \in L^1$.*

(i) If $u(x) = \lim_n u_n(x)$ for a.e. $x \in \Omega$ and if $|u_n| \leqslant g$ for all n, then $u \in L^1$ and $u_n \to u$ in L^1. [Lebesgue]

(ii) If $u_n \to u$ in L^1, then, up to a subsequence, $u_n(x) \to u(x)$ for a.e. $x \in \Omega$ (i.e. there is a subsequence (v_n) of (u_n) such that $v_n(x) \to u(x)$ for a.e. $x \in \Omega$).

(iii) If $g \leqslant u_n$ for all n, then $\int_\Omega \liminf_n u_n \leqslant \liminf_n \int_\Omega u_n$. [Fatou] ○

Concerning mappings between Lebesgue spaces, Carathéodory functions play an important role:

3.67 Proposition *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, and suppose there are $p, q \in [1, \infty)$, a function $a \in L^q$ and a constant $b \geqslant 0$ such that*

$$|f(x,s)| \leqslant |a(x)| + b|s|^{p/q} \quad \text{for a.e. } x \in \Omega \text{ and all } s \in \mathbb{R}.$$

Then the superposition operator (the so called ***Nemitskij operator****)*

$$N_f\colon L^p \to L^q, \quad u \mapsto [x \mapsto f(x, u(x))]$$

is well-defined, bounded and continuous. ○

Upper Carathéodory multifunctions play an important role, too. They allow to take limits under weak conditions. The proof of the following proposition follows the ideas presented in [116, Th. 3.1.2]).

3.68 Proposition *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $G\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be an upper Carathéodory multifunction with closed and convex values and such that $G(x,s)$ is bounded for almost every $x \in \Omega$ and all $s \in \mathbb{R}$. Suppose furthermore $g_n \rightharpoonup g$ in L^1 and $u_n \to u$ in L^0 pointwise a.e. If*

$$\text{(i)} \quad g_n \subset G(\cdot, u_n) \qquad \text{or} \qquad \text{(ii)} \quad g_n \subset (\tau_n(G))(\cdot, u_n),$$

where $\tau_n\colon s \to \max(-n, \min(s, n))$ is the truncation at level n, then $g \subset G(\cdot, u)$.

Proof: Since closed convex sets are weakly closed, we have $g \in \operatorname{cl}\operatorname{cv}\{g_n, g_{n+1}, \ldots\}$, where cl denotes the closure and cv the convex hull. Take $h_n \in \operatorname{cv}\{g_n, g_{n+1}, \ldots\}$ such that $\|h_n - g\|_1 \leqslant 1/n$, then we have $h_n \to g$ in L^1 and thus $h_{n_k} \to g$ pointwise a.e. as $k \to \infty$ for some subsequence $(h_{n_k}) \subset (h_n)$.

Let Ω_0 be the set of all $x \in \Omega$ such that $G(x, \cdot)$ is upper semicontinuous, $G(x,s)$ is bounded for all $s \in \mathbb{R}$, $h_{n_k}(x) \to g(x)$, $u_n(x) \to u(x)$, $g_n(x) \in G(x, u_n(x))$ in case (i) and $g_n(x) \in \tau_n(G(x, u_n(x)))$ in case (ii) for all n. Clearly, $\Omega \setminus \Omega_0$ is a null-set.

Now suppose $x \in \Omega_0$ and let E be an arbitrary bounded open convex set with $G(x, u(x)) \subset E$. Since $G(x, \cdot)$ is upper semicontinuous and $u_n(x) \to u(x)$, there is some smallest $n_E \in \mathbb{N}$ with $G(x, u_n(x)) \subset E$ provided $n \geqslant n_E$.

In case (i), we have for all $n_k \geqslant n_E$

$$h_{n_k}(x) \in \operatorname{cv} \bigcup_{n \geqslant n_k} G(x, u_n(x)) \subset E. \tag{3.4}$$

In case (ii), we have for all $n_k \geqslant n_E$

$$h_{n_k}(x) \in \operatorname{cv}\{g_{n_k}(x), g_{n_k+1}(x), \dots\},$$

where $g_n = \tau_n(\widetilde{g}_n)$ for some $\widetilde{g}_n$ with $\widetilde{g}_n(x) \in G(x, u_n(x))$. We claim that in fact $\tau_n(\widetilde{g}_n(x)) = \widetilde{g}_n(x)$ for n large enough, that is, $g_n(x) \in G(x, u_n(x))$. Otherwise, we would have $|\widetilde{g}_n(x)| > n$ for infinitely many $n \in \mathbb{N}$ and thus E wouldn't be bounded. Thus, (3.4) holds also in case (ii) for all sufficiently large n_k.

In both cases it follows that $g(x) \in \operatorname{cl} E$. Since the closed, bounded and convex set $G(x, u(x))$ is the intersection of all such E (to see this, consider the open and convex sets $G(x, u(x)) + B_\varepsilon$), we obtain $g(x) \in G(x, u(x))$ and thus $g \subset G(\cdot, u)$ as desired. ○

Finally, let us introduce an important example of a pseudomonotone operator. The basic idea is written down in [65, Lemma 3.6]; note, however, that we do not need any knowledge about *Hausdorff upper semicontinuous* multifunctions.

3.69 Lemma *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $g\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be an upper Carathéodory multifunction with closed and convex values. Further, let $L = L^{q'}$ for some $q \in (1, \infty)$, and let W be a normed space such that the embedding $i\colon L \hookrightarrow W'$ is compact, and suppose that there is some $b \in L$ such that g satisfies, for a.e. $x \in \Omega$,*

$$\sup\{|y| : y \in g(x, s), s \in \mathbb{R}\} \leqslant b(x). \tag{3.5}$$

Then the selection mapping $G\colon W \to \mathcal{P}(W')$, defined by

$$G(u) := \{i\eta : \eta \text{ is a measurable selection of } g(\cdot, u)\},$$

is well-defined, pseudomonotone and bounded, and has closed and convex values.

Proof: Let $u \in W$ be given. Due to Theorem 3.47, there are measurable selections of $g(\cdot, u)$, and due to the growth condition (3.5), we see that the measurable selections of $g(\cdot, u)$ are uniformly bounded in L independent of u. Since $i\colon L \hookrightarrow W'$ is a compact embedding, we conclude that $G(u)$ is well-defined and uniformly bounded in W', again independent of u. Further, $G(u)$ is convex and closed in W', since g has convex and closed values. Indeed, let $(\eta_n) \subset g(\cdot, u)$ be some sequence such that the sequence $(i\eta_n)$ converges in W'. Since $g(\cdot, u)$ is bounded in the reflexive space L, we conclude, up to a subsequence, $\eta_n \rightharpoonup \eta$ in L for some η, and due to Proposition 3.68, we have $\eta \subset g(\cdot, u)$. Because i is linear and bounded, it follows $i\eta_n \rightharpoonup i\eta$ in W' and we infer that $G(u)$ is closed.

Now, let us apply Proposition 2.54 to ensure that G is pseudomonotone; we only have to verify that $\operatorname{gr} G$ is sequentially weakly closed and that the duality pairing is $(\mathrm{w}\times\mathrm{w})$-continuous on $\operatorname{gr} G$. To this end, let $(u_n) \subset W$ and $(i\eta_n) \subset W'$ be sequences such that $\eta_n \subset g(\cdot, u_n)$ for all n, and, for some $u \in W$, $\eta^* \in W'$, $u_n \rightharpoonup u$ in W and $i\eta_n \rightharpoonup \eta^*$ in W'. Since also the embedding $W \hookrightarrow L'$ is compact, and since (η_n) is bounded in L, we have, up to a subsequence, $u_n \to u$ in L', $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, and $\eta_n \rightharpoonup \eta$ in L for some $\eta \in L$. We apply once more Proposition 3.68 to conclude $\eta \subset g(\cdot, u)$. Since i is linear and bounded, we infer $i\eta_n \rightharpoonup i\eta$, and thus $\eta^* = i\eta \in G(u)$. Further, we apply the identity $\langle i\theta, v\rangle_W = \langle \theta, v\rangle_{L'}$ (for $\theta \in L$, $v \in W$) and the triangle-inequality to conclude

$$0 \leqslant |\langle \eta_n^*, u_n\rangle_W - \langle \eta^*, u\rangle_W| \leqslant \|\eta_n\|_L \|u_n - u\|_{L'} + |\langle \eta_n - \eta, u\rangle_{L'}| \to 0.$$

Thus, G fulfills the requirements of Proposition 2.54 and is thus pseudomonotone. ○

3.3.3 Sobolev Spaces

To introduce Sobolev spaces, we have first to define weak derivatives. To this end, we denote by $C_c^\infty = C_c^\infty(\Omega)$ the space of infinitely often differentiable functions $\varphi\colon \Omega \to \mathbb{R}$ with compact support in Ω. For $\varphi \in C_c^\infty$ we denote by $D_i\varphi := \frac{\partial}{\partial x_i}\varphi$, $i = 1, \dots, N$, the usual partial derivatives of φ.

3.70 Definition Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $u, w \in L^1_{\mathrm{loc}}$. If

$$\int_\Omega u D_i\varphi = -\int_\Omega w\varphi \quad \text{for all } \varphi \in C_c^\infty(\Omega),$$

then we define $D_i u := w$ and call w **weak derivative** of u. ○

3.71 Proposition *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $u \in L^1_{\mathrm{loc}}$.*

(i) If $D_i u$ exists, it is uniquely defined in L^1_{loc}.

(ii) If $u \in C_c^\infty$, then both definitions of $D_i u$ coincide. ○

3.72 Definition Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $1 \leqslant p < \infty$.

(i) We denote by $W^{1,p} = W^{1,p}(\Omega)$ the space of all $u \in L^p$ such that $D_i u \in L^p$ for all $i = 1, \dots, N$.

(ii) For $u \in W^{1,p}$ we define $\nabla u\colon \Omega \to \mathbb{R}^N$ via $(\nabla u)(x) := (D_1 u(x), \dots, D_N u(x))$.

(iii) On $W^{1,p}$ we define $\|\cdot\|_{1,p}$ via $\|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p$.

(iv) By $W_0^{1,p} = W_0^{1,p}(\Omega)$ we denote the closure of C_c^∞ in $W^{1,p}$ with respect to $\|\cdot\|_{1,p}$. ○

We have the following structural properties of the Sobolev spaces $W^{1,p}$ and $W_0^{1,p}$:

3.73 Proposition *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and let $1 \leqslant p < \infty$.*

(i) *Let* $1 \leqslant p < \infty$*. Then* $W^{1,p}$ *and* $W_0^{1,p}$ *are partial ordered subsets and subspaces of* L^0*, and* $W^{1,p} = (W^{1,p}, \|\cdot\|_{1,p})$ *and* $W_0^{1,p} = (W_0^{1,p}, \|\cdot\|_{1,p})$ *are ordered Banach spaces.*

(ii) *On* $W_0^{1,p}$*,* $u \mapsto \|\nabla u\|_p$ *is an equivalent norm of* $\|\cdot\|_{1,p}$*, which is a consequence of the Poincaré-Friedrichs Inequality.*

(iii) *Let* $1 < p < \infty$*. Then* $W^{1,p}$ *and* $W_0^{1,p}$ *are reflexive.* ○

In order to have more structural properties of $W^{1,p}$, in the following we restrict our considerations to bounded domains Ω with **Lipschitz boundary** $\partial\Omega$, i.e. $\partial\Omega$ is locally the graph of a Lipschitz continuous function. (See, e.g., [28, Definition 2.71] for a more detailed definition.)

3.74 Lemma (Chain Rule) *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with Lipschitz boundary, and let* $1 \leqslant p < \infty$*. Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a Lipschitz continuous function and* $u \in W^{1,p}$*. Then* $f \circ u \in W^{1,p}$ *and*

$$D_i(f \circ u) = (f_B \circ u) D_i u,$$

where $f_B\colon \mathbb{R} \to \mathbb{R}$ *is a Borel measurable function such that* $f_B = f'$ *a.e. in* Ω*.* ○

3.75 Corollary *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with Lipschitz boundary, and let* $1 \leqslant p < \infty$*. Then* $W^{1,p}$ *and* $W_0^{1,p}$ *are sublattices of* L^0*, and it holds, e.g.,*

$$(D_i u^+)(x) = \begin{cases} D_i u(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) \leqslant 0, \end{cases} \qquad (D_i(u \wedge v))(x) = \begin{cases} D_i u(x) & \text{if } u(x) > v(x), \\ D_i v(x) & \text{if } u(x) \leqslant v(x). \end{cases}$$ ○

We furthermore have not only $W^{1,p} \subset L^p$ as ordered sets, but there is also a compact embedding:

3.76 Theorem *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with Lipschitz boundary, and let* $1 \leqslant p < \infty$*. Then we denote by* p^* *the* ***criticial Sobolev exponent****, defined via*

$$p^* := \frac{Np}{N-p} \quad \text{if } p < N \quad \textit{and} \quad p^* := \infty \quad \text{if } p \geqslant N.$$

Then, for $q \in [1, \infty]$*, the embedding* $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *is continuous if* $q \leqslant p^*$*, and compact if* $q < p^*$*.*

Especially, we have the compact embedding $W^{1,p} \hookrightarrow L^p$*.* ○

3.77 Remark Recall that for a linear and continuous operator $A\colon X \to Y$ on Banach spaces its **dual operator** $A'\colon Y' \to X'$ is defined via $\langle A'\varphi, u\rangle = \langle \varphi, Au\rangle$ for all $\varphi \in Y'$. As a matter of fact, A' is linear and continuous, too, with norm $\|A'\| = \|A\|$, and if A is compact, then so is A'.

Thus, Theorem 3.76 gives us, by this version of duality, compact embeddings on the duals of various Lebesgue and Sobolev spaces. ○

Now, as promised on Page 64, let us introduce an important second-order quasilinear differential operator on Sobolev spaces of **Leray-Lions type**:

3.78 Theorem *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain with Lipschitz boundary* $\partial\Omega$*, let* $1 < p < \infty$*, and let functions* $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$*,* $i = 0, \dots, N$*, be given satisfying the following conditions:*

(A1) Each a_i *is a* ***Carathéodory function****, i.e.* $x \mapsto a_i(x, s, \xi)$ *is measurable for all* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ *and* $(s, \xi) \mapsto a_i(x, s, \xi)$ *is continuous for a.e.* $x \in \Omega$*.*

(A2) The vector-valued function $a = (a_1, \dots, a_N)$ *is* p*-**coercive** in the last argument, i.e. there is a constant* $\alpha_2 > 0$ *and a function* $k_2 \in L^1$ *such that*

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \geqslant \alpha_2 |\xi|^p - k_2(x)$$

for a.e. $x \in \Omega$ *and for all* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$*.*

(A3) There exists a constant $\alpha_3 > 0$ *and a function* $k_3 \in L^{p'}$ *such that each* a_i *satisfies the growth condition*

$$|a_i(x, s, \xi)| \leqslant \alpha_3(|s|^{p-1} + |\xi|^{p-1}) + k_3(x).$$

(A4) Either *the vector-valued function* $a = (a_1, \dots, a_N)$ *is* ***strictly monotone*** *with respect to its last argument, i.e.*

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0$$

for a.e. $x \in \Omega$*, all* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ *and all* $\xi' \in \mathbb{R}^N$ *with* $\xi \neq \xi'$*,*

or *there are functions* $\tilde{a}_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ or $\overline{a}_0 : \Omega \times \mathbb{R} \to \mathbb{R}^N$ *such that, for a.e.* $x \in \Omega$ *and all* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$*,* $a_0(x, s, \xi) = \tilde{a}_0(x, s)$ *or* $a_0(x, s, \xi) = \overline{a}_0(x, s)\xi$*.*

Further, let V *be a closed subspace of* $W^{1,p}$ *such that* $W_0^{1,p} \subset V \subset W^{1,p}$*, define the operators* $A_0, A_1 : V \to V'$ *via*

$$\langle A_0 u, v \rangle := \int_\Omega a_0(\cdot, u, \nabla u)v \quad \textit{and} \quad \langle A_1 u, v \rangle := \sum_{i=1}^{N} \int_\Omega a_i(\cdot, u, \nabla u) D_i v$$

for all $u, v \in V$*, and let* $A = A_0 + A_1$*. Then the second-order quasilinear differential operator* $A : V \to V'$ *is continuous, bounded, pseudomonotone and coercive.* ○

For a proof, we refer to [98, Chapter 2], where also slightly more general conditions are presented for A to be pseudomonotone. Note also that A_1 is only pseudomonotone if a is monotone with respect to its last argument, see [9].

3.3.4 Radon Measures

In this final subsection, let us shortly inspect a very useful connection between Functional Analysis and Measure Theory. To this end, let us define the following spaces of continuous functions:

3.79 Definition Let X be an open subset of $\mathbb{R}^N$ (or, more generally, a Hausdorff space which is **locally compact**, i.e. every point $x \in X$ has a compact neighborhood).

(i) The **support** $\operatorname{supp} f$ is the closure of the set $\{x \in \Omega : f(x) \neq 0\}$.

(ii) By $C_c(X)$ we denote the space of all continuous functions $f\colon X \to \mathbb{R}$ with compact support, equipped with the norm $\|\cdot\|_\infty\colon f \mapsto \sup\{|f(x)| : x \in X\}$.

(iii) By $C_0(X)$ we denote the space of all continuous functions $f\colon X \to \mathbb{R}$ such that, for any $\varepsilon > 0$, $\{x \in X : |f(x)| \geqslant \varepsilon\}$ is compact, equipped with $\|\cdot\|_\infty$. ○

As we know, continuous functions on a topological space X are Borel measurable. This gives rise to linear functionals on $C_0(X)$ and its dense subspace $C_c(X)$ which are defined via integration against **Borel measures**, i.e. measures $\mu\colon \mathcal{B}(X) \to [0,\infty]$. The study of such functionals reveals an isometric isomorphism between $C_0(X)'$ and the space $\mathcal{M}_b(X)$ of signed Radon measures. To be more precise, let us give some definitions and results (for more information, we refer, e.g., to [42, Chapter 7]).

3.80 Definition Let X be an open subset of $\mathbb{R}^N$ (or, more generally, a locally compact Hausdorff space), and let μ be a Borel measure on X.

(i) μ is called **inner regular** on $A \in \mathcal{B}(X)$ if

$$\mu(A) = \sup\{\mu(K) : K \subset A, K \text{ is compact}\}.$$

(ii) μ is called **outer regular** on $A \in \mathcal{B}(X)$ if

$$\mu(A) = \inf\{\mu(U) : A \subset U, U \text{ is open}\}.$$

(iii) μ is called **Radon measure** if it is finite on compact sets, inner regular on open sets and outer regular on $\mathcal{B}(X)$. ○

Those definitions can be extended to **signed Borel measures** on X, i.e. σ-*additive* functions $\mu\colon \mathcal{B}(X) \to \mathbb{R}$. To this end, let us recall the *Jordan decomposition*: For a signed Borel measure μ we define two measures μ^+ and μ^- on $\mathcal{B}(X)$ via

$$\mu^+(A) := \sup\{\mu(B) : B \in \mathcal{B}(X), B \subset A\}, \quad \mu^-(A) := \sup\{-\mu(B) : B \in \mathcal{B}(X), B \subset A\}.$$

Then it holds $\mu = \mu^+ - \mu^-$ and we define:

3.81 Definition Let X be an open subset of $\mathbb{R}^N$ (or, more generally, a locally compact Hausdorff space), and let μ be a signed Borel measure over X.

(i) μ is called **signed Radon measure** if μ^+ and μ^- are Radon measures. We denote by $\mathcal{M}_b(X)$ the set of all signed Radon measures over X.

(ii) The measure $|\mu| := \mu^+ + \mu^-$ is called **total variation** of μ.

(iii) On $\mathcal{M}_b(X)$ we define a norm via $\|\mu\| := |\mu|(X)$, which makes $\mathcal{M}_b(X)$ to a Banach space.

(iv) If $u\colon X \to \mathbb{R}$ is integrable with respect to $|\mu|$, we define

$$\int_X u \, d\mu := \int_X u \, d\mu^+ - \int_X u \, d\mu^-,$$

from which it follows especially $|\int_X u \, d\mu| \leqslant \int_X |u| \, d|\mu|$. ○

Finally, we have the Riesz Representation Theorem:

3.82 Theorem *Let X be an open subset of $\mathbb{R}^N$ (or, more generally, a locally compact Hausdorff space). Then for each bounded and linear functional $f \in C_0(X)'$ there is a unique signed Radon measure $\mu \in \mathcal{M}_b(X)$ such that*

$$\langle f, \varphi \rangle = \int_X \varphi \, d\mu \quad \text{for all } \varphi \in C_0(X).$$

Further, we have $\|\mu\| = |\mu|(X) = \|f\|$. ○

The representation as integral against a Borel measure is especially useful in combination with Fubini's or Tonelli's theorem. The latter is given as follows:

3.83 Theorem *Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be **σ-finite** measure spaces (i.e. they are the union of countably many subsets with finite measure), and let $f\colon X \times Y \to [0, \infty]$ be product-measurable. Then it holds*

$$\int_X \int_Y f(x, y) \, d\mu(x) \, d\nu(y) = \int_Y \int_X f(x, y) \, d\nu(y) \, d\mu(x).$$ ○

We will use this representation in order to treat variational equations whose right-hand sides are given by measures.

Part II

Applications

Introduction to Part II

In Part I, we have demonstrated the interplay between order theory, topology and measure theory on a theoretical level. Now, the goal of Part II is to demonstrate how Theorem 2.33 and the knowledge about multivalued bifunctions can be applied to solve a wide range of multivalued variational inequalities. This study provides a unifying framework for known, recently developed, and future existence and enclosure results.

Variational Inequality Problems

Although there is no precise definition of what exactly a variational inequality is, mathematicians will surly agree if a given problem is of variational nature or not. Let us give two illustrating examples from [62] (we also refer to [115]), which illustrate the evolution of this notion.

II.1 Example Let $f\colon [0,1] \subset \mathbb{R} \to \mathbb{R}$ be a smooth function. Then there surely is $x_0 \in [0,1]$ such that

$$f(x_0) = \min\{f(x) : x \in [0,1]\}. \tag{II.1}$$

Since f is smooth, it is elementary to deduce that from (II.1) it follows

$$f'(x_0)(x - x_0) \geqslant 0 \quad \text{for all } x \in [0,1] \tag{II.2}$$

(but unfortunately not vice versa), which is a basic variational inequality. ○

II.2 Example Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and closure $\overline{\Omega}$, and let $\varphi\colon \overline{\Omega} \to \mathbb{R}$ be a function such that $\max\{\varphi(x) : x \in \overline{\Omega}\} \geqslant 0$ and $\varphi \leqslant 0$ on $\partial\Omega$. Define further the convex set

$$K := \{u \in C^1(\overline{\Omega}) : u \geqslant \varphi \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\}.$$

Let us assume that $K \neq \emptyset$ and that there is $u \in K$ such that

$$\int_\Omega |\nabla u|^2 = \min\{\int_\Omega |\nabla v|^2 : v \in K\}.$$

Then, the function

$$f\colon [0,1] \to \mathbb{R}, \quad t \mapsto \int_\Omega |\nabla(u + t(v-u))|^2$$

attains its minimum at $t = 0$. It follows $f'(0) \geqslant 0$, i.e.

$$\int_\Omega \nabla u \nabla(v - u) \geqslant 0 \quad \text{for all } v \in K, \tag{II.3}$$

which is a more general variational inequality than (II.2). ○

Equations (II.2) and (II.3) can be generalized in several aspects. To this end, note that Equation (II.3) makes sense if K is a subset of the Sobolev space $W := W_0^{1,2}(\Omega)$. Thus, let us define the operator

$$A\colon W \to W', \quad u \mapsto [v \mapsto \int_\Omega \nabla u \nabla v]$$

and let us formulate the following problem:

$$\text{Find } u \in K \subset W \text{ s.t.} \quad \langle Au, v-u\rangle \geqslant 0 \quad \text{for all } v \in K. \tag{II.4}$$

This problem is our prototype of a **variational inequality problem**, and it is solved by application of Corollary 2.56.

By using more involved deductions, one can show that Problem (II.4) has solutions if A, W and K are specified in the following way:

II.3 Theorem *Let X be a reflexive Banach space, let K be a non-empty, closed and convex subset of X, and let $A\colon K \to X'$ be monotone, and continuous on finite dimensional subspaces. If K is bounded or if A is coercive, then there exists a solution of* (II.4).

Proof: See [62, Theorem 1.4, Corollary 1.8]. ○

The basic variational problem can be generalized further. For instance, if the feasible set K is not constant, but depends on u, we obtain **quasi-variational inequalities**, and if the operator A is multivalued, we obtain **multivalued variational inequalities**. Further, the solutions u and the test functions $v \in K$ may not belong to the same space, leading to **variational inequalities with measures**.

It is the goal of the current research (see, e.g., [5, 27, 60, 68, 84, 89, 90]) to extend Theorem II.3 to a wide range of variational problems.

Sub-Supersolution Method

A mathematical problem occurs if the feasible set K is unbounded *and* if the operator A is non-coercive. One way to handle such problems is given by the famous sub-supersolution method (see [104] for "historical highlights of the theory of sub-supersolutions"). It depends on the existence of **subsolutions** and **supersolutions** which are appropriately defined in such a way that the following principle holds true:

Let $\underline{u}$ be a subsolution and let $\overline{u}$ be a supersolution such that $\underline{u} \leqslant \overline{u}$.
Then there is a solution u such that $\underline{u} \leqslant u \leqslant \overline{u}$.

In [28] the application of the sub-supersolution method to a wide range of variational problems is presented and systematically studied. There, the following basic case is given:

II.4 Example Let $\Omega \subset \mathbb{R}^N$, $W = W_0^{1,2}(\Omega)$, $K \subset W$ non-empty, closed and convex, and consider the following variational inequality problem:

$$\text{Find } u \in K \text{ s.t.} \quad \int_\Omega \nabla u \nabla(v-u) \geqslant 0 \quad \text{for all } v \in K.$$

Then, $\underline{u} \in W$ is called a **subsolution** if

$$\int_\Omega \nabla \underline{u} \nabla (v - \underline{u}) \geqslant 0 \quad \text{for all } v \in \underline{u} \wedge K,$$

and, as a dual notion, $\overline{u} \in W$ is called a **supersolution** if

$$\int_\Omega \nabla \overline{u} \nabla (v - \overline{u}) \geqslant 0 \quad \text{for all } v \in \overline{u} \vee K. \qquad \bigcirc$$

In proofs, it is often useful—if not even required—that each solution of a variational problem is both a subsolution and a supersolution. To this end, the convex set K is assumed to be a lattice, which holds true in many applications. We have the following examples from [28]:

(i) $K = W$, then K is a lattice due to Corollary 3.75.

(ii) $K = \{u \in W : u \geqslant \psi\} = \psi^{\uparrow}_W$ for given $\psi \in L^0$.

(iii) $K = \{u \in W : \psi_1 \leqslant u \leqslant \psi_2\} = [\psi_1, \psi_2]_W$ for given $\psi_i \in L^0$.

(iv) $K = \{u \in W : |\nabla u(x)| \leqslant c$ for a.e. $x \in \Omega\}$ for given $c \geqslant 0$.

(v) $K = \{u \in W : D_i u \leqslant \psi_i, i = 1, \dots, n\}$ for given $\psi_i \in L^0$.

The main idea of the sub-supersolution method is to reduce the variational problem to an auxiliary problem whose operator is coercive. This is achieved by truncating the involved operators in such a way that only their behaviour on the order-interval $[\underline{u}, \overline{u}]$ is of relevance. Then, in a second step, it is shown that solutions of the auxiliary problem are solutions of the original problem, located between the given pair of sub-supersolutions.

Of course, the disadvantage of this approach is that one has to find a pair of sub-supersolutions in order to apply the developed theory. However, such pairs of sub-supersolutions can often be constructed as solutions of appropriately defined auxiliary problems which are known to have solutions beforehand. In this thesis, we do not treat this question in full generality. However, we will present two particular approaches in Section 5.5 and Subsection 8.3.4.

Framework for Variational Inequalities with Multivalued Bifunctions

Although the order-theoretical fixed point theorems of Chapter 1 and their topological corollaries of Chapter 2 are not restricted in their applications, the general framework presented in Theorem 2.33 was developed to be eventually applied to multivalued variational problems with bifunctions. The basic idea goes as follows:

Let X be a reflexive Banach space, let $K \subset X$ be a subset of X, let $A \colon K \times X \to X'$ be an operator and let sub-supersolutions $\underline{u} \in X$ and $\overline{u} \in X$ be given. Then, define the multivalued operator

$$S \colon [\underline{u}, \overline{u}] \subset X \to \mathcal{P}(X), \quad v \mapsto \{u \in K : \langle A(u, v), w - u \rangle \geqslant 0 \text{ for all } w \in K\},$$

and *prove* that S has a fixed point u^*. Then, from $u^* \in S(u^*)$ we have

$$u^* \in K \quad \text{and} \quad \langle A(u^*, u^*), w - u^* \rangle \geqslant 0 \quad \text{for all } w \in K.$$

In order to carry out this plan, we have three tasks:

(i) Understand, for all $v \in [\underline{u}, \overline{u}]$, the following problem:

$$\text{Find } u \in K \text{ such that} \quad \langle A(u, v), w - u \rangle \geqslant 0 \quad \text{for all } w \in K.$$

(ii) Choose some fixed point theorem that seems suitable for finding a fixed point of S. (Unsurprisingly, we will use Theorem 2.33. We refer to, e.g., [60] for another approach which is not order-theoretical.)

(iii) Provide all properties needed for the application of the chosen fixed point theorem. In case of Theorem 2.33, we will define the suboperator $\underline{S}$ in such a way that its fixed points are subsolutions of the variational problem (so that, e.g., $\underline{u} \in \underline{S}(\underline{u})$). Further, it is often useful to introduce, as the dual counterpart of $\underline{S}$, the superoperator $\overline{S}$, whose fixed points are supersolutions.

Above all, point (iii) of the plan is by no means trivial and involves the usage of specialized analytic tools. We will present them in the rest of this thesis while implementing the sub-supersolution method for the following variational problems:

- multivalued variational inequalities with bifunctions in Chapter 4,
- multivalued quasi-variational inequalities with bifunctions in Chapter 5,
- multivalued inclusions with measures in Chapter 6,
- multivalued inclusions with measures and bifunctions in Chapter 7,
- systems of the before mentioned variational problems in Chapter 8.

Many more applications of the general framework and real-world applications are possible, see Chapter 9.

4 Multivalued Variational Inequalities with Nonsmooth Bifunctions

4.1 Introduction

In the last years, Carl and Le, among others, have developed an approach that covers a wide range of variational problems, see, e.g., [20, 22, 23, 24, 27, 28, 68]. They built on theorems presented in [50], see Lemma 1.58 above, and combined them with the theory of multivalued pseudomonotone operators, see Theorem 2.57, and the concept of sub-supersolutions. This was the starting point of this thesis.

Indeed, the origin of the ideas presented in this section is the effort to extend existence theorems for the following multivalued variational inequality problem: Find $u \in K$ such that

$$\langle Au, w-u \rangle + \int_\Omega \eta(w-u) \geqslant 0 \quad \text{for all } w \in K \text{ and some } \eta \subset f(\cdot, u), \tag{4.1}$$

where W is some Sobolev space of functions with zero boundary values in the sense of traces, $A\colon W \to W'$ an elliptic differential operator of Leray-Lions type, $\Omega \subset \mathbb{R}^N$ a bounded domain with Lipschitz boundary, $K \subset W$ a convex set and $f\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ an upper Carathéodory multifunction. If one allows f in (4.1) to depend on a further real argument, one obtains the following multivalued variational inequality problem: Find

$$u \in K \text{ s.t. } \quad \langle Au, w-u \rangle + \int_\Omega \eta(w-u) \geqslant 0 \quad \text{for all } w \in K \text{ and some } \eta \subset f(\cdot, u, u), \tag{4.2}$$

that has a solution if $t \mapsto f(x, s, t)$ is a decreasing multifunction in the sense of Definition 1.22. (This setting allows, e.g., for a treatment of generalized gradients of a locally Lipschitz function.)

As a matter of fact, the methods used to provide the existence theorem for Problem (4.2) are order-theoretical ones that can be formulated in more generality, which gave rise to an early form of the general framework presented in Theorem 2.33. As we will see in the following Chapters 5, 7 and 8, this framework can be used to obtain existence results for more complicated variational inequalities. It is very likely that this holds also for a wide range of problems not considered in this thesis.

The goal of the current chapter is to demonstrate how Theorem 2.33 leads to an existence theorem for a multivalued variational inequality. In order to emphasize the main ideas, we confine us to Problem (4.2) (which is rather simple, but, nevertheless, covers many special cases studied in the past) and skip some analytic details which will be presented along other applications in the forthcoming chapters.

Before we start, let us give a few comments about the data of this problem and the connection to the theory presented in Part I of this mathematical endeavour:

(i) The operator A is a single-valued Leray-Lions operator as considered in Theorem 3.78, which is pseudomonotone and thus allows for an application of Theorem 2.57. In this thesis, we will not change this setting dramatically, but it would be interesting to study multivalued leading terms, as is was done very recently in [69, 70, 71, 72].

(ii) The set K is convex, which allows for a reformulation of the variational problem in the spirit of Example 2.60. We will see in Chapter 5 that there is room for a more general formulation involving convex functionals.

(iii) The multifunction $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is chosen such that a few properties hold. For instance, we need that the values of the solution operator S are weakly closed, for which we need some closure property as presented in Proposition 3.68, and we need $\underline{S}$ to be increasing upward, for which it is needed that, for each $u, v \in L^0$, the function $x \mapsto f(x, u(x), v(x))$ has at least one measurable selection, whence Corollary 3.50 comes into play. It is kind of lucky that all those properties hold if f satisfies only a few assumptions, namely:

  (i) $(x, t) \to f(x, s, t)$ is superpositionally measurable,

  (ii) $s \mapsto f(x, s, t)$ is upper semicontinuous,

  (iii) $t \mapsto f(x, s, t)$ is decreasing,

  (iv) f satisfies some local growth condition to ensure integrability of its selectors.

  We currently do not know if the assumptions on f can be further relaxed so that, e.g., the variational problem has only maximal solutions between each given pair of sub-supersolutions, but not a greatest one.

(iv) The functions of the Sobolev space W have zero boundaries in the sense of traces, that is $W = W_0^{1,p}(\Omega)$ for some $p \in [1, \infty)$. This is by no means a necessary condition: We could incorporate boundary value conditions at the cost of some more involved calculations (cf. [27]). However, we decided to confine ourselves to zero boundary values in order to present the main ideas.

(v) The test functions w belong to $K \subset W$, so that it is possible to take solutions or subsolutions themselves as test functions, which simplifies some arguments. We will see in Chapters 6 and 7 how to deal with the more general case in which test functions belong to a function space with higher regularity.

Let us formulate the precise conditions on the data in the next section, followed by the abstract formulation of Problem (4.2) and the detailed application of our framework.

4.2 Setting

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded domain with Lipschitz boundary, and let $p \in (1, \infty)$ and $q \in [1, p^*)$ be fixed, where p^* denotes the critical Sobolev exponent associated with p and the dimension N. Recall that we denote the norm of the Lebesgue space $L^r = L^r(\Omega)$ by $\|\cdot\|_r$. Further, we introduce the following abbreviations:

$$L := L^{q'}, \qquad V := L^{p^*}, \qquad W := W_0^{1,p},$$

where q' is the Hölder conjugate of q and $W_0^{1,p}$ the usual Sobolev space with zero boundary values in the sense of traces. Note that the embedding $W \hookrightarrow V$ is continuous, that the embedding $L \hookrightarrow W'$ is compact, and that $u \mapsto \|\nabla u\|_p$ is an equivalent norm on W. Recall further that L^0 and all its subspaces are equipped with the natural partial order $\leqslant$, defined by $u \leqslant v$ if $u(x) \leqslant v(x)$ for a.e. $x \in \Omega$, by which L^0 is known to be a lattice-ordered linear space with sub-lattices L, V and W (which are all ordered Banach spaces and reflexive provided $q \in (1, p^*)$).

Throughout this chapter, we impose the following conditions on the data:

4.1 Assumption We call a pair $(\underline{u}, \overline{u})$ of a subsolution $\underline{u}$ and a supersolution $\overline{u}$ (which will be defined in Definition 4.8 below) **ordered pair of sub-supersolutions** if $\underline{u} \leqslant \overline{u}$. We assume:

(S) There is an ordered pair $(\underline{u}, \overline{u})$ of sub-supersolutions. ○

4.2 Assumption Let $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a function defining the (single-valued) differential operator A of Leray-Lions type via $Au = -\operatorname{div} a(\cdot, \nabla u)$. The following standard assumptions on a are meant to hold for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

(A1) a is a **Carathéodory function**, i.e. $x \mapsto a(x, \xi)$ is measurable and $\xi \mapsto a(x, \xi)$ is continuous.

(A2) a is **p-coercive** in the second argument, i.e. there is a constant $\alpha_2 > 0$ and a function $k_2 \in L^1(\Omega)$ such that $a(x, \xi)\xi \geqslant \alpha_2 |\xi|^p - k_2(x)$.

(A3) There exists a constant $\alpha_3 > 0$ and a function $k_3 \in L^{p'}(\Omega)$ such that a satisfies the **growth** condition $|a(x, \xi)| \leqslant \alpha_3 |\xi|^{p-1} + k_3(x)$.

(A4) a is **monotone** in the second argument, i.e. it holds, for all $\xi, \xi' \in \mathbb{R}^N$,

$$(a(x, \xi) - a(x, \xi'))(\xi - \xi') \geqslant 0. \qquad ○$$

4.3 Remark From Theorem 3.78 we know that under conditions (A1)—(A4) the elliptic differential operator $A \colon W \to W'$, defined by

$$\langle Au, w \rangle := \int_\Omega a(\cdot, \nabla u) \nabla w, \quad u, w \in W = W_0^{1,p},$$

is continuous, bounded, pseudomonotone, and coercive. An example is given by the famous p-Laplacian defined by $a(x, \xi) = |\xi|^{p-2}\xi$. See, e.g., [98, Sec. 2.4] for an exhaustive treatment. ○

4.4 Assumption Let $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multifunction whose values are compact intervals. The following conditions are meant to hold for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$.

(F1) The function $(x, t) \mapsto f(x, s, t)$ is **superpositionally measurable**, i.e. for all $v \in L^0$ the multivalued function $x \mapsto f(x, s, v(x))$ is measurable.

(F2) The function $s \mapsto f(x, s, t)$ is **upper semicontinuous** on $\mathbb{R}$, i.e. for each open $U \subset \mathbb{R}$ from $f(x, s, t) \subset U$ it follows $f(x, s', t) \subset U$ for all s' near s.

