On the Navier–Stokes equations on surfaces

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Dedicated to Matthias Hieber on the occasion of his 60th birthday

Abstract. We consider the motion of an incompressible viscous fluid that completely covers a smooth, compact and embedded hypersurface Σ without boundary and flows along Σ. Local-in-time well-posedness is established in the framework of L_p-L_q-maximal regularity. We characterize the set of equilibria as the set of all Killing vector fields on Σ, and we show that each equilibrium on Σ is stable. Moreover, it is shown that any solution starting close to an equilibrium exists globally and converges at an exponential rate to a (possibly different) equilibrium as time tends to infinity.

1. Introduction

Suppose Σ is a smooth, compact, connected, embedded (oriented) hypersurface in $\mathbb{R}^{d+1}$ without boundary. Then, we consider the motion of an incompressible viscous fluid that completely covers Σ and flows along Σ.

Fluid equations on manifolds appear in the literature as mathematical models for various physical and biological processes, for instance in the modeling of emulsions and biological membranes. The reader may also think of an aquaplanet whose surface is completely covered by a fluid. The case of a planet with oceans and landmass will be considered in future work.

Fluid equations on manifolds have also been studied as mathematical problems in their own right, see for instance [1,4,5,7,8,11,23,24] and the references cited therein.

In this paper, we model the fluid by the 'surface Navier–Stokes equations' on Σ, using as constitutive law the *Boussinesq–Scriven surface stress tensor*

$$\mathcal{T}_\Sigma = \mathcal{T}_\Sigma(u, \pi) = 2\mu_s \mathcal{D}_\Sigma(u) + (\lambda_s - \mu_s)(\mathrm{div}_\Sigma u)\mathcal{P}_\Sigma - \pi \mathcal{P}_\Sigma, \qquad (1.1)$$

where μ_s is the surface shear viscosity, λ_s the surface dilatational viscosity, u the velocity field, π the pressure, and

$$\mathcal{D}_\Sigma(u) := \frac{1}{2}\mathcal{P}_\Sigma \left(\nabla_\Sigma u + [\nabla_\Sigma u]^\mathsf{T} \right) \mathcal{P}_\Sigma \qquad (1.2)$$

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the surface rate-of-strain tensor. Here, $\mathcal{P}_\Sigma$ denotes the orthogonal projection onto the tangent bundle $\mathsf{T}\Sigma$ of Σ, $\operatorname{div}_\Sigma$ the surface divergence, and ∇_Σ the surface gradient. We refer to Prüss and Simonett [13] and Appendix for more background information on these objects.

Boussinesq [3] first suggested to consider surface viscosity to account for intrinsic frictional forces within an interface. Several decades later, Scriven [22] generalized Boussinesq's approach to material surfaces having arbitrary curvature. The resulting tensor is nowadays called the Boussinesq–Scriven stress tensor.

For an incompressible fluid, i.e., $\operatorname{div}_\Sigma u = 0$, the Boussinesq–Scriven surface tensor simplifies to

$$\mathcal{T}_\Sigma = 2\mu_s \mathcal{D}_\Sigma(u) - \pi\mathcal{P}_\Sigma. \tag{1.3}$$

We then consider the following *surface Navier–Stokes equations* for an incompressible viscous fluid

$$\begin{aligned}
\varrho\big(\partial_t u + \mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u)\big) - \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{T}_\Sigma &= 0 \quad &&\text{on } \Sigma \\
\operatorname{div}_\Sigma u &= 0 \quad &&\text{on } \Sigma \\
u(0) &= u_0 \quad &&\text{on } \Sigma,
\end{aligned} \tag{1.4}$$

where ϱ is a positive constant. In the sequel, we will always assume that $u_0 \in \mathsf{T}\Sigma$, i.e., u_0 is a tangential field.

Remark 1.1. Suppose $u_0 \in \mathsf{T}\Sigma$. If $(u(t), \pi(t))$ is a (sufficiently) smooth solution to (1.4) on some time interval $[0, T]$, then we also have $u(t) \in \mathsf{T}\Sigma$ for all $t \in [0, T]$. This can readily be seen by taking the inner product of the first equation in (1.4) with $\nu_\Sigma(p)$, yielding $(\partial_t u(t, p)|\nu_\Sigma(p)) = 0$ for $(t, p) \in [0, T) \times \Sigma$, where ν_Σ is the unit normal field of Σ. Hence, $(u(t, p)|\nu_\Sigma(p)) = (u_0(p)|\nu_\Sigma(p)) = 0$ for $(t, p) \in [0, T) \times \Sigma$.

It will be shown in Appendix that (1.4) can we written in the form

$$\begin{aligned}
\varrho\big(\partial_t u + \mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u)\big) - \mu_s \Delta_\Sigma u - \mu_s(\kappa_\Sigma L_\Sigma - L_\Sigma^2)u + \nabla_\Sigma \pi &= 0 \quad &&\text{on } \Sigma \\
\operatorname{div}_\Sigma u &= 0 \quad &&\text{on } \Sigma \\
u(0) &= u_0 \quad &&\text{on } \Sigma,
\end{aligned} \tag{1.5}$$

where Δ_Σ is the (negative) Bochner-Laplacian, κ_Σ the d-fold mean curvature of Σ (the sum of the principal curvatures), and L_Σ the Weingarten map. Here, we use the convention that a sphere has negative mean curvature.

We would like to emphasize that the tensor $(\kappa_\Sigma L_\Sigma - L_\Sigma^2)$ is an intrinsic quantity. In fact, we shall show in Proposition A.2 that

$$\begin{aligned}
\kappa_\Sigma L_\Sigma - L_\Sigma^2 &= \operatorname{Ric}_\Sigma \\
\kappa_\Sigma L_\Sigma - L_\Sigma^2 &= K_\Sigma \quad \text{in case } d = 2,
\end{aligned} \tag{1.6}$$

where $\operatorname{Ric}_\Sigma$ is the Ricci tensor and K_Σ the Gaussian curvature of Σ (the product of the principal curvatures).

The formulation (1.4) coincides with [7, formula (3.2)]. In that paper, the equations for the motion of a viscous incompressible fluid on a surface were derived from fundamental continuum mechanical principles. The same equations were also derived in [8, formula (4.4)], based on global energy principles. We mention that the authors of [7,8] also consider material surfaces that may evolve in time.

Here, we would like to point out that several formulations for the 'surface Navier–Stokes equations' have been used in the literature, see [4] for a comprehensive discussion, and also [7, Section 3.2]. It turns out that the model based on the Boussinesq–Scriven surface stress tensor leads to the same equations as, for instance, in [5, Note added to Proof] and [23]. Indeed, this follows from (1.5)-(1.6) and the relation

$$\Delta_\Sigma u = \Delta_H u + \mathrm{Ric}_\Sigma u, \tag{1.7}$$

where Δ_H denotes the Hodge Laplacian (acting on 1-forms).

The plan of this paper is as follows. In Sect. 2, we show that kinetic energy is dissipated by the fluid system (1.4), and we characterize all the equilibrium solutions of (1.4). It is shown that at equilibrium, the gradient of the pressure is completely determined by the velocity field. Moreover, it is shown that the equilibrium (that is, the stationary) velocity fields correspond exactly to the Killing fields of Σ. We finish Sect. 2 with some observations concerning the motion of fluid particles in the case of a stationary velocity field.

In Sect. 3, we prove that the linearization of (1.5) enjoys the property of L_p-L_q-maximal regularity. We rely on results contained in [13, Sections 6 and 7]. Moreover, we introduce the Helmholtz projection on Σ and we prove interpolation results for divergence-free vector fields on Σ. We then establish local well-posedness of (1.5) in the (weighted) class of L_p-L_q-maximal regularity, see Theorem 3.5.

In Sect. 4, we prove that all equilibria of (1.5) are stable. Moreover, we show that any solution starting close to an equilibrium exists globally and converges at an exponential rate to a (possibly different) equilibrium as time tends to infinity. In order to prove this result, we show that each equilibrium is normally stable. Let us recall that the set of equilibria $\mathcal{E}$ coincides with the vector space of all Killing fields on Σ. It then becomes an interesting question to know how many Killing fields a given manifold can support. In Sect. 4.1, we include some remarks about the dimension of $\mathcal{E}$ and we discuss some examples.

In forthcoming work, we plan to use the techniques introduced in this manuscript to study the Navier–Stokes equations on manifolds with boundary.

We would like to briefly compare the results of this paper with previous results by other authors. Existence of solutions for the Navier–Stokes equations (1.5) has already been established in [23], see also [11] and the comprehensive list of references in [4]. The authors in [11,23] employ techniques of pseudo-differential operators, and they make use of the property that the Hodge Laplacian commutes with the Helmholtz projection. Under the assumption that the spectrum of the linearization is contained in the negative real axis, stability of the zero solution is shown in [23]. The author remarks

that this assumption implies that the isometry group of Σ is discrete. In contrast, our stability result in Theorem 4.3 applies to any manifold.

The Boussinesq–Scriven surface stress tensor has also been employed in the situation of two incompressible fluids which are separated by a free surface, where surface viscosity (accounting for internal friction within the interface) is included in the model, see [2].

Finally, we mention [7,12,18,19] and the references contained therein for interesting numerical investigations. These authors also observed that the equilibria velocities correspond to Killing fields.

2. Energy dissipation and equilibria

In the following, we set $\varrho = 1$. Let

$$\mathsf{E}(t) := \int_\Sigma \frac{1}{2} |u(t)|^2 \, \mathrm{d}\Sigma \tag{2.1}$$

be the (kinetic) energy of the fluid system. We show that the energy is dissipated by the fluid system (1.4).

Proposition 2.1. *Suppose (u, π) is a sufficiently smooth solution of (1.4) with initial value $u_0 \in \mathsf{T}\Sigma$, defined on some interval $(0, T)$. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{E}(t) = -2\mu_s \int_\Sigma |\mathcal{D}_\Sigma(u(t))|^2 \, \mathrm{d}\Sigma, \quad t \in (0, T). \tag{2.2}$$

Proof. By Remark 1.1, we know that $u(t) \in \mathsf{T}\Sigma$ for each $t \in (0, T)$. In order not to overburden the notation, we suppress the variables $(t, p) \in (0, T) \times \Sigma$ in the following computation. It follows from (A.17) and Lemma A.1 that

$$\left(\mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u) \big| u\right) = \frac{1}{2} \left(\nabla_\Sigma |u|^2 \big| u\right), \quad \left(\mathcal{P}_\Sigma \mathrm{div}_\Sigma(\pi \mathcal{P}_\Sigma) \big| u\right) = \left(\nabla_\Sigma \pi \big| u\right),$$

$$\left(\mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) \big| u\right) = \mathrm{div}_\Sigma \left(\mathcal{D}_\Sigma(u)u\right) - |\mathcal{D}_\Sigma(u)|^2,$$

where $(\cdot | \cdot)$ denotes the Euclidean inner product. Hence, the surface divergence theorem (A.14) and the relation $\mathrm{div}_\Sigma u = 0$ yield

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{E}(t) &= \int_\Sigma (\partial_t u | u) \, \mathrm{d}\Sigma \\ &= \int_\Sigma \left(-\left(P_\Sigma(u \cdot \nabla_\Sigma u) \big| u\right) + 2\mu_s \left(P_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u) \big| u\right) \right. \\ &\quad \left. - \left(P_\Sigma \mathrm{div}_\Sigma(\pi \mathcal{P}_\Sigma) \big| u\right)\right) \mathrm{d}\Sigma \\ &= -2\mu_s \int_\Sigma |\mathcal{D}_\Sigma(u)|^2. \end{aligned}$$

$\square$

We now characterize the equilibria of (1.4). It will turn out that at equilibrium, the gradient of the pressure is completely determined by the velocity. Moreover, the equilibrium velocity fields correspond exactly to the Killing fields of Σ.

