# Approximated Adaptive Explicit Parametric Optimal Control

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"If in other sciences we should arrive at certainty without doubt and truth without error, it behooves us to place the foundations of knowledge in mathematics."

***Roger Bacon***, medieval English philosopher and Franciscan friar (ca. 1220 - 1292)

# Contents

# List of Abbreviations

| | |
|---|---|
| AM | Al'brekht's Method |
| BDP | Bellman Dynamic Programming |
| CARE | continuous-time algebraic Riccati equation |
| CAS | Computer Algebra System |
| CLN | Class Library for Numbers |
| CRP | culture redox potential |
| DARE | discrete-time algebraic Riccati equation |
| Fig. | Figure |
| GiNaC | GiNaC is Not a CAS |
| GPS | Global Positioning System |
| HJB | Hamilton-Jacobi-Bellman |
| HJBE | Hamilton-Jacobi-Bellman equation |
| i. e. | "id est", "in other words" |
| MPC | Model Predictive Control |
| ODE | ordinary differential equation |
| PDE | partial derivative equation |
| PMP | Pontryagin's Minimum Principle |
| resp. | respectively |
| SAM | Solver for Al'brekht's Method |
| s.t. | subject to |
| w.l.o.g. | without loss of generality |

# List of Mathematical Notations

| | |
|---|---|
| $\wedge$ | Mathematical „and“ |
| $\subseteq$ | Subset, not necessarily strict |
| $\subsetneq$ | Strict subset |
| $\mathbb{N}$ | $\{1, 2, 3, \ldots\}$ |
| $\mathbb{N}_0$ | $\{0, 1, 2, 3, \ldots\}$ |
| $\mathbb{R}^n$, $\mathbb{C}^n$ | Sets containing all real resp. complex-valued vectors in $n \in \mathbb{N}$ dimensions, $\mathbb{R} = \mathbb{R}^1$, $\mathbb{C} = \mathbb{C}^1$ |
| $\Re(x)$ | Real part of a complex number $x \in \mathbb{C}$ |
| $\mathbb{R}^n_{>0}$ | Set containing all positive real-valued vectors in $n \in \mathbb{N}$ dimensions, $\mathbb{R}_{>0} = \mathbb{R}^1_{>0}$ |
| $\mathbb{R}^n_{\geq 0}$ | Set containing all non-negative real-valued vectors in $n \in \mathbb{N}$ dimensions, $\mathbb{R}_{\geq 0} = \mathbb{R}^1_{\geq 0}$ |
| $\mathbb{K}^{n \times m}$ | Set containing all matrices with $n$ rows, $m$ columns and entries in $\mathbb{K} = \mathbb{R}$ resp. $\mathbb{K} = \mathbb{C}$, $n, m \in \mathbb{N}$ |
| $M \succ 0$ | Matrix $M$ that is positive definite |
| $M \succeq 0$ | Matrix $M$ that is positive semidefinite |
| $M^{\mathrm{T}}$ | Transposed of a Matrix $M$ |
| $\|\|.\|\|$ | $\|\|.\|\|_2$, Norm of a vector or matrix |
| $\mathbb{S}^{n-1}$ | $\{x \in \mathbb{R}^n : \|\|x\|\| = 1\}$, $n \in \mathbb{N}$ |
| $B_r(y)$ | $\{x \in \mathbb{R}^n : \|\|x - y\|\| < r\}$, $y \in \mathbb{R}^n$, $r \in \mathbb{R}_{>0}$, $n \in \mathbb{N}$ |
| $\overline{S}$ | Closure of a set $S$ |
| $[n]$ | $\{i \in \mathbb{N} : 1 \leq i < n + 1\}$, $n \in \mathbb{N}_0 \cup \{\infty\}$ |
| $[n]_0$ | $[n] \cup \{0\}$, $n \in \mathbb{N} \cup \{\infty\}$ |
| $I_n$ | Identity matrix in $n \in \mathbb{N}$ dimensions |
| $0_{n \times m}$ | $n$-by-$m$ matrix containing only zeros $n, m \in \mathbb{N}$ |
| $e_n$ | $n$-th unit vector in a vetor space, $n \in \mathbb{N}$ |
| $\mathrm{GL}(n; \mathbb{K})$ | Set containing all invertible $n \times n$ matrices with entries in |
| $\mathrm{rank}(M)$ | Rank of a matrix $M$ |
| | $\mathbb{K} = \mathbb{R}$ resp. $\mathbb{K} = \mathbb{C}$, $n \in \mathbb{N}$ |
| $C^k$ | Space containing $k$ times continuous differentiable mappings, $k \in \mathbb{N} \cup \{\infty\}$ |
| $f^{[k]}(x)$ | Denotes all terms homogeneous in $x \in \mathbb{R}^n$ with degree $k$ of a power series expansion of a map $f \colon \mathbb{R}^n \to \mathbb{R}^m$, $n, m \in \mathbb{N}$ |
| $f^{[j;k]}(x)$ | $\sum\limits_{i=j}^{k} f^{[i]}(x),\ k, j \in \mathbb{N}$ |

| | |
|---|---|
| $o(g)$ | Set of all functions $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $\lim\limits_{x\to 0} \left\lvert \frac{f(x)}{g(x)} \right\rvert = 0$ for a given $g\colon \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$ |
| $\nabla_x f(x)$ | Jacobian matrix with respect to the variables $x$ of a map $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $n, m \in \mathbb{N}$ |

# Abstract

Optimal control problems are among the most important formulations and elements in control engineering. Many methods for control, estimation, and monitoring are based on the repeated solution of optimal control problems while the process is running. This leads to high computational effort, especially on the so-called embedded systems with reduced computational power. Therefore control in real-time is not always possible. Despite numerous advances in the field of explicit solutions of optimal control problems, most existing methods are limited to linear systems with quadratic cost functions.

In this work, an explicit solution approach based on a power series approach, which was first shown by E. G. Al'brekht, is extended to non-linear parametric systems and the output feedback case. In particular, the case of variable system parameters, which lead to parametric explicit solutions, is considered. In addition, inequality constraints regarding the control input, system's states, and output values, as well as the regulation without the full state information is examined. Proofs for the existence of optimal solutions in the form of a power series are given. The proofs take up the results and ideas of E. G. Al'brekht and extend them considerably.

A significant advantage of polynomial, parametric solutions is their real-time capability, as well as the possibility of explicitly verifying the properties of the resulting control in advance. Furthermore, the parameters can be updated while the process is running.

The developed methods and results are verified and validated using simulation studies. In addition to the methodical results, the approaches are implemented in form of a software toolbox.