(F3) The function $t \mapsto f(x, s, t)$ is **decreasing** on $\mathbb{R}$, i.e. $f(x, s, t) \leqslant^*_* f(x, s, t')$ for $t' \leqslant t$.

(F4) There is some $k_4 \in L = L^{q'}(\Omega)$ such that f satisfies the **growth** condition

$$\sup\{|y| : y \in f(x, s, t), s, t \in [\underline{u}(x), \overline{u}(x)]\} \leqslant k_4(x).$$ ○

4.5 Remark Assumption (F1) implies that the function $x \mapsto f(x, s, t)$ is measurable on Ω. Thanks to this and to (F2), $(x, s) \mapsto f(x, s, t)$ is upper Carathéodory, which is together with (F4) the set of standard assumptions in the case that f is not a bifunction but only a function on $\Omega \times \mathbb{R}$.

In our case, f is weakly superpositionally measurable, as we have seen in Corollary 3.50, and we would like to stress that no continuity in the last argument is assumed. ○

4.6 Assumption Let $K \subset W$ be a non-empty, closed and convex subset and suppose the following lattice condition:

(K) K is a **sub-lattice** of W, i.e. for all $u, v \in K$ one has $u \vee v \in K$ and $u \wedge v \in K$. ○

4.7 Remark This assumption allows us to establish a sub-supersolution method for variational inequalities. It is shared by many convex sets of interest, especially by order-intervals, the order cone, and the whole space. ○

Finally, let us state Problem (4.2) in more detail: Find

$$\begin{aligned} u \in K \subset W_0^{1,p}(\Omega),\ \eta \in L^{p^*}(\Omega) \text{ s.t.} \quad \eta(x) &\in f(x, u(x), u(x)) \quad \text{for a.e. } x \in \Omega, \\ \langle Au, w - u\rangle + \int_\Omega \eta(w - u) &\geqslant 0 \quad \text{for all } w \in K. \end{aligned} \tag{4.3}$$

In the next section, we will reformulate Problem (4.3) with the help of abstract operators, and then we will apply the framework of Theorem 2.33 in order to prove that Problem (4.3) has a greatest solution in $[\underline{u}, \overline{u}]$.

4.3 Abstract Formulation

We are now in a position to formulate precise definitions for solutions, sub- and supersolutions in a unifying framework. To this end, we use the bifunction $F\colon W \times V \to \mathcal{P}(L^0)$ that assigns to each pair (u, v) the set of measurable selectors

$$F(u, v) := \{\eta \in L^0 : \eta \subset f(\cdot, u, v)\},$$

which is due to Corollary 3.50 non-empty. Further, for all functions $u \in W$, $v \in V$ and each subset $M \subset W$ we say a function $\eta \in L$ is **associated** to (u, v) with respect to M if

$$\eta \in F(u, v) \quad \text{and} \quad \langle Au, w - u\rangle + \int_\Omega \eta(w - u) \geqslant 0 \quad \text{for all } w \in M.$$

We write $\eta \sim (u, v, M)$ for short. Further, by abuse of notation, u is called **associated** to v with respect to M if there is at least one function $\eta \sim (u, v, M)$. We write $u \sim (v, M)$ for short.

With help of these abbreviations, we finally define the following mappings on the domain $D := [\underline{u}, \overline{u}]_V = [\underline{u}, \overline{u}] \cap V$ (whose values are subsets of D):

$$\begin{aligned}
S\colon D &\to \mathcal{P}_\emptyset(W), \quad v \mapsto S(v) := \{u \in D \cap K : u \sim (v, K)\},\\
\underline{S}(v)\colon D &\to \mathcal{P}_\emptyset(W), \quad v \mapsto \underline{S}(v) := \{u \in D \cap W : u \vee K \subset K \text{ and } u \sim (v, u \wedge K)\},\\
\overline{S}(v)\colon D &\to \mathcal{P}_\emptyset(W), \quad v \mapsto \overline{S}(v) := \{u \in D \cap W : u \wedge K \subset K \text{ and } u \sim (v, u \vee K)\}.
\end{aligned}$$

4.8 Definition We call $u \in W$ a **solution**, **subsolution** or **supersolution** of Problem (4.3) if $u \in \operatorname{Fix} S$, $u \in \operatorname{Fix} \underline{S}$ or $u \in \operatorname{Fix} \overline{S}$, respectively. ○

4.9 Remark Subsolutions and supersolutions are related by duality, but their definitions are not a result of purly order-theoretical abstract arguments, but emerged from a long history of similar definitions for other analytic problems. Thus, Definition 4.8 extends the definitions of solutions, sub- and supersolutions of special cases, as is pointed out, e.g., in [24, 28]. ○

4.4 Existence of Solutions

As said before, we are going to apply Theorem 2.33. To this end, we will provide several propositions that are of interest in their own right, but to keep the presentation simple, we will refer to Chapter 5 for some sophisticated proofs.

Recall that we have to provide the following analytic and order-theoretical properties to conclude that S has a greatest fixed point in D:

(i) D is bounded in V and it holds $\underline{u} \in \underline{S}(\underline{u})$.

(ii) $S(D)$ is bounded in W and the values of S are weakly sequentially closed in W.

(iii) $\underline{S}$ is permanent upward, its values are directed upward and for all $v \in D$ it holds $S(v) \subset \underline{S}(v) \leqslant^* S(v)$.

First of all, let us check the monotonicity of $\underline{S}$ and $\overline{S}$:

4.10 Proposition *The operator $\underline{S}\colon D \to \mathcal{P}_\emptyset(W)$ is permanent upward, whereas the operator $\overline{S}\colon D \to \mathcal{P}_\emptyset(W)$ is permanent downward.*

Proof: Let $v_1, v_2 \in D$ be such that $v_1 \leqslant v_2$, and let $u \in \underline{S}(v_1)$ be given. Then there is some $\eta_1 \sim (u, v_1, u \wedge K)$, and since f is weakly superpositionially measurable, there is some $\eta_2 \in F(u, v_2)$, which belongs to L due to (F4). Consequently, $\eta_3 := \eta_1 \wedge \eta_2 \in L$ and we claim that $\eta_3 \sim (u, v_2, u \wedge K)$, by which $u \in \underline{S}(v_2)$ is readily seen.

Let us first prove that $\eta_3 \in F(u, v_2)$, i.e. $\eta_3(x) \in f(x, u(x), v_2(x))$ for a.e. $x \in \Omega$. To this end, recall that for a.e. $x \in \Omega$ we have

$$\eta_1(x) \in f(x, u(x), v_1(x)), \quad \eta_2(x) \in f(x, u(x), v_2(x)) \quad \text{and} \quad v_1(x) \leqslant v_2(x).$$

Due to (F3), f is decreasing upward in the last argument and thus there is some element $\alpha \in f(x, u(x), v_2(x))$ such that $\alpha \leqslant \eta_1(x)$. This implies

$$\alpha \wedge \eta_2(x) \leqslant \eta_1(x) \wedge \eta_2(x) \leqslant \eta_2(x).$$

Since the values of f are order-convex downward, it follows $\eta_3(x) \in f(x, u(x), v_2(x))$, i.e. $\eta_3 \in F(u, v_2)$.

To conclude the proof of $\eta_3 \sim (u, v_2, u \wedge K)$, it suffices to note that

$$\langle A(u), w - u\rangle + \int_\Omega \eta_3(w - u) \geqslant \langle A(u), w - u\rangle + \int_\Omega \eta_1(w - u) \geqslant 0, \quad w \in u \wedge K,$$

where we have used that $\eta_3 \leqslant \eta_1$ and $w - u \leqslant 0$ for all $w \in u \wedge K$.

Consequently, $\underline{S}$ is permanent upward. By duality and since f is also decreasing downward in the last argument, $\overline{S}\colon D \to \mathcal{P}_\emptyset(W)$ is seen to be permanent downward. ○

In particular, we conclude $\underline{u} \in \underline{S}(v)$ for all $v \in D$. This means, we have $\underline{u} \in W$ and there is $\underline{\eta} \in L$ such that $\underline{\eta} \subset f_v(\cdot, \underline{u}) := f(\cdot, \underline{u}, v)$ and

$$\langle A\underline{u}, w - \underline{u}\rangle + \int_\Omega \underline{\eta}(w - \underline{u}) \geqslant 0 \quad \text{for all } w \in \underline{u} \wedge K,$$

which is the natural definition of a subsolution in case the bifunction f is replaced by the multifunction $f_v\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$. Since f_v is upper Carathéodory and (F4) holds, we can use the ideas of [27, 30] to prove that for all $v \in D$ there is $u \in S(v)$ such that $\underline{u} \leqslant u \leqslant \overline{u}$. However, the proof of this existence result uses sophisticated truncation techniques that rely on the ordered pair of sub-supersolutions. Thus, we present only the following condensed proposition and refer for a proof to Theorem 5.13 below.

4.11 Theorem *Let $v \in D$ be arbitrary, and let $\underline{u}_i \in \underline{S}(v)$ and $\overline{u}_i \in \overline{S}(v)$, $i = 1, 2$, be such that*

$$\underline{u}_1 \vee \underline{u}_2 \leqslant \overline{u}_1 \wedge \overline{u}_2.$$

Then there is $u \in S(v)$ such that $\underline{u}_1 \vee \underline{u}_2 \leqslant u \leqslant \overline{u}_1 \wedge \overline{u}_2$. ○

From Theorem 4.11 it follows the crucial relation $\underline{S}(v) \leqslant^* S(v)$. Since (K) implies easily $S(v) \subset \underline{S}(v) \cap \overline{S}(v)$, we conclude also that the values of $\underline{S}$ are directed upward.

Next, let us provide some topological properties of S.

4.12 Proposition *The operator* $S\colon D \to \mathcal{P}(W)$ *has uniformly bounded values.*

Proof: Let $v \in D = [\underline{u}, \overline{u}]_V$ and any $u \in S(v)$ be given and suppose $\eta \sim (u, v, K)$. Then it holds, where $w \in K$ is any fixed element,

$$\langle Au, w\rangle + \int_\Omega \eta(w-u) \geqslant \langle Au, u\rangle. \tag{4.4}$$

Estimating both sides in (4.4) by use of (A3), Hölder's inequality and (A2) gives

$$(\alpha_3\|\nabla u\|_p^{p-1} + \|k_3\|_{p'})\|\nabla w\|_p + \|\eta\|_{q'}(\|w\|_q + \|u\|_q) \geqslant \alpha_2\|\nabla u\|_p^p - \|k_2\|_1. \tag{4.5}$$

Since $\eta \in F(u,v)$ and $u, v \in [\underline{u}, \overline{u}]$, $\|\eta\|_{q'}$ is bounded by $\|k_4\|_{q'}$ due to (F4), and we have $\|u\|_q \leqslant \|\underline{u}\|_q + \|\overline{u}\|_q$. Further, w is some fixed element, so that from (4.5) we deduce

$$\|\nabla u\|_p^p \leqslant c(\|\nabla u\|_p^{p-1} + 1)$$

for some $c > 0$ not depending on u or v, which implies $\|\nabla u\|_p \leqslant 2c + 1$. ○

We finally have to check the topological properties of the values of S, which are even better than demanded.

4.13 Proposition *The operator* $S\colon D \to \mathcal{P}(W)$ *has weakly compact values.*

Proof: Suppose $v \in [\underline{u}, \overline{u}]_V$ and $(u_n) \subset S(v) \subset K$. Due to Proposition 4.12, (u_n) is bounded, and since W is reflexive and compactly embedded in V, we can assume $u_n \rightharpoonup u$ in W and $u_n \to u$ in V for some $u \in W$, and as K is weakly closed, we obtain $u \in K$. Furthermore, there exists a sequence $(\eta_n) \subset L$ such that $\eta_n \sim (u_n, v, K)$ and we can assume $\eta_n \rightharpoonup \eta$ in L for some $\eta \in V$ since (η_n) is bounded in L due to (F4). Thanks to Proposition 3.68, we have $\eta \in F(u, v)$ and we claim $\eta \sim (u, v, K)$. To this end, note that $u \in K$ and $\eta_n \sim (u_n, v, K)$ imply

$$\langle Au_n, u - u_n\rangle + \int_\Omega \eta_n(u - u_n) \geqslant 0. \tag{4.6}$$

Passing to the limit in (4.6) yields $\limsup_n\langle Au_n, u_n - u\rangle \leqslant 0$ and since A is pseudomonotone, we infer $\langle Au, w - u\rangle \geqslant \limsup_n\langle Au_n, w - u_n\rangle$ for all $w \in W$. Since (4.6) holds also if u is replaced by any $w \in K$, it follows

$$\langle Au, w-u\rangle + \int_\Omega \eta(w-u) \geqslant \limsup\nolimits_n\langle Au_n, w - u_n\rangle + \lim\nolimits_n \int_\Omega \eta_n(w - u_n) \geqslant 0,$$

which proves $\eta \sim (u, v, K)$ and thus $u \in S(v)$. ○

This said, we can apply Theorem 2.33 to the multifunctions S and $\underline{S}$ to deduce the main result of this chapter:

4.14 Theorem *Suppose (S), (A1)—(A4), (F1)—(F4) and (K). Then Problem (4.3) has the greatest (and by duality the smallest) solution in* $[\underline{u}, \overline{u}]$. ○

4.15 Remark The ideas presented in this section can be extended to a number of related problems.

(i) Problem (4.3) can be written equivalently as

$$Au + F(u, u) + \partial I_K(u) \ni 0,$$

where I_K is the indicator function of the closed convex set K and ∂I_K its subdifferential in the sense of Convex Analysis. In [68] this problem was generalized by replacing ∂I_K with the subdifferential of a general convex functional $K_u = K(\cdot, u)$, such that the problem reads as

$$Au + F(u, u) + \partial K_u(u) \ni 0.$$

By means of the abstract framework presented here, one can drop some technical conditions while obtaining the same conclusion as in [68]. This will be the content of the next chapter.

(ii) The abstract framework can be used to extend results concerning differential equations with measures presented in Chapter 6, where we approximate the main problem with well-behaving classical auxiliary problems such that solutions are limits of classical solutions. This extension will be the content of Chapter 7.

(iii) The bifunctions can be extended to formulate and to solve systems of multivalued variational inequalities such as

$$A_i u_i + F_i(u_i, u_1, u_2) + \partial I_{K_i}(u_i) \ni 0, \quad i = 1, 2.$$

To this end, consider the multifunction S that maps pairs (v_1, v_2) to solutions (u_1, u_2) of

$$A_i u_i + F_i(u_i, v_1, v_2) + \partial I_{K_i}(u_i) \ni 0, \quad i = 1, 2.$$

Again, a fixed point of S is the desired solution. As a consequence, some results of [20] can be extended to nonsmooth systems. We will provide details in Chapter 8.

(iv) There exist more applications and generalizations not covered in this thesis. We refer to the final Chapter 9 for some inspiration for further work. ○

5 Multivalued Quasi-Variational Inequalities with Bifunctions

5.1 Introduction

The combination of order-theoretical fixed point theorems, surjectivity results for pseudomonotone operators and the concept of sub-supersolutions generates a very powerful method to solve a wide range of variational inequality problems, as presented in [24, 27, 68]. In [68], the following multivalued quasi-variational inequality problem (that generalizes the problem treated in Chapter 4) was considered:

$$\begin{aligned}&\text{Find } u \in W \text{ such that for some } \eta \subset f(\cdot, u, u) \text{ it holds}\\ &\langle Au, w-u\rangle + \int_{\Omega} \eta(w-u) + K(w,u) - K(u,u) \geqslant 0 \quad \text{for all } w \in W.\end{aligned} \tag{5.1}$$

There, as in the last chapter, $W = W_0^{1,p}$ is the usual Sobolev space over a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, $A\colon W \to W'$ is a Leray-Lions operator, $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued bifunction and $K(\cdot, u)\colon W \to \mathbb{R} \cup \{+\infty\}$ is a convex functional for each $u \in W$, which allows especially for the functional $\partial I_{K(u)}\colon W \to \mathbb{R} \cup \{+\infty\}$ and so for a treatment of quasi-variational inequalities.

The key feature in these problems are the weak assumptions on f and K such that a wide range of applications is covered. For example, f is assumed to be upper semicontinuous in the second and increasing in the third argument, so that the mapping $s \mapsto f(x, s, s)$ is in general neither upper semicontinuous nor lower semicontinuous nor monotone. The lack of continuity allows the solution set to be not compact, but still one can show that there are extremal (i.e. smallest and greatest) solutions between every ordered pair of sub-supersolutions.

Problem (5.1) was considered in [68] (which bases on the ideas of [65, 66, 67]). However, in [68] no fixed point theorem was used but a more direct proof for the existence of solutions of the quasi-variational inequality was presented. This inspired us to raise the question whether this problem could be solved with direct use of a multivalued order-theoretical fixed point theorem. Our study gave rise to an abstract framework for solving variational inequalities with multivalued bifunctions, which is Theorem 2.33 presented above. The aim of this chapter is to apply this framework in order to provide a systematic and unified exposition of an existence theory for Problem (5.1). As a bonus, it turns out that, thanks to the powerful order-theoretical fixed point theorem, we can get rid of some technical assumptions used in [68].

We would like to mention that the method developed here can be used to treat the following even more general problem: Find $u \in W^{1,p}(\Omega)$ such that there are selections

$\eta \subset f(\cdot, u, u)$ and $\xi \subset \hat{f}(\cdot, \gamma u, \gamma u)$ such that

$$\langle Au, w - u\rangle + \int_\Omega \eta(w-u)dx + \int_{\partial\Omega} \xi(\gamma w - \gamma u)d\sigma \\ + K(w, u) - K(u, u) + \hat{K}(\gamma w, \gamma u) - \hat{K}(\gamma u, \gamma u) \geqslant 0 \quad \text{for all } w \in W^{1,p}(\Omega),$$

where γu denotes the trace of u, σ the boundary measure on the boundary $\partial\Omega$ of Ω, and where $\hat{f} \colon \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and $\hat{K} \colon L^p(\partial\Omega) \times L^p(\partial\Omega) \to \mathbb{R} \cup \{+\infty\}$ have similar properties as f and K. (For further inspiration, see [27].)

5.2 Setting

In this section, we will use the same notations as presented in Subsection 4.2. Especially, $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, is a bounded domain with Lipschitz boundary, $p \in (1, \infty)$ and $q \in (1, p^*)$ are fixed, and we we use the abbreviations

$$L = L^{q'}, \qquad V = L^{p^*}, \qquad W = W_0^{1,p}.$$

Further, we will work with functionals of

$$\Gamma := \{k \colon W \to \mathbb{R} \cup \{+\infty\} : \mathcal{D}(k) = \{u \in W : k(u) < \infty\} \neq \emptyset\}.$$

In what follows, we equip Γ with the relation $\preccurlyeq_*^*$ defined in Definition 1.32, and although $\preccurlyeq_*^*$ is not a partial order, we will use the notions $K^{\downarrow} := \{k \in \Gamma : k \preccurlyeq_*^* K\}$ and $K^{\uparrow} := \{k \in \Gamma : K \preccurlyeq_*^* k\}$.

Next, let us give precise assumptions on the data, which will guarantee that all conditions of Theorem 2.33 are fulfilled, so that Problem (5.1) has extremal solutions between each ordered pair of sub-supersolutions. To this end, we will restrict our considerations to the set $D := [\underline{u}, \overline{u}]_V$, where $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution as defined below in Definition 5.9. Since we do not provide them, we have to assume:

5.1 Assumption (S) There is an ordered pair $(\underline{u}, \overline{u})$ of sub-supersolutions. ○

5.2 Remark See Proposition 5.24 below for a way to find an ordered pair of sub-supersolutions under appropriate conditions. ○

5.3 Assumption Let $a \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a function that satisfies conditions (A1)—(A4) given in Subsection 4.2, and let the operator $A \colon W \to W'$ be defined via

$$\langle Au, w\rangle := \int_\Omega a(\cdot, \nabla u)\nabla w. \qquad ○$$

5.4 Assumption Let $f \colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multifunction that satisfies conditions (F1)—(F4) given in Subsection 4.2. ○

5.5 Assumption Let $K \colon W \times V \to \mathbb{R} \cup \{+\infty\}$ be a functional satisfying the following conditions:

(K1) For all $v \in V$ the function $K_v := K(\cdot, v)\colon W \to \mathbb{R} \cup \{+\infty\}$ is **proper** (i.e. $\mathcal{D}(K_v) \neq \emptyset$, thus $K_v \in \Gamma$), **convex** and **lower semicontinuous**.

(K2) The mapping $v \mapsto K_v$ is **lattice-increasing**, i.e. $v_1 \leqslant v_2$ in V implies $K_{v_1} \preccurlyeq^*_* K_{v_2}$.

(K3) There is some constant $c_3 > 0$ such that for all $v \in V$ there is some $w_v \in \mathcal{D}(K_v)$ such that $\|\nabla w_v\|_p \leqslant c_3$ and $K_v(w_v) \leqslant c_3$ and such that for all $u \in W$ it holds $K_v(u) \geqslant -c_3(\|\nabla u\|_p^{p-1} + 1)$. ○

5.6 Remark From Proposition 2.49 and (K1) we have that K_v is weakly sequentially lower semicontinuous and from Theorem 2.61 we have that ∂K_v is maximal monotone.

As an example, we have $K_v = I_{K(v)}$, the indicator function of a non-empty closed convex set $K(v) \subset W$. In this case, $\mathcal{D}(K_v) = K(v)$ and (K2) becomes $K(v_1) \preccurlyeq^*_* K(v_2)$ for $v_1 \leqslant v_2$. (K3) holds if, e.g., all $K(v)$, $v \in V$, have a common element, or if all $K(v)$ are contained in a bounded subset of W. ○

5.7 Remark Instead of (K3) we could use the following more general condition that combines order and topology (see Remark 5.17 below):

(K3') There is some constant $c_3' > 0$ such that for all increasing or decreasing sequences $(v_n), (u_n) \subset [\underline{u}, \overline{u}]_V$ there is some sequence $(w_n) \subset W$ such that, for all n,

$$\|\nabla w_n\|_p \leqslant c_3', \qquad K_{v_n}(w_n) \leqslant c_3', \qquad K_{v_n}(u_n) \geqslant -c_3'(\|\nabla u\|_p^{p-1} + 1). \qquad ○$$

5.8 Remark Note, that in (F4) and (K3) we have no global growth condition but only a local growth condition between sub-supersolutions $\underline{u}$ and $\overline{u}$. It is often possible to verify such a local growth even if $\underline{u}$ and $\overline{u}$ are not known explicitly. ○

5.3 Abstract Formulation

The aim of this chapter is to provide extremal solutions of the following multivalued quasi-variational inequality problem:

$$\begin{gathered}\text{Find } u \in W \text{ such that there is } \eta \subset f(\cdot, u, u) \text{ such that for all } w \in W \text{ it holds} \\ \langle Au, w - u\rangle + \int_\Omega \eta(w - u) + K(w, u) - K(u, u) \geqslant 0.\end{gathered} \tag{5.2}$$

By use of the embedding operator $i_q^*\colon L \hookrightarrow W'$, the multifunction $F\colon L^0 \times L^0 \to \mathcal{P}_\emptyset(L)$, defined by

$$F(u, v) := \{\eta \in L : \eta \subset f(\cdot, u, v)\},$$

and the subdifferential $\partial K_u(u)$ of the functional $K_u = K(\cdot, u)$, we can rewrite Problem (5.2) as a multivalued operator equation in W':

$$\text{Find } u \in W \text{ such that} \quad Au + i_q^* F(u, u) + \partial K_u(u) \ni 0 \quad \text{in } W'.$$

If f and K do not depend on their last argument, we can find solutions by means of Theorem 2.57. Further, in this case the solution set has good properties and the method of sub-supersolutions applies. Thus, we can apply our fixed point result if we formulate Problem (5.2) as a fixed point problem with help of the following notations:

For any sets $L_0 \subset L$, $\Gamma_0 \subset \Gamma$ and $W_0 \subset W$, we define the set $R(L_0, \Gamma_0, W_0)$ as consisting of those functions $u \in W$ such that there are $\eta \in L_0$ and $k \in \Gamma_0$ such that

$$\langle Au, w - u\rangle + \int_\Omega \eta(w - u) + k(w) - k(u) \geqslant 0 \quad \text{for all } w \in W_0. \tag{5.3}$$

(Note, that (5.3) can only hold if $u \in \mathcal{D}(k)$.)

Further, we define the mappings $S, \underline{S}, \overline{S} \colon D \to \mathcal{P}_\emptyset(W)$ as follows:

$$\begin{aligned}
S(v) &:= \{u \in D : u \in R(F(u, v), \{K_v\}, W)\},\\
\underline{S}(v) &:= \{u \in D : u \in R(F(u, v), K_v^{\downarrow}, u \wedge \mathcal{D}(K_v))\},\\
\overline{S}(v) &:= \{u \in D : u \in R(F(u, v), K_v^{\uparrow}, u \vee \mathcal{D}(K_v))\}.
\end{aligned}$$

Then, clearly, fixed points of S are solutions of Problem (5.2) located in D and vice versa. Furthermore, we can finally define sub- and supersolutions:

5.9 Definition We call fixed points of $\underline{S}$ and $\overline{S}$ **subsolution** and **supersolution** of Problem (5.2), respectively. ○

5.4 Existence of Solutions

In the following, we provide the properties needed to apply Theorem 2.33. The proofs are inspired by [27, 67, 68] and use new ideas presented in [30, 110]. In particular, we don't assume that $(x, s) \mapsto f(x, s, t)$ is superpositionally measurable as in [68], but only (by (F1) and (F2)) that $(x, s) \mapsto f(x, s, t)$ is weakly superpositionally measurable.

First, let us inspect the monotone dependence of $\underline{S}$ on the data f and K. To this end, we write $\underline{S}_{f,K}$ instead of $\underline{S}$, and for given mappings $g \colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ and $J \colon W \times V \to \mathbb{R} \cup \{+\infty\}$ (which do not need to satisfy the same conditions as f and K), we write $\underline{S}_{g,J}$ for the corresponding subsolution operator of Problem (5.2) with f and K replaced by g and J, respectively, i.e.

$$\underline{S}_{g,J} \colon D \to \mathcal{P}_\emptyset(W), \quad v \mapsto \{u \in D : u \in R(G(u, v), J_v^{\downarrow}, u \wedge \mathcal{D}(J_v))\},$$

where $G \colon L^0 \times L^0 \to \mathcal{P}_\emptyset(L)$ and $J_v \colon W \to \mathbb{R} \cup \{+\infty\}$ are defined analogous to F and K_v.

5.10 Proposition *Let $u, v_1, v_2 \in D$ be given such that $v_1 \leqslant v_2$, and suppose that the data f, g, K and J satisfy the following relations:*

(i) It holds $f(x, s, t) \leqslant_* g(x, s, t)$ for a.e. $x \in \Omega$ and all $s, t \in [\underline{u}(x), \overline{u}(x)]$,

(ii) it holds $J_v \preccurlyeq_*^* K_v$ for all $v \in V$.

Suppose further $\mathcal{D}(J_v) \neq \emptyset$ and $J_v \preccurlyeq_*^* J_v$ for all $v \in V$. Then $\underline{S}_{g,J}(v_1) \subset \underline{S}_{f,K}(v_2)$.

Proof: Let $u \in \underline{S}_{g,J}(v_1)$ be given. Then there are $\eta_1 \in G(u, v_1)$ and $k \in J^{\downarrow}_{v_1}$ such that

$$\langle Au, w-u\rangle + \int_\Omega \eta(w-u) + k(w) - k(u) \geqslant 0 \quad \text{for all } w \in u \wedge \mathcal{D}(J_{v_1}). \tag{5.4}$$

Let further $\eta_2 \in F(u, v_2)$ be arbitrary (such η_2 exists, since f is superpositionally measurable and satisfies (F4)).

First, we claim that $\eta_3 := \eta_1 \wedge \eta_2$ belongs to $F(u, v_2)$. To this end, recall that for a.e. $x \in \Omega$ we have

$$\eta_1(x) \in g(x, u(x), v_1(x)), \quad \eta_2(x) \in f(x, u(x), v_2(x)) \quad \text{and} \quad v_1(x) \leqslant v_2(x).$$

From (F3) and (i) we have $f(x, u(x), v_2(x)) \leqslant_* f(x, u(x), v_1(x)) \leqslant_* g(x, u(x), v_1(x))$ and thus there is $\beta \in f(x, u(x), v_2(x))$ such that

$$\beta \wedge \eta_2(x) \leqslant \eta_1(x) \wedge \eta_2(x) \leqslant \eta_2(x).$$

Since the values of f are order-convex downward, it follows $\eta_3(x) \in f(x, u(x), v_2(x))$, i.e. $\eta_3 \in F(u, v_2)$.

Second, we have $k \preccurlyeq^*_* J_{v_1} \preccurlyeq^*_* K_{v_1} \preccurlyeq^*_* K_{v_2}$, thus by Proposition 1.33 it follows $k \in K^{\downarrow}_{v_2}$.

Third, we finally claim that it holds

$$\langle Au, w-u\rangle + \int_\Omega \eta(w-u) + k(w) - k(u) \geqslant 0 \quad \text{for all } w \in u \wedge \mathcal{D}(K_{v_2}).$$

To deduce this from (5.4), two comments are in order: On the one hand, we have

$$\int_\Omega \eta_3(w-u) \geqslant \int_\Omega \eta_1(w-u) \quad \text{for all } w \in u \wedge \mathcal{D}(J_{v_1}),$$

for which we have used that $\eta_3 \leqslant \eta_1$ and $w - u \leqslant 0$ for all $w \in u \wedge \mathcal{D}(J_{v_1})$. On the other hand, we have $u \wedge \mathcal{D}(K_{v_2}) \subset u \wedge \mathcal{D}(J_{v_1})$, which follows readily from the relations

$$u \wedge w_2 = u \wedge ((u \vee w_1) \wedge w_2) \qquad \text{and} \qquad \mathcal{D}(k) \preccurlyeq^* \mathcal{D}(J_{v_1}) \preccurlyeq_* \mathcal{D}(K_{v_2}).$$

Consequently, we have $u \in \underline{S}_{f,K}(v_2)$. ○

This result can be used to find subsolutions of Problem (5.2) as solutions of simpler auxiliary problems (see Corollary 5.22). Furthermore, if $f = g$ and $K = J$, it provides the needed monotonicity of $\underline{S}$:

5.11 Corollary *The operator $\underline{S}\colon D \to \mathcal{P}_\emptyset(W)$ is permanent upward, whereas (by duality) the operator $\overline{S}\colon D \to \mathcal{P}_\emptyset(W)$ is permanent downward.* ○

5.12 Remark To prove by duality that $\overline{S}$ is permanent downward, we only need $\mathcal{D}(K_{v_1}) \preccurlyeq^* \mathcal{D}(K_{v_2}) \preccurlyeq_* \mathcal{D}(k)$. That means, that in Corollary 5.11 we have used $\mathcal{D}(K_{v_1}) \preccurlyeq^*_* \mathcal{D}(K_{v_2})$, but only $\mathcal{D}(k) \preccurlyeq^* \mathcal{D}(K_{v_1}))$ and $\mathcal{D}(K_{v_2}) \preccurlyeq_* \mathcal{D}(k)$. Those relations alone are enough to prove some transitiveness. But we don't currently know if this can be used to weaken the definition of sub- and supersolutions, since the full inequality $k \preccurlyeq^*_* K_v$ is needed in the proof of the next theorem. ○

In particular, from Corollary 5.11 we conclude that $\underline{u} \in \underline{S}(v)$ and $\overline{u} \in \overline{S}(v)$ for all $v \in D = [\underline{u}, \overline{u}]_V$. This is why we only need one pair $(\underline{u}, \overline{u})$ of sub-supersolutions to prove that all values of $\underline{S}$ and $\overline{S}$ are non-empty. Next, let us illuminate the interplay between the operators S, $\underline{S}$ and $\overline{S}$ (from which it follows that also S has non-empty values):

5.13 Theorem *Let $v \in D$ be arbitrary, and let $\underline{v}_i \in \underline{S}(v)$ and $\overline{v}_i \in \overline{S}(v)$, $i = 1, 2$, be such that*

$$\underline{u} \leqslant \underline{v} := \underline{v}_1 \vee \underline{v}_2 \leqslant \overline{v}_1 \wedge \overline{v}_2 =: \overline{v} \leqslant \overline{u}.$$

Then there is $u \in S(v)$ such that $\underline{v} \leqslant u \leqslant \overline{v}$.

Proof: **Step 1: Auxiliary functions** We are going to define an auxiliary problem whose solutions will be the desired elements of $S(v)$. To this end, recall that, by definition, for $i = 1, 2$ there are $\underline{\eta}_i \in F(\underline{v}_i, v)$ and $\underline{k}_i \in \Gamma$ as well as $\overline{\eta}_i \in F(\overline{v}_i, v)$ and $\overline{k}_i \in \Gamma$ such that we have the relations

$$\underline{v}_i \in \mathcal{D}(\underline{k}_i) \quad \text{and} \quad \underline{k}_i \preccurlyeq^*_* K_v, \qquad \text{and} \qquad K_v \preccurlyeq^*_* \overline{k}_i \quad \text{and} \quad \mathcal{D}(\overline{k}_i) \ni \overline{v}_i$$

and such that the following inequalities hold:

$$\langle A\underline{v}_i, w - \underline{v}_i \rangle + \int_\Omega \underline{\eta}_i (w - \underline{v}_i) + \underline{k}_i(w) - \underline{k}_i(\underline{v}_i) \geqslant 0 \quad \text{for all } w \in \underline{v}_i \wedge \mathcal{D}(K_v), \tag{5.5}$$

$$\langle A\overline{v}_i, w - \overline{v}_i \rangle + \int_\Omega \overline{\eta}_i (w - \overline{v}_i) + \overline{k}_i(w) - \overline{k}_i(\overline{v}_i) \geqslant 0 \quad \text{for all } w \in \overline{v}_i \wedge \mathcal{D}(K_v). \tag{5.6}$$

Define the functions $\underline{\eta}, \overline{\eta}$ on Ω pointwise a.e. by

$$\underline{\eta}(x) := \begin{cases} \underline{\eta}_1(x) & \text{if } \underline{v}_1(x) \geqslant \underline{v}_2(x), \\ \underline{\eta}_2(x) & \text{if } \underline{v}_1(x) < \underline{v}_2(x), \end{cases} \qquad \overline{\eta}(x) := \begin{cases} \overline{\eta}_1(x) & \text{if } \overline{v}_1(x) \leqslant \overline{v}_2(x), \\ \overline{\eta}_2(x) & \text{if } \overline{v}_1(x) > \overline{v}_2(x), \end{cases}$$

and note that $\underline{\eta} \in F(\underline{v}, v)$ and $\overline{\eta} \in F(\overline{v}, v)$. Further, we need three auxiliary functions d, g and h, which will be introduced subsequently.

First, let us introduce the cut-off function $d \colon \Omega \times \mathbb{R} \to \mathbb{R}$, defined pointwise by

$$d(x, s) := \begin{cases} -(\underline{v}(x) - s)^{p-1} & \text{if } s < \underline{v}(x), \\ 0 & \text{if } \underline{v}(x) \leqslant s \leqslant \overline{v}(x), \\ (s - \overline{v}(x))^{p-1} & \text{if } \overline{v}(x) < s. \end{cases}$$

Obviously, d is a Carathéodory function and satisfies the growth condition

$$|d(x, s)| \leqslant d_0 \left(|\underline{v}(x)|^{p-1} + |s|^{p-1} + |\overline{v}(x)|^{p-1} \right) \tag{5.7}$$

for some constant $d_0 > 0$. Hence, the Nemytskij operator $v \mapsto d(\cdot, v)$ is known to be continuous and bounded from $L^p(\Omega)$ to its dual space. Thanks to the compact embedding $W \hookrightarrow L^p(\Omega)$ we conclude that the composed mapping $D \colon W \to W'$, defined by

$$\langle Dv, w \rangle := \int_\Omega d(\cdot, v) w,$$

is bounded and completely continuous, thus pseudomonotone. For further use, let us note that there are constants $d_1, d_2 > 0$ such that both $-(t-s)^{p-1}s$ for $s \leqslant t$ and $(s-t)^{p-1}s$ for $s \geqslant t$ are bounded below by $d_1|s|^p - d_2|t|^{p-1}|s|$. One may take $d_1 = 1$ and $d_2 = 2^{2-p}$ if $p \leqslant 2$ and $d_1 = 2^{2-p}$ and $d_2 = 1$ if $p \geqslant 2$. By these estimates and by use of Young's Inequality with Epsilon, we obtain for all $v \in W$

$$\langle Dv, v\rangle \geqslant d_3\|v\|_p^p - d_3(\|\underline{v}\|_p^p + \|\overline{v}\|_p^p), \tag{5.8}$$

where $d_3 > 0$ is some constant not depending on v.

Second, let us truncate f in the following sense, to define the multivalued function $g\colon \Omega \times \mathbb{R} \to \mathcal{P}(R)$ by

$$g(x,s) := \begin{cases} \{\underline{\eta}(x)\} & \text{if } s < \underline{v}(x), \\ f(x,s,v(x)) & \text{if } \underline{v}(x) \leqslant s \leqslant \overline{v}(x), \\ \{\overline{\eta}(x)\} & \text{if } \overline{v}(x) < s. \end{cases}$$

From (F1) and (F2) we have that $(x,s) \mapsto f(x,s,v(x))$ is upper Carathéodory and it is readily seen (by invoking $\underline{\eta} \subset f(\cdot,\underline{v},v)$ and $\overline{\eta} \subset f(\cdot,\overline{v},v)$) that g is upper Carathéodory, too. Since g has closed and convex values, and due to (F4) and Theorem 3.69, the mapping $G\colon W \to \mathcal{P}(W')$, defined by

$$G(u) := \{i_q^*\eta : \eta \text{ is a measurable selection of } g(\cdot,u)\}$$

has closed and convex values, is well-defined, pseudomonotone and bounded.

Third, in order to define h, let us introduce the following notation: For any real numbers $x_1 < x_2$ and y_1, y_2, denote by

$$l = \big[(x_1,y_1) \rightsquigarrow (x_2,y_2)\big]$$

the continuous piecewise linear function $l\colon \mathbb{R} \to \mathbb{R}$ that satisfies $l(x) = y_1$ for $x \leqslant x_1$, $l'(x) = (y_2 - y_2)/(x_2 - x_1)$ for $x_1 < x < x_2$, and $l(x) = y_2$ for $x \geqslant x_2$.

