Existence Results for Vector Quasi-Variational Problems

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To Luisa. To Ingrid and Thomas.

Contents

List of Symbols and Abbreviations

$:=$	equal by definition,
$\doteq$	rounding,
$\mathbb{N}$	set of natural numbers,
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$,
$k, l \in \mathbb{N}$	two specific natural numbers,
$\mathbb{Q}$	set of rational numbers,
$\mathbb{R}$	set of real numbers,
$\mathbb{R}_>$	set of positive real numbers,
$\mathbb{R}_\geq$	set of non-negative real numbers,
$\mathbb{R}^k$	k-dimensional Euclidean space,
e_j	jth unit vector in $\mathbb{R}^k$,
$\\|\cdot\\|_2$	Euclidean norm in $\mathbb{R}^l$,
$a^1, \ldots, a^k$	points in $\mathbb{R}^2$,
$\mathcal{A}$	convex hull of the locations $a^1, \ldots, a^k$,
X, Y	real Banach spaces with corresponding norms $\\|\cdot\\|_X$ and $\\|\cdot\\|_Y$, respectively,
X^*, Y^*	dual spaces of X and Y, respectively,
$\mathrm{L}(X, Y)$	space of linear and bounded mappings from X to Y,
$\mathrm{Mat}_{k\times l}(\mathbb{R})$	space of $k \times l$ matrices with entries in $\mathbb{R}$,
C	non-empty, closed and convex subset of X,
C_n	discrete subset of $\mathbb{R}^l$ with cardinality $n \in \mathbb{N}$,
$\langle\cdot,\cdot\rangle$	Euclidean scalar product, duality pairing, inner product, or evaluation brackets,
$x^\top y$	Euclidean scalar product of two vectors $x, y \in \mathbb{R}^l$,
$\varepsilon_n \downarrow 0$	right-sided convergence of a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_>$ to 0,
$x_n \to x$	(strong) convergence of a sequence $\{x_n\}$ to x,
$x_n \rightharpoonup x$	weak convergence of a sequence $\{x_n\}$ to x,
ψ	vector-valued mapping $\psi : X \to Y$,

$\psi(C)$	image of C under ψ,
Proj	(orthogonal) projection onto the set C,
$\partial\psi(x)$	(weak) subdifferential of ψ at $x \in X$,
$\delta\psi(x;h)$	directional derivative of ψ at $x \in X$ in direction $h \in X$,
$D_{\mathrm{G}}\psi(x)$	Gâteaux-derivative of ψ at $x \in X$,
F, R	two mappings from X to $\mathrm{L}(X,Y)$,
F^{-}	(shifted) adjoint of F,
$L^p(\mu)$	Lebesgue space,
$\emptyset$	empty set,
A, B	non-empty subsets of Y,
$\lvert A\rvert$	cardinality of A,
$A \subseteq B$	A is a subset of B,
$A \subsetneqq B$	A is a proper subset of B, that is, $A \subseteq B$ and $A \neq B$,
$A \not\subseteq B$	A is not a subset of B,
$A \cup B$	union of the sets A and B,
$A \cap B$	intersection of the sets A and B,
$A \setminus B$	complement of B in A,
$A + B$	Minkowski sum of A and B,
$A - B$	Minkowski difference of A and B,
$a \pm B$	$a \pm B := \{a\} \pm B$, $B \pm a := a \pm B$,
λA	multiplication of the scalar $\lambda \in \mathbb{R}$ with A,
$\operatorname{bd} A$	topological boundary of the set A,
$\operatorname{cl} A$	topological closure of the set A,
$\operatorname{conv} A$	convex hull of the set A,
$\operatorname{int} A$	topological interior of the set A,
$d_H(A,B)$	Hausdorff distance of the sets A and B,
$B(x_0, r)$	closed ball centered at $x_0 \in X$ with radius $r > 0$,
$\mathbb{R}^k_{\geq}$	natural ordering cone in $\mathbb{R}^k$,
K	cone in Y,
K^p	natural ordering cone in $L^p(\mu)$,
K^*	dual cone of K,
$\operatorname{qi} K^*$	quasi-interior of the dual cone of K,
$\leq_K$	partial ordering induced by the proper, convex and pointed cone K,
$\leq_{\operatorname{int} K}$	strict partial ordering induced by the proper, solid, convex and pointed cone K,
$A \preccurlyeq^{1,2}_{\operatorname{int} K} B$,	weak binary set relations w.r.t. the solid convex cone K,

$\mathrm{Min}(A, K)$	set of minimal elements w.r.t. the convex cone K,
$\mathrm{Max}(A, K)$	set of maximal elements w.r.t. the convex cone K,
$\mathrm{WMin}(A, K)$	set of weakly minimal elements w.r.t. the convex and solid cone K,
$\mathrm{WMax}(A, K)$	set of weakly maximal elements w.r.t. the convex and solid cone K,
$\mathrm{Eff}(\psi(C), K)$	set of efficient elements w.r.t. the convex and pointed cone K,
$\mathrm{WEff}(\psi(C), K)$	set of weakly efficient elements w.r.t. the solid, convex and pointed cone K,
$\mathcal{K}$	variable domination structure, i.e., $\mathcal{K} : X \rightrightarrows Y$ is a set-valued mapping whose values are proper, closed, convex, pointed and solid cones in Y,
$\mathrm{Eff}(\psi(C), \mathcal{K})$	set of efficient elements w.r.t. the variable domination structure $\mathcal{K}$,
$\mathrm{WEff}(\psi(C), \mathcal{K})$	set of weakly efficient elements w.r.t. the variable domination structure $\mathcal{K}$,
$-\infty_Y, +\infty_Y$	smallest and greatest element in the extended space $Y \cup \{\pm\infty_Y\}$,
χ_C	(generalized) indicator mapping of the non-empty set C,
$\mathcal{D}(\varphi)$	effective domain of an extended mapping $\varphi : X \to Y \cup \{\pm\infty_Y\}$,
$\varphi^*(U)$	weak conjugate of φ at $U \in \mathrm{L}(X, Y)$,
S	set-valued mapping, variational selection,
S^{-1}	inverse of S,
$\mathcal{D}(S)$	domain of S,
$\mathcal{R}(S)$	range of S,
$\mathcal{G}(S)$	graph of S,
$S(C)$	image of C under S,
Sol (VVI)	solution set of vector variational inequality (3.1.1),
Sol (VI_s)	solution set of variational inequality (3.2.3) w.r.t. s,
Sol (VQVI)	solution set of vector quasi-variational inequality (5.0.5),
Sol (QVI_s)	solution set of quasi-variational inequality (5.1.12) w.r.t. s,
Sol (VVI_n)	solution set of the discrete finite-dimensional vector variational inequality (6.2.1).

Chapter 1

Introduction

The origin of variational inequalities goes back to the work of Fichera [65], who formulated a contact problem in elasticity, introduced by his friend and teacher Signorini and nowadays known as *Signorini problem*, as variational inequality. One year later, in 1964, the first cornerstone for the theory of variational inequalities was posed by Stampacchia [152]. In his paper, Stampacchia coined the name *variational inequality* for all problems dealing with inequalities of this kind. Two years later, Hartman and Stampacchia [89] used variational inequalities as an analytic tool for solving partial differential equations with applications in mechanics and elasticity. The latter works started an intensive study of the subject by numerous celebrated researchers, such as Brezis, Browder, Kinderlehrer, Lions, and many others; see [22, 23, 135]. Some prolific applications of variational inequalities can be found, for example, in economics [88, 100, 142], structural mechanics [125, 145], optimization [76], physics [112, 141, 167], and in many other fields of pure and applied mathematics. The reader can also be referred to the book of Kinderlehrer and Stampacchia [121] and the books of Facchinei and Pang [57, 58].

Later, in 1980, Giannessi [71] extended the notion of variational inequalities to the one of finite-dimensional *vector variational inequalities*. Within the last 40 years, vector variational inequalities have turned out to be a powerful tool for studying numerous mathematical models in applied and industrial mathematics, for example, in multi-objective optimization and related fields which consist of the simultaneous investigation of contrary tasks; see Ansari, Köbis and Yao [10], Elster, Hebestreit, Khan and Tammer [56], Giannessi [73], Giannessi and Mastroeni [77] and Göpfert, Tammer and Zălinescu [78]. Furthermore, a detailed introduction to some of the recent developments in the field of vector variational inequalities and related problems can be found in the survey papers by Giannessi, Mastroeni and Yang [75] and Hebestreit [91].

Motivating examples ([91, Section 1])**. 1.** Let us investigate the following prominent problem: Given a non-empty, closed and convex subset C of $\mathbb{R}^2$ and a point $a^1 \in \mathbb{R}^2 \setminus C$, we want to find the *best approximation* of a^1 (to the set C w.r.t. the Euclidean norm $\|\cdot\|_2$), that is, we consider the following optimization problem: Find $x \in C$ such that

$$\|x - a^1\|_2 \leq \|y - a^1\|_2, \qquad \text{for every} \quad y \in C.$$

Figure 1.1 indicates that the best approximation of a^1 is characterized by the following observation: An element $x \in C$ is the best approximation of a^1 if and only if for all $y \in C$ the angle between $x - a^1$ and $x - y$ is greater than 90°, or equivalently,

$$\measuredangle(x - a^1, x - y) = \frac{\langle x - a^1, x - y\rangle}{\|x - a^1\|_2 \|x - y\|_2} \leq 0, \qquad \text{for every} \quad y \in C \setminus \{x\},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^l$. Therefore, finding the best approximation of a^1 is equivalent to solving the following *variational inequality*: Find $x \in C$ such that

$$\langle x - a^1, y - x\rangle \geq 0, \qquad \text{for every} \quad y \in C. \tag{1.0.1}$$

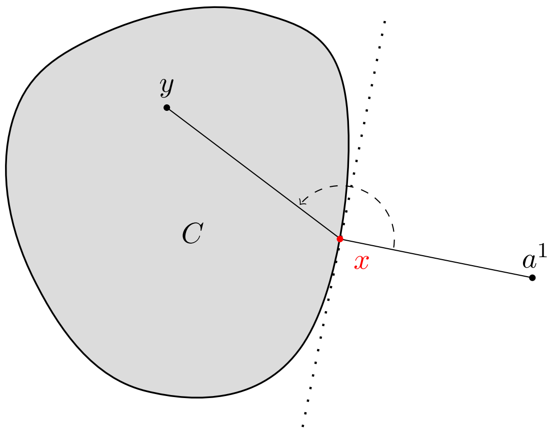

Figure 1.1: Geometric interpretation of variational inequality (1.0.1)

2. Let $a^2 \in C$ be another point. We are now looking for all points $x \in C$ such that the Euclidean distance between $x - a^1$ and $x - a^2$ is minimal simultaneously. Here, minimization is understood in the sense that it is impossible to decrease the distance to a^1 (or a^2, respectively) without increasing the distance to a^2 (or a^1, respectively) at the same time. As illustrated in Figure 1.2, the element $\tilde{x}$ is non-minimal since one can shift it upwards which would cause a simultaneous decrease of the distance between $\tilde{x}$ to the points a^1 and a^2. We further observe than an element $x \in C$ is optimal if for every $y \in C$ the angles between $x - a^1$ and $x - y$ as well as between $x - a^2$ and $x - y$ are not bigger than 90° at the same time, that is, either $\measuredangle(x - a^1, x - y) \leq 0$ or $\measuredangle(x - a^2, x - y) \leq 0$ for every $y \in C$. Consequently, this is equivalent to saying that the element $x \in C$ is a solution of the following *vector variational inequality*: Find $x \in C$ such that

$$\begin{pmatrix} \langle x - a^1, y - x\rangle \\ \langle x - a^2, y - x\rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^2_{\geq}, \qquad \text{for every} \quad y \in C. \tag{1.0.2}$$

Notice that the solution set of vector variational inequality (1.0.2) is given by the line segment $[a^1, a^2] \cap C$; compare Figure 1.2.

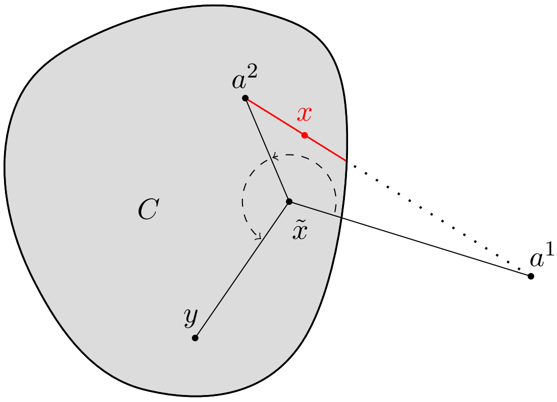

Figure 1.2: Geometric interpretation of vector variational inequality (1.0.2)

To be precise, consider a mapping $F : X \to \mathrm{L}(X,Y)$, where X and Y are real Banach spaces and $\mathrm{L}(X,Y)$ denotes the space of linear and bounded operators from X to Y. If C is a non-empty, closed and convex subset of X and K is a proper, closed, convex and solid cone in Y, then the *vector variational inequality* over the feasible set C consists of finding $x \in C$ such that

$$\langle Fx, y - x\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C. \tag{1.0.3}$$

Here, the abbreviation $\langle Fx, y - x\rangle := (F(x))(y - x)$ is used. Besides vector variational inequality (1.0.3), several extension of it have been studied in the literature, e.g. Ansari, Köbis and Yao [10], Chen [31], Giannessi [73], Huang, Ma, O'Regan and Wu [95] and Kim and Kum [115]. However, without proposing additional assumptions to the data of problem (1.0.3), it is not clear whether the vector variational inequality attains a solution or not. In the field of vector variational inequalities, there are numerous papers focusing on existence theorems for problem (1.0.3) and extensions of it, e.g. Ceng and Huang [28], Chen and Yang [39], Hebestreit, Khan, Köbis and Tammer [93], Khan and Usman [109], Konnov and Yao [124], Kim, Lee, Lee and Yen [119], Yang [159] and Yao and Zeng [161]. Further, an extensive survey of existence results can be found in the book of Ansari, Köbis and Yao [10] and in the survey papers of Giannessi, Mastroeni and Yang [75] and Hebestreit [91].

In order to ensure that vector variational inequality (1.0.3) attains a solution, the data of problem (1.0.3) have to be regular in some certain sense. A condition that ensures regularity of the data is called *coercivity condition*. Some coercivity conditions, which have frequently been used in the literature [10, 36, 39, 62, 124, 148], are:

1. The constraining set C is *bounded*.
2. F is *v-coercive*, that is, there exists a non-empty and compact subset B of X and an element $y_0 \in B \cap C$ such that

$$\langle Fy, y_0 - y\rangle \in \operatorname{int} K, \qquad \text{for every} \quad y \in C \setminus B.$$

3. C is unbounded and F is *weakly coercive*, that is, there exists an element $x_0 \in C$

and a functional s in the quasi-interior of K^* such that

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\langle s \circ Fx - s \circ Fx_0, x - x_0 \rangle}{\|x - x_0\|_X} = +\infty.$$

Recently, Hebestreit, Khan, Köbis and Tammer [93] proposed a new existence theorem for vector variational inequality (1.0.3), which uses the following novel coercivity condition; see [93, Definition 2.11]:

4. C is unbounded and F is *κ-coercive*, that is, there exists a (non-negative) mapping $\kappa : X \to \mathbb{R}_{\geq}$ and a functional $s \in K^* \setminus \{0\}$ with

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\kappa(x)}{\|x\|_X} = +\infty$$

and

$$\langle s \circ Fx, x \rangle \geq \|s\|_{Y^*} \, \kappa(x), \qquad \text{for every} \quad x \in C.$$

However, in the absence of any coercivity condition, the existence results in the literature cannot be applied; compare Example 3.8 in [93]. For this purpose, Luong [136] proposed to study a family of so-called unconstrained penalized problems instead. He further showed that every penalized problem has a solution and that the sequence of penalized solutions converges to a solution of the original problem. Unfortunately, he considered the finite-dimensional case only, where $X = \mathbb{R}^l$ and $Y = \mathbb{R}^k$. In addition to that, the feasible set C is given by inequality constraints and in the main results [136, Theorem 3.10, Theorem 3.12], the objective mapping F is assumed to be coercive in a certain sense. However, by imposing such a regularity condition, the penalization approach is superfluous. Recently, Hebestreit, Khan, Köbis and Tammer [93] proposed an extension of the well-known Browder-Tikhonov regularization method [121] which has been extensively used for variational and quasi-variational inequalities, e.g. Alber [1], Alber and Ryazantseva [2], Giannessi and Khan [74] and Théra and Tichatschke [153].

To shed more light on this idea, assume that problem (1.0.3) is non-coercive and let a mapping $R : X \to \mathrm{L}(X, Y)$ and a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_>$ of positive parameters be given. Instead of problem (1.0.3), the authors in [93] considered the following family of *regularized vector variational inequalities*: Find $x_n = x(\varepsilon_n) \in C$ such that

$$\langle Fx_n + \varepsilon_n R x_n, y - x_n \rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C. \tag{1.0.4}$$

In the above, R is the *regularization mapping* and ε_n is the *regularization parameter* to problem (1.0.4). It should be noted that the above family evolves from problem (1.0.3) by replacing F with the perturbed mapping $F + \varepsilon_n R : X \to \mathrm{L}(X, Y)$. Due to some nice features of R only, the regularized mapping $F + \varepsilon_n R$ has significantly better properties than F and every regularized vector variational inequality (1.0.4) has a solution. Hence, one can study the sequence $\{x_n\}$ of regularized solutions, which, under some boundedness conditions, has a weakly or strongly convergent subsequence. It turns out that the (weak) limit point of $\{x_n\}$ is a solution of vector variational inequality (1.0.3). In other words, the regularization method, proposed in [93], enables one to approximate any non-coercive vector variational inequality by a family of (well-behaving) regularized vector variational inequalities.

Besides existence theorems, another research interest are inverse (or, dual) results for vector variational inequalities. The fundamental idea goes back to the work of Mosco [138]. In 1972, he introduced a *dual variational inequality*, using the Fenchel conjugate for convex functions. For this purpose, Mosco used the term *dual* in order to point out similarities to the duality principle in optimization, e.g. Boţ, Grad and Wanka [20], Gao [69], Göpfert, Tammer, Zălinescu [78] and Goh and Yang [80]. Some years later, the first attempt to extend Mosco's idea to the vector case has been made by Yang [159]. Unfortunately, the main result in the paper of Yang, compare [159, Theorem 3], contains some crucial errors, which cannot be fixed offhand. Consequently, the results in Chen, Huang and Yang [35], Chen, Kim, Lee and Lee [36], Chen and Li [38] and Chen and Yang [40], which copied the errors, are also incorrect. Nevertheless, the ideas in [159] have been carefully adapted in [56], where Elster, Hebestreit, Khan and Tammer introduced two *inverse vector variational inequalities*. The fundamental idea in [56] is to embed the (generalized) vector variational inequality into the two inverse problems in the sense, that, under suitable assumptions, every solution of the first inverse problem generates one of the vector variational inequality, and every solution of the vector variational inequality generates a solution of the second inverse problem.

To be precise, assume again that X and Y are real Banach spaces and let C be a non-empty, closed and convex subset of X. Suppose further that $F : X \rightrightarrows \mathrm{L}(X, Y)$ is a set-valued mapping, $\varphi : X \to Y \cup \{+\infty_Y\}$ and $\mathcal{K} : X \rightrightarrows Y$ is a variable domination structure. For corresponding definitions, see Chapter 4 of this thesis. Then, the *generalized vector variational inequality* w.r.t. the variable domination structure $\mathcal{K}$ consists of finding $x \in C \cap \mathcal{D}(\varphi)$ such that for some $U \in F(x)$ it holds that

$$\langle U, y - x\rangle \nleq_{\operatorname{int} \mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X. \tag{1.0.5}$$

Evidently, by suitably adjusting the data, problem (1.0.5) recovers vector variational inequality (1.0.3) as a special case. Denoting the vector conjugate of φ by φ^*, the two inverse problems [56, Section 4] for generalized vector variational inequality (1.0.5) are:

1. *First inverse vector variational inequality.* Find an operator $U \in \mathcal{D}(F^{-1}(-\,\cdot))$ and $x \in F^{-1}(-U) \cap C \cap \mathcal{D}(\varphi)$ such that

$$\begin{aligned} &\langle V - U, -x\rangle \not\preccurlyeq^1_{\operatorname{int} \mathcal{K}(x)} \varphi^*(U) - \varphi^*(V), \\ &\qquad\qquad\qquad \text{for every} \quad V \in \mathrm{L}(X, Y) \;\text{ with }\; \varphi^*(V) \neq \emptyset. \end{aligned} \tag{1.0.6}$$

2. *Second inverse vector variational inequality.* Find an operator $U \in \mathcal{D}(F^{-1}(-\,\cdot))$

and $x \in F^{-1}(-U) \cap C \cap \mathcal{D}(\varphi)$ such that

$$\langle V - U, -x \rangle \not\leq^2_{\operatorname{int} \mathcal{K}(x)} \varphi^*(U) - \varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X,Y). \tag{1.0.7}$$

In [56], the authors showed that, if (x, U) is a solution of generalized vector variational inequality (1.0.5) and it holds that $\mathcal{K}(x) \subseteq \mathcal{K}(y)$ for every $y \in X$, then $(-U, x)$ solves the first inverse problem (1.0.6). Conversely, if $(-U, x)$ solves problem (1.0.7) and a certain domination property is satisfied, then (x, U) is a solution of problem (1.0.5). The latter relations have been used to characterize solutions of the beam intensity optimization problem in radiotherapy treatment, see [54, 130], which is currently used to treat cancer in prostate, head and neck, breast and many others; compare Section 5.2 in [56].

In recent years, the theory of variational and vector variational inequalities has become a promising domain of applied and industrial mathematics. In 1973, Bensoussan and Lions [18] introduced the notion of *quasi-variational inequalities* in connection with a stochastic impulse control problem. Since then, quasi-variational inequalities have been investigated by several authors and emerged as a standard tool for the modeling of various equilibrium-type scenarios. The resulting applications include, for example, elastohydrodynamics [94], equilibrium problems [16, 49], image processing [17], probability theory [111], production management [21] and solid and continuum mechanics [19, 90], and many others. For a recent account of the theory of quasi-variational inequalities, the reader is referred to the books of Baiocchi and Capelo [13] and Kravchuk and Neittaanmäki [125].

The fundamental forms of all kinds of quasi-variational inequalities can be captured in the following abstract setting [15]: Let C be a non-empty, closed and convex subset of a real Banach space X. Further, let $E : C \rightrightarrows C$ and $P : C \rightrightarrows C$ be set-valued mappings with non-empty values. Then, the *quasi-variational-like problem* consists of finding $x \in C$ such that

$$x \in E(x) \qquad \text{and} \qquad x \in P(y), \qquad \text{for every} \quad y \in E(x). \tag{1.0.8}$$

Since the constraining set $E(x)$ depends upon the unknown x, problem (1.0.8) requires that the set-valued fixed-point problem $x \in C: \ x \in E(x)$ and the variational-like problem $x \in C: \ x \in P(y)$, for every $y \in E(x)$, should be solved simultaneously. By suitably adjusting the data, problem (1.0.8) recovers numerous problems of interest; compare Section 1 in [15]. Therefore, depending on the data, various solution methods for problem (1.0.8) have been proposed in the literature, e.g. Altangerel [6], Chan and Pang [29], Fukushima [68] and Mosco [139]. A widely used technique to tackle quasi-variational problems consists of finding fixed-points of the associated variational selection, e.g. Kano, Kenmochi and Murase [102], Khan, Tammer and Zălinescu [107] and Le [129].

To shed some light on this idea, let $u \in C$ and consider the following *parametric* *variational-like problem* [15] with element u as the parameter: Find $x_u \in C$ such that

$$x_u \in E(u) \qquad \text{and} \qquad x_u \in P(y), \qquad \text{for every} \quad y \in E(u). \tag{1.0.9}$$

Denoting the solution set of problem (1.0.9) by $S(u)$, when the parameter u varies in C, $S(u)$ induces a mapping $S : C \rightrightarrows C$, which is known as *variational selection*. As an immediate consequence, any fixed-point of S, that is, any element such that

$$x \in C : \quad x \in S(x), \tag{1.0.10}$$

is a solution of quasi-variational-like problem (1.0.8).

However, if the constraining set C is unbounded and the values of S are non-convex, the investigation of problem (1.0.10) is not helpful since applying fixed-point results is too restrictive for the data of problem (1.0.8). This is due to the fact that mostly all known set-valued fixed-point results require either the boundedness of C or the convexity of the values of S; see [4, 82, 165]. Recently, a new approach was used by Bruckner [24, 25, 26], then further developed by Bao, Hebestreit and Tammer [15] and Jadamba, Khan and Sama [97]: Instead of finding fixed-points of the variational selection S, the authors minimize the difference between inputs and outputs of the mapping S, that is, they study the following optimization problem: Find $(u, x) \in \mathcal{G}(S)$ such that

$$\|x - u\|_X^2 \le \|y - v\|_X^2, \qquad \text{for every} \quad (v, y) \in \mathcal{G}(S). \tag{1.0.11}$$

Here, $\mathcal{G}(S)$ denotes the graph of S. An element $x \in C$ is called a *generalized solution* (of quasi-variational-like problem (1.0.8)) if there is a parameter $u \in C$ such that (u, x) is a minimizer of optimization problem (1.0.11). Thus, if problem (1.0.11) is solvable with $(u, x) \in \mathcal{G}(S)$ as solution and $\|x - u\|_X = 0$, then quasi-variational-like problem (1.0.8) is solvable. Conversely, if problem (1.0.8) is solvable with $x \in C$ as solution, then $(x, x) \in \mathcal{G}(S)$ solves optimization problem (1.0.11). Consequently, if $\mathcal{G}(S)$ is non-empty, then $M := \{(u, x) \in \mathcal{G}(S) \mid \|x - u\|_X \le \|\bar{x} - \bar{u}\|_X\}$ is non-empty, where $(\bar{u}, \bar{x}) \in \mathcal{G}(S)$ is fixed, and it is enough to study the following relaxed optimization problem: Find $(u, x) \in M$ such that

$$\|x - u\|_X = \inf_{(v,y) \in M} \|y - v\|_X. \tag{1.0.12}$$

Moreover, as a consequence of Weierstraß' theorem, problem (1.0.12) attains a solution provided M is weak sequentially compact. It is important to emphasize that the solvability of the relaxed optimization problem (1.0.12) does not require the convexity of the mapping S. In addition, one does not need to assume that the constraining set C is bounded and it is further possible to relax the boundedness of C by some suitable coercivity conditions.

Within the last years, hundreds of papers were devoted to various and very important aspects of vector variational inequalities and generalizations, like existence results, scalarization methods, inverse results, gap functions, image space analysis, stability and sensitivity analysis and many others; see [91]. To the best of our knowledge, there are only a handful of papers dealing with numerical methods, which either use very restrictive assumptions or are incorrect, e.g. Chen [32], Chen, Pu and Wang [42] and Goh and Yang [79]. Therefore, the last chapter of this thesis is dedicated to the study of three projection based algorithms for vector variational inequality (1.0.3), depending on monotonicity and Lipschitz continuity properties.

To be precise, assume that C is a non-empty, closed and convex subset of the real Hilbert space X. Suppose further that K is a proper, closed, convex and solid cone in the real Banach space Y and let $F : X \to \mathrm{L}(X, Y)$ be a given mapping. Using a scalarization method, the author of this thesis showed that every element $x \in C$ with

$$\mathrm{Proj}(x - \rho^{-1} F_s x) = x, \tag{1.0.13}$$

that is, any fixed-point of $\mathrm{Proj}(I - \rho^{-1} F_s) : X \to X$, is a solution of problem (1.0.3). Thereby, $s \in K^* \setminus \{0\}$, $F_s = s \circ F$, $\rho > 0$ and Proj denotes the orthogonal projection onto the set C. By using the fixed-point formulation (1.0.13), the author derived a basic projection method (with variable step size) and an Extragradient method. The latter algorithms make it possible to calculate solutions of vector variational inequality (1.0.3). Indeed, if $F_s : X \to X$ is strongly monotone and Lipschitz continuous with modulus $c > 0$ and $L > 0$, respectively, and it holds $L^2 < 2c\rho$, then the sequence $\{x_n\}$, given for every $n \in \mathbb{N}_0$ by

$$x_{n+1} := x_{n+1}(\rho, s) := \mathrm{Proj}(x_n - \rho^{-1} F_s x_n),$$

converges for every $x_0 \in C$ to a solution of problem (1.0.3).

Besides that, the author of this thesis considered discrete finite-dimensional vector variational inequalities. In order to compute the whole solution set of these discrete problems he proposed a naive and a reduction based method, where the latter one uses the Graef-Younes procedure; see [85, 98].

Structure of the thesis

This thesis is structured as follows:

In Chapter 2, the required mathematical background is provided. For this purpose, fundamental results from the fields of linear and non-linear functional analysis, monotone operators and variational inequalities, single- and set-valued fixed-point results as well as basic concepts from the field of multi-objective optimization will be recalled.

Chapter 3 is devoted to the study of vector variational inequalities. After a detailed survey of the literature, a handy coercivity condition and a novel existence result are provided. Furthermore, a new regularization method for non-coercive problems is proposed, allowing to derive existence results for vector variational inequalities even in the absence of any coercivity (or regularity) condition. As a consequence, by using the latter regularization technique, some new existence results for generalized vector variational inequalities are developed.

In Chapter 4, inverse results for (generalized) vector variational inequalities are investigated. To this aim, some novel existence results and a generalized Minty lemma for the latter problem class are derived. Using a vector conjugate and a perturbation approach, it is shown that every generalized vector variational inequality can be embedded into two inverse problems. This technique allows the deduction of necessary and sufficient conditions for the beam intensity optimization problem in radiotherapy treatment which is used to treat cancer in prostate, head and neck, breast and many others.

Chapter 5 is devoted to the study of existence results for quasi-variational-like problems. However, in the absence of convexity or boundedness properties of the corresponding variational selection, it is not possible to apply classic solution methods. To this aim, so-called generalized solutions are considered and a closely related optimization problem is investigated. By this, novel existence results for quasi-variational and vector quasi-variational inequalities are derived. Then, some applications of the latter results to a multi-objective optimization problem with respect to forbidden regions justify the theoretical framework and show the usefulness of the results in this chapter.

Finally, in Chapter 6, some novel projection based algorithms for vector variational inequalities are investigated. Besides that, so-called finite-dimensional discrete vector variational inequalities are introduced. Lastly, a naive as well as a reduction based method are proposed that allow the calculation of the whole solution set of the discrete problems.

The author's main contributions

The author's main contributions to each chapter of this thesis are as follows:

In Chapter 3, the author investigated vector variational inequalities of the form (1.0.3). He derived a novel existence result for problem (1.0.3) using a linear scalarization method; see Theorem 3.3.26. To this end, he introduced a novel coercivity condition that can be checked easily, compared to other conditions in the literature. However, in the absence of any known coercivity condition, the author of this thesis proposed to study the family of well-behaving and regularized vector variational inequalities (1.0.4) instead; compare also Example 3.4.1. He then showed that the corresponding sequence of regularized solutions is well-defined and converges to a solution of the initial problem; see the Theorems 3.4.2 and 3.4.3 as well as Corollary 3.4.5. In order to relax certain convergence assumptions, he further provided some alternative conditions for the convergence of regularized solutions; compare the Corollaries 3.4.7 and 3.4.8. Finally, in Section 3.5, the author applied his regularization method to derive a novel existence result for a non-coercive generalized vector variational inequality; see Theorem 3.5.4.

Chapter 4 is devoted to the study of the inverse (generalized) vector variational inequalities (1.0.6) and (1.0.7). In Section 4.1, the author of this thesis derived a novel existence result for problem (1.0.5), using a generalized Minty lemma; compare the Lemmas 4.1.7 and 4.1.11 as well as Theorem 4.1.10. He then investigated inverse results for problem (1.0.5) that are based on a vector conjugate approach; see Theorem 4.2.1. By using a perturbation approach, the author developed new inverse results that provide necessary and sufficient conditions for problem (1.0.3); see the Theorems 4.3.2

and 4.3.3. Finally, as a main application of the previous results, the author of this thesis investigated a multi-objective optimization problem that arises in intensity modulated radiotherapy treatment; compare the Theorems 4.4.2 and 4.4.3.

In Chapter 5, the author of this thesis derived several novel existence results for quasi-variational and vector quasi-variational inequalities; see the Theorems 5.1.2, 5.1.3, 5.1.13 and 5.1.14. Further, in the absence of, for example, convexity properties of the corresponding variational selection, he investigated an optimization problem that is closely related to the latter problems. This approach made it possible to derive several new existence results for the latter problem classes by using the famous Weierstraß theorem; see the Theorems 5.2.4, 5.2.7 and 5.2.13 and the Corollaries 5.2.9, 5.2.10 and 5.2.11. In Section 5.3, the author applied his abstract results to a multi-objective optimization problem with forbidden areas. Compared to several results in the literature, his method requires very mild conditions for the data of the multi-objective optimization problem.

In Chapter 6 of this thesis, the author derived three new algorithms for the calculation of solutions of vector variational inequality (1.0.3). These are the basic projection method, the basic projection method with variable step size and the Extragradient method. To this aim, he proposed the study of a necessary fixed-point problem. Depending on monotonicity, Lipschitz continuity or co-coercivity assumptions, he proved that his proposed algorithms converge to a solution of vector variational inequality (1.0.3); compare the Theorems 6.1.1, 6.1.6 and 6.1.11. Furthermore, the latter methods have been implemented in Python 3.7.4 and applied to some finite-dimensional problems; see the Examples 6.1.3, 6.1.8 and 6.1.13. In the second part of this chapter, the author introduced for the first time finite-dimensional discrete vector variational inequalities. He then investigated a naive and an improved solution method and applied his implemented algorithms to some test problems.

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It is my great pleasure to offer warm thanks to my parents who have always supported and encouraged me. Finally, I am deeply thankful to Luisa for her infinite love and support. You make my life worth living.

Chapter 2

Mathematical Background

In this chapter, we provide the mathematical background as it will be used in the following chapters. The results can be found, for example, in the standard textbooks [5, 53, 64, 108, 126, 151, 166, 167].

2.1 Functional analysis

Definition 2.1.1. Let X be a real linear space. A function $\|\cdot\|_X : X \to \mathbb{R}$ is called a *norm* in X if the following conditions hold:

(i) $\|x\|_X = 0$ if and only if $x = 0$.

(ii) $\|\lambda x\|_X = |\lambda|\,\|x\|_X$ for every $x \in X$ and $\lambda \in \mathbb{R}$.

(iii) For every $x, y \in X$ it holds that $\|x + y\|_X \leq \|x\|_X + \|y\|_X$.

The pair $X := (X, \|\cdot\|_X)$ is called *normed space*. Here and everywhere else in this thesis, we will simply say that X is a normed space when the definition of the norm is understood from the context. If $\{x_n\}$ is a sequence in X, then we say that x_n *converges* (strongly) to $x \in X$ if it holds $\|x_n - x\|_X \to 0$. We simply write $\lim_{n\to+\infty} x_n = x$ or $x_n \to x$. A sequence $\{x_n\}$ in X is said to be *bounded* if there exists $M > 0$ such that $\|x_n\|_X \leq M$. for all $n \in \mathbb{N}$. Further, the sequence $\{x_n\}$ is called *Cauchy sequence* if for every $\varepsilon > 0$, there exists an index $n(\varepsilon) \in \mathbb{N}$ such that for all integers $n, m \geq n(\varepsilon)$ it holds $\|x_n - x_m\|_X < \varepsilon$. The normed space X is called *complete* if any Cauchy sequence in X converges. A complete normed space is called a *Banach space*.
Let again X be a real linear space. The mapping $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ is called *inner product* if it enjoys the following properties:

(i) $\langle x, x\rangle = 0$ if and only if $x = 0$.

(ii) $\langle x, y\rangle = \langle y, x\rangle$ for every $x, y \in X$.

(iii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, x\rangle$ for every $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.

The pair $(X, \langle\cdot,\cdot\rangle)$ is called *inner product space*. When the definition of the inner product is clear from the context, we simply say that X is an inner product space. Next, it is well known that the the inner product $\langle \cdot, \cdot \rangle$ induces a norm through $x \mapsto \sqrt{\langle x, x\rangle}$. This norm is called *Hilbert norm.* If the inner product space X is complete with respect to the Hilbert norm, then X is called *Hilbert space.*

Example 2.1.2. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, that is, Ω is a set, $\mathcal{A} \subseteq 2^{\Omega}$ is a σ-algebra, and $\mu : \mathcal{A} \to [0, +\infty]$ is a measure. Fix a constant $1 \leq p < +\infty$. A measurable function $f : \Omega \to \mathbb{R}$ is called p-integrable if $\int_\Omega |f|^p \, \mathrm{d}\mu < +\infty$ and the space of p-integrable functions on Ω will be denoted by

$$\mathcal{L}^p(\mu) := \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is measurable and } \int_\Omega |f|^p \, \mathrm{d}\mu < +\infty \right\}.$$

The function $\mathcal{L}^p(\mu) \to \mathbb{R}$ with $f \mapsto \|f\|_p$ defined by

$$\|f\|_p := \left(\int_\Omega |f|^p \, \mathrm{d}\mu \right)^{1/p}$$

is non-negative and satisfies the triangle inequality. Unfortunately, it holds $\|f\|_p = 0$ if and only if f vanishes almost everywhere, that is, $\|\cdot\|_p$ does not defined a norm in $\mathcal{L}^p(\mu)$. However, to obtain a normed space, one considers the quotient space $L^p(\mu) := \mathcal{L}^p(\mu)/\sim$, where $f \sim g$ if and only if $f = g$ almost everywhere. The function $\|\cdot\|_p$ descends to the quotient space and, with this norm, $L^p(\mu)$ becomes a Banach space. It is often convenient to abuse notation and use the same letter f to denote a function in $\mathcal{L}^p(\mu)$ and its equivalence class in the quotient space $L^p(\mu)$.

Theorem 2.1.3 (Riesz)**.** *Let X be a real normed space. Then the closed unit ball in X is compact if and only if X is finite-dimensional.*

Definition 2.1.4. Let X and Y be real normed spaces and let $A : \mathcal{D}(A) \subseteq X \to Y$ be an operator with domain $\mathcal{D}(A)$. When $\mathcal{D}(A) = X$, we write $A : X \to Y$. It is further convenient to write Ax instead of $A(x)$, where $x \in \mathcal{D}(A)$.

(i) The operator $A : \mathcal{D}(A) \subseteq X \to Y$ is called *linear* if for all $x, y \in \mathcal{D}(A)$ and $\alpha, \beta \in \mathbb{R}$ it holds $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$.

(ii) A is *continuous at the point* $x \in \mathcal{D}(A)$ if for each sequence $\{x_n\}$ in $\mathcal{D}(A)$, $x_n \to x$ implies $Ax_n \to Ax$. The operator A is called *continuous* if it is continuous at each point in $\mathcal{D}(A)$.

(iii) A is said to be *bounded* if there is a constant $c > 0$ such that $\|Ax\|_Y \leq c\|x\|_X$ for all $x \in \mathcal{D}(A)$.

(iv) A is called *compact* if A is continuous, and A maps bounded sets into relatively compact sets, that is, $M \subseteq X$ bounded implies $\operatorname{cl} A(M)$ is compact in Y.

Proposition 2.1.5. *Let X and Y be real normed spaces and let $A : X \to Y$ be a linear operator. Then A, is continuous if and only it is bounded.*

Definition 2.1.6. Let X and Y be real normed spaces. The linear space of linear and bounded operators from X to Y will be denoted by $\mathrm{L}(X,Y)$. The operator norm $\mathrm{L}(X,Y) \to \mathbb{R}$ with $A \mapsto \|A\|_{\mathrm{L}(X,Y)}$ is defined by

$$\|A\|_{\mathrm{L}(X,Y)} := \sup_{\|x\|_X \leq 1} \|Ax\|_Y.$$

We have the following useful proposition.

Proposition 2.1.7. *Let X be a real normed space and let Y be a real Banach space. Then, $\mathrm{L}(X,Y)$ is a real Banach space with respect to the operator norm.*

Definition 2.1.8. Let X be a real normed space. A linear and continuous functional on X is a linear and continuous operator from X to $\mathbb{R}$. The set of all linear and continuous functionals on X is called the *dual space* X^* of X, that is, $X^* := \mathrm{L}(X,\mathbb{R})$. Recall that X^* is a real Banach space with respect to the norm $\|\cdot\|_{X^*} \to \mathbb{R}$ with $f \mapsto \sup_{\|x\|_X \leq 1} |\langle f, x\rangle|$. We further set $X^{**} := (X^*)^* = \mathrm{L}(X^*,\mathbb{R})$, where X^{**} is called the *bidual space* and which consists of all linear and continuous functionals from X^* to $\mathbb{R}$. For the image $f(x)$ of the functional $f \in X^*$ at $x \in X$, we write $\langle f, x\rangle := f(x)$. $\langle \cdot, \cdot \rangle$ is called the *duality pairing*. If X is a real Banach space and $\{x_n\}$ is a sequence in X, then we say that $\{x_n\}$ *converges weakly* to the element $x \in X$ if $\langle f, x_n\rangle \to \langle f, x\rangle$ for every $f \in X^*$. The weak convergence is denoted by $x_n \rightharpoonup x$.

Example 2.1.9. Let $\Omega \subseteq \mathbb{R}^d$ be a domain, that is, Ω is open and connected, and denote by μ the Lebesgue measure. Then for $1 < p < +\infty$ the dual space of $L^p(\mu)$ is (isometrically isomorph to) $L^q(\mu)$, where q satisfies $1/p + 1/q = 1$.

Definition 2.1.10. Let X be a normed linear space. The operator $J : X \to X^{**}$, defined by $J(x)(f) := \langle f, x\rangle$ for all $x \in X$ and $f \in X^*$ is called the *canonical embedding* of X into X^{**}. We call X *reflexive* if J is surjective, that is, $J(X) = X^{**}$.

Example 2.1.11. For $1 < p < +\infty$, $L^p(\mu)$ is a reflexive Banach space.

Theorem 2.1.12 (Eberlein-Smulian Theorem)**.** *Let X be a reflexive Banach space. Then each bounded sequence in X has a weakly convergent subsequence.*

We next recall some useful convergence principles.

Proposition 2.1.13. *Suppose that X and Y are real Banach spaces and let $\{x_n\}$, $\{f_n\}$ and $\{A_n\}$ be sequence in X, X^* and $\mathrm{L}(X,Y)$, respectively. Then it holds:*

(i) *The strong convergence $x_n \to x$ implies the weak convergence $x_n \rightharpoonup x$.*

(ii) *If $\dim X < +\infty$, then the weak convergence $x_n \rightharpoonup x$ implies the strong convergence $x_n \to x$.*

(iii) *If $x_n \rightharpoonup x$, then $\{x_n\}$ is bounded and*

$$\|x\|_X \leq \liminf_{n\to+\infty} \|x_n\|_X.$$

(iv) *It follows from* $x_n \rightharpoonup x$ *and* $f_n \to f$ *that*

$$\langle f_n, x_n\rangle \to \langle f, x\rangle.$$

(v) *If* X *is reflexive in addition,* $f_n \rightharpoonup f$ *and* $x_n \to x$*, then it follows that*

$$\langle f_n, x_n\rangle \to \langle f, x\rangle.$$

(vi) *It follows from* $x_n \to x$ *and* $A_n \to A$ *that*

$$A_n(x_n) \to A(x).$$

(vii) *Let* X *be reflexive in addition and assume that* $\{x_n\}$ *is bounded. If all convergent subsequences of* $\{x_n\}$ *have the same weak limit* x*, then the whole sequence converges weakly to* x*.*

Definition 2.1.14. A subset M of a real normed space X is called *weak sequentially closed* if the limit of every weakly convergent sequence in M belongs to M.

Proposition 2.1.15 (Mazur)**.** *Let* M *be a convex subset of a real normed space. Then,* M *is closed if and only if* M *is weak sequentially closed.*

The following result by Weierstraß will play a crucial role in Section 5.2 of this thesis.

Theorem 2.1.16 (Weierstraß)**.** *Let* M *be a non-empty subset of the real reflexive Banach space* X*. Further, let* $f : M \to \mathbb{R} \cup \{\pm\infty\}$ *be given. Then, the optimization problem*

$$\min_{x\in M} f(x)$$

has a solution in case the following hold:

(i) M *is weakly compact, that is, bounded and weak sequentially closed.*

(ii) f *is weak sequentially lower semicontinuous on* M*, that is, for every* $x \in M$ *and every sequence* $\{x_n\}$ *in* M *with* $x_n \rightharpoonup x$ *it holds that* $f(x) \le \liminf_{n\to+\infty} f(x_n)$*.*

The following results can be found in [164, Section 1]. Compare also Example 1 for a geometric interpretation of the orthogonal projection.

Theorem 2.1.17. *Let* C *be a non-empty, closed and convex subset of a real Hilbert space* X *and let* $x \in C$*. Then there exists exactly one element* $p(x) \in C$ *with*

$$\|x - p(x)\|_X = \inf_{y\in C} \|x - y\|_X. \tag{2.1.1}$$

The operator Proj : $X \to C$ *with* $x \mapsto p(x)$ *is called (orthogonal) projection and has the following properties:*

(i) Proj *is non-expansive, that is, it holds* $\|\operatorname{Proj}(x)-\operatorname{Proj}(y)\|_X \leq \|x-y\|_X$ *for every* $x, y \in X$.

(ii) *It holds* $\langle \operatorname{Proj}(x) - \operatorname{Proj}(y), x - y\rangle \geq \|\operatorname{Proj}(x) - \operatorname{Proj}(y)\|_X^2$ *for every* $x, y \in X$.

(iii) *It holds* $\|\operatorname{Proj}(x)\|_X \leq \|x\|_X$ *for every* $x \in X$.

(iv) *Condition* (2.1.1) *is equivalent to* $\langle x - \operatorname{Proj}(x), \operatorname{Proj}(x) - y\rangle \geq 0$ *for every* $y \in C$. *The latter condition is called variational characterization.*

Definition 2.1.18. Let X and Y be real Banach spaces, and let $f : U \subseteq X \to Y$ be a mapping whose domain $\mathcal{D}(f) = U$ is an open subset of X. The *directional derivative* of f at $x \in U$ in the direction $h \in X$ is given by

$$\delta f(x; h) := \lim_{t\to 0} \frac{f(x + th) - f(x)}{t},$$

provided this limit exists. If $\delta f(x; h)$ exist for every $h \in X$, and if the mapping $D_{\mathrm{G}}f(x) : X \to Y$ defined by $D_{\mathrm{G}}f(x)h := \delta f(x; h)$ is linear and continuous, then we say that f is *Gâteaux-differentiable* at x, and we call $D_{\mathrm{G}}f(x)$ the *Gâteaux-derivative* of f at x.

2.2 Theory of monotone operators and variational inequalities

In what follows, we will collect some useful results from the field of monotone operators and variational inequalities.