# Deutsche Kurzfassung

## Approximative adaptive explizite parametrische optimale Regelung

Optimalsteuerungsprobleme gehören zu den wichtigsten Formulierungen und Elementen in der Regelungstechnik. Viele Methoden der Regelung, Schätzung und Überwachung basieren auf der wiederholten Lösung von Optimalsteuerungsproblemen während der Prozess läuft. Dies stellt insbesondere auf Systemen mit reduzierten Rechenleistungen, sogenannten eingebetteten Systemen, oftmals einen erheblichen Rechenaufwand dar, wodurch Regelung in Echtzeit nicht in jedem Fall möglich ist. Trotz zahlreicher Fortschritte im Bereich expliziter Lösungen von Optimalsteuerungsproblemen, sind diese oftmals auf lineare Systeme mit quadratischen Kostenfunktionen begrenzt.
In dieser Arbeit wird ein expliziter Lösungsansatz, basierend auf einem Potenzreihenansatz, der erstmals durch E. G. Al'brekht aufgezeigt wurde, auf nicht lineare parametrische Systeme und den Ausgangsrückführungsfall erweitert. Hierzu wird der optimale Systemeingang mittels einer lokalen Potenzreihe approximiert. Speziell wird der Fall variabler Systemparameter betrachtet, was zu parametrischen expliziten Lösungen führt. Daneben werden Ungleichungsbeschränkungen bezüglich der Eingänge, Zustände und Ausgänge, sowie die Regelung ohne Kenntnis aller Zustände untersucht. Es werden Beweise für die Existenz optimaler Lösungen in Form einer Potenzreihe angegeben, welche die Ergebnisse und Ideen von E. G. Al'brekht aufgreifen, sie aber erheblich erweitern.
Ein bedeutender Vorteil von polynomialen, parametrischen Lösungen ist ihre Echtzeitfähigkeit, sowie die prinzipielle Möglichkeit die Eigenschaften der sich ergebenden Regelung explizit vorab zu verifizieren. Weiterhin können die Parameter während des laufenden Prozesses aktualisiert werden.
Die entwickelten Methoden und Ergebnisse werden anhand von Simulationsstudien verifiziert und validiert. Neben den methodischen Ergebnissen werden die Ansätze in Form einer Softwaretoolbox implementiert.

# 1 Introduction and Motivation

The earliest formulations of optimal control problems (OCP) date back to Galileo Galilei (1564-1641), who presented two problems in 1638 [87]. For a long time, optimal control stayed rather unexplored. Its breakthrough only came with the availability of computers, which made the solution of optimal control problems possible, even in the case where analytic solutions are difficult or impossible to derive.
The solution techniques changed from geometric ones to variational calculus resp. analytic techniques. Two fundamental results in optimal control date back to the 1950s and 1960s [87]. Optimality criteria were stated by Lev Semyonovich Pontryagin (1908-1988) in his well-known maximum principle [12, 79] and the Hamilton-Jacobi-Bellman (HJB) equation [8], which roots date back to works of William Rowan Hamilton (1805-1865), Carl Gustav Jacob Jacobi (1804-1851) and Richard Ernest Bellman (1920-1984). Throughout this work, the Hamilton-Jacobi-Bellman equation, an approach to analyze and solve optimal control problems, will be of main interest.
After the establishment of necessary and sufficient conditions for optimal control laws, the uniqueness and existence results using convergence arguments were of major interest, see [13]. Over the past decades, the theory of optimal control was expanded to handle constraints, uncertainties, and output regulation, to name just a few expansions. For an overview, the reader is referred to the books of Vinter [94], Kirk [46], and Berkovitz [10]. Many solution approaches - numerical [80, 84] and analytical - were proposed.
In 1961 E. G. Al'brekht proposed a power series approach to solve the Hamilton-Jacobi-Bellman equation for nonlinear systems [3]. The control law and the optimized cost are stated as local power series around the origin. Both series can be approximated up to a chosen degree. One big advantage of Al'brekht's Method is its analytic character. The solution can be obtained offline, while only the evaluation of the polynomial control law is done online. Having an analytic expression of the value function also allows determining beforehand whether the closed-loop system is stable or not, or to validate the achievable region of attraction.
This thesis extends Al'brekht's Method to output control with parametric uncertainties and constraints. Over time several extensions and improvements have been made. Lukes [69] applied Al'brekht's Method to systems with $C^2$ dynamics and Hölder continuous derivatives instead of analytic dynamics, which then leads to weaker approximations. Isidori and Byrnes [41] used Al'brekht's Method to solve output regulation problems with a fully known external signal and linear input. Krener [51] used a similar setup but allowed a nonlinear input. Furthermore, Al'brekht's Method was

applied to the Francis-Byrnes-Isidor PDE or used in Pontryagins Maximum Principle as optimality condition [53]. Discrete-time versions of Al'brekht's Method have also been developed and different extensions and combinations exist, see e. g. [54–56]. Furthermore, Al'brekht's Method has been extended to systems with disturbances in form of white noise effecting the linear part of the dynamics [57, 58]. To overcome the issue of small convergence areas, C. Aguilar and A. Krener [2] used different control laws for different regions that are patched together. Regional stability results were derived using sum-of-squares methods on even order approximations of the optimal cost to obtain a local Lyapunov function [65]. However, extended convergence proofs are still rare. Furthermore, handling constraints on states and inputs is still challenging. Apart from Al'brekht's Method, there are numerous approaches to apply optimal control. Model Predictive Control (MPC) solves the optimal control problem repeatedly, predicting the state over a finite time in the future. It provides feedback and reaction to disturbances by the repeated solution of the optimal control problem using updated state and parameter information. An introduction can be found in [85] and [26]. MPC is capable of handling nonlinear dynamics and constraints. Drawbacks are the verifiability and online computation, which can still be challenging on embedded systems despite significant advancements in the last decades [27]. There are several toolbox like ACADO [34], ACADOS [93], do-MPC [68] and $\mu$AO-MPC [66, 103], which solve Model Predictive Control problems or generate code to use it efficiently on embedded systems, see also [78, 88]. Embedded systems with limited computation capability or which demand strict verification of the resulting application are possible application areas for Al'brekht's Method. The offline solution facilitates applicability in real-time. Further methods for the efficient and real-time application of optimal and predictive control, for example, exploit the monotonicity properties of the system [67]. While monotonicity requirements limit the application, they simplify the optimal control problem. In [35], Houska et. al. propose to use polynomial expansions for online computations in optimal control. Huh and Sejnowski [37] found analytic solutions for optimal control problems subject to multiplicative noise in one dimension with invertible dynamics. Krstic [59, 60] provided algorithms to calculate explicit but not optimal control laws for classes of linearizable systems using a diffeomorphic transformation to obtain a chain of integrators. For nonlinear systems with linear input, Margaliot and Langholz [71] utilized Young's inequality to find control laws. For a similar class of systems, Mylvaganam and Sassano [74] used an approximation of control laws based on linear quadratic regulators. Ying's et. al. [101] approach is also based on an LQR and solving the corresponding Riccati equation. The design of an LQR is extended by Rafikov et. al. [82]. He focused on bounded time-varying systems with linear input. Furthermore, local asymptotic stability results are given. An explicit control law with discrete decision variables for linear systems is given by Sakizlis et. al. [86]. Another explicit approach was introduced by Wu et. al. [100]. The approach uses polynomial chaos expansions and the Chebyshev interval method to solve an optimization problem with nonlinear objective and constraints. There are also results for application cases using explicit optimal control laws for direct examples, e. g. for hybrid electric powertrains [4]. All the mentioned solutions to explicit optimal control are challenged by a series of points. Typically they are restricted to a rather small class of systems, or the control law is linear, whereas Al'brekht's Method applies to a broader class of systems and gives a nonlinear approximation of the real optimal solution with arbitrary precision.

Apart from the methods mentioned so far, there is a lot of work done on explicit MPC. Wen et. al. [97], Kouramas et. al. [49] and Bayat [7] state their control laws as lattice piecewise-affine functions, which are defined in different regions. Darup and Mönnigmann [18] found for nonlinear systems suboptimal control laws as piecewise-affine functions. The idea of getting an explicit controller in the form of piecewise-affine functions in different regions, e. g. polytopes, is very common, see [32]. Mönningmann and Kastsian [72] further used multiway instead of binary trees to determine the current region more efficiently. Mönnigmann and Jost [73] developed a method that only requires the vertices of the polytopes to determine the affine control law for linear systems. In [20] active sets are enumerated offline, and matrices for the Karush-Kuhn-Tucker condition are prefactored. Christophersen et. al. [16] and Cychowski et. al. [17] show how to determine efficiently the active region and how to perform the partitioning. State-space partitioning for piece-wise linear feedback with a priori guarantee of asymptotic stability is done by Johansen [42]. Suboptimal solutions were also investigated via branch-and-bound methods for parametric mixed-integer quadratic programs, see [5, 6]. Constraint handling for suboptimal solutions was proposed by Bemporad and Filippi [9]. Goebel et. al. [28] simplified the online computation of linear MPC via a tailored subspace clustering algorithm to train data consisting of states and corresponding solutions. Ding et. al. [19] and later Oberdieck et. al. [77] considered explicit MPC for linear time-variant systems.

