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On systems of parabolic variational inequalities with multivalued terms

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Abstract

In this paper we present an analytical framework for the following system of multivalued parabolic variational inequalities in a cylindrical domain $Q = \Omega \times (0, \tau)$: For $k = 1, \ldots, m$, find $u_k \in K_k$ and $\eta_k \in L^{p'_k}(Q)$ such that

$$u_k(\cdot, 0) = 0 \ \text{ in } \Omega, \quad \eta_k(x,t) \in f_k(x,t,u_1(x,t), \ldots, u_m(x,t)),$$
$$\langle u_{kt} + A_k u_k, v_k - u_k \rangle + \int_Q \eta_k \, (v_k - u_k) \, dxdt \geq 0, \quad \forall \, v_k \in K_k,$$

where K_k is a closed and convex subset of $L^{p_k}(0, \tau; W_0^{1,p_k}(\Omega))$, A_k is a time-dependent quasilinear elliptic operator, and $f_k : Q \times \mathbb{R}^m \to 2^{\mathbb{R}}$ is an upper semicontinuous multivalued function with respect to $s \in \mathbb{R}^m$. We provide an existence theory for the above system under certain coercivity assumptions. In the noncoercive case, we establish an appropriate sub-supersolution method that allows us to get existence and enclosure results. As an application, a multivalued parabolic obstacle system is treated. Moreover, under a lattice condition on the constraints K_k, systems of evolutionary variational-hemivariational inequalities are shown to be a subclass of the above system of multivalued parabolic variational inequalities.

Keywords System of parabolic variational inequalities · Multivalued parabolic variational inequality · Upper semicontinuous multivalued operator · Pseudomonotone multivalued operator · Sub-supersolution · Obstacle problem · Evolutionary variational-hemivariational inequalities

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $Q = \Omega \times (0, \tau)$ a space-time cylindrical domain with base Ω, and $\Gamma = \partial\Omega \times (0, \tau)$ its lateral boundary with $\tau > 0$. For $p \in (1, \infty)$, we denote by $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ the usual Sobolev spaces with dual spaces $(W^{1,p}(\Omega))^*$ and $W^{-1,p'}(\Omega)$, respectively, where p' is the Hölder conjugate of p satisfying $1/p + 1/p' = 1$. Note that if $2 \le p < \infty$, then $W^{1,p}(\Omega) \subset L^2(\Omega) \subset (W^{1,p}(\Omega))^*$ form an evolution triple with all the imbeddings being dense and compact (see e.g. [18]) .

Let $m \in \mathbb{N}$ and $p_1, \ldots, p_m \in [2, \infty)$. We are concerned in this paper with the following system of m multivalued parabolic variational inequalities: For each $k = 1, \ldots, m$, find $u_k \in W_{0k} \cap K_k$ and $\eta_k \in L^{p_k'}(Q)$ such that

$$u_k(\cdot, 0) = 0 \text{ in } \Omega, \quad \eta_k(x, t) \in f_k(x, t, u_1(x, t), \ldots, u_m(x, t)), \tag{1.1}$$

$$\langle u_{kt} + A_k u_k, v_k - u_k\rangle + \int_Q \eta_k\,(v_k - u_k)\,dxdt \ge 0, \quad \forall\, v_k \in K_k, \tag{1.2}$$

where K_k is a closed, convex subset of $X_{0k} := L^{p_k}(0, \tau; W_0^{1,p_k}(\Omega))$, $W_{0k} = \{u_k \in X_{0k} : u_{kt} \in X_{0k}^*\}$, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X_{0k}^* and X_{0k}. The operator $A_k : X_{0k} \to X_{0k}^*$ is a second order quasilinear differential operator of Leray-Lions type, given by

$$A_k(u_k)(x, t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i^{(k)}(x, t, \nabla u_k(x, t)),$$

and $f_k : Q \times \mathbb{R}^m \to 2^{\mathbb{R}}$, $(x, t, s_1, \ldots, s_m) \mapsto f_k(x, t, s_1, \ldots, s_m) \in 2^{\mathbb{R}}$, is an upper semicontinuous multivalued function with respect to $s := (s_1, \ldots, s_m) \in \mathbb{R}^m$, that will be specified later.

The main goal of this article is to present a mathematical theory for systems of parabolic variational inequalities with upper semicontinuous multivalued functions of the form (1.1)–(1.2) in both coercive and noncoercive cases, and to provide existence and enclosure principles when subsolutions and supersolutions of (1.1)–(1.2), defined in certain appropriate sense, exist. To the best of our knowledge, systems of parabolic multivalued variational inequalities have not been studied before in a systematic way by sub-supersolution (lattice) approaches. Moreover, we point out here that the closed and convex sets K_k's that represent constraints in system (1.1)–(1.2) are not supposed to have nonempty interior parts or to satisfy some conditions of similar type. Such assumptions typically allow the application of Rockafellar's theorem about sums of maximal monotone operators, which facilitates the study of parabolic variational inequalities considerably by the implementation of arguments and results for elliptic variational inequalities to parabolic variational inequalities. However, assumptions of these types would exclude the investigation of certain most important classes of evolutionary variational inequalities such as parabolic obstacles problems, in which the associated closed and convex sets representing the obstacles have empty interior parts. As will be seen later, our approach here applies also to obstacle problems. We also remark that (1.1)–(1.2) covers a wide range of parabolic systems when specifying K and/or f such as the special cases mentioned above including parabolic initial-boundary value problem in the case when $K = X_0$, and $f : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Moreover, under a lattice condition on the constraints K_k, systems of evolutionary variational-hemivariational inequalities will be shown to be a subclass of the above system of multivalued parabolic variational inequalities (1.1)–(1.2).

The paper is organized as follows. After introducing necessary assumptions and notations and providing auxiliary results, including one on the pseudomonotonicity of multivalued Nemytskij operators with respect to the graph norm of the time derivative operator, in Sects. 2 and 3, we present our main results in Section 4. In the first part (Sect. 4.1) we treat the coercive case, where some relative growth condition of A_k and f_k for u with large norm is imposed. In this case the existence of solutions of (1.1)–(1.2) follows from penalty arguments and the solvability of systems of equations with multivalued pseudomonotone operators. In the second part (Sect. 4.2), we deal with the noncoercive case where such growth condition is not assumed. We establish in that section a sub-supersolution method that will allow us to prove existence and enclosure results. The concepts of sub- and supersolutions and the arguments in our case here are combinations of those for parabolic multivalued variational inequalities in [5] and those for systems of multivalued elliptic variational inequalities in [10]. In Sect. 5, as an application of the theory developed in the preceding sections, we treat an obstacle problem by explicitly constructing an ordered pair of sub- and supersolutions. Finally, we show in Section 6 that under a lattice condition on the constraints, systems of evolutionary variational-hemivariational inequalities turn out to be only a subclass of system (1.1)–(1.2).

2 Assumptions: setting of the problem

Let us begin with some needed notation and assumptions. Let Ω, Q, X_{0k}, and W_{0k} be defined as in Sect. 1, and $L^0(\Omega)$ (resp. $L^0(Q)$) be the set of all (equivalent classes of) measurable functions from Ω (resp. from Q) to $\mathbb{R}$.

For $k = 1, \dots, m$, let W_k be defined by

$$W_k = \{u \in X_k : u_t \in X_k^*\},$$

where $X_k = L^{p_k}(0, \tau; W^{1,p_k}(\Omega))$ with its dual $X_k^* = L^{p_k'}(0, \tau; (W^{1,p_k}(\Omega))^*)$, and the derivative $u_t := \partial u/\partial t$ is understood in the sense of vector-valued distributions.

The space W_k endowed with the graph norm of the operator $\partial/\partial t$

$$\|u\|_{W_k} = \|u\|_{X_k} + \|u_t\|_{X_k^*}$$

is a Banach space which is separable and reflexive due to the separability and reflexivity of X_k and X_k^*, where $\|\cdot\|_{X_k}$ and $\|\cdot\|_{X_{0k}}$ are the usual norms defined on X_k and X_{0k} (and similarly on X_k^* and X_{0k}^*) :

$$\|u\|_{X_k} = \left(\int_0^\tau \|u(t)\|^{p_k}_{W^{1,p_k}(\Omega)}\, dt\right)^{1/p_k}, \quad \|u\|_{X_{0k}} = \left(\int_0^\tau \|u(t)\|^{p_k}_{W_0^{1,p_k}(\Omega)}\, dt\right)^{1/p_k}.$$

For any $k \in \{1, \ldots, m\}$, W_k is continuously embedded into $C([0, \tau],\ L^2(\Omega))$. Thus, by Aubin's lemma, the embedding $W_k \hookrightarrow\hookrightarrow L^{p_k}(Q)$ is compact due to the compact embedding $W^{1,p_k}(\Omega) \hookrightarrow\hookrightarrow L^{p_k}(\Omega)$. Similar properties hold true for the space W_{0k},

$$W_{0k} = \{u \in X_{0k} : u_t \in X^*_{0k}\},$$

introduced in Sect. 1.

For $k = 1, \ldots, m$, we denote by $L_k := \partial/\partial t$, where its domain of definition, $D(L_k)$, is given by

$$D(L_k) = \left\{u \in X_{0k} : u_t \in X^*_{0k} \text{ and } u(\cdot, 0) = 0\right\}. \tag{2.1}$$

It is known that the linear operator $L_k : D(L_k) \subset X_{0k} \to X^*_{0k}$ is closed, densely defined and maximal monotone, e.g., cf. [18, Chap. 32].

For $u, v \in \mathbb{R}^m$, we denote $u \le v$ if $u_k \le v_k$, $\forall k \in \{1, \ldots, m\}$. This ordering is extended to functions $u, v \in [L^0(\Omega)]^m$ (resp. $u, v \in [L^0(Q)]^m$) in a natural way: $u \le v$ if and only if $u(x) \le v(x)$ for a.e. $x \in \Omega$ (resp. $u(x,t) \le v(x,t)$ for a.e. $(x,t) \in Q$). If $u_j \in \mathbb{R}$ with $j \in \{1, \ldots, m\}\backslash\{k\}$, and $t \in \mathbb{R}$, then we denote

$$[u]_k = (u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_m) \in \mathbb{R}^{m-1},$$
$$(t, [u]_k) = (u_1, \ldots, u_{k-1}, t, u_{k+1}, \ldots, u_m) \in \mathbb{R}^m,$$

For $u \in \mathbb{R}^m$, we also use the same notation $[u]_k$ for $(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_m) \in \mathbb{R}^{m-1}$. Let $u, v \in \mathbb{R}^m$ such that $u \le v$, we put

$$[u, v] = \{w \in \mathbb{R}^m : u \le w \le v\}.$$

Similarly, for $k \in \{1, \ldots, m\}$ and $[u]_k, [v]_k \in \mathbb{R}^{m-1}$ with $[u]_k \le [v]_k$, we denote

$$[u, v]_k = [[u]_k, [v]_k] = \{[w]_k \in \mathbb{R}^{m-1} : [u]_k \le [w]_k \le [v]_k\}.$$

We use the same notation for vector functions, that is, for $u, v \in [L^0(Q)]^m$ or $u, v \in \prod_{j=1}^m X_j$ and for $[u]_k, [v]_k \in [L^0(Q)]^{m-1}$ or $[u]_k, [v]_k \in \prod_{j\in\{1,\ldots,m\}\backslash\{k\}} X_j$. For example, if $u, v \in \prod_{j=1}^m X_j = X$ and $u \le v$, then

$$[u, v] = \{w \in X : u \le w \le v\},$$

and if $[u]_k, [v]_k \in \prod_{j\in\{1,\ldots,m\}\backslash\{k\}} X_j$ and $u \le v$, then

$$[u, v]_k = [[u]_k, [v]_k] = \left\{[w]_k \in \prod_{j\in\{1,\ldots,m\}\backslash\{k\}} X_j : [u]_k \le [w]_k \le [v]_k\right\}. \tag{2.2}$$

For a normed vector space Z, we denote by $\mathcal{K}(Z)$ the collection of all nonempty, closed, and convex subsets of Z. Let $Z_1, \dots, Z_m$ be Banach spaces with the corresponding norms $\|\cdot\|_{Z_1}, \dots, \|\cdot\|_{Z_m}$. The product $Z = \prod_{k=1}^m Z_k$ is a Banach space with the product norm: $\|u\|_Z = \sum_{k=1}^m \|u_k\|_{Z_k}$ for $u = (u_1, \dots, u_m) \in Z$.

We use here the standard identification of $u^* \in Z^*$ with $(u_1^*, \dots, u_m^*) \in \prod_{k=1}^m Z_k^*$ by

$$\langle u_k^*, u_k\rangle_{Z_k^*, Z_k} = \langle u^*, (u_k, [0]_k)\rangle_{Z^*, Z}, \ \forall u_k \in Z_k, \ \forall k \in \{1, \dots, m\}, \tag{2.3}$$

and

$$\langle u^*, u\rangle_{Z^*, Z} = \langle (u_1^*, \dots, u_m^*), (u_1, \dots, u_m)\rangle_{Z^*, Z} = \sum_{k=1}^m \langle u_k^*, u_k\rangle_{Z_k^*, Z_k}, \ \forall u \in Z. \tag{2.4}$$

In this pattern, we consider the following product spaces:

$$X = \prod_{k=1}^m X_k, \ X_0 = \prod_{k=1}^m X_{0k}, \ W = \prod_{k=1}^m W_k, \ W_0 = \prod_{k=1}^m W_{0k},$$

and their dual spaces,

$$X^* \equiv \prod_{k=1}^m X_k^*, \ X_0^* \equiv \prod_{k=1}^m X_{0k}^*, \ W^* \equiv \prod_{k=1}^m W_k^*, \ W_0^* \equiv \prod_{k=1}^m W_{0k}^*.$$

For simplicity of notation and when there is no confusion, we use $\|\cdot\|$ for the norms in X, X_0, X_k, and X_{0k}. By the same token, $\langle\cdot,\cdot\rangle$ stands for any of the dual pairings between any of the spaces X_k, X_{0k}, $W^{1,p_k}(\Omega)$, $W_0^{1,p_k}(\Omega)$, X, X_0, $\prod_{k=1}^m W^{1,p_k}(\Omega)$, $\prod_{k=1}^m W_0^{1,p_k}(\Omega)$, and its corresponding dual space. For example, if $u^* \in X^*$ and $u \in X$, then

$$\langle u^*, u\rangle = \int_0^\tau \langle u^*(t), u(t)\rangle \, dt = \sum_{k=1}^m \int_0^\tau \langle u_k^*(t), u_k(t)\rangle \, dt.$$

However, indices will be used in the above norms and dual pairings wherever clarification is needed.