This said, we introduce the functions $\underline{\theta}_i, \overline{\theta}_i\colon \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, defined by

$$\begin{aligned} \underline{\theta}_i(x,\cdot) &:= \big[\big(\underline{v}_i(x), \underline{\eta}(x) - \underline{\eta}_i(x)\big) \rightsquigarrow \big(\underline{v}(x), 0\big)\big], \\ \overline{\theta}_i(x,\cdot) &:= \big[\big(\overline{v}(x), 0\big) \rightsquigarrow \big(\overline{v}_i(x), \overline{\eta}_i(x) - \overline{\eta}(x)\big)\big]. \end{aligned}$$

It is easy to check that $\underline{\theta}_i$ and $\overline{\theta}_i$ are measurable in the first argument and, of course, continuous in the second. Let us combine them in order to form the Carathéodory function $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$, defined by

$$h(x,s) := |\underline{\theta}_1(x,s)| + |\underline{\theta}_2(x,s)| - |\overline{\theta}_1(x,s)| - |\overline{\theta}_2(x,s)|.$$

We have constructed h in such a way, that, for all $u \in L^0$, $i = 1,2$, the inequalities

$$\underline{\eta} - \underline{\eta}_i - h(\cdot,u) \leqslant 0 \quad \text{on } \{u < \underline{v}_i\}, \qquad \overline{\eta}_i - \overline{\eta} + h(\cdot,u) \leqslant 0 \quad \text{on } \{\overline{v}_i < u\} \tag{5.9}$$

hold true. Indeed, if, e.g., $v(x) < \underline{v}_i(x) \leqslant \underline{v}(x)$ for some $x \in \Omega$, then we obtain per definition $\underline{\theta}_i(x, v(x)) = \underline{\eta}(x) - \underline{\eta}_i(x)$ and $\overline{\theta}_i(x, v(x)) = 0$, $i = 1, 2$, which implies the first pair of inequalities in (5.9). Moreover, $h(x, s) = 0$ if $\underline{v}(x) \leqslant s \leqslant \overline{v}(x)$.

Further, $h(x, \cdot)$ is obviously bounded by some function $\eta \in L$, so that the Nemytskij operator $v \mapsto h(\cdot, v)$ is known to be continuous and bounded from L to its dual space. Thanks to the compact embedding $W \hookrightarrow L$ we conclude that the composed mapping $H\colon W \to W'$, defined by

$$\langle Hv, w\rangle := \int_\Omega h(\cdot, v)w,$$

is bounded and completely continuous, thus pseudomonotone.

Step 2: Solutions of auxiliary problem Let us consider the multivalued operator

$$A + D + G - H + \partial K_v\colon W \to \mathcal{P}(W'),$$

which is the sum of the bounded, pseudomonotone single-valued operators A, D and $-H$, the bounded, pseudomonotone operator G with closed and convex values, and the maximal monotone operator ∂K_v. Further, A is coercive with respect to any $u_0 \in \mathcal{D}(\partial K_v)$, since by (A2) and (A3) we have for all $u, u_0 \in W$

$$\langle Au, u - u_0\rangle \geqslant \alpha_2\|\nabla u\|_p^p - \|k_2\|_1 - (\alpha_3\|\nabla u\|_p^{p-1} + \|k_3\|_{p'})\|\nabla u_0\|_p, \tag{5.10}$$

which implies

$$\frac{\langle Au, u - u_0\rangle}{\|u\|_W} \to \infty \quad \text{as} \quad \|u\|_W \to \infty$$

(note that $\|\nabla u\|_p$ defines an equivalent norm on W). Further, for all $u, u_0 \in W$ and any $\eta \subset g(\cdot, u)$ we have, since the selections of $g(\cdot, u)$ and Hu are uniformly bounded,

$$\langle i^*\eta - Hu, u - u_0\rangle_W \geqslant -\|\eta - Hu\|_L\|u - u_0\|_{L'} \geqslant -c_0\|u\|_W - c_1 \tag{5.11}$$

for some constants c_0, c_1. In addition, estimates (5.7) and (5.8) imply

$$\langle Du, u - u_0\rangle \geqslant -d_3(\|\underline{v}\|_p^p + \|\overline{v}\|_p^p) - d_0(\|\underline{v}\|_p^{p-1} + \|u\|_p^{p-1} + \|\overline{v}\|_p^{p-1}) \geqslant d_4 - d_5\|u\|_W^{p-1} \tag{5.12}$$

for constants $d_4, d_5 > 0$. From (5.10), (5.11) and (5.12) it follows that the operator $A + D + G - H\colon W \to \mathcal{P}(W')$ is coercive with respect to u_0.

Taken together, all conditions of Theorem 2.57 are fulfilled, and thus there is some $u \in W$ such that $(A + D + G - H + \partial K_v)(u) \ni 0$. This means, $u \in \mathcal{D}(K_v)$ and there is $\eta \subset g(\cdot, u)$ such that, for all $w \in W$,

$$\langle Au, w - u\rangle + \int_\Omega d(\cdot, u)(w - u) + \int_\Omega (\eta - h(\cdot, u))\,(w - u) + K_v(w) - K_v(u) \geqslant 0. \tag{5.13}$$

Step 3: Solution of the MQVIP In this last step, let us check that for any $u \in W$ and $\eta \subset g(\cdot, u)$ with (5.13) we have $\underline{v} \leqslant u \leqslant \overline{v}$. To this end, for $i = 1, 2$, take

$$w = \underline{v}_i - (\underline{v}_i - u)^+ = \underline{v}_i \wedge u \in \underline{v}_i \wedge \mathcal{D}(K_v)$$

as test function in (5.5), take

$$w = u + (\underline{v}_i - u)^+ = \underline{v}_i \vee u \in \mathcal{D}(\underline{k}_i) \vee \mathcal{D}(K_v) \subset \mathcal{D}(K_v)$$

as test function in (5.13), and add the resulting inequalities to obtain

$$\langle Au - A\underline{v}_i, (\underline{v}_i - u)^+ \rangle + \int_\Omega d(\cdot, u)(\underline{v}_i - u)^+ + \int_\Omega \left(\eta - \underline{\eta}_i - h(\cdot, u)\right)(\underline{v}_i - u)^+ \\ + K_v(\underline{v}_i \vee u) - K_v(u) + \underline{k}_i(\underline{v}_i \wedge u) - \underline{k}_i(\underline{v}_i) \geqslant 0. \tag{5.14}$$

By (A4) and the identity $\nabla(\underline{v}_i - u)^+ = \chi_{\{\underline{v}_i \geqslant u\}} \nabla(\underline{v}_i - u)$ (where χ_M denotes the characteristic function of a set M) we deduce

$$\langle Au - A\underline{v}_i, (\underline{v}_i - u)^+ \rangle \leqslant 0. \tag{5.15}$$

Further, it follows from (5.9)

$$\int_\Omega \left(\eta - \underline{\eta}_i - h(\cdot, u)\right)(\underline{v}_i - u)^+ \leqslant 0, \tag{5.16}$$

and $\underline{v}_i \in \mathcal{D}(\underline{k}_i)$, $u \in \mathcal{D}(K_v)$ and $\underline{k}_i \preccurlyeq^*_* K_v$ imply

$$K_v(\underline{v}_i \wedge u) - K_v(u) + \underline{k}_i(\underline{v}_i \vee u) - \underline{k}_i(\underline{v}_i) \leqslant 0. \tag{5.17}$$

Combining (5.14)—(5.17), we deduce

$$\int_\Omega d(\cdot, u)(\underline{v}_i - u)^+ \geqslant 0. \tag{5.18}$$

By definition of d, (5.18) implies $\|(\underline{v}_i - u)^+\|_p^p \leqslant 0$, which in turn implies $\underline{v}_i \leqslant u$, $i = 1, 2$. Since $\underline{v} = \underline{v}_1 \vee \underline{v}_2$, we have $\underline{v} \leqslant u$.

By dual reasoning, we conclude $u \leqslant \bar{v}$ by taking $w = \bar{v}_i + (u - \bar{v}_i)^+ = \bar{v}_i \vee u$ as test function in (5.6) and $w = u - (u - \bar{v}_i)^+ = u \wedge \bar{v}_i$ as test function in (5.13).

In view of $\underline{v} \leqslant u \leqslant \bar{v}$, (5.13) reduces to

$$\langle Au, w - u \rangle + \int_\Omega \eta(w - u) + K_v(w) - K_v(u) \geqslant 0 \quad \text{for all } w \in W,$$

and as $\eta \subset g(\cdot, u) = f(\cdot, u, v)$, we have $u \in S(v)$. ○

The results of Theorem 5.13 immediately imply the following corollary:

5.14 Corollary *Let $v \in D$. Then $\underline{S}(v)$ is directed upward, $\overline{S}(v)$ is directed downward and it holds*

$$S(v) \subset \underline{S}(v) \leqslant^* S(v) \quad \textit{and} \quad S(v) \leqslant_* \overline{S}(v) \supset S(v).$$ ○

5.15 Remark Let $\underline{v}$ be the supremum of finitely many $\underline{v}_i \in \underline{S}(v)$ and let $\bar{v}$ be the infimum of finitely many $\bar{v}_i \in \overline{S}(v)$. Then the above proof can be easily modified to show that under the same conditions as in Theorem 5.13 there is a solution $u \in S(v)$ between $\underline{v}$ and $\bar{v}$ provided $\underline{v} \leqslant \bar{v}$. (See [67] for inspiration.) ○

The next two propositions provide topological properties of S.

5.16 Proposition *The operator* $S\colon D \to \mathcal{P}(W)$ *has uniformly bounded values.*

Proof: Let $v \in [\underline{u}, \overline{u}]_V$ and any $u \in S(v)$ be given. Then it holds, for some $\eta \in F(u, v)$ and $w_v \in \mathcal{D}(K_v)$ as given by (K3),

$$\langle Au, w_v\rangle + \int_\Omega \eta(w_v - u) + K_v(w_v) - K_v(u) \geqslant \langle Au, u\rangle. \tag{5.19}$$

By (K3) we have $K_v(w_v) \leqslant c_3$, and $-K_v(u) \leqslant c_3(\|\nabla u\|_p^{p-1} + 1)$. Combining these estimates with (A3), Hölder's inequality and (A2) gives from (5.19) the estimate

$$\begin{aligned}(\alpha_3\|\nabla u\|_p^{p-1} + \|k_3\|_{p'})\|\nabla w_v\|_p \;+\; \|\eta\|_{q'}(\|w_v\|_q + \|u\|_q) \;+\; c_3(\|\nabla u\|_p^{p-1} + 2)\\ \geqslant \alpha_2\|\nabla u\|_p^p - \|k_2\|_1.\end{aligned} \tag{5.20}$$

The terms $\|\nabla w_v\|_p$ and $\|w_v\|_q$ can be estimated due to (K3), $\|\eta\|_{q'}$ is bounded due to (F4), and we have $\|u\|_q \leqslant \|\underline{u}\|_q + \|\overline{u}\|_q$, since $u \in [\underline{u}, \overline{u}]$. Thus, from (5.20) we deduce

$$\|\nabla u\|_p^p \leqslant c(\|\nabla u\|_p^{p-1} + 1)$$

for some $c > 0$ not depending on u or v, which implies $\|\nabla u\|_p \leqslant 2c + 1$. ○

5.17 Remark We note that the estimates above are valid if (K3) is replaced by the weaker assumption (K3'), but not for any $v \in [\underline{u}, \overline{u}]_V$, but only for each member $v = v_n$ of an increasing sequence $(v_n) \subset [\underline{u}, \overline{u}]_V$. This boundedness was considered in Remark 2.32. ○

5.18 Proposition *The operator* $S\colon D \to \mathcal{P}(W)$ *has weakly compact values.*

Proof: Suppose $v \in [\underline{u}, \overline{u}]_V$ and $(u_n) \subset S(v)$. Due to Proposition 5.16, (u_n) is bounded, and since W is reflexive and compactly embedded in V, we can assume $u_n \rightharpoonup u$ in W and $u_n \to u$ in V (and thus $u_n \to u$ in L') for some $u \in W$. Recall that, thanks to (K1), K_v is weakly sequentially lower semicontinuous. Thus we have from $(u_n) \subset \mathcal{D}(K_v)$ that $u \in \mathcal{D}(K_v)$.

Now, let $(\eta_n) \subset L$ be a sequence such that $\eta_n \in F(u_n, v)$ and

$$\langle Au_n, w - u_n\rangle + \int_\Omega \eta_n(w - u_n) \geqslant K_v(u_n) - K_v(w) \quad \text{for all } w \in W. \tag{5.21}$$

We can assume $\eta_n \rightharpoonup \eta$ in L for some $\eta \in V$ since (η_n) is bounded in L due to (F4), and thanks to Proposition 3.68, we have $\eta \in F(u, v)$. Letting $w = u$ in (5.21) and passing to the limit yields, using again that K_v is weakly sequentially lower semicontinuous,

$$\limsup_n \langle Au_n, u_n - u\rangle \leqslant 0.$$

Since A is pseudomonotone, we infer $\langle Au, w - u \rangle \geqslant \limsup_n \langle Au_n, w - u_n \rangle$ for all $w \in W$. Using again (5.21), it follows, for all $w \in W$,

$$\begin{aligned}\langle Au, w - u \rangle &+ \int_\Omega \eta(w - u) + K_\nu(w) - K_\nu(u) \\ &\geqslant \limsup_n \langle Au_n, w - u_n \rangle + \lim_n \int_\Omega \eta_n(w - u_n) + K_\nu(w) - \liminf_n K_\nu(u_n) \\ &\geqslant \limsup_n \left(\langle Au_n, w - u_n \rangle + \int_\Omega \eta_n(w - u_n) + K_\nu(w) - K_\nu(u_n) \right) \geqslant 0,\end{aligned}$$

which proves $u \in S(\nu)$. ○

Finally, the main theorem of this chapter follows from Theorem 2.33:

5.19 Theorem *Suppose (S), (A1)—(A4), (F1)—(F4) and (K1)—(K3). Then Problem* (5.2) *has both the smallest and the greatest solution in* $[\underline{u}, \overline{u}]_V$.

5.20 Remark The same conclusion holds if (K3) is replaced by (K3'). If (K3') holds only for increasing sequences, then we can only prove that Problem (5.2) has a greatest solution (or a smallest solution if (K3') holds for decreasing sequences). Note, however, that it is not an option to restrict the monotonicity of f to $f(x, s, t) \leqslant_* f(x, s, t')$ or to $f(x, s, t) \leqslant^* f(x, s, t')$ (for $t' \leqslant t$), since we need both of those properties to provide the required monotonicity of $\underline{S}$ and $\overline{S}$, respectively. ○

5.21 Remark In view of the preceding proofs, we could restrict some conditions to hold with respect to $[\underline{u}, \overline{u}]_V$ (which is useful if $\underline{u}$ and $\overline{u}$ or at least some properties of them are explicitly known).

Another modification would be to restrict our considerations to $[\underline{u}, \overline{u}]_Y \cap B$, where $Y \subset W$ is a non-empty, weakly closed lattice containing $\underline{u}$ and $\overline{u}$, and $B \subset W$ is an appropriately large closed ball. In this case, we would apply Theorem 2.33 and its dual with $V = W$.

Finally, we could set $W = W^{1,p}(\Omega)$ at the cost of some slightly more complicated calculations. (See [68] for inspiration.) ○

5.5 Construction of Subsolutions

If mappings f and K are given explicitly, one may construct appropriate auxiliary problems whose solutions are subsolutions of Problem (5.2). In the following, let us present abstract results in this direction, which are inspired by the results of [68] for more concrete examples.

To this end, from Proposition 5.10 we have at once the following special case (for $u = \nu_1 = \nu_2$):

5.22 Corollary *Let the assumptions of Proposition 5.10 hold true. Then it holds* $\operatorname{Fix} \underline{S}_{g,J} \subset \operatorname{Fix} \underline{S}_{f,K}$. ○

5.23 Remark In the proof of Proposition 5.10 we used the growth condition (F4) which holds true only in the order-interval $[\underline{u}, \overline{u}]$ generated by the given sub-supersolutions $\underline{u}$ and $\overline{u}$. However, the proof does not use that $\underline{u}$ and $\overline{u}$ are sub-supersolutions, so that we may assume in this section that $\underline{u}$ and $\overline{u}$ are only some functions belonging to W. ○

Now, the idea is to find particular simple mappings g and J such that some elements $\underline{S}_{g,J}$ (or even solutions of the corresponding problem) can be found more easily than elements of $\underline{S}_{f,K}$. Especially, this is the case if g and J depend on fewer arguments:

5.24 Proposition *Let mappings* $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *and* $J\colon W \to \mathbb{R} \cup \{+\infty\}$ *be given such that the following hypotheses are satisfied:*

(i) It holds $f(x, s, s) \leqslant_* g(x, s)$ *for a.e.* $x \in \Omega$ *and all* $s \in [\underline{u}(x), \overline{u}(x)]$.

(ii) J *is decreasing, it holds* $\mathcal{D}(J) \neq \emptyset$, *and* $J \preccurlyeq_*^* J \preccurlyeq_*^* K_v$ *for all* $v \in V$.

Let furthermore $u \in W$ *be given such that* $g(\cdot, u) \in L$ *and such that one of the following hypotheses holds true:*

(a) $u \vee \mathcal{D}(J) \subset \mathcal{D}(J)$ *and*

$$\langle Au, w - u\rangle + \int_\Omega g(\cdot, u)(w - u) \geqslant 0 \quad \textit{for all } w \in u \wedge \mathcal{D}(J) \tag{5.22}$$

(b) $u \in \mathcal{D}(J)^{\downarrow}$, $\mathcal{D}(J)$ *is order-convex upward in* W, *and*

$$\langle Au, w\rangle + \int_\Omega g(\cdot, u) w \leqslant 0 \quad \textit{for all } w \in W_+. \tag{5.23}$$

Then u *is a subsolution of Problem* (5.2).

Proof: Define the functional $j := I_{u \wedge \mathcal{D}(J)}$. Then we have $j \preccurlyeq_*^* J$, i.e.

$$j(v_1 \wedge v_2) + J(v_1 \vee v_2) \leqslant j(v_1) + J(v_2) \quad \text{for all } v_1, v_2 \in W. \tag{5.24}$$

Indeed, we may assume $v_1 \in \mathcal{D}(j) = u \wedge \mathcal{D}(J)$ and $v_2 \in \mathcal{D}(J)$. From $J \preccurlyeq_*^* J$ we have $\mathcal{D}(J) \preccurlyeq_* \mathcal{D}(J)$ and thus $v_1 \wedge v_2 \in u \wedge \mathcal{D}(J) = \mathcal{D}(j)$. Thus, (5.24) reads as $J(v_1 \vee v_2) \leqslant J(v_2)$, which is true since J is decreasing.

Now assume that (b) holds true. Then (a) holds true, too. Indeed, there is $v \in \mathcal{D}(J)$ such that $u \leqslant v$. Thus, for each $w \in \mathcal{D}(J)$ it follows $u \vee w \in [w, v \vee w]_W$, and since $\mathcal{D}(J)$ is order-convex upward, we have $u \vee w \in \mathcal{D}(J)$, and consequently $u \vee \mathcal{D}(J) \subset \mathcal{D}(J)$. Further, from (5.23) we infer readily (5.22), since for $w \in u \wedge \mathcal{D}(J)$ we have $u - w \in W_+$.

Thus, we may assume (a). Then from $u \vee \mathcal{D}(J) \subset \mathcal{D}(J)$ it follows (for any $w \in \mathcal{D}(J)$) $u = u \wedge (u \vee w) \in u \wedge D(J)$. Thus, we have $j(u) = 0$, and likewise $j(w) = 0$ for $w \in u \wedge \mathcal{D}(J)$, so that from $g(\cdot, u) \in L$, $j \preccurlyeq_*^* J$ and (5.22) it follows $u \in \operatorname{Fix} \underline{S}_{g,J}$ (where we have interpreted g as a multifunction on $\Omega \times \mathbb{R} \times \mathbb{R}$ and J as a functional on $W \times V$).

It now follows from Corollary 5.22 that u is a subsolution of Problem (5.2). ○

6 | Variational Inclusions with Measures

6.1 Introduction

Let us study the multivalued quasilinear elliptic problem

$$Au + G(\cdot, u) \ni f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{P}$$

where $A\colon W \to W'$ is a quasilinear elliptic divergence form operator of Leray-Lions type on a Sobolev space W, and $s \mapsto G(\cdot, s)$ is an upper semicontinuous multifunction, such that $(x, s) \mapsto G(x, s)$ becomes upper Carathéodory. The right-hand side f of (P) is assumed to be a signed Radon measure, which makes the study of solutions of (P) rather involved. We will overcome this difficulty by use of an approximation scheme. To this end, we will consider (P) with a more regular right-hand side f which belongs to the dual space of $W_0^{1,p}(\Omega)$. Existence and comparison results for this regular case have been obtained recently in [27].

It should be noted that the multifunction $s \mapsto G(\cdot, s)$ includes as a special case the generalized gradient $s \mapsto \partial j(\cdot, s)$ of some locally Lipschitz function $s \mapsto j(\cdot, s)$. This is because any generalized gradient $s \mapsto \partial j(\cdot, s)$ is an upper semicontinuous multifunction. Thus, in this case (P) reduces to a hemivariational inequality with measure right-hand side of the form

$$Au + \partial j(\cdot, u) \ni f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{6.1}$$

First attempts for dealing with the special case (6.1) can be found in [26].

The main goals and the novelties of this chapter are as follows: First, in Section 6.2, we develop an existence theory for the above multivalued elliptic problem with measure right-hand side. To this end, we provide notations and assumptions, present a precise formulation for Problem (P), and state, in Subsections 6.2.3 and 6.2.4, various preparatory lemmas, some of which are of interest in its own right. Then, we prove existence of solutions. Second, in Section 6.3, we propose concepts of sub-supersolutions for this problem and establish an existence and comparison principle. Third, we topologically characterize the solution set enclosed by sub-supersolutions.

6.2 Coercive Case

6.2.1 Setting

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. As usual, for every $p \in [1, \infty]$, $L^p(\Omega)$ denotes the Lebesgue space of p-integrable (for $p = \infty$: essentially bounded) functions, and in this chapter we use the abbreviation $V_p := W_0^{1,p}(\Omega)$.

Recall further that p' denotes the Hölder conjugate of p, and p^* denotes its critical Sobolev exponent, which is given by $p^* = Np/(N-p)$ for $p < N$. In particular, we have $1^* = N/(N-1)$.

As for the data, let us suppose that $f \in \mathcal{M}_b(\Omega)$, where $\mathcal{M}_b(\Omega)$ denotes the set of all signed Radon measures on Ω (see Subsection 3.3.4), let $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a function defining the (single-valued) differential operator $Au = -\operatorname{div} a(\cdot, u, \nabla u)$ of Leray-Lions type, and let $G\colon \Omega \times \mathbb{R} \to P(\mathbb{R})$ be a multifunction.

6.1 Assumption Let $p \in (2 - 1/N, N]$ be a fixed constant, $p' = p/(p-1)$. The following assumptions are meant to hold for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and a.e. $x \in \Omega$ and are standard assumptions on the Leray-Lions operators A.

(A1) The function $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a **Carathéodory function**, i.e.

$$\begin{aligned} x \mapsto a(x, s, \xi) \quad &\text{is measurable on } \Omega, \\ (s, \xi) \mapsto a(x, s, \xi) \quad &\text{is continuous on } \mathbb{R} \times \mathbb{R}^N. \end{aligned}$$

(A2) The function $\xi \mapsto a(x, s, \xi)$ is **p-coercive**, in the sense that there is $\alpha > 0$ with

$$a(x, s, \xi)\xi \geqslant \alpha |\xi|^p.$$

Together with (A1), this implies $a(x, s, 0) = 0$.

(A3) There exists a function $b_1 \in L^{p'}(\Omega)$ and a constant $b_2 \geqslant 0$ such that

$$|a(x, s, \xi)| \leqslant b_1(x) + b_2(|s|^{p-1} + |\xi|^{p-1}).$$

(A4) For every $\xi, \xi' \in \mathbb{R}^N$, $\xi' \neq \xi$, it holds the **strict monotonicity**

$$(a(x, s, \xi) - a(x, s, \xi'))(\xi - \xi') > 0. \qquad \circ$$

6.2 Assumption Let $\delta \geqslant 1$ be such that there is $q_0 \in [1, (p-1)1^*)$ with $\delta = q_0^*$. (Such a q_0 exists, since $2 - 1/N < p$.) This is equivalent to $\delta < (p-1)N/(N-p) = ((p-1)1^*)^*$. The following assumptions are meant to hold for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

(G1) The multifunction G is **upper Carathéodory**, i.e. $s \mapsto G(x, s)$ is upper semicontinuous and $x \mapsto G(x, s)$ is measurable, and has closed and convex values.

(G2) There exist $\beta_1 \in L^1_{\mathrm{loc}}(\Omega)$ and $\beta_2 \in L^\infty_{\mathrm{loc}}(\Omega)$ such that

$$\sup\{|y| : y \in G(x, s)\} \leqslant \beta_1(x) + \beta_2(x)|s|^\delta.$$

(G3) For all $y \in G(x, s)$ one has the **sign condition**

$$ys \geqslant 0$$

and furthermore there is some $\beta_3 \in L^1(\Omega)$ such that

$$\sup\{|y| : y \in G(x, 0)\} \leqslant \beta_3(x). \qquad \circ$$

6.2.2 Formulation of the Problem

The main purpose of this chapter is to prove existence and comparison results of the following problem, which includes the lower order multivalued upper Carathéodory operator G. Inspired by [10], we denote by $C_c^\infty(\Omega)$ the linear space of infinitely often differentiable functions $\varphi\colon \Omega \to \mathbb{R}$ with compact support, and define a solution as follows:

6.3 Definition A pair (u, g) is called **solution** of Problem (P) if

$$\begin{aligned} u \in V_1 &\quad \text{with} \quad a(\cdot, u, \nabla u) \in L^1_{loc}(\Omega), \\ g \subset G(\cdot, u) &\quad \text{with} \quad g \in L^1_{loc}(\Omega), \\ \int_\Omega a(\cdot, u, \nabla u)\nabla\varphi + \int_\Omega g\varphi = \int_\Omega \varphi\, df &\quad \text{for all } \varphi \in C_c^\infty(\Omega). \end{aligned} \tag{6.2}$$

Since the integrals (6.2) define linear mappings $L\colon C_c^\infty(\Omega) \to \mathbb{R}$, we sometimes will shorten (6.2) as

$$\langle Au, \varphi\rangle + \langle g, \varphi\rangle = \langle f, \varphi\rangle.$$

○

6.4 Remark To emphasize the main ideas, we have restricted our considerations to parameters $p \in (2-1/N, N]$. For $p > N$, we have the continuous embedding $V_p \subset C_0(\overline{\Omega})$ (where $C_0(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$ with zero boundary), thus, for every $f \in \mathcal{M}_b(\Omega)$, the mapping $\varphi \mapsto \int_\Omega \varphi\, df$ belongs to V_p' and the methods of [27] apply. For $p \leqslant 2 - 1/N$, distributional solutions of (6.2) may not belong to $L^1(\Omega)$ and one has to generalize the notion of solutions (see, e.g., [35]). ○

To prove existence of solutions, we proceed as follows:

(i) We approximate f by a sequence $(f_n) \subset C_c^\infty(\Omega)$ and truncate the multivalued nonlinearity G to obtain a sequence of multivalued operators (G_n). Then we consider the auxiliary problems (P_n), where f and G are replaced by f_n and G_n, respectively, and ensure the existence of a solution (u_n, g_n) without taking the growth (G2) into account.

(ii) We investigate convergence properties of the sequences (u_n) and (g_n).

(iii) We take limits in the distributional equation $Au_n + g_n = f_n$ to obtain $Au + g = f$. Since $g \subset G(\cdot, u)$, (u, g) is the desired solution of (P).

This procedure will lead to our first main theorem:

Main Theorem (Coercive Case) *Let hypotheses (A1)—(A4) and (G1)—(G3) be satisfied. Then Problem* (P) *has a solution* (u, g) *which satisfies even* $u \in V_q$ *for all* $q \in [1, (p-1)1^*)$, $a(\cdot, u, \nabla u) \in L^r(\Omega)$ *for all* $r \in [1, 1^*)$ *and* $g \in L^1(\Omega)$. *Furthermore, the defining equation* (6.2) *holds for all* $\varphi \in V_{r'}$, $r' > N$.

Our second main result is formulated in terms of appropriately defined sub-supersolutions $(\underline{u}, \underline{g})$ and $(\overline{u}, \overline{g})$, respectively, see subsection 6.3.3. To this end we strengthen the hypotheses on f and a as follow: we assume $f \in L^1(\Omega)$ and let a be independent of its second argument. We then provide the following result:

Main Theorem (Sub-supersolution) *Assume* $f \in L^1(\Omega)$ *and let hypotheses (S), (A5) and (G4) (see below) be satisfied. Then there is a solution* (u, g) *of* (P') *which is located in the order-interval* $[\underline{u}, \overline{u}]$ *and has regularity* $u \in V_q$ *for all* $q \in [1, (p-1)1^*)$, $a(\cdot, \nabla u) \in L^r(\Omega)$ *for all* $r \in [1, 1^*)$ *and* $g \in L^{q_0'}(\Omega)$.

To prove our main results, next we are going to provide various preparatory lemmas, which are also of interest in its own right. These lemmas are concerned with existence results for regular multivalued equations $Au + G(\cdot, u) \ni f$ and some a priori estimates and compactness results of solutions of unperturbated, single-valued problems $Au = f$.

6.2.3 Auxiliary Problems

Let us define some auxiliary problems by approximating f and truncating G. For the readers' convenience, let us recall and prove the following lemma, which can be found among others in [95, Lemma 3.4].

6.5 Lemma *Suppose* $f \in \mathcal{M}_b(\Omega)$. *Then there is a sequence* $(f_n) \subset C_c^\infty(\Omega)$ *which converges to* f *in the* ***distributional sense*** *(i.e.* $\langle f_n, \varphi\rangle \to \langle f, \varphi\rangle$ *for all* $\varphi \in C_c^\infty(\Omega)$*) and satisfies* $\|f_n\|_{L^1(\Omega)} \leqslant \|f\|_{\mathcal{M}_b(\Omega)}$.

Proof: Via $\langle f, \varphi\rangle := \int_\Omega \varphi \, df$, each $f \in \mathcal{M}_b(\Omega)$ is a continuous linear functional on the space of continuous real functions with compact support in Ω, that is, $f \in C_c(\Omega)'$. Especially, f is a distribution and the real-valued functions

$$f_n = (\xi_n f) * \varphi_n \in C_c^\infty(\Omega) \tag{6.3}$$

approximate f in the distributional sense, where $(\xi_n) \subset C_c^\infty(\Omega)$ is a sequence of non-negative real cut-off functions which are pointwise bounded by 1, and $(\varphi_n) \subset C_c^\infty(\Omega)$ is a sequence of non-negative functions of integral 1 which defines, via convolution, a mollifier (see, e.g., [38, Cor. 11.7]). It remains to prove the estimate $\|f_n\|_{L^1(\Omega)} \leqslant \|f\|_{\mathcal{M}_b(\Omega)}$. Since we have $\|f\|_{\mathcal{M}_b(\Omega)} = |f|(\Omega)$, where $|f|$ is the total variation of f, this is, however, a direct consequence of (6.3) and Tonelli's theorem:

$$\|f_n\|_{L^1(\Omega)} = \int_\Omega \left| \int_\Omega \xi_n(y)\varphi_n(x-y)\, df(y) \right| dx \leqslant \int_\Omega \int_\Omega \varphi_n(x)\, dx\, d|f| = |f|(\Omega). \qquad \circ$$

6.6 Definition Using the usual truncations $\tau_n \colon s \mapsto \max(-n, \min(s, n))$ at level n, $n \in \mathbb{N}$, we define the multivalued truncated operator

$$G_n := \tau_n(G) \colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R}), \quad (x, s) \mapsto \{\tau_n(y) : y \in G(x, s)\}.$$

That is, we can evaluate G_n pointwise as

$$G_n(x, s) = \begin{cases} \{n\} & \text{if } y > n \text{ for all } y \in G(x, s), \\ \{-n\} & \text{if } y < n \text{ for all } y \in G(x, s), \\ G(x, s) \cap [-n, n] & \text{otherwise.} \end{cases} \qquad \circ$$

6.7 Definition Let (f_n) be some fixed sequence converging to f in distributional sense provided by Lemma 6.5. The auxiliary problem is to find some pair (u_n, g_n) that solves

$$Au_n + G_n(\cdot, u_n) \ni f_n \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{P_n}$$

in the sense that

$$\begin{gathered} u_n \in V_p, \qquad g_n \subset G_n(\cdot, u_n), \\ \int_\Omega a(\cdot, u_n, \nabla u_n)\nabla\varphi + \int_\Omega g_n\varphi = \int_\Omega f_n\varphi \quad \text{for all } \varphi \in V_p \end{gathered} \tag{6.4}$$

(where the test functions φ are allowed to have lower regularity than in (6.2) since the solutions (u_n, g_n) have higher regularity). ○

To provide the existence of solutions of the auxiliary Problems (P_n), we need to know the mapping properties of the multivalued operator G_n.

6.8 Proposition *For each* $n \in \mathbb{N}$, $G_n : \Omega\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has closed and convex values, is upper Carathéodory, and satisfies (G3). Furthermore,* G_n *has uniformly bounded values.*

Proof: Since $\tau_n(s)s \geqslant 0$ and $|\tau_n(s)| \leqslant |s|$, (G3) is fulfilled for G_n. Furthermore, G_n has non-empty, closed and convex values, as G has, and all values are bounded by n. It remains to prove the upper Carathéodory property.

Let $x \in \Omega$ be such that $G(x, \cdot)$ is upper semicontinuous, and let $U \subset \mathbb{R}$ be some open set such that $G_n(x, s) \subset U$. Then there is some open set U' such that $G(x, s) \subset U \cup U'$ and $U' \cap [-n, n] = \emptyset$. Since $G(x, \cdot)$ is upper semicontinuous, one has $G(x, s') \subset U \cup U'$ and $G_n(x, s') \subset U$ for all s' near s. Thus, $G_n(x, \cdot)$ is upper semicontinuous, too.

Now, fix $s \in \mathbb{R}$ and let $U \subset \mathbb{R}$ be open. Then the pre-image

$$G_n(\cdot, s)^{-1}_{-}(U) = \{x \in \Omega : G_n(x, s) \cap U \neq \emptyset\}$$

equals, depending on the structure of U, the union of some of the pre-images

$$G(\cdot, s)^{-1}_{-}(U \cap (-n, n)), \quad G(\cdot, s)^{-1}_{-}((-\infty, -n]) \quad \text{and} \quad G(\cdot, s)^{-1}_{-}([n, \infty)),$$

which are all measurable. (Note that $\mathbb{R}$ is σ-compact and that G has non-empty and closed values, which is why weak and strong measurability of $G(\cdot, s)$ coincide.) Consequently, G_n is upper Carathéodory. ○

Following [27], we next formulate Problem (P_n) as a multivalued operator equation in order to apply abstract surjectivity results. To this end, we introduce the following operator:

6.9 Definition We define the multivalued Nemytskij operator N_{G_n} by

$$N_{G_n} : V_p \to \mathcal{P}(L^0(\Omega)), \quad u \mapsto \{g \in L^0(\Omega) : g \subset G_n(\cdot, u)\},$$

that is, $N_{G_n}(u)$ is the set of measurable selections of $G_n(\cdot, u)$. ○

Due to Theorem 3.47, G_n is weakly superpositionally measurable, thus the operator N_{G_n} is well-defined. Furthermore, G_n has uniformly bounded values, so that we have $N_{G_n}(u) \subset L^\infty(\Omega)$ for every $u \in V_p$. Therefore, the next definition makes sense.

6.10 Definition We define the multivalued Nemytskij operator $\mathcal{N}_{G_n}$ by

$$\mathcal{N}_{G_n}: V_p \to \mathcal{P}(V_p'), \quad u \mapsto i^* N_{G_n}(u) = \{i^* g : g \in N_{G_n}(u)\},$$

where $i^*: L^{p'}(\Omega) \to V_p'$ is the linear and compact adjoint operator to the compact embedding $i: V_p \to L^p(\Omega)$. ○

Now, Problem (P_n) reads as follows: Find a function $u_n \in V_p$ satisfying

$$Au_n + \mathcal{N}_{G_n}(u_n) \ni f_n \quad \text{in } V_p', \tag{6.5}$$

which means that there is some $i^* g_n \in \mathcal{N}_{G_n}(u_n)$ such that

$$Au_n + i^* g_n = f_n \quad \text{in } V_p'.$$

Then (u_n, g_n) is a solution of (P_n).

Such a solution u_n with corresponding selection g_n exists if $A + \mathcal{N}_{G_n}$ is surjective in the sense that $\bigcup_{u \in V_p} (A + \mathcal{N}_{G_n})(u) = V_p'$. But this is the case:

6.11 Proposition *The operator* $A + \mathcal{N}_{G_n}: V_p \to P(V_p')$ *is surjective, thus Problem* (P_n) *has a solution* (u_n, g_n) *with* $u_n \in V_p$.

Proof: Due to (A1)—(A4), A is known to be a single-valued continuous, bounded and pseudomonotone mapping. Furthemore, we readily apply Proposition 3.69 to conclude that $\mathcal{N}_{G_n}: V_p \to \mathcal{P}(V_p')$ is a well-defined pseudomonotone multifunction with closed and convex values. Thus, the sum $A + \mathcal{N}_{G_n}$ is pseudomonotone, too. Furthermore, A is coercive and $\mathcal{N}_{G_n}$ has uniformly bounded values, so their sum is a coercive multifunction with respecto to $u_0 := 0$. Since $M: u \mapsto \{0\}$ is maximal monotone on V_p, $A + \mathcal{N}_{G_n}$ is surjective due to Theorem 2.57. ○

6.2.4 Compactness Results

Suppose $f \in V_p' \cap L^1(\Omega)$ and let us consider the problem

$$Au = f \quad \text{in } V_p', \qquad u \in V_p \tag{P_f}$$

under conditions (A1)—(A4). We will assume that solutions u_n with respect to various right-hand sides f_n are given and are interested in proving the compactness of (u_n) in a suitable Sobolev space. The proofs follow the ideas given in [10] and [8], modified as proposed there.

6.12 Lemma *Let* $(f_n) \subset V_p' \cap L^1(\Omega)$ *be bounded in* $L^1(\Omega)$ *and let* u_n *be a solution of* (P_f) *with right-hand side* f_n, $n \in \mathbb{N}$. *Then, for every* $q \in [1, (p-1)1^*)$, (u_n) *is bounded in* V_q.