Definition 2.2.1. Let X be a real Banach space and let $A : X \to X^*$. Then A is called

(i) *continuous* at the point $x \in X$ if $x_n \to x$ implies $Ax_n \to Ax$. A is called *continuous* if it is continuous at each point in X,

(ii) *hemicontinuous* if the real function $t \mapsto \langle A(x + ty), z\rangle$ is continuous on $[0, 1]$ for all $x, y, z \in X$.

Definition 2.2.2. Let X be a real Banach space and let $A : X \to X^*$. Then A is called

(i) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0$ for all $x, y \in X$,

(ii) *strictly monotone* if $\langle Ax - Ay, x - y\rangle > 0$ for all $x, y \in X$ with $x \neq y$,

(iii) *strongly monotone* if there exists a constant $c > 0$ such that $\langle Ax - Ay, x - y\rangle \geq c\|x - y\|_X^2$ for all $x, y \in X$,

(iv) *pseudomonotone* if $x_n \rightharpoonup x$ and $\limsup_{n\to+\infty}\langle Ax_n, x_n - x\rangle \leq 0$ implies for all $y \in X$

$$\langle Ax, x - y\rangle \leq \liminf_{n\to+\infty}\langle Ax_n, x_n - y\rangle.$$

Definition 2.2.3. Let X be a real Banach space. A set-valued operator $A : X \rightrightarrows X^*$ is called

(i) *monotone* if it holds

$$\langle x^* - y^*, x - y \rangle \geq 0, \qquad \text{for every} \quad (x, x^*),\ (y, y^*) \in \mathcal{G}(A),$$

(ii) *maximal monotone* if A is monotone, and it follows from $(x, x^*) \in X \times X^*$ and

$$\langle x^* - y^*, x - y \rangle \geq 0, \qquad \text{for every} \quad (y, y^*) \in \mathcal{G}(A)$$

that $(x, x^*) \in \mathcal{G}(A)$.

Definition 2.2.4. Let X be a real Banach space. A set-valued operator $A : X \times X \rightrightarrows X^*$ is called *semi-monotone*, if $\mathcal{D}(A) = X \times X$ and the following conditions are satisfied:

(SM1) For any $u \in X$, $A(u, \cdot) : X \rightrightarrows X^*$ is maximal monotone with $\mathcal{D}(A(u, \cdot)) = X$.

(SM2) Let $x \in X$ and $\{u_n\} \subseteq X$ be a sequence such that $u_n \rightharpoonup u$. Then, for every $w \in A(u, x)$, there exists a sequence $\{w_n\}$ in X^* such that $w_n \in A(u_n, x)$ and $w_n \to w$.

Definition 2.2.5. Let X be a real Banach space and let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function. A functional $x^* \in X^*$ is called *subgradient* of f at the point $x \in X$ if $f(x) \neq \pm\infty$ and

$$f(y) \geq f(x) + \langle x^*, y - x \rangle, \qquad \text{for every} \quad y \in X.$$

The set of all subgradients of f at x is called the *subdifferential* $\partial f(x)$ at x. If it holds that $f(x) \in \{-\infty, +\infty\}$, then put $\partial f(x) = \emptyset$.

Lemma 2.2.6 (Minty)**.** *Let C be a non-empty, closed and convex subset of the real Banach space X and let $A : X \to X^*$ be monotone and hemicontinuous. Then $x \in C$ satisfies*

$$\langle Ax, y - x \rangle \geq 0, \qquad \textit{for every} \quad y \in C, \tag{2.2.1}$$

if and only if it satisfies

$$\langle Ay, y - x \rangle \geq 0, \qquad \textit{for every} \quad y \in C. \tag{2.2.2}$$

Theorem 2.2.7 (Hartmann-Stampacchia Theorem)**.** *Let C be a non-empty, closed and convex subset of the real Banach space X and let $A : X \to X^*$ be monotone and hemicontinuous. If in addition either the set C is bounded or A is coercive, that is, there is $x_0 \in C$ such that*

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\langle Ax - Ax_0, x - x_0 \rangle}{\|x - x_0\|_X} = +\infty,$$

then variational inequality (2.2.1) *has a solution.*

Remark 2.2.8. If in addition, A is strictly (or ,strongly) monotone, then it is easily seen that the solution of variational inequality (2.2.1) is unique, if it exists.

2.3 Fixed-point results

In nearly all fields of mathematics, fixed-point results play an important role for proving the existence and uniqueness of solutions to various mathematical models such as differential, integral, ordinary and partial differential equations, variational inequalities and numerous others.

Historically, one of the most important results under all fixed-point theorems is the famous theorem of L. E. J. Brouwer, which has been been published in 1910 by Brouwer; see Theorem 2.3.1. Brouwer proved his famous result later in 1912 using a degree theoretical approach. Several other proofs, using analytical or topological methods were given by amongst others by Lefschetz, Leray, Kakutani, Klee and Browder. Since mostly all problems in functional analysis are concerned with infinite-dimensional spaces, Birkhoff and Kellogg gave in 1922 the first infinite-dimensional fixed-point theorem. Some years later, in 1930, J. P. Schauder extended Brouwer's theorem to the case of infinite-dimensional Banach spaces; see the Corollaries 2.3.3 and 2.3.4. Since the therein used compactness (boundedness) conditions are very strong, A. N. Tychonoff proved in 1935 a generalization of Schauder's fixed-point theorem for compact operators; see Theorem 2.3.2. Such an extensions is crucial since mostly all problems in functional analysis do not have a compact setting. In the meantime, S. Banach introduced in 1922 a so-called contraction principle, where he considered Lipschitz-continuous mappings with constant strictly smaller 1, so-called contractions. Due to the convergence property of the successive iterates to the unique fixed-point, several generalizations of Banach's fixed-point results have been published within the last years.

The study of fixed-point results for set-valued mappings was initiated by S. Kakutani in 1941 for finite-dimensional spaces. Some years later, H. F. Bohnenblust and S. Karlin extended Kakutani's result to locally convex spaces; see Theorem 2.3.10. Browder then provided a fixed-point theorem where the compactness of the underlying set is dropped and replaced with a geometrical coercivity condition; compare Theorem 2.3.8. At almost the same time, in 1969, S. B. Nadler extended Banach's fixed-point theorem to the set-valued case. The result is frequently called generalized Banach fixed-point result; compare [4, 99, 165] for an extensive historical overview of fixed-point results for single- and set-valued mappings.

2.3.1 Single-valued fixed-point results

Let S be a self mapping on the non-empty set C, that is, $S : C \to C$. An element $x \in C$ is said to be a *fixed-point* of the mapping S if

$$S(x) = x.$$

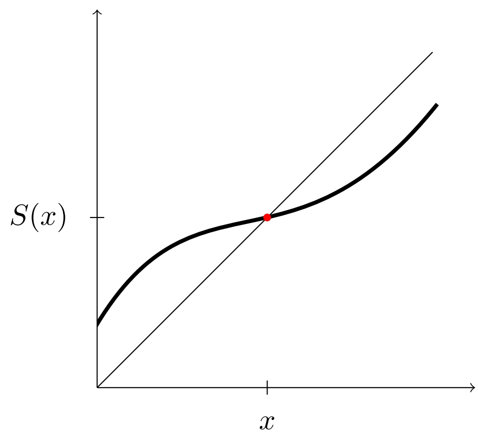


Figure 2.1: Illustration of a fixed-point

The following fixed-point results can be found in [165].

Theorem 2.3.1 (Brouwer, [165, Proposition 2.6])**.** *Let C be a non-empty, convex and compact subset of a finite-dimensional Euclidean space and let $S : C \to C$ be a continuous mapping. Then S has a fixed-point.*

The next theorem by Schauder and Tychonoff is the direct translation of Brouwer's fixed-point theorem to Banach spaces.

Theorem 2.3.2 (Schauder, Tychonoff)**.** *Let C be a non-empty, closed, convex and bounded subset of the real Banach space X. If $S : C \to C$ is compact, then S admits a fixed-point.*

The next corollary is also known as Tychonoff's fixed-point result.

Corollary 2.3.3 (Schauder, Alternative version)**.** *Let C be a non-empty, convex and compact subset of the real Banach space X. If $S : C \to C$ is continuous, then S admits a fixed-point.*

Corollary 2.3.4 (Schauder, Second alternative version)**.** *Let C be a non-empty, closed, convex and bounded subset of the real, reflexive and separable Banach space X, and suppose that $S : C \to C$ is a weak sequentially continuous operator. Then, S has a fixed-point.*

We will use the following fixed-point result by Banach for the investigation of projection based algorithms for vector variational inequalities; compare Theorem 6.1.1.

Theorem 2.3.5 (Banach)**.** *Let C be a non-empty, closed and convex subset of the real Banach space X. Suppose further that $S : C \to C$ is a contraction, that is, there exists $k \in [0, 1)$ such that $\|S(x) - S(y)\|_X \leq k\|x - y\|_X$ for all $x, y \in C$. Then for every $x_0 \in C$, the sequence $\{x_n\}$, given for $n \in \mathbb{N}_0$ by $x_{n+1} = S(x_n)$, converges to the unique fixed-point of S.*

2.3.2 Set-valued fixed-point results

Let X and Y be topological spaces and let C be a non-empty subset of X. By a *set-valued mapping* (or, multi-valued mapping) $S : C \rightrightarrows Y$, we mean a mapping which assigns to each point $x \in C$ a subset $S(x) \subseteq Y$. Every single-valued mapping $\tilde{S} : C \to Y$ can be identified with a set-valued mapping by setting $S(x) = \{\tilde{S}(x)\}$ for all $x \in C$. Thus, $S(x)$ is a singleton, consisting of the image point $\tilde{S}(x)$ only. The *domain* of a set-valued mapping S is denoted by

$$\mathcal{D}(S) := \big\{x \in C \mid S(x) \neq \emptyset\big\},$$

and we write $S : X \rightrightarrows Y$ if $\mathcal{D}(S) = X$. If for every $x \in \mathcal{D}(S)$, the set $S(x)$ has a certain property $\mathcal{P}$, then we say that S is *$\mathcal{P}$-valued.* Further, the *range* of S is the set

$$\mathcal{R}(S) := \bigcup_{x \in \mathcal{D}(S)} S(x).$$

Naturally, $S(C)$ denotes the union of all sets $S(x)$ over $x \in C$, that is, $S(C) := \bigcup_{x\in C} S(x)$. The *graph* of S is the set

$$\mathcal{G}(S) := \big\{(y, x) \in Y \times X \mid y \in \mathcal{D}(S) \text{ and } x \in S(y)\big\},$$

and the *inverse* of S is the set-valued mapping $S^{-1} : Y \rightrightarrows X$ with

$$S^{-1}(y) := \big\{x \in X \mid x \in S(y)\big\}.$$

It should be noted that the inverse of a set-valued mapping always exists. We evidently have $\mathcal{D}(S^{-1}) = \mathcal{R}(S)$ and $(y, x) \in \mathcal{G}(S)$ if and only if $(x, y) \in \mathcal{G}(S^{-1})$. Further, if $S : C \rightrightarrows X$ is a set-valued mapping, then an element $x \in C$ with

$$x \in S(x)$$

is called a *fixed-point* of S.

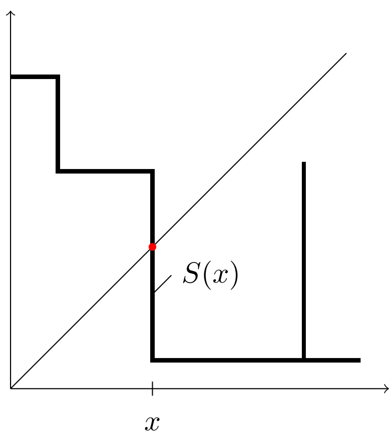


Figure 2.2: Illustration of a fixed-point

In what follows, we recall some famous fixed-point results for set-valued mappings, which can be found in [122, 165].

Recall that in a metric space (X, d), the Hausdorff distance d_H of two non-empty subset A, B in X is defined by

$$d_H(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(a, B) := \inf_{b\in B} d(a, b)$ is the distance of $a \in A$ to the set B. Notice that the next result provides a generalization of Theorem 2.3.5.

Theorem 2.3.6 (Nadler)**.** *Let C be a non-empty and closed subset of X, where (X, d) is a complete metric space. Assume further that S has non-empty and closed values and there is $k \in [0, 1)$ such that $d_H(S(x), S(y)) \leq kd(x, y)$ for all $x, y \in C$, where d_H denotes the Hausdorff distance. Then S has a fixed-point.*

We further need the following definition.

Definition 2.3.7. Let C be a non-empty subset of the real Banach space X. A set-valued mapping is said to have *open lower sections* if for every $y \in C$, the set $S^{-1}(y) = \{x \in C \mid y \in S(x)\}$ is open.

Theorem 2.3.8 (Browder)**.** *Let C be a non-empty, convex and compact subset of the real Banach space X. Assume that $S : C \rightrightarrows X$ is a set-valued mapping with non-empty, closed and convex values and with open lower sections. Suppose further that one of the following boundary conditions is satisfied:*

(i) *For every $x \in \operatorname{bd} C$ there are points $y \in S(x)$ and $z \in C$, and a number $\lambda > 0$ such that $y = x + \lambda(z - x)$.*

(ii) *For every $x \in \operatorname{bd} C$ there are points $y \in S(x)$ and $z \in C$, and a number $\lambda < 0$ such that $y = x + \lambda(z - x)$.*

Then, S has a fixed-point.

Some geometric interpretations of the conditions (i) and (ii) in Theorem 2.3.8 can be found, for example, in [165].

Theorem 2.3.9 (Ky-Fan, Glicksberg)**.** *Let C be a non-empty, convex and compact subset of the real Banach space X. Assume further that the set-valued mapping $S : C \rightrightarrows C$ has non-empty, closed and convex values and has open lower sections. Then, S has a fixed-point.*

Kakutani proved this theorem for the finite-dimensional case. The generalization is due to Ky-Fan (1952) and Glicksberg (1952). It should be noted that the fixed-point results of Ky-Fan and Glicksberg is a special case of Browder's fixed-point theorem, since we have $S(C) \subseteq C$, such that we can chose the point $u = y$ for a fixed $y \in S(x)$ and $\lambda = 1$.

Theorem 2.3.10 (Bohnenblust, Karlin)**.** *Let C be a non-empty, closed and convex subset of the real Banach space X. Assume further that $S : C \rightrightarrows C$ is a set-valued mapping with non-empty, closed and convex values and with open lower sections. If the set $S(C)$ is relatively compact, then, S has a fixed-point.*

We will use the following result by Kluge to derive novel existence results for vector quasi-variational inequalities; compare Section 5.1.2 of this thesis. See also [107] for some applications of Theorem 2.3.11 in the field of (set-valued) quasi-variational inequalities.

Theorem 2.3.11 (Kluge)**.** *Let C be a non-empty, closed and convex subset of the real reflexive Banach space X. Assume that $S : C \rightrightarrows C$ is a set-valued mapping with non-empty, closed and convex values, and the graph of S is weak sequentially closed. If either the set C is bounded or the image $S(C)$ is bounded, then, S has a fixed-point.*

2.4 Functional analysis over cones

Definition 2.4.1. Let Y be a real linear space and let A and B be non-empty subsets. Then, the *Minkowski sum* and *Minowski difference* of A and B will be denoted by $A + B := \{a + b \mid a \in A \text{ and } b \in B\}$ and $A - B := \{a - b \mid a \in A \text{ and } b \in B\}$, respectively, where the multiplication by a scalar $\lambda \in \mathbb{R}$ with A will be denoted by $\lambda A := \{\lambda a \mid a \in A\}$. We further let $A \pm \emptyset := \emptyset \pm A := \emptyset$ for any set A in Y.

In order to compare elements of abstract spaces, it is convenient to recall the notion of a cone and corresponding cone properties.

Definition 2.4.2. Let Y be a real topological linear space. A non-empty set K in Y is a *cone* if $\lambda K \subseteq K$ for every $\lambda \geq 0$. The cone K is called

(i) *convex* if $K + K \subseteq K$,

(ii) *proper* (or *non-trivial*) if $K \neq \{0\}$ and $K \neq Y$,

(iii) *closed* if $\operatorname{cl} K = K$,

(iv) *pointed* if $K \cap (-K) = \{0\}$,

(v) *solid* if $\operatorname{int} K \neq \emptyset$.

Remark 2.4.3. (i) Clearly, if K is a cone, then $0 \in K$. If in addition K is proper and solid, then we always have $0 \notin \operatorname{int} K$.
(ii) Obviously, the cone K satisfies condition (i) if and only if K is a convex set. The cone further satisfies $K \subseteq K + \{0\} \subseteq K + K$. Therefore, K is convex if and only if $K + K = K$.
(iii) A commonly used cone in $\mathbb{R}^k$, which enjoys all properties of Definition 2.4.2, is the so-called non-negative ordering cone or Pareto cone

$$\mathbb{R}^k_{\geq} := \{y \in \mathbb{R}^k \mid y_j \geq 0 \text{ for } j = 1, \dots, k\}.$$

(iv) In contrast, the non-negative ordering cone

$$K^p := \{f \in L^p(\mu) \mid f \geq 0 \;\; \mu\text{-a.e. on } \Omega\}$$

in $L^p(\mu)$, where $1 \leq p < +\infty$ and μ is the Lebesgue measure, is a proper, closed, convex and pointed cone which is non-solid; compare the next example.

Example 2.4.4. It is easily seen that K^p is a proper, closed, convex and pointed cone in $L^p(\mu)$. It remains to show that the interior of K^p is empty. Let $f \in K^p$ and let $\{\varepsilon_n\} \subseteq \mathbb{R}$ be a sequence with $\varepsilon_n > 0$ and $\varepsilon_n \downarrow 0$. In order to show $f \notin \operatorname{int} K^p$, we will construct a sequence $\{f_n\} \not\subseteq K$ with $\|f_n - f\|_p \to 0$. Define $A_n := \{x \in \Omega \mid f(x) \geq \varepsilon_n^{-1}\}$. Since $\varepsilon_n^{-1}\chi_{A_n} \leq f$, we have

$$\varepsilon_n^{-1}\mu(A_n) = \int_\Omega \varepsilon_n^{-1}\chi_{A_n} \,\mathrm{d}\mu \leq \int_\Omega f \,\mathrm{d}\mu < +\infty,$$

that is, $\mu(A_n) < +\infty$ and $\mu(A_n) \downarrow 0$. Consequently, given $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$

$$0 < \varepsilon \leq \mu(\Omega \setminus A_n) = \mu\left(\bigcup_{q \in \mathbb{Q}^d} B(q, \varepsilon_n^2) \cap (\Omega \setminus A_n)\right) \leq \sum_{q \in \mathbb{Q}^d} \mu\big(B(q, \varepsilon_n^2) \cap (\Omega \setminus A_n)\big).$$

The inequality implies that we can find $\tilde{q} \in \mathbb{Q}^d$ such that for all $n \geq N(\varepsilon)$ it holds that $0 < \varepsilon \leq \mu\big(B(\tilde{q}, \varepsilon_n^2) \cap (\Omega \setminus A_n)\big)$. Define $C_n := B(\tilde{q}, \varepsilon_n^2) \cap (\Omega \setminus A_n)$ for $n \geq N(\varepsilon)$ and introduce a sequence $\{f_n\}$ by

$$f_n := f - \varepsilon_n^{1 - \frac{2d}{p}} \chi_{C_n}.$$

Since $\|\chi_{C_n}\|_p^p \leq \varepsilon_n^{2d}\mu(B(0,1))$, we immediately have $f_n \in L^p(\mu)$. However, due to the fact that f_n is negative on C_n with $\mu(C_n) > 0$, it holds that $f_n \notin K^p$. It remains to show that $f_n \to f$ in $L^p(\mu)$. Indeed, we have

$$\begin{aligned}
&\|f_n - f\|_p^p \\
&= \int_{\Omega \setminus C_n} |f_n - f|^p \,\mathrm{d}\mu + \int_{C_n} |f_n - f|^p \,\mathrm{d}\mu \;\leq\; \varepsilon_n^{1 - \frac{2d}{p}} \int_{B(\tilde{q}, \varepsilon_n^2)} 1 \,\mathrm{d}\mu \;\leq\; \varepsilon_n \mu\big(B(0,1)\big),
\end{aligned}$$

where we used $\mu\big(B(\tilde{q}, \varepsilon_n^2)\big) = \varepsilon_n^{2d}\mu\big(B(0,1)\big)$. Thus, $f_n \to f$ which shows that the interior of K^p is empty.

Definition 2.4.5. Let Y be a real linear space with a convex cone K.

(i) The cone $K^* := \{y^* \in Y^* \mid \langle y^*, y\rangle \geq 0 \text{ for every } y \in K\}$ is called *dual cone* for K.

(ii) The set $\operatorname{qi} K^* := \{y^* \in Y^* \mid \langle y^*, y\rangle > 0 \text{ for every } y \in K \setminus \{0\}\}$ is called the *quasi-interior* of the dual cone for K.

Remark 2.4.6. Note that K^* is indeed a convex cone, that is, the previous definition makes sense. For $K = \{0\}$ and $K = Y$ one obtains $K^* = Y^*$ and $K^* = \{0\}$, respectively.

Example 2.4.7. It holds $(\mathbb{R}^l_{\geq})^* = \mathbb{R}^l_{\geq}$ and $\operatorname{qi}(\mathbb{R}^l_{\geq})^* = \operatorname{int}(\mathbb{R}^l_{\geq})^* = \mathbb{R}^l_{>}$.

Lemma 2.4.8. *Let K be a convex cone in the real linear space Y. Then we have:*

(i) *If $\operatorname{qi} K^*$ is non-empty, then K is pointed.*

(ii) *If in addition Y is a real locally convex space and K has a base, then the quasi-interior $\operatorname{qi} K^*$ of the dual cone for K is non-empty.*

(iii) *If Y is locally convex and separated where the topology gives Y as the topological dual space of Y and K is closed and solid, then we have $\operatorname{int} K^* = \operatorname{qi} K^*$.*

(iv) *If K is closed, then $K = \{y \in Y \mid \langle y^*, y\rangle \geq 0 \text{ for every } y^* \in K^*\}$.*

(v) *If K is solid, then it holds that $\operatorname{int} K = \{y \in Y \mid \langle y^*, y\rangle > 0 \text{ for every } y^* \in K^* \setminus \{0\}\}$.*

Theorem 2.4.9 (Krein-Rutman)**.** *In a real separable normed space Y with a closed, convex and pointed cone K the quasi-interior $\operatorname{qi} K^*$ of the dual cone is non-empty.*

Example 2.4.10. Let $1 \leq p < +\infty$. Then the quasi-interior of the natural ordering cone K^p in $L^p(\mu)$, given by

$$\operatorname{qi}(K^p)^* = \left\{ g \in L^q(\mu) \mid \int_\Omega fg \,\mathrm{d}\mu > 0 \text{ for every } f \in K^p \setminus \{0\} \right\},$$

is non-empty.

Definition 2.4.11. Let K be a proper, closed and convex cone in the linear space Y and let $a, b \in Y$ be given elements. We define binary relations in the following way:

$$\begin{aligned} a \leq_K b \quad &:\Longleftrightarrow \quad b \in a + K, \\ a \not\leq_K b \quad &:\Longleftrightarrow \quad b \notin a + K. \end{aligned}$$

If in addition K is solid, then we define two weak binary relations in the following way:

$$\begin{aligned} a \leq_{\operatorname{int} K} b \quad &:\Longleftrightarrow \quad b \in a + \operatorname{int} K, \\ a \not\leq_{\operatorname{int} K} b \quad &:\Longleftrightarrow \quad b \notin a + \operatorname{int} K. \end{aligned}$$

Proposition 2.4.12. *Let K be a cone in the linear space Y. The binary relations defined in the previous definition have the following properties:*

(i) *The relation $\leq_K$ is a partial ordering, that is, $\leq_K$ is reflexive, transitive and antisymmetric, if and only if K is a proper, convex and pointed cone.*

(ii) *The relation $\leq_K$ is compatible with scalar multiplication and addition, that is, for all $a, b, c \in Y$ and $\lambda \geq 0$, it holds that $a \leq_K b$ implies $\lambda a \leq_K \lambda b$ and $a \leq_K b$ implies $a + c \leq_K b + c$.*

(iii) *Let in addition K be solid. The relation $\not\leq_{\operatorname{int} K}$ is compatible with addition, that is, for all $a, b, c \in Y$ and $\lambda \geq 0$, it holds that $a \not\leq_{\operatorname{int} K} b$ implies $a + c \not\leq_{\operatorname{int} K} b + c$.*

(iv) *If in addition K is convex and solid, then for all $a, b, \in Y$ it holds that $a \leq_K b$ and $a \not\leq_{\operatorname{int} K} 0$ implies $b \not\leq_{\operatorname{int} K} 0$.*

(v) *If in addition K is solid, then for all $a \in Y$ and $\lambda \geq 0$, it holds that $\lambda a \not\geq_{\operatorname{int} K} 0$ implies $a \not\geq_{\operatorname{int} K} 0$.*

Proof. We will show part (iv) only. The proof of the statement follows from the useful identity

$$K + \operatorname{int} K = \operatorname{int} K. \tag{2.4.1}$$

Notice that $0 \in K$ and therefore it holds $\operatorname{int} K = \operatorname{int} K + 0 \subseteq \operatorname{int} K + K$. For the converse inclusion, let $x \in \operatorname{int} K$, $y \in K$ and $z \in Y$. Since K is convex and $\operatorname{int} K \neq \emptyset$, it holds that $\operatorname{int} K = \operatorname{cor} K$, see [98], where $\operatorname{cor} K := \{k \in K \mid \forall\, y \in Y\ \exists\, \varepsilon' > 0,\ \forall\, \varepsilon \in [0, \varepsilon'],\ k + \varepsilon y \in K\}$ denotes the algebraic interior of K. From $x \in \operatorname{int} K$ we therefore conclude that there is $\varepsilon' > 0$ such that $x + \varepsilon z \in K$ for every $\varepsilon \in [0, \varepsilon']$. The convexity of K implies $x + y + \varepsilon z \in K$ for every $\varepsilon \in [0, \varepsilon']$, that is, $x + y$ is an interior point of $\operatorname{cor} K$. This shows (2.4.1).
Now let $a, b \in Y$ and assume to the contrary that it holds that $-b \in \operatorname{int} K$, where $b - a \in K$ and $a \notin -\operatorname{int} K$. Then, we deduce from (2.4.1) that $-a = b - a - b \in \operatorname{int} K$, which is impossible. The proof is complete. □

Definition 2.4.13. Let K be a proper, closed, convex and solid cone in the linear space Y and suppose that A and B are non-empty subsets of Y. Then we define the following weak binary relations for sets:

$$\begin{aligned}
A \preccurlyeq^1_{\operatorname{int} K} B \quad &:\Longleftrightarrow \quad \exists a \in A,\ \forall b \in B :\ a \leq_{\operatorname{int} K} b,\\
A \not\preccurlyeq^1_{\operatorname{int} K} B \quad &:\Longleftrightarrow \quad \forall a \in A,\ \exists b \in B :\ a \not\leq_{\operatorname{int} K} b,\\
A \preccurlyeq^2_{\operatorname{int} K} B \quad &:\Longleftrightarrow \quad \forall a \in A,\ \exists b \in B :\ a \leq_{\operatorname{int} K} b,\\
A \not\preccurlyeq^2_{\operatorname{int} K} B \quad &:\Longleftrightarrow \quad \exists a \in A,\ \forall b \in B :\ a \not\leq_{\operatorname{int} K} b.
\end{aligned}$$

Now let $\sim$ denote one of the four set relations. If the set A is a singleton, that is, $A = \{a\}$ then we write $a \sim B$ instead of $\{a\} \sim B$. Similar, if B is a singleton, that is, $B = \{b\}$ then we abbreviate $A \sim \{b\}$ by $A \sim b$.

Remark 2.4.14. It should be noted that set-relation $\preccurlyeq^2_{\operatorname{int} K}$ is known in the literature as (weak) upper set less order relation; compare [108, 116, 117]. It further holds $A \preccurlyeq^2_{\operatorname{int} K} B$ if and only if $A \subseteq B - \operatorname{int} K$, provided A and B are non-empty. Some useful characterizations of several set relations by means of a non-linear scalarization function have been investigated, for example, by Hebestreit and Köbis in [92].

2.5 Solution concepts in multi-objective optimization

In this section, we recall some basic notions and concepts from the field of multi-objective optimization (vector optimization) that will be used through the thesis.

Numerous real word optimization problems require the minimization of multiple, in general conflicting, objectives. For example, intensity modulated radiotherapy treatment (IMRT) aims at applying to the patient a suitable radiation dose to treat cancer in prostate, head and neck, breast and many others. Therefore, the corresponding multi-objective optimization problem consists of minimizing the radiation dose through critical organs while the dose in the infected structures is increased; compare Section 4.4.2 of this thesis. See also Section 1 in [53] and Section 11 in [98] for some introductory examples of multi-objective optimization problems.

In order to investigate the optimality notion in abstract spaces, the following definition is needed.

Definition 2.5.1. Let A be a non-empty subset of a linear topological space Y with proper, closed and convex cone K. Then the set of *minimal elements* of A with respect to the cone K is defined by

$$\mathrm{Min}(A, K) := \big\{x \in A \mid (x - K) \cap A \subseteq x + K\big\}.$$

In a similar way, the set of *maximal elements* of A with respect to the cone K is defined by

$$\mathrm{Max}(A, K) := \big\{x \in A \mid (x + K) \cap A \subseteq x + K\big\}.$$

If in addition K is pointed, then it holds that $\mathrm{Min}(A, K) = \{x \in A \mid (x-K)\cap A = \{x\}\}$ and $\mathrm{Max}(A, K) = \{x \in A \mid (x + K) \cap A = \{x\}\}$. Moreover, if K is proper, closed, convex, pointed and solid cone, the set of *weakly minimal elements* and *weakly maximal elements* of A with respect to K is defined by

$$\mathrm{WMin}(A, K) := \big\{x \in A \mid (x - \operatorname{int} K) \cap A = \emptyset\big\}.$$

and

$$\mathrm{WMax}(A, K) := \big\{x \in A \mid (x + \operatorname{int} K) \cap A = \emptyset\big\},$$

respectively. It should be noted that we always have

$$\mathrm{Min}(A, K) \subseteq \mathrm{WMin}(A, K) \subseteq A \cap \operatorname{bd} A.$$

Remark 2.5.2. Evidently, if$K$ is proper, closed, convex, pointed and solid cone, then it holds that $x \in \mathrm{WMin}(A, K)$ if and only if $y \not\leq_{\operatorname{int} K} x$ for every $y \in A$.

In multi-objective optimization, one aims at minimizing a vector-valued objective mapping

$$\psi : C \to Y$$

over a non-empty subset C in X, where X and Y are linear topological spaces and Y is partially ordered by a proper, closed, convex and pointed cone K. In what follows, we denote for any non-empty subset C of X, the image of ψ of C by

$$\psi(C) := \{\psi(x) \mid x \in C\}.$$

Definition 2.5.3. Let C be a non-empty subset of a linear topological space X and suppose that K is a proper, closed, convex and pointed cone in the linear topological space Y. Then we call an element $\psi(x)$, $x \in C$, *efficient* if $\psi(x)$ is a minimal element of the image set $\psi(C)$. Thus, the set of *efficient elements* is given by

$$\begin{aligned} \operatorname{Eff}(\psi(C), K) &:= \big\{x \in C \mid \psi(x) \in \operatorname{Min}(\psi(C), K)\big\} \\ &= \big\{x \in C \mid \psi(C) \cap \big(\psi(x) - K\big) = \{\psi(x)\}\big\}. \end{aligned}$$

Similar, if in addition K is solid, the set of *weakly efficient elements* is given by

$$\begin{aligned} \operatorname{WEff}(\psi(C), K) &:= \big\{x \in C \mid \psi(x) \in \operatorname{WMin}(\psi(C), K)\big\} \\ &= \big\{x \in C \mid \psi(C) \cap \big(\psi(x) - \operatorname{int} K\big) = \emptyset\big\}. \end{aligned}$$

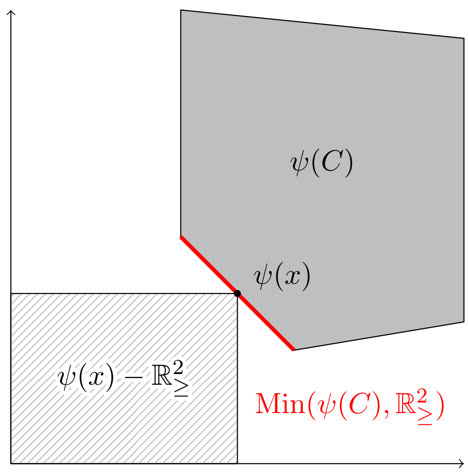


Figure 2.3: Illustration of minimal elements of $\psi(C) \subseteq \mathbb{R}^2$ w.r.t. $\mathbb{R}^2_{\geq}$

Remark 2.5.4. It should be noted that in the finite-dimensional case, where $Y = \mathbb{R}^k$ and $K = \mathbb{R}^k_{\geq}$, efficient and weakly efficient elements can be characterized in the following way:

$$x \in \mathrm{Eff}(\psi(C), \mathbb{R}^k_{\geq}) \iff \nexists y \in C \text{ s.t. } \begin{cases} \forall\, i \in \{1, \dots, k\}: \ \psi_i(y) \leq \psi_i(x), \\ \exists\, j \in \{1, \dots, k\}: \ \psi_j(y) < \psi_j(x); \end{cases}$$
$$x \in \mathrm{WEff}(\psi(C), \mathbb{R}^k_{\geq}) \iff \nexists y \in C \text{ s.t. } \forall i \in \{1, \dots, k\}: \ \psi_i(y) < \psi_i(x).$$

Definition 2.5.5 ([15, Definition 7])**.** Let X and Y be real normed spaces. A set-valued mapping $\mathcal{K} : X \rightrightarrows Y$ is called a *variable domination structure* on Y if for every $x \in X$, the set $\mathcal{K}(x)$ is a proper, closed, convex, pointed and solid cone in Y.

Definition 2.5.6. Let X and Y be linear topological spaces and let C be a non-empty subset of Y. Suppose further that $\mathcal{K} : X \rightrightarrows Y$ is a variable domination structure on Y. Then, the set of *efficient elements* with respect to the variable domination structure $\mathcal{K}$ is given by

$$\begin{aligned} \mathrm{Eff}(\psi(C), \mathcal{K}) &:= \big\{x \in C \mid \psi(x) \in \mathrm{Min}(\psi(C), \mathcal{K}(x))\big\} \\ &= \big\{x \in C \mid \psi(C) \cap \big(\psi(x) - \mathcal{K}(x)\big) = \{\psi(x)\}\big\}. \end{aligned}$$

Further, the set of *weakly efficient elements* with respect to the variable domination structure $\mathcal{K}$ is given by

$$\begin{aligned} \mathrm{WEff}(\psi(C), \mathcal{K}) &:= \big\{x \in C \mid \psi(x) \in \mathrm{WMin}(\psi(C), \mathcal{K}(x))\big\} \\ &= \big\{x \in C \mid \psi(C) \cap \big(\psi(x) - \operatorname{int} \mathcal{K}(x)\big) = \emptyset\big\}. \end{aligned}$$

Remark 2.5.7. (i) Besides the above solution concept, Yu introduced in 1974 the notion of non-dominated elements; compare, for example, [163].
(ii) Clearly, if the set-valued mapping $\mathcal{K} : X \rightrightarrows Y$ in Definition 2.5.6 is constant, that is, $\mathcal{K}(x) = K$, where K is a proper, closed, convex, pointed and solid cone in Y, then the above definition collapses to Definition 2.5.3.

In analogy to the extended real space $\mathbb{R} \cup \{\pm\infty\}$, it is useful to attach to the linear topological space Y a greatest and smallest element, denoted by $+\infty_Y$ and $+\infty_Y$; compare Section 2.1.1 in [20]. Then for $y \in Y \cup \{\pm\infty_Y\}$ it holds that $-\infty_Y \leq_K y \leq +\infty_Y$ and similar for $y \in Y$ it holds that $-\infty_Y \leq_{\operatorname{int} K} y \leq_{\operatorname{int} K} +\infty_Y$. Here, K is a proper, closed, convex and solid cone in Y. On $Y \cup \{\pm\infty_Y\}$ we consider the following operations:

$$\begin{aligned} y + (+\infty_Y) = (+\infty_Y) + y &:= +\infty_Y, && \text{for all} \quad y \in Y \cup \{+\infty_Y\}, \\ y + (-\infty_Y) = (-\infty_Y) + y &:= -\infty_Y, && \text{for all} \quad y \in Y \cup \{-\infty_Y\}, \\ \lambda \cdot (+\infty_Y) &:= +\infty_Y, && \text{for all} \quad \lambda > 0, \\ \lambda \cdot (-\infty_Y) &:= -\infty_Y, && \text{for all} \quad \lambda < 0, \\ (+\infty_Y) + (-\infty_Y) = (-\infty_Y) + (+\infty_Y) &:= +\infty_Y. \end{aligned}$$

We will further use the following conventions, which have been proposed in [56]:

$$
\begin{aligned}
+\infty_Y &\not\leq_{\operatorname{int} K} y, && \text{for all} \quad y \in Y,\\
y &\not\leq_{\operatorname{int} K} -\infty_Y, && \text{for all} \quad y \in Y.
\end{aligned}
$$

The above operations and conventions are useful since one frequently looks for minimal or efficient elements of a non-empty set $C \subseteq X$, where the objective mapping is $\psi : C \to Y$. However, by considering the new objective mapping

$$\tilde{\psi}(x) = \begin{cases} \psi(x), & x \in C, \\ +\infty_Y, & \text{else,} \end{cases}$$

it is possible to reformulate the above problems in the form of an unconstrained problem.

Definition 2.5.8. Let X and Y be linear topological spaces. Suppose further that $\psi : X \to Y \cup \{\pm\infty_Y\}$ and let K be a convex and pointed and solid cone in Y. Then the set

$$\mathcal{D}(\psi) := \big\{x \in C \mid \psi(x) \in Y\big\}$$

is called the (effective) domain of ψ. Until otherwise stated we assume that the effective domain of any extended mapping is non-empty. In analogy to the previous definitions, the set of efficient and weakly efficient elements are given by

$$\operatorname{Min}(\psi(X), K) = \big\{x \in \mathcal{D}(\psi) \mid \psi(\mathcal{D}(\psi)) \cap (\psi(x) - K) = \{\psi(x)\}\big\}.$$

and

$$\operatorname{WMin}(\psi(X), K) = \big\{x \in \mathcal{D}(\psi) \mid \psi(\mathcal{D}(\psi)) \cap (\psi(x) - \operatorname{int} K) = \{\psi(x)\}\big\},$$

respectively.

Chapter 3

Existence Results for Vector Variational Inequalities

Abstract. This chapter is devoted to the investigation of existence results for vector variational inequalities. Therefore, the second section consists of some preliminary definitions and results, such as the famous Minty lemma, scalarization techniques and some applications of vector variational inequalities in the field of multi-objective optimization. Besides that, we will introduce a new coercivity condition which allows us to prove a novel existence result for vector variational inequalities. However, in the absence of any coercivity condition, the existence of solutions of vector variational inequalities cannot be guaranteed. We therefore propose a regularization approach, which aims at approximating solutions of non-coercive problems by a family of regularized vector variational inequalities. In the end, we discuss some alternative conditions for the convergence of regularized solutions and apply our results to derive existence results for generalized vector variational inequalities.

3.1 Vector variational inequalities

In 1980, F. Giannessi introduced vector variational inequalities in a finite-dimensional setting; see [71]. He further provided some applications to alternative theorems, quadratic programs and complementarity problems. Since then, numerous researchers have proposed generalized vector variational inequalities and provided several existence results, which we recall in this chapter. Some of these existence theorems can be found, for example, in [7, 8, 10, 11, 28, 33, 36, 39, 47, 61, 62, 77, 84, 93, 106, 118, 119, 124, 133, 134, 148, 157, 161] and the references therein.

Let us consider some introductory details. Let X and Y be real Banach spaces, let C be a non-empty, closed and convex subset of X, and let K be proper, closed, convex and solid cone in Y. Given a mapping $F : X \to \mathrm{L}(X, Y)$, which maps into the space of linear and bounded operators from X to Y, the vector variational inequality, which will be studied in this chapter, consists of finding $x \in C$ such that

$$\langle Fx, y-x\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C. \tag{3.1.1}$$

In the above, $\langle\cdot,\cdot\rangle$ denotes the evaluation brackets of an operator $A \in \mathrm{L}(X,Y)$ at $x \in X$, that is, $\langle A, x\rangle := A(x)$. Clearly, if $Y = \mathbb{R}$, then the evaluation brackets coincide with the duality pairing in $X^* = \mathrm{L}(X,\mathbb{R})$. Consequently, by letting $Y = \mathbb{R}$ and $K = \mathbb{R}_{\geq}$, the above vector variational inequality recovers variational inequality (2.2.1) as special case. Besides that, we could also let $X = \mathbb{R}^l$, $Y = \mathbb{R}^k$ and $K = \mathbb{R}^k_{\geq}$. By identifying $\mathrm{L}(\mathbb{R}^l,\mathbb{R}^k)$ with the space of real $k \times l$ matrices $\mathrm{Mat}_{k\times l}(\mathbb{R})$, problem (3.1.1) recovers the following so-called finite-dimensional vector variational inequality: Find $x \in C$ such that

$$\begin{pmatrix} \langle F_1 x, y-x\rangle \\ \vdots \\ \langle F_k x, y-x\rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^k_{\geq}, \qquad \text{for every} \quad y \in C. \tag{3.1.2}$$

Here, for $j \in \{1,\dots,k\}$, $F_j : \mathbb{R}^l \to \mathbb{R}^l$ are the components of F, that is, $F = (F_1,\dots,F_k)^\top$ and, with some abuse of the notation, $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^l$. Recall that the scalar product of two vectors $x, y \in \mathbb{R}^l$ is defined by $\langle x, y\rangle := \sum_{j=1}^{l} x_j y_j$.

Remark 3.1.1. (i) Vector variational inequality (3.1.1) is a special case of the following vector equilibrium problem: Find $x \in C$ such that

$$T(x,y) \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C.$$

In the above, C is a non-empty, closed and convex subset of a real Banach space X, K is a proper, closed, convex and solid cone in the Banach space Y and $T : X \times X \to Y$ is a given mapping with $T(x,x) = 0$ for all $x \in C$. For further details, we refer to [10, 73] and the references therein.
(ii) If in addition to the setting of (i), the constraining set C is a convex cone and $T : X \to \mathrm{L}(X,Y)$ is a mapping with $\langle Tx, x\rangle \in -K$ for every $x \in C$, then vector variational inequality (3.1.1) is equivalent to the following vector complementary problem: Find $x \in C$ such that

$$\langle Tx, x\rangle \notin \operatorname{int} K \qquad \text{and} \qquad \langle Tx, y\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C.$$

See Proposition 4.2 in [39] and [73, 75] for some of the recent developments in the field of vector complementary problems.

In what follows, we will abbreviate by (A) the following assumptions:

(A1) X is a real reflexive Banach space. Y is a real Banach space.

(A2) The constraining set C is a non-empty, closed and convex subset of X.

(A3) K is a proper, closed, convex, and solid cone in Y.

Remark 3.1.2. It should be noted that we do not assume that the cone K is pointed.

Example 3.1.3. Let $a^1, \ldots, a^k$ be different vectors in $\mathbb{R}^l$ and consider the following finite-dimensional vector variational inequality: Find $x \in \mathbb{R}^l$ such that

$$\begin{pmatrix} \langle x - a^1, y - x \rangle \\ \vdots \\ \langle x - a^k, y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^k_{\geq}, \qquad \text{for every} \quad y \in \mathbb{R}^l. \tag{3.1.3}$$

The constraining set C of problem (3.1.3) is the whole space $\mathbb{R}^l$ while the objective mapping $F : \mathbb{R}^l \to \mathrm{Mat}_{k \times l}(\mathbb{R})$ is given by

$$Fx = \begin{pmatrix} F_1 x \\ \vdots \\ F_k x \end{pmatrix} := \begin{pmatrix} x - a^1 \\ \vdots \\ x - a^k \end{pmatrix}, \qquad \text{for every} \quad x \in \mathbb{R}^l. \tag{3.1.4}$$

In what follows, we denote the solution set of vector variational inequality (3.1.3) by S. Now let $j \in \{1, \ldots, k\}$ be arbitrarily chosen and consider the variational inequality of finding $x \in \mathbb{R}^l$ such that

$$\langle x - a^j, y - x \rangle \geq 0, \qquad \text{for every} \quad y \in \mathbb{R}^l.$$

Let us show that a^j is the unique solution of the above variational inequality. Clearly, a^j solves the variational inequality. Now, assume to the contrary that there is a second solution $\tilde{a}^j$. But this is impossible, since this would imply

$$\|\tilde{a}^j - a^j\|_2^2 \;=\; \langle \tilde{a}^j - a^j, a^j - \tilde{a}^j \rangle \;\leq\; 0,$$

and consequently, $a^j = \tilde{a}^j$. Therefore, by the representation of $\operatorname{int} \mathbb{R}^k_{\geq}$, we have

$$\{a^1, \ldots, a^k\} \;\subseteq\; \mathrm{S}\,.$$

The following examples will show that S is equivalent to the convex hull of the vectors $a^1, \ldots, a^k$.

3.2 Preliminary results and concepts

In this section, we are going to recall some preliminary results and concepts, which will be used in this thesis.

3.2.1 Minty lemma for vector variational inequalities

In order to make the following sections self contained, we briefly set forth below some important notations, definitions and results which we use here.

Definition 3.2.1 ([10, Section 5.2])**.** Besides assumption (A), let $F : X \to \mathrm{L}(X, Y)$.

(i) The mapping F is called K*-monotone* if for every $x, y \in X$ it holds that

$$\langle Fx - Fy, x - y \rangle \in K.$$

(ii) F is called K*-pseudomonotone* if for every $x, y \in X$, $\langle Fx, y - x \rangle \notin -\operatorname{int} K$ implies $\langle Fy, y - x \rangle \notin -\operatorname{int} K$.

(iii) We say that F is *v-hemicontinuous* if for every $x, y, z \in X$ the mapping $[0, 1] \to Y$, given by $t \mapsto \langle F(x + ty), z \rangle$, is continuous.

(iv) The mapping F is called *continuous at* $x \in X$ if for each sequence $\{x_n\}$ in X, $x_n \to x$ implies $Fx_n \to Fx$. F is called *continuous* if it is continuous at each point in X.

Remark 3.2.2. (i) Using Proposition 2.4.12 (iv), it is easily seen that any K-monotone mapping is K-pseudomonotone.
(ii) Any continuous mapping $F : X \to \mathrm{L}(X, Y)$ is v-hemicontinuous. However, the converse does not hold in general; compare [63, Example 2.4].
(iii) If we let $Y = \mathbb{R}$ and $K = \mathbb{R}_{\geq}$, then the above notions recover the notions of hemicontinuous and monotone operators from X to $\mathrm{L}(X, \mathbb{R}) = X^*$; compare the Definitions 2.2.1 and 2.2.2.

We next recall the vector analogue of Minty's lemma for vector variational inequalities.