We consider next some assumptions imposed on the principal and lower order terms in (1.1)–(1.2). For $k = 1, \dots, m$, let us assume the following Leray–Lions conditions on the coefficient $a_i^{(k)}$, $i = 1, \dots, N$, of the operator A_k.

(A1) $a_i^{(k)} : Q \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, i.e., $a_i^{(k)}(\cdot,\cdot,\xi) : Q \to \mathbb{R}$ is measurable for all $\xi \in \mathbb{R}^N$ and $a_i^{(k)}(x,t,\cdot) : \mathbb{R}^N \to \mathbb{R}$ is continuous for a.e. $(x,t) \in Q$. In addition, the following growth condition holds:

$$|a_i^{(k)}(x,t,\xi)| \le c_1^{(k)} |\xi|^{p_k - 1} + c_2^{(k)}(x,t)$$

for a.e. $(x,t) \in Q$ and for all $\xi \in \mathbb{R}^N$, for some constant $c_1^{(k)} > 0$ and some function $c_2^{(k)} \in L_+^{p_k'}(Q)$.

(A2) *(Strict monotonicity)* For a.e. $(x,t) \in Q$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$ the following monotonicity in ξ holds:

$$\sum_{i=1}^{N} (a_i^{(k)}(x,t,\xi) - a_i^{(k)}(x,t,\xi'))(\xi_i - \xi_i') > 0.$$

(A3) There is some constant $c_3^{(k)} > 0$ such that for a.e. $(x,t) \in Q$ and for all $\xi \in \mathbb{R}^N$ the inequality

$$\sum_{i=1}^{N} a_i^{(k)}(x,t,\xi)\xi_i \geq c_3^{(k)} |\xi|^{p_k} - c_4^{(k)}(x,t)$$

is satisfied for some function $c_4^{(k)} \in L^1(Q)$.

In view of (A1), the operator A_k defined by

$$\langle A_k u, \varphi \rangle := \int_Q \sum_{i=1}^{N} a_i^{(k)}(x,t,\nabla u) \frac{\partial \varphi}{\partial x_i}\, dx\, dt, \quad \forall \varphi \in X_{0k}, \tag{2.5}$$

is continuous and bounded from X_{0k} into X_{0k}^*.

For functions $w,\ z$ and sets W and Z of functions we use the notations: $w \wedge z = \min\{w,z\}$, $w \vee z = \max\{w,z\}$, $W \wedge Z = \{w \wedge z : w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z : w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. In particular, we denote $w^+ = w \vee 0$.

For $k = 1, \ldots, m$, let us introduce the multivalued Nemytskij operator F_k associated with the multivalued function $f_k : Q \times \mathbb{R} \to \mathcal{K}(\mathbb{R})$ by

$$\begin{aligned} F_k(u) = \{\eta : Q \to \mathbb{R} : \eta \text{ is measurable on } Q \text{ and} \\ \eta(x,t) \in f_k(x,t,u(x,t)) \text{ for a.e. } (x,t) \in Q\}. \end{aligned} \tag{2.6}$$

For each $k \in \{1, \ldots, m\}$, we impose the following conditions on f_k:

(F1) $f_k : Q \times \mathbb{R}^m \to \mathcal{K}(\mathbb{R})$ is graph measurable on $Q \times \mathbb{R}^m$, that is,

$$\mathrm{Gr}(f_k) := \{(x,t,u,\eta) \in Q \times \mathbb{R}^m \times \mathbb{R} : \eta \in f(x,t,u)\}$$

belongs to $[\mathcal{L}(Q) \times \mathcal{B}(\mathbb{R}^m)] \times \mathcal{B}(\mathbb{R})$, where $\mathcal{L}(Q)$ is the family of Lebesgue measurable subsets of Q and $\mathcal{B}(\mathbb{R}^m)$ (resp. $\mathcal{B}(\mathbb{R})$) is the σ-algebra of Borel sets in $\mathbb{R}^m$ (resp. in $\mathbb{R}$).

(F2) For a.e. $(x,t) \in Q$, the function $f_k(x,t,\cdot) : \mathbb{R}^m \to \mathcal{K}(\mathbb{R})$ is upper semicontinuous.

(F3) f_k satisfies the growth condition

$$\sup\{|\eta| : \eta \in f_k(x,t,s)\} \leq \alpha_k(x,t) + \beta_k \sum_{j=1}^{m} |s_j|^{\frac{p_j}{p'_k}}, \tag{2.7}$$

for a.e. $(x,t) \in Q$, $\forall\, s \in \mathbb{R}^m$, where $\alpha_k \in L^{p'_k}(Q)$, and $\beta_k \geq 0$.

For any $u \in [L^0(Q)]^m$, it follows from (F1) that the function $(x,t) \mapsto f_k(x,t,u(x,t))$ is also a measurable function from Q to $\mathcal{K}(\mathbb{R})$, which implies that $F_k(u) \neq \emptyset$. Moreover, as a consequence of (F3), we see that $F_k(u) \subset L^{p'_k}(Q)$ whenever $u \in \prod_{j=1}^m L^{p_j}(Q)$. Hence, the Nemytskij operator F_k is a well defined mapping from $\prod_{j=1}^m L^{p_j}(Q)$ to $2^{L^{p'_k}(Q)} \setminus \{\emptyset\}$.

Let $i_k : X_{0k} \hookrightarrow L^{p_k}(Q)$ be the (continuous) embedding of X_{0k} into $L^{p_k}(Q)$, and let $i_k^* : L^{p'_k}(Q) \hookrightarrow X_{0k}^*$ be its adjoint. The mapping i_k^* is the natural restriction on X_{0k} in the following sense:

$$i_k^*(w_k^*) = w_k^*|_{X_{0k}}, \ \ \forall w_k^* \in L^{p'_k}(Q) (\equiv [L^{p_k}(Q)]^*).$$

Let $i = i_1 \times \cdots \times i_m : X_0 \to \prod_{k=1}^m L^{p_k}(Q)$, $u \mapsto u$, $\forall u \in X_0$ be the embedding of X_0 into $\prod_{k=1}^m L^{p_k}(Q)$. Hence, its adjoint $i^* : \prod_{k=1}^m L^{p'_k}(Q) \to X_0^*$ is the natural restriction on X_0, i.e.,

$$\begin{aligned} i^*(w^*) = i^*(w_1^*, \ldots, w_m^*) &= (i_1^*(w_1^*), \ldots, i_m^*(w_m^*)) = (w_1^*|_{X_{01}}, \ldots, w_m^*|_{X_{0m}}) \\ &= w^*|_{X_0}. \end{aligned}$$

Let us define $F = (F_1, \ldots, F_m) : \prod_{k=1}^m L^{p_k}(Q) \to 2^{\prod_{k=1}^m L^{p'_k}(Q)}$, $F(u) = \prod_{k=1}^m F_k(u)$, and its corresponding composed operator

$$\mathcal{F} = i^* \circ F \circ i : X_0 \to 2^{X_0^*}. \tag{2.8}$$

In the next step, we shall formulate the system (1.1)–(1.2) as a single variational inequality. Let us define $A : X_0 \to X_0^*$ by

$$Au = (A_1 u_1, \ldots, A_m u_m), \ \forall u = (u_1, \ldots, u_m) \in X_0, \tag{2.9}$$

with $A_1, \ldots, A_m$ given by (2.5). It follows from the corresponding property of $A_1, \ldots, A_m$ that A is a continuous and bounded operator from X_0 to X_0^*. Next, we define

$$D(L) = \prod_{k=1}^m D(L_k),$$

which can be easily seen as

$$D(L) = \left\{u \in X_0 : u_t \in X_0^* \text{ and } u(\cdot, 0) = 0\right\}. \tag{2.10}$$

The time derivative for vector-valued functions is defined by $L : D(L) \to X_0^*$, $L = L_1 \times \cdots \times L_k$, that is,

$$Lu = (L_1 u_1, \ldots, L_m u_m) = (u_{1t}, \ldots, u_{mt}) = u_t \in \prod_{k=1}^{m} X_{0k}^* \equiv X_0^*, \tag{2.11}$$

for all $u = (u_1, \ldots, u_m) \in D(L)$.

Lastly, let

$$K = \prod_{k=1}^{m} K_k, \tag{2.12}$$

which is a closed and convex subset of X_0. With these definitions and settings, we see that the system (1.1)–(1.2) can be formulated as the following multivalued evolutionary variational inequality: Find $u \in D(L) \cap K$ and $\eta \in \mathcal{F}(u)$ such that

$$\langle Lu + Au + \eta, v - u \rangle \geq 0, \ \forall v \in K.$$

We study in the sequel the existence of solutions of this variational inequality in both coercive and noncoercive cases.

3 Auxiliary results

We first have the following simple, yet essential, property of the time derivative operator in the vector case.

Proposition 3.1 *The operator L given in (2.11) is a linear, closed, densely defined and maximal monotone operator from $D(L) \subset X_0$ to X_0^*.*

Proof By mathematical induction, the above properties of L immediately follow from the corresponding properties of the component operators L_k ($k = 1, \ldots, m$), which are well known for the time derivative operator. □

We are now ready to state and prove a crucial property of $\mathcal{F}$, which is its pseudomonotonicity with respect to the graph norm topology of the domain $D(L)$ of L. Let us recall the following definition of a multivalued pseudomonotone operator with respect to the graph norm topology of the domain $D(L)$ (w.r.t. $D(L)$ for short) of a linear, closed, densely defined and maximal monotone operator $L : D(L) \subset Y \to Y^*$ (cf. [3], [16], [8]).

Definition 3.1 Let Y be a reflexive Banach space, and let $L : D(L) \subset Y \to Y^*$ be a linear, closed, densely defined and maximal monotone operator. The operator $\mathcal{T} : Y \to 2^{Y^*}$ is called **pseudomonotone w.r.t.** $D(L)$ if the following conditions are satisfied:

(i) The set $\mathcal{T}(u)$ is nonempty, bounded, closed and convex for all $u \in Y$.
(ii) $\mathcal{T}$ is upper semicontinuous from each finite dimensional subspace of Y to Y^* equipped with the weak topology.

(iii) If $\{u_n\} \subset D(L)$ with $u_n \rightharpoonup u$ in Y, $Lu_n \rightharpoonup Lu$ in Y^*, $u_n^* \in \mathcal{T}(u_n)$ with $u_n^* \rightharpoonup u^*$ in Y^* and $\limsup \langle u_n^*, u_n - u\rangle \le 0$, then $u^* \in \mathcal{T}(u)$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

Similarly, we have the following definition of operators of class (S_+) with respect to the graph norm topology of the domain $D(L)$ (w.r.t. $D(L)$ for short).

Definition 3.2 Let Y be a reflexive Banach space, and let $L : D(L) \subset Y \to Y^*$ be a linear, closed, densely defined and maximal monotone operator. The operator $\mathcal{T} : Y \to Y^*$ is said to be of **class** (S_+) **w.r.t.** $D(L)$ if for any sequences $\{u_n\} \subset D(L)$, the conditions $u_n \rightharpoonup u$ in X_0, $Lu_n \rightharpoonup Lu$ in X_0^* and $\limsup \langle Tu_n, u_n - u\rangle \le 0$ imply that $u_n \to u$ in X_0.

Proposition 3.2 *Under conditions (A1)–(A3), the operator $A : X_0 \to X_0^*$ defined by (2.5) and (2.9) is of class (S_+) w.r.t. $D(L)$, where L and $D(L)$ are given by (2.10)-(2.11).*

Proof It is known (cf. e.g. [1,2,4]) that under conditions (A1)–(A3), each operator A_k given by (2.5) is of class (S_+) on X_{0k} w.r.t. $D(L_k)$. By mathematical induction, we see directly from the definition of A in (2.9) that A is also of class (S_+) w.r.t. $D(L)$. □

We have the following result about the pseudomonotonicity of $\mathcal{F}$, which is a vector version of Proposition 2.2, [5].

Proposition 3.3 *Under conditions (F1)–(F3), the mapping $\mathcal{F} = i^* \circ F \circ i : X_0 \to 2^{X_0^*}$ is pseudomonotone with respect to $D(L)$, where L and $D(L)$ are given by (2.10)–(2.11).*

Proof The proof of this proposition is divided into three steps.

Step 1: Property (i) of Definition 3.1

We prove in this step that $\mathcal{F}$ is a bounded mapping from X_0 to $\mathcal{K}(X_0^*)$.

First, we prove that for any $u \in \prod_{k=1}^m L^{p_k}(Q)$, $F(u)$ is a nonempty, bounded, closed, and convex subset of $\prod_{k=1}^m L^{p_k'}(Q)$ and in particular,

$$F(u) \in \mathcal{K}\Big(\prod_{k=1}^m L^{p_k'}(Q)\Big).$$

Moreover, we will prove next that the mapping

$$F : \prod_{k=1}^m L^{p_k}(Q) \to \mathcal{K}\Big(\prod_{k=1}^m L^{p_k'}(Q)\Big)$$

is bounded. The convexity of $F(u)$ follows from the fact that $f_k(x, t, u)$ is a closed interval in $\mathbb{R}$ for any $k \in \{1, \ldots, m\}$. Let $\eta = (\eta_1, \ldots, \eta_m) \in F(u)$. As a consequence of (2.7), for each $k \in \{1, \ldots, m\}$,

$$|\eta_k(x,t)| \le \alpha_k(x,t) + \beta_k \sum_{j=1}^m |u_j(x,t)|^{\frac{p_j}{p_k'}}, \quad \text{a.e. } (x,t) \in Q. \tag{3.1}$$

Since $|u_j|^{\frac{p_j}{p_k'}} \in L^{p_k'}(Q)$, we immediately obtain the boundedness of $F(u)$ in $\prod_{k=1}^m L^{p_k'}(Q)$. To prove that $F(u)$ is closed in $\prod_{k=1}^m L^{p_k'}(Q)$, let $\{\eta_n\}$ be a sequence in $F(u)$ such that $\eta_n \to \eta$ in $\prod_{k=1}^m L^{p_k'}(Q)$. By passing to a subsequence, we can assume without loss of generality that $\eta_n(x,t) \to \eta(x,t)$ for a.e. $(x,t) \in Q$. Since $\eta_{nk}(x,t) \in f_k(x,t,u(x,t))$ for a.e. $(x,t) \in Q$, all $n \in \mathbb{N}$, all $k \in \{1,\ldots,m\}$, and $f_k(x,t,u(x,t))$ is a closed interval in $\mathbb{R}$, we have $\eta_k(x,t) \in f_k(x,t,u(x,t))$, $\forall k \in \{1,\ldots,m\}$. As this holds for a.e. $(x,t) \in Q$, it follows that $\eta_k \in F_k(u)$, $\forall k \in \{1,\ldots,m\}$, i.e., $\eta \in F(u)$, which proves the closedness of $F_k(u)$ in $L^{p_k'}(Q)$ and thus of $F(u)$ in $\prod_{k=1}^m L^{p_k'}(Q)$. Due to the reflexivity of $L^{p_k'}(Q)$ (resp. of $\prod_{k=1}^m L^{p_k'}(Q)$), we see from these properties that $F_k(u)$ (resp. $F(u)$) is a weakly closed, and thus a weakly compact subset of $L^{p_k'}(Q)$ (resp. of $\prod_{k=1}^m L^{p_k'}(Q)$).