Proposition 2.2.

(a) *Let* $\mathfrak{E} := \{(u, \pi) \in C^2(\Sigma, \mathsf{T}\Sigma) \times C^1(\Sigma) : (u, \pi) \text{ is an equilibrium for (1.4)}\}$. *Then,*

$$\mathfrak{E} = \left\{ (u, \pi) : \operatorname{div}_{\Sigma} u = 0, \ \mathcal{D}_{\Sigma}(u) = 0, \ \pi = \frac{1}{2} |u|^2 + c \right\},$$

where c is an arbitrary constant.

(b) *Suppose* $\mathcal{D}_{\Sigma}(u) = 0$ *for* $u \in C^1(\Sigma, \mathsf{T}\Sigma)$. *Then,* $\operatorname{div}_{\Sigma} u = 0$.

(c) *The* C^1*-tangential fields satisfying the relation* $\mathcal{D}_{\Sigma}(u) = 0$ *correspond exactly to the Killing fields of* Σ.

Proof. (a) Suppose (u, π) is an equilibrium of (1.4). By the second line in (1.4), $\operatorname{div}_{\Sigma} u = 0$. It follows from Proposition 2.1 that $\mathcal{D}_{\Sigma}(u) = 0$ and the first line in (1.4) then implies

$$\mathcal{P}_{\Sigma}(u \cdot \nabla_{\Sigma} u) + \nabla_{\Sigma} \pi = 0.$$

This, together with $\mathcal{D}_{\Sigma}(u) = 0$ and Lemma A.1(b),(c), yields

$$\nabla_{\Sigma} \pi = -\mathcal{P}_{\Sigma}(u \cdot \nabla_{\Sigma} u) = -\mathcal{P}_{\Sigma}((\nabla_{\Sigma} u)^{\mathsf{T}} u) = \mathcal{P}_{\Sigma}((\nabla_{\Sigma} u) u) = \frac{1}{2} \nabla_{\Sigma} |u|^2.$$

Analogous arguments show that the inverse implication also holds true.

(b) This is a direct consequence of (A.17)₃.

(c) This follows from Remark A.3(e).

$\square$

Suppose that (u, π) is an equilibrium solution of (1.4). Then, we point out the following interesting observation.

Let $\gamma(s)$ be the trajectory of a fluid particle on Σ. Then, γ satisfies the differential equation

$$\dot{\gamma}(s) = u(\gamma(s)), \quad s \in \mathbb{R}, \quad \gamma(0) = \gamma_0 \in \Sigma.$$

Using the assertion in Proposition 2.2(a), we obtain

$$\frac{d}{ds}(\pi \circ \gamma)(s) = (\nabla_{\Sigma} \pi | u)(\gamma(s)) = \frac{1}{2} (\nabla_{\Sigma} |u|^2 | u)(\gamma(s))$$
$$= ((\nabla_{\Sigma} u)^{\mathsf{T}} u | u)(\gamma(s)) = (\mathcal{D}_{\Sigma}(u) u | u)(\gamma(s)) = 0,$$

as $\mathcal{D}_{\Sigma}(u) = 0$. Hence, π is constant along stream lines of the flow.

Furthermore,

$$\ddot{\gamma}(s) = u(\gamma(s)) \cdot \nabla_{\Sigma} u(\gamma(s)) = u^i(\gamma(s))(\partial_i u)(\gamma(s))$$
$$= u^i(\gamma(s))(\partial_i u_j - \Lambda_{ij}^k u_k)(\gamma(s))\tau^j(\gamma(s)) + (l_i^j u^i u_j)(\gamma(s)) \nu_{\Sigma}(\gamma(s))$$
$$= [\mathcal{P}_{\Sigma}(u \cdot \nabla_{\Sigma} u)](\gamma(s)) + [(L_{\Sigma} u | u) \nu_{\Sigma}](\gamma(s)).$$

A short computation shows that

$$(\nabla_\Sigma P_\Sigma)u = l_i^j(\tau_j \otimes \nu_\Sigma)u^i,$$

for tangential vector fields u, hence $(u|\nabla_\Sigma P_\Sigma u) = (L_\Sigma u|u)\nu_\Sigma$. Therefore, we obtain the relation

$$\begin{aligned}
\ddot{\gamma}(s) &= [P_\Sigma(u \cdot \nabla_\Sigma u)](\gamma(s)) + [(u|\nabla_\Sigma P_\Sigma u)](\gamma(s)) \\
&= -[\nabla_\Sigma \pi](\gamma(s)) + [(u|\nabla_\Sigma P_\Sigma u)](\gamma(s)),
\end{aligned} \tag{2.3}$$

since $\nabla_\Sigma \pi = -P_\Sigma(u \cdot \nabla_\Sigma u)$ in an equilibrium. Let us compare this ODE with the following second-order system with constraints:

$$\ddot{x} = f(x, \dot{x}), \quad g(x) = 0.$$

Here, $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d-m}$ are smooth with rank $g'(x) = d - m$ for each $x \in g^{-1}(0)$. In general, the ODE $\ddot{x} = f(x, \dot{x})$ does not leave $\Sigma := g^{-1}(0)$ invariant. However, it can be shown that the movement of a particle under the constraint $g(x) = 0$ in the force field f is governed by the ODE

$$\ddot{x} = P_\Sigma(x)f(x, \dot{x}) + (\dot{x}|\nabla_\Sigma P_\Sigma(x)\dot{x}). \tag{2.4}$$

In other words, the effective force field is the sum of the tangential part of f on Σ and the constraint force $(\dot{x}|\nabla_\Sigma P_\Sigma(x)\dot{x})$, which results from the geodesic flow, see also [17, Section 13.5].

Observe that the structure of (2.3) and (2.4) is the same. Of course, in our situation, it follows from the ODE $\dot{\gamma} = u(\gamma)$ that $\gamma(s) \in \Sigma$, $\dot{\gamma}(s) \in \mathsf{T}_{\gamma(s)}\Sigma$, provided $\gamma(0) = \gamma_0 \in \Sigma$, since u is a tangential vector field on Σ.

Finally, note that the energy $E(s) := \frac{1}{2}|\dot{\gamma}(s)|^2 + \pi(\gamma(s))$ is conserved, i.e., $\dot{E}(s) = 0$, since $(u|\nabla_\Sigma P_\Sigma u)$ is perpendicular to $\mathsf{T}\Sigma$ and $\dot{\gamma} \in \mathsf{T}\Sigma$. This known as Bernoulli's principle.

3. Existence of solutions

In this section, we show that there exists a unique solution

$$\begin{aligned}
&u \in H_{p,\mu}^1((0, a); L_q(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}((0, a); H_q^2(\Sigma, \mathsf{T}\Sigma)), \\
&\pi \in L_{p,\mu}((0, a); \dot{H}_q^1(\Sigma))
\end{aligned}$$

of (1.4) resp. (1.5) for some suitable number $a > 0$. To this end, we first consider the principal linearization of (1.5) and show that the corresponding linear operator has L_p-L_q-maximal regularity in suitable function spaces. This will enable us to apply the contraction mapping principle to prove the existence and uniqueness of a strong solution to (1.5).

3.1. The principal linearization

We consider the following linear problem

$$\begin{aligned}
\partial_t u + \omega u - \mu_s \Delta_\Sigma u + \nabla_\Sigma \pi &= f && \text{on } \Sigma \\
\operatorname{div}_\Sigma u &= g && \text{on } \Sigma \\
u(0) &= u_0 && \text{on } \Sigma,
\end{aligned} \tag{3.1}$$

where $\omega > 0$. Here and in the sequel, we assume without loss of generality that $\varrho = 1$. The main result of this section reads as follows.

Theorem 3.1. *Suppose Σ is a smooth, compact, connected, embedded (oriented) hypersurface in $\mathbb{R}^{d+1}$ without boundary and let $1 < p, q < \infty$, $\mu \in (1/p, 1]$. Then, there exists $\omega_0 > 0$ such that for each $\omega > \omega_0$, problem (3.1) admits a unique solution*

$$u \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\Sigma, \mathsf{T}\Sigma)) =: \mathbb{E}_{1,\mu}(\Sigma),$$
$$\pi \in L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\Sigma)),$$

if and only if the data (f, g, u_0) are subject to the following conditions

(1) $f \in L_{p,\mu}(\mathbb{R}_+; L_q(\Sigma, \mathsf{T}\Sigma)) =: \mathbb{E}_{0,\mu}(\Sigma)$
(2) $g \in L_{p,\mu}(\mathbb{R}_+; H^1_q(\Sigma))$, $g \in H^1_{p,\mu}(\mathbb{R}_+; \dot{H}^{-1}_q(\Sigma))$
(3) $u_0 \in B^{2\mu-2/p}_{qp}(\Sigma, \mathsf{T}\Sigma)$
(4) $g(0) = \operatorname{div}_\Sigma u_0$.

Moreover, the solution (u, π) depends continuously on the given data (f, g, u_0) in the corresponding spaces.

Remark 3.2.

(a) In Theorem 3.1, we use the notations

$$\dot{H}^1_q(\Sigma) := \{w \in L_{1,loc}(\Sigma) : \nabla_\Sigma w \in L_q(\Sigma, \mathsf{T}\Sigma)\}, \quad \dot{H}^{-1}_q(\Sigma) := (\dot{H}^1_{q'}(\Sigma))^*$$

and we identify g with the functional $[\phi \mapsto \int_\Sigma g\phi \, d\Sigma]$ on $\dot{H}^1_{q'}(\Sigma)$.
(b) Note that the assumption $g \in \dot{H}^{-1}_q(\Sigma)$ includes the condition $\int_\Sigma g \, d\Sigma = 0$.
(c) The assertion $\pi \in L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\Sigma))$ means that π is unique up to a constant.
(d) Necessity of the conditions (1)–(4) in Theorem 3.1 is well known, we refer the reader, e.g., to the monograph [13, Chapter 7].

3.2. Pressure Regularity

It is a remarkable fact that the pressure π has additional time-regularity in some special cases.

Proposition 3.3. *In the situation of Theorem 3.1, assume further*

$$u_0 = 0, \quad g = 0, \quad \operatorname{div}_\Sigma f = 0 \quad \text{on } \Sigma.$$

Then, $P_0\pi \in {}_0H^\alpha_{p,\mu}(\mathbb{R}_+; L_q(\Sigma))$, *for* $\alpha \in (0, 1/2]$, *where*

$$P_0 v := v - \frac{1}{|\Sigma|} \int_\Sigma v \, d\Sigma$$

for $v \in L_1(\Sigma)$. *Furthermore, there exists a constant* $C > 0$ *such that the estimate*

$$|P_0\pi|_{L_{p,\mu}(L_q(\Sigma))} \leq C|u|_{L_{p,\mu}(H^1_q(\Sigma))}$$

is valid.

