Error estimates for a finite difference approximation of mean curvature flow for surfaces of torus type

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Abstract

Subject of this work is the geometric evolution equation called mean curvature flow. It evolves a surface pointwise into the direction of its normal with a velocity that is given by the mean curvature at that point. We restrict our attention to surfaces in $\mathbb{R}^3$ of torus type. The aim is to approximate a reparametrized version of the flow and derive error estimates for the fully discrete problem. Our strategy is to apply a variant of the well-known DeTurck trick for the reparametrization. The generated evolution equation depends on a parameter α that determines a tangential velocity. Using a finite difference method, we discretize this flow and derive a family of semi-implicit fully discrete approximations. In the convergence proof we obtain optimal-order error bounds in discrete analogs to different Sobolev norms like a discrete H^2-norm as well as a discrete L^∞-norm. This analysis is complemented with a numerical simulation of the approximated flow. We compute the experimental order of convergence to support the theoretical results and provide an illustration of the influence of α on the approximation.

Kurzzusammenfassung

Gegenstand dieser Arbeit ist der Mean Curvature Flow. Diese geometrische Evolutionsgleichung bewegt eine Fläche punktweise in Richtung ihrer Normalen mit einer Geschwindigkeit, die gleich der mittleren Krümmung in diesem Punkt ist. Wir beschränken uns auf Flächen im $\mathbb{R}^3$ vom Typ des Torus. Das Ziel ist, eine Umparametrisierung des Flusses zu approximieren und Fehlerabschätzungen für das vollständig diskrete Problem herzuleiten. Die Vorgehensweise besteht dabei darin, eine Variante des bekannten DeTurck Tricks für die Umparamterisierung anzuwenden. Dies führt auf eine Evolutionsgleichung, welche von einem Parameter α abhängt und eine durch diesen Parameter bestimmte Geschwindigkeit in tangentialer Richtung besitzt. Der modifizierte Fluss wird mittels einer Finite-Differenzen-Methode diskretisiert und somit eine Familie von semi-impliziten, vollständig diskreten Approximationen erzeugt. Der Konvergenzbeweis liefert Fehlerschranken optimaler Ordnung in verschiedenen diskreten Normen, welche als Analogon von Sobolev-Normen wie der H^2- und der L^∞-Norm betrachtet werden können. Die Analyse wird durch eine numerische Simulation des Flusses ergänzt. Wir berechnen zum einen die experimentelle Konvergenzordnung, um die theoretischen Ergebnisse zu stützen, zum anderen illustrieren wir den Einfluss von α auf die Approximation.

Contents

1 Introduction

A family of hypersurfaces $\{\Gamma(t)\}_{t\in[0,T]} \subset \mathbb{R}^{n+1}$ is said to move according to mean curvature flow, if at each point it moves into the direction of the normal with a velocity that is given by the mean curvature. That is, if

$$V = H, \tag{1.1}$$

where H denotes the mean curvature, defined as the sum of the principal curvatures, and V is the normal velocity of $\Gamma(t)$. For $n = 1$, i.e. for curves, one obtains the curve shortening flow, where the mean curvature is replaced by the curvature. Note that for the unit outward normal ν we let the sign of the mean curvature of a sphere be negative in our sign convention. One can show that the first variation of the area functional of a surface in direction of the normal can be represented by the mean curvature. This gives rise to the interpretation of mean curvature flow as the L^2-gradient flow of the area functional, moving a surface pointwise into the direction of the steepest descent of its area. In particular, for the total area it holds that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Gamma(t)| = -\int\limits_{\Gamma(t)} H^2 \,\mathrm{d}A,$$

compare e.g. [37]. The flow therefore appears naturally in situations where a surface energy is involved. Imagine for example boundaries between the phases of a system that are formed in a way such as to minimize their energy given by the area of the interfaces. Applications to the flow by mean curvature are various, and for further explanations we refer to [16].

As a matter of fact, the sphere is one of the few exact solutions of the mean curvature flow. It shrinks to a point without changing shape, which can be seen in the following way: Let $\Gamma(t) = \partial B_{R(t)}(x) \subset \mathbb{R}^{n+1}$ be a family of spheres. On $\partial B_{R(t)}(x)$, we have that $V = R'(t)$ and $H = -n/R(t)$, and so (1.1) reduces to the ordinary differential equation

$$R'(t) = -\frac{n}{R(t)},$$

for which we assume the initial datum $R(0) = R$. The solution to the mean curvature flow are hence spheres with radius $R(t) = \sqrt{R^2 - 2nt}$ that tends to zero as t approaches the maximal time of existence $R^2/(2n)$. The absence of a broader range of exact solutions is one reason for an interest in numerical approximations of the flow, which we will make the subject of our discussion later.
Huisken's work [31], where he actually proved that convex hypersurfaces shrink to points with asymptotically spherical shape in finite time, was seminal. In the case of curves the result is even stronger. By the curve shortening flow, closed embedded

plane curves become convex, [25], and then shrink to a 'round' point in finite time, [24]. This does not hold for surfaces, where non-convexity can lead to the formation of singularities before the extinction. Indeed, developing a singularity in finite time is a rather typical behaviour. It occurs when the curvature at a point becomes unbounded for some reason. Grayson in [26] was the first to rigorously prove the existence of a surface with such evolution, the dumbbell shape: Two spheres, connected by a thin tube that shrinks much faster than the spheres do. The tube, also called neck, then pinches off, and the two spheres each continue to flow by their mean curvature until they vanish. A collection of results on the study of singularities can be found in [33].

Investigating the problem of the flow by mean curvature can be approached in several ways. One is the parametric formulation that will be used throughout this work. It is based on tracking the evolution of a parametrization $F(\cdot,t) : \mathcal{M} \to \mathbb{R}^{n+1}$, where $t \in [0,T)$, such that $\Gamma(t) = F(\cdot,t)(\mathcal{M})$, where $\mathcal{M} \subset \mathbb{R}^{n+1}$ is a reference manifold. This fixes the topological type of the hypersurface and is thus a limitation since we know that topological changes can occur, like in the example of the dumbbell. Describing $\mathcal{M}$ by a local parametrization $\hat{x} : \Omega \to \mathbb{R}^{n+1}$, where $\Omega \subset \mathbb{R}^n$, and setting $x = F \circ \hat{x}$ we can write

$$x_t = H\nu \tag{1.2}$$

to define the flow by mean curvature. Multiplication by ν restores the description in (1.1).
Defining the metric on $\Gamma(t)$ through $g_{ij}(u,t) = x_{u_i}(u,t) \cdot x_{u_j}(u,t)$ for $(u,t) \in \Omega \times [0,T]$ and $i,j \in \{1,\dots,n\}$, Mean Curvature Flow can equivalently be represented by

$$x_t = \sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \left(g^{ij}\sqrt{g}x_{u_j}\right)_{u_i} = \sum_{i,j=1}^{n} g^{ij} x_{u_i u_j} - \sum_{i,j,k,l=1}^{n} g^{ij} g^{kl} (x_{u_i u_j} \cdot x_{u_k}) x_{u_l}, \tag{1.3}$$

where $(g_{ij}) = (g_{ij})_{i,j\in\{1,\dots,n\}}$ denotes the matrix of the induced metric coefficients, $g = \det(g_{ij})$ the area element and g^{ij} the components of the inverse matrix of (g_{ij}). A derivation of the identity in (1.3) and how this follows from (1.2) will be given in the subsequent chapter of this work.
The spatial operator in (1.3) clearly depends on the metric and the metric evolves in time. In particular, the evolution equation is nonlinear and degenerate in the tangential direction. By degeneration we mean that the partial differential equation describing mean curvature flow is not strongly, but weakly parabolic. For a formal derivation of the link between the degeneration and the notion of weak parabolicity see [3]. The author also provides a proof of the invariance of the flow under tangential reparametrization. More detailed information on the mean curvature flow from a parametric point of view, including important analytic results and references to further literature, are thoroughly prepared in [37]. Ecker in [21] also provides a comprehensive introduction to Mean Curvature Flow including some examples and basic results.

Another possibility to study the flow is given if the surfaces can be written as the graph of a function u, i.e. $\Gamma(t) = \{(x, u(x,t)) \mid x \in \Omega \subset \mathbb{R}^n\}$. Note that it is not sufficient if this is true for the initial surface since this feature can be lost during the evolution. The flow is given by a scalar nonlinear parabolic partial differential equation for u in non-divergence form. It has for instance been investigated in [32], where the author proved a global existence result under some regularity assumptions.
Brakke in [9] considered mean curvature flow from the viewpoint of geometric measure theory. By extending the evolution from manifolds to varifolds, he introduces weak solutions that can be described through and beyond singularities.
A further approach is to represent the hypersurface and its evolution by the level sets of a function u, i.e. $\Gamma(t) = \{x \in \mathbb{R}^{n+1} \mid u(x,t) = 0\}$. This turns the evolution equation into a nonlinear degenerate and singular partial differential equation for u. Both the works of Chen, Giga and Goto, [10], and Evans and Spruck, [23], supplied the basis for the theory which utilizes the notion of viscosity solutions. The level set formulation of the motion by mean curvature also allows to define a global solution including times after the appearance of a singularity.
Another way that enables a global description of solutions is the phase field approach. Like the level set formulation, it is implicit and makes use of a level set function u^ε, here called phase field function. However, one considers a diffuse interface $\Gamma_\varepsilon(t)$ of width $\mathcal{O}(\varepsilon)$ that approximates $\Gamma(t)$ as ε tends to zero. The corresponding evolution equation is formulated for u^ε and reads

$$u_t^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon (1 - (u^\varepsilon)^2).$$

It is a nonlinear equation of reaction-diffusion type that appears as a model equation in many applications.

As stated above, research has also been highly interested in approximating mean curvature flow. Dziuk in [18] has achieved pioneering work in presenting the first numerical approximation of mean curvature flow in the parametric setting. His finite element method is based on approximating the surface $\Gamma(t^m)$ by a polyhedron Γ_h^m and on finding, in each time step, $x_h^{m+1} \in X_h^m$ such that

$$\frac{1}{\tau} \int_{\Gamma_h^m} (x_h^{m+1} - id) \cdot \varphi_h \, \mathrm{d}A + \int_{\Gamma_h^m} \nabla_{\Gamma_h^m} x_h^{m+1} \cdot \nabla_{\Gamma_h^m} \varphi_h \, \mathrm{d}A = 0 \qquad \forall \varphi_h \in X_h^m,$$

where X_h^m is the space of all continuous, piecewise linear functions on the polyhedral surface Γ_h^m. Here, τ denotes the time step size, id the identity map on Γ_h^m and $\nabla_{\Gamma_h^m}$ is the tangential gradient. The new surface is defined as $\Gamma_h^{m+1} = x_h^{m+1}(\Gamma_h^m)$. The author presented numerical examples that can be computed to a point of time very close to singularities. Still, to our knowledge, up to now no convergence proof has been found for this algorithm.

The scheme in [6] by Barrett, Garcke and Nürnberg is related to the approximation of Dziuk in [18] but adds intrinsic discrete tangential motion. This does not change the given evolution because the evolution in (1.1) is defined only by the normal part of the velocity. The authors have employed this technique in a series of papers including works on the curve shortening flow, e.g. in [7], and on the mean curvature flow, e.g. in [6]. Their motivation is that, when simulating geometric evolution equations such as the mean curvature flow, mesh degeneration seems to be a typical issue. In fact, as the evolution equation for the flow only determines the normal component of the velocity, numerical schemes based on (1.2) only move nodes in normal direction. As a consequence, in regions of high curvature nodes can concentrate, while in other regions the opposite can occur and as a result the mesh degenerates or causes lower accuracy. To avoid this, the algorithm in [6] includes a remeshing within the computation of the flow. Like in Dziuk's method, polyhedral surfaces Γ_h^m are constructed to approximate $\Gamma(t^m)$ and, in each time step, the new surface is parametrized via $\Gamma_h^{m+1} = x^{m+1}(\Gamma_h^m)$. The idea of the scheme is to find x_h^{m+1} and κ_h^{m+1} such that

$$
\begin{aligned}
\frac{1}{\tau}\int_{\Gamma_h^m}(x_h^{m+1} - id)\cdot(\phi_h \nu^m)\,\mathrm{d}A - \int_{\Gamma_h^m}\kappa_h^{m+1}\phi_h\,\mathrm{d}A = 0 \qquad \forall \phi_h \in X_h^m,\\
\int_{\Gamma_h^m}\kappa_h^{m+1}\nu^m\cdot\varphi_h\,\mathrm{d}A + \int_{\Gamma_h^m}\nabla_{\Gamma_h^m}x_h^{m+1}\cdot\nabla_{\Gamma_h^m}\varphi_h\,\mathrm{d}A = 0 \qquad \forall \varphi_h \in (X_h^m)^3,
\end{aligned}
$$

where X_h^m is the space of all continuous, piecewise linear, scalar functions on Γ_h^m and ν^m is the outward unit normal to Γ_h^m. The second equation imposes a condition on the normal that induces the desired tangential motion. The authors provide a proof for the good redistribution of mesh points for the semidiscrete problem. In the fully discrete case, they give numerical examples that show a good mesh behaviour. As the tangential motion in the scheme is added artificially, the discrete solution cannot converge toward the solution of the mean curvature flow, whose tangential velocity is zero. Whether the approximation converges to another limit, for example a flow by mean curvature with a prescribed tangential velocity, is an open question.

Recently in [35], Kovács, Li and Lubich chose a different approach to derive a surface finite element approximation of mean curvature flow. Instead of using (1.2) combined with a weak formulation of $\Delta_\Gamma id$ as done in [6] and [18], they discretize a weak formulation of the system of (1.2) together with the evolution equations of the unit normal and the mean curvature found by [31],

$$
\begin{aligned}
v &= H\nu, \\
\partial^\bullet \nu &= \Delta_{\Gamma(t)}\nu + |\nabla_{\Gamma(t)}\nu|^2 \nu, \\
\partial^\bullet H &= \Delta_{\Gamma(t)} H + |\nabla_{\Gamma(t)}\nu|^2 H,
\end{aligned}
$$

where $v(x,t)$ is the velocity at a point $x \in \Gamma(t)$ and $\partial^\bullet$ denotes the material derivative. ∇_Γ again is the tangential gradient and Δ_Γ is the Laplace-Beltrami operator on Γ. For the discretization, they use an evolving surface finite element method for the spatial variable together with a linearly implicit backward difference formula for the time variable. The authors derive convergence results for x, v, ν and H in the semi- as well as the fully discrete case and are the first to present an error analysis for an approximated mean curvature flow in the parametric setting. The estimate for the fully discrete version with respect to the spatial grid size h and the time step size τ in a norm of the error in the position on the surface then reads

$$
\|(x_h^n)^L - id\|_{H^1(\Gamma(t^n))} \leq c(h^k + \tau^q),
$$

where q is given by the choice of the q-step backward difference formula for $q \leq 5$ and $k \geq 2$ is the degree of the polynomials used in the finite element method. Here, $(x_h^n)^L$ is the lift of the position on the discrete surface onto the exact surface Γ at time t^n. The condition $k \geq 2$ arises from the requirement of controling the $W^{1,\infty}$-norm of the position error, where the H^1-error bound and an inverse inequality enter. Thus, for second order convergence with respect to h in the H^1-norm, quadratic polynomials are necessary. Still, for the cost of solving four additional equations, Kovács, Li and Lubich get error estimates for ν and H that do not result from an approach that is only based on (1.2). Yet the authors remark that it would be desirable to improve their algorithm with the help of tangential redistribution of mesh points, which is not provided by their approximation.

Because of the described issues connected to mesh degeneration and the question of convergence of an approximation of the flow by mean curvature with an additional tangential velocity, Elliott and Fritz in [22] presented a “built-in” approach. That means, they introduce intrinsic tangential movement to the evolution equation before the discretization. The invariance of the flow under tangential reparametrization implies that specifying a tangential velocity in the mean curvature flow leads to a solution that can be traced back to the solution of the original problem by reparametrization. In the literature, this has been exploited by making use of the so called DeTurck trick. It originates in a paper by DeTurck, [17], and was further specified in [28]. The concept was introduced in order to prove short-time existence for the Ricci flow, but has been used for similar results on other flows like the mean curvature flow, see for instance [3] and references therein. The idea of the trick consists in reparametrizing the mean curvature flow with diffeomorphisms that are solutions to the harmonic map heat flow.

Combining the heat flow with the reparametrized equation generates a flow which still represents the evolution of mean curvature flow but which is non-degenerate and strongly parabolic. It can be written as

$$x_t = \sum_{i,j=1}^{n} g^{ij} x_{u_i u_j} + \left(\frac{1}{\alpha} - 1\right) \sum_{i,j,k,l=1}^{n} g^{ij} g^{kl} (x_{u_i u_j} \cdot x_{u_k}) x_{u_l} \tag{1.4}$$

and will be called Mean Curvature DeTurck Flow in the following. The parameter α has the meaning of an inverse diffusion constant in the heat flow and will determine the scale of the tangential motion in the resulting equation.
Although the DeTurck trick originally was an analytic tool, it supplies a nice option that is of interest for numerical analysis and numerics of mean curvature flow, too, as the work [22] of Elliott and Fritz demonstrates. Note that the elliptic part of the Mean Curvature DeTurck Flow is not in divergence form and does not admit a weak formulation. For realizing a finite element approximation, the authors in [22] consequently aimed to simplify the spatial term. A first step in achieving this is another trick they performed on (1.4), by which the parameter α is shifted onto the time derivative and this way weights and separates the time components. This is motivated by the special case of the curve shortening flow, where this shifting leads to an equation from which a weak formulation can be derived directly. However, in the general case, this trick does not immediately yield a divergence form and some computations are necessary beforehand. For this reason, Elliott and Fritz also introduced a second variant of the DeTurck trick for which the resulting flow is almost in divergence form after the shifting of the parameter α. For both variants, the authors obtained a weak formulation and implemented it, but did not give an analytical convergence proof. Both algorithms induce tangential motion that leads to a redistribution of mesh points on the discrete surface. The experiments in [22] illustrate how this increases the mesh quality of their schemes, e.g. compared to the algorithm of Barrett, Garcke and Nürnberg in [6]. The latter performs well in many cases, but yet fails to converge in an example found by Elliott and Fritz, where the approximation in [22] leads to a good behaviour of the mesh.
In the case of curves, the authors in [22] provided a convergence analysis. In this case, the scheme can be considered as a generalization of the approximation in [14], which in turn results from [22] for $\alpha = 1$. Error bounds have been given in [14] and the proof has been adapted by the authors in [22]. The choice $\alpha = 0$, which is only possible after the trick that splits the time derivative, unveils a link to the scheme in [7] for curves. As mentioned above, no error analysis exists for the schemes of Barrett, Garcke and Nürnberg. The error bound in [22] indicates a possible reason: The constant depends exponentially on α^{-1} and thus blows up for α approaching zero.

The above approximations all have in common that they are based on the finite element method. Seemingly, discretizing via finite differences has not yet been used for parametric mean curvature flow. However, in [11], the authors applied the finite difference method to the level set formulation of the flow. A corresponding error estimate has been proved in [12]. Other convergence results have been obtained for the mean curvature flow of graphs, see e.g. [13] and [15]. In both works, optimal error bounds were obtained. A general overview of numerical approximations of geometric evolution equations like the mean curvature flow can be found in [5], [16] and [20].

In this thesis, we let $n = 2$ and consider the flow of hypersurfaces of torus type. We choose the reference manifold $\mathcal{M}$ to be the standard torus and parametrize it globally on $\Omega = [0, 2\pi]^2$ so that $\Gamma(t) = x(\Omega, t)$ for a map $x : \Omega \times [0, T] \to \mathbb{R}^3$, which we suppose to be 2π-periodic in both spatial variables. The aim is to construct an approximation based on the Mean Curvature DeTurck Flow (1.4), which we use in order to prescribe a tangential velocity and thus to overcome the obstacles connected to the degeneration. Moreover, the goal is to prove error estimates for the approximation of the reparametrized flow.
Our approach differs widely from that of Elliott and Fritz in [22]. First of all, we do not make use of their trick that separates the time components but discretize (1.4) directly despite the, to some extent, difficult spatial differential operator. Secondly, in order to avoid a weak formulation, we discretize via a finite difference method. For a spatial grid size $h = \frac{2\pi}{N}$ and a time grid size $\tau = \frac{T}{M}$ with $u_{k,l} = (kh, lh)$ and $t^s = s\tau$ we define the mesh $\mathcal{G} = \{(u_{k,l}, t^s)\}_{k,l\in\{0,\dots,N\},s\in\{0,\dots,M\}} \subset [0, 2\pi]^2 \times [0, T]$. The fully discrete problem reads:
For a function $x_h : \mathcal{G} \to \mathbb{R}^3$ that is 2π-periodic with respect to the spatial grid we demand that for all $k, l \in \{0, \dots, N\}$ and $s \in \{0, \dots, M-1\}$: $x^0_{k,l} = x^0$ and

$$\frac{x^{s+1}_{k,l} - x^s_{k,l}}{\tau} = \sum_{i,j=1}^{2} g^{ij,s}_{k,l} \Delta_{ij} x^{s+1}_{k,l} + \left(\frac{1}{\alpha} - 1\right) \sum_{i,j,m,n=1}^{2} g^{ij,s}_{k,l} g^{mn,s}_{k,l} (\Delta_{ij} x^{s+1}_{k,l} \cdot \overline{\Delta}_m x^s_{k,l}) \overline{\Delta}_n x^s_{k,l},$$

where Δ_{ij} denote second order and $\overline{\Delta}_r$ denote first order difference operators, $x^s_{k,l}$ is the evaluation of x_h at a gridpoint $(u_{k,l}, t^s) \in \mathcal{G}$ and $g^{ij,s}_{k,l}$ are suitable discrete inverse metric coefficients. By choosing a semi-implicit time differencing, the nonlinearity in the unknown of the numerical scheme is removed. Hence, in each time step, a linear system of equations has to be solved.
We assume that (1.4) has a smooth solution $x : \Omega \times [0, T] \to \mathbb{R}^3$ that is regular in the sense that $g \geq 2\bar{c} > 0$. We denote by $e^s_{k,l}$ the evaluation of the error $e_h := x - x_h$ at the grid points. Our main results are the following optimal order error estimates in discrete integral norms.

Theorem. *Let $\alpha \in (0, 1]$. There exist positive constants c, c' and h^*, such that for all $0 < h \leq h^*$ and $\tau \leq c'h^2$ the estimates*

$$\max_{s\in\{0,\dots,M\}}\left(h^2\sum_{k,l=1}^{N}|e_{k,l}^s|^2+h^2\sum_{k,l=1}^{N}\sum_{r=1}^{2}\frac{|\Delta_r^- e_{k,l}^s|^2}{h^2}\right)^{1/2}\le c(h^2+\tau),$$
$$\left(\tau\sum_{s=1}^{M}\left(h^2\sum_{k,l=1}^{N}\frac{|e_{k,l}^{s+1}-e_{k,l}^s|^2}{\tau^2}+h^2\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}\frac{|\Delta_{ij}^* e_{k,l}^s|^2}{h^4}\right)\right)^{1/2}\le c(h^2+\tau),$$
$$\max_{s\in\{0,\dots,M\}}\max_{k,l\in\{1,\dots,N\}}|e_{k,l}^s|\le c|\ln(h)|^{\frac12}(h^2+\tau)$$

hold and the constants only depend on x, T and α^{-1}.

We use energy methods to derive these estimates from the error equation for our fully discrete approximation. For the sake of convenience, we supply the reader with an illustration of some key steps in a continuous example. Consider the solution (1.4) for $\alpha = 1$ and test the equation with $-\Delta x$:

$$-\sum_{r=1}^{2}\int_\Omega x_t\cdot x_{u_ru_r}=-\sum_{i,j,r=1}^{2}\int_\Omega g^{ij}x_{u_iu_j}\cdot x_{u_ru_r}.$$

Integration by parts yields

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\sum_{r=1}^{2}\int_\Omega|x_{u_r}|^2=\sum_{r=1}^{2}\int_\Omega x_{tu_r}\cdot x_{u_r}\quad=-\sum_{i,j,r=1}^{2}\int_\Omega g^{ij}x_{u_iu_r}\cdot x_{u_ju_r}+\int_\Omega G(x),$$

where $G(x)$ summarizes lower order terms. The sum of second spatial derivatives of x can then be estimated by the smallest eigenvalue λ of (g^{ij}) and a norm of the second derivatives,

$$\lambda\sum_{i,r=1}^{2}|x_{u_iu_r}|^2\le\sum_{i,j,r=1}^{2}g^{ij}x_{u_iu_r}\cdot x_{u_ju_r},$$

so that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\sum_{r=1}^{2}\int_\Omega|x_{u_r}|^2+\lambda\sum_{i,j=1}^{2}\int_\Omega|x_{u_iu_j}|^2\le\int_\Omega G(x).$$

By controlling the integral of G, integrating over t and applying the lemma of Gronwall we can infer bounds on the $L^2(\Omega)$-norms of first and second derivatives. To obtain the discrete L^2-norms of first and second order differences displayed above, we transfer this proceeding onto a discrete level. That means testing the error equation with a suitable discrete Laplacian, summation by parts and applying the lemma of Gronwall in a discrete version. The most interesting and involved of the results are the estimates for the discrete first order spatial derivatives, where superconvergence effects lead to a second order convergence in space. This enables us to control the geometry of the discrete surface within an inductive argument. In particular, due to the quadratic convergence in the discrete H_0^1-norm, an inverse estimate yields a $W^{1,\infty}$- bound for e_h that is linear in the spatial grid size h. This is crucial to infer a uniform in space bound on the discrete area element by imposing a smallness condition on h.

In the numerical illustrations of this work we are interested in the tangential motion induced by our scheme, especially in a comparison for different choices of α. Because of the good results in [22] for small α, we expect similar results in our experiments. Note that, in contrast to [22], we are not able to choose $\alpha = 0$ in our equations. In the case $\alpha = 1$, our scheme approximates the differential equation

$$x_t = \sum_{i,j=1}^{2} g^{ij} x_{u_i u_j}.$$

Although an analog equation for curves has been approximated and analysed in [14], this seems not to be the case for surfaces. Apparently, no existing approximation with a convergence proof for mean curvature flow of surfaces can be linked to the DeTurck trick so far.

The outline of this thesis is as follows. Chapter 2 supplies basic notation and sketches how the reparametrization works. In Chapter 3 some information on the chosen finite differences in combination with the setting are collected. This explains how the periodic boundary conditions are integrated into the analysis. In particular, some formulae for summation by parts are presented since they are one important instrument in the proof of convergence. The fully discrete approximation is presented and its consistency is shown. This estimate makes for a first contribution to the convergence proof which is conducted in Chapter 4. We set up an induction to derive some necessary bounds from the induction hypothesis in the first paragraph. The error estimates for discrete first and second order derivatives, where superconvergence effects occur, are achieved in the induction step and can be considered as the main part of the work. Estimates in other discrete norms are of interest, too, but are less involved and can be derived directly once the convergence of the first and second discrete derivatives is established. Chapter 5 is devoted to numerical examples for torus-like surfaces. These comprise an illustration of the shrinking property of the flow by mean curvature, a study of the mesh quality for different choices of the parameter α and the experimental order of convergence. The results are recapulated and resumed in the last part of the thesis, where finally a short outlook on possible pursuing work is given.

2 Background information

The first part of this chapter is devoted to providing basic knowledge that is needed for the approximation and the convergence analysis presented in this work. In the second part, a derivation of the Mean Curvature DeTurck Flow is given.

2.1 Basics

Elementary inequalities

The following inequality is known in a more general formulation and proved as such in the literature given below. We state a special case that we frequently use in our analysis.

2.1 Theorem (A Young inequality)**.** *For $a, b \geq 0$ and $\varepsilon > 0$ we have*

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2. \tag{Y}$$

Proof. Follows from IV.2.15 in [1] for $\xi = \sqrt{2\varepsilon} a$, $\eta = \frac{1}{\sqrt{2\varepsilon}} b$ and $p = p' = 2$. □

We now formulate a discrete analog of the lemma of Gronwall.

2.2 Theorem (A discrete lemma of Gronwall)**.** *Let $m \in \mathbb{N}$, $I = \{0, \dots, m\}$ and K be a nonnegative constant. Suppose $(z_i)_{i \in I}$ and $(w_i)_{i \in I}$ are nonnegative sequences in $\mathbb{R}$. If for all $l \in I$*

$$z_l \leq K + \sum_{i=0}^{l-1} w_i z_i,$$

then for all $l \in I$

$$z_l \leq K \, \exp\left(\sum_{i=0}^{l-1} w_i \right).$$

Proof. See [29]. □

Basic differential geometry

We next aim to introduce some basic facts from differential geometry and thereby settle some notation. These explanations are restricted on formulations in local coordinates. For further information we refer to [36].
In what follows, let $\Omega \subset \mathbb{R}^2$ and Γ an immersed surface that is locally parametrized by $x : \Omega \to \mathbb{R}^3$. That means $\Gamma = x(\Omega)$ and at each $u \in \Omega$ the Jacobi matrix $Dx(u)$ has rank 2. We suppose $x \in C^1(\Omega)$. Let $p = x(u)$ with coordinates $u = (u_1, u_2) \in \Omega$. We make use of the convention to sum over repeated indices. By $\cdot$ we denote the

Euclidean scalar product in $\mathbb{R}^3$ and by $|\cdot|$ its induced norm. To shorten the notation of the partial derivative of f, we index the function with the corresponding variable: $f_{u_i} := \frac{\partial f}{\partial u_i}$. Sometimes, $\partial^\gamma = \partial^{(\gamma_1,\gamma_2,\gamma_3)}$ is used as a differential operator with multi-index when partial derivatives of order three or four occur to shorten the notation.

2.3 Definition (Tangent plane)**.** *At each $u \in \Omega$ we define the tangent plane as*

$$T_u x := span\{x_{u_1}(u), x_{u_2}(u)\}.$$

Formally, the first fundamental form maps each point $p \in \Gamma$ at the restriction of $\cdot$ on the tangent plane at p. Because we always have a basis of $T_u x$ due to the parametrization $x(u) = p$, we can represent the scalar product by a matrix and thus formulate the following definition that is not the most general, but sufficient for our setting.

2.4 Definition (First fundamental form)**.** *At each $p = x(u) \in \Gamma$, the first fundamental form can be represented by the positive definite matrix (g_{ij}). For $i, j \in \{1, 2\}$ the matrix entries are*

$$g_{ij} = x_{u_i} \cdot x_{u_j}.$$

We will also refer to the g_{ij} as metric coefficients. The determinant of (g_{ij}) is denoted by g, i.e.

$$g = \det((g_{ij})) = |x_{u_1}|^2|x_{u_2}|^2 - (x_{u_1} \cdot x_{u_2})^2.$$

With a small abuse of notation we will call g the area element instead of $\sqrt{g}$.
Of importance is also the matrix $(g^{ij}) = (g_{ij})^{-1}$, where

$$(g^{ij}) = \frac{1}{g}\begin{pmatrix} |x_{u_2}|^2 & -(x_{u_1} \cdot x_{u_2}) \\ -(x_{u_1} \cdot x_{u_2}) & |x_{u_1}|^2 \end{pmatrix},$$

containing what we call inverse metric coefficients. Note that this relation implies that $g_{ij}g^{jk} = \delta_{ik}$, where δ_{ik} is the Kronecker-Delta.

2.5 Definition (Unit normal field)**.** *A map $\nu : \Omega \to \mathbb{R}^3$ is called unit normal field of Γ, if $\nu(u) \perp T_u x$ and $|\nu(u)| = 1$. In terms of local coordinates we can compute*

$$\nu = \pm \frac{x_{u_1} \times x_{u_2}}{\|x_{u_1} \times x_{u_2}\|}.$$

Like for the first fundamental form, also for the second fundamental form we do not give a general definition, but restrict it to a formulation that is possible due to the local parametrization.

2.6 Definition (Second fundamental form)**.** *At each $p = x(u) \in \Gamma$, the second fundamental form can be represented by the matrix (h_{ij}). For $i, j \in \{1, 2\}$ the matrix entries* *are*

$$h_{ij} = \nu \cdot x_{u_i u_j}.$$

In the literature, the mean curvature H is defined as the sum (or the arithmetic mean) of the principal curvatures of a surface, that is the sum (or the arithmetic mean) of the eigenvalues of the Weingarten map. This way of introducing the curvature is not necessary here: Having the second fundamental form at hand and working in local coordinates, we can give the following formula.

2.7 Definition (Mean curvature)**.** *At each $p = x(u) \in \Gamma$, the mean curvature H of a surface Γ is given as*

$$H = g^{ij} h_{ij}.$$

2.8 Remark. *ν and hence H are up to a sign independent of the choice of parametrization. Since a change of the sign in ν implies a change of sign in H, the product $H\nu$ is independent of parametrization and sign.*

The following statements justify the use of the definitions of mean curvature flow given in (1.2) and (1.3).

2.9 Lemma. *With H, ν and g^{ij} given as above, the following identity holds:*

$$H\nu = g^{ij} x_{u_i u_j} - g^{ij} g^{kl} (x_{u_i u_j} \cdot x_{u_k}) x_{u_l}. \tag{2.1}$$

Proof. Note that $g^{ij} g^{kl} (x_{u_i u_j} \cdot x_{u_k}) x_{u_l} \in T_u x$ is the projection of $g^{ij} x_{u_i u_j}$ onto the tangent plane. To see this, we show that for any vector $w \in \mathbb{R}^3$ the term $w - g^{kl}(w \cdot x_{u_k}) x_{u_l}$ is orthogonal to $T_u x$:

$$(w - g^{kl}(w \cdot x_{u_k}) x_{u_l}) \cdot x_{u_m} = w \cdot x_{u_m} - g^{kl}(w \cdot x_{u_k}) g_{lm} = w \cdot x_{u_m} - \delta_{km}(w \cdot x_{u_k}) = 0.$$

The right hand-side of (2.1) is thus a vector of normal direction and we can write

$$g^{ij} x_{u_i u_j} - g^{ij} g^{kl} (x_{u_i u_j} \cdot x_{u_k}) x_{u_l} = \beta \nu$$

for some $\beta \in \mathbb{R}$. Multiplying by ν we infer

$$\beta = g^{ij} x_{u_i u_j} \cdot \nu - g^{ij} g^{kl} (x_{u_i u_j} \cdot x_{u_k})(x_{u_l} \cdot \nu) = g^{ij} x_{u_i u_j} \cdot \nu.$$

The above definitions of H and h_{ij} yield

$$H = g^{ij} x_{u_i u_j} \cdot \nu = \beta$$

and thus the proof is completed. □

2.10 Lemma. *With g and g^{ij} given as above, the following identity holds:*

$$\frac{1}{\sqrt{g}}\left(g^{ij}\sqrt{g}x_{u_j}\right)_{u_i} = g^{ij}x_{u_iu_j} - g^{ij}g^{kl}(x_{u_iu_j}\cdot x_{u_k})x_{u_l}. \tag{2.2}$$

Proof. Because of

$$\frac{1}{\sqrt{g}}\left(g^{ij}\sqrt{g}x_{u_j}\right)_{u_i} = g^{ij}x_{u_iu_j} + (g^{ij})_{u_i}x_{u_j} + \frac{1}{\sqrt{g}}g^{ij}(\sqrt{g})_{u_i}x_{u_j}, \tag{2.3}$$

it remains to find the derivatives of g^{ij} and $\sqrt{g}$. Since

$$(g^{ij})_{u_r} = (g^{ik})_{u_r}\delta_{kj} = (g^{ik})_{u_r}g_{kl}g^{lj} = -g^{ik}(g_{kl})_{u_r}g^{lj},$$

where equality in the last step holds through $0 = (g^{ik}g_{kl})_{u_r} = (g^{ik})_{u_r}g_{kl} + g^{ik}(g_{kl})_{u_r}$, we have that

$$(g^{ij})_{u_i} = -g^{ik}g^{lj}(g_{kl})_{u_i}.$$

Furthermore, from

$$g_{u_i} = \frac{\partial g}{\partial g_{kl}}\frac{\partial g_{kl}}{\partial u_i} = g^{kl}g(g_{kl})_{u_i}$$

it follows that

$$(\sqrt{g})_{u_i} = \frac{1}{2\sqrt{g}}g_{u_i} = \frac{1}{2}\sqrt{g}g^{kl}(g_{kl})_{u_i}.$$

Thus,

$$\begin{aligned}(g^{ij})_{u_i}x_{u_j} + \frac{1}{\sqrt{g}}g^{ij}(\sqrt{g})_{u_i}x_{u_j} =& - g^{ik}g^{lj}(g_{kl})_{u_i}x_{u_j} + \frac{1}{\sqrt{g}}g^{ij}\frac{1}{2}\sqrt{g}g^{kl}(g_{kl})_{u_i}x_{u_j}\\ =& - g^{ik}g^{lj}(x_{u_ku_i}\cdot x_{u_l})x_{u_j} - g^{ik}g^{lj}(x_{u_k}\cdot x_{u_lu_i})x_{u_j}\\ &+ \frac{1}{2}g^{ij}g^{kl}(x_{u_ku_i}\cdot x_{u_l})x_{u_j} + \frac{1}{2}g^{ij}g^{kl}(x_{u_k}\cdot x_{u_lu_i})x_{u_j}.\end{aligned}$$

By renaming indices in the summand $g^{ik}g^{lj}(x_{u_k}\cdot x_{u_lu_i})x_{u_j}$ successively, precisely i to k, k to l and then l to i, the term is reformulated to $g^{kl}g^{ij}(x_{u_l}\cdot x_{u_iu_k})x_{u_j}$. Hence

$$\begin{aligned}(g^{ij})_{u_i}x_{u_j} + \frac{1}{\sqrt{g}}g^{ij}(\sqrt{g})_{u_i}x_{u_j} =& - g^{ik}g^{lj}(x_{u_ku_i}\cdot x_{u_l})x_{u_j}\\ &- \frac{1}{2}g^{ij}g^{kl}(x_{u_ku_i}\cdot x_{u_l})x_{u_j} + \frac{1}{2}g^{ij}g^{kl}(x_{u_k}\cdot x_{u_lu_i})x_{u_j}.\end{aligned}$$

Switching k and l in the last term and making use of the symmetry of g^{kl} we obtain

$$(g^{ij})_{u_i}x_{u_j} + \frac{1}{\sqrt{g}}g^{ij}(\sqrt{g})_{u_i}x_{u_j} = -g^{ik}g^{lj}(x_{u_ku_i}\cdot x_{u_l})x_{u_j} \tag{2.4}$$

and the assertion of the lemma holds. □

2.11 Corollary. *Equations (1.2) and (1.3) yield equivalent definitions of the motion by mean curvature.*

Proof. The assertion is a direct consequence of (2.1) and (2.2). □

2.2 Mean Curvature DeTurck Flow

In what follows we explain the derivation of the Mean Curvature DeTurck Flow as given in (1.4). The flow arises when the mean curvature flow is reparametrized with a solution to the harmonic map heat flow. Showing every detail exceeds the intention of this work, we rather want to give the idea of this procedure, which is a variant of the so called the DeTurck trick. In contrast to a derivation which is given by the authors in [22], we only work with local coordinates here, as our further computations are entirely based on these fomulations.

Let $x : [0, 2\pi]^2 \times [0, T] \to \mathbb{R}^3$ be a smooth parametrization of a family of hypersurfaces $\Gamma(t)$ of torus type that move according to mean curvature flow. Recall that we choose the reference manifold $\mathcal{M}$ to be the standard torus that can be parametrized globally on $[0, 2\pi]^2$. Let g_{ij} denote the metric coefficients with respect to the coordinates u_1, u_2, i.e. $g_{ij}(u,t) = x_{u_i}(u,t) \cdot x_{u_j}(u,t)$, g^{ij} the corresponding inverse metric coefficients and $g = \det((g_{ij}))$.

The Harmonic Map Heat Flow is given by

$$\alpha\eta_t^\alpha = \frac{1}{\sqrt{g}} \left(g^{ij} \sqrt{g} \eta_{u_j}^\alpha \right)_{u_i}, \tag{2.5}$$

where α is a positive constant. We assume $\eta^\alpha(\cdot, t) : [0, 2\pi]^2 \to [0, 2\pi]^2$ to be a diffeomorphism for every $t \in [0, T]$ with the initial condition $\eta^\alpha(\cdot, 0) = id(\cdot)$ for the identical map id on $[0, 2\pi]^2$. We also require the Jacobi matrix of η^α to fulfill $\det(D\eta^\alpha) > 0$.