Overall explicit MPC approaches focus on linear systems and do not provide solutions for the general nonlinear case in a structured way. All the papers mentioned deal with linear or piece-wise affine control laws with possible modifications. Combining Al'brekht's Method with explicit MPC could allow overcoming this challenges, see [54–56].

This thesis is organized as the following. Chapter 2 introduces Al'brekht's Method in continuous and discrete time. The convergence proof from the original paper [3] is adapted and generalized to allow more dimensional and nonlinear input. Al'brekht's Method is extended towards parameter-dependent systems in Chapter 3. The existence of the solution resp. the convergence is proven, and an inequality to estimate the region of validity is given. Chapter 4 deals with a rigorous approach to include inequality constraints. The output-feedback problem is investigated in Chapter 5. Again the convergence proof and an estimation of the valid region are given. Chapter 6 deals with the discrete-time counterpart. A toolbox where all the proposed methods are implemented is introduced in Chapter 7. It the end, in Chapter 8 a brief summary of the thesis is provided and future research directions are pointed out.

# 2 Al'brekht's Method

Al'brekht's Method [3] allows to approximately solving optimal control problems with nonlinear system dynamics and possibly non-quadratic cost functions. While linear control and linear quadratic regulators are already well studied, their nonlinear pendants can still be challenging. Especially when the need for offline or analytic solutions arises. The method requires analytic dynamics and cost function and assumes that the control law and the optimal cost are also analytic. In general, this can not be guaranteed, not even locally. In fact, regularity theory [21, 64] for solutions of the Hamilton-Jacobi-Bellman equation shows that the optimal cost is two times continuously differentiable with Hölder continuous second derivative if the cost and dynamics are at least $C^1$ and no constraints are present. Nevertheless, under the requirements of Al'brekht's Method, it is, in fact, possible to prove for special cases that the solution is analytic. Furthermore, it is always possible to calculate the corresponding power series. Section 2.1 reviews the continuous time case of Al'brekht's Method [3]. The discrete-time case can be found in the literature, e. g. [56].

## 2.1 Al'brekht's Method for continuous-time systems

Throughout this section, Al'brekht's first version of his power series method will be outlined in detail for continuous-time systems, whereas the discrete-time equivalent will be considered in Chapter 6 for a more general class of optimal control problems. It requires analytic system dynamics and an analytic cost function. Under the assumption that the optimal input and the optimal cost are analytic functions with respect to the states, the OCP is solved using the Hamilton-Jacobi-Bellman equation and its derivative with respect to the input. Doing so, all involved functions are developed as power series. The HJBE leads then to infinitely many conditions resp. equations, namely, one for each degree. Under further assumptions, the resulting equations can be solved degree-wise, starting with the lowest. Having this calculation done, the first findings are summarized in Theorem 1. Afterward, the local stability of the closed-loop system is proven. The effectiveness of this method is then illustrated via two examples.

Al'brekht's Method [3] considers optimal control problems

$$\begin{aligned} \pi\big(x(0)\big) &= \min_{u(.)} \int\limits_0^\infty \ell\big(x(\tau), u(\tau)\big)\, \mathrm{d}\tau \\ \text{s.t. } \dot{x} &= f(x,u), \end{aligned} \tag{OCP}$$

where it is assumed that the system dynamics $f\colon \mathbb{X}\times\mathbb{U} \to \mathbb{R}^{n_x}$ and the cost function $\ell\colon \mathbb{X}\times\mathbb{U}\to\mathbb{R}$ are analytic in an open set $\mathbb{X}\times\mathbb{U}\subseteq\mathbb{R}^{n_x}\times\mathbb{R}^{n_u}$, which contains the origin. Thus their power series also exist in a neighborhood around the origin. Furthermore, it is assumed that the origin is a fixed point for both functions and that the cost function does not contain a linear part:

$$f(0,0) = 0 \quad \wedge \quad \ell(0,0) = 0 \quad \wedge \quad \nabla_{\binom{x}{u}}\ell(0,0) = 0.$$

Thus the power series of $f$ and $\ell$ can be written as

$$f(x,u) = \sum_{k=1}^{\infty} f^{[k]}(x,u) = Fx + Gu + f^{[2]}(x,u) + \dots, \tag{$f$}$$

$$\ell(x,u) = \sum_{k=2}^{\infty} \ell^{[k]}(x,u) = \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + x^{\mathrm{T}}\ell_{xu}u + \frac{1}{2}u^{\mathrm{T}}\ell_{uu}u + \ell^{[3]}(x,u) + \dots, \tag{$\ell$}$$

where the matrices $F, \ell_{xx} \in \mathbb{R}^{n_x\times n_x}$, $G, \ell_{xu}\in\mathbb{R}^{n_x\times n_u}$, and $\ell_{uu}\in\mathbb{R}^{n_u\times n_u}$ need to fulfill further conditions, which will be outlined when needed. The superscript $[k]$ indicates all terms of the power series, which are homogeneous of degree $k$ in $(x,u)$.

Al'brekht's Method is based on the assumption that the optimal cost/value function $\pi\colon \mathbb{X}\to\mathbb{R}$ as well as the minimizing control input, if written in terms of the system's states $x$, $u_{\min}\colon \mathbb{X}\to\mathbb{R}^{n_u}$ are also analytic. For simplicity we consider in the following that $\mathbb{X} = \mathbb{R}^{n_x}$ and $\mathbb{U} = \mathbb{R}^{n_u}$. This also avoids the condition $u_{\min}(\mathbb{X})\subseteq\mathbb{U}$. Note that the general case will be considered in the case studies.

The optimal cost function $\pi$ and the optimal control input $u_{\min}$ are also written as power series.

$$\pi(x) = \sum_{k=0}^{\infty} \pi^{[k]}(x) = \pi_0 + \pi_x x + \frac{1}{2}x^{\mathrm{T}}\pi_{xx}x + \pi^{[3]}(x) + \dots, \tag{$\pi$}$$

$$u_{\min}(x) := \kappa(x) = \sum_{k=0}^{\infty} \kappa^{[k]}(x) = \kappa_0 + Kx + \kappa^{[2]}(x) + \dots \tag{$\kappa$}$$

The existence/convergence of those will be shown afterward in Section 2.2. The domain of convergence must be contained in $\mathbb{X}$ but might be very small.