Lemma 3.2.3 (Minty, [39, 72])**.** *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X, Y)$ *be a* K*-monotone and v-hemicontinuous mapping. Then, an element* $x \in C$ *is a solution of the vector variational inequality* (3.1.1)*, that is, satisfies*

$$\langle Fx, y - x \rangle \notin -\operatorname{int} K, \qquad \textit{for every} \quad y \in C,$$

if and only if it satisfies

$$\langle Fy, y - x \rangle \notin -\operatorname{int} K, \qquad \textit{for every} \quad y \in C. \tag{3.2.1}$$

Remark 3.2.4. (i) A generalized Minty lemma for generalized vector variational inequalities is provided in Lemma 4.1.7.
(ii) Problem (3.2.1) is frequently called Minty vector variational inequality; see [72]. For a kind of symmetry, problem (3.1.1) is called Stampacchia vector variational inequality; see [10, Chapter 5].

Example 3.2.5 ([91, Example 4.1])**.** The mapping F, given by (3.1.4), is $\mathbb{R}^k_{\geq}$-monotone and v-hemicontinuous; see Example 4.1 in [91]. Thus, vector variational inequality

(3.1.3) is equivalent to finding $x \in \mathbb{R}^l$ such that

$$\begin{pmatrix} \langle y - a^1, y - x \rangle \\ \vdots \\ \langle y - a^k, y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^k_{\geq}, \qquad \text{for every} \quad y \in \mathbb{R}^l.$$

3.2.2 Linear and non-linear scalarization

Let us come back to the finite-dimensional problem (3.1.2). Clearly, if there exists an index $j \in \{1, \ldots, k\}$ such that $x \in C$ satisfies

$$\langle F_j x, y - x \rangle \geq 0, \qquad \text{for every} \quad y \in C,$$

then x is a solution of problem (3.1.2). It should be noted that we have $\langle \mathrm{e}_j^\top Fx, y - x \rangle = \langle F_j x, y - x \rangle$, where e_j denotes the jth unit vector in $\mathbb{R}^k$. In what follows, we will transfer this observation to the case where the underlying problem is vector variational inequality (3.1.1) and e_j is replaced with a suitable functional.

To be precise, assume that assumption (A) holds and let $F : X \to \mathrm{L}(X, Y)$. Further, let

$$s : Y \to \mathbb{R}$$

be given. We will study the following family of scalar variational inequalities: Find $x = x(s) \in C$ such that

$$s\big(\langle Fx, y - x \rangle\big) \geq 0, \qquad \text{for every} \quad y \in C. \tag{3.2.2}$$

One of the most powerful and most used approaches for vector variational inequalities is to apply scalarization techniques; see [30, 33, 39, 41, 43, 44, 47, 62, 84, 93, 119, 124, 149, 154, 160]. These methods are very important from the theoretical as well as computational point of view; see, for example, Chapter 6 of this thesis. Indeed, both necessary and sufficient optimality conditions and even equivalent formulations for problem (3.1.1) can be derived. In the context of vector variational inequalities, two types of scalarizing functions have turned out to be of great use:

1. *Linear scalarizing.* If $s \in Y^* \setminus \{0\}$ is a linear and continuous functional then problem (3.2.2) becomes: Find $x = x(s) \in C$ such that

   $$\langle s \circ Fx, y - x \rangle \geq 0, \qquad \text{for every} \quad y \in C. \tag{3.2.3}$$

   Note that for every $x \in X$, the composition $s \circ Fx$ belongs to X^*, that is, $s \circ F$ defines an operator from X to X^*. In what follows, the solution set of problem (3.2.3) will be denoted by $\mathrm{Sol}(\mathrm{VI}_s)$.

2. Non-linear scalarizing: If $s : Y \to \mathbb{R}$ is the non-linear Tammer-Weidner scalarization function, that is, $s(y) = \inf\{t \in \mathbb{R} \mid y \in te - K\}$ for $y \in Y$, where $e \in \operatorname{int} K$

is fixed, then problem (3.2.2) becomes: Find $x = x(s) \in C$ such that

$$s\big(\langle Fx, y - x\rangle\big) \geq 0, \qquad \text{for every} \quad y \in C. \tag{3.2.4}$$

The next results show that vector variational inequality (3.1.1) can be completely characterized by the scalar problems (3.2.3) and (3.2.4). Recall that we denote the solution set of problem (3.1.1) by Sol (VVI).

Proposition 3.2.6 ([47, Proposition 2.1])**.** *Suppose that assumption* (A) *holds and let* $F : X \to \mathrm{L}(X,Y)$*. Assume further that the quasi-interior of* K^* *is non-empty. Then it holds that*

$$\bigcup_{s \in \mathrm{qi}\, K^*} \mathrm{Sol}(\mathrm{VI}_s) \subseteq \mathrm{Sol}\,(\mathrm{VVI}) = \bigcup_{s \in K^* \setminus \{0\}} \mathrm{Sol}(\mathrm{VI}_s).$$

Example 3.2.7 ([91, Example 3.1])**.** Let us come back to Example 3.1.3. In what follows, we will characterize the solution set S of problem (3.1.3) using the previous proposition. Therefore, let $s = (s_1, \ldots, s_k)^\top \in \mathbb{R}^k_{\geq} \setminus \{0\}$. Then the scalar variational inequality with respect to s consists of finding $x = x(s) \in \mathbb{R}^l$ such that

$$\langle \sum_{j=1}^{k} s_j (x - a^j), y - x\rangle \geq 0, \qquad \text{for every} \quad y \in \mathbb{R}^l. \tag{3.2.5}$$

It is not hard to check that

$$x = \frac{\sum_{j=1}^{k} s_j a^j}{\sum_{j=1}^{k} s_j} = \sum_{j=1}^{k} \frac{s_j}{\sum_{i=1}^{k} s_i} a^j \tag{3.2.6}$$

is the unique solution of problem (3.2.5). Consequently, denoting the solution set of problem (3.2.5) w.r.t. the vector $s \in \mathbb{R}^k_{\geq} \setminus \{0\}$ by $\mathrm{S}(s)$, we immediately observe

$$\mathrm{S}(s) \subseteq \mathrm{conv}\{a^1, \ldots, a^k\}.$$

Thus, Proposition 3.2.6 yields

$$\mathrm{S} = \bigcup_{s \in \mathbb{R}^k_{\geq} \setminus \{0\}} \mathrm{S}(s) \subseteq \mathrm{conv}\,\{a^1, \ldots, a^k\}. \tag{3.2.7}$$

In order to show the converse inclusion in (3.2.7) assume there is $x \in \mathrm{conv}\{a^1, \ldots, a^k\}$, which is not a solution of problem (3.1.3). Thus we can find real numbers $\lambda_1, \ldots, \lambda_k$ with $\lambda_j \geq 0$ for $j = 1, \ldots, k$ and $\sum_{j=1}^{k} \lambda_j = 1$ such that $x = \sum_{j=1}^{k} \lambda_j a^j$. Without any loss of generality assume $\lambda_j \neq 0$ for $j = 1, \ldots, k$. Then there exists a vector $y \in \mathbb{R}^l$ such that

$$\langle x - a^j, y - x\rangle < 0, \qquad \text{for} \quad j = 1, \ldots, k.$$

By multiplying every inequality with λ_j and summing up the resulting inequalities, we obtain

$$0 > \sum_{j=1}^{k} \lambda_j \langle x - a^j, y - x\rangle = \sum_{j=1}^{k} \lambda_j \langle x, y - x\rangle - \sum_{j=1}^{k} \lambda_j \langle a^j, y - x\rangle$$
$$= \langle x, y - x\rangle - \langle x, y - x\rangle = 0,$$

which is impossible. Therefore, the inverse inclusion in (3.2.7) holds and consequently

$$\mathrm{S} = \operatorname{conv}\{a^1, \dots, a^k\}. \tag{3.2.8}$$

The next result follows from the fact that the Tammer-Weidner function $s : Y \to \mathbb{R}$ satisfies the so-called representability condition

$$\{y \in Y \mid s(y) < 0\} = -\operatorname{int} K;$$

compare Theorem 5.2.3 in [108].

Proposition 3.2.8 ([91, Section 3.2]). *Suppose that assumption* (A) *holds and let* $F : X \to \mathrm{L}(X, Y)$. *Then, problem* (3.2.4) *is equivalent to vector variational inequality* (3.1.1).

Example 3.2.9 ([91, Example 3.2]). Let us again come back to Example 3.1.3. Let $e = (e_1, \dots, e_k)^\top \in \operatorname{int} \mathbb{R}^k_{\geq}$ and note that for every $x, y \in \mathbb{R}^l$, it holds that

$$s\big(\langle Fx, y - x\rangle\big) = \min\left\{ t \in \mathbb{R} \mid \begin{pmatrix} \langle x - a^1, y - x\rangle \\ \vdots \\ \langle x - a^k, y - x\rangle \end{pmatrix} \in t \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} - \mathbb{R}^k_{\geq} \right\}$$
$$= \min\left\{ t \in \mathbb{R} \mid \frac{\langle x - a^j, y - x\rangle}{e_j} \leq t, \quad \text{for} \quad j = 1, \dots, k \right\}$$
$$= \max_{j=1,\dots,k} \langle x - a^j, y - x\rangle,$$

where $s : \mathbb{R}^k \to \mathbb{R}$ is the Tammer-Weidner function. By the previous calculations, the non-linear scalar variational inequality (3.2.4) becomes: Find $x \in \mathbb{R}^l$ such that

$$\max_{j=1,\dots,k} \langle x - a^j, y - x\rangle \geq 0, \qquad \text{for every} \quad y \in \mathbb{R}^l.$$

It is not hard to show that the solution set of the above problem is equivalent to the convex hull of $a^1, \dots, a^k$, which again confirms formula (3.2.8).

3.2.3 Applications in multi-objective optimization

In this section, we are going to show that vector variational inequalities can be used to study multi-objective optimization problems. Such relations are very useful since one can transfer techniques and ideas from one field to the other; see [56, 79].

For further use, we need the definition of K-convex mappings.

Definition 3.2.10 ([98, Definition 2.4])**.** Let X and Y be real linear spaces and let K be a convex cone in Y. A mapping $\psi : X \to Y$ is said to be K-*convex* if for all $x, y \in X$ and every $\lambda \in [0,1]$ it holds that $\lambda\psi(x) + (1-\lambda)\psi(y) - \psi\big(\lambda x + (1-\lambda)y\big) \in K$.

Remark 3.2.11. A mapping $\psi : \mathbb{R}^l \to \mathbb{R}^k$ is $\mathbb{R}^k_{\geq}$-convex if and only if all its components are convex.

The following result shows that smooth multi-objective optimization problems are equivalent to vector variational inequalities of the type (3.1.1).

Theorem 3.2.12 ([35, Proposition 3.3])**.** *Besides assumption* (A)*, let* $\psi : X \to Y$*. If in addition, the cone* K *is pointed, then we have:*

(i) *If* ψ *is right-handed Gâteaux-differentiable at* $x \in C$ *with derivative* $D^+_{\mathrm{G}}\psi(x)$ *and*

$$x \in \mathrm{WEff}(\psi(C), K), \tag{3.2.9}$$

then, $x \in C$ *solves the vector variational inequality*

$$\langle D^+_{\mathrm{G}}\psi(x), y - x\rangle \notin -\operatorname{int} K, \qquad \textit{for every} \quad y \in C. \tag{3.2.10}$$

(ii) *Conversely, if* ψ *is* K*-convex and* $x \in C$ *is a solution of vector variational inequality* (3.2.10)*, then* x *satisfies* (3.2.9)*.*

Remark 3.2.13. (i) Theorem 3.2.12 is known as Fermat's rule for multi-objective optimization problems; see Section 2 in [162].
(ii) Investigations of relations between vector variational inequalities and (set-valued) multi-objective problems can also be found in Chapter 4 of this thesis and in [10, 11, 48, 66, 81, 127, 131, 132, 144, 154].

Example 3.2.14 ([93, Example 3.7])**.** Let $a^1, \dots, a^k$ be different vectors in $\mathbb{R}^l$. In this example we are going to calculate the set of (weakly) efficient points

$$\mathrm{WEff}\,\big(\psi(\mathbb{R}^l), \mathbb{R}^k_{\geq}\big), \tag{3.2.11}$$

where the objective mapping $\psi : \mathbb{R}^l \to \mathbb{R}^k$ is given by

$$\psi(x) := \begin{pmatrix} \frac{1}{2}\|x - a^1\|_2^2 \\ \vdots \\ \frac{1}{2}\|x - a^k\|_2^2 \end{pmatrix}, \qquad \text{for every} \quad x \in \mathbb{R}^l.$$

As always, $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^l$. In order to calculate (3.2.11), we will use Theorem 3.2.12 and study an equivalent vector variational inequality. Let us therefore show that the objective mapping ψ is $\mathbb{R}^k_{\geq}$-convex and Gâteaux-differentiable. Indeed, since all components of ψ are convex functions from $\mathbb{R}^l$ to $\mathbb{R}$, the $\mathbb{R}^k_{\geq}$-convexity of ψ follows; compare Remark 3.2.11. Note that it holds that

$$\psi_j(x + th) - \psi_j(x) = \|x + th - a^j\|_2^2 - \|x - a^j\|_2^2 = t\langle x - a^j, h\rangle + \frac{1}{2}t^2\|h\|_2^2$$

for every $x, y \in \mathbb{R}^l$, $t \in \mathbb{R}$ and $j \in \{1, \ldots, k\}$. Thus, we conclude

$$\delta\psi(x;h) = \lim_{t\to 0} \frac{\psi(x+th)-\psi(x)}{t} = \begin{pmatrix} \langle x-a^1, h\rangle \\ \vdots \\ \langle x-a^k, h\rangle \end{pmatrix} \quad \text{and} \quad D_{\mathrm{G}}\psi(x) = \begin{pmatrix} x-a^1 \\ \vdots \\ x-a^k \end{pmatrix}.$$

Consequently, Theorem 3.2.12 states that problem (3.2.11) is equivalent to the following finite-dimensional vector variational inequality: Find $x \in \mathbb{R}^l_{\geq}$ such that

$$\langle D_{\mathrm{G}}\psi(x), y-x\rangle \;=\; \begin{pmatrix} \langle x-a^1, y-x\rangle \\ \vdots \\ \langle x-a^k, y-x\rangle \end{pmatrix} \notin -\operatorname{int}\mathbb{R}^k_{\geq}, \qquad \text{for every} \quad y \in \mathbb{R}^l.$$

However, we have already shown that the solution set of the above vector variational inequality is given by the convex hull of $a^1, \ldots, a^k$; compare the previous examples. Thus,

$$\mathrm{WEff}\left(\psi(\mathbb{R}^l), \mathbb{R}^k_{\geq}\right) \;=\; \mathrm{conv}\left\{a^1, \ldots, a^k\right\}.$$

Remark 3.2.15. A multi-objective optimization problem of the type (3.2.11) is known in the literature as *location problem* and has various real-life applications; see [78, Chapter 4] or Section 5.3 of this thesis.

3.3 Classic existence results

In this section, we recall classic existence results for vector variational inequality (3.1.1). We further provide a new coercivity condition, which we use to prove a new existence results for problem (3.1.1); compare Theorem 3.3.26.

In what follows, the main tools for deriving existing results for problem (3.1.1) are the Fan-KKM lemma, the Hartmann-Stampacchia theorem, Ky-Fan and Glicksberg's fixed-point theorem for set-valued mappings, Fan's section lemma and Brouwer's fixed-point theorem.

3.3.1 Existence results for monotone problems

Let us recall the famous Fan-KKM lemma by Fan, Knaster, Kuratowsik and Mazurkiewicz, which is one of the main tools for deriving existence results for vector variational inequalities.

Lemma 3.3.1 (Fan-KKM, [59, Lemma 1])**.** *Let C be a non-empty subset of the topological vector space X and let $G : C \rightrightarrows X$ be a set-valued mapping with non-empty values. Then it holds that*

$$\bigcap_{y\in C} G(y) \neq \emptyset,$$

provided G satisfies the following properties:

(i) *G is a KKM-mapping, that is, for any $k \in \mathbb{N}$ and any finite subset $\{x_1, \dots, x_k\}$ in C it holds that $\operatorname{conv}\{x_1, \dots, x_k\} \subseteq \bigcup_{j=1}^{k} G(x_j)$.*

(ii) *G has closed values.*

(iii) *There exists $y_0 \in C$ such that $G(y_0)$ is compact in X.*

Remark 3.3.2. (i) An excellent overview about applications of the Fan-KKM lemma in the field of set-valued fixed-point results, minimax equalities and inequalities and variational inequalities can be found in [128, 146].
(ii) It is easily seen that a set-valued mapping $G : C \rightrightarrows X$ is a KKM-mapping if it holds that $y \in G(y)$ and $X \setminus G^{-1}(y)$ is convex for every $y \in C$; see [82, (66.2) Proposition].

The next theorem provides a first existence result for vector variational inequality (3.1.1).

Theorem 3.3.3 ([39, Theorem 2.1])**.** *Besides assumption* (A)*, let $F : X \to \mathrm{L}(X, Y)$ be a K-monotone and v-hemicontinuous mapping. If C is bounded in addition, then, vector variational inequality* (3.1.1) *has a solution.*

Remark 3.3.4. (i) The proof of Theorem 3.3.3 uses Fan-KKM's lemma and is based on the following observation: Vector variational inequality (3.1.1) has a solution if and only if one of the sets

$$\bigcap_{y \in C} G(y) \qquad \text{and} \qquad \bigcap_{y \in C} G'(y)$$

is non-empty, where $G, G' : C \rightrightarrows X$ are set-valued mappings, given by $G(y) = \{x \in C \mid \langle Fx, y - x\rangle \notin -\operatorname{int} K\}$ and $G'(y) = \{x \in C \mid \langle Fy, y - x\rangle \notin -\operatorname{int} K\}$, for every $y \in C$.
(ii) Evidently, the requirements for the constraining set C in Theorem 3.3.3 can be replaced in the following way: The set C is non-empty, convex and weakly compact; compare [124, Theorem 3.1]. Further, if C is assumed to be compact, then the reflexivity of X can be dropped.
(iii) The Fan-KKM Lemma 3.3.1 cannot be applied to the establishment of the existence of vector variational inequalities of the following type: Find $x \in C$ such that

$$\langle Fx, y - x\rangle \notin -K \setminus \{0\}, \qquad \text{for every} \quad y \in C.$$

This is due to the fact that the corresponding set-valued mappings G and G', compare part (i) of this remark, do not have closed values in general.
(iv) Several other existence results for vector variational inequality (3.1.1) and generalizations of it, based on the Fan-KKM lemma, can be found, for example, in [3, 12, 36, 46, 60, 62, 67, 87, 104, 106, 114, 118, 123, 134, 148, 157, 158, 161].

In the following theorem, the K-monotonicity of F is replaced with a so-called L-condition. See [10, Chapter 5] for several other existence results based on the L-condition.

Theorem 3.3.5 ([159, Proposition 1]). *Besides assumption* (A), *let* $F : X \to \mathrm{L}(X, Y)$ *be a* v*-hemicontinuous mapping, which satisfies the following L-condition: For any natural number* $k \in \mathbb{N}$, *for any finite subset* $\{x_1, \ldots, x_k\} \subseteq C$ *and for all* $\lambda_1, \ldots, \lambda_k$ *with* $\lambda_j \geq 0$ *for* $j = 1, \ldots, k$ *and* $\sum_{j=1}^{k} \lambda_j = 1$ *it holds that*

$$\sum_{j=1}^{k} \lambda_j \langle Fx_j, x_j \rangle - \sum_{j=1}^{k} \lambda_j \langle Fx_j, \bar{x} \rangle \in K, \tag{3.3.1}$$

where $\bar{x} := \sum_{j=1}^{k} \lambda_j x_j$. *If in addition,* C *is bounded, then vector variational inequality* (3.1.1) *has a solution.*

Recall that the following definition has already been introduced in the introduction of this thesis.

Definition 3.3.6 ([159, Section 3]). Besides assumption (A), let $F : X \to \mathrm{L}(X, Y)$ be a given mapping. F is said to be v*-coercive* if there exists a non-empty and compact subset $B \subseteq X$ and an element $y_0 \in B \cap C$ such that

$$\langle Fy, y_0 - y \rangle \in \operatorname{int} K, \qquad \text{for every} \quad y \in C \setminus B. \tag{3.3.2}$$

Remark 3.3.7. (i) Clearly, F is v-coercive with $B = C$ provided that C is compact.
(ii) See [10, Definition 5.7] for several other v-coercivity conditions, which are related to the previous definition.

In the next result, the boundedness of C in Theorem 3.3.3 is replaced with the v-coercivity of F.

Theorem 3.3.8 ([159, Theorem 1]). *Besides assumption* (A), *let* $F : X \to \mathrm{L}(X, Y)$ *be a* K*-monotone,* v*-hemicontinuous and* v*-coercive mapping. Then, vector variational inequality* (3.1.1) *has a solution.*

Remark 3.3.9. (i) The reflexivity of X can be dropped in Theorem 3.3.8.
(ii) In comparison to Theorem 3.3.3, the v-coercivity condition ensures the compactness of the set $G'(y_0)$ only, where $y_0 \in C$ is given by the v-coercivity condition, while the boundedness of C would imply the weak compactness of all sets $G'(y)$, $y \in C$.
(iii) Similar existence results for vector variational inequalities can be found in [11, 36, 47, 134, 157] and the references therein.

Theorem 3.3.10 ([124, Theorem 3.3]). *Besides assumption* (A), *let* $F : X \to \mathrm{L}(X, Y)$ *be a* K*-monotone and* v*-hemicontinuous mapping. If it holds that* $0 \in C$ *and there exists an element* $y_0 \in C$ *and a number* $d > 0$ *such that*

$$\langle Fy, y_0 - y \rangle \in - \operatorname{int} K, \tag{3.3.3}$$

for every $y \in C$ *with* $\|y_0 - y\|_X > d$, *then, vector variational inequality* (3.1.1) *has a solution.*

Remark 3.3.11. Note that coercivity condition (3.3.3) can be equivalently stated in the following way: There exists an element $y_0 \in C$ and a number $d > 0$ such that

$$\langle Fy, y_0 - y\rangle \in -\operatorname{int} K, \qquad \text{for every} \quad y \in C \setminus B(y_0, d).$$

Thus, if X is a finite-dimensional Euclidean space, the previous theorem recovers Theorem 3.3.10; compare also Theorem 2.1.3.

The next result can be found in [109, Theorem 3.2]. A similar coercivity condition is also used in [28, 161].

Theorem 3.3.12. *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X, Y)$ *be a* K*-monotone and* v*-hemicontinuous mapping. If it holds that* $0 \in C$ *and there is a number* $r > 0$ *such that*

$$\langle Fx, x\rangle \in \operatorname{int} K, \qquad \textit{for every} \quad x \in C \cap \operatorname{bd} B(0, r), \tag{3.3.4}$$

then, vector variational inequality (3.1.1) *has a solution.*

Remark 3.3.13. If F is K-monotone and $F(0) = 0$, then we have $\langle Fx, x\rangle \in K$ for every $x \in X$.

In order to formulate the next existence result, we recall the following coercivity condition for F which has already been introduced in Chapter 1.

Definition 3.3.14 ([39, Section 2])**.** Besides assumption (A), assume that the quasi-interior of K^* is non-empty and let $F : X \to \mathrm{L}(X, Y)$ be a given mapping. F is said to be *weakly coercive* if there exist an element $x_0 \in C$ and a functional $s \in \operatorname{qi} K^*$ such that

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\langle s \circ Fx - s \circ Fx_0, x - x_0\rangle}{\|x - x_0\|_X} = +\infty. \tag{3.3.5}$$

We have the following result, which is Theorem 2.1 in [39]. The proof is using the fact that the scalar problem (3.2.3) is necessary for vector variational inequality (3.1.1); see Proposition 3.2.6. It is therefore enough to ensure that problem (3.2.3), where $s \in \operatorname{qi} K^*$ is given by the weak coercivity of F, has a solution.

Theorem 3.3.15. *Besides assumption* (A)*, assume that the quasi-interior of* K^* *is non-empty and let* $F : X \to \mathrm{L}(X, Y)$ *be a* K*-monotone,* v*-hemicontinuous and weakly coercive mapping. Then, vector variational inequality* (3.1.1) *has a solution.*

Remark 3.3.16. (i) Notice that the above results recovers Theorem 2.2.7 if we let $Y = \mathbb{R}$ and $K = \mathbb{R}_{\geq}$.
(ii) Some other existence results, which are based on a scalarization technique, can be found in [62, 84, 119, 124, 160]. Further, an overview of linear scalarization methods for the finite-dimensional problem (3.1.2) can be found in Chapter 6 of [10].

3.3.2 Existence results for non-monotone problems

This section is devoted to the study of existence results for vector variational inequality (3.1.1), where, in comparison to the previous section, the K-monotonicity of F is dropped.

Theorem 3.3.17 ([39, Theorem 2.2])**.** *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X, Y)$ *be a continuous mapping. If* C *is bounded in addition, then, vector variational inequality* (3.1.1) *has a solution.*

Remark 3.3.18. In [34], the authors apply Ky-Fan's and Glickberg's fixed-point result to prove Theorem 3.3.17. The proof makes use of the fact that solving vector variational inequality (3.1.1) is equivalent to finding a solution of problem (3.2.4). Thus, introducing a family $S_n : C \rightrightarrows C$ of set-valued mappings by

$$S_n(x) := \left\{ y \in C \mid s\big(\langle Fx, y - x\rangle\big) - \min_{\tilde{y} \in C} s\big(\langle Fx, \tilde{y} - x\rangle\big) < \frac{1}{n} \right\}, \qquad \text{for every} \quad x \in C,$$

where $s : Y \to \mathbb{R}$ is the Tammer-Weidner function, it remains to show that every S_n has a fixed-point, that is, there is $x_n \in C$ with

$$x_n \in S_n(x_n). \tag{3.3.6}$$

This can be done by using Theorem 2.3.9. Indeed, if (3.3.6) holds, then, due to the compactness of C, there exists a subsequence, again denoted by $\{x_n\}$, with $x_n \to x$ and $x \in C$. Finally, passing in (3.3.6) to the limit yields

$$\min_{y \in C} s\big(\langle Fx, y - x\rangle\big) \geq 0.$$

Thus, in view of Proposition 3.2.8, the limit point x is a solution of vector variational inequality (3.1.1).

Similar to the previous results, one can replace the boundedness (or compactness) condition for C with a strong v-coercivity condition of F.

Definition 3.3.19 ([10, Definition 5.7])**.** Besides assumption (A), let $F : X \to \mathrm{L}(X, Y)$ be a given mapping. F is said to be *strongly v-coercive* if there exists a non-empty and compact subset $B \subseteq X$ and an element $y_0 \in B \cap C$ such that

$$\langle Fy_0, y_0 - y\rangle \in \operatorname{int} K, \qquad \text{for every} \quad y \in C \setminus B.$$

Theorem 3.3.20 ([159, Theorem 1])**.** *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X, Y)$ *be a continuous and strongly v-coercive mapping. Then, vector variational inequality* (3.1.1) *has a solution.*

Note that in the following theorem the mapping F is neither assumed to be K-monotone nor continuous.

Theorem 3.3.21 ([109, Theorem 4.1])**.** *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X, Y)$ *be a given mapping. If* C *is compact and for every* $y \in C$*, the set*

$$H(y) := \{x \in C \mid \langle Fx, y - x\rangle \in -\operatorname{int} K\}$$

is open in X*, then vector variational inequality* (3.1.1) *has a solution.*

Remark 3.3.22. (i) The proof in [109] uses Brouwer's fixed-point theorem; compare Theorem 2.3.1.
(ii) Some other results using similar assumptions can be found in [61, Theorem 2.1] and [148, Theorem 4.1].

3.3.3 A new existence result for vector variational inequalities

In what follows we will introduce a new coercivity condition for F and derive a novel existence result for vector variational inequality (3.1.1). There results of this sections are based on the paper [93] by Hebestreit, Khan, Köbis and Tammer.

Definition 3.3.23 ([93, Definition 2.4])**.** Besides assumption (A), let $F : X \to \mathrm{L}(X, Y)$ be a given mapping. F is said to be *strongly continuous* if $x_n \rightharpoonup x$ in X implies $Fx_n \to Fx$ in $\mathrm{L}(X, Y)$.

Remark 3.3.24. (i) Strongly continuous mappings are sometimes called completely continuous; see [35, Definition 3.13].
(ii) Clearly, if X and Y are finite-dimensional Euclidean spaces, then the notion of completely continuous and continuous mappings coincide; compare Proposition 2.1.13 (ii).

In order to formulate the next result, we need the following coercivity condition, which has been introduced in [93]. Recall that Definition 3.3.25 has already been introduced in the introduction of this thesis.

Definition 3.3.25. Besides assumption (A), let $F : X \to \mathrm{L}(X, Y)$ be a given mapping. F is said to be κ*-coercive* if there exists a mapping $\kappa : X \to \mathbb{R}_{\geq}$ and a functional $s \in K^* \setminus \{0\}$ such that

$$\langle s \circ Fx, x\rangle \geq \|s\|_{Y^*}\, \kappa(x), \qquad \text{for every} \quad x \in C,$$
$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\kappa(x)}{\|x\|_X} = +\infty.$$

Theorem 3.3.26 ([93, Theorem 6])**.** *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X, Y)$ *be a strongly continuous and* κ*-coercive mapping. If it holds that* $0 \in C$*, then, vector variational inequality* (3.1.1) *has a solution.*

Proof. The key idea of this proof is to show that scalar variational inequality (3.2.3) has a solution, where $s \in K^* \setminus \{0\}$ is given by the κ-coercivity condition, since every solution of problem (3.2.3) is one of (3.1.1); see Proposition 3.2.6. For this purpose, we define a set-valued mapping $G_s : C \rightrightarrows X$ by

$$G_s(y) := \{x \in C \mid \langle s \circ Fx, y - x\rangle \geq 0\}, \qquad \text{for every} \quad y \in C.$$

Note that for every $y \in C$, it holds that $y \in G_s(y)$, that is, G_s has non-empty values. Evidently, every element belonging to the intersection

$$I_{G_s} := \bigcap_{y \in C} G_s(y)$$

is a solution of the scalar variational inequality (3.2.3). In order to prove that I_{G_s} is non-empty, we are going to show that all requirements of Lemma 3.3.1 are satisfied.

(I). Let us show that G_s is a KKM-mapping. Indeed, arguing by contradiction, let $k \in \mathbb{N}$ and $\{x_1, \ldots, x_k\} \subseteq C$ and suppose that

$$\operatorname{conv}\{x_1, \ldots, x_k\} \not\subseteq \bigcup_{j=1}^{k} G_s(x_j).$$

Then there are real numbers $\lambda_1, \ldots, \lambda_k$ with $\sum_{j=1}^{k} \lambda_j = 1$ and $\lambda_j \geq 0$ for $j = 1, \ldots, k$ such that $\bar{x} \notin \bigcup_{j=1}^{k} G_s(x_j)$, where $\bar{x} := \sum_{j=1}^{k} \lambda_j x_j$. In other words, we have

$$\langle s \circ F\bar{x}, x_j - \bar{x}\rangle < 0, \qquad \text{for} \quad j = 1, \ldots, k.$$

By multiplying every inequality by λ_j and summing them up, we derive the contradiction

$$\langle s \circ F\bar{x}, \bar{x}\rangle = \langle s \circ F\bar{x}, \sum_{j=1}^{k} \lambda_j x_j\rangle \;<\; \sum_{j=1}^{k} \lambda_j \langle s \circ F\bar{x}, \bar{x}\rangle = \langle s \circ F\bar{x}, \bar{x}\rangle,$$

showing that G_s is a KKM-mapping.

(II). Let us show that G_s has weakly closed values. Indeed, let $y \in C$ be arbitrarily chosen and consider a sequence $\{x_n\}$ in $G_s(y)$ with $x_n \rightharpoonup x$. Since F is strongly continuous, we have $Fx_n \to Fx$ in $\mathrm{L}(X, Y)$ which implies

$$\langle s \circ Fx_n, y - x_n\rangle \;\to\; \langle s \circ Fx, y - x\rangle;$$

see Proposition 2.1.13. Note that we have $x \in C$ since the set is assumed to be closed and convex; compare Proposition 2.1.15. This shows $x \in G_s(y)$. Hence, G_s has weakly closed values.

(III). Let us show that $G_s(0)$ is weakly compact. Since we have already shown that $G_s(0)$ is weakly closed, it remains to show that $G_s(0)$ is bounded. Assume by contradiction that $G_s(0)$ is unbounded. Then, we can find a sequence $\{x_n\}$ in $G_s(0)$ such that

$$\|x_n\|_X \to +\infty.$$

Consequently, using the κ-coercivity of F and the fact that the sequence $\{x_n\}$ lies in $G_s(0)$, it follows

$$\langle s \circ Fx_n, x_n \rangle \leq 0,$$

and consequently

$$\|s\|_{Y^*} \frac{\kappa(x_n)}{\|x_n\|_X} \leq \frac{\langle s \circ Fx_n, x_n \rangle}{\|x_n\|_X} \leq 0.$$

The above inequality leads to a contradiction when passing to the limit since the left-hand side is unbounded. Thus, $G_s(0)$ is weakly compact.

(IV). We are finally in position to apply Lemma 3.3.1, ensuring that the set I_{G_s} is non-empty. Consequently,

$$\emptyset \neq I_{G_s} = \mathrm{Sol}(\mathrm{VI}_s) \subseteq \mathrm{Sol}\,(\mathrm{VVI}),$$

where the inclusion follows from Proposition 3.2.6. We therefore have shown that vector variational inequality (3.1.1) has a solution. The proof is complete. □

As a special case of Theorem 3.3.26, we have the following new existence result for variational inequality (2.2.1).

Corollary 3.3.27. *Let C be a non-empty, closed and convex subset of the real reflexive Banach space X with $0 \in C$. Further let $F : X \to X^*$ be strongly continuous and assume there exists a mapping $\kappa : X \to \mathbb{R}_{\geq}$ such that*

$$\langle Fx, x \rangle \geq \kappa(x), \qquad \textit{for every} \quad x \in C,$$
$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\kappa(x)}{\|x\|_X} = +\infty.$$

Then, variational inequality (2.2.1) has a solution.

The next example is Example 3.7 in [93].

Example 3.3.28. Let us come back to problem (3.1.3). In what follows, we will use Theorem 3.3.26 to show that vector variational inequality (3.1.3) has a solution. Let us assume without any loss of generality that $a^1 = 0$. Since the mapping F, given by (3.1.4), is obviously continuous, it remains to show that F is κ-coercive. Indeed, let $s = \mathrm{e}_1$ and define $\kappa(x) = \|x\|_2^2$ for $x \in \mathbb{R}^l$. It then holds that

$$\langle s^\top Fx, x \rangle = \langle x, x \rangle = \|x\|_2^2 = \|s\|_2 \kappa(x), \qquad \text{for every} \quad x \in \mathbb{R}^l,$$

and

$$\lim_{\|x\|_2 \to +\infty} \frac{\kappa(x)}{\|x\|_2} = \lim_{\|x\|_2 \to +\infty} \|x\|_2 = +\infty.$$

Thus, F is κ-coercive, that is, by applying Theorem 3.3.26, we deduce that problem (3.1.3) has a solution.

3.4 Regularization of non-coercive vector variational inequalities

The previous sections have extensively shown that coercivity conditions of the data of vector variational inequality (3.1.1) are crucial to ensure the existence of solutions of it. However, the solvability of problem (3.1.1) can be still studied, even though its data do not satisfy any coercivity condition. For this purpose, Hebestreit, Khan, Köbis and Tammer [93] proposed an extension of the well-known Browder-Tikhonov regularization method [121], which has been used extensively for (quasi) variational inequalities; see [2, 14, 121] and the references therein.

To be precise, assume again that vector variational inequality (3.1.1) is non-coercive and let a mapping $R : X \to \mathrm{L}(X, Y)$ and a sequence $\{\varepsilon_n\} \subseteq \mathbb{R}_>$ of parameters with $\varepsilon_n \downarrow 0$ be given. Instead of problem (3.1.1), we consider the family of regularized vector variational inequalities of finding $x_n = x(\varepsilon_n) \in C$ such that

$$\langle Fx_n + \varepsilon_n Rx_n, y - x_n\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C. \tag{3.4.1}$$

Any solution of problem (3.4.1) is said to be a regularized solution and the solution set of problem (3.4.1) will be denoted by Sol (RVVI). In the above, R is the regularizing mapping and ε_n is the regularization parameter to problem (3.4.1). Note that this family of problems evolves from vector variational inequality (3.1.1) by replacing F with the perturbed mapping

$$F + \varepsilon_n R : X \to \mathrm{L}(X, Y).$$

Due to the nice features of R, which will be specified shortly, the mapping $F + \varepsilon_n R$ has significantly better properties than F and every regularized problem (3.4.1) has a solution x_n. This allows us to study the sequence $\{x_n\}$ of regularized solutions, which, under some boundedness conditions, has a weakly or strongly convergent subsequence and any limit point is a solution of vector variational inequality (3.1.1). By this, we can ensure the existence of solutions of non-coercive vector variational problems.

Since scalarization techniques are commonly used to study vector variational inequalities, we introduce for $s_n \in Y^* \setminus \{0\}$ the following family of regularized variational inequalities: Find $x_n = x(\varepsilon_n, s_n) \in C$ such that

$$\langle s_n \circ Fx_n + \varepsilon_n s_n \circ Rx_n, y - x_n\rangle \geq 0, \qquad \text{for every} \quad y \in C. \tag{3.4.2}$$

In what follows, we will denote the solution set of the regularized problem depending on ε_n and s_n by Sol (RVI_{s_n}).

3.4.1 Motivation

The following example is based on the one in [93]. It presents a finite-dimensional vector variational inequality of the type (3.1.2), which has a solution although the data of the problem do not satisfy any coercivity condition of the previous section.

Example 3.4.1 ([93, Example 2]). Consider the finite-dimensional vector variational inequality (3.1.2) where $l = k = 2$ and $C = \mathbb{R}^2_{\geq}$. Assume that the mapping $F : \mathbb{R}^2 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is given by

$$Fx := \begin{pmatrix} f(x_1) & 0 \\ 0 & g(x_2) \end{pmatrix}, \qquad \text{for every} \quad x \in \mathbb{R}^2,$$

where the monotone and continuous mappings $f, g : \mathbb{R} \to \mathbb{R}$ enjoy the following properties:

$$f(x) = 0 \;\; \text{for} \;\; x \geq 0, \quad g(0) = 0 \quad \text{and} \quad \lim_{x \to \pm\infty} g(x) < +\infty.$$

By this special choice of the data, problem (3.1.2) becomes: Find $x \in \mathbb{R}^2_{\geq}$ such that

$$\begin{pmatrix} f(x_1)(y_1 - x_1) \\ g(x_2)(y_2 - x_2) \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^2_{\geq}, \qquad \text{for every} \quad y \in \mathbb{R}^2_{\geq}. \tag{3.4.3}$$

It is easily seen that the mapping $F : \mathbb{R}^2 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is $\mathbb{R}^2_{\geq}$-monotone and continuous. Indeed, this follows directly from the fact that f and g are monotone and continuous. However, we will show that none of the coercivity conditions of the previous sections hold:

1. C is unbounded. This is obviously fulfilled. Thus, we cannot apply the Theorems 3.3.3, 3.3.5, 3.3.17 and 3.3.21.

2. F is not v-coercive. Indeed, let B be a non-empty and compact subset of $\mathbb{R}^2$. Then for every $y_0 \in B$ and $y \in \mathbb{R}^2_{\geq} \setminus B$ it holds that

   $$\langle Fy, y - y_0 \rangle = \begin{pmatrix} 0 \\ g(y_2)(y_2^0 - y_2) \end{pmatrix} \notin \operatorname{int} \mathbb{R}^2_{\geq},$$

   where we used $f(y_2) = 0$. In a similar way, one can show that F is not strongly v-coercive as well. Thus, we cannot apply the Theorems 3.3.8 and 3.3.20.

3. Condition (3.3.3) does not hold. This follows from Remark 3.3.11. Thus, we cannot apply Theorem 3.3.10.

4. Condition (3.3.4) does not hold. Let $r > 0$ and $x \in \mathbb{R}^2_{\geq}$ with $\|x\|_2 = r$. Then

   $$\langle Fx, x \rangle = \begin{pmatrix} 0 \\ g(x_2)x_2 \end{pmatrix} \notin \operatorname{int} \mathbb{R}^2_{\geq},$$

   that is, we cannot apply Theorem 3.3.12.

5. F is not weakly coercive. Let $s \in \operatorname{qi} \mathbb{R}^2_{\geq}$ and $x_0 \in \mathbb{R}^2_{\geq}$. An easy calculation shows

$$\lim_{\substack{\|x\|_2 \to +\infty \\ x \in \mathbb{R}^2_{\geq}}} \frac{\langle s^\top Fx - s^\top Fx_0, x - x_0\rangle}{\|x - x_0\|_2} \leq \lim_{\substack{\|x\|_2 \to +\infty \\ x \in \mathbb{R}^2_{\geq}}} s_2 g(x_2) < +\infty.$$

   Thus, we cannot apply Theorem 3.3.15.

6. F is not κ-coercive. Let $s \in \mathbb{R}^2_{\geq} \setminus \{0\}$ and $x \in \mathbb{R}^2_{\geq}$. Then it hold $\langle s^\top Fx, x\rangle = s_2 g(x_2) x_2$ and consequently

$$\lim_{\substack{\|x\|_2 \to +\infty \\ x \in \mathbb{R}^2_{\geq}}} \frac{\langle s^\top Fx, x\rangle}{\|x\|_2} = \lim_{\substack{\|x\|_2 \to +\infty \\ x \in \mathbb{R}^2_{\geq}}} \frac{s_2 g(x_2) x_2}{\|x\|_2} \leq \lim_{\substack{\|x\|_2 \to +\infty \\ x \in \mathbb{R}^2_{\geq}}} s_2 g(x_2) < +\infty.$$

   Thus, we cannot apply Theorem 3.3.26.

The above calculations show that the conditions of all existence theorems in the previous sections are violated. But nevertheless, using the fact that $f(0) = g(0) = 0$, it is easy to check that the zero vector is a solution of vector variational inequality (3.4.3).

3.4.2 Regularization results

The next theorem ensures that every regularized problem (3.4.1) has a solution even if we do not assume that F satisfies any coercivity condition.

Theorem 3.4.2 ([93, Theorem 3.9]). *Besides assumption* (A), *assume that* $F, R : X \to \mathrm{L}(X, Y)$ *are given mappings.*

(i) *Suppose that F and R are K-monotone and v-hemicontinuous. If either R is weakly coercive or C is bounded, then, regularized vector variational inequality* (3.4.1) *has a solution.*

(ii) *Suppose that F and R are strongly continuous. If R is κ-coercive with respect to the functional $s \in K^* \setminus \{0\}$ and it holds that*

$$\langle s \circ Fx, x\rangle \geq 0, \qquad \textit{for every} \quad x \in C,$$

then, regularized vector variational inequality (3.4.1) *has a solution.*

(iii) *Suppose that F and R are K-monotone and v-hemicontinuous. If $F(0) = 0$, $0 \in C$ and R satisfies coercivity condition* (3.3.4), *then, regularized vector variational inequality* (3.4.1) *has a solution.*

Proof. In what follows, we abbreviate the mapping $F + \varepsilon_n R$ by F_n.
(i) Since F_n is the sum of K-monotone mappings and K is a cone, F_n is K-monotone. Note further that the v-hemicontinuity of F and R imply that of F_n. If C is bounded, then the regularized problem has a solution; see Theorem 3.3.3. Now, let C be unbounded but R weakly coercive. To apply Theorem 3.3.15, we will show that F_n is

weakly coercive. Since R is weakly coercive, there are $x_0 \in C$ and $s \in \operatorname{qi} K^*$ such that

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \frac{\langle s \circ Rx - s \circ Rx_0, x - x_0 \rangle}{\|x - x_0\|_X} = +\infty.$$

Since F is K-monotone, we conclude that the operator $s \circ F : X \to X^*$ is monotone. Indeed, since $s \in \operatorname{qi} K^*$, we have for every $x, y \in X$

$$\langle s \circ Fx - s \circ Fy, x - y \rangle = s\big(\langle Fx - Fy, x - y \rangle\big) \geq 0.$$

Finally, using the monotonicity of $s \circ F$, we conclude

$$\langle s \circ F_n x - s \circ F_n x_0, x - x_0 \rangle \;\geq\; \varepsilon_n \langle s \circ Rx - s \circ Rx_0, x - x_0 \rangle,$$

which ensures that the mapping F_n is weakly coercive.
(ii) Let us show that F_n satisfies the requirements of Theorem 3.3.26. It obviously remains to show that F_n is κ-coercive. Indeed, there is $s \in K^* \setminus \{0\}$ such that

$$\langle s \circ F_n x, x \rangle \;\geq\; \varepsilon_n \langle s \circ Rx, x \rangle \;\geq\; \varepsilon_n \|s\|_{Y^*} \kappa(x), \qquad \text{for every} \quad x \in C.$$

(iii) Let us show that F_n satisfies the requirements of Theorem 3.3.12. Indeed, since $F(0) = 0$, the K-monotonicity of F implies $\langle Fx, x \rangle \in K$ for every $x \in X$. Due to condition (3.3.4), there is $r > 0$ such that $\langle Rx, x \rangle \in \operatorname{int} K$ for all $x \in C \cap \operatorname{bd} B(0, r)$. However, due to (2.4.1), we conclude

$$\langle F_n x, x \rangle = \langle Fx, x \rangle + \varepsilon_n \langle Rx, x \rangle \in \operatorname{int} K, \qquad \text{for every} \quad x \in C \cap \operatorname{bd} B(0, r),$$

which completes the proof. □

We now come to the main result of this section which states that a non-coercive vector variational inequality can be approximated by a family of regularized vector variational inequalities.