Proof. Let $\phi \in L_{q'}(\Sigma)$, $1/q + 1/q' = 1$ and solve the equation

$$\Delta^L_\Sigma \psi = P_0\phi \quad \text{on } \Sigma.$$

Here, Δ^L_Σ denotes the (scalar) Laplace-Beltrami operator on Σ. This yields a unique solution $\psi \in H^2_{q'}(\Sigma)$ with

$$|\nabla_\Sigma \psi|_{L_{q'}(\Sigma)} + |\nabla^2_\Sigma \psi|_{L_{q'}(\Sigma)} \leq C|\phi|_{L_{q'}(\Sigma)}.$$

This follows, for instance, from [13, Theorem 6.4.3 (i)], and the fact 0 is in the resolvent set of Δ^L_Σ, acting on functions with zero average. We then obtain from the surface divergence theorem (A.14), (A.17), Proposition A.2 (a), and the fact that $(\text{div}_\Sigma f, g) = 0$

$$
\begin{aligned}
(\pi | P_0\phi)_\Sigma = (\pi | \Delta^L_\Sigma \psi)_\Sigma &= (\pi | \text{div}_\Sigma (\nabla_\Sigma \psi))_\Sigma = -(\nabla_\Sigma \pi | \nabla_\Sigma \psi)_\Sigma \\
&= (\partial_t u + \omega u - f | \nabla_\Sigma \psi)_\Sigma - \mu_s (\Delta_\Sigma u | \nabla_\Sigma \psi)_\Sigma \\
&= \mu_s ((\kappa_\Sigma L_\Sigma - L^2_\Sigma) u | \nabla_\Sigma \psi)_\Sigma - 2\mu_s (P_\Sigma \text{div}_\Sigma \mathcal{D}_\Sigma(u) | \nabla_\Sigma \psi)_\Sigma \\
&= \mu_s ((\kappa_\Sigma L_\Sigma - L^2_\Sigma) u | \nabla_\Sigma \psi)_\Sigma + 2\mu_s \int_\Sigma \mathcal{D}_\Sigma(u) : \nabla^2_\Sigma \psi \, d\Sigma,
\end{aligned}
$$

where $(\cdot|\cdot)_\Sigma$ denotes the inner product in $L_2(\Sigma)$ or $L_2(\Sigma, \mathsf{T}\Sigma)$.

Noting that $\mathcal{D}_\Sigma(u) \in {}_0H^{1/2}_{p,\mu}(\mathbb{R}_+; L_q(\Sigma))$, we may apply the fractional time-derivative ∂^α_t to the result

$$(\partial^\alpha_t \pi | P_0\phi)_\Sigma = \mu_s ((\kappa_\Sigma L_\Sigma - L^2_\Sigma) \partial^\alpha_t u | \nabla_\Sigma \psi)_\Sigma + 2\mu_s \int_\Sigma \partial^\alpha_t \mathcal{D}_\Sigma(u) : \nabla^2_\Sigma \psi \, d\Sigma,$$

since ∂^α_t and L_Σ commute. This yields the claim. $\qquad\square$

Without loss of generality, we may always assume that $(\text{div}_\Sigma f, g, u_0) = 0$. To see this, let (u, π) be a solution of (3.1) and solve the parabolic problem

$$
\begin{aligned}
\partial_t v + \omega v - \mu_s \Delta_\Sigma v &= f \quad \text{on } \Sigma \\
v(0) &= u_0 \quad \text{on } \Sigma,
\end{aligned}
\tag{3.2}
$$

by [13, Theorem 6.4.3 (ii)] to obtain a unique solution $v \in \mathbb{E}_{1,\mu}(\Sigma)$. Next, we solve $\Delta_\Sigma^L \Phi = \mathrm{div}_\Sigma v - g$ in $\dot{H}_q^{-1}(\Sigma)$ by [13, Theorem 6.4.3 (i)] to obtain a solution Φ such that $\nabla_\Sigma \Phi$ is unique with regularity

$$\nabla_\Sigma \Phi \in {}_0 H_{p,\mu}^1(\mathbb{R}_+; L_q(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; H_q^2(\Sigma, \mathsf{T}\Sigma)).$$

Note that $\nabla_\Sigma \Phi(0) = 0$ by the compatibility condition $\mathrm{div}_\Sigma u_0 = g(0)$. Define

$$\tilde{u} = u - v + \nabla_\Sigma \Phi \quad \text{and} \quad \tilde{\pi} = \pi - (\partial_t + \omega)\Phi + \psi,$$

where $\nabla_\Sigma \psi \in L_q(\Sigma, \mathsf{T}\Sigma)$ is the unique solution of

$$(\nabla_\Sigma \psi | \nabla_\Sigma \phi)_\Sigma = \mu_s (\Delta_\Sigma \nabla_\Sigma \Phi | \nabla_\Sigma \phi)_\Sigma, \quad \phi \in \dot{H}_{q'}^1(\Sigma).$$

Then, $(\tilde{u}, \tilde{\pi})$ solves (3.1) with $(\mathrm{div}_\Sigma f, g, u_0) = 0$.

Of course, the converse is also true. If $(\tilde{u}, \tilde{\pi})$ solves (3.1) with $(\mathrm{div}_\Sigma f, g, u_0) = 0$, then one may construct a solution (u, π) of (3.1) with prescribed data (f, g, u_0) being subject to the conditions in Theorem 3.1, by reversing the above procedure.

3.3. Localization

In this subsection, we prove the existence and uniqueness of a solution to (3.1). We start with the proof of uniqueness. To this end, let (u, π) be a solution of (3.1) with $(\mathrm{div}_\Sigma f, g, u_0) = 0$.

By compactness of Σ, there exists a family of charts $\{(U_k, \varphi_k) : k \in \{1, \dots, N\}\}$ such that $\{U_k\}_{k=1}^N$ is an open covering of Σ. Let $\{\psi_k\}_{k=1}^N \subset C^\infty(\Sigma)$ be a partition of unity subordinate to the open covering $\{U_k\}_{k=1}^N$. Note that without loss of generality, we may assume that $\varphi_k(U_k) = B_{\mathbb{R}^d}(0, r)$. We call $\{(U_k, \varphi_k, \psi_k) : k \in \{1, \dots, N\}\}$ a *localization system* for Σ.

Let $\{\tau_{(k)j}(p)\}_{j=1}^d$ denote a local basis of the tangent space $\mathsf{T}_p\Sigma$ of Σ at $p \in U_k$ and denote by $\{\tau_{(k)}^j(p)\}_{j=1}^d$ the corresponding dual basis of the cotangent space $\mathsf{T}_p^*\Sigma$ at $p \in U_k$. Accordingly, we define $g_{(k)}^{ij} = (\tau_{(k)}^i | \tau_{(k)}^j)$ and $g_{(k)ij}$ is defined in a very similar way, see also Appendix. Then, with $\bar{u} = u \circ \varphi_k^{-1}$, $\bar{\pi} = \pi \circ \varphi_k^{-1}$ and so on, the system (3.1) with respect to the local charts (U_k, φ_k), $k \in \{1, \dots, N\}$, reads as follows.

$$\partial_t \bar{u}_{(k)}^\ell + \omega \bar{u}_{(k)}^\ell - \mu_s \bar{g}_{(k)}^{ij} \partial_i \partial_j \bar{u}_{(k)}^\ell + \bar{g}_{(k)}^{i\ell} \partial_i \bar{\pi}_{(k)} = \bar{f}_{(k)}^\ell + F_{(k)}^\ell(\bar{u}, \bar{\pi}) \quad \text{in } \mathbb{R}^d$$

$$\partial_i \bar{u}_{(k)}^i = H_{(k)}(\bar{u}) \quad \text{in } \mathbb{R}^d \quad (3.3)$$

$$\bar{u}_{(k)}^\ell(0) = 0 \quad \text{in } \mathbb{R}^d,$$

where

$$\bar{u}_{(k)}^\ell = (\bar{u}\bar{\psi}_k | \bar{\tau}_{(k)}^\ell), \quad \bar{\pi}_{(k)} = \bar{\pi}\bar{\psi}_k,$$

$$\bar{f}_{(k)}^\ell = (\bar{f}\bar{\psi}_k | \bar{\tau}_{(k)}^\ell), \quad F_{(k)}^\ell(\bar{u}, \bar{\pi}) = \bar{\pi}\bar{g}_{(k)}^{i\ell} \partial_i \bar{\psi}_k + (B_{(k)}\bar{u} | \bar{\tau}_{(k)}^\ell), \quad (3.4)$$

$\ell \in \{1, \ldots, d\}$, $B_{(k)}$ is a linear differential operator of order one and

$$H_{(k)}(\bar{u}) = \bar{u}^i \partial_i \bar{\psi}_k - \bar{u}^j_{(k)}(\bar{\tau}^i_{(k)} | \partial_i \bar{\tau}_{(k)j}).$$

Here, upon translation and rotation, $\bar{g}^{ij}_{(k)}(0) = \delta^i_j$ and the coefficients have been extended in such a way that $|\bar{g}^{ij}_{(k)} - \delta^i_j|_{L_\infty(\mathbb{R}^d)} \leq \eta$, where $\eta > 0$ can be made as small as we wish, by decreasing the radius $r > 0$ of the ball $B_{\mathbb{R}^d}(0, r)$.

In order to handle system (3.3), we define vectors in $\mathbb{R}^d$ as follows:

$$\bar{u}_{(k)} := (\bar{u}^1_{(k)}, \ldots, \bar{u}^d_{(k)}), \quad \bar{f}_{(k)} := (\bar{f}^1_{(k)}, \ldots, \bar{f}^d_{(k)})$$

and

$$F_{(k)}(\bar{u}, \bar{\pi}) := (F^1_{(k)}(\bar{u}, \bar{\pi}), \ldots, F^d_{(k)}(\bar{u}, \bar{\pi})).$$

Moreover, we define the matrix $G_{(k)} = (\bar{g}^{ij}_{(k)})^d_{i,j=1} \in \mathbb{R}^{d \times d}$. With these notations, system (3.3) reads as

$$\begin{aligned}
\partial_t \bar{u}_{(k)} + \omega \bar{u}_{(k)} - \mu_s(G_{(k)}\nabla | \nabla)\bar{u}_{(k)} + G_{(k)}\nabla \bar{\pi}_{(k)} = \bar{f}_{(k)} + F_{(k)}(\bar{u}, \bar{\pi}) \quad &\text{in } \mathbb{R}^d \\
\operatorname{div} \bar{u}_{(k)} = H_{(k)}(\bar{u}) \quad &\text{in } \mathbb{R}^d \\
\bar{u}_{(k)}(0) = 0 \quad &\text{in } \mathbb{R}^d.
\end{aligned}$$
(3.5)

For each $k \in \{1, \ldots, N\}$, we define operators $L_{k,\omega}$ by the first two lines on the left side of (3.5). Then, each operator is invertible and bounded. This can be seen by first freezing the coefficients at $x = 0$, leading to full-space Stokes problems, which enjoy the property of L_p-L_q-maximal regularity by [13, Theorem 7.1.1]. Secondly, a Neumann series argument yields the claim, since $G_{(k)}$ is a perturbation of the identity in $\mathbb{R}^{d \times d}$. With the operator $L_{k,\omega}$ at hand, we may rewrite (3.5) in the more condensed form

$$L_{k,\omega}(\bar{u}_{(k)}, \bar{\pi}_{(k)}) = (F_{(k)}, H_{(k)}) + (\bar{f}_{(k)}, 0),$$

Next, we remove the term $H_{(k)}$, since it is not of lower order. For that purpose, solve the equation $\operatorname{div}(G_{(k)}\nabla \phi_k) = H_{(k)}(\bar{u})$. Since $\int_{\mathbb{R}^d} H_{(k)}(\bar{u})\mathrm{d}x = 0$ ($H_{(k)}(\bar{u})$ is compactly supported), there exists a solution ϕ_k such that $\nabla \phi_k$ is unique, with regularity

$$\nabla \phi_k \in {}_0H^1_{p,\mu}(\mathbb{R}_+; H^1_q(\mathbb{R}^d)^d) \cap L_{p,\mu}(\mathbb{R}_+; H^3_q(\mathbb{R}^d)^d).$$