We reparametrize by defining, for every fixed t, $\bar{x}^\alpha := x \circ (\eta^\alpha)^{-1}$. That means, with $\eta^\alpha = (\eta_1^\alpha, \eta_2^\alpha)^T$ we have that

$$x(u_1, u_2, t) = \bar{x}^\alpha(\eta_1^\alpha(u_1, u_2, t), \eta_2^\alpha(u_1, u_2, t), t) \qquad \forall (u_1, u_2, t) \in [0, 2\pi]^2 \times [0, T].$$

Let $\bar{u}_1 = \eta_1^\alpha(u_1, u_2, t)$ and $\bar{u}_2 = \eta_2^\alpha(u_1, u_2, t)$ denote the new coordinates. In what follows, we compute both flows with respect to $\bar{u}_1$ and $\bar{u}_2$. We have

$$x_{u_j} = (\bar{x}^\alpha \circ \eta^\alpha)_{u_j} = \eta_{k,u_j}^\alpha (\bar{x}_{\bar{u}_k}^\alpha \circ \eta^\alpha) \tag{2.6}$$

as well as

$$x_t = \bar{x}_t^\alpha + \eta_{k,t}^\alpha (\bar{x}_{\bar{u}_k}^\alpha \circ \eta^\alpha).$$

For the sake of simplicity, we omit the composition with η^α in the subsequent calculations and advise the reader to keep in mind that x and $\bar{x}^\alpha$ are defined for different parameters.
Let $\bar{g} = \det(\bar{g}_{ij})$ and $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$ be defined through the parametrization $\bar{x}^\alpha$ in the coordinates $\bar{u}_i$, in particular $\bar{g}_{ij} = \bar{x}^\alpha_{\bar{u}_1} \cdot \bar{x}^\alpha_{\bar{u}_2}$. Let $\bar{g}$ denote the corresponding area element. Note that the metric depends on α.

2.12 Lemma. *The reparametrized mean curvature flow is given as*

$$\bar{x}^\alpha_t + \eta^\alpha_{k,t}\bar{x}^\alpha_{\bar{u}_k} = \frac{1}{\sqrt{\bar{g}}}\left(\bar{g}^{ij}\sqrt{\bar{g}}\bar{x}^\alpha_{\bar{u}_j}\right)_{\bar{u}_i}. \tag{2.7}$$

Proof. The time derivative of x in the mean curvature flow is given as the product of the mean curvature H with the normal field ν and this product is independent of the choice of parametrization, compare Remark 2.8. Thus,

$$x_t = \frac{1}{\sqrt{g}}\left(g^{ij}\sqrt{g}x_{u_j}\right)_{u_i} = H\nu = \frac{1}{\sqrt{\bar{g}}}\left(\bar{g}^{ij}\sqrt{\bar{g}}\bar{x}^\alpha_{\bar{u}_j}\right)_{\bar{u}_i}. \tag{2.8}$$

□

2.13 Lemma. *The harmonic map heat flow can be expressed by*

$$\alpha\eta^\alpha_{k,t}\bar{x}^\alpha_{\bar{u}_k} = (\bar{g}^{ij})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j} + \frac{1}{\sqrt{\bar{g}}}\bar{g}^{ij}(\sqrt{\bar{g}})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j}. \tag{2.9}$$

Sketch of the proof. We will not execute every computation in detail. To begin with, we explain how to prove the following intermediate result:

$$g^{ij}x_{u_iu_j} = g^{ij}\eta^\alpha_{k,u_iu_j}\bar{x}^\alpha_{\bar{u}_k} + \bar{g}^{ij}\bar{x}^\alpha_{\bar{u}_i\bar{u}_j}. \tag{2.10}$$

We use (2.6) to calculate

$$x_{u_iu_j} = \eta^\alpha_{k,u_iu_j}\bar{x}^\alpha_{\bar{u}_k} + \eta^\alpha_{k,u_i}\eta^\alpha_{l,u_j}\bar{x}^\alpha_{\bar{u}_k\bar{u}_l}.$$

Because of $x = \bar{x}^\alpha \circ \eta^\alpha$, for the induced metric we have $(g_{ij}) = (D\eta^\alpha)^T(\bar{g}_{ij})D\eta^\alpha$ and for the inverse $(g^{ij}) = (D\eta^\alpha)^{-1}(\bar{g}^{ij})(D\eta^\alpha)^{-T}$. Thus, $(\bar{g}^{ij}) = D\eta^\alpha(g^{ij})(D\eta^\alpha)^T$ and, since $D\eta^\alpha = (\eta^\alpha_{i,u_j})_{i,j\in\{1,2\}}$,

$$\bar{g}^{ij} = \eta^\alpha_{i,u_k}g^{kl}\eta^\alpha_{j,u_l}.$$

We infer

$$\begin{aligned} g^{ij}x_{u_iu_j} - \bar{g}^{ij}\bar{x}^\alpha_{\bar{u}_i\bar{u}_j} &= g^{ij}\eta^\alpha_{k,u_iu_j}\bar{x}^\alpha_{\bar{u}_k} + g^{ij}\eta^\alpha_{k,u_i}\eta^\alpha_{l,u_j}\bar{x}^\alpha_{\bar{u}_k\bar{u}_l} - \eta^\alpha_{i,u_k}g^{kl}\eta^\alpha_{j,u_l}\bar{x}^\alpha_{\bar{u}_i\bar{u}_j} \\ &= g^{ij}\eta^\alpha_{k,u_iu_j}\bar{x}^\alpha_{\bar{u}_k}, \end{aligned}$$

i.e. the above claim (2.10) is true.

Reformulating the harmonic map heat flow and thereby applying (2.3) repeatedly we finally have that

$$\begin{aligned}
\alpha\eta^\alpha_{k,t}\bar{x}^\alpha_{\bar{u}_k} &\overset{(2.5)}{=} g^{ij}\eta^\alpha_{k,u_iu_j}\bar{x}^\alpha_{\bar{u}_k} + (g^{ij})_{u_i}\eta^\alpha_{k,u_j}\bar{x}^\alpha_{\bar{u}_k} + \frac{1}{\sqrt{g}}g^{ij}(\sqrt{g})_{u_i}\eta^\alpha_{k,u_j}\bar{x}^\alpha_{\bar{u}_k} \\
&\overset{(2.6)}{=} g^{ij}\eta^\alpha_{k,u_iu_j}\bar{x}^\alpha_{\bar{u}_k} + (g^{ij})_{u_i}x_{u_j} + \frac{1}{\sqrt{g}}g^{ij}(\sqrt{g})_{u_i}x_{u_j} \\
&\overset{(2.8)}{=} g^{ij}\eta^\alpha_{k,u_iu_j}\bar{x}^\alpha_{\bar{u}_k} + \frac{1}{\sqrt{\bar{g}}}\left(\bar{g}^{ij}\sqrt{\bar{g}}\bar{x}^\alpha_{\bar{u}_j}\right)_{\bar{u}_i} - g^{ij}x_{u_iu_j} \\
&\overset{(2.10)}{=} (\bar{g}^{ij})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j} + \frac{1}{\sqrt{\bar{g}}}\bar{g}^{ij}(\sqrt{\bar{g}})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j}.
\end{aligned}$$

□

Proceeding with the DeTurck trick, we combine the reparametrized mean curvature flow (2.7) with the harmonic map heat flow (2.9).

2.14 Corollary. *The Mean Curvature DeTurck Flow can be written as*

$$\bar{x}^\alpha_t = \bar{g}^{ij}\bar{x}^\alpha_{\bar{u}_i\bar{u}_j} + \left(\frac{1}{\alpha} - 1\right)\bar{g}^{ij}\bar{g}^{kl}(\bar{x}^\alpha_{\bar{u}_i\bar{u}_j}\cdot\bar{x}^\alpha_{\bar{u}_k})\bar{x}^\alpha_{\bar{u}_l}.$$

Proof. Combining the reparametrized evolution equations we conclude

$$\begin{aligned}
\bar{x}^\alpha_t &\overset{(2.7)}{=} -\eta^\alpha_{k,t}\bar{x}^\alpha_{\bar{u}_k} + \frac{1}{\sqrt{\bar{g}}}\left(\bar{g}^{ij}\sqrt{\bar{g}}\bar{x}^\alpha_{\bar{u}_j}\right)_{\bar{u}_i} \\
&\overset{(2.9)}{=} -\frac{1}{\alpha}\left((\bar{g}^{ij})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j} + \frac{1}{\sqrt{\bar{g}}}\bar{g}^{ij}(\sqrt{\bar{g}})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j}\right) + \frac{1}{\sqrt{\bar{g}}}\left(\bar{g}^{ij}\sqrt{\bar{g}}\bar{x}^\alpha_{\bar{u}_j}\right)_{\bar{u}_i} \\
&\overset{(2.3)}{=} \bar{g}^{ij}\bar{x}^\alpha_{\bar{u}_i\bar{u}_j} + \left(1 - \frac{1}{\alpha}\right)\left((\bar{g}^{ij})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j} + \frac{1}{\sqrt{\bar{g}}}\bar{g}^{ij}(\sqrt{\bar{g}})_{\bar{u}_i}\bar{x}^\alpha_{\bar{u}_j}\right).
\end{aligned}$$

Applying (2.4) yields the evolution equation for the Mean Curvature DeTurck Flow presented in (1.4). □

2.15 Remark.

- *Note that by Corollary 2.11, the non-reparametrized mean curvature evolution is given by*
$$x_t = g^{ij}x_{u_iu_j} - g^{ij}g^{kl}(x_{u_iu_j}\cdot x_{u_k})x_{u_l}.$$
In comparison with the Mean Curvature DeTurck Flow, observe the additional term that has tangential direction.

- *As the Mean Curvature DeTurck Flow is a strongly parabolic equation, a short time existence result follows from the theory of parabolic partial differential equations. We mentioned in our introduction that this fact can then be used to prove* *short time existence for the original mean curvature flow problem, since a solution to (1.3) exists if a solution to (1.4) exists, compare Proposition* 3.27 *in [3]. The same applies for the uniqueness of this solution.*

In what follows, we set $\bar{u}_i = u_i$ and write x and instead of $\bar{x}^\alpha$, but keep in mind that a whole family of solutions of (1.4) depending on α exists.

3 Finite Difference Approximation

In this chapter, a finite difference approximation of the Mean Curvature DeTurck Flow is introduced. In the first section some basic properties of the difference operators in use are explained. After presenting the scheme for the discretization of the flow in the second section, we calculate the consistency error.

3.1 Difference operators on the spatial grid

In the following section we introduce notations for some finite differences which are relevant in this work as well as relations between them. The latter is connected to the given setting of periodic boundary conditions and only concerns the spatial variable. Since in addition to that the time discretization which we want to employ is rather simple, we restrict our attention to the spatial one in this section.

First of all, we need a grid in $[0, 2\pi]^2$ on which the discrete equations can be defined. That means a domain for the so called grid function that approximates the smooth solution x of (1.4).
To begin with, we consider one space dimension case. Let $\{u_k\}_{k\in\{0,\dots,N\}} \subset [0, 2\pi]$ with $N \in \mathbb{N}$ be a grid dividing $[0, 2\pi]$ into subintervals of equal length h, i.e. $u_k = kh$ and $h = 2\pi/N$. For a function $f : \{u_k\}_{k\in\{0,\dots,N\}} \to \mathbb{R}^3$ we denote by $f_k := f(u_k)$ the evaluation of f at the grid points. For the forward and backward differences we write

$$\Delta^+ f_k := f_{k+1} - f_k \qquad \text{and} \qquad \Delta^- f_k := f_k - f_{k-1}.$$

Note that we have

$$\Delta^+ f_k = \Delta^- f_{k+1} \qquad \text{and} \qquad \Delta^- f_k = \Delta^+ f_{k-1}. \tag{3.1}$$

By addition respectively composition of Δ^+ and Δ^- we derive useful central differences, namely

$$\begin{aligned} \overline{\Delta} f_k &:= \tfrac{1}{2}(\Delta^+ + \Delta^-) f_k = \tfrac{1}{2}(f_{k+1} - f_{k-1}), \\ \Delta^+\Delta^- f_k &= f_{k+1} - 2f_k + f_{k-1}. \end{aligned} \tag{3.2}$$

In two dimensions, a second direction is added to the one-dimensional grid in order to divide $[0, 2\pi]^2$ into squares of equal edge length. The resulting mesh is denoted by $\{u_{k,l}\}_{k,l\in\{0,\dots,N\}}$, where $u_{k,l} = (kh, lh)$. By $f_{k,l}$ we denote the evaluation $f(u_{k,l})$ of a function f defined on the grid. We call h the mesh size of the grid. An index $i \in \{1, 2\}$ is attached to the forward and backward difference operators $\Delta^\pm$ as well as $\overline{\Delta}$, referring to the first respectively second component of the vector $u_{k,l}$ and therefore indicating for which direction the difference is formed.
We aim to construct a finite difference method that is consistent of order two. For x_{u_i} and $x_{u_i u_i}$, $i \in \{1, 2\}$, this order can be achieved by using the central differences in

(3.2). We approximate the mixed derivative $x_{u_1u_2}$ with the help of the difference

$$\begin{aligned}\Delta_{12} :=& \tfrac{1}{2}(\Delta_1^+\Delta_2^+ + \Delta_1^-\Delta_2^-)f_{k,l}\\ =& \tfrac{1}{2}(f_{k+1,l+1} - f_{k+1,l} - f_{k,l+1} + 2f_{k,l} - f_{k,l-1} - f_{k-1,l} + f_{k-1,l-1})\end{aligned}$$

since it has the required order of consistency. Moreover, it can be analyzed because of the grid points that are involved. For alternative approximations of $x_{u_1u_2}$ as well as general information on finite differences we refer to [27].

3.1 Remark. *In general, an explicit treatment of the mixed derivative via finite differences is not trivial. As the author in [27] points out, it is therefore desirable to have spatial operators in divergence form. This is of course also the case for finite element methods, where a weak formulation has to be found. We explained in our introduction how this has been handled in the literature for the mean curvature flow, which is not in divergence form. Compared to the issues that occured there, choosing a finite difference method with the above approximation for $x_{u_1u_2}$ seems to be a rather practicable way.*

Despite the necessity to use a relatively complex approximation that includes seven grid points for the mixed derivative, in the formulation of our convergence result only the backward difference $\Delta_1^-\Delta_2^-$ is used. This is due to the fact that the forward part can later be traced to the backward part (as shown in 2. in Lemma 3.5). The orders of the approximations will be proved in the next section when the time variable is considered. This needs to be taken into account because of our semi-implicit choice of time discretization.

As a compact notation for difference operators of second order we introduce

$$\Delta_{ij} = \begin{cases} \Delta_i^+\Delta_i^- & \text{for } i = j \in \{1,2\},\\ \frac{1}{2}(\Delta_1^+\Delta_2^+ + \Delta_1^-\Delta_2^-) & \text{for } i \neq j. \end{cases} \tag{3.3}$$

3.2 Remark. *All difference operators work componentwise. They commute in the sense that*

$$\Delta_i^{\circ_1}\Delta_j^{\circ_2} f_{k,l} = \Delta_j^{\circ_2}\Delta_i^{\circ_1} f_{k,l},$$

where $\circ_1$ and $\circ_2$ stand for $+$ or $-$. This can be seen by the symmetry of the differences they produce in the case $i = j$ or $\circ_1 = \circ_2$. A short computation shows the commutativity in the remaining cases. Furthermore, the product rule

$$\Delta^-(f_{1,k}f_{2,k}) = f_{1,k}f_{2,k} - f_{1,k-1}f_{2,k-1} = \Delta^-(f_{1,k})f_{2,k} + f_{1,k-1}\Delta^-(f_{2,k}) \tag{3.4}$$

holds.

In order to understand the effect of periodic operands, we first give the following two definitions.

3.3 Definition.

- *A function* $f : \{u_k\}_{k\in\{0,\dots,N\}} \to \mathbb{R}^n$ *is called* 2π*-periodic on* $\{u_k\}_{k\in\{0,\dots,N\}}$*, if* $f_0 = f_N$*. We then define* $f_{N+1} := f_1$ *and* $f_{-1} := f_{N-1}$*.*
- *A function* $f : \{u_{k,l}\}_{k,l\in\{0,\dots,N\}} \to \mathbb{R}^n$ *is called* 2π*-periodic on* $\{u_{k,l}\}_{k,l\in\{0,\dots,N\}}$*, if for all* $k,l \in \{0,\dots,N\}$ *the identities* $f_{0,l} = f_{N,l}$ *and* $f_{k,0} = f_{k,N}$ *hold.*
  We then define $f_{k,N+1} := f_{k,1}$*,* $f_{N+1,l} := f_{1,l}$*,* $f_{k,-1} := f_{k,N-1}$ *and* $f_{-1,l} := f_{N-1,l}$ *for all* $k,l \in \{0,\dots,N\}$ *as well as* $f_{N+1,N+1} := f_{1,1}$ *and* $f_{-1,-1} := f_{N-1,N-1}$*.*

3.4 Definition.

$C^0_{per}([0,2\pi];\mathbb{R}^n) = \{f : [0,2\pi] \to \mathbb{R}^n \text{ continuous} \mid f(0) = f(2\pi)\}$,
$C^0_{per}([0,2\pi]^2;\mathbb{R}^n) = \{f : [0,2\pi]^2 \to \mathbb{R}^n \text{ continuous} \mid f(u_1,\cdot), f(\cdot,u_2) \in C^0_{per}([0,2\pi];\mathbb{R}^n)\}$.

Thus, elements of the just defined spaces fulfill Definition 3.3 when being restricted on the grid $\{u_k\}_{k\in\{0,\dots,N\}}$ or $\{u_{k,l}\}_{k,l\in\{0,\dots,N\}}$, respectively, and can be extended accordingly.

The main advantage of periodicity is that by the periodic extension of a function f to points outside the domain, we can apply the established difference operators to $f_{k,l}$ for all grid points $u_{k,l}$, $k,l \in \{0,\dots,N\}$. This also yields that the sum over all mesh points is independent of the concrete index (i.e. grid point). For a 2π-periodic function f on $\{u_k\}_{k\in\{0,\dots,N\}}$ that means

$$\begin{aligned}
\sum_{k=1}^{N} f_{k+1} &= \sum_{k=2}^{N+1} f_k = \sum_{k=1}^{N} f_k + \underbrace{f_{N+1} - f_1}_{=0} = \sum_{k=1}^{N} f_k, \\
\sum_{k=1}^{N} f_{k-1} &= \sum_{k=0}^{N-1} f_k = \sum_{k=1}^{N} f_k + \underbrace{f_0 - f_N}_{=0} = \sum_{k=1}^{N} f_k.
\end{aligned} \tag{3.5}$$

Adding the second variable, i.e. an index over which we sum, does not change the asserted equations since we can fix the value of the index for the second variable as shown in the proof of 1. in Lemma 3.5.

From (3.5) we can deduce some important relations between the operators, as it is shown in the following lemmata.

3.5 Lemma.

1. *For a* 2π*-periodic function* $f : \{u_{k,l}\}_{k,l\in\{0,\dots,N\}} \to \mathbb{R}^n$ *we have for* $r \in \{1,2\}$

$$\sum_{k,l=1}^{N} |\Delta_r^+ f_{k,l}|^2 = \sum_{k,l=1}^{N} |\Delta_r^- f_{k,l}|^2.$$

2. For a 2π-periodic function $f : \{u_{k,l}\}_{k,l \in \{0,\dots,N\}} \to \mathbb{R}^n$ we have

$$\sum_{k,l=1}^{N} |\Delta_1^+ \Delta_2^+ f_{k,l}|^2 = \sum_{k,l=1}^{N} |\Delta_1^- \Delta_2^- f_{k,l}|^2.$$

Proof. 1. Let $r = 1$. Applying (3.5) to $\tilde{f}_{k+1,l} := |f_{k+1,l} - f_{k,l}|^2$ for fixed l yields

$$\sum_{k,l=1}^{N} |\Delta_1^+ f_{k,l}|^2 = \sum_{l=1}^{N} \sum_{k=1}^{N} |f_{k+1,l} - f_{k,l}|^2 = \sum_{l=1}^{N} \sum_{k=1}^{N} |f_{k,l} - f_{k-1,l}|^2 = \sum_{k,l=1}^{N} |\Delta_1^- f_{k,l}|^2.$$

For $r = 2$ the steps are the same after renaming indices.

2. Using (3.5) for each direction of the mesh we obtain

$$\begin{aligned} \sum_{k,l=1}^{N} |\Delta_1^+ \Delta_2^+ f_{k,l}|^2 &= \sum_{k,l=1}^{N} |f_{k+1,l+1} - f_{k+1,l} - f_{k,l+1} + f_{k,l}|^2 \\ &= \sum_{k,l=1}^{N} |f_{k,l+1} - f_{k,l} - f_{k-1,l+1} + f_{k-1,l}|^2 \\ &= \sum_{k,l=1}^{N} |f_{k,l} - f_{k,l-1} - f_{k-1,l} + f_{k-1,l-1}|^2 = \sum_{k,l=1}^{N} |\Delta_1^- \Delta_2^- f_{k,l}|^2. \end{aligned}$$

□

For brevity, the following lemma is formulated for a one-dimensional grid. As demonstrated above, this is easily transferred to a two-dimensional grid.

3.6 Lemma. *Let $f, \bar{f} : \{u_k\}_{k \in \{0,\dots,N\}} \to \mathbb{R}^n$, $\beta : \{u_k\}_{k \in \{0,\dots,N\}} \to \mathbb{R}$ be 2π-periodic grid functions. Then the differences $\Delta^+ f_k$ and $-\Delta^- f_k$ result from each other by summation by parts. Two variants are possible, precisely*

1.

$$\sum_{k=1}^{N} \beta_k f_k \cdot (\Delta^+ \bar{f}_k) = -\sum_{k=1}^{N} \beta_k (\Delta^- f_k) \cdot \bar{f}_k - \sum_{k=1}^{N} (\Delta^- \beta_k) f_{k-1} \cdot \bar{f}_k. \tag{3.6}$$

In particular, for $\beta \equiv 1$ we have

$$\sum_{k=1}^{N} f_k \cdot (\Delta^+ \bar{f}_k) = -\sum_{k=1}^{N} (\Delta^- f_k) \cdot \bar{f}_k. \tag{3.7}$$

2.

$$\sum_{k=1}^{N} \beta_k f_k \cdot (\Delta^+ \bar{f}_k) = -\sum_{k=1}^{N} \beta_{k-1} (\Delta^- f_k) \cdot \bar{f}_k - \sum_{k=1}^{N} (\Delta^- \beta_k) f_k \cdot \bar{f}_k. \tag{3.8}$$

Proof. The sum on the left-hand side is the same in the first and second claim and can be rewritten as follows:

$$\begin{aligned}
\sum_{k=1}^{N} \beta_k f_k \cdot (\Delta^+ \bar{f}_k) &= \sum_{k=1}^{N} \beta_k f_k \cdot (\bar{f}_{k+1} - \bar{f}_k) \\
&= \sum_{k=1}^{N} \beta_k f_k \cdot \bar{f}_{k+1} - \sum_{k=1}^{N} \beta_k f_k \cdot \bar{f}_k \\
&\overset{(3.5)}{=} \sum_{k=1}^{N} \beta_{k-1} f_{k-1} \cdot \bar{f}_k - \sum_{k=1}^{N} \beta_k f_k \cdot \bar{f}_k.
\end{aligned}$$

For the resulting terms, there are two possibilities to rejoin them. On the one hand

$$\begin{aligned}
&\sum_{k=1}^{N} \beta_{k-1} f_{k-1} \cdot \bar{f}_k - \sum_{k=1}^{N} \beta_k f_k \cdot \bar{f}_k \\
= &- \sum_{k=1}^{N} \beta_k (f_k - f_{k-1}) \cdot \bar{f}_k - \sum_{k=1}^{N} (\beta_k - \beta_{k-1}) f_{k-1} \cdot \bar{f}_k \\
= &- \sum_{k=1}^{N} \beta_k (\Delta^- f_k) \cdot \bar{f}_k - \sum_{k=1}^{N} (\Delta^- \beta_k) f_{k-1} \cdot \bar{f}_k,
\end{aligned}$$

which corresponds to the assertion in 1. On the other hand

$$\begin{aligned}
&\sum_{k=1}^{N} \beta_{k-1} f_{k-1} \cdot \bar{f}_k - \sum_{k=1}^{N} \beta_k f_k \cdot \bar{f}_k \\
= &- \sum_{k=1}^{N} \beta_{k-1} (f_k - f_{k-1}) \cdot \bar{f}_k - \sum_{k=1}^{N} (\beta_k - \beta_{k-1}) f_k \cdot \bar{f}_k \\
= &- \sum_{k=1}^{N} \beta_{k-1} (\Delta^- f_k) \cdot \bar{f}_k - \sum_{k=1}^{N} (\Delta^- \beta_k) f_k \cdot \bar{f}_k,
\end{aligned}$$

by which the equation in 2. is proved. □

3.2 Approximation and consistency

In the following we present the approximation of the Mean Curvature DeTurck Flow

$$x_t = g^{ij} x_{u_i u_j} + (\frac{1}{\alpha} - 1) g^{ij} g^{mn} (x_{u_i u_j} \cdot x_{u_m}) x_{u_n}$$

which will be studied in the further course of this work. We will also estimate the corresponding consistency error. That is, the error that results from inserting the solution x of the differential equation into the difference equation.

In addition to the established spatial grid we introduce the grid $\{t^s\}_{s\in\{0,\dots,M\}}$ in the time interval $[0,T]$. Let $T = M\tau$ for a time step size τ and $t^s = s\tau$ be the grid points for $s \in \{0,\dots,M\}$. We choose a forward difference quotient to approximate the time derivative and propose the following

Finite Difference Approximation of Mean Curvature DeTurck Flow.
Find $x_h : \{u_{k,l}\}_{k,l\in\{0,\dots,N\}} \times \{t^s\}_{s\in\{0,\dots,M\}} \to \mathbb{R}^3$, with evaluation $x^s_{k,l} = x_h(u_{k,l}, t^s)$ at the mesh points, such that for each $k,l \in \{1,\dots,N\}$ and $s \in \{0,\dots,M-1\}$ the function solves

$$\frac{x^{s+1}_{k,l} - x^s_{k,l}}{\tau} = g^{ij,s}_{k,l}\Delta_{ij}x^{s+1}_{k,l} + \left(\frac{1}{\alpha} - 1\right) g^{ij,s}_{k,l} g^{mn,s}_{k,l} (\Delta_{ij}x^{s+1}_{k,l} \cdot \overline{\Delta}_m x^s_{k,l})\overline{\Delta}_n x^s_{k,l} \tag{3.9}$$

where

$$\begin{aligned}
g^{11,s}_{k,l} &= \frac{|\Delta^+_2 x^s_{k,l}||\Delta^-_2 x^s_{k,l}|}{g^s_{k,l}}, g^{22,s}_{k,l} = \frac{|\Delta^+_1 x^s_{k,l}||\Delta^-_1 x^s_{k,l}|}{g^s_{k,l}},\\
g^{12,s}_{k,l} &= -\frac{\overline{\Delta}_1 x^s_{k,l} \cdot \overline{\Delta}_2 x^s_{k,l}}{g^s_{k,l}} = g^{21,s}_{k,l},\\
g^s_{k,l} &= |\Delta^+_1 x^s_{k,l}||\Delta^-_1 x^s_{k,l}||\Delta^+_2 x^s_{k,l}||\Delta^-_2 x^s_{k,l}| - (\overline{\Delta}_1 x^s_{k,l} \cdot \overline{\Delta}_2 x^s_{k,l})^2.
\end{aligned} \tag{3.10}$$

We obtain a complete difference scheme by specifying the initial and boundary conditions: For all $s \in \{0,\dots,M\}$

$$\begin{aligned}
x^0_{k,l} &= x(u_{k,l}, t^0),\ \forall k,l \in \{0,\dots,N\},\\
x^s_{0,l} &= x^s_{N,l},\ \forall l \in \{0,\dots,N\},\\
x^s_{k,0} &= x^s_{k,N},\ \forall k \in \{0,\dots,N\}.
\end{aligned} \tag{3.11}$$

3.7 Remark. *Note that the operators defined at the beginning of this chapter are differences, not difference quotients. Division by some corresponding power of the spatial grid size h to approximate the derivatives is of course taken into account in all calculations. Still, when describing the objects in words, for simplicity we do not make a distinction. For instance, $|\Delta^+_i x_{k,l}||\Delta^-_i x_{k,l}|$ are called the discrete versions of the squared length elements though they need to be divided by h^2 to approximate the squared length elements, as can be seen in Lemma 3.10.*

3.8 Remark. *The equations in (3.9) are formulated for $u_{k,l}$ with $k,l \in \{1,\dots,N\}$, while information on $u_{k,0}, u_{0,l} \in \{u_{k,l}\}_{k,l\in\{0,\dots,N\}}$ is given via (3.11). In (3.9) the nature of the chosen differences additionally requires involving the points $u_{k,N+1}$ and $u_{N+1,l}$ for $k,l \in \{1,\dots,N\}$ as well as $u_{N+1,N+1}$ when operating on functions evaluated at $u_{k,l}$ for $k = N$ or $l = N$. As given in definition 3.3, these values are provided by* *the periodicity of the function. This definition also allows for function evaluations at* $u_{k,-1}$ *und* $u_{-1,l}$*, which are not directly involved in the above difference equation. Their application will become more obvious later on within the scope of summation by parts and some preliminary considerations in the next chapter. There we have differences of the quantities* g^{ij} *and we apply our difference operators to functions which are evaluated at* $u_{k-1,l}$ *and* $u_{k,l-1}$*, respectively.*

In order to avoid an extra index, just like in the continuous case we omit the fact that the discrete solution x_h depends on the parameter α and we thus actually have a family of solutions. During the analysis of the approximation, the dependence on α will enter into the constant of the estimate. In the numerical part of this work, the influence of the parameter will be considered explicitly.
Scheme (3.9) is semi-implicit and requires a linear system of equations to be solved. Under certain assumptions on the discrete solution at t^s this system has a unique solution $(x_{1,1}^{s+1}, \ldots, x_{N,N}^{s+1})$, as we show in the subsequent lemma. These assumptions will be ensured by conducting an induction within the convergence proof, where it becomes apparent that the constants $\bar{C}_m$, C_O^{ij}, $i, j, m \in \{1, 2\}$, depend on x but not on h or τ. Bounding τ by h^2 as postulated below is necessary for the convergence analysis as well and is thus no further restriction.

3.9 Lemma. *Let* $s \in \{0, \ldots, M-1\}$ *and* $x_h^s : \{u_{k,l}\}_{k,l \in \{0,\ldots,N\}} \to \mathbb{R}^3$ *the discrete solution at* t^s*. Assume that there are constants* $\bar{C}_m$*,* C_O^{ij}*,* $i, j, m \in \{1, 2\}$*, such that* $|\overline{\Delta}_m x_{k,l}^s| \leq 2\bar{C}_m h$ *and* $|g_{k,l}^{ij,s}| \leq C_O^{ij} h^{-2}$*. Then there exists a constant* $c' > 0$ *only depending on* x_h^s *such that (3.9) has a unique solution provided that* $\tau \leq c' h^2$*.*

Proof. The system is of the form

$$(x_{1,1}^{s+1}, \ldots, x_{N,N}^{s+1}) - \tau A(x_{1,1}^{s+1}, \ldots, x_{N,N}^{s+1}) = (x_{1,1}^{s}, \ldots, x_{N,N}^{s})$$

for a matrix $A \in \mathbb{R}^{3N^2 \times 3N^2}$ with entries A_{ij}. For $X \in \mathbb{R}^{3N^2}$ the corresponding homogeneous system reads $X - \tau AX = 0$, from which we infer that for each $i \in \{1, \ldots, 3N^2\}$

$$|X_i| = \tau \left| \sum_{j=1}^{3N^2} A_{ij} X_j \right| \leq c\tau \max_j |A_{ij}| \max_j |X_j|.$$

Note that c corresponds to the number of non-zero entries of A in a row and is independent of h. Thus, if we can bound

$$c\tau \max_j |A_{ij}| \leq \tfrac{1}{2},$$

we obtain that $|X_i| \leq \frac{1}{2} \max_j |X_j|$. In particular, this holds for the maximum over all $i \in \{1, \ldots, 3N^2\}$, which yields

$$\tfrac{1}{2} \max_i |X_i| \leq 0.$$

Hence, the homogeneous system has the unique solution $X = 0 \in \mathbb{R}^{3N^2}$ and the inhomogeneous system has the unique solution $(x_{1,1}^{s+1}, \ldots, x_{N,N}^{s+1}) \in \mathbb{R}^{3N^2}$.
It remains to show that A can be bounded as required. The non-zero entries of A define the coefficients of the points on the surface. Let $x_{k,l}^{s,(q)}$ denote the components of such a point $x_{k,l}^s \in \mathbb{R}^3$. Depending on k, l and q, the entries of A are, except constants that are determined by the coefficients in Δ_{ij}, given by

$$\sum_{i,j=1}^{2} g_{k,l}^{ij,s} + \left(\frac{1}{\alpha} - 1\right) \sum_{i,j=1}^{2} \sum_{r=1}^{3} g_{k,l}^{ij,s} g_{k,l}^{mn,s} \overline{\Delta}_m x_{k,l}^{s,(r)} \overline{\Delta}_n x_{k,l}^{s,(q)}$$

or

$$\left(\frac{1}{\alpha} - 1\right) \sum_{i,j=1}^{2} \sum_{r=1}^{3} g_{k,l}^{ij,s} g_{k,l}^{mn,s} \overline{\Delta}_m x_{k,l}^{s,(r)} \overline{\Delta}_n x_{k,l}^{s,(q)}.$$

All in all, each entry of A is hence bounded by $c_A h^{-2}$ and c_A only depends on x_h^s. Making use of the condition $\tau \leq c' h^2$, where $c' = \frac{1}{2cc_A}$, leads to the desired estimate. $\square$

In what follows, by a solution to (3.9), we mean a 2π-periodic function that can be extended as in Definition 3.3 and consequently automatically fulfills the boundary conditions in (3.11). They are integrated into the error estimate because they allow the use of one equation for all grid points including those on the boundary.

The choice of the above discretization is motivated on the following pages. For this aim, first a consistency estimate for the length elements is given. Since the assertion is valid for both directions of the spatial grid and independently of time, the considerations are restricted on a one-dimensional spatial grid. We then continue to prove consistency for the area element, where we also omit the time dependence.

3.10 Lemma. *Let $x \in C^0_{per}([0, 2\pi]; \mathbb{R}^3) \cap C^3([0, 2\pi]; \mathbb{R}^3)$ with $\partial^\gamma x \in C^0_{per}([0, 2\pi]; \mathbb{R}^3)$ for $|\gamma| \leq 2$ and let $|x_u| \geq c_L > 0$. Then the restriction of x onto the grid $\{u_k\}_{k \in \{0,\ldots,N\}}$, $\tilde{x}_k = x(u_k)$, satisfies for all $k \in \{1, \ldots, N\}$*

$$|\Delta^- \tilde{x}_k||\Delta^+ \tilde{x}_k| = h^2 |x_u(u_k)|^2 + Q_k, \tag{3.12}$$

where $|Q_k| \leq ch^4$ and c only depends on x.

Proof. Considering the periodic boundary conditions described above, we can see that the following assertions are valid for all $k \in \{1, \ldots, N\}$, where the periodicity requirement for its derivatives is necessary to ensure that they can be evaluated at grid points on the boundary. Let thus $u_k \in \{u_k\}_{k \in \{0,\ldots,N\}}$ for arbitrary $k \in \{1, \ldots, N\}$. According to Taylor's formula, we have

$$\tilde{x}_{k\pm1} = x(u_k \pm h) = x(u_k) \pm h x_u(u_k) + \frac{1}{2}h^2 x_{uu}(u_k) \pm \frac{1}{6}h^3 x_{uuu}(\zeta_k^\pm), \qquad (3.13)$$

where $x_{uuu}(\zeta_k^\pm)$, with a little abuse of notation, denotes the evaluation of the vector x_{uuu} at (possibly) different points for each component, i.e.

$$x_{uuu}(\zeta_k^\pm) = (x_{uuu}^{(1)}(\zeta_k^{(1),\pm}), x_{uuu}^{(2)}(\zeta_k^{(2),\pm}), x_{uuu}^{(3)}(\zeta_k^{(3),\pm}))$$

and $\zeta_k^{(q),-} \in (u_{k-1}, u_k)$ respectively $\zeta_k^{(q),+} \in (u_k, u_{k+1})$ for $q \in \{1,2,3\}$. For the sake of brevity, the argument u_k of the derivatives is omitted, whereas the evaluation of functions at points in between is always given in the abbreviated notation $x_{uuu}(\zeta_k^\pm)$. From (3.13) we infer

$$|\Delta^- \tilde{x}_k| = |\tilde{x}_k - \tilde{x}_{k-1}| = |h x_u - \frac{1}{2}h^2 x_{uu} + \frac{1}{6}h^3 x_{uuu}(\zeta_k^-)|$$

and

$$|\Delta^+ \tilde{x}_k| = |\tilde{x}_{k+1} - \tilde{x}_k| = |h x_u + \frac{1}{2}h^2 x_{uu} + \frac{1}{6}h^3 x_{uuu}(\zeta_k^+)|.$$

In order to further rewrite $|\Delta^- \tilde{x}_k|$ as well as $|\Delta^+ \tilde{x}_k|$ we make use of the fact that the euclidean norm of a vector can be defined as the square root of the standard scalar product of the vector with itself:

$$\begin{aligned}
&|\Delta^- \tilde{x}_k| \\
&= \left(\left(h x_u - \frac{1}{2}h^2 x_{uu} + \frac{1}{6}h^3 x_{uuu}(\zeta_k^-)\right) \cdot \left(h x_u - \frac{1}{2}h^2 x_{uu} + \frac{1}{6}h^3 x_{uuu}(\zeta_k^-)\right)\right)^{1/2} \\
&= \Bigg(h^2 |x_u|^2 - \frac{1}{2}h^3 x_u \cdot x_{uu} + \frac{1}{6}h^4 x_u \cdot x_{uuu}(\zeta_k^-) - \frac{1}{2}h^3 x_u \cdot x_{uu} + \frac{1}{4}h^4 |x_{uu}|^2 \\
&\quad - \frac{1}{12}h^5 x_{uu} \cdot x_{uuu}(\zeta_k^-) + \frac{1}{6}h^4 x_u \cdot x_{uuu}(\zeta_k^-) - \frac{1}{12}h^5 x_{uu} \cdot x_{uuu}(\zeta_k^-) + \frac{1}{36}h^6 |x_{uuu}(\zeta_k^-)|^2\Bigg)^{1/2} \\
&= h|x_u| \Bigg(1 - h\frac{x_u \cdot x_{uu}}{|x_u|^2} + h^2 \left(\frac{1}{4}\frac{|x_{uu}|^2}{|x_u|^2} + \frac{1}{3}\frac{x_u}{|x_u|^2} \cdot x_{uuu}(\zeta_k^-)\right) \\
&\qquad - \frac{1}{6}h^3 \frac{x_{uu}}{|x_u|^2} \cdot x_{uuu}(\zeta_k^-) + \frac{1}{36}h^4 \frac{|x_{uuu}(\zeta_k^-)|^2}{|x_u|^2}\Bigg)^{1/2} \\
&= h|x_u| \left(1 - h\frac{x_u \cdot x_{uu}}{|x_u|^2} + Q_k^-\right)^{1/2},
\end{aligned}$$

where in the radicand terms of order two and higher are resumed in Q_k^-, i.e. $|Q_k^-| \le ch^2$ holds. Note that this requires $|x_u| > c_L$ as assumed and also note that the constant c only depends on x. Abbreviations of this kind will be introduced repeatedly in order to keep the calculations as clear as possible.