Theorem 3.4.3 ([93, Theorem 3.10]). *Besides assumption* (A)*, assume that* $F, R : X \to \mathrm{L}(X, Y)$ *are given mappings. Then, the following statements hold:*

(i) *Suppose that F and R are K-monotone and v-hemicontinuous and R is weakly coercive. If there is a strongly convergent sequence of regularized solutions, then, vector variational inequality* (3.1.1) *has a solution.*

(ii) *Suppose that F and R are strongly continuous, R is κ-coercive with respect to the functional $s \in K^* \setminus \{0\}$ and it holds that*

$$\langle s \circ Fx, x \rangle \;\geq\; 0, \qquad \textit{for every} \quad x \in C. \tag{3.4.4}$$

If there is a strongly convergent sequence of regularized solutions, then, vector variational inequality (3.1.1) *has a solution.*

(iii) *Suppose that* F *and* R *are* K*-monotone and continuous. Assume further that* $F(0) = 0$, $0 \in C$ *and* R *satisfies coercivity condition* (3.3.4)*. If there is a strongly convergent sequence of regularized solutions, then, vector variational inequality* (3.1.1) *has a solution.*

Proof. Let us denote the sequence of regularized solutions by $\{x_n\}$ and its limit point by x. Note that in all cases, the sequence is well-defined in the sense that every regularized vector variational inequality (3.4.1) has a solution; compare Theorem 3.4.2.
(i) Since by definition x_n solves problem (3.4.1), we infer $x_n \in C$ and

$$\langle Fx_n + \varepsilon_n Rx_n, y - x_n\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C. \tag{3.4.5}$$

From the K-monotonicity and v-hemicontinuity of $F + \varepsilon_n R$ follows that x_n satisfies equivalently

$$\langle Fy + \varepsilon_n Ry, y - x_n\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C;$$

see Lemma 3.2.3. Further, $\varepsilon_n \downarrow 0$ and $x_n \to x$ implies that for fixed $y \in C$ it holds that

$$\langle Fy + \varepsilon_n Ry, y - x_n\rangle \;\to\; \langle Fy, y - x\rangle \;\text{ in }\; Y;$$

compare Proposition 2.1.13 (vi). Since $\langle Fy + \varepsilon_n Ry, y - x_n\rangle \in Y \setminus (-\operatorname{int} K)$ and the set $Y \setminus (-\operatorname{int} K)$ is closed, we conclude $\langle Fy, y - x\rangle \notin -\operatorname{int} K$. Finally, applying Lemma 3.2.3 once again, we have shown that $x \in C$ fulfills $\langle Fx, y - x\rangle \notin -\operatorname{int} K$ for every $y \in C$, that is, the limit point $x \in C$ is a solution of the vector variational inequality (3.1.1).
(ii) Due to the strong continuity of $F + \varepsilon_n R$, it holds that $Fx_n + \varepsilon_n Rx_n \to Fx$ in $\mathrm{L}(X, Y)$. Therefore, we can pass in (3.4.5) to the limit, that is, we have

$$\langle Fx_n + \varepsilon_n Rx_n, y - x_n\rangle \;\to\; \langle Fx, y - x\rangle \;\text{ in }\; Y.$$

We have therefore shown that the limit point $x \in C$ is a solution of vector variational inequality (3.1.1).
(iii) This part follows similar to the previous one. The proof is complete. □

Remark 3.4.4. (i) The above proof shows that the strong limit point of any convergent sequence of regularized solutions solves vector variational inequality (3.1.1).
(ii) A similar technique for finite-dimensional vector variational inequalities has also been proposed in [136]. In his paper, Luong proposes to consider a family of so-called penalized finite-dimensional vector variational inequalities. Note that the results in [136] still require the coercivity of F which makes a penalization approach superfluous. Luong shows that every penalized problem has a solution and that the sequence of penalized solutions converges to a solution of the original problem (3.1.2), where the set C is given by

$$C := \{x \in \mathbb{R}^l \mid g_j(x) \leq 0 \text{ for } j = 1, \ldots, k\},$$

with continuous functions $g_j : \mathbb{R}^l \to \mathbb{R}$ for $j = 1, \ldots, k$. Due to this special choice of the data, Luong defines a penalization mapping $R : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$ by $R := (\nabla P, \ldots, \nabla P)^\top$, where $\nabla P \in \mathbb{R}^l$ denotes the Fréchet-derivative of $P : \mathbb{R}^l \to \mathbb{R}$, given by

$$P(x) := \sum_{j=1}^{k} [\max\{0, g_j(x)\}]^2, \qquad \text{for every} \quad x \in \mathbb{R}^l.$$

Therefore, the proofs in [136] extensively use the fact that the convex and Fréchet differentiable mapping P enjoys the property of a penalty mapping for C, namely

$$P(x) = 0 \ \text{ for } \ x \in C \qquad \text{and} \qquad P(x) > 0 \ \text{ else.}$$

(iii) Condition (3.4.4) holds for any functional in $K^* \setminus \{0\}$ provided F is K-monotone with $F(0) = 0$.

We have the following corollary, where we equip the finite-dimensional Euclidean space $\mathbb{R}^k$ with the Pareto cone $\mathbb{R}^k_{\geq}$. The result is Corollary 3.11 in [93].

Corollary 3.4.5. *Let C be a non-empty, closed and convex set of $\mathbb{R}^l$ and let $F, R : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$ be given mappings. Assume that one of the following conditions holds:*

(i) *F and R are $\mathbb{R}^k_{\geq}$-monotone and v-hemicontinuous and R is weakly coercive.*

(ii) *F and R are continuous, R is κ-coercive with respect to $s \in \mathbb{R}^k_{\geq} \setminus \{0\}$ and it holds that*

$$\langle s^\top Fx, x\rangle \geq 0, \qquad \textit{for every} \quad x \in C.$$

(iii) *F and R are K-monotone and continuous. It further holds $F(0) = 0$, $0 \in C$ and R satisfies coercivity condition (3.3.4).*

If there is a bounded sequence of regularized solutions, then the finite-dimensional vector variational inequality (3.1.2) has a solution.

Proof. Of course, the proof of this corollary follows directly from Theorem 3.4.3. Nevertheless, we are giving a short proof in order to show how one can use the finite-dimensional structure. Note again that in all cases (i), (ii) and (iii), the sequence of regularized solutions is well-defined in the sense that every problem (3.4.1) has a solution. In this finite-dimensional setting, we can use the fact that it holds that

$$\mathrm{Sol}\,(\mathrm{RVVI}) = \bigcup_{s\in\partial B(0,1)\cap\mathbb{R}^k_{\geq}} \mathrm{Sol}(\mathrm{RVI}_s), \tag{3.4.6}$$

where $\partial B(0,1) := \{x \in \mathbb{R}^k \mid \|x\|_2 = 1\}$; see [119, Theorem 2.1]. Using relation (3.4.6), for every regularized solution $x_n \in C$ exists a functional $s_n \in \partial B(0,1) \cap \mathbb{R}^k_{\geq}$ such that

$$\langle s_n^\top Fx_n + \varepsilon_n s_n^\top Rx_n, y - x_n\rangle \geq 0, \qquad \text{for every} \quad y \in C. \tag{3.4.7}$$

Since $\partial B(0,1) \cap \mathbb{R}^k_{\geq}$ is compact and $\{x_n\}$ is bounded, we can assume without loss of generality that $s_n \to s$ and $x_n \to x$, where $s \in \partial B(0,1) \cap \mathbb{R}^k_{\geq}$ and $x \in C$. Using assumption (i), (ii) or (iii), we can pass in (3.4.7) to the limit, which implies that x solves problem (3.1.2). The proof is complete. □

Example 3.4.6 ([93, Example 3.12])**.** Let us come back to Example 3.4.1. Using Corollary 3.4.5, we are going to show that the finite-dimensional vector variational inequality (3.4.3) has a solution. In order to do so, we introduce a regularization mapping $R : \mathbb{R}^2 \to \mathrm{Mat}_{2\times 2}(\mathbb{R})$ by

$$Rx := \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \qquad \text{for every} \quad x \in \mathbb{R}^2.$$

Further, let $\{\varepsilon_n\} \subseteq \mathbb{R}_>$ be a sequence such that $\varepsilon_n \downarrow 0$. It is easily seen that R is $\mathbb{R}^2_{\geq}$-monotone and v-hemicontinuous. If we let $s = (1,1)^\top$ and $x_0 = (0,0)^\top$, then it holds that $\langle s^\top Rx - s^\top Rx_0, x - x_0\rangle = \|x\|_2^2$ for every $x \in \mathbb{R}^2$ which implies the weak coercivity of R. From Theorem 3.4.2 follows that the following regularized vector variational inequality has a solution: Find $x_n \in \mathbb{R}^2_{\geq}$ such that

$$\begin{pmatrix} (f(x_1^n) + \varepsilon_n x_1^n)(y_1 - x_1^n) \\ (g(x_2^n) + \varepsilon_n x_2^n)\,(y_2 - x_2^n) \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^2_{\geq}, \qquad \text{for every} \quad y \in \mathbb{R}^2_{\geq}.$$

Indeed, a solution is given by $x_n = (0, \varepsilon_n^2)^\top$ and it holds that

$$(0, \varepsilon_n^2)^\top \to (0,0)^\top.$$

We finally conclude from Corollary 3.4.5 that the limit point $(0,0)^\top$ is a solution of vector variational inequality (3.4.3).

3.4.3 Alternative conditions for the convergence of regularized solutions

The following two corollaries give conditions to relax the convergence assumption of regularized solutions in Theorem 3.4.3. For further use, we define the possibly empty sets

$$S' := \bigcap_{n\in\mathbb{N}} \left\{ s_n \in \operatorname{qi} K^* \mid \mathrm{Sol}\left(\mathrm{RVI}_{s_n}\right) \neq \emptyset \right\}, \quad S := \bigcap_{n\in\mathbb{N}} \left\{ s_n \in K^* \setminus \{0\} \mid \mathrm{Sol}\left(\mathrm{RVI}_{s_n}\right) \neq \emptyset \right\}.$$

Corollary 3.4.7 ([93, Corollary 3.15])**.** *Besides assumption* (A)*, let* $F, R : X \to \mathrm{L}(X,Y)$ *be given mappings. Assume that one of the following conditions hold:*

(i) *It holds* $S' \neq \emptyset$*,* F *and* R *are* K*-monotone and* v*-hemicontinuous and* R *is weakly coercive.*

(ii) *It holds* $S \neq \emptyset$*,* F *and* R *are strongly continuous,* R *is* κ*-coercive with respect to* *the functional $s \in K^* \setminus \{0\}$ and it holds that*

$$\langle s \circ Fx, x \rangle \geq 0, \qquad \text{for every} \quad x \in C.$$

(iii) *It holds $S \neq \emptyset$, F and R are K-monotone and continuous with $F(0) = 0$, $0 \in C$ and R satisfies coercivity condition* (3.3.4)*.*

If there is a bounded sequence of regularized solutions, then, vector variational inequality (3.1.1) *has a solution.*

Proof. Let us denote the sequence of regularized solutions by $\{x_n\}$.
(i) Let $s \in S'$. Then $x_n \in C$ satisfies in particular

$$\langle s \circ Fx_n + \varepsilon_n\, s \circ Rx_n, y - x_n \rangle \geq 0, \qquad \text{for every} \quad y \in C,$$

or equivalently, compare Lemma 2.2.6,

$$\langle s \circ Fy + \varepsilon_n\, s \circ Ry, y - x_n \rangle \geq 0, \qquad \text{for every} \quad y \in C. \tag{3.4.8}$$

Since the sequence $\{x_n\}$ is bounded and X is reflexive, there exists a subsequence, which we again denote by $\{x_n\}$, such that $x_n \rightharpoonup x$. We further have $x \in C$ since the set is closed and convex and therefore weakly closed; see Proposition 2.1.15. Since for fixed $y \in C$, $s \circ Fy + \varepsilon_n\, s \circ Ry \to s \circ Fy$ in X^*, passing in (3.4.8) to the limit shows $\langle s \circ Fy, y - x \rangle \geq 0$ for every $y \in C$. We finally conclude from Lemma 2.2.6 and Proposition 3.2.6 that the weak limit point belongs to $\mathrm{Sol}\,(\mathrm{VVI})$.
(ii) Let $s \in S$. Then, since $x_n \rightharpoonup x$ for a subsequence, due to the strong convergence of F and R, it holds that $Fx_n + \varepsilon_n Rx_n \to Fx$ in $\mathrm{L}(X,Y)$ and consequently

$$\langle s \circ Fx_n + \varepsilon_n\, s \circ Rx_n, y - x_n \rangle \;\to\; \langle s \circ Fx, y - x \rangle,$$

compare Proposition 2.1.13 (iv). Again, in view of Proposition 3.2.6, the element $x \in C$ is a solution of problem (3.1.1).
(iii) This part follows similar to (i). The proof is complete. □

Adapting the proof of the previous corollary, we have the following result.

Corollary 3.4.8 ([93, Corollary 3.14])**.** *Besides assumption* (A)*, let $F, R : X \to \mathrm{L}(X,Y)$ be given mappings. Assume that one of the following conditions holds:*

(i) *F and R are K-monotone and v-hemicontinuous and R is weakly coercive. For every sequence $\{x_n\}$ in C with $x_n \in \mathrm{Sol}\,(\mathrm{RVVI})$ and $x_n \rightharpoonup x$, there exists a sequence $\{s_n\}$ in $K^* \setminus \{0\}$ such that $\mathrm{Sol}(\mathrm{RVI}_{s_n}) \neq \emptyset$ and $s_n \to s$, where $s' \in K^* \setminus \{0\}$.*

(ii) *F and R are strongly continuous. R is κ-coercive with respect to the functional $s \in K^* \setminus \{0\}$ and it holds that*

$$\langle s \circ Fx, x \rangle \geq 0, \qquad \text{for every} \quad x \in C.$$

> *For every sequence* $\{x_n\}$ *in* C *with* $x_n \in \mathrm{Sol}\,(\mathrm{RVVI})$ *and* $x_n \rightharpoonup x$*, there exists a sequence* $\{s_n\}$ *in* $K^* \setminus \{0\}$ *such that* $\mathrm{Sol}(\mathrm{RVI}_{s_n}) \neq \emptyset$ *and* $s_n \to s'$*, where* $s' \in K^* \setminus \{0\}$*.*

If there is a bounded sequence of regularized solutions, then, vector variational inequality (3.1.1) *has a solution.*

Example 3.4.9. We now come back to the previous example. Let $s_n \in \operatorname{int}\mathbb{R}^2_{\geq}$ or $s_n \in \mathbb{R}^2_{\geq} \setminus \{0\}$. We consider the following regularized variational inequality: Find $x_n \in \mathbb{R}^2_{\geq}$ such that

$$s_1^n\big(f(x_1^n) + \varepsilon_n x_1^n\big)(y_1 - x_1^n) + s_2^n\big(g(x_2^n) + \varepsilon_n x_2^n\big)(y_2 - x_2^n) \geq 0, \qquad \text{for every} \quad y \in \mathbb{R}^2_{\geq}.$$

Clearly, the solution set of every regularized problem is non-empty. Thus, $S' = \operatorname{int}\mathbb{R}^2_{\geq}$ and $S = \mathbb{R}^2_{\geq} \setminus \{0\}$ and in view of Corollary 3.4.7, vector variational inequality (3.4.3) attains a solution.

3.5 A new existence result for generalized vector variational inequalities based on a regularization approach

In this section, we focus on generalized vector variational inequalities and use the previous regularization method to derive a new existence result for the latter problem class. To be precise, assume again that assumption (A) holds and let $F : X \rightrightarrows \mathrm{L}(X,Y)$ be a set-valued mapping, that is, for every $x \in X$, $F(x)$ is a subset of $\mathrm{L}(X,Y)$. To simplify our notation, we assume that F has non-empty values, that is, $\mathcal{D}(F) = X$. Then, the generalized vector variational inequality consists of finding $x \in C$ and $U \in F(x)$ such that

$$\langle U, y - x\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C. \tag{3.5.1}$$

Obviously, if F is a point-to-point mapping, then problem (3.5.1) recovers vector variational inequality (3.1.1). For further use, we need the following definitions, which can be found in [10].

Definition 3.5.1. Besides assumption (A), let $F : X \rightrightarrows \mathrm{L}(X,Y)$ be a set-valued mapping.

(i) F is said to be K*-monotone* if it holds that

$$\langle U - V, x - y\rangle \in K, \qquad \text{for every} \quad x, y \in X,\ U \in F(x),\ V \in F(y).$$

(ii) F is called *v-hemicontinuous* if for every $x, y, z \in X$ the set-valued mapping $\mathbb{R} \rightrightarrows Y$ with $t \rightrightarrows \langle F(x + t(y-x)), z\rangle := \big\{\langle w, z\rangle \mid w \in F(x + t(y-x)),\ z \in X\big\}$ is upper semicontinuous at 0.

Definition 3.5.2. Besides assumption (A), let $F : X \rightrightarrows \mathrm{L}(X, Y)$ be a set-valued mapping with non-empty values. A mapping $F_{\mathrm{sel}} : X \to \mathrm{L}(X, Y)$ such that

$$F_{\mathrm{sel}}(x) \in F(x)$$

for every $x \in X$, is called a *selection* of F.

The next theorem ensures the existence of solutions of problem (3.5.1) under a variety of coercivity conditions; see [124].

Theorem 3.5.3. *Besides assumption* (A)*, let $F : X \rightrightarrows \mathrm{L}(X, Y)$ be a given K-monotone and v-hemicontinuous set-valued mapping with non-empty values. Assume further that one of the following conditions holds:*

(i) *The set C is bounded.*

(ii) *F is generalized v-coercive in the sense that there exists a weakly compact subset $B \subseteq X$ and an element $y_0 \in B \cap C$ such that for every $w \in F(x)$ it holds that*

$$\langle w, y_0 - x \rangle \in -\operatorname{int} K, \qquad \textit{for every} \quad x \in C \setminus B.$$

(iii) *F is weakly coercive in the sense that there exists $x_0 \in C$ and $s \in K^* \setminus \{0\}$ such that*

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in C}} \inf_{w \in s \circ F(x)} \frac{\langle w, x - x_0 \rangle}{\|x - x_0\|_X} = +\infty,$$

where $s \circ F(x) := \{s \circ U \mid U \in F(x)\}$.

(iv) *There exists an element $y_0 \in C$ and a number $d > 0$ such that, for every $U \in F(x)$, $\langle U, y_0 - x \rangle \in -\operatorname{int} K$ if $x \in C$ and $\|y_0 - x\|_X > d$.*

Then, generalized vector variational inequality (3.5.1) *has a solution.*

The following existence theorem for problem (3.5.1) is based on a regularization approach. Thus, we do not have to assume that F fulfills one of the above mentioned coercivity conditions.

Theorem 3.5.4 ([93, Theorem 4.4])**.** *Besides assumption* (A)*, let $F : X \rightrightarrows \mathrm{L}(X, Y)$ be a K-monotone and v-hemicontinuous set-valued mapping. Suppose further that there exists a v-hemicontinuous selection $F_{\mathrm{sel}} : X \to \mathrm{L}(X, Y)$ of F. Further, let $R : X \to \mathrm{L}(X, Y)$ be a K-monotone, v-hemicontinuous and weakly coercive mapping. If there is a strongly convergent sequence $\{x_n\}$ in C, where every $x_n \in C$ is a solution of the vector variational inequality*

$$\langle F_{\mathrm{sel}}\, x_n + \varepsilon_n R x_n, y - x_n \rangle \notin -\operatorname{int} K, \qquad \textit{for every} \quad y \in C, \tag{3.5.2}$$

then, generalized vector variational inequality (3.5.1) *has a solution.*

Proof. Since F_{sel} is a selection of the K-monotone set-valued mapping F, we conclude that $F_{\text{sel}} : X \to \mathrm{L}(X,Y)$ is K-monotone. Therefore, in view of Theorem 3.4.2 (i), the sequence $\{x_n\}$ is well-defined in the sense that every problem (3.5.2) has a solution. Let us denote the strong limit point of $\{x_n\}$ by x. Following Theorem 3.4.3, we conclude that $x \in C$ satisfies

$$\langle F_{\text{sel}}\, x, y - x\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in C.$$

Thus, $x \in C$ and $U = F_{\text{sel}}(x) \in F(x)$ solve the generalized problem (3.5.1). The proof is complete. $\square$

Chapter 4

Inverse Generalized Vector Variational Inequalities

Abstract. In this chapter, we consider a generalized vector variational inequality with respect to a variable domination structure. We introduce two inverse vector variational inequalities and explore the relationships between the original vector variational inequality and the corresponding inverse problems. Further, we derive new existence results for generalized vector variational inequalities and apply the inverse results to two different vector approximation problems with respect to a variable domination structure to justify the theoretical framework.

Notice that the results of this chapter are based on he joint work [56] by Elster, Hebestreit, Khan and Tammer.

4.1 Preliminary results and concepts

This section is concerned with the collection of preliminary results and concepts which will be used in the sequel. Besides that, we will derive some new existence results for generalized vector variational inequalities.

4.1.1 Weak subgradients and weak conjugates

We shall now collect some definitions and results for later use. In what follows, we will abbreviate by (B) the following assumptions:

(B1) X and Y are real Banach spaces.

(B2) C is a non-empty, closed and convex subset of X.

(B3) The set-valued mapping $\mathcal{K} : X \rightrightarrows Y$ is a variable domination structure on Y.

Definition 4.1.1 ([56, Definition 3.7]). Besides assumption (B), let $\varphi : X \to Y \cup \{+\infty_Y\}$. An operator $U \in \mathrm{L}(X, Y)$ is called a *weak subgradient* of φ at $x \in \mathcal{D}(\varphi)$ w.r.t. the variable domination structure $\mathcal{K}$ if it holds that

$$\varphi(y) - \varphi(x) - \langle U, y - x\rangle \not\leq_{\operatorname{int}\mathcal{K}(x)} 0, \qquad \text{for every} \quad y \in X.$$

The set of *weak subgradients* of φ at x will be denoted by $\partial\varphi(x)$. If $\partial\varphi(x)$ is non-empty, then φ is said to be *weakly subdifferentiable at* x. Further, φ is said to be *weakly subdifferentiable* w.r.t. the variable domination structure $\mathcal{K}$ if φ is weakly subdifferentiable w.r.t. the variable domination structure $\mathcal{K}$ at every point of its domain.

Definition 4.1.2 ([108, Definition 8.1.1]). Besides assumption (B), let $\varphi : X \to Y \cup \{+\infty_Y\}$. The set-valued mapping $\varphi^* : \mathrm{L}(X, Y) \rightrightarrows Y$, defined by

$$\varphi^*(U) := \operatorname{WMax}\big(\{\langle U, x\rangle - \varphi(x) \mid x \in \mathcal{D}(\varphi)\}, \mathcal{K}\big), \qquad \text{for every} \quad U \in \mathrm{L}(X, Y),$$

is called the *weak conjugate* of φ w.r.t. the moving domination structure $\mathcal{K}$.

Remark 4.1.3. (i) It is easily seen that the generalized indicator mapping $\chi_C : X \to Y \cup \{+\infty_Y\}$ is weakly subdifferentiable on C w.r.t. any variable domination structure. (ii) Definition 4.1.2 recovers the well-known notion of Fenchel conjugate for functions $\psi : X \to \mathbb{R} \cup \{+\infty\}$, compare [20, Definition 2.3.1], if we let $Y = \mathbb{R}$ and $K = \mathbb{R}_{\geq}$. Recall that the Fenchel conjugate of ψ is the mapping $\psi^* : X^* \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\psi^*(x^*) := \sup_{x\in X}\{\langle x^*, x\rangle - \psi(x)\} = \sup_{x\in\mathcal{D}(\psi)}\{\langle x^*, x\rangle - \psi(x)\}, \qquad \text{for every} \quad x^* \in X^*,$$

where $\mathcal{D}(\psi) := \{x \in X \mid \psi(x) \neq +\infty\}$ denotes the effective domain of ψ.

Lemma 4.1.4 ([56, Lemma 3.11]). *Besides assumption* (B)*, let* $\varphi : X \to Y \cup \{+\infty_Y\}$ *be weakly subdifferentiable w.r.t.* $\mathcal{K}$ *at* $x \in \mathcal{D}(\varphi)$. *Then we have*

$$U \in \partial\varphi(x) \qquad \textit{if and only if} \qquad \langle U, x\rangle - \varphi(x) \in \varphi^*(U).$$

Proof. Let $x \in \mathcal{D}(\varphi)$ and $U \in \partial\varphi(x)$. By the definition of the weak subdifferential, we have

$$\langle U, x\rangle - \varphi(x) \not\leq_{\operatorname{int}\mathcal{K}(x)} \langle U, y\rangle - \varphi(y), \qquad \text{for every} \quad y \in X.$$

But this is equivalent to $\langle U, x\rangle - \varphi(x) \in \varphi^*(U)$. The proof is complete. □

4.1.2 A new existence result for generalized vector variational inequalities

In this section, we investigate novel existence results for generalized vector variational inequalities. In what follows, if assumption (B) holds, we define a closed and convex cone by

$$K := \bigcap_{x \in X} \mathcal{K}(x) \tag{4.1.1}$$

compare also Definition 2.5.5. The following definition can be found in [10].

Definition 4.1.5. Besides assumption (B), let $F : X \rightrightarrows \mathrm{L}(X,Y)$ be a set-valued mapping.

(i) F is said to be *K-monotone* if it holds that

$$\langle U - V, x - y \rangle \in K, \qquad \text{for every} \quad x, y \in X,\ U \in F(x),\ V \in F(y).$$

(ii) If in addition, the values of F are compact, then we call F *generalized v-hemicontinuous* if for every $x, y \in X$, the mapping $[0,1] \to \mathbb{R}$, $t \mapsto d_H(F(x + t(y - x)), F(x))$ is continuous at 0, where d_H denotes the Hausdorff distance.

Besides assumption (B), let $F : X \rightrightarrows \mathrm{L}(X,Y)$ be a set-valued mapping. Suppose further that $\varphi : X \to Y \cup \{+\infty_Y\}$ is a mapping with

$$C \cap \mathcal{D}(\varphi) \neq \emptyset \qquad \text{and} \qquad \varphi \not\equiv +\infty_Y.$$

Then the generalized vector variational inequality w.r.t. the variable domination structure $\mathcal{K}$ consists of finding $x \in C \cap \mathcal{D}(\varphi)$ such that for some $U \in F(x)$ it holds that

$$\langle U, y - x \rangle \not\leq_{\operatorname{int} \mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X. \tag{4.1.2}$$

In particular, if F is single-valued, $\varphi = \chi_C$ is the generalized indicator mapping and $\mathcal{K} : X \rightrightarrows Y$ is constant, that is, $\mathcal{K}(x) = K$ for every $x \in X$, then problem (4.1.2) recovers vector variational inequality (3.1.1) as special case.

Lemma 4.1.6 ([140, Remark 4])**.** *Let Z be a real normed space and suppose that A and B are non-empty and compact subsets of Z. Then, for each $a \in A$, there exists an element $b \in B$ such that*

$$\|a - b\|_Z \leq d_H(A, B),$$

where d_H denotes the Hausdorff-distance; compare Section 2.3.2.

The following lemma shows that, under suitable conditions, problem (4.1.2) is equivalent to another vector variational inequality; compare also Lemma 2.2.6.

Lemma 4.1.7 (Minty, Generalized version, [56, Lemma 3.20])**.** *Besides assumption* (B)*, let $F : X \rightrightarrows \mathrm{L}(X,Y)$ and $\varphi : X \to Y \cup \{+\infty_Y\}$. Suppose further that the following conditions hold:*

(i) *The cone K, given by (4.1.1), is solid.*

(ii) *F is K-monotone and generalized v-hemicontinuous with compact values.*

(iii) *φ is K-convex with convex effective domain $\mathcal{D}(\varphi)$.*

Then the following generalized vector variational inequalities are equivalent:

1. *Find $x \in \mathcal{D}(\varphi)$ and $U \in F(x)$ such that*

$$\langle U, y - x\rangle \not\leq_{\operatorname{int}\mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \textit{for every} \quad y \in X. \tag{4.1.3}$$

2. *Find $x \in \mathcal{D}(\varphi)$ such that*

$$\langle U, y - x\rangle \not\leq_{\operatorname{int}\mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \textit{for every} \quad y \in X,\ U \in F(y). \tag{4.1.4}$$

Proof. (I). Let $x \in \mathcal{D}(\varphi)$ and $U \in F(x)$ be a solution of problem (4.1.3). Since F is K-monotone and $\mathcal{K}(x) \subseteq K$, we have for every $y \in \mathcal{D}(\varphi)$ and $V \in F(y)$

$$\langle U, y - x\rangle + \varphi(y) - \varphi(x) \geq_{\mathcal{K}(x)} \langle V, y - x\rangle + \varphi(y) - \varphi(x).$$

Thus, adding the previous inequality to problem (4.1.3) we conclude using Proposition 2.4.12 (iv) that

$$\langle V, y - x\rangle + \varphi(y) - \varphi(x) \not\leq_{\operatorname{int}\mathcal{K}(x)} 0, \qquad \text{for every} \quad y \in \mathcal{D}(\varphi),\ V \in F(y).$$

It should be noted that the previous inequality also holds for all $y \in X$ with $\varphi(y) = +\infty_Y$; compare Section 2.5. Thus, $x \in \mathcal{D}(\varphi)$ is a solution of problem (4.1.4).

(II). Conversely, let $x \in \mathcal{D}(\varphi)$ be a solution of problem (4.1.4). Further let $y \in \mathcal{D}(\varphi)$, let $t \in [0, 1]$ and define $y_t = (1 - t)x + ty$. Note that $y_t \in \mathcal{D}(\varphi)$; see assumption (iii). Let $V_t \in F(y_t)$. Inserting all these elements into problem (4.1.4), we have in particular

$$\langle V_t, y_t - x\rangle + \varphi(y_t) - \varphi(x) \not\leq_{\operatorname{int}\mathcal{K}(x)} 0, \qquad \text{for every} \quad t \in [0, 1].$$

Again using the fact that $\mathcal{K}(x) \subseteq K$, the K-convexity of φ further implies

$$\langle V_t, y_t - x\rangle + \varphi(y_t) - \varphi(x) \leq_{\mathcal{K}(x)} t\big[\langle V_t, y - x\rangle + \varphi(y) - \varphi(x)\big].$$

With the help of Proposition 2.4.12 (iv) and (v) we deduce that $x \in \mathcal{D}(\varphi)$ satisfies

$$\langle V_t, y - x\rangle + \varphi(y) - \varphi(x) \not\leq_{\operatorname{int}\mathcal{K}(x)} 0, \qquad \text{for every} \quad V_t \in F(y_t),\ t \in [0, 1]. \tag{4.1.5}$$

In what follows, we will show that it is possible to pass in problem (4.1.5) to the limit $t \downarrow 0$. Indeed, since F is compact-valued, for every $t \in [0, 1]$, $y \in \mathcal{D}(\varphi)$ and $V_t \in F(y_t)$ there exists $U_t \in F(x)$ with

$$\|U_t - V_t\|_{\mathrm{L}(X,Y)} \leq d_H\big(F(x), F(y_t)\big);$$

see Lemma 4.1.6. Since $F(x)$ is compact, we may assume without loss of generality that

$U_t \to U$ in $\mathrm{L}(X,Y)$ and $U \in F(x)$. Further, the inequality

$$\begin{aligned}\|U - V_t\|_{\mathrm{L}(X,Y)} &\leq \|U_t - V_t\|_{\mathrm{L}(X,Y)} + \|U - U_t\|_{\mathrm{L}(X,Y)} \\ &\leq d_H\big(F(x), F(y_t)\big) + \|U - U_t\|_{\mathrm{L}(X,Y)}\end{aligned}$$

shows $V_t \to U$ in $\mathrm{L}(X,Y)$, where we used the generalized v-hemicontinuity of F; compare assumption (ii). Since the set $Y \setminus (-\operatorname{int}\mathcal{K}(x))$ is closed, we are able to pass in problem (4.1.5) to the limit $t \downarrow 0$. Consequently, $x \in \mathcal{D}(\varphi)$ and $U \in F(x)$ satisfy

$$\langle U, y - x\rangle \not\leq_{\operatorname{int}\mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in \mathcal{D}(\varphi).$$

However, the above inequality also holds true for all $y \in X$ with $\varphi(y) = +\infty_Y$. The proof is complete. □

We further need the following definition.

Definition 4.1.8. Besides assumption (B), let $W : X \rightrightarrows Y$ be a set-valued mapping. We call W *closed* if $\mathcal{G}(W) \subseteq X \times Y$ is sequentially closed, that is, for every sequence $\{(x_n, y_n)\} \subseteq \mathcal{G}(W)$ with $x_n \to x$ and $y_n \to y$, we have $(x, y) \in \mathcal{G}(W)$.

Example 4.1.9 ([15, Example 1])**.** Let X be a real Banach space and let $T : X \to X^*$. Then the values of $\mathcal{K} : X \rightrightarrows X$, given by

$$\mathcal{K}(x) := \big\{y \in X \mid \|y\|_X \leq \langle Tx, y\rangle\big\}, \qquad \text{for every} \quad x \in X, \tag{4.1.6}$$

are closed and convex cones in X. For every $x \in X$, the cone $\mathcal{K}(x)$ is called Bishop-Phelps cone; compare Section 1.2.1 in [55]. If in addition for every $x \in X$ it holds that $\|Tx\|_{X^*} > 1$, then $\mathcal{K}(x)$ is proper and solid, and consequently, $\mathcal{K} : X \rightrightarrows X$ defines a variable domination structure on X; compare Lemma 1.16 in [55]. We now define a set-valued mapping $W : X \rightrightarrows X$ by

$$W(x) := X \setminus (-\operatorname{int}\mathcal{K}(x)) = \big\{y \in X \mid \langle Tx, y\rangle \geq -\|y\|_X\big\}, \tag{4.1.7}$$

for every $x \in X$, where $\mathcal{K} : X \rightrightarrows X$ is given by (4.1.6). If $T : X \to X^*$ is continuous, then the set-valued mapping W is closed. Indeed, let $\{(x_n, y_n)\} \subseteq \mathcal{G}(W)$ be a sequence with $x_n \to x$ and $y_n \to y$. We then have $\langle Tx_n, y_n\rangle \geq -\|y_n\|_X$ and the continuity of T and $\|\cdot\|_X$ imply $\langle Tx, y\rangle \geq -\|y\|_X$. Thus, $(x, y) \in \mathcal{G}(W)$.

Using the generalized Minty Lemma 4.1.7, we have the following new existence result for problem (4.1.2), which uses the ideas in [28, 50, 133].

Theorem 4.1.10 ([56, Theorem 3.25])**.** *Besides assumption* (B)*, let* $F : X \rightrightarrows \mathrm{L}(X,Y)$ *and* $\varphi : X \to Y \cup \{+\infty_Y\}$*. Suppose further that the following conditions hold:*

(i) *The cone* K*, given by* (4.1.1)*, is solid.*

(ii) *The set-valued mapping* $W : X \rightrightarrows Y$*, given by* $W(x) := Y \setminus (-\operatorname{int}\mathcal{K}(x))$ *for every* $x \in X$*, is closed.*

(iii) *φ is K-convex and continuous with closed and convex domain $\mathcal{D}(\varphi)$.*

(iv) *F is K-monotone and generalized v-hemicontinuous with compact values.*

(v) *F is generalized v-coercive in the sense that there exists $y_0 \in C \cap \mathcal{D}(\varphi)$ and a non-empty compact subset B_0 of X such that*

$$\big\{x \in X \mid \langle V, y_0 - x\rangle + \varphi(y_0) - \varphi(x) \not\leq_{\operatorname{int} \mathcal{K}(x)} 0, \text{ for every } V \in F(y_0)\big\} \subseteq B_0.$$

Then, generalized vector variational inequality (4.1.2) has a solution.

Proof. Let $\tilde{C} := C \cap \mathcal{D}(\varphi)$ and define set-valued mappings $G, G' : \tilde{C} \rightrightarrows X$ by

$$G(y) := \big\{x \in \tilde{C} \mid \text{there exists } U \in F(x) \text{ with } \langle U, y - x\rangle + \varphi(y) - \varphi(x) \not\leq_{\operatorname{int} \mathcal{K}(x)} 0\big\},$$

and

$$G'(y) := \big\{x \in \tilde{C} \mid \langle V, y - x\rangle + \varphi(y) - \varphi(x) \not\leq_{\operatorname{int} \mathcal{K}(x)} 0, \text{ for every } V \in F(y)\big\},$$

for every $y \in \tilde{C}$. The proof of this theorem is based on the following observation: Any element belonging to the set

$$I_{G'} := \bigcap_{y \in \tilde{C}} G'(y)$$

is a solution of generalized vector variational inequality (4.1.2); compare Lemma 4.1.7. Thus, we should show that $I_{G'}$ is non-empty. In what follows, we will prove that G' satisfies the conditions of Lemma 3.3.1.

(I). Let us show that G' is a KKM-mapping. In order to do so, note that due to the K-monotonicity of F, we have

$$G(y) \subseteq G'(y), \qquad \text{for every} \quad y \in X.$$

Thus, it is enough to show that G is a KKM mapping. Indeed, assume to the contrary that there are a number $k \in \mathbb{N}$ and elements $y_1, \ldots, y_k \in \tilde{C}$ such that

$$\sum_{j=1}^{k} \lambda_j y_j \notin \bigcup_{j=1}^{k} G(y_j),$$

where $\sum_{j=1}^{k} \lambda_j = 1$ and $\lambda_j \geq 0$ for $j = 1, \ldots, k$. Let $\bar{y} := \sum_{j=1}^{k} \lambda_j y_j$. Since $\bar{y} \notin G(y_j)$ for $j = 1, \ldots, k$, we deduce

$$\langle V, y_j - \bar{y}\rangle + \varphi(y_j) - \varphi(\bar{y}) \leq_{\operatorname{int} \mathcal{K}(\bar{y})} 0, \qquad \text{for every} \quad V \in F(\bar{y}),\ j = 1, \ldots, k. \quad (4.1.8)$$

By the K-convexity of φ and relation (4.1.8) we further conclude that for all $V \in F(\bar{y})$

it holds that

$$0 = \langle V, \bar{y} - \bar{y} \rangle + \varphi(\bar{y}) - \varphi(\bar{y}) \geq_{\mathcal{K}(\bar{y})} \sum_{j=1}^{k} \lambda_j \big[\langle V, \bar{y} - y_j \rangle + \varphi(\bar{y}) - \varphi(y_j) \big] \geq_{\operatorname{int} \mathcal{K}(\bar{y})} 0.$$

This shows $0 \in \operatorname{int} \mathcal{K}(\bar{y})$, which is impossible. Hence, G is a KKM mapping and so is G'.

(II). Let us show that G' has closed values. Let $y \in \tilde{C}$ and let $\{x_n\}$ be a sequence in $G'(y)$ with $x_n \to x$. Let us show that the limit point x belongs to $G'(y)$. We infer that

$$\langle V, y - x_n \rangle + \varphi(y) - \varphi(x_n) \notin - \operatorname{int} \mathcal{K}(x_n), \qquad \text{for every} \quad V \in F(y). \tag{4.1.9}$$

Since $\{x_n\}$ converges strongly to x and φ is continuous, we have

$$\langle V, y - x_n \rangle + \varphi(y) - \varphi(x_n) \; \to \; \langle V, y - x \rangle + \varphi(y) - \varphi(x), \qquad \text{for every} \quad V \in F(y).$$

Finally, using assumption (ii), we conclude $x \in G'(y)$. This shows that $G'(y)$ is closed.

(III). Evidently, the coercivity assumption states the existence of an element $y_0 \in \tilde{C}$ and a non-empty and compact subset B_0 of X with

$$G'(y_0) \subseteq B_0.$$

Since $G'(y_0)$ is closed, compare the previous step, $G'(y_0)$ is compact. Thus, all requirements of Lemma 3.3.1 are satisfied and $I_{G'}$ is non-empty. The proof is complete. □

We have the following special case for single-valued F.

Corollary 4.1.11 ([56, Corollary 3.26]). *Besides assumption* (B), *let* $F : X \to \mathrm{L}(X, Y)$ *and* $\varphi : X \to Y \cup \{+\infty_Y\}$. *Suppose further that the following conditions hold:*

(i) *The cone* K*, given by* (4.1.1)*, is solid.*

(ii) *The set-valued mapping* $W : X \rightrightarrows Y$*, given by* $W(x) := Y \setminus (-\operatorname{int} \mathcal{K}(x))$ *for every* $x \in X$*, is closed.*

(iii) φ *is* K*-convex and continuous with closed and convex domain* $\mathcal{D}(\varphi)$*.*

(iv) F *is* K*-monotone and* v*-hemicontinuous.*

(v) F *is* v*-coercive in the sense that there exists* $y_0 \in C \cap \mathcal{D}(\varphi)$ *and a non-empty compact subset* $B_0 \subseteq X$ *such that*

$$\big\{ x \in X \mid \langle F y_0, y_0 - x \rangle + \varphi(y_0) - \varphi(x) \not\leq_{\operatorname{int} \mathcal{K}(x)} 0 \big\} \subseteq B_0.$$

Then the following vector variational inequality has a solution: Find $x \in C \cap \mathcal{D}(\varphi)$ *such that*

$$\langle F x, y - x \rangle \not\leq_{\operatorname{int} \mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \textit{for every} \quad y \in X.$$

Definition 4.1.12 ([56, Definition 3.14])**.** Besides assumption (B), let K be a proper, closed, convex and solid cone in Y. Further let A and B be non-empty subsets of $Y \cup \{\pm\infty_Y\}$. We say that A and B satisfy the *weak* (A, B)*-domination property* w.r.t. the cone K if for every $b \in B \setminus \{+\infty_Y\}$ there exists $a^0 \in \mathrm{WMax}(A, K)$ with $a^0 \geq_K b$.

Remark 4.1.13. Evidently, the weak (A, B)-domination property of A and B holds if and only if $B \setminus \{+\infty_Y\} \subseteq \mathrm{WMax}(A, K) - K$.

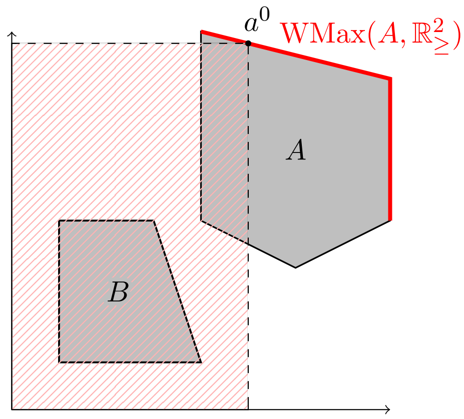


Figure 4.1: Illustration of the weak (A, B)-domination property of two sets A and B w.r.t. $\mathbb{R}^2_{\geq}$

4.2 Inverse results based on a vector conjugate approach

This section is devoted to the study of inverse generalized vector variational inequalities. The fundamental idea goes back to the work of Mosco [138]. In 1972, he introduced a dual variational inequality for variational inequality (2.2.1), using the Fenchel conjugate for convex functions. For this purpose, Mosco used the term dual in order to point out similarities to the duality principle in optimization; see [20, 69, 78, 80] and the references therein. However, it should be mentioned that the concept of duality has not been defined for a variational inequality yet, which is why one should call it *inverse variational inequality* instead; see [56, 75]. In the last years, Mosco's idea has been adapted by several authors [35, 36, 38, 40, 56, 159] in order to derive inverse results for vector variational inequalities and related problems. Let us recall some of the main reasons to study inverse generalized vector variational inequalities; compare [80, Chapter 1]:

1. It has a tremendous aesthetic appeal.
2. It deepens the theoretical understanding of vector variational inequalities.
3. It provides the insight for devising effective computational methods and algorithms.

To be precise, suppose that assumption (B) holds and let $F : X \rightrightarrows \mathrm{L}(X, Y)$ and $\varphi : X \to Y \cup \{+\infty_Y\}$ be given mappings. Recall that the generalized vector variational inequality (4.1.2) w.r.t. the variable domination structure $\mathcal{K}$ consists of finding an element $x \in C \cap \mathcal{D}(\varphi)$ such that for some $U \in F(x)$ it holds that

$$\langle U, y - x\rangle \nleq_{\operatorname{int} \mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X.$$

If $x \in C \cap \mathcal{D}(\varphi)$ and $U \in F(x)$ satisfy the generalized vector variational inequality, we will briefly say that the pair (x, U) solves problem (4.1.2). In what follows, we will consider the shifted inverse $F^{-1}(-\,\cdot)$ of F, which is given for $U \in \mathcal{D}(F^{-1}(-\,\cdot))$ by $F^{-1}(-U) = \{x \in X \mid U \in F(-x)\}$.

1. *First inverse vector variational inequality.* The first inverse problem consists of finding $U \in \mathcal{D}(F^{-1}(-\,\cdot))$ and $x \in F^{-1}(-U) \cap C \cap \mathcal{D}(\varphi)$ such that

$$\begin{aligned}\langle V - U, -x\rangle \not\preccurlyeq^1_{\operatorname{int} \mathcal{K}(x)} \varphi^*(U) - \varphi^*(V),& \\ \text{for every} \quad V \in \mathrm{L}(X, Y) \ \text{ with } \ \varphi^*(V) \neq \emptyset.&\end{aligned} \tag{4.2.1}$$

   It should be noted that the right-hand side of problem (4.2.1) is the Minkowski difference of the weak conjugates of φ; compare the Definitions 2.4.1 and 4.1.1.

2. *Second inverse vector variational inequality.* Conversely, the second inverse problem consists of finding $U \in \mathcal{D}(F^{-1}(-\,\cdot))$ and $x \in F^{-1}(-U) \cap C \cap \mathcal{D}(\varphi)$ such that

$$\langle V - U, -x\rangle \not\preccurlyeq^2_{\operatorname{int} \mathcal{K}(x)} \varphi^*(U) - \varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X, Y). \tag{4.2.2}$$

Again, if $U \in \mathcal{D}(F^{-1}(-\,\cdot))$ and $x \in F^{-1}(-U) \cap C \cap \mathcal{D}(\varphi)$ satisfy problem (4.2.1) (respectively, problem (4.2.2)), then we briefly call the pair (U, x) a solution of problem (4.2.1) (respectively, problem (4.2.2)).

Recall that, given a proper, closed, convex and solid cone K in Y, the set relations $\not\preccurlyeq^1_{\operatorname{int} K}$ and $\not\preccurlyeq^2_{\operatorname{int} K}$ are defined for non-empty subset A and B of Y in the following way; compare Definition 2.4.13: $A \not\preccurlyeq^1_{\operatorname{int} K} B$ if and only if for all $a \in A$, there is $b \in B$ such that $a \nleq_{\operatorname{int} K} b$; $A \not\preccurlyeq^2_{\operatorname{int} K} B$ if and only if there is $a \in A$, such that for all $b \in B$ it holds that $a \nleq_{\operatorname{int} K} b$. We further use the convention $A \not\preccurlyeq^2_{\operatorname{int} K} \emptyset$ for every non-empty subset A of Y.

To the best of our knowledge, the first attempt to extend Mosco's idea to the vector case has been proposed in [159]. However, the main result in the paper of Yang, see Theorem 3 in [159], contains some errors. Consequently, the results in [35, 36, 38, 40], which copied the errors, are incorrect as well. Nevertheless, the ideas of [159] have been adapted in [56], where Elster, Hebestreit, Khan and Tammer introduced the inverse vector variational inequalities (4.2.1) and (4.2.2). The main idea in [56] is to embed vector variational inequality (4.1.2) into the two inverse problems in the sense, that, under suitable assumptions, every solution of problem (4.2.1) generates one of (4.1.2), and every solution of problem (4.1.2) generates a solution of problem (4.2.2).

First inverse vector variational inequality (4.2.1)	$\longleftarrow$	Generalized vector variational inequality (4.1.2)	$\longleftarrow$	Second inverse vector variational inequality (4.2.2)

We now come to the main results of this chapter.