Inequality (3.1) also implies that if S is a bounded set in $\prod_{k=1}^m L^{p_k}(Q)$ then $F(S)$ is a bounded set in $\prod_{k=1}^m L^{p_k'}(Q)$, that is, F is a bounded mapping from $\prod_{k=1}^m L^{p_k}(Q)$ to $2^{\prod_{k=1}^m L^{p_k'}(Q)}$ and thus to $\mathcal{K}(\prod_{k=1}^m L^{p_k'}(Q))$.

For $u \in X_0$, from the boundedness of i^* and the above arguments we see that $\mathcal{F}(u)$ is a nonempty, convex and bounded subset of X_0^*. Moreover, since

$$\|i^*\eta\|_{X_0^*} \le C\|\eta\|_{\prod_{k=1}^m L^{p_k'}(Q)}, \forall\, \eta \in \prod_{k=1}^m L^{p_k'}(Q)$$

for some constant $C > 0$, it follows from the boundedness of F that $\mathcal{F}$ is also a bounded mapping.

Next, let us prove that $\mathcal{F}(u)$ is closed in X_0^*. For this purpose, suppose that $\{\eta_n\} \subset \mathcal{F}(u)$, $\eta_n = i^*\tilde{\eta}_n$ with $\tilde{\eta}_n \in F(iu) = F(u)$, $\forall n \in \mathbb{N}$, and that

$$\eta_n \to \eta \text{ in } X_0^*. \tag{3.2}$$

Because $\{\tilde{\eta}_n : n \in \mathbb{N}\} \subset F(u)$, $\{\tilde{\eta}_n\}$ is a bounded sequence in $\prod_{k=1}^m L^{p_k'}(Q)$. By passing to a subsequence if necessary, we can assume without loss of generality that

$$\tilde{\eta}_n \rightharpoonup \tilde{\eta}_0 \text{ in } \prod_{k=1}^m L^{p_k'}(Q). \tag{3.3}$$

Since $F(u)$ is weakly closed in $\prod_{k=1}^m L^{p_k'}(Q)$, $\tilde{\eta}_0 \in F(u)$ and thus $i^*\tilde{\eta}_0 \in i^*F(u) = \mathcal{F}(u)$. On the other hand, since i^* is continuous from $\prod_{k=1}^m L^{p_k'}(Q)$ to X_0^* both with weak topologies, we have from (3.3) that

$$\eta_n = i^*\tilde{\eta}_n \rightharpoonup i^*\tilde{\eta}_0 \text{ in } X_0^*,$$

which, combined with (3.2), yields $\eta = i^*\tilde{\eta}_0 \in \mathcal{F}(u)$. Hence, $\mathcal{F}(u)$ is closed in X_0^*.

Step 2: Property (ii) of Definition 3.1

Let V be a finite dimensional subspace of X_0. We prove in this step that the restriction $\mathcal{F}|_V$ of $\mathcal{F}$ on V is upper semicontinuous from V into $2^{X_0^*}$ with respect to the weak topology of X_0^*.

In fact, assume $u_0 \in V$. To prove the upper semicontinuity of $\mathcal{F}|_V$ at u_0, we assume by contradiction that there are a weakly open neighborhood U of $\mathcal{F}(u_0)$ in X_0^* and sequences $\{u_n\} \subset V, \{\eta_n\} \subset X_0^*$ such that $u_n \to u_0$ in V and $\eta_n \in \mathcal{F}(u_n)\backslash U,\ \forall n \in \mathbb{N}$. We see that $\tilde{U} = (i^*)^{-1}(U)$ is a weakly open neighborhood of $F(u_0)$ in $\prod_{k=1}^m L^{p_k'}(Q)$. Moreover, since $\eta_n \in i^* F(u_n)$, there exists $\tilde{\eta}_n \in F(u_n)$ such that

$$\eta_n = i^* \tilde{\eta}_n. \tag{3.4}$$

We have $\tilde{\eta}_n \notin \tilde{U}$ for all $n \in \mathbb{N}$. As $\{u_n\}$ is a bounded sequence in $\prod_{k=1}^m L^{p_k}(Q)$, it follows from Step 1 that $\{\tilde{\eta}_n\}$ is a bounded sequence in $\prod_{k=1}^m L^{p_k'}(Q)$. Also, as above by passing to a subsequence we can assume that

$$\tilde{\eta}_n \rightharpoonup \tilde{\eta}_0 \text{ in } \prod_{k=1}^m L^{p_k'}(Q). \tag{3.5}$$

Since $u_n \to u_0$ in $\prod_{k=1}^m L^{p_k}(Q)$, we have from conditions (F1)–(F3) that all assumptions of Lemma 3.3, [10], are satisfied. According to this result, we have for all $k \in \{1, \ldots, m\}$, $h^*_{L^{p_k'}(Q)}(F_k(u_n), F_k(u_0)) \to 0$ where

$$h^*_{L^{p_k'}(Q)}(A, B) = \sup_{u \in A} \left(\inf_{v \in B} \|u - v\|_{L^{p_k'}(Q)} \right)$$

is the Hausdorff distance between subsets A, B of $L^{p_k'}(Q)$. As

$$\begin{aligned} h^*_{L^{p_k'}(Q)}(F_k(u_n), F_k(u_0)) &\geq \operatorname{dist}_{L^{p_k'}(Q)}(\tilde{\eta}_{nk}, F_k(u_0)) \\ &= \inf\{\|\tilde{\eta}_{nk} - v\|_{L^{p_k'}(Q)} : v \in F_k(u_0)\}, \end{aligned}$$

there is a sequence $\{\overline{\eta}_n^{(k)}\} \subset F_k(u_0)$ such that $\|\tilde{\eta}_{nk} - \overline{\eta}_n^{(k)}\|_{L^{p_k'}(Q)} \to 0$. As $F_k(u_0)$ is a convex, closed, and bounded subset of $L^{p_k'}(Q)$, it is weakly compact in $L^{p_k'}(Q)$. Hence, by passing to a subsequence if necessary, we can assume that $\overline{\eta}_n^{(k)} \rightharpoonup \overline{\eta}_0^{(k)}$ in $L^{p_k'}(Q)$ for some $\overline{\eta}_0^{(k)} \in F_k(u_0)$. It follows that $\tilde{\eta}_{nk} \rightharpoonup \overline{\eta}_0^{(k)}$ in $L^{p_k'}(Q)$ for all $k = 1, \ldots, m$, that is, $\tilde{\eta}_n \rightharpoonup (\overline{\eta}_0^{(1)}, \ldots, \overline{\eta}_0^{(m)})$ in $\prod_{k=1}^m L^{p_k'}(Q)$ with $(\overline{\eta}_0^{(1)}, \ldots, \overline{\eta}_0^{(m)}) \in F(u_0)$.

From (3.5), we have $\tilde{\eta}_0 = (\overline{\eta}_0^{(1)}, \ldots, \overline{\eta}_0^{(m)}) \in F(u_0)$ and thus $\tilde{\eta}_0 \in \tilde{U}$. Again from (3.5) we have $\tilde{\eta}_n \in \tilde{U}$ for all n sufficiently large, contradicting (3.4) and the assumption on η_n, and therefore proving the upper semicontinuity of $\mathcal{F}|_V$.

Step 3: Property (iii) of Definition 3.1

First, let us prove that $\mathcal{F}$ is sequentially weakly closed from $D(L)(\subset X_0)$ with respect to the $D(L)$-graph norm topology into $2^{X_0^*}\backslash\{\emptyset\}$ with X_0^* equipped with its weak topology, that is, if $\{u_n\}$ and $\{\eta_n\}$ are sequences in $D(L)$ and X_0^* respectively such that

$$u_n \rightharpoonup u \text{ in } X_0, u_{nt} \rightharpoonup u_t \text{ in } X_0^*, \tag{3.6}$$

$$\eta_n \rightharpoonup \eta \text{ in } X_0^*, \tag{3.7}$$

and

$$\eta_n \in \mathcal{F}(u_n), \ \forall n \in \mathbb{N}, \tag{3.8}$$

then,

$$\eta \in \mathcal{F}(u). \tag{3.9}$$

In fact, assume (3.6)–(3.8). From (3.8), for each $n \in \mathbb{N}$, there exists $\tilde{\eta}_n \in F(i(u_n)) = F(u_n)$ such that $\eta_n = i^*(\tilde{\eta}_n) = \tilde{\eta}_n|_{X_0^*}$. From (3.6) and Aubin's lemma (cf. [13]), we have

$$u_n = i(u_n) \to i(u) = u \text{ in } \prod_{k=1}^{m} L^{p_k}(Q). \tag{3.10}$$

As in Step 2, for each $k = 1, \dots, m$, it follows from (F1)–(F3) and Lemma 3.3 in [10] that

$$h^*_{L^{p'_k}(Q)}(F_k(u_n), F_k(u)) \to 0. \tag{3.11}$$

Since $\tilde{\eta}_{nk} \in F_k(u_n)$,

$$\inf_{v \in F_k(u)} \|\tilde{\eta}_{nk} - v\|_{L^{p'_k}(Q)} \le h^*_{L^{p'_k}(Q)}(F_k(u_n), F_k(u)).$$

Hence, $\inf_{v \in F_k(u)} \|\tilde{\eta}_{nk} - v\|_{L^{p'_k}(Q)} \to 0$ as $n \to \infty$, and there exists a sequence $\{\eta_n^{(k)}\} \subset F_k(u)$ such that

$$\lim_{n\to\infty} \|\tilde{\eta}_{nk} - \eta_n^{(k)}\|_{L^{p'_k}(Q)} = 0. \tag{3.12}$$

Since $\{\eta_n^{(k)}\} \subset F_k(u)$ and, as noted in Steps 1, $F_k(u)$ is a weakly compact subset of $L^{p'_k}(Q)$, by passing to a subsequence if necessary, we can assume that

$$\eta_n^{(k)} \rightharpoonup \eta_0^{(k)} \text{ in } L^{p'_k}(Q) \tag{3.13}$$

for some $\eta_0^{(k)} \in F_k(u)$. Hence, (3.12) implies that $\tilde{\eta}_{nk} \rightharpoonup \eta_0^{(k)}$ in $L^{p'_k}(Q)$ for $k = 1, \dots, m$. Putting $\eta_0 = (\eta_0^{(1)}, \dots, \eta_0^{(m)})$, we see that $\eta_0 \in F(u)$ and

$$\tilde{\eta}_n \rightharpoonup \eta_0 \text{ in } \prod_{k=1}^{m} L^{p'_k}(Q). \tag{3.14}$$

Since i^* is continuous in the weak topologies of both $\prod_{k=1}^m L^{p'_k}(Q)$ and X_0^*, it follows from (3.14) that

$$\eta_n = i^*(\tilde{\eta}_n) = \tilde{\eta}_n|_{X_0^*} \rightharpoonup i^*(\eta_0) = \eta_0|_{X_0^*} \tag{3.15}$$

weakly in X_0^*. From (3.7) and (3.15), we have $\eta = i^*(\eta_0) \in i^*F(u)$, since $\eta_n \rightharpoonup \eta$ and $\eta_n \rightharpoonup i^*(\eta_0)$ both in the sense of distribution. The inclusion (3.9) is thus verified, which completes our proof of the weakly closed property of $\mathcal{F}$.

Next, we prove that if $\{u_n\} \subset D(L)$, $\{\eta_n\} \subset X_0^*$ are sequences satisfying (3.6)-(3.8) then

$$\langle \eta_n, u_n \rangle_{X_0^*, X_0} \to \langle \eta, u \rangle_{X_0^*, X_0}. \tag{3.16}$$

In fact, let $\{\tilde{\eta}_n\}$ and η_0 be as above. We have

$$\begin{aligned}
\langle \eta_n, u_n \rangle_{X_0^*, X_0} &= \langle \tilde{\eta}_n|_{X_0^*}, u_n \rangle_{X_0^*, X_0} \\
&= \langle i^*(\tilde{\eta}_n), u_n \rangle_{X_0^*, X_0} \\
&= \langle \tilde{\eta}_n, i(u_n) \rangle_{\prod_{k=1}^m L^{p_k'}(Q), \prod_{k=1}^m L^{p_k}(Q)} \\
&= \langle \tilde{\eta}_n, u_n \rangle_{\prod_{k=1}^m L^{p_k'}(Q), \prod_{k=1}^m L^{p_k}(Q)}.
\end{aligned} \tag{3.17}$$

From (3.10) and (3.14), we have

$$\begin{aligned}
\langle \tilde{\eta}_n, u_n \rangle_{\prod_{k=1}^m L^{p_k'}(Q), \prod_{k=1}^m L^{p_k}(Q)} &\to \langle \eta_0, u \rangle_{\prod_{k=1}^m L^{p_k'}(Q), \prod_{k=1}^m L^{p_k}(Q)} \\
&= \langle \eta_0, i(u) \rangle_{\prod_{k=1}^m L^{p_k'}(Q), \prod_{k=1}^m L^{p_k}(Q)} \\
&= \langle i^*(\eta_0), u \rangle_{X_0^*, X_0} \\
&= \langle \eta, u \rangle_{X_0^*, X_0}.
\end{aligned}$$

This limit, together with (3.17), proves (3.16).

The weakly closed property of $\mathcal{F}$ and (3.16) show that $\mathcal{F}$ satisfies condition (iii) in Definition 3.1, which together with the results proved in Steps 1 and 2, shows that $\mathcal{F}$ is pseudomonotone from X_0 to $\mathcal{K}(X_0^*)$ with respect to $D(L)$. □

4 Main results

In this section we prove our main results about problem (1.1)–(1.2), which is equivalently rewritten in Sect. 2 as the following evolutionary multivalued variational inequality: Find $u \in D(L) \cap K$ and $\eta \in \mathcal{F}(u)$ such that

$$\langle Lu + Au + \eta, v - u \rangle \geq 0, \ \forall v \in K, \tag{4.1}$$

where L, $D(L)$, A, $\mathcal{F}$, and K are defined in (2.10), (2.11), (2.5), (2.9), (2.8), and (2.12).