To calculate the power series ($\pi$) and ($\kappa$), the Hamilton-Jacobi-Bellman equations, see Remark 18 ($d$) Formulas (HJBE-1"') and (HJBE-2"'), are solved. As outlined in the Appendix A, a similar approach can be used for Pontryagin's Minimum Principle of cause leading to the same results, see Remark 19.

$$0 = \nabla_x \pi(x) \cdot f\Big(x, \kappa(x)\Big) + \ell\Big(x, \kappa(x)\Big) \tag{HJBE-1}$$

$$0 = \nabla_x \pi(x) \cdot \nabla_u f\Big(x, \kappa(x)\Big) + \nabla_u \ell\Big(x, \kappa(x)\Big) \tag{HJBE-2}$$

Since $f^{[0]}(x,u)$ is vanishing and $\ell(.,.)$ does neither contain a constant nor a linear part, it is clear that $\kappa_0$ and $\pi_0$ are also vanishing if the cost function has a local minimum in $(0,0)$.
Since polynomials with different degree are linearly independent, every degree of the Hamilton-Jacobi-Bellman equations will give an independent equation for certain parts of the power series of $\pi$ and $\kappa$. Since $f$ and $\ell$ do not contain a constant, collecting all terms of degree 0 from equation (HJBE-1) leads to the following equality:

$$\text{(HJBE-1)}^{[0]}: \;\; 0 = 0.$$

Collecting all terms of degree 0 from equation (HJBE-2) results in the first condition for $\pi_x$:

$$\text{(HJBE-2)}^{[0]}: \;\; 0 = \pi_x \cdot G. \tag{$\pi_x$-1}$$

Taking into account the next higher degree, i. e. degree 1 from (HJBE-1), gives a second condition for the calculation of $\pi_x$.

$$\text{(HJBE-1)}^{[1]}: \;\; 0 = \pi_x \cdot (F + GK) \cdot x \overset{(\pi_x\text{-}1)}{=} \pi_x \cdot Fx$$

This equation has to hold for all state vectors $x$ in a neighborhood of the origin. Therefore they can be neglected.

$$\Rightarrow 0 = \pi_x \cdot F \tag{$\pi_x$-2}$$

Combining the conditions ($\pi_x$-1) and ($\pi_x$-2), the matrix $\begin{pmatrix} F & G \end{pmatrix}$ has to have full rank to imply $\pi_x$ equals 0. In the following, this case is assumed to guarantee the solvability of the following equations.
In the next step, all terms of degree 1 from equation (HJBE-2) are collected.

$$\text{(HJBE-2)}^{[1]}: \;\; 0 = x^{\mathrm{T}} \pi_{xx} \cdot G + x^{\mathrm{T}} \ell_{xu} + x^{\mathrm{T}} K^{\mathrm{T}} \ell_{uu}$$

Again this has to hold for all $x$ in a neighborhood of the origin, which makes it possible to find a formula for the matrix $K$ that represents the first degree of the control law ($\kappa$).

$$\Rightarrow K = -\ell_{uu}^{-1} \cdot \Big(G^{\mathrm{T}} \pi_{xx} + \ell_{xu}^{\mathrm{T}}\Big) \tag{$K$}$$

This formula still depends on $\pi_{xx}$ resp. the second degree of the value function ($\pi$), which is fixed by the following equality.

$$\text{(HJBE-1)}^{[2]}: \ 0 = x^{\mathrm{T}}\pi_{xx} \cdot (F + GK) \cdot x + \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + x^{\mathrm{T}}\ell_{xu}Kx + \frac{1}{2}x^{\mathrm{T}}K^{\mathrm{T}}\ell_{uu}Kx$$

This equality is scalar, but has to hold in a neighborhood of the origin. This leads to a matrix equation, which is symmetric since, for example, there is no difference between terms $x_1 \cdot x_2$ and $x_2 \cdot x_1$. Therefore one needs to look for a solution $\pi_{xx}$, which is also symmetric.

$$0 = \pi_{xx} \cdot (F + GK) + (F + GK)^{\mathrm{T}} \cdot \pi_{xx} + \ell_{xx} + \ell_{xu}K + K^{\mathrm{T}}\ell_{xu}^{\mathrm{T}} + K^{\mathrm{T}}\ell_{uu}K$$

After plugging in the formula ($K$) and simplifying the continuous time algebraic Riccati equation (CARE) is derived.

$$0 = \pi_{xx}F + F^{\mathrm{T}}\pi_{xx} + \ell_{xx} - \left(\pi_{xx}G + \ell_{xu}\right) \cdot \ell_{uu}^{-1} \cdot \left(G^{\mathrm{T}}\pi_{xx} + \ell_{xu}^{\mathrm{T}}\right) \qquad (\pi_{xx})$$

The Riccati equation is uniquely solvable if

(I) the second-order part of the cost function is convex in $(x, u)$ and strictly convex in $u$ that is

$$\begin{pmatrix} \ell_{xx} & \ell_{xu} \\ \ell_{xu}^{\mathrm{T}} & \ell_{uu} \end{pmatrix} \succeq 0 \quad \text{and} \quad \ell_{uu} \succ 0,$$

(II) the linearized system resp. the pair $(F, G)$ is stabilizable,

(III) and the pair $(F, \ell_{xx})$ is detectable,

see for example [62, 83]. If the linearized system is controllable, then condition (II) is fulfilled and $\begin{pmatrix} F & G \end{pmatrix}$ has full rank, which implies that the later one does not have to be checked. If (I)-(III) are fulfilled, then the matrix $F + GK$ only has eigenvalues in the left half-plane. This is the key to show the solvability of the following equations. Before the general case is taken care of, the equations that define all second-order terms $\kappa^{[2]}(x)$ of the control law and all third-order terms $\pi^{[3]}(x)$ of the value function are presented in more detail. Starting again with (HJBE-2) and collecting all terms of degree two leads to a formula for $\kappa^{[2]}(x)$ in dependence of $\pi^{[3]}(x)$.

$$\begin{aligned} \text{(HJBE-2)}^{[2]}: \ 0 = {} & \nabla_x\pi^{[3]}(x) \cdot G + x^{\mathrm{T}}\pi_{xx} \cdot \nabla_u f^{[2]}(x, Kx) \\ & + \nabla_u \ell^{[3]}(x, Kx) + \kappa^{[2]}(x)^{\mathrm{T}} \cdot \ell_{uu} \end{aligned}$$

Leaving out the state vector would be more difficult here and is therefore omitted at this point. But it can be done to obtain the coefficients of the polynomial $\kappa^{[2]}(x)$, see Chapter 7. Nevertheless, an explicit formula can be given.

$$\kappa^{[2]}(x) = -\ell_{uu}^{-1} \cdot \left(G^{\mathrm{T}} \cdot \nabla_x\pi^{[3]}(x)^{\mathrm{T}} + \nabla_u f^{[2]}(x, Kx)^{\mathrm{T}} \cdot \pi_{xx}x + \nabla_u \ell^{[3]}(x, Kx)^{\mathrm{T}}\right) \qquad (\kappa^{[2]})$$

The coefficients of $\pi^{[3]}(x)$ do not depend on $\kappa^{[2]}(x)$. This can be seen after applying $(K)$ to the third degree of (HJBE-1).

$$\begin{aligned}(\text{HJBE-1})^{[3]}:\ 0 &= \nabla_x\pi^{[3]}(x)\cdot(F+GK)\cdot x + x^{\mathrm{T}}\pi_{xx}\cdot\Big(f^{[2]}(x,Kx)+G\kappa^{[2]}(x)\Big)\\ &\quad+\ell^{[3]}(x,Kx)+\Big(x^{\mathrm{T}}\ell_{xu}+x^{\mathrm{T}}K^{\mathrm{T}}\ell_{uu}\Big)\cdot\kappa^{[2]}(x,p)\end{aligned}$$

Now collecting all terms that contain $\kappa^{[2]}(x)$ and using

$$(\text{HJBE-2})^{[1]}:\ 0 = x^{\mathrm{T}}\pi_{xx}G + x^{\mathrm{T}}\ell_{xu} + x^{\mathrm{T}}K^{\mathrm{T}}\ell_{uu},$$

the equation above is simplified to

$$0 = \nabla_x\pi^{[3]}(x)\cdot(F+GK)\cdot x + \ell^{[3]}(x,Kx) + x^{\mathrm{T}}\pi_{xx}\cdot f^{[2]}(x,Kx). \qquad (\pi^{[3]})$$