Theorem 4.2.1 ([56, Theorem 4.1]). *Besides assumption* (B)*, let* $F : X \rightrightarrows \mathrm{L}(X,Y)$ *and* $\varphi : X \to Y \cup \{+\infty_Y\}$ *be given mappings and suppose that* φ *is subdifferentiable w.r.t. the moving domination structure* $\mathcal{K}$*. Then it holds:*

(i) *If* (x,U) *is a solution of the generalized vector variational inequality* (4.1.2) *and it holds*

$$\mathcal{K}(x) \subseteq \mathcal{K}(y), \qquad \textit{for every} \quad y \in X, \tag{4.2.3}$$

then, $(-U,x)$ *solves the first inverse vector variational inequality* (4.2.1)*.*

(ii) *Conversely, let* $(-U,x)$ *be a solution of the second inverse vector variational inequality* (4.2.2)*. Suppose that* $\partial\varphi(x) \neq \emptyset$ *and assume that the sets*

$$A := \big\{\langle U,y\rangle - \varphi(y) \mid y \in X\big\} \qquad \textit{and} \qquad B := \big\{\langle U,x\rangle - \varphi(x)\big\}$$

satisfy the weak (A,B)*-domination property w.r.t. the cone* $\mathcal{K}(x)$*. Then,* (x,U) *is a solution of the generalized vector variational inequality* (4.1.2)*.*

Proof. (i) Let (x,U) be a solution of problem (4.1.2) such that (4.2.3) holds. In particular, $(x,U) \in \mathcal{G}(F)$, or equivalently, $(-U,x) \in \mathcal{G}(F^{-1}(-\,\cdot))$, showing that $(-U,x)$ is feasible for problem (4.2.1). Since (x,U) solves the generalized vector variational inequality (4.1.2) we have after some rearrangement

$$-\langle U,x\rangle - \varphi(x) \not\leq_{\mathrm{int}\,\mathcal{K}(x)} -\langle U,y\rangle - \varphi(y), \qquad \text{for every} \quad y \in X.$$

Using the definition of the weak conjugate of φ, the previous line is equivalent to

$$-\langle U,x\rangle - \varphi(x) \;\in\; \mathrm{WMax}(\tilde{A},\mathcal{K}) \;=\; \varphi^*(-U), \tag{4.2.4}$$

where $\tilde{A} := \{-\langle U,y\rangle - \varphi(y) \mid y \in X\}$. Assume to the contrary that $(-U,x)$ does not solve problem (4.2.1). Hence, we can find $\tilde{V} \in \mathrm{L}(X,Y)$ with $\varphi^*(\tilde{V}) \neq \emptyset$ such that

$$-\langle \tilde{V},x\rangle - \langle U,x\rangle = \langle \tilde{V}+U,-x\rangle \preccurlyeq^1_{\mathrm{int}\,\mathcal{K}(x)} \varphi^*(-U) - \varphi^*(\tilde{V}). \tag{4.2.5}$$

Since (4.2.5) holds in particular for the element $-\langle U,x\rangle - \varphi(x) \in \varphi^*(-U)$, compare relation (4.2.4), we have after some rearrangement

$$\varphi(x) - \langle \tilde{V},x\rangle \preccurlyeq^1_{\mathrm{int}\,\mathcal{K}(x)} -\varphi^*(\tilde{V}),$$

or equivalently

$$\varphi(x) - \langle \tilde{V}, x\rangle \preccurlyeq^1_{\operatorname{int} K(x)} -\tilde{v}, \qquad \text{for every} \quad \tilde{v} \in \varphi^*(\tilde{V}). \tag{4.2.6}$$

Now, let $\tilde{v} \in \varphi^*(\tilde{V})$ be arbitrarily chosen, that is, $\tilde{v} = \langle \tilde{V}, \tilde{x}\rangle - \varphi(\tilde{x})$ for some $\tilde{x} \in \mathcal{D}(\varphi)$. By the definition of the weak conjugate of φ, we conclude

$$\langle \tilde{V}, y\rangle - \varphi(y) - \langle \tilde{V}, \tilde{x}\rangle + \varphi(\tilde{x}) \notin \operatorname{int} \mathcal{K}(\tilde{x}), \qquad \text{for every} \quad y \in X. \tag{4.2.7}$$

Since (4.2.6) holds in particular for the element $\langle \tilde{V}, \tilde{x}\rangle - \varphi(\tilde{x}) \in \varphi^*(\tilde{V})$, we conclude

$$\varphi(x) - \langle \tilde{V}, x\rangle \leq_{\operatorname{int} \mathcal{K}(x)} \varphi(\tilde{x}) - \langle \tilde{V}, \tilde{x}\rangle.$$

Finally, using relation (4.2.3), we deduce that

$$\langle \tilde{V}, x\rangle - \varphi(x) - \langle \tilde{V}, \tilde{x}\rangle + \varphi(\tilde{x}) \in \operatorname{int} \mathcal{K}(\tilde{x}),$$

which contradicts inequality (4.2.7). Therefore, $(-U, x)$ is a solution of the first inverse vector variational inequality (4.2.1).

(ii) Conversely, let $(-U, x)$ be a solution of problem (4.2.2). Similar to the previous part, it is easy to see that $(x, U) \in \mathcal{G}(F)$ is feasible for generalized vector variational inequality (4.1.2). Since $(-U, x)$ solves problem (4.2.2), we deduce

$$-\langle U, x\rangle - \langle V, x\rangle \not\preccurlyeq^2_{\operatorname{int} \mathcal{K}(x)} \varphi^*(-U) - \varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X, Y),$$

Let $\tilde{U} \in \partial\varphi(x)$. By Lemma 4.1.4, it holds that $\langle \tilde{U}, x\rangle - \varphi(x) \in \varphi^*(\tilde{U})$ and the previous inequality leads to

$$-\langle U, x\rangle - \varphi(x) \not\preccurlyeq^2_{\operatorname{int} \mathcal{K}(x)} \varphi^*(-U).$$

Consequently, using the definition of the set relation $\not\preccurlyeq^2_{\operatorname{int} \mathcal{K}(x)}$, the previous inequality is equivalent to

$$-\langle U, x\rangle - \varphi(x) \notin \varphi^*(-U) - \operatorname{int} \mathcal{K}(x)$$

and the weak (A, B)-domination property for the sets A and B w.r.t. the fixed cone $\mathcal{K}(x)$ implies

$$-\langle U, x\rangle - \varphi(x) \in \varphi^*(-U).$$

Now, suppose to the contrary that (x, U) is not a solution of problem (4.1.2). Then there exists an element $\tilde{y} \in X$ such that

$$\langle U, \tilde{y} - x\rangle \leq_{\operatorname{int} \mathcal{K}(x)} \varphi(x) - \varphi(\tilde{y}).$$

However, this means $-\langle U, x\rangle - \varphi(x) \notin \varphi^*(-U)$, compare the previous part of this proof, which is impossible. The proof is complete. □

Remark 4.2.2. Notice that condition (4.2.3) is equivalent to saying that $x \in X$ satisfies $\mathcal{K}(x) = K$, where K is given by (4.1.1).

Assume now that $F : X \to \mathrm{L}(X, Y)$ is single-valued. Thus, problem (4.1.2) becomes: Find $x \in C \cap \mathcal{D}(\varphi)$ such that

$$\langle Fx, y - x\rangle \nleq_{\operatorname{int}\mathcal{K}(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X. \tag{4.2.8}$$

In this case, the inverse of F does not exist in general. However, if we assume that F is injective, we may consider the so-called (shifted) adjoint mapping $F^{-} : \mathrm{L}(X, Y) \to X$, defined by

$$F^{-}U := F^{-1}(-U), \qquad \text{for every} \quad U \in \mathcal{D}(F^{-});$$

see [56]. Here, with some abuse of the notation, F^{-1} denotes the inverse mapping of the bijection $F : X \to \mathcal{R}(F)$. Note that the above definition of the adjoint mapping differs slightly from that in [159, Section 4]. Thus, in this setting, the inverse problem appear as follows:

1. *First inverse vector variational inequality.* Find $U \in \mathcal{D}(F^{-})$ such that

$$\langle V - U, -F^{-}U\rangle \nprec^{1}_{\operatorname{int}\mathcal{K}(F^{-}U)} \varphi^{*}(U) - \varphi^{*}(V),$$
$$\text{for every} \quad V \in \mathrm{L}(X, Y) \ \text{ with } \ \varphi^{*}(V) \neq \emptyset. \tag{4.2.9}$$

2. *Second inverse vector variational inequality.* Conversely, the second inverse problem consists of finding $U \in \mathcal{D}(F^{-})$ such that

$$\langle V - U, -F^{-}U\rangle \nprec^{2}_{\operatorname{int}\mathcal{K}(F^{-}U)} \varphi^{*}(U) - \varphi^{*}(V), \qquad \text{for every} \quad V \in \mathrm{L}(X, Y). \tag{4.2.10}$$

We have the following special case of Theorem 4.2.1.

Corollary 4.2.3 ([56, Theorem 4.3])**.** *Besides assumption* (B)*, assume that* $F : X \to \mathrm{L}(X, Y)$ *is injective. Suppose further that* $\varphi : X \to Y \cup \{+\infty_Y\}$ *is subdifferentiable w.r.t. the moving domination structure* $\mathcal{K}$*. Then it holds:*

(i) *If* $x \in C \cap \mathcal{D}(\varphi)$ *is a solution of the generalized vector variational inequality* (4.2.8) *and it holds* (4.2.3)*, then,* $-Fx$ *solves the first inverse vector variational inequality* (4.2.9)*.*

(ii) *Conversely, let* $U \in \mathcal{D}(F^{-})$ *be a solution of the second inverse vector variational inequality* (4.2.10) *and define* $x = F^{-}U$*. If* $x \in C \cap \mathcal{D}(\varphi)$*,* $\partial\varphi(x) \neq \emptyset$ *and the sets*

$$A := \big\{\langle U, y\rangle - \varphi(y) \mid y \in X\big\} \qquad \textit{and} \qquad B := \big\{\langle U, x\rangle - \varphi(x)\big\}$$

satisfy the weak (A, B)*-domination property w.r.t. the cone* $\mathcal{K}(x)$*, then,* x *is a solution of the generalized vector variational inequality* (4.2.8)*.*

Remark 4.2.4. If we let $Y = \mathbb{R}$ and $K(x) = \mathbb{R}_{\geq}$ for every $x \in X$, then problem (4.2.8) becomes the following variational inequality: Find $x \in C \cap \mathcal{D}(\varphi)$ such that

$$\langle Fx, y - x \rangle \geq \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X.$$

Further, both the inverse problems (4.2.9) and (4.2.10) coincide to the following inverse problem: Find a functional $x^* \in \mathcal{D}(F^-)$ such that

$$\langle y^* - x^*, F^- x^* \rangle \geq \varphi^*(x^*) - \varphi^*(y^*), \qquad \text{for every} \quad y^* \in X^*.$$

Here, φ^* denotes the well-known Fenchel conjugate of $\varphi : X \to \mathbb{R} \cup \{+\infty\}$; compare Remark 4.1.3. By this special choice of the data, the previous theorem recovers the results for variational inequalities in [80, 138].

4.3 Inverse results based on a perturbation approach

Suppose that assumption (A) holds and let $F : X \to \mathrm{L}(X, Y)$ be an injective mapping. We then introduce a perturbation mapping $\Psi : X \times X \times X \to Y \cup \{+\infty_Y\}$ such that

$$\Psi(x, y, 0) = \langle Fx, y \rangle + \chi_C(y), \qquad \text{for every} \quad x, y \in X,$$

where $\chi_C : X \to Y \cup \{+\infty_Y\}$ denotes the generalized indicator mapping of the set C, that is, $\chi_C(y) = 0$ for $y \in C$ and $\chi_C(y) = +\infty_Y$ else. This allows us to embed vector variational inequality (3.1.1) into a family of so-called perturbed multi-objective optimization problems, which, given a parameter $z \in X$, consist of finding $x \in X$ such that

$$\Psi(x, x, z) \not\geq_{\operatorname{int} K} \Psi(x, y, z), \qquad \text{for every} \quad y \in X. \tag{4.3.1}$$

Obviously, if we let $z = 0$, then problem (4.3.1) recovers vector variational inequality (3.1.1) as special case.

In what follows, we will need the following definition.

Definition 4.3.1 ([56, Definition 4.5])**.** Let assumption (A) hold and suppose that the perturbation mapping $\Psi : X \times X \times X \to Y \cup \{+\infty_Y\}$ is given as above. Then the *weak conjugate* of Ψ is the set-valued mapping $\Psi^* : \mathrm{L}(X, Y) \times \mathrm{L}(X, Y) \times \mathrm{L}(X, Y) \rightrightarrows Y$, given for every triple $(U, V, W) \in \mathrm{L}(X, Y) \times \mathrm{L}(X, Y) \times \mathrm{L}(X, Y)$ by $\langle U, x \rangle + \langle V, y \rangle + \langle W, z \rangle - \Psi(\tilde{x}, \tilde{y}, \tilde{z}) \in \Psi^*(U, V, W)$ for some $\tilde{x}, \tilde{y}, \tilde{z} \in X$ if and only if for every $x, y, z \in X$

$$\langle U, \tilde{x} \rangle + \langle V, \tilde{y} \rangle + \langle W, \tilde{z} \rangle - \Psi(x, y, z) \leq_{\operatorname{int} K} \langle U, x \rangle + \langle V, y \rangle + \langle W, z \rangle - \Psi(x, y, z).$$

Using this notation, we state the following inverse problem for the perturbed multi-objective optimization problem (4.3.1): Find $W \in D(F^-)$ such that

$$-\Psi^*(0,0,W) \cap \mathrm{WMax}\left(\bigcup_{W' \in \mathrm{L}(X,Y)} -\Psi^*(0,0,W'),\ K \right) \neq \emptyset. \tag{4.3.2}$$

Here, $W \in \mathrm{L}(X,Y)$ is a solution of problem (4.3.2) if there exist elements $\tilde{x}, \tilde{y}, \tilde{z} \in X$ and $\tilde{W} \in \mathrm{L}(X,Y)$ such that for every $x, y, z \in X$ it holds that

$$\langle \tilde{W}, \tilde{z} \rangle - \Psi(\tilde{x}, \tilde{y}, \tilde{z}) \not\leq_{\mathrm{int}\, K} \langle W, z \rangle - \Psi(x, y, z). \tag{4.3.3}$$

Theorem 4.3.2 ([56, Theorem 4.6])**.** *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X,Y)$ *be an injective mapping. Then we have*

$$\Psi^*(0,0,W) \not\preceq^1_{\mathrm{int}\, K} -\Psi(x,x,0), \qquad \textit{for every} \quad x \in C,\ W \in \mathrm{L}(X,Y).$$

Proof. Let $x \in C$, $W \in \mathrm{L}(X,Y)$ and $\langle \tilde{W}, \tilde{z} \rangle - \Psi(\tilde{x}, \tilde{y}, \tilde{z}) \in \Psi^*(0,0,W)$ for some $\tilde{x}, \tilde{y}, \tilde{z} \in X$. Thus,

$$\langle W, \tilde{z} \rangle - \Psi(\tilde{x}, \tilde{y}, \tilde{z}) \not\leq_{\mathrm{int}\, K} \langle W, z \rangle - \Psi(x, y, z),$$

for every $x, y, z \in X$. Inserting $x = y = \tilde{x}$ and $z = 0$ finishes the proof. □

Theorem 4.3.3 ([56, Theorem 4.7])**.** *Besides assumption* (A)*, let* $F : X \to \mathrm{L}(X,Y)$ *be an injective mapping. Then, any element* $\tilde{x} \in C$ *with*

$$-\Psi(\tilde{x}, \tilde{x}, 0) \in \Psi^*(0,0,-F\tilde{x}) \tag{4.3.4}$$

is a solution of vector variational inequality (3.1.1) *while* $-F\tilde{x}$ *solves the inverse problem* (4.3.2)*.*

Proof. Let $\tilde{x} \in C$ such that (4.3.4) holds. We therefore have

$$-\Psi(\tilde{x}, \tilde{x}, 0) \not\leq_{\mathrm{int}\, K} -\langle F\tilde{x}, z \rangle - \Psi(x, y, z), \qquad \text{for every} \quad x, y, z \in X.$$

However, inserting $x = \tilde{x}$ and $z = 0$, we consequently deduce

$$-\Psi(\tilde{x}, \tilde{x}, 0) \not\leq_{\mathrm{int}\, K} -\Psi(\tilde{x}, y, 0),$$

which shows that $\tilde{x} \in C$ is a solution of vector variational inequality (3.1.1). For the second assertion define $\tilde{W} = -F\tilde{x}$. Thus, $F^-\tilde{W} = \tilde{x}$ and by relation (4.3.4) we deduce

$$-\Psi(F^-\tilde{W}, F^-\tilde{W}, 0) \not\leq_{\mathrm{int}\, K} \langle \tilde{W}, z \rangle - \Psi(x, y, z),$$

for every $x, y, z \in X$, which shows that $-F\tilde{x}$ solves problem (4.3.2). The proof is complete. □

4.4 Applications

In what follows, we present some applications of the inverse results for generalized vector variational inequalities, which have been presented in the previous sections.

4.4.1 A multi-objective location problem

Let us consider Example 3.1.3 once again. It is easily seen that $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$, given by relation (3.1.4), is injective. Thus, the adjoint $F^- : \mathrm{Mat}_{k\times l}(\mathbb{R}) \to \mathbb{R}^l$ is given by $F^-U = F^{-1}(-U)$, for every $U \in \mathcal{D}(F^-)$, where

$$\mathcal{D}(F^-) = \left\{U \in \mathrm{Mat}_{k\times l}(\mathbb{R}) \mid \text{there is } x \in \mathbb{R}^l \text{ such that } U = (a^1 - x, \dots, a^k - x)^\top\right\}.$$

Therefore, the first inverse vector variational inequality for problem (3.1.3) consists of finding a matrix $U = (a^1 - x, \dots, a^k - x)^\top \in D(F^-)$, where $x \in \mathbb{R}^l$, such that

$$\langle V - U, -x\rangle \not\preccurlyeq^1_{\operatorname{int}\mathbb{R}^k_\geq} \chi^*_{\mathbb{R}^l}(U) - \chi^*_{\mathbb{R}^l}(V), \qquad \text{for every} \quad V \in \mathrm{Mat}_{k\times l}(\mathbb{R}), \quad \chi^*_{\mathbb{R}^l}(V) \neq \emptyset. \tag{4.4.1}$$

Note that $\chi_{\mathbb{R}^l} \equiv 0$ is the zero mapping. An easy calculation shows that it holds $\chi^*_{\mathbb{R}^l}(U) = \mathbb{R}^l$ if and only if $U \in \mathrm{Mat}_{k\times l}(\mathbb{R})$ has a zero-row and $\chi^*_{\mathbb{R}^l}(U) = \emptyset$ else. Therefore, problem (4.4.1) may be written in the following way: Find a matrix $U = (a^1 - x, \dots, a^k - x)^\top \in \mathcal{D}(F^-)$, where $x \in \mathbb{R}^l$, such that

$$\langle V - U, -x\rangle \not\preccurlyeq^1_{\operatorname{int}\mathbb{R}^k_\geq} \mathbb{R}^l \qquad \text{for every} \quad V \in \mathrm{Mat}_{k\times l}(\mathbb{R}) \ \text{ with zero-row.}$$

Recall that the solution set S of problem (3.1.3) is equivalent to the convex hull of $a^1, \dots, a^k$. Finally, by Theorem 4.2.3 (i), any $U = (a^1 - x, \dots, a^k - x)^\top \in \mathcal{D}(F^-)$, where $x \in \operatorname{conv}\{a^1, \dots, a^k\}$, is a solution of problem (4.4.1), which can easily be confirmed.

In a similar way, the second inverse vector variational inequality for problem (3.1.3) consists of finding a matrix $U = (a^1 - x, \dots, a^k - x)^\top \in \mathcal{D}(F^-)$, where $x \in \mathbb{R}^l$, such that

$$\langle V - U, -x\rangle \not\preccurlyeq^2_{\operatorname{int}\mathbb{R}^k_\geq} \chi^*_{\mathbb{R}^l}(U) - \chi^*_{\mathbb{R}^l}(V), \qquad \text{for every} \quad V \in \mathrm{Mat}_{k\times l}(\mathbb{R}). \tag{4.4.2}$$

Let us define the set

$$\operatorname{conv}^\circ\left\{a^1, \dots, a^k\right\} := \operatorname{conv}\left\{a^1, \dots, a^k\right\} \setminus \left\{a^1, \dots, a^k\right\}.$$

It is easily seen that the sets $A := \{-\langle Fx, y\rangle_{\mathbb{R}^k} \mid y \in \mathbb{R}^l\}$ and $B := \{-\langle Fx, x\rangle_{\mathbb{R}^k}\}$ satisfy the weak domination property, provided $x \in \operatorname{conv}^\circ\{a^1, \dots, a^k\}$. Using the conventions $\emptyset - A = \emptyset$ and $A \not\preccurlyeq^2_{\operatorname{int}\mathbb{R}^k_\geq} \emptyset$ for every subset A of $\mathbb{R}^k$, it follows that any matrix $(a^1 - x, \dots, a^k - x)^\top \in \mathcal{D}(F^-)$, where $x \in \operatorname{conv}^\circ\{a^1, \dots, a^k\}$, is a solution of problem (4.4.2). Finally, Theorem 4.2.3 (ii) implies that any point in $\operatorname{conv}^\circ\{a^1, \dots, a^k\}$ is a solution of vector variational inequality (3.1.3). In other words, we have shown that it holds

$$\operatorname{conv}\left\{a^1,\dots,a^k\right\} \setminus \left\{a^1,\dots,a^k\right\} \subseteq \mathrm{S}\,.$$

4.4.2 Beam intensity optimization problem in radiotherapy treatment

As a second application of our results, we present a multi-objective optimization problem, which arises in radio therapy treatment; compare [54, 130]. The intensity modulated radiotherapy treatment (IMRT) is currently used to treat cancer in prostate, head and neck, breast and many others; see [55, 130]. The main idea of IMRT is to apply to the patient a suitable radiation dose, that is, the intensity of rays going through sensitive critical structures is reduced while the dose in the infected structures is increased. This problem is modeled in [130] as multi-objective optimization problem with respect to a variable domination structure, which describes the dose of beam intensity.

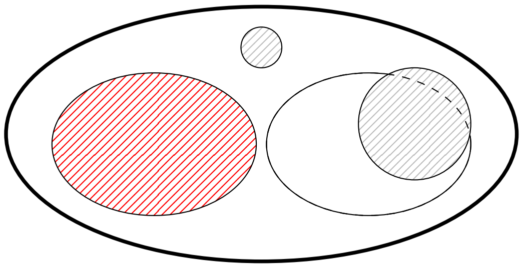


Figure 4.2: Schematic axial body cut: Lunge cancer (red) and critical organs spinal cord and heart (gray)

To be precise, assume we are given $k-1$ critical organs, where $k \geq 3$. Suppose further that a threshold vector $\theta \in \mathbb{R}^n$ is given, where $\theta_1 = 0$ and for every $j \in \{2,\dots,k\}$ the component θ_j is defined as the dose of radiation, below which the organism j does not suffer from any effect. In the following, we assume that the dose delivered to the tumor organ is given by $A_T x$, where $A_T \in \mathrm{Mat}_{k\times k}(\mathbb{R})$ is the regular dose deposition matrix, and $x \in \mathbb{R}^k$ is the beam intensity. The dose delivered to the $k-1$ critical organs $C_2,\dots,C_k$ is given by $A_{C_2}x,\dots,A_{C_k}x$, where for $j \in \{2,\dots,k\}$, the matrices $A_{C_j} \in \mathrm{Mat}_{k\times k}(\mathbb{R})$ are assumed to be regular. Notice that in [96] the author claims that the Moore–Penrose generalized inverse of the involved matrices exist. The composite matrix

$$A = (A_T, A_{C_2},\dots,A_{C_k})^\top \in \mathrm{Mat}_{k^2\times k}(\mathbb{R})$$

is called the dose deposition matrix and we have the following relationship

$$Ax = d,$$

where $d \in \mathbb{R}^{k^2}$ is a dose vector. Since different tissues tolerate different amounts of radiation, the radiation oncologist needs to determine a target dose $a^{\mathrm{tar}} \in \mathbb{R}^k$ for the tumor, lower and upper bounds to tumor voxels $a, b \in \mathbb{R}^k$ and upper bounds on the dose to normal voxels which are divided into vectors $c_j \in \mathbb{R}^k$ for $j \in \{2, \dots, k\}$. Here, the variable domination structure is constructed by using the following [130] practical perspective: The dose delivered to a critical organ C_j should be reduced when it exceeds its threshold θ_j. If not, one can increase this dose in favor of an improvement in the value of another critical organ. Therefore, given the threshold vector θ, the variable domination structure $\mathcal{K} : \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ is given by

$$\mathcal{K}(x) := \left\{ y \in \mathbb{R}^k \mid y_j \geq 0 \text{ for } j \in I^{>}(x) \right\}, \tag{4.4.3}$$

where $I^{>}(x) := \{j \in \{1, \dots, k\} \mid x_j > \theta_j\}$ for every $x \in \mathbb{R}^k$.

The next lemma states important properties of $\mathcal{K}$.

Lemma 4.4.1 ([130, Proposition 3.1])**.** *Let the set-valued mapping $\mathcal{K} : \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ be given by (4.4.3). Then the following assertions hold true:*

(i) *$\mathcal{K}$ defines a variable domination structure on $\mathbb{R}^k$, that is, for every $x \in \mathbb{R}^k$, $\mathcal{K}(x)$ is a proper, closed, convex and solid cone in $\mathbb{R}^k$.*

(ii) *For every $x \in \mathbb{R}^k$, the cone $\mathcal{K}(x)$ is pointed if and only if $x - \theta \in \operatorname{int} \mathbb{R}^k_{\geq}$, or equivalently, $I^{>}(x) = \{1, \dots, k\}$.*

(iii) *For every $x \in \mathbb{R}^k$, it holds that $\mathbb{R}^k \subseteq \mathcal{K}(x)$.*

In order to describe the multi-objective optimization problem, which has been considered in [130], we introduce the following set of bound conditions for beam intensity

$$\begin{aligned} C^{\mathrm{bc}} := \Big\{ x \in \mathbb{R}^k \mid 0 \leq_{\mathbb{R}^k_{\geq} \setminus \{0\}} x, \;\; a \leq_{\mathbb{R}^k_{\geq} \setminus \{0\}} A_T x \leq_{\mathbb{R}^k_{\geq} \setminus \{0\}} b, \\ A_{C_j} x \leq_{\mathbb{R}^k_{\geq} \setminus \{0\}} c_j \;\; \text{for} \;\; j \in \{2, \dots, k\} \Big\}. \end{aligned}$$

Thus, the beam intensity optimization problem may be formulated as the multi-objective optimization problem

$$\mathrm{WEff}(\psi(C^{\mathrm{bc}}), \mathcal{K}), \tag{4.4.4}$$

where the variable domination structure $\mathcal{K}$ is given by (4.4.3) and the objective mapping $\psi : \mathbb{R}^k \to \mathbb{R}^k$ is defined for every $x \in \mathbb{R}^k$ by

$$\psi(x) := \begin{pmatrix} \frac{1}{2} \|A_T x - a^{\mathrm{tar}}\|_2^2 \\ \frac{1}{2} \|A_{C_2} x\|_2^2 \\ \vdots \\ \frac{1}{2} \|A_{C_k} x\|_2^2 \end{pmatrix}.$$

Recall that $\| \cdot \|_2$ denotes the Euclidean norm in $\mathbb{R}^k$. Notice that the first component of ψ can be interpreted as the deviation from the prescribed dose to the dose delivered to tumor, while the other components describe the average dose delivered to the critical organs.

In what follows, we will need the following result which allows us to treat problem (4.4.4) as an equivalent vector variational inequality w.r.t. the variable domination structure $\mathcal{K}$, given by (4.4.3).

Theorem 4.4.2 ([56, Theorem 3.29]). *Let X and Y be real Banach spaces and let C be a non-empty, closed and convex subset of X. Assume further that $\mathcal{K} : X \rightrightarrows Y$ is a variable domination structure on Y and $\psi : X \to Y$. Then we have:*

(i) *If ψ is right-handed Gâteaux-differentiable at $x \in C$ with derivative $D_{\mathrm{G}}^{+}\psi(x)$, the cone $\mathcal{K}(x)$ is pointed and it holds that*

$$x \in \mathrm{WEff}(\psi(C), \mathcal{K}), \tag{4.4.5}$$

then, $x \in C$ solves vector variational inequality

$$\langle D_{\mathrm{G}}^{+}\psi(x), y - x\rangle \notin -\operatorname{int}\mathcal{K}(x), \qquad \text{for every} \quad y \in C. \tag{4.4.6}$$

(ii) *Conversely, if ψ is $\mathcal{K}(x)$-convex, the cone $\mathcal{K}(x)$ is pointed and $x \in C$ is a solution of vector variational inequality (4.4.6), then, x satisfies (4.4.5).*

Proof. (i) Let $x \in \mathrm{WEff}(\psi(C), \mathcal{K})$. Since C is convex, we deduce

$$\frac{1}{t}\big[\psi(x + t(y-x)) - \psi(x)\big] \notin -\operatorname{int}\mathcal{K}(x),$$

for every $y \in C$ and $t \in (0,1)$. Passing to the limit $t \downarrow 0$ yields $\langle D_{\mathrm{G}}^{+}\psi(x), y - x\rangle \notin -\operatorname{int}\mathcal{K}(x)$ for every $y \in C$, that is, $x \in C$ solves the vector variational inequality (4.4.6).
(ii) Conversely, let $x \in C$ be a solution of the vector variational inequality (4.4.6). Due to the $\mathcal{K}(x)$-convexity of ψ, it holds in particular

$$\psi(y) - \psi(x) - \frac{1}{t}\big[\psi(x + t(y-x)) - \psi(x)\big] \in \operatorname{int}\mathcal{K}(x),$$

for every $y \in C$. Passing to the limit $t \downarrow 0$ and combining the resulting inequality with (4.4.6) implies $\psi(x) - \psi(y) \notin \operatorname{int}\mathcal{K}(x)$. The proof is complete. □

In view of this theorem, problem (4.4.4) is equivalent to the following vector variational inequality: Find $x \in C^{\mathrm{bc}}$ with $I^{>}(x) = \mathbb{R}^k_{\geq}$ such that

$$\langle D_{\mathrm{G}}^{+}\psi\, x, y - x\rangle = \begin{pmatrix} \langle A_T x - a^{\mathrm{tar}}, A_T(y-x)\rangle \\ \langle A_{C_2}x, A_{C_2}(y-x)\rangle \\ \vdots \\ \langle A_{C_k}x, A_{C_k}(y-x)\rangle \end{pmatrix} \notin -\operatorname{int}\mathcal{K}(x), \qquad \text{for every} \quad y \in C.$$

However, if we denote by $\chi_{C^{\mathrm{bc}}}$ the generalized indicator mapping of the constraining set C, it is easily seen that the above problem becomes the following generalized vector variational inequality of type (4.2.8): Find $x \in C$ with $I^{>}(x) = \mathbb{R}^k_{\geq}$ such that

$$\langle D_{\mathrm{G}}^{+}\psi\, x, y - x\rangle \not\leq_{\operatorname{int}\mathcal{K}(x)} \chi_{C^{\mathrm{bc}}}(x) - \chi_{C^{\mathrm{bc}}}(y), \qquad \text{for every} \quad y \in \mathbb{R}^k. \tag{4.4.7}$$

It should be noted that $\mathcal{D}(\chi_{C^{\mathrm{bc}}}) = C^{\mathrm{bc}}$ and the mapping $D_{\mathrm{G}}^{+} : \mathbb{R}^k \to \mathrm{Mat}_{k\times k}(\mathbb{R})$ is injective. Further, it is easy to check that $\chi_{C^{\mathrm{bc}}}$ is weakly subdifferentiable w.r.t. the variable domination structure $\mathcal{K}$, given by (4.4.3). Thus, applying Corollary 4.2.3, we have the following result.

Theorem 4.4.3 ([56, Theorem 5.6])**.** *Using our previous notations and observations and denoting the adjoint mapping of $D_{\mathrm{G}}^{+}\psi$ by $D_{\mathrm{G}}^{\pm}\psi$, the following assertions hold:*

(i) *If $x \in C^{\mathrm{bc}}$ is a solution of the generalized vector variational inequality (4.4.7), then, $-D_{\mathrm{G}}^{+}\psi(x)$ solves the following first inverse vector variational inequality: Find $U \in \mathcal{D}(D_{\mathrm{G}}^{\pm}\psi)$ such that*

$$\langle V - U, -D_{\mathrm{G}}^{\pm}\psi(U)\rangle \not\preceq^{1}_{\operatorname{int}\mathcal{K}(D_{\mathrm{G}}^{\pm}\psi(U))} \chi^{*}_{C^{\mathrm{bc}}}(U) - \chi^{*}_{C^{\mathrm{bc}}}(V),$$
$$\textit{for every} \quad V \in \mathrm{Mat}_{k\times k}(\mathbb{R}), \quad \chi^{*}_{C^{\mathrm{bc}}}(V) \neq \emptyset.$$

(ii) *Conversely, let $U \in \mathcal{D}(D_{\mathrm{G}}^{\pm}\psi)$ be a solution of the second inverse vector variational inequality*

$$\langle V - U, -D_{\mathrm{G}}^{\pm}\psi(U)\rangle \not\preceq^{2}_{\operatorname{int}\mathcal{K}(D_{\mathrm{G}}^{\pm}\psi(U))} \chi^{*}_{C^{\mathrm{bc}}}(U) - \chi^{*}_{C^{\mathrm{bc}}}(V),$$
$$\textit{for every} \quad V \in \mathrm{Mat}_{k\times k}(\mathbb{R}).$$

and define $x = D_{\mathrm{G}}^{\pm}\psi(U)$. If $x \in C^{\mathrm{bc}}$, $\partial\chi_{C^{\mathrm{bc}}}(x) \neq \emptyset$ and the sets

$$A := \big\{\langle U, y\rangle - \chi_{C^{\mathrm{bc}}}(y) \mid y \in \mathbb{R}^k\big\} \qquad \textit{and} \qquad B := \big\{\langle U, x\rangle - \chi_{C^{\mathrm{bc}}}(x)\big\}$$

satisfy the weak (A, B)-domination property w.r.t. the cone $\mathcal{K}(x)$, then, x is a solution of the generalized vector variational inequality (4.4.7).

Chapter 5

Existence Results for Quasi-Variational-Like Problems

Abstract. In this chapter, we study existence results and alternative solution techniques for quasi-variational-like problems. A commonly used technique to tackle the latter problem class is by formulating it as an equivalent single- or set-valued fixed-point problem. In order to apply appropriate fixed-point results, the corresponding variational selection S has to have convex values and the constraining set must be bounded. However, in the absence of such properties, the latter approach requires very stringent conditions for the data of the quasi-variational-like problem. We therefore aim at minimizing inputs and outputs of S, which leads to the concept of generalized solutions. In that regard, we investigate a closely related optimization problem, whose solvability does not require convexity or boundedness assumptions.

Several questions in applied and industrial mathematics take the form of a quasi-variational-like problem; see [13, 110, 121, 125]. It consists of a set-valued fixed-point problem and a parametric variational-like problem; one depends on the other. The interdependence of these two problems challenges techniques, which are available for variational-like problems only and set-valued fixed-point problems only.

To be precise, assume that C is a non-empty, closed and convex subset of the real Banach space X. Further, let $E : C \rightrightarrows C$ and $P : C \rightrightarrows C$ be set-valued mappings with non-empty values. Then the quasi-variational-like problem consists of finding $x \in C$ such that

$$x \in E(x) \qquad \text{and} \qquad x \in P(y), \qquad \text{for every} \quad y \in E(x). \tag{5.0.1}$$

Notice that in problem (5.0.1), the constraining set $E(x)$ depends upon the unknown x. In fact, due to this dependence, the existence of solutions of problem (5.0.1) are challenging and require that a set-valued fixed-point problem and a variational-like problem should be solved simultaneously. Depending on properties of P, various solution methods for quasi-variational-like problem (5.0.1) have been proposed in the literature; see, for example, [6, 29, 68, 139]. If additionally E is constant, that is, $E(x) = C$ for every $x \in C$, then problem (5.0.1) recovers the following variational-like problem: Find $x \in C$ such that

$$x \in P(y), \qquad \text{for every} \quad y \in C. \tag{5.0.2}$$

It should be noted that problem (5.0.2) covers, for example, variational inequality (2.2.1) as special case if P is given by $P(y) := \big\{x \in C \mid \langle Fx, y - x\rangle \geq 0\big\}$, for $y \in C$, where $F : X \to X^*$. Thus, in a similar fashion, quasi-variational-like problem (5.0.1) also covers many important problems of interest as particular case; compare also Section 1 in [56]. Indeed, by suitably adjusting the data in problem (5.0.1), we have the following special cases, which will be studied in this chapter:

1. *Quasi-variational inequality.* Let $F : X \to X^*$ and suppose that the set-valued mapping P is defined by

$$P(y) := \big\{x \in C \mid \langle Fx, y - x\rangle \geq 0\big\},$$

for every $y \in C$. Then problem (5.0.1) recovers the following quasi-variational inequality: Find $x \in C$ such that

$$x \in E(x) \qquad \text{and} \qquad \langle Fx, y - x\rangle \geq 0, \qquad \text{for every} \quad y \in E(x). \tag{5.0.3}$$

The above problem was introduced by Bensoussan and Lions [18] in connection with a problem of impulse control. In fact, by suitably adjusting the data, it can be shown that problem (5.0.3) includes variational inequalities, non-linear inverse problems, split-feasibility problems, and many others; see [107, 121].

2. *Quasi-variational inequality.* Let $F : X \times X \rightrightarrows X^*$ be a set-valued mapping and let $f \in X^*$. If the set-valued mapping P is defined by

$$P(y) := \big\{x \in C \mid \text{ there exists } w(y) \in F(y, x) \text{ such that } \langle w(y) - f, y - x\rangle \geq 0\big\},$$

for every $y \in C$, then problem (5.0.1) recovers the following quasi-variational inequality: Find $x \in C$ such that $x \in E(x)$ and for some $w \in F(x, x)$ it holds that

$$\langle w - f, y - x\rangle \geq 0, \qquad \text{for every} \quad y \in E(x). \tag{5.0.4}$$

The above problem has been introduced by Joly and Mosco [101] and is frequently called set-valued quasi-variational inequality. In [102, 103], the authors state the elastic-plastic torsion problem for visco-elastic material by a quasi-variational inequality of type (5.0.4).

3. *Vector quasi-variational inequality.* Let Y be a real Banach space and let $F : X \to \mathrm{L}(X, Y)$ be a single-valued mapping. Suppose further that K is a proper, closed,

convex and solid cone in Y. If the set-valued mapping P is given by

$$P(y) := \{x \in C \mid \langle Fx, y - x\rangle \notin -\operatorname{int} K\},$$

for every $y \in C$, then problem (5.0.1) recovers the following vector quasi-variational inequality: Find $x \in C$ such that

$$x \in E(x) \qquad \text{and} \qquad \langle Fx, y - x\rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in E(x). \tag{5.0.5}$$

In [93], the authors used vector quasi-variational inequality (5.0.5) for the investigation of an multi-objective optimization problem with respect to forbidden areas.

4. *Generalized vector quasi-variational inequality.* Let Y be a real Banach space, let $F : X \rightrightarrows \mathrm{L}(X, Y)$ be a set-valued mapping and let $f \in \mathrm{L}(X, Y)$. Assume further that $\mathcal{K} : X \rightrightarrows Y$ is a variable domination structure on Y. If the set-valued mapping P is given by

$$P(y) := \{x \in C \mid \text{there exists } w \in F(x) \text{ such that } \langle w - f, y - x\rangle \notin -\operatorname{int} \mathcal{K}(x)\},$$

for every $y \in C$, then problem (5.0.1) recovers the following generalized vector quasi-variational inequality with respect to a variable domination structure: Find $x \in C$ such that $x \in E(x)$ and for some $w \in F(x)$ it holds that

$$\langle w - f, y - x\rangle \notin -\operatorname{int} \mathcal{K}(x), \qquad \text{for every} \quad y \in E(x). \tag{5.0.6}$$

Clearly, problem (5.0.5) is a special case of the generalized vector quasi-variational inequality (5.0.6). The vector quasi-variational inequalities (5.0.5) and (5.0.6) have been studied by numerous authors, whereby a common used technique to solve the problems consists of finding minimal elements or equilibrium choices of a certain set-valued mapping; see, for example, [9, 37, 70, 83, 115, 120, 147, 156].

5.1 Classic existence results

The purpose of this section is the investigation of classic existence results for quasi-variational and vector quasi-variational inequalities.

5.1.1 Existence results for quasi-variational inequalities

Within the last years, the theory of variational and quasi-variational inequalities emerged as one of the most promising branches of pure and applied mathematics. This theory provides us with a powerful mathematical apparatus for studying numerous types of problems arising in diverse fields such as economics, elasticity, financial mathematics, mechanics, optimization and many others; see [13, 16, 21, 94, 125] and the references therein. Solution techniques for quasi-variational inequalities of type (5.0.3) are challenging and require that the set-valued fixed-point problem

$$x \in C : \qquad x \in E(x)$$

and the variational inequality

$$x \in C : \qquad \langle Fx, y - x \rangle \geq 0, \qquad \text{for every} \quad y \in E(x)$$

should be solved simultaneously. One of the most commonly used techniques for the solvability of quasi-variational inequalities is finding fixed-points of the associated variational selection; see [102, 107, 129, 155]. This approach is quite natural since problem (5.0.3) already contains a set-valued fixed-point problem.

To shed some light on this idea, let $u \in C$ and consider the following parametric variational inequality with element u as the parameter: Find $x_u \in E(u)$ such that

$$\langle Fx_u, y - x_u \rangle \geq 0, \qquad \text{for every} \quad y \in E(u). \tag{5.1.1}$$

Let us denote the solution set of problem (5.1.1) by $S(u)$. When the parameter u varies in C, $S(u)$ defines the so-called variational selection or solution mapping $S : C \rightrightarrows C$. Evidently, if x is a fixed-point of the set-valued mapping S, that is, if

$$x \in C : \qquad x \in S(x), \tag{5.1.2}$$

then x solves quasi-variational inequality (5.0.3). However, depending on, for example, monotonicity properties of the operator F, see Definition 2.2.2, the corresponding variational selection S has different properties:

1. Suppose F is strictly monotone. Then, for every $u \in C$, parametric variational inequality (5.1.1) has a unique solution $x_u \in C$, if it exists. Thus, the variational selection $S : C \to C$ with $u \mapsto x_u$ is a single-valued mapping and problem (5.1.2) collapses to the (single-valued) fixed-point problem

$$x \in C : \qquad S(x) = x. \tag{5.1.3}$$

   Consequently, in order to solve problem (5.1.3), one can apply appropriate fixed-point results of Section 2.3.1.

2. Suppose F is monotone. Then, for every $u \in C$, parametric variational inequality (5.1.1) has multiple solutions, if they exist. Therefore, for every $u \in C$, $S(u)$ is a set and $S : C \rightrightarrows C$ with $u \mapsto S(u)$ defines a set-valued mapping. One can further show that, under a weak continuity assumption, the values of S are convex. Thus, in order to investigate problem (5.1.2), one could apply all of the set-valued fixed-point results in Section 2.3.2.

3. Suppose F is pseudomonotone. Then, for every $u \in C$, problem (5.1.1) has multiple solutions, if they exist, and the corresponding variational selection has

non-convex values. In this case, rewriting quasi-variational inequality (5.0.3) as an equivalent fixed-point problem is no longer useful.

Let us recall the following notion of set-convergence, which has been introduced by Mosco for the study of quasi-variational inequalities; see [138].

Definition 5.1.1. Let C be a non-empty subset of the real Banach space X. A set-valued mapping $E : C \rightrightarrows C$ is called *Mosco-continuous* if the following conditions hold:

(M1) The graph of E is weak sequentially closed, that is, for all sequences $\{x_n\}$ and $\{u_n\}$ with $x_n \in E(u_n)$, $x_n \rightharpoonup x$ and $u_n \rightharpoonup u$, we have $x \in E(u)$.

(M2) If $\{u_n\}$ is a sequence in C with $u_n \rightharpoonup u$, then for every $z \in E(u)$ there exists a sequence $\{z_n\}$ such that $z_n \in E(u_n)$ and $z_n \to z$.

The following novel existence result for quasi-variational inequality (5.0.3) is motivated by Theorem 2.2 in [105].

Theorem 5.1.2. *Let C be a non-empty, closed, convex and bounded subset of the real reflexive Banach space X. Suppose that $E : C \rightrightarrows C$ is a Mosco-continuous mapping with non-empty, closed and convex values, and let $F : X \to X^*$ be monotone and continuous. Then, quasi-variational inequality* (5.0.3) *has a solution.*

Proof. In what follows, we will use Kluge's fixed-point theorem to show that problem (5.1.2) has a solution. Recall that in this context, the variational selection $S : C \rightrightarrows C$ is given for every $u \in C$ by

$$S(u) = \big\{x \in C \mid x \in E(u) \text{ and } \langle Fx, y - x\rangle \geq 0, \text{ for every } y \in E(u)\big\}.$$

Clearly, for every $u \in C$, $S(u)$ is non-empty; see Theorem 2.2.7. Further, using Lemma 2.2.6, it is easy to see that S has closed and convex values. It remains to show that the graph of S is weak sequentially closed. Indeed, let $\{x_n\}$ and $\{u_n\}$ be sequences in X with $x_n \in S(u_n)$, $x_n \rightharpoonup x$ and $u_n \rightharpoonup u$. Since $x_n \in S(u_n)$, we infer that $x_n \in E(u_n)$ and

$$\langle Fx_n, y - x_n\rangle \geq 0, \qquad \text{for every} \quad y \in E(u_n).$$

Taking into account the Mosco-continuity of E, $x_n \rightharpoonup x$ and $u_n \rightharpoonup u$ implies that $x \in E(u)$. However, since F is monotone and continuous, we conclude from Lemma 2.2.6 that it holds

$$\langle Fy, y - x_n\rangle \geq 0, \qquad \text{for every} \quad y \in E(u_n).$$

Now let $z \in E(u)$. By using the Mosco-continuity of E, we can find a sequence $\{z_n\}$ with $z_n \in E(u_n)$ and $z_n \to z$. Inserting this sequence in the above variational inequality and passing to the limit yields

$$\langle Fz_n, z_n - x_n\rangle \to \langle Fz, z - x\rangle,$$

which is a consequence of the continuity of F and Proposition 2.1.13 (iv). Applying the Minty lemma once again, we obtain $x \in S(u)$, that is, the graph of S is weak sequentially closed. Thus, all requirements of Theorem 2.3.11 are satisfied and S attains a fixed-point. The proof is complete. □

The next result uses a regularization technique, which allows to study the single-valued fixed-point problem (5.1.3) instead of (5.1.2). The proof is motivated by the proof of Theorem 2.1 in [102].

Theorem 5.1.3. *Let C be a non-empty, convex and compact subset of the real, strictly convex, separable and reflexive Banach space X. Suppose that the dual space X^* is locally uniformly convex, assume that $E : C \rightrightarrows C$ is a Mosco-continuous mapping with non-empty, closed and convex values, and let $F : X \to X^*$ be monotone and continuous. Then, quasi-variational inequality (5.0.3) has a solution.*

Proof. This proof is using a regularization argument. It is provided in the following two steps: We will first assume that F is strictly monotone and show that problem (5.0.3) attains a (unique) solution. The general case is treated in the second step.