Proof: For each $t > 0$ let us define the following cut-off functions τ_t and σ_t by

$$\tau_t(s) := \begin{cases} t & \text{if } s > t, \\ s & \text{if } s \in [-t, t], \\ -t & \text{if } s < -t, \end{cases} \qquad \sigma_t(s) := \begin{cases} \operatorname{sgn}(s) & \text{if } |s| > t+1, \\ \operatorname{sgn}(s)(|s| - t) & \text{if } |s| \in [t, t+1], \\ 0 & \text{if } s \in (-t, t), \end{cases}$$

where sgn denotes the usual sign function. Clearly, both τ_t and σ_t are increasing, bounded and Lipschitz. Furthermore, define for each $u \in V_p$ the corresponding sets

$$\Omega_t^\tau(u) := \{x \in \Omega : |u(x)| < t\}, \qquad \Omega_t^\sigma(u) := \{x \in \Omega : t \leqslant |u(x)| < t+1\}.$$

In this setting, $\tau_t(u_n)$ belongs to V_p with gradient $\nabla\tau_t(u_n) = \tau_t'(u_n)\nabla(u_n)$ and thus

$$\langle Au_n, \tau_t(u_n)\rangle = \int_\Omega a(\cdot, u_n, \nabla u_n)\nabla\tau_t(u_n) = \int_{\Omega_t^\tau(u_n)} a(\cdot, u_n, \nabla u_n)\nabla u_n, \tag{6.6}$$

since $\tau_t'(u_n) = 1$ on $\Omega_t^\tau(u_n)$ and $\tau_t'(u_n) = 0$ otherwise. Likewise, $\sigma_t(u_n) \in V_p$ and

$$\langle Au_n, \sigma_t(u_n)\rangle = \int_\Omega a(\cdot, u_n, \nabla u_n)\nabla\sigma_t(u_n) = \int_{\Omega_t^\sigma(u_n)} a(\cdot, u_n, \nabla u_n)\nabla u_n. \tag{6.7}$$

Now, let (f_n) be bounded in $L^1(\Omega)$ by c. Then we have for any $t > 0$

$$\langle Au_n, \tau_t(u_n)\rangle = \langle f_n, \tau_t(u_n)\rangle \leqslant ct. \tag{6.8}$$

Combining (6.6) with (6.8), (A2) and Hölder's inequality with conjugates $p/q > 1$ (note: $p \leqslant N$) and $p/(p-q)$ easily gives

$$\int_{\Omega_t^\tau(u_n)} |\nabla u_n|^q \leqslant \left(\frac{ct}{\alpha}\right)^{q/p} |\Omega_t^\tau(u_n)|^{(p-q)/p}. \tag{6.9}$$

Taking into account $t \leqslant |u_n|$ on $\Omega_t^\sigma(u_n)$, we obtain likewise from $\langle Au_n, \sigma_t(u_n)\rangle \leqslant c$ and (6.7) the useful estimate

$$\int_{\Omega_t^\sigma(u_n)} |\nabla u_n|^q \leqslant \left(\frac{c}{\alpha}\right)^{q/p} \left(\int_{\Omega_t^\sigma(u_n)} \frac{|u_n|^{q^*}}{t^{q^*}}\right)^{(p-q)/p}. \tag{6.10}$$

Now, let us specialize t as natural numbers $k, k+1, \ldots$ corresponding to the disjoint partition $\Omega = \Omega_k^\tau(u_n) \cup \Omega_k^\sigma(u_n) \cup \Omega_{k+1}^\sigma(u_n) \cup \ldots$ This partition and the vectorial Hölder's inequality along with (6.9) and (6.10) give

$$\|\nabla u_n\|_{L^q(\Omega)}^q \leqslant c_1(k) + c_2(k)\|u_n\|_{L^{q^*}(\Omega)}^{q^*(p-q)/p},$$

where $c_1(k)$ and $c_2(k)$ satisfy

$$c_1(k)^{p/q} = \frac{ck}{\alpha}|\Omega|^{(p-q)/q}, \quad c_2(k)^{p/q} = \frac{c}{\alpha}\sum_{m=k}^\infty m^{-q^*(p-q)/q}.$$

Since $\|u_n\|_{L^{q^*}(\Omega)} \leqslant c_3\|\nabla u_n\|_{L^q(\Omega)}$ for some embedding constant c_3, we deduce under the given hypotheses on p and q that $\|u_n\|_{V_q}$ is bounded by a constant depending neither on u_n nor f_n but only on c, q and the data p, α and $|\Omega|$. ○

The proof of Lemma 6.15 is far more involved. For this purpose, first let us state a trivial auxiliary result and the convergence theorem of Vitali for a finite measure space (cf., e.g., [96, Th. 11]).

6.13 Lemma *Let (f_n) be bounded in $L^q(\Omega)$, $q > 1$, and let $(\Omega_n) \subset \Omega$ be a sequence of sets with $|\Omega_n| \to 0$. Then*

$$\int_{\Omega_n} |f_n| \leqslant |\Omega_n|^{1/q'} \|f_n\|_{L^q(\Omega)} \to 0 \quad \text{as } n \to \infty. \qquad \circ$$

6.14 Theorem (Vitali) *Let $S = (X, \mathcal{A}, \mu)$ be a finite measure space, $1 \leqslant p < \infty$, $(f_n) \subset L^p(S)$, $f \in L^0(S)$ and $f_n \to f$ a.e. Then $f \in L^p(S)$, and we have the convergence $\|f_n - f\|_{L^p(S)} \to 0$ if and only if (f_n) is p-equi-integrable, that is (since S is a finite measure space), $\lim_{\mu(A)\to 0} \int_A |f_n|^p \, d\mu = 0$ uniformly in n.* $\circ$

6.15 Lemma *Let $(f_n) \subset V_p' \cap L^1(\Omega)$ be bounded in $L^1(\Omega)$ and let u_n be a solution of (P_f) with right-hand side f_n, $n \in \mathbb{N}$. Then there is a function u such that for every $q \in [1, (p-1)1^*)$ and every $r \in [1, 1^*)$ there is a subsequence of (u_n) (we do not relabel) for which $u_n \to u$ in V_q and $a(\cdot, u_n, \nabla u_n) \to a(\cdot, u, \nabla u)$ in $L^r(\Omega)$ hold.*

Proof: **Step 1. Setting**

Let $(f_n) \subset V_p' \cap L^1(\Omega)$ be bounded in $L^1(\Omega)$ by c and fix $q \in (1, (p-1)1^*)$. By Lemma 6.12, (u_n) is bounded in V_q, thus, up to a subsequence, $\nabla u_n \rightharpoonup \nabla u$ in $L^q(\Omega)$, $u_n \to u \in L^q(\Omega)$ and $u_n \to u$ a.e. We shall prove that even $\nabla u_n \to \nabla u$ in $L^q(\Omega)$. To this end, we consider $F_n \in L^1(\Omega)$, defined by

$$F_n := [a(\cdot, u_n, \nabla u_n) - a(\cdot, u_n, \nabla u)][\nabla u_n - \nabla u]. \tag{6.11}$$

Note that $F_n \geqslant 0$ due to (A4). Furthermore, we will need the truncated function $\tau_k(u)$, where τ_k is the usual truncation at level k. Taking $\tau_k(u_n)$ as test function in $Au_n = f_n$, we obtain, since $\tau_k'(s) \in \{0, 1\}$,

$$\begin{aligned} \int_\Omega \alpha |\nabla \tau_k(u_n)|^p &= \int_\Omega \alpha |\nabla u_n|^p \tau_k'(u_n) \\ &\leqslant \int_\Omega a(\cdot, u_n, \nabla u_n) \nabla u_n \tau_k'(u_n) \\ &= \int_\Omega a(\cdot, u_n, \nabla u_n) \nabla \tau_k(u_n) = \int_\Omega f_n \tau_k(u_n) \leqslant ck \end{aligned} \tag{6.12}$$

and thus the boundedness of $(\tau_k(u_n))$ in V_p. Therefore, we have $\tau_k(u_{n_l}) \rightharpoonup \tau$ in V_p and $\tau_k(u_{n_l}) \to \tau$ a.e. for some subsequence (u_{n_l}) of (u_n) and some $\tau \in V_p$. Since also $\tau_k(u_{n_l}) \to \tau_k(u)$ a.e., we have $\tau = \tau_k(u)$ and thus $\tau_k(u) \in V_p$ (and of course $\tau_k(u) \in L^\infty(\Omega)$). Since the limit is independent of the chosen subsequence, it follows $\tau_k(u_n) \rightharpoonup \tau_k(u)$ in V_p. Similar, one proves $\tau_\varepsilon(u_n - \tau_k(u)) \rightharpoonup \tau_\varepsilon(u - \tau_k(u))$ in V_p for every $\varepsilon > 0$ by using $\nabla \tau_\varepsilon(u_n - \tau_k(u)) = \tau_\varepsilon'(u_n - \tau_k(u)) \nabla(\tau_{\varepsilon+k}(u_n) - \tau_k(u))$.

Step 2. $\lim_n \int_\Omega F_n^s = 0$ for some $s \in (0,1)$

We will split Ω two times in two complementary subsets which are constructed first by considering $|u|$ and second by considering $|u_n - \tau_k(u)|$. This will give rise to an estimate

$$0 \leqslant \int_\Omega F_n^s \leqslant \int_{\Omega_u^k} F_n^s + \int_{\Omega_{n,k}^\varepsilon} F_{n,k}^s + \int_{\Omega\setminus\Omega_{n,k}^\varepsilon} F_{n,k}^s, \tag{6.13}$$

where the sets of integration and the integrand $F_{n,k}$ are defined as below. These three integrals will be estimated separately and the conclusion will be $\lim_n \int_\Omega F_n^s = 0$.

First, consider for all $k \in \mathbb{N}$ the set

$$\Omega_u^k := \{x \in \Omega : |u(x)| \geqslant k\},$$

whose measure tends to 0 as $k \to \infty$. By this, (6.14) below follows, but we need uniform convergence with respect to n. To this end, employ Hölder's inequality to obtain

$$\int_{\Omega_u^k} F_n^s \leqslant \left(\int_{\Omega_u^k} |a(\cdot, u_n, \nabla u_n) - a(\cdot, u_n, \nabla u)|^{sq'}\right)^{1/q'} \left(\int_{\Omega_u^k} |\nabla u_n - \nabla u|^{sq}\right)^{1/q}.$$

Since (u_n) is bounded in V_q, $(a(\cdot, u_n, \nabla u_n))$ is bounded in $L^r(\Omega)$ for all r which satisfy $r \leqslant q/(p-1) < p'$ due to (A3), and so we can apply Lemma 6.13 if both $sq' < q/(p-1)$ and $sq < q$. Thus, for all $s < (q-1)/(p-1) < 1$, we obtain

$$\lim_k J(n,k) = 0, \qquad \text{where } J(n,k) := \int_{\Omega_u^k} F_n^s, \tag{6.14}$$

the convergence being uniform in n.

By (A4) the remaining integral satisfies

$$\int_{\Omega\setminus\Omega_u^k} F_n^s \leqslant \int_\Omega \left([a(\cdot, u_n, \nabla u_n) - a(\cdot, u_n, \nabla \tau_k(u))]\,[\nabla u_n - \nabla \tau_k(u)]\right)^s =: \int_\Omega F_{n,k}^s.$$

Now, for fixed $\varepsilon > 0$ and all $n \in \mathbb{N}$ consider the sets

$$\Omega_{n,k}^\varepsilon := \{x \in \Omega : |u_n(x) - \tau_k(u(x))| \geqslant \varepsilon\}.$$

Thanks to Lemma 6.13 and the boundedness of the sequences involved, we obtain in the same way as before the following estimate:

$$K(n,k,\varepsilon) := \int_{\Omega_{n,k}^\varepsilon} F_{n,k}^s \leqslant c_1 |\Omega_{n,k}^\varepsilon|^r$$

for some constants $c_1 > 0$ and $r > 0$. The right-hand side depends on all ε, n and k, but can, by considering sets $\Omega_m := \{x \in \Omega : |u(x) - \tau_k(u(x))| \geqslant \varepsilon - 1/m\}$ and the convergence $u_n \to u$ in measure, easily be estimated in the following way:

$$\limsup_n K(n,k,\varepsilon) \leqslant c_1 \left|\{x \in \Omega : |u(x) - \tau_k(u(x))| \geqslant \varepsilon\}\right|^r. \tag{6.15}$$

Indeed, from $|u_n(x) - \tau_k(u(x))| \geqslant \varepsilon$ it follows

$$x \in \Omega_m \quad \text{or} \quad x \in \{x \in \Omega : |u_n(x) - u(x)| \geqslant 1/m\}. \tag{6.16}$$

Since $u_n \to u$ in measure, the second set in (6.16) has measure less any given $\delta > 0$ provided n is sufficiently large, so that $\limsup_n |\Omega^\varepsilon_{n,k}| \leqslant |\Omega_m|$. Furthermore, we have

$$\Omega_0 := \{x \in \Omega : |u(x) - \tau_k(u(x))| \geqslant \varepsilon\} = \bigcap\nolimits_m \Omega_m,$$

and since (Ω_m) is a decreasing sequence (with respect to inclusion), we conclude finally $\limsup_n |\Omega^\varepsilon_{n,k}| \leqslant \lim_m |\Omega_m| = |\Omega_0|$.

It remains the integral $L(n, k, \varepsilon)$, defined as

$$\int_{\Omega\setminus\Omega^\varepsilon_{n,k}} F^s_{n,k} = \int_{\Omega\setminus\Omega^\varepsilon_{n,k}} \left([a(\cdot, u_n, \nabla u_n) - a(\cdot, u_n, \nabla\tau_k(u))] \, [\nabla u_n - \nabla\tau_k(u)]\right)^s .$$

Since $\tau'_\varepsilon(u_n - \tau_k(u)) = 1$ on $\complement\Omega^\varepsilon_{n,k}$ and $\tau'_\varepsilon(u_n - \tau_k(u)) = 0$ on $\Omega^\varepsilon_{n,k}$, we can write

$$L(n, k, \varepsilon) = \int_\Omega \big([a(\cdot, u_n, \nabla u_n) - a(\cdot, u_n, \nabla\tau_k(u))] \, [\nabla(u_n - \tau_k(u))] \, \tau'_\varepsilon(u_n - \tau_k(u))\big)^s ,$$

and since $\nabla\tau_\varepsilon(u_n - \tau_k(u)) = \tau'_\varepsilon(u_n - \tau_k(u))\nabla(u_n - \tau_k(u))$, we can simplify $L(n, k, \varepsilon)$ further. By employing Hölder's inequality and by taking $\tau_\varepsilon(u_n - \tau_k(u)) \in V_p \cap L^\infty(\Omega)$ as test function in $Au_n = f_n$, we obtain

$$\begin{aligned} &L(n, k, \varepsilon) \\ &\leqslant \left(\int_\Omega [a(\cdot, u_n, \nabla u_n) - a(\cdot, u_n, \nabla\tau_k(u))] \, \nabla\tau_\varepsilon(u_n - \tau_k(u))\right)^s |\Omega|^{1-s} \\ &= \left(\int_\Omega f_n \tau_\varepsilon(u_n - \tau_k(u)) - \int_\Omega a(\cdot, \tau_{k+\varepsilon}(u_n), \nabla\tau_k(u))\nabla\tau_\varepsilon(u_n - \tau_k(u))\right)^s |\Omega|^{1-s}. \end{aligned} \tag{6.17}$$

To justify the last line in (6.17), note that $u_n = \tau_{k+\varepsilon}(u_n)$ on $\{x \in \Omega : |u_n| \leqslant k + \varepsilon\}$ and that $\nabla\tau_\varepsilon(u_n - \tau_k(u)) = 0$ on $\{x \in \Omega : |u_n| > k + \varepsilon\}$. Now, letting $n \to \infty$ in (6.17) leads to

$$\limsup\nolimits_n L(n, k, \varepsilon) \leqslant \left(c\varepsilon - \int_\Omega a(\cdot, \tau_{k+\varepsilon}(u), \nabla\tau_k(u))\nabla\tau_\varepsilon(u - \tau_k(u))\right)^s |\Omega|^{1-s}.$$

Since $a(\cdot, s, 0) = 0$ for all $s \in \mathbb{R}$, we immediately infer

$$\limsup\nolimits_n L(n, k, \varepsilon) \leqslant c^s \varepsilon^s \, |\, \Omega \,|^{1-s} , \tag{6.18}$$

the right-hand side of which is independent of k.

Now we are able to prove $\lim_n \int_\Omega F^s_n = 0$ by using estimate (6.13), which reads now as

$$0 \leqslant \int_\Omega F^s_n \leqslant J(n, k) + K(n, k, \varepsilon) + L(n, k, \varepsilon),$$

and carefully combining it with (6.14), (6.15) and (6.18): Let $\tilde{\varepsilon} > 0$ be given and let $\varepsilon > 0$ be so small that $c^s \varepsilon^s |\Omega|^{1-s} < \tilde{\varepsilon}$. By (6.14), $J(n,k) \leqslant \tilde{\varepsilon}$ if k is sufficiently large. We may choose k so large that $|\{x \in \Omega : |u(x)| \geqslant k\}|^r < \tilde{\varepsilon}/c_1$. As $\{x \in \Omega : |u(x) - \tau_k(u(x))| \geqslant \varepsilon\}$ is a subset of $\{x \in \Omega : |u(x)| \geqslant k\}$, (6.15) implies $K(n,k,\varepsilon) \leqslant \tilde{\varepsilon}$ if n is sufficiently large. We may choose n so large that, by (6.18), $L(n,k,\varepsilon) \leqslant \tilde{\varepsilon}$. Inserting these estimates in (6.13), we arrive at $0 \leqslant \int_\Omega F_n^s \leqslant 3\tilde{\varepsilon}$ for all sufficiently large n.

Step 3. Applying Leray-Lions theory

The following is standard, see, e.g., [74, Lemme 3.3] or [106, Lemma 6.3].

Due to Step 2, we have $F_n^s \to 0$ in $L^1(\Omega)$, and so, up to a subsequence, $F_n^s \to 0$ a.e., and thus $F_n \to 0$ a.e. Furthermore, for a.e. $x \in \Omega$ we have $u_n(x) \to u(x)$ (thus $(u_n(x))$ is bounded) and $b_1(x) < \infty$, and due to (A3) there is a constant c_x such that

$$F_n(x) \geqslant a(x, u_n(x), \nabla u_n(x)) \nabla u_n(x) - c_x(1 + |\nabla u_n(x)|^{p-1} + |\nabla u_n(x)|). \tag{6.19}$$

Combining (6.19)) with (A2), we derive boundedness of $(\nabla u_n(x))$ and thus we have $\nabla u_{n_k}(x) \to \xi(x) \in \mathbb{R}^N$ for some subsequence $(u_{n_k}(x)) \subset (u_n(x))$. Taking limits in (6.11) at the point x, we obtain

$$0 = [a(x, u(x), \xi(x)) - a(x, u(x), \nabla u(x))][\xi(x) - \nabla u(x)]$$

and, due to (A4), $\xi(x) = \nabla u(x)$. Since the limit $\nabla u(x)$ is independent of the chosen subsequence (u_{n_k}), we have in fact $\nabla u_n \to \nabla u$ a.e. Since (∇u_n) is uniformly bounded in $L^{\tilde{q}}(\Omega)$ for all $\tilde{q} \in (q, (p-1)1^*)$, we have due to Vitali's convergence theorem and Lemma 6.13 $\nabla u_n \to \nabla u$ in $L^q(\Omega)$.

As regards other exponents $\tilde{q} \in (q, (p-1)1^*)$, one can repeat the proof while starting with the current sequence (u_n) (which is a subsequence of the original given one) to obtain, up to a subsequence, $\nabla u_n \to \nabla u$ in $L^{\tilde{q}}(\Omega)$.

Due to A3, $(a(\cdot, u_n, \nabla u_n))$ is uniformly bounded in $L^r(\Omega)$, $r < 1^*$, and since (A1) holds, we have for each $r < 1^*$, again due to Vitali's convergence theorem and up to a subsequence, the strong convergence $a(\cdot, u_n, \nabla u_n) \to a(\cdot, u, \nabla u)$ in $L^r(\Omega)$. This completes the proof. ○

6.2.5 Existence of Solutions

We are in a position now to prove the first main result of this chapter. Its proof will follow the path outlined in Subsection 6.2.2.

6.16 Theorem *Let hypotheses (A1)—(A4) and (G1)—(G3) be satisfied. Then Problem* (P) *has a solution* (u, g) *which satisfies even* $u \in V_q$ *for all* $q \in [1, (p-1)1^*)$, $a(\cdot, u, \nabla u) \in L^r(\Omega)$ *for all* $r \in [1, 1^*)$ *and* $g \in L^1(\Omega)$. *Furthermore, the defining equation* (6.2) *holds for all* $\varphi \in V_{r'}$, $r' > N$.

Proof: Let (u_n, g_n) be a solution of the auxiliary problem (P_n), $n \in \mathbb{N}$, which exists due to Proposition 6.11. By definition, $u_n \in V_p$, $g_n \subset G_n(\cdot, u_n)$ and

$$\langle Au_n, \varphi \rangle + \langle g_n, \varphi \rangle = \langle f_n, \varphi \rangle \quad \text{for all } \varphi \in V_p. \tag{6.20}$$

To take limits in (6.20), we want to apply Lemma 6.15, hence, we have to ensure that both (f_n) and (g_n) are bounded in $L^1(\Omega)$. Concerning (f_n), this holds by Lemma 6.5, and concerning (g_n), this holds true due to the following procedure:

Since $g_n \subset G_n(\cdot, u_n)$, the sign condition (G3) ensures

$$|g_n(x)| = g_n(x)\,\mathrm{sgn}(u_n(x)) \quad \text{for a.e. } x \in \Omega_n := \{x \in \Omega : u_n(x) \neq 0\}.$$

By considering test functions $\varphi_k(u_n)$ in (6.20), where the increasing Lipschitz functions $\varphi_k \colon \mathbb{R} \to \mathbb{R}$ are defined by

$$\varphi_k(s) := \mathrm{sgn}(s)\left[1 - (|s|+1)^{-k}\right] \in [-1, 1],$$

considering (A2) together with $\nabla\varphi_k(u_n) = \varphi_k'(u_n)\nabla u_n$, and passing to the limit as $k \to \infty$, we obtain

$$\int_{\Omega_n} |g_n| = \int_\Omega g_n\,\mathrm{sgn}(u_n) \leqslant \int_\Omega f_n\,\mathrm{sgn}(u_n) \leqslant \int_\Omega |f_n| \leqslant \|f\|_{\mathcal{M}_b(\Omega)}. \tag{6.21}$$

Combining estimate (6.21) with the second statement of (G3) yields

$$\|g_n\|_{L^1(\Omega)} \leqslant \|f\|_{\mathcal{M}_b(\Omega)} + \|\beta_3\|_{L^1(\Omega)},$$

that is, (g_n) is uniformly bounded in $L^1(\Omega)$ as demanded.

Thanks to Lemma 6.15, there is some function u such that for all $q \in [1, (p-1)1^*)$ and $r \in [1, 1^*)$, up to a subsequence, $u_n \to u$ in V_q and $a(\cdot, u_n, \nabla u_n) \to a(\cdot, u, \nabla u)$ in $L^r(\Omega)$. Due to the continuous embedding $V_q \subset L^{q^*}(\Omega)$, for all $s \in [1, ((p-1)1^*)^*)$ we can also assume $u_n \to u$ in $L^s(\Omega)$. Fix now such q and r.

To take limits in (6.20), it remains to prove some convergence of (g_n). This step depends on the growth of G. Since $|\tau_n(s)| \leqslant |s|$, we can use condition (G2): There is some $q_0 \in [1, (p-1)1^*)$ and $\delta = q_0^*$ such that for every compact set $\Omega_k \subset \mathbb{R}^N$ with $\Omega_k \subset \Omega$ one has

$$\int_{\Omega_k} |g_n| \leqslant \int_{\Omega_k} \beta_1 + \|\beta_2\|_{L^\infty(\Omega_k)} \int_{\Omega_k} |u_n|^\delta.$$

Since for all $s \in [1, ((p-1)1^*)^*)$, (u_n) can be chosen to be convergent in $L^s(\Omega)$ and so in $L^s(\Omega_k)$ as well, (u_n) is δ-equi-integrable by Vitali's convergence theorem, thus (g_n) is equi-integrable in $L^1(\Omega_k)$. By the well-known Dunford-Pettis theorem, (g_n) has a weakly convergent subsequence in $L^1(\Omega_k)$.

Let (Ω_k) be a sequence of compact subsets of $\mathbb{R}^N$ such that $\Omega_k \subset \operatorname{int}\Omega_{k+1}$ (where int denotes the interior) for all $k \in \mathbb{N}$ and $\bigcup_{k=1}^\infty \Omega_k = \Omega$. For each k, we select by means of the calculations we have just done inductively a weakly convergent subsequence (g_{k_n}) of (g_n) with weak limit $g^k \in L^1(\Omega_k)$ such that

$$g_{k_n} \rightharpoonup g^k \quad \text{in } L^1(\Omega_k) \quad \text{and} \quad (g_{(k+1)_n}) \subset (g_{k_n}) \quad \text{for all } k \in \mathbb{N}.$$

Observe that for each k both $g_{(k+1)_n}|_{\Omega_k} \rightharpoonup g^k$ and $g_{(k+1)_n}|_{\Omega_k} \rightharpoonup g^{k+1}|_{\Omega_k}$ in $L^1(\Omega_k)$ and thus $g^{k+1}|_{\Omega_k} = g^k$. By extending each g^k trivially to a function on Ω, the pointwise limit $g = \lim_{k\to\infty} g^k$ is well-defined, measurable and we have by Fatou's lemma

$$\|g\|_{L^1(\Omega)} \leqslant \liminf_k \int_{\Omega_k} |g^k| \leqslant \|f\|_{\mathcal{M}_b(\Omega)} + \|\beta_3\|_{L^1(\Omega)},$$

since (g_n) is bounded in $L^1(\Omega)$ by $\|f\|_{\mathcal{M}_b(\Omega)} + \|\beta_3\|_{L^1(\Omega)}$. Thus, $g \in L^1(\Omega)$ and, by Lebesgue's convergence theorem, $g^k \to g$ in $L^1(\Omega)$.

Up to a subsequence, we have ensured the following convergence properties:

$$\begin{aligned} f_n &\to f && \text{in distributional sense,} \\ u_n &\to u && \text{in } V_q \text{ for some } q \in [1, (p-1)1^*) \text{ and a.e.,} \\ a(\cdot, u_n, \nabla u_n) &\to a(\cdot, u, \nabla u) && \text{in } L^r(\Omega) \text{ for some } r \in [1, 1^*), \\ g_{k_n} &\rightharpoonup g^k && \text{in } L^1(\Omega_k), \quad k \in \mathbb{N}, \\ g^k &\to g && \text{in } L^1(\Omega). \end{aligned}$$

Furthermore, we have $g_{k_n} \subset G_{k_n}(\cdot, u_{k_n})$ for all $n, k \in \mathbb{N}$. By Proposition 3.68, we obtain $g^k(x) \in G(x, u(x))$ for a.e. $x \in \Omega_k$ (clearly, $G\colon \Omega_k \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is upper Carathéodory) and thus, since $g = g^k$ on Ω_k, $g \subset G(\cdot, u)$. Passing to the limit as $n \to \infty$ in (6.20) yields, for fixed $k \in \mathbb{N}$,

$$\langle Au, \varphi \rangle + \langle g^k, \varphi \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in C_c^\infty(\Omega) \text{ with } \operatorname{supp} \varphi \subset \Omega_k.$$

Since each $\varphi \in C_c^\infty(\Omega)$ is supported in some Ω_k, we have in fact

$$\langle Au, \varphi \rangle + \langle g, \varphi \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{6.22}$$

that is, (u, g) is a solution of (P) with better regularity.

Since $a(\cdot, u, \nabla u) \in L^r(\Omega)$ for all $r \in [1, 1^*)$, we have $Au \in V'_{r'}$ for all $r' > N$ (note that $(1^*)' = (N/(N-1))' = N$). Now take $\varphi \in V_{r'}$ arbitrary, let $(\varphi_n) \subset C_c^\infty(\Omega)$ be some sequence with $\varphi_n \to \varphi$ in $V_{r'}$ and consider (6.22) for those φ_n. Since we have the continuous embedding $V_{r'} \subset C_0(\overline{\Omega})$ and since $f \in \mathcal{M}_b(\Omega)$ induces a linear, continuous functional on $C_0(\overline{\Omega})$, we can take limits and obtain

$$\langle Au, \varphi \rangle + \langle g, \varphi \rangle = \langle f, \varphi \rangle \quad \text{for all } \varphi \in V_{r'}, \quad r' > N.$$

This completes the proof. ○

6.17 Remark Instead of the sign condition (G3), every condition that ensures uniformly boundedness of (g_n) would be enough. For instance, we could assume that in (G2) one has $\beta_2 = 0$ and $\beta_1 \in L^1(\Omega)$. ○

6.3 Sub-Supersolution Method

6.3.1 New Setting

Let us restrictively assume in the following that the measure f has regularity $f \in L^1(\Omega)$ and that the generating function a of A does not depend on its second argument. Problem (P) now reads as follows:

6.18 Definition A pair (u, g) is a **solution** of the problem

$$Au + G(\cdot, u) \ni f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{P'}$$

if and only if for all $q \in [1, (p-1)1^*)$ and all $r \in [1, 1^*)$

$$\begin{aligned} u \in V_q \quad &\text{with} \quad a(\cdot, \nabla u) \in L^r(\Omega), \\ g \subset G(\cdot, u) \quad &\text{with} \quad g \in L^1(\Omega), \\ \int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega g\varphi = \int_\Omega f\varphi \quad &\text{for all } \varphi \in V_{r'}. \end{aligned}$$

(Note that $r < 1^*$ implies $r' > N$ and thus $V_{r'} \subset L^\infty(\Omega)$.) ○

The regularity $f \in L^1(\Omega)$ will be needed to use truncated sub-supersolutions as test functions. Those sub-supersolutions are appropriately defined as follows with respect to some parameter q_0 which depends on G (see condition (G4) below).

6.19 Definition A pair $(\underline{u}, \underline{g})$ is called **subsolution** of (P') with respect to some parameter q_0 if and only if

$$\begin{aligned} \underline{u} \in V_p \quad &\text{with} \quad a(\cdot, \nabla\underline{u}) \in L^{p'}(\Omega), \\ \underline{g} \subset G(\cdot, \underline{u}) \quad &\text{with} \quad \underline{g} \in L^{q_0'}(\Omega), \\ \int_\Omega a(\cdot, \nabla\underline{u})\nabla\varphi + \int_\Omega \underline{g}\varphi \leqslant \int_\Omega f\varphi \quad &\text{for all } \varphi \in V_p \cap L^\infty_+(\Omega). \end{aligned} \tag{6.23}$$

Similarly, $(\overline{u}, \overline{g})$ is called **supersolution** of (P') with respect to q_0 if and only if

$$\begin{aligned} \overline{u} \in V_p \quad &\text{with} \quad a(\cdot, \nabla\overline{u}) \in L^{p'}(\Omega), \\ \overline{g} \subset G(\cdot, \overline{u}) \quad &\text{with} \quad \overline{g} \in L^{q_0'}(\Omega), \\ \int_\Omega a(\cdot, \nabla\overline{u})\nabla\varphi + \int_\Omega \overline{g}\varphi \geqslant \int_\Omega f\varphi \quad &\text{for all } \varphi \in V_p \cap L^\infty_+(\Omega). \end{aligned} \tag{6.24}$$

(Recall that $L^\infty_+(\Omega)$ denotes the set of all functions $u \in L^\infty(\Omega)$ with $u \geqslant 0$.) ○

Throughout this section, we will assume the following conditions:

- (S) There are a subsolution $(\underline{u}, \underline{g})$ and a supersolution $(\overline{u}, \overline{g})$ of (P') with respect to some parameter $q_0 \in [1, p^*)$ which are ordered in the sense that $\underline{u} \leqslant \overline{u}$.
- (A5) Conditions (A1)—(A4) hold with $a = a(x, s, \xi)$ independent of its second argument s.
- (G4) Condition (G1) holds. Furthermore, there exists $\beta_1 \in L^{q_0'}(\Omega)$, q_0 given by (S), such that
$$\sup\{|y| : y \in G(x, s)\} \leqslant \beta_1(x)$$
for a.e. $x \in \Omega$ and all $s \in [\underline{u}(x), \overline{u}(x)]$.

Note that, due to (A3), $\underline{u} \in V_p$ and $\overline{u} \in V_p$ immediately imply $a(\cdot, \nabla\underline{u}) \in L^{p'}(\Omega)$ and $a(\cdot, \nabla\overline{u}) \in L^{p'}(\Omega)$, respectively, and that we dropped the sign condition on G. Thanks to the sub-supersolutions, we will get a solution u of (P'), located even in the order-interval $[\underline{u}, \overline{u}]$, that is, $\underline{u} \leqslant u \leqslant \overline{u}$. To prove this, we will follow roughly the same path as in the previous section, but modify the auxiliary problems as in [26] to carry out the well-known sub-supersolution method.

6.3.2 Auxiliary Problems

Let us introduce the Carathéodory function

$$d(x,s) := \begin{cases} (s - \overline{u}(x))^{p-1} & \text{if } \ s > \overline{u}(x), \\ 0 & \text{if } \ s \in [\underline{u}(x), \overline{u}(x)], \\ -(\underline{u}(x) - s)^{p-1} & \text{if } \ s < \underline{u}(x). \end{cases}$$

From

$$|d(x,s)| \leqslant c\left(|\underline{u}(x)|^{p-1} + |\overline{u}(x)|^{p-1} + |s|^{p-1}\right) \tag{6.25}$$

and by use of the compact embedding $V_p \to L^p(\Omega)$, we have that the Nemytskij operator $D\colon V_p \to V_p'$, defined by $\langle Du, \varphi\rangle := \int_\Omega d(\cdot, u)\varphi$, is bounded and completely continuous, thus pseudomonotone.

Furthermore, consider the truncated multivalued operator

$$\widetilde{G}\colon \Omega \times \mathbb{R} \to P(\mathbb{R}), \quad (x,s) \mapsto \begin{cases} \{\overline{g}(x)\} & \text{if } \ s > \overline{u}(x), \\ G(x,s) & \text{if } \ s \in [\underline{u}(x), \overline{u}(x)], \\ \{\underline{g}(x)\} & \text{if } \ s < \underline{u}(x). \end{cases}$$

Obviously, one has for every $u \in L^0(\Omega)$ and all $g \subset \widetilde{G}(\cdot, u)$

$$|g(x)| \leqslant |\overline{g}(x)| + |\beta_1(x)| + |\underline{g}(x)| \quad \text{for a.e. } x \in \Omega. \tag{6.26}$$

Furthermore, it is readily seen that $\widetilde{G}$ is upper Carathéodory and has closed and convex values. The only question that arises is, whether the inclusion $\widetilde{G}(x,s) \subset U$ for some open set $U \subset \mathbb{R}$ implies $\widetilde{G}(x,s') \subset U$ for a.e. $x \in \Omega$ and all s' near s if $s = \underline{u}(x)$ or $s = \overline{u}(x)$. However, this holds true because $\underline{g} \subset G(\cdot, \underline{u})$, $\overline{g} \subset G(\cdot, \overline{u})$ and $G(x, \cdot)$ is upper semicontinuous for a.e. $x \in \Omega$.

The following auxiliary problem plays a crucial role in the proof of the main result of this section: Find a pair (u, g) that solves

$$Au + \widetilde{G}(\cdot, u) + Du \ni f, \quad u = 0 \quad \text{on } \partial\Omega \tag{$\widetilde{P}$}$$

in the sense of Definition 6.18.

Similarly as in the preceding section, we will find a solution of $(\widetilde{P})$ by approximation. The approximating sequences (u_n) and (g_n) will be of importance in proving that a solution of $(\widetilde{P})$ is in fact a solution of the original Problem (P'), so we state:

6.20 Proposition *For each* $f_n \in V_p'$, *there is a pair* $(u_n, g_n) \in V_p \times L^{q_0'}(\Omega)$ *such that*

$$\int_\Omega a(\cdot, \nabla u_n)\nabla\varphi + \int_\Omega g_n\varphi + \int_\Omega d(\cdot, u_n)\varphi = \int_\Omega f_n\varphi \quad \textit{for all } \varphi \in V_p, \tag{6.27}$$

where $g_n \subset \widetilde{G}(\cdot, u_n)$ *is the corresponding selection.*

Proof: Due to estimate (6.26), the values of $\mathcal{N}_{\widetilde{G}}(u)$ are uniformly majorized in $L^{q_0'}$ by $|\underline{g}| + \beta_1 + |\overline{g}|$, thus, the operator

$$\mathcal{N}_{\widetilde{G}}\colon V_p \to P(V_p'), \quad u \mapsto \{i_{q_0}^* g : g \in L^{q_0'}(\Omega),\ g \subset \widetilde{G}(\cdot, u)\},$$

is well-defined, where $i_{q_0}^*$ denotes the adjoint operator of the compact embedding operator $i_{q_0}\colon V_p \to L^{q_0}(\Omega)$. Since Lemma 3.69 holds true also for $\widetilde{G}$, we have that $\mathcal{N}_{\widetilde{G}}$ is pseudomonotone. Thus, as in the proof of Proposition 6.11, we obtain the existence of the solution (u_n, g_n) we looked for. ○

To make up for the absent sign condition, we provide one last preliminary result.

6.21 Proposition *Let* $u \in V_p$ *and* $h \in V_p' \cap L^1(\Omega)$ *be such that*

$$\int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega d(\cdot, u)\varphi = \int_\Omega h\varphi \quad \textit{for all } \varphi \in V_p. \tag{6.28}$$

Then $\|d(\cdot, u)\|_{L^1(\Omega)} \leqslant \|h\|_{L^1(\Omega)} + c$, *where* c *is independent of* u *and* h.