$(\pi^{[3]})$ is a partial differential equation in terms of the states and thus it is in general not easy to solve but in our case it is linear in the coefficients defining the polynomial $\pi^{[3]}(x)$. Those coefficients are the actual unknowns. Using Theorem 9 and the fact that $F+GK$ is stable, one obtains the solvability of the linear equation system. The proof also shows how to derive the linear equations from the polynomial. Having $\pi^{[3]}$, the calculation of $\kappa^{[2]}$ is straight forward.
The next step is to generalize the calculation that has been done for the second degree of the control law and the third degree of the value function. Therefore, for a given arbitrary $k\in\mathbb{N}$ with $k\geq 2$ $(\text{HJBE-2})^{[k]}$ and $(\text{HJBE-1})^{[k+1]}$ will be studied. The first one can be written as follows.

$$(\text{HJBE-2})^{[k]}:\ 0 = \sum_{i=1}^{k}\nabla_x\pi^{[i+1]}(x)\cdot\Big[\nabla_u f\big(x,\kappa(x)\big)\Big]^{[k-i]} + \Big[\nabla_u\ell\big(x,\kappa(x)\big)\Big]^{[k]}$$

The only unknowns are $\kappa^{[k]}(x)$ and $\pi^{[k+1]}(x)$. After splitting up the different degrees of the cost function, $\kappa^{[k]}(x)$ can be derived:

$$\begin{aligned}\kappa^{[k]}(x) = -\ell_{uu}^{-1}\cdot\Bigg(&\sum_{i=1}^{k}\Big[\nabla_u f\big(x,\kappa(x)\big)^{\mathrm{T}}\Big]^{[k-i]}\cdot\nabla_x\pi^{[i+1]}(x)^{\mathrm{T}}\\ &+\Big[\nabla_u\ell^{[3;k+1]}\big(x,\kappa(x)\big)^{\mathrm{T}}\Big]^{[k]}\Bigg).\end{aligned} \qquad (\kappa^{[k]})$$

Next, the $(k+1)$-th degree of (HJBE-1) is stated. Again the only unknowns are $\kappa^{[k]}(x)$ and $\pi^{[k+1]}(x)$ since all lower degrees of the control law and the value function have been calculated before.

$$(\text{HJBE-1})^{[k+1]}:\ 0 = \sum_{i=1}^{k}\nabla_x\pi^{[i+1]}(x)\cdot\Big[f\big(x,\kappa(x)\big)\Big]^{[k+1-i]} + \Big[\ell\big(x,\kappa(x)\big)\Big]^{[k+1]}$$

To simplify this equation, first, the unknowns have to be written in a more explicit manner.

$$0 = \nabla_x \pi^{[k+1]}(x) \cdot (F + GK) \cdot x + \sum_{i=2}^{k-1} \nabla_x \pi^{[i+1]}(x) \cdot \Big[f\Big(x, \kappa(x)\Big)\Big]^{[k+1-i]}$$
$$+ x^{\mathrm{T}} \pi_{xx} \cdot G\kappa^{[k]}(x) + x^{\mathrm{T}} \pi_{xx} \cdot \Big[f^{[2;k]}\Big(x, \kappa(x)\Big)\Big]^{[k]}$$
$$+ \Big(x^{\mathrm{T}} \ell_{xu} + x^{\mathrm{T}} K^{\mathrm{T}}\Big) \cdot \ell_{uu} \cdot \kappa^{[k]}(x) + \Big[\ell^{[3;k+1]}\Big(x, \kappa(x)\Big)\Big]^{[k+1]}$$

As in the exemplary case before, all terms which contain $\kappa^{[k]}(x)$ are collected, and $(K)$ is used to derive 0.

$$\begin{aligned} 0 = \nabla_x \pi^{[k+1]}(x) \cdot (F + GK) \cdot x + \sum_{i=2}^{k-1} \nabla_x \pi^{[i+1]}(x) \cdot \Big[f\Big(x, \kappa(x)\Big)\Big]^{[k+1-i]} \\ + x^{\mathrm{T}} \pi_{xx} \cdot \Big[f^{[2;k]}\Big(x, \kappa(x)\Big)\Big]^{[k]} + \Big[\ell^{[3;k+1]}\Big(x, \kappa(x)\Big)\Big]^{[k+1]} \end{aligned} \tag{$\pi^{[k+1]}$}$$

The resulting polynomial equality is again linear in the unknown coefficients of $\pi^{[k+1]}(x)$ and can be found using Theorem 9 and its proof since $F + GK$ is stable. Thus each degree of the power series $(\pi)$ and $(\kappa)$ can be calculated if the conditions (I)-(III) are fulfilled. But it is unclear if the power series exist resp. converge at all. A proof of local existence for a less general case is given in Section 2.2.

The calculation shown above is now summarized in Theorem 1.

**Theorem 1** (Determinability of $\pi$ and $\kappa$)**.**
*Consider an optimal control problem (OCP) where the function that defines the system dynamics and the cost function are analytic such that they can be written as in $(f)$ and $(\ell)$. Furthermore, the conditions (I)-(III) are holding. Then each part of the power series given in $(\pi)$ and $(\kappa)$ is uniquely defined.*

*Proof.* According to the Hautus Lemma 3 [90, 102], (II) implies rank $\begin{pmatrix} F & G \end{pmatrix} = n_x$. Therefore all conditions are fulfilled under the given assumptions and the claim follows from the previous calculation. □

**Corollary 1** (Local stability)**.**
*Under the requirements of Theorem 1, local stability is achieved if the power series $(\pi)$ and $(\kappa)$ converge.*

*Proof.* $\pi(x)$ is used as Lyapunov function. Using the little-o-notation for the higher-order terms the value function can be written as

$$\pi(x) = \frac{1}{2} x^{\mathrm{T}} \pi_{xx} x + o(||x||^3).$$

Thus there exists an $\varepsilon > 0$ such that

$$\pi(x) \geq 0$$

for all $x \in B_\varepsilon(0)$. Equality only holds for vanishing $x$ since $\pi_{xx}$ is positive definite as the solution of a Riccati equation. Therefore $\pi(.)$ is locally positive definite. Using the Hamilton-Jacobi-Bellman equation (HJBE-1), it is seen that $\dot{\pi}\big(x(t)\big)$ is locally negative definite.

$$\begin{aligned}
\dot{\pi}\big(x(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\pi\big(x(t)\big) = \nabla_x \pi\big(x(t)\big) \cdot f\big(x(t), \kappa(x(t))\big) \overset{\text{(HJBE-1)}}{=} -\ell\big(x(t), \kappa(x(t))\big) \\
&= -\frac{1}{2}x(t)^{\mathrm{T}}\ell_{xx}x(t) - x(t)^{\mathrm{T}}\ell_{xu}Kx(t) - \frac{1}{2}x(t)^{\mathrm{T}}K^{\mathrm{T}}\ell_{uu}Kx(t) + o\big(||x(t)||^3\big)
\end{aligned}$$

Since $\ell^{[2]}(x, Kx)$ is positive definite, there exists an $\varepsilon > 0$ such that

$$\dot{\pi}\big(x(t)\big) \leq 0$$

for all $x(t) \in B_\varepsilon(0)$. Again equality holds only for $x(t) = 0$. Thus $\pi(x)$ is a local Lyapunov function for

$$\dot{x} = f\big(x, \kappa(x)\big).$$

□

The region where local stability can be guaranteed has been investigated by Rumschinski, see [65]. He used the approximations of $\pi$ and added terms to obtain a polynomial, which is a sum-of-squares [61]. Thus local Lyapunov function candidates are constructed. Those candidates are then used to estimate the region of attraction.