(I). Assume that F is strictly monotone. Due to the strict monotonicity of F, the parametric variational inequality with respect to the parameter $u \in C$ has a unique solution; compare again Theorem 2.2.7. Therefore, the corresponding variational selection S is a single-valued mapping from C to C. In what follows, we are going to study the fixed-point problem (5.1.3). Since S is single-valued, we are in position to use a classic fixed-point result, such as the one of Schauder; compare Corollary 2.3.4. Let us show that S is weak sequentially continuous. Indeed, let $\{u_n\}$ be a sequence in C with $u_n \rightharpoonup u$. We need to show that $x_n \rightharpoonup x$, where we let $x_n = S(u_n)$ and $x = S(u)$. However, this follows similar to the previous proof and will be therefore omitted.

(II). We now prove the general case. Let us denote by $J : X \to X^*$ the duality mapping from X to X^*. Note that, due to the strict convexity of X, J is a strictly monotone, continuous and single-valued operator; see Proposition 32.22 in [167]. Further, define the regularized operator $F_n := F + \varepsilon_n J$, where $\{\varepsilon_n\} \subseteq \mathbb{R}_>$ is a sequence with $\varepsilon_n \downarrow 0$. Since F_n is strictly monotone, the achievement of step (I) provides that the following family of regularized quasi-variational inequalities attains a (unique) solution: Find $x_n \in C$ such that

$$x_n \in E(x_n) \qquad \text{and} \qquad \langle F_n x_n, y - x_n\rangle \geq 0, \qquad \text{for every} \quad y \in E(x_n). \tag{5.1.4}$$

Due to the compactness of C, there exists a subsequence of $\{x_n\}$, again denoted by $\{x_n\}$, with $x_n \to x$ and $x \in C$. From the Mosco-continuity of E follows $x \in E(x)$. Further, given $z \in E(x)$, there exists a sequence $\{z_n\}$ with $z_n \in E(x_n)$ and $z_n \to z$. Inserting this sequence in problem (5.1.4), we finally conclude

$$\langle F x_n + \varepsilon_n J x_n, z_n - x_n\rangle \to \langle F x, z - x\rangle.$$

Since $z \in E(x)$ was chosen arbitrarily, we have shown that the limit point x is a solution of quasi-variational inequality (5.0.3). The proof is complete. □

In what follows, we will study the following quasi-variational inequality of the second kind: Find $x \in C$ such that

$$a(x, y-x) + J(x, y) - J(x, x) \geq \langle f, y-x \rangle, \qquad \text{for every} \quad y \in C. \tag{5.1.5}$$

In the above, C is a non-empty, closed and convex subset of the real Hilbert space X with inner product $\langle \cdot, \cdot \rangle$. Further, $J : X \times X \to [0, +\infty]$ is a function, $a : X \times X \to \mathbb{R}$ is an elliptic bilinear form and $f \in X^*$. Recall that a is called elliptic [121, Chapter II] if it is bounded and coercive, that is, there exist constants $\alpha, \beta > 0$ such that

$$\begin{aligned} &a(x, y) \leq \beta \|x\|_X \|y\|_X, \qquad \text{for every} \quad x, y \in X, \\ &a(x, x) \geq \alpha \|x\|_X^2, \qquad \text{for every} \quad x \in X. \end{aligned}$$

In order to prove that problem (5.1.5) has a solution, we need the following auxiliary result. The proof uses the ideas in [45].

Theorem 5.1.4. *Let C be a non-empty, closed and convex subset of the Hilbert space X. Suppose that $a : X \times X \to \mathbb{R}$ is an elliptic bilinear form and $f \in X^*$. If $j : X \to [0, +\infty]$ is proper, convex and weak sequentially lower semicontinuous, then the following variational inequality of the second kind has a unique solution: Find $x \in C$ such that*

$$a(x, y-x) + j(y) - j(x) \geq \langle f, y-x \rangle, \qquad \textit{for every} \quad y \in C. \tag{5.1.6}$$

Proof. The proof of this theorem consists of several parts.

(I). Assume in addition that a is symmetric and define a function $I : X \to \mathbb{R} \cup \{+\infty\}$ by

$$I(x) := \frac{1}{2} a(x, x) - \langle f, x \rangle + j(x),$$

for $x \in X$. Recall that a is said to be symmetric if $a(x, y) = a(y, x)$ for all $x, y \in X$.

(a). Let us first show that variational inequality (5.1.6) is equivalent to finding $x \in C$ such that

$$I(x) = \inf_{y \in C} I(y). \tag{5.1.7}$$

Indeed, let $x \in C$ be a solution of problem (5.1.6), that is, it holds for every $y \in C$

$$\frac{1}{2} a(x, x) - \langle f, x \rangle + j(x) \;\leq\; \frac{1}{2} a(x, x) + a(x, y-x) - \langle f, y \rangle + j(y).$$

Since $\frac{1}{2} a(x, x) + a(x, y-x) = \frac{1}{2} a(y, y) - \frac{1}{2} a(x-y, x-y)$, we deduce

$$I(x) \;\leq\; I(y) - \frac{1}{2} a(x-y, x-y), \qquad \text{for every} \quad y \in C.$$

The coercivity of a further implies $I(x) \leq I(y)$ for every $y \in C$, that is, x solves the optimization problem (5.1.7). Conversely, let x be a solution of problem (5.1.7). Since C is convex, $I(x) \leq I(x + t(y - x))$ for every $t \in (0, 1)$ and $y \in C$. Therefore,

$$\frac{1}{2}a(x,x) - \langle f, x\rangle + j(x) \;\leq\; \frac{1}{2}a\big(x_t, x_t\big) - \langle f, x_t\rangle + j\big(x_t\big), \qquad \text{for every} \quad y \in C,$$

where $x_t = x + t(y - x)$. Since a is assumed to be symmetric, rearranging the previous inequality leads to

$$t\langle f, y - x\rangle \;\leq\; ta(x, y - x) + \frac{1}{2}t^2 a(x - y, x - y) + tj(y) - tj(x), \qquad \text{for every} \quad y \in C.$$

Dividing the previous inequality by t and passing to the limit $t \downarrow 0$, we conclude

$$\langle f, y - x\rangle \;\leq\; a(x, y - x) + j(y) - j(x), \qquad \text{for every} \quad y \in C.$$

Thus, x is a solution of variational inequality (5.1.6).

(b). In what follows, we will show that problem (5.1.7) attains a solution. Indeed, I is bounded from below since for any $\varepsilon > 0$ and $x \in X$

$$\begin{aligned} I(x) \;\geq\; \frac{1}{2}a(x,x) - \langle f, x\rangle \;&\geq\; \frac{\alpha}{2}\|x\|_X^2 - \|f\|_{X^*}\|x\|_X \\ &\geq\; \frac{\alpha}{2}\|x\|_X^2 - \varepsilon\|x\|_X^2 - c(\varepsilon)\|f\|_{X^*}^2 \;\geq\; \text{const}, \end{aligned}$$

where the last inequality follows from Young's inequality; see Section 2.2.4 in [27]. Thus, we can find a minimizing sequence $\{x_n\}$ in C with $I(x_n) \to I_0$, where $I_0 := \inf_{y \in C} I(y)$. Due to the properness of j, there exists a constant $c > 0$ and $N \in \mathbb{N}$ with $j(x_n) \leq c$ for all $n \geq N$. Thus, using the positivity of j, we have

$$\frac{1}{2}\alpha\|x_n\|_X^2 \;\leq\; \frac{1}{2}a(x_n, x_n) \;\leq\; c + \langle f, x_n\rangle \;\leq\; c + \|f\|_{X^*}\|x_n\|_X,$$

for $n \geq N$. Consequently, $\{x_n\}$ is bounded and we can find a subsequence, again denoted by $\{x_n\}$, with $x_n \rightharpoonup x$ and $x \in C$. Let us show that the limit point x is a solution of optimization problem (5.1.7). The weak sequentially lower semicontinuity of $x \mapsto a(x,x)$ and j yield

$$I_0 \;=\; \lim_{n \to +\infty} I(x_n) \;=\; \liminf_{n \to +\infty} I(x_n) \;\geq\; \frac{1}{2}a(x,x) - \langle f, x\rangle - j(x) \;=\; I(x).$$

Thus, $I_0 \geq I(x)$ while the converse inequality trivially holds. This shows that x is a solution of problem (5.1.7).

(c). The uniqueness of problem (5.1.6) is easily seen and follows from the coercivity of a. Finally, step (a) and (b) show that variational inequality (5.1.6) attains a unique solution provided the bilinear form is symmetric.

(II). We should show that problem (5.1.6) has a unique solution, even if a is not necessarily symmetric. For this purpose, we introduce the symmetric and antisymmetric part of a given by $a_S(x, y) := \frac{1}{2}\big(a(x, y) + a(y, x)\big)$ and $a_A(x, y) := \frac{1}{2}\big(a(x, y) - a(y, x)\big)$

for every $x, y \in X$. Let us further introduce for $t \in \mathbb{R}$ the bilinear form

$$a_t := a_S + a_A$$

and the set

$$T := \{t \in \mathbb{R} \mid \text{for every } f \in X^* \text{ the problem (5.1.8) has a unique solution}\},$$

where problem (5.1.8) is the variational inequality of finding $x \in C$ such that

$$a_t(x, y - x) + j(y) - j(x) \geq \langle f, y - x\rangle, \qquad \text{for every} \quad y \in C. \tag{5.1.8}$$

In view of step (I), the set T is non-empty since $0 \in T$. Let $t_0 \in T$. Then, we can claim, adapting the proof of Theorem 4.1 in [45], that T contains the segment $[t_0 - \frac{\alpha}{2\beta}, t_0 + \frac{\alpha}{2\beta}]$ where α, β are the constants given by the ellipticity condition of a. Consequently, this implies

$$T = \mathbb{R}$$

and for $t = 1$, problem (5.1.8) has a unique solution. This finally shows that problem (5.1.6) has a unique solution. The proof is complete. □

Remark 5.1.5. (i) An alternative proof for Theorem 5.1.4, using the projection operator, can be found in [137, Theorem 2.8].
(ii) The previous proof shows that variational inequality (5.1.6) is equivalent to the optimization problem (5.1.7), provided the bilinear form a is symmetric.

The following theorem is a special case of Theorem 2.1 in [86] and uses Banach's fixed-point theorem; see Theorem 2.3.5.

Theorem 5.1.6. *Let C be a non-empty, closed and convex subset of the real Hilbert space X. Suppose that $a : X \times X \to \mathbb{R}$ is an elliptic bilinear form, $f \in X^*$ and for every $x \in X$, $J(x, \cdot) : X \to [0, +\infty]$ is a proper, convex and weak sequentially lower semicontinuous function. If there is a constant $\delta > 0$ with $\alpha - \delta > 0$ and*

$$|J(x, \tilde{y}) + J(\tilde{x}, y) - J(x, y) - J(\tilde{x}, \tilde{y})| \leq \delta \|x - \tilde{x}\|_X \|y - \tilde{y}\|_X,$$

for every $x, \tilde{x}, y, \tilde{y} \in X$, then, quasi-variational inequality of the second kind (5.1.5) has a unique solution.

Proof. Let $u \in C$ and consider the parametric problem of finding $x_u \in C$ such that

$$a(x_u, y - x_u) + J(u, y) - J(u, x_u) \geq \langle f, y - x_u\rangle, \qquad \text{for every} \quad y \in C. \tag{5.1.9}$$

In view of Theorem 5.1.4, problem (5.1.9) has a unique solution. This permits us to define the variational selection $S : C \to C$ such that for every $u \in C$, $S(u) = x_u$ is the unique solution of problem (5.1.9). Recall that any fixed-point of the variational selection S solves problem (5.1.5). Let $u, v \in C$ and let x_u and x_v be the corresponding solutions of problem (5.1.9). That is, the following parametric variational inequalities hold:

$$a(x_u, y - x_u) + J(u, y) - J(u, x_u) \geq \langle f, y - x_u \rangle, \qquad \text{for every} \quad y \in C.$$
$$a(x_v, y - x_v) + J(v, y) - J(v, x_v) \geq \langle f, y - x_v \rangle, \qquad \text{for every} \quad y \in C.$$

We rearrange these inequalities with $y = x_v$ in the first and $y = x_u$ in the second to get

$$a(x_u - x_v, x_u - x_v) \leq J(u, x_v) + J(v, x_u) - J(u, x_u) - J(v, x_v),$$

which implies

$$\alpha \|x_u - x_v\|_X^2 \leq \delta \|u - v\|_X \|x_u - x_v\|_X.$$

In other words, we have shown $\|S(u) - S(v)\|_X \leq \frac{\delta}{\alpha}\|u - v\|_X$, and hence the mapping $S : C \to C$ is a contraction. Therefore, Banach's fixed-point theorem ensures that there is a unique fixed-point of S and hence quasi-variational inequality (5.1.5) is uniquely solvable. The proof is complete. □

5.1.2 Existence results for vector quasi-variational inequalities

In this section, we investigate existence results for vector quasi-variational inequality (5.0.5). Notice that the ideas and methods of this section can be adapted to derive existence results for generalized problems of the type (5.0.6). In comparison to the previous section, rewriting problem (5.0.5) as an equivalent fixed-point problem is not useful. This is because, even by assuming that $F : X \to \mathrm{L}(X, Y)$ is K-monotone, or even $\operatorname{int} K$-monotone, the corresponding variational selection is a set-valued mapping with non-convex values. Thus, applying set-valued fixed-point results is too restrictive for the data of problem (5.0.5) and (5.0.6), respectively. However, a commonly used technique, which we use here, is to apply so-called minimal element theorems, which are closely related to set-valued fixed-point problems.

The following lemma is a special case of Theorem 2 in [51].

Lemma 5.1.7 (Minimal element)**.** *Let C be a non-empty, convex and compact subset of a real Hausdorff topological vector space X and let $V, W : C \rightrightarrows C$ be set-valued mappings. Assume further that the following hold:*

(i) *V has non-empty, closed and convex values. V is upper semicontinuous.*

(ii) *V and W have open lower sections.*

(iii) *For every $x \in C$, $x \notin \operatorname{conv} W(x)$.*

Then there exists an element $x \in C$ such that

$$V(x) \cap W(x) = \emptyset. \tag{5.1.10}$$

Remark 5.1.8. (i) Notice that the set-valued mapping W is not assumed to have convex values.
(ii) An element $x \in C$ such that (5.1.10) holds is frequently called minimal element or equilibrium choice.

We have the following existence result for problem (5.0.5).

Theorem 5.1.9 ([120, Theorem 1])**.** *Suppose that X and Y are real Banach spaces and let C be a non-empty, convex and compact subset of X. Let $E : C \rightrightarrows C$ be a set-valued mapping and denote by K a proper, closed, convex and solid cone in Y. Assume further that the following hold:*

(i) *E has non-empty, closed and convex values. E is upper semicontinuous.*

(ii) *E has open lower sections.*

(iii) *$F : X \to \mathrm{L}(X, Y)$ is a continuous mapping.*

Then, vector quasi-variational inequality (5.0.5) *has a solution.*

Proof. Let us define two set-valued mappings $V, W : C \rightrightarrows C$ by $V := E$ and

$$W(x) := \big\{y \in C \mid \langle Fx, y - x\rangle \in -\operatorname{int} K\big\}, \qquad \text{for every} \quad x \in C.$$

The proof is complete if we can ensure the existence of an element $x \in C$ such that it holds

$$x \in E(x) \qquad \text{and} \qquad E(x) \cap W(x) = \emptyset,$$

since this is equivalent to saying that the element $x \in E(x)$ does not belong to $W(x)$. In what follows, we shall show that the assumptions of Lemma 5.1.7 hold. Let us show that W has open lower sections first. Indeed, let $y \in C$ and note that

$$W^{-1}(y) = \big\{x \in C \mid \langle Fx, y - x\rangle \in -\operatorname{int} K\big\}.$$

Let $\{x_n\}$ be a sequence in $C \setminus W^{-1}(y)$, depending on $y \in C$, with $x_n \to x$ and $x \in C$. Since $x_n \in C \setminus W^{-1}(y)$ is equivalent to $\langle Fx_n, y - x_n\rangle \notin -\operatorname{int} K$ and F is assumed to be continuous, we may pass to the limit. Thus, $\langle Fx_n, y - x_n\rangle \to \langle Fx, y - x\rangle$ in Y; compare Proposition 2.1.13. Due to the fact that the set $Y \setminus (-\operatorname{int} K)$ is closed, we consequently have $\langle Fx, y - x\rangle \notin -\operatorname{int} K$. It follows that $x \in C \setminus W^{-1}(y)$, that is, W has open lower sections. Let us finally show that assumption (iii) of Lemma 5.1.7 holds. Suppose to the contrary that there is some element $x \in C$ such that $x \in \operatorname{conv} W(x)$. Then we can find a positive integer $k \in \mathbb{N}$, $x_1, \dots, x_k \in W(x)$ and $\lambda_1, \dots, \lambda_k \in [0, 1]$ with $\sum_{j=1}^{k} \lambda_j = 1$ such that $x = \sum_{j=1}^{k} \lambda_j x_j$. Consequently, for any $j \in \{1, \dots, k\}$ it holds that $\langle Fx, x_j - x\rangle \in -\operatorname{int} K$. Thus, multiplying every relation by λ_j and summing them up, we obtain

$$\sum_{j=1}^{k} \lambda_j \langle Fx, x_j\rangle \in \langle Fx, x\rangle - \operatorname{int} K.$$

However, the left-hand side of the above relation is equivalent to $\langle Fx, x\rangle$. We therefore deduce $0 \in \operatorname{int} K$, which is impossible; see Remark 2.4.3 (i). By the assumptions of this theorem, the rest of the hypothesis of Lemma 5.1.7 are also satisfied. The proof is complete. □

Lemma 5.1.10 ([113, Lemma]). *Let C be a non-empty and convex subset of a real Hausdorff topological vector space X and let $U : C \rightrightarrows C$ be a set-valued mapping. Assume further that the following hold:*

(i) *U has open lower sections.*

(ii) *For every $x \in C$, $x \notin \operatorname{conv} U(x)$.*

(iii) *There exists a non-empty and compact subset $B \subseteq C$ and a non-empty, convex and compact subset $\tilde{B} \subseteq C$ such that $\tilde{B} \cap \operatorname{conv} U(x) \neq \emptyset$ for all $x \in C \setminus B$.*

Then there exists an element $x \in C$ such that

$$U(x) = \emptyset.$$

The following existence result for problem (5.0.5) is a special case of Theorem 2.1 in [52].

Theorem 5.1.11. *Suppose that X and Y are real Banach spaces, let C be a non-empty and convex subset of X, let $E : C \rightrightarrows C$ be a set-valued mapping and denote by K a proper, closed, convex and solid cone in Y. Assume further that the following hold:*

(i) *E has non-empty, closed and convex values. E has open lower sections and the set of fixed-points $\operatorname{Fix} E := \{x \in C \mid x \in E(x)\}$ is non-empty and closed.*

(ii) *$F : X \to \mathrm{L}(X, Y)$ is a continuous mapping.*

(iii) *There exists a non-empty and compact subset $B \subseteq X$ and a non-empty, convex and compact subset $\tilde{B} \subseteq X$ such that for all $x \in X \setminus B$, there exists an element $y \in \tilde{B}$ such that $x \in E(x)$ and $\langle Fx, y - x\rangle \in -\operatorname{int} K$.*

Then, vector quasi-variational inequality (5.0.5) has a solution.

Proof. In what follows, we are going to consider the set-valued mapping $U : C \rightrightarrows C$, defined by

$$U(x) := E(x) \cap \operatorname{conv} W(x) \qquad \text{for} \quad x \in \operatorname{Fix} E, \qquad U(x) := E(x) \qquad \text{else},$$

where $W : C \rightrightarrows C$ is given as in the proof of Theorem 5.1.9. In order to show that problem (5.0.5) has a solution, it is enough to show the existence of an element $x \in C$ such that

$$U(x) = \emptyset. \tag{5.1.11}$$

Thus, since $E(x)$ is non-empty, this would imply $x \in E(x)$ and $E(x) \cap W(x) = \emptyset$. In other words, similar to the proof of Theorem 5.1.9, the element $x \in C$ would be a solution of vector quasi-variational inequality (5.0.5). It therefore remains to show that U satisfies all assumptions of Lemma 5.1.10. However, using the previous arguments, it is easily seen that $x \notin \operatorname{conv} W(x)$ for all $x \in C$ and W has open lower sections. Thus, since $\operatorname{conv} U = U$, the set-valued mapping U also has open lower sections. Here, the fact that $X \setminus \operatorname{Fix} E$ is open is crucial. Finally, condition (iii) of this theorem implies that there exists a non-empty and compact set $B \subseteq X$ and a non-empty, convex and compact set $\tilde{B} \subseteq X$ such that

$$\tilde{B} \cap U(x) = \tilde{B} \cap \operatorname{conv} U(x) \neq \emptyset,$$

for all $x \in C \setminus B$. Thus, by Lemma 5.1.10, we can find an element $x \in C$ such that (5.1.11) holds. The proof is complete. $\square$

Let $s \in Y^* \setminus \{0\}$. In what follows, we will consider the following quasi-variational inequality with respect to the scalarization functional s: Find $x = x(s) \in C$ such that

$$x \in E(x) \qquad \text{and} \qquad \langle s \circ Fx, y - x\rangle \geq 0, \qquad \text{for every} \quad y \in E(x). \tag{5.1.12}$$

The solution set of problem (5.0.5) and (5.1.12) will be denoted by $\operatorname{Sol}(\mathrm{VQVI})$ and $\operatorname{Sol}(\mathrm{QVI}_s)$, respectively.

Similar to Proposition 3.2.6, we have the following useful result.

Proposition 5.1.12 ([93, Proposition 5.1]). *Suppose that X and Y are real Banach spaces, let C be a non-empty, closed and convex subset of X, let $E : C \rightrightarrows C$ be a set-valued mapping with non-empty values and denote by K a proper, closed, convex and solid cone in Y. If the quasi-interior of K^* is non-empty and $F : X \to \mathrm{L}(X, Y)$ is any mapping, then it holds that*

$$\bigcup_{s \in \operatorname{qi} K^*} \operatorname{Sol}(\mathrm{QVI}_s) \subseteq \operatorname{Sol}(\mathrm{VQVI}).$$

Proof. Assume the statement does not hold. Then, there exists $s \in \operatorname{qi} K^*$ and $x \in C$ with $x \in \operatorname{Sol}(\mathrm{QVI}_s)$ but $x \notin \operatorname{Sol}(\mathrm{VQVI})$. Consequently, we can find $y \in E(x)$ with $\langle Fx, y - x\rangle \in -\operatorname{int} K$. Taking into account $s \in \operatorname{qi} K^*$ and $\operatorname{int} K \subseteq K \setminus \{0\}$, we conclude $\langle s \circ Fx, y - x\rangle < 0$, which is impossible. The proof is complete. $\square$

The next result has been proposed by Hebestreit, Khan, Köbis and Tammer [93]. The proof is based on a scalarization technique and a regularization approach.

Theorem 5.1.13 ([93, Theorem 5.4]). *Suppose that X is a real, reflexive and separable Banach space. Let Y be a real Banach space, let C be a non-empty, closed, convex and bounded subset of X, let $E : C \rightrightarrows C$ be a set-valued mapping with non-empty, closed and convex values and denote by K a proper, closed, convex and solid cone in Y. Assume further that the quasi-interior of K^* is non-empty and suppose that the following conditions hold:*

(i) *$F : X \to \mathrm{L}(X, Y)$ is a K-monotone mapping and it holds that $x_n \to x$ and $y_n \rightharpoonup y$ in X imply $\langle Fx_n, y_n\rangle \to \langle Fx, y\rangle$ in Y.*

(ii) *E is Mosco-continuous.*

Then, vector quasi-variational inequality (5.0.5) *has a solution.*

Proof. Let $s \in \operatorname{qi} K^*$ be arbitrarily chosen. In what follows, we are going to study the set-valued mapping $S = S(s) : C \rightrightarrows C$, given for $u \in C$ by

$$S(u) := \big\{x \in C \mid \langle s \circ Fx, y - x\rangle \geq 0, \text{ for every } y \in C(u)\big\}.$$

In view of Proposition 5.1.12, it is sufficient to show that S has a fixed-point. The proof of this theorem is divided into two parts.

(I). Suppose that $s \circ F : X \to X^*$ is strictly monotone. As a consequence of assumption (i), the operator $s\circ F$ is hemicontinuous. Together with the K-monotonicity of F, it follows from Theorem 2.2.7 that the values of S are non-empty. Further, since $s \circ F$ is strictly monotone, the values of S are singletons. In other words, the solution mapping $S : C \to C$ is a single-valued operator. In what follows, we shall show that S satisfies the conditions of Schauder's fixed-point theorem. It obviously remains to show the weak sequentially continuity of S. For this purpose, let $\{u_n\}$ be a sequence in C and define $x_n := S(u_n)$. In other words, $x_n \in E(u_n)$ is the unique solution of the variational inequality

$$\langle s \circ Fx_n, y - x_n\rangle \geq 0, \qquad \text{for every} \quad y \in E(u_n). \tag{5.1.13}$$

Due to the Mosco-continuity of E, it holds that $x \in E(u)$. Now let $z \in E(u)$. Then we can find a sequence $\{z_n\}$ with $z_n \in E(u_n)$ and $z_n \to z$. Inserting this sequence into the Minty version of problem (5.1.13) and passing to the limit, we deduce that $x \in E(u)$ satisfies

$$\langle s \circ Fz, z - x\rangle \geq 0, \qquad \text{for every} \quad z \in E(u).$$

Consequently, applying Lemma 2.2.6 once again, we have shown that x is the unique solution of problem (5.1.12). Following the convergence principle in Proposition 2.1.13 (vii), we have $Su_n \rightharpoonup Su$. Finally, applying Corollary 2.3.4, S attains a fixed-point, which solves the scalar problem (5.1.12) and consequently vector quasi-variational inequality (5.0.5).

(II). Suppose that $s \circ F : X \to X^*$ is not strictly monotone. Let $R : X \to \mathrm{L}(X, Y)$ be an $\operatorname{int} K$-monotone mapping such that $x_n \to x$ and $y_n \rightharpoonup y$ implies $\langle Rx_n, y_n\rangle \to \langle Rx, y\rangle$. In other words, R satisfies assumption (i) of this theorem. Since it holds that $K + \operatorname{int} K \subseteq \operatorname{int} K$, it is easily seen that the operator $s \circ F_n : X \to X^*$ is strictly monotone and hemicontinuous, where $F_n := F + \varepsilon_n R$ and $\{\varepsilon_n\} \subseteq \mathbb{R}_>$ is a sequence such that $\varepsilon_n \downarrow 0$. Thus, we are in position to apply step (I) such that the following family of vector quasi-variational inequalities has a solution: Find an element $x_n = x(\varepsilon_n) \in C$

such that

$$x_n \in E(x_n) \qquad \text{and} \qquad \langle F_n x_n, y - x_n \rangle \notin -\operatorname{int} K, \qquad \text{for every} \quad y \in E(x_n).$$

Since X is reflexive and C is bounded, we can find a subsequence, again denoted by $\{x_n\}$, with $x_n \rightharpoonup x$ and $x \in E(x)$. Let $z \in E(x)$. Due to the Mosco-continuity of E, there is a sequence $\{z_n\}$ with $z_n \in E(x_n)$ and $z_n \to z$. Inserting this sequence into the Minty version of the above vector problem and passing to the limit, we have

$$\langle F_n z_n, z_n - x_n \rangle \to \langle Fz, x - z \rangle.$$

Finally, applying the Minty lemma once again, it follows that the weak limit $x \in E(x)$ is a solution of vector quasi-variational inequality (5.0.5). The proof is complete. □

The following results is based on Kluge's fixed-point theorem. Again, the main idea is to show that the set-valued variational selection $S : C \rightrightarrows C$ attains a fixed-point.

Theorem 5.1.14 ([93, Theorem 16])**.** *Suppose that X is a real reflexive Banach space. Let Y be a real Banach space, let C be a non-empty, closed, convex and bounded subset of X, let $E : C \rightrightarrows C$ be a set-valued mapping with non-empty, closed and convex values and denote by K a proper, closed, convex and solid cone in Y. Assume further that the quasi-interior of K^* is non-empty and suppose that the following conditions hold:*

(i) *$F : X \to \mathrm{L}(X, Y)$ is K-monotone and continuous.*

(ii) *E is Mosco-continuous.*

Then, vector quasi-variational inequality (5.0.5) has a solution.

We have the following corollary, where we equip the finite-dimensional Euclidean space $\mathbb{R}^k$ with the Pareto cone $\mathbb{R}^k_{\geq}$.

Corollary 5.1.15. *Let C be a non-empty, convex and compact subset of $\mathbb{R}^l$ and let $E : C \rightrightarrows C$ be a Mosco continuous set-valued mapping with non-empty, closed and convex values. Suppose further that $F : \mathbb{R}^l \to \mathrm{Mat}_{k \times l}(\mathbb{R})$ is a $\mathbb{R}^k_{\geq}$-monotone and continuous mapping. Then the following finite-dimensional vector quasi-variational inequality has a solution: Find $x \in C$ such that*

$$x \in E(x) \qquad \text{and} \qquad \begin{pmatrix} \langle F_1 x, y - x \rangle \\ \vdots \\ \langle F_k x, y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^k_{\geq}, \qquad \text{for every} \quad y \in E(x). \tag{5.1.14}$$

5.2 Generalized solutions

The previous sections have shown that solution methods for quasi-variational-like problems highly depend on properties of the underlying variational selection. In case, the corresponding parametric variational-like problem has a unique solution, single-valued fixed-point results can be applied. However, in case the values of the variational selection are non-convex and unbounded sets, there are no suitable fixed-point or minimal point theorems in the literature; compare Section 2.3.

Recently, a new approach was used in [24, 25, 26], then further developed by Bao, Hebestreit and Tammer [15] and Khan and Jadamba [97]. Instead of finding fixed-points of the variational selection S, one minimizes the difference between inputs and outputs of S, that is, the authors studied the following optimization problem: Find $(u, x) \in \mathcal{G}(S)$ such that

$$\|x - u\|_X^2 \le \|y - v\|_X^2, \qquad \text{for every} \quad (v, y) \in \mathcal{G}(S). \tag{5.2.1}$$

Definition 5.2.1 ([15, Definition 1])**.** An element $x \in C$ is called a *generalized solution* of quasi-variational-like problem (5.0.1) if there is $u \in C$ such that (u, x) is a minimizer of problem (5.2.1).

The following connections between generalized solutions and classical solutions are straightforward.

Lemma 5.2.2 ([15, Lemma 1])**.** *If problem (5.2.1) is solvable with $(u, x) \in \mathcal{G}(S)$ as solution and $\|x - u\|_X = 0$, then quasi-variational-like problem (5.0.1) is solvable. Conversely, if problem (5.0.1) is solvable with $x \in C$ as solution, then $(x, x) \in \mathcal{G}(S)$ solves optimization problem (5.2.1).*

Lemma 5.2.2 leads us to study the existence of solutions of problem (5.2.1). Assume that there is a pair $(\bar{u}, \bar{x}) \in \mathcal{G}(S)$. Then, the set defined by

$$M := \left\{ (u, x) \in \mathcal{G}(S) \mid \|x - u\|_X \le \|\bar{x} - \bar{u}\|_X \right\} \tag{5.2.2}$$

is non-empty and every solution of problem (5.2.1) solves the following relaxed optimization problem: Find $(u, x) \in M$ such that

$$\|x - u\|_X = \inf_{(v,y) \in M} \|y - v\|_X. \tag{5.2.3}$$

Since the objective function of problem (5.2.3), $f : X \times X \to [0, +\infty)$ with $f(u, x) := \|u - x\|_X$, is weak sequentially continuous, the Weierstraß Theorem 2.1.16 provides the following existence result for the relaxed optimization problem.

Lemma 5.2.3 ([15, Lemma 2])**.** *Problem (5.2.3) has a solution if M is a non-empty and weak sequentially compact subset in the reflexive Banach space $X \times X$.*

It is important to emphasize that the solvability of problem (5.2.3) does not require that S has convex values. In addition, we must not assume that the constraining set C is bounded.

5.2.1 Generalized solutions of quasi-variational inequalities

We have the following existence result for generalized solutions of quasi-variational inequality (5.0.4).

Theorem 5.2.4 ([15, Theorem 3]). *Let C be a non-empty, closed and convex subset of the reflexive Banach space X and let $E : C \rightrightarrows C$ be a set-valued mapping with non-empty, closed and convex values. Further, define the set-valued mapping $S : C \rightrightarrows C$ by*

$$S(u) := \big\{x \in C \mid x \in E(u) \text{ and there exists } w(u) \in F(u,x) \\ \text{such that } \langle w(u) - f, y - x\rangle \geq 0, \text{ for every } y \in E(u)\big\},$$

for every $u \in C$, where we assume that the following conditions hold:

(i) *$F : X \times X \rightrightarrows X^*$ is a semi-monotone and bounded mapping.*

(ii) *$E : C \rightrightarrows C$ is Mosco-continuous and there exists $\bar{u} \in C$ such that $\operatorname{int} E(\bar{u}) \neq \emptyset$.*

(iii) *For every $u \in C$, there exist constants $\tau_1, \tau_2 > 0$ and $m(u) \in E(u)$ such that*

$$\|m(u)\|_X \leq \tau_1 \|u\|_X + \tau_2,$$

and

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in S(u)}} \inf_{\substack{w \in F(u,x) \\ \tilde{w} \in \partial\chi_{E(u)}(x)}} \frac{\langle w + \tilde{w}, x - m(u)\rangle}{\|x\|_X} = +\infty.$$

Under the above assumptions, quasi-variational inequality (5.0.4) has a generalized solution, that is, problem (5.2.1) has a solution.

Proof. The proof of this theorem is based on Lemma 5.2.3. We will therefore show that the set M is non-empty and weak sequentially compact.

(I). Let us show that M is well-defined and therefore non-empty. Let $\bar{u} \in C$ be given by assumption (ii). Let us show that there exists $\bar{x} \in C$ with $\bar{x} \in S(\bar{u})$. In other words, we have to show that the following variational inequality with parameter $\bar{u}$ has a solution: Find $\bar{x} \in E(\bar{u})$ and $w(\bar{u}) \in F(\bar{u}, \bar{x})$ such that

$$\langle w(\bar{u}) - f, y - \bar{x}\rangle \geq 0, \qquad \text{for every} \quad y \in E(\bar{u}). \tag{5.2.4}$$

It is easily seen that problem (5.2.4) is equivalent to the following inclusion problem:

$$\bar{x} \in E(\bar{u}) : \quad f \in F(\bar{u}, \bar{x}) + \partial\chi_{E(\bar{u})}(\bar{x}), \tag{5.2.5}$$

where $\partial\chi_{E(\bar{u})}(\bar{x})$ denotes the subdifferential of the indicator mapping $\chi_{E(\bar{u})} : X \to \mathbb{R} \cup \{+\infty\}$, that is, $\chi_{E(\bar{u})}(x) = 0$ for $x \in E(\bar{u})$ and $\chi_{E(\bar{u})}(x) = +\infty$ else. Recall that $\partial\chi_{E(\bar{u})} : X \rightrightarrows X^*$ is maximal monotone; compare Proposition 32.17 in [167]. Since A is semi-monotone, $F(\bar{u}, \cdot)$ is maximal monotone. Further, due to

$$\mathcal{D}(F(\bar{u}, \cdot)) \cap \operatorname{int} \mathcal{D}(\partial\chi_{E(\bar{u})}(\cdot)) = X \cap \operatorname{int} E(\bar{u}) \neq \emptyset,$$

see (SM1) and assumption (ii), the sum $F(\bar{u},\cdot)+\partial\chi_{E(\bar{u})}(\cdot)$ is maximal monotone; compare Theorem 32.I in [167]. Consequently, assumption (iii) yields

$$\mathcal{R}\big(F(\bar{u},\cdot)+\partial\chi_{E(\bar{u})}(\cdot)\big)=X^*,$$

see [167, Corollary 32.35], ensuring that problem (5.2.5) has a solution. Thus, M is non-empty.

(II). Let us show that M is weak sequentially closed. For this purpose, let $\{(u_n,x_n)\}\subseteq M$ be a sequence such that $u_n\rightharpoonup u$ and $x_n\rightharpoonup x$. In other words, we have $x_n\in S(u_n)$, $x_n\in C$, $u_n\in C$ and $\|x_n-u_n\|_X\leq\|\bar{x}-\bar{u}\|_X$. Since C is closed and convex, the set is weak sequentially closed and thus $x\in C$ and $u\in C$; compare Proposition 2.1.15. The weak sequentially lower semicontinuity of $\|\cdot\|_X$ further yields

$$\|x-u\|_X\leq\liminf_{n\to+\infty}\|x_n-u_n\|_X\leq\|\bar{x}-\bar{u}\|_X.$$

Since it holds that $x_n\in S(u_n)$, we infer $x_n\in E(u_n)$ and there exists $w_n\in F(u_n,x_n)$ such that

$$\langle w_n-f,y-x_n\rangle\geq 0,\qquad\text{for every}\quad y\in E(u_n).\tag{5.2.6}$$

Note that $w_n\rightharpoonup w$ for an appropriate subsequence since F is bounded. Since E is Mosco-continuous, by (M1), $x\in E(u)$. Further, by (M2), there exists a sequence $\{\tilde{x}_n\}$ with $\tilde{x}_n\in E(u_n)$ and $\tilde{x}_n\to x$. We therefore have

$$\begin{aligned}\limsup_{n\to+\infty}\langle w_n,x_n\rangle&=\limsup_{n\to+\infty}\left[\langle w_n,x_n-\tilde{x}_n\rangle+\langle w_n,\tilde{x}_n\rangle\right]\\&\leq\limsup_{n\to+\infty}\left[\langle f,x_n-\tilde{x}_n\rangle+\langle w_n,\tilde{x}_n\rangle\right]\\&=\langle w,x\rangle.\end{aligned}$$

The previous relation and the semi-continuity of F imply [102]

$$\langle w_n,x_n\rangle\ \to\ \langle w,x\rangle\qquad\text{and}\qquad w\in F(u,x).$$

Now, let $z\in E(u)$ be arbitrarily chosen. Then, there exists a sequence $\{z_n\}$ such that $z_n\in E(u_n)$ and $z_n\to z$. Inserting this sequence in the parametric variational inequality (5.2.6) and passing to the limit yields

$$\begin{aligned}\lim_{n\to+\infty}\left[\langle w_n-f,z_n-x_n\rangle\right]&=\lim_{n\to+\infty}\left[\langle w_n,z_n\rangle-\langle w_n,x_n\rangle-\langle f,z_n-x_n\rangle\right]\\&=\langle w-f,z-x\rangle.\end{aligned}$$

Since $z\in E(u)$ was chosen arbitrarily, we conclude

$$\langle w-f,z-x\rangle\geq 0,\qquad\text{for every}\quad z\in E(u).$$

But this shows $x\in S(u)$, that is, M is weak sequentially closed.

(III). Let us show that M is bounded. For this purpose, let $(u,x) \in M$, that is, $x \in S(u)$ and $\|x-u\|_X \leq \|\bar{x}-\bar{u}\|_X$. It is easily seen from (5.2.6), that for $x \in E(u)$, there exists $w \in F(u,x)$ and $\tilde{w} \in \partial\chi_{E(u)}(x)$ such that

$$\langle w+\tilde{w}, x-y\rangle \leq \langle f, x-y\rangle, \qquad \text{for every} \quad y \in E(u). \tag{5.2.7}$$

Further, the inequality

$$\|u\|_X \leq \|x-u\|_X + \|x\|_X \leq \|\bar{x}-\bar{u}\|_X + \|x\|_X.$$

shows that necessary $\|x\|_X \to +\infty$ if M is unbounded. Assume to the contrary that M is unbounded. Inserting the element $m(u) \in E(u)$, given by assumption (iii), in inequality (5.2.7) and rearranging yields

$$\begin{aligned}
\frac{\langle w+\tilde{w}, x-m(u)\rangle}{\|x\|_X} &\leq \frac{\langle f, x-m(u)\rangle}{\|x\|_X} \\
&\leq \|f\|_{X^*}\frac{\|x-m(u)\|_X}{\|x\|_X} \\
&\leq \|f\|_{X^*}\left(1+\frac{\tau_1\|u\|_X+\tau_2}{\|x\|_X}\right) \\
&\leq \|f\|_{X^*}\left(1+\frac{\tau_1\big(\|\bar{x}-\bar{u}\|_X+\|x\|_X\big)+\tau_2}{\|x\|_X}\right).
\end{aligned}$$

Consequently, the above inequality implies

$$\lim_{\|x\|_X\to+\infty}\frac{\langle w+\tilde{w}, x-m(u)\rangle}{\|x\|_X} \leq \text{const} < +\infty,$$

which contradicts the coercivity condition (iii). Thus, M is bounded. We have finally shown that the set M is weak sequentially closed and bounded. Since X is reflexive, M is weak sequentially compact and the proof is complete. $\square$

5.2.2 Generalized solutions of generalized vector quasi-variational inequalities

In this section, we investigate generalized solutions of generalized vector quasi-variational inequality (5.0.6).

Definition 5.2.5 ([15, Definition 3]). Let X and Y be real Banach spaces and let $W : X \rightrightarrows Y$ be a given set-valued mapping. W is said to be *weak-strong sequentially closed* if for every sequence $\{(x_n,y_n)\} \subseteq \mathcal{G}(W)$ with $x_n \rightharpoonup x$ and $y_n \to y$, it holds that $(x,y) \in \mathcal{G}(W)$.

Example 5.2.6 ([15, Example 1]). Let X be a real Banach space. Then the set-valued mapping $W : X \rightrightarrows X$, given by (4.1.7), is weak-strong sequentially closed if the operator $T : X \to X^*$ is strongly continuous, that is, $x_n \rightharpoonup x$ implies $Tx_n \to Tx$, for example linear and compact.

We have the following existence result for generalized solutions of generalized vector quasi-variational inequality (5.0.6).

Theorem 5.2.7 ([15, Theorem 1])**.** *Let X and Y be real reflexive Banach spaces, let C be a non-empty, closed and convex subset of X. Let $\mathcal{K} : X \rightrightarrows Y$ be a variable ordering structure and let $E : C \rightrightarrows C$ be a set-valued mapping with non-empty, closed and convex values and let $f \in \mathrm{L}(X,Y)$. Further, define the set-valued mapping $S : C \rightrightarrows C$ by*

$$S(u) := \big\{x \in C \mid x \in E(u) \text{ and there exists } w \in F(x) \text{ such that } \\ \langle w - f, y - x\rangle \notin -\operatorname{int}\mathcal{K}(u) \text{ for every } y \in E(u)\big\},$$

for every $u \in C$, where we assume that the following conditions hold:

(i) *$F : X \rightrightarrows \mathrm{L}(X,Y)$ is a bounded set-valued mapping. For every sequence $\{z_n\} \subseteq X$ with $z_n \to z$ and every sequence $\{(x_n, w_n)\} \subseteq \mathcal{G}(F)$ with $x_n \rightharpoonup x$ and $w_n \rightharpoonup w$, we have*

$$\langle w_n - f, z_n - x_n\rangle \to \langle w - f, z - x\rangle \qquad \textit{and} \qquad (x,w) \in \mathcal{G}(F).$$

(ii) *E is Mosco-continuous.*

(iii) *The set-valued mapping $W : X \rightrightarrows Y$, defined by $W(x) := Y \setminus (-\operatorname{int}\mathcal{K}(x))$ for every $x \in E$, is weak-strong sequentially closed.*

(iv) *For every $u \in C$, there exist constants $\tau_1, \tau_2 > 0$ and an element $m(u) \in E(u)$ such that*

$$\|m(u)\|_X \leq \tau_1\|u\|_X + \tau_2. \tag{5.2.8}$$

Further it holds that

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in S(u)}} \frac{\langle s \circ w, x - m(u)\rangle}{\|x\|_X} = +\infty, \tag{5.2.9}$$

for every $w \in F(x) \setminus \{0\}$ and every $s \in \mathcal{K}^*(u) \setminus \{0\}$, where $\mathcal{K}^*(u)$ denotes the dual cone of $\mathcal{K}(u)$.

(v) *The set M, defined in (5.2.2), is well-defined and therefore non-empty.*

Under the above assumptions, generalized vector quasi-variational inequality (5.0.6) has a generalized solution, that is, problem (5.2.1) has a solution.

Proof. The proof of this theorem is again based on Lemma 5.2.3. We will therefore show that M is weak sequentially closed and bounded. Notice that M is non-empty; see assumption (v).

(I). Let us show that M is weak sequentially closed. For this purpose, let $\{(u_n, x_n)\} \subseteq M$ be a sequence such that $u_n \rightharpoonup u$ and $x_n \rightharpoonup x$. In other words, we have $x_n \in S(u_n)$, $x_n \in C$, $u_n \in C$ and $\|x_n - u_n\|_X \leq \|\bar{x} - \bar{u}\|_X$. Adapting the arguments in the proof of Theorem 5.2.4, we have $u \in C$, $x \in C$ and $\|x-u\|_X \leq \|\bar{x}-\bar{u}\|_X$. Thus, it remains to show that $x \in S(u)$. We further have $x_n \in E(u_n)$ and there exists some operator $w_n \in F(x_n)$ such that

$$\langle w_n - f, y - x_n \rangle \notin -\operatorname{int} \mathcal{K}(u_n), \qquad \text{for every} \quad y \in E(u_n).$$

Since $w_n \in F(x_n)$ and F is bounded, the reflexivity of $\mathrm{L}(X,Y)$ ensures that there exists a weakly convergent subsequence, again denoted by $\{w_n\}$, with $w_n \rightharpoonup w$. Since we have $x_n \in E(u_n)$, $x_n \rightharpoonup x$ and $u_n \rightharpoonup u$, the Mosco-continuity of E, see (M1), yields $x \in E(u)$. Let $z \in E(u)$ be arbitrarily chosen. By (M2), we can find a sequence $\{z_n\} \subseteq C$ such that $z_n \in E(u_n)$ and $z_n \to z$. Inserting z_n in the above parametric vector variational inequality yields

$$\langle w_n - f, z_n - x_n \rangle \notin -\operatorname{int} \mathcal{K}(u_n).$$

By assumption (i), we are able to pass in the above inequality to the limit. We therefore have

$$\langle w_n - f, z_n - x_n \rangle \ \to \ \langle w - f, z - x \rangle \qquad \text{and} \qquad (x, w) \in \mathcal{G}(F).$$

The weak-strong sequential continuity of W, see assumption (iii), ensures

$$\langle w - f, z - x \rangle \notin -\operatorname{int} \mathcal{K}(u).$$

Since $z \in E(u)$ was chosen arbitrarily, we have shown that for some $w \in F(x)$ it holds that

$$\langle w - f, z - x \rangle \notin -\operatorname{int} \mathcal{K}(u), \qquad \text{for every} \quad z \in E(u),$$

that is, $x \in S(u)$. This show the weak sequentially closedness of M.