Moreover, we have the estimates

$$\begin{aligned}
|\nabla \phi_k|_{L_{p,\mu}(H^1_q(\mathbb{R}^d))} &\leq C|\bar{u}|_{\mathbb{E}_{0,\mu}(\mathbb{R}^d)} \\
|\nabla \phi_k|_{\mathbb{E}_{1,\mu}(\mathbb{R}^d)} + |\nabla^2 \phi_k|_{\mathbb{E}_{1,\mu}(\mathbb{R}^d)} &\leq C|\bar{u}|_{\mathbb{E}_{1,\mu}(\mathbb{R}^d)} \\
|\nabla \phi_k|_{L_{p,\mu}(H^2_q(\mathbb{R}^d))} &\leq C\omega^{-1/2}(\omega|\bar{u}|_{\mathbb{E}_{0,\mu}(\mathbb{R}^d)} + |\bar{u}|_{\mathbb{E}_{1,\mu}(\mathbb{R}^d)}),
\end{aligned}$$
(3.6)

see also [13, (7.41)]. Define

$$\tilde{u}_{(k)} = \bar{u}_{(k)} - G_{(k)} \nabla \phi_k - u_{(k)}^0 \quad \text{and} \quad \tilde{\pi}_{(k)} = \bar{\pi}_{(k)} + (\partial_t + \omega)\phi_k - \Phi_k - \pi_{(k)}^0,$$

where $(u_{(k)}^0, \pi_{(k)}^0) = L_{k,\omega}^{-1}(\bar{f}_{(k)}, 0)$ and Φ_k satisfies $\operatorname{div}(G_{(k)} \nabla \Phi_k) = \operatorname{div} \tilde{F}_{(k)}(\bar{u}, \bar{\pi})$ in $\dot{H}_q^{-1}(\mathbb{R}^d)$, with

$$\tilde{F}_{(k)}(\bar{u}, \bar{\pi}) := F_{(k)}(\bar{u}, \bar{\pi}) + \mu_s(G_{(k)} \nabla | \nabla)(G_{(k)} \nabla \phi_k).$$

The couple $(\tilde{u}_{(k)}, \tilde{\pi}_{(k)})$ then solves the equation

$$L_{k,\omega}(\tilde{u}_{(k)}, \tilde{\pi}_{(k)}) = (\tilde{F}_{(k)}(\bar{u}, \bar{\pi}) - G_{(k)} \nabla \Phi_k, 0).$$

We note on the go that $\operatorname{div}(\tilde{F}_{(k)}(\bar{u}, \bar{\pi}) - G_{(k)} \nabla \Phi_k) = 0$ and that the pressure $\tilde{\pi}_{(k)}$ enjoys additional time regularity. This can be seen exactly as in the proof of [13, Proposition 7.3.5 (ii)] with an obvious modification concerning the matrix $G_{(k)}$. In particular, there exists a constant $C > 0$, such that

$$|P_0 \tilde{\pi}_{(k)}|_{L_{p,\mu}(L_q(B_{\mathbb{R}^d}(0,r)))} \leq C |\tilde{u}_{(k)}|_{L_{p,\mu}(H_q^1(\mathbb{R}^d))}. \tag{3.7}$$

Let us now introduce a norm for the solution, taking the parameter ω into account. Set

$$\|(\bar{u}_{(k)}, \bar{\pi}_{(k)})\|_\omega = \omega |\bar{u}_{(k)}|_{\mathbb{E}_{0,\mu}(\mathbb{R}^d)} + |\bar{u}_{(k)}|_{\mathbb{E}_{1,\mu}(\mathbb{R}^d)} + |\nabla \bar{\pi}_{(k)}|_{\mathbb{E}_{0,\mu}(\mathbb{R}^d)},$$

and similarly for $\|(u, \pi)\|_\omega$ on Σ.

For each $k \in \{1, \dots, N\}$, there exists $\omega_0 > 0$ such that the operator $L_{k,\omega}$ has the property of L_p-L_q-maximal regularity, provided $\omega > \omega_0$. In particular, there exists a constant $C > 0$ such that

$$\|(u_{(k)}^0, \pi_{(k)}^0)\|_\omega \leq C |\bar{f}_{(k)}|_{\mathbb{E}_{0,\mu}(\mathbb{R}^d)} \leq C |f|_{\mathbb{E}_{0,\mu}(\Sigma)}$$

and

$$\|(\tilde{u}_{(k)}, \tilde{\pi}_{(k)})\|_\omega \leq C \omega^{-\gamma} \|(u, \pi)\|_\omega,$$

where for the last inequality, we made use of Proposition 3.3, implying the estimate

$$\|\bar{\pi} G_{(k)} \nabla \bar{\psi}_k\|_{\mathbb{E}_{0,\mu}(\mathbb{R}^d)} \leq C \omega^{-\gamma} (\omega |u|_{\mathbb{E}_{0,\mu}(\Sigma)} + |u|_{\mathbb{E}_{1,\mu}(\Sigma)}),$$

for some constant $\gamma > 0$, by interpolation between

$$L_{p,\mu}(\mathbb{R}_+; L_q(\Sigma)) \quad \text{and} \quad L_{p,\mu}(\mathbb{R}_+; H_q^2(\Sigma)).$$

In the same way, making also use of (3.6) and the definition of Φ_k, we obtain

$$|\nabla \Phi_k|_{\mathbb{E}_{0,\mu}(\mathbb{R}^d)} \leq C \omega^{-\gamma} \|(u, \pi)\|_\omega.$$

Furthermore, we have

$$
\begin{aligned}
|(\partial_t + \omega)\phi_k|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} &\leq |\tilde{\pi}_{(k)}|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} + |\bar{\pi}_{(k)}|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} \\
&\quad + |\pi_{(k)}^0 + \Phi_k|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} \\
&\leq |\tilde{\pi}_{(k)}|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} + |\bar{\pi}_{(k)}|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} \\
&\quad + C(|\nabla\pi_{(k)}^0|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} + |\nabla\Phi_k|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))}) \\
&\leq C(|f|_{\mathbb{E}_{0,\mu}(\Sigma)} + \omega^{-\gamma}\|(u,\pi)\|_\omega),
\end{aligned}
$$

by (3.7), Proposition 3.3 for $\bar{\pi}$ and the Poincaré inequality for $\pi_{(k)}^0$ and Φ_k, since we may assume without loss of generality, that $\pi_{(k)}^0$ as well as Φ_k have mean value zero on $B_{\mathbb{R}^d}(0,r)$. By interpolation with (3.6), this yields

$$
|(\partial_t + \omega)\nabla\phi_k|_{\mathbb{E}_{0,\mu}(B_{\mathbb{R}^d}(0,r))} \leq C(|f|_{\mathbb{E}_{0,\mu}(\Sigma)} + \omega^{-\gamma/2}\|(u,\pi)\|_\omega).
$$

In conclusion, we obtain the estimate

$$
\|\bar{\chi}_k(\bar{u}_{(k)}, \bar{\pi}_{(k)})\|_\omega \leq C(|f|_{\mathbb{E}_{0,\mu}(\Sigma)} + \omega^{-\gamma}\|(u,\pi)\|_\omega)
$$

valid for each $k \in \{1, \ldots, N\}$ and $C > 0$ does not depend on $\omega > 0$. Here, $\{\chi_k\}_{k=1}^N \subset C^\infty(\Sigma)$ such that $\chi_k = 1$ on $\mathrm{supp}(\psi_k)$ and $\mathrm{supp}(\chi_k) \subset U_k$. As usual, we have set $\bar{\chi}_k = \chi_k \circ \varphi_k^{-1}$.

For the components $\bar{u}_{(k)}^\ell \circ \varphi_k$ of the vector $\bar{u}_{(k)} \circ \varphi_k \in \mathbb{R}^d$, we derive from (3.4)

$$
\bar{u}_{(k)}^\ell \circ \varphi_k = (u\psi_k | \tau_{(k)}^\ell) = (u^j \tau_{(k)j} \psi_k | \tau_{(k)}^\ell) = u^\ell \psi_k,
$$

hence

$$
u = \sum_{k=1}^N \psi_k u = \sum_{k=1}^N \psi_k u^\ell \tau_{(k)\ell} = \sum_{k=1}^N (\bar{u}_{(k)}^\ell \bar{\tau}_{(k)\ell}) \circ \varphi_k.
$$

Since $\chi_k = 1$ on $\mathrm{supp}(\psi_k)$, this finally yields the estimate

$$
\|(u,\pi)\|_\omega \leq C(|f|_{\mathbb{E}_{0,\mu}(\Sigma)} + \omega^{-\gamma}\|(u,\pi)\|_\omega)
$$

valid for all $\omega > \omega_0$. Choosing $\omega_0 > 0$ sufficiently large, we conclude

$$
\|(u,\pi)\|_\omega \leq C|f|_{\mathbb{E}_{0,\mu}(\Sigma)}.
$$

This in turn implies uniqueness of a solution to (3.1).

It remains to prove the existence of a solution to (3.1). To this end, we may assume that $(\mathrm{div}_\Sigma f, g, u_0) = 0$. Solve the parabolic problem

$$
\begin{aligned}
\partial_t v + \omega v - \mu_s \Delta_\Sigma v &= f \quad \text{on } \Sigma \\
v(0) &= 0 \quad \text{on } \Sigma,
\end{aligned}
\tag{3.8}
$$

by [13, Theorem 6.4.3 (ii)] to obtain a unique solution $v \in {}_0\mathbb{E}_{1,\mu}(\Sigma)$. Next, we solve $\Delta_\Sigma^L \phi = \operatorname{div}_\Sigma v$ by [13, Theorem 6.4.3 (i)] to obtain a solution ϕ such that $\nabla_\Sigma \phi$ is unique with regularity

$$\nabla_\Sigma \phi \in {}_0H^1_{p,\mu}(\mathbb{R}_+; L_q(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\Sigma, \mathsf{T}\Sigma)).$$

Define $\tilde{u} = v - \nabla_\Sigma \phi$ and $\tilde{\pi} = (\partial_t + \omega)\phi$. It follows that

$$L_\omega(\tilde{u}, \tilde{\pi}) = f + \mu_s \Delta_\Sigma \nabla_\Sigma \phi,$$

where

$$L_\omega : \{u \in {}_0\mathbb{E}_{1,\mu}(\Sigma) : \operatorname{div}_\Sigma u = 0\} \times L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\Sigma)) \to \mathbb{E}_{0,\mu}(\Sigma)$$

is defined by

$$L_\omega(u, \pi) = \partial_t u + \omega u - \mu_s \Delta_\Sigma u + \nabla_\Sigma \pi.$$

Making use of local coordinates, one can show that

$$\Delta_\Sigma \nabla_\Sigma \phi = \nabla_\Sigma \Delta_\Sigma^L \phi + \mathcal{A}_\Sigma \phi,$$

where $\mathcal{A}_\Sigma$ is a second-order operator. Setting $\hat{u} = \tilde{u}$, $\hat{\pi} = \tilde{\pi} - \mu_s \Delta_\Sigma^L \phi$ and $Sf :=$ $(\hat{u}, \hat{\pi})$, we obtain

$$L_\omega Sf = L_\omega(\hat{u}, \hat{\pi}) = f + Rf,$$

with $Rf := \mu_s \mathcal{A}_\Sigma \phi$. Since $\mathcal{A}_\Sigma$ is of second order, this yields

$$
\begin{aligned}
|Rf|_{\mathbb{E}_{0,\mu}(\Sigma)} = |\mu_s \mathcal{A}_\Sigma \phi|_{\mathbb{E}_{0,\mu}(\Sigma)} &\leq C|\phi|_{L_{p,\mu}(\mathbb{R}_+; H^2_q(\Sigma))} \\
&= |(\Delta_\Sigma^L)^{-1} \Delta_\Sigma^L \phi|_{L_{p,\mu}(\mathbb{R}_+; H^2_q(\Sigma))} \leq C|\operatorname{div}_\Sigma v|_{\mathbb{E}_{0,\mu}(\Sigma)} \\
&\leq \omega^{-1/2} C|f|_{\mathbb{E}_{0,\mu}(\Sigma)},
\end{aligned}
$$

where the constant $C > 0$ does not depend on ω. We note that $(\Delta_\Sigma^L)^{-1}$, acting on functions with average zero, is well defined. A Neumann series argument implies that $(I + R)$ is invertible provided $\omega > 0$ is sufficiently large. Hence, the operator $S(I+R)^{-1}$ is a right inverse for L_ω, which means that L_ω is surjective. This completes the proof of Theorem 3.1.