## Effectiveness of Al'brekht's Method and application examples

To show the effectiveness of Al'brekht's Method, it is applied to two quadcopter examples. Example 1 considers the optimal control of a quadcopter using a model with 10 states. Example 2 expands the results to a more complex model that contains 12 states and complicated nonlinearities. Both examples will be used as running examples in the next three chapters, where parameters, constraints, and output variables are added. The general control scheme is depicted in Figure 2.1 and will also be extended throughout this thesis.

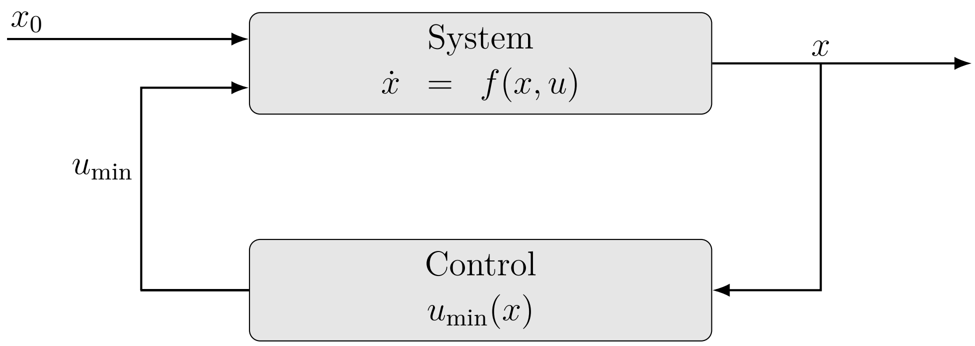


**Figure 2.1:** General control scheme

**Example 1** (Quadcopter: 10 states)**.**
*Considered is the control of a quadcopter as shown in Figure 2.2. The system dynamics are given by*

$$
\begin{aligned}
\dot{p}_x &= v_x & \dot{v}_x &= g \cdot \tan\left(\frac{\phi}{\mathrm{rad}}\right) \\
\dot{p}_y &= v_y & \dot{v}_y &= g \cdot \tan\left(\frac{\theta}{\mathrm{rad}}\right) \\
\dot{p}_z &= v_z & \dot{v}_z &= -g + k_t \cdot u_z \\
\dot{\phi} &= -d_1 \cdot \phi + v_\phi & \dot{v}_\phi &= -d_0 \cdot \phi + n_0 \cdot u_\phi \\
\dot{\theta} &= -d_1 \cdot \theta + v_\theta & \dot{v}_\theta &= -d_0 \cdot \theta + n_0 \cdot u_\theta,
\end{aligned}
$$

*where* $x = \begin{pmatrix} p_x & p_y & p_z & v_x & v_y & v_z & \phi & \theta & v_\phi & v_\theta \end{pmatrix}^{\mathrm{T}}$ *is the vector that contains all the state variables and* $u = \begin{pmatrix} u_z & u_\phi & u_\theta \end{pmatrix}^{\mathrm{T}}$ *the input. The first three states, namely,* $p_x$*,* $p_y$*, and* $p_z$ *represent the position of the quadcopter in x resp. y-direction and the altitude in the world coordinate system. The world and the quadcopter coordinate systems are visualized in Figure 2.2.* $v_x$*,* $v_y$*, and* $v_z$ *are the velocities in x resp. y-direction and*

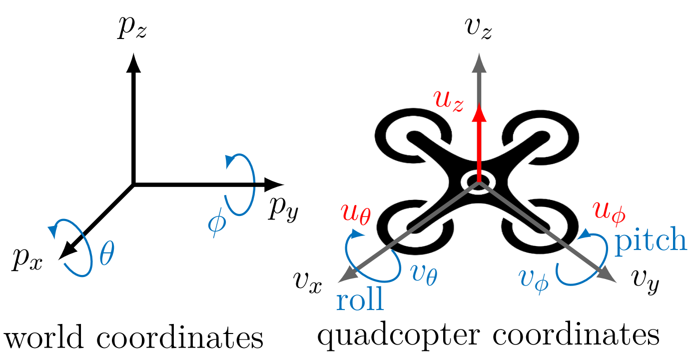


**Figure 2.2:** Coordinate systems for the 10 state quadcopter model

*altitude in the quadcopter coordinate system.* $\phi$ *and* $\theta$ *are the roll and pitch angles, while* $v_\phi$ *and* $v_\theta$ *represent the rotational velocities. The input* $u_z$ *states the vertical*

*acceleration while* $u_\phi$ *and* $u_\theta$ *are the rotational accelerations. Furthermore, there are several fixed parameters, namely,* $g = 9.81\,\mathrm{m/s^2}$, $k_t = 0.91$, $n_0 = 10$, $d_0 = 10\,/\mathrm{s^2}$, *and* $d_1 = 8\,\mathrm{s^{-1}}$. *This model is also used in multiple publications [36, 43, 45, 48] and, therefore, delivers comparability with other methods.*
*The reader may realize that one condition, which is needed to apply Al'brekht's Method, is not fulfilled.* $f(0,0) = 0$ *is not satisfied for the sixth component. Thus the following input transformation is introduced.*

$$u_z = \tilde{u}_z + \frac{g}{k_t}$$

*Using* $u_T = \begin{pmatrix} \tilde{u}_z & u_\phi & u_\theta \end{pmatrix}^\mathrm{T}$ *as new input and the quadratic cost function*

$$\ell(x, u_T) = \frac{1}{2} x^\mathrm{T} \ell_{xx} x + \frac{1}{2} u_T^\mathrm{T} \ell_{uu} u_T,$$

*with matrices* $\ell_{xx} \in \mathbb{R}^{10\times 10}$, $\ell_{uu} \in \mathbb{R}^{3\times 3}$ *given by* $\mathrm{diag}(1,1,1,1,1,1,100,100,100,100)$ *resp.* $\frac{1}{10} \cdot I_3$ *(all units are neglected in the cost), all the conditions (I) to (III) are also holding. Thus Al'brekht's Method can be applied, but one should keep in mind that the calculation of the power series can only be valid as long as* $\phi$ *and* $\theta$ *stay in* $\left(-\frac{\pi}{2}\mathrm{rad}, \frac{\pi}{2}\mathrm{rad}\right)$ *since the power series of the tangents can not be extended further.*
*To determine* $\pi(x)$ *and* $\kappa(x)$, *own software called SAM (Solver for Al'brekht's Method) has been used. This software has been developed in the frame of this work. For more explanations towards SAM and details towards the calculation, the reader is referred to Chapter 7. The control law has been approximated up to degree four and the value function up to degree five. In total,* 5992 *coefficients from which* 119 *are non-zero are determined, see Table 7.1. Since the system dynamics are mainly linear, the cost is quadratic, and the power series of the tangents only contains monomials with an odd degree,* $\kappa^{[2]}(x)$, $\kappa^{[4]}(x)$, $\pi^{[3]}(x)$, *and* $\pi^{[5]}(x)$ *are vanishing. Thus the following equalities are clear.*

$$\begin{aligned} \kappa^{[1;2]}(x) &= \kappa^{[1]}(x) \qquad & \kappa^{[1;4]}(x) &= \kappa^{[1;3]}(x) \\ \pi^{[2;3]}(x) &= \pi^{[2]}(x) \qquad & \pi^{[2;5]}(x) &= \pi^{[2;4]}(x) \end{aligned}$$