For further approximation, the square root is reformulated by the Taylor series expansion

$$\sqrt{1+y} = 1 + \frac{1}{2}y + r(y),$$

where $|r(y)| \leq cy^2$ for $|y| \leq \frac{1}{2}$. This yields

$$|\Delta^- \tilde{x}_k| = h|x_u| \left(1 - \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2} + \frac{1}{2}Q_k^- + r_k^-\right)$$

with $|r_k^-| \leq ch^2$ and $|Q_k^-| \leq ch^2$, if $|-h\frac{x_u \cdot x_{uu}}{|x_u|^2} + Q_k^-| \leq ch \leq \frac{1}{2}$, i.e. if h is sufficiently small.
Analogously,

$$\begin{aligned}
|\Delta^+ \tilde{x}_k| &= |hx_u + \frac{1}{2}h^2 x_{uu} + \frac{1}{6}h^3 x_{uuu}(\zeta_k^+)| \\
&= h|x_u| \left(1 + h\frac{x_u \cdot x_{uu}}{|x_u|^2} + h^2\left(\frac{1}{4}\frac{|x_{uu}|^2}{|x_u|^2} + \frac{1}{3}\frac{x_u}{|x_u|^2} \cdot x_{uuu}(\zeta_k^+)\right)\right. \\
&\qquad \left. + \frac{1}{6}h^3\frac{x_{uu}}{|x_u|^2} \cdot x_{uuu}(\zeta_k^+) + \frac{1}{36}h^4\frac{|x_{uuu}(\zeta_k^+)|^2}{|x_u|^2}\right)^{1/2} \\
&= h|x_u| \left(1 + h\frac{x_u \cdot x_{uu}}{|x_u|^2} + Q_k^+\right)^{1/2} \\
&= h|x_u| \left(1 + \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2} + \frac{1}{2}Q_k^+ + r_k^+\right),
\end{aligned}$$

where $|r_k^+| \leq ch^2$ as well as $|Q_k^+| \leq ch^2$.
We therefore have

$$\begin{aligned}
&|\Delta^- \tilde{x}_k||\Delta^+ \tilde{x}_k| \\
&= h^2|x_u|^2 \left(1 - \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2} + \frac{1}{2}Q_k^- + r_k^-\right)\left(1 + \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2} + \frac{1}{2}Q_k^+ + r_k^+\right) \\
&= h^2|x_u|^2 \left(1 + \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2} + \frac{1}{2}Q_k^+ + r_k^+\right. \\
&\qquad - \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2} - \frac{1}{4}h^2\left(\frac{x_u \cdot x_{uu}}{|x_u|^2}\right)^2 - \frac{1}{4}h\frac{x_u \cdot x_{uu}}{|x_u|^2}Q_k^+ - \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2}r_k^+ \\
&\qquad + \frac{1}{2}Q_k^- + \frac{1}{4}h\frac{x_u \cdot x_{uu}}{|x_u|^2}Q_k^- + \frac{1}{4}Q_k^- Q_k^+ + \frac{1}{2}Q_k^- r_k^+ \\
&\qquad \left. + r_k^- + \frac{1}{2}h\frac{x_u \cdot x_{uu}}{|x_u|^2}r_k^- + \frac{1}{2}r_k^- Q_k^+ + r_k^- r_k^+\right) \\
&= h^2|x_u|^2 + Q_k,
\end{aligned}$$

where $|Q_k| \leq ch^4$. $\square$

3.11 Lemma. *Let* $x \in C^3([0,2\pi]^2;\mathbb{R}^3) \cap C^0_{per}([0,2\pi]^2;\mathbb{R}^3)$ *with* $\partial^\gamma x \in C^0_{per}([0,2\pi]^2;\mathbb{R}^3)$ *for* $|\gamma| \leq 2$ *and let* $\tilde{x}_{k,l} := x(u_{k,l})$ *denote the restriction of* x *onto the mesh* $\{u_{k,l}\}_{k,l\in\{0,\dots,N\}} \subset [0,2\pi]^2$ *with* $(u_{k,l}) = (kh, lh)$, $h = \frac{2\pi}{N}$. *Furthermore, let*

$$\tilde{g}_{k,l} = |\Delta_1^- \tilde{x}_{k,l}||\Delta_1^+ \tilde{x}_{k,l}||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| - (\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l})^2$$

denote the approximation of the area element g. *Under the regularity assumption that* $0 < 2\bar{c} \leq g$, *we have that for all* $k,l \in \{1,\dots,N\}$

$$\tilde{g}_{k,l} = h^4 g(u_{k,l}) + R_{k,l}, \tag{3.14}$$

where $|R_{k,l}| \leq ch^6$ *and* c *only depends on* x. *In particular, if* $h \leq h_0$ *for some* $h_0 > 0$, *the restriction of the area element satisfies*

$$\tilde{g}_{k,l} \geq \bar{c} h^4. \tag{3.15}$$

Proof. Due to the periodicity of x, we can formulate the following computations for all $k,l \in \{1,\dots,N\}$. The central differences $\overline{\Delta}_r$, which were defined at the beginning of this chapter in (3.2), can be expressed by the Taylor expansion in (3.13): For all $k,l \in \{1,\dots,N\}$ we have that

$$\begin{aligned} 2\overline{\Delta}_1 \tilde{x}_{k,l} &= \tilde{x}_{k+1,l} - \tilde{x}_{k-1,l} = 2h x_{u_1}(u_{k,l}) + R^{(1)}_{k,l}, \\ 2\overline{\Delta}_2 \tilde{x}_{k,l} &= \tilde{x}_{k,l+1} - \tilde{x}_{k,l-1} = 2h x_{u_2}(u_{k,l}) + R^{(2)}_{k,l}, \end{aligned} \tag{3.16}$$

with $|R^{(1)}_{k,l}| \leq ch^3$, $|R^{(2)}_{k,l}| \leq ch^3$. This yields

$$\begin{aligned} \overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l} &= \frac{1}{2}(\tilde{x}_{k+1,l} - \tilde{x}_{k-1,l}) \cdot \frac{1}{2}(\tilde{x}_{k,l+1} - \tilde{x}_{k,l-1}) \\ &= (h x_{u_1} + \frac{1}{2} R^{(1)}_{k,l}) \cdot (h x_{u_2} + \frac{1}{2} R^{(2)}_{k,l}) \\ &= h^2 (x_{u_1} \cdot x_{u_2}) + \frac{1}{2} h x_{u_1} \cdot R^{(2)}_{k,l} + \frac{1}{2} h x_{u_2} \cdot R^{(1)}_{k,l} + \frac{1}{4} R^{(1)}_{k,l} \cdot R^{(2)}_{k,l} \\ &= h^2 (x_{u_1} \cdot x_{u_2}) + R^{(3)}_{k,l}, \end{aligned} \tag{3.17}$$

where $|R^{(3)}_{k,l}| \leq ch^4$.
We use the statement of the preceding lemma, which says that

$$\begin{aligned} |\Delta_1^- \tilde{x}_{k,l}||\Delta_1^+ \tilde{x}_{k,l}| &= h^2 |x_{u_1}|^2 + R^{(4)}_{k,l}, \\ |\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| &= h^2 |x_{u_2}|^2 + R^{(5)}_{k,l} \end{aligned} \tag{3.18}$$

with $|R^{(4)}_{k,l}| \leq ch^4$, $|R^{(5)}_{k,l}| \leq ch^4$ and $k,l \in \{1,\dots,N\}$. The boundedness

$$|x_{u_r}| \geq c_{L_r}, \tag{3.19}$$

which is necessary for the application of Lemma 3.10, follows from the assumed lower boundedness of g by means of a compactness argument. Consequently,

$$
\begin{aligned}
|\Delta_1^- \tilde{x}_{k,l}||\Delta_1^+ \tilde{x}_{k,l}||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| &= (h^2|x_{u_1}|^2 + R_{k,l}^{(4)})(h^2|x_{u_2}|^2 + R_{k,l}^{(5)}) \\
&= h^4|x_{u_1}|^2|x_{u_2}|^2 + h^2|x_{u_1}|^2 R_{k,l}^{(5)} + h^2|x_{u_2}|^2 R_{k,l}^{(4)} + R_{k,l}^{(4)} R_{k,l}^{(5)} \\
&= h^4|x_{u_1}|^2|x_{u_2}|^2 + R_{k,l}^{(6)},
\end{aligned}
$$

where $|R_{k,l}^{(6)}| \leq ch^6$.
For the approximation $\tilde{g}_{k,l}$ of the area element it follows that

$$
\begin{aligned}
\tilde{g}_{k,l} &= |\Delta_1^- \tilde{x}_{k,l}||\Delta_1^+ \tilde{x}_{k,l}||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| - \left(\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l}\right)^2 \\
&= h^4|x_{u_1}|^2|x_{u_2}|^2 + R_{k,l}^{(6)} - \Big(h^4(x_{u_1} \cdot x_{u_2})^2 + 2h^2(x_{u_1} \cdot x_{u_2}) R_{k,l}^{(3)} + (R_{k,l}^{(3)})^2\Big) \\
&= h^4(|x_{u_1}|^2|x_{u_2}|^2 - (x_{u_1} \cdot x_{u_2})^2) + R_{k,l}^{(7)}, \\
&= h^4 g + R_{k,l}^{(7)},
\end{aligned}
$$

where $|R_{k,l}^{(7)}| \leq ch^6$. This way we can also derive a lower bound for $\tilde{g}_{k,l}$. Since we presumed $g \geq 2\bar{c}$, we have

$$
\tilde{g}_{k,l} \geq h^4 g - |R_{k,l}^{(7)}| \geq 2\bar{c}h^4 - ch^6 \geq \bar{c}h^4,
$$

in case $h \leq \sqrt{c/\bar{c}}$.

□

We advise the reader to bear in mind that this result means that $g(u_{k,l})$ is approximated by $\tilde{g}_{k,l}/h^4$, compare Remark 3.7. Despite this fact, we often call $\tilde{g}_{k,l}$ the approximated area element for simplicity.
After these preliminary results we are ready to formulate the consistency statement for the whole difference scheme.

3.12 Theorem. *Let $x \in C^4([0, 2\pi]^2 \times [0, T]; \mathbb{R}^3) \cap C^0([0, T]; C^0_{per}([0, 2\pi]^2; \mathbb{R}^3))$ be the solution of the continuous problem (1.4) with $\partial^\gamma x \in C^0([0, T]; C^0_{per}([0, 2\pi]^2; \mathbb{R}^3))$ for $|\gamma| \leq 3$. For $k, l \in \{0, \ldots, N\}$ and $s \in \{0, \ldots, M\}$ let $\tilde{x}_{k,l}^s := x(u_{k,l}, t^s)$ denote the restriction of x to the mesh $\{(u_{k,l}, t^s)\}_{k,l \in \{0,\ldots,N\}, s \in \{0,\ldots,M\}}$ with $(u_{k,l}) = (kh, lh)$, $h = \frac{2\pi}{N}$, and $t^s = s\tau$, $\tau = \frac{T}{M}$, as well as*

$$
\begin{aligned}
\tilde{g}_{k,l}^{11,s} &= \frac{|\Delta_2^- \tilde{x}_{k,l}^s||\Delta_2^+ \tilde{x}_{k,l}^s|}{\tilde{g}_{k,l}^s}, \quad \tilde{g}_{k,l}^{22,s} = \frac{|\Delta_1^- \tilde{x}_{k,l}^s||\Delta_1^+ \tilde{x}_{k,l}^s|}{\tilde{g}_{k,l}^s}, \\
\tilde{g}_{k,l}^{12,s} &= -\frac{\overline{\Delta}_1 \tilde{x}_{k,l}^s \cdot \overline{\Delta}_2 \tilde{x}_{k,l}^s}{\tilde{g}_{k,l}^s} = \tilde{g}_{k,l}^{21,s}, \\
\tilde{g}_{k,l}^s &= |\Delta_1^- \tilde{x}_{k,l}^s||\Delta_1^+ \tilde{x}_{k,l}^s||\Delta_2^- \tilde{x}_{k,l}^s||\Delta_2^+ \tilde{x}_{k,l}^s| - (\overline{\Delta}_1 \tilde{x}_{k,l}^s \cdot \overline{\Delta}_2 \tilde{x}_{k,l}^s)^2.
\end{aligned}
\tag{3.20}
$$

If $0 < 2\bar{c} \leq g$, $h \leq h_0$ *for some* $h_0 > 0$ *and* $\tau \leq c'h^2$ *for some* $c' > 0$, *then for all* $k, l \in \{1, \dots, N\}$ *and* $s \in \{0, \dots, M-1\}$ *the following equation holds:*

$$\frac{\tilde{x}_{k,l}^{s+1} - \tilde{x}_{k,l}^{s}}{\tau} = \tilde{g}_{k,l}^{ij,s} \Delta_{ij} \tilde{x}_{k,l}^{s+1} + \left(\frac{1}{\alpha} - 1\right) \tilde{g}_{k,l}^{ij,s} \tilde{g}_{k,l}^{mn,s} (\Delta_{ij} \tilde{x}_{k,l}^{s+1} \cdot \overline{\Delta}_m \tilde{x}_{k,l}^{s}) \overline{\Delta}_n \tilde{x}_{k,l}^{s} + \tilde{R}_{k,l}^{\alpha,s}, \tag{3.21}$$

where $|\tilde{R}_{k,l}^{\alpha,s}| \leq c_{\tilde{R}}(h^2 + \tau)$ *and* $c_{\tilde{R}}$ *only depends on* x *and* α.

Proof. Again, the subsequent assertions hold for all points of the spatial grid, i.e. for all $k, l \in \{1, \dots, N\}$, because of the periodicity of the solution function and its derivatives. In the previous lemmata, the difference operators of first order have been investigated. In order to examine the remaining differences in view of their consistency, we need Taylor polynomials of higher degree and therefore now additionally require the derivatives of third order to fulfill the periodic boundary conditions. The evaluation of differences of second order at time t^{s+1}, $s \in \{0, \dots, M-1\}$, also requires to include the time variable in the expansion. Note that we did not consider a time dependence so far and that the quantities we studied only appear at time t^s in our difference scheme. We can thus directly transfer these results without addressing the further variable in the Taylor expansions. For clarification, we add the index s to the remainder terms obtained earlier.

Since we always expand around $(u_{k,l}, t^s)$, we do not note the argument of derivatives except from the case of the Lagrangian remainder. For the remainder, analogously to (3.13), we write $\partial^\gamma x(\xi)$ but mean that each component of the derivative is evaluated at a different ξ. Let $s \in \{0, \dots, M-1\}$ if not stated otherwise. We have

$$\begin{aligned}
\tilde{x}_{k\pm1,l}^{s+1} &= x(u_{k,l} \pm he_1, t^s + \tau) = x((k \pm 1)h, lh, (s+1)\tau) \\
&= \tilde{x}_{k,l}^{s} + \tau x_t \pm h x_{u_1} + \frac{1}{2}(h^2 x_{u_1u_1} \pm 2h\tau x_{u_1 t} + \tau^2 x_{tt}) \\
&\quad + \frac{1}{6}\big(\pm h^3 \partial^{(3,0,0)} x + 3h^2\tau \partial^{(2,0,1)} x \pm 3h\tau^2 \partial^{(1,0,2)} x + \tau^3 \partial^{(0,0,3)} x\big) \\
&\quad + \sum_{|(\gamma_1,0,\gamma_3)|=4} \frac{1}{\gamma_1! \gamma_3!} (\pm h)^{\gamma_1} \tau^{\gamma_3} \partial^{(\gamma_1,0,\gamma_3)} x(\xi_{k,s}^{\pm}),
\end{aligned}$$

where $\xi_{k,s}^{-} \in ((k-1)h, kh) \times \{lh\} \times (s\tau, (s+1)\tau)$ and $\xi_{k,s}^{+} \in (kh, (k+1)h) \times \{lh\} \times (s\tau, (s+1)\tau)$. Likewise it holds that

$$\begin{aligned}
\tilde{x}_{k,l\pm1}^{s+1} &= \tilde{x}_{k,l}^{s} + \tau x_t \pm h x_{u_2} + \frac{1}{2}(h^2 x_{u_2u_2} \pm 2h\tau x_{u_2 t} + \tau^2 x_{tt}) \\
&\quad + \frac{1}{6}\big(\pm h^3 \partial^{(0,3,0)} x + 3h^2\tau \partial^{(0,2,1)} x \pm 3h\tau^2 \partial^{(0,1,2)} x + \tau^3 \partial^{(0,0,3)} x\big) \\
&\quad + \sum_{|(0,\gamma_2,\gamma_3)|=4} \frac{1}{\gamma_2! \gamma_3!} (\pm h)^{\gamma_2} \tau^{\gamma_3} \partial^{(0,\gamma_2,\gamma_3)} x(\xi_{l,s}^{\pm}),
\end{aligned}$$

where $\xi_{l,s}^{-} \in \{kh\} \times ((l-1)h, lh) \times (s\tau, (s+1)\tau)$ and $\xi_{l,s}^{+} \in \{kh\} \times (lh, (l+1)h) \times$

$(s\tau,(s+1)\tau)$. The expansions in both spatial variables and in time of $\tilde{x}^{s+1}_{k+1,l+1}$ and $\tilde{x}^{s+1}_{k-1,l-1}$ at $(u_{k,l},t^s)$ are

$$\begin{aligned}\tilde{x}^{s+1}_{k\pm1,l\pm1} &= x(u_{k,l}\pm h(1,1),t^s+\tau) = x((k\pm1)h,(l\pm1)h,(s+1)\tau)\\ &= \tilde{x}^s_{k,l}+\tau x_t \pm h x_{u_1} \pm h x_{u_2}\\ &\quad+\frac{1}{2}(h^2x_{u_1u_1}+2h^2x_{u_1u_2}+h^2x_{u_2u_2}\pm 2h\tau x_{u_2t}\pm 2h\tau x_{u_1t}+\tau^2x_{tt})\\ &\quad+\frac{1}{6}\big(\pm h^3\partial^{(3,0,0)}x+3h^2\tau\partial^{(2,0,1)}x\pm 3h\tau^2\partial^{(1,0,2)}x\pm 3h^3\partial^{(2,1,0)}x\pm 3h^3\partial^{(1,2,0)}x\\ &\qquad+h^2\tau\partial^{(1,1,1)}x\pm h^3\partial^{(0,3,0)}x+3h^2\tau\partial^{(0,2,1)}x\pm 3h\tau^2\partial^{(0,1,2)}x+\tau^3\partial^{(0,0,3)}x\big)\\ &\quad+\sum_{|\gamma|=4}\frac{1}{\gamma!}(\pm h,\pm h,\tau)^\gamma\partial^\gamma x(\xi^{\pm}_{k,l,s}),\end{aligned}$$

where for the arguments of the remainders $\xi^-_{k,l,s}\in((k-1)h,kh)\times((l-1)h,lh)\times(s\tau,(s+1)\tau)$ and $\xi^+_{k,l,s}\in(kh,(k+1)h)\times(lh,(l+1)h)\times(s\tau,(s+1)\tau)$ hold, respectively. Besides we have

$$\tilde{x}^{s+1}_{k,l}=\tilde{x}^s_{k,l}+\tau x_t+\frac{1}{2}\tau^2x_{tt}(u_{k,l},\vartheta^s), \tag{3.22}$$

where $\vartheta^s\in(s\tau,(s+1)\tau)$.

We obtain the order of consistency of the operators Δ_{ij}, $i\in\{1,2\}$, compare (3.3) and previous notations, by inserting the above expressions into the corresponding differences:

$$\begin{aligned}\Delta_{11}\tilde{x}^{s+1}_{k,l} &= \tilde{x}^{s+1}_{k+1,l}-2\tilde{x}^{s+1}_{k,l}+\tilde{x}^{s+1}_{k-1,l}\\ &= \tilde{x}^s_{k,l}+\tau x_t+hx_{u_1}+\frac{1}{2}(h^2x_{u_1u_1}+2h\tau x_{u_1t}+\tau^2x_{tt})\\ &\quad+\frac{1}{6}(h^3\partial^{(3,0,0)}x+3h^2\tau\partial^{(2,0,1)}x+3h\tau^2\partial^{(1,0,2)}x+\tau^3\partial^{(0,0,3)}x)\\ &\quad-2\tilde{x}^s_{k,l}-2\tau x_t-\tau^2x_{tt}(u_{k,l},\vartheta^s)\\ &\quad+\tilde{x}^s_{k,l}+\tau x_t-hx_{u_1}+\frac{1}{2}(h^2x_{u_1u_1}-2h\tau x_{u_1t}+\tau^2x_{tt})\\ &\quad+\frac{1}{6}\big(-h^3\partial^{(3,0,0)}x+3h^2\tau\partial^{(2,0,1)}x-3h\tau^2\partial^{(1,0,2)}x+\tau^3\partial^{(0,0,3)}x\big)\\ &\quad+\sum_{|(\gamma_1,0,\gamma_3)|=4}\frac{1}{\gamma_1!\gamma_3!}(\pm h)^{\gamma_1}\tau^{\gamma_3}\left(\partial^{(\gamma_1,0,\gamma_3)}x(\xi^+_{k,s})+\partial^{(\gamma_1,0,\gamma_3)}x(\xi^-_{k,s})\right)\\ &= h^2x_{u_1u_1}+R^{(8)}_{k,l,s},\end{aligned}\tag{3.23}$$

where $|R^{(8)}_{k,l,s}|\le c(\tau^2+h^2\tau+h^4)$, and analogously

$$\Delta_{22}\tilde{x}^{s+1}_{k,l}=\tilde{x}^{s+1}_{k,l+1}-2\tilde{x}^{s+1}_{k,l}+\tilde{x}^{s+1}_{k,l-1}=h^2x_{u_2u_2}+R^{(9)}_{k,l,s}, \tag{3.24}$$

where $|R^{(9)}_{k,l,s}|\le c(\tau^2+h^2\tau+h^4)$.

Furthermore,

$$
\begin{aligned}
&2\Delta_{12}\tilde{x}^{s+1}_{k,l}\\
&= \tilde{x}^{s+1}_{k+1,l+1} - \tilde{x}^{s+1}_{k+1,l} - \tilde{x}^{s+1}_{k,l+1} + 2\tilde{x}^{s+1}_{k,l} - \tilde{x}^{s+1}_{k,l-1} - \tilde{x}^{s+1}_{k-1,l} + \tilde{x}^{s+1}_{k-1,l-1}\\
&= \tilde{x}^{s}_{k,l}(1-1-1+2-1-1+1) + \tau x_t(1-1-1+2-1-1+1)\\
&\quad + hx_{u_1}(1-1-(-1)+(-1)) + hx_{u_2}(1-1-(-1)+(-1))\\
&\quad + \frac{1}{2}h\tau x_{u_1t}(2-2-(-2)+(-2)) + \frac{1}{2}h\tau x_{u_2t}(2-2-(-2)+(-2))\\
&\quad + \frac{1}{2}h^2x_{u_1u_1}(1-1-1+1) + \frac{1}{2}h^2x_{u_1u_2}(2+2) + \frac{1}{2}h^2x_{u_2u_2}(1-1-1+1)\\
&\quad + 2\frac{1}{2}\tau^2x_{tt}(u_{k,l},\vartheta^s) + (\frac{1}{2}\tau^2x_{tt} + \frac{1}{6}\tau^3(\partial^{(0,0,3)}x))(1-1-1-1-1+1)\\
&\quad + \frac{1}{6}h^3(\partial^{(3,0,0)}x)(1-1-(-1)+(-1)) + \frac{1}{6}h^3(\partial^{(0,3,0)}x)(1-1-(-1)+(-1))\\
&\quad + \frac{1}{6}h^3((\partial^{(2,1,0)}x) + (\partial^{(1,2,0)}x))(3+(-3))\\
&\quad + \frac{1}{6}h^2\tau((\partial^{(2,0,1)}x) + (\partial^{(0,2,1)}x))(3-3-3+3) + \frac{1}{6}h^2\tau(\partial^{(1,1,1)}x)(1+1)\\
&\quad + \frac{1}{6}h\tau^2((\partial^{(1,0,2)}x) + (\partial^{(0,1,2)}x))(3-3-(-3)+(-3))\\
&\quad + \sum_{|(\gamma_1,0,\gamma_3)|=4} \frac{1}{\gamma_1!\gamma_3!}(\pm h)^{\gamma_1}\tau^{\gamma_3}\left(\partial^{(\gamma_1,0,\gamma_3)}x(\xi^+_{k,s}) + \partial^{(\gamma_1,0,\gamma_3)}x(\xi^-_{k,s})\right)\\
&\quad + \sum_{|(0,\gamma_2,\gamma_3)|=4} \frac{1}{\gamma_2!\gamma_3!}(\pm h)^{\gamma_2}\tau^{\gamma_3}\left(\partial^{(0,\gamma_2,\gamma_3)}x(\xi^+_{l,s}) + \partial^{(0,\gamma_2,\gamma_3)}x(\xi^-_{l,s})\right)\\
&\quad + \sum_{|\gamma|=4} \frac{1}{\gamma!}(\pm h,\pm h,\tau)^{\gamma}\left(\partial^{\gamma}x(\xi^+_{k,l,s}) + \partial^{\gamma}x(\xi^-_{k,l,s})\right)\\
&= 2h^2x_{u_1u_2} + R^{(10)}_{k,l,s},
\end{aligned}
\tag{3.25}
$$

where $|R^{(10)}_{k,l,s}| \le c(\tau^2 + h^2\tau + h^4)$.

Inserting the continuous solution function x into the difference equation we obtain for each $k,l \in \{1,\dots,N\}$ the consistency error

$$
\tilde{R}^{\alpha,s}_{k,l} := \frac{\tilde{x}^{s+1}_{k,l} - \tilde{x}^{s}_{k,l}}{\tau} - \tilde{g}^{ij,s}_{k,l}\Delta_{ij}\tilde{x}^{s+1}_{k,l} - \left(\frac{1}{\alpha} - 1\right)\tilde{g}^{ij,s}_{k,l}\tilde{g}^{mn,s}_{k,l}(\Delta_{ij}\tilde{x}^{s+1}_{k,l}\cdot\overline{\Delta}_m\tilde{x}^{s}_{k,l})\overline{\Delta}_n\tilde{x}^{s}_{k,l}. \tag{3.26}
$$

In view of the definitions of $\tilde{g}^{ij,s}_{k,l}$ and the appearance of the product $\tilde{g}^{ij,s}_{k,l}\tilde{g}^{mn,s}_{k,l}$, we multiply this equation by $(\tilde{g}^{s}_{k,l})^2$. By replacing the differences by the expressions we obtained from the Taylor expansions, we calculate the consistency error of the approximated Mean Curvature DeTurck Flow. Remember that, compared to Lemmas 3.10 and 3.11, we indicate the time dependence of x and the remainder terms by a further index. We start with the following intermediate computation using the definition of

$\tilde{g}^{ij,s}_{k,l}$ in (3.20) as well as (3.17), (3.18) and (3.24), (3.25):

$$\begin{aligned}
&\tilde{g}^s_{k,l}\tilde{g}^{ij,s}_{k,l}\Delta_{ij}\tilde{x}^{s+1}_{k,l}\\
&= |\Delta_2^+\tilde{x}^s_{k,l}||\Delta_2^-\tilde{x}^s_{k,l}|\Delta_{11}\tilde{x}^{s+1}_{k,l} - 2(\overline{\Delta}_1\tilde{x}^s_{k,l}\cdot\overline{\Delta}_2\tilde{x}^s_{k,l})\Delta_{12}\tilde{x}^{s+1}_{k,l} + |\Delta_1^+\tilde{x}^s_{k,l}||\Delta_1^-\tilde{x}^s_{k,l}|\Delta_{22}\tilde{x}^{s+1}_{k,l}\\
&= (h^2|x_{u_2}|^2 + R^{(5)}_{k,l,s})(h^2 x_{u_1u_1} + R^{(8)}_{k,l,s})\\
&\quad - 2(h^2(x_{u_1}\cdot x_{u_2}) + R^{(3)}_{k,l,s})(h^2 x_{u_1u_2} + R^{(10)}_{k,l,s})\\
&\quad + (h^2|x_{u_1}|^2 + R^{(4)}_{k,l,s})(h^2 x_{u_2u_2} + R^{(9)}_{k,l,s})\\
&= h^4|x_{u_2}|^2 x_{u_1u_1} + h^2|x_{u_2}|^2 R^{(8)}_{k,l,s} + h^2 R^{(5)}_{k,l,s}x_{u_1u_1} + R^{(5)}_{k,l,s}R^{(8)}_{k,l,s}\\
&\quad - 2(h^4(x_{u_1}\cdot x_{u_2})x_{u_1u_2} + h^2(x_{u_1}\cdot x_{u_2})R^{(10)}_{k,l,s} + h^2 R^{(3)}_{k,l,s}x_{u_1u_2} + R^{(3)}_{k,l,s}R^{(10)}_{k,l,s})\\
&\quad + h^4|x_{u_1}|^2 x_{u_2u_2} + h^2|x_{u_1}|^2 R^{(9)}_{k,l,s} + h^2 R^{(4)}_{k,l,s}x_{u_2u_2} + R^{(4)}_{k,l,s}R^{(9)}_{k,l,s}\\
&= h^4|x_{u_2}|^2 x_{u_1u_1} - 2h^4(x_{u_1}\cdot x_{u_2})x_{u_1u_2} + h^4|x_{u_1}|^2 x_{u_2u_2} + R^{(11)}_{k,l,s}\\
&= h^4 g g^{ij} x_{u_iu_j} + R^{(11)}_{k,l,s},
\end{aligned} \tag{3.27}$$

where $|R^{(11)}_{k,l,s}| \le ch^2(\tau^2 + h^2\tau + h^4)$. From this we infer with the help of (3.14) that

$$\begin{aligned}
(\tilde{g}^s_{k,l})^2\tilde{g}^{ij,s}_{k,l}\Delta_{ij}\tilde{x}^{s+1}_{k,l} &\overset{(3.27)}{=} \tilde{g}^s_{k,l}(h^4 g g^{ij}x_{u_iu_j} + R^{(11)}_{k,l,s})\\
&\overset{(3.14)}{=} (h^4 g + R^{(7)}_{k,l,s})(h^4 g g^{ij}x_{u_iu_j} + R^{(11)}_{k,l,s})\\
&= h^8 g^2 g^{ij}x_{u_iu_j} + h^4 g(R^{(11)}_{k,l,s} + R^{(7)}_{k,l,s}g^{ij}x_{u_iu_j}) + R^{(7)}_{k,l,s}R^{(11)}_{k,l,s}\\
&= h^8 g^2 g^{ij}x_{u_iu_j} + R^{(12)}_{k,l,s},
\end{aligned} \tag{3.28}$$

where $|R^{(12)}_{k,l,s}| \le ch^6(\tau^2 + h^2\tau + h^4)$. Likewise we have for $m, n \in \{1,2\}$ because of (3.27) and (3.16), that

$$\begin{aligned}
&\tilde{g}^s_{k,l}\tilde{g}^{ij,s}_{k,l}(\Delta_{ij}\tilde{x}^{s+1}_{k,l}\cdot\overline{\Delta}_m\tilde{x}^s_{k,l})\\
&= (h^4 g g^{ij}x_{u_iu_j} + R^{(11)}_{k,l,s})\cdot(hx_{u_m} + \tfrac{1}{2}R^{(m)}_{k,l,s})\\
&= h^5 g g^{ij}(x_{u_iu_j}\cdot x_{u_m}) + \tfrac{1}{2}h^4 g g^{ij}(x_{u_iu_j}\cdot R^{(m)}_{k,l,s}) + h(R^{(11)}_{k,l,s}\cdot x_{u_m}) + \tfrac{1}{2}(R^{(11)}_{k,l,s}\cdot R^{(m)}_{k,l,s})\\
&= h^5 g g^{ij}(x_{u_iu_j}\cdot x_{u_m}) + R^{(13)}_{k,l,s},
\end{aligned}$$

where $|R^{(13)}_{k,l,s}| \le ch^3(\tau^2 + h^2\tau + h^4)$, and hence

$$\begin{aligned}
&\tilde{g}^s_{k,l}\tilde{g}^{ij,s}_{k,l}(\Delta_{ij}\tilde{x}^{s+1}_{k,l}\cdot\overline{\Delta}_m\tilde{x}^s_{k,l})\overline{\Delta}_n\tilde{x}^s_{k,l}\\
&= \left(h^5 g g^{ij}(x_{u_iu_j}\cdot x_{u_m}) + R^{(13)}_{k,l,s}\right)\left(hx_{u_n} + \tfrac{1}{2}R^{(n)}_{k,l,s}\right)\\
&= h^6 g g^{ij}(x_{u_iu_j}\cdot x_{u_m})x_{u_n} + \tfrac{1}{2}h^5 g g^{ij}(x_{u_iu_j}\cdot x_{u_m})R^{(n)}_{k,l,s} + hR^{(13)}_{k,l,s}x_{u_n} + \tfrac{1}{2}R^{(13)}_{k,l,s}R^{(n)}_{k,l,s}\\
&= h^6 g g^{ij}(x_{u_iu_j}\cdot x_{u_m})x_{u_n} + R^{(14,m,n)}_{k,l,s},
\end{aligned}$$

where $|R_{k,l,s}^{(14,m,n)}| \leq ch^4(\tau^2 + h^2\tau + h^4)$.

Note that (3.17) and (3.18) can be resumed to

$$\tilde{g}_{k,l}^{s}\tilde{g}_{k,l}^{mn,s} = h^2 g g^{mn} + R_{k,l,s}^{(15,m,n)}, \ \ m,n \in \{1,2\}, \tag{3.29}$$

where $|R_{k,l,s}^{(15,m,n)}| \leq ch^4$. Thus

$$\begin{aligned}
&(\tilde{g}_{k,l}^{s})^2 \tilde{g}_{k,l}^{ij,s}\tilde{g}_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\overline{\Delta}_m\tilde{x}_{k,l}^{s})\overline{\Delta}_n\tilde{x}_{k,l}^{s} \\
&= \tilde{g}_{k,l}^{s}\tilde{g}_{k,l}^{mn,s}\tilde{g}_{k,l}^{s}\tilde{g}_{k,l}^{ij,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\overline{\Delta}_m\tilde{x}_{k,l}^{s})\overline{\Delta}_n\tilde{x}_{k,l}^{s} \\
&= (h^2 g g^{mn} + R_{k,l,s}^{(15,m,n)})\left(h^6 g g^{ij}(x_{u_iu_j}\cdot x_{u_m})x_{u_n} + R_{k,l,s}^{(14,m,n)}\right) \\
&= h^8 g g^{mn} g g^{ij}(x_{u_iu_j}\cdot x_{u_m})x_{u_n} + h^2 g g^{mn} R_{k,l,s}^{(14,m,n)} \\
&\quad + R_{k,l,s}^{(15,m,n)} h^6 g g^{ij}(x_{u_iu_j}\cdot x_{u_m})x_{u_n} + R_{k,l,s}^{(15,m,n)}R_{k,l,s}^{(14,m,n)} \\
&= h^8 g g^{mn}(g g^{ij} x_{u_iu_j}\cdot x_{u_m})x_{u_n} + R_{k,l,s}^{(16)},
\end{aligned} \tag{3.30}$$

where $|R_{k,l,s}^{(16)}| \leq ch^6(\tau^2 + h^2\tau + h^4)$.

From (3.14) we conclude that

$$(\tilde{g}_{k,l}^{s})^2 = h^8(g)^2 + 2h^4 g R_{k,l,s}^{(7)} + (R_{k,l,s}^{(7)})^2 = h^8(g)^2 + R_{k,l,s}^{(17)},$$

where $|R_{k,l,s}^{(17)}| \leq ch^{10}$. So, using (3.22), (3.26), (3.28) and (3.30)

$$\begin{aligned}
&(\tilde{g}_{k,l}^{s})^2 \tilde{R}_{k,l}^{\alpha,s} \\
&= (\tilde{g}_{k,l}^{s})^2 \frac{\tilde{x}_{k,l}^{s+1} - \tilde{x}_{k,l}^{s}}{\tau} - (\tilde{g}_{k,l}^{s})^2 \tilde{g}_{k,l}^{ij,s}\Delta_{ij}\tilde{x}_{k,l}^{s+1} - (\frac{1}{\alpha} - 1)(\tilde{g}_{k,l}^{s})^2\tilde{g}_{k,l}^{ij,s}\tilde{g}_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\overline{\Delta}_m\tilde{x}_{k,l}^{s})\overline{\Delta}_n\tilde{x}_{k,l}^{s} \\
&= \left(h^8 g^2 + R_{k,l,s}^{(17)}\right)\left(x_t + \frac{1}{2}\tau x_{tt}(kh, lh, \vartheta^s)\right) - h^8 g^2 g^{ij} x_{u_iu_j} - R_{k,l,s}^{(12)} \\
&\qquad - (\frac{1}{\alpha} - 1)(h^8 g g^{mn} g g^{ij}(x_{u_iu_j}\cdot x_{u_m})x_{u_n} + R_{k,l,s}^{(16)}) \\
&= h^8 g^2\left(x_t - g^{ij}x_{u_iu_j} - (\frac{1}{\alpha} - 1)g^{ij}g^{mn}(x_{u_iu_j}\cdot x_{u_m})x_{u_n}\right) + \frac{1}{2}\tau h^8 g^2 x_{tt}(kh, lh, \vartheta^s) \\
&\qquad + \frac{1}{2}\tau R_{k,l,s}^{(17)} x_{tt}(kh, lh, \vartheta^s) + R_{k,l,s}^{(17)} x_t - R_{k,l,s}^{(12)} - (\frac{1}{\alpha} - 1)R_{k,l,s}^{(16)} \\
&= \frac{1}{2}\tau h^8 g^2 x_{tt}(kh, lh, \vartheta^s) + \frac{1}{2}\tau R_{k,l,s}^{(17)} x_{tt}(kh, lh, \vartheta^s) + R_{k,l,s}^{(17)} x_t - R_{k,l,s}^{(12)} - (\frac{1}{\alpha} - 1)R_{k,l,s}^{(16)},
\end{aligned}$$

where in the last step we inserted the differential equation for the parametrization x being evaluated at $(u_{k,l}, t^s)$. Finally,

$$|\tilde{R}_{k,l}^{\alpha,s}| \leq \frac{1}{|\tilde{g}_{k,l}^{s}|^2}(ch^8\tau + ch^{10} + ch^6(\tau^2 + h^2\tau + h^4)) \leq c(h^{-2}\tau^2 + \tau + h^2), \tag{3.31}$$

since $|\tilde{g}^s_{k,l}| \geq \bar{c}h^4 > 0$ by Lemma 3.11. Recalling that the relation $\tau \leq c'h^2$ is assumed, $|\tilde{R}^{\alpha,s}_{k,l}| \leq c_{\tilde{R}}(h^2 + \tau)$ as stated in (3.21). The constant clearly depends on x and, since $|\frac{1}{\alpha} - 1|$ needs to be bounded in (3.31), also on α^{-1}. □

The uniform boundedness of the area element $g(u)$ (and hence of its approximation $\tilde{g}^s_{k,l}$ according to (3.15)) is one of the essential requirements for our study of convergence. The discrete area element $g^s_{k,l}$ can be expected to fulfill a similar bound as $\tilde{g}^s_{k,l}$ as we will explain in the course of the next chapter.

4 Convergence analysis

Recalling the notation $\tilde{x}^s_{k,l}$ for the restriction of the solution x of the Mean Curvature DeTurck Flow onto the grid $\mathcal{G} = \{u_{k,l}\}_{k,l\in\{0,\dots,N\}} \times \{t^s\}_{s\in\{0,\dots,M\}}$ as well as $x^s_{k,l}$ for the evaluation of the solution x_h of the fully discrete problem at mesh points, we define the error function $e_h : \mathcal{G} \to \mathbb{R}^3$ by

$$e_h := x - x_h$$

and denote by $e^s_{k,l}$ its evaluation at $(u_{k,l}, t^s)$.

The aim of this chapter is to prove convergence for the fully discrete scheme given in (3.9) for fixed $\alpha \in (0, 1]$. To this end, in the first section we investigate how to control several differences and discrete geometric quantities, respectively. In the second section, optimal order of convergence is proved in different norms, starting with discrete L^2-norms on $\{u_{k,l}\}_{k,l\in\{0,\dots,N\}}$ of the first and second discrete spatial derivatives of e_h. Assertions on convergence in other norms on the spatial grid, namely discrete L^2-norms of e_h and its discrete time derivative as well as an L^∞-norm of e_h, follow.

As mentioned before, in the convergence theorem the second order differences will not be estimated in the form in which they are given in (3.3). The central difference for approximating $x_{u_1u_2}$ was chosen to the benefit of a consistency of order 2. In the convergence estimate, however, instead of Δ_{12} we only make use of the backward difference contained in Δ_{12}. The forward difference can be turned into a backward difference when summed over all mesh points, compare Lemma 3.5. This will be carried out in relevant situations. Thus we are going to switch to the notation

$$\Delta^*_{ij} = \begin{cases} \Delta_{ij} & \text{for } i = j = 1, 2, \\ \Delta^-_1\Delta^-_2 & \text{for } i \neq j \end{cases} \tag{4.1}$$

when appropriate.

The primary goal of the convergence analysis is to show that the estimates

$$\begin{aligned} \max_{s\in\{0,\dots,M\}} \left(h^2 \sum_{k,l=1}^{N} \sum_{r=1}^{2} \frac{|\Delta^-_r e^s_{k,l}|^2}{h^2} \right)^{1/2} &\leq c(h^2 + \tau) \\ \left(\tau \sum_{s'=0}^{M} h^2 \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} \frac{|\Delta^*_{ij} e^{s'}_{k,l}|^2}{h^4} \right)^{1/2} &\leq c(h^2 + \tau) \end{aligned} \tag{4.2}$$

hold. Error bounds for other norms then follow from (4.2). Recall that the spatial difference operators are not difference quotients, but merely differences and are thus divided by powers of h in the presented estimates. For the analysis it is crucial to control certain quantities on the grid, as shown in the following section.