Proof: Consider the function $w \in L^{p'}(\Omega)$, defined by

$$w(x) := \begin{cases} -(\overline{u}^-(x))^{p-1} & \text{if } u(x) > \overline{u}(x), \\ 0 & \text{if } u(x) \in [\underline{u}(x), \overline{u}(x)], \\ (\underline{u}^+(x))^{p-1} & \text{if } u(x) < \underline{u}(x). \end{cases}$$

We have $\|w\|_{L^{p'}(\Omega)}^{p'} \leqslant \|\underline{u}\|_{L^p(\Omega)}^p + \|\overline{u}\|_{L^p(\Omega)}^p$, so $\|w\|_{L^1(\Omega)} \leqslant c$. Furthermore,

$$|d(x, u(x)) + w(x)| = (d(x, u(x)) + w(x)) \operatorname{sgn}(u(x)) \quad \text{for a.e. } x \in \Omega. \tag{6.29}$$

For $k > 0$, let φ_k be the increasing Lipschitz function, defined by

$$\varphi_k(s) = \operatorname{sgn}(s)\left[1 - (|s| + 1)^{-k}\right].$$

Obviously, $|\varphi_k| \leqslant 1$ and (φ_k) converges pointwise to the sign function sgn as $k \to \infty$. Let us take the function $v_k := \varphi_k(u) \in V_p$ as a test function in (6.28). In view of $\nabla v_k = \varphi_k'(u)\nabla u$ and (A2), we obtain

$$\int_\Omega (d(\cdot, u) + w)\varphi_k(u) \leqslant \int_\Omega (h + w)\varphi_k(u). \tag{6.30}$$

Passing to the limit in (6.30) as $k \to \infty$ and combining the result with (6.29), we get

$$\int_\Omega |d(\cdot, u) + w| \leqslant \int_\Omega (h + w)\operatorname{sgn}(u) \leqslant \|h + w\|_{L^1(\Omega)}.$$

By this, the assertion follows. ○

6.3.3 Existence of Solutions

We are now in the position to prove the second main theorem of this chapter. This will be done in two steps: First, we show, with the same methods as in the preceding section, that $(\widetilde{P})$ has a solution u. Second, we verify that, due to the special structure of $\widetilde{G}$ and D, u is actually a solution of (P').

6.22 Theorem *Assume* $f \in L^1(\Omega)$ *and let hypotheses (S), (A5) and (G4) be satisfied. Then there is a solution* (u, g) *of* (P') *which is located in* $[\underline{u}, \overline{u}]$ *and has regularity* $u \in V_q$ *for all* $q \in [1, (p-1)1^*)$, $a(\cdot, \nabla u) \in L^r(\Omega)$ *for all* $r \in [1, 1^*)$ *and* $g \in L^{q_0'}(\Omega)$.

Proof: Let $(f_n) \subset C_c^\infty(\Omega)$ be a sequence with $f_n \to f$ in $L^1(\Omega)$ and let u_n and g_n be the functions derived in Proposition 6.20 with respect to f_n, $n \in \mathbb{N}$. Obviously, (f_n) is bounded in $L^1(\Omega)$, and since $g_n \subset \widetilde{G}(\cdot, u_n)$, (g_n) is bounded in $L^{q_0'}(\Omega)$ due to estimate (6.26). Employing Proposition 6.21 with $h = f_n - g_n$ shows that also the sequence $(d(\cdot, u_n))$ is bounded in $L^1(\Omega)$.

Fix $r \in [1, 1^*)$ and $q \in [1, (p-1)1^*)$ such that $q^* \geqslant p - 1$, then thanks to Lemma 6.15 we have, up to a subsequence, the following convergences:

$$\begin{aligned} u_n &\to u \quad \text{in } V_q \text{ and a.e.,} \\ a(\cdot, \nabla u_n) &\to a(\cdot, \nabla u) \quad \text{in } L^r(\Omega). \end{aligned}$$

Further, due to (6.25) and (6.26) and up to a subsequence, we get

$$\begin{aligned} g_n &\rightharpoonup g \quad \text{in } L^{q_0'}(\Omega), \\ d(\cdot, u_n) &\to d(\cdot, u) \quad \text{in } L^1(\Omega). \end{aligned}$$

Taking limits in (6.27) yields

$$\int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega g\varphi + \int_\Omega d(\cdot, u)\varphi = \int_\Omega f\varphi \quad \text{for all } \varphi \in V_{r'},$$

that is $Au + \widetilde{G}(\cdot, u) + Du \ni f$. Again by Proposition 3.68 we see that $g \subset \widetilde{G}(\cdot, u)$ and thus (u, g) is a solution of the auxiliary Problem $(\widetilde{P})$.

We will see in the sequel that (u, g) is even a solution of Problem (P'). For this to come true, we only have to show $u \in [\underline{u}, \overline{u}]$, which implies $\widetilde{G}(\cdot, u) = G(\cdot, u)$ and $d(\cdot, u) = 0$. Let us only show $u \leqslant \overline{u}$; the proof for $\underline{u} \leqslant u$ is quite similar.

Subtracting the corresponding inequality (6.24) for the supersolution $\overline{u}$ from equation (6.27) corresponding to (u_n, g_n) yields

$$\int_\Omega \left[a(\cdot, \nabla u_n) - a(\cdot, \nabla \overline{u})\right] \nabla\varphi + \int_\Omega \left[g_n - \overline{g}\right] \varphi + \int_\Omega d(\cdot, u_n)\varphi \leqslant \int_\Omega \left[f_n - f\right] \varphi, \quad (6.31)$$

which holds for all $\varphi \in V_p \cap L_+^\infty(\Omega)$. For all $\varepsilon > 0$, let ψ_ε be the increasing Lipschitz function defined by

$$\psi_\varepsilon(s) := \begin{cases} 0 & \text{if } s \leqslant 0, \\ \varepsilon^{-1}s & \text{if } s \in (0, \varepsilon), \\ 1 & \text{if } s \geqslant \varepsilon. \end{cases}$$

Taking $\varphi = \psi_\varepsilon(u_n - \overline{u})$ as test function in (6.31), we obtain by involving (A4)

$$\int_\Omega [g_n - \overline{g}]\,\psi_\varepsilon(u_n - \overline{u}) + \int_\Omega d(\cdot, u_n)\psi_\varepsilon(u_n - \overline{u}) \leqslant \int_\Omega [f_n - f]\,\psi_\varepsilon(u_n - \overline{u}). \qquad (6.32)$$

We have $\psi_\varepsilon(u_n - \overline{u}) \to \psi_\varepsilon(u - \overline{u})$ in $L^{q_0}(\Omega)$ as $n \to \infty$, and we can assume that $(d(\cdot, u_n))$ and (f_n) are majorised in $L^1(\Omega)$ and converge pointwise a.e. Therefore, the derived convergence properties and Lebesgue's convergence theorem allow us to pass to the limit in (6.32) as $n \to \infty$, which yields

$$\int_\Omega [g - \overline{g}]\,\psi_\varepsilon(u - \overline{u}) + \int_\Omega d(\cdot, u)\psi_\varepsilon(u - \overline{u}) \leqslant 0.$$

Again be applying Lebesgue's theorem we may pass to the limit as $\varepsilon \to 0$ and obtain

$$\int_{\{u > \overline{u}\}} g - \overline{g} + \int_{\{u > \overline{u}\}} d(\cdot, u) \leqslant 0.$$

Since $\widetilde{G}(x, s) = \{\overline{g}\}$ for $s > \overline{u}(x)$ and by definition of d, we obtain

$$\int_\Omega |(u - \overline{u})^+|^{p-1} \leqslant 0$$

and thus $u \leqslant \overline{u}$ as desired. ○

6.3.4 Properties of the Solution Set

If the right-hand side f is more regular, say, $f \in L^{p'}(\Omega)$, then the set $\mathcal{S}$ of all solutions $u \in [\underline{u}, \overline{u}] \cap V_p$ of Problem (P') is non-empty, compact and directed, see, e.g., [28, Sec. 4.2.2]. The usual proof relies on the fact that solutions itself can be used as test functions in their defining equations and that solutions and sub-supersolutions have the same regularity. However, this is not the case if we only have $f \in L^1(\Omega)$, as solutions u are only guaranteed to satisfy $u \in V_q$. Still, by approximation like in the preceding subsection, we are able to prove at least compactness and thanks to Zorn's lemma the existence of maximal and minimal solutions. As for smallest and greatest solutions within the order-intervall $[\underline{u}, \overline{u}]$, we refer to the next chapter.

6.23 Definition Let $\mathcal{S}$ be the set of all $u \in [\underline{u}, \overline{u}]$ such that there is a solution (u, g) of problem (P') which can be approximated in the sense we did above, i.e. there are, for all $q \in [1, (p-1)1^*)$, sequences $(u_n) \subset V_p$, $(g_n) \subset L^{q_0'}(\Omega)$ and $(f_n) \subset C_c^\infty(\Omega)$ such that equation (6.27) is satisfied, $g_n \subset \widetilde{G}(\cdot, u_n)$ for all n, $u_n \to u$ in V_q, $g_n \rightharpoonup g$ in $L^{q_0'}(\Omega)$ and $f_n \to f$ in $L^1(\Omega)$. ○

6.24 Lemma *For each* $q \in [1, (p-1)1^*)$, $\mathcal{S}$ *is a compact subset of* V_q

Proof: Fix some $q \in [1, (p-1)1^*)$ such that $q^* \geqslant p-1$, let $(u_n) \subset \mathcal{S}$ be some sequence in V_q, and let $(u_{n,k})$, $(g_{n,k})$ and $(f_{n,k})$ be sequences of functions associated to u_n, $n \in \mathbb{N}$, in the sense of Definition 6.23 with $u_{n,k} \to u_n$ in V_q as $k \to \infty$. Further, let $\varphi \in V_{r'}$ be some test function, $r' > N$.

Up to a subsequence, we may assume $\|u_n - u_{n,n}\|_{V_q} \leqslant 1/n$, $\|d(\cdot, u_{n,n})\|_{L^1(\Omega)} \leqslant 1/n$ (since $d(\cdot, u_n) = 0$), and $\|f_{n,n} - f\|_{L^1(\Omega)} \leqslant 1/n$. Since $g_{n,n} \subset \widetilde{G}(\cdot, u_{n,n})$, the sequence $(g_{n,n})$ is bounded in $L^{q_0'}(\Omega)$ and thus, up to a subsequence, $g_{n,n} \rightharpoonup g$ in $L^{q_0'}(\Omega)$.

By definition, we have

$$\int_\Omega a(\cdot, \nabla u_{n,n}) \nabla \varphi + \int_\Omega g_{n,n} \varphi + \int_\Omega d(\cdot, u_{n,n}) \varphi = \int_\Omega f_{n,n} \varphi \tag{6.33}$$

for all n and the sequences $(g_{n,n})$, $(d(\cdot, u_{n,n}))$ and $(f_{n,n})$ are bounded in $L^1(\Omega)$. Due to Lemma 6.15, we have, up to a subsequence, $u_{n,n} \to u$ in V_q, and for any other $\widetilde{q} \in (q, (p-1)1^*)$, there is a subsequence (u_{n_k,n_k}) with $u_{n_k,n_k} \to u$ in $V_{\widetilde{q}}$. Let us take limits as $n \to \infty$ in (6.33) to obtain

$$\int_\Omega a(\cdot, \nabla u) \nabla \varphi + \int_\Omega g \varphi + \int_\Omega d(\cdot, u) \varphi = \int_\Omega f \varphi.$$

Since $d(\cdot, u_{n,n}) \to d(\cdot, u)$ and $d(\cdot, u_{n,n}) \to 0$ in $L^1(\Omega)$, we have $d(\cdot, u) = 0$ and thus $u \in [\underline{u}, \overline{u}]$. Furthermore, Proposition 3.68 guaranties $g \subset \widetilde{G}(\cdot, u) = G(\cdot, u)$. Altogether, (u, g) is a solution of (P') which can be approximated in the demanded sense, thus $u \in \mathcal{S}$. Since $u_n \to u$ in V_q, the proof is completed. ○

6.25 Corollary *The set $\mathcal{S}$ contains a maximal element and a minimal element, and for every $u \in \mathcal{S}$, the sets $\{v \in \mathcal{S} : u \leqslant v\}$ and $\{v \in \mathcal{S} : v \leqslant u\}$ contain a maximal and a minimal element.*

Proof: Let $C \subset \mathcal{S}$ be a chain. Since $\mathcal{S} \subset V_q$ is compact for all $q \in (1, (p-1)1^*)$, C is bounded in V_q. Since V_q is a reflexive ordered Banach space, there is an increasing sequence $(u_n) \subset C$ that converges weakly to $\sup C$ in V_q. As $\mathcal{S} \subset V_q$ is compact, we conclude $\sup C \in \mathcal{S}$, that is, every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$. Thanks to the Lemma of Zorn, there is a maximal element of $\mathcal{S}$.

The remaining assertions are proved similarly. ○

6.26 Remark By inspecting the proofs of this chapter, one realizes that some of the results have generalizations for variational inequalities

$$\int_\Omega a(\cdot, \nabla u) \nabla(\varphi - u) + \int_\Omega g(\varphi - u) = \int_\Omega f(\varphi - u) \quad \text{for all } \varphi \in K,$$

provided the set $K \subset V_p$ is defined appropriately. For instance, in order to take truncations as test functions in the proofs of Lemmas 6.12 and 6.15, one could choose $K = [a, b]_{V_p}$ with $a, b \in V_p$ such that $a \leqslant 0 \leqslant b$, and if $\varphi - u \in L^\infty$, then $\int_\Omega f(\varphi - u)$ is well-defined. ○

7 Variational Inclusions with Measures and Nonsmooth Bifunctions

7.1 Introduction

In Chapter 6, we considered quasilinear multivalued elliptic equations with right-hand sides $f \in \mathcal{M}_b(\Omega)$ of the form

$$Au + G(\cdot, u) \ni f,$$

where A is an elliptic operator in divergence form and where G is upper Carathéodory. The novelty was the combination of a measure right-hand side with rather general multivalued terms. The obtained existence of maximal and minimal solutions between each ordered pair of sub-supersolutions extended the existing literature with a new class of differential inclusions.

In contrast, in Chapter 5, we considered multivalued quasi-variational inequalities of the form

$$Au + F(u, u) + \partial K_u(u) \ni 0, \tag{7.1}$$

where the interesting part is the twofold dependence of both terms $F(u, u)$ and $\partial K_u(u)$ on the solution u and, notably, that $t \mapsto f(x, s, t)$ was only assumed to be a (multivalued) decreasing function, which is possibly nonsmooth. Proceeding from pioneering works [24, 68], we developed an abstract framework for such problems that merged the famous concept of sub-supersolutions with an order-theoretical fixed point theorem. As a result, we were able to solve Problem (7.1) under weak assumptions and we even found smallest and greatest solutions between in each interval $[\underline{u}, \overline{u}]$ formed by sub-supersolutions.

The aim of the current chapter is to combine the approximation methods of Chapter 6 with the abstract framework of Theorem 2.33 in order to deal with multivalued elliptic equations with bifunctions of the form

$$Au + f(\cdot, u, u) \ni \mu, \tag{P}$$

where A is an elliptic Leray-Lions operator, $(x, s) \mapsto f(x, s, t)$ is upper Carathéodory and $t \mapsto f(x, s, t)$ is a decreasing, possibly nonsmooth multifunction, and $\mu \in L^1$.

Since the solutions considered in Chapter 6 are (due to regularity issues) in general neither sub- nor supersolutions—which is a key element in the abstract framework—, we are going to extend the concept of limit-solutions, considered at the end of Chapter 6, to the new concept of limit-subsolutions. Then, the abstract framework applies to limit-problems and we can prove as our main result that there exist smallest and greatest limit-solutions between each ordered pair of sub-supersolutions.

The rest of this chapter is organized as follows: First, we recall concepts and results that are used to deal with nonsmooth differential equations with L^1-functions. Second, we introduce the notion of limit-subsolutions and prove with help of the abstract framework that Problem (P) has extremal limit-solutions between each ordered pair of sub-supersolutions.

7.2 Setting

This section serves as a reference for notations, definitions and assumptions used in the rest of the chapter. The exact formulation of the considered problem and some preliminary results will be given hereafter in Section 7.3.

Let, as in Chapter 6, $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with Lipschitz boundary and let $p \in (2 - 1/N, N]$ and $q \in (1, p^*)$ be fixed, where p^* denotes the critical Sobolev exponent associated with p and the dimension N, which is given by $p^* = Np/(N-p)$ for $p < N$ and can be chosen arbitrarily large if $p = N$. In particular, $1^* = N/(N-1)$.

Now, we set $W_r := W_0^{1,r}(\Omega)$, and furthermore we use the following abbreviations (where q' is the Hölder conjugate of q):

$$L := L^{q'}, \qquad V := L^{p^*}, \qquad W := \bigcap \{W_r : 1 \leqslant r < (p-1)1^*\}.$$

Thus, the embedding $W_p \hookrightarrow V$ is continuous and the embedding $L \hookrightarrow W_p'$ is compact.

Concerning W, we say a subset $M \subset W$ is **bounded** if it is bounded in every W_r, $r \in [1, (p-1)1^*)$, and a sequence $(u_n) \subset W$ is said to **converge (weakly)** to $u \in W$ (we write $u_n \to u$ ($u_n \rightharpoonup u$)) if $u_n \to u$ ($u_n \rightharpoonup u$) in every W_r, $r \in [1, (p-1)1^*)$.

Next, let us state the conditions on the data used later.

7.1 Assumption From the right-hand side μ, we assume only the following:

(M) $\mu \in L^1(\Omega)$. ○

7.2 Assumption Let $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a function. The following standard assumptions on a are meant to hold for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

(A1) The function a is a **Carathéodory function**.

(A2) There is a constant $\alpha_2 > 0$ such that $a(x, \xi)\xi \geqslant \alpha_2 |\xi|^p$.

(A3) There exists a constant $\alpha_3 > 0$ and a function $a_3 \in L^{p'}(\Omega)$ such that a satisfies the **growth** condition $|a(x, \xi)| \leqslant a_3(x) + \alpha_3 |\xi|^{p-1}$.

(A4) The function a is **strictly monotone** in the second argument. ○

7.3 Remark Under conditions (A1)—(A4), $A\colon W_p \to W_p'$, defined by

$$\langle Au, \varphi \rangle := \int_\Omega a(\cdot, \nabla u) \nabla \varphi,$$

is known to be bounded, coercive, monotone, continuous, and pseudomonotone. ○

7.4 Assumption The following assumption refers to the notion of sub-supersolutions that will be introduced in Definition 7.7 below.

(S) There is a subsolution $\underline{u}$ and a supersolution $\overline{u}$ of problem (P) such that $\underline{u} \leqslant \overline{u}$. ○

7.5 Assumption Let $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multifunction whose values are compact intervals. The following conditions are meant to hold for a.e. $x \in \Omega$ and all $s, t \in \mathbb{R}$.

(F1) The function $(x, t) \mapsto f(x, s, t)$ is **superpositionally measurable**.

(F2) The function $s \mapsto f(x, s, t)$ is **upper semicontinuous** on $\mathbb{R}$.

(F3) The function $t \mapsto f(x, s, t)$ is **decreasing** on $\mathbb{R}$.

(F4) There is some $b_4 \in L_+$ such that f satisfies the **growth** condition

$$\sup\{|y| : y \in f(x, s, t), s, t \in [\underline{u}(x), \overline{u}(x)]\} \leqslant b_4(x).$$ ○

7.6 Remark It follows that $(x, s) \mapsto f(x, s, t)$ is upper Carathéodory for all $t \in \mathbb{R}$, and that f is weakly superpositionally measurable (see Proposition 3.50). Further, we emphasize again that no continuity in the last argument of f is assumed, and that (F4) assumes only a local growth condition between sub-supersolutions $\underline{u}$ and $\overline{u}$. Such a condition is often easily checked, even if $\underline{u}$ and $\overline{u}$ are not explicitly known. ○

7.3 Preliminary Results

Similar as in the preceding chapters, a **solution** of Problem (P) is a function $u \in W$, i.e.

$$u \in W_0^{1,r}(\Omega) \quad \text{for all } r \in [1, (p-1)1^*),$$

such that there is a corresponding measurable selection $\eta \subset f(\cdot, u, u)$ for which it holds

$$\int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega \eta\varphi = \int_\Omega \mu\varphi \quad \text{for all } \varphi \in W_r, \quad r > N. \tag{7.2}$$

(Note that $W_r \subset L^\infty$ for $r > N$, so that the right-hand side in (7.2) is well-defined.) This problem extends the problem considered in Chapter 6 to the case of bifunctions f. Our main result states that there is a smallest and a greatest solution of Problem (P) between each given pair of sub-supersolutions. These semi-solutions are given as follows:

7.7 Definition A function $\underline{u} \in W_p$ is called **subsolution** of Problem (P) if there is some measurable $\underline{\eta} \subset f(\cdot, \underline{u}, \underline{u})$ such that

$$\int_\Omega a(\cdot, \nabla\underline{u})\nabla\varphi + \int_\Omega \underline{\eta}\varphi \leqslant \int_\Omega \mu\varphi \quad \text{for all } \varphi \in W_p \cap L_+^\infty. \tag{7.3}$$

Similar, a function $\overline{u} \in W_p$ is called **supersolution** of Problem (P) if there is some measurable $\overline{\eta} \subset f(\cdot, \overline{u}, \overline{u})$ such that

$$\int_\Omega a(\cdot, \nabla\overline{u})\nabla\varphi + \int_\Omega \overline{\eta}\varphi \geqslant \int_\Omega \mu\varphi \quad \text{for all } \varphi \in W_p \cap L_+^\infty. \tag{7.4}$$

(Note, that from (A3) and (F4) we obtain that all terms in (7.3) and (7.4) are well-defined.) ○

Those definitions of solutions, sub- and supersolutions are a straightforward generalization of those of the problem considered in Chapter 6. Thus, it comes by no surprise that results of Chapter 6 can be applied if we fix one argument of f. To be more precise, let any $v \in [\underline{u}, \overline{u}]_{L^0}$ be given and define the function

$$f_v \colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R}), \quad (x, s) \mapsto f(x, s, v(x)).$$

Then, let us consider the following problem:

$$Au + f_v(\cdot, u) \ni \mu, \tag{P(v)}$$

whose solutions are $u \in W$ such that there is $\eta \subset f_v(\cdot, u)$ for which it holds

$$\int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega \eta\varphi = \int_\Omega \mu\varphi \quad \text{for all } \varphi \in W_r, \quad r > N. \tag{7.5}$$

One readily checks that the results of Section 6.3 apply to Problem $(P(v))$. One only has to make sure that $\underline{u}$ and $\overline{u}$ are sub-supersolutions of $(P(v))$ (according to Definition 6.19). However, this can be proved with the same idea used in Proposition 7.23 below. Thus, Problem $(P(v))$ has a solution u that is located in $[\underline{u}, \overline{u}]$. This solution is obtained as the limit of solutions of appropriately defined auxiliary problems. For further reference let us state the existence result for solutions of the auxiliary problem in a slightly more general form (which is easily seen to be true by considering the original result Proposition 6.20).

7.8 Proposition *Let* $g \colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *be an upper Carathéodory multifunction with closed and convex values such that all of its measurable selections are majorized in* L, *let* $B \colon W_p \to W_p'$ *be bounded and completely continuous and such that* $\langle Bv, v\rangle_{W_p}$ *is bounded from below, and let* $h \in W_p'$ *be given. Then there is* $u \in W_p$ *and* $\eta \subset g(\cdot, u)$ *such that*

$$\int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega \eta\varphi + \langle Bu, \varphi\rangle = \int_\Omega h\varphi \quad \textit{for all } \varphi \in W_p. \qquad ○$$

In order to define auxiliary problems, we specify both g and B. Concerning g, truncate f_v in the following way with respect to the given sub-supersolutions $\underline{u}$ and $\overline{u}$ and fixed associated selections $\underline{\eta}$ and $\overline{\eta}$:

$$g := g_v \colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R}), \quad (x, s) \mapsto \begin{cases} \{\overline{\eta}(x)\} & \text{if } s > \overline{u}(x), \\ f(x, s, v(x)) & \text{if } s \in [\underline{u}(x), \overline{u}(x)], \\ \{\underline{\eta}(x)\} & \text{if } s < \underline{u}(x). \end{cases} \tag{7.6}$$

Then $g = g_\nu$ has closed and convex values, is readily seen to be upper Carathéodory, and all its measurable selections are majorized by $|\underline{\eta}| + b_4 + |\overline{\eta}| \in L$.

Concerning B, one defines the cut-off function $b\colon \Omega \times \mathbb{R} \to \mathbb{R}$ pointwise by

$$b(x,s) := \begin{cases} -(\underline{u}(x) - s)^{p-1} & \text{if } s < \underline{u}(x), \\ 0 & \text{if } \underline{u}(x) \leqslant s \leqslant \overline{u}(x), \\ (s - \overline{u}(x))^{p-1} & \text{if } \overline{u}(x) < s \end{cases} \tag{7.7}$$

and $B\colon W_p \to W_p'$ via $\langle B\nu, \varphi\rangle := \int_\Omega b(\cdot,\nu)\varphi$. By standard arguments (see, e.g., [28]), B is readily seen to be bounded and completely continuous, and there are constants $b_1, b_2 > 0$ such that $\langle B\nu, \nu\rangle \geqslant b_1 |\nu|_p^p - b_2$ for all $\nu \in W_p$.

Now, for given $h \in L$, we consider the following auxiliary problem:

$$Au + g_\nu(\cdot,u) + Bu \ni h, \tag{$Q_h(\nu)$}$$

whose solutions are $u \in W_p$ such that there is some measurable selection $\eta \subset g_\nu(\cdot,u)$ for which it holds

$$\int_\Omega a(\cdot,\nabla u)\nabla\varphi + \int_\Omega \eta\varphi + \int_\Omega b(\cdot,u)\varphi = \int_\Omega h\varphi \quad \text{for all } \varphi \in W_p. \tag{7.8}$$

By Proposition 7.8 there is a solution of Problem $(Q_h(\nu))$. Furthermore, from Lemma 6.12 and Proposition 6.21 we deduce the following a priori bound:

7.9 Proposition *Let $\nu \in [\underline{u},\overline{u}]_{L^0}$ and $c > 0$ be given, and let u be a solution of $(Q_h(\nu))$ for some $h \in L$ with $\|h\|_{L^1} \leqslant c$. Then for each $r \in [1,(p-1)1^*)$ there is a constant $c_r > 0$ depending only on c and the data of the problem, but neither on ν nor on u or h, such that $\|u\|_{W_r} \leqslant c_r$.* ○

To establish the existence of solutions of $(P(\nu))$, one takes a sequence $(h_n) \subset L$ that approximates μ in $L^1(\Omega)$, chooses solutions u_n of $(Q_{h_n}(\nu))$ and shows, via the crucial compactness result Lemma 6.15, the following variant of Theorem 6.22:

7.10 Proposition *Let $\nu \in [\underline{u},\overline{u}]_{L^0}$ be given, let $(h_n) \subset L$ be a sequence that approximates μ in $L^1(\Omega)$, and let u_n be a solution of $(Q_{h_n}(\nu))$ with corresponding selection $\eta_n \subset g(\cdot,u_n)$. Then there are a solution u of $(P(\nu))$ located in $[\underline{u},\overline{u}]$ and a corresponding measurable selection $\eta \subset f(\cdot,u,\nu)$ such that, for a sequence (n_k), $u_{n_k} \to u$ in W and $\eta_{n_k} \rightharpoonup \eta$ in L.* ○

7.11 Remark Because of the low regularity of the right-hand side μ, one can only show that the limit u belongs to W. If μ has higher regularity, one can show that also u has higher regularity, but if μ does not belong to W_p', u will not belong to W_p, in general. ○

7.12 Remark The mentioned crucial compactness result Lemma 6.15, which is also an important tool in proving Proposition 7.16 below, only states that for a specified sequence (u_n) there is $u \in W$ such that, for every $r \in [1,(p-1)1^*)$, there is a subsequence of (u_n) that converges weakly in W_r to u. However, it follows by the convergence principle that the subsequence can be chosen independently of r, meaning that there is $u \in W$ and a subsequence of (u_n) which converges weakly to u in W_r, $r \in [1,(p-1)1^*)$, and thus in W. (See Proposition 2.28.) ○

7.13 Remark In the proof of Proposition 7.10 one has to show that the limit of the auxiliary solutions lies in the interval $[\underline{u}, \overline{u}]$ (by which it follows that $g(\cdot, u) = f(\cdot, u, v)$ and $d(\cdot, u) = 0$). To this end, one chooses special test functions in (7.8) which only belong to W_p if also $\underline{u}$ and $\overline{u}$ belong to W_p. This explains why sub-supersolutions have by definition a higher regularity than solutions. ○

Proposition 7.10 motivates the following notion:

7.14 Definition Let $v \in [\underline{u}, \overline{u}]$ be given. Then a function $u \in W$ is called **limit-solution of** $(P(v))$ if there are $(h_n) \subset L$ and solutions u_n of $(Q_{h_n}(v))$ such that $h_n \to \mu$ in L^1 and $u_n \to u$ in W. ○

From Propositions 7.8 and 7.10 we conclude the following:

7.15 Corollary *Let $v \in [\underline{u}, \overline{u}]$ be given. Then there is at least one limit-solution of $(P(v))$ and each limit-solution of $(P(v))$ is a solution of $(P(v))$ located in $[\underline{u}, \overline{u}]$.* ○

Further, we have the following compactness result, which can be proven essentially like Lemma 6.24 via a diagonal argument.

7.16 Proposition *Let $v \in [\underline{u}, \overline{u}]$ be given. Then the set $\mathcal{S}(v)$ of limit-solutions with respect to v is sequentially compact in W.* ○

In this chapter, let us make use of a slightly more general version of Theorem 2.33. There, both V and W were assumed to be reflexive ordered Banach spaces such that $W \subset V$ as ordered sets. However, by inspecting the proof, we see that the assertions hold also for W as defined in this chapter as the intersection of W_r, $r \in [1,(p-1)1^*)$. One only has to use that each weakly convergent subsequence of an increasing sequence $(u_n) \subset W$ converges weakly against the supremum of (u_n), and that each bounded sequence $(u_n) \subset W$ has a subsequence converging weakly in W. (Those properties follow readily from the corresponding properties of the spaces W_r.) Thus, we have the following theorem (with the notions of this chapter):

7.17 Theorem *Let $S\colon D \subset V \to \mathcal{P}(D \cap W)$ and $\underline{S}\colon D \to \mathcal{P}(V)$ be multifunctions such that the following conditions are satisfied:*

(i) D is a lattice, bounded and weakly sequentially closed in V, and there is $\underline{u} \in D$ such that $\underline{u} \leqslant^* \underline{S}(\underline{u})$.

(ii) $S(D)$ is bounded in W and S has weakly sequentially closed values in W.

(iii) $\underline{S}$ is permanent upward, its values are directed upward, and for all $v \in D$ it holds $S(v) \subset \underline{S}(v) \leqslant^* S(v)$.

Then S has a greatest fixed point u^* and it holds $\underline{u} \leqslant u^*$. ○

7.18 Remark It is readily seen that W is a *Fréchet space* with topology defined by the norms $\|\cdot\|_{W_r}$. For such spaces, a general theory holds true, and it might be interesting to formulate Theorem 7.17 in this setting or an even more general one. ○

7.4 Abstract Formulation

In this section, we are going to show that Problem (P), as defined in Section 7.3, has, under the conditions provided in Section 7.2, a solution u in the interval $[\underline{u}, \overline{u}]$ generated by the given sub-supersolutions. That is, there is $u \in [\underline{u}, \overline{u}]_W$ and a corresponding measurable selection $\eta \subset f(\cdot, u, u)$ such that

$$\int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega \eta\varphi = \int_\Omega \mu\varphi \quad \text{for all } \varphi \in W_0^{1,r}(\Omega), \quad r > N.$$

Inspired by the notions so far, the straightforward approach to show existence of at least one solution would be to define operators S and $\underline{S}$ such that fixed points of S coincide with solutions of (P) and such that fixed points of $\underline{S}$ are subsolutions of (P). Unfortunately, we face the problem that solutions of (P) may be less regular than sub-supersolutions, so that an inclusion $S(v) \subset \underline{S}(v)$ is not guaranteed to hold. Further, as far as we know, not even $S(v) \leqslant^* \underline{S}(v)$ holds (which would be a good enough replacement). This is the reason why we restrict our considerations to the following kind of limit-solutions of (P):

7.19 Definition A function $u \in W$ is called **limit-solution of** (P) if there are $(h_n) \subset L$ and solutions u_n of $(Q_{h_n}(u))$ such that $h_n \to \mu$ in L^1 and $u_n \to u$ in W. ○

This means, that a limit-solution u of (P) is a limit-solution of $(\mathsf{P}(u))$, and since the latter is known to be a solution of $(\mathsf{P}(u))$, we obtain that u is a solution of (P).

In the following we are going to show that (P) has not only a limit-solution, but even that there is a greatest limit-solution u^* of (P) located in $[\underline{u}, \overline{u}]$, meaning that for each limit-solution u' of (P) in $[\underline{u}, \overline{u}]$ one has $u' \leqslant u^*$.

The key idea is to generalize the notion of limit-solutions to the notion of limit-subsolutions. To this end, for given $h \in L$, we consider **subsolutions** of Problem $(Q_h(v))$, which are functions $u \in W_p$ such that there is some measurable selection $\eta \subset g_v(\cdot, u)$ for which it holds

$$\int_\Omega a(\cdot, \nabla u)\nabla\varphi + \int_\Omega \eta\varphi + \int_\Omega b(\cdot, u)\varphi \leqslant \int_\Omega h\varphi \quad \text{for all } \varphi \in (W_p)_+.$$

Then, we proceed analogously to Definition 7.14:

7.20 Definition Let $v \in [\underline{u}, \overline{u}]$ be given. Then a function $u \in W$ is called **limit-subsolution of** $(\mathsf{P}(v))$ if there are $(h_n) \subset L$ and subsolutions u_n of $(Q_{h_n}(v))$ such that $h_n \to \mu$ in L^1 and $u_n \to u$ in W. ○

A few comments are in order:

(i) Each limit-solution of $(P(v))$ is a limit-subsolution of $(P(v))$. Thus we know by Corollary 7.15 that limit-subsolutions exists. We only have to assume the existence of one ordered pair $(\underline{u}, \overline{u})$ of sub-supersolutions of (P) (see Condition (S)).

(ii) We don't have a convergence result about a sequence $(\underline{u}_n)$ of subsolutions $\underline{u}_n$ of $(Q_{h_n}(v))$ like Proposition 7.10. Thus, it is difficult to construct limit-subsolutions per hand. However, in the sequel there is no need for such a construction.

(iii) Due to regularity issues, a limit-subsolution of $(P(v))$ is in general no subsolution of $(P(v))$.

The concept of limit-subsolutions is a theoretical one that we use to bring Problem (P) in a form that goes well with Theorem 7.17. The set $D \subset V$ is given by $D := [\underline{u}, \overline{u}]_V$ and the operators S and $\underline{S}$ are defined in the following simple way:

$$S\colon [\underline{u}, \overline{u}]_V \to \mathcal{P}([\underline{u}, \overline{u}]_W), \quad v \mapsto \{u \in [\underline{u}, \overline{u}]_W : u \text{ is a limit-solution of } (P(v))\}$$
$$\underline{S}\colon [\underline{u}, \overline{u}]_V \to \mathcal{P}([\underline{u}, \overline{u}]_W), \quad v \mapsto \{u \in [\underline{u}, \overline{u}]_W : u \text{ is a limit-subsolution of } (P(v))\}.$$

Obviously, the fixed points $u \in S(u)$ of S are exactly the limit-solutions of Problem (P) which are located in $[\underline{u}, \overline{u}]_W$. Thus, with help of Theorem 7.17, we can prove our main existence theorem, which will be done in the next section.

7.5 Existence of Solutions

In order to apply Theorem 7.17, let us first provide some properties of S and $\underline{S}$. To this end, we use results for the auxiliary problems $(Q_h(v))$ and ideas presented in [28]. However, due to the new notions of limit-solution and limit-subsolution, a careful treatment is needed.

7.21 Proposition *The operator* $S\colon [\underline{u}, \overline{u}]_V \to \mathcal{P}(W)$ *is uniformly bounded.*

Proof: Let $v \in [\underline{u}, \overline{u}]_V$ be arbitrary and take any $u \in S(v)$. Then, by definition of limit-solutions, there are a sequence $(h_n) \subset L$ and solutions u_n of $(Q_{h_n}(v))$ such that $h_n \to \mu$ in L^1 and $u_n \to u$ in W. W.l.o.g. we can assume $\|h_n\|_{L^1} \leqslant \|\mu\|_{L^1} + 1$. Now, by Proposition 7.9 there is, for each $r \in [1, (p-1)1^*)$, a constant $c_r > 0$ (independent of v, u_n and h_n) such that $\|u_n\|_{W_r} \leqslant c_r$. Since $u_n \to u$ in W_r, also $\|u\|_{W_r} \leqslant c_r$. ○

7.22 Proposition *The operator* $S\colon [\underline{u}, \overline{u}]_V \to \mathcal{P}(W)$ *has weakly sequentially closed values.*

Proof: From Proposition 7.16 we know that the values of S are even sequentially compact in W. ○

7.23 Proposition *The operator* $\underline{S}\colon [\underline{u}, \overline{u}]_V \to \mathcal{P}(W)$ *is permanent upward.*

Proof: Let $v, v' \in [\underline{u}, \overline{u}]_V$ be given, suppose $v \leqslant v'$ and take any $u \in \underline{S}(v)$. Then there are, according to Definition 7.20, sequences $(u_n) \subset W_p$, $(\eta_n) \subset L$ and $(h_n) \subset L$ such that $u_n \to u$ in W, $\eta_n \subset g_v(\cdot, u_n)$, $h_n \to \mu$ in L^1, and

$$\int_\Omega a(\cdot, \nabla u_n)\nabla\varphi + \int_\Omega \eta_n \varphi + \int_\Omega b(\cdot, u_n)\varphi \leqslant \int_\Omega h_n \varphi \quad \text{for all } \varphi \in (W_p)_+. \tag{7.9}$$

Let, for every n, $\eta'_n \subset g_{v'}(\cdot, u_n)$ be arbitrary given and set $\eta''_n := \eta_n \wedge \eta'_n$. We claim that $\eta''_n \subset g_{v'}(\cdot, u_n)$. To this end, recall first that for a.e. $x \in \Omega$ we have

$$\eta_n(x) \in g_v(x, u_n(x)), \quad \eta'_n(x) \in g_{v'}(x, u_n(x)) \quad \text{and} \quad v(x) \leqslant v'(x).$$

Due to (F3), f is decreasing in the last argument and thus there is some $\alpha \in g_{v'}(x, u_n(x))$ such that $\alpha \leqslant \eta_n(x)$. This implies

$$\alpha \wedge \eta'_n(x) \leqslant \eta_n(x) \wedge \eta'_n(x) \leqslant \eta'_n(x).$$

Since the values of $g_{v'}$ are closed real intervals, it follows $\eta''_n(x) \in g_{v'}(x, u_n(x))$, i.e. $\eta''_n \subset g_{v'}(\cdot, u_n)$.