(II). Let us show that M is bounded. Let $(u,x) \in M$ be arbitrarily chosen. In other words, we have $x \in E(u)$, $\|x-u\|_X \leq \|\bar{x}-\bar{u}\|_X$ and there exists $w \in F(x)$ such that

$$\langle w - f, y - x \rangle \notin -\operatorname{int} \mathcal{K}(u), \qquad \text{for every} \quad y \in E(u). \tag{5.2.10}$$

This obviously implies

$$\langle w - f, x \rangle \notin \langle w - f, E(u) \rangle + \operatorname{int} \mathcal{K}(u), \tag{5.2.11}$$

where we define the convex set $\langle w - f, E(u) \rangle := \{\langle w - f, y \rangle \mid y \in E(u)\}$. Note that the right-hand side of (5.2.11) is a convex set with non-empty interior. By a separation theorem for convex sets [98], we can find a linear functional $s \in Y^* \setminus \{0\}$, depending on $u \in C$, such that

$$s\left(\langle w - f, x \rangle\right) \leq s\left(z\right) + s(k), \qquad \text{for every} \quad z \in \langle w - f, E(u) \rangle, \ k \in \operatorname{int} \mathcal{K}(u). \tag{5.2.12}$$

Since s is continuous and $\operatorname{int}\mathcal{K}(u)$ is closed and solid, it is easily seen that s belongs to the dual cone of $\mathcal{K}(u)$; compare also [10]. In particular, (5.2.12) holds for every element in $\mathcal{K}(u)$, implying that we have

$$\langle s \circ w - s \circ f, y - x \rangle \geq 0, \qquad \text{for every} \quad y \in E(u).$$

By the coercivity assumption (iv), there exists an element $m(u) \in E(u)$ and positive constants τ_1 and τ_2 such that

$$\|m(u)\|_X \leq \tau_1 \|u\|_X + \tau_2.$$

Inserting $m(u)$ in the above variational inequality, we have

$$\begin{aligned}
\frac{\langle s \circ w, x - m(u) \rangle}{\|x\|_X} &\leq \frac{\langle s \circ f, x - m(u) \rangle}{\|x\|_X} \\
&\leq \|s \circ f\|_{X^*} \frac{\|x - m(u)\|_X}{\|x\|_X} \\
&\leq \|s \circ f\|_{X^*} \left(1 + \frac{\tau_1 \|u\|_X + \tau_2}{\|x\|_X} \right) \\
&\leq \|s \circ f\|_{X^*} \left(1 + \frac{\tau_1 \big(\|\bar{x} - \bar{u}\|_X + \|x\|_X \big) + \tau_2}{\|x\|_X} \right),
\end{aligned} \tag{5.2.13}$$

where we again used

$$\|u\|_X \leq \|x - u\|_X + \|x\|_X \leq \|\bar{x} - \bar{u}\|_X + \|x\|_X.$$

Now, assume to the contrary that M is unbounded. Clearly, the previous inequality already shows that we have $\|x\|_X \to +\infty$ necessarily. However, passing in (5.2.13) to the limit yields

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in E(u)}} \frac{\langle s \circ w, x - m(u) \rangle}{\|x\|_X} \leq \text{const} < +\infty.$$

This contradicts the coercivity condition (5.2.9), implying that the set M is bounded.

(III). We have finally shown that the set M is weak sequentially closed and bounded. Since X is reflexive, M is weak sequentially compact. The proof is complete. □

Remark 5.2.8. (i) Evidently, assumption (iv) in the previous theorem can be dropped if the constraining set C is bounded. In this case, the boundedness of $M \subseteq C \times C$ trivially holds.

(ii) Since the proof of Theorem 5.2.7 uses a linear separation for the parametric vector variational inequality (5.2.10), we have to assume that (5.2.9) holds for all non-trivial functionals s in the dual cone of $\mathcal{K}(u)$. Let us denote by $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$, $\varphi(y) := \inf\{t \in \mathbb{R} \mid y \in te - \mathcal{K}(u)\}$ for $y \in Y$, the non-linear Tammer-Weidner function, where $e \in \operatorname{int}\mathcal{K}(u)$ is a fixed element. By using φ, we can replace (5.2.9) with the following coercivity condition: For all $u \in C$ there exists $m(u) \in E(u)$ such that

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in S(u)}} \frac{\varphi\big(\langle w - f, x - m(u)\rangle\big)}{\|x\|_X} = +\infty,$$

for every $w \in F(x)$. This follows from the fact that the parametric vector variational inequality (5.2.10) is equivalent to the following scalar problem: Find $x \in E(u)$ such that for some $w \in F(x)$ it holds that

$$\varphi\big(\langle w - f, y - x\rangle\big) \geq 0, \qquad \text{for every} \quad y \in E(u);$$

compare also Proposition 3.2.8.

We further have the following corollary.

Corollary 5.2.9. *Theorem 5.2.7 remains correct if in addition $F : X \rightrightarrows \mathrm{L}(X, Y)$ is K-monotone, where $K := \bigcap_{z \in X} \mathcal{K}(z)$, and condition (5.2.9) is relaxed in the following way: For all $u \in C$ it holds that*

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in S(u)}} \frac{\langle s \circ \tilde{w}, x - m(u)\rangle}{\|x\|_X} = +\infty,$$

for every $\tilde{w} \in F(u) \setminus \{0\}$ and every $s \in \mathcal{K}^*(u) \setminus \{0\}$.

In what follows, $\|\cdot\|$ denotes any norm in $\mathbb{R}^l$.

Corollary 5.2.10. *Let C be a non-empty, closed and convex subset of $\mathbb{R}^l$ and suppose that $E : C \rightrightarrows C$ is a set-valued mapping with non-empty, closed and convex values. Further, define the set-valued mapping $S : C \rightrightarrows C$ by*

$$S(u) := \left\{ x \in C \mid x \in E(u) \text{ and } \begin{pmatrix} \langle F_1 x, y - x\rangle \\ \vdots \\ \langle F_k x, y - x\rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^k_{\geq}, \text{ for every } \; y \in E(u) \right\},$$

for every $u \in C$, where we assume that the following conditions hold:

(i) *$F : \mathbb{R}^l \to \mathrm{Mat}_{k \times l}(\mathbb{R})$ is continuous.*

(ii) *E is Mosco-continuous.*

(iii) *For every $u \in C$, there exist constants $\tau_1, \tau_2 > 0$ and an element $m(u) \in E(u)$ such that*

$$\|m(u)\| \leq \tau_1 \|u\| + \tau_2. \tag{5.2.14}$$

Further it holds that

$$\lim_{\substack{\|x\| \to +\infty \\ x \in S(u)}} \frac{\langle s^\top F x, x - m(u)\rangle}{\|x\|} = +\infty, \tag{5.2.15}$$

for every $s \in \mathbb{R}^k_{\geq} \setminus \{0\}$.

(iv) *The set* M*, defined in* (5.2.2)*, is well-defined and therefore non-empty.*

Under the above assumptions, the finite-dimensional vector quasi-variational inequality (5.1.14) *has a generalized solution, that is, problem* (5.2.1) *has a solution.*

In the following, we are going to consider the particular case of Theorem 5.2.7, where $Y = \mathbb{R}$ and $K(x) = \mathbb{R}_{\geq}$ for every $x \in X$. The next corollary turns out to be a special case of the results in [97]; compare the next remark.

Corollary 5.2.11. *Let* C *be a non-empty, closed and convex subset of the reflexive Banach space* X *and let* $E : C \rightrightarrows C$ *be a set-valued mapping with non-empty, closed and convex values. Further, define the set-valued mapping* $S : C \rightrightarrows C$ *by*

$$S(u) := \big\{x \in C \mid x \in E(u) \text{ and there exists } w \in F(x) \text{ such that } \langle w - f, y - x\rangle \geq 0 \text{ for every } y \in E(u)\big\},$$

for every $u \in C$*, where we assume that the following conditions hold:*

(i) $F : X \rightrightarrows X^*$ *is a bounded set-valued mapping with non-empty values. For every sequence* $\{(x_n, w_n)\} \subseteq \mathcal{G}(F)$ *with* $x_n \rightharpoonup x$ *and* $w_n \rightharpoonup w$*, we have*

$$\langle w_n, x_n\rangle \to \langle w, x\rangle \qquad \text{and} \qquad (x, w) \in \mathcal{G}(F).$$

(ii) $E : C \rightrightarrows C$ *is Mosco-continuous.*

(iii) *For every* $u \in C$*, there exist constants* $\tau_1, \tau_2 > 0$ *and* $m(u) \in E(u)$ *such that*

$$\|m(u)\|_X \leq \tau_1 \|u\|_X + \tau_2.$$

Further it holds that

$$\lim_{\substack{\|x\|_X \to +\infty \\ x \in S(u)}} \frac{\langle w, x - m(u)\rangle}{\|x\|_X} = +\infty,$$

for every $w \in F(x) \setminus \{0\}$.

(iv) *The set* M*, defined in* (5.2.2)*, is well-defined and therefore non-empty.*

Under the above assumptions, quasi-variational inequality (5.0.3) *has a generalized solution, that is, problem* (5.2.1) *has a solution.*

Remark 5.2.12. In [97], Jadamba and Khan studied the existence of generalized solutions of pseudomonotone quasi-variational inequalities. In their paper, compare Theorem 2 in [97], the authors assumed the following conditions instead of the conditions (i) and (iv) in Corollary 5.2.11:

(P1) The mapping $F : X \rightrightarrows X^*$ has closed, convex and bounded values.

(P2) F is finitely continuous, that is, for any finite-dimensional subspace $Z \subseteq X$, the restriction $F : X \cap Z \rightrightarrows X^*$ is upper semicontinuous.

(P3) For any sequence $\{(x_n, w_n)\}$ in $\mathcal{G}(F)$ with $x_n \rightharpoonup x$ and $\limsup_{n\to+\infty}\langle w_n, x_n - x\rangle \leq 0$, it holds that for each $y \in \mathcal{D}(F)$, there exists $w(y) \in F(x)$ satisfying

$$\liminf_{n\to+\infty} \langle w_n, x_n - y\rangle \geq \langle w(y), x - y\rangle.$$

(P4) For each $y \in X$ and each bounded subset $B \subseteq X$, there exists a constant $N(B, y)$ such that $\langle w, x - y\rangle \geq N(B, y)$ for every $(w, x) \in \mathcal{G}(F)$ with $x \in B$.

However, one can show [103] that a mapping F satisfying (P1), (P3) and (P4) is generalized pseudomonotone in the following sense: The mapping $F : X \rightrightarrows X^*$ is generalized pseudomonotone, if it satisfies (P1) and (P3) defined above, and for any sequence $\{(x_n, w_n)\} \subseteq \mathcal{G}(F)$ with $x_n \rightharpoonup x$ and $w_n \rightharpoonup w$ such that $\limsup_{n\to+\infty}\langle w_n, x_n - x\rangle \leq 0$, we have

$$\langle w_n, x_n\rangle \to \langle w, x\rangle \qquad \text{and} \qquad (x, w) \in \mathcal{G}(F).$$

The following result gives a necessary condition for the non-emptyness of the set M, defined by (5.2.2), in Theorem 5.2.7.

Theorem 5.2.13. *Let X and Y be real reflexive Banach spaces and let C be a non-empty, closed and convex subset of X. Let $\mathcal{K} : X \rightrightarrows Y$ be a variable ordering structure. Further let $E : C \rightrightarrows C$ be a set-valued mapping with non-empty, closed and convex values and let $f \in \mathrm{L}(X, Y)$. Let $\bar{u} \in C$ and assume that the following conditions hold:*

(i) *The set-valued mapping $F : X \rightrightarrows \mathrm{L}(X, Y)$ has non-empty and compact values and is $\mathcal{K}(\bar{u})$-monotone and generalized v-hemicontinuous.*

(ii) *There exists a non-empty and compact subset $B(\bar{u}) \subseteq X$ and an element $y(\bar{u}) \in B(\bar{u}) \cap E(\bar{u})$ such that for every $y \in B(\bar{u}) \setminus E(\bar{u})$, there exists $w(y) \in F(y)$ such that*

$$\langle w(y) - f, y(\bar{u}) - y\rangle \in -\operatorname{int}\mathcal{K}(\bar{u}).$$

Then, there exists an element $\bar{x} \in E(\bar{u})$ such that $\bar{x} \in S(\bar{u})$, where S is given as in Theorem 5.2.7. In other words, the set M, defined in Theorem 5.2.7, is non-empty.

Proof. Let $\bar{u} \in C$. For simplicity of notation, we define

$$\tilde{K} := \mathcal{K}(\bar{u}), \quad \tilde{E} := E(\bar{u}) \quad \text{and} \quad \tilde{B} := B(\bar{u}),$$

where the element $\bar{u} \in C$ is given by the assumptions of this theorem. In order to show that M is non-empty, it is sufficient to show that the following vector variational inequality has a solution: Find $\bar{x} \in \tilde{E}$ such that for some $w \in F(\bar{x})$ we have

$$\langle w - f, y - \bar{x}\rangle \notin -\operatorname{int}\tilde{K}, \qquad \text{for every} \quad y \in \tilde{E}. \tag{5.2.16}$$

Let us define two set-valued mappings $G, G' : \tilde{E} \rightrightarrows X$ by

$$G(y) := \{x \in \tilde{E} \mid \text{there is } w \in F(x) \text{ such that } \langle w - f, y - x\rangle \notin -\operatorname{int} \tilde{K}\},$$
$$G'(y) := \{x \in \tilde{E} \mid \text{there is } \tilde{w} \in F(y) \text{ such that } \langle \tilde{w} - f, y - x\rangle \notin -\operatorname{int} \tilde{K}\},$$

for every $y \in \tilde{E}$. In what follows, we are going to show that the set

$$\bigcap_{y \in \tilde{E}} G'(y) \tag{5.2.17}$$

is non-empty, using the Fan-KKM Lemma 3.3.1.

(I). For this purpose, let us first show that G is a KKM-mapping whose definition is given in Lemma 3.3.1 (i). Indeed, suppose to the contrary that there are a natural number k and elements $x_1, \ldots, x_k \in \tilde{E}$, and $\lambda_j \geq 0$ for $j = 1, \ldots, k$ with $\sum_{j=1}^k \lambda_j = 1$ such that

$$\sum_{j=1}^k \lambda_j x_j \notin \bigcup_{j=1}^k G(x_j).$$

Let us define $x := \sum_{j=1}^k \lambda_j x_j$. Since $x \notin G(x_j)$ for $j = 1, \ldots, k$, we consequently obtain for every $w \in F(x)$,

$$\langle w - f, x_j - x\rangle \in -\operatorname{int} \tilde{K}, \qquad \text{for} \quad j = 1, \ldots, k.$$

Thus, summing up all relations, we have

$$\sum_{j=1}^k \lambda_j \langle w - f, x_j\rangle \in \langle w - f, x\rangle - \operatorname{int} \tilde{K}. \tag{5.2.18}$$

However, the left hand side of the above relation is equivalent to $\langle w - f, x\rangle$. Relation (5.2.18) consequently states $0 \in \operatorname{int} \tilde{K}$, which is impossible since $\tilde{K}$ is a proper cone. We have therefore shown that G is a KKM-mapping. As an easy consequence,

$$G(y) \subseteq G'(y), \qquad \text{for every} \quad y \in \tilde{E}.$$

Indeed, let $y \in \tilde{E}$ and $x \in G(y)$. Then there is $w \in F(x)$ such that $\langle w - f, y - x\rangle \notin -\operatorname{int} \tilde{K}$. Since F is $\tilde{K}$-monotone, see assumption (i), we further have $\langle \tilde{w} - w, y - x\rangle \in \tilde{K}$ for every $\tilde{w} \in F(y)$. Adding these inequalities yields

$$\langle \tilde{w} - f, y - x\rangle \notin -\operatorname{int} \tilde{K};$$

compare Proposition 2.4.12. Thus, $x \in G'(y)$. Consequently, G' is a KKM-mapping as well. Note that $G'(y) \neq \emptyset$ for every $y \in \tilde{E}$ since $y \in G'(y)$.

(II). Let us show that the values of G' are closed. Indeed, let $y \in \tilde{E}$ and let $\{x_n\} \subseteq G'(y)$ be a sequence such that $x_n \to x$. Obviously $x \in \tilde{E}$. By definition of x_n, there exists $\tilde{w}_n \in F(y)$ such that

$$\langle \tilde{w}_n - f, y - x_n \rangle \notin -\operatorname{int} \tilde{K}.$$

Since the set $F(y)$ is compact in $\mathrm{L}(X,Y)$, there is a subsequence, again denoted by $\{\tilde{w}_n\}$, such that $\tilde{w}_n \to \tilde{w}$ w.r.t. $\|\cdot\|_{\mathrm{L}(X,Y)}$ and $\tilde{w} \in F(y)$. We further have

$$\begin{aligned}&\big\|\langle \tilde{w}_n - f, y - x_n \rangle - \langle \tilde{w} - f, y - x \rangle\big\|_Y \\ &\leq \|\tilde{w}_n - f\|_{\mathrm{L}(X,Y)} \|x_n - x\|_X + \|\tilde{w}_n - \tilde{w}\|_{\mathrm{L}(X,Y)} \|y - x\|_X.\end{aligned}$$

Consequently, passing in the above inequality to the limit, we have

$$\langle \tilde{w}_n - f, y - x_n \rangle \rightarrow \langle \tilde{w} - f, y - x \rangle,$$

that is, $x \in G'(y)$. This shows that G' has closed values.

(III). Let us show that $G'(y(\bar{u}))$ is compact in X, where the element $y(\bar{u}) \in C$ is given by assumption (ii) of this theorem. Indeed, by the same assumption, we can find a set $\tilde{B} \subseteq X$ such that

$$G'(y(\bar{u})) \subseteq \tilde{B}.$$

As a closed subset of the compact set $\tilde{B}$, we obtain that $G'(y(\bar{u}))$ is compact.

(IV). The previous parts (I), (II) and (III) have shown that G' satisfies all requirements of Lemma 3.3.1. Thus, the set in (5.2.17) is non-empty, that is, there exists an element $\bar{x} \in \tilde{E}$ such that

$$\bar{x} \in \bigcap_{y \in \tilde{E}} G'(y). \tag{5.2.19}$$

In the following we are going to show that $\bar{x}$ is a solution of problem (5.2.16). Indeed, from (5.2.19) follows that for every $y \in \tilde{E}$, there exists $w \in F(y)$ such that

$$\langle w - f, y - \bar{x} \rangle \notin -\operatorname{int} \tilde{K}.$$

In particular, for every $z \in \tilde{E}$ and $\lambda \in (0,1)$, there exists $w_\lambda \in F(\lambda\bar{x} + (1-\lambda)z)$ such that

$$\langle w_\lambda - f, z - \bar{x} \rangle \notin -\operatorname{int} \tilde{K}.$$

The generalized hemicontinuity of F now ensures the existence of an operator $w \in F(\bar{x})$ such that

$$\langle w_\lambda - f, z - \bar{x} \rangle \rightarrow \langle w - f, z - \bar{x} \rangle$$

for $\lambda \downarrow 0$. Since $Y \setminus (-\operatorname{int} \tilde{K})$ is closed, we have finally shown that there is an operator

$w \in F(\bar{x})$ such that

$$\langle w - f, z - \bar{x} \rangle \notin -\operatorname{int} \tilde{K}, \qquad \text{for every} \quad z \in \tilde{E}.$$

In other words, we have shown that $\bar{x} \in \tilde{E}$ solves problem (5.2.16), that is, $\bar{x} \in S(\bar{u})$. Thus, the set M is non-empty and the proof is complete. □

5.3 Applications

Let us come back to Example 3.2.14, where we studied the multi-objective optimization problem (3.2.11). As pointed out in Remark 3.2.15, we may interpret the problem in the following way: Denoting the locations of all hospitals in the city Halle (Saale) by $a^1, \ldots, a^8 \in \mathbb{R}^2$, we are looking for a new location of a main hospital such that the air-line distance between the unknown location and all hospitals a^j, $j \in \{1, \ldots, 8\}$, is minimal simultaneously. As we have seen in Example 3.2.14, the above multi-objective optimization problem can equivalently be stated in the following way: Find a location $x \in \mathbb{R}^2$ such that

$$\begin{pmatrix} \langle x - a^1, y - x \rangle \\ \vdots \\ \langle x - a^8, y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^8_{\geq}, \qquad \text{for every} \quad y \in \mathbb{R}^2. \tag{5.3.1}$$

Furthermore, we have shown that the solution set of problem (3.2.11) with $k = 8$ and problem (5.3.1), respectively, is given by

$$\mathcal{A} := \operatorname{conv}\{a^1, \ldots, a^8\},$$

compare the next figure.

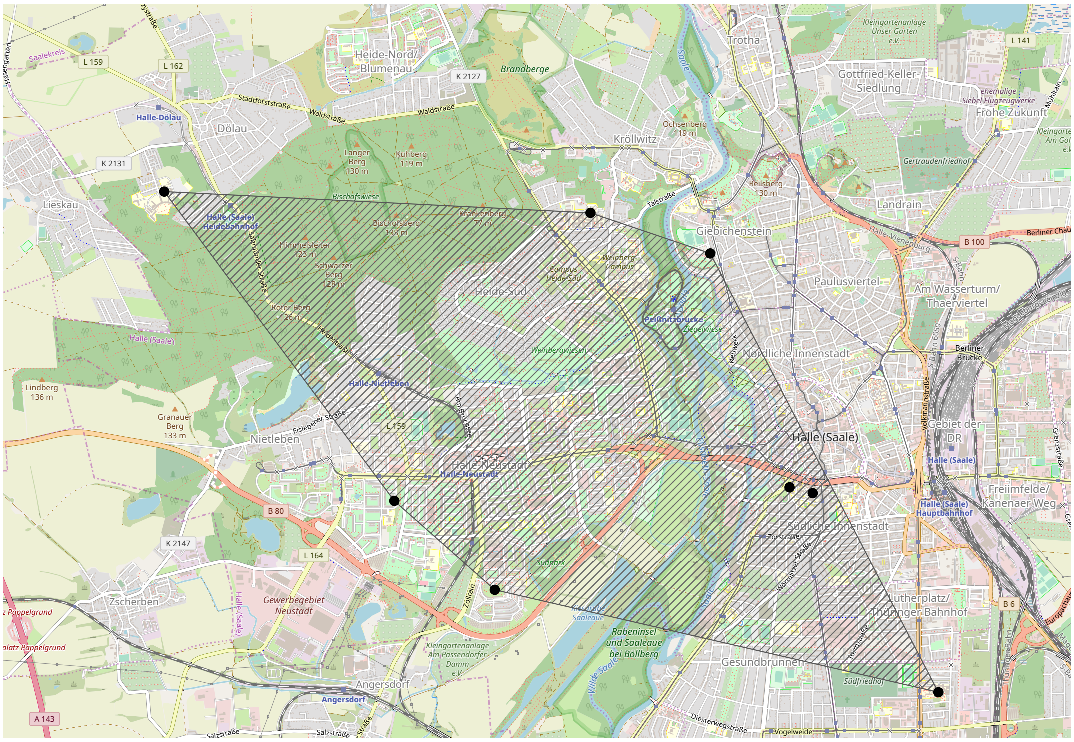


Figure 5.1: Illustration of eight hospitals in Halle (Saale) (black dots) and the corresponding solution set of problem (5.3.1) (gray grid)

From a practical point of view, we cannot expect that every solution in $\mathcal{A}$ is suited, in the sense that one can build a main hospital. As we can see, the solution set $\mathcal{A}$ contains parts of the river Saale or some wooded areas which are surely not suited to build a main hospital; compare Figure 5.1. In order to do so, we will replace problem (5.3.1) with a vector quasi-variational inequality. We therefore decompose $\mathbb{R}^2$ using a finite number of non-empty and disjoint sets $\Omega_1, \ldots, \Omega_m$, where $m \geq 2$, such that

$$\mathbb{R}^2 = \bigcup_{j=1}^{m} \Omega_j.$$

In order to compare the above sets, we introduce a rating function $\mu : \mathbb{R}^2 \to \mathbb{R}_{\geq}$. In what follows, given two sets Ω_j and Ω_k, $j, k \in \{1, \ldots, m\}$, we say that Ω_j is better than Ω_k if and only if $\mu(\Omega_j) \leq \mu(\Omega_k)$. For a non-empty set Ω in $\mathbb{R}^2$, $\mu(\Omega)$ denotes the image of Ω under μ, that is, $\mu(\Omega) := \{\mu(\omega) \in \mathbb{R}_{\geq} \mid \omega \in \Omega\}$. Clearly, there might be some areas which are neither perfectly suited nor completely inappropriate. We should therefore introduce a threshold $\mu_0 > 0$ such that any area Ω_j, $j \in \{1, \ldots, m\}$, with

$$\mu(\Omega_j) \leq \mu_0$$

will be accepted. Conversely, if $\mu_0 < \mu(\Omega_j)$ then Ω_j may be interpreted as forbidden area and therefore rejected.

Now let C be a non-empty subset of $\mathbb{R}^2$, which will be specified in the following, and define a set-valued mapping $E : C \rightrightarrows C$ by

$$E(y) := \{x \in C \mid \mu(y) + \|x\| \leq \mu_0 + \|y\|\}, \qquad \text{for every} \quad y \in C. \tag{5.3.2}$$

Here, $\| \cdot \|$ denotes any norm in $\mathbb{R}^2$. Recall that a fixed-point of E is a point $x \in C$ such that $x \in E(x)$. But this means $\mu(x) + \|x\| \leq \mu_0 + \|x\|$, or equivalently, $\mu(x) \leq \mu_0$. Therefore, any fixed-point of E represents a location which is suited to build a main hospital. Thus, we may replace problem (5.3.1) with the following vector quasi-variational inequality: Find $x \in C$ such that

$$x \in E(x) \qquad \text{and} \qquad \begin{pmatrix} \langle x - a^1, y - x \rangle \\ \vdots \\ \langle x - a^8, y - x \rangle \end{pmatrix} \notin -\operatorname{int} \mathbb{R}^8_{\geq}, \qquad \text{for every} \quad y \in E(x). \tag{5.3.3}$$

In contrast to problem (5.3.1) the above vector quasi-variational inequality aims at finding a suited location $x \in \mathbb{R}^2$ whose distance to a^j, $j \in \{1, \dots, 8\}$, is minimal simultaneously.

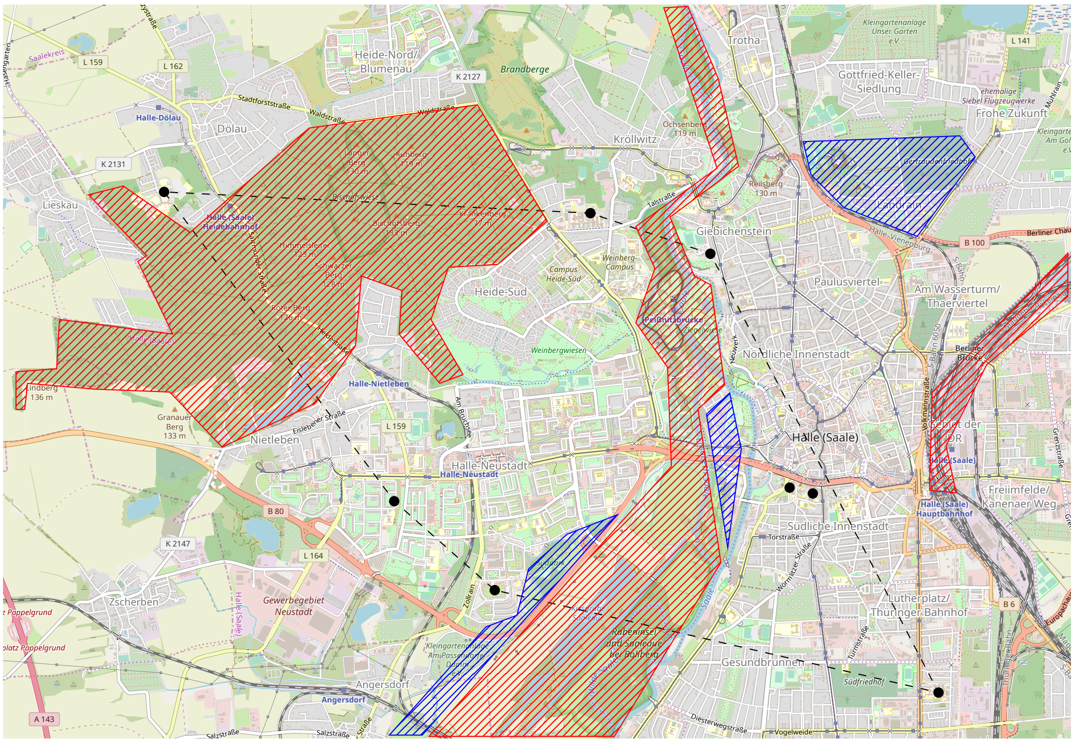


Figure 5.2: Illustration of critical areas: Forbidden areas (red and blue), suitable areas (without color)

Using Corollary 5.1.15, we have the following existence result for problem (5.3.3).

Theorem 5.3.1. *Assume that the following assertions hold:*

(H1) *The utility mapping* $\mu : \mathbb{R}^2 \to \mathbb{R}_{\geq}$ *is continuous.*

(H2) *There exist a number* $r > 0$ *and a ball* $B(0, r)$ *in* $\mathbb{R}^2$ *with* $\mathcal{A} \subseteq B(0, r)$ *and* $0 < \mu_0 + \|y\| - \mu(y) \leq r$ *for every* $y \in \mathbb{R}^2$.

(H3) *The set-valued mapping* $E : C \rightrightarrows C$ *is defined by* (5.3.2)*, where* $C := B(0, r)$ *and* $r > 0$ *is given by hypothesis* (H2).

Then, vector quasi-variational inequality (5.3.3) *has a solution.*

Proof. Let us show that all conditions of Corollary 5.1.15 are satisfied. Clearly, since the mapping F, given by (3.1.4) is $\mathbb{R}^8_{\geq}$-monotone and continuous and C is non-empty, convex and compact, it remains to show that E is Mosco-continuous. Indeed, let $\{x_n\}$ and $\{y_n\}$ be sequences in C with $x_n \in E(y_n)$, $x_n \to x$ and $y_n \to y$. Since $x_n \in E(y_n)$, we infer that

$$\mu(y_n) + \|x_n\| \leq \mu_0 + \|y_n\|.$$

Since μ and $\|\cdot\|$ are continuous, compare (H2), passing to the limit yields $\mu(y) + \|x\| \leq \mu_0 + \|y\|$, or equivalently, $x \in E(y)$. For the second part, let $\{y_n\}$ be a sequence in C with $y_n \to y$. Let $x \in E(y)$. We define a sequence $\{x_n\}$ in $\mathbb{R}^2$ by

$$x_n := \frac{\mu_0 + \|y_n\| - \mu(y_n)}{\mu_0 + \|y\| - \mu(y)}\, x.$$

The sequence is well-defined since the denominator never vanishes, compare hypothesis (H2). Since $x \in E(y)$, we have $\|x\| \leq \mu_0 + \|y\| - \mu(y)$ and consequently

$$\|x_n\| = \frac{\mu_0 + \|y_n\| - \mu(y_n)}{\mu_0 + \|y\| - \mu(y)}\, \|x\| \leq \mu_0 + \|y_n\| - \mu(y_n),$$

which shows $x_n \in E(y_n)$. Finally, by the continuity of μ and $\|\cdot\|$ we conclude $x_n \to x$, showing the Mosco-continuity of E. The proof is complete. $\square$

Remark 5.3.2. (i) It should be noted that the above approach does not require any convexity or compactness of the sets Ω_j, $j \in \{1, \ldots, m\}$.
(ii) In order to apply Corollary 5.1.15, we need to ensure that the values of $E : C \rightrightarrows C$ are non-empty, convex and compact subsets of $\mathbb{R}^2$. Clearly, for every $y \in \mathbb{R}^2$ it holds that $E(y) = B(0, r(y))$ where $r(y) := \mu_0 + \|y\| - \mu(y)$. Evidently, condition (H2) ensures that the values of E are non-empty and bounded.

Chapter 6

Algorithms for Vector Variational Inequalities

Abstract. In this chapter, we study projection type algorithms for vector variational inequalities. Depending on monotonicity and Lipschitz continuity or co-coercivity properties, we propose three algorithms and apply them to some test problems. Then, in the second section of this chapter, we introduce finite-dimensional discrete vector variational inequalities and investigate solution methods. In order to calculate the whole solution set, we propose a naive algorithm and an improved method that uses the Graef-Younes reduction procedure.

6.1 Algorithms for vector variational inequalities

Until now, mostly all research in the field of vector variational inequalities has focused on theoretical results only; see also the comments in Section 1 of [32]. To the best of our knowledge, there are only a handful of papers in which the authors proposed algorithms for the computation of solutions of the finite-dimensional vector variational inequality (3.1.2). In [79, Section 4], Goh and Yang proposed an active-set method where the objective mapping F is assumed to be affine. The fundamental idea of their algorithm is to associate to every scalarized variational inequality a related optimization problem and to apply some active-set method. In contrast, the algorithms in Chen [32] and Chen, Pu and Wang [42] are based on proximal-type methods where for $n \in \mathbb{N}_0$ an element x_{n+1} is generated by the following inclusion problem:

$$0 \in s_n^\top F(x_{n+1}) + s_n^\top G_C(x_{n+1}) + \varepsilon_n d(x_{n+1} - x_n).$$

In the above, $G_C : \mathbb{R}^l \rightrightarrows \mathrm{Mat}_{k\times l}(\mathbb{R})$ with $G(x) := \{U \in \mathrm{Mat}_{k\times l}(\mathbb{R}) \mid U(y-x) \notin \operatorname{int} \mathbb{R}^k_{\geq}$, for every $y \in C\}$ for $x \in C$ denotes the (set-valued) weak normal mapping of the non-empty, closed and convex set C, see [32, Definition 2.8], $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$,

$s_n \in \mathbb{R}^k_{\geq} \cap B(0,1)$, $\varepsilon_n > 0$ and $d : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}_{\geq}$ is some function. For example, in [32, Section 3], d is given by $d(x,y) := x - y$ for $x, y \in \mathbb{R}^l$. The author(s) claim that every accumulation point of the sequence $\{x_n\}$ solves finite-dimensional vector variational inequality (3.1.2). Unfortunately, the results [32, Theorem 3.2] and [42, Theorem 3.1] are incorrect without further modifications. This is because the proofs treat s_n as a separating vector for some sets by mistake.

The previous papers have shown that scalarization techniques for vector variational inequalities can result in powerful methods. The purpose of this section is therefore to derive three algorithms for problem (3.1.1) by using a scalarization method. It should be noted that by simple modifications, the algorithms in this section can be used to study several generalizations of problem (3.1.1). Roughly spoken, the main idea of all three algorithms is to apply a linear scalarization and a projection based method for scalar variational inequalities, which are necessary for problem (3.1.1).

To be precise, assume that C is a non-empty, closed and convex subset of the real Hilbert space X with inner product $\langle \cdot, \cdot \rangle$. Suppose further that K is a proper, closed, convex and solid cone in the real Banach space Y and let $F : X \to \mathrm{L}(X, Y)$ be a given mapping. By Proposition 3.2.6, every solution of the following variational inequality solves vector variational inequality (3.1.1), provided it holds that $s \in K^* \setminus \{0\}$: Find $x = x(s) \in C$ such that

$$\langle F_s x, y - x \rangle \geq 0, \qquad \text{for every} \quad y \in C.$$

Here we use the notation $F_s := s \circ F$, that is, F_s maps from X to $X^* \cong X$. Evidently, for any $\rho > 0$, the above variational inequality is equivalent to finding $x \in C$ such that

$$\langle x, y - x \rangle \geq \langle x - \rho^{-1} F_s x, y - x \rangle, \qquad \text{for every} \quad y \in C.$$

Thus, by applying the orthogonal projection $\mathrm{Proj} : X \to X$, the latter variational inequality can equivalently be stated as the following fixed-point problem: Find $x \in C$ such that

$$\mathrm{Proj}(x - \rho^{-1} F_s x) = x. \tag{6.1.1}$$

In other words, any solution of problem (6.1.1) with fixed-point operator $\mathrm{Proj}(I - \rho^{-1} F_s) : X \to X$ solves problem (3.2.3) w.r.t. s and consequently vector variational inequality (3.1.1); compare again Proposition 3.2.6.

The following three projection type algorithms, which will be presented in this section, require the evaluation of the orthogonal projection Proj on the constraining set C. In general, such projection methods are conceptually simple and do not require the use of derivatives. Besides that, if the orthogonal projection on C is easily computable, the methods become very cheap and fast. This is, in particular, the case if $C = X$.

We have the following new method for vector variational inequality (3.1.1), which is motivated by [58, Theorem 12.1.2]. Recall that F_s denotes the composition of a linear and continuous functional $s \in K^* \setminus \{0\}$ with $F : X \to \mathrm{L}(X, Y)$.

Theorem 6.1.1 (Basic projection method)**.** *Let C be a non-empty, closed and convex subset of the real Hilbert space X, let K be a proper, closed, convex and solid cone in the real Banach space Y, let $F : X \to \mathrm{L}(X, Y)$ and let $\rho > 0$. Further let $x_0 \in C$ and $s \in K^* \setminus \{0\}$ and assume that $F_s : X \to X$ is strongly monotone and Lipschitz continuous with modulus $c > 0$ and $L > 0$, respectively. If*

$$L^2 < 2c\rho, \tag{6.1.2}$$

and variational inequality (3.2.3) w.r.t. s has a (unique) solution, then the iterative $\{x_n\}$, given for every $n \in \mathbb{N}_0$ by

$$x_{n+1} := x_{n+1}(\rho, s) := \mathrm{Proj}(x_n - \rho^{-1} F_s x_n), \tag{6.1.3}$$

converges to a solution of vector variational inequality (3.1.1).

Proof. The proof of this theorem is based on Banach's fixed-point theorem and the linear scalarization method for vector variational inequalities. We will therefore show that the operator $\mathrm{Proj}(I - \rho^{-1} F_s) : X \to X$ is a contraction. Let $x, y \in X$. Since Proj is non-expansive, see Theorem 2.1.17 (i), we get

$$\| \mathrm{Proj}(x - \rho^{-1} F_s x) - \mathrm{Proj}(y - \rho^{-1} F_s y)\|_X \leq \|x - \rho^{-1} F_s x - y + \rho^{-1} F_s y\|_X.$$

Further, by the strong monotonicity and Lipschitz continuity of F_s, we deduce

$$\begin{aligned} \|x - \rho^{-1} F_s x - y + \rho^{-1} F_s y\|_X^2 &= \|x - y\|_X^2 - 2\rho^{-1} \langle F_s x - F_s y, x - y\rangle \\ &\quad + \rho^{-2} \|F_s x - F_s y\|_X^2 \\ &\leq \left(1 - 2\rho^{-1} c + \rho^{-2} L^2\right) \|x - y\|_X^2. \end{aligned}$$

Finally, using relation (6.1.2), we have $1 - 2\rho^{-1} c + \rho^{-2} L^2 < 1$, showing that $\mathrm{Proj}(I - \rho^{-1} F_s)$ is a contraction. Applying Banach's fixed-point theorem, see Theorem 2.3.5, the sequence $\{x_n\}$, given by (6.1.3), converges to the unique fixed-point of $\mathrm{Proj}(I - \rho^{-1} F_s)$. This fixed-point is the unique solution of variational inequality (3.2.3) w.r.t. $s \in K^* \setminus \{0\}$ and therefore one of vector variational inequality (3.1.1); see Proposition 3.2.6. The proof is complete. □

Remark 6.1.2. (i) The proof of Theorem 6.1.1 shows that the iterate $\{x_n\}$, given by (6.1.3), converges to the unique fixed-point of $\mathrm{Proj}(I - \rho^{-1} F_s) : X \to X$.
(ii) The convergence rate of $\{x_n\}$ is linear: Indeed, denoting the limit point by x, it follows

$$\begin{aligned} \|x_{n+1} - x\|_X &= \| \mathrm{Proj}(x_n - \rho^{-1} F_s x_n) - \mathrm{Proj}(x - \rho^{-1} F_s x)\|_X \\ &\leq (1 - 2\rho^{-1} c + \rho^{-2} L^2)^{\frac{1}{2}} \|x_n - x\|_X \end{aligned}$$

and therefore

$$\limsup_{n\to+\infty} \frac{\|x_{n+1} - x\|_X}{\|x_n - x\|_X} = (1 - 2\rho^{-1}c + \rho^{-2}L^2)^{\frac{1}{2}} < 1.$$

(iii) In case we have $C = X$, the iterate $\{x_n\}$ becomes $x_{n+1} = x_n - \rho^{-1}F_s x_n$ for $n \in \mathbb{N}_0$.
(iv) If in addition to the assumptions of Theorem 6.1.1 the set C is compact, variational inequality (3.2.3) has a unique solution; compare Theorem 2.2.7. Under the above assumptions, the existence of a solution can be proven by applying Brouwer's fixed-point Theorem 2.3.1 to problem (6.1.1). The uniqueness of the fixed-point follows from the strong monotonicity of F_s and the equivalence of the problems (3.2.3) and (6.1.1).

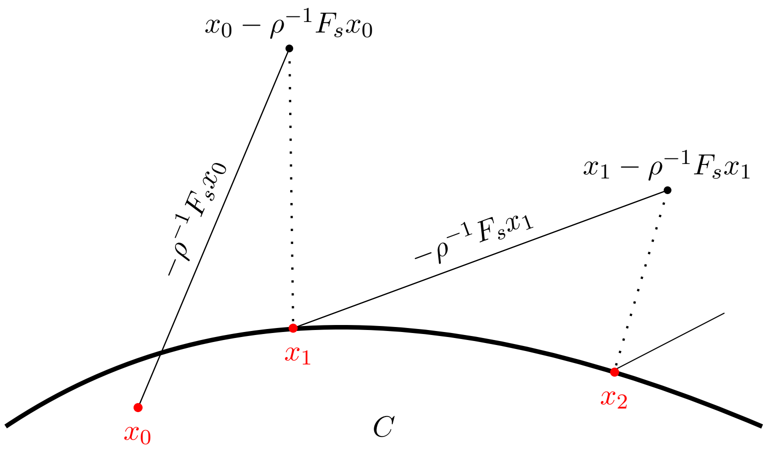


Figure 6.1: Illustration of the basic projection method

Under the assumptions of Theorem 6.1.1, the latter method can be summarized in the following algorithm:

Algorithm 1: Basic projection method.

Result: Calculation of a solution of vector variational inequality (3.1.1).
1 **Input:** The set C, $x_0 \in C$, $s \in K^* \setminus \{0\}$, $F_s : X \to X$ and $\rho > 0$.
2 $\mathrm{Sol(VVI)} \leftarrow \emptyset$.
3 $n \leftarrow 0$.
4 **if** $x_n = \mathrm{Proj}(x_n - \rho^{-1}F_s x_n)$ **then**
5     $\mathrm{Sol(VVI)} \leftarrow \mathrm{Sol(VVI)} \cup \{x_n\}$ **stop**.
6 **else**
7     $x_{n+1} \leftarrow \mathrm{Proj}(x_n - \rho^{-1}F_s x_n)$.
8     $n \leftarrow n + 1$.
9   Go to line 4.
10 **end**
11 **Output:** An element of the solution set Sol(VVI).

Example 6.1.3. Once again we consider the finite-dimensional vector variational inequality (3.1.3). It is easily seen that for $s \in \mathbb{R}^k_{\geq}$ the mapping $F_s : \mathbb{R}^l \to \mathbb{R}^l$ with $F_s x = \sum_{j=1}^k s_j(x - a^j)$ for $x \in \mathbb{R}^l$ is strongly monotone and Lipschitz continuous with modulus $c = \|s\|_1$ and $L = \|s\|_1$, respectively, where $\|s\|_1 = \sum_{j=1}^k s_j$. Let $l = 2$, $k = 4$, $a^1 = (0,0)^\top$, $a^2 = (0,10)^\top$, $a^3 = (10,10)^\top$ and $a^4 = (10,0)^\top$. Further let $s = (1,2,0,3)^\top$, $x_0 = (-1,-1)^\top$ and $\rho = 10$. Thus, (6.1.2) holds and the first elements of $\{x_n\}$, given by (6.1.3), are: $x_0 = (-1,-1)^\top$, $x_1 = (2.6, 1.6)^\top$, $x_2 \doteq (4.04, 2.64)^\top$, $x_3 \doteq (4.616, 3.056)^\top$, $x_4 \doteq (4.8464, 3.2224)^\top$, $x_5 \doteq (4.93856, 3.28896)^\top$, $x_6 = (4.975424, 3.315584)^\top$ and $x_7 = (4.9901696, 3.3262336)^\top$. It follows from the previous theorem that $\{x_n\}$ converges to the unique solution x of variational inequality (3.2.5) w.r.t. the vector $s = (1,2,0,3)^\top$. We further have $\|x_n - x\|_2 < \frac{1}{1000}$ for $n \geq 10$, where

$$x = \sum_{j=1}^{4} \frac{s_j}{\|s\|_1} a^j = (5, 3\tfrac{1}{3})^\top;$$

compare relation (3.2.6) in Example 3.2.7.

Recall that, using the above data the solution set of problem (3.1.3) is given by $\mathrm{S} = \mathrm{conv}\{a^1, a^2, a^3, a^4\} = [0,10]^2$. We have randomly computed 10 000 and 50 000 vectors in $\mathbb{R}^4_{\geq} \setminus \{0\}$, respectively, and applied Theorem 6.1.1. The resulting 10 000 and 50 000 solutions of vector variational inequality (3.1.3) are shown in the next figure.

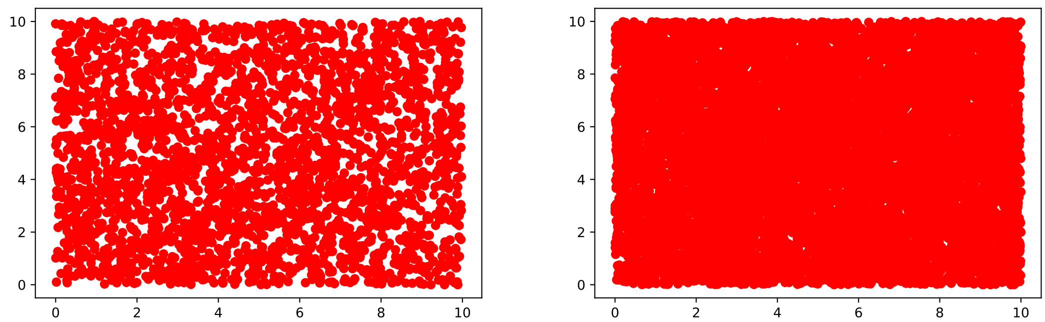


Figure 6.2: Illustration of 10 000 and 50 000 calculated solutions (red dots) of vector variational inequality (3.1.3) by multiple application of the basic projection method

The basic projection method (6.1.3) requires the strong monotonicity and Lipschitz continuity of F_s. But the constants c and L are not typically known in practice. It is therefore desirable to introduce a projection scheme, that does not require the knowledge of such constants. In what follows, the constants are substituted by a so-called co-coercivity constant; compare Section 12.1.2 in [58].