By identifying $\eta = (\eta_1, \ldots, \eta_m) \in \prod_{k=1}^m L^{p_k'}(Q)$ with $i^*\eta = \eta|_{X_0} \in X_0^*$, we see that (4.1) is also written in the form: Find $u \in D(L) \cap K$ and an $\eta \in F(u)$ such that

$$\eta \in F(u), \ \langle Lu + Au, v - u \rangle + \int_Q \eta\, (v - u)\, dxdt \geq 0, \quad \forall\, v \in K. \tag{4.2}$$

In Sect. 4.1 we deal with the coercive case for (4.1) and in Sect. 4.2 a version of the method of sub-supersolution for (4.1) is established to treat the noncoercive case.

We first recall the following definition of a penalty operator associated with a convex set.

Definition 4.1 Let $C \neq \emptyset$ be a closed and convex subset of a reflexive Banach space Y. A bounded, hemicontinuous and monotone operator $P : Y \to Y^*$ is called a *penalty operator* associated with $C \subset Y$ if

$$P(u) = 0 \Longleftrightarrow u \in C.$$

In what follows, we assume that for each $k \in \{1, \dots, m\}$, there exists a penalty operator $P_k : X_{0k} \to X_{0k}^*$ associated with $K_k \subset X_{0k}$ with the following properties:

(P) For each $u_k \in D(L_k)$, there exists $w_k = w_k(u_k) \in X_{0k}$ such that

$$\begin{aligned} &\text{(i) } \langle u_{kt} + A_k u_k, w_k \rangle \geq 0, \quad \text{and} \\ &\text{(ii) } \langle P_k u_k, w_k \rangle \geq D_k \|P_k u_k\|_{X_{0k}^*} \|w_k\|_{L^{p_k}(Q)}, \end{aligned} \tag{4.3}$$

for some constant $D_k > 0$ independent of u_k and w_k.

For $u \in X_0$, let

$$Pu = (P_1 u_1, \dots, P_m u_m) \in X_0^*. \tag{4.4}$$

It is clear that P is a penalty operator associated with K.

4.1 Coercive case

In this subsection, we prove the existence of solutions of (4.1) under certain coercivity condition. More precisely, we have the following result.

Theorem 4.1 *Assume (A1)–(A3) and that f satisfies hypotheses (F1)–(F3). Suppose $D(L) \cap K \neq \emptyset$ and $u_0 \in D(L) \cap K$, and assume the existence of a penalty operator associated with K satisfying (P). Then, under the coercivity condition*

$$\lim_{\|u\|_{X_0} \to \infty} \left[\inf_{\eta \in \mathcal{F}(u)} \frac{\langle Au + \eta, u - u_0 \rangle}{\|u\|_{X_0}} \right] = \infty, \tag{4.5}$$

the multivalued parabolic variational inequality (4.1) has solutions.

Proof For $\varepsilon > 0$, let us consider the following penalized equation:

$$u \in D(L), \eta \in \mathcal{F}(u) : \langle u_t, v \rangle + \langle A(u) + \eta, v \rangle + \frac{1}{\varepsilon} \langle Pu, v \rangle = 0, \ \forall v \in X_0, \tag{4.6}$$

where P is a penalty operator (associated to K) defined in (4.4).

From Proposition 3.3, $\mathcal{F}$ is pseudomonotone with respect to $D(L)$. Since A and $\varepsilon^{-1} P$ are monotone and hemicontinuous, they are pseudomonotone and thus pseudomonotone with respect to $D(L)$ (cf. e.g. Proposition 27.6, [18]). As a consequence, $A + \mathcal{F} + \varepsilon^{-1} P$ is pseudomonotone with respect to $D(L)$. Moreover, it is bounded since A, P and $\mathcal{F}$ are bounded mappings. From the coercivity condition (4.5) and the monotonicity of $\varepsilon^{-1}P$, it is easy to see that $A+\mathcal{F}+\varepsilon^{-1}P$ is coercive on X_0 in the following sense:

$$\lim_{\|u\|_{X_0}\to\infty}\left[\inf_{\eta\in\mathcal{F}(u)}\frac{\langle (A+\varepsilon^{-1}P)(u)+\eta, u-u_0\rangle}{\|u\|_{X_0}}\right]=\infty. \tag{4.7}$$

According to the surjectivity result of [8, Theorem 1.3.73, p. 62], (4.6) has solutions for each $\varepsilon>0$. Let $u_\varepsilon\in D(L)$ and $\eta_\varepsilon\in\mathcal{F}(u_\varepsilon)$ satisfy (4.6). Let us show that the family $\{u_\varepsilon : \varepsilon>0,\ \text{small}\}$ is bounded with respect to the graph norm of $D(L)$. In fact, let u_0 be a (fixed) element of $D(L)\cap K$. Putting $v=u_\varepsilon-u_0$ into (4.6) (with u_ε) and noting the monotonicity of L and that $Pu_0=0$, one gets

$$\begin{aligned}\langle -u_{0t}, u_\varepsilon-u_0\rangle &= \langle u_{\varepsilon t}-u_{0t}, u_\varepsilon-u_0\rangle+\langle Au_\varepsilon+\eta_\varepsilon, u_\varepsilon-u_0\rangle\\ &\quad+\frac{1}{\varepsilon}\langle Pu_\varepsilon-Pu_0, u_\varepsilon-u_0\rangle\\ &\ge \langle Au_\varepsilon+\eta_\varepsilon, u_\varepsilon-u_0\rangle.\end{aligned}$$

Thus,

$$\frac{\langle Au_\varepsilon+\eta_\varepsilon, u_\varepsilon-u_0\rangle}{\|u_\varepsilon-u_0\|_{X_0}}\le\|u_{0t}\|_{X_0^*},$$

for all $\varepsilon>0$. From (4.5), we have that the set $\{\|u_\varepsilon\|_{X_0} : \varepsilon>0\}$ is bounded. As a consequence, we see that Au_ε stays bounded in X_0^*. Moreover, from the growth condition (2.7), we see that the set $\{\eta_\varepsilon : \varepsilon>0\}$ is bounded in $\prod_{k=1}^m L^{p_k'}(Q)$.

Next, let us check that the set $\{(\varepsilon^{-1}Pu_\varepsilon) : \varepsilon>0\}$ is also bounded in X_0^*. To see this, for each $k=1,\dots,m$ and $\varepsilon>0$, we choose $w_k=w_{\varepsilon k}$ to be an element satisfying (4.3) with $u_k=u_{\varepsilon k}$. From (4.6) with $v=(w_{\varepsilon k},[0]_k)$, we obtain

$$\langle u_{\varepsilon kt}, w_{\varepsilon k}\rangle+\langle A_k u_{\varepsilon k}+\eta_{\varepsilon k}, w_{\varepsilon k}\rangle+\frac{1}{\varepsilon}\langle P_k u_{\varepsilon k}, w_{\varepsilon k}\rangle=0.$$

From (4.3)(i), we see that $\langle u_{\varepsilon kt}, w_{\varepsilon k}\rangle+\langle A_k u_{\varepsilon k}, w_{\varepsilon k}\rangle\ge 0$. Therefore,

$$\frac{1}{\varepsilon}\langle P_k u_{\varepsilon k}, w_{\varepsilon k}\rangle\le\langle -\eta_{\varepsilon k}, w_{\varepsilon k}\rangle. \tag{4.8}$$

Since the set $\{\|\eta_\varepsilon\|_{\prod_{k=1}^m L^{p_k'}(Q)} : \varepsilon>0\}$ is bounded, there exists a constant $c>0$ such that

$$|\langle\eta_{\varepsilon k}, w_{\varepsilon k}\rangle|\le c\|w_{\varepsilon k}\|_{L^{p_k}(Q)},\ \ \forall\varepsilon.$$

This and (4.3)(ii) imply that for all $k\in\{1,\dots,m\}$,

$$\frac{1}{\varepsilon}\|P_k u_{\varepsilon k}\|_{X_{0k}^*}\le\frac{c}{D_k},\ \ \forall\varepsilon>0,$$

which proves the boundedness of the set $\{(\varepsilon^{-1}Pu_\varepsilon) : \varepsilon > 0\}$ in X_0^*. On the other hand, since

$$u_{\varepsilon t} = -(A + \varepsilon^{-1}P)(u_\varepsilon) - \eta_\varepsilon$$

in X_0^*, the above estimate implies that $(u_{\varepsilon t})$ is also bounded in X_0^*. Thus, we have shown that $\{u_\varepsilon : \varepsilon > 0\}$ is bounded with respect to the graph norm of $D(L)$. Hence, there exist $u \in X_0$, with $u_t \in X_0^*$, and a sequence $\{u_{\varepsilon_n}\}$, which is still denoted by $\{u_\varepsilon\}$, for simplicity of notation, such that

$$u_\varepsilon \rightharpoonup u \text{ in } X_0,\ \ u_{\varepsilon t} \rightharpoonup u_t \text{ in } X_0^* \ (\varepsilon \to 0^+). \tag{4.9}$$

Since $D(L)$ is closed in W_0 and convex, it is weakly closed in W_0, and thus $u \in D(L)$. Now, let us prove that u is a solution of the variational inequality (4.1). First, note that $Pu = 0$. In fact, we have $Pu_\varepsilon \to 0$ in X_0^*. It follows from the monotonicity of P that

$$\langle Pv, v - u\rangle \geq 0,\ \ \forall v \in X_0.$$

As in the proof of Minty's lemma (cf. [9]), one obtains from this inequality that

$$\langle Pu, v\rangle \geq 0,\ \ \forall v \in X_0.$$

Hence, $Pu = 0$ in X_0^*, that is, $u \in K$. On the other hand, (4.9) and Aubin's lemma imply that

$$u_\varepsilon \to u \ \text{ in } \prod_{k=1}^{m} L^{p_k}(Q). \tag{4.10}$$

As a consequence, we have

$$\langle \eta_\varepsilon, u_\varepsilon - u\rangle \to 0 \text{ as } \varepsilon \to 0^+. \tag{4.11}$$

For $w \in K$, letting $v = w - u_\varepsilon$ in (4.6) (with $u = u_\varepsilon$), one gets

$$\langle u_{\varepsilon t}, w - u_\varepsilon\rangle + \langle Au_\varepsilon + \eta_\varepsilon, w - u_\varepsilon\rangle = \frac{1}{\varepsilon}\langle -Pu_\varepsilon, w - u_\varepsilon\rangle \geq 0. \tag{4.12}$$

By choosing $w = u$ in (4.12), we have

$$\begin{aligned}\langle Au_\varepsilon, u - u_\varepsilon\rangle &\geq -\langle \eta_\varepsilon, u - u_\varepsilon\rangle - \langle u_t, u - u_\varepsilon\rangle + \langle u_t - u_{\varepsilon t}, u - u_\varepsilon\rangle \\ &\geq -\langle \eta_\varepsilon, u - u_\varepsilon\rangle - \langle u_t, u - u_\varepsilon\rangle.\end{aligned}$$

As a consequence, one gets

$$\liminf_{\varepsilon \to 0^+} \langle Au_\varepsilon, u - u_\varepsilon\rangle \geq 0.$$

Note that A is of class (S_+) with respect to $D(L)$, according to Proposition 3.2, we deduce from (4.9) and the above limit that

$$u_\varepsilon \to u \text{ in } X_0. \tag{4.13}$$

On the other hand, since $\{\eta_\varepsilon : \varepsilon > 0\}$ is bounded in X_0^*, by passing to a subsequence still denoted by $\{\eta_\varepsilon\}$ for simplicity of notation, we have

$$\eta_\varepsilon \rightharpoonup \eta \text{ in } X_0^*. \tag{4.14}$$

From (4.9) and the weak closedness of the mapping $\mathcal{F}$ with respect to $D(L)$ proved in Step 3 of Proposition 3.3, we have

$$\eta \in \mathcal{F}(u). \tag{4.15}$$

Letting $\varepsilon \to 0$ in (4.12) and taking (4.9), (4.13), (4.14) and the continuity of the operator A into account, we obtain

$$\langle u_t, w - u\rangle + \langle Au + \eta, w - u\rangle \geq 0.$$

This holds for all $w \in K$ which together with (4.15) proves that u is in fact a solution of (4.1). ◻

Penalty operators associated with obstacle problems

For $k = 1, \dots, m$, let $A_k = -\Delta_{p_k}$ $(p_k \geq 2)$ be the p_k-Laplacian. For an upward obstacle constraint, the convex set K_k is given by

$$K_k = \{u_k \in X_{0k} : u_k \leq \psi_k \ \text{ a.e. on } \ Q\}, \tag{4.16}$$

with ψ_k a given function in W_k such that $\psi_k(\cdot, 0) \geq 0$ on Ω, $\psi_k \geq 0$ on Γ, and $\psi_{kt} + A_k\psi_k \geq 0$ in X_{0k}^*, in the sense that

$$\langle \psi_{kt} + A_k\psi_k, v_k\rangle \geq 0, \ \forall v_k \in X_{0k} \cap L_+^{p_k}(Q).$$

In other words, the obstacle function ψ_k is assumed to be a (weak) supersolution of the following parabolic initial boundary value problem:

$$w_{kt} - \Delta_{p_k} w_k = 0, \ \ w_k(\cdot, 0) = 0 \text{ on } \Omega, \ \ w_k = 0 \text{ on } \Gamma.$$

Then the operator P_k given by

$$\langle P_k u_k, v_k\rangle = \int_Q [(u_k - \psi_k)^+]^{p_k - 1}\, v_k \, dxdt, \ \ \forall\, u_k, v_k \in X_{0k},$$

is easily seen to be a penalty operator, and, moreover, property (P) can be verified with $w_k(u_k) = (u_k - \psi_k)^+$.

Analogously, for a downward obstacle constraint, the convex set K_k is given by

$$K_k = \{u_k \in X_{0k} : u_k \geq \vartheta_k \ \text{ a.e. on } \ Q\}, \tag{4.17}$$

with $\vartheta_k \in W_k$, $\vartheta_k(\cdot, 0) \leq 0$ on Ω, $\vartheta_k \leq 0$ on Γ, and $\vartheta_{kt} + A_k\vartheta_k \leq 0$ in X_{0k}^*, i.e.,

$$\langle \vartheta_{kt} + A_k\vartheta_k, v_k\rangle \leq 0, \ \forall v_k \in X_{0k} \cap L_+^{p_k}(Q).$$

In the case of a lower obstacle constraint, the operator P_k given by

$$\langle P_k u_k, v_k\rangle = -\int_Q [(u_k - \vartheta_k)^-]^{p_k-1}\, v_k\, dxdt, \ \ \forall\, u_k, v_k \in X_{0k},$$

is a penalty operator for K_k that satisfies property (P), where $w_k = w_k(u_k)$ corresponding to $u_k \in D(L_k)$ is chosen as $w_k(u_k) = -(u_k - \vartheta_k)^-$.