Furthermore, we have

$$\int_\Omega \eta''_n \varphi \leqslant \int_\Omega \eta_n \varphi, \quad \text{for all } \varphi \in (W_p)_+,$$

and thus, from (7.9),

$$\int_\Omega a(\cdot, \nabla u_n)\nabla\varphi + \int_\Omega \eta''_n \varphi + \int_\Omega b(\cdot, u_n)\varphi \leqslant \int_\Omega h_n \varphi \quad \text{for all } \varphi \in (W_p)_+.$$

Consequently, u_n is a subsolution of $(Q_{h_n}(v'))$, and $u \in \underline{S}(v')$. ○

7.24 Proposition *Let $v \in [\underline{u}, \overline{u}]_V$ be arbitrary, and let $u_i \in \underline{S}(v)$, $i = 1, 2$, be given. Then there is $u \in S(v)$ such that $u_1 \vee u_2 =: u_3 \leqslant u$.*

Proof: **Step 1: Setting** According to Definition 7.20 there are, for $i = 1, 2$, sequences $(u_{n,i}) \subset W_p$, $(\eta_{n,i}) \subset L$ and $(h_{n,i}) \subset L$ such that $u_{n,i} \to u_i$ in W, $\eta_{n,i} \subset g_v(\cdot, u_{n,i})$, $h_{n,i} \to \mu$ in L^1, and

$$\int_\Omega a(\cdot, \nabla u_{n,i})\nabla\varphi + \int_\Omega \eta_{n,i}\varphi + \int_\Omega b(\cdot, u_{n,i})\varphi \leqslant \int_\Omega h_{n,i}\varphi \quad \text{for all } \varphi \in (W_p)_+. \tag{7.10}$$

Since

$$\int_\Omega h_{n,i}\varphi \leqslant \int_\Omega (h_{n,1} \vee h_{n,2})\varphi \quad \text{for all } \varphi \in (W_p)_+,$$

$h_{n,1} \vee h_{n,2} \in L$ and $h_{n,1} \vee h_{n,2} \to \mu$ in L^1, we can assume $h_{n,1} = h_{n,2} =: h_n$ so that $u_{n,i}$ is a subsolution of $(Q_{h_n}(v))$, $i = 1, 2$.

Further, set $u_{n,1} \vee u_{n,2} := u_{n,3} \in W_p$ and let us define the measurable selections $\eta_{n,3} \subset g_\nu(\cdot, u_{n,3})$ by

$$\eta_{n,3}(x) := \begin{cases} \eta_{n,1}(x) & \text{if } u_{n,1}(x) \geqslant u_{n,2}(x), \\ \eta_{n,2}(x) & \text{if } u_{n,1}(x) < u_{n,2}(x). \end{cases}$$

In the following, we are going to prove that there is, for each n, some solution (u_n) of $(Q_{h_n}(\nu))$ such that $u_{n,3} \leqslant u_n$, and it will follow that (u_n) generates the desired limit-solution of $(P(\nu))$.

Step 2: Auxiliary Problem In order to define the sequence (u_n), we rely on Proposition 7.8 and appropriately adapted mappings g and B.

Concerning g, recall the defining equation (7.6) of g_ν and set

$$g_{\nu,n} \colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R}), \quad (x, s) \mapsto \begin{cases} g_\nu(x, s) & \text{if } s \geqslant u_{n,3}(x), \\ \{\eta_{n,3}(x)\} & \text{if } s < u_{n,3}(x). \end{cases} \tag{7.11}$$

Then $g_{\nu,n}$ has closed and convex values, is upper Carathéodory like g_ν, and all its measurable selections are majorized in L.

Further, recall the notation $\big[(x_1, y_1) \rightsquigarrow (x_2, y_2)\big]$ introduced at Page 121, and define functions $\theta_{n,i} \colon \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, by

$$\theta_{n,i}(x, \cdot) := \Big[\big(u_{n,i}(x), \eta_{n,3}(x) - \eta_{n,i}(x)\big) \rightsquigarrow \big(u_{n,3}(x), 0\big)\Big].$$

By this construction, for all $u \in L^0$ it holds, for $i = 1, 2$,

$$\eta_{n,3} - \eta_{n,i} - |\theta_{n,1}(\cdot, u)| - |\theta_{n,2}(\cdot, u)| \leqslant 0 \quad \text{on } \{x \in \Omega : u(x) < u_{n,i}(x)\}. \tag{7.12}$$

Furthermore, it is easy to check that $\theta_{n,i}$ is measurable in the first argument, continuous in the second, and bounded in L. Thus, the function

$$g := g_{\nu,n} - |\theta_{n,1}| - |\theta_{n,2}| \tag{7.13}$$

fulfills the requirements of Proposition 7.8.

Concerning B, let us define the cut-off function $b_n \colon \Omega \times \mathbb{R} \to \mathbb{R}$ analogously to (7.7), now with respect to $u_{n,3}$:

$$b_n(x, s) := -\left[(u_{n,3}(x) - s)^+\right]^{p-1} = \begin{cases} 0, & \text{if } s \geqslant u_{n,3}(x), \\ -(u_{n,3}(x) - s)^{p-1} & \text{if } s < u_{n,3}(x). \end{cases}$$

Clearly, b_n is a Carathéodory function, and it is known that it satisfies the growth conditions

$$|b_n(x, s)| \leqslant d_0 \left(|u_{n,3}(x)|^{p-1} + |s|^{p-1}\right), \qquad b_n(x, s)s \geqslant -d_0|u_{n,3}(x)|^p$$

for some constant $b_0 > 0$. Since we have the analogous growth conditions for b, defined in (7.7), the Nemtytskij operator $\nu \mapsto (b_n + b)(\cdot, \nu)$ is known to be continuous and bounded from L^p to its dual space. Thanks to the compact embedding $W_p \hookrightarrow L^p$ we conclude that the mapping

$$B\colon W_p \to W_p', \quad \text{defined via } \langle Bv, \varphi \rangle := \int_\Omega (b_n + b)(\cdot, v)\varphi,$$

is bounded and completely continuous. Due to the second growth condition, we have that $\langle Bv, v \rangle_{W_p}$ is bounded from below.

Consequently, from Proposition 7.8 we know that there are functions $u_n \in W_p$ and $\eta_n \subset g(\cdot, u_n)$ such that

$$\int_\Omega a(\cdot, \nabla u_n)\nabla\varphi + \int_\Omega \eta_n \varphi + \int_\Omega (b_n + b)(\cdot, u_n)\varphi = \int_\Omega h_n \varphi \quad \text{for all } \varphi \in W_p. \tag{7.14}$$

Step 3: Comparison In this step, we are going to show that $u_{n,3} \leqslant u_n$. To this end, take the special test function $\varphi = (u_{n,i} - u_n)^+ \in (W_p)_+$, $i = 1, 2$, in (7.10) and (7.14) and combine them to obtain

$$\begin{aligned} \int_\Omega (a(\cdot, \nabla u_n) - a(\cdot, \nabla u_{n,i}))\nabla(u_{n,i} - u_n)^+ + \int_\Omega (\eta_n - \eta_{n,i})(u_{n,i} - u_n)^+ + \\ \int_\Omega (b_n(\cdot, u_n) + b(\cdot, u_n) - b(\cdot, u_{n,i}))(u_{n,i} - u_n)^+ \geqslant 0. \end{aligned} \tag{7.15}$$

Next, let us estimate the terms of (7.15) separately: By (A4), we deduce

$$\int_\Omega (a(\cdot, \nabla u_n) - a(\cdot, \nabla u_{n,i}))\nabla(u_{n,i} - u_n)^+ \leqslant 0. \tag{7.16}$$

Further, on $\{x \in \Omega : u_n(x) < u_{n,i}(x)\}$ we have $u_n(x) < u_{n,3}(x)$ and thus it follows from (7.11) and (7.13)

$$\eta_n = \eta_{n,3} - |\theta_{n,1}(\cdot, u_n)| - |\theta_{n,2}(\cdot, u_n)|.$$

Consequently, it follows from (7.12)

$$\int_\Omega (\eta_n - \eta_{n,i})(u_{n,i} - u_n)^+ \leqslant 0. \tag{7.17}$$

Finally, note that $s \mapsto d(x, s)$ is increasing. Thus, it follows readily

$$\int_\Omega (b(\cdot, u_n) - b(\cdot, u_{n,i}))(u_{n,i} - u_n)^+ \leqslant 0. \tag{7.18}$$

Combining (7.15)—(7.18), we deduce

$$\int_\Omega b_n(\cdot, u_n)(u_{n,i} - u_n)^+ \geqslant 0. \tag{7.19}$$

By definition of b_n, (7.19) implies $\|(u_{n,i} - u)^+\|_p^p \leqslant 0$, which in turn implies $u_{n,i} \leqslant u$, $i = 1, 2$, and thus $u_{n,3} \leqslant u$.

Step 4: Conclusion Since $u_{n,3} \leqslant u_n$, we have $\eta_n \subset g_{\nu,n}(\cdot, u_n) = g_\nu(\cdot, u_n)$ and $b_n(\cdot, u_n) = 0$. Thus, from (7.14) we know that u_n is a solution of $(Q_{h_n}(\nu))$. Now, by Proposition 7.10 there is a solution u of $(P(\nu))$ located in $[\underline{u}, \overline{u}]_W$ such that $u_n \to u$ in W, and thus we have $u \in S(\nu)$. Since $u_{n,1} \vee u_{n,2} = u_{n,3} \leqslant u_n$, it follows further $u_1 \vee u_2 = u_3 \leqslant u$, which concludes the proof. ○

The results of Proposition 7.24 immediately imply the following corollary.

7.25 Corollary *The values of* $\underline{S}\colon [\underline{u}, \overline{u}]_V \to \mathcal{P}(W)$ *are directed upward and for all* $\nu \in [\underline{u}, \overline{u}]_V$ *it holds* $S(\nu) \subset \underline{S}(\nu) \leqslant^* S(\nu)$.

Finally, we are in the position to prove the main theorem of this chapter:

7.26 Theorem *Suppose (M), (A1)—(A4), (S) and (F1)—(F4). Then Problem* (P) *has the greatest limit-solution in* $[\underline{u}, \overline{u}]_V$.

Proof: By Corollary 7.15 both S and $\underline{S}$ are well-defined and we have $\underline{u} \leqslant^* \underline{S}(\underline{u})$. Further, $D = [\underline{u}, \overline{u}]_V \subset V$ is readily seen to be a bounded, weakly sequentially closed lattice. Thus, by Propositions 7.21, 7.22, 7.23 and Corollary 7.25, all conditions of Theorem 7.17 are fulfilled, and it follows that S has the greatest fixed point in D, which is the greatest limit-solution of (P) in $[\underline{u}, \overline{u}]_V$. ○

7.27 Remark By duality, one can introduce limit-supersolutions, which can be used to prove that Problem (P) has, under the same conditions, also a smallest limit-solution.

Note, however, that we do not need limit-supersolutions (but only a supersolution of (P)) to provide the existence of greatest limit-solutions. This shows that, in Theorem 7.17, D needs not to be defined with help of $\underline{S}$ (although there is a connection). Further, we have seen an application of Theorem 7.17 in which we have $\underline{u} \leqslant^* \underline{S}(\underline{u})$, but in general not $\underline{u} \in \underline{S}(\underline{u})$. This justifies the general formulation of Theorems 2.33 and 7.17. ○

8 Systems of Variational Inclusions

8.1 Introduction

In Chapter 4 we combined the method of sub-supersolutions with a multivalued order-theoretical fixed point theorem to obtain a general framework for the study of multivalued differential inclusions with nonsmooth bifunctions of the type

$$A(u) + F(u, u) \ni 0, \tag{8.1}$$

where $A \colon W \to W'$ is a differential operator on a Sobolev space W, and the elements of $F(u, u)$ are, roughly speaking, selections of some multivalued function $f(\cdot, u, u)$.

In Chapter 5 we generalized those results to multivalued quasi-variational inequalities of the form

$$A(u) + F(u, u) + \partial K_u(u) \ni 0 \quad \text{in } W'. \tag{8.2}$$

In (8.2), a nonsmooth term is given by ∂K_u, and in both inclusion (8.1) and (8.2) the multivalued term

$$F(u, u) = \{\eta : \eta \text{ is a measurable selection of } x \mapsto f(x, u(x), u(x)\}$$

is, in general, nonsmooth, because the multifunction $t \mapsto f(x, s, t)$ is only assumed to be decreasing. Further, $(x, s) \mapsto f(x, s, t)$ is only assumed to be upper Carathéodory, so that (8.1) and (8.2) cover a wide range of special problems, e.g. problems with Clarke's generalized gradient like variational-hemivariational inequalities.

In Chapter 7 we combined the ideas of Chapter 4 and 6 to treat nonsmooth elliptic inclusion with L^1-measure right-hand sides over a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$. To be more precise, we provided an existence and enclosure result for inclusions of the form

$$A(u) + F(u, u) \ni \mu \tag{8.3}$$

with $\mu \in L^1(\Omega)$.

Now, we are going to extend all those results to systems of elliptic inclusions, in which the single inclusions are coupled via the vector-valued multivalued operator $\mathbf{F}$. This results, e.g., in the problem of finding $u_i \in W$, $i = 1, 2$, such that

$$A_1(u_1) + F_1(u_1, u_1, u_2) \ni 0, \tag{8.4$_1$}$$

$$A_2(u_2) + F_2(u_2, u_1, u_2) \ni 0, \tag{8.4$_2$}$$

where $A_i \colon W \to W'$ are differential operators on a Sobolev space W, and the elements of $F_i(u_i, u_1, u_2)$ are selections of $f_i(\cdot, u_i, u_1, u_2)$. The leading terms $A_i(u_i)$ depend only on u_i (which means that we have there a diagonal structure), and the coupling of the inclusions is established via the multivalued terms $F_i(u_i, u_1, u_2)$. Like in the scalar case, the multifunction $(t_1, t_2) \mapsto f_i(x, s, t_1, t_2)$ is only assumed to be decreasing, whereas $(x, s) \mapsto f_i(x, s, t_1, t_2)$ has to be upper Carathéodory.

The idea for solving such systems is to fix the last two arguments in $F_i(u_i, u_1, u_2)$ and to find fixed points of the operator $\mathbf{S}\colon W \times W \to \mathcal{P}(W \times W)$ which maps each pair (v_1, v_2) to solutions (u_1, u_2) of the decoupled inclusions

$$A_1(u_1) + F_1(u_1, v_1, v_2) \ni 0,$$
$$A_2(u_2) + F_2(u_2, v_1, v_2) \ni 0.$$

Clearly, a fixed point of the multifunction $\mathbf{S}$, i.e. a pair $\mathbf{u} = (u_1, u_2)$ such that $\mathbf{u} \in \mathbf{S}(\mathbf{u})$, is a solution of System (8.4). In order to solve those fixed point problems, we are going to apply the basic framework of Theorem 2.33 to the special case of product spaces and vector-valued multifunctions.

The rest of this chapter is organized as follows: First, we recall some definitions and the basic framework—and extend it to systems. Second, we present the application of the new framework to a general model problem and to systems of inclusions like (8.2) and (8.3). Many more applications are possible whenever the method of sub-supersolutions applies.

8.2 Framework for Systems

Recall the general framework given in Theorem 2.33:

Theorem *Let V and W be reflexive ordered Banach spaces such that $W \subset V$ as posets, and let $S\colon D \subset V \to \mathcal{P}_\emptyset(D \cap W)$ and $\underline{S}\colon D \to \mathcal{P}_\emptyset(V)$ be multifunctions such that the following conditions are satisfied:*

- *(i) D is a lattice, bounded and weakly sequentially closed in V, and there is $\underline{u} \in D$ such that $\underline{u} \leqslant^* \underline{S}(\underline{u})$.*
- *(ii) $S(D)$ is bounded in W and S has weakly sequentially closed values in W.*
- *(iii) $\underline{S}$ is permanent upward, its values are directed upward (i.e. for all $a, b \in \underline{S}(v)$ there is $c \in \underline{S}(v)$ such that $a, b \leqslant c$) and for all $v \in D$ it holds $S(v) \subset \underline{S}(v) \leqslant^* S(v)$.*

Then S has the greatest fixed point u^* and it holds $\underline{u} \leqslant u^*$.

In order to solve systems of differential inclusions, we are going to apply this basic framework to the special case of product spaces and vector-valued multifunctions. Since there is no need to restrict our considerations on just two equations, let us introduce and note the following basics. In what follows, the index i will always range between 1 and a natural number n.

(i) For sets M_i let $\prod_i M_i$ denote the usual Cartesian product. We will take the bold letters $\boldsymbol{M}$ and $\mathbf{u}$ to denote $\prod_i M_i$ and its elements $(u_1, \ldots, u_n)$, respectively.

(ii) If D_i are posets, $\mathbf{D} = \prod_i D_i$ is partially ordered via $\mathbf{u} \leqslant \mathbf{v}$ if $u_i \leqslant v_i$ for all i. If all D_i are lattices, then so is $\mathbf{D}$.

(iii) If V_i are reflexive Banach spaces, $\mathbf{V} = \prod_i V_i$ is a reflexive Banach space with norm given by $\|\mathbf{u}\| = \sum_i \|u_i\|_{V_i}$. Further, it is known that

$$J\colon \prod_i V_i' \to \mathbf{V}', \quad \boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \mapsto [\mathbf{u} \mapsto \sum_i \langle \eta_i, u_i \rangle].$$

is an isometric isomorphism, where $\langle \eta_i, u_i \rangle$ denotes the duality pairing on (V_i', V_i).

(iv) For multifunctions $F_i\colon \mathbf{V} \to \mathcal{P}_\emptyset(W_i)$, we use the following product-function:

$$F = \prod_i F_i\colon \mathbf{V} \to \mathcal{P}_\emptyset(\mathbf{W}), \quad \mathbf{u} \mapsto \prod_i F_i(\mathbf{u}).$$

Now, from Theorem 2.33 we deduce the following theorem as a special case (since all relevant topological and order-theoretical properties used there are inherited to Cartesian products).

8.1 Theorem *Let V_i and W_i, $i = 1, \dots, n$, be reflexive ordered Banach spaces such that $W_i \subset V_i$ as posets, and let $D_i \subset V_i$ be lattices which are bounded and weakly sequentially closed in V_i. Suppose further that, for all i, the multivalued mappings*

$$S_i\colon \mathbf{D} \to \mathcal{P}_\emptyset(D_i \cap W_i) \quad \textit{and} \quad \underline{S}_i\colon \mathbf{D} \to \mathcal{P}_\emptyset(V_i)$$

satisfy the following conditions:

(i) There is $\underline{\mathbf{u}} \in \mathbf{D}$ such that $\underline{u}_i \leqslant^* \underline{S}_i(\underline{\mathbf{u}})$.

(ii) $S_i(\mathbf{D})$ is bounded in W_i and S_i has weakly sequentially closed values in W_i.

(iii) $\underline{S}_i$ is increasing upward, its values are directed upward and for all $\mathbf{v} \in \mathbf{D}$ it holds $S_i(\mathbf{v}) \subset \underline{S}_i(\mathbf{v}) \leqslant^* S_i(\mathbf{v})$.

Then the multifunction $\mathbf{S}\colon \mathbf{D} \to \mathcal{P}_\emptyset(\mathbf{D} \cap \mathbf{W})$ has the greatest fixed point $\mathbf{u}^*$ and it holds $\underline{\mathbf{u}} \leqslant \mathbf{u}^*$.

Proof: We are going to apply Theorem 2.33. As noted before, $\mathbf{V}$ and $\mathbf{W}$ are ordered reflexive Banach spaces, and we have $\mathbf{V} \subset \mathbf{W}$ as ordered sets. Further, $\mathbf{D} \subset \mathbf{V}$ is readily seen to be a lattice which is bounded and weakly sequentially closed in $\mathbf{V}$. Further, from (i) it follows $\underline{\mathbf{u}} \leqslant^* \underline{\mathbf{S}}(\underline{\mathbf{u}})$, and from (ii) it follows that $\mathbf{S}(\mathbf{D})$ is uniformly bounded in $\mathbf{W}$ and that $\mathbf{S}$ has weakly sequentially closed values in $\mathbf{W}$. Finally, we conclude easily from (iii) that $\underline{\mathbf{S}}$ is increasing upward, that the values of $\underline{\mathbf{S}}$ are directed upward, and $\mathbf{S}(\mathbf{v}) \subset \underline{\mathbf{S}}(\mathbf{v}) \leqslant^* \mathbf{S}(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{D}$. Thus, all assumptions of Theorem 2.33 hold true and thus $\mathbf{S}$ has a greatest fixed point $\mathbf{u}^*$ and it holds $\underline{\mathbf{u}} \leqslant \mathbf{u}^*$. ○

In Theorem 8.1 we considered the case in which the suboperator $\underline{S}$ and thus, due to the compatibility condition (iii), the main operator S are increasing upward. The question arises, if there are similar fixed point results for operators which are increasing downward, decreasing or mixed-monotone. In the following, we will give some partial answers to this question.

First, let us formulate the dual assertion to Theorem 8.1 for increasing downward operators. That is, let us replace the partial order $\leqslant$ with its dual partial order $\geqslant$ and let us introduce the super-operator $\overline{S}$ to obtain the following corollary:

8.2 Corollary *Let* D_i, V_i *and* W_i *given as in Theorem 8.1 and suppose that the mappings* $S_i\colon \mathbf{D} \to \mathcal{P}_\emptyset(D_i \cap W_i)$ *and* $\overline{S}_i\colon \mathbf{D} \to \mathcal{P}_\emptyset(V_i)$ *satisfy the following conditions:*

(i) *There is* $\overline{\mathbf{u}} \in \mathbf{D}$ *such that* $\overline{S}_i(\overline{\mathbf{u}}) \leqslant_* \overline{u}_i$.

(ii) $S_i(\mathbf{D})$ *is bounded in* W_i *and* S_i *has weakly sequentially closed values in* W_i.

(iii) $\overline{S}_i$ *is increasing downward, its values are directed downward, and for all* $\mathbf{v} \in \mathbf{D}$ *it holds* $S_i(\mathbf{v}) \leqslant_* \overline{S}_i(\mathbf{v}) \supset S_i(\mathbf{v})$.

Then $\mathbf{S}$ *has the smallest fixed point* $\mathbf{u}_*$ *and it holds* $\mathbf{u}_* \leqslant \overline{\mathbf{u}}$. ○

In the applications below, we will use Corollary 8.2 together with Theorem 8.1 and the set $D = [\underline{\mathbf{u}}, \overline{\mathbf{u}}]$. Therefore, we will obtain smallest and greatest solutions within the interval $[\underline{\mathbf{u}}, \overline{\mathbf{u}}]$.

Note, however, that all operators $\underline{S}_i$ are increasing upward, and all operators $\overline{S}_i$ are increasing downward (from which it follows that all operators S_i are increasing). If we dualize the order only in some spaces V_i, we obtain the following more general result, which allows to treat some mixed-monotone systems.

8.3 Corollary *Let* D_i, V_i *and* W_i *given as in Theorem 8.1 and suppose that the mappings* $S_i\colon \mathbf{D} \to \mathcal{P}_\emptyset(D_i \cap W_i)$ *and* $\underline{S}_i, \overline{S}_i\colon \mathbf{D} \to \mathcal{P}_\emptyset(V_i)$ *satisfy the following conditions with respect to some index* $r \geqslant 0$*:*

(i) *There are* $\underline{\mathbf{u}}, \overline{\mathbf{u}} \in \mathbf{D}$ *such that*

$$\begin{aligned} \underline{u}_i &\leqslant^* \underline{S}_i(\underline{\mathbf{v}}) \quad \textit{and} \quad \overline{S}_i(\overline{\mathbf{v}}) \leqslant_* \overline{u}_i \quad \textit{if } i \leqslant r, \\ \underline{u}_i &\leqslant^* \underline{S}_i(\overline{\mathbf{v}}) \quad \textit{and} \quad \overline{S}_i(\underline{\mathbf{v}}) \leqslant_* \overline{u}_i \quad \textit{if } i > r, \end{aligned}$$

where $\underline{\mathbf{v}}, \overline{\mathbf{v}} \in \mathbf{D}$ *are given by*

$$\begin{aligned} \underline{v}_i &:= \underline{u}_i \quad \textit{and} \quad \overline{v}_i := \overline{u}_i \quad \textit{if } i \leqslant r, \\ \underline{v}_i &:= \overline{u}_i \quad \textit{and} \quad \overline{v}_i := \underline{u}_i \quad \textit{if } i > r. \end{aligned}$$

(ii) $S_i(\mathbf{D})$ *is bounded in* W_i *and* S_i *has weakly sequentially closed values in* W_i.

(iii) *If* $i \leqslant r$, $\underline{S}_i$ *is increasing upward in the first* r *arguments and decreasing downward in the other ones. If* $i > r$, $\underline{S}_i$ *is decreasing downward in the first* r *arguments and increasing upward in the other ones. In both cases, the values of* $\underline{S}_i$ *are directed upward and for all* $\mathbf{v} \in \mathbf{D}$ *it holds* $S_i(\mathbf{v}) \subset \underline{S}_i(\mathbf{v}) \leqslant^* S_i(\mathbf{v})$.

(iv) If $i \leqslant r$, $\overline{S}_i$ is increasing downward in the first r arguments and decreasing upward in the other ones. If $i > r$, $\overline{S}_i$ is decreasing upward in the first r arguments and increasing downward in the other ones. In both cases, the values of $\overline{S}_i$ are directed downward and for all $\boldsymbol{v} \in \mathbf{D}$ it holds $S_i(\boldsymbol{v}) \leqslant_* \overline{S}_i(\boldsymbol{v}) \supset S_i(\boldsymbol{v})$.

Then $\mathbf{S}$ has fixed points $\mathbf{u}_*$ and $\mathbf{u}^*$ that are extremal in the sense that for all fixed points $\mathbf{u}$ of $\mathbf{S}$ it holds

$$u_{*i} \leqslant u_i \leqslant u_i^* \quad \text{if } i \leqslant r, \qquad u_i^* \leqslant u_i \leqslant u_{*i} \quad \text{if } i > r.$$

In particular, $\mathbf{u}$ is located in $[\mathbf{u}_* \wedge \mathbf{u}^*, \mathbf{u}_* \vee \mathbf{u}^*]$.

Proof: We are going to apply both Theorem 8.1 and Corollary 8.2. To this end, let us define posets

$$\mathcal{V}_i := (V_i, \leqslant) \quad \text{if } i \leqslant r, \qquad \mathcal{V}_i := (V_i, \geqslant) \quad \text{if } i > r.$$

The subsets $\mathcal{W}_i$ and $\mathcal{D}_i$ of $\mathcal{V}_i$ are defined analogously. Further, let us define operators $\underline{T}_i, \overline{T}_i \colon \boldsymbol{\mathcal{D}} \to \mathcal{P}(\mathcal{V}_i)$ via

$$\underline{T}_i := \underline{S}_i \quad \text{and} \quad \overline{T}_i := \overline{S}_i \quad \text{if } i \leqslant r,$$
$$\underline{T}_i := \overline{S}_i \quad \text{and} \quad \overline{T}_i := \underline{S}_i \quad \text{if } i > r.$$

Then it is readily seen that all conditions of Theorem 8.1 and Corollary 8.2 hold true (where we use the sets $\mathcal{D}_i$, $\mathcal{V}_i$ and $\mathcal{W}_i$ in place of D_i, V_i and W_i, the operators $\underline{T}$ and $\overline{T}$ in place of $\underline{S}$ and $\overline{S}$, and the sub-supersolutions $\underline{v}$ and $\overline{v}$ in place of $\underline{u}$ and $\overline{u}$, respectively).

Indeed, first, from (i) we obtain

$$\underline{v}_i \leqslant^* \underline{S}_i(\underline{v}) = \underline{T}_i(\underline{v}) \quad \text{and} \quad \overline{T}_i(\overline{v}) = \overline{S}_i(\overline{v}) \leqslant_* \overline{v}_i \quad \text{if } i \leqslant r,$$
$$\underline{v}_i \geqslant^* \overline{S}_i(\underline{v}) = \underline{T}_i(\underline{v}) \quad \text{and} \quad \overline{T}_i(\overline{v}) = \underline{S}_i(\overline{v}) \geqslant_* \overline{v}_i \quad \text{if } i > r.$$

Second, the topological properties are independent of the chosen partial order, so $S_i(\boldsymbol{\mathcal{D}})$ is bounded in $\mathcal{W}_i$ and S_i has sequentially closed values.

Third, $\underline{T}_i$ is increasing upward with respect to the given order. To see this, let $\boldsymbol{v}, \boldsymbol{v}' \in \boldsymbol{\mathcal{D}}$ be given such that they only differ in their j-th component, and suppose $v_j \leqslant v_j'$ if $j \leqslant r$ and $v_j \geqslant v_j'$ if $j \geqslant r$. Then it follows from (iii) and (iv)

$$\underline{T}_i(\boldsymbol{v}) = \underline{S}_i(\boldsymbol{v}) \leqslant^* \underline{S}_i(\boldsymbol{v}') = \underline{T}_i(\boldsymbol{v}') \quad \text{if } i \leqslant r,$$
$$\underline{T}_i(\boldsymbol{v}) = \overline{S}_i(\boldsymbol{v}) \geqslant^* \overline{S}_i(\boldsymbol{v}') = \underline{T}_i(\boldsymbol{v}') \quad \text{if } i > r.$$

Since both $\leqslant^*$ and $\geqslant^*$ are transitive, the claim follows. Analogously, one shows that $\overline{T}_i$ is increasing downward: Let $\boldsymbol{v}, \boldsymbol{v}' \in \boldsymbol{\mathcal{D}}$ be given such that they only differ in their j-th component, and suppose $v_j' \leqslant v_j$ if $j \leqslant r$ and $v_j' \geqslant v_j$ if $j \geqslant r$. Then

$$\overline{T}_i(\boldsymbol{v}) = \overline{S}_i(\boldsymbol{v}) \geqslant^* \overline{S}_i(\boldsymbol{v}') = \overline{T}_i(\boldsymbol{v}') \quad \text{if } i \leqslant r,$$
$$\overline{T}_i(\boldsymbol{v}) = \underline{S}_i(\boldsymbol{v}) \leqslant^* \underline{S}_i(\boldsymbol{v}') = \overline{T}_i(\boldsymbol{v}') \quad \text{if } i > r.$$

Fourth, the values of $\underline{T}_i$ are directed upward and that of $\overline{T}_i$ are directed downward and it holds

$$\underline{T}_i(\boldsymbol{v}) = \underline{S}_i(\boldsymbol{v}) \leqslant^* S_i(\boldsymbol{v}) \quad \text{and} \quad S_i(\boldsymbol{v}) \leqslant_* \overline{S}_i(\boldsymbol{v}) = \overline{T}_i(\boldsymbol{v}) \quad \text{if } i \leqslant r,$$
$$\underline{T}_i(\boldsymbol{v}) = \overline{S}_i(\boldsymbol{v}) \geqslant^* S_i(\boldsymbol{v}) \quad \text{and} \quad S_i(v) \geqslant_* \underline{S}_i(\boldsymbol{v}) = \overline{T}_i(\boldsymbol{v}) \quad \text{if } i > r.$$

All in all, all conditions of Theorem 8.1 and Corollary 8.2 are fulfilled. Thus it follows that $\mathbf{S}\colon \mathcal{D} \to \mathcal{P}_\emptyset(\mathcal{D} \cap \mathcal{W})$ has both a smallest fixed point $\boldsymbol{u}_*$ and a greatest one $\boldsymbol{u}^*$ with respect to the order in $\mathcal{V}$. That means, for every fixed point $\boldsymbol{u}$ of $\mathbf{S}$ we have $u_{*i} \leqslant u_i \leqslant u_i^*$ if $i \leqslant r$, and $u_i^* \leqslant u_i \leqslant u_{*i}$ if $i > r$. ○

8.4 Remark Due to the compatibility condition, the operator S_i has the same monotonicity properties as $\underline{S}_i$ and $\overline{S}_i$. Due to the symmetry in conditions (iii) and (iv), $v_j \mapsto S_i(\boldsymbol{v})$ is either increasing or decreasing. One can represent the monotonicity of the operators S_i by a matrix $A = (a_{ij}) \in \{0,1\}^{n \times n}$ that is interpreted in such a way that S_i is increasing in the j-th argument if $a_{ij} = 0$, and decreasing otherwise.

If we allow for renaming of the arguments $v_1, \ldots, v_n$, then a short algebraic calculation reveals that Corollary 8.3 covers the case of all matrices that a generated by any vector $a \in \{0,1\}^n$ and the rule $a_{ij} \equiv_2 a_i + a_j$. Possible matrices are

$$(0)\,, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Note that we always have $a_{ii} = 0$ and that we have a special block structure. It would be interesting to study the case in which the monotonicity-matrix has not such a special form. ○

In the following, let us consider three special cases of Corollary 8.3.

First, by letting $n = 1$ and $r = 0$ or $r = 1$ in Corollary 8.3, we reobtain Theorem 2.33 in which $S = S_1$ is an increasing operator. That is, even with dualization we have no assertion about a decreasing operator $S = S_1$. Furthermore, there are simple single-valued functions $S\colon [0,1] \subset \mathbb{R} \to \mathbb{R}$ which are decreasing and have no fixed points, e.g. the characteristic function of the set $[0, 1/2]$. We currently do not know if methods similar to those used in the proof of Theorem 8.3 can be used to provide a fixed point result for a class of purely decreasing operators.

Next, let $n = 2$ and $r = 1$ and assume that $v_i \mapsto S_i(v_1, v_2)$ is constant (and thus increasing). Then, from Corollary 8.3 we obtain the following corollary, which we will use in the next section.

8.5 Corollary *Let $n = 2$, and let D_i, V_i and W_i given as in Theorem 8.1. Suppose furthermore that the mappings $R_i\colon \mathbf{D} \to \mathcal{P}_\emptyset(D_i \cap W_i)$ and $\underline{R}_i, \overline{R}_i\colon D_{3-i} \to \mathcal{P}_\emptyset(V_i)$ satisfy the following conditions:*

(i) There are $\underline{\mathbf{u}}, \overline{\mathbf{u}} \in \mathbf{D}$ such that

$$\underline{u}_1 \in \underline{R}_1(\overline{u}_2), \quad \underline{u}_2 \in \underline{R}_2(\overline{u}_1), \quad \overline{u}_1 \in \overline{R}_1(\underline{u}_2), \quad \overline{u}_2 \in \overline{R}_2(\underline{u}_1).$$

(ii) $R_i(\mathbf{D})$ *is bounded in* W_i *and* R_i *has weakly sequentially closed values in* W_i.

(iii) $\underline{R}_i$ *is decreasing downward, its values are directed upward, and for all* $\boldsymbol{v} \in \mathbf{D}$ *it holds* $R_i(\boldsymbol{v}) \subset \underline{R}_i(v_{3-i}) \leqslant^* R_i(\boldsymbol{v})$.

(iv) $\overline{R}_i$ *is decreasing upward, its values are directed downward and for all* $\boldsymbol{v} \in \mathbf{D}$ *it holds* $R_i(\boldsymbol{v}) \leqslant_* \overline{R}_i(v_{3-i}) \supset R_i(\boldsymbol{v})$.

Then $\mathbf{R}\colon \mathbf{D} \to \mathcal{P}_\emptyset(\mathbf{D} \cap \boldsymbol{W})$ *has fixed points* $\boldsymbol{u}_*$ *and* $\boldsymbol{u}^*$ *such that for all fixed points* $\boldsymbol{u}$ *of* $\mathbf{R}$ *it holds* $u_{*1} \leqslant u_1 \leqslant u_1^*$ *and* $u_2^* \leqslant u_2 \leqslant u_{*2}$. ○

Finally, by letting $n = 2$, $r = 1$ and $S_1 = S_2$, we obtain from Corollary 8.3 the existence of a **coupled fixed point** of a multivalued bifunction F, that is, a pair (u, v) such that $u \in F(u, v)$ and $v \in F(v, u)$.

8.6 Corollary *Let* V *and* W *be reflexive ordered Banach spaces such that* $W \subset V$ *as ordered sets, let* $D \subset V$ *be a bounded and weakly sequentially closed lattice, and let* $F\colon D \times D \to \mathcal{P}_\emptyset(D \cap W)$ *and* $\underline{F}, \overline{F}\colon D \times D \to \mathcal{P}_\emptyset(V)$ *be multifunctions such that the following hypotheses are satisfied:*

(i) There are $\underline{u}, \overline{u} \in V$ *such that* $\underline{u} \leqslant^* \underline{F}(\underline{u}, \overline{u})$ *and* $\overline{F}(\overline{u}, \underline{u}) \leqslant_* \overline{u}$.

(ii) $F(D \times D)$ *is bounded in* W *and* F *has weakly sequentially closed values in* W.

(iii) $\underline{F}$ *is increasing upward in the first argument and decreasing downward in the second, its values are directed upward, and it holds* $F(u_1, u_2) \subset \underline{F}(u_1, u_2) \leqslant^* F(u_1, u_2)$.

(iv) $\overline{F}$ *is increasing downward in the first argument and decreasing upward in the second, its values are directed downward and it holds* $F(u_2, u_1) \leqslant_* \overline{F}(u_2, u_1) \supset F(u_2, u_1)$.

Then F *has an ordered coupled fixed point* (u_*, u^*) *that is extremal in the sense that for all coupled fixed points* (v, w) *of* F *it holds* $v, w \in [u_*, u^*]$.