3.4. The surface Stokes operator

By Proposition A.2, $\mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(u)$ is a lower perturbation of $\Delta_\Sigma u$ if $\operatorname{div}_\Sigma u$ is prescribed. This implies the following result for the system

$$
\begin{aligned}
\partial_t u + \omega u - 2\mu_s \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(u) + \nabla_\Sigma \pi &= f &&\text{on } \Sigma \\
\operatorname{div}_\Sigma u &= g &&\text{on } \Sigma \\
u(0) &= u_0 &&\text{on } \Sigma.
\end{aligned}
\tag{3.9}
$$

Corollary 3.4. *Under the assumptions in Theorem* 3.1, *there exists* $\omega_0 > 0$ *such that for each* $\omega > \omega_0$, *problem* (3.9) *admits a unique solution*

$$u \in H^1_{p,\mu}(\mathbb{R}_+; L_q(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}(\mathbb{R}_+; H^2_q(\Sigma, \mathsf{T}\Sigma)), \quad \pi \in L_{p,\mu}(\mathbb{R}_+; \dot{H}^1_q(\Sigma)),$$

if and only if the data (f, g, u_0) *are subject to the conditions* (1)–(4) *in Theorem* 3.1. *Moreover, the solution* (u, π) *depends continuously on the given data* (f, g, u_0) *in the corresponding spaces.*

Proof. Without loss of generality, we may assume $(\mathrm{div}_\Sigma f, g, u_0) = 0$. With the operator L_ω defined above, we rewrite (3.9) as

$$(u, \pi) = L_\omega^{-1} f + \mu_s L_\omega^{-1}(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u.$$

For the term of order zero on the right hand side, we have the estimate

$$|(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u|_{\mathbb{E}_{0,\mu}(\Sigma)} \le \omega^{-1} C \|(u, \pi)\|_\omega,$$

where $C > 0$ does not depend on $\omega > 0$. By Theorem 3.1, the solution depends continuously on the data; hence, there exists a constant $M = M(\omega_0) > 0$ such that

$$\|L_\omega^{-1}(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u\|_\omega \le M |(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u|_{\mathbb{E}_{0,\mu}(\Sigma)} \le \omega^{-1} M C \|(u, \pi)\|_\omega.$$

Therefore, a Neumann series argument yields the claim, if $\omega > 0$ is chosen sufficiently large. $\qquad\square$

We will now define the Stokes operator on surfaces. Let $P_{H,\Sigma}$ denote the *surface Helmholtz projection*, defined by

$$P_{H,\Sigma} v := v - \nabla_\Sigma \psi, \quad v \in L_q(\Sigma, \mathsf{T}\Sigma),$$

where $\nabla_\Sigma \psi \in L_q(\Sigma, \mathsf{T}\Sigma)$ is the unique solution of

$$(\nabla_\Sigma \psi | \nabla_\Sigma \phi)_\Sigma = (v | \nabla_\Sigma \phi)_\Sigma, \quad \phi \in \dot{H}^1_{q'}(\Sigma).$$

We note that $(P_{H,\Sigma} u | v)_\Sigma = (u | P_{H,\Sigma} v)_\Sigma$ for all $u \in L_q(\Sigma, \mathsf{T}\Sigma)$, $v \in L_{q'}(\Sigma, \mathsf{T}\Sigma)$, which follows directly from the definition of $P_{H,\Sigma}$ (and for smooth functions from the surface divergence theorem (A.14)). Define

$$X_0 := L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma) := P_{H,\Sigma} L_q(\Sigma, \mathsf{T}\Sigma)$$

and $X_1 := H^2_q(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. The *surface Stokes operator* is defined by

$$A_{S,\Sigma} u := -2\mu_s P_{H,\Sigma} \mathcal{P}_\Sigma \mathrm{div}_\Sigma \mathcal{D}_\Sigma(u), \quad u \in D(A_{S,\Sigma}) := X_1. \tag{3.10}$$

We would also like to refer to the survey article [6] for the Stokes operator in various other geometric settings.

Making use of the projection $P_{H,\Sigma}$, (3.9) with $(\mathrm{div}_\Sigma f, g) = 0$ is equivalent to the equation

$$\partial_t u + \omega u + A_{S,\Sigma} u = f, \quad t > 0, \quad u(0) = u_0. \tag{3.11}$$

By Corollary 3.4, the operator $A_{S,\Sigma}$ has L_p-maximal regularity, hence $-A_{S,\Sigma}$ generates an analytic C_0-semigroup in X_0, see for instance [13, Proposition 3.5.2].

3.5. Interpolation spaces

In this subsection, we will determine the real and complex interpolation spaces $(X_0, X_1)_{\alpha, p}$ and $(X_0, X_1)_\alpha$, respectively. To this end, let $A_\Sigma u := -2\mu_s \mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(u)$ with domain $D(A_\Sigma) := H_q^2(\Sigma, \mathsf{T}\Sigma)$ and define a linear mapping Q on $D(A_\Sigma)$ by

$$Q = (\omega + A_{S,\Sigma})^{-1} P_{H,\Sigma}(\omega + A_\Sigma),$$

for some fixed and sufficiently large $\omega > 0$.

Then, $Q : D(A_\Sigma) \to X_1$ is a bounded projection, as $Qu \in X_1$ and

$$Q^2 u = (\omega + A_{S,\Sigma})^{-1} P_{H,\Sigma}(\omega + A_\Sigma) Qu = (\omega + A_{S,\Sigma})^{-1}(\omega + A_{S,\Sigma}) Qu = Qu,$$

for all $u \in D(A_\Sigma)$. Furthermore, $Q|_{X_1} = I_{X_1}$ and therefore $Q : D(A_\Sigma) \to X_1$ is surjective. By a duality argument, there exists some constant $C > 0$ such that

$$\|Qu\|_{L_q(\Sigma)} \le C \|u\|_{L_q(\Sigma)} \tag{3.12}$$

for all $u \in D(A_\Sigma)$. In fact,

$$
\begin{aligned}
(Qu|\phi)_\Sigma &= ((\omega + A_{S,\Sigma})^{-1} P_{H,\Sigma}(\omega + A_\Sigma)u|\phi)_\Sigma \\
&= (P_{H,\Sigma}(\omega + A_{S,\Sigma})^{-1} P_{H,\Sigma}(\omega + A_\Sigma)u|\phi)_\Sigma \\
&= (P_{H,\Sigma}(\omega + A_\Sigma)u|(\omega + A_{S,\Sigma})^{-1} P_{H,\Sigma}\phi)_\Sigma \\
&= (u|(\omega + A_\Sigma)(\omega + A_{S,\Sigma})^{-1} P_{H,\Sigma}\phi)_\Sigma
\end{aligned}
$$

implies

$$|(Qu|\phi)_\Sigma| \le C \|u\|_{L_q(\Sigma)} \|\phi\|_{L_{q'}(\Sigma)}$$

for all $u \in D(A_\Sigma)$ and $\phi \in L_{q'}(\Sigma, \mathsf{T}\Sigma)$, with

$$C := \|(\omega + A_\Sigma)(\omega + A_{S,\Sigma})^{-1} P_{H,\Sigma}\|_{\mathcal{B}(L_{q'}(\Sigma); L_{q'}(\Sigma))} > 0.$$

Since $D(A_\Sigma)$ is dense in $L_q(\Sigma, \mathsf{T}\Sigma)$, there exists a unique bounded extension $\tilde{Q} : L_q(\Sigma, \mathsf{T}\Sigma) \to X_0$ of Q. Clearly, $\tilde{Q}$ is a projection and as X_1 is dense in X_0, $\tilde{Q}|_{X_0} = I_{X_0}$.

It follows that

$$L_q(\Sigma, \mathsf{T}\Sigma) = X_0 \oplus N(\tilde{Q}) \quad \text{and} \quad D(A_\Sigma) = X_1 \oplus [D(A_\Sigma) \cap N(\tilde{Q})]$$

since $\tilde{Q}D(A_\Sigma) = D(A_\Sigma) \cap R(\tilde{Q}) = D(A_{S,\Sigma}) = X_1$. Moreover, with the help of the projection $\tilde{Q}$ and the relation $R(\tilde{Q}) = L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$, we may now compute

$$
\begin{aligned}
(X_0, X_1)_\alpha &= (\tilde{Q}L_q(\Sigma, \mathsf{T}\Sigma), \tilde{Q}D(A_\Sigma))_\alpha \\
&= \tilde{Q}(L_q(\Sigma, \mathsf{T}\Sigma), D(A_\Sigma))_\alpha = H_q^{2\alpha}(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)
\end{aligned}
$$

as well as

$$
\begin{aligned}
(X_0, X_1)_{\alpha, p} &= (\tilde{Q}L_q(\Sigma, \mathsf{T}\Sigma), \tilde{Q}D(A_\Sigma))_{\alpha, p} \\
&= \tilde{Q}(L_q(\Sigma, \mathsf{T}\Sigma), D(A_\Sigma))_{\alpha, p} = B_{qp}^{2\alpha}(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)
\end{aligned}
$$

for $\alpha \in (0, 1)$ and $p \in (1, \infty)$, see [25, Theorem 1.17.1.1].

3.6. Nonlinear well-posedness

We will show that there exists a unique local-in-time solution to (1.5). Observe that the semilinear problem (1.5) is equivalent to the abstract semilinear evolution equation

$$\partial_t u + A_{S,\Sigma} u = F_\Sigma(u), \quad t > 0, \quad u(0) = u_0, \tag{3.13}$$

where $F_\Sigma(u) := -P_{H,\Sigma} \mathcal{P}_\Sigma(u \cdot \nabla_\Sigma u)$. In order to solve this equation in the maximal regularity class $\mathbb{E}_{1,\mu}(\Sigma)$, we will apply Theorem 2.1 in [9]. To this end, let $q \in (1, d)$ and

$$\mu_c := \frac{1}{2}\left(\frac{d}{q} - 1\right) + \frac{1}{p}, \tag{3.14}$$

with $2/p + d/q < 3$, so that $\mu_c \in (1/p, 1)$. We will show that for each $\mu \in (\mu_c, 1]$ there exists $\beta \in (\mu - 1/p, 1)$ with $2\beta - 1 < \mu - 1/p$ such that F_Σ satisfies the estimate

$$|F_\Sigma(u) - F_\Sigma(v)|_{X_0} \leq C(|u|_{X_\beta} + |v|_{X_\beta})|u - v|_{X_\beta} \tag{3.15}$$

for all $u, v \in X_\beta := (X_0, X_1)_{\beta,p}$.

By Hölders inequality, the estimate

$$|F_\Sigma(u)|_{L_q(\Sigma, \mathsf{T}\Sigma)} \leq C|u|_{L_{qr'}(\Sigma, \mathsf{T}\Sigma)}|u|_{H^1_{qr}(\Sigma, \mathsf{T}\Sigma)}$$

holds. We choose $r, r' \in (1, \infty)$ in such a way that

$$1 - \frac{d}{qr} = -\frac{d}{qr'} \quad \text{or equivalently} \quad \frac{d}{qr} = \frac{1}{2}\left(1 + \frac{d}{q}\right),$$

which is feasible if $q \in (1, d)$. Next, by Sobolev embedding, we have

$$(X_0, X_1)_{\beta,p} \subset B^{2\beta}_{qp}(\Sigma, \mathsf{T}\Sigma) \hookrightarrow H^1_{qr}(\Sigma, \mathsf{T}\Sigma) \cap L_{qr'}(\Sigma, \mathsf{T}\Sigma),$$

provided

$$2\beta - \frac{d}{q} > 1 - \frac{d}{qr} \quad \text{or equivalently} \quad \beta > \frac{1}{4}\left(\frac{d}{q} + 1\right).$$

The condition $\beta < 1$ requires $q > d/3$, hence $q \in (d/3, d)$. Note that

$$\frac{1}{2}\left(\frac{d}{q} + 1\right) - 1 < \mu - 1/p,$$

since $\mu > \mu_c$. This implies that $1 > \beta > (d/q + 1)/4$ can be chosen in such a way that the inequalities $2\beta - 1 < \mu - 1/p$ and $\mu - 1/p < \beta$ are satisfied.