4.1 Control of the geometry of the discrete surfaces

The proof of the estimates in (4.2) is conducted by means of an inductive argument with regard to the time grid. To begin with, we present the precise estimates of the induction claim without specifying any prerequisites yet. We then draw important conclusions from the induction hypothesis to control the geometry of the discrete surface. The induction step is then proved in the subsequent section.

4.1 Induction claim. *There exists a constant $W > 0$ depending on x, T and α such that*

$$\begin{aligned} h^2 \sum_{k,l=1}^{N} \sum_{r=1}^{2} \frac{|\Delta_r^- e_{k,l}^s|^2}{h^2} &\leq W(h^2+\tau)^2, \\ \tau \sum_{s'=0}^{s} h^2 \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} \frac{|\Delta_{ij}^* e_{k,l}^{s'}|^2}{h^4} &\leq (h^2+\tau)^{3/2} \end{aligned} \tag{4.3}$$

for all $s \in \{0, \dots, M\}$.

In consequence of the C^4-regularity of the solution x, the periodocity of its derivatives and the condition $0 < 2\bar{c} \leq g$ on the area element, which were both postulated in the consistency estimation of the last chapter and shall be kept throughout the analysis, the following can be assumed to hold:
The approximations of the length elements as well as the approximation of the area element are uniformly bounded, i.e. for all $k, l \in \{1, \dots, N\}$ and for all $s \in \{0, \dots, M\}$ the corresponding differences satisfy

$$\begin{aligned} \bar{c}_1 h \leq |\Delta_1^- \tilde{x}_{k,l}^s| \leq \bar{C}_1 h, \\ \bar{c}_2 h \leq |\Delta_2^- \tilde{x}_{k,l}^s| \leq \bar{C}_2 h, \end{aligned} \tag{4.4}$$

as well as

$$\bar{c} h^4 \leq \tilde{g}_{k,l}^s \leq \bar{C} h^4. \tag{4.5}$$

Furthermore,

$$|\Delta_{ij}^* \tilde{x}_{k,l}^s| \leq \hat{c} h^2. \tag{4.6}$$

The constants $\bar{c}_1, \bar{c}_2, \bar{C}_1, \bar{C}_2, \bar{C}, \hat{c}$ only depend on x. Due to (3.1) and the fact that the inequalities in (4.4) hold for all mesh points, the same bounds are valid for $|\Delta_i^+ \tilde{x}_{k,l}^s|$, $i \in \{1, 2\}$. A proof for the lower bound of $\tilde{g}_{k,l}^s$ for the given bound on g can be found in the previous chapter, see (3.14) and (3.15). In the same manner lower bounds on $|\Delta_i^- \tilde{x}_{k,l}^s|$ follow from bounds on x_{u_i}, $i \in \{1, 2\}$. More precisely, we can choose $\bar{c}_i = 2c_{L_i}$ with c_{L_i} as in (3.19) and impose a smallness condition on h similar to that in the proof of (3.15).

We assume that the norms of interest of the error function fulfil estimates the form in (4.3) at a point of time t^s, where $s \in \{0, \ldots, M-1\}$ is arbitrary but fixed. This is formulated in the following induction hypothesis (IH). The bounds are trivially satisfied in the base case $s = 0$ because $x^0_{k,l} = \tilde{x}^0_{k,l}$.

4.2 Induction hypothesis. *We assume there exists a constant $W > 0$ depending on x, T and α such that*

$$\begin{aligned} h^2 \sum_{k,l=1}^{N} \sum_{r=1}^{2} \frac{|\Delta_r^- e^s_{k,l}|^2}{h^2} &\le W(h^2+\tau)^2, \\ \tau \sum_{s'=0}^{s} h^2 \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} \frac{|\Delta^*_{ij} e^{s'}_{k,l}|^2}{h^4} &\le (h^2+\tau)^{3/2} \end{aligned} \tag{4.7}$$

for one $s \in \{0, \ldots, M-1\}$.

Note that this implies that the second inequality holds for all preceding $s_0 \in \{0, \ldots, s\}$ since for all $i, j \in \{1, 2\}$, $k, l \in \{1, \ldots, N\}$ we have

$$\sum_{s'=0}^{s_0} |\Delta^*_{ij} e^{s'}_{k,l}|^2 \le \sum_{s'=0}^{s} |\Delta^*_{ij} e^{s'}_{k,l}|^2.$$

For the grid point t^s constraints on the discrete length elements $|\Delta_r^- x^s_{k,l}|$ and discrete area element $g^s_{k,l}$ follow. The corresponding bounds in (4.4) and (4.5) only need to be weakened slightly as presented in the next corollary. It is also important to control discrete second derivatives of x_h as formulated in corollary 4.4.

4.3 Corollary. *Let s be chosen as in the induction hypothesis. Then there exists a constant $h_1 > 0$, such that for $h \le h_1$ and for all $k, l \in \{1, \ldots, N\}$*

$$\begin{aligned} \frac{\bar{c}_1}{2} h \le |\Delta_1^{\pm} x^s_{k,l}| &\le 2\bar{C}_1 h, \\ \frac{\bar{c}_2}{2} h \le |\Delta_2^{\pm} x^s_{k,l}| &\le 2\bar{C}_2 h, \\ \frac{\bar{c}}{2} h^4 \le g^s_{k,l} &\le 2\bar{C} h^4. \end{aligned} \tag{4.8}$$

4.4 Corollary. *Let s be chosen as in the induction hypothesis and $k, l \in \{1, \ldots, N\}$ arbitrary. If $\tau \le c' h^2$ for a constant $c' > 0$, then*

1. *there exists a constant $c_2 > 0$ such that*

$$\tau h^{-4} \sum_{i,j=1}^{2} |\Delta^*_{ij} x^s_{k,l}|^2 \le c_2 h; \tag{4.9}$$

2. there exists a constant $h_2 > 0$, such that for $h \le h_2$

$$\tau \sum_{s'=1}^{s} h^{-4} \sum_{i,j=1}^{2} |\Delta_{ij}^{*} x_{k,l}^{s'-1}|^2 \le 8\hat{c}^2 T + 1. \tag{4.10}$$

We begin with the first statement.

Proof of Corollary 4.3. (4.7) together with $\tau \le c' h^2$ implies

$$|\Delta_r^- e_{k,l}^s|^2 \le \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^s|^2 \le c^* h^4. \tag{4.11}$$

On the one hand, it follows for each $r \in \{1,2\}$ that

$$|\Delta_r^- x_{k,l}^s| \le |\Delta_r^- e_{k,l}^s| + |\Delta_r^- \tilde{x}_{k,l}^s| \le \sqrt{c^*} h^2 + \bar{C}_r h \le 2\bar{C}_r h,$$

if $\sqrt{c^*} h^2 \le \bar{C}_r h$, i.e. if $h \le \bar{C}_r / \sqrt{c^*}$, and on the other hand for $r \in \{1,2\}$

$$|\Delta_r^- x_{k,l}^s| \ge \big| |\Delta_r^- e_{k,l}^s| - |\Delta_r^- \tilde{x}_{k,l}^s| \big| \ge |\Delta_r^- \tilde{x}_{k,l}^s| - |\Delta_r^- e_{k,l}^s| \ge \bar{c}_r h - \sqrt{c^*} h^2 \ge \tfrac{1}{2} \bar{c}_r h,$$

if $\sqrt{c^*} h^2 \le \frac{1}{2} \bar{c}_r h$, i.e. if $h \le \bar{c}_r / 2\sqrt{c^*}$. The first two lines of the asserted inequalities are thus satisfied. Hence, for the central differences $\overline{\Delta}_i x_{k,l}^s$ of x_h, and analogously for those of x because of (4.4), we get

$$\begin{aligned} |\overline{\Delta}_i x_{k,l}^s| &= \tfrac{1}{2} |\Delta_i^+ x_{k,l}^s + \Delta_i^- x_{k,l}^s| \le \tfrac{1}{2}(2\bar{C}_i h + 2\bar{C}_i h) = 2\bar{C}_i h, \\ |\overline{\Delta}_i \tilde{x}_{k,l}^s| &= \tfrac{1}{2} |\Delta_i^+ \tilde{x}_{k,l}^s + \Delta_i^- \tilde{x}_{k,l}^s| \le \tfrac{1}{2}(\bar{C}_i h + \bar{C}_i h) = \bar{C}_i h, \end{aligned} \tag{4.12}$$

both of which will be frequently used during the convergence analysis.
The established estimates are now used to prove the remaining inequalities for the discrete area element in (4.8) by finding further upper bounds on the spatial grid size h. To this end, at first the difference between $\tilde{g}_{k,l}^s$ and $g_{k,l}^s$ is examined in such a way as to trace it back to the difference between x and x_h. More precisely, in what follows we will show that

$$\begin{aligned} |\tilde{g}_{k,l}^s - g_{k,l}^s| = \big| &|\Delta_1^+ \tilde{x}_{k,l}^s| |\Delta_1^- \tilde{x}_{k,l}^s| |\Delta_2^+ \tilde{x}_{k,l}^s| |\Delta_2^- \tilde{x}_{k,l}^s| - (\overline{\Delta}_1 \tilde{x}_{k,l}^s \cdot \overline{\Delta}_2 \tilde{x}_{k,l}^s)^2 \\ &- |\Delta_1^+ x_{k,l}^s| |\Delta_1^- x_{k,l}^s| |\Delta_2^+ x_{k,l}^s| |\Delta_2^- x_{k,l}^s| + (\overline{\Delta}_1 x_{k,l}^s \cdot \overline{\Delta}_2 x_{k,l}^s)^2 \big| \\ \le ch^3 &\left(|\Delta_1^- e_{k,l}^s| + |\Delta_1^+ e_{k,l}^s| + |\Delta_2^- e_{k,l}^s| + |\Delta_2^+ e_{k,l}^s| \right). \end{aligned} \tag{4.13}$$

For simplicity, within the scope of this intermediate demonstration we will omit the time index s since no other point of time is treated here.

Using (4.4) and the estimates from (4.8) which have already been proved we have

$$\begin{aligned}
&\big||\Delta_r^- x_{k,l}||\Delta_r^+ x_{k,l}| - |\Delta_r^- \tilde{x}_{k,l}||\Delta_r^+ \tilde{x}_{k,l}|\big| \\
\leq\ &\big||\Delta_r^- x_{k,l}| - |\Delta_r^- \tilde{x}_{k,l}|\big|\,|\Delta_r^+ x_{k,l}| + |\Delta_r^- \tilde{x}_{k,l}|\,\big||\Delta_r^+ x_{k,l}| - |\Delta_r^+ \tilde{x}_{k,l}|\big| \\
\leq\ &\big||\Delta_r^- x_{k,l}| - |\Delta_r^- \tilde{x}_{k,l}|\big|\,2\bar{C}_r h + \bar{C}_r h\,\big||\Delta_r^+ x_{k,l}| - |\Delta_r^+ \tilde{x}_{k,l}|\big| \\
\leq\ &2\bar{C}_r h|\Delta_r^-(x_{k,l} - \tilde{x}_{k,l})| + \bar{C}_r h|\Delta_r^+(x_{k,l} - \tilde{x}_{k,l})| \\
\leq\ &2\bar{C}_r h\left(|\Delta_r^- e_{k,l}| + |\Delta_r^+ e_{k,l}|\right).
\end{aligned} \tag{4.14}$$

This implies, again together with (4.4) and the first two lines of (4.8), that

$$\begin{aligned}
&\big||\Delta_1^- \tilde{x}_{k,l}||\Delta_1^+ \tilde{x}_{k,l}||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| - |\Delta_1^- x_{k,l}||\Delta_1^+ x_{k,l}||\Delta_2^- x_{k,l}||\Delta_2^+ x_{k,l}|\big| \\
= &\big||\Delta_1^- \tilde{x}_{k,l}||\Delta_1^+ \tilde{x}_{k,l}||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| - |\Delta_1^- x_{k,l}||\Delta_1^+ x_{k,l}||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| \\
&+ |\Delta_1^- x_{k,l}||\Delta_1^+ x_{k,l}||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| - |\Delta_1^- x_{k,l}||\Delta_1^+ x_{k,l}||\Delta_2^- x_{k,l}||\Delta_2^+ x_{k,l}|\big| \\
\leq &\big||\Delta_1^- \tilde{x}_{k,l}||\Delta_1^+ \tilde{x}_{k,l}| - |\Delta_1^- x_{k,l}||\Delta_1^+ x_{k,l}|\big|\ |\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| \\
&+ |\Delta_1^- x_{k,l}||\Delta_1^+ x_{k,l}|\ \big||\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| - |\Delta_2^- x_{k,l}||\Delta_2^+ x_{k,l}|\big| \\
\overset{(4.14)}{\leq} &2\bar{C}_1 h\left(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}|\right)|\Delta_2^- \tilde{x}_{k,l}||\Delta_2^+ \tilde{x}_{k,l}| \\
&+ |\Delta_1^- x_{k,l}||\Delta_1^+ x_{k,l}|2\bar{C}_2 h\left(|\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|\right) \\
\leq &2\bar{C}_1 h\left(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}|\right)(\bar{C}_2 h)^2 + (2\bar{C}_1 h)^2 2\bar{C}_2 h\left(|\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|\right) \\
\leq &ch^3\left(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|\right).
\end{aligned}$$

Furthermore, since

$$|\overline{\Delta}_r \tilde{x}_{k,l} - \overline{\Delta}_r x_{k,l}| = |\overline{\Delta}_r(\tilde{x}_{k,l} - x_{k,l})| = |\overline{\Delta}_r e_{k,l}| \leq \tfrac{1}{2}\left(|\Delta_r^+ e_{k,l}| + |\Delta_r^- e_{k,l}|\right),$$

applying (4.12) we obtain

$$\begin{aligned}
&\big|\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l} - \overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l}\big| \\
\leq\ &|\overline{\Delta}_1 \tilde{x}_{k,l}||\overline{\Delta}_2 \tilde{x}_{k,l} - \overline{\Delta}_2 x_{k,l}| + |\overline{\Delta}_1 \tilde{x}_{k,l} - \overline{\Delta}_1 x_{k,l}||\overline{\Delta}_2 x_{k,l}| \\
\leq\ &\bar{C}_1 h\tfrac{1}{2}\left(|\Delta_2^+ e_{k,l}| + |\Delta_2^- e_{k,l}|\right) + \tfrac{1}{2}\left(|\Delta_1^+ e_{k,l}| + |\Delta_1^- e_{k,l}|\right)2\bar{C}_2 h \\
\leq\ &ch\left(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|\right)
\end{aligned} \tag{4.15}$$

which in combination with (4.4) and their analogs for the discrete solution in (4.8) yields

$$\begin{aligned}
&|(\overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l})^2 - (\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l})^2| \\
= &|\overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l} + \overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l}||\overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l} - \overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l}| \\
\leq &((2\bar{C}_1 h)(2\bar{C}_2 h) + (\bar{C}_1 h)(\bar{C}_2 h))\Big(\big|\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l} - \overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l}\big|\Big) \\
\leq &ch^3(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|).
\end{aligned}$$

That means the claimed estimate in (4.13) is valid at t^s.

Because Δ_r^+ can be traced back to Δ_r^- by means of periodicity within the summation over all k and l, compare Lemma 3.5, the above estimate together with (4.11) implies

$$|g_{k,l}^s - \tilde{g}_{k,l}^s|^2 \leq ch^6 \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^s|^2 \leq cc^* h^{10}.$$

Therefore using (4.5) gives,

$$g_{k,l}^s \leq |g_{k,l}^s - \tilde{g}_{k,l}^s| + |\tilde{g}_{k,l}^s| \leq \sqrt{cc^*} h^5 + \bar{C} h^4 \leq 2\bar{C} h^4,$$

if $\sqrt{cc^*} h^5 \leq \bar{C} h^4$, i.e. if $h \leq \bar{C}/\sqrt{cc^*}$, as well as

$$g_{k,l}^s \geq |\tilde{g}_{k,l}^s| - |g_{k,l}^s - \tilde{g}_{k,l}^s| \geq \bar{c} h^4 - \sqrt{cc^*} h^5 \geq \tfrac{\bar{c}}{2} h^4,$$

if $\sqrt{cc^*} h^5 \leq \frac{\bar{c}}{2} h^4$, i.e. if $h \leq \bar{c}/2\sqrt{cc^*}$. □

Proof of Corollary 4.4. Observe that the second part of (4.7) together with $\tau \leq c'h^2$ implies

$$\tau h^{-2} \sum_{s'=0}^{s} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s'}|^2 \leq c^{**} h^3, \tag{4.16}$$

where $c^{**} = (1 + c')^{3/2}$ and the constant c' was determined in the proof of Lemma 3.9.

1. Recalling the bound of second order differences of x in (4.6), for the discrete second order derivatives of x_h we infer that for all $k, l \in \{1, \ldots, N\}$

$$\begin{aligned} \tau h^{-4} \sum_{i,j=1}^{2} |\Delta_{ij}^* x_{k,l}^s|^2 &\leq 2h^{-4}\tau \sum_{i,j=1}^{2} |\Delta_{ij}^* \tilde{x}_{k,l}^s|^2 + 2\tau h^{-4} \sum_{s'=0}^{s} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s'}|^2 \\ &\leq 8\hat{c}^2 c' h^2 + 2c^{**} h^{-4} h^5 \\ &\leq c_2 h. \end{aligned}$$

2. Though the aim is to control differences of x_h of second order as already done before, the summation over s' makes a slight difference. This is briefly recorded in the following estimate where we obtain another smallness condition for the spatial mesh size h. We make use of (4.6) and (4.16) again to infer

$$\tau \sum_{s'=1}^{s} h^{-4} \sum_{i,j=1}^{2} |\Delta_{ij}^* x_{k,l}^{s'-1}|^2 \leq 8\hat{c}^2 s\tau + 2c^{**} h \leq 8\hat{c}^2 T + 1,$$

if $h \leq (2c^{**})^{-1}$.

□

Note that the estimate for the discrete solution x_h^s in (4.12) was used to prove existence for the discrete solution at t^{s+1} in Lemma 3.9 and that the bound does not depend on h or τ. The same holds for the estimates in the first claim of the next lemma.

4.5 Lemma. *Let $k,l \in \{1,\dots,N\}$ be arbitrary and s be chosen as in the induction hypothesis. Then there exists a constant $h_3 > 0$ such that for $h \le h_3$ the following holds:*

1. *There exist positive constants such that*

$$\begin{aligned} \tilde{C}_U^{11} h^{-2} &\le \tilde{g}_{k,l}^{11,s} \le \tilde{C}_O^{11} h^{-2}, & C_U^{11} h^{-2} &\le g_{k,l}^{11,s} \le C_O^{11} h^{-2}, \\ \tilde{C}_U^{22} h^{-2} &\le \tilde{g}_{k,l}^{22,s} \le \tilde{C}_O^{22} h^{-2}, & C_U^{22} h^{-2} &\le g_{k,l}^{22,s} \le C_O^{22} h^{-2}, \\ |\tilde{g}_{k,l}^{12,s}| &\le \tilde{C}_O^{12} h^{-2}, & |g_{k,l}^{12,s}| &\le C_O^{12} h^{-2}. \end{aligned} \tag{4.17}$$

2. *For all $i,j \in \{1,2\}$ we have*

$$|g_{k,l}^{ij,s} - \tilde{g}_{k,l}^{ij,s}| \le c h^{-3} \sum_{r=1}^{2} \left(|\Delta_r^+ e_{k,l}^s| + |\Delta_r^- e_{k,l}^s| \right). \tag{4.18}$$

3. *Let $\lambda_{k,l}^s$ be an eigenvalue of $(g_{k,l}^{ij,s})$. Then there exists a constant $c_0 > 0$ such that*

$$c_0 h^{-2} \le \lambda_{k,l}^s. \tag{4.19}$$

   In particular, $(g_{k,l}^{ij,s})$ is positive definite.

4. *For all $i,j,r \in \{1,2\}$ we have*

$$|\Delta_r^- g_{k,l}^{ij,s}| \le c h^{-3} \left(\max_{k,l} |\Delta_{rr}^* x_{k,l}^s| + \max_{k,l} |\Delta_{12}^* x_{k,l}^s| \right). \tag{4.20}$$

Proof. Since s is the only time index occuring here, for simplification we will not indicate the time dependence throughout the whole proof.

1. Follows directly from the definitions of $g_{k,l}^{ij}$ (see (3.10)) and their analogs $\tilde{g}_{k,l}^{ij}$ as well as from (4.4), (4.5), (4.8) and (4.12), respectively.

2. We first consider the case $i \ne j$. Here we have

$$\begin{aligned} |g_{k,l}^{12} - \tilde{g}_{k,l}^{12}| &= \left| \frac{\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l}}{\tilde{g}_{k,l}} - \frac{\overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l}}{g_{k,l}} \right| \\ &= \left| \frac{g_{k,l}(\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l}) - \tilde{g}_{k,l}(\overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l})}{g_{k,l} \tilde{g}_{k,l}} \right| \\ &\le \frac{|g_{k,l} - \tilde{g}_{k,l}| \, |\overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l}| + \tilde{g}_{k,l} \left| \overline{\Delta}_1 \tilde{x}_{k,l} \cdot \overline{\Delta}_2 \tilde{x}_{k,l} - \overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l} \right|}{g_{k,l} \tilde{g}_{k,l}}, \end{aligned}$$

which is why we can make use of the intermediate result (4.13) giving an estimate for the difference between restricted and discrete area elements. Together with (4.5) and (4.8) as well as (4.12), (4.13) and (4.15) we obtain

$$
\begin{aligned}
&|g^{12}_{k,l} - \tilde{g}^{12}_{k,l}| \\
&\leq \frac{|g_{k,l} - \tilde{g}_{k,l}|\,|\overline{\Delta}_1\tilde{x}_{k,l}\cdot\overline{\Delta}_2\tilde{x}_{k,l}| + \tilde{g}_{k,l}\,\left|\overline{\Delta}_1\tilde{x}_{k,l}\cdot\overline{\Delta}_2\tilde{x}_{k,l} - \overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l}\right|}{g_{k,l}\tilde{g}_{k,l}} \\
&\overset{(4.12)}{\leq} ch^{-8}\left(|g_{k,l} - \tilde{g}_{k,l}|(\bar{C}_1 h)(\bar{C}_2 h) + \bar{C}h^4\left|\overline{\Delta}_1\tilde{x}_{k,l}\cdot\overline{\Delta}_2\tilde{x}_{k,l} - \overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l}\right|\right) \\
&\overset{(4.13),(4.15)}{\leq} ch^{-8}\big(ch^{3+2}(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|) \\
&\qquad\quad + ch^{4+1}(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|)\big) \\
&\leq ch^{-3}(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|).
\end{aligned}
$$

With the help of the (intermediate) estimates that have been proved so far, the assertions for $|g^{11}_{k,l} - \tilde{g}^{11}_{k,l}|$ and $|g^{22}_{k,l} - \tilde{g}^{22}_{k,l}|$ can be seen as well: For $r \in \{1,2\}$, using the essential bounds on the approximated and discrete area elements in (4.5) and (4.8) again, yields

$$
\begin{aligned}
&\left|\frac{|\Delta_r^-\tilde{x}_{k,l}||\Delta_r^+\tilde{x}_{k,l}|}{\tilde{g}_{k,l}} - \frac{|\Delta_r^- x_{k,l}||\Delta_r^+ x_{k,l}|}{g_{k,l}}\right| \\
&= \left|\frac{g_{k,l}|\Delta_r^-\tilde{x}_{k,l}||\Delta_r^+\tilde{x}_{k,l}| - \tilde{g}_{k,l}|\Delta_r^- x_{k,l}||\Delta_r^+ x_{k,l}|}{g_{k,l}\tilde{g}_{k,l}}\right| \\
&\leq \frac{|g_{k,l} - \tilde{g}_{k,l}|\,|\Delta_r^-\tilde{x}_{k,l}||\Delta_r^+\tilde{x}_{k,l}| + \tilde{g}_{k,l}\big||\Delta_r^-\tilde{x}_{k,l}||\Delta_r^+\tilde{x}_{k,l}| - |\Delta_r^- x_{k,l}||\Delta_r^+ x_{k,l}|\big|}{g_{k,l}\tilde{g}_{k,l}} \\
&\overset{(4.13),(4.14)}{\leq} ch^{-8}\big(ch^3(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|)|\Delta_r^-\tilde{x}_{k,l}||\Delta_r^+\tilde{x}_{k,l}| \\
&\qquad\quad + \tilde{g}_{k,l} ch(|\Delta_r^- e_{k,l}| + |\Delta_r^+ e_{k,l}|)\big) \\
&\overset{(4.4)}{\leq} ch^{-3}(|\Delta_1^- e_{k,l}| + |\Delta_1^+ e_{k,l}| + |\Delta_2^- e_{k,l}| + |\Delta_2^+ e_{k,l}|).
\end{aligned}
$$

3. Before considering the matrix $(g^{ij}_{k,l})$ that corresponds to the metric induced by the discrete solution x_h, we treat the continuous case. As the matrix $(g^{ij}(u))$, representing the inverse metric tensor, is positive definite, by a compactness argument we can deduce that for any $u \in [0,2\pi]^2$ and $w \in \mathbb{R}^2$ with $|w| = 1$,

$$w^T \cdot (g^{ij}(u)) \cdot w \geq 4c_0$$

for a suitable constant $c_0 > 0$: If the product was not bounded from below, we would have $\inf_{u\in[0,2\pi]^2} w(g^{ij}(u))w = 0$. Moreover, the infimum would be attained by some $u^* \in [0,2\pi]^2$, which contradicts the positive definiteness.

We now show that the matrix $(\tilde{g}^{ij}_{k,l})$ of the approximated inverse metric coefficients satisfies a similar estimate. From (3.14) and (3.29) we know that

$$\begin{aligned}\tilde{g}_{k,l} &= h^4 g(u_{k,l}) + R_{k,l},\\ \tilde{g}_{k,l}\tilde{g}^{ij}_{k,l} &= h^2 g(u_{k,l}) g^{ij}(u_{k,l}) + R^{(15,i,j)}_{k,l},\ i,j \in \{1,2\},\end{aligned}$$

where $|R_{k,l}| \le ch^6$ and $|R^{(15,i,j)}_{k,l}| \le ch^4$. Hence

$$(h^4 g(u_{k,l}) + R_{k,l})\tilde{g}^{ij}_{k,l} = h^2 g(u_{k,l}) g^{ij}(u_{k,l}) + R^{(15,i,j)}_{k,l}$$

and further

$$h^4 g(u_{k,l})\tilde{g}^{ij}_{k,l} = h^2 g(u_{k,l}) g^{ij}(u_{k,l}) + R^{(15,i,j)}_{k,l} - R_{k,l}\tilde{g}^{ij}_{k,l}.$$

We conclude

$$\tilde{g}^{ij}_{k,l} = h^{-2} g^{ij}(u_{k,l}) + G^{(i,j)}_{k,l},$$

where $|G^{(i,j)}_{k,l}| \le c$ due to the constraints obtained in (4.17) and the boundedness of g. Since $|w| = 1$, it follows that

$$\begin{aligned} w^T \cdot (\tilde{g}^{ij}_{k,l}) \cdot w = \sum_{i,j=1}^{2} w_i w_j \tilde{g}^{ij}_{k,l} &\ge h^{-2} \sum_{i,j=1}^{2} w_i w_j g^{ij}(u_{k,l}) - \left| \sum_{i,j=1}^{2} w_i w_j G^{(i,j)}_{k,l} \right| \\ &\ge 4c_0 h^{-2} - 2c \\ &\ge 2c_0 h^{-2}, \end{aligned}$$

if $h \le \sqrt{c_0/c}$. Combining this bound with the estimate for $|g^{ij}_{k,l} - \tilde{g}^{ij}_{k,l}|$ in (4.18) and the error bound in (4.11), we obtain

$$\begin{aligned} w^T \cdot (g^{ij}_{k,l}) \cdot w = \sum_{i,j=1}^{2} w_i w_j g^{ij}_{k,l} &\ge \sum_{i,j=1}^{2} w_i w_j \tilde{g}^{ij}_{k,l} - \sum_{i,j=1}^{2} |w_i||w_j| \left| g^{ij}_{k,l} - \tilde{g}^{ij}_{k,l} \right| \\ &\ge 2c_0 h^{-2} - 2ch^{-3} \sum_{r=1}^{2} \left(|\Delta^+_r e_{k,l}| + |\Delta^-_r e_{k,l}| \right) \\ &\ge 2c_0 h^{-2} - 4c\sqrt{c^*} h^{-1} \\ &\ge c_0 h^{-2}, \end{aligned}$$

in case $h \le c_0/(4c\sqrt{c^*})$. Note that this also implies the positive definiteness of $(g^{ij}_{k,l})$, which we will need later on.

According to the formula of Rayleigh, [30], the eigenvalues $\lambda_{k,l}$ of the matrix $(g^{ij}_{k,l})$ can, with the help of their corresponding eigenvectors $w_{k,l}$, be estimated by

$$\lambda_{k,l} = \frac{w_{k,l}(g^{ij}_{k,l})w_{k,l}}{w_{k,l} \cdot w_{k,l}} = \frac{w_{k,l}}{|w_{k,l}|}(g^{ij}_{k,l})\frac{w_{k,l}}{|w_{k,l}|} \ge c_0 h^{-2}.$$

4. In the following proof we sometimes need to shift grid functions by $\pm h$ in one direction of the grid, where the direction depends on that of the operator Δ_r^-. In order to avoid a case differentiation and conduct the proof for both $r \in \{1,2\}$ at once, we introduce the notation $E_r^{\pm}$, $r \in \{1,2\}$, for shift operators. Although for a function f we would have to write $(E_r^{\pm} f)_{k,l}$, we use the following notation for convenience and shortness since these operators are often applied to a product of several factors:
$$E_r^{\pm}(f_{k,l}) := \begin{cases} f_{k\pm 1,l} & \text{for } r=1, \\ f_{k,l\pm 1} & \text{for } r=2. \end{cases} \tag{4.21}$$

Thus we have $\Delta_r^- f_{k,l} = f_{k,l} - E_r^-(f_{k,l})$ and $\Delta_r^+ f_{k,l} = E_r^+(f_{k,l}) - f_{k,l}$.

We study the effect of Δ_r^-, $r \in \{1,2\}$, on the entries of the inverse of the metric corresponding to the discrete solution x_h, that is $g_{k,l}^{ij}$. Recalling the definitions in (3.10) we therefore consider the expressions $(g_{k,l})^{-1}(g_{ij})_{k,l}$, where
$$(g_{ij})_{k,l} = \begin{cases} |\Delta_i^+ x_{k,l}||\Delta_i^- x_{k,l}| & \text{for } i=j, \\ \overline{\Delta}_1 x_{k,l} \cdot \overline{\Delta}_2 x_{k,l} & \text{for } i \neq j. \end{cases}$$

With the help of the bounds from (4.8) and (4.12) we find that for $i,j \in \{1,2\}$
$$|(g_{ij})_{k,l}| \leq (2\bar{C}_i h)(2\bar{C}_j h). \tag{4.22}$$

The differences of interest can be rewritten as
$$\begin{aligned} \Delta_r^-\left(\frac{(g_{ij})_{k,l}}{g_{k,l}}\right) &= \frac{(g_{ij})_{k,l}}{g_{k,l}} - \frac{E_r^-((g_{ij})_{k,l})}{E_r^-(g_{k,l})} \\ &= \frac{(g_{ij})_{k,l}E_r^-(g_{k,l}) - E_r^-((g_{ij})_{k,l})g_{k,l}}{E_r^-(g_{k,l})g_{k,l}} \\ &= \frac{-(g_{ij})_{k,l}\Delta_r^-(g_{k,l}) + \Delta_r^-((g_{ij})_{k,l})g_{k,l}}{E_r^-(g_{k,l})g_{k,l}}, \end{aligned} \tag{4.23}$$

and hence we need to have a closer look at the expressions $\Delta_r^-((g_{ij})_{k,l})$ and $\Delta_r^-(g_{k,l})$. For the first one, respresenting the difference of discrete metric coefficients at neighboured mesh points, we distinguish g_{ij} for $i=j$ and $i \neq j$. In the case $i=j$ we have
$$\begin{aligned} |\Delta_r^-((g_{ii})_{k,l})| &= \big||\Delta_i^+ x_{k,l}||\Delta_i^- x_{k,l}| - E_r^-(|\Delta_i^+ x_{k,l}||\Delta_i^- x_{k,l}|)\big| \\ &= \big||\Delta_i^+ x_{k,l}|\Delta_r^-(|\Delta_i^- x_{k,l}|) + \Delta_r^-(|\Delta_i^+ x_{k,l}|)E_r^-(|\Delta_i^- x_{k,l}|)\big| \\ &\overset{(4.8)}{\leq} 2\bar{C}_i h|\Delta_r^-(|\Delta_i^- x_{k,l}|)| + |\Delta_r^-(|\Delta_i^+ x_{k,l}|)|2\bar{C}_i h. \end{aligned}$$

For the resulting differences we obtain

$$\begin{aligned}\left|\Delta_r^-(|\Delta_i^\pm x_{k,l}|)\right| &= \left||\Delta_i^\pm x_{k,l}| - E_r^-(|\Delta_i^\pm x_{k,l}|)\right|\\ &\le \left|\Delta_i^\pm x_{k,l} - E_r^-(\Delta_i^\pm x_{k,l})\right|\\ &= \left|\Delta_r^-\Delta_i^\pm x_{k,l}\right|.\end{aligned}$$

Note that for $i = r$, $\Delta_r^-\Delta_i^\pm x_{k,l}$ is defined for all $k, l \in \{1, \ldots, N\}$ due to the periodic boundary conditions, see (3.11) and Remark 3.8. Applying this is necessary, since for example for $r = i = 2$ we have that $\Delta_2^-\Delta_2^- x_{k,l} = x_{k,l} - 2x_{k,l-1} + x_{k,l-2}$ and for $l = 1$ we thus evaluate at points which are not contained in the grid.

Prior to an examination of $\Delta_r^-\Delta_i^\pm x_{k,l}$, let us see how the case $i \neq j$ results in the same expression. For $i \neq j$ we compute

$$\begin{aligned}&|\Delta_r^-((g_{ij})_{k,l})|\\ &= \left|\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l} - E_r^-(\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l})\right|\\ &= \left|\overline{\Delta}_1 x_{k,l}\cdot\Delta_r^-(\overline{\Delta}_2 x_{k,l}) + \Delta_r^-(\overline{\Delta}_1 x_{k,l})\cdot E_r^-(\overline{\Delta}_2 x_{k,l})\right|\\ &= \tfrac{1}{2}\left|\overline{\Delta}_1 x_{k,l}\cdot\Delta_r^-(\Delta_2^+ + \Delta_2^-)x_{k,l} + \Delta_r^-(\Delta_1^+ + \Delta_1^-)x_{k,l}\cdot E_r^-(\overline{\Delta}_2 x_{k,l})\right|\\ &\overset{(4.12)}{\le} \tfrac{1}{2}\big(2\bar{C}_1 h(|\Delta_r^-\Delta_2^+ x_{k,l}| + |\Delta_r^-\Delta_2^- x_{k,l}|) + (|\Delta_r^-\Delta_1^+ x_{k,l}| + |\Delta_r^-\Delta_1^- x_{k,l}|)2\bar{C}_2 h\big).\end{aligned}$$

For making sense of $\Delta_r^-\Delta_i^\pm x_{k,l}$ and tracing it back to the established differences of second order Δ_{ij}, we now are in need of a case differentiation. To begin, remember that $\Delta_r^+ = \Delta_r^- E_r^+$ (see (3.1)) and recall (3.3) and (4.1). This yields the identities

$$\begin{aligned}\Delta_r^-\Delta_i^+ x_{k,l} &= \begin{cases}\Delta_r^-\Delta_r^+ x_{k,l} = \Delta_{rr}^* x_{k,l} & \text{for } i = r \in \{1,2\},\\ \Delta_r^-\Delta_i^- E_i^+(x_{k,l}) = \Delta_{12}^* E_i^+(x_{k,l}) & \text{for } i \neq r,\end{cases}\\ \Delta_r^-\Delta_i^- x_{k,l} &= \begin{cases}\Delta_r^-\Delta_r^+ E_r^-(x_{k,l}) = \Delta_{rr}^* E_r^-(x_{k,l}) & \text{for } i = r \in \{1,2\},\\ \Delta_1^-\Delta_2^- x_{k,l} = \Delta_{12}^* x_{k,l} & \text{for } i \neq r.\end{cases}\end{aligned} \tag{4.24}$$

The shifted terms can be bounded by the maximum over all indices k, l, i.e.

$$\begin{aligned}|\Delta_{12}^* E_i^+(x_{k,l})| &\le \max_{k,l}|\Delta_{12}^* E_i^+(x_{k,l})| = \max_{k,l}|\Delta_{12}^* x_{k,l}|,\\ |\Delta_{rr}^* E_r^-(x_{k,l})| &\le \max_{k,l}|\Delta_{rr}^* E_r^-(x_{k,l})| = \max_{k,l}|\Delta_{rr}^* x_{k,l}|,\end{aligned}$$

and so some of the single cases can be condensed again. For the estimation of $\Delta_r^-((g_{ij})_{k,l})$ with $i = j$ this means in practice

$$
\begin{aligned}
|\Delta_r^-((g_{ii})_{k,l})| &\le 2\bar{C}_i h|\Delta_r^-\Delta_i^- x_{k,l}| + |\Delta_r^-\Delta_i^+ x_{k,l}|2\bar{C}_i h \\
&\le \begin{cases} 4\bar{C}_i h \max_{k,l}|\Delta_{rr}^* x_{k,l}| & \text{for } i = r \in \{1,2\}, \\ 4\bar{C}_i h \max_{k,l}|\Delta_{12}^* x_{k,l}| & \text{for } i \ne r. \end{cases}
\end{aligned}
$$

If $i \ne j$ we obtain

$$
\begin{aligned}
&|\Delta_r^-((g_{ij})_{k,l})| \\
&\le \tfrac{1}{2}\big(2\bar{C}_1 h(|\Delta_r^-\Delta_2^+ x_{k,l}| + |\Delta_r^-\Delta_2^- x_{k,l}|) + (|\Delta_r^-\Delta_1^+ x_{k,l}| + |\Delta_r^-\Delta_1^- x_{k,l}|)2\bar{C}_2 h\big) \\
&\le ch(\max_{k,l}|\Delta_{rr}^* x_{k,l}| + \max_{k,l}|\Delta_{12}^* x_{k,l}|)
\end{aligned}
$$

and finally

$$
|\Delta_r^-((g_{ij})_{k,l})| \le \begin{cases} ch\max_{k,l}|\Delta_{rr}^* x_{k,l}| & \text{for } i = j = r, \\ ch\max_{k,l}|\Delta_{12}^* x_{k,l}| & \text{for } i = j \ne r, \\ ch(\max_{k,l}|\Delta_{rr}^* x_{k,l}| + \max_{k,l}|\Delta_{12}^* x_{k,l}|) & \text{for } i \ne j. \end{cases} \tag{4.25}
$$

Because of (4.23), it remains to estimate $|\Delta_r^- g_{k,l}|$. Using the product rule (3.4) it holds

$$
\begin{aligned}
&\Delta_r^- g_{k,l} \\
&= \Delta_r^-\left(|\Delta_1^+ x_{k,l}||\Delta_1^- x_{k,l}||\Delta_2^+ x_{k,l}||\Delta_2^- x_{k,l}|\right) - \Delta_r^-\left((\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l})^2\right) \\
&= |\Delta_1^+ x_{k,l}||\Delta_1^- x_{k,l}|\Delta_r^-\left(|\Delta_2^+ x_{k,l}||\Delta_2^- x_{k,l}|\right) \\
&\quad + \Delta_r^-\left(|\Delta_1^+ x_{k,l}||\Delta_1^- x_{k,l}|\right) E_r^-(|\Delta_2^+ x_{k,l}||\Delta_2^- x_{k,l}|) \\
&\quad - \left(\Delta_r^-(\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l})\right)\left(\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l}\right) \\
&\quad - \left(E_r^-(\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l})\right)\left(\Delta_r^-(\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l})\right).
\end{aligned}
$$

Again, the appearance of $\Delta_r^-\Delta_r^\pm x_{k,l}$ and E_r^- requires to apply the implications of the periodic boundary conditions.