Proof: Let us set $S_1 := F$, $\underline{S}_1 := \underline{F}$ and $\overline{S}_1 := \overline{F}$, and let us define three multifunctions $S_2\colon D \times D \to \mathcal{P}(D \cap W)$ and $\underline{S}_2, \overline{S}_2\colon D \times D \to \mathcal{P}(V)$ via

$$S_2(u_1, u_2) := F(u_2, u_1), \quad \underline{S}_2(u_1, u_2) := \underline{F}(u_2, u_1), \quad \overline{S}_2(u_1, u_2) := \overline{F}(u_2, u_1).$$

Evidently, a fixed point of $\mathbf{S}$ is a coupled fixed point of F and vice versa, so it suffices to understand the fixed points of $\mathbf{S}$. To this end, let us apply Corollary 8.3 with $n = 2$, $r = 1$, $D_i = D$, $V_i = V$, $W_i = W$, $\underline{u}_1 = \underline{u}_2 = \underline{u}$, and $\overline{u}_1 = \overline{u}_2 = \overline{u}$. One readily verifies that all assumptions are fulfilled, and thus there is $(u_*, u^*) \in D \times D$ which is the smallest fixed point of $\mathbf{S}$ with respect to the partial order defined via

$$\boldsymbol{u} \leqslant_{\boldsymbol{V}} \boldsymbol{v} \quad \Longleftrightarrow \quad u_1 \leqslant v_1 \text{ and } v_2 \leqslant u_2.$$

That is, we have

$$u_* \in F(u_*, u^*) \quad \text{and} \quad u^* \in F(u^*, u_*)$$

and for each coupled fixed point (v, w) of F it holds $u_* \leqslant v$ and $w \leqslant u^*$. Since (w, v) is a coupled fixed point of F, too, we infer even $v, w \in [u_*, u^*]$, and we notice that (u^*, u_*) is the greatest fixed point of $\mathbf{S}$ with respect to $\leqslant_{\boldsymbol{V}}$. ○

Let us conclude this section with a simple example that illustrates the application of Corollary 8.6 to real-valued functions:

8.7 Example Let $g, h\colon [0,1] \to [0,1/2]$ be increasing functions and define

$$f\colon [0,1] \times [0,1] \to [0,1], \quad (s,t) \mapsto g(s) - h(t) + 1/2.$$

Then f is well-defined, increasing in the first and decreasing in the second argument. Since $0 \leqslant f(0,1)$ and $f(1,0) \leqslant 1$, there is, due to Corollary 8.6, an extremal coupled fixed point of f.

If g and h are continuous, then $s \mapsto f(s,s)$ is continuous, too, and has a fixed point s^*, such that (s^*, s^*) is a coupled fixed point (which is only extremal if there are no other coupled fixed points). However, consider the case

$$g(s) = \begin{cases} 0 & \text{if } s \leqslant 1/2, \\ 1/2 & \text{if } s > 1/2, \end{cases} \qquad h(t) = \begin{cases} 0 & \text{if } t < 1/2, \\ 1/2 & \text{if } x \geqslant 1/2. \end{cases}$$

Then $f(s,s) = 1/2$ if $s \neq 1/2$, and $f(1/2,1/2) = 0$. Thus, $s \mapsto f(s,s)$ has *no* fixed point. However, due to Corollary 8.6, there are at least ordered coupled fixed points (s_*, s^*). In this example, the only two such pairs are $(s_*, s^*) = (0, 1/2)$ and $(s_*, s^*) = (0,1)$, and the second one is extremal. ○

In the next sections, we will present applications of the developed framework to different systems of multivalued variational inequalities and to systems of elliptic inclusions with L^1-measure right-hand side, which extends the study done so far in the last chapters. Similar applications are possible for a wide range of differential inclusions (or a mix of them) for which the method of sub-supersolutions is established.

8.3 Systems of Variational Inequalities

In this section, we consider a basic model problem that can be generalized in various ways. It reads as follows: Find $\boldsymbol{u} \in \boldsymbol{W}$ such that

$$A_i(u_i) + F_i(u_i, \boldsymbol{u}) + \partial I_{K_i}(u_i) \ni 0 \quad \text{in } W_i', \tag{8.5$_i$}$$

where $i = 1, \dots, n$, $A_i\colon W_i \to W_i'$ are differential operators of Leray-Lions type from a Sobolev space W_i to its dual space W_i', and $\boldsymbol{W} = \prod_i W_i$. The elements of $F_i(u_i, \boldsymbol{u})$ are selections of some multivalued function $f_i(\cdot, u_i, \boldsymbol{u})$ (with $f_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$) and ∂I_{K_i} denotes the subdifferential (in the sense of Convex Analysis) of a nonempty, closed and convex set $K_i \subset W_i$. That means, $\boldsymbol{u} \in \mathbf{K}$ is a solution of System (8.5) if there are measurable selections $\eta_i \subset f_i(\cdot, u_i, \boldsymbol{u})$ such that

$$\int_\Omega a_i(\cdot, \nabla u_i) \nabla(\varphi_i - u_i) + \int_\Omega \eta_i(\varphi_i - u_i) \geqslant 0 \quad \text{for all } \varphi_i \in K_i.$$

The emphasis in this thesis lies on the perturbations f_i: We assume that $s \mapsto f(x,s,t)$ is upper semicontinuous and that $t \mapsto f(x,s,t)$ is decreasing (in particular, no continuity in the last arguments is assumed). Furthermore, we only assume a local growth condition between given sub-supersolutions.

The exact setting will be given in the next subsection. We also refer to Chapter 4 for a more detailed treatment of the case $n = 1$.

8.8 Remark To simplify the notations, we are going to formulate the most general results only for the case in which all mappings $t \mapsto f_i(x, s, t)$ are decreasing. Special mixed-monotone systems can be treated analogously by use of Corollary 8.3. ○

8.9 Remark In [20, 21], a special case of System (8.5) with $n = 2$ and single-valued f_i was considered. There, no monotonicity assumptions on f_i were used, but, in order to apply variational methods, the functions f_i were considered to be smooth. The novelty of the approach presented here is twofold: First, we deal with a multivalued problem. Second, we replace the smoothness-condition by the weak conditions (F1)—(F4) below, which allow for nonsmooth perturbations while allowing for the construction of sub-supersolutions as presented in a special application in [20]. We will provide more details in Subsection 8.3.4 below.

In comparison of the methods of [20] and our approach, one sees that there is a trade-off between monotonicity and smoothness. We refer further to [25, 57], were also monotone and mixed-monotone systems were considered, but with a slightly different approach and under more restrictive conditions. ○

8.10 Remark By use of the operator $J\colon \prod_i W_i' \to \boldsymbol{W}'$, we can define the diagonal operator

$$J\boldsymbol{A}\colon \boldsymbol{W} \to \boldsymbol{W}', \quad \boldsymbol{u} \mapsto [\boldsymbol{\varphi} \mapsto \textstyle\sum_i \langle A_i u_i, \varphi_i \rangle]$$

and the selection operator

$$JF\colon \boldsymbol{W} \to \boldsymbol{W}', \quad \boldsymbol{u} \mapsto \{\boldsymbol{\eta} \in \boldsymbol{W}' : \exists \eta_i \subset f_i(\cdot, u_i, \boldsymbol{u}), \eta_i \in L_i \text{ s.t. } \langle \boldsymbol{\eta}, \boldsymbol{\varphi} \rangle = \textstyle\sum_i \langle \eta_i, \varphi_i \rangle\},$$

where η_i acts on φ_i via $\langle \eta_i, \varphi_i \rangle := \int_\Omega \eta_i \varphi_i$. Using them, we can write System (8.5) in compact form as

$$\boldsymbol{u} \in \boldsymbol{W} : \quad J\boldsymbol{A}(\boldsymbol{u}) + JF(\boldsymbol{u}) + \partial I_{\boldsymbol{K}}(\boldsymbol{u}) \ni \boldsymbol{0} \quad \text{in } \boldsymbol{W}'$$

or, equivalently,

$$\boldsymbol{u} \in \boldsymbol{K} : \quad \exists \boldsymbol{\eta} \in JF(\boldsymbol{u}) : \quad \langle J\boldsymbol{A}(u), \boldsymbol{\varphi} - \boldsymbol{u} \rangle + \langle \boldsymbol{\eta}, \boldsymbol{\varphi} - \boldsymbol{u} \rangle \geqslant 0 \quad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{K}.$$

Note, that by letting $\varphi_j = u_j$ for all $j \neq i$ one recovers that u_i is indeed a solution of the single inclusion (8.5_i). ○

8.3.1 Setting

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 1$, be a bounded domain with Lipschitz boundary. If not otherwise stated, the following holds for all $i = 1, \dots, n$. Let $p_i \in (1, \infty)$ and $q_i \in [1, p_i^*)$ be fixed, where p_i^* denotes the critical Sobolev exponent associated with p and the dimension N.

We use the abbreviations

$$L_i := L^{q_i'}(\Omega), \quad V_i := L^{p_i^*}(\Omega), \quad W_i := W_0^{1,p_i}(\Omega) \quad \text{and} \quad \mathbf{V} := \textstyle\prod_i V_i \text{ etc.}$$

Let $K_i \subset W_i$ be a non-empty, closed and convex lattice, and suppose that the following conditions on the data hold true:

8.11 Assumption With reference to Definition 8.15 we assume the following:

(S) There is a pair $(\underline{\mathbf{u}}, \overline{\mathbf{u}})$ of sub-supersolutions of System (8.5) such that $\underline{\mathbf{u}} \leqslant \overline{\mathbf{u}}$. ○

8.12 Assumption Let $a_i\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a function defining the (single-valued) differential operator A_i of Leray-Lions type via $A_i(u_i) = -\operatorname{div} a_i(\cdot, \nabla u_i)$. The following standard assumptions on a are meant to hold for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^N$.

(A1) a_i is a **Carathéodory function**.

(A2) There are $\alpha_2 > 0$ and $k_{2,i} \in L^1(\Omega)$ such that $a_i(x,\xi)\xi \geqslant \alpha_2|\xi|^{p_i} - k_{2,i}(x)$.

(A3) There are $\alpha_3 > 0$ and $k_{3,i} \in L^{p_i'}(\Omega)$ such that $|a_i(x,\xi)| \leqslant \alpha_3|\xi|^{p_i-1} + k_{3,i}(x)$.

(A4) For all $\xi, \xi' \in \mathbb{R}^N$, it holds $(a_i(x,\xi) - a_i(x,\xi'))(\xi - \xi') \geqslant 0$. ○

8.13 Assumption Let $f_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R})$ be a multifunction whose values are compact intervals. The following conditions are meant to hold for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $t \in \mathbb{R}^n$.

(F1) The function $(x,t) \mapsto f_i(x,s,t)$ is **superpositionally measurable**.

(F2) The function $s \mapsto f_i(x,s,t)$ is **upper semicontinuous**.

(F3) The function $t \mapsto f_i(x,s,t)$ is **decreasing**.

(F4) There is $k_{4,i} \in L_i$ such that f_i satisfies

$$\sup\{|y| : y \in f_i(x,s,t), s \in [\underline{u}_i(x), \overline{u}_i(x)], t \in [\underline{\mathbf{u}}(x), \overline{\mathbf{u}}(x)]\} \leqslant k_{4,i}(x). \qquad ○$$

8.14 Remark Recall that f_i is called weakly superpositionally measurable if for any measurable function $u\colon \Omega \to \mathbb{R}^{n+1}$ the multifunction $x \mapsto f_i(x, u(x))$ has a measurable selection $\eta \subset f_i(\cdot, u)$. Now, according to Theorem 3.47 it follows from (F1) and (F2) that f_i is weakly superpositionally measurable, and we would like to stress that upper semicontinuity is only assumed in one argument of f_i, but not in the other ones. ○

8.3.2 Abstract Formulation

We adopt the notation from Chapter 4 to systems. To this end, set $\mathbf{D} := [\underline{\mathbf{u}}, \overline{\mathbf{u}}]_{\mathbf{V}}$ and let $F_i \colon D_i \times \mathbf{D} \to \mathcal{P}(L_i)$ be the selection mapping defined by

$$F_i(u_i, \mathbf{v}) = \{\eta_i \in L_i : \eta_i \subset f_i(\cdot, u_i, \mathbf{v})\}.$$

(Since f_i is weakly superpositionally measurable and due to (F4), F_i is well-defined.) Further, for all functions $u_i \in W_i \cap D_i$, $\mathbf{v} \in \mathbf{V}$ and each subset $T_i \subset W_i$ we write $u_i \sim (\mathbf{v}, T_i)$ if and only if there is a function $\eta_i \in F_i(u_i, \mathbf{v})$ such that

$$\langle A_i u_i, \varphi - u_i \rangle + \int_\Omega \eta_i (\varphi_i - u_i) \geqslant 0 \quad \text{for all } \varphi_i \in T_i.$$

With help of this abbreviation, we define the operators $S_i, \underline{S}_i, \overline{S} \colon \mathbf{D} \to \mathcal{P}_\emptyset(D_i \cap W_i)$ via

$$\begin{aligned} S_i(\mathbf{v}) &:= \{u_i : u_i \in K_i \text{ and } u_i \sim (\mathbf{v}, K_i)\}, \\ \underline{S}_i(\mathbf{v}) &:= \{u_i : u_i \vee K_i \subset K_i \text{ and } u_i \sim (\mathbf{v}, u_i \wedge K_i)\}, \\ \overline{S}_i(\mathbf{v}) &:= \{u_i : u_i \wedge K_i \subset K_i \text{ and } u_i \sim (\mathbf{v}, u_i \vee K_i)\}. \end{aligned}$$

Recall that $\mathbf{S} \colon \mathbf{D} \to \mathcal{P}_\emptyset(\mathbf{D} \cap \mathbf{W})$ is given by $\mathbf{S}(\mathbf{v}) = \prod_i S_i(\mathbf{v})$. Then it is clear that the fixed points of $\mathbf{S}$ coincide with the solutions of System (8.5) which are located in the interval $\mathbf{D} = [\underline{\mathbf{u}}, \overline{\mathbf{u}}]_{\mathbf{V}}$ generated by the given sub-supersolutions. These semi-solutions are, finally, defined as follows:

8.15 Definition We call $\mathbf{u} \in \mathbf{W}$ **subsolution** or **supersolution** of System (8.5) if $\mathbf{u} \in \underline{\mathbf{S}}(\mathbf{u})$ or $\mathbf{u} \in \overline{\mathbf{S}}(\mathbf{u})$, respectively. ○

8.3.3 Existence of Solutions

The following Propositions are straight forward generalizations of the results of Chapter 4. We thus omit the proofs.

8.16 Proposition *The operator* $\underline{S}_i \colon \mathbf{D} \to \mathcal{P}_\emptyset(\mathbf{V})$ *is permanent upward, whereas the operator* $\overline{S}_i \colon \mathbf{D} \to \mathcal{P}_\emptyset(\mathbf{V})$ *is permanent downward. In particular it follows that* $\underline{S}_i$ *and* $\overline{S}_i$ *have non-empty values.* ○

8.17 Proposition *Let* $\mathbf{v} \in \mathbf{D}$ *be arbitrary, and let* $\underline{v}_i, \underline{w}_i \in \underline{S}_i(\mathbf{v})$ *and* $\overline{v}_i, \overline{w}_i \in \overline{S}_i(\mathbf{v})$ *be such that*

$$\underline{v}_i \vee \underline{w}_i \leqslant \overline{v}_i \wedge \overline{w}_i.$$

Then there is $u_i \in S_i(\mathbf{v})$ *such that* $\underline{v}_i \vee \underline{w}_i \leqslant u_i \leqslant \overline{v}_i \wedge \overline{w}_i$*. In particular,* S *has non-empty values,* $\underline{S}_i(\mathbf{v})$ *is directed upward,* $\overline{S}_i(\mathbf{v})$ *is directed downward, and it holds* $\underline{S}_i(\mathbf{v}) \leqslant^* S_i(\mathbf{v}) \leqslant_* \overline{S}_i(\mathbf{v})$*.* ○

8.18 Proposition *The operator* $S_i \colon \mathbf{D} \to \mathcal{P}(\mathbf{W})$ *is uniformly bounded and has weakly compact values.* ○

Now, from Theorem 8.1 and Corollary 8.2 we have the following existence theorem:

8.19 Theorem *Suppose (S), (A1)—(A4) and (F1)—(F4). Then System* (8.5) *has both the smallest and the greatest solution in* $[\underline{\mathbf{u}}, \overline{\mathbf{u}}]_{\mathbf{V}}$*.* ○

Next, let us investigate a concrete example.

8.3.4 Example

The monotonicity imposed in (F3) allows for discontinuous behavior of f_i. However, without monotonicity, in general, smoothness as well as a more restrictive definition of sub-supersolutions are required. Then, as demonstrated in [20, 21], variational methods can be used to find solutions. In what follows, we are going to inspect the applicability of our approach to the following special case of (8.5) which was treated in [20]: Find $u_i \in K_i \subset W_i$ such that

$$\langle -\Delta_{p_i} u_i + f_i(\cdot, u_i, \boldsymbol{u}), \varphi_i - u_i \rangle \geqslant 0 \quad \text{for all } \varphi_i \in K_i, \tag{8.6$_i$}$$

where $n = 2$, Δ_{p_i} is the p_i-Laplacian and $f_i \colon \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a single-valued function. To be more precise, in [20] functions $g_i \colon \mathbb{R}^2 \to \mathbb{R}$ were given such that

$$f_1(x, s, t_1, t_2) = -g_1(s, t_2) \quad \text{and} \quad f_2(x, s, t_1, t_2) = -g_2(t_1, s).$$

Thus, System (8.6) reads as

$$u_i \in K_i: \quad \langle -\Delta_{p_i} u_i - g_i(\boldsymbol{u}), \varphi_i - u_i \rangle \geqslant 0 \quad \text{for all } \varphi_i \in K_i. \tag{8.7$_i$}$$

In [20, 21], the main assumption on (g_1, g_2) was that it has a potential in $C^2(\mathbb{R}^2)$. In particular, no monotonicity was assumed. However, this resulted in a more involved definition of sub-supersolutions $(\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}})$. For example, $\underline{\boldsymbol{u}}$ is a subsolution of System (8.7) in the sense of [20, Def. 2.1] if $u_i \vee K_i \subset K_i$ and

$$\langle -\Delta_{p_1} \underline{u}_1 - g_1(\underline{u}_1, w_2), \varphi_1 - \underline{u}_1 \rangle \geqslant 0 \quad \text{for all } \varphi_1 \in \underline{u}_1 \wedge K_1,\ w_2 \in [\underline{u}_2, \overline{u}_2], \tag{8.8$_1$}$$

$$\langle -\Delta_{p_2} \underline{u}_2 - g_2(w_1, \underline{u}_2), \varphi_2 - \underline{u}_2 \rangle \geqslant 0 \quad \text{for all } \varphi_2 \in \underline{u}_2 \wedge K_2,\ w_1 \in [\underline{u}_1, \overline{u}_1]. \tag{8.8$_2$}$$

As long as no monotonicity is assumed, one has to make sure that inequalities (8.8$_1$) and (8.8$_2$) hold not only for a special choice of $\boldsymbol{w}$, but for all $\boldsymbol{w} \in [\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}}]$. This task is much easier if $v_2 \mapsto g_1(\boldsymbol{v})$ and $v_1 \mapsto g_2(\boldsymbol{v})$ are monotonous. With a view to the example below, let us assume in the sequel that g_1 is decreasing in the second argument and g_2 is decreasing in the first. Then from

$$\langle -\Delta_{p_1} \underline{u}_1 - g_1(\underline{u}_1, \overline{u}_2), \varphi_1 - \underline{u}_1 \rangle \geqslant 0 \quad \text{for all } \varphi_1 \in \underline{u}_1 \wedge K_1, \tag{8.9$_1$}$$

$$\langle -\Delta_{p_2} \underline{u}_2 - g_2(\overline{u}_1, \underline{u}_2), \varphi_2 - \underline{u}_2 \rangle \geqslant 0 \quad \text{for all } \varphi_2 \in \underline{u}_2 \wedge K_2, \tag{8.9$_2$}$$

we easily deduce (8.8$_1$) and (8.8$_2$). Since analogous deductions can be made for the supersolution $\overline{\boldsymbol{u}}$, we arrive at a simpler definition for sub-supersolutions. To formulate it in abstract terms, take any $\boldsymbol{v} \in \mathbf{D} = [\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}}]_{\mathbf{V}}$ and let us define the operators $\underline{R}_i, \overline{R}_i \colon D_{3-i} \to \mathcal{P}_\emptyset(V_i)$ via

$$\underline{R}_1 := \underline{S}_1(v_1, \cdot), \quad \underline{R}_2 := \underline{S}_2(\cdot, v_2), \quad \overline{R}_1 := \overline{S}_1(v_1, \cdot), \quad \overline{R}_2 := \overline{S}_2(\cdot, v_2).$$

Evidently, $(\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}})$ is a pair of sub-supersolutions in the sense of [20, Def. 2.1] if and only if

$$\underline{u}_1 \in \underline{R}_1(\overline{u}_2), \quad \underline{u}_2 \in \underline{R}_2(\overline{u}_1), \quad \overline{u}_1 \in \overline{R}_1(\underline{u}_2), \quad \overline{u}_2 \in \overline{R}_2(\underline{u}_1),$$

which is exactly condition (i) of Corollary 8.5. Thus, our approach allows for a treatment of the systems considered in [20] if they are not as smooth as required while building on the same kinds of sub-supersolutions.

8.20 Theorem *Suppose that there is an ordered pair* $\underline{\mathbf{u}}, \overline{\mathbf{u}} \in \mathbf{W}$ *of sub-supersolutions of System* (8.7) *in the sense of [20, Def. 2.1] and suppose that* $g_i \colon \mathbb{R}^2 \to \mathbb{R}$ *satisfy the following conditions:*

(G1) The functions $t_2 \mapsto g_1(s, t_2)$ *and* $t_1 \mapsto g_2(t_1, s)$ *are decreasing.*

(G2) The functions $s \mapsto g_1(s, t_2)$ *and* $s \mapsto g_2(t_1, s)$ *are continuous.*

(G3) There is $k_{4,i} \in L_i$ *such that* $|g_i(t_1, t_2)| \leqslant k_{4,i}(x)$ *for all* $(t_1, t_2) \in [\underline{\mathbf{u}}(x), \overline{\mathbf{u}}(x)]$.

Then System (8.7) *has solutions* $\mathbf{u}_*$ *and* $\mathbf{u}^*$ *located in* $[\underline{\mathbf{u}}, \overline{\mathbf{u}}]$ *which are extremal in the sense that for all solutions* $\mathbf{u}$ *of System* (8.7) *located in* $[\underline{\mathbf{u}}, \overline{\mathbf{u}}]$ *it holds* $u_{*1} \leqslant u_1 \leqslant u_1^*$ *and* $u_2^* \leqslant u_2 \leqslant u_{*2}$.

Proof: We are going to apply Corollary 8.5, and we have already seen that condition (i) holds true. Moreover, it follows easily that $\underline{R}_i$ is decreasing downward and that $\overline{R}_i$ is decreasing upward. For instance, suppose $\underline{u}_1 \in \underline{R}_1(\nu_2)$ and $\nu_2' \leqslant \nu_2$. Then we have

$$\langle -\Delta_{p_1}\underline{u}_1 - g_1(\underline{u}_1, \nu_2'), \varphi_1 - \underline{u}_1 \rangle \geqslant \langle -\Delta_{p_1}\underline{u}_1 - g_1(\underline{u}_1, \nu_2), \varphi_1 - \underline{u}_1 \rangle \geqslant 0 \quad \text{for all } \varphi_1 \in \underline{u}_1 \wedge K_1$$

and thus $\underline{u}_1 \in \underline{R}_1(\nu_2')$, which means that $\underline{R}_1$ is even permanent downward. Since $\underline{R}_i(\overline{u}_{3-i})$ and $\overline{R}_i(\underline{u}_{3-i})$ are non-empty, it follows that $\underline{R}_i$ and $\overline{R}_i$ have non-empty values. The remaining conditions hold true due to Propositions 8.17 and 8.18, since the proofs of them are independent of the monotonic behaviour of $t \mapsto f_i(x, s, t)$ and conditions (A1)—(A4), (F1), (F2) and (F4) hold true. ○

8.21 Example In [20], the following example was given: Let $\alpha > p_1$, $\beta > p_2$, $\lambda > 0$, $\mu > 0$, and set

$$\begin{aligned} g_1(s, t_2) &:= \lambda |s|^{p_1 - 2} s - \alpha |s|^{\alpha - 2} s (1 + h_1(t_2)), \\ g_2(t_1, s) &:= \mu |s|^{p_2 - 2} s - \beta |s|^{\beta - 2} s (1 + h_2(t_1)), \end{aligned}$$

where $h_1(t_2) := |t_2|^\beta$ and $h_2(t_1) := |t_1|^\alpha$. Set furthermore $K_i := W_i$, then System (8.7) results in the following Dirichlet problem for systems: Find $u_i \in W_i$ such that

$$\int_\Omega |\nabla u_i|^{p_i - 2} \nabla u_i \nabla \varphi_i = \int_\Omega g_i(u_1, u_2) \varphi_i \quad \text{for all } \varphi_i \in W_i. \tag{8.10$_i$}$$

Since $u_1 = u_2 = 0$ is a solution of System (8.10), the question arises if there are non-trival solutions, e.g. positive ones. To this end, we recall [20, Corollary 2.3]:

8.22 Proposition *Let* ψ_i *be the (normalized, positive) eigenfunction corresponding to the first eigenvalue* $\lambda_i > 0$ *of* $-\Delta_{p_i}$ *on* W_i, *and let* $h_i \in W_i$ *be the unique solution of* $-\Delta_{p_i} h_i = 1$ *in* W_i'. *Suppose further* $\lambda > \lambda_1$, $\mu > \lambda_2$. *Then*

$$\underline{\mathbf{u}} := (c\psi_1, c\psi_2) \quad \textit{and} \quad \overline{\mathbf{u}} := (Mh_1, Mh_2)$$

form an ordered pair of positive sub-supersolutions of System (8.10) *in the sense of [20, Def. 2.1] provided* $c > 0$ *is small and* $M > 0$ *is large.* ○

By inspecting the proof of [20, Corollary 2.3] one realizes that the result holds not only for the special choice $h_1(t_2) = |t_2|^\beta$ and $h_2(t_1) = |t_1|^\alpha$, but for all $h_i\colon \mathbb{R} \to \mathbb{R}$ satisfying the following condition:

(H) h_i is increasing on $\mathbb{R}_+$ and satisfies $h_i(t) = 0$ for $t \leqslant 0$.

In [20], the special choice of h_i was needed in order to find a potential of (g_1, g_2). Since we do not build on variational methods, we have more freedom. Since one easily deduces from (H) that g satisfies (G1)—(G3) with respect to $(\underline{u}, \overline{u})$ given by Proposition 8.22, we have the following corollary of Theorem 8.20:

8.23 Corollary *Suppose (H) and* $\lambda > \lambda_1$*,* $\mu > \lambda_2$*. Then System* (8.10) *has at least one non-trivial, positive solution.* ○

This concludes the example. ○

8.4 Systems of Quasi-Variational Inequalities

In Chapter 5, we applied the general framework to multivalued quasi-variational inequalities. In this section, we are going to use the results established there in order to treat the following problem: Find $\mathbf{u} \in \boldsymbol{W}$ such that

$$A_i(u_i) + F_i(u_i, \mathbf{u}) + \partial K_{i,\mathbf{u}}(u_i) \ni 0 \quad \text{in } W_i', \tag{8.11$_i$}$$

where $\boldsymbol{W}$, A_i and F_i are defined as in Section 8.3 above, and where $\partial K_{i,\mathbf{u}}$ is the subdifferential in the sense of Convex Analysis of the functional $K_{i,\mathbf{u}} = K_i(\cdot, \mathbf{u})$, which is assumed to be proper, convex and lower semicontinuous.

8.24 Remark Problem (8.11) specializes to Problem (8.5) if $K_{i,\mathbf{u}}$ equals the indicator function of a non-empty, closed and convex set K_i as considered in Section 8.3. ○

8.25 Remark We are going to formulate the results only for the case in which $t \mapsto f_i(x, s, t)$ is decreasing and $\boldsymbol{v} \mapsto K_{i,\boldsymbol{v}}$ is increasing. Special mixed-monotone systems can be treated analogously by use of Corollary 8.3. ○

8.4.1 Setting

In this section, we use the same setting and assumptions (S), (A1)—(A4) and (F1)—(F4) as in Subsection 8.3.1. In particular, we set $\mathbf{D} := [\underline{\mathbf{u}}, \overline{\mathbf{u}}]_{\boldsymbol{V}}$ for an ordered pair $(\underline{\mathbf{u}}, \overline{\mathbf{u}})$ of sub-supersolutions. However, we redefine the operators S, $\underline{S}$ and $\overline{S}$ in the next subsection (and so we redefine sub- and supersolutions), and K_i denotes no more a convex lattice, but a functional $K_i\colon D_i \times \mathbf{D} \to \mathbb{R} \cup \{+\infty\}$.

In order to compare functionals, we introduce the spaces

$$\Gamma_i := \{k\colon D_i \to \mathbb{R} \cup \{+\infty\} \text{ with } \mathcal{D}(k) \neq \emptyset\},$$

and like in Chapter 5, we equip Γ_i with the relation $\preccurlyeq_*^*$.

On K_i, we impose the following assumptions:

8.26 Assumption Let $K_i\colon D_i \times \mathbf{D} \to \mathbb{R} \cup \{+\infty\}$ be a functional such that the following conditions hold:

(K1) For all $\boldsymbol{v} \in \mathbf{D}$ the function $K_{i,\boldsymbol{v}}\colon D_i \to \mathbb{R} \cup \{+\infty\}$, defined by $K_{i,\boldsymbol{v}}(u_i) = K(u_i, \boldsymbol{v})$, is **proper**, **convex** and **lower semicontinuous**.

(K2) The mapping $\boldsymbol{v} \mapsto K_{i,\boldsymbol{v}}$ is **increasing**, i.e. $\boldsymbol{v} \leqslant \boldsymbol{v}'$ in $\mathbf{D}$ implies $K_{i,\boldsymbol{v}} \preccurlyeq^*_* K_{i,\boldsymbol{v}'}$.

(K3) There is some constant $c_3 > 0$ such that for all $\boldsymbol{v} \in \mathbf{V}$ there is some $\varphi_{i,\boldsymbol{v}} \in \mathcal{D}(K_{i,\boldsymbol{v}})$ such that for all $u_i \in W_i$ it holds

$$\|\nabla\varphi_{i,\boldsymbol{v}}\|_{p_i} \leqslant c_3, \qquad K_{i,\boldsymbol{v}}(\varphi_{i,\boldsymbol{v}}) \leqslant c_3, \qquad K_{i,\boldsymbol{v}}(u_i) \geqslant -c_3(\|\nabla u_i\|_{p_i}^{p_i-1} + 1). \quad \circ$$

8.4.2 Abstract Formulation

We generalize the notion of Subsection 8.3.2 along the lines of Chapter 5: For all functions $u_i \in W_i \cap D_i$, $\boldsymbol{v} \in \mathbf{D}$, and all subsets $W_{0,i} \subset W_i$ and $\Gamma_{0,i} \subset \Gamma_i$, we write $u_i \sim (\boldsymbol{v}, \Gamma_{0,i}, W_{0,i})$ if there is a function $\eta_i \in F_i(u_i, \boldsymbol{v})$ and a functional $k_i \in \Gamma_{0,i}$ such that

$$\langle A_i(u_i), \varphi_i - u_i\rangle + \int_\Omega \eta_i(\varphi_i - u_i) + k_i(\varphi_i) - k_i(u_i) \geqslant 0 \quad \text{for all } \varphi_i \in W_{0,i}.$$

In this section, let us define the operators $S_i, \underline{S}_i, \overline{S}_i\colon \mathbf{D} \to \mathcal{P}(\mathbf{D} \cap \boldsymbol{W})$ as follows:

$$\begin{aligned}
S_i(\boldsymbol{v}) &:= \{u_i : u_i \sim (\boldsymbol{v}, \{K_{i,\boldsymbol{v}}\}, W_i)\},\\
\underline{S}_i(\boldsymbol{v}) &:= \{u_i : u_i \sim (\boldsymbol{v}, K_{i,\boldsymbol{v}}^{\downarrow}, u_i \wedge \mathcal{D}(K_{i,\boldsymbol{v}}))\},\\
\overline{S}_i(\boldsymbol{v}) &:= \{u_i : u_i \sim (\boldsymbol{v}, K_{i,\boldsymbol{v}}^{\uparrow}, u_i \vee \mathcal{D}(K_{i,\boldsymbol{v}}))\}.
\end{aligned}$$

Like above, $\boldsymbol{u} \in \boldsymbol{W}$ is a solution, sub- or supersolution of System (8.11) if $\boldsymbol{u}$ is a fixed point of $\mathbf{S} = \prod_i S_i$, $\underline{\mathbf{S}} = \prod_i \underline{S}_i$ or $\overline{\mathbf{S}} = \prod_i \overline{S}_i$, respectively.

8.4.3 Existence of Solutions

From the results in Chapter 5 we deduce that Propositions 8.16—8.18 also hold true in the setting of this section. Thus, from Theorem 8.1 and Corollary 8.2 we have the following existence and enclosure theorem:

8.27 Theorem *Suppose (S), (A1)—(A4), (F1)—(F4) and (K1)—(K3). Then System* (8.11) *has both the smallest and the greatest solution in* $[\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}}]_{\boldsymbol{W}}$. $\circ$

8.5 Systems with Measures

With reference to Chapter 7, let us consider finally the following problem: Find $\boldsymbol{u} \in W$ such that

$$A_i(u_i) + F_i(u_i, \boldsymbol{u}) \ni \mu_i. \qquad (8.12_i)$$

Here, A_i and F_i are defined as in Section 8.3, and the right-hand side μ_i belongs to $L^1(\Omega)$. Thus, as in the scalar case $n = 1$, the regularity of solutions will be lower and that of test functions will be higher than in Sections 8.3 and 8.4. To be more precise, $\mathbf{u}$ belongs to $\boldsymbol{W}$ if $u_i \in W_0^{1,r}(\Omega)$ for all $r \in [1, (p_i - 1)1^*)$ (recall: $1^* = N/(N-1)$), and $\mathbf{u} \in \boldsymbol{W}$ is a solution of System (8.12) if there are measurable selections $\eta_i \subset f_i(\cdot, u_i, \mathbf{u})$ for which it holds

$$\int_\Omega a_i(\cdot, \nabla u_i)\nabla\varphi_i + \int_\Omega \eta_i\varphi_i = \int_\Omega \mu_i\varphi_i \quad \text{for all } \varphi_i \in W_0^{1,r}(\Omega), \quad r > N.$$

The investigation of this problem is more involved than in the previous applications, since solutions are only obtained via an approximation procedure. However, we will see that the results of Chapter 7 for the scalar case together with our general fixed point result allow to treat the system case.

8.28 Remark As before, we will formulate the results only for the case in which $t \mapsto f_i(x, s, t)$ is decreasing. Special mixed-monotone systems can be treated analogously by use of Corollary 8.3. ○

8.5.1 Setting

We use the setting of Chapter 7, generalized to the system case, which differs in some crucial details from the setting used in Sections 8.3 and 8.4.

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded domain with Lipschitz boundary, and let the constants $p_i \in (2 - 1/N, N]$ and $q_i \in (1, p_i^*)$ be fixed. In this section, let us use the following abbreviations:

$$L_i := L^{q_i'}, \qquad V_i := L^{p_i^*}, \qquad W_i := \bigcap\{W_0^{1,r} : 1 \leqslant r < (p_i - 1)1^*\}.$$

As we noted in Chapter 7, our fixed point results remain valid for the Fréchet spaces W_i. The same argumentation holds for the product space $\boldsymbol{W} = \prod_i W_i$. In the following, we will use this without further notice. Besides $\mathbf{L} := \prod_i L_i$, $\mathbf{V} := \prod_i V_i$ and $\boldsymbol{W}$, we will further use the space $\boldsymbol{W}_\mathbf{p} := \prod_i W_0^{1,p_i}$.

8.29 Assumption Like in Assumption 8.12 above, we define $A_i u = -\operatorname{div} a_i(\cdot, \nabla u)$ for a function $a_i \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (A1) and (A3) and the following slightly more restrictive variants of (A2) and (A4), which are meant to hold for a.e. $x \in \Omega$ and all $\xi, \xi' \in \mathbb{R}^N$.

(A2') There are $\alpha_2 > 0$ such that $a_i(x, \xi)\xi \geqslant \alpha_2 |\xi|^{p_i}$.

(A4') It holds $(a_i(x, \xi) - a_i(x, \xi'))(\xi - \xi') > 0$. ○

Further, we suppose in the following that f_i satisfies conditions (F1)—(F4) as above and that (S) holds with respect to the redefinition of sub-supersolutions below, such that we can define $\mathbf{D} := [\underline{\mathbf{u}}, \overline{\mathbf{u}}]_\mathbf{V}$.