We note that in an upward (resp. downward) obstacle system, all constraint sets K_k in (1.2) are of the form (4.16) (resp. (4.17)), while in mixed system of upward-downward obstacle problems, some of constraint sets K_k in (1.2) are given by (4.16), while the others are given by (4.17).

4.2 Noncoercive case

Note that when the growth condition (2.7) or the coercivity condition (4.5) is not fulfilled then the inequality (4.1) may not have solutions. However, without these conditions, we can still have the existence and other properties of solutions of (4.1) if sub- and supersolutions of (4.1), defined in a certain appropriate sense, exist. In this subsection we establish a sub-supersolution method for (4.1), which will allow us to derive existence and enclosure results for (4.1).

Let us first introduce our basic notion of sub-supersolution for the system of parabolic MVI (1.1)–(1.2). Let $\underline{u}, \overline{u} \in X_0$ be a pair of functions such that $\underline{u} \leq \overline{u}$. For $k = 1, \ldots, m$, we use the notation $Q_k = Q_{k,\underline{u},\overline{u}}$ for the cylinder based on Q and lying between $[\underline{u}]_k$ and $[\overline{u}]_k$:

$$Q_k = \{(x, t, [s]_k) \in Q \times \mathbb{R}^{m-1} : [\underline{u}(x, t)]_k \leq [s]_k \leq [\overline{u}(x, t)]_k \ \text{ for a.e. } \ (x, t) \in Q\}.$$

Definition 4.2 A pair of functions $\underline{u}, \overline{u} \in W$ is said to form an ordered pair of subsolution–supersolution of (4.1) if $\underline{u} \leq \overline{u}$ and the following conditions are satisfied.

(i) $\underline{u} \vee K \subset K, \overline{u} \wedge K \subset K$,
(ii) $\underline{u}_k(\cdot, 0) \leq 0$ in Ω, $\overline{u}_k(\cdot, 0) \geq 0$ in Ω $(k = 1, \ldots, m)$, and
(iii) for each $k \in \{1, \ldots, m\}$, there exist functions $\underline{\eta}_k, \overline{\eta}_k : Q_k \to \mathbb{R}$ such that for any $[w]_k \in [\underline{u}, \overline{u}]_k$, the functions $(x, t) \mapsto \underline{\eta}_k(x, t, [w(x, t)]_k)$ and $(x, t) \mapsto \overline{\eta}_k(x, t, [w(x, t)]_k)$ belongs to $L^{p_k'}(Q)$,

$$\underline{\eta}_k(x, t, [w(x, t)]_k) \in f_k(x, t, \underline{u}_k(x, t), [w(x, t)]_k), \tag{4.18}$$

$$\overline{\eta}_k(x,t,[w(x,t)]_k) \in f_k(x,t,\overline{u}_k(x,t),[w(x,t)]_k), \tag{4.19}$$

for a.e. $(x,t) \in Q$, and

$$\langle \underline{u}_{kt} + A_k\underline{u}_k, v_k - \underline{u}_k\rangle + \int_Q \underline{\eta}_k(\cdot,\cdot,[w]_k)\,(v_k - \underline{u}_k)\,dxdt \geq 0,\ \forall v_k \in \underline{u}_k \wedge K_k, \tag{4.20}$$

and

$$\langle \overline{u}_{kt} + A_k\overline{u}_k, v_k - \overline{u}_k\rangle + \int_Q \overline{\eta}_k(\cdot,\cdot,[w]_k)\,(v_k - \overline{u}_k)\,dxdt \geq 0,\ \forall v_k \in \overline{u}_k \vee K_k. \tag{4.21}$$

Throughout this subsection instead of the growth condition (F3) of the preceding section we assume the following local growth assumption with respect to the ordered interval of sub-supersolutions.

(F4) Assume that there exists a pair of sub-supersolutions $\underline{u}$ and $\overline{u}$ of (4.1) such that for all $k \in \{1,\ldots,m\}$, f_k has the following growth between $\underline{u}$ and $\overline{u}$:

$$|\eta| \leq c_5^{(k)}(x,t),\quad \forall\, \eta \in f_k(x,t,s), \tag{4.22}$$

for a.e. $(x,t) \in Q$, and all $s \in [\underline{u}(x,t), \overline{u}(x,t)]$, for some $c_5^{(k)} \in L^{p_k'}(Q)$.

We note that (F3) implies (F4), that is, the local growth condition (F4) is a weaker condition.
We are now ready to state and prove our main existence and enclosure result.

Theorem 4.2 *Assume (A1)–(A3) and that (4.1) has an ordered pair of sub- and supersolutions $\underline{u}$ and $\overline{u}$, and that (F1)–(F2), (F4) are satisfied. Suppose furthermore that $D(L) \cap K \neq \emptyset$, and that there exists a penalty operator associated with K satisfying (P). Then, (4.1) has a solution u such that $\underline{u} \leq u \leq \overline{u}$.*

Proof For $k = 1,\ldots,m$, we define the following cut-off function $b_k : Q \times \mathbb{R} \to \mathbb{R}$:

$$b_k(x,t,s) = \begin{cases} [s - \overline{u}_k(x,t)]^{p_k-1} & \text{if } s > \overline{u}_k(x,t) \\ 0 & \text{if } \underline{u}_k(x,t) \leq s \leq \overline{u}_k(x,t) \\ -[\underline{u}_k(x,t) - s]^{p_k-1} & \text{if } s < \underline{u}_k(x,t), \end{cases}$$

for $(x,t) \in Q \times \mathbb{R}$. It is easy to check that b_k is a Carathéodory function with the growth condition

$$|b_k(x,t,s)| \leq c_6^{(k)}(x,t) + c_7^{(k)}|s|^{p_k-1},\quad \text{for a.e. } (x,t) \in Q,\ \text{all } s \in \mathbb{R}, \tag{4.23}$$

with $c_6^{(k)} \in L^{p_k'}(Q)$, $c_7^{(k)} > 0$. Hence, the Nemytskij operator $B_k : u \mapsto b_k(\cdot,\cdot,u)$ is a continuous and bounded mapping from $L^{p_k}(Q)$ to $L^{p_k'}(Q)$ and $\mathcal{B}_k = i_k^* \circ B_k \circ i_k$:

$X_{0k} \to X_{0k}^*$ is given by

$$\langle \mathcal{B}_k u, v\rangle = \int_Q b_k(\cdot,\cdot,u)\, v\, dxdt, \ \forall u, v \in X_{0k}. \tag{4.24}$$

Moreover, there are $c_8^{(k)}, c_9^{(k)} > 0$ such that

$$\int_Q b_k(\cdot,\cdot,u)u\, dxdt \geq c_8^{(k)} \|u\|_{L^{p_k}(Q)}^{p_k} - c_9^{(k)}, \ \forall u \in L^{p_k}(Q). \tag{4.25}$$

Let $\mathcal{B} : X_0 \to X_0^*$ be defined by $\mathcal{B}u = (\mathcal{B}_1 u_1, \ldots, \mathcal{B}_m u_m)$ for $u \in X_0$. We have from (4.25) that

$$\langle \mathcal{B}u, u\rangle \geq c_8 \sum_{k=1}^m \|u\|_{L^{p_k}(Q)}^{p_k} - c_9, \ \forall u \in X_0, \tag{4.26}$$

for some constants $c_8, c_9 > 0$. For $k \in \{1, \ldots, m\}$, $(x,t) \in Q$, $u_k \in \mathbb{R}$, let us define the truncation function T_k as follows:

$$(T_k u_k)(x,t) = \begin{cases} \overline{u}_k(x,t) & \text{if } u_k > \overline{u}_k(x,t), \\ u_k & \text{if } \underline{u}_k(x,t) \leq u_k \leq \overline{u}_k(x,t), \\ \underline{u}_k(x,t) & \text{if } u_k < \underline{u}_k(x,t). \end{cases} \tag{4.27}$$

In other words,

$$(T_k u_k)(x,t) = [u_k \wedge \overline{u}_k(x,t)] \vee \underline{u}_k(x,t) = [u_k \vee \underline{u}_k(x,t)] \wedge \overline{u}_k(x,t).$$

Straightforward calculations show that T_k is continuous and bounded from $L^{p_k}(Q)$ (resp. X_{0k}) into itself. The corresponding truncated vector function for $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$, Tu is given by

$$(Tu)(x,t) = ((T_1 u_1)(x,t), \ldots, (T_m u_m)(x,t)), \tag{4.28}$$

and as above,

$$[Tu]_k(x,t) = ((T_j u_j)(x,t) : j \in \{1, \ldots m\} \setminus \{k\}). \tag{4.29}$$

For $k = 1, \ldots, m$, we define next the truncated function $f_{0k} : Q \times \mathbb{R}^m \to 2^{\mathbb{R}}$ of f_k as follows:

$$f_{0k}(x,t,u) = \begin{cases} \{\underline{\eta}_k(x,t,[Tu(x,t)]_k)\} & \text{if } u_k < \underline{u}_k(x,t) \\ f_k(x,t,u_k,[Tu(x,t)]_k) & \text{if } \underline{u}_k(x,t) \leq u_k \leq \overline{u}_k(x,t) \\ \{\overline{\eta}_k(x,t,[Tu(x,t)]_k)\} & \text{if } u_k > \overline{u}_k(x,t), \end{cases} \tag{4.30}$$

for $(x,t,u) \in Q \times \mathbb{R}^m$, where $\underline{\eta}$ and $\overline{\eta}$ correspond to $\underline{u}$ and $\overline{u}$ as in Definition 4.2.

Let $f_0 = (f_{01}, \ldots, f_{0m})$. Since f satisfies (F1) and (F2), in view of (4.18) and (4.19), we can check that f_0 satisfies (F1) and (F2) as well. Moreover, as a consequence of (4.27), (4.18), (4.19), and the growth condition (4.22) in (F4), f_0 also satisfies (2.7) of (F3) with $\beta_k = 0$ and $\alpha_k = c_5^{(k)} \in L^{p_k'}(Q)$. For $u : Q \to \mathbb{R}$ measurable, let

$$F_{0k}(u) = \{\eta : Q \to \mathbb{R} : \eta \text{ is measurable on } Q \text{ and } \eta(x,t) \in f_{0k}(x,t,u(x,t))\},$$

for $k = 1, \ldots, m$, and

$$\begin{aligned} F_0(u) &= \prod_{k=1}^{m} F_{0k}(u) \\ &= \{\eta : Q \to \mathbb{R}^m : \eta \in [L^0(Q)]^m \text{ and } \eta(x,t) \in f_0(x,t,u(x,t))\}. \end{aligned}$$

From (F4) it follows that $F_0(u) \subset \prod_{k=1}^{m} L^{p_k'}(Q)$ for any measurable function u defined on Q, which allows us to define the Nemytskij operator of f_0,

$$F_0 : \prod_{k=1}^{m} L^{p_k}(Q) \to 2^{\prod_{k=1}^{m} L^{p_k'}(Q)}, \quad u \mapsto F_0(u),$$

and its related mapping

$$\mathcal{F}_0 : X_0 \to 2^{X_0^*}, \quad \mathcal{F}_0 = i^* \circ F_0 \circ i.$$

For any $u \in X_0$, we have $\mathcal{F}_0(u) = \prod_{k=1}^{m} \mathcal{F}_{0k}(u)$, where $\mathcal{F}_{0k} = i_k^* \circ F_{0k} \circ i_k$. We see that $\mathcal{F}_0$ is pseudomonotone with respect to $D(L)$, according to Proposition 3.3. Let us consider the following auxiliary variational inequality:

$$u \in D(L) \cap K, \eta \in \mathcal{F}_0(u) : \langle Lu + Au + \mathcal{B}u + \eta, v - u \rangle \geq 0, \ \forall v \in K. \quad (4.31)$$

It is clear from its definition that $\mathcal{B}$ is a (single-valued) pseudomonotone mapping w.r.t. $D(L)$ from X_0 to X_0^*. Moreover, $f_1 = b + f_0$ satisfies (F1)–(F3), and thus $\mathcal{F}_1 = \mathcal{B} + \mathcal{F}_0$ is pseudomonotone with respect to $D(L)$ according to Proposition 3.3.

Now, let us verify that $A + \mathcal{B} + \mathcal{F}_0$ is coercive on X_0 in the following sense:

$$\lim_{\|u\|_{X_0} \to \infty} \left[\inf_{\eta \in \mathcal{F}_0(u)} \frac{\langle Au + \mathcal{B}u + \eta, u - \varphi \rangle}{\|u\|_{X_0}} \right] = \infty, \quad (4.32)$$

for any (fixed) $\varphi \in X_0$. In fact, from (A3), we have

$$\langle Au, u \rangle \geq c_3 \sum_{k=1}^{m} \| \, |\nabla u_k| \, \|_{L^{p_k}(Q)}^{p_k} - c_{10}, \ \forall u \in X_0, \quad (4.33)$$

with some constants $c_3, c_{10} > 0$. For $\eta \in \mathcal{F}_0(u)$, $\eta = i^*\tilde{\eta}$ with $\tilde{\eta} \in F_0(u)$, we have

$$\begin{aligned} |\langle \eta, u\rangle| &\le \sum_{k=1}^{m} \left| \int_Q \tilde{\eta}_k u_k \, dxdt \right| \\ &\le \sum_{k=1}^{m} \|c_5^{(k)}\|_{L^{p_k'}(Q)} \|u_k\|_{L^{p_k}(Q)}. \end{aligned} \tag{4.34}$$

Combining (4.26) with (4.33) and (4.34), one gets for all $u \in X_0$

$$\begin{aligned} \langle (Au + \mathcal{B}u + \eta, u\rangle \ge c_3 \sum_{k=1}^{m} \| \, |\nabla u_k| \, \|_{L^{p_k}(Q)}^{p_k} + c_8 \sum_{k=1}^{m} \|u\|_{L^{p_k}(Q)}^{p_k} \\ - \sum_{k=1}^{m} \|c_5^{(k)}\|_{L^{p_k'}(Q)} \|u_k\|_{L^{p_k}(Q)} - c_9 - c_{10}. \end{aligned} \tag{4.35}$$

For $\varphi \in X_0$ fixed, it is inferred from (A1), (4.23), and (4.22) that

$$|\langle Au + \mathcal{B}u + \eta, \varphi\rangle| \le c_{11} \left(\sum_{k=1}^{m} \|u\|_{X_{0k}}^{p_k - 1} + 1 \right), \ \forall u \in X_0, \tag{4.36}$$

for some constant $c_{11} > 0$. From (4.35) and (4.36), we obtain (4.32). Let $u_0 \in D(L) \cap K$ be fixed. With the particular choice of $\varphi = u_0$, we see that all conditions of Theorem 4.1 are fulfilled with $\mathcal{F}_1 = \mathcal{B} + \mathcal{F}_0$ in place of $\mathcal{F}$. According to Theorem 4.1, (4.31) has solutions.