In case $q \geq d$, we may choose any $\mu \in (1/p, 1]$, since

$$B^{2\beta}_{qp}(\Sigma, \mathsf{T}\Sigma) \hookrightarrow H^1_q(\Sigma, \mathsf{T}\Sigma) \cap L_\infty(\Sigma, \mathsf{T}\Sigma),$$

provided $2\beta > 1$.

Since F_Σ is bilinear, it follows that the estimate (3.15) holds and, moreover, that $F_\Sigma \in C^\infty(X_\beta, X_0)$. Therefore, Theorem 2.1 in [9] yields the following result.

Theorem 3.5. *Let $p, q \in (1, \infty)$. Suppose that one of the following conditions holds:*

(a) $q \in (d/3, d)$, $2/p + d/q < 3$ and $\mu \in (\mu_c, 1]$, where μ_c is defined in (3.14).

(b) $q \geq d$ and $\mu \in (1/p, 1]$.

Then, for any initial value $u_0 \in B_{qp}^{2\mu-2/p}(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$, there exists a number $a = a(u_0) > 0$ such that (1.5) admits a unique solution

$$u \in H_{p,\mu}^1((0, a); L_q(\Sigma, \mathsf{T}\Sigma)) \cap L_{p,\mu}((0, a); H_q^2(\Sigma, \mathsf{T}\Sigma)),$$
$$\pi \in L_{p,\mu}((0, a); \dot{H}_q^1(\Sigma)).$$

Moreover,

$$u \in C([0, a]; B_{qp}^{2\mu-2/p}(\Sigma, \mathsf{T}\Sigma)) \cap C((0, a]; B_{qp}^{2-2/p}(\Sigma, \mathsf{T}\Sigma)).$$

Remark 3.6.

(a) The number $\mu_c \in (1/p, 1]$ defined in (3.14) is called the critical weight and was introduced in [14–16]. It has been shown in [14] that the 'critical spaces' $(X_0, X_1)_{\mu_c-1/p,p}$ correspond to scaling invariant spaces in case the underlying equations enjoy scaling invariance.

(b) In future work, we plan to show that $A_{S,\Sigma}$ has a bounded H^∞-calculus. Then, one may set $\mu = \mu_c$ in Theorem 3.5, thereby obtaining well-posedness in critical spaces.

(c) In case $d = 2$, global existence has been obtained by Taylor [23, Proposition 6.5]. An alternative proof can be based on the approach via critical spaces mentioned above.

4. Stability of equilibria, examples

Consider the semilinear evolution equation

$$\partial_t u + A_{S,\Sigma} u = F_\Sigma(u), \ t > 0, \quad u(0) = u_0, \tag{4.1}$$

in $X_0 = L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$. Define the set

$$\mathcal{E} := \{u_* \in H_q^2(\Sigma; \mathsf{T}\Sigma) : \mathrm{div}_\Sigma u_* = 0, \ \mathcal{D}_\Sigma(u_*) = 0\}. \tag{4.2}$$

We show that the set $\mathcal{E}$ corresponds exactly to the set of equilibria for (4.1). To this end, let u_* be an equilibrium of (4.1), i.e., $u_* \in X_1$ satisfies $A_{S,\Sigma} u_* = F_\Sigma(u_*)$. Multiplying this equation by u_* and integrating over Σ yields

$$0 = 2\mu_s (\mathrm{div}_\Sigma \mathcal{D}_\Sigma(u_*)|u_*)_\Sigma + (u_* \cdot \nabla_\Sigma u_*|u_*)_\Sigma.$$

By Lemma A.1 and the surface divergence theorem (A.14), the last term vanishes, since $\mathrm{div}_\Sigma u_* = 0$. Furthermore, (A.17) and again (A.14) show that

$$(\mathrm{div}_\Sigma \mathcal{D}_\Sigma(u_*)|u_*)_\Sigma = -\int_\Sigma |\mathcal{D}_\Sigma(u_*)|^2 \, d\Sigma,$$

which implies $\mathcal{D}_\Sigma(u_*) = 0$, hence $u_* \in \mathcal{E}$.

Conversely, let $u_* \in \mathcal{E}$ be given. Then, $A_{S,\Sigma} u_* = 0$ and from Lemma A.1, we obtain that

$$F_\Sigma(u_*) = -P_{H,\Sigma} \mathcal{P}_\Sigma (u_* \cdot \nabla_\Sigma u_*) = \frac{1}{2} P_{H,\Sigma} (\nabla_\Sigma |u_*|^2) = 0.$$

Summarizing, we have shown that

$$\mathcal{E} = \{u_* \in H_q^2(\Sigma; \mathsf{T}\Sigma) : \operatorname{div}_\Sigma u_* = 0, \ A_{S,\Sigma} u_* = F_\Sigma(u_*)\}.$$

Observe that the set $\mathcal{E}$ is a linear manifold, consisting exactly of the *Killing fields* on Σ, see Remark A.3.

Define an operator $A_0 : X_1 \rightarrow X_0$ by

$$A_0 v = A_{S,\Sigma} v - F'_\Sigma(u_*) v, \tag{4.3}$$

where $F'_\Sigma(u_*) v := -P_{H,\Sigma} \mathcal{P}_\Sigma (v \cdot \nabla_\Sigma u_* + u_* \cdot \nabla_\Sigma v)$. This operator is the full linearization of (4.1) at the equilibrium $u_* \in \mathcal{E}$. We collect some properties of A_0 in the following

Proposition 4.1. *Suppose $u_* \in \mathcal{E}$ and let A_0 be given by (4.3). Then $-A_0$ generates a compact analytic C_0-semigroup in X_0 which has L_p-maximal regularity. The spectrum of A_0 consists only of eigenvalues of finite algebraic multiplicity and the kernel $N(A_0)$ is given by*

$$N(A_0) = \{u \in H_q^2(\Sigma, \mathsf{T}\Sigma) : \operatorname{div}_\Sigma u = 0, \ \mathcal{D}_\Sigma(u) = 0\}.$$

If $\mathcal{E} \neq \{0\}$, then $u_* \in \mathcal{E}$ is normally stable, i.e.,

(i) *Re $\sigma(-A_0) \leq 0$ and $\sigma(A_0) \cap i\mathbb{R} = \{0\}$.*
(ii) *$\lambda = 0$ is a semi-simple eigenvalue of A_0.*
(iii) *The kernel $N(A_0)$ is isomorphic to $\mathsf{T}_{u_*}\mathcal{E}$.*

In case $\mathcal{E} = \{0\}$, it holds that Re $\sigma(-A_0) < 0$.

Proof. By Sect. 3.4, the surface Stokes operator $A_{S,\Sigma}$ has the property of L_p-maximal regularity in X_0, hence $-A_{S,\Sigma}$ is $\mathcal{R}$-sectorial in X_0. Furthermore, the linear mapping $[v \mapsto F'_\Sigma(u_*) v]$ is relatively bounded with respect to $A_{S,\Sigma}$. An application of [13, Proposition 4.4.3] yields that $-A_0$ generates an analytic C_0-semigroup in X_0 having L_p-maximal regularity. Since the domain X_1 of A_0 is compactly embedded into X_0, the spectrum $\sigma(A_0)$ is discrete and consists solely of eigenvalues of A_0 having finite algebraic multiplicity.

We first consider the case $\mathcal{E} \neq \{0\}$. Let $\lambda \in \sigma(-A_0)$ and denote by $v \in X_1$ a corresponding eigenfunction. Multiplying the equation $\lambda v + A_0 v = 0$ by the complex conjugate $\bar{v}$ and integrating over Σ yields

$$\operatorname{Re}\lambda |v|_{L_2(\Sigma)}^2 = 2\mu_s \operatorname{Re}(\mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(v)|\bar{v})_\Sigma - \operatorname{Re}(\mathcal{P}_\Sigma(v \cdot \nabla_\Sigma u_*) + \mathcal{P}_\Sigma(u_* \cdot \nabla_\Sigma v)|\bar{v})_\Sigma$$

$$= -2\mu_s \int_\Sigma |\mathcal{D}_\Sigma(v)|^2 \, d\Sigma. \tag{4.4}$$

Here, we used the identities

$$(v \cdot \nabla_\Sigma u_* | \bar{v})_\Sigma = -\overline{(v \cdot \nabla_\Sigma u_* | \bar{v})_\Sigma}$$

and

$$(u_* \cdot \nabla_\Sigma v | \bar{v})_\Sigma = (u_* | \nabla_\Sigma |v|^2)_\Sigma - (u_* | (\nabla_\Sigma \bar{v}) v)_\Sigma$$
$$= -\overline{(u_* \cdot \nabla_\Sigma v | \bar{v})_\Sigma}$$

since $\mathcal{D}_\Sigma(u_*) = 0$ and $\operatorname{div}_\Sigma u_* = 0$, employing the surface divergence theorem (A.14). It follows that $\operatorname{Re}\lambda \leq 0$ and if $\operatorname{Re}\lambda = 0$, then $\mathcal{D}_\Sigma(v) = 0$. Observe that the equations $\mathcal{D}_\Sigma(v) = 0 = \mathcal{D}_\Sigma(u_*)$ then lead to the identity

$$\mathcal{P}_\Sigma(v \cdot \nabla_\Sigma u_* + u_* \cdot \nabla_\Sigma v) = -\nabla_\Sigma(u_* | v),$$

hence $F'_\Sigma(u_*)v = P_{H,\Sigma}(\nabla_\Sigma(u_*|v)) = 0$ and therefore $A_0 v = 0$.

The above calculations show that $\sigma(A_0) \cap i\mathbb{R} = \{0\}$ and

$$N(A_0) = \{v \in X_1 : \mathcal{D}_\Sigma(v) = 0\},$$

wherefore $N(A_0) \cong \mathcal{E} \cong \mathsf{T}_{u_*}\mathcal{E}$.

We will now prove that $\lambda = 0$ is semi-simple. To this end, it suffices to prove that $N(A_0^2) \subset N(A_0)$. Let $w \in N(A_0^2)$ and $v := A_0 w$. Then, $v \in N(A_0)$ and we obtain

$$|v|^2_{L_2(\Sigma)} = (A_0 w | v)_\Sigma = 2\mu_s \int_\Sigma \mathcal{D}_\Sigma(w) : \mathcal{D}_\Sigma(v) \, d\Sigma = 0,$$

by Lemma A.1, (A.17) and the property $\mathcal{D}_\Sigma(v) = 0$, which implies $w \in N(A_0)$.

Finally, we consider the case $\mathcal{E} = \{0\}$. If $\lambda \in \sigma(-A_0)$ with eigenfunction $v \neq 0$, it follows from (4.4) that $\operatorname{Re}\lambda \leq 0$ and if $\operatorname{Re}\lambda = 0$, then $\mathcal{D}_\Sigma(v) = 0$ by (4.4), hence $v \in \mathcal{E} = \{0\}$, a contradiction. Therefore, in this case, $\operatorname{Re}\sigma(-A_0) < 0$. □

Remark 4.2. The above computations show that the operator A_0 from (4.3) is not necessarily symmetric in case $u_* \neq 0$. In fact, we have

$$(A_0 v | \bar{v})_\Sigma = 2\mu_s \int_\Sigma |\mathcal{D}_\Sigma(v)|^2 \, d\Sigma + i \operatorname{Im}[(v \cdot \nabla_\Sigma u_* | \bar{v})_\Sigma + (u_* \cdot \nabla_\Sigma v | \bar{v})_\Sigma], \quad v \in X_1.$$

Since F_Σ is bilinear, we obtain from [13, Theorem 5.3.1] and the proof of Proposition 5.1 as well as Theorem 5.2 in [10] the following result.