Definition 6.1.4. Let C be a non-empty, closed and convex subset of the real finite-dimensional Hilbert space X. An operator $A : X \to X$ is said to be *co-coercivity* on C

with modulus $\gamma > 0$ if for all $x, y \in C$ it holds that

$$\langle Ax - Ay, x - y\rangle \geq \gamma \|Ax - Ay\|_X^2.$$

Remark 6.1.5. (i) It is easily seen that any strongly monotone and Lipschitz continuous mapping is co-coercive. Conversely, any co-coercive mapping is Lipschitz continuous (but not necessarily strongly monotone).
(ii) The projection operator Proj is co-coercive on X with modulus 1; compare Theorem 2.1.17 (ii).

The following projection method extends the one in Theorem 6.1.1. However, the method is not based on a fixed-point argument and does not require the strong monotonicity and Lipschitz continuity of the operator F_s. In contrast to method (6.1.3), the following scheme uses a variable step size ρ_n. It should be noted that the projection method with variable step size is not a line-search method.

The next result is motived by [58, Theorem 12.1.8]. Notice that we use ρ_n instead of ρ_n^{-1} in formula (6.1.5).

Theorem 6.1.6 (Basic projection method with variable step size)**.** *Let C be a non-empty, closed and convex subset of the real finite-dimensional Hilbert space X, let K be a proper, closed, convex and solid cone in the real Banach space Y, let $F : X \to \mathrm{L}(X,Y)$ and let $\{\rho_n\} \subseteq \mathbb{R}_{>}$. Further let $x_0 \in C$ and $s \in K^* \setminus \{0\}$ and assume that $F_s : X \to X$ is co-coercive on C with modulus $\gamma > 0$. If*

$$0 < \inf_{n\in\mathbb{N}_0} \rho_n \leq \sup_{n\in\mathbb{N}_0} \rho_n < 2\gamma, \tag{6.1.4}$$

and variational inequality (3.2.3) w.r.t. s has a solution, then the iterate $\{x_n\}$, given for every $n \in \mathbb{N}_0$ by

$$x_{n+1} := x_{n+1}(\rho_n, s) := \mathrm{Proj}(x_n - \rho_n F_s x_n), \tag{6.1.5}$$

produces a sequence converging to a solution of vector variational inequality (3.1.1).

Proof. Define the operator $P_n := P_n(\rho_n, s) := I - \mathrm{Proj}(I - \rho_n F_s)$. Since Proj is co-coercive on C with modulus 1, an easy calculation shows that P_n is co-coercive on C with modulus $\gamma_n := 1 - \frac{\rho_n}{4\gamma}$; compare Lemma 12.1.7 in [58]. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by (6.1.5) and starting from $x_0 \in X$ and $y_0 \in \mathrm{Sol}(\mathrm{VI}_s)$, respectively. We then have $y_0 = \mathrm{Proj}(y_0 - \rho_n F_s y_0)$ and therefore $y_n = y_0$ for every $n \in \mathbb{N}$, that is, $\{y_n\}$ is constant. Taking into account the co-coercivity of F_s, a direct calculation yields

$$\|x_{n+1} - y_{n+1}\|_X^2 = \|x_{n+1} - y_0\|_X^2 \leq \|x_n - y_0\|_X^2 - (2\gamma_0 - 1)\|P_n x_n - P_n y_0\|_X^2, \tag{6.1.6}$$

where we let $\gamma_0 := \inf_{n\in\mathbb{N}_0} \gamma_n$. Thus, $\{\|x_n - y_n\|_X\}$ is non-increasing and therefore convergent. By multiple applying inequality (6.1.6), it follows

$$\|x_{n+1} - y_{n+1}\|_X^2 \le \|x_0 - y_0\|_X^2 - (2\gamma_0 - 1)\sum_{j=0}^{n-1} \|P_j x_j - P_j y_0\|_X^2,$$

implying the convergence of the series $\sum_{j=0}^{+\infty} \|P_j x_j\|_X^2$. Notice that due to (6.1.4), we have $0 < 2\gamma_0 - 1 < 1$. In particular, $\{\|P_n x_n\|_X\}$ is a null sequence. Further, since $\{\|x_n - y_0\|_X\}$ is convergent, $\{x_n\}$ is bounded. Thus, using the fact that X is a finite-dimensional Hilbert space, we can find a subsequence, again denoted by $\{x_n\}$, which converges to an element $x \in C$. Due to the fact that P_n is co-coercive with constant γ_n, P_n is Lipschitz continuous with constant γ_n^{-1}. We therefore have

$$\|P_n x_n - P_n x\|_X \le \frac{1}{\gamma_0}\|x_n - x\|_X.$$

Consequently, since the subsequence $\{x_n\}$ converges to x, we obtain

$$\lim_{n\to+\infty} \|P_n x_n - P_n x\|_X = 0.$$

Combining the previous relation with the observation that $\{\|P_n x_n\|_X\}$ is a null sequence, we deduce

$$\lim_{n\to+\infty} \|P_n x\|_X = 0. \tag{6.1.7}$$

By definition of P_n we have $P_n \tilde{x} = 0$ if and only if $\tilde{x} \in \mathrm{Sol}(\mathrm{VI}_s)$ and therefore

$$\inf_{n\in\mathbb{N}_0} \|P_n \tilde{x}\|_X > 0, \qquad \text{for every} \quad \tilde{x} \notin \mathrm{Sol}(\mathrm{VI}_s). \tag{6.1.8}$$

Finally, using (6.1.7) and (6.1.8), x is a solution of problem (3.2.3) w.r.t. s and consequently one of vector variational inequality (3.1.1); see Proposition 3.2.6. The proof is complete. □

Remark 6.1.7. (i) Notice $\{x_n\}$, generated by (6.1.5), produces a convergent sequence only.
(ii) Since F_s is not necessarily strongly monotone, the solution of variational inequality (3.2.3) is non-unique in general.
(iii) If in addition to the preliminaries of the previous theorem F_s is monotone and there exists an element $x_0 \in C$ such that

$$\lim_{\substack{\|x\|_X\to+\infty \\ x\in C}} \frac{\|F_s x - F_s x_0\|_X^2}{\|x - x_0\|_X} = +\infty,$$

then scalar variational inequality (3.2.3) w.r.t. s attains a solution. This immediately follows from the co-coercivity of F_s and Theorem 2.2.7.
(iv) If F_s is strongly monotone and Lipschitz continuous, then the iterate, given by

(6.1.5), converges to the unique solution of problem (3.2.3) w.r.t. $s \in K^* \setminus \{0\}$ provided

$$\rho_n \geq 0, \qquad \lim_{n\to+\infty} \rho_n = 0, \qquad \text{and} \qquad \sum_{n=0}^{+\infty} \rho_n = +\infty;$$

compare [58, Example 12.8.2]. In this case, no knowledge of the monotonicity and Lipschitz constants $c > 0$ and $L > 0$ are needed.

Under the assumptions of Theorem 6.1.6, the latter method can be summarized in the following way:

Algorithm 2: Basic projection method with variable step size.

Result: Calculation of a solution of vector variational inequality (3.1.1).

1 **Input:** The set C, $x_0 \in C$, $s \in K^* \setminus \{0\}$, $F_s : X \to X$ and a sequence $\{\rho_n\}$.

2 $\mathrm{Sol(VVI)} \leftarrow \emptyset$.

3 $n \leftarrow 0$.

4 **if** $x_n = \mathrm{Proj}(x_n - \rho_n^{-1} F_s x_n)$ **then**

5 | $\mathrm{Sol(VVI)} \leftarrow \mathrm{Sol(VVI)} \cup \{x_n\}$ **stop**.

6 **else**

7 | $x_{n+1} \leftarrow \mathrm{Proj}(x_n - \rho_n^{-1} F_s x_n)$.

8 | $n \leftarrow n + 1$.

9 | Go to line 4.

10 **end**

11 **Output:** A bounded sequence $\{x_n\}$ having a subsequence that converges to an element in Sol(VVI).

Example 6.1.8. Let us come back to Example 6.1.3. It is easy to see that for every vector $s \in \mathbb{R}^k_{\geq} \setminus \{0\}$ the mapping $F_s : \mathbb{R}^l \to \mathbb{R}^l$ is co-coercive on $\mathbb{R}^l$ with modulus $\gamma = \frac{1}{\|s\|_1}$. We again let $l = 2$, $k = 4$, $a^1 = (0,0)^\top$, $a^2 = (0,10)^\top$, $a^3 = (10,10)^\top$, $a^4 = (10,0)^\top$, $x_0 = (-1,-1)^\top$ and $s = (1,2,0,3)^\top$. Further, consider $\rho_n = \frac{1}{5} + \frac{(-1)^n}{10}$ for $n \in \mathbb{N}_0$. Evidently, $\inf_{n\in\mathbb{N}_0} \rho_n = \frac{1}{10}$ and $\sup_{n\in\mathbb{N}_0} \rho_n = \frac{3}{10}$, and condition (6.1.4) holds with $\gamma = \frac{1}{6}$. Therefore, following Theorem 6.1.6, the sequence $\{x_n\}$ with $x_{n+1} = x_n - \rho_n F_s x_n$ for $n \in \mathbb{N}_0$ produces a sequence that converges to a solution of vector variational inequality (3.1.3). Indeed, the first elements of $\{x_n\}$ are: $x_0 = (-1,-1)^\top$, $x_1 = (9.8, 6.8)^\top$, $x_2 = (6.92, 4.72)^\top$, $x_3 = (3.464, 2.224)^\top$, $x_4 = (4.3856, 2.8896)^\top$, $x_5 = (5.49152, 3.68832)^\top$, $x_6 = (5.196608, 3.475328)^\top$ and $x_7 = (4.8427136, 3.2197376)^\top$.

Until now, the proposed schemes have executed one projection per iteration. This execution is needed since every solution $x \in C$ of variational inequality (3.2.3) w.r.t. the functional s is characterized by $\mathrm{Proj}(x - \rho F_s x) = x$. We now introduce a new variable y and write the latter problem as

$$x = \mathrm{Proj}(x - \rho F_s y), \qquad x = y.$$

This seems artificial but it allows the introduction of the so-called Extragradient method (with fixed step size). Although the method will require the calculation of two projections, the benefits are significant since only a weak monotonicity assumption for F_s is needed, which we recall here; compare [58, Section 12.1.2].

Definition 6.1.9. Let C be a non-empty, closed and convex subset of the real finite-dimensional Hilbert space X. Let $A : X \to X$ and assume that the solution set $S = \{x \in C \mid \langle Ax, y - x\rangle \geq 0, \text{ for every } y \in C\}$ is non-empty. Then A is said to be *pseudomonotone* on C with respect to S if for every $x \in S$ it holds that

$$\langle Ay, y - x\rangle \geq 0, \qquad \text{for every} \quad y \in C.$$

Remark 6.1.10. If S is non-empty, then any monotone operator is pseudomonotone on C w.r.t. S; compare also Lemma 2.2.6.

The following theorem is motivated by [58, Theorem 12.1.11].

Theorem 6.1.11 (Extragradient method)**.** *Let C be a non-empty, closed and convex subset of the real finite-dimensional Hilbert space X, let K be a proper, closed, convex and solid cone in the real Banach space Y, let $F : X \to \mathrm{L}(X, Y)$ and let $\rho > 0$. Further let $x_0 \in C$ and $s \in K^* \setminus \{0\}$ and assume that variational inequality (3.2.3) w.r.t. s attains a solution. Suppose further that $F_s : X \to X$ is Lipschitz continuous with modulus $L > 0$ and pseudomonotone on C w.r.t. the solution set $\mathrm{Sol(VI}_s)$ of problem (3.2.3). If*

$$0 < \rho < \frac{1}{L}, \tag{6.1.9}$$

then the iterative $\{x_n\}$, given for every $n \in \mathbb{N}_0$ by

$$\begin{aligned} y_n &:= y_n(\rho, s) := \mathrm{Proj}(x_n - \rho F_s x_n), \\ x_{n+1} &:= x_{n+1}(\rho, s) := \mathrm{Proj}(x_n - \rho F_s y_n), \end{aligned} \tag{6.1.10}$$

converges to a solution of vector variational inequality (3.1.1).

Proof. Using the pseudomonotonicity of F_s w.r.t. the solution set of variational inequality (3.2.3) and the variational characterization of the projection operator, compare Theorem 2.1.17 (iv), one can show that for every $\tilde{x} \in \mathrm{Sol(VI}_s)$ and every $n \in \mathbb{N}_0$ it holds that

$$\|x_{n+1} - \tilde{x}\|_X^2 \leq \|x_n - \tilde{x}\|_X^2 - (1 - \rho^2 L^2)\|y_n - x_n\|_X^2, \tag{6.1.11}$$

with $\{x_n\}$ and $\{y_n\}$ generated by formula (6.1.10); compare Lemma 12.1.10 in [58]. From relation (6.1.9) follows $0 < 1 - \rho^2 L^2 < 1$ and combined with (6.1.11), this shows that $\{x_n\}$ is bounded. According to (6.1.11), we further have for every $\tilde{x} \in \mathrm{Sol(VI}_s)$ and $N \in \mathbb{N}$

$$(1 - \rho^2 L^2) \sum_{n=0}^{N} \|x_n - y_n\|_X^2 \leq \sum_{n=0}^{N} \|x_n - \tilde{x}\|_X^2 - \|x_{n+1} - \tilde{x}\|_X^2 \leq \|x_0 - \tilde{x}\|_X^2,$$

implying the convergence of the series $\sum_{n=0}^{+\infty} \|x_n - y_n\|_X^2$. We thus get

$$\lim_{n\to+\infty} \|x_n - y_n\|_X = 0.$$

Now let $\{x_n\}$ be an appropriate subsequence converging to an element $x \in C$, that is,

$$\lim_{n\to+\infty} x_n = x. \tag{6.1.12}$$

We then also have

$$\lim_{n\to+\infty} y_n = x.$$

Finally, passing in (6.1.10) to the limit yields

$$x = \lim_{n\to+\infty} y_n = \lim_{n\to+\infty} \mathrm{Proj}(x_n - \rho F_s x_n) = \mathrm{Proj}(x - \rho F_s x),$$

where the last equation follows from the continuity of $\mathrm{Proj}(I - \rho F_s)$. Thus, $x \in \mathrm{Sol}(\mathrm{VI}_s)$. However, in order to complete the proof, we have to show that the whole sequence $\{x_n\}$ converges to x. To this end it is enough to apply (6.1.11) with $\tilde{x}$ replaced with x. Following the previous arguments, the sequence $\{\|x_n - x\|_X\}$ is monotonically decreasing and therefore convergent. We conclude from (6.1.12) that the whole sequence $\{x_n\}$ converges to a solution x of variational inequality (3.2.3) and consequently to one of vector variational inequality (3.1.1). The proof is complete. □

Remark 6.1.12. (i) Within the last years, several Extragradient based methods for variational inequalities, quasi-variational inequalities, optimization problems and many other classes have been proposed. Some of them are, for instance, the Hyperplane projection method and the Tikhonov regularization method; compare [57, 58, 167].
(ii) Under some local error bound assumptions for the solution set $\mathrm{Sol}(\mathrm{VI}_s)$, one can show that the sequence $\{x_n\}$, generated by (6.1.10), converges to a solution $x \in C$ of vector variational inequality (3.1.1) at least R-linearly; see Section 12.6 in [58]. Here, R-linearity is understood in the sense that it holds

$$0 < \limsup_{n\to+\infty} \|x_n - x\|_X^{\frac{1}{n}} < 1.$$

(iii) If $C = X$, then the iterates $\{x_n\}$ and $\{y_n\}$, given by (6.1.10), become

$$\begin{aligned} y_n &= x_n - \rho F_s x_n, \\ x_{n+1} &= x_n - \rho F_s y_n, \end{aligned}$$

for $n \in \mathbb{N}_0$.

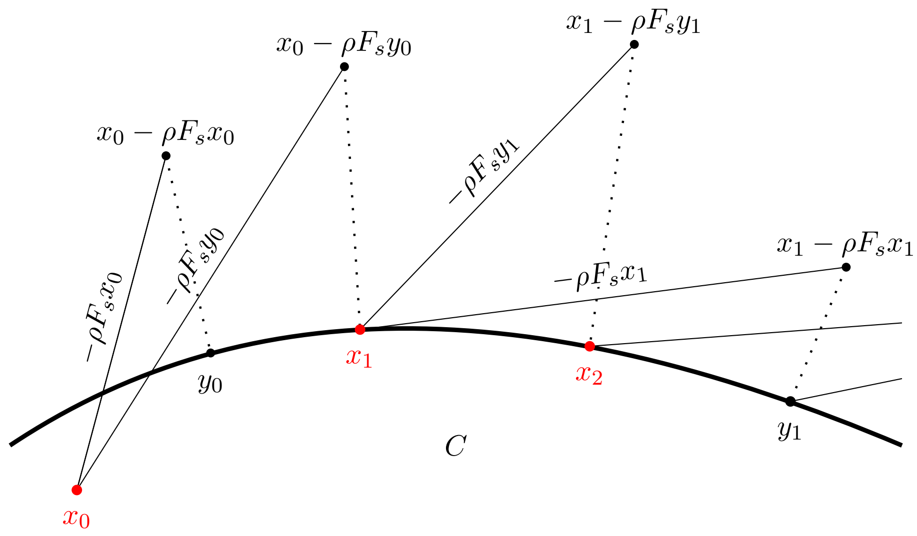


Figure 6.3: Illustration of the Extragradient method

Under the assumptions of Theorem 6.1.11, we have the following new algorithm for vector variational inequality (3.1.1):

Algorithm 3: Extragradient method.

Result: Calculation of a solution of vector variational inequality (3.1.1).
1 **Input:** The set C, $x_0 \in C$, $s \in K^* \setminus \{0\}$, $F_s : X \to X$ and $\rho > 0$.
2 $\mathrm{Sol(VVI)} \leftarrow \emptyset$.
3 $n \leftarrow 0$.
4 **if** $x_n = \mathrm{Proj}(x_n - \rho F_s x_n)$ **then**
5 $\quad$ $\mathrm{Sol(VVI)} \leftarrow \mathrm{Sol(VVI)} \cup \{x_n\}$ **stop**.
6 **else**
7 $\quad$ $y_n \leftarrow \mathrm{Proj}(x_n - \rho F_s x_n)$.
8 $\quad$ $x_{n+1} \leftarrow \mathrm{Proj}(x_n - \rho F_s y_n)$.
9 $\quad$ $n \leftarrow n + 1$.
10 $\quad$ Go to line 4.
11 **end**
12 **Output:** An element of the solution set Sol(VVI).

Example 6.1.13. We use the notation of the previous two examples. Since for every $s \in \mathbb{R}^k_{\geq} \setminus \{0\}$ the mapping $F_s : \mathbb{R}^l \to \mathbb{R}^l$ is monotone, it is already pseudomonotone w.r.t. the solution set of problem (3.2.5). The Lipschitz constant of F_s is $L = \|s\|_1$ and by letting $\rho = \frac{1}{10}$, the first elements of $\{x_n\}$, given by (6.1.10) with initial value $x_0 = (-1, -1)^\top$, are: $x_0 = (-1, -1)^\top$, $x_1 \doteq (0.44, 0.04)^\top$, $x_2 \doteq (1.5344, 0.8304)^\top$, $x_3 = (2.3661440000000002, 1.431104)^\top$, $x_4 = (2.9982694400000005, 1.8876390399999998)^\top$, $x_5 = (3.4786847744, 2.23460567039)^\top$, $x_6 = (3.8438004285440006, 2.4983003095039)^\top$ and $x_7 = (4.12128832569344, 2.6987082352230396)^\top$. This sequence converges slowly in comparison to the previous sequences and it holds that $\|x_n - x\|_2 < \frac{1}{1000}$ for $n \geq 32$, where $x = (5, 3\frac{1}{3})^\top$ is the (unique) solution of problem (3.2.5) w.r.t. to s.

6.2 Discrete finite-dimensional vector variational inequalities

In this section, we investigate discrete (finite-dimensional) vector variational inequalities. To be precise, let $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$, where $l, k \in \mathbb{N}$, and denote by C_n a discrete subset of $\mathbb{R}^l$ with cardinality $n \in \mathbb{N}$, that is, $|C_n| = n$ and

$$C_n = \{x^1, \dots, x^n\}$$

for some pairwise distinct elements $x^1, \dots, x^n \in \mathbb{R}^l$. Then the discrete vector variational inequality consists of finding $x \in C_n$ such that

$$\langle Fx, x^j - x\rangle \notin -\operatorname{int}\mathbb{R}^k_{\geq}, \qquad \text{for every} \quad j \in \{1, \dots, n\}. \tag{6.2.1}$$

In what follows, we will denote the solution set of problem (6.2.1) by $\mathrm{Sol}(\mathrm{VVI}_n)$.

Remark 6.2.1. Since the constraining set C_n is non-convex, mostly all known methods and results in the literature for vector variational inequalities cannot be applied. However, it is easy to see that $g_n : C_n \to \mathbb{R}$ with

$$g_n(x) := \max_{i\in\{1,\dots,n\}} \min_{j\in\{1,\dots,k\}} \langle F_j x, x - x^i\rangle, \qquad \text{for every} \quad x \in C_n,$$

is a (single-valued) gap function for problem (6.2.1), that is, $g_n \geq 0$ and $g_n(x) = 0$ if and only if $x \in \mathrm{Sol}(\mathrm{VVI}_n)$; compare Section 8 in [91]. Thus, discrete vector variational inequality (6.2.1) is equivalent to the optimization problem of finding $x \in C_n$ with

$$x \in \operatorname*{argmin}_{j\in\{1,\dots,n\}} g_n(x^j).$$

In what follows, we will focus on effective algorithms for the calculation of $\mathrm{Sol}(\mathrm{VVI}_n)$ only. The simplest one, yet naive, is captured in the following algorithm:

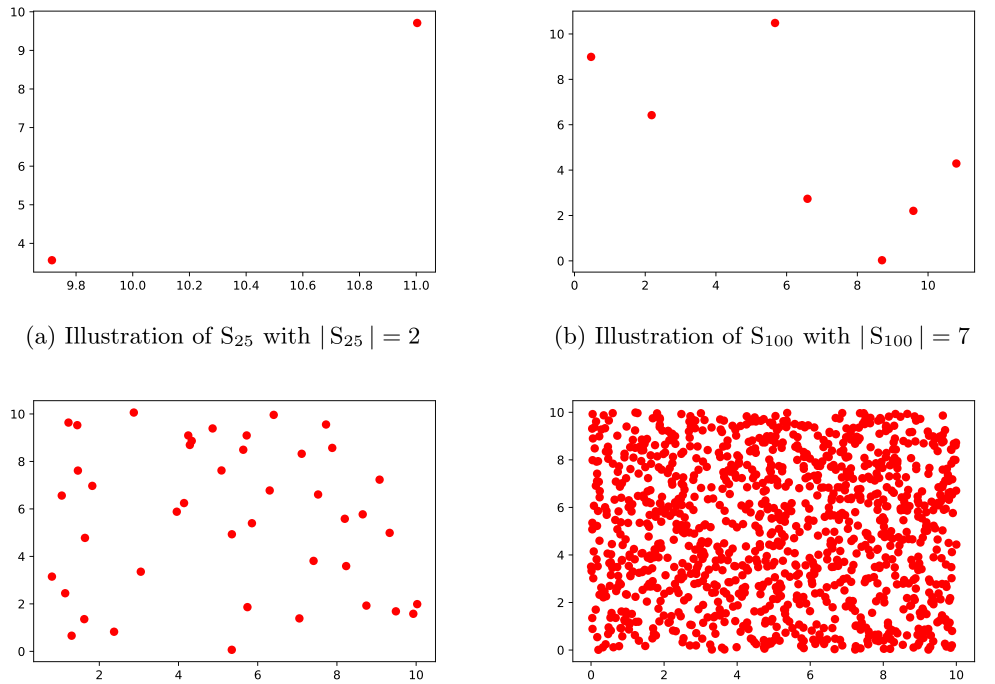
Algorithm 4: Naive method.

Result: Calculation of the solution set $\mathrm{Sol}(\mathrm{VVI}_n)$ of problem (6.2.1).

Input: Natural numbers $n, k, l \in \mathbb{N}$, the set $C_n = \{x^1, \dots, x^n\}$ and a mapping $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$.
$\mathrm{Sol}(\mathrm{VVI}_n) \leftarrow \emptyset$.
for $i \leftarrow 1$ **to** n **do**
  if $\langle Fx^i, x^j - x^i\rangle \notin -\operatorname{int}\mathbb{R}^k_{\geq}$ **for** $j \leftarrow 1$ **to** n **then**
    $\mathrm{Sol}(\mathrm{VVI}_n) \leftarrow \mathrm{Sol}(\mathrm{VVI}_n) \cup \{x^i\}$.
  end
end
Output: The set $\mathrm{Sol}(\mathrm{VVI}_n)$.

Remark 6.2.2. The computational complexity of Algorithm 4 is highly depending on the cardinality of C_n. This is mainly due to line 4 of the algorithm, where every point in C_n is compared with all elements of C_n.

Example 6.2.3. In this example, we will consider the discrete version of vector variational inequality (3.1.3) where we let $l = 2$, $k = 4$ and $a^1 = (0,0)^\top$, $a^2 = (0,10)^\top$, $a^3 = (10,10)^\top$ and $a^4 = (10,0)^\top$; compare the examples in the previous section. We have randomly computed discrete subsets C_n of $\mathbb{R}^2$ with cardinalities $n \in \{25, 100, 1000, 5000\}$. The corresponding solution sets S_n for $n \in \{25, 100, 1000, 5000\}$ are shown in the figures below.



(a) Illustration of S_{25} with $|\,\mathrm{S}_{25}\,| = 2$

(b) Illustration of S_{100} with $|\,\mathrm{S}_{100}\,| = 7$

(c) Illustration of S_{1000} with $|\,\mathrm{S}_{1000}\,| = 43$

(d) Illustration of S_{5000} with $|\,\mathrm{S}_{5000}\,| = 1126$

Figure 6.4: Illustration of the solution sets S_n for $n \in \{25, 100, 1000, 5000\}$

Since Algorithm 4 becomes ineffective when the cardinality of C_n increases, it is important to reduce the cost in line 4. Therefore, the following observation is crucial: An element x^i with $i \in \{1, \dots, n\}$ solves discrete vector variational inequality (6.2.1) if and only if

$$\langle Fx^i, x^i \rangle \in \mathrm{WMin}\left(A^i, \mathbb{R}^k_{\geq}\right), \tag{6.2.2}$$

where $A^i := \{\langle Fx^i, x^j \rangle \mid j \in \{1, \dots, n\}\}$ for $i \in \{1, \dots, n\}$. Thus, the for loop in Algorithm 4 can be written in the following way:

for $i \leftarrow 1$ **to** n **do**
  if $\mathrm{WMin}\left(A^i, \mathbb{R}^k_{\geq}\right) \neq \emptyset$ **and** $\langle Fx^i, x^i \rangle \in \mathrm{WMin}\left(A^i, \mathbb{R}^k_{\geq}\right)$ **then**
    $\mathrm{Sol(VVI}_n) \leftarrow \mathrm{Sol(VVI}_n) \cup \{x^i\}$.
  end
end

Even though the complexity remains high, a natural idea arises, namely to apply in every step $i \in \{1, \ldots, n\}$ a reduction procedure that generates a set $B^i \subseteq A^i$ with

$$\mathrm{WMin}(B^i, \mathbb{R}^k_{\geq}) = \mathrm{WMin}(A^i, \mathbb{R}^k_{\geq}). \tag{6.2.3}$$

Such a procedure is known in the literature as Graef-Younes method [98, Section 12.4] and has currently been used to generate all minimal elements of a finite set with respect to an ordering cone; compare [85].

In what follows, we will recall some slightly modified definitions and results from Section 2 in [85]. It should be noted that the results of this section still hold if we replace the Euclidean space $\mathbb{R}^k$ with an arbitrary linear space.

Definition 6.2.4. A subset A of $\mathbb{R}^k$ satisfies the *weak domination property* (w.r.t. the cone $\mathbb{R}^k_{\geq}$) if $\mathrm{WMin}(A, \mathbb{R}^k_{\geq}) \neq \emptyset$ and for all $x \in A$ there exists $x_0 \in \mathrm{WMin}(A, \mathbb{R}^k_{\geq})$ such that $x_0 \leq_{\mathbb{R}^k_{\geq}} x$, that is, $x \in x_0 + \mathbb{R}^k_{\geq}$.

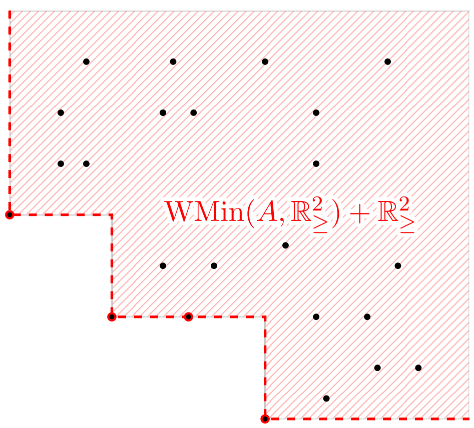


Figure 6.5: Illustration of the weak domination property of a set A in $\mathbb{R}^2$ with cardinality 24

Remark 6.2.5. Let $A \subseteq \mathbb{R}^k$ and consider the following three conditions:

(i) A satisfies the weak domination property.

(ii) It holds that $A \subseteq \mathrm{WMin}(A, \mathbb{R}^k_{\geq}) + \mathbb{R}^k_{\geq}$.

(iii) It holds that $A + \operatorname{int} \mathbb{R}^k_{\geq} = \mathrm{WMin}(A, \mathbb{R}^k_{\geq}) + \operatorname{int} \mathbb{R}^k_{\geq}$.

Then, (i) and (ii) are equivalent (by definition) while (ii) implies (iii).

Indeed, the equivalence of (i) and (ii) is trivial. Now assume that (ii) holds. By relation (2.4.1) we deduce $A+\operatorname{int}\mathbb{R}^k_{\geq} \subseteq \operatorname{WMin}(A,\mathbb{R}^k_{\geq})+\mathbb{R}^k_{\geq}+\operatorname{int}\mathbb{R}^k_{\geq} = \operatorname{WMin}(A,\mathbb{R}^k_{\geq})+\operatorname{int}\mathbb{R}^k_{\geq}$. Since it always holds $\operatorname{WMin}(A,\mathbb{R}^k_{\geq}) \subseteq A$, the inverse inclusion $\operatorname{WMin}(A,\mathbb{R}^k_{\geq})+\operatorname{int}\mathbb{R}^k_{\geq} \subseteq A+\operatorname{int}\mathbb{R}^k_{\geq}$ trivially holds, which shows (iii).

The next lemma is motivated by [150, Proposition 2.2.1'].

Lemma 6.2.6. *Every finite set $A \subseteq \mathbb{R}^k$ with $\operatorname{WMin}(A,\mathbb{R}^k_{\geq}) \neq \emptyset$ satisfies the weak domination property.*

Proof. If $A=\emptyset$, then the weak domination property trivially holds. Assume $A \neq \emptyset$ and let $x \in A$ be arbitrarily chosen. If $x \in \operatorname{WMin}(A,\mathbb{R}^k_{\geq})$ the proof is complete since $\leq_{\mathbb{R}^k_{\geq}}$ is reflexive. Therefore let $x \in A$ but $x \notin \operatorname{WMin}(A,\mathbb{R}^k_{\geq})$. Thus, we can find $x^1 \in A$ with $x^1 \leq_{\operatorname{int}\mathbb{R}^k_{\geq}} x$. If $x^1 \in \operatorname{WMin}(A,\mathbb{R}^k_{\geq})$, we are again done. Else we can find $x^2 \in A$ with $x^2 \leq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^1$. By induction, we generate a sequence $\{x^n\} \subseteq A$ with $x^0 = x$ and $x^{j+1} \leq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^j$ for $j \in \mathbb{N}_0$, that is,

$$x \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^1 \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^2 \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^3 \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} \cdots$$

Clearly, the elements of $\{x^n\}$ are pairwise distinct since $\leq_{\operatorname{int}\mathbb{R}^k_{\geq}}$ is irreflexive. Therefore, there exists a natural number $m \in \mathbb{N}$ with $m \leq |A|$ such that

$$x \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^1 \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^2 \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^3 \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} \cdots \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^m.$$

We thus have $x^m \in \operatorname{WMin}(A,\mathbb{R}^k_{\geq})$ since otherwise the construction of the finite sequence could have been continued with $x^m \geq_{\operatorname{int}\mathbb{R}^k_{\geq}} x^{m+1}$ for some $x^{m+1} \in A$. Since $\leq_{\operatorname{int}\mathbb{R}^k_{\geq}}$ is transitive, we get $x^m \leq_{\operatorname{int}\mathbb{R}^k_{\geq}} x$ and therefore $x^m \leq_{\mathbb{R}^k_{\geq}} x$. The proof is complete. □

Lemma 6.2.7. *Let $A \subseteq \mathbb{R}^k$. It holds that* $\operatorname{WMin}(A,\mathbb{R}^k_{\geq}) = \operatorname{WMin}(A+\mathbb{R}^k_{\geq},\mathbb{R}^k_{\geq})$.

Proof. Let $x \in \operatorname{WMin}(A,\mathbb{R}^k_{\geq})$ but $x \notin \operatorname{WMin}(A+\mathbb{R}^k_{\geq},\mathbb{R}^k_{\geq})$. Thus, we can find elements $y \in A+\mathbb{R}^k_{\geq}$ and $z \in \operatorname{int}\mathbb{R}^k_{\geq}$ with $z = x-y$. Since $y \in A+\mathbb{R}^k_{\geq}$, there are $y' \in A$ and $z' \in \mathbb{R}^k_{\geq}$ with $y = y'+z'$. We consequently have

$$x = y+z = y'+z'+z \in A+\mathbb{R}^k_{\geq}+\operatorname{int}\mathbb{R}^k_{\geq} = A+\operatorname{int}\mathbb{R}^k_{\geq},$$

compare again relation (2.4.1), which contradicts the fact that $x \in \operatorname{WMin}(A,\mathbb{R}^k_{\geq})$. The converse inclusion follows from the fact that $A \subseteq A+\{0\} \subseteq A+\mathbb{R}^k_{\geq}$. Indeed, let $x \in \operatorname{WMin}(A+\mathbb{R}^k_{\geq},\mathbb{R}^k_{\geq})$, that is, $(x-\operatorname{int}\mathbb{R}^k_{\geq}) \cap (A+\mathbb{R}^k_{\geq}) = \emptyset$. This implies $(x-\operatorname{int}\mathbb{R}^k_{\geq}) \cap A = \emptyset$, that is, $x \in \operatorname{WMin}(A,\mathbb{R}^k_{\geq})$. The proof is complete. □

Theorem 6.2.8. *Let A and B be subsets of $\mathbb{R}^k$ with $B \subseteq A$ and consider the following two assertions:*

(i) *It holds* $\operatorname{WMin}(A,\mathbb{R}^k_{\geq}) \subseteq B$.

(ii) *It holds* $\operatorname{WMin}(A,\mathbb{R}^k_{\geq}) = \operatorname{WMin}(B,\mathbb{R}^k_{\geq})$.

Then, (ii) *implies* (i)*. If in addition A satisfies the weak domination property then* (i) *implies* (ii)*.*

Proof. Assume that (ii) holds. Since by definition $\mathrm{WMin}(B, \mathbb{R}^k_\geq) \subseteq B$, we conclude

$$\mathrm{WMin}(A, \mathbb{R}^k_\geq) = \mathrm{WMin}(B, \mathbb{R}^k_\geq) \subseteq B,$$

which shows assertion (i).
For the converse let assertion (i) hold and suppose that A satisfies the weak domination property. Let $x \in \mathrm{WMin}(A, \mathbb{R}^k_\geq)$, that is, $(x - \operatorname{int} \mathbb{R}^k_\geq) \cap A = \emptyset$. Since $B \subseteq A$, we immediately get $(x - \operatorname{int} \mathbb{R}^k_\geq) \cap B = \emptyset$, that is, $x \in \mathrm{WMin}(B, \mathbb{R}^k_\geq)$. This shows $\mathrm{WMin}(A, \mathbb{R}^k_\geq) \subseteq \mathrm{WMin}(B, \mathbb{R}^k_\geq)$. In oder to prove the reverse inclusion, let $x \in \mathrm{WMin}(B, \mathbb{R}^k_\geq)$. Invoking Lemma 6.2.7, we have $\mathrm{WMin}(B, \mathbb{R}^k_\geq) = \mathrm{WMin}(B + \mathbb{R}^k_\geq, \mathbb{R}^k_\geq)$ and therefore $(x - \operatorname{int} \mathbb{R}^k_\geq) \cap (B + \mathbb{R}^k_\geq) = \emptyset$. Using the weak domination property of A and assertion (i), we deduce

$$(x - \operatorname{int} \mathbb{R}^k_\geq) \cap A \subseteq (x - \operatorname{int} \mathbb{R}^k_\geq) \cap \left(\mathrm{WMin}(A, \mathbb{R}^k_\geq) + \mathbb{R}^k_\geq\right) \subseteq (x - \operatorname{int} \mathbb{R}^k_\geq) \cap (B + \mathbb{R}^k_\geq),$$

and consequently $(x - \operatorname{int} \mathbb{R}^k_\geq) \cap A = \emptyset$. Thus, $x \in \mathrm{WMin}(A, \mathbb{R}^k_\geq)$ and the proof is complete. $\square$

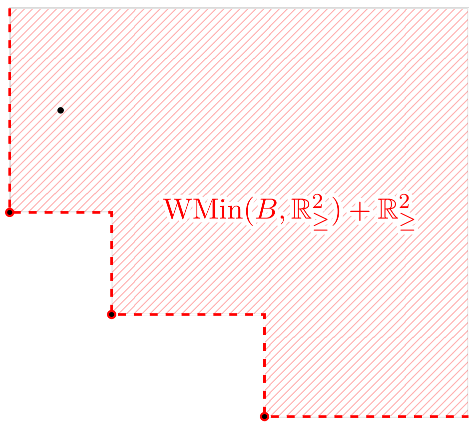


Figure 6.6: Illustration of a set B in $\mathbb{R}^2$ with $\mathrm{WMin}(A, \mathbb{R}^2_\geq) + \mathbb{R}^2_\geq = \mathrm{WMin}(B, \mathbb{R}^2_\geq) + \mathbb{R}^2_\geq$ and cardinality 4 (reduction procedure)

The previous results enable us to replace line 4 of Algorithm 4 with a Graef-Younes reduction method. Its formulation is given in Algorithm 5.

Algorithm 5: Graef-Younes based reduction method.

Result: Calculation of the solution set $\mathrm{Sol}(\mathrm{VVI}_n)$ of problem (6.2.1).
Input: Natural numbers $n, k, l \in \mathbb{N}$, the set $C_n = \{x^1, \dots, x^n\}$ and a mapping $F : \mathbb{R}^l \to \mathrm{Mat}_{k\times l}(\mathbb{R})$.

```
Sol(VVI_n) ← ∅.
for i ← 1 to n do
    /* Reduction procedure: Computation of a set B^i having the
    /* property WMin(B^i, R^k_≥) = WMin(A^i, R^k_≥) ⊆ B^i ⊆ A^i.
    B^i ← ⟨Fx^i, x^1⟩.
    for j ← 2 to n do do
        if ⟨Fx^i, x^j⟩ ∉ B^i + int R^k_≥ then
            B^i ← B^i ∪ {⟨Fx^i, x^j⟩}.
        end
    end
    if WMin(B^i, R^k_≥) ≠ ∅ and ⟨Fx^i, x^i⟩ ∈ WMin(B^i, R^k_≥) then
        Sol(VVI_n) ← Sol(VVI_n) ∪ {x^i}.
    end
end
```

Output: The set $\mathrm{Sol}(\mathrm{VVI}_n)$.

Remark 6.2.9. (i) Since every set A^i, where $i \in \{1, \dots, n\}$, is finite, it enjoys the weak domination property; see Lemma 6.2.6. Thus, according to Theorem 6.2.8, the set B^i, generated by the reduction procedure, satisfies relation (6.2.3).
(ii) In general, the set B^i exceeds the set $\mathrm{WMin}(A^i, \mathbb{R}^k_{\geq})$, that is, $\mathrm{WMin}(A^i, \mathbb{R}^k_{\geq}) \subsetneqq B^i$. An approach which is fixing this issue is the so-called Graef-Younes method with backward iteration; see [98, Algorithm 12.20].
(iii) Numerical tests with the data of Example 6.2.3 show that the Graef-Younes method can reduce a set A^i containing 100 000 points to a set B^i containing 157 points only, among which 82 are weakly minimal elements of A^i. Unfortunately, some modifications of the latter example show that the cardinality of B^i may be very large. It is even possible that we have $B^i = A^i$ and hence the computation of $\mathrm{WMin}(B^i, \mathbb{R}^k_{\geq})$ is not significantly easier than the computation of $\mathrm{WMin}(A^i, \mathbb{R}^k_{\geq})$.

Conclusion and Outlook

Conclusion

This thesis studied vector quasi-variational problems covering several other problems of interest. The results of this thesis can be summarized as follows:

- In Chapter 3, we introduced a novel coercivity condition and gave a new existence result for vector variational inequalities. We further developed a regularization method for non-coercive vector variational inequalities by replacing the latter problem with a family of well-behaving and coercive problems. We then used the family of regularized vector variational inequalities to derive new existence results for vector variational inequalities whose data do not satisfy any known coercivity condition. We finally investigated alternative conditions for the convergence of regularized solutions and, by application of our results, derived new existence results for generalized vector variational inequalities.

- By introducing a novel Minty-type lemma for generalized vector variational inequalities, we developed a new existence result for the latter problems using the famous Fan-KKM lemma.

- Motivated by the duality principle in optimization, we introduced two inverse vector variational inequalities, using the vector conjugate approach, and studied their connections to (generalized) vector variational inequalities. We then investigated inverse problems based on a perturbation approach. Applications of our results allowed us to derive necessary and sufficient conditions for the beam intensity optimization problem in radiotherapy treatment.

- Besides variational problems, the fifth chapter of this thesis focused on quasi-variational-like problems. The abstract setting allowed us to study several special cases, like quasi-variational inequalities and vector quasi-variational inequalities. We gave a detailed overview of some classic solution methods and proposed some novel existence results for the latter problem classes by applying suitable fixed-point theorems and scalarization techniques. We then used the concept of generalized solutions for the study of several problem classes of interest. We therefore considered a closely related optimization problem, which enabled us to relax some restrictive assumptions for the data of the latter problems.

- In the last chapter of this thesis, we introduced three projection based algorithms for vector variational inequalities. These were the basic projection method, the basic projection method with variable step size and the Extragradient method. Our procedures heavily relied on a scalarization method and used the strong monotonicity and Lipschitz continuity (or, co-coerciveness) of the underlying mapping.

- Furthermore, we investigated discrete finite-dimensional vector variational inequalities and proposed a naive and a reduction based method, where the latter one uses the well-known Graef-Younes procedure.

Outlook

During this present work, we have discovered several topics of interest, which should be investigated in the future. Some of them are as follows:

- As already pointed out in Chapter 6, in the field of vector variational inequalities, most of the research has been dedicated to theoretical results only. In order to effectively compute solutions of vector variational problems, powerful algorithms must be developed.

- Until now, to the best of our knowledge, discrete vector variational inequalities have not been studied in the literature yet. Since Example 6.2.3 indicates that vector variational inequality (3.1.3) is closely related to its discrete version, it is of interest to investigate connections of the latter problems. Such investigations will possibly result in necessary optimality conditions and new solution methods for vector variational inequalities.

- In comparison to Section 6.1, it would be desirable to develop algorithms for vector quasi-variational inequality (5.0.5). This can be done using linear scalarization techniques and projection based methods. Indeed, from Proposition 5.1.12 and Theorem 9 in [143] follows that every element $x \in C$ with

  $$\mathrm{Proj}_{E(x)}(x - \rho^{-1} F_s x) = x,$$

  where $s \in \mathrm{qi}\, K^*$, $F_s : X \to X$ and $\rho > 0$, is a solution of problem (5.0.5). Thus, by adapting the main ideas in Chapter 6 of this thesis, it is possible to derive for the first time algorithms for problem (5.0.5).

- In order to test algorithms for (discrete) vector variational and vector quasi-variational inequalities, the creation of a collection of appropriate test problems is required. In order to do so, the examples in this thesis and in the survey paper [91] can serve as a solid base.

Summary of Contributions

The main results of this thesis are mainly based on the following four articles [15, 56, 91, 93] that are published or accepted in peer-reviewed international journals:

1. T. Q. Bao, N. Hebestreit, C. Tammer, Generalized solutions of quasi-variational-like problems, accepted.
2. R. Elster, N. Hebestreit, A. A. Khan, C. Tammer, Inverse generalized vector variational inequalities with respect to variable domination structures and applications to vector approximation problems, Appl. Anal. Optim. 2 (2018), 341-372.
3. N. Hebestreit, Vector variational inequalities and related topics: A survey of theory and applications, Appl. Set-Valued Anal. Optim. 1 (2019), 231-305.
4. N. Hebestreit, A. A. Khan, E. Köbis, C. Tammer, Existence theorems and regularization methods for non-coercive vector variational and vector quasi-variational inequalities, J. Nonlinear Convex Anal. 20 (2019), 565-591.

In what follows, we summarize the author's main contributions to each chapter of this thesis:

- In Chapter 3, the discussion of the Examples 3.1.3, 3.2.5, 3.2.7, 3.2.9 and 3.2.14 and the application of all results from the field of vector variational inequalities are novel and based on the paper [91], which is the author's sole work. Further, in the Sections 3.3.1 and 3.3.2, the author of this thesis takes an extended view on the existence results for vector variational inequalities in the literature. Sections 3.3.3, 3.4 and 3.5 are based on the joint work [93] with A. A. Khan, E. Köbis and C. Tammer.
- Chapter 4 is based on the joint work [56] with R. Elster, A. A. Khan and C. Tammer while Section 4.4.1 is new.
- The discussions and novel results in the Sections 5.1.1 and 5.1.2 are the sole work of the author. However, the Theorems 5.1.13 and 5.1.14 as well as Corollary 5.1.15 can be found in [93]. Section 5.2 is mainly based on the joint work [15] with T. Q. Bao and C. Tammer. The applications in Section 5.3 are new and were motivated by Example 5.8 in [93].
- Chapter 6 is the sole work of the author.

List of Figures

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit

Existence Results for Vector Quasi-Variational Problems

selbstständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

Halle (Saale), den 09.11.2020

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Publikationen

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2020	T. Q. Bao, N. Hebestreit, C. Tammer, Generalized solutions of vector quasi-variational-like problems, Vietnam J. Math. 48 (2020), 509–526.
2019	N. Hebestreit, Vector variational inequalities and related topics: A survey of theory and applications, Appl. Set-Valued Anal. Optim. 1 (2019), 231-305.
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