*MATLAB has been utilized to simulate the system. The initial values were chosen as* $p_x = 25\,\mathrm{m}$, $p_z = 5\,\mathrm{m}$, *and* $\theta = 0.1745\,\mathrm{rad}$ *(*$\approx 10°$*), while all the other states are zero. All controllers resulting from different degrees of approximation are able to control the system, see Fig. 2.3. There is a small overshoot in* $p_x$ *but not in the altitude* $p_z$. *Both depicted controls (Fig. 2.4) are nearly the same due to the „weak“ nonlinearity. The reduction of the cost (Fig. 2.5) using higher-order approximations is not significant. This fact will change for more complicated examples or when variable parameters or constraints are present.*

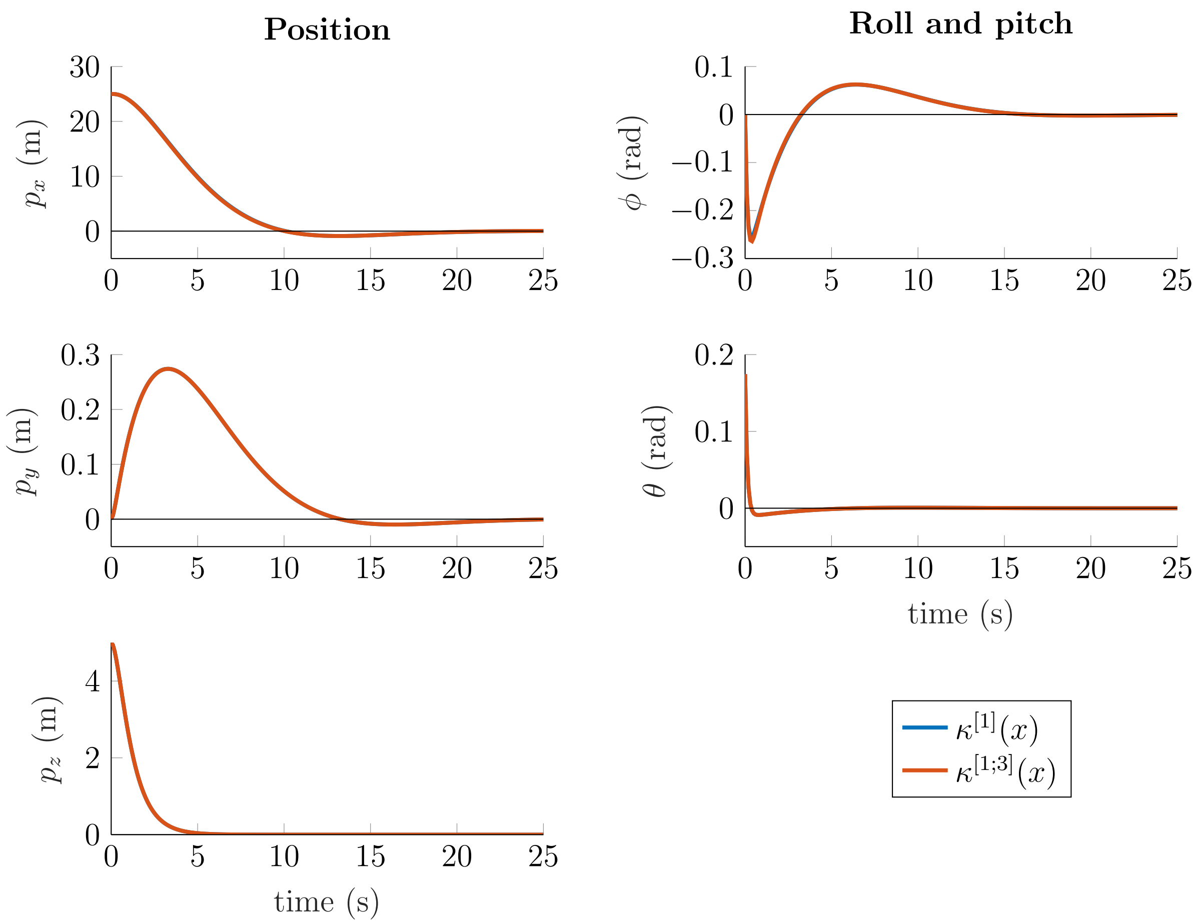


**Figure 2.3:** Propagation of five out of ten states (10 states model)

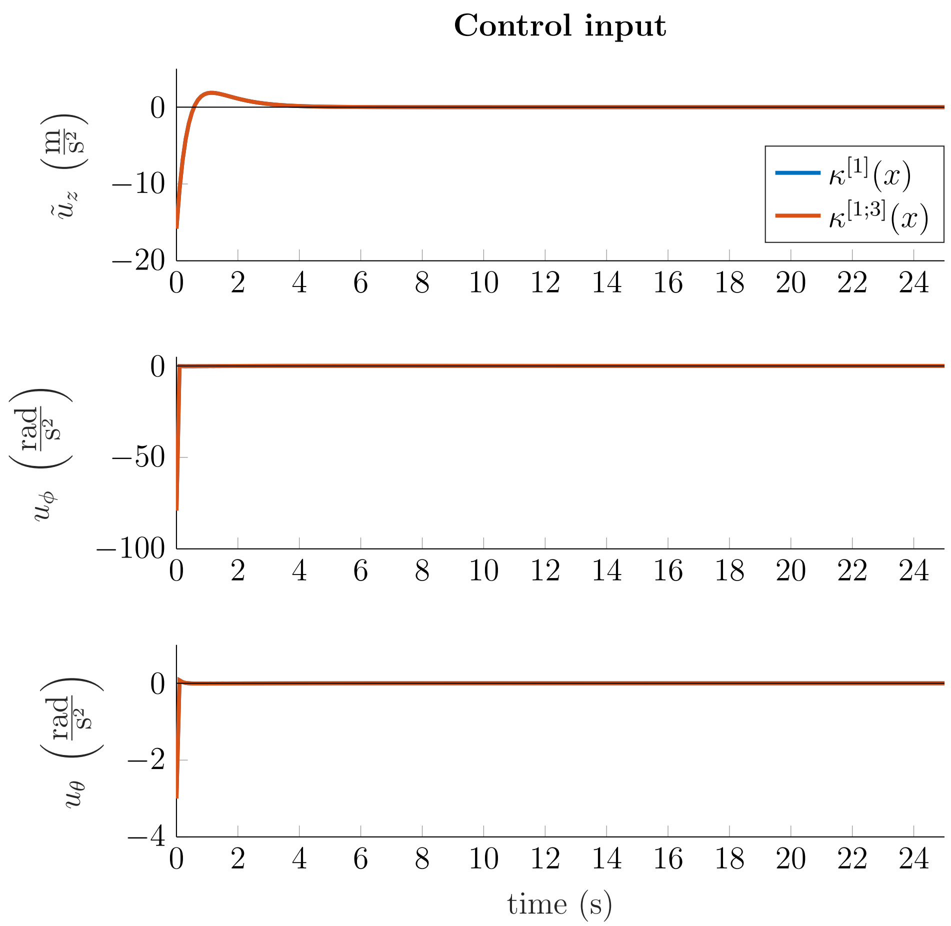


**Figure 2.4:** Control input for the vertical and angular accelerations (10 states model)

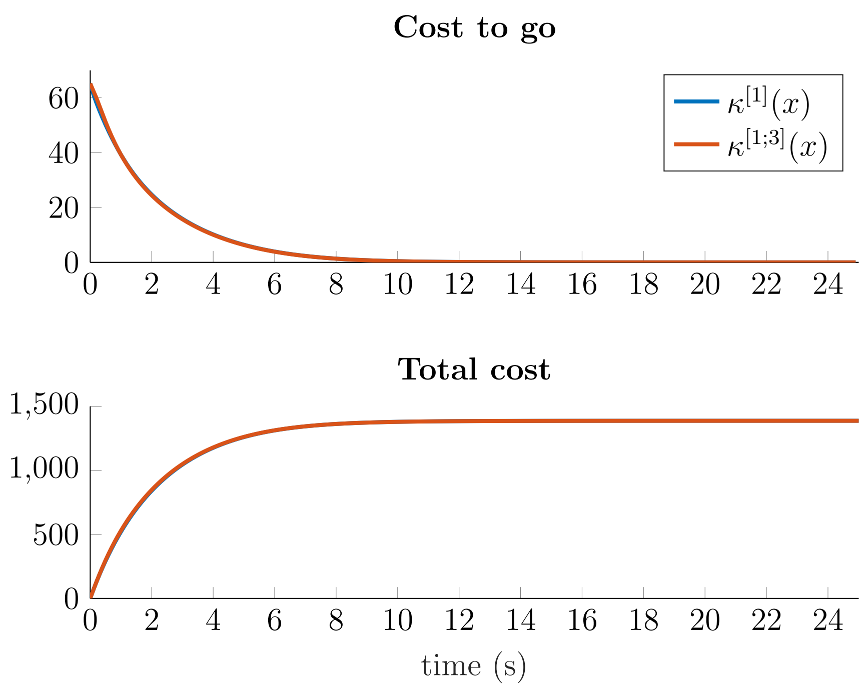


**Figure 2.5:** Resulting cost (10 states model)

**Example 2** (Quadcopter: 12 states)**.**
*In the following, Al'brekht's Method is applied to another more complex quadcopter model as presented in [39, 40]. The considered system dynamics are given by*

$$
\begin{aligned}
\dot{p}_x &= \cos(\theta)\cos(\psi)\cdot v_x + \Big(\sin(\phi)\sin(\theta)\cos(\psi) - \cos(\phi)\sin(\psi)\Big)\cdot v_y \\
&\quad + \Big(\cos(\phi)\sin(\theta)\cos(\psi) + \sin(\phi)\sin(\psi)\Big)\cdot v_z \\
\dot{p}_y &= \cos(\theta)\sin(\psi)\cdot v_x + \Big(\sin(\phi)\sin(\theta)\sin(\psi) + \cos(\phi)\cos(\psi)\Big)\cdot v_y \\
&\quad + \Big(\cos(\phi)\sin(\theta)\sin(\psi) - \sin(\phi)\cos(\psi)\Big)\cdot v_z \\
\dot{p}_z &= -\sin(\theta)\cdot v_x + \sin(\phi)\cos(\theta)\cdot v_y + \cos(\phi)\cos(\theta)\cdot v_z \\
\dot{v}_x &= \frac{v_\psi\cdot v_y - v_\theta\cdot v_z}{\mathrm{rad}} - g\cdot\sin(\theta) \\
\dot{v}_y &= \frac{v_\phi\cdot v_z - v_\psi\cdot v_x}{\mathrm{rad}} + g\cdot\sin(\phi)\cos(\theta) \\
\dot{v}_z &= \frac{v_\theta\cdot v_x - v_\phi\cdot v_y}{\mathrm{rad}} + g\cdot\cos(\phi)\cos(\theta) - \frac{u_z}{m_t} \\
\dot{\phi} &= v_\phi + \tan(\theta)\cdot\Big(\sin(\phi)\cdot v_\theta + \cos(\phi)\cdot v_\psi\Big) \\
\dot{\theta} &= \cos(\phi)\cdot v_\theta - \sin(\phi)\cdot v_\psi \\
\dot{\psi} &= \frac{\sin(\phi)}{\cos(\theta)}\cdot v_\theta + \frac{\cos(\phi)}{\cos(\theta)}\cdot v_\psi \\