Theorem 4.3. *Suppose p, q and μ satisfy the assumptions of Theorem 3.5.*
Then, each equilibrium $u_* \in \mathcal{E}$ is stable in $X_{\gamma,\mu} := B_{qp}^{2\mu-2/p}(\Sigma, \mathsf{T}\Sigma) \cap L_{q,\sigma}(\Sigma, \mathsf{T}\Sigma)$ and there exists $\delta > 0$ such that the unique solution $u(t)$ of (4.1) with initial value $u_0 \in X_{\gamma,\mu}$ satisfying $|u_0 - u_*|_{X_{\gamma,\mu}} < \delta$ exists on $\mathbb{R}_+$ and converges at an exponential rate in $X_{\gamma,1}$ to a (possibly different) equilibrium $u_\infty \in \mathcal{E}$ as $t \to \infty$.

4.1. Existence of equilibria

According to (4.2), see also Proposition 2.2, the equilibria $\mathcal{E}$ of the evolution equation (4.1) correspond to the Killing fields of Σ.

It is then an interesting question to know how many Killing fields a given manifold can support, or to be more precise, what the dimension of the vector space of all Killing fields of Σ is. (In fact, it turns out that the Killing fields on a Riemannian manifold form a sub Lie-algebra of the Lie-algebra of all tangential fields).

It might also be worthwhile to recall that the Killing fields of a Riemannian manifold (M, g) are the infinitesimal generators of the isometries $I(M, g)$ on (M, g), that is, the generators of flows that are isometries on (M, g). Moreover, in case (M, g) is complete, the Lie-algebra of Killings fields is isometric to the Lie-algebra of $I(M, g)$, see for instance Corollary III.6.3 in [20].

It then follows from [20, Proposition III.6.5] that dim $\mathcal{E} \leq d(d + 1)/2$, where d is the dimension of Σ. For compact manifolds, equality holds if and only if Σ is isometric to $\mathbb{S}^d$, the standard d-dimensional Euclidean sphere in $\mathbb{R}^{d+1}$.

On the other hand, if (M, g) is compact and the Ricci tensor is negative definite everywhere, then any Killing field on M is equal to zero and $I(M, g)$ is a finite group, see [20, Proposition III.6.6]. In particular, if (M, g) is a two-dimensional Riemannian manifold with negative Gaussian curvature then any Killing field is 0.

Example 4.4.

(a) Let $\Sigma = \mathbb{S}^2$. Then, dim $\mathcal{E} = 3$ and each equilibrium $u_* \in \mathcal{E}$ corresponds to a rotation about an axis spanned by a vector $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$. Therefore, $u_* \in \mathcal{E}$ is given by

$$u_*(x) = \omega \times x, \quad x \in \mathbb{S}^2,$$

for some $\omega \in \mathbb{R}^3$. According to Theorem 4.3, each equilibrium $u_* = \omega \times x$ is stable and each solution u of (4.1) that starts out close to u_* converges at an exponential rate toward a (possibly different) equilibrium $u_\infty = \omega_\infty \times x$ for some $\omega_\infty \in \mathbb{R}^3$.

(b) Suppose $\Sigma = \mathbb{T}^2$, say with parameterization

$$\begin{aligned}
x_1 &= (R + r \cos \phi) \cos \theta \\
x_2 &= (R + r \cos \phi) \sin \theta \\
x_3 &= r \sin \phi,
\end{aligned} \tag{4.5}$$

where $\phi, \theta \in [0, 2\pi)$ and $0 < r < R$. Then, one readily verifies that the velocity field $u_* = \omega e_3 \times x$, with $\omega \in \mathbb{R}$, is an equilibrium. Hence, the fluid on the torus rotates about the x_3-axis with angular velocity ω. According to Theorem 4.3, all of these equilibria are stable.

With the above parameterization, one shows that

$$K = \frac{\cos \phi}{r(R + r \cos \phi)},$$

where K is the Gauss curvature. By Gauss' Theorema Egregium, K is invariant under (local) isometries, and this implies that rotations around the x_3-axis are the only isometries on $\mathbb{T}^2$.

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Appendix A

In this appendix, we collect some results from differential geometry that are employed throughout the manuscript. We also refer to [13, Chapter 2] for complementary information.

We will assume throughout that Σ is a smooth, compact, closed (that is, without boundary) hypersurface embedded in $\mathbb{R}^{d+1}$. We mention on the go that these assumptions imply that Σ is orientable, see for instance [21].

Let ν_Σ be the unit normal field of Σ (which is compatible with the chosen orientation). Then, the orthogonal projection $\mathcal{P}_\Sigma$ onto the tangent bundle of Σ is defined by $\mathcal{P}_\Sigma = I - \nu_\Sigma \otimes \nu_\Sigma$.

We use the notation $\{\tau_1(p), \cdots, \tau_d(p)\}$ to denote a local basis of the tangent space $\mathsf{T}_p\Sigma$ of Σ at p, and $\{\tau^1(p), \cdots, \tau^d(p)\}$ to denote the corresponding dual basis of the cotangent space $\mathsf{T}_p^*\Sigma$ at p. Hence, we have $(\tau^i(p)|\tau_j(p)) = \delta_j^i$, the Kronecker delta function. Note that

$$\{\tau_1, \ldots, \tau_d\} = \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d} \right\}, \quad \{\tau^1, \ldots, \tau^d\} = \left\{ dx^1, \ldots, dx^d \right\}.$$

In this manuscript, we will occasionally not distinguish between vector fields and covector fields, that is, we identify

$$u = u^i \tau_i = u_i \tau^i, \tag{A.1}$$

where, as usual, the Einstein summation convention is employed throughout. The metric tensor is given by $g_{ij} = (\tau_i | \tau_j)$, where $(\cdot | \cdot)$ is the Euclidean inner product of $\mathbb{R}^{d+1}$, and the dual metric g^* on the cotangent bundle $\mathsf{T}^* \Sigma$ is given by $g^{ij} = (\tau^i | \tau^j)$. It holds that

$$g^{ik} g_{kj} = \delta^i_j, \quad \tau^i = g^{ij} \tau_j, \quad \tau_j = g_{jk} \tau^k. \tag{A.2}$$

Hence, g is induced by the inner product $(\cdot | \cdot)$, that is, we have

$$g(u, v) = g_{ij} u^i v^j = (u^i \tau_i | v^j \tau_j) = (u | v), \quad u = u^i \tau_i, \quad v = v^j \tau_j, \tag{A.3}$$

whereas

$$g^*(u, v) = g^{ij} u_i v_j = (u_i \tau^i | v_j \tau^j) = (u | v), \quad u = u_i \tau^i, \quad v = v_j \tau^j. \tag{A.4}$$

It holds that

$$\partial_i \tau_j = \Lambda^k_{ij} \tau_k + l_{ij} \nu_\Sigma, \quad \partial_i \tau^j = -\Lambda^j_{ik} \tau^k + l^j_i \nu_\Sigma, \tag{A.5}$$

where Λ^k_{ij} are the Christoffel symbols, l_{ij} are the components of the second fundamental form, and l^i_j are the components of the Weingarten tensor L_Σ; that is, we have

$$l_{ij} = (\partial_j \tau_i | \nu_\Sigma) = -(\tau_i | \partial_j \nu_\Sigma), \quad l^j_i = g^{jk} l_{ki}, \tag{A.6}$$

and

$$L_\Sigma \tau_j = -\partial_j \nu_\Sigma, \quad L_\Sigma = l^j_i \tau^i \otimes \tau_j. \tag{A.7}$$

If $\varphi \in C^1(\Sigma, \mathbb{R})$, the surface gradient of φ is defined by $\nabla_\Sigma \varphi = \partial_i \varphi \tau^i$. If u is a C^1-vector field on Σ (not necessarily tangential), we define the surface gradient of u by

$$\nabla_\Sigma u = \tau^i \otimes \partial_i u. \tag{A.8}$$

It follows from (A.5) that

$$\partial_i u = \partial_i (u^j \tau_j) = (\partial_i u^j + \Lambda^j_{ik} u^k) \tau_j + l_{ij} u^j \nu_\Sigma \tag{A.9}$$

for a tangential vector field u. The *covariant derivative* $\nabla_i u$ of a tangential vector field is defined by $\nabla_i u = P_\Sigma \partial_i u$. Hence, we have

$$\nabla_i u = (\partial_i u^j + \Lambda^j_{ik} u^k) \tau_j \quad \text{for} \quad u = u^j \tau_j. \tag{A.10}$$

The surface divergence $\mathrm{div}_\Sigma u$ for a (not necessarily tangential) vector field u is defined by

$$\mathrm{div}_\Sigma u = (\tau^i | \partial_i u). \tag{A.11}$$

Then, the d-fold mean curvature κ_Σ of Σ is given by

$$\kappa_\Sigma = -\mathrm{div}_\Sigma \nu_\Sigma = -(\tau^i|\partial_i\nu_\Sigma) = (\tau^i|L_\Sigma\tau_i) = \mathrm{tr}L_\Sigma = l_i^i. \qquad (\text{A.12})$$

Hence, κ_Σ is the trace of L_Σ (which equals the sum of the principal curvatures). For a vector field $u = v^j\tau_j + w\nu_\Sigma$, it follows from (A.9), (A.12) and the fact that ν_Σ and τ^i are orthogonal

$$\mathrm{div}_\Sigma u = (\tau^i|\partial_i u) = (\partial_i v^i + \Lambda_{ik}^i v^k) - w\kappa_\Sigma. \qquad (\text{A.13})$$

For a tangent vector field u and a scalar function φ, the surface divergence theorem states that

$$\int_\Sigma (\nabla_\Sigma\varphi|u)\,\mathrm{d}\Sigma = -\int_\Sigma \varphi\,\mathrm{div}_\Sigma u\,\mathrm{d}\Sigma. \qquad (\text{A.14})$$

For a tensor $K = k_i^j\tau^i \otimes \tau_j$, the surface divergence is defined by

$$\mathrm{div}_\Sigma K = (\partial_i K)^\mathsf{T}\tau^i. \qquad (\text{A.15})$$

Hence,

$$\mathrm{div}_\Sigma \mathcal{P}_\Sigma = \partial_i(\tau_j \otimes \tau^j)\tau^i = \kappa_\Sigma\nu_\Sigma. \qquad (\text{A.16})$$

Lemma A.1. *Suppose φ is a C^1-scalar function and u, v, w are C^1-tangential vector fields on Σ. Then,*

(a) $\mathrm{div}_\Sigma(\varphi\mathcal{P}_\Sigma) = \nabla_\Sigma\varphi + \varphi\kappa_\Sigma\nu_\Sigma$.
(b) $(u \cdot \nabla_\Sigma v) = (\nabla_\Sigma v)^\mathsf{T}u$.
(c) $\nabla_\Sigma(u|v) = (\nabla_\Sigma u)v + (\nabla_\Sigma v)u$.
(d) $\big(u|\nabla_\Sigma(v|w)\big) = \big(u \cdot \nabla_\Sigma v\big|w\big) + \big(u \cdot \nabla_\Sigma w\big|v\big)$.