Next, we show that any solution u of (4.31) satisfies: $\underline{u} \le u \le \overline{u}$ a.e. in Q. We verify that $\underline{u} \le u$, the second inequality is proved in the same way. Let u be a solution of (4.31), which is equivalent to the system

$$u_k \in D(L_k) \cap K_k, \eta_k \in \mathcal{F}_{0k}(u) : \langle u_{kt} + A_k u_k + \mathcal{B}_k u_k + \eta_k, v_k - u_k\rangle \ge 0, \ \forall v_k \in K_k, \tag{4.37}$$

with $k = 1, \ldots, m$. Because $u_k \in K_k$, it follows that

$$u_k + (\underline{u}_k - u_k)^+ = \underline{u}_k \vee u_k \in K_k.$$

Letting $v_k = u_k + (\underline{u}_k - u_k)^+$ into (4.37), one gets

$$\langle u_{kt}, (\underline{u}_k - u_k)^+\rangle + \langle A_k u_k + \mathcal{B}_k u_k + \eta_k, (\underline{u}_k - u_k)^+\rangle \ge 0. \tag{4.38}$$

On the other hand, let $\underline{\eta}$ be associated with the subsolution $\underline{u}$ as in Definition 4.2. For $[w]_k = [Tu]_k \in [\underline{u}, \overline{u}]_k$, and

$$v_k = \underline{u}_k - (\underline{u}_k - u_k)^+ = \underline{u}_k \wedge u_k \in \underline{u}_k \wedge K_k,$$

we have from (4.20) that

$$-\langle \underline{u}_{kt}, (\underline{u}_k - u_k)^+\rangle - \langle A_k \underline{u}_k, (\underline{u}_k - u_k)^+\rangle - \langle i_k^* \underline{\eta}_k(\cdot, \cdot, [Tu]_k), (\underline{u}_k - u_k)^+\rangle \geq 0. \quad (4.39)$$

Adding (4.38) and (4.39) yields

$$\begin{aligned} &\langle (u_k - \underline{u}_k)_t, (\underline{u}_k - u_k)^+\rangle + \langle A_k u_k - A_k \underline{u}_k + \mathcal{B}_k u_k, (\underline{u}_k - u_k)^+\rangle \\ &\quad + \langle \eta_k - i_k^* \underline{\eta}_k(\cdot, \cdot, [Tu]_k), (\underline{u}_k - u_k)^+\rangle \geq 0. \end{aligned} \quad (4.40)$$

We have $\underline{u}_k - u_k \in W_k$ and $(\underline{u}_k - u_k)^+(\cdot, 0) = 0$, and thus

$$\langle (\underline{u}_k - u_k)_t, (\underline{u}_k - u_k)^+\rangle = \frac{1}{2}\|(\underline{u}_k - u_k)^+(\cdot, \tau)\|^2_{L^2(\Omega)} \geq 0. \quad (4.41)$$

On the other hand, it is easy to check from (A2) that

$$\langle A_k \underline{u}_k - A_k u_k, (\underline{u}_k - u_k)^+\rangle \geq 0. \quad (4.42)$$

Moreover, with $\eta_k = i_k^* \tilde{\eta}_k$, $\tilde{\eta}_k \in F_{0k}(u)$, we have

$$\begin{aligned} &\langle \eta_k - i_k^* \underline{\eta}_k(\cdot, \cdot, [Tu]_k), (\underline{u}_k - u_k)^+\rangle \\ &\quad = \int_Q (\tilde{\eta}_k(x,t) - \underline{\eta}_k(x,t,[Tu(x,t)]_k))(\underline{u}_k(x,t) - u_k(x,t))^+ \, dxdt \\ &\quad = \int_{\{\underline{u}_k > u_k\}} (\tilde{\eta}_k(x,t) - \underline{\eta}_k(x,t,[Tu(x,t)]_k))(\underline{u}_k(x,t) - u_k(x,t)) \, dxdt, \end{aligned}$$

where $\{\underline{u}_k > u_k\} = \{(x,t) \in Q : \underline{u}_k(x,t) > u_k(x,t)\}$. But because of (4.30), we have

$$\tilde{\eta}_k(x,t) = \underline{\eta}_k(x,t,[Tu(x,t)]_k)) \text{ for a.e. } (x,t) \in \{\underline{u}_k > u_k\}.$$

Therefore

$$\langle \eta_k - i_k^* \underline{\eta}_k(\cdot, \cdot, [Tu]_k), (\underline{u}_k - u_k)^+\rangle = 0. \quad (4.43)$$

Combining (4.41)–(4.43) with (4.40), we obtain

$$0 \leq \langle \mathcal{B}_k u_k, (\underline{u}_k - u_k)^+\rangle = -\int_{\{\underline{u}_k > u_k\}} (\underline{u}_k - u_k)^{p_k} \, dxdt \leq 0.$$

This proves that $\underline{u}_k - u_k = 0$ a.e. on $\{\underline{u}_k > u_k\}$, i.e., $\{\underline{u}_k > u_k\}$ has measure zero, and thus $\underline{u}_k \leq u_k$ a.e. on Q. Since this holds true for all $k = 1, \ldots, m$, we have $\underline{u} \leq u$. A similar proof shows that $u \leq \overline{u}$. From $\underline{u} \leq u \leq \overline{u}$, we have $\mathcal{B}u = 0$ and $\mathcal{F}_0(u) \subset \mathcal{F}(u)$. Consequently, a solution u of (4.31) is also a solution of (4.1). □

5 Application: obstacle problem

In this section we deal with the system of multivalued parabolic variational inequalities (1.1)–(1.2) with $A_k = -\Delta_{p_k}$ ($p_k \geq 2$) being the p_k-Laplacian, and under upward obstacle constraints K_k given by (4.16), that is,

$$K_k = \{u_k \in X_{0k} : u_k \leq \psi_k \ \text{ a.e. on } \ Q\}, \tag{5.1}$$

with ψ_k a given function in W_k such that $\psi_k(\cdot, 0) \geq 0$ on Ω, $\psi_k \geq 0$ on Γ, and $\psi_{kt} + A_k\psi_k \geq 0$ in X_{0k}^*, in the sense that

$$\langle \psi_{kt} + A_k\psi_k, v_k\rangle \geq 0, \ \forall v_k \in X_{0k} \cap L_+^{p_k}(Q).$$

In other words, the obstacle function ψ_k is assumed to be a (weak) supersolution of the parabolic initial boundary value problem:

$$v_t - \Delta_{p_k} v = 0, \quad v(\cdot, 0) = 0 \text{ on } \Omega, \quad v = 0 \text{ on } \Gamma.$$

Thus, by comparison we have $\psi_k(x, t) \geq 0$ for a.a. $(x, t) \in Q$. Moreover, it has been shown in Sect. 4 that there exists a penalty operator P_k associated with K_k satisfying property (P).

Assuming hypotheses (F1)–(F3) for the multivalued lower order terms $f_k : Q \times \mathbb{R}^m \to 2^{\mathbb{R}}\setminus\{\emptyset\}$, our main goal is to construct an ordered pair of sub-supersolutions for the obstacle problem. Only for simplifying the presentation in this section, we assume $\Omega = B(0, 1)$ with $B(0, 1)$ being the unit ball in $\mathbb{R}^N$. Further, let $\Omega_R = B(0, R)$ be the ball with radius $R > 1$. For $k = 1, \ldots, m$, let $h_k \in W_0^{1,p_k}(\Omega_R)$ be the unique weak solution of

$$-\Delta_{p_k} h_k = 1 \quad \text{in } \Omega_R, \quad h_k = 0 \quad \text{on } \partial\Omega_R, \tag{5.2}$$

which means

$$h_k \in W_0^{1,p_k}(\Omega_R) : -\Delta_{p_k} h_k = 1 \quad \text{in } (W_0^{1,p_k}(\Omega_R))^*. \tag{5.3}$$

Let $s^- = \max\{-s, 0\}$ for $s \in \mathbb{R}$, and using $-h_k^- \in W_0^{1,p_k}(\Omega_R)$ as a test function in (5.3), we see that

$$\langle -\Delta_{p_k} h_k, -h_k^-\rangle = \|\nabla h_k^-\|_{L^p(\Omega_R)}^p = -\int_{\Omega_R} h_k^-(x)\,dx \leq 0,$$

which implies that $h_k^- = 0$, and thus $h_k \geq 0$. From the nonlinear regularity theory (cf., e.g. [12]) we have $h_k \in C_0^1(\overline{\Omega_R})$, and from the nonlinear strong maximum principle due to Vazquez (see [17]) we infer that $h_k \in \operatorname{int}(C_0^1(\overline{\Omega_R})_+)$. Here $\operatorname{int}(C_0^1(\overline{\Omega_R})_+)$ denotes the interior of the positive cone $C_0^1(\overline{\Omega_R})_+ = \{u \in C_0^1(\overline{\Omega_R}) : u(x) \geq 0, \ \forall x \in \Omega_R\}$ in the Banach space $C_0^1(\overline{\Omega_R}) = \{u \in C^1(\overline{\Omega_R}) : u(x) = 0, \ \forall x \in \partial\Omega_R\}$, given by

$$\operatorname{int}(C_0^1(\overline{\Omega_R})_+) = \left\{ u \in C_0^1(\overline{\Omega_R}) \ : \ u(x) > 0, \ \forall x \in \Omega_R, \text{ and } \frac{\partial u}{\partial n}(x) < 0, \ \forall x \in \partial\Omega_R \right\},$$

where $n = n(x)$ is the outer unit normal at $x \in \partial\Omega_R$. We are going to construct a pair of sub-supersolutions by means of the solutions h_k of the Dirichlet problem (5.3) on $\Omega_R = B(0, R)$ with $R > 0$. Since the lower order multivalued nonlinearities $f_k : Q \times \mathbb{R}^m \to \mathcal{K}(\mathbb{R})$ satisfy (F1)–(F3), we have the following representation of f_k for a.a. $(x, t) \in Q = B(0, 1) \times (0, \tau)$ and for all $s \in \mathbb{R}^m$

$$f_k(x, t, s) = [\underline{f_k}(x, t, s), \overline{f_k}(x, t, s)]. \tag{5.4}$$

By means of [11, Proposition 4.2] we see that the (single-valued) functions $\underline{f_k}$, $\overline{f_k}$: $Q \times \mathbb{R}^m \to \mathbb{R}$ have the following properties for a.a. $(x, t) \in Q$ and for all $s \in \mathbb{R}^m$:

$$\begin{aligned}
&(x, t) \mapsto \underline{f_k}(x, t, s), \ (x, t) \mapsto \overline{f_k}(x, t, s) \text{ are measurable on } Q,\\
&s \mapsto \underline{f_k}(x, t, s) \text{ is lower semicontinuous on } \mathbb{R}^m,\\
&s \mapsto \overline{f_k}(x, t, s) \text{ is upper semicontinuous on } \mathbb{R}^m.
\end{aligned}$$

Thus $\underline{f_k}$, $\overline{f_k}$: $Q \times \mathbb{R}^m \to \mathbb{R}$ belong to a Baire–Carathéodory class, and are therefore superpositionally measurable, that is, the associated Nemytskij operators $\underline{F_k}(u)(x, t) = \underline{f_k}(x, t, u(x, t))$, and $\overline{F_k}(u)(x, t) = \overline{f_k}(x, t, u(x, t))$ map measurable functions into measurable functions. We now make the following assumption on the (single-valued) functions $\underline{f_k}$, $\overline{f_k} : Q \times \mathbb{R}^m \to \mathbb{R}$:

(H) There exist functions $\underline{c_k} \in L^\infty(Q)$ and $\overline{c_k} \in L^\infty(Q)$ such that for a.a. $(x, t) \in Q$ and for all $s \in \mathbb{R}^m$ we have

$$\underline{f_k}(x, t, s) \le \underline{c_k}(x, t), \quad \overline{f_k}(x, t, s) \ge \overline{c_k}(x, t), \ \ k = 1, \dots, m. \tag{5.5}$$

Assume $0 \notin f_i(x, t, 0)$ for at least one $i \in \{1, \dots, m\}$.

We note that hypothesis (H) excludes the trivial solution, and the one-sided bounds in (5.5) still allow the multi-valued functions f_k to be unbounded.

We are now in the position to explicitly construct an ordered pair of sub-supersolution for the (upward) obstacle problem (1.1)–(1.2) with $A_k = -\Delta_{p_k}$ ($p_k \ge 2$), and K_k given by (5.1).

Theorem 5.1 *Assume (F1)–(F3) for the multivalued lower order terms f_k and let hypothesis (H) on the single-valued functions $\underline{f_k}$, $\overline{f_k}$ generating f_k through (5.4) be satisfied. Then*

$$\begin{aligned}
&\underline{u}(x, t) = (-M_1\phi_1(t)h_1(x), \dots, -M_m\phi_m(t)h_m(x)) \ \text{ and}\\
&\overline{u}(x, t) = (M_1\phi_1(t)h_1(x), \dots, M_m\phi_m(t)h_m(x)), \ \ (x, t) \in Q,
\end{aligned}$$

form an ordered pair of sub- and supersolution for $M_k > 0$ sufficiently large, where h_k are the positive solutions of problem (5.2) on Ω_R, and $\phi_k \in C^1([0, \tau])$ are supposed to satisfy $\phi_k(0) = 0$, and $\phi_k(t) \ge 0$, $\phi_k'(t) \ge d_k > 0$, $\forall\, t \in [0, \tau]$, $k = 1, \dots, m$.