8.5.2 Abstract Formulation

The natural concept of sub-supersolutions of System (8.12) is the following one:

(i) A function $\underline{\boldsymbol{u}} \in \boldsymbol{W_p}$ is called **subsolution** of System (8.12) if there are measurable selections $\underline{\eta}_i \subset f_i(\cdot, \underline{u}_i, \underline{\boldsymbol{u}})$ such that

$$\int_\Omega a_i(\cdot, \nabla \underline{u}_i)\nabla\varphi_i + \int_\Omega \underline{\eta}_i \varphi_i \leqslant \int_\Omega \mu_i \varphi_i \quad \text{for all } \varphi_i \in W_0^{1,p_i}(\Omega) \cap L_+^\infty(\Omega). \quad (8.13_i)$$

(ii) A function $\overline{\boldsymbol{u}} \in \boldsymbol{W_p}$ is called **supersolution** of System (8.12) if there are measurable selections $\overline{\eta}_i \subset f_i(\cdot, \overline{u}_i, \overline{\boldsymbol{u}})$ such that

$$\int_\Omega a_i(\cdot, \nabla \overline{u}_i)\nabla\varphi_i + \int_\Omega \overline{\eta}_i \varphi_i \geqslant \int_\Omega \mu_i \varphi_i \quad \text{for all } \varphi_i \in W_0^{1,p_i}(\Omega) \cap L_+^\infty(\Omega). \quad (8.14_i)$$

As pointed out in Chapter 7, this concept of sub-supersolutions is to broad to be combined with the abstract framework. Thus, we have to extend the concepts of solutions and subsolutions to limit-solutions (which are, as a matter of fact, also solutions) and limit-subsolutions (which are, in general, no subsolutions due to regularity issues), respectively. In order to do so, we introduce the following functions with respect to the given pair of sub-supersolutions $(\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}})$ and fixed corresponding selections $\underline{\boldsymbol{\eta}}$ and $\overline{\boldsymbol{\eta}}$:

For any given $\boldsymbol{v} \in \boldsymbol{V}$, we truncate f_i to obtain $g_{i,\boldsymbol{v}}\colon \Omega \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, which is defined by

$$g_{i,\boldsymbol{v}}(x,s) := \begin{cases} \{\underline{\eta}_i(x)\} & \text{if } s < \underline{u}_i(x), \\ f_i(x, s, \boldsymbol{v}(x)) & \text{if } s \in [\underline{u}_i(x), \overline{u}_i(x)], \\ \{\overline{\eta}_i(x)\} & \text{if } s > \overline{u}_i(x). \end{cases}$$

Further, we define the cut-off function $d_i\colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$d_i(x,s) := \begin{cases} -(\underline{u}_i(x) - s)^{p_i - 1} & \text{if } s < \underline{u}_i(x), \\ 0 & \text{if } s \in [\underline{u}_i(x), \overline{u}_i(x)], \\ (s - \overline{u}_i(x))^{p_i - 1} & \text{if } s > \overline{u}_i(x). \end{cases}$$

By use of these functions, let us introduce Problem $(Q_{i,h_i}(\boldsymbol{v}))$: Find, for given $\boldsymbol{v} \in \boldsymbol{V}$ and $h_i \in L_i$, a function $u_i \in W_0^{1,p_i}(\Omega)$ such that there is some measurable selection $\eta_i \subset f_i(\cdot, u_i, \boldsymbol{v})$ for which it holds

$$\int_\Omega a_i(\cdot, \nabla u_i)\nabla\varphi_i + \int_\Omega \eta_i \varphi_i + \int_\Omega d_i(\cdot, u_i)\varphi_i = \int_\Omega h_i \varphi_i \quad \text{for all } \varphi_i \in W_0^{1,p_i}(\Omega). \quad (8.15_i)$$

As usual, a subsolution of Problem $(Q_{i,h_i}(\boldsymbol{v}))$ is a function $u_i \in W_0^{1,p_i}(\Omega)$ such that there is some measurable selection $\eta_i \subset f_i(\cdot, u_i, \boldsymbol{v})$ for which it holds

$$\int_\Omega a_i(\cdot, \nabla u_i)\nabla\varphi_i + \int_\Omega \eta_i \varphi_i + \int_\Omega d_i(\cdot, u_i)\varphi_i \leqslant \int_\Omega h_i \varphi_i \quad \text{for all } \varphi_i \in W_0^{1,p_i}(\Omega)_+. \quad (8.16_i)$$

This said, let us define the operators $S_i, \underline{S}_i \colon \mathbf{D} \to \mathcal{P}(\boldsymbol{W} \cap \mathbf{D})$ as follows:

$$S_i(v) := \{u : \text{ there are a sequence } (h_{i,k}) \subset L_i \text{ and solutions } u_{i,k} \text{ of } (Q_{i,h_{i,k}}(v)) \\ \text{such that } h_{i,k} \to \mu_i \text{ in } L^1 \text{ and } u_{i,k} \to u_i \text{ in } W_i\},$$

$$\underline{S}_i(v) := \{u : \text{ there are a sequence } (h_{i,k}) \subset L_i \text{ and subsolutions } u_{i,k} \text{ of } (Q_{i,h_{i,k}}(v)) \\ \text{such that } h_{i,k} \to \mu_i \text{ in } L^1 \text{ and } u_{i,k} \to u_i \text{ in } W_i\}.$$

Consequently, a **limit-solution** of System (8.12) is defined to be a fixed point of $\boldsymbol{S}$. That is, a function $\boldsymbol{u} \in \boldsymbol{W}$ is called limit-solution of System (8.12) if there is a sequence $(\boldsymbol{h}_k) \subset \mathbf{L}$ and solutions $u_{i,k}$ of $(Q_{i,h_{i,k}}(\boldsymbol{u}))$ such that $\boldsymbol{h}_k \to \boldsymbol{\mu}$ in $L^1(\Omega)^N$ and $\boldsymbol{u}_k \to \boldsymbol{u}$ in $\boldsymbol{W}$. From the results of Chapter 7 it follows that such limit-solutions are indeed solutions of System (8.12) as defined at the beginning of this section.

Our main result states that there is a largest limit-solution between each ordered pair $(\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}})$ of sub-supersolutions as defined above. In particular, note that sub-supersolutions are not defined to be fixed points of $\underline{\boldsymbol{S}}$, but solutions of System (8.13) and (8.14), respectively. However, like in the sections above, the operator $\underline{\boldsymbol{S}}$ plays a crucial role in the proof of our main theorem, and its fixed points are called **limit-subsolutions**. Let us also mention that, in general, $\underline{\boldsymbol{u}} \in \underline{\boldsymbol{S}}(\underline{\boldsymbol{u}})$ does not hold, but that it holds trivially $\underline{\boldsymbol{u}} \leqslant^* \underline{\boldsymbol{S}}(\underline{\boldsymbol{u}})$ provided $\underline{\boldsymbol{S}}$ has non-empty values.

8.5.3 Existence of Solutions

From the results in Chapter 7 we deduce that Propositions 8.16—8.18 hold true in the setting of this section. Thus, from Theorem 8.1 and Corollary 8.2 we have the following existence theorem:

8.30 Theorem *Suppose (S), (A1), (A2'), (A3), (A4') and (F1)—(F4). Then System* (8.12) *has the greatest limit-solution (which is a solution of System* (8.12)*) in* $[\underline{\boldsymbol{u}}, \overline{\boldsymbol{u}}]_{\boldsymbol{V}}$.

8.31 Remark In [94] it was demonstrated that the method of sub-supersolutions might fail if the right-hand side of a differential equation is a general bounded measure, e.g. the Dirac delta function. In order to avoid technical difficulties, we restricted our considerations to L^1 right-hand sides.

However, it is possible to use our approach to generalize the existence results of [94] for systems of the form

$$-\Delta u_i + g_i(\boldsymbol{u}) = \mu_i \quad \text{in } \Omega, \qquad u_i = 0 \quad \text{on } \partial\Omega \tag{8.17$_i$}$$

with $n = 2$, where μ_i is even allowed to be a so called *diffuse measure*. For more information, we refer to [94]. Here, let us only sketch the approach used there: System (8.17) is formulated as a fixed point problem, which is solved by Schauder's fixed point theorem. However, to apply this special fixed point theorem, $g_i \colon \mathbb{R}^2 \to \mathbb{R}$ is assumed to be continuous in both arguments *and* to be increasing in one argument. Then, the monotone behavior of g_i guarantees that a certain auxiliary problem has unique solutions, which in turn implies that the fixed point operator is continuous.

Though the presentation in [94] is sophisticated, the approach is limited by the use of Schauder's fixed point theorem, so that a trade-off between smoothness and monotonicity is not possible. However, we are positive that our approach can be used to obtain far reaching generalizations of the results in [94]. ○

9 | Further Improvements

In this thesis, we have provided a unifying framework for the study of multivalued non-smooth variational inequalities, which we have applied to various examples. For future work, there are three topics which allow for decades of research.

9.1 Improvements in Theory

In Chapter 1, we have tried our best to state a general order-theoretical fixed point theorem for multifunctions. This study can be extended to answer strongly related questions about

(i) **coupled fixed points** (see Corollary 8.6 for a first result),

(ii) **common fixed points** of a family of multifunctions $\{F_i : i \in I\}$, i.e. some u such that $u \in F_i(u)$ for all $i \in I$,

(iii) and **fixed sets** of a multifunction F, i.e. sets S such that $S = F(S)$.

The theory of fixed sets is in particular interesting, since on the power set $\mathcal{P}_\emptyset(X)$ of any set X, we have a natural partial ordering via $A \leqslant B$ if and only if $A \subset B$. If we call, in analogy to subpoints and by extending the usual definition of a subset, a set $S \in \mathcal{P}_\emptyset(X)$ a **subset** of a multifunction $F\colon X \to \mathcal{P}_\emptyset(X)$ if $S \subset F(S)$, then we have the following result, which is essentially [93, Proposition 1.2]:

9.1 Proposition *Let X be a set, let $F\colon X \to \mathcal{P}_\emptyset(X)$ be a multifunction, and let S be the greatest subset of F. Then S is the greatest fixed set of F.*

Proof: First, let us note that F has indeed a greatest subset which is given by

$$S := \bigcup\{A \in \mathcal{P}_\emptyset(X) : A \text{ is a subset of } F\}.$$

Now, assume that S is not a fixed set of F. Then there is $x \in F(S) \setminus S$. But then, $S \cup \{x\}$ is a subset of F which is greater than S (with respect to set inclusion), which is a contradiction. Thus, S is a fixed set of F. Since every fixed set of F is also a subset of F, S is indeed the greatest fixed set of F. ○

Note, however, that the greatest fixed set of a multifunction $F\colon X \to \mathcal{P}_\emptyset(X)$ could be $\emptyset$ (which is always a fixed set). But if F has a fixed point $u \in F(u)$, then $\{u\}$ is a subset of F and from Proposition 9.1 we have that F has a non-trivial fixed set. Since there are plenty of multifunctions which have a fixed set but no fixed point, the question arises which multifunctions have a non-trivial fixed set (which satisfies a given property). For further information, see [93].

In Chapter 2, we used topological methods to find a general fixed point theorem on reflexive ordered Banach spaces. As we have noted in Theorem 7.17, this framework can be extended to reflexive Fréchet spaces. The question arises if there are other connections between order-theory and topology which can be of use. The theory of Riesz spaces presented in [120] and the theory of convergence structures in [6] are a good starting point for future research.

In [6], also generalizations of well-known results in Functional Analysis are given. Maybe this gives rise to a more general formulation of Theorem 2.57 for pseudomonotone operators. Furthermore, it would be interesting to find connections between order-theoretical and topological results (like the connection between 2.38 and 2.39) which can be used to unify and extend different fixed point and existence results. In the epilog, we will give a first simple result in this direction.

In Chapter 3, we introduced the concept of ordered measurable spaces, but we gave only a few simple results. The question arises if there are deeper results and in particular such ones that can be used in the study of variational inequalities.

9.2 Improvements in Applications

Our general framework of Theorem 2.33 is designed in such a way that it can be applied to a wide range of variational problems. Whenever the method of sub-supersolutions can be applied, a next possible step in the study is to introduce bifunctions to allow for nonsmooth multifunctions. Even if the variational problem is very general, it should be possible to adjust the framework so that it can be applied (like we have done in Chapter 7). For inspiration, let us state a few possible generalizations:

In Chapters 4 and 5, we have provided an application of Theorem 2.33 to multivalued variational and quasi-variational inequalities whose leading term is a single-valued differential operator. Those results can be extended to multivalued leading terms. We refer to the recent papers [69, 70, 71, 72] for information about such multivalued problems.

Furthermore, there is no need to confine our considerations to the Sobolev space $W_0^{1,p}(\Omega)$. Most of the results can be generalized to more general function spaces, e.g. spaces of Sobolev functions with non-trivial boundary values, Sobolev spaces with respect to variable exponents, or Orlicz-Sobolev spaces.

In Chapters 6 and 7, we studied variational problems with measures on the right-hand side. Our study was inspired by [10], so that we considered bounded Radon measures and $p > 2 - 1/N$ in order to obtain solutions in Sobolev spaces. The restriction of p can be dropped if one generalizes the notion of solutions to so called **entropy solutions**, as done in [7, 35]. In such spaces, truncation methods are possible, and so the open question remains, if we can extend the results to bifunctions.

Another approach for problems involving a Radon measure μ comes into play if a problem has *no* solution. Then there might be a largest measure $\mu^* \leqslant \mu$ such that the problem with μ^* instead of μ has a solution. We refer to [15] for details and pose the question if this approach can be combined with our general framework.

In Chapter 8, we considered systems of variational problems. By fixing some arguments, we decoupled the system, then we used that there are solutions for the case $n = 1$, and finally we applied our fixed point theorem to find a solution of the whole system. This approach works also if we generalize the case $n = 1$ as descriped above. But there is another approach: If we know that a system without bifunctions has solutions (e.g. by application of topological results like in [29, 90, 91]) and the method of sub-supersolutions applies, then we may extend the results to the case of bifunctions.

Furthermore, we could extend our results to nonsmooth multivalued evolutionary problems or even to Banach-valued differential equations. See [24] for more information in this direction. Also the case of unbounded Ω is open.

Finally, it would be interesting to extend our results to variable set relations as described in [39] to treat problems of vector-valued optimization.

9.3 Real-World Applications

An important aspect not covered in this thesis is the application of the results to problems in Mechanics and Engineering and the numerical treatment. However, this is beyond the scope of this thesis, thus we refer the interested reader to the detailed introduction in [92] and the more recent expositions in [1, 19, 43, 73].

Epilog

What is provable will be proved. Why? Because I am here!
— All Math, *My Hero Mathematica*

Whereof one cannot speak, thereof one must be silent.
— Ludwig Wittgenstein, *Tractatus Logico-Philosophicus 7*

A long, long time ago ...

... there was a function $f\colon \mathbb{R} \to \mathbb{R}$*, watching over a sleeping curious mind. Suddenly—another basic question pushed the curious mind back into the mathematical realm:*

Does the equation $f(x) = x$ *have a solution?*

Zeros and Fixed Points

One could argue that this question is even more basic than the first one, especially in case the curious mind has temporarily forgotten about the special role of 0 in $\mathbb{R}$—but there is a more interesting and moreover well-defined issue: the stunning isomorphy between the underlying mathematics of the two basic questions.

Indeed, if we map each real function f to the function s, defined by

$$s(x) := x - f(x),$$

then it is readily seen that x is a zero of f if and only if x is a fixed point of s. Further, f satisfies (C^*) and (C_*) if and only if s has the following properties, respectively:

(M^*) If $(x_n) \subset \mathbb{R}$ is an increasing sequence converging to x and if $x_n \leqslant s(x_n)$ for all n, then $x \leqslant s(x)$.

(M_*) If $(x_n) \subset \mathbb{R}$ is a decreasing sequence converging to x and if $s(x_n) \leqslant x_n$ for all n, then $s(x) \leqslant x$.

Furthermore, $\underline{x}$ is a subsolution of f if and only if $\underline{x}$ is a subpoint of s, analogously for supersolutions and superpoints. Thus, we have the following corollary from our celebrated Theorem A:

Corollary *Let* $s\colon \mathbb{R} \to \mathbb{R}$ *be an increasing function, and let* $\underline{x}, \overline{x} \in \mathbb{R}$ *be such that* $\underline{x} \leqslant s(\underline{x}) \leqslant s(\overline{x}) \leqslant \overline{x}$. *Then* s *has the greatest and the smallest fixed point in* $[\underline{x}, \overline{x}]$. ○

Note, however, that this corollary is nothing more than Tarski's fixed point theorem. This points to some underlying structure for all those elementary results.

Theorem of Everything

Indeed, if we generalize the ideas used in the proofs so far, we obtain the following framework:

Definition Let X be a set, and let S, $\underline{S}$ and $\overline{S}$ be subsets of X. Then $(S, \underline{S}, \overline{S})$ is called a **Sub-Super-Problem** on X (SSP for short).

(i) If $(S, \underline{S}, \overline{S})$ is an SSP, then the elements of S are called **solutions**, the elements of $\underline{S}$ are called **subsolutions**, and the elements of $\overline{S}$ are called **supersolutions**.

(ii) An SSP $(S, \underline{S}, \overline{S})$ on X is called **natural** if $\underline{S} \cup \overline{S} = X$ and $\underline{S} \cap \overline{S} = S$. ○

Of course, even if $(S, \underline{S}, \overline{S})$ is a natural SSP, there might be no solution. But we want to find them! Thus, we build a trap:

Definition Let X be an ordered topological space.

(i) A set $A \subset X$ is called **closed upward** if for each converging and increasing sequence $(x_n) \subset A$ one has $\lim_n x_n \in A$.

Let $(S, \underline{S}, \overline{S})$ be an SSP on X.

(ii) If $\underline{x} \in \underline{S}$ and $\overline{x} \in \overline{S}$, then $[\underline{x}, \overline{x}]$ is called a **trap**.

(iii) A trap T is called **good** if $\underline{S} \cap T$ is **closed upward** and if $\overline{S} \cap T$ is **closed downward** (defined by duality). ○

Now, it is readily seen that the search for zeros of a continuous function and the search for fixed points of an increasing function both can be modelled as an SSP. Furthermore, the ideas so far give us at once the following theorem:

Theorem *Let* P *be a natural SSP on* $\mathbb{R}$. *Then each non-empty good trap* T *contains a solution, and there are even smallest and greatest solutions in* T. ○

Thus, order-theoretical fixed point results and topological existence results which we have treated as if they belong to separated fields of mathematics, are in fact two sides of the same coin. Naturally, the following question arises:

To what extend can this connection be generalized?

That was a Question ...

... that neither the curious mind nor any of his friends could answer. And so they thought, and thought, and—lived happily ever after.

Index

Bibliography

[1] V. Acary, B. Brogliato, **Numerical Methods for Nonsmooth Dynamical Systems: Applications in Mechanics and Electronics**, Lecture Notes in Applied and Computational Mechanics, Vol. 35, Springer-Verlag, Berlin, 2008.

[2] C. D. Aliprantis, K. C. Border, **Infinite Dimensional Analysis. A Hitchhiker's Guide**, 2nd Ed., Springer-Verlag Berlin, 1999.

[3] J. Appell, E. De Pascale, H. T. Nguyen, P. P. Zabrejko, **Multivalued superpositions**, Dissertationes Mathematicae (1995), Vol. 345, 1–97.

[4] J. Appell, P. P. Zabrejko, **Nonlinear Superposition Operators**, Cambridge Tracts in Mathematics (1990), Vol. 95, Cambridge University Press.

[5] D. Averna, D. Motreanu E. Tornatore, **Existence and asymptotic properties for quasilinear elliptic equations with gradient dependence**, Applied Mathematics Letters (2016), Vol. 61, 102–107.

[6] R. Beattie, H.-P. Butzmann, **Convergence Structures and Applications to Functional Analysis**, Kluwer Academic Publishers, Dordrecht, 2002.

[7] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez, **An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations**, Annali della Scuola Normale Superiore di Pisa – Classe di Scienze (1955), Vol. 22(2), 241–273.

[8] L. Boccardo, **Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form**, Progress in Elliptic and Parabolic Partial Differential Equations (Capri, 1994), 43–57, Pitman Res. Notes Math. Ser., Vol. 350, Longman, Harlow, 1996.

[9] L. Boccardo, B. Dacorogna, **A characterization of pseudomonotone differential operators in divergence form**, Communications in Partial Differential Equations (1984), Vol. 9(11), 1107–1117.

[10] L. Boccardo, T. Gallouët, **Non-linear elliptic and parabolic equations involving measure data**, Journal of Functional Analysis (1989), Vol. 87, 149–169.

[11] V. I. Bogachev, **Measure Theory: Volume I**, Springer-Verlag, Berlin, 2007.

[12] V. I. Bogachev, **Measure Theory: Volume II**, Springer-Verlag, Berlin, 2007.

[13] N. Bourbaki, **Éléments de mathématique: Théorie des ensembles**, Hermann, 1970.

[14] H. Brézis, **Équations et inéquations non-linéaires dans les espaces vectoriel en dualité**, Annales de l'Institut Fourier (1986), Vol. 18, 155–176.

[15] H. Brezis, M. Marcus, A. C. Ponce, **Nonlinear elliptic equations with measures revisited**, Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, S. Klainerman, eds.), Annals of Mathematics Studies (2007), Vol. 163, Princeton University Press, Princeton, 55–110.

[16] D. S. Bridges, G. B. Mehta, **Representations of Preferences Orderings**, Springer-Verlag, Berlin, 1995.

[17] F. Browder, **Nonlinear elliptic boundary value problems**, Bulletin of the American Mathematical Society (1963), Vol. 69(6), 862–874.

[18] T. Bühler, D. A. Salamon, **Functional Analysis**, Graduate Studies in Mathematics 191, American Mathematical Society, Providence (Rhode Island), 2018.

[19] A. Capatina, **Variational Inequalities and Frictional Contact Problems**, Advances in Mechanics and Mathematics, Vol. 31, Springer, Cham, 2014.

[20] S. Carl, **Variational approach to noncoercive systems of elliptic variational inequalities via trapping region**, Complex Variables and Elliptic Equations (2012), Vol. 57(2–4), 169–183.

[21] S. Carl, **Addendum to: Variational approach to noncoercive systems of elliptic variational inequalities via trapping region**, Complex Variables and Elliptic Equations (2012), Vol. 57(7–8), 921–924.

[22] S. Carl, **Elliptic Variational Inequalities with Discontinuous Multi-Valued Lower Order Terms**, Advanced Nonlinear Studies (2013), Vol. 13, 55–78.

[23] S. Carl, **Signorini type variational inequality with state-dependent discontinuous multi-valued boundary operators**, Nonlinear Analysis (2013), Vol. 92, 138–152.

[24] S. Carl, S. Heikkilä, **Fixed Point Theory in Ordered Sets and Applications**, Springer, New York, 2011.

[25] S. Carl, J. W. Jerome, **Trapping region for discontinuous quasilinear elliptic systems of mixed monotone type**, Nonlinear Analysis (2002), Vol. 51, 843–863.

[26] S. Carl, V. K. Le, **Existence results for hemivariational inequalities with measures**, Applicable Analysis (2007), Vol. 86, 735–753.

[27] S. Carl, V. K. Le, **Elliptic inequalities with multi-valued operators: Existence, comparison and related variational-hemivariational type inequalities**, Nonlinear Analysis: Theory, Methods & Applications (2015), Vol. 121, 130–152.

[28] S. Carl, V. K. Le, D. Motreanu, **Nonsmooth Variational Problems and Their Inequalities: Comparison Principles and Applications**, Springer Science+Business Media, New York, 2007.

[29] S. Carl, D. Motreanu, **Extremal solutions for nonvariational quasilinear elliptic systems via expanding trapping regions**, Monatshefte für Mathematik (2017), Vol. 182, 801–821.

[30] S. Carl, C. Tietz, **Quasilinear elliptic equations with measures and multi-valued lower order terms**, Discrete & Continuous Dynamical Systems – S (2018), Vol. 11(2), 193–212.

[31] C. Castaing, M. Valadier, **Convex Analysis and Measurable Multifunctions**, Springer, Berlin, 2006.

[32] F.H. Clarke, **Optimization and nonsmooth analysis**, 2^{nd} edition, Classics in Applied Mathematics, Vol. 5, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1990.

[33] F. Clément, V. Martin, **The Lax-Milgram Theorem: A detailed proof to be formalized in Coq**, arXiv:1607.03618, 2016.

[34] R. Dacić, **Fixed Points Of Monotone Multifunctions In Partially Ordered Sets**, Publications de l'Institut Mathématique (1980), Vol. 27(41), 41–50.

[35] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, **Renormalized solutions of elliptic equations with general measure data**, Annali della Scuola Normale Superiore di Pisa – Classe di Scienze (4) (1999), Vol. 28, 741–808.

[36] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, **Lebesgue and Sobolev Spaces with Variable Exponents**, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011.

[37] P. Drábek, J. Milota, **Methods of Nonlinear Analysis: Applications to Differential Equations**, Birkhäuser Advances Texts, Birkhäuser Verlag, Basel, 2007.

[38] J. J. Duistermaat and J. A. C. Kolk, **Distributions: Theory and Applications**,

Birkhäuser, Boston, 2010.

[39] G. Eichfelder, M. Pilecka, **Set Approach for Set Optimization with Variable Ordering Structures: Part I: Set Relations and Relationship to Vector Approach**, Journal of Optimization Theory and Applications (2016), Vol. 171(3), 931–946.

[40] J. Elstrodt, **Maß- und Integrationstheorie**, 7. Auflage, Springer-Verlag, Berlin, 2011.

[41] Y. Feng, Y. Wang, **Fixed points of multi-valued monotone operators and the solvability of a fractional integral inclusion**, Fixed Point Theory and Applications (2016), 2016:64.

[42] G. B. Folland, **Real Analysis. Modern Techniques and their Applications**, 2nd edition, John Wiley & Sons, Inc, New York, 1999.

[43] C. Glocker, **Set-Valued Force Laws: Dynamics of Non-Smooth Systems**, Lecture Notes in Applied Mechanics, Vol. 1, Springer-Verlag, Berlin, 2001.

[44] L. Górniewicz, **Topological Fixed Point Theory of Multivalued Mappings**, Kluwer Academic Publisher, Dordrecht, 1999.

[45] A. Granas, J. Dugunji, **Fixed Point Theory**, Springer-Verlag, New York, 2003.

[46] G. Grätzer, **Lattice Theory: First Concepts and Distributive Lattices**, W. H. Freeman and Company, San Francisco, 1971.

[47] P. Halmos, **Naive Set Theory**, Van Nostrand, 1960.

[48] A. H. Hamel, F. Heyde, A. Löhne, B. Rudloff, C. Schrage (Eds.), **Set Optimization and Applications: The State of the Art**, Springer-Verlag, Berlin, 2015.

[49] P. Harjulehto, P. Hästö, **Orlicz Spaces and Generalized Orlicz Spaces**, manuscript, June 2018.

[50] S. Heikkilä, V. Lakshmikantham, **Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations**, Marcel Dekker, Inc., New York, 1994.

[51] C. J. Himmelberg, **Measurable relations**, Fundamenta Mathematicae (1975), Vol. 87(1), 53–72.

[52] P. Hitzler, A. Seda, **Mathematical Aspects of Logic Programming Semantics**, Taylor & Francis Group, Boca Raton, 2011.

[53] S. Hu, N. S. Papageorgiou, **Handbook of Multivalued Analysis: Volume I: Theory**, Kluwer Academic Publishers, Dordrecht, 1997.

[54] N. B. Huy, T. T. Binh, V. V. Tri, **The monotone minorant method and eigenvalue problem for multivalued operators in cones**, Fixed Point Theory (2018), Vol. 19(1), 275-286.

[55] R. Israel on http://math.stackexchange.com/questions/128703/joint-measurability-of-a-function?rq=1.

[56] G. Jameson, **Ordered Linear Spaces**, Springer-Verlag, Berlin, 1970.

[57] J. W. Jerome, **A Trapping Principle for Discontinuous Elliptic Systems of Mixed Monotone Type**, Journal of Mathematical Analysis and Applications (2001), Vol. 262(2), 700–721.

[58] M. Kamenskii, V. Obukhovskii, P. Zecca, **Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces**, de Gruyter Series in Nonlinear Analysis and Applications 7, Walter de Gruyter, Berlin, 2011.

[59] A. Kechris, **Classical Descriptive Set Theory**, Springer-Verlag, New York, 1995.

[60] A. A. Khan, D. Motreanu, **Existence Theorems for Elliptic and Evolutionary Variational and Quasi-Variational Inequalities**, Journal of Optimization Theory and Applications (2015), Vol. 167(3), 1136–1161.

[61] A. Khan, C. Tammer, C. Zalinescu, **Regularization of quasi-variational inequalities**,

Optimization (2015), Vol. 64(8), 1703–1724.

[62] D. Kinderlehrer, G. Stampacchia, **An Introduction to Variational Inequalities and Their Applications**, Academic Press, New York, 1980.

[63] W. Kirk, B. Sims (Eds.), **Handbook of Metric Fixed Point Theory**, Kluwer Academic Publishers, Dordrecht, 2001.

[64] D. Kuroiwa, **The natural criteria in set-valued optimization**, Sūrikaisekikenkyūsho Kōkyūroku (1998), 1031:85–90.

[65] V. K. Le, **A range and existence theorem for pseudomonotone perturbations of maximal monotone operators**, Proceedings of the American Mathematical Society (2011), Vol. 139(5), 1645–1658.

[66] V. K. Le, **Variational Inequalites with General Multivalued Lower Order Terms Given by Integrals**, Advanced Nonlinear Studies (2011), Vol. 11, 1–24.

[67] V. K. Le, **On variational and quasi-variational inequalities with multivalued lower order terms and convex functionals**, Nonlinear Analysis (2014), Vol. 94, 12–31.

[68] V. K. Le, **On quasi-variational inequalities with discontinuous multivalued lower order terms given by bifunctions**, Differential Integral Equations (2015), Vol. 28(11/12), 1197–1232.

[69] V. K. Le, **Existence Results for Quasi-Variational Inequalities with Multivalued Perturbations of Maximal Monotone Mappings**, Results in Mathematics (2016), Vol. 71(1), 423–453.

[70] V. K. Le, **On the convergence of solutions of inclusions containing maximal monotone and generalized pseudomonotone mappings**, Nonlinear Analysis (2016), Vol. 143, 64–88.

[71] V. K. Le, **On Variational Inequalities with Multivalued Perturbing Terms Depending on Gradients**, Differential Equations and Dynamical Systems (2017).

[72] V. K. Le, **On the Solvability of Inclusions with Multivalued Compact Perturbations of Bi-Mappings**, Set-Valued and Variational Analysis (2017).

[73] R. Leine, N. van de Wouw, **Stability and Convergence of Mechanical Systems with Unilateral Constraints**, Springer-Verlag, Berlin, 2008.

[74] J. Leray, J.-L. Lions, **Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder**, Bulletin de la Société Mathématique de France (1965), Vol. 93, 97–107.

[75] J. Lewin, **A simple proof of zorn's lemma**, The American Mathematical Monthly (1991), 98(4), 353–354.

[76] J. Li, **Several extensions of the Abian–Brown Fixed Point Theorem and their applications to extended and generalized Nash equilibria on chain-complete posets**, Journal of Mathematical Analysis and Applications (2014), Vol. 409, 1084–1092.

[77] J. Li, **Fixed point theorems on partially ordered topological vector spaces and their applications to equilibrium problems with incomplete preferences**, Fixed Point Theory and Applications (2014), 2014:192.

[78] J. Li, **Inductive properties of fixed point sets of mappings on posets and partially ordered topological spaces**, Fixed Point Theory and Applications (2015), 2015:211.

[79] J. Li, **On the Normality, Regularity and Chain-completeness of Partially Ordered Banach Spaces and Applications**, arXiv:1705.01580, 2017.

[80] J. Li, E. A. Ok, **Optimal solutions to variational inequalities on Banach lattices**, Journal of Mathematical Analysis and Applications(2012), Vol. 388(2), 1157–1165.

[81] J.-L. Lions, **Quelques Méthodes de Résolutions des Problèmes aux Limites Nonlinéaires**, Dunod, Paris, 1969.

[82] N. H. Loc, K. Schmitt, **Bernstein–Nagumo conditions and solutions to nonlinear differential inequalities**, Nonlinear Analysis: Theory, Methods & Applications (2012), Vol. 75(12), 4664–4671.

[83] A. Löhne, **Vector Optimization with Infimum and Supremum**, Springer-Verlag, Berlin, 2011.

[84] S. Migórski, A. Ochal, M. Sofonea, **A Class of Variational-Hemivariational Inequalities in Reflexive Banach Spaces**, Journal of Elasticity (2017), Vol. 127(2), 151–178.

[85] G. J. Minty, **On a "monotonicity" method for the solution of nonlinear equations in Banach spaces**, Proceedings of the National Academy of Sciences of the United States of America (1963), Vol. 50(6), 1038–1041.

[86] G. H. Moore, **Zermelo's Axiom of Choice: Its Origins, Development, and Influence**, Springer-Verlag, New York, 1982.

[87] B. S. Mordukhovich, Y. Shao, **Nonsmooth sequential analysis in Asplund spaces**, Transactions of the American Mathematical Society (1996), Vol. 348(4), 1235–1280.

[88] S. A. Morris, E. S. Noussair, **The Schauder-Tychonoff Fixed Point Theorem and Applications**, Matematický č asopis (1975), Vol. 25(2), 165–172.

[89] D. Motreanu, V. V. Motreanu, A. Moussaoui, **Location of nodal solutions for quasilinear elliptic equations with gradient dependence**, Discrete & Continuous Dynamical Systems – S (2018), Vol. 11(2), 293–307.

[90] D. Motreanu, A. Moussaoui, D. S. Pereira, **Multiple Solutions for Nonvariational Quasilinear Elliptic Systems**, Mediterranean Journal of Mathematics (2018), 15:88.

[91] D. Motreanu, C. Vetro, F. Vetro, **Systems of quasilinear elliptic equations with dependence on the gradient via subsolution-supersolution method**, Discrete & Continuous Dynamical Systems – S (2018), Vol. 11(2), 293–307.

[92] Z. Naniewicz, P. D. Panagiotopoulos, **Mathematical Theory of Hemivariational Inequalities and Applications**, Marcel Dekker, Inc., New York, 1995.

[93] E. A. Ok, **Fixed set theory for closed correspondences with applications to self-similarity and games**, Nonlinear Analysis: Theory, Methods & Applications (2004), Vol. 56(3), 309–330.

[94] L. Orsina, A. C. Ponce, **Semilinear elliptic equations and systems with diffuse measures**, Journal of Evolution Equations (2008), Vol. 8(4), 781–812.

[95] A. C. Ponce, **Selected problems on elliptic equations involving measures**, preprint, arXiv1204.0668v2.

[96] M. M. Rao, **Measure Theory and Integration**, Marcel Dekker, Inc., New York, Basel, 2004.

[97] R. T. Rockafellar, **Characterization of the subdifferentials of convex functions**, Pacific Journal of Mathematics (1966), Vol. 17(3), 497–510.

[98] T. Roubíček, **Nonlinear Partial Differential Equations with Applications**, 2nd edition, Springer, Basel, 2013.

[99] H. L. Royden, **Real Analysis**, 3rd edition, Macmillan, New York, 1988.

[100] W. Rudin, **Functional Analysis**, McGrawHill, New York, 1973.

[101] T. Runst, W. Sickel, **Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations**, de Gruyter Series in Nonlinear Analysis and Applications 3, Walter de Gruyter, Berlin, 1996.

[102] A. Sapounakls (a.k.a. A. Sapounakis), **Properties of measures on totally ordered spaces**, Rendiconti Del Circolo Matematico di Palermo Serie II (1989), Vol. 38, 277–296.

[103] H. H. Schaefer, **Topological Vector Spaces**, Springer-Verlag, New York, 1999.

[104] K. Schmitt, **Revisiting the method of sub- and supersolutions for nonlinear elliptic problems**, Electronic Journal of Differential Equations (2007), Conference 15, 377–385.

[105] B. S. W. Schröder, **Ordered Sets: An Introduction**, Birkhäuser, Boston, 2003.

[106] R. E. Showalter, **Monotone Operators in Banach Space and Nonlinear Partial Differential Equations**, American Mathematical Society, Providence, RI, 1997.

[107] R. E. Smithson, **Fixed Points of Order Preserving Multifunctions**, Proceedings of the American Mathematical Society (1971), Vol. 28(1), 304–310.

[108] T. Tao, **Well-ordered sets, ordinals, and Zorn's lemma (optional)**, 245B, Notes 7, https://terrytao.wordpress.com/2009/01/28/245b-notes-7-well-ordered-sets-ordinals-and-zorns-lemma-optional/.

[109] A. Tarski, **A lattice-theoretical fixpoint theorem and its applications**, Pacific Journal of Mathematics (1955), Vol. 5(2), 285–309.

[110] C. Tietz, **An order theoretic fixed point theorem with application to multivalued variational inequalities**, Journal of Fixed Point Theory and Applications (2019), 21:5.

[111] C. Tietz, **Correction to: An order theoretic fixed point theorem with application to multivalued variational inequalities with nonsmooth bifunctions**, Journal of Fixed Point Theory and Applications (2019), 21:30.

[112] C. Tietz, **Multivalued Quasi-Variational Inequalities with Nonsmooth Bifunctions**, 2020. In preparation.

[113] C. Tietz, **Multivalued Elliptic Inclusions with Nonsmooth Bifunctions and L^1 Right-Hand Sides**, 2020. In preparation.

[114] C. Tietz, **Systems of Quasilinear Elliptic Equations with L^1-Functions and of Quasi-Variational Multivalued Variational Inequalities**, 2020. In preparation.

[115] G. M. Troianiello, **Elliptic Differential Equations and Obstacle Problems**, Plenum Press, New York, 1987.

[116] I. I. Vrabie, **Compactness Methods for Nonlinear Evolutions**, Pitman Monographs and Surveys in Pure and Applied Mathematikcs 75, 2^{nd} edition, Longman, New York, 1995.

[117] A. Wieczorek, **Ordering probabilities on an ordered measurable space**, Fundamenta Mathematicae (1978), Vol. 101(1), 53–78.

[118] R. Whitley, **An Elementary Proof of the Eberlein-Šmulian Theorem**, Mathematische Annalen (1967), Vol. 172, 116–118.

[119] J. Yeh, **Real Analysis: Theory of Measure and Integration**, World Scientific Publishing, New Jersey, 2006.

[120] A. C. Zaanen, **Introduction to Operator Theory in Riesz Spaces**, Springer-Verlag, Berlin, 1997.

[121] E. Zeidler, **Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators**, Springer-Verlag, New York, 1990.

[122] E. Zeidler, **Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators**, Springer-Verlag, New York, 1990.

[123] E. Zeidler, **Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization**, Springer-Verlag, New York, 1985.

[124] M. Zorn, **A remark on method in transfinite algebra**, Bulletin of the American Mathematical Society (1935), Vol. 41(10), 667–670.

[125] W. Zygmunt, **On superpositional measurability of semi-Carathéodory multifunctions**, Commentationes Mathematicae Universitatis Carolinae (1994), Vol. 35(4), 741–744.

Eigenständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit mit dem Thema

Variational Inequalities with Multivalued Bifunctions

eigenständig, ohne unzulässige Hilfe Dritter und nur unter Verwendung der angegebenen Quellen und Hilfsmittel verfasst habe. Alle Passagen, die ich wörtlich oder sinngemäß aus der Literatur übernommen habe, wurden als Zitat mit Angabe der Quelle kenntlich gemacht.

Dies ist mein erster Promotionsversuch. Die eingereichte Arbeit ist in keiner Fassung einer anderen Fakultät vorgelegt worden.

Christoph Tietz
Dresden, 19. Februar 2020

Lebenslauf

Ob es mehr zu wissen gibt? Selbstverständlich!

Persönliche Angaben

Name	Christoph Tietz
Geburtsdatum	18.07.1990
Geburtsort	Merseburg (Saale)
Anschrift	Stollestr. 40, 01159 Dresden
E-Mail	christoph.tietz.mathematik@gmail.com
Nationalität	deutsch

Akademischer Werdegang

ab 03/2019	**Wissenschaftlicher Mitarbeiter** an der Technischen Universität Dresden
10/2015 – 03/2016	**Wissenschaftlicher Mitarbeiter** an der Martin-Luther-Universität Halle-Wittenberg
10/2010 – 12/2015	**Wissenschaftliche Hilfskraft** an der Martin-Luther-Universität Halle-Wittenberg (mit zeitlichen Unterbrechungen)

Studium

seit 02/2016	**Promotion** an der Martin-Luther-Universität Halle-Wittenberg Thema: *Quasilineare elliptische Variationsungleichungen mit unstetigen mengenwertigen Nichtlinearitäten*
11/2013 – 09/2015	**Master** an der Martin-Luther-Universität Halle-Wittenberg Mathematik mit Anwendungsfach Wirtschaftswissenschaften Masterarbeit: *Existenz und Einschließung bei nichtlokalen elliptischen Randwertproblemen*
10/2009 – 11/2013	**Bachelor** an der Martin-Luther-Universität Halle-Wittenberg Mathematik mit Anwendungsfach Informatik Bachelorarbeit: *Der Spektralsatz für sektorielle Operatoren*

Christoph Tietz
Dresden, 19. Februar 2020