Proof. (a) It follows from (A.15) that

$$\mathrm{div}_\Sigma(\varphi\mathcal{P}_\Sigma) = \partial_i(\varphi\mathcal{P}_\Sigma)\tau^i = (\partial_i\varphi)\mathcal{P}_\Sigma\tau^i + \varphi\partial_i(\mathcal{P}_\Sigma)\tau^i = \partial_i\varphi\tau^i + \varphi\mathrm{div}_\Sigma\mathcal{P}_\Sigma.$$

The assertion is now a consequence of (A.16).

(b) Using local coordinates, we obtain

$$u \cdot \nabla_\Sigma v = u^i\partial_i v = (\partial_i v \otimes \tau^i)u = (\nabla_\Sigma v)^\mathsf{T}u.$$

(c) In local coordinates, $\partial_i(u|v) = (\partial_i u|v) + (u|\partial_i v)$. It is now easy to conclude that

$$\nabla_\Sigma(u|v) = \partial_i(u|v)\tau^i = (\tau^i \otimes \partial_i u)v + (\tau^i \otimes \partial_i v)u = (\nabla_\Sigma u)v + (\nabla_\Sigma v)u.$$

(d) This follows from the assertions in (b) and (c). $\qquad\square$

Let

$$\mathcal{D}_\Sigma(u) := \frac{1}{2}\mathcal{P}_\Sigma\big(\nabla_\Sigma u + [\nabla_\Sigma u]^\mathsf{T}\big)\mathcal{P}_\Sigma.$$

Suppose $u \in C^2(\Sigma, \mathsf{T}\Sigma)$, $v \in C^1(\Sigma, \mathsf{T}\Sigma)$. Then, one shows that

$$(\operatorname{div}_\Sigma \mathcal{D}_\Sigma(u)|v) = \operatorname{div}_\Sigma(\mathcal{D}_\Sigma(u)v) - \mathcal{D}_\Sigma(u) : \nabla_\Sigma v$$

$$(\operatorname{div}_\Sigma \mathcal{D}_\Sigma(u)|u) = \operatorname{div}_\Sigma(\mathcal{D}_\Sigma(u)u) - |\mathcal{D}_\Sigma(u)|^2 \qquad\qquad (A.17)$$

$$\operatorname{tr} \mathcal{D}_\Sigma(u) = \operatorname{div}_\Sigma u,$$

where $\mathcal{D}_\Sigma(u) : \nabla_\Sigma v = (\mathcal{D}_\Sigma(u)\tau^j|(\nabla_\Sigma v)^\mathsf{T}\tau_j)$, $|\mathcal{D}_\Sigma(u)|^2 = \mathcal{D}_\Sigma(u) : \mathcal{D}_\Sigma(u)$, and $\operatorname{tr} \mathcal{D}_\Sigma(u) = (\mathcal{D}_\Sigma(u)\tau^j|\tau_j)$.

Proposition A.2. *Suppose $u \in C^2(\Sigma, \mathsf{T}\Sigma)$.*

(a) *Then, we have the following representation:*

$$2\mathcal{P}_\Sigma \operatorname{div}_\Sigma \mathcal{D}_\Sigma(u) = \Delta_\Sigma u + \nabla_\Sigma \operatorname{div}_\Sigma u + (\kappa_\Sigma L_\Sigma - L_\Sigma^2)u,$$

where Δ_Σ is the Bochner-Laplacian (also called the conformal Laplacian), defined in local coordinates by

$$\Delta_\Sigma = g^{ij}(\nabla_i \nabla_j - \Lambda_{ij}^k \nabla_k).$$

(b) *It holds that*

$$(\kappa_\Sigma L_\Sigma - L_\Sigma^2)u = \operatorname{Ric}_\Sigma u$$

$$(\kappa_\Sigma L_\Sigma - L_\Sigma^2)u = K_\Sigma u \quad \text{in case } d = 2,$$

where $\operatorname{Ric}_\Sigma$ is the Ricci tensor and K_Σ the Gaussian curvature of Σ, respectively.

Proof. (a) We note that in local coordinates, $\mathcal{D}_\Sigma(u)$ is given by

$$2\mathcal{D}_\Sigma(u) = \tau^i \otimes \mathcal{P}_\Sigma \partial_i u + \mathcal{P}_\Sigma \partial_i u \otimes \tau^i = \tau^i \otimes \nabla_i u + \nabla_i u \otimes \tau^i. \qquad (A.18)$$

From (A.15) and the relation, $\mathcal{P}_\Sigma = I - \nu_\Sigma \otimes \nu_\Sigma$ follows

$$\begin{aligned}
\mathcal{P}_\Sigma \operatorname{div}_\Sigma(\nabla_i u \otimes \tau^i) &= \mathcal{P}_\Sigma \partial_j(\tau^i \otimes \mathcal{P}_\Sigma \partial_i u)\tau^j \\
&= (\mathcal{P}_\Sigma \partial_j \tau^i \otimes \mathcal{P}_\Sigma \partial_i u)\tau^j + (\tau^i \otimes \partial_j \mathcal{P}_\Sigma \partial_i u)\tau^j \\
&= (\partial_i u|\tau^j)\mathcal{P}_\Sigma \partial_j \tau^i + \tau^i(\partial_j(\partial_i u - [\nu_\Sigma \otimes \nu_\Sigma]\partial_i u)|\tau^j) \\
&= (\partial_i u|\tau^j)\mathcal{P}_\Sigma \partial_j \tau^i + \tau^i(\partial_j \partial_i u|\tau^j) - \tau^i(\partial_j \nu_\Sigma|\tau^j)(\nu_\Sigma|\partial_i u),
\end{aligned}$$

where, in the last line, we employed the relation $(\nu_\Sigma|\tau^j) = 0$. Next, we observe that

$$\begin{aligned}
(\partial_j \partial_i u|\tau^j)\tau^i &= \partial_i(\partial_j u|\tau^j)\tau^i - (\partial_j u|\partial_i \tau^j)\tau^i \\
&= \nabla_\Sigma \operatorname{div}_\Sigma u - (\partial_j u| - \Lambda_{ik}^j \tau^k + l_i^j \nu_\Sigma)\tau^i \\
&= \nabla_\Sigma \operatorname{div}_\Sigma u - (\partial_j u|\tau^k)\mathcal{P}_\Sigma \partial_k \tau^j - l_i^j(\partial_j u|\nu_\Sigma)\tau^i \\
&= \nabla_\Sigma \operatorname{div}_\Sigma u - (\partial_j u|\tau^k)\mathcal{P}_\Sigma \partial_k \tau^j - L_\Sigma^2 u,
\end{aligned}$$

where we used (A.9) and the relations $L_\Sigma \tau^m = l_r^m \tau^r$ as well as $L_\Sigma \tau_m = l_{mr} \tau^r$ to deduce

$$l_i^j (\partial_j u | v_\Sigma) \tau^i = l_i^j l_{jk} u^k \tau^i = L_\Sigma (u^k l_{jk} \tau^j) = L_\Sigma^2 u_k \tau^k = L_\Sigma^2 u. \tag{A.19}$$

From (A.9) and (A.12) follows

$$- (\partial_j v_\Sigma | \tau^j)(v_\Sigma | \partial_i u) \tau^i = l_j^j l_{ik} u^k \tau^i = \kappa_\Sigma L_\Sigma u^k \tau_k = \kappa_\Sigma L_\Sigma u. \tag{A.20}$$

Summarizing we have shown that

$$\mathcal{P}_\Sigma \mathrm{div}_\Sigma (\nabla_i u \otimes \tau^i) = \nabla_\Sigma \mathrm{div}_\Sigma u + \kappa_\Sigma L_\Sigma u - L_\Sigma^2 u.$$

Moreover, we have

$$\begin{aligned}
\mathcal{P}_\Sigma \mathrm{div}_\Sigma (\tau^i \otimes \nabla_i u) &= \mathcal{P}_\Sigma \partial_j (\nabla_i u \otimes \tau^i) \tau^j \\
&= g^{ij} \mathcal{P}_\Sigma \partial_j \nabla_i u + (\partial_j \tau^i | \tau^j) \nabla_i u \\
&= g^{ij} (\nabla_i \nabla_j u - \Lambda_{ij}^k \nabla_k u) = \Delta_\Sigma u.
\end{aligned}$$

(b) The computations in (A.19), (A.20) show that in local coordinates

$$(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u = g^{jm} (l_{jm} l_{ik} - l_{im} l_{jk}) u^k \tau^i.$$

By the Gauss equation, see for instance [20, Proposition II.3.8], this yields

$$(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u = g^{jm} R_{jikm} u^k \tau^i = R_{ik} u^k \tau^i = \mathrm{Ric}_\Sigma u,$$

where R_{jikm} are the components of the curvature tensor and R_{ik} the components of the Ricci $(0, 2)$-tensor. In case that Σ is a surface embedded in $\mathbb{R}^3$, one obtains

$$(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u = K_\Sigma u,$$

where K_Σ is the Gauss curvature of Σ. This can, for instance, be seen as follows:

$$\begin{aligned}
(\kappa_\Sigma L_\Sigma - L_\Sigma^2) u &= g^{jm} (l_{jm} l_{ik} - l_{im} l_{jk}) u^k \tau^i \\
&= (l_j^j l_i^k - l_i^j l_j^k) u_k \tau^i = \det(L_\Sigma) \, \delta_i^k u_k \tau^i = K_\Sigma u.
\end{aligned}$$

The proof of Proposition A.2 is now complete.

$\square$

Remark A.3. (a) We note that in local coordinates,

$$2\mathcal{D}_\Sigma (u) = (\partial_i u_j - \Lambda_{ij}^k u_k) \tau^i \otimes \tau^j + (\partial_i u_j - \Lambda_{ij}^k u_k) \tau^j \otimes \tau^i$$

for $u = u_j \tau^j \in C^1(\Sigma, \mathrm{T}^*\Sigma)$ and

$$2\mathcal{D}_\Sigma (u) = (\partial_i u^j + \Lambda_{ik}^j u^k) \tau^i \otimes \tau_j + (\partial_i u^j + \Lambda_{ik}^j u^k) \tau_j \otimes \tau^i$$

for $u = u^j \tau_j \in C^1(\Sigma, \mathrm{T}\Sigma)$.

(b) Suppose u, v are tangential fields on Σ. Then,

$$\nabla_v u := (\nabla_i u \otimes \tau^i)v \qquad (A.21)$$

coincides with the Levi–Civita connection ∇ of Σ.

We note that $\nabla_i = \nabla_{\tau_i} = \nabla_{\frac{\partial}{\partial x^i}}$ and $\nabla u = \nabla_i u \otimes \tau^i$ in local coordinates.

(c) A straightforward computation shows that in local coordinates

$$g^{ij}(\nabla_i \nabla_j u - \Lambda_{ij}^k \nabla_k u) = g^{ij}(\nabla^2 u)(\tau_i, \tau_j), \quad u \in C^2(\Sigma, \mathsf{T}\Sigma).$$

Hence, $\Delta_\Sigma = \mathrm{tr}_g(\nabla^2 u)$.

(d) Let ∇ be the Levi–Civita connection of Σ. Then, it follows from (b) and (A.18)

$$\mathcal{D}_\Sigma(u) = \frac{1}{2}\left(\nabla u + [\nabla u]^\mathsf{T}\right).$$

(e) Suppose u, v, w are C^1-tangential fields. Employing (A.18) and (A.21), one readily verifies that

$$(\mathcal{D}_\Sigma(u)v|w) + (\mathcal{D}_\Sigma(u)w|v) = (\nabla_v u|w) + (\nabla_w u|v).$$

We remind that a tangential field u on Σ is called a *Killing field* if

$$(\nabla_v u|w) + (\nabla_w u|v) = 0 \quad \text{for all tangential fields } v, w \text{ on } \Sigma,$$

see for instance [20, Lemma III.6.1]. This implies for a C^1-tangent field u

$$\mathcal{D}_\Sigma(u) = 0 \iff u \text{ is a Killing field.}$$

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