Proof Let us verify that $\underline{u}$ and $\overline{u}$ satisfy Definition 4.2. Clearly, we have $\underline{u} \le \overline{u}$ and properties (i) and (ii) of Definition 4.2 with K_k given by (5.1) are satisfied. So it remains to check property (iii) in Definition 4.2, that is, we need to show the existence of functions $\underline{\eta}_k, \overline{\eta}_k : Q_k \to \mathbb{R}$ such that for any $[w]_k \in [\underline{u}, \overline{u}]_k$, the functions $(x,t) \mapsto \underline{\eta}_k(x,t,[w(x,t)]_k)$ and $(x,t) \mapsto \overline{\eta}_k(x,t,[w(x,t)]_k)$ belong to $L^{p_k'}(Q)$, and

$$\underline{\eta}_k(x,t,[w(x,t)]_k) \in f_k(x,t,\underline{u}_k(x,t),[w(x,t)]_k), \tag{5.6}$$
$$\overline{\eta}_k(x,t,[w(x,t)]_k) \in f_k(x,t,\overline{u}_k(x,t),[w(x,t)]_k), \tag{5.7}$$

for a.e. $(x,t) \in Q$, and

$$\langle \underline{u}_{kt} - \Delta_{p_k}\underline{u}_k, v_k - \underline{u}_k\rangle + \int_Q \underline{\eta}_k(\cdot,\cdot,[w]_k)\,(v_k - \underline{u}_k)\,dxdt \ge 0,\ \forall\, v_k \in \underline{u}_k \wedge K_k, \tag{5.8}$$

and

$$\langle \overline{u}_{kt} - \Delta_{p_k}\overline{u}_k, v_k - \overline{u}_k\rangle + \int_Q \overline{\eta}(\cdot,\cdot,[w]_k)\,(v_k - \overline{u}_k)\,dxdt \ge 0,\ \forall\, v_k \in \overline{u}_k \vee K_k, \tag{5.9}$$

where $\underline{u}_k(x,t) = -M_k\phi_k(t)h_k(x)$ and $\overline{u}_k(x,t) = M_k\phi_k(t)h_k(x)$. Let $\varphi_k \in K_k$, then $v_k \in \underline{u}_k \wedge K_k$ has the representation $v_k = \underline{u}_k - (\underline{u}_k - \varphi_k)^+$, and thus (5.8) becomes

$$\langle \underline{u}_{kt} - \Delta_{p_k}\underline{u}_k, (\underline{u}_k - \varphi_k)^+\rangle + \int_Q \underline{\eta}_k(\cdot,\cdot,[w]_k)\,(\underline{u}_k - \varphi_k)^+\,dxdt \le 0,\ \forall\, \varphi_k \in K_k. \tag{5.10}$$

Similarly, $v_k \in \overline{u}_k \vee K_k$ can be written as $v_k = \overline{u}_k + (\varphi_k - \overline{u}_k)^+$ with $\varphi_k \in K_k$, and thus (5.9) becomes

$$\langle \overline{u}_{kt} - \Delta_{p_k}\overline{u}_k, (\varphi_k - \overline{u}_k)^+\rangle + \int_Q \overline{\eta}(\cdot,\cdot,[w]_k)\,(\varphi_k - \overline{u}_k)^+\,dxdt \ge 0,\ \forall\, \varphi_k \in K_k. \tag{5.11}$$

We are going to verify (5.10) with $\underline{\eta}_k$ given by

$$\underline{\eta}_k(x,t,[w(x,t)]_k) = \underline{f_k}(x,t,-M_k\phi_k(t)h_k(x),[w(x,t)]_k) \text{ for } (x,t) \in Q. \tag{5.12}$$

Since $\underline{f_k}$ is superpositionally measurable, the growth condition (F3) on f_k implies that $\underline{\eta}_k$ given by (5.12) belongs to $L^{p_k'}(Q)$. Applying hypothesis (H) we have for all $[w(x,t)]_k$

$$\underline{f_k}(x,t,-M_k\phi_k(t)h_k(x),[w(x,t)]_k) \le \underline{c_k}(x,t),$$

and thus we get the following inequalities (in the weak sense)

$$\begin{aligned}
&\underline{u}_{kt} - \Delta_{p_k}\underline{u}_k + \underline{\eta}_k(\cdot,\cdot,[w]_k)\\
&\quad = -M_k\phi_k' h_k - (M_k\phi_k)^{p_k-1} + \underline{f_k}(x,t,-M_k\phi_k(t)h_k(x),[w(x,t)]_k)
\end{aligned}$$

$$\leq -M_k d_k h_k + \underline{c_k}(x,t)$$

We note that h_k is the positive solution on the bigger ball Ω_R with $R > 1$, and therefore the restriction of h_k on $\overline{\Omega} = \overline{B(0,1)}$ has a positive minimum, that is,

$$\min_{x \in \overline{B(0,1)}} h_k(x) = \delta_k > 0,$$

which yields the estimate

$$\begin{aligned}
&\underline{u}_{kt} - \Delta_{p_k} \underline{u}_k + \underline{\eta}_k(\cdot,\cdot,[w]_k) \\
&\quad = -M_k \phi_k' h_k - (M_k \phi_k)^{p_k - 1} + \underline{f_k}(x,t,-M_k\phi_k(t)h_k(x),[w(x,t)]_k) \\
&\quad \leq -M_k d_k h_k + \underline{c_k}(x,t) \\
&\quad \leq -M_k d_k \delta_k + \|\underline{c_k}\|_{L^\infty(Q)} \leq 0, \quad \text{for} \quad M_k \text{ large}
\end{aligned}$$

and thus (5.10) is verified. Let us next check (5.11). To this end we take

$$\overline{\eta}_k(x,t,[w(x,t)]_k) = \overline{f_k}(x,t,M_k\phi_k(t)h_k(x),[w(x,t)]_k) \text{ for } (x,t) \in Q. \tag{5.13}$$

By the same arguments as for $\underline{\eta}_k$ we have that $\overline{\eta}_k \in L^{p_k'}(Q)$, and from hypothesis (H) we get, for all $[w(x,t)]_k$,

$$\overline{f_k}(x,t,M_k\phi_k h_k,[w(x,t)]_k) \geq \overline{c_k}(x,t).$$

Using the definition of h_k we obtain the following inequalities (in the weak sense) with $\overline{u}_k(x,t) = M_k\phi_k(t)h_k(x)$

$$\begin{aligned}
&\overline{u}_{kt} - \Delta_{p_k} \overline{u}_k + \overline{\eta}(\cdot,\cdot,[w]_k) \\
&\quad = M_k \phi_k' h_k + (M_k \phi_k)^{p_k - 1} + \overline{f_k}(x,t,M_k\phi_k h_k,[w(x,t)]_k) \\
&\quad \geq M_k d_k h_k - \|\overline{c_k}\|_{L^\infty(Q)} \geq 0, \quad \text{for } M_k \text{ large,}
\end{aligned}$$

which proves (5.11). This completes the proof of Theorem 5.1. □

An immediate consequence is the following corollary.

Corollary 5.1 *Under the hypotheses of Theorem 5.1 there exists a solution u of the (upward) obstacle problem (1.1)–(1.2) with $A_k = -\Delta_{p_k}$ ($p_k \geq 2$), and K_k given by (5.1) satisfying*

$$\underline{u} \leq u \leq \overline{u} \wedge \psi,$$

for $M_k > 0$ sufficiently large, where $\underline{u}$ and $\overline{u}$ are as in Theorem 5.1, and ψ is the obstacle function.

Proof Since $(\underline{u}, \overline{u})$ is an ordered pair of sub-supersolution, by Theorem 4.2 there exists a solution $u \in [\underline{u}, \overline{u}]$ of the obstacle problem with K_k given by (5.1), and thus $\underline{u} \leq u \leq \overline{u} \wedge \psi$. □

6 Systems of evolutionary variational-hemivariational inequalitiesI

Variational-hemivariational inequalities have been proved a powerful tool to describe relevant models in mechanical engineering, and have been first introduced by Panagiotopoulos see e.g. [14],[15]. With the notations of the preceding sections, in this section we consider the following system of evolutionary variational-hemivariational inequalities: Find $u \in D(L) \cap K$ such that

$$\langle Lu + Au, v - u\rangle + \int_Q \sum_{k=1}^m j_k^o(x, t, u_k, [u]_k; v_k - u_k)\, dxdt \ge 0, \quad \forall\, v \in K, \quad (6.1)$$

which is equivalent to ($k = 1, \dots, m$)

$$\langle u_{kt} + A_k u_k, v_k - u_k\rangle + \int_Q j_k^o(x, t, u_k, [u]_k; v_k - u_k)\, dxdt \ge 0, \quad \forall\, v_k \in K_k. \quad (6.2)$$

The functions $j_k : Q \times \mathbb{R}^m \to \mathbb{R}$, $k = 1, \dots, m$, are supposed to be Carathéodory functions with $s_k \mapsto j_k(x, t, s_k, [s]_k)$ being locally Lipschitz for a.a. $(x, t) \in Q$ and for all $[s]_k \in \mathbb{R}^{m-1}$, and $s_k \mapsto j_k^o(x, t, s_k, [s]_k; \varrho_k)$ denotes Clarke's partial generalized directional derivative with respect to the s_k component of $s \in \mathbb{R}^m$ in the direction $\varrho_k \in \mathbb{R}$, which is defined by

$$j_k^o(x, t, s_k, [s]_k; \varrho_k) = \limsup_{h \to s_k,\ \varepsilon \downarrow 0} \frac{j_k(x, t, h + \varepsilon\, \varrho_k, [s]_k) - j_k(x, t, h, [s]_k)}{\varepsilon}, \quad (6.3)$$

(cf., e.g., [7, Chap. 2]). Further, let us introduce Clarke's partial generalized gradient $\partial_k j_k$ of the locally Lipschitz function $s_k \mapsto j_k(x, t, s_k, [s]_k)$ given by

$$\partial_k j_k(x, t, s) = \{\eta \in \mathbb{R} : j_k^o(x, t, s_k, [s]_k; \varrho_k) \ge \eta\, \varrho_k, \quad \forall\, \varrho_k \in \mathbb{R}\}. \quad (6.4)$$

We assume the following hypotheses on j_k.

- (J1) The functions $j_k : Q \times \mathbb{R}^m \to \mathbb{R}$, $k = 1, \dots, m$, are supposed to be Carathéodory functions, that is, $(x, t) \mapsto j_k(x, t, s)$ is measurable in Q for all $s \in \mathbb{R}^m$, and $s \mapsto j_k(x, t, s)$ is continuous in $\mathbb{R}^m$ for a.a. $(x, t) \in Q$, and $s_k \mapsto j_k(x, t, s_k, [s]_k)$ is locally Lipschitz for a.a. $(x, t) \in Q$ and for all $[s]_k \in \mathbb{R}^{m-1}$.
- (J2) The functions $s \mapsto j_k^o(x, t, s_k, [s]_k; \varrho_k)$, $k = 1, \dots, m$, are upper semicontinuous for $\varrho_k = \pm 1$.
- (J3) Clarke's partial generalized gradient $\partial_k j_k$ satisfies the growth condition

$$\sup\{|\eta| : \eta \in \partial_k j_k(x, t, s)\} \le \alpha_k(x, t) + \beta_k \sum_{j=1}^m |s_j|^{\frac{p_j}{p_k'}},$$

  for a.e. $(x, t) \in Q$, $\forall\, s \in \mathbb{R}^m$, where $\alpha_k \in L^{p_k'}(Q)$, and $\beta_k \ge 0$.

Remark 6.1 Regarding assumption (J2) on Clarke's partial generalized directional derivative $s \mapsto j_k^o(x, t, s_k, [s]_k; \varrho_k)$ a few comments are in order. One may ask for sufficient conditions on the function $j_k = j_k(x, t, s)$ itself such that the general condition (J2) is satisfied. Here, we provide such sufficient conditions for functions $j_k : Q \times \mathbb{R}^m \to \mathbb{R}$ of the following class:

$$j_k(x, t, s) = g_k(x, t, s_k)\, h_k(x, t, [s]_k), \tag{6.5}$$

for $(x, t) \in Q$, $s = (s_k, [s]_k) \in \mathbb{R}^m$.

Corollary 6.1 *Assume that $g_k : Q \times \mathbb{R} \to \mathbb{R}$ and $h_k : Q \times \mathbb{R}^{m-1} \to \mathbb{R}$ are Carathéodory functions such that for a.e. $(x, t) \in Q$, $s_k \to g_k(x, t, s_k)$ is locally Lipschitz, and $h_k(x, t, [s]_k) \geq 0$ for a.e. $(x, t) \in Q$, all $[s]_k \in \mathbb{R}^{m-1}$. Then j_k given by (6.5) fulfills (J2), that is, $s \mapsto j_k^o(x, t, s_k, [s]_k; \varrho_k)$ is upper semicontinuous for $\varrho_k = \pm 1$.*

Proof Let $(s^{(j)}) \subset \mathbb{R}^m$ such that $s^{(j)} \to s$ as $j \to \infty$. To prove that $s \mapsto j_k^o(x, t, s_k, [s]_k; \varrho_k)$ is upper semicontinuous, we need to show that

$$\limsup_{j\to\infty} j_k^o(x, t, s_k^{(j)}, [s^{(j)}]_k; \varrho_k) \leq j_k^o(x, t, s_k, [s]_k; \varrho_k). \tag{6.6}$$

In fact, we have, for any $\varrho_k \in \mathbb{R}$,

$$\begin{aligned}
j_k^o(x, t, s_k, [s]_k; \varrho_k) &= \limsup_{h\to s_k,\ \varepsilon\downarrow 0} \frac{j_k(x, t, h + \varepsilon\,\varrho_k, [s]_k) - j_k(x, t, h, [s]_k)}{\varepsilon} \\
&= \limsup_{h\to s_k,\ \varepsilon\downarrow 0} \left[\frac{g_k(x, t, h + \varepsilon\,\varrho_k) - g_k(x, t, h)}{\varepsilon} h_k(x, t, [s]_k)\right] \\
&= \limsup_{h\to s_k,\ \varepsilon\downarrow 0} \left[\frac{g_k(x, t, h + \varepsilon\,\varrho_k) - g_k(x, t, h)}{\varepsilon}\right] h_k(x, t, [s]_k) \\
&= g_k^o(x, t, s_k; \varrho_k) h_k(x, t, [s]_k)\,.
\end{aligned}$$

As $s^{(j)} \to s$ in $\mathbb{R}^m$, it follows that $s_k^{(j)} \to s_k$ in $\mathbb{R}$ and $[s^{(j)}]_k \to [s]_k$ in $\mathbb{R}^{m-1}$. Thus for a.e. $(x, t) \in Q$, all $\varrho_k \in \mathbb{R}$, we have from a basic property of Clarke's generalized directional derivative (see [7, Chap. 2]) that

$$\limsup_{j\to\infty} g_k^o(x, t, s_k^{(j)}; \varrho_k) \leq g_k^o(x, t, s_k; \varrho_k).$$

By the Carathéodory property we have

$$\lim_{j\to\infty} h_k(x, t, [s^{(j)}]_k) = h_k(x, t, [s]_k),$$

and along with $h_k(x,t,[s]_k) \geq 0$ we obtain

$$\begin{aligned}\limsup_{j\to\infty} j_k^o(x,t,s_k^{(j)},[s^{(j)}]_k;\varrho_k) &= \limsup_{j\to\infty}[g_k^o(x,t,s_k^{(j)};\varrho_k)h_k(x,t,[s^{(j)}]_k)]\\ &= \limsup_{j\to\infty} g_k^o(x,t,s_k^{(j)};\varrho_k)\lim_{j\to\infty} h_k(x,t,[s^{(j)}]_k)]\\ &\leq g_k^o(x,t,s_k;\varrho_k)h_k(x,t,[s]_k)\\ &= j_k^o(x,t,s_k,[s]_k;\varrho_k)\end{aligned}$$

which proves (6.6). □

We note that these arguments also hold when j_k is a finite sum of functions of the above form.
Let us introduce the multivalued functions $f_k : Q \times \mathbb{R}^m \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ defined by

$$f_k(x,t,s) = \partial_k j_k(x,t,s). \tag{6.7}$$

Our main goal in this section is to show that under some lattice condition on the constraint K and assuming (J1)–(J3), the system of evolutionary variational-hemivariational inequalities (6.1) is equivalent to the system of multi-valued parabolic variational inequalities (1.1)–(1.2) with f_k specified by (6.7). Thus, system (6.1) is only a particular case of system (1.1)–(1.2).