With the help of the last intermediate result (4.25) together with some basic results obtained earlier we derive

$$
\begin{aligned}
&|\Delta_r^- g_{k,l}| \\
&\overset{(4.8),(4.12)}{\le} (2\bar{C}_1 h)^2\left|\Delta_r^-\left(|\Delta_2^+ x_{k,l}||\Delta_2^- x_{k,l}|\right)\right| + \left|\Delta_r^-\left(|\Delta_1^+ x_{k,l}||\Delta_1^- x_{k,l}|\right)\right|(2\bar{C}_2 h)^2 \\
&\quad + \left((2\bar{C}_1 h)(2\bar{C}_2 h) + (2\bar{C}_1 h)(2\bar{C}_2 h)\right)\left|\Delta_r^-\left(\overline{\Delta}_1 x_{k,l}\cdot\overline{\Delta}_2 x_{k,l}\right)\right| \\
&\overset{(4.25)}{\le} ch^3(\max_{k,l}|\Delta_{rr}^* x_{k,l}| + \max_{k,l}|\Delta_{12}^* x_{k,l}|).
\end{aligned}
$$

Alltogether we arrive at

$$
\begin{aligned}
\left|\Delta_r^-\left(\frac{(g_{ij})_{k,l}}{g_{k,l}}\right)\right| \leq & \frac{|\Delta_r^-(g_{ij})_{k,l}||g_{k,l}| + |(g_{ij})_{k,l}||\Delta_r^- g_{k,l}|}{E_r^-(g_{k,l})g_{k,l}} \\
\stackrel{(4.8),(4.22)}{\leq} & \frac{|\Delta_r^-(g_{ij})_{k,l}|2\bar{C}h^4 + (2\bar{C}_i h)(2\bar{C}_j h)|\Delta_r^- g_{k,l}|}{ch^8} \\
\leq & ch^{5-8}(\max_{k,l}|\Delta_{12}^* x_{k,l}| + \max_{k,l}|\Delta_{rr}^* x_{k,l}|)
\end{aligned}
$$

and the assertion in (4.20) holds.

□

With these conclusions from the induction hypothesis we can prove the induction step in the next section.

4.2 Error estimation

Apart from the completion of the induction, this section adresses the derivation of further estimates for different discrete norms of the error function. Nevertheless, proving an analog estimate to (4.7) for t^{s+1} constitutes the largest part of it.

By subtracting the difference equations for the functions x and x_h for an arbitrary but fixed position $u_{k,l}$ in the spatial grid,

$$
\begin{aligned}
\frac{\tilde{x}_{k,l}^{s+1} - \tilde{x}_{k,l}^s}{\tau} &= \tilde{g}_{k,l}^{ij,s}\Delta_{ij}\tilde{x}_{k,l}^{s+1} + (\frac{1}{\alpha} - 1)\tilde{g}_{k,l}^{ij,s}\tilde{g}_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1} \cdot \overline{\Delta}_m \tilde{x}_{k,l}^s)\overline{\Delta}_n \tilde{x}_{k,l}^s + \tilde{R}_{k,l}^{\alpha,s}, \\
\frac{x_{k,l}^{s+1} - x_{k,l}^s}{\tau} &= g_{k,l}^{ij,s}\Delta_{ij}x_{k,l}^{s+1} + (\frac{1}{\alpha} - 1)g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}x_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)\overline{\Delta}_n x_{k,l}^s,
\end{aligned}
$$

we obtain the starting point of our estimates:

$$
\begin{aligned}
\frac{e_{k,l}^{s+1} - e_{k,l}^s}{\tau} = & g_{k,l}^{ij,s}\Delta_{ij}e_{k,l}^{s+1} + (\tilde{g}_{k,l}^{ij,s} - g_{k,l}^{ij,s})\Delta_{ij}\tilde{x}_{k,l}^{s+1} + \tilde{R}_{k,l}^{\alpha,s} \\
& + (\frac{1}{\alpha} - 1)\Big[g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)\overline{\Delta}_n x_{k,l}^s \\
& \quad + g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1} \cdot \overline{\Delta}_m e_{k,l}^s)\overline{\Delta}_n x_{k,l}^s \\
& \quad + g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1} \cdot \overline{\Delta}_m \tilde{x}_{k,l}^s)\overline{\Delta}_n e_{k,l}^s \\
& \quad + (\tilde{g}_{k,l}^{ij,s}\tilde{g}_{k,l}^{mn,s} - g_{k,l}^{ij,s}g_{k,l}^{mn,s})(\Delta_{ij}\tilde{x}_{k,l}^{s+1} \cdot \overline{\Delta}_m \tilde{x}_{k,l}^s)\overline{\Delta}_n \tilde{x}_{k,l}^s\Big].
\end{aligned}
\tag{4.26}
$$

Testing the error equation with $-(\Delta_{11}+\Delta_{22})e_{k,l}^{s+1}$ and summation over k and l yields

$$
\begin{aligned}
& -\frac{1}{\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}(e_{k,l}^{s+1}-e_{k,l}^{s})\cdot\Delta_{rr}e_{k,l}^{s+1} \\
=& -\sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{ij,s}\Delta_{ij}e_{k,l}^{s+1}\cdot\Delta_{rr}e_{k,l}^{s+1} - \sum_{r=1}^{2}\sum_{k,l=1}^{N}(\tilde{g}_{k,l}^{ij,s}-g_{k,l}^{ij,s})\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\Delta_{rr}e_{k,l}^{s+1} \\
& -(\frac{1}{\alpha}-1)\Bigg[\sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^{s})(\overline{\Delta}_n x_{k,l}^{s}\cdot\Delta_{rr}e_{k,l}^{s+1}) \\
& \quad + \sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\overline{\Delta}_m e_{k,l}^{s})(\overline{\Delta}_n x_{k,l}^{s}\cdot\Delta_{rr}e_{k,l}^{s+1}) \\
& \quad + \sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\overline{\Delta}_m \tilde{x}_{k,l}^{s})(\overline{\Delta}_n e_{k,l}^{s}\cdot\Delta_{rr}e_{k,l}^{s+1}) \\
& \quad + \sum_{r=1}^{2}\sum_{k,l=1}^{N} (\tilde{g}_{k,l}^{ij,s}\tilde{g}_{k,l}^{mn,s}-g_{k,l}^{ij,s}g_{k,l}^{mn,s})(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\overline{\Delta}_m \tilde{x}_{k,l}^{s})(\overline{\Delta}_n \tilde{x}_{k,l}^{s}\cdot\Delta_{rr}e_{k,l}^{s+1})\Bigg] \\
& -\sum_{r=1}^{2}\sum_{k,l=1}^{N}\tilde{R}_{k,l}^{\alpha,s}\cdot\Delta_{rr}e_{k,l}^{s+1} \\
=:& \; S_1+\cdots+S_7.
\end{aligned}
\tag{4.27}
$$

In the following assertion, the constant c' does not have to meet any further condition than being positive. Yet, it is determined by an estimate in Lemma 3.9, where $\tau \le c'h^2$ has to be presumed to prove existence of the discrete solution at t^{s+1}. This yields that c' only depends on the discrete solution at t^s and thus on x.

4.6 Induction step. *Let $\alpha \in (0,1]$ and let s be chosen as in the induction hypothesis. If $\tau \le c'h^2$ for some $c' > 0$, then there exists a constant $h^* > 0$, such that the estimates*

$$
\begin{aligned}
h^2\sum_{k,l=1}^{N}\sum_{r=1}^{2}\frac{|\Delta_r^- e_{k,l}^{s+1}|^2}{h^2} &\le W(h^2+\tau)^2 \\
\tau\sum_{s'=0}^{s+1}h^2\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}\frac{|\Delta_{ij}^* e_{k,l}^{s'}|^2}{h^4} &\le (h^2+\tau)^{3/2}
\end{aligned}
\tag{4.28}
$$

hold provided that $0 < h \le h^*$.

We present the main steps of the proof of (4.28) in form of the following auxiliary results.

4.7 Lemma. *With $S_2, \dots, S_7$ as in (4.27) we have that*

$$\frac{1}{2\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}(|\Delta_r^- e_{k,l}^{s+1}|^2 - |\Delta_r^- e_{k,l}^{s}|^2) + c_0 h^{-2}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}^* e_{k,l}^{s+1}|^2 \le S_2 + \dots + S_7 + S_{12}, \tag{4.29}$$

where c_0 depends on x, T and α and where

$$|S_{12}| \le c\varepsilon h^{-2}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon}h^{-4}\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^* x_{k,l}^{s}|^2\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_r^- e_{k,l}^{s+1}|^2.$$

4.8 Lemma. *For S_3 in (4.27) we have that*

$$S_3 \le (\tfrac{1}{\alpha} - 1)(P_1 + P_2), \tag{4.30}$$

where

$$|(\tfrac{1}{\alpha}-1)(P_1+P_2)| \le c\varepsilon h^{-2}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon}h^{-4}\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^* x_{k,l}^{s}|^2\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_r^- e_{k,l}^{s+1}|^2.$$

4.9 Lemma. *The following estimate holds*

$$|S_2 + S_4 + S_5 + S_6 + S_7| \le \frac{c}{\varepsilon}(h^2+\tau)^2 + c\varepsilon h^{-2}\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_{rr}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon}\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_r^- e_{k,l}^{s}|^2$$

Proof of Lemma 4.7. With the help of summation by parts according to (3.7), the product of the test function with the discrete time derivative on the left hand-side of equation (4.27) can be rewritten. That means, with respect to the notation $\Delta_{rr} = \Delta_r^+\Delta_r^-$ we have for $r \in \{1,2\}$

$$\begin{aligned}
&-\frac{1}{\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}(e_{k,l}^{s+1} - e_{k,l}^{s})\cdot\Delta_{rr}e_{k,l}^{s+1}\\
&= \frac{1}{\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}\Delta_r^-(e_{k,l}^{s+1} - e_{k,l}^{s})\cdot\Delta_r^- e_{k,l}^{s+1}\\
&= \frac{1}{2\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}(|\Delta_r^- e_{k,l}^{s+1}|^2 - |\Delta_r^- e_{k,l}^{s}|^2) + \frac{1}{2\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}|\Delta_r^- e_{k,l}^{s+1} - \Delta_r^- e_{k,l}^{s}|^2\\
&\ge \frac{1}{2\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}(|\Delta_r^- e_{k,l}^{s+1}|^2 - |\Delta_r^- e_{k,l}^{s}|^2).
\end{aligned}$$

This implies

$$\frac{1}{2\tau}\sum_{r=1}^{2}\sum_{k,l=1}^{N}(|\Delta_r^- e_{k,l}^{s+1}|^2 - |\Delta_r^- e_{k,l}^{s}|^2) \le S_1 + \cdots + S_7 \tag{4.31}$$

and to proceed, S_1 is estimated and reformulated as follows in order to carry out summation by parts for some assignments of indices. At the same time, the notation Δ_{12} is resolved into $\frac{1}{2}(\Delta_1^+\Delta_2^+ + \Delta_1^-\Delta_2^-)$. We obtain

$$\begin{aligned}
S_1 &= -\sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{ij,s}\Delta_{ij}e_{k,l}^{s+1}\cdot\Delta_{rr}e_{k,l}^{s+1} \\
&= -\sum_{k,l=1}^{N}(g_{k,l}^{11,s}\Delta_{11}e_{k,l}^{s+1} + 2g_{k,l}^{12,s}\Delta_{12}e_{k,l}^{s+1} + g_{k,l}^{22,s}\Delta_{22}e_{k,l}^{s+1})\cdot(\Delta_{11}e_{k,l}^{s+1} + \Delta_{22}e_{k,l}^{s+1}) \\
&= -\sum_{k,l=1}^{N}\sum_{r=1}^{2}\left[g_{k,l}^{rr,s}|\Delta_{rr}e_{k,l}^{s+1}|^2 + g_{k,l}^{12,s}\Delta_1^-\Delta_2^- e_{k,l}^{s+1}\cdot\Delta_{rr}e_{k,l}^{s+1}\right] \\
&\quad -\sum_{k,l=1}^{N}\sum_{r=1}^{2}\left[g_{k,l}^{rr,s}\Delta_{22}e_{k,l}^{s+1}\cdot\Delta_{11}e_{k,l}^{s+1} + g_{k,l}^{12,s}\Delta_1^+\Delta_2^+ e_{k,l}^{s+1}\cdot\Delta_{rr}e_{k,l}^{s+1}\right].
\end{aligned} \tag{4.32}$$

The summands in the last row are now each reformulated by applying (3.6) twice. To start with, for each $r \in \{1, 2\}$ the operator Δ_1^+ in

$$g_{k,l}^{rr,s}\Delta_{22}e_{k,l}^{s+1}\cdot\Delta_{11}e_{k,l}^{s+1} = g_{k,l}^{rr,s}\Delta_{22}e_{k,l}^{s+1}\cdot\Delta_1^+(\Delta_1^- e_{k,l}^{s+1})$$

is turned into $-\Delta_1^-$ by shifting it to the other factors. The resulting difference of third order becomes one of second order again by using the summation by parts formula again, this time applying it to Δ_2^+. In detail this means

$$\begin{aligned}
&-\sum_{k,l=1}^{N} g_{k,l}^{rr,s}\Delta_{22}e_{k,l}^{s+1}\cdot\Delta_{11}e_{k,l}^{s+1} \\
&\overset{(3.6)}{=} \sum_{k,l=1}^{N}\Delta_1^-(g_{k,l}^{rr,s})\Delta_2^+\Delta_2^- e_{k-1,l}^{s+1}\cdot\Delta_1^- e_{k,l}^{s+1} + \sum_{k,l=1}^{N} g_{k,l}^{rr,s}\underbrace{\Delta_1^-\Delta_2^+\Delta_2^- e_{k,l}^{s+1}}_{=\Delta_2^+(\Delta_1^-\Delta_2^- e_{k,l}^{s+1})}\cdot\Delta_1^- e_{k,l}^{s+1} \\
&\overset{(3.6)}{=} \sum_{k,l=1}^{N}\Delta_1^-(g_{k,l}^{rr,s})\Delta_2^+\Delta_2^- e_{k-1,l}^{s+1}\cdot\Delta_1^- e_{k,l}^{s+1} \\
&\quad -\sum_{k,l=1}^{N}\Delta_2^-(g_{k,l}^{rr,s})\Delta_1^-\Delta_2^- e_{k,l}^{s+1}\cdot\Delta_1^- e_{k,l-1}^{s+1} - \sum_{k,l=1}^{N} g_{k,l}^{rr,s}\underbrace{\Delta_1^-\Delta_2^- e_{k,l}^{s+1}\cdot\Delta_2^-\Delta_1^- e_{k,l}^{s+1}}_{=|\Delta_1^-\Delta_2^- e_{k,l}^{s+1}|^2}.
\end{aligned}$$

In this calculation, the fact that the $g_{k,l}^{ij,s}$, just like x_h, are periodic in space is of importance. Firstly, this is the main feature in the proof of (3.6), which leads to a cancelling of the boundary terms. Secondly, expressions like $\Delta_2^-(g_{k,l}^{rr,s})$ require to evaluate x_h at points outside of the spatial grid which are defined with the help of periodic extension.

Although we treat the second term in the last row of (4.32),

$$\sum_{r=1}^{2} g_{k,l}^{12,s}\Delta_1^+\Delta_2^+ e_{k,l}^{s+1}\cdot\Delta_{rr}e_{k,l}^{s+1} = g_{k,l}^{12,s}\Delta_1^+\Delta_2^+ e_{k,l}^{s+1}\cdot(\Delta_1^+(\Delta_1^- e_{k,l}^{s+1}) + \Delta_2^+(\Delta_2^- e_{k,l}^{s+1})),$$

analogously, this is carried out in detail to further illustrate summation by parts in principle. This will help understanding the method when applied to S_3, where there are more factors and a more compact notation. We obtain

$$\begin{aligned}
&-\sum_{k,l=1}^{N} g_{k,l}^{12,s}\Delta_1^+\Delta_2^+ e_{k,l}^{s+1}\cdot(\Delta_1^+(\Delta_1^- e_{k,l}^{s+1}) + \Delta_2^+(\Delta_2^- e_{k,l}^{s+1}))\\
=&\sum_{k,l=1}^{N}\Delta_1^-(g_{k,l}^{12,s})\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}\cdot\Delta_1^- e_{k,l}^{s+1} + \sum_{k,l=1}^{N}\Delta_2^-(g_{k,l}^{12,s})\Delta_1^+\Delta_2^+ e_{k,l-1}^{s+1}\cdot\Delta_2^- e_{k,l}^{s+1}\\
&+\sum_{k,l=1}^{N} g_{k,l}^{12,s}\underbrace{\Delta_1^-\Delta_1^+\Delta_2^+ e_{k,l}^{s+1}}_{=\Delta_2^+(\Delta_1^-\Delta_1^+ e_{k,l}^{s+1})}\cdot\Delta_1^- e_{k,l}^{s+1} + \sum_{k,l=1}^{N} g_{k,l}^{12,s}\underbrace{\Delta_2^-\Delta_1^+\Delta_2^+ e_{k,l}^{s+1}}_{=\Delta_1^+(\Delta_2^-\Delta_2^+ e_{k,l}^{s+1})}\cdot\Delta_2^- e_{k,l}^{s+1}\\
=&\sum_{k,l=1}^{N}\Delta_1^-(g_{k,l}^{12,s})\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}\cdot\Delta_1^- e_{k,l}^{s+1} + \sum_{k,l=1}^{N}\Delta_2^-(g_{k,l}^{12,s})\Delta_1^+\Delta_2^+ e_{k,l-1}^{s+1}\cdot\Delta_2^- e_{k,l}^{s+1}\\
&-\sum_{k,l=1}^{N}\Delta_2^-(g_{k,l}^{12,s})\Delta_1^-\Delta_1^+ e_{k,l}^{s+1}\cdot\Delta_1^- e_{k,l-1}^{s+1} - \sum_{k,l=1}^{N}\Delta_1^-(g_{k,l}^{12,s})\Delta_2^-\Delta_2^+ e_{k,l}^{s+1}\cdot\Delta_2^- e_{k-1,l}^{s+1}\\
&-\sum_{k,l=1}^{N} g_{k,l}^{12,s}\underbrace{\Delta_1^-\Delta_1^+ e_{k,l}^{s+1}}_{=\Delta_{11}e_{k,l}^{s+1}}\cdot\Delta_2^-\Delta_1^- e_{k,l}^{s+1} - \sum_{k,l=1}^{N} g_{k,l}^{12,s}\underbrace{\Delta_2^-\Delta_2^+ e_{k,l}^{s+1}}_{=\Delta_{22}e_{k,l}^{s+1}}\cdot\Delta_1^-\Delta_2^- e_{k,l}^{s+1}.
\end{aligned}$$

Combining these results with the unaltered terms in (4.32) yields

$$\begin{aligned}
S_1 =& -\sum_{k,l=1}^{N}\sum_{r=1}^{2}\left[g_{k,l}^{rr,s}|\Delta_{rr}e_{k,l}^{s+1}|^2 + 2g_{k,l}^{12,s}\Delta_1^-\Delta_2^- e_{k,l}^{s+1}\cdot\Delta_{rr}e_{k,l}^{s+1} + g_{k,l}^{rr,s}|\Delta_1^-\Delta_2^- e_{k,l}^{s+1}|^2\right]\\
&+\sum_{k,l=1}^{N}\Big[\sum_{r=1}^{2}\left(\Delta_1^-(g_{k,l}^{rr,s})\Delta_{22}e_{k-1,l}^{s+1}\cdot\Delta_1^- e_{k,l}^{s+1} - \Delta_2^-(g_{k,l}^{rr,s})\Delta_1^-\Delta_2^- e_{k,l}^{s+1}\cdot\Delta_1^- e_{k,l-1}^{s+1}\right)\\
&\qquad + \Delta_1^- g_{k,l}^{12,s}\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}\cdot\Delta_1^- e_{k,l}^{s+1} + \Delta_2^- g_{k,l}^{12,s}\Delta_1^+\Delta_2^+ e_{k,l-1}^{s+1}\cdot\Delta_2^- e_{k,l}^{s+1}\\
&\qquad - \Delta_2^- g_{k,l}^{12,s}\Delta_{11}e_{k,l}^{s+1}\cdot\Delta_1^- e_{k,l-1}^{s+1} - \Delta_1^- g_{k,l}^{12,s}\Delta_{22}e_{k,l}^{s+1}\cdot\Delta_2^- e_{k-1,l}^{s+1}\Big]\\
=&: -S_{11} + S_{12}.
\end{aligned} \tag{4.33}$$

Recalling notation (4.1), which replaces Δ_{rr} by Δ^*_{rr} and $\Delta^-_1\Delta^-_2$ by Δ^*_{12}, we can reformulate

$$\begin{aligned}
S_{11} = \sum_{k,l=1}^{N} \Big[& g^{11,s}_{k,l}|\Delta_{11}e^{s+1}_{k,l}|^2 + 2g^{12,s}_{k,l}\Delta^*_{12}e^{s+1}_{k,l}\cdot\Delta_{11}e^{s+1}_{k,l} + g^{22,s}_{k,l}|\Delta^*_{12}e_{k,l}|^2 \\
& + g^{22,s}_{k,l}|\Delta_{22}e^{s+1}_{k,l}|^2 + 2g^{12,s}_{k,l}\Delta^*_{12}e^{s+1}_{k,l}\cdot\Delta_{22}e^{s+1}_{k,l} + g^{11,s}_{k,l}|\Delta^*_{12}e_{k,l}|^2\Big] \\
= \sum_{k,l=1}^{N}\sum_{r=1}^{2} & g^{ij,s}_{k,l}\Delta^*_{ir}e^{s+1}_{k,l}\cdot\Delta^*_{jr}e^{s+1}_{k,l}.
\end{aligned}$$

For an estimate, the following well-known result from linear algebra is helpul:
Let $(\lambda^s_{k,l})_{\min}$ be the smallest eigenvalue of the symmetric matrix $(g^{ij,s}_{k,l})$. Then

$$w^T\cdot(g^{ij,s}_{k,l})\cdot w \geq (\lambda^s_{k,l})_{\min}\|w\|^2 \tag{4.34}$$

for all $w\in\mathbb{R}^2$.

Applying this as well as (4.19) we conclude that

$$\begin{aligned}
\sum_{r=1}^{2} g^{ij,s}_{k,l}\Delta^*_{ir}e^{s+1}_{k,l}\cdot\Delta^*_{jr}e^{s+1}_{k,l} &= \sum_{r=1}^{2}(\Delta^*_{1r}e^{s+1}_{k,l},\Delta^*_{2r}e^{s+1}_{k,l})\cdot(g^{ij,s}_{k,l})\cdot(\Delta^*_{1r}e^{s+1}_{k,l},\Delta^*_{2r}e^{s+1}_{k,l})^T \\
&\geq (\lambda^s_{k,l})_{\min}\sum_{r=1}^{2}(|\Delta^*_{1r}e^s_{k,l}|^2 + |\Delta^*_{2r}e^s_{k,l}|^2) \\
&= (\lambda^s_{k,l})_{\min}\sum_{i,j=1}^{2}|\Delta^*_{ij}e^s_{k,l}|^2 \\
&\geq c_0h^{-2}\sum_{i,j=1}^{2}|\Delta^*_{ij}e^s_{k,l}|^2
\end{aligned}$$

holds for any fixed $k,l\in\{1,\dots,N\}$. That means

$$S_1 = -S_{11}+S_{12} \leq -c_0h^{-2}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta^*_{ij}e^{s+1}_{k,l}|^2 + S_{12},$$

and, together with (4.31), the inequality in (4.29) follows.

It remains to show the estimate for S_{12}, which uses the inequalities of Cauchy-Schwarz, denoted by (CS), and Young, marked by (Y). The differences of $g^{ij,s}_{k,l}$ with respect to each direction of the variable $u_{k,l}$, which arise in

$$
\begin{aligned}
S_{12} = \sum_{k,l=1}^{N} \Big[& \Delta_1^-(g_{k,l}^{22,s} + g_{k,l}^{11,s})\Delta_{22} e_{k-1,l}^{s+1} \cdot \Delta_1^- e_{k,l}^{s+1} - \Delta_2^-(g_{k,l}^{22,s} + g_{k,l}^{11,s})\Delta_1^-\Delta_2^- e_{k,l}^{s+1} \cdot \Delta_1^- e_{k,l-1}^{s+1} \\
& + \Delta_1^- g_{k,l}^{12,s} \Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1} \cdot \Delta_1^- e_{k,l}^{s+1} + \Delta_2^- g_{k,l}^{12,s} \Delta_1^+\Delta_2^+ e_{k,l-1}^{s+1} \cdot \Delta_2^- e_{k,l}^{s+1} \\
& - \Delta_2^- g_{k,l}^{12,s} \Delta_{11} e_{k,l}^{s+1} \cdot \Delta_1^- e_{k,l-1}^{s+1} - \Delta_1^- g_{k,l}^{12,s} \Delta_{22} e_{k,l}^{s+1} \cdot \Delta_2^- e_{k-1,l}^{s+1} \Big],
\end{aligned}
$$

are bounded by different discrete second derivatives of x_h, see (4.20). For instance

$$
|\Delta_r^- g_{k,l}^{12,s}| \le ch^{-3}(\max_{k,l} |\Delta_{rr} x_{k,l}^s| + \max_{k,l} |\Delta_{12}^* x_{k,l}^s|),
$$

which appears in the second and third line of S_{12}. For the whole second line we thus have

$$
\begin{aligned}
& \left| \sum_{k,l=1}^{N} \left[\Delta_1^- g_{k,l}^{12,s} \Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1} \cdot \Delta_1^- e_{k,l}^{s+1} + \Delta_2^- g_{k,l}^{12,s} \Delta_1^+\Delta_2^+ e_{k,l-1}^{s+1} \cdot \Delta_2^- e_{k,l}^{s+1} \right] \right| \\
& \le \sum_{k,l=1}^{N} \Big[ch^{-3}(\max_{k,l} |\Delta_{11} x_{k,l}^s| + \max_{k,l} |\Delta_{12}^* x_{k,l}^s|) |\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}| |\Delta_1^- e_{k,l}^{s+1}| \\
& \quad + ch^{-3}(\max_{k,l} |\Delta_{22} x_{k,l}^s| + \max_{k,l} |\Delta_{12}^* x_{k,l}^s|) |\Delta_1^+\Delta_2^+ e_{k,l-1}^{s+1}| |\Delta_2^- e_{k,l}^{s+1}| \Big].
\end{aligned}
$$

Instead of estimating all its terms seperately, we consider one summand exemplarily. In addition to applying the inequalities of Cauchy-Schwarz and Young, $\Delta_1^+\Delta_2^+$ is changed into $\Delta_1^-\Delta_2^- = \Delta_{12}^*$, see (4.1), according to Lemma 3.5. After that we make use of the periodicity of e_h on the spatial grid as demonstrated in (3.5).

$$
\begin{aligned}
& \sum_{k,l=1}^{N} ch^{-3} \max_{k,l} |\Delta_{12}^* x_{k,l}^s| |\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}| |\Delta_1^- e_{k,l}^{s+1}| \\
& \overset{\text{(CS)}}{\le} \left(h^{-2} \sum_{k,l=1}^{N} |\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}|^2 \right)^{1/2} \left(ch^{-6} h^2 \max_{k,l} |\Delta_{12}^* x_{k,l}^s|^2 \sum_{k,l=1}^{N} |\Delta_1^- e_{k,l}^{s+1}|^2 \right)^{1/2} \\
& \overset{\text{(Y)}}{\le} \varepsilon h^{-2} \sum_{k,l=1}^{N} |\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}|^2 + \frac{c}{\varepsilon} h^{-4} \max_{k,l} |\Delta_{12}^* x_{k,l}^s|^2 \sum_{k,l=1}^{N} |\Delta_1^- e_{k,l}^{s+1}|^2 \qquad (4.35) \\
& = \varepsilon h^{-2} \sum_{k,l=1}^{N} |\Delta_1^-\Delta_2^- e_{k-1,l}^{s+1}|^2 + \frac{c}{\varepsilon} h^{-4} \max_{k,l} |\Delta_{12}^* x_{k,l}^s|^2 \sum_{k,l=1}^{N} |\Delta_1^- e_{k,l}^{s+1}|^2 \\
& = \varepsilon h^{-2} \sum_{k,l=1}^{N} |\Delta_{12}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon} h^{-4} \max_{k,l} |\Delta_{12}^* x_{k,l}^s|^2 \sum_{k,l=1}^{N} |\Delta_1^- e_{k,l}^{s+1}|^2.
\end{aligned}
$$

Proceeding like this for the whole second line of S_{12}, we infer

$$\left|\sum_{k,l=1}^{N}\sum_{r=1}^{2}\Delta_r^- g_{k,l}^{12,s}\Delta_1^+\Delta_2^+ e_{k-1,l}^{s+1}\cdot\Delta_r^- e_{k,l}^{s+1}\right|$$
$$\leq \varepsilon h^{-2}\sum_{k,l=1}^{N}|\Delta_{12}^* e_{k,l}^{s+1}|^2+\frac{c}{\varepsilon}h^{-4}\sum_{r=1}^{2}(\max_{k,l}|\Delta_{rr}x_{k,l}^s|^2+\max_{k,l}|\Delta_{12}^* x_{k,l}^s|^2)\sum_{k,l=1}^{N}|\Delta_r^- e_{k,l}^{s+1}|^2.$$

Note that, compared to this example, for other terms in S_{12} we obtain summands in the bound like $\max_{k,l}|\Delta_{ii}^* x_{k,l}^s|^2|\Delta_r^- e_{k,l}^{s+1}|^2$ for $i\neq r$. This is due to the coupling of Δ_1^- and Δ_2^- within one product in some of the terms in S_{12}, e.g.

$$\left|\sum_{k,l=1}^{N}\Delta_2^- g_{k,l}^{12,s}\Delta_{11}e_{k,l}^{s+1}\cdot\Delta_1^- e_{k,l-1}^{s+1}\right|$$
$$\leq \varepsilon h^{-2}\sum_{k,l=1}^{N}|\Delta_{11}e_{k,l}^{s+1}|^2+\frac{c}{\varepsilon}h^{-4}(\max_{k,l}|\Delta_{22}x_{k,l}^s|^2+\max_{k,l}|\Delta_{12}^* x_{k,l}^s|^2)\sum_{k,l=1}^{N}|\Delta_1^- e_{k,l}^{s+1}|^2.$$

Approaching the rest of S_{12} in the same manner and thereby replacing Δ_{rr} by Δ_{rr}^*, see (4.1) again, we get

$$|S_{12}|\leq c\varepsilon h^{-2}\sum_{k,l=1}^{N}\left(|\Delta_{11}^* e_{k,l}^{s+1}|^2+|\Delta_{12}^* e_{k,l}^{s+1}|^2+|\Delta_{22}^* e_{k,l}^{s+1}|^2\right)$$
$$+\frac{c}{\varepsilon}h^{-4}\left(\max_{k,l}|\Delta_{11}^* x_{k,l}^s|^2+\max_{k,l}|\Delta_{12}^* x_{k,l}^s|^2+\max_{k,l}|\Delta_{22}^* x_{k,l}^s|^2\right)\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_r^- e_{k,l}^{s+1}|^2.$$

□

The proofs of Lemma 4.8 and Lemma 4.9 are conducted both at once since we wish to postpone the estimation of the term $|P_1+P_2|$ to the end of the argumentation.

Proof of Lemma 4.8 and Lemma 4.9. We examine S_3 similarly to S_1 by summation by parts. This time, we make use of the formula in 2. in Lemma 3.6. Since summation by parts involves a shifting into neighboured mesh points and since we now apply the lemma for both variables u_1 and u_2, we want to take up a notation from the proof of Lemma 4.5, where we defined

$$E_r^{\pm}(f_{k,l}):=\begin{cases}f_{k\pm1,l} & \text{for } r=1,\\ f_{k,l\pm1} & \text{for } r=2.\end{cases}$$

Note that $\Delta_r^- f_{k,l}=f_{k,l}-E_r^-(f_{k,l})$ and $\Delta_r^+ f_{k,l}=E_r^+(f_{k,l})-f_{k,l}$ and that, again, we actually mean $(E_r^{\pm}f)_{k,l}$, but use the above notation as an abbreviation since the operators will be applied to products. This enables us to carry out the summation by parts with respect to $\Delta_r^{\pm}$ without specifiying $r \in \{1,2\}$. To this end, the following re-sorting is helpful, where, without loss of generality, we suppose that $\alpha \neq 1$:

$$
\begin{aligned}
(\frac{1}{\alpha}-1)^{-1}S_3 &= -\sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{ij,s} g_{k,l}^{mn,s} (\Delta_{ij} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1}) \\
&= -\sum_{r=1}^{2}\sum_{k,l=1}^{N} \Big[g_{k,l}^{rr,s} g_{k,l}^{mn,s} (\Delta_{rr} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1}) \\
&\qquad + g_{k,l}^{12,s} g_{k,l}^{mn,s} (\Delta_1^- \Delta_2^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1}) \Big] \\
&\quad - \sum_{i\neq r}\sum_{k,l=1}^{N} \Big[g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\Delta_{ii} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1}) \\
&\qquad + g_{k,l}^{ir,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_r^+ e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1}) \Big] \\
&=: S_{31} + S_{32}.
\end{aligned}
$$

Terms where $i = j = r$ as well as terms with $\Delta_1^- \Delta_2^-$ when $i \neq j$ are summarized in S_{31} and are not reformulated. For each of the other terms, i.e. for those where $i = j \neq r$ and those with $\Delta_i^+ \Delta_r^+ e_{k,l}$ for $i \neq j = r$, we perform summation by parts twice, as presented below. We apply (3.8) for a first time to find

$$
\begin{aligned}
S_{32} &\overset{(3.8)}{=} \sum_{i\neq r}\sum_{k,l=1}^{N} E_r^- \left(g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_i^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \right) (\Delta_r^- \overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&\quad + \sum_{i\neq r}\sum_{k,l=1}^{N} E_r^- \left(g_{k,l}^{ir,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_r^+ e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \right) (\Delta_r^- \overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&\quad + \sum_{i\neq r}\sum_{k,l=1}^{N} \Delta_r^- \left(g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_i^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \right) (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&\quad + \sum_{i\neq r}\sum_{k,l=1}^{N} \Delta_r^- \left(g_{k,l}^{ir,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_r^+ e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \right) (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}).
\end{aligned}
$$

Note that here, the additionally defined evaluations of x_h and x at points outside of the spatial grid (see Remark 3.8), appear, e.g. in $E_1^-(\overline{\Delta}_1 x_{k,l}^s) = x_{k,l} - x_{k-2,l}$ for $k = 1$. Moreover, the periodicity of the discrete length elements is used again.