To this end, first we are going to show the following lemma.

Lemma 6.1 *Under the assumptions (J1)–(J3), the multivalued functions* $f_k : Q \times \mathbb{R}^m \to 2^{\mathbb{R}}\backslash\{\emptyset\}$ *defined by (6.7) satisfy hypotheses (F1)–(F3).*

Proof Clearly, (F3) follows directly from (J3). As for the proof of the graph measurability of f_k and the upper semicontinuity of $s \mapsto f_k(x,t,s)$ we follow an idea from [6, Sect.5].

By definition of Clarke's gradient $\partial_k j_k(x,t,s)$ and the positive homogeneity of the mapping $\varrho_k \mapsto j_k^o(x,t,s_k,[s]_k;\varrho_k) = j_k^o(x,t,s;\varrho_k)$, we see that for almost all $(x,t) \in Q$, and all $s \in \mathbb{R}^m$,

$$\partial_k j_k(x,t,s) = [-j_k^o(x,t,s;-1), j_k^o(x,t,s;1)].$$

Hence,

$$\begin{aligned}\mathrm{Gr}(f_k) &= \{(x,t,s,\eta) \in Q \times \mathbb{R}^m \times \mathbb{R} : \eta \in \partial_k j_k(x,t,s)\}\\ &= \{(x,t,s,\eta) \in Q \times \mathbb{R}^m \times \mathbb{R} : -j_k^o(x,t,s;-1) \leq \eta \leq j_k^o(x,t,s;1)\}\\ &= \{(x,t,s,\eta) \in Q \times \mathbb{R}^m \times \mathbb{R} : \eta \geq -j_k^o(x,t,s;-1)\}\\ &\quad \bigcap \{(x,t,s,\eta) \in Q \times \mathbb{R}^m \times \mathbb{R} : \eta \leq j_k^o(x,t,s;1)\}.\end{aligned}$$

For each $\varrho_k \in \mathbb{R}$, it follows from (J1) that the function $(x,t,s) \mapsto j_k^o(x,t,s;\varrho_k)$ is measurable on $Q \times \mathbb{R}^m$ with respect to the measure $\mathcal{L}(Q) \times \mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{R})$, as "countable limit superior" of measurable functions there. Hence the functions $(x,t,s) \mapsto j_k^o(x,t,s;1)$ and $(x,t,s) \mapsto j_k^o(x,t,s;-1)$ are measurable on $Q \times \mathbb{R}^m$

with respect to the measure $\mathcal{L}(Q) \times \mathcal{B}(\mathbb{R}^m)$. This implies that $\mathrm{Gr}(f_k)$ belongs to $[\mathcal{L}(Q) \times \mathcal{B}(\mathbb{R}^m)] \times \mathcal{B}(\mathbb{R})$, i.e., f_k satisfies (F1).

As for the proof of (F2), let $(x, t) \in Q$ be a point such that the functions $s \mapsto j_k^o(x, t, s; \pm 1)$ are upper semicontinuous on $\mathbb{R}^m$. Let $s_0 \in \mathbb{R}^m$ and U be an open neighborhood of $\partial_k j_k(x, t, s_0)$. Then there exists $\varepsilon > 0$ such that

$$(-j_k^o(x, t, s_0; -1) - \varepsilon, j_k^o(x, t, s_0; 1) + \varepsilon) \subset U.$$

From the upper semicontinuity of the (single-valued) functions $s \mapsto j_k^o(x, t, s; \pm 1)$ at s_0, there exists an open neighborhood O of s_0 such that

$$\begin{cases} j_k^o(x, t, s; 1) < j_k^o(x, t, s_0; 1) + \varepsilon, \text{ and} \\ j_k^o(x, t, s; -1) < j_k^o(x, t, s_0; -1) + \varepsilon, \ \forall s \in O. \end{cases}$$

Hence, for all $s \in O$,

$$\begin{aligned} \partial_k j_k(x, t, s) &= [-j_k^o(x, t, s; -1), j_k^o(x, t, s; 1)] \\ &\subset (-j_k^o(x, t, s_0; -1) - \varepsilon, j_k^o(x, t, s_0; 1) + \varepsilon) \\ &\subset U. \end{aligned}$$

which shows the upper semicontinuity of f_k at s_0. □

With the multivalued functions f_k specified by (6.7), let us consider the system (1.1)–(1.2), that is, we consider the following system of multivalued parabolic variational inequalities: For each $k = 1, \ldots, m$, find $u_k \in W_{0k} \cap K_k$ and $\eta_k \in L^{p_k'}(Q)$ such that

$$u_k(\cdot, 0) = 0 \text{ in } \Omega, \ \ \eta_k(x, t) \in f_k(x, t, u_1(x, t), \ldots, u_m(x, t)), \tag{6.8}$$

$$\langle u_{kt} + A_k u_k, v_k - u_k \rangle + \int_Q \eta_k (v_k - u_k)\, dxdt \geq 0, \ \ \forall\, v_k \in K_k, \tag{6.9}$$

The following equivalence result of system (6.1) and (6.8)–(6.9) holds true.

Theorem 6.1 *Let (A1)–(A3) and (J1)–(J3) be satisfied and assume the following lattice condition for the constraint K:*

$$K \vee K \subset K \ \ \textit{and} \ \ K \wedge K \subset K. \tag{6.10}$$

Then u is a solution of the system of evolutionary variational-hemivariational inequalities (6.1) if and only if u is a solution of the system of multi-valued parabolic variational inequalities (6.8)–(6.9) with the multi-functions f_k given by (6.7).

Proof Let u be a solution of (6.8)–(6.9) , which means $u \in D(L) \cap K$ and there are $\eta_k \in L^{p_k'}(Q)$ with

$$\eta_k(x, t) \in f_k(x, t, u_1(x, t), \ldots, u_m(x, t)) = \partial_k j_k(x, t, u_1(x, t), \ldots, u_m(x, t))$$

such that

$$\langle u_{kt} + A_k u_k, v_k - u_k\rangle + \int_Q \eta_k\,(v_k - u_k)\,dxdt \geq 0, \quad \forall\, v_k \in K_k, \tag{6.11}$$

By definition of $\partial_k j_k(x, t, u)$ we get for any $v \in K_k$

$$j_k^o(x, t, u_k, [u]_k; v_k - u_k) \geq \eta_k(x, t)(v_k - u_k). \tag{6.12}$$

From (J1) and (J3) it follows that the left-hand side of inequality (6.12) is integrable, which by combining with (6.11) yields (6.2) or equivalently (6.1). We have seen by this way that any solution of (6.8)–(6.9) is a solution of the system of evolutionary variational-hemivariational inequalities (6.1).

Now let us show the reverse, and assume u is a solution of (6.1). In order to show that u is a solution of (6.8)–(6.9), we are going to show that u is both a subsolution and a supersolution for (6.8)–(6.9) which, via Theorem 4.2, completes the proof. In fact, according to Theorem 4.2, there exists a solution $\hat{u}$ within the interval of sub- and supersolutions, that is, $u \leq \hat{u} \leq u$, and therefore $u = \hat{u}$ must be a solution of (6.8)–(6.9), completing the proof. We note that Theorem 4.2 can be applied in this situation, since by Lemma 6.1 the hypotheses (F1)–(F3) for f_k (defined by (6.7)) are fulfilled and (F3) implies (F4).

Let u be a solution of (6.1), that is, of (6.2). By assumption, K has the lattice property (6.10), so K_k has the same property. In particular, we can use in (6.2) $v_k \in u_k \wedge K_k \subset K_k$, i.e., $v_k = u_k \wedge \varphi_k = u_k - (u_k - \varphi_k)^+$ with $\varphi_k \in K_k$, which yields for all $\varphi_k \in K_k$,

$$\langle u_{kt} + A_k u_k, -(u_k - \varphi_k)^+\rangle + \int_Q j_k^o(x, t, u_k, [u]_k; -(u_k - \varphi_k)^+)\,dxdt \geq 0.$$

Because $\varrho \mapsto j_k^o(x, t, u_k, [u]_k; \varrho_k)$ is positively homogeneous, the last inequality is equivalent to

$$\langle u_{kt} + A_k u_k, -(u_k - \varphi_k)^+\rangle + \int_Q j_k^o(x, t, u_k, [u]_k; -1)(u_k - \varphi_k)^+\,dxdt \geq 0,$$

for all $\varphi_k \in K_k$. Using again $v_k = u_k \wedge \varphi_k = u_k - (u_k - \varphi_k)^+$, the last inequality is equivalent to

$$\begin{cases} \langle u_{kt} + A_k u_k, v_k - u_k\rangle + \displaystyle\int_Q -j_k^o(x, t, u_k, [u]_k; -1)(v_k - u_k)\,dxdt \geq 0, \\ \forall\, v_k \in u_k \wedge K_k. \end{cases} \tag{6.13}$$

In view of [7, Proposition 2.1.2] we have

$$\begin{aligned} & j_k^o(x,t,u_k(x,t),[u(x,t)]_k;-1) \\ &= \max\{-\theta_k(x,t) : \theta_k(x,t) \in \partial_k j_k(x,t,u_k(x,t),[u(x,t)]_k)\} \\ &= -\min\{\theta_k(x,t) : \theta_k(x,t) \in \partial_k j_k(x,t,u_k(x,t),[u(x,t)]_k)\} \\ &=: -\underline{\eta}_k(x,t), \end{aligned} \tag{6.14}$$

where

$$\underline{\eta}_k(x,t) \in \partial_k j_k(x,t,u_k(x,t),[u(x,t)]_k) = f_k(x,t,u_1(x,t),\dots,u_m(x,t)). \tag{6.15}$$

Since $(x,t) \mapsto j_k^o(x,t,u_k(x,t),[u(x,t)]_k;-1)$ is measurable, it follows that $(x,t) \mapsto \underline{\eta}_k(x,t)$ is measurable in Q as well. Thus, in view of the growth conditions (J3) on the Clarke's gradients, we infer that $\underline{\eta}_k \in L^{p_k'}(Q)$. Taking (6.14)–(6.15) into account, from (6.13) we get ($k = 1,\dots,m$)

$$\begin{cases} \langle u_{kt} + A_k u_k, v_k - u_k\rangle + \int_Q \underline{\eta}_k(x,t)(v_k - u_k)\,dxdt \ge 0, \\ \forall\, v_k \in u_k \wedge K_k. \end{cases} \tag{6.16}$$

which shows that u is a subsolution for (6.8)–(6.9)(with respect to the interval $[u, u]$). By similar arguments one shows that u is also a supersolution, which completes the proof. □

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References

1. Berkovits, J., Mustonen, V.: Topological degree for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems. Re. Mat. Serie VII **12**, 597–621 (1992)
2. Berkovits, J., Mustonen, V.: Monotonicity methods for nonlinear evolution equations. Nonlinear Anal. **27**, 1397–1405 (1996)
3. Browder, F.E., Hess, P.: Nonlinear mappings of monotone type in Banach spaces. J. Funct. Anal. **11**, 251–294 (1972)
4. Carl, S., Le, V.K., Motreanu, D.: Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications. Springer, New York (2007)

5. Carl, S., Le, V.K.: Quasilinear parabolic variational inequalities with multi-valued lower-order terms. Z. Angew. Math. Phys. **65**(5), 845–864 (2014)
6. Carl, S., Le, V.K.: Multi-valued parabolic variational inequalities and related variational-hemivariational inequalities. Adv. Nonlinear Stud. **14**(3), 631–659 (2014)
7. Clarke, F.H.: Optimization and Nonsmooth Analysis. SIAM, Philadelphia (1990)
8. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis. Applications. Kluwer Academic Publishers, Boston (2003)
9. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
10. Le, V.K.: Existence and enclosure of solutions to noncoercive systems of inequalities with multivalued mappings and non-power growths. Discret. Contin. Dyn. Syst. **33**, 255–276 (2013)
11. Le, V.K.: On quasi-variational inequalities with discontinuous multivalued lower order terms given by bifunctions. Differ. Integr. Equ. **28**, 1197–1232 (2015)
12. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. **12**, 1203–1219 (1988)
13. Lions, J.L.: Quelques méthodes de résolution des problèmes aus limites non linéaires. Dunod, Gauthier-Villars, Paris (1969)
14. Naniewicz, Z., Panagiotopoulos, P.D.: Mathematical Theory of Hemivariational Inequalities and Applications. Marcel Dekker, New York (1995)
15. Panagiotopoulos, P.D.: Hemivariational Inequalities and Applications in Mechanics and Engineering. Springer, New York (1993)
16. Papageorgiou, N.S., Papalini, F., Renzacci, F.: Existence of solutions and periodic solutions for non-linear evolution inclusions. Rend. Circ. Mat. Palermo (2) **48**(2), 341–364 (1999)
17. Vázquez, J.: A strong maximum principle for some quasilinear elliptic equations. Appl. Math. Optim. **12**(3), 191–202 (1984)
18. Zeidler, E.: Nonlinear Functional Analysis and its Applications, vol. 2A/B. Springer, Berlin (1990)

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