The product rule for the operator Δ_r^-, which we recorded in (3.4), gives

$$
\begin{aligned}
&\Delta_r^- \left((g_{k,l}^{mn,s} g_{k,l}^{ii,s} \Delta_i^+ \Delta_i^- + g_{k,l}^{mn,s} g_{k,l}^{ir,s} \Delta_i^+ \Delta_r^+) e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s \right) \\
&= E_r^- \left((g_{k,l}^{mn,s} g_{k,l}^{ii,s} \Delta_i^+ \Delta_i^- + g_{k,l}^{mn} g_{k,l}^{ir,s} \Delta_i^+ \Delta_r^+) e_{k,l}^{s+1} \right) \cdot \Delta_r^- \overline{\Delta}_m x_{k,l}^s \\
&\quad + \Delta_r^- \left((g_{k,l}^{mn,s} g_{k,l}^{ii,s} \Delta_i^+ \Delta_i^- + g_{k,l}^{mn,s} g_{k,l}^{ir,s} \Delta_i^+ \Delta_r^+) e_{k,l}^{s+1} \right) \cdot \overline{\Delta}_m x_{k,l}^s \\
&= E_r^- \left((g_{k,l}^{mn,s} g_{k,l}^{ii,s} \Delta_i^+ \Delta_i^- + g_{k,l}^{mn} g_{k,l}^{ir,s} \Delta_i^+ \Delta_r^+) e_{k,l}^{s+1} \right) \cdot \Delta_r^- \overline{\Delta}_m x_{k,l}^s \\
&\quad + \left(\Delta_r^- (g_{k,l}^{mn,s} g_{k,l}^{ii,s}) E_r^- (\Delta_i^+ \Delta_i^- e_{k,l}^{s+1}) + \Delta_r^- (g_{k,l}^{mn,s} g_{k,l}^{ir,s}) E_r^- (\Delta_i^+ \Delta_r^+ e_{k,l}^{s+1}) \right) \cdot \overline{\Delta}_m x_{k,l}^s \\
&\quad + \left((g_{k,l}^{mn,s} g_{k,l}^{ii,s} \Delta_r^- \Delta_i^+ \Delta_i^- + g_{k,l}^{mn,s} g_{k,l}^{ir,s} \Delta_r^- \Delta_i^+ \Delta_r^+) e_{k,l}^{s+1} \right) \cdot \overline{\Delta}_m x_{k,l}^s,
\end{aligned}
$$

where the last row now contains differences of third order that are of special interest in our further steps. Thus

$$
\begin{aligned}
S_{32} = P_1 &+ \sum_{i \neq r} \sum_{k,l=1}^{N} g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\underbrace{\Delta_r^- \Delta_i^+ \Delta_i^-}_{=\Delta_i^+ (\Delta_r^- \Delta_i^-)} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+ \sum_{i \neq r} \sum_{k,l=1}^{N} g_{k,l}^{ir,s} g_{k,l}^{mn,s} (\underbrace{\Delta_r^- \Delta_i^+ \Delta_r^+}_{=\Delta_i^+ (\Delta_r^- \Delta_r^+)} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}),
\end{aligned}
$$

where

$$
\begin{aligned}
P_1 = &\sum_{i \neq r} \sum_{k,l=1}^{N} E_r^- \left(g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_i^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \right) (\Delta_r^- \overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+ \sum_{i \neq r} \sum_{k,l=1}^{N} E_r^- \left(g_{k,l}^{ir,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_r^+ e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \right) (\Delta_r^- \overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+ \sum_{i \neq r} \sum_{k,l=1}^{N} E_r^- (g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_i^- e_{k,l}^{s+1} \cdot \Delta_r^- \overline{\Delta}_m x_{k,l}^s))(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+ \sum_{i \neq r} \sum_{k,l=1}^{N} E_r^- (g_{k,l}^{ir,s} g_{k,l}^{mn,s} (\Delta_i^+ \Delta_r^+ e_{k,l}^{s+1} \cdot \Delta_r^- \overline{\Delta}_m x_{k,l}^s))(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+ \sum_{i \neq r} \sum_{k,l=1}^{N} \Delta_r^- (g_{k,l}^{mn,s} g_{k,l}^{ii,s}) E_r^- (\Delta_i^+ \Delta_i^- e_{k,l}^{s+1}) \cdot \overline{\Delta}_m x_{k,l}^s (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+ \sum_{i \neq r} \sum_{k,l=1}^{N} \Delta_r^- (g_{k,l}^{mn,s} g_{k,l}^{ir,s}) E_r^- (\Delta_i^+ \Delta_r^+ e_{k,l}^{s+1}) \cdot \overline{\Delta}_m x_{k,l}^s (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}).
\end{aligned}
\tag{4.36}
$$

Note that again we apply the periodic boundary conditions. This concerns for instance $E_2^-(g_{k,l}^{11})$, which leads to the difference

$$
E_2^-(|\Delta_2^+ x_{k,l}||\Delta_2^- x_{k,l}|) = |x_{k,l} - x_{k,l-1}||x_{k,l-1} - x_{k,l-2}|.
$$

S_{32} contains the discrete third derivatives $\Delta_i^+(\Delta_r^-\Delta_i^-)e_{k,l}^{s+1}$ and $\Delta_i^+(\Delta_r^-\Delta_r^+)e_{k,l}^{s+1}$ of e_h. The order of the difference is reduced again by the second summation by parts which yields

$$\begin{aligned}
&\sum_{i\neq r}\sum_{k,l=1}^{N} g_{k,l}^{ii,s} g_{k,l}^{mn,s}(\Delta_i^+(\Delta_r^-\Delta_i^-)e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1})\\
&+\sum_{i\neq r}\sum_{k,l=1}^{N} g_{k,l}^{ir,s} g_{k,l}^{mn,s}(\Delta_i^+\Delta_{rr} e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1})\\
\overset{(3.8)}{=}&-\sum_{i\neq r}\sum_{k,l=1}^{N}(\Delta_r^-\Delta_i^- e_{k,l}^{s+1}\cdot\Delta_i^-\overline{\Delta}_m x_{k,l}^s)E_i^-\left(g_{k,l}^{mn,s} g_{k,l}^{ii,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1})\right)\\
&-\sum_{i\neq r}\sum_{k,l=1}^{N}(\Delta_{rr} e_{k,l}^{s+1}\cdot\Delta_i^-\overline{\Delta}_m x_{k,l}^s)E_i^-\left(g_{k,l}^{mns,} g_{k,l}^{ir,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1})\right)\\
&-\sum_{i\neq r}\sum_{k,l=1}^{N}\Delta_i^-\left(g_{k,l}^{ii,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1})\right)(\Delta_r^-\Delta_i^- e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s)\\
&-\sum_{i\neq r}\sum_{k,l=1}^{N}\Delta_i^-\left(g_{k,l}^{ir,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1})\right)(\Delta_{rr} e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s).
\end{aligned}$$

Here, among others, $E_1^-(g_{k,l}^{11,s})$ requires the evaluation of x_h in $u_{k-2,l}$ and thus uses the periodic extension again. The next step is to use the product rule for the remaining difference Δ_i^-. We calculate a general case:

$$\begin{aligned}
\Delta_i^-(g_{k,l}^{ij,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1})) = {}&\Delta_i^-(g_{k,l}^{ij,s} g_{k,l}^{mn,s})E_i^-((\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^- e_{k,l}^{s+1}))\\
&+g_{k,l}^{ij,s} g_{k,l}^{mn,s}(\Delta_i^-\overline{\Delta}_n x_{k,l}^s\cdot E_i^-(\Delta_r^- e_{k,l}^{s+1}))\\
&+g_{k,l}^{ij,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_i^-\Delta_r^- e_{k,l}^{s+1}).
\end{aligned}$$

Applying this for $j=i$ and $j=r$ we can therefore write

$$\begin{aligned}
S_{32} = &-\sum_{i\neq r}\sum_{k,l=1}^{N} g_{k,l}^{ii,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^-\Delta_i^- e_{k,l}^{s+1})(\Delta_r^-\Delta_i^- e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s)\\
&-\sum_{i\neq r}\sum_{k,l=1}^{N} g_{k,l}^{ir,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^-\Delta_i^- e_{k,l}^{s+1})(\Delta_{rr} e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s)+P_1+P_2\\
=&-\sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{rr,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_1^-\Delta_2^- e_{k,l}^{s+1})(\Delta_1^-\Delta_2^- e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s)\\
&-\sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{12,s} g_{k,l}^{mn,s}(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_1^-\Delta_2^- e_{k,l}^{s+1})(\Delta_{rr} e_{k,l}^{s+1}\cdot\overline{\Delta}_m x_{k,l}^s)+P_1+P_2,
\end{aligned}$$

where

$$
\begin{aligned}
P_2 = &- \sum_{i \neq r} \sum_{k,l=1}^{N} (\Delta_1^- \Delta_2^- e_{k,l}^{s+1} \cdot \Delta_i^- \overline{\Delta}_m x_{k,l}^s) E_i^- \left(g_{k,l}^{mn,s} g_{k,l}^{ii,s} (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \right) \\
&- \sum_{i \neq r} \sum_{k,l=1}^{N} (\Delta_{rr} e_{k,l}^{s+1} \cdot \Delta_i^- \overline{\Delta}_m x_{k,l}^s) E_i^- \left(g_{k,l}^{mns,} g_{k,l}^{ir,s} (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \right) \\
&- \sum_{i \neq r} \sum_{k,l=1}^{N} \Delta_i^- (g_{k,l}^{ii,s} g_{k,l}^{mn,s}) E_i^- ((\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}))(\Delta_1^- \Delta_2^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \\
&- \sum_{i \neq r} \sum_{k,l=1}^{N} g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\Delta_i^- \overline{\Delta}_n x_{k,l}^s \cdot E_i^- (\Delta_r^- e_{k,l}^{s+1}))(\Delta_1^- \Delta_2^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \\
&- \sum_{i \neq r} \sum_{k,l=1}^{N} \Delta_i^- (g_{k,l}^{ir,s} g_{k,l}^{mn,s}) E_i^- ((\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}))(\Delta_{rr} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \\
&- \sum_{i \neq r} \sum_{k,l=1}^{N} g_{k,l}^{12,s} g_{k,l}^{mn,s} (\Delta_i^- \overline{\Delta}_n x_{k,l}^s \cdot E_i^- (\Delta_r^- e_{k,l}^{s+1}))(\Delta_{rr} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s).
\end{aligned} \tag{4.37}
$$

Recall that we split S_3 into S_{31} and S_{32}. Reassembling the parts we get

$$
\begin{aligned}
(\frac{1}{\alpha} - 1)^{-1} S_3 = &- \sum_{r=1}^{2} \sum_{k,l=1}^{N} \Big[g_{k,l}^{rr,s} g_{k,l}^{mn,s} (\Delta_{rr} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1}) \\
&\qquad + g_{k,l}^{12,s} g_{k,l}^{mn,s} (\Delta_1^- \Delta_2^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1}) \Big] \\
&- \sum_{r=1}^{2} \sum_{k,l=1}^{N} \Big[g_{k,l}^{rr,s} g_{k,l}^{mn,s} (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_1^- \Delta_2^- e_{k,l}^{s+1})(\Delta_1^- \Delta_2^- e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \\
&\qquad + g_{k,l}^{12,s} g_{k,l}^{mn,s} (\overline{\Delta}_n x_{k,l}^s \cdot \Delta_1^- \Delta_2^- e_{k,l}^{s+1})(\Delta_{rr} e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \Big] + P_1 + P_2.
\end{aligned}
$$

In what follows next we concentrate on the summands of S_3 that are not listed in P_1 or P_2. We utilize notation (4.1) to replace Δ_{rr} by Δ_{rr}^* and $\Delta_1^- \Delta_2^-$ by Δ_{12}^*. Summarizing terms in a suitable way we arrive at

$$
\begin{aligned}
(\frac{1}{\alpha} - 1)^{-1} S_3 = &- \sum_{k,l=1}^{N} g_{k,l}^{ij,s} g_{k,l}^{mn,s} (\Delta_{i1}^* e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{j1}^* e_{k,l}^{s+1}) \\
&- \sum_{k,l=1}^{N} g_{k,l}^{ij,s} g_{k,l}^{mn,s} (\Delta_{i2}^* e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{j2}^* e_{k,l}^{s+1}) + (P_1 + P_2) \\
= &- \sum_{k,l=1}^{N} \sum_{r=1}^{2} g_{k,l}^{ij,s} g_{k,l}^{mn,s} (\Delta_{ir}^* e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_{jr}^* e_{k,l}^{s+1}) + (P_1 + P_2) \\
=: &- P_3 + P_1 + P_2.
\end{aligned}
$$

Hence, we aim show that

$$-(\frac{1}{\alpha} - 1)P_3 \leq 0.$$

Remember that we assumed $0 < \alpha \leq 1$, which reformulates to $\frac{1}{\alpha} - 1 \geq 0$.
Since (g^{ij}) is symmetric and positive definite, see the proof of Lemma 4.5, the matrix can be expressed as the square of another, likewise symmetric and positive definite, matrix B with entries b^{ij}:

$$g^{ij} = b^{iv}\delta_{vt}b^{tj}.$$

For reasons of clarity, we introduce the notations $a_{im} := (\Delta^*_{ir}e^{s+1}_{k,l} \cdot \overline{\Delta}_m x^s_{k,l})$ as well as $b^{iv}a_{im} = b^{vi}a_{im} =: c^v_m$, with $r \in \{1,2\}$ arbitrary but fixed, and do not make use of the sum convention for the rest of this part of the proof. For all $k, l \in \{1, \dots, N\}$ we infer

$$\begin{aligned}
&- \sum_{i,j,m,n=1}^{2} g^{ij,s}_{k,l} g^{mn,s}_{k,l} (\Delta^*_{ir}e^{s+1}_{k,l} \cdot \overline{\Delta}_m x^s_{k,l})(\overline{\Delta}_n x^s_{k,l} \cdot \Delta^*_{jr}e^{s+1}_{k,l}) \\
&= - \sum_{i,j,m,n=1}^{2} g^{ij}_{k,l} g^{mn}_{k,l} a_{im}a_{jn} \\
&= - \sum_{i,j,m,n,v,t=1}^{2} b^{iv}\delta_{vt}b^{tj} g^{mn}_{k,l} a_{im}a_{jn} \\
&= - \sum_{m,n,v,t=1}^{2} c^v_m \delta_{vt} c^t_n g^{mn}_{k,l} \\
&= - \sum_{m,n,v=1}^{2} c^v_m c^v_n g^{mn}_{k,l}.
\end{aligned}$$

Together with (4.34) we derive

$$\begin{aligned}
- \sum_{m,n,v=1}^{2} c^v_m c^v_n g^{mn}_{k,l} &\leq - \lambda_{\min} \sum_{m,v=1}^{2} (c^v_m)^2 \\
&= - \lambda_{\min} \sum_{m,n,v,t=1}^{2} \delta_{vt}\delta^{mn} c^v_m c^t_n \\
&= - \lambda_{\min} \sum_{i,j,m,n,v,t=1}^{2} \delta_{vt}\delta^{mn} b^{iv} a_{im} b^{tj} a_{jn} \\
&= - \lambda_{\min} \sum_{i,j,m=1}^{2} g^{ij}_{k,l} a_{im}a_{jm} \\
&\leq - \lambda^2_{\min} \sum_{i,m=1}^{2} a^2_{im} \\
&\leq 0.
\end{aligned}$$

Thus, $-P_3$ can be bounded from above by zero as claimed and (4.30) holds. Before starting to estimate $P_1 + P_2$, we prove Lemma 4.9. The proceeding is then the same for the terms in $P_1 + P_2$, which seem to be more complex at first sight.

This analysis is again based on the inequalities of Cauchy-Schwarz, which is denoted by (CS), and Young, which is marked by (Y). Apart from that, specific arguments are used to control some of the differences. For the examination of S_2 we make use of (4.18) in Lemma 4.5. This gives an upper bound for the difference between the discrete and approximated inverse metric coefficients. For $i, j \in \{1, 2\}$ we have

$$|g_{k,l}^{ij,s} - \tilde{g}_{k,l}^{ij,s}| \leq ch^{-3} \left(|\Delta_2^+ e_{k,l}^s| + |\Delta_2^- e_{k,l}^s| + |\Delta_1^+ e_{k,l}^s| + |\Delta_1^- e_{k,l}^s| \right).$$

This yields

$$\begin{aligned} |S_2| &= \left| \sum_{k,l=1}^{N} \sum_{r=1}^{2} (g_{k,l}^{ij,s} - \tilde{g}_{k,l}^{ij,s}) \Delta_{ij} \tilde{x}_{k,l}^{s+1} \cdot \Delta_{rr} e_{k,l}^{s+1} \right| \\ &\leq \sum_{k,l=1}^{N} ch^{-3} \left(|\Delta_2^+ e_{k,l}^s| + |\Delta_2^- e_{k,l}^s| + |\Delta_1^+ e_{k,l}^s| + |\Delta_1^- e_{k,l}^s| \right) \sum_{i,j,r=1}^{2} |\Delta_{ij} \tilde{x}_{k,l}^{s+1}| |\Delta_{rr} e_{k,l}^{s+1}| \end{aligned}$$

and the further estimation is again demonstrated for one summand exemplarily. The consistency proof in the last chapter entails assertions on the order of approximation with respect to the spatial grid size h for the discrete second derivatives of x. More precisely, (3.23), (3.24) and (3.25) can be summarized to

$$|\Delta_{ij} \tilde{x}_{k,l}^{s+1}| \leq h^2 |x_{u_i u_j}(u_{k,l}, t^s)| + c(\tau^2 + h^2 \tau + h^4) \leq ch^2 \tag{4.38}$$

for $i, j \in \{1, 2\}$. For a comparison with a second derivative $x_{u_i u_j}$ we advise the reader again to take into account that the difference operators Δ_{ij} have to be divided by h^2 for an approximation, which is thus bounded by c. Hence for $r \in \{1, 2\}$

$$\begin{aligned} &\sum_{k,l=1}^{N} ch^{-3} |\Delta_2^+ e_{k,l}^s| |\Delta_{ij} \tilde{x}_{k,l}^{s+1}| |\Delta_{rr} e_{k,l}^{s+1}| \\ &\overset{\text{(CS)}}{\leq} \left(\sum_{k,l=1}^{N} |\Delta_2^+ e_{k,l}^s|^2 \right)^{1/2} \left(\sum_{k,l=1}^{N} ch^{-6} |\Delta_{ij} \tilde{x}_{k,l}^{s+1}|^2 |\Delta_{rr} e_{k,l}^{s+1}|^2 \right)^{1/2} \\ &\overset{\text{(Y)}}{\leq} \frac{1}{4\varepsilon} \sum_{k,l=1}^{N} |\Delta_2^+ e_{k,l}^s|^2 + ch^{-6} \varepsilon \sum_{k,l=1}^{N} |\Delta_{ij} \tilde{x}_{k,l}^{s+1}|^2 |\Delta_{rr} e_{k,l}^{s+1}|^2 \\ &\leq \frac{1}{4\varepsilon} \sum_{k,l=1}^{N} |\Delta_2^+ e_{k,l}^s|^2 + ch^{-2} \varepsilon \sum_{k,l=1}^{N} |\Delta_{rr} e_{k,l}^{s+1}|^2. \end{aligned}$$

Obtaining the backward difference of e_h is achieved through Lemma 3.5 due to the summation over all mesh points and so

$$\sum_{k,l=1}^{N} ch^{-3}|\Delta_2^+ e_{k,l}^s||\Delta_{ij}\tilde{x}_{k,l}^{s+1}||\Delta_{rr}e_{k,l}^{s+1}| \leq \frac{1}{4\varepsilon}\sum_{k,l=1}^{N}|\Delta_2^- e_{k,l}^s|^2 + ch^{-2}\varepsilon\sum_{k,l=1}^{N}|\Delta_{rr}^* e_{k,l}^{s+1}|^2.$$

Using the same strategy for the rest of S_2 yields

$$|S_2| \leq \frac{c}{\varepsilon}\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_r^- e_{k,l}^s|^2 + ch^{-2}\varepsilon\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_{rr}^* e_{k,l}^{s+1}|^2.$$

In each S_4 and S_5, the operator denoted by $\overline{\Delta}_r = \frac{1}{2}(\Delta_r^+ + \Delta_r^-)$ appears twice. Altough it consists of two different operators, $\overline{\Delta}_r x_{k,l}^s$ is not split for it can be estimated by (4.12). When applied to $e_{k,l}^s$, we will decompose $\overline{\Delta}_r$ and study only one resulting term. As demonstrated before, this can easily be transferred to the other operator by means of the given periodicity. So for

$$S_4 = -(\frac{1}{\alpha}-1)\sum_{r=1}^{2}\sum_{k,l=1}^{N} g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\frac{1}{2}(\Delta_m^+ + \Delta_m^-)e_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_{rr}e_{k,l}^{s+1})$$

we will show how to analyse the part that contains Δ_m^+. Again we use the inequalities of Cauchy-Schwarz und Young to obtain L^2-norms. Furthermore, the prerequisites from (4.8), (4.12) and (4.17) as well as (4.38) can be applied. Using the periodicity relation between Δ_m^+ and Δ_m^- from Lemma 3.5, for all $i,j,m,n,r \in \{1,2\}$ the following holds

$$\begin{aligned}
&\left|-\frac{1}{2}(\frac{1}{\alpha}-1)\sum_{k,l=1}^{N} g_{k,l}^{ij,s}g_{k,l}^{mn,s}(\Delta_{ij}\tilde{x}_{k,l}^{s+1}\cdot\Delta_m^+ e_{k,l}^s)(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_{rr}e_{k,l}^{s+1})\right| \\
&\overset{(CS)}{\leq}\frac{1}{2}(\frac{1}{\alpha}-1)\left(\sum_{k,l=1}^{N}|\Delta_m^+ e_{k,l}^s|^2\right)^{1/2}\left(\sum_{k,l=1}^{N}|g_{k,l}^{ij,s}|^2|g_{k,l}^{mn,s}|^2|\Delta_{ij}\tilde{x}_{k,l}^{s+1}|^2|\overline{\Delta}_n x_{k,l}^s|^2|\Delta_{rr}e_{k,l}^{s+1}|^2\right)^{1/2} \\
&\overset{(Y)}{\leq}\frac{c}{\varepsilon}(\frac{1}{\alpha}-1)\sum_{k,l=1}^{N}|\Delta_m^+ e_{k,l}^s|^2 + c\varepsilon(\frac{1}{\alpha}-1)\sum_{k,l=1}^{N}|g_{k,l}^{ij,s}|^2|g_{k,l}^{mn,s}|^2|\Delta_{ij}\tilde{x}_{k,l}^{s+1}|^2|\overline{\Delta}_n x_{k,l}^s|^2|\Delta_{rr}e_{k,l}^{s+1}|^2 \\
&\leq \frac{c}{\varepsilon}(\frac{1}{\alpha}-1)\sum_{k,l=1}^{N}|\Delta_m^+ e_{k,l}^s|^2 + c\varepsilon(\frac{1}{\alpha}-1)h^{-4-4+4+2}\sum_{k,l=1}^{N}|\Delta_{rr}e_{k,l}^{s+1}|^2 \\
&= \frac{c}{\varepsilon}(\frac{1}{\alpha}-1)\sum_{k,l=1}^{N}|\Delta_m^- e_{k,l}^s|^2 + c\varepsilon(\frac{1}{\alpha}-1)h^{-2}\sum_{k,l=1}^{N}|\Delta_{rr}^* e_{k,l}^{s+1}|^2.
\end{aligned}$$

An analog proceeding including the decomposing of $\overline{\Delta}_n e_{k,l}^s$, application of the mentioned inequalities and constraints as well as the usage of (4.4) instead of (4.8) in

$$S_5 = -(\frac{1}{\alpha} - 1) \sum_{r=1}^{2} \sum_{k,l=1}^{N} g_{k,l}^{ij,s} g_{k,l}^{mn,s} (\Delta_{ij} \tilde{x}_{k,l}^{s+1} \cdot \overline{\Delta}_m \tilde{x}_{k,l}^{s}) (\frac{1}{2}(\Delta_n^+ + \Delta_n^-) e_{k,l}^s \cdot \Delta_{rr} e_{k,l}^{s+1})$$

provides

$$|S_4 + S_5| \leq \frac{c}{\varepsilon} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^s|^2 + c\varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_{rr}^* e_{k,l}^{s+1}|^2.$$

Here we had to bound $|\frac{1}{\alpha} - 1|$, so that the constants on the right depends on α^{-1}. This is also the case in the estimation of the next summand. Apart from the established techniques, S_6 requires to control a further difference between discrete and approximated inverse metric coefficients. From (4.17) and (4.18) we infer that for $i, j, m, n \in \{1, 2\}$

$$\begin{aligned} |\tilde{g}_{k,l}^{ij,s} \tilde{g}_{k,l}^{mn,s} - g_{k,l}^{ij,s} g_{k,l}^{mn,s}| &\leq |\tilde{g}_{k,l}^{ij,s}||\tilde{g}_{k,l}^{mn,s} - g_{k,l}^{mn,s}| + |\tilde{g}_{k,l}^{ij,s} - g_{k,l}^{ij,s}||g_{k,l}^{mn,s}| \\ &\leq ch^{-2-3} \sum_{r=1}^{2} \left(|\Delta_r^+ e_{k,l}^s| + |\Delta_r^- e_{k,l}^s| \right). \end{aligned} \tag{4.39}$$

That means

$$\begin{aligned} |S_6| &= \left| -(\frac{1}{\alpha} - 1) \sum_{r=1}^{2} \sum_{k,l=1}^{N} (\tilde{g}_{k,l}^{ij,s} \tilde{g}_{k,l}^{mn,s} - g_{k,l}^{ij,s} g_{k,l}^{mn,s}) (\Delta_{ij} \tilde{x}_{k,l}^{s+1} \cdot \overline{\Delta}_m \tilde{x}_{k,l}^{s}) (\overline{\Delta}_n \tilde{x}_{k,l}^{s} \cdot \Delta_{rr} e_{k,l}^{s+1}) \right| \\ &\leq ch^{-5} \sum_{i,j,m,n,r=1}^{2} \sum_{k,l=1}^{N} \left(|\Delta_r^+ e_{k,l}^s| + |\Delta_r^- e_{k,l}^s| \right) |\Delta_{ij} \tilde{x}_{k,l}^{s+1}||\overline{\Delta}_m \tilde{x}_{k,l}^{s}||\overline{\Delta}_n \tilde{x}_{k,l}^{s}||\Delta_{rr} e_{k,l}^{s+1}| \end{aligned}$$

and here, too, we illustrate the further estimate with the help of an example. Using (4.12) together with (4.38) and Lemma 3.5, for any $i, j, m, n, r \in \{1, 2\}$ we arrive at

$$\begin{aligned} & ch^{-5} \sum_{k,l=1}^{N} |\Delta_r^+ e_{k,l}^s||\Delta_{ij} \tilde{x}_{k,l}^{s+1}||\overline{\Delta}_m \tilde{x}_{k,l}^{s}||\overline{\Delta}_n \tilde{x}_{k,l}^{s}||\Delta_{rr} e_{k,l}^{s+1}| \\ &\overset{(CS)}{\leq} c \left(\sum_{k,l=1}^{N} |\Delta_r^+ e_{k,l}^s|^2 \right)^{1/2} \left(ch^{-10} \sum_{k,l=1}^{N} |\Delta_{ij} \tilde{x}_{k,l}^{s+1}|^2 |\overline{\Delta}_m \tilde{x}_{k,l}^{s}|^2 |\overline{\Delta}_n \tilde{x}_{k,l}^{s}|^2 |\Delta_{rr} e_{k,l}^{s+1}|^2 \right)^{1/2} \\ &\overset{(Y)}{\leq} \frac{c}{\varepsilon} \sum_{k,l=1}^{N} |\Delta_r^+ e_{k,l}^s|^2 + c\varepsilon h^{-10} \sum_{k,l=1}^{N} |\Delta_{ij} \tilde{x}_{k,l}^{s+1}|^2 |\overline{\Delta}_m \tilde{x}_{k,l}^{s}|^2 |\overline{\Delta}_n \tilde{x}_{k,l}^{s}|^2 |\Delta_{rr} e_{k,l}^{s+1}|^2 \\ &\leq \frac{c}{\varepsilon} \sum_{k,l=1}^{N} |\Delta_r^+ e_{k,l}^s|^2 + c\varepsilon h^{-10+4+2+2} \sum_{k,l=1}^{N} |\Delta_{rr} e_{k,l}^{s+1}|^2 \\ &= \frac{c}{\varepsilon} \sum_{k,l=1}^{N} |\Delta_r^- e_{k,l}^s|^2 + c\varepsilon h^{-2} \sum_{k,l=1}^{N} |\Delta_{rr}^* e_{k,l}^{s+1}|^2. \end{aligned}$$

Therefore, S_6 can be controlled by our discrete first and second derivatives of e_h just like in the former estimates:

$$|S_6| \leq \frac{c}{\varepsilon} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^s|^2 + c\varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_{rr}^* e_{k,l}^{s+1}|^2.$$

From the bound of the consistency error we derived in Theorem 3.12 in the previous chapter, i.e. from $|\tilde{R}_{k,l}^{\alpha,s}| \leq c_{\tilde{R}}(h^2 + \tau)$, it follows

$$\begin{aligned}
|S_7| &= \left| \sum_{r=1}^{2} \sum_{k,l=1}^{N} \tilde{R}_{k,l}^{\alpha,s} \cdot \Delta_{rr} e_{k,l}^{s+1} \right| \\
&\overset{\text{(CS)}}{\leq} \left(\sum_{r=1}^{2} \sum_{k,l=1}^{N} h^2 |\tilde{R}_{k,l}^{\alpha,s}|^2 \right)^{1/2} \left(h^{-2} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_{rr} e_{k,l}^{s+1}|^2 \right)^{1/2} \\
&\overset{\text{(Y)}}{\leq} \frac{2}{4\varepsilon} \sum_{k,l=1}^{N} h^2 |\tilde{R}_{k,l}^{\alpha,s}|^2 + \varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_{rr} e_{k,l}^{s+1}|^2 \\
&\leq \frac{1}{2\varepsilon} N^2 h^2 (c_{\tilde{R}})^2 (h^2 + \tau)^2 + \varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_{rr} e_{k,l}^{s+1}|^2 \\
&= \frac{c}{\varepsilon} (h^2 + \tau)^2 + \varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_{rr}^* e_{k,l}^{s+1}|^2,
\end{aligned}$$

where we inserted $N = \frac{2\pi}{h}$. Note that $c_{\tilde{R}}$ and thus c in $\frac{c}{\varepsilon}(h^2 + \tau)^2$ depends on α^{-1}.

Alltogether,

$$|S_2 + S_4 + S_5 + S_6 + S_7| \leq \frac{c}{\varepsilon}(h^2 + \tau)^2 + c\varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_{rr}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^s|^2$$

as claimed in lemma 4.9.

For the proof of Lemma 4.8, it remains to look at $(\frac{1}{\alpha} - 1)P_1$ and $(\frac{1}{\alpha} - 1)P_2$, both of which originated during the summation by parts of S_3 and are given in (4.36) and (4.37), respectively. We will estimate P_1 and P_2 and keep in mind that the factor $(\frac{1}{\alpha} - 1)$ again has to be bounded by a suitable constant. After re-sorting of the terms we have that for all $m, n \in \{1, 2\}$

$$\begin{aligned}
P_1 + P_2 =& \sum_{i\neq r}\sum_{k,l=1}^{N} E_r^- \left(g_{k,l}^{ii,s} g_{k,l}^{mn,s}(\Delta_{ii}^* e_{k,l}^{s+1}\cdot \overline{\Delta}_m x_{k,l}^s)\right)(\Delta_r^-\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+\sum_{i\neq r}\sum_{k,l=1}^{N} E_r^- \left(g_{k,l}^{ir,s} g_{k,l}^{mn,s}(\Delta_i^+\Delta_r^+ e_{k,l}^{s+1}\cdot \overline{\Delta}_m x_{k,l}^s)\right)(\Delta_r^-\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+\sum_{i\neq r}\sum_{k,l=1}^{N} E_r^- (g_{k,l}^{ii,s} g_{k,l}^{mn,s}(\Delta_{ii}^* e_{k,l}^{s+1}\cdot \Delta_r^-\overline{\Delta}_m x_{k,l}^s))(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+\sum_{i\neq r}\sum_{k,l=1}^{N} E_r^- (g_{k,l}^{ir,s} g_{k,l}^{mn,s}(\Delta_i^+\Delta_r^+ e_{k,l}^{s+1}\cdot \Delta_r^-\overline{\Delta}_m x_{k,l}^s))(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&-\sum_{i\neq r}\sum_{k,l=1}^{N} (\Delta_1^-\Delta_2^- e_{k,l}^{s+1}\cdot \Delta_i^-\overline{\Delta}_m x_{k,l}^s) E_i^- \left(g_{k,l}^{mn,s} g_{k,l}^{ii,s}(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1})\right) \\
&-\sum_{i\neq r}\sum_{k,l=1}^{N} (\Delta_{rr} e_{k,l}^{s+1}\cdot \Delta_i^-\overline{\Delta}_m x_{k,l}^s) E_i^- \left(g_{k,l}^{mns,} g_{k,l}^{ir,s}(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1})\right) \\
&-\sum_{i\neq r}\sum_{k,l=1}^{N} g_{k,l}^{ii,s} g_{k,l}^{mn,s}\left(\Delta_i^-\overline{\Delta}_n x_{k,l}^s \cdot E_i^-(\Delta_r^- e_{k,l}^{s+1})\right)(\Delta_{12}^* e_{k,l}^{s+1}\cdot \overline{\Delta}_m x_{k,l}^s) \\
&-\sum_{i\neq r}\sum_{k,l=1}^{N} g_{k,l}^{12,s} g_{k,l}^{mn,s}\left(\Delta_i^-\overline{\Delta}_n x_{k,l}^s \cdot E_i^-(\Delta_r^- e_{k,l}^{s+1})\right)(\Delta_{rr}^* e_{k,l}^{s+1}\cdot \overline{\Delta}_m x_{k,l}^s) \\
&+\sum_{i\neq r}\sum_{k,l=1}^{N} \Delta_r^-(g_{k,l}^{mn,s} g_{k,l}^{ii,s})\left(E_r^-(\Delta_{ii}^* e_{k,l}^{s+1})\cdot \overline{\Delta}_m x_{k,l}^s\right)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&+\sum_{i\neq r}\sum_{k,l=1}^{N} \Delta_r^-(g_{k,l}^{mn,s} g_{k,l}^{ir,s})\left(E_r^-(\Delta_i^+\Delta_r^+ e_{k,l}^{s+1})\cdot \overline{\Delta}_m x_{k,l}^s\right)(\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \\
&-\sum_{i\neq r}\sum_{k,l=1}^{N} \Delta_i^-(g_{k,l}^{ii,s} g_{k,l}^{mn,s}) E_i^-\left((\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1})\right)(\Delta_{12}^* e_{k,l}^{s+1}\cdot \overline{\Delta}_m x_{k,l}^s) \\
&-\sum_{i\neq r}\sum_{k,l=1}^{N} \Delta_i^-(g_{k,l}^{ir,s} g_{k,l}^{mn,s}) E_i^-\left((\overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1})\right)(\Delta_{rr}^* e_{k,l}^{s+1}\cdot \overline{\Delta}_m x_{k,l}^s) \\
=:& \tilde{P}_1 + \cdots + \tilde{P}_{12}.
\end{aligned}$$

First of all, it is helpful to understand the discrete second derivatives $\Delta_r^-\overline{\Delta}_n x_{k,l}^s$, whose precise form depends on r and n as well as on the part of $\overline{\Delta}_n = \frac{1}{2}(\Delta_n^+ + \Delta_n^-)x_{k,l}^s$ being considered. In order to restore the variants of discrete derivatives that were introduced at the beginning, similarly to (4.24) in the proof of Lemma 4.5, the shift operators $E_r^\pm$ defined in (4.21) are exerted again. With these we have

$$\Delta_r^- \overline{\Delta}_n x_{k,l}^s = \begin{cases} \frac{1}{2}(\Delta_r^-\Delta_r^+ + \Delta_r^-\Delta_r^-)x_{k,l}^s = \frac{1}{2}(\Delta_{rr}^* + E_r^-(\Delta_{rr}^*))x_{k,l}^s & \text{if } n = r, \\ \frac{1}{2}(\Delta_r^-\Delta_n^+ + \Delta_r^-\Delta_n^-)x_{k,l}^s = \frac{1}{2}(E_n^+(\Delta_{12}^*) + \Delta_{12}^*)x_{k,l}^s & \text{if } n \neq r. \end{cases}$$

In what follows we will study those summands of $P_1 + P_2$ containing the second derivatives that we have just characterized, namely $\tilde{P}_1$ to $\tilde{P}_8$. Whilst proceeding to the maximum as in

$$|E_r^-(\Delta_{rr}^*)x_{k,l}^s| \leq \max_{k,l} |E_r^-(\Delta_{rr}^* x_{k,l}^s)| = \max_{k,l} |\Delta_{rr}^* x_{k,l}^s|,$$

the dependence on the spatial grid point can be eliminated. For the bounds on the differences of first order and on the metric coefficients such a dependence does not exist. Thus, (4.17) holds despite a shifting along grid lines. Making use of this together with (4.12) after the usual application of the Cauchy-Schwarz and Young inequalities to $\tilde{P}_1$ yields

$$\begin{aligned} |\tilde{P}_1| &= \left| \sum_{i \neq r} \sum_{k,l=1}^{N} E_r^- \left(g_{k,l}^{ii,s} g_{k,l}^{mn,s} (\Delta_{ii}^* e_{k,l}^{s+1} \cdot \overline{\Delta}_m x_{k,l}^s) \right) (\Delta_r^- \overline{\Delta}_n x_{k,l}^s \cdot \Delta_r^- e_{k,l}^{s+1}) \right| \\ &\overset{\text{(CS),(Y)}}{\leq} c\varepsilon \sum_{i \neq r} \sum_{m,n=1}^{2} \sum_{k,l=1}^{N} |E_r^-(g_{k,l}^{ii,s})|^2 |E_r^-(\Delta_{ii}^* e_{k,l}^{s+1})|^2 |\overline{\Delta}_m x_{k,l}^s|^2 \\ &\qquad + \frac{c}{\varepsilon} \sum_{m,n,r=1}^{2} \max_{k,l} |\Delta_{rn}^* x_{k,l}^s|^2 \sum_{k,l=1}^{N} |E_r^-(g_{k,l}^{mn,s})|^2 |\Delta_r^- e_{k,l}^{s+1}|^2 \\ &\leq c\varepsilon h^{-2} \sum_{r=1}^{2} \sum_{k,l=1}^{N} |\Delta_{rr}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon} h^{-4} \sum_{r,n=1}^{2} \max_{k,l} |\Delta_{rn}^* x_{k,l}^s|^2 \sum_{k,l=1}^{N} |\Delta_r^- e_{k,l}^{s+1}|^2. \end{aligned}$$

The approach is the same for $\tilde{P}_2$ to $\tilde{P}_8$. Note that additional terms occur in the constraint on the right hand-side of the resulting inequality. Apart from the examination of other second order differences of e_h, this is because of the simultaneous appearance of the factors $\Delta_i^- \overline{\Delta}_m x_{k,l}^s$ and $E_i^-(\Delta_r^- e_{k,l}^{s+1})$ or, more precisely, of the operators Δ_i^- and Δ_r^- for $i \neq r$, for instance in $\tilde{P}_5$. Out of this results a summation over all indices $i, j \in \{1, 2\}$ in the product of $\max_{k,l} |\Delta_{ij}^* x_{k,l}^s|^2$ with $|\Delta_r^- e_{k,l}^{s+1}|^2$ for both $r \in \{1, 2\}$:

$$|\tilde{P}_1 + \cdots + \tilde{P}_8| \leq c\varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon} h^{-4} \sum_{i,j=1}^{2} \max_{k,l} |\Delta_{ij}^* x_{k,l}^s|^2 \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^{s+1}|^2.$$

The remaining terms in $P_1 + P_2$ all have the factor $\Delta_r^-(g_{k,l}^{mn,s} g_{k,l}^{ij,s})$ in common. Observe that, by means of (4.17) and (4.20),

$$\begin{aligned}|\Delta_r^-(g_{k,l}^{mn,s}g_{k,l}^{ij,s})| &= |\Delta_r^-(g_{k,l}^{mn,s})g_{k,l}^{ij,s} + E_r^-(g_{k,l}^{mn,s})\Delta_r^-(g_{k,l}^{ij,s})| \\ &\le ch^{-2}(|\Delta_r^-(g_{k,l}^{mn,s})| + |\Delta_r^-(g_{k,l}^{ij,s})|) \\ &\le ch^{-2}h^{-3}\left(\max_{k,l}|\Delta_{rr}^*x_{k,l}^s| + \max_{k,l}|\Delta_{12}^*x_{k,l}^s|\right).\end{aligned}$$

We look exemplarily at $\tilde{P}_9$. With the help of the inequalities of Cauchy-Schwarz and Young and a suitable splitting of the factor $h^{-2}h^{-3}$ as well as with (4.12) we derive

$$\begin{aligned}|\tilde{P}_9| &= \left|\sum_{i\neq r}\sum_{k,l=1}^{N}\Delta_r^-(g_{k,l}^{mn,s}g_{k,l}^{ii,s})\left(E_r^-(\Delta_{ii}^*e_{k,l}^{s+1})\cdot\overline{\Delta}_m x_{k,l}^s\right)\left(\overline{\Delta}_n x_{k,l}^s\cdot\Delta_r^-e_{k,l}^{s+1}\right)\right| \\ &\overset{\text{(CS),(Y)}}{\le} c\varepsilon h^{-4}\sum_{i\neq r}\sum_{k,l=1}^{N}\sum_{m,n=1}^{2}|E_r^-(\Delta_{ii}^*e_{k,l}^{s+1})|^2|\overline{\Delta}_m x_{k,l}^s|^2 \\ &\quad + \frac{c}{\varepsilon}h^{-6}\sum_{m,n,r=1}^{2}\sum_{k,l=1}^{N}|\Delta_r^-e_{k,l}^{s+1}|^2|\overline{\Delta}_n x_{k,l}^s|^2\left(\max_{k,l}|\Delta_{rr}^*x_{k,l}^s|^2 + \max_{k,l}|\Delta_{12}^*x_{k,l}^s|^2\right) \\ &\le c\varepsilon h^{-2}\sum_{i=1}^{2}\sum_{k,l=1}^{N}|\Delta_{ii}^*e_{k,l}^{s+1}|^2 \\ &\quad + \frac{c}{\varepsilon}h^{-4}\sum_{r=1}^{2}\left(\max_{k,l}|\Delta_{rr}^*x_{k,l}^s|^2 + \max_{k,l}|\Delta_{12}^*x_{k,l}^s|^2\right)\sum_{k,l=1}^{N}|\Delta_r^-e_{k,l}^{s+1}|^2.\end{aligned}$$

Note that the shifting of $\Delta_{ii}^*e_{k,l}^{s+1}$ is reversed by means of the periodicity. This also applies to $E_r^-(\Delta_i^+\Delta_r^+e_{k,l}^{s+1})$ in $\tilde{P}_{10}$, which is treated like the second order difference of e_h in (4.35).
The given estimate is representative for $\tilde{P}_9,\dots,\tilde{P}_{12}$. Again, further combinations of i,j and r are obtained following the same principle as before, so that we have

$$|\tilde{P}_9+\cdots+\tilde{P}_{12}| \le c\varepsilon h^{-2}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}^*e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon}h^{-4}\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^*x_{k,l}^s|^2\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_r^-e_{k,l}^{s+1}|^2.$$

The proof of Lemma 4.8 is thus completed.

□

We finally combine the estimates from Lemmas 4.7, 4.8 and 4.9 in the

Proof of the induction step. Applying Lemmas 4.7 and 4.8 we infer

$$\begin{aligned}
&c_0 h^{-2} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s+1}|^2 + \frac{1}{2\tau} \sum_{r=1}^{2} \sum_{k,l=1}^{N} (|\Delta_r^- e_{k,l}^{s+1}|^2 - |\Delta_r^- e_{k,l}^{s}|^2) \\
&\overset{(4.29)}{\leq} S_{12} + S_2 + \cdots + S_7 \\
&\overset{(4.30)}{\leq} S_{12} + S_2 + S_4 + \cdots + S_7 + (\tfrac{1}{\alpha} - 1)(P_1 + P_2),
\end{aligned}$$

so that Lemma 4.9 gives

$$\begin{aligned}
&c_0 h^{-2} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s+1}|^2 + \frac{1}{2\tau} \sum_{r=1}^{2} \sum_{k,l=1}^{N} (|\Delta_r^- e_{k,l}^{s+1}|^2 - |\Delta_r^- e_{k,l}^{s}|^2) \\
&\leq \frac{c}{\varepsilon}(h^2+\tau)^2 + c\varepsilon h^{-2} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s+1}|^2 + \frac{c}{\varepsilon} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^{s}|^2 \\
&\quad + \frac{c}{\varepsilon} h^{-4} \sum_{i,j=1}^{2} \max_{k,l} |\Delta_{ij}^* x_{k,l}^{s}|^2 \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^{s+1}|^2.
\end{aligned}$$

Summation over all points of the time grid $t^{s'}$, $s' \in \{0, \ldots, s\}$, and multiplication by τ yields

$$\begin{aligned}
&\tau \sum_{s'=0}^{s} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} c_0 h^{-2} |\Delta_{ij}^* e_{k,l}^{s'+1}|^2 + \frac{1}{2} \sum_{s'=0}^{s} \sum_{k,l=1}^{N} \sum_{r=1}^{2} (|\Delta_r^- e_{k,l}^{s'+1}|^2 - |\Delta_r^- e_{k,l}^{s'}|^2) \\
&\leq \frac{c}{\varepsilon}(s+1)\tau(h^2+\tau)^2 + \frac{c}{\varepsilon}\tau \sum_{s'=0}^{s} \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^{s'}|^2 + c\varepsilon\tau h^{-2} \sum_{s'=0}^{s} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s'+1}|^2 \\
&\quad + \frac{c}{\varepsilon}\tau h^{-4} \sum_{s'=0}^{s} \sum_{i,j=1}^{2} \max_{k,l} |\Delta_{ij}^* x_{k,l}^{s'}|^2 \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^{s'+1}|^2.
\end{aligned}$$

This inequality can be rewritten as

$$\begin{aligned}
&(c_0 - c\varepsilon)\tau h^{-2} \sum_{s'=0}^{s+1} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} |\Delta_{ij}^* e_{k,l}^{s'}|^2 \\
&\quad + (\frac{1}{2} - \frac{c}{\varepsilon} h^{-4}\tau \sum_{i,j=1}^{2} \max_{k,l} |\Delta_{ij}^* x_{k,l}^{s}|^2) \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^{s+1}|^2 \\
&\leq \frac{c}{\varepsilon} T(h^2+\tau)^2 + \frac{c}{\varepsilon}\tau \sum_{s'=1}^{s} \big(1 + h^{-4} \sum_{i,j=1}^{2} \max_{k,l} |\Delta_{ij}^* x_{k,l}^{s'-1}|^2\big) \sum_{k,l=1}^{N} \sum_{r=1}^{2} |\Delta_r^- e_{k,l}^{s'}|^2,
\end{aligned}$$

since $|e_{k,l}^0|^2 = 0$. Choosing ε such that $0 < c_1 \leq c_0 - c\varepsilon$ provides the following lower bound:

$$\sum_{s'=0}^{s+1}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2} c_1\tau h^{-2}|\Delta_{ij}^{*}e_{k,l}^{s'}|^2 \leq \sum_{s'=0}^{s+1}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}(c_0-c\varepsilon)\tau h^{-2}|\Delta_{ij}^{*}e_{k,l}^{s'}|^2.$$

In order to find constraints for the factors containing second order derivatives of x_h on either sides of the estimate, we take up the induction hypothesis in form of Corollary 4.4. The bound in (4.9) implies

$$\frac{1}{2}-ch^{-4}\tau\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^{*}x_{k,l}^{s}|^2 \geq \frac{1}{2}-cc_2h \geq c_3,$$

if $h \leq (\frac{1}{2}-c_3)(cc_2)^{-1}$, and for the overall estimate we infer

$$\begin{aligned} &c_1\tau h^{-2}\sum_{s'=0}^{s+1}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}^{*}e_{k,l}^{s'}|^2 + c_3\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_{r}^{-}e_{k,l}^{s+1}|^2 \\ &\leq c_4(h^2+\tau)^2 + c_5\tau\sum_{s'=1}^{s}\big(1+h^{-4}\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^{*}x_{k,l}^{s'-1}|^2\big)\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_{r}^{-}e_{k,l}^{s'}|^2. \end{aligned} \tag{4.40}$$

Note that $c_4 = c_4(x,T,\alpha^{-1})$, $c_5 = c_5(x,\alpha^{-1})$ and all of the other constants only depend on x. This is important to note because the lemma of Gronwall gives

$$\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_{r}^{-}e_{k,l}^{s+1}|^2 \leq \frac{c_4}{c_3}(h^2+\tau)^2\exp\left(\frac{c_5}{c_3}\tau\sum_{s'=1}^{s}\big(1+h^{-4}\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^{*}x_{k,l}^{s'-1}|^2\big)\right)$$

and the right hand-side consequently depends exponentially on α^{-1}.
To proceed we make use of the induction hypothesis in form of Corollary 4.4 again. We apply (4.10) to estimate

$$\tau\sum_{s'=1}^{s}(1+h^{-4}\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^{*}x_{k,l}^{s'-1}|^2) \leq s\tau + 8\hat{c}^2T + 1 \leq (1+8\hat{c}^2)T+1.$$

Therefore

$$\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_{r}^{-}e_{k,l}^{s+1}|^2 \leq \frac{c_4}{c_3}(h^2+\tau)^2\exp\left(\frac{c_5}{c_3}(1+8\hat{c}^2)T\right)\exp(1) = W(h^2+\tau)^2$$

as claimed in (4.28). So the first inequality in (4.3) is true for each $s \in \{0,\ldots,M\}$ under the assumption that for the preceding point of time the second part of (4.3) holds. Since the latter is true for all $s' \in \{0,\ldots,s\}$ according to the induction hypothesis, the estimate for the discrete first derivatives holds for all $s' \in \{0,\ldots,s+1\}$. Inserting this into (4.40) we infer

$$
\begin{aligned}
&\tau h^{-2}\sum_{s'=0}^{s+1}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}^{*}e_{k,l}^{s'}|^{2}\\
\leq&\frac{c_4}{c_1}(h^2+\tau)^2+\frac{c_5}{c_1}W(h^2+\tau)^2\tau\sum_{s'=1}^{s}\big(1+h^{-4}\sum_{i,j=1}^{2}\max_{k,l}|\Delta_{ij}^{*}x_{k,l}^{s'-1}|^2\big)\\
\leq&\frac{c_4}{c_1}(h^2+\tau)^2+\frac{c_5}{c_1}((1+8\hat{c}^2)T+1)W(h^2+\tau)^2\\
\leq&(h^2+\tau)^{3/2},
\end{aligned}
$$

if $h \leq (2c^{**})^{-1}$, see the proof of (4.10), and $(h^2+\tau)^{1/2} \leq (\frac{c_4}{c_1}+\frac{c_5}{c_1}((1+8\hat{c}^2)T+1)W)^{-1}$. Consequently, also the second part of the induction step is proved. □

Thus, the induction is completed. Note that this also means that auxiliary results derived from the induction hypothesis for the concrete point t^s are now valid for all $s \in \{0,\dots,M\}$. In particular, this is true for (4.8), (4.12) and (4.17)-(4.20). We can proceed with improving the estimate for the second order differences.

4.10 Theorem. *Let $\alpha \in (0,1]$ and let $x \in C^4([0,2\pi]^2 \times [0,T];\mathbb{R}^3)$ be the solution of the continuous problem (1.4) with $x(\cdot,t) \in C^0_{per}([0,2\pi]^2;\mathbb{R}^3)$ for all $t \in [0,T]$ and $\partial^\gamma x \in C^0([0,T];C^0_{per}([0,2\pi]^2;\mathbb{R}^3))$ for $|\gamma| \leq 3$. Let also $2\bar{c} \leq g$ for a constant $\bar{c} > 0$. For $k,l \in \{1,\dots,N\}$ and $s \in \{0,\dots,M\}$ let $\tilde{x}^s_{k,l} = x(u_{k,l},t^s)$ denote the restriction of x to the mesh $\{(u_{k,l},t^s)\}_{k,l\in\{0,\dots,N\},s\in\{0,\dots,M\}}$ with $(u_{k,l}) = (kh,lh)$, $h = \frac{2\pi}{N}$, and $t^s = s\tau$, $\tau = \frac{T}{M}$. Let x_h be the solution of the discrete problem (3.9) with $x^s_{k,l} = x_h(u_{k,l},t^s)$ and let $e^s_{k,l} = e_h(u_{k,l},t^s) = \tilde{x}^s_{k,l} - x^s_{k,l}$. Then there exist positive constants c, c' and h^*, such that for all $0 < h \leq h^*$ and $\tau \leq c'h^2$ the estimates*

$$
\begin{aligned}
\max_{s\in\{0,\dots,M\}}\left(h^2\sum_{k,l=1}^{N}\sum_{r=1}^{2}\frac{|\Delta_r^- e_{k,l}^s|^2}{h^2}\right)^{1/2} &\leq c(h^2+\tau),\\
\left(\tau\sum_{s=0}^{M}h^2\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}\frac{|\Delta_{ij}^{*}e_{k,l}^s|^2}{h^4}\right)^{1/2} &\leq c(h^2+\tau)
\end{aligned}
\tag{4.41}
$$

hold and the constants only depend on x, T and α^{-1}.

Proof. It remains to show that the order of convergence for the discrete second derivatives is the same as for the first derivatives. We insert the first assertion of (4.28) into (4.40) again. This time, we can use the full exponent of $(h^2+\tau)$ and so we infer for each $s \in \{0, \dots, M\}$

$$c_1 \sum_{s=0}^{M} \sum_{k,l=1}^{N} \sum_{i,j=1}^{2} \tau h^{-2} |\Delta_{ij}^{*} e_{k,l}^{s}|^2 \le c_4(h^2+\tau)^2 + c_5((1+8\hat{c}^2)T+1)W(h^2+\tau)^2$$
$$\le c(h^2+\tau)^2.$$

□

4.11 Remark. *Reducing the order of convergence for the second order differences within the induction statement (4.3) compared to the final convergence result presented in Theorem 4.10 is necessary for technical reasons: The splitting $(h^2+\tau)^{3/2+1/2}$ makes it possible to impose another smallness condition on the mesh sizes in order to obtain exactly the constant in the constraint of the induction hypothesis again in the induction step.*

Apart from the discrete spatial derivatives we can also control the discrete time derivative of e_h in a L^2-norm on the grid by some term of second order in h and first order in τ, as we display in the following.

4.12 Theorem. *Under the assumptions of Theorem 4.10 there exists a constant $c > 0$ as well as an $h_0 > 0$, such that for all $0 < h \le h_0$ the estimate*

$$\left(\tau \sum_{s=0}^{M-1} h^2 \sum_{k,l=1}^{N} \frac{|e_{k,l}^{s+1} - e_{k,l}^{s}|^2}{\tau^2} \right)^{1/2} \le c(h^2+\tau) \tag{4.42}$$

holds and the constant c only depends on x, T and α^{-1}.

Proof. Let $k,l \in \{0,\dots,N\}$ and $s \in \{0,\dots,M-1\}$. Equation (4.26) yields

$$\begin{aligned}\left|\frac{e_{k,l}^{s+1}-e_{k,l}^{s}}{\tau}\right| \le{}& |g_{k,l}^{ij,s}||\Delta_{ij} e_{k,l}^{s+1}| + |\tilde{g}_{k,l}^{ij,s} - g_{k,l}^{ij,s}||\Delta_{ij}\tilde{x}_{k,l}^{s+1}| + |\tilde{R}_{k,l}^{\alpha,s}| \\ &+ \left|\frac{1}{\alpha}-1\right| \Big[|g_{k,l}^{ij,s} g_{k,l}^{mn,s}||\Delta_{ij} e_{k,l}^{s+1}||\overline{\Delta}_m x_{k,l}^{s}||\overline{\Delta}_n x_{k,l}^{s}| \\ &\quad + |g_{k,l}^{ij,s} g_{k,l}^{mn,s}||\Delta_{ij}\tilde{x}_{k,l}^{s+1}||\overline{\Delta}_m e_{k,l}^{s}||\overline{\Delta}_n x_{k,l}^{s}| \\ &\quad + |g_{k,l}^{ij,s} g_{k,l}^{mn,s}||\Delta_{ij}\tilde{x}_{k,l}^{s+1}||\overline{\Delta}_m \tilde{x}_{k,l}^{s}||\overline{\Delta}_n e_{k,l}^{s}| \\ &\quad + |\tilde{g}_{k,l}^{ij,s} \tilde{g}_{k,l}^{mn,s} - g_{k,l}^{ij,s} g_{k,l}^{mn}||\Delta_{ij}\tilde{x}_{k,l}^{s+1}||\overline{\Delta}_m \tilde{x}_{k,l}^{s}||\overline{\Delta}_n \tilde{x}_{k,l}^{s}| \Big].\end{aligned}$$

Using (4.17) and (4.18), to control $\tilde{g}_{k,l}^{ij,s}$, $g_{k,l}^{ij,s}$ and their difference, together with

$$|\tilde{g}_{k,l}^{ij,s} \tilde{g}_{k,l}^{mn,s} - g_{k,l}^{ij,s} g_{k,l}^{mn,s}| \le c h^{-2-3} \sum_{r=1}^{2} \left(|\Delta_r^{+} e_{k,l}^{s}| + |\Delta_r^{-} e_{k,l}^{s}| \right),$$

compare (4.39), we obtain

$$
\begin{aligned}
&\frac{1}{\tau}|e_{k,l}^{s+1}-e_{k,l}^{s}| \\
&\le ch^{-2}\sum_{i,j=1}^{2}|\Delta_{ij}e_{k,l}^{s+1}| + ch^{-3}\left(\sum_{i,j=1}^{2}|\Delta_{ij}\tilde{x}_{k,l}^{s+1}|\right)\left(\sum_{r=1}^{2}(|\Delta_r^+e_{k,l}^s|+|\Delta_r^-e_{k,l}^s|)\right)+|\tilde{R}_{k,l}^{\alpha,s}| \\
&\quad + ch^{-4}\left(\sum_{m,n=1}^{2}|\overline{\Delta}_m x_{k,l}^s||\overline{\Delta}_n x_{k,l}^s|\right)\left(\sum_{i,j=1}^{2}|\Delta_{ij}e_{k,l}^{s+1}|\right) \\
&\quad + ch^{-4}\left(\sum_{i,j=1}^{2}|\Delta_{ij}\tilde{x}_{k,l}^{s+1}|\right)\left(\sum_{r=1}^{2}(|\overline{\Delta}_r x_{k,l}^s|+|\overline{\Delta}_r\tilde{x}_{k,l}^s|)\right)\left(\sum_{r=1}^{2}(|\Delta_r^+e_{k,l}^s|+|\Delta_r^-e_{k,l}^s|)\right) \\
&\quad + ch^{-5}\left(\sum_{i,j=1}^{2}|\Delta_{ij}\tilde{x}_{k,l}^{s+1}|\right)\left(\sum_{m,n=1}^{2}|\overline{\Delta}_m\tilde{x}_{k,l}^s||\overline{\Delta}_n\tilde{x}_{k,l}^s|\right)\left(\sum_{r=1}^{2}(|\Delta_r^+e_{k,l}^s|+|\Delta_r^-e_{k,l}^s|)\right).
\end{aligned}
$$

The constraints on first order differences of x_h and x, see (4.4), (4.8) and (4.12) as well as the boundedness of second order differences of x as given in (4.38) yield

$$
\begin{aligned}
\frac{1}{\tau}|e_{k,l}^{s+1}-e_{k,l}^{s}| &\le c(h^{-2}+h^{-4+1+1})\sum_{i,j=1}^{2}|\Delta_{ij}e_{k,l}^{s+1}| \\
&\quad + c(h^{-3+2}+h^{-4+2+1}+h^{-5+2+1+1})\sum_{r=1}^{2}(|\Delta_r^+e_{k,l}^s|+|\Delta_r^-e_{k,l}^s|)+|\tilde{R}_{k,l}^{\alpha,s}|.
\end{aligned}
$$

After squaring and summation over k and l we have

$$
\begin{aligned}
&\frac{1}{\tau^2}\sum_{k,l=1}^{N}|e_{k,l}^{s+1}-e_{k,l}^{s}|^2 \\
&\le ch^{-4}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}e_{k,l}^{s+1}|^2 + ch^{-2}\sum_{k,l=1}^{N}\sum_{r=1}^{2}(|\Delta_r^+e_{k,l}^s|^2+|\Delta_r^-e_{k,l}^s|^2)+c\sum_{k,l=1}^{N}|\tilde{R}_{k,l}^{\alpha,s}|^2.
\end{aligned}
$$

We replace Δ_r^+ by Δ_r^- as well as $\Delta_1^+\Delta_2^+$ by $\Delta_1^-\Delta_2^-$ (and therefore also Δ_{ij} by Δ_{ij}^*), compare Lemma 3.5. Furthermore, with the help of the estimate for the consistency error

$$
\sum_{k,l=1}^{N}|\tilde{R}_{k,l}^{\alpha,s}|^2 \le c_{\tilde{R}}^2 N^2(h^2+\tau)^2 = ch^{-2}(h^2+\tau)^2
$$

it follows that

$$
\frac{1}{\tau^2}\sum_{k,l=1}^{N}|e_{k,l}^{s+1}-e_{k,l}^{s}|^2 \le ch^{-4}\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}|\Delta_{ij}^*e_{k,l}^{s+1}|^2 + ch^{-2}\sum_{k,l=1}^{N}\sum_{r=1}^{2}|\Delta_r^-e_{k,l}^s|^2 + ch^{-2}(h^2+\tau)^2.
$$

Multiplying by h^2 and inserting (4.28) for the differences of first order yields

$$\frac{h^2}{\tau^2}\sum_{k,l=1}^{N}|e_{k,l}^{s+1}-e_{k,l}^{s}|^2 \le c(h^2+\tau)^2 + ch^2\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}\frac{|\Delta_{ij}^{*}e_{k,l}^{s+1}|^2}{h^4}.$$

Summation over all $s \in \{0,\dots,M-1\}$ and multiplication by τ allows to estimate the differences of second order analogously, which is why we have

$$\begin{aligned}\tau\sum_{s=0}^{M-1}h^2\sum_{k,l=1}^{N}\frac{|e_{k,l}^{s+1}-e_{k,l}^{s}|^2}{\tau^2} &\le cM\tau(h^2+\tau)^2 + c\tau\sum_{s=0}^{M-1}h^2\sum_{k,l=1}^{N}\sum_{i,j=1}^{2}\frac{|\Delta_{ij}^{*}e_{k,l}^{s+1}|^2}{h^4}\\ &\le cT(h^2+\tau)^2 + c(h^2+\tau)^2\\ &\le c(h^2+\tau)^2\end{aligned}$$

as asserted. □

With the aid of this result, we can show the convergence of second order in h and first order in τ for the discrete L^2-norm of the error function e_h itself in a few steps. This is demonstrated in the following theorem.

4.13 Theorem. *Under the assumptions of Theorem 4.10 there exists a constant $c > 0$ as well as an $h_0 > 0$, such that for all $0 < h \le h_0$ the estimate*

$$\max_{s\in\{0,\dots,M\}}\left(h^2\sum_{k,l=1}^{N}|e_{k,l}^{s}|^2\right)^{1/2} \le c(h^2+\tau) \tag{4.43}$$

holds and the constant c only depends on x, T and α^{-1}.

Proof. Let $s \in \{0,\dots,M-1\}$ arbitrary but fixed. $e_{k,l}^{0} = 0$ implies

$$\begin{aligned}\sum_{k,l=1}^{N}|e_{k,l}^{s+1}|^2 &= \sum_{s'=0}^{s}\sum_{k,l=1}^{N}\left(|e_{k,l}^{s'+1}|^2 - |e_{k,l}^{s'}|^2\right)\\ &= \sum_{s'=0}^{s}\sum_{k,l=1}^{N}\left(2(e_{k,l}^{s'+1}-e_{k,l}^{s'})\cdot e_{k,l}^{s'+1} - |e_{k,l}^{s'+1}-e_{k,l}^{s'}|^2\right)\\ &\le \sum_{s'=0}^{s}\sum_{k,l=1}^{N}\left(2(e_{k,l}^{s'+1}-e_{k,l}^{s'})\cdot e_{k,l}^{s'+1}\right)\\ &\overset{\text{(CS)}}{\le} 2\sum_{s'=0}^{s}\left(\sum_{k,l=1}^{N}\frac{1}{\tau}|e_{k,l}^{s'+1}-e_{k,l}^{s'}|^2\right)^{1/2}\left(\tau\sum_{k,l=1}^{N}|e_{k,l}^{s'+1}|^2\right)^{1/2}\\ &\overset{\text{(Y)}}{\le} \frac{1}{2\varepsilon}\sum_{s'=0}^{s}\sum_{k,l=1}^{N}\frac{1}{\tau}|e_{k,l}^{s'+1}-e_{k,l}^{s'}|^2 + 2\varepsilon\tau\sum_{s'=0}^{s}\sum_{k,l=1}^{N}|e_{k,l}^{s'+1}|^2.\end{aligned}$$

Subtraction of $2\varepsilon\tau|e_{k,l}^{s+1}|^2$ for every $k,l \in \{1,\ldots,N\}$ and a suitable choice of ε so that $c_6 \leq 1 - 2\varepsilon\tau$ for a constant $c_6 > 0$ we have

$$c_6 \sum_{k,l=1}^{N} |e_{k,l}^{s+1}|^2 \leq 2 \sum_{s'=0}^{M-1} \sum_{k,l=1}^{N} \frac{1}{\tau} |e_{k,l}^{s'+1} - e_{k,l}^{s'}|^2 + 2\tau \sum_{s'=0}^{s} \sum_{k,l=1}^{N} |e_{k,l}^{s'}|^2$$
$$\overset{(4.42)}{\leq} ch^{-2}(h^2+\tau)^2 + 2\tau \sum_{s'=0}^{s} \sum_{k,l=1}^{N} |e_{k,l}^{s'}|^2,$$

where we made use of the result of the preceding theorem to estimate the difference of the error function regarding time.
The lemma of Gronwall then gives

$$\sum_{k,l=1}^{N} |e_{k,l}^{s+1}|^2 \leq ch^{-2}(h^2+\tau)^2 \exp(c\sum_{s'=0}^{s}\tau) \leq ch^{-2}(h^2+\tau)^2 \exp(cT),$$

so that after multiplication by h^2 and extracting the square root

$$\left(h^2 \sum_{k,l=1}^{N} |e_{k,l}^{s+1}|^2\right)^{1/2} \leq c(h^2+\tau).$$

The estimate holds for all $s \in \{0,\ldots,M-1\}$ and so does (4.43). □

The next theorem states an estimate in a discrete maximum norm. Since $n = 2$, an embedding $H^1(\Omega) \hookrightarrow C(\bar{\Omega})$ does not exist and so the H^1-bound does not immediately imply an L^∞-estimate. Still, we can show a slightly weaker estimate that originates from the theory of finite element approximations.

4.14 Theorem. *Under the assumptions of Theorem 4.10 there exists a constant $c > 0$ as well as an $h_0 > 0$, such that for all $0 < h \leq h_0$ the estimate*

$$\max_{s\in\{0,\ldots,M\}} \max_{k,l\in\{1,\ldots,N\}} |e_{k,l}^{s}| \leq c|\ln(h)|^{\frac{1}{2}}(h^2+\tau) \tag{4.44}$$

holds and the constant c only depends on x, T and α^{-1}.

Proof. To begin with we consider the domain $[0,2\pi]^2$, which was divided into squares of edge length h by defining the grid $\{u_{k,l}\}_{k,l\in\{0,\ldots,N\}}$. These squares can be divided in half by their diagonals from the top left to the bottom right corner. That means, by connecting $u_{k-1,l}$ to $u_{k,l-1}$ for all $k,l \in \{1,\ldots,N\}$ we add new grid lines and obtain a mesh of traingles $\{\kappa^{k,l}, \kappa_{k,l}\}_{k,l\in\{1,\ldots,N\}}$. We denote

$$\begin{aligned}\kappa^{k,l} &= \{(u_1,u_2) \in [0,2\pi]^2 | (k-1)h \leq u_1 \leq kh,\ (l+k-1)h - u_1 \leq u_2 \leq lh\},\\ \kappa_{k,l} &= \{(u_1,u_2) \in [0,2\pi]^2 | (k-1)h \leq u_1 \leq kh,\ (l-1)h \leq u_2 \leq (l+k-1)h - u_1\}.\end{aligned} \tag{4.45}$$

Let $I_h : [0, 2\pi]^2 \to \mathbb{R}^3$ be the continuous linear interpolant that coincides with e_h at the corners of such a triangle and is extended to be linear in between. For arbitrary k, l and $u = (u_1, u_2) \in \kappa^{k,l}$ we thus have that $I_h^{k,l}[e_h](u) := a^{k,l}u_1 + b^{k,l}u_2 + c^{k,l}$ and the coefficients are uniquely defined by

$$\begin{aligned} I_h^{k,l}[e_h](u_{k,l}, t^s) &= e_{k,l}^s, \\ I_h^{k,l}[e_h](u_{k-1,l}, t^s) &= e_{k-1,l}^s, \\ I_h^{k,l}[e_h](u_{k,l-1}, t^s) &= e_{k,l-1}^s. \end{aligned} \tag{4.46}$$

Analogously, for $u = (u_1, u_2) \in \kappa_{k,l}$ we let $I_{h;k,l}[e_h](u) := a_{k,l}u_1 + b_{k,l}u_2 + c_{k,l}$ so that it coincides with e_h in $u_{k-1,l}$, $u_{k,l-1}$ and $u_{k-1,l-1}$ and the coefficients are given by a corresponding system of equations.
For all $s \in \{0, \ldots, M\}$ we have

$$\max_{k,l \in \{1,\ldots,N\}} |e_{k,l}^s| = \max_{u \in [0,2\pi]^2} |I_h[e_h](u, t^s)|.$$

Since $I_h[e_{k,l}^s]$ is continuous and piecewise linear, Lemma 6.4 in [41] implies that the estimate

$$\max_{u \in [0,2\pi]^2} |I_h[e_h](u, t^s)| \le c|\ln(h)|^{\frac{1}{2}} \left(\int\limits_{[0,2\pi]^2} (|I_h[e_h(t^s)]|^2 + |\nabla I_h[e_h(t^s)]|^2) \, \mathrm{d}u \right)^{\frac{1}{2}}$$

holds. Hence (4.44) is true if

$$\int\limits_{[0,2\pi]^2} (|I_h[e_h(t^s)]|^2 + |\nabla I_h[e_h(t^s)]|^2) \, \mathrm{d}u \le ch^2 \sum_{k,l=1}^{N} \left(|e_{k,l}^s|^2 + \sum_{r=1}^{2} \frac{|\Delta_r^- e_{k,l}^s|^2}{h^2} \right), \tag{4.47}$$

for we have already shown the convergence of second order in h and first order in τ in the discrete H^1-norm of e_h. In what follows, we thus aim to prove this inequality and therefore make use of the triangulation of $[0, 2\pi]^2$.
Due to the fact that $\nabla I_h^{k,l}[e_h](u) = (a^{k,l}, b^{k,l})$, the evalution of the L^2-norm of the gradient in the upper triangle $\kappa^{k,l}$ yields

$$\int\limits_{\kappa^{k,l}} |\nabla I_h^{k,l}[e_h]|^2 \, \mathrm{d}u = (|a^{k,l}|^2 + |b^{k,l}|^2) \int\limits_{\kappa^{k,l}} \mathrm{d}u = (|a^{k,l}|^2 + |b^{k,l}|^2)\frac{1}{2}h^2.$$

Solving the linear equation system (4.46) gives

$$a^{k,l} = \frac{e_{k,l} - e_{k-1,l}}{h} = \frac{\Delta_1^- e_{k,l}}{h} \text{ and } b^{k,l} = \frac{e_{k,l} - e_{k,l-1}}{h} = \frac{\Delta_2^- e_{k,l}}{h}$$

and we make use of the same proceeding for $\kappa_{k,l}$ to find

$$a_{k,l} = \frac{e_{k,l-1} - e_{k-1,l-1}}{h} = \frac{\Delta_1^- e_{k,l-1}}{h} \text{ and } b_{k,l} = \frac{e_{k-1,l} - e_{k-1,l-1}}{h} = \frac{\Delta_2^- e_{k-1,l}}{h}.$$

This implies

$$\begin{aligned}\int\limits_{[0,2\pi]^2} |\nabla I_h[e_h(t^s)]|^2 \, \mathrm{d}u =& \frac{1}{2} h^2 \sum_{k,l=1}^{N} \left(\frac{|\Delta_1^- e^s_{k,l}|^2 + |\Delta_2^- e^s_{k,l}|^2}{h^2} + \frac{|\Delta_1^- e^s_{k,l-1}|^2 + |\Delta_2^- e^s_{k-1,l}|^2}{h^2} \right) \\ =& h^2 \sum_{k,l=1}^{N} \frac{|\Delta_1^- e^s_{k,l}|^2 + |\Delta_2^- e^s_{k,l}|^2}{h^2},\end{aligned}$$

where we used the periodicity of e_h in the last step, compare the first result in Lemma 3.5.

The L^2-norm of I_h itself is evaluated by another interpolation. Let $J_h : [0, 2\pi]^2 \to \mathbb{R}$ be the continuous, piecewise linear function which coincides with $|I_h[e_h]|^2$ in each of the corners of a triangle κ. Since J_h is linear, we can make use of the quadrature rule ([39])

$$\int\limits_{\kappa} J_h[|I_h[e_h]|^2](u_1, u_2) \, \mathrm{d}u = |\kappa| J_h[|I_h[e_h]|^2](S_x, S_y), \tag{4.48}$$

where $|\kappa|$ is the area of κ and (S_x, S_y) are the coordinates of the centroid of κ. It holds that $|\kappa| = \frac{1}{2}h^2$ and

$$(S_x, S_y) = \begin{cases} (\frac{1}{3}h + (k-1)h, \frac{1}{3}h + (l-1)h) & \text{for } \kappa = \kappa_{k,l}, \\ (\frac{2}{3}h + (k-1)h, \frac{2}{3}h + (l-1)h) & \text{for } \kappa = \kappa^{k,l}. \end{cases}$$

Determining the coefficients of $J_h^{k,l}[|I_h[e_h]|^2](u)$ for the triangle and inserting the corresponding centroid into the quadrature rule yields

$$\int\limits_{\kappa^{k,l}} J_h^{k,l}[|I_h[e_h]|^2](u_1, u_2) \, \mathrm{d}u = h^2 \frac{1}{6} \left(|e_{k,l}|^2 + |e_{k-1,l}|^2 + |e_{k,l-1}|^2 \right).$$

Using the same method to calculate the integral of $J_{h;k,l}$ over $\kappa_{k,l}$ we arrive at

$$\begin{aligned}\int\limits_{[0,2\pi]^2} J_h \left[|I_h[e_h(t^s)]|^2 \right] \, \mathrm{d}u &= h^2 \sum_{k,l=1}^{N} \left(\frac{1}{6}|e^s_{k,l}|^2 + \frac{1}{3}|e^s_{k-1,l}|^2 + \frac{1}{3}|e^s_{k,l-1}|^2 + \frac{1}{6}|e^s_{k-1,l-1}|^2 \right) \\ &= ch^2 \sum_{k,l=1}^{N} |e^s_{k,l}|^2,\end{aligned}$$

where equality in the last step again holds because of the periodicity of the solution functions x and x_h on the spatial grid. Observing that $I_h[e_h]$ is continuous and piecewise linear, and letting $\kappa = \kappa^{k,l}$ and $\kappa = \kappa_{k,l}$, respectively, we have the following well-known estimate, which is used e.g. in [8]:

$$\int_{\kappa} |I_h^{k,l}[e_h]|^2 \,\mathrm{d}u \leq \int_{\kappa} J_h^{k,l}[|I_h[e_h]|^2] \,\mathrm{d}u.$$

This way, (4.47) is verified as well as is the asserted error estimation. $\square$

5 Numerical experiments

In what follows we want to demonstrate some numerical results for the family of fully discrete approximations introduced in (3.9). Firstly, we compute the experimental order of convergence for several discrete norms to give numerical evidence for the estimates obtained in the previous chapter. In the second section, we consider examples illustrating the evolution of a torus and the shrinking of the surface area for different choices of α. Furthermore, we are interested in a visualization of the quality of the generated mesh, where we focus on the influence of α as the parameter that determines the tangential velocity rather than on producing meshes with small mesh sizes h and τ.

5.1 Experimental order of convergence

In the first part of this chapter we compute the experimental convergence rate of our approximation. Since the torus is not an exact solution to the Mean Curvature DeTurck Flow, we are going to modify evolution equation (1.4) accordingly. The parametrization

$$x(u_1,u_2,t) = \begin{pmatrix} (r(t)\cos(u_1)+R)\cos(u_2) \\ (r(t)\cos(u_1)+R)\sin(u_2) \\ r(t)\sin(u_1) \end{pmatrix}, (u_1,u_2,t) \in (0,2\pi]^2 \times [0,T),$$

satisfies the inhomogeneous equation

$$x_t - g^{ij}x_{u_iu_j} + \left(\frac{1}{\alpha}-1\right) g^{ij}g^{mn}(x_{u_iu_j}\cdot x_{u_m})x_{u_n} = f(u_1,u_2,t), \tag{5.1}$$

where

$$f(u_1,u_2,t) = \frac{1}{r(t)\cos(u_1)+R}\begin{pmatrix} \cos(u_2) \\ \sin(u_2) \\ 0 \end{pmatrix} + \left(\frac{1}{\alpha}-1\right)\frac{\sin(u_1)}{r(t)\cos(u_1)+R}\begin{pmatrix} \sin(u_1)\cos(u_2) \\ \sin(u_1)\sin(u_2) \\ -\cos(u_1) \end{pmatrix},$$

if we choose $r_t = -\frac{1}{r}$ with $r(0)=1$, i.e. $r(t)=\sqrt{1-2t}$.
Corresponding to the analytical results the difference between exact solution $x(u_{k,l},t^s)$ and numerical solution $x^s_{k,l}$ is computed in the following discrete norms:

$$E_{H^1_0}(h) := \max_{s\in\{0,\ldots,M\}} \left(h^2 \sum_{k,l=1}^{N}\sum_{r=1}^{2} \frac{|\Delta_r^- e^s_{k,l}|^2}{h^2}\right)^{1/2},$$

$$E_{\Delta_t}(h) := \left(\tau \sum_{s=0}^{M-1} h^2 \sum_{k,l=1}^{N} \frac{|e^{s+1}_{k,l}-e^s_{k,l}|^2}{\tau^2}\right)^{1/2},$$

$$E_{L^2}(h) := \max_{s\in\{0,\dots,M\}} \left(h^2 \sum_{k,l=1}^{N} |e^s_{k,l}|^2 \right)^{1/2},$$
$$E_{L^\infty}(h) := \max_{s\in\{0,\dots,M\}} \max_{k,l\in\{1,\dots,N\}} |e^s_{k,l}|.$$

For each of these quantities measuring the absolute errors, we define the experimental order of convergence as

$$\mathrm{EOC}(h_1,h_2) = \log\left(\frac{E(h_1)}{E(h_2)}\right)\left(\log\left(\frac{h_1}{h_2}\right)\right)^{-1}.$$

In Table 1, the corresponding values are gathered for the radii $R = 2$ and $r(0) = 1$. $r(t) = \sqrt{1-2t}$ implies the maximal time of existence $T = 0.5$. Thus, for reliable results away from the extinction, we let $M\tau = 0.4$ in our computations. The parameter α was chosen to be 0.01 and we let $\tau = h^2/25$. These results confirm the theoretical estimates that have been established in this work. The experiments for the discrete L^2-norm of the discrete second derivatives yield similar values as those for the discrete L^2-norm of the discrete time derivative. In the maximum norm over both spatial and time variable, compared to the other norms, the order of convergence with respect to h is reduced by the factor $|\ln(h)|^{\frac{1}{2}}$. This apparently does not manifest in the computations. For different choices of radii, i.e. also in the case of a fat torus with an appropriate choice of T, as well as different values of α, similar results are obtained and thus not listed here.

h_i	E_{H^1}	EOC	E_{Δ_t}	EOC	E_{L^2}	EOC	E_{L^∞}	EOC
0.2094	1.0749	-	1.6354	-	0.9240	-	0.1936	-
0.1571	0.6093	1.97	0.9236	1.99	0.5216	1.99	0.1091	1.99
0.1257	0.3910	1.99	0.5924	1.99	0.3340	2.00	0.06978	2.00
0.1047	0.2716	2.00	0.4116	2.00	0.2318	2.00	0.04840	2.01
0.08378	0.1743	1.99	0.2641	1.99	0.1486	1.99	0.03100	2.00
0.06981	0.1212	1.99	0.1837	1.99	0.1033	1.99	0.02160	1.98

Table 1: Error in different discrete norms of e_h and of its dicrete time derivative as well as convergence rate in each norm for $R = 2$, $r = 1$, $\tau = h^2/25$, $T = 0.4$ and $\alpha = 0.01$.

5.2 Surface evolution, mesh quality and area decrease

From [40] it is known that the different evolutions of a torus that flows by its mean curvature can be grouped into three: a family of “thin” tori shrinking to a circle, a family of “fat” tori trying to merge to a sphere and one torus at the limit. Although the classification of the evolution is effected by the ratio of the small radius and the big radius, no exact value for the limit case has been determined yet. In [34], the evo-

lution of tori as rotational surfaces of some generating curve is considered. The author derives a condition on the shape of the generating curve that assures the asymptotic tranformation to a circle before it becomes a point and the torus thus becomes a circle. This gives a lower bound for the critical radius.
Numerical examples showing the two families of tori are for instance given in Figure 4.7 in [16] and also in Figures 5 and 6 in [6] and in Figures 2 and 3 in [4]. A numerical approximation for the critical radius at the transition between the thin and the fat torus is presented in [38]. In [4], where surfaces of rotation are considered, the authors also delimit the critical radius numerically. In what follows, we will both verify that our approximation leads to the evolution of thin and fat tori as well as determine an interval for the critical radius for two choices of α.
A torus that evolves by its mean curvature can have a self-similar shape, found by and named after Angenent, see [2]. Self-similarity plays an important role in the study of singularities and is hence of huge interest. Producing the Angenent torus numerically would be an intersting task to try with our approach. Yet it exceeds the scope of this work and is not considered in the following.

In this experimental chapter we are mainly interested in the influence of α on the approximation and its solution. The parameter α appeared within the reparametrization, which was introduced in order to specify a tangential velocity in the Mean Curvature Flow. This induces a tangential movement of nodes on the discrete surfaces when simulating the Mean Curvature DeTurck Flow. In particular, small choices of α are expected to produce meshes with a good behaviour, that is, meshes that do not degenerate or have distorted cells. However, the constant of the error bounds obtained in our analysis depend exponentially on α^{-1} and thus small values of alpha could be a disadvantage, too. In order to evaluate the mesh behaviour systematically, we turn the rectangular mesh on the surface into a triangular one and measure the skewness of its angles. Dividing each rectangle with edges $x_{k-1,l-1}$, $x_{k-1,l}$, $x_{k,l-1}$ and $x_{k,l}$ into two triangles $\mathcal{T}^{k,l}$ and $\mathcal{T}_{k,l}$ by connecting $x_{k-1,l}$ with $x_{k,l-1}$, allows us to compute

$$\sigma_{max} = \max_{\mathcal{T}\in\{\mathcal{T}^{k,l},\mathcal{T}_{k,l}\}_{k,l\in\{1,\dots,N\}}} \frac{L(\mathcal{T})}{R(\mathcal{T})}$$

where $L(\mathcal{T})$ is the longest side of $\mathcal{T}$ and $R(\mathcal{T})$ denotes the radius of the inscribed circle of the triangle. If σ_{max} is small, there is no triangle with sharp angles and the mesh quality is considered as good. This quantity seems to be a typical measure in the literature and can be found in several works on the numerical analysis of partial differential equations, e.g. in [19] and [22].

A characteristic property of the flow by mean curvature is the decreasing of the surface area. We computed the area with the help of linear interpolants and the quadrature rule (4.48) we used in the proof of Theorem 4.14. That means, we divide the parameter domain $[0, 2\pi]^2$ into triangles $\kappa^{k,l}, \kappa_{k,l}$ by connecting certain points of the grid $\{u_{k,l}\}_{k,l\in\{0,\ldots,N\}}$, compare (4.45). We define interpolants $I_h^{k,l}, I_{h;k,l}$ that coincide with $\sqrt{g^s_{k,l}}$ in the corners of the corresponding triangle. This yields the formula

$$\text{Area}(t^s) \approx \frac{1}{3}\sum_{k,l=1}^{N}\left(\sqrt{g^s_{k,l}} + 2\sqrt{g^s_{k-1,l}} + 2\sqrt{g^s_{k,l-1}} + \sqrt{g^s_{k-1,l-1}}\right).$$

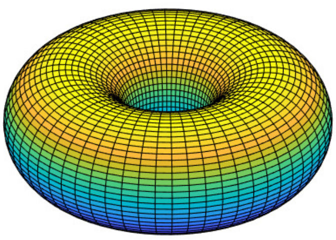
Since our focus lies on a visualization of the influence of α, with few exceptions we restrict our computations on a spatial grid size $h \approx 0.1$ and a time grid size $\tau = 10^{-4}$ to keep the expense fair. We implemented the algorithm in MATLAB. The linear system of equations that results from the fully discrete scheme in (3.9) has $3N^2$ variables with a matrix that is nearly tridiagonal. Additional non-zero-entries result from scalar products and the periodic boundary conditions.

Example 1

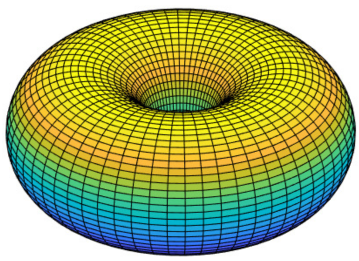
As initial surface we consider the parametrization

$$x_0(u_1, u_2) = \begin{pmatrix} (r\cos(u_1) + R)\cos(u_2) \\ (r\cos(u_1) + R)\sin(u_2) \\ r\sin(u_1) \end{pmatrix}, (u_1, u_2) \in (0, 2\pi]^2, \tag{5.2}$$

of a torus. Its projection onto the grid for the choices $R = 1$ and $r = 0.6$ as well as $R = 1$ and $r = 0.7$ is shown in Figure 1. The characteristic evolution of these surfaces can be observed in Figures 2 and 3, where we let $N = 60$, i.e. $h \approx 0.1047$, and $\tau = 10^{-4}$.

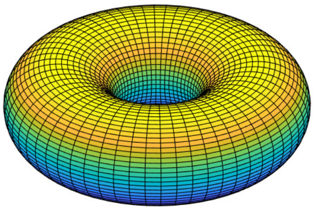


(a) Discretized surface (5.2) for $R = 1$ and $r = 0.6$

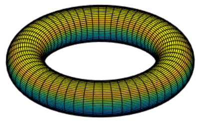


(b) Discretized surface (5.2) for $R = 1$ and $r = 0.7$

Figure 1: Initial surface (5.2), discretized with $N = 60$, for different initial radii representing the two families of evolution.



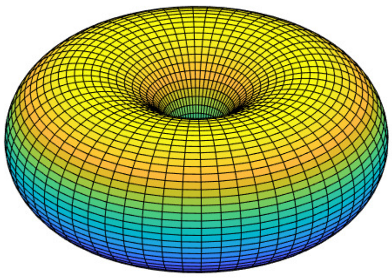
(a) Surface at $t = 0.1$



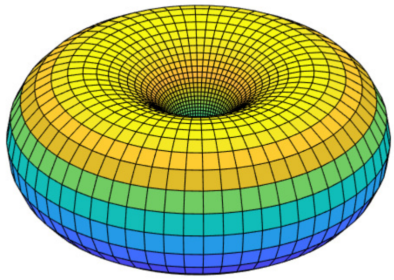
(b) Surface at $t = 0.2$

Figure 2: Simulation of the scheme in (3.9) for the initial surface in Figure 1(a). We chose $N = 60$, $\tau = 10^{-4}$ and $\alpha = 1$.

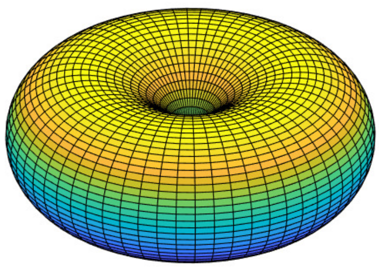
In Figure 2 we show two steps of the evolution of a thin torus for $\alpha = 1$. The surface appears to be only marginally different for other values of α, which we therefore do not display. A consideration of σ_{max} still shows differences in the mesh behaviour as explained below.
The change in topology in the case of a fat torus can not be calculated by means of our parametric approach, but we can observe how the torus pinches around the x_3-axis, see Figures 3 and 4. The latter is an enlarged section of Figures 3(e) and 3(f), where we show a projection of the discrete surfaces from Figures 3(c) and 3(d) onto the x_1x_3-plane at $x_2 = 0$. We compare two values of α that illustrate the tangential movement of nodes induced by our scheme. For $\alpha = 0.001$ this movement is much larger than for $\alpha = 1$ and the resulting mesh for $\alpha = 0.001$ produces rectangles with perceiveably differing sizes of grid cells. Still, as the enlarged part of the torus in Figure 4 indicates, the skewness is relatively small for $\alpha = 0.001$. In a study of σ_{max} in Figure 6, one can see that the triangles contained within the rectangles indeed do not have any sharp angles. We display σ_{max} for the whole time interval to give an overview, but also show the behaviour for times away from the singularity to emphasise the difference between the graphs. For the thin torus an analog is shown, see Figure 5. Although for this choice of radii no significant difference is visible on the surface, the mesh quality clearly depends on α. The behaviour of the mesh is similar in both types of development in the beginning of the evolution. Despite the maintaining of the good mesh quality for $\alpha = 0.001$ during the whole time of existence, the singularity occurs faster for this value of the parameter in the case of a fat torus, but slower in the case of a thin torus. In general and for both families of tori, smaller values of α lead to smaller values of σ_{max}.



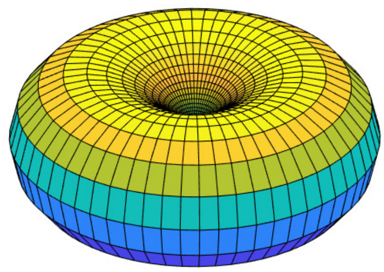
(a) Surface at $t = 0.05$



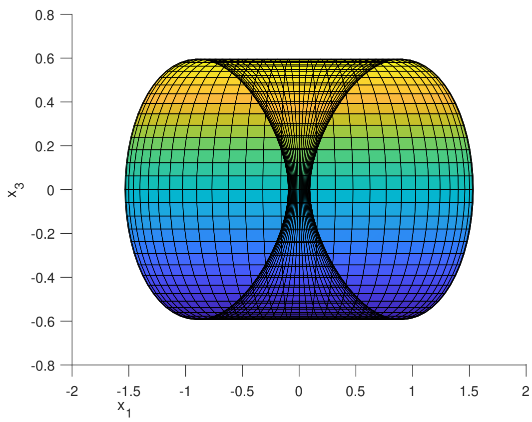
(b) Surface at $t = 0.05$



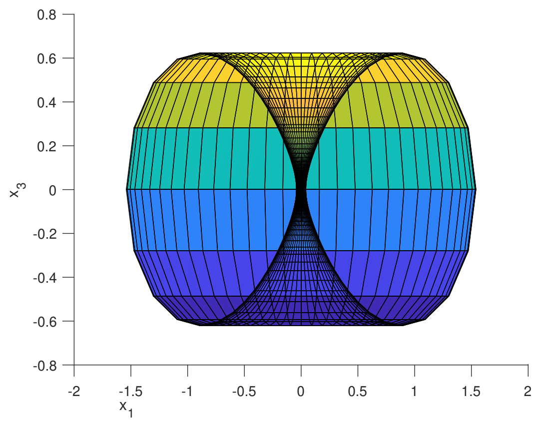
(c) Surface at $t = 0.075$



(d) Surface at $t = 0.075$

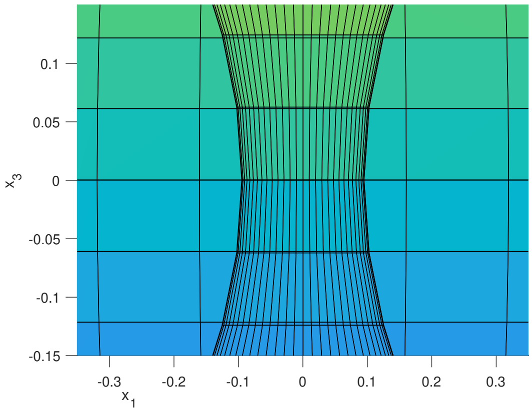


(e) Cut surface at $t = 0.075$

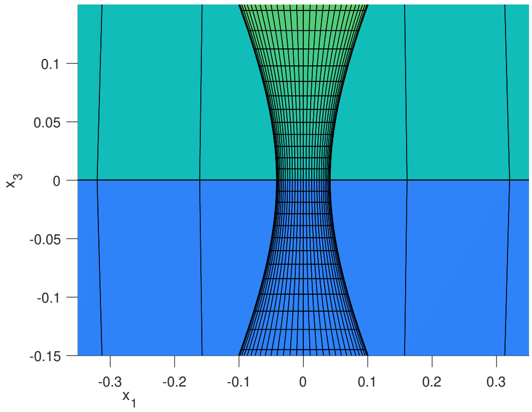


(f) Cut surface at $t = 0.075$

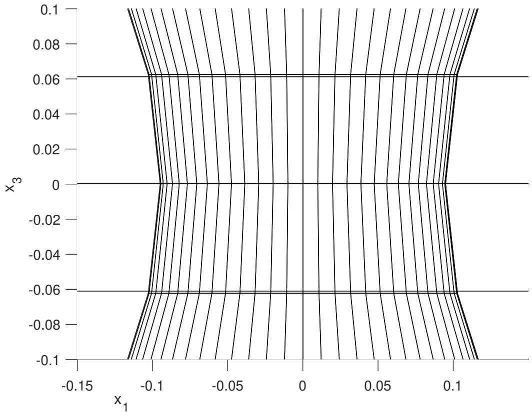
Figure 3: Simulation of the scheme in (3.9) for the initial surface in Figure 1(b). We chose $N = 60$, $\tau = 10^{-4}$ and $\alpha = 1$ in the left column as well as $\alpha = 0.001$ in the right column. In (e) and (f), the projection of the discrete surfaces in (c) and (d) onto the plane at $x_2 = 0$ is shown.



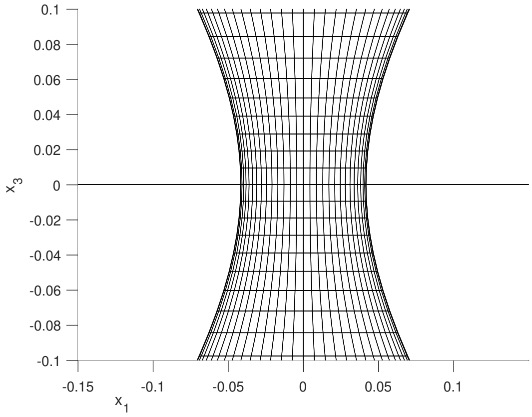
(a) Enlarged section of the surface at $t = 0.075$ for $\alpha = 1$



(b) Enlarged section of the surface at $t = 0.075$ for $\alpha = 0.001$



(c) Enlarged section of the surface at $t = 0.075$ for $\alpha = 1$



(d) Enlarged section of the surface at $t = 0.075$ for $\alpha = 0.001$

Figure 4: Simulation of the scheme in (3.9) for the initial surface in Figure 1(b). We chose $N = 60$, $\tau = 10^{-4}$ and $\alpha = 1$ on the left as well as $\alpha = 0.001$ on the right.

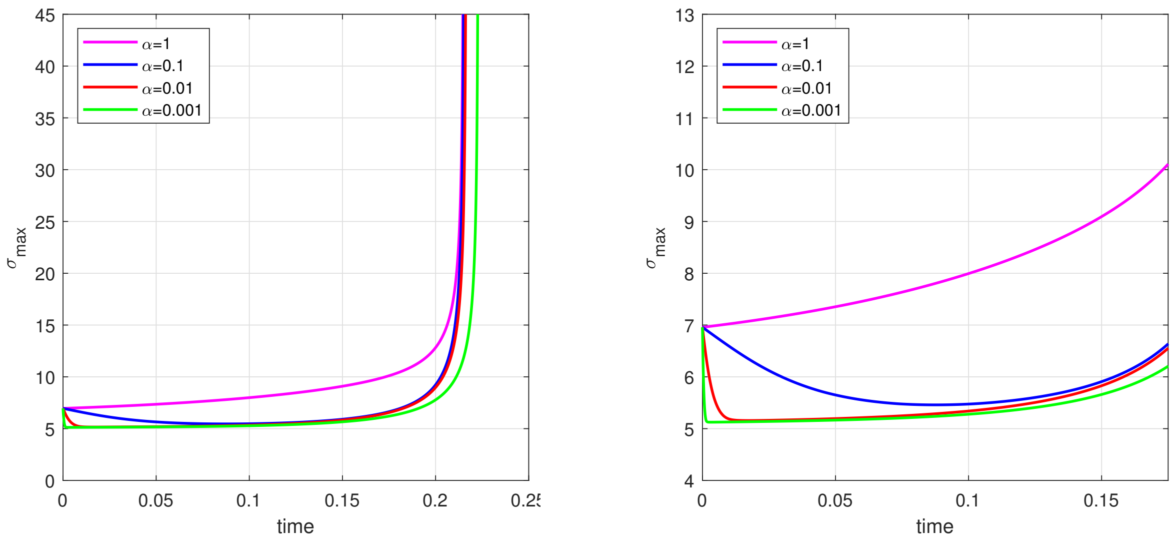


Figure 5: Mesh quality σ_{max} during the evolution of a torus with initial radii $R = 1$ and $r = 0.6$ of the scheme in (3.9), comparing $\alpha \in \{1, 0.1, 0.01, 0.001\}$. We have $N = 60$ and $\tau = 10^{-4}$.

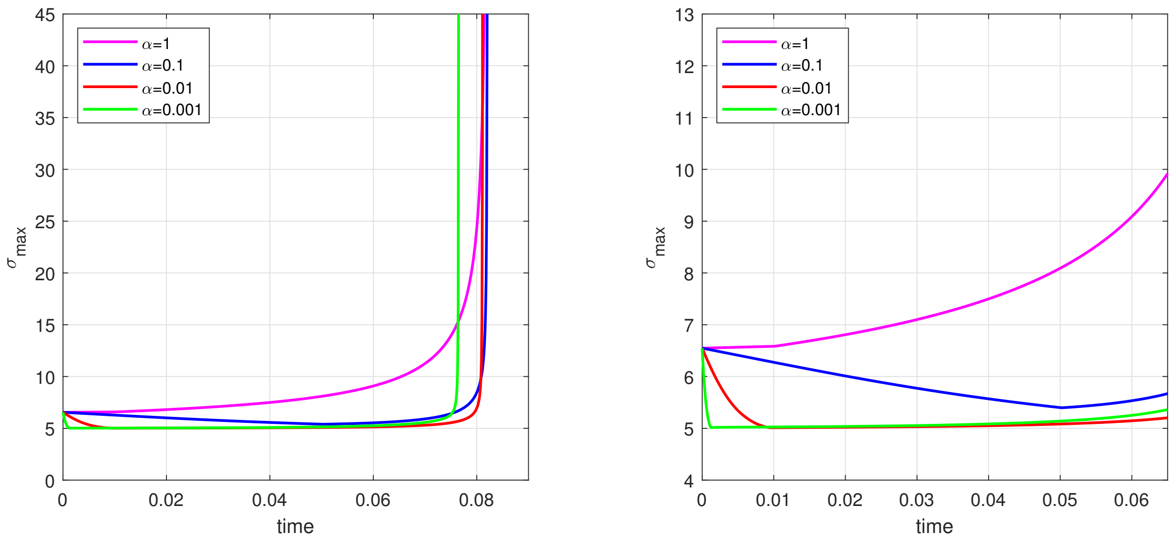
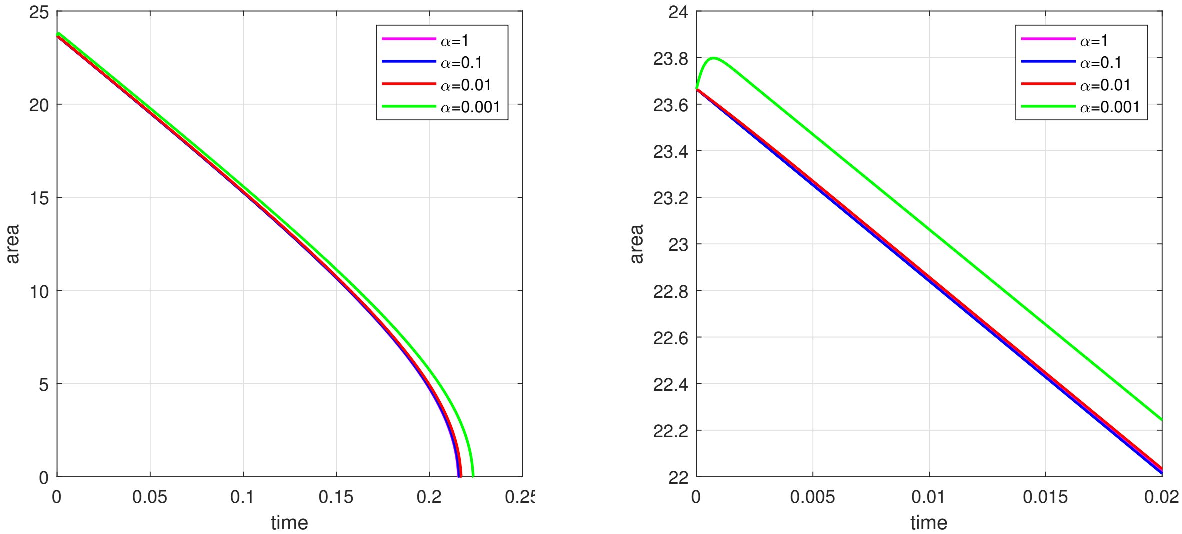


Figure 6: Mesh quality σ_{max} during the evolution of a torus with initial radii $R = 1$ and $r = 0.7$ of the scheme in (3.9), comparing $\alpha \in \{1, 0.1, 0.01, 0.001\}$. We have $N = 60$ and $\tau = 10^{-4}$.

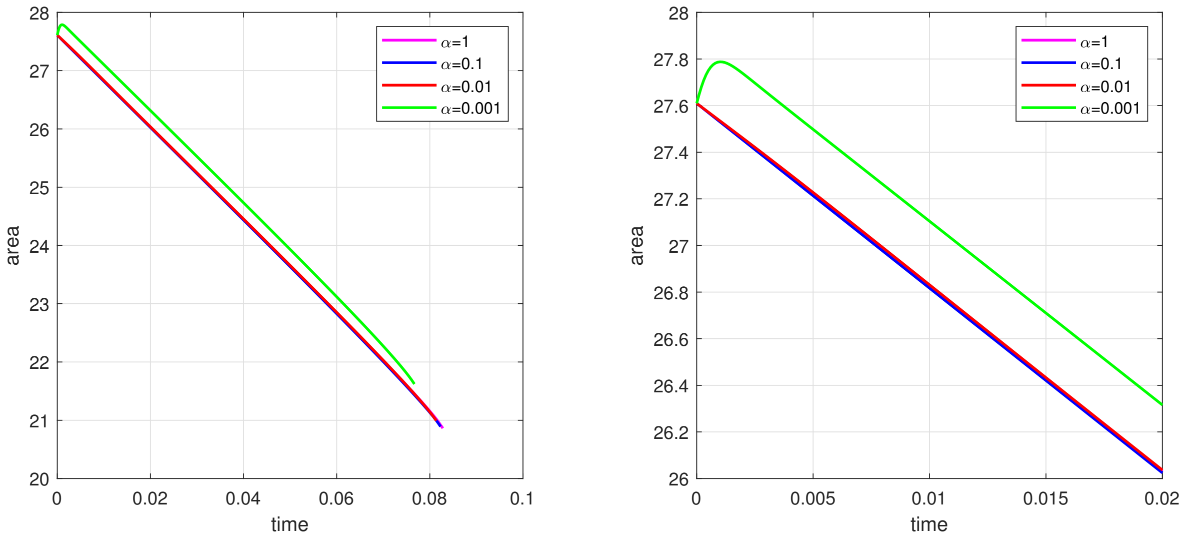
In Figures 7 and 8 we study the influence of the parameter α on the process of area decrease. Again, we performed the computations for the thin and the fat torus. Note the increase of the surface area for small times when $\alpha = 0.001$. This phenonemon appeared as well in [22], Example 1, for small α.



(a) Area for the initial surface in Figure 1(a)

(b) Enlarged section of area evolution

Figure 7: Surface area as a function of time under the approximated Mean Curvature DeTurck Flow for a thin torus with radii $R = 1$ and $r = 0.6$, comparing $\alpha \in \{1, 0.1, 0.01, 0.001\}$ with $N = 60$ and $\tau = 10^{-4}$.



(a) Area for the initial surface in Figure 1(b)

(b) Enlarged section of area evolution

Figure 8: Surface area as a function of time under the approximated Mean Curvature DeTurck Flow for a fat torus with radii $R = 1$ and $r = 0.7$, comparing $\alpha \in \{1, 0.1, 0.01, 0.001\}$ with $N = 60$ and $\tau = 10^{-4}$.

We also determined an interval for the critical radius between the two families of thin and fat tori and thereby tested the influence of α on the development. As we can see in Figures 9 and 10, the critical radius r^* changes for different choices of the parameter. For $\alpha = 1$, our simulations indicate that $r^* \in (0.641, 0.642)$, while for $\alpha = 0.01$ we obtained that $r^* \in (0.64, 0.641)$.

(a) Surface at $t = 0.28$ for $r = 0.641$ (b) Surface at $t = 0.2395$ for $r = 0.642$

Figure 9: Simulation of the scheme in (3.9) with $\alpha = 1$ for a torus with different initial radii. For $R = 1$ and $r = 0.641$ (left) we obtain a thin torus, while for $R = 1$ and $r = 0.642$ (right) we obtain a fat torus. We chose $N = 60$ and $\tau = 10^{-5}$.

(a) Surface at $t = 0.282$ for $r = 0.64$ (b) Surface at $t = 0.2583$ for $r = 0.641$

Figure 10: Simulation of the scheme in (3.9) with $\alpha = 0.01$, $N = 60$ and $\tau = 10^{-5}$ for a torus with different initial radii. For $R = 1$ and $r = 0.640$ (left) we obtain a thin torus, while for $R = 1$ and $r = 0.641$ (right) we obtain a fat torus.

Example 2

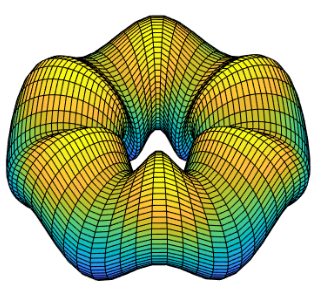
For the second example, we let the initial surface be parametrized by

$$x_0(u_1, u_2) = \begin{pmatrix} (r\cos(u_1) + R)\cos(u_2) \\ (r\cos(u_1) + R)\sin(u_2) \\ r\sin(u_1) + \frac{1}{5}\sin(6u_2) \end{pmatrix}, (u_1, u_2) \in (0, 2\pi]^2. \tag{5.3}$$

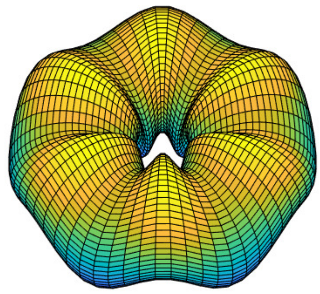
Note that compared to the parametrization in Example 1, a term is added in the third component. In Figure 11, we present the initial surfaces for $R = 1$ and $r = 0.6$ or $r = 0.7$ and two subsequent steps in their evolution. Like for the torus in Example 1 we observe a shrinking toward a circle and a merging to a sphere. For a comparison with the results obtained by Elliott and Fritz in [22], where this surface was investigated as well, we chose $N = 90$, which leads to a similar number of vertices on the surface. This example demonstrates the smoothing effect of geometric flows, i.e. the flattening of the surface during the process of shrinking.

The authors in [22] also considered a surface with an initial radius between that of the thin and the fat torus observed above, more precisely they chose $R = 1$ and $r = 0.65$. Their algorithm converges in a situation where the algorithm by [6] leads to a degenerate mesh, see Figure 27 in [22], where the authors chose $\alpha = 1$ and $\tau = 10^{-5}$. In Figure 12 we see that the mesh produced by our approximation for $\alpha = 1$, $\tau = 10^{-5}$ and $N = 90$ does not degenerate either. Note that although we used the DeTurck trick like the authors in [22], their method leads to a completely different approximation than ours, also in the special case $\alpha = 1$. Still, both approximations have in common that they induce tangential motion that is advantageous for the mesh properties, especially for small α.

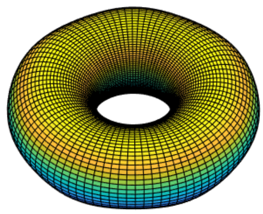
A study of the mesh property σ_{max} for our scheme with $R = 1$ and $r = 0.65$ is given in Figure 14, where we returned to the choice of $N = 60$ and $\tau = 10^{-4}$. We observe that, as long as the surface does not become singular, the mesh quality for $\alpha = 0.001$ is better than for any other choice of the parameter. Apparently, for different α, i.e. different numerical schemes, different singularities can occur since we chose an initial radius in between $r = 0.6$ and $r = 0.7$, which both lead to different evolutions. This can also be guessed by Figure 13, which shows the decreasing of the area of the computed surface. For the two smallest values of α, the computations stop at a point of time when the surface has positive area, while for the two larger values of α, the algorithm converges and the surfaces vanish. To be able to compare σ_{max} for different α for one kind of singularity, we repeat the computations for $r = 0.635$. The results are shown in Figures 15 and 16. Here, in all cases the surface converges toward a circle and a better mesh quality can be observed for smaller α.



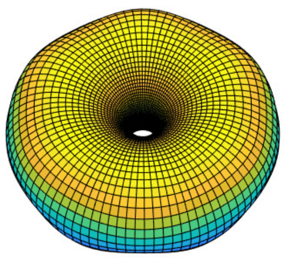
(a) Discrete initial surface ($r = 0.6$).



(b) Discrete initial surface ($r = 0.7$).

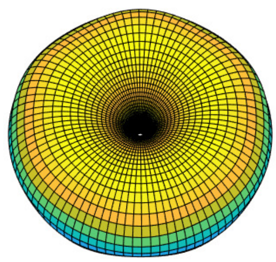


(c) Surface at $t = 0.1$.



(d) Surface at $t = 0.08$.

(e) Surface at $t = 0.2$.



(f) Surface at $t = 0.09$.

Figure 11: Evolution of (5.3) under the scheme in (3.9) for $\alpha = 0.01$ with different initial radii. We chose $N = 90$, $\tau = 10^{-4}$, $R = 1$. In (a),(c),(e), the small radius is $r = 0.6$, in (b),(d),(f) $r = 0.7$.

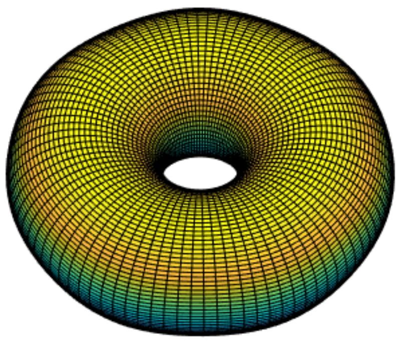


Figure 12: Evolution of (5.3) with initial radii $R = 1$ and $r = 0.65$ at $t = 0.11$ for $\alpha = 1$, $N = 90$ and $\tau = 10^{-5}$.

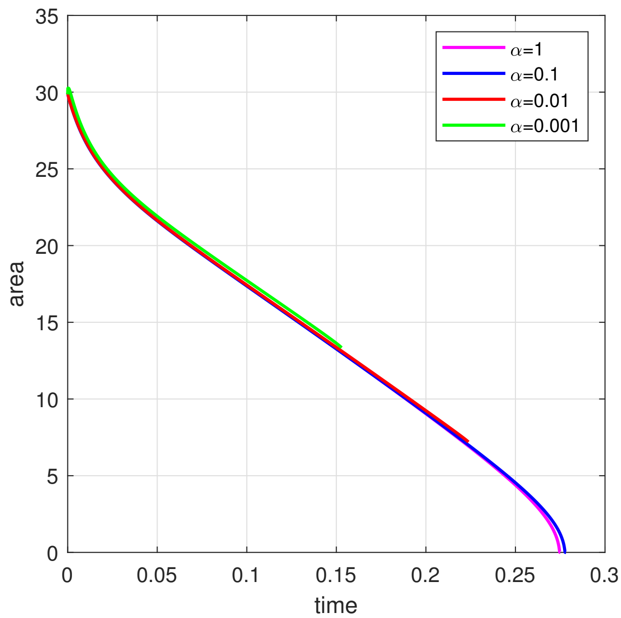


Figure 13: Surface area as a function of time under the approximated Mean Curvature DeTurck Flow for the initial surface given by (5.3) where $R = 1$ and $r = 0.65$, including a comparison for $\alpha \in \{1, 0.1, 0.01, 0.001\}$. We have $N = 60$ and $\tau = 10^{-4}$.

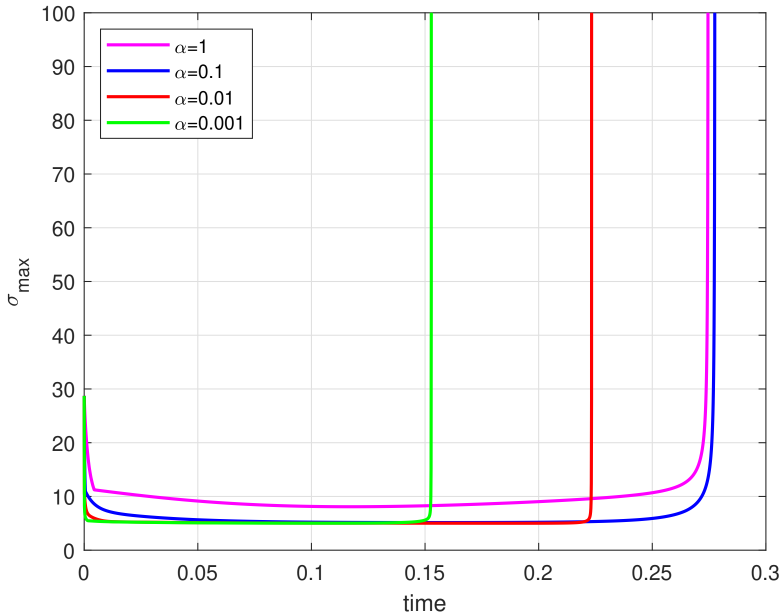


Figure 14: Mesh quality measured by the quantity σ_{max} during the evolution of the initial surface given by (5.3) where $R = 1$ and $r = 0.65$ under the approximated MCDTF, including a comparison for $\alpha \in \{1, 0.1, 0.01, 0.001\}$. We have $N = 60$ and $\tau = 10^{-4}$.

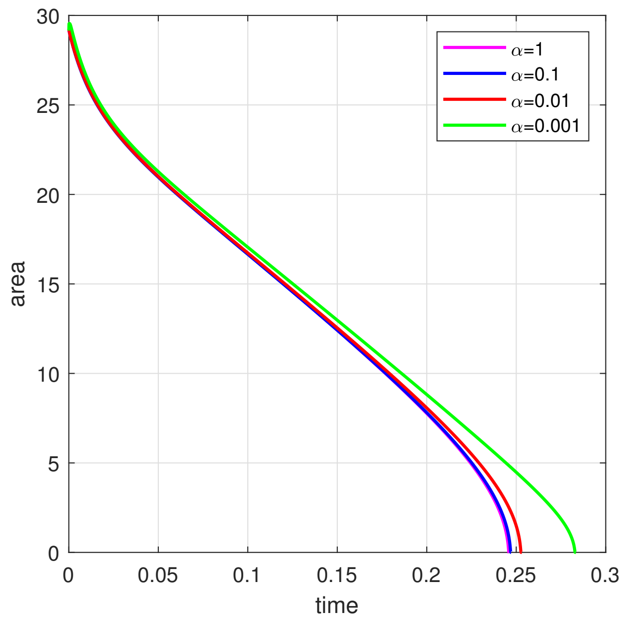


Figure 15: Surface area as a function of time under the approximated Mean Curvature DeTurck Flow for the initial surface given by (5.3) where $R = 1$ and $r = 0.635$, including a comparison for $\alpha \in \{1, 0.1, 0.01, 0.001\}$. We have $N = 60$ and $\tau = 10^{-4}$.

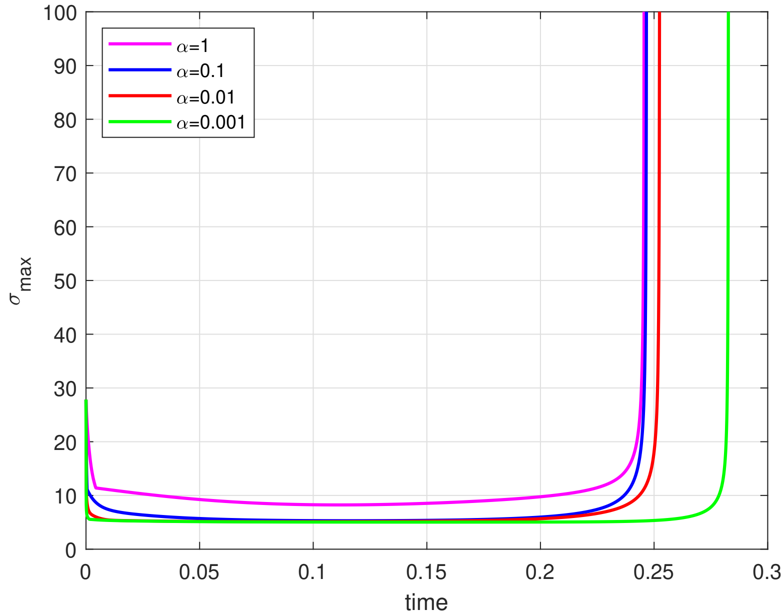


Figure 16: Mesh quality measured by the quantity σ_{max} during the evolution of the initial surface given by (5.3) where $R = 1$ and $r = 0.635$ under the approximated MCDTF, including a comparison for $\alpha \in \{1, 0.1, 0.01, 0.001\}$. We have $N = 60$ and $\tau = 10^{-4}$.

6 Conclusions

In the present work, we approximated a reparametrization of the evolution equation describing the flow by mean curvature. The reparametrized flow, which depends on a parameter α, is generated by applying a variant of the DeTurck trick. It is called Mean Curvature DeTurck Flow and has the desirable property of being strongly parabolic. That means, in contrast to the original mean curvature flow, it has a prescribed tangential velocity, which seems to be advantageous not only from an analytic point of view but also for the numerical analysis. We introduced the trick in order to derive a finite difference approximation for surfaces of torus type in $\mathbb{R}^3$ that allows for a convergence analysis. The choice of a finite difference method, though requiring a high smoothness assumption on the solution to the Mean Curvature DeTurck Flow, enables to handle the spatial operator, which is not in divergence form. The resulting family of fully discrete schemes presented in this work is semi-implicit. With the help of energy methods, we proved optimal order error bounds in several discrete integral norms. The crucial regularity assumption of a uniformly bounded area element g can be ensured as long as the curvature stays bounded. Together with superconvergence effects in the first spatial derivatives this yields a $W^{1,\infty}$-bound by an inverse estimate which was essential to control the geometry on the discrete surface via smallness conditions on the mesh sizes h and τ.

To our knowledge, this is only the second convergence proof for the mean curvature flow problem for surfaces. Compared to the first convergence result obtained by Kovács, Li and Lubich in [35] for a surface finite element method, we do not have to introduce further variables to our scheme like the mean curvature and the normal vector to solve the system. In addition, our approximation has a built-in tangential motion that is an advantage in the simulation of the flow because it can prevent mesh degeneration. The latter is also true for the scheme presented by Elliott and Fritz in [22], where the DeTuck trick is used in combination with a finite element method. Still, no convergence proof has been given for this approximation of mean curvature flow yet.

The present error estimates were confirmed by numerical computations. Experiments with varying choices of α showed that, in particular, small values of the parameter lead to good mesh properties in the sense that the generated meshes do not exhibit any skewed angles. Yet, we learned that the constants in our error estimates depend exponentially on α^{-1}. Hence, small values of alpha have a negative effect on the error in the computed solution, too. In any case, a rigorous proof that the quality of the generated meshes is good and that this feature is maintained during the evolution is an open problem as well.

Another interesting question for future research is, whether our approach could be used to produce self-similar solutions of the flow by mean curvature numerically. A solution x that maintains its shape throughout the evolution has to satisfy the stationary equation

$$\frac{1}{2T}(x \cdot \nu)\nu = H(x)\nu,$$

where T is the time of the singularity of the surface, compare e.g. [21]. Defining a tangential component and constructing an approximation with the help of the methods presented in this work would lead to the problem to find $x_h(u_{k,l})$ such that for all $k, l \in \{1, \ldots, N\}$

$$\frac{1}{2T}\, x_{k,l} = \; g^{ij}_{k,l}\Delta_{ij} x_{k,l} + \left(\frac{1}{\alpha} - 1\right) g^{ij}_{k,l} g^{mn}_{k,l} (\Delta_{ij} x_{k,l} \cdot \overline{\Delta}_m x_{k,l}) \overline{\Delta}_n x_{k,l}.$$

This system of equations is nonlinear and solving it would require to apply a Newton method.

Furthermore, since our considerations are restricted to surfaces of torus type, it is natural to ask whether our approach is suitable for surfaces of the type of the sphere. While tori can be treated by means of the periodic boundary conditions which we imposed on the domain $[0, 2\pi]^2$, mapping a rectangle onto a sphere is impossible without singularities. In addition, choosing a finite difference method implies rectangular mesh cells, which might cause issues such as mesh degeneration near the poles of a sphere. It is thus not clear how our approximation could be transferred to the case of a spherical surface.

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List of symbols

$\cdot$	Euclidean scalar product
$\lvert \cdot \rvert$	Euclidean norm
$\frac{\partial}{\partial v} f = f_v$	partial derivative with respect to the variable v
$\partial^\gamma = \partial^{(\gamma_1,\gamma_2,\gamma_3)}$	differential operator with multi-index
C^m	space of m times continuously differentiable functions
L^p, H^p	Sobolev spaces
$C^0_{per}([0,2\pi]^2;\mathbb{R}^n)$	space of continuous functions on $[0,2\pi]^2$
u_i, $i \in \{1,2\}$	parameters
$[0,2\pi]^2$	domain of parameters
h	spatial mesh size, $h = \frac{2\pi}{N}$
$\{u_{k,l}\}_{k,l\in\{0,\dots,N\}}$	spatial grid with $u_{k,l} = (kh, lh)$
τ	time step size, $\tau = \frac{T}{M}$
$\{t^s\}_{s\in\{0,\dots,M\}}$	time grid with $t^s = s\tau$
$f_{k,l}$	function evaluation at $u_{k,l}$
$\Delta_1^+ f_{k,l}$	$= f_{k+1,l} - f_{k,l}$
$\Delta_1^- f_{k,l}$	$= f_{k,l} - f_{k-1,l}$
$\Delta_2^+ f_{k,l}$	$= f_{k,l+1} - f_{k,l}$
$\Delta_2^- f_{k,l}$	$= f_{k,l} + f_{k,l-1}$
$\overline{\Delta}_1 x_{k,l}$	$= \frac{1}{2}(f_{k+1,l} - f_{k-1,l})$
$\overline{\Delta}_2 x_{k,l}$	$= \frac{1}{2}(f_{k,l+1} - f_{k,l-1})$
$\Delta_{11} f_{k,l}$	$= f_{k+1,l} - 2f_{k,l} + f_{k-1,l}$
$\Delta_{12} f_{k,l}$	$= \frac{1}{2}(f_{k+1,l+1} - f_{k+1,l} - f_{k,l+1} + 2f_{k,l} - f_{k,l-1} - f_{k-1,l} + f_{k-1,l-1})$
$\Delta_{22} f_{k,l}$	$= f_{k,l+1} - 2f_{k,l} + f_{k,l-1}$
$\Delta_{11}^* f_{k,l}$	$= \Delta_{11} f_{k,l}$
$\Delta_{12}^* f_{k,l}$	$= f_{k,l} - f_{k,l-1} - f_{k-1,l} + f_{k-1,l-1}$
$\Delta_{22}^* f_{k,l}$	$= \Delta_{22} f_{k,l}$
$E_r^\pm f_{k,l}$	$= \begin{cases} f_{k\pm1,l} & \text{for } r = 1, \\ f_{k,l\pm1} & \text{for } r = 2 \end{cases}$

ν	unit normal field of a surface
H	mean curvature of a surface
x	solution of the Mean Curvature DeTurck Flow
g_{ij}	coefficients of the induced metric
g^{ij}	entries of $(g^{ij}) = (g_{ij})^{-1}$
g	determinant of (g_{ij})
$\tilde{x}^s_{k,l} = x(u_{k,l}, t^s)$	restriction of x onto the grid
$(\tilde{g}_{ij})^s_{k,l}$	approximation of $g_{ij}(u_{k,l}, t^s)$, compare (3.20)
$\tilde{g}^{ij,s}_{k,l}$	approximation of $g^{ij}(u_{k,l}, t^s)$, compare (3.20)
$\tilde{g}^s_{k,l}$	approximation of $g(u_{k,l}, t^s)$, compare (3.20)
$\tilde{R}^{\alpha,s}_{k,l}$	consistency error at $(u_{k,l}, t^s)$
x_h	solution of the approximation of the Mean Curvature DeTurck Flow
$x^s_{k,l} = x_h(u_{k,l}, t^s)$	evaluation of x_h in the mesh points $(u_{k,l}, t^s)$
$(g^s_{ij})_{k,l}$	discrete version of g_{ij} that corresponds to x_h, compare (3.10)
$g^{ij,s}_{k,l}$	discrete version of g^{ij} that corresponds to x_h, compare (3.10)
$g^s_{k,l}$	discrete version of g that corresponds to x_h, compare (3.10)
$e_h = x - x_h$	error function
$e^s_{k,l} = e(u_{k,l}, t^s)$	evaluation of e_h at grid points

Ehrenerklärung

Ich versichere hiermit, dass ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; verwendete fremde und eigene Quellen sind als solche kenntlich gemacht.

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