\dot{v}_\phi &= \frac{J_y - J_z}{J_x}\cdot\frac{v_\theta\cdot v_\psi}{\mathrm{rad}} + \frac{u_\phi}{J_x} \\
\dot{v}_\theta &= \frac{J_z - J_x}{J_y}\cdot\frac{v_\phi\cdot\ v_\psi}{\mathrm{rad}} + \frac{u_\theta}{J_y} \\
\dot{v}_\psi &= \frac{J_x - J_y}{J_z}\cdot\frac{v_\phi\cdot v_\theta}{\mathrm{rad}} + \frac{u_\psi}{J_z}.
\end{aligned}
$$

*They contain 12 states* $x = \begin{pmatrix} p_x & p_y & p_z & v_x & v_y & v_z & \phi & \theta & \psi & v_\phi & v_\theta & v_\psi \end{pmatrix}^{\mathrm{T}}$*, where* $p_x$*,* $p_y$*, and* $p_z$ *represent again the position in x resp. y-direction and the altitude in the world coordinate system (Fig. 2.6). The velocities in the three directions are stated as* $v_x$*,* $v_y$*, and* $v_z$*.* $\phi$*,* $\theta$*, and* $\psi$ *are the roll, pitch, and yaw (*rad*) with velocities* $v_\phi$*,* $v_\theta$*, and* $v_\psi$*. The sine and cosine functions also have to cancel out the unit* rad*. For simplicity of the equations, it is not shown here. The moments of inertia are known parameters with values* $J_x = 0.053\,\mathrm{kg}\cdot\mathrm{m}^2$*,* $J_y = 0.053\,\mathrm{kg}\cdot\mathrm{m}^2$*, and* $J_z = 0.098\,\mathrm{kg}\cdot\mathrm{m}^2$*.* $g = 9.81\,\mathrm{m/s}^2$ *is the gravity acceleration and* $m_t = 3\,\mathrm{kg}$ *the total mass of the quadcopter. The control input* $u = \begin{pmatrix} u_z & u_\phi & u_\theta & u_\psi \end{pmatrix}^{\mathrm{T}}$ *represents the thrust and the moments about the axis. As in the 10 state model from Example 1, the control input has to be shifted to meet all the necessary conditions.*

$$u_z = \tilde{u}_z - m_t\cdot g \qquad \Big(u_T = \begin{pmatrix} \tilde{u}_z & u_\phi & u_\theta & u_\psi \end{pmatrix}^{\mathrm{T}}\Big)$$

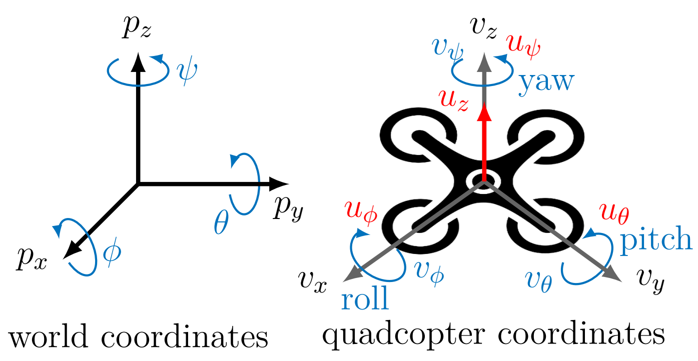


**Figure 2.6:** Coordinate systems for the 12 state quadcopter model

*A quadratic cost function*

$$\ell(x, u_T) = \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + \frac{1}{2}u_T^{\mathrm{T}}\ell_{uu}u_T,$$

*is used. The diagonal matrices* $\ell_{xx} = \mathrm{diag}(1, 1, 1, 1, 1, 1, 100, 100, 100, 100, 100, 100)$ *and* $\ell_{uu} = \frac{1}{10} \cdot I_4$ *(units are left out) are used this time. The considered initial values for the simulation are* $p_x = 25\,\mathrm{m}$*,* $p_z = 5\,\mathrm{m}$*, and* $\phi = 0.1745\,\mathrm{rad} \approx 10°$*. Other states are set to zero. The state* $p_z$ *can be shifted to achieve any altitude as setpoint.*
*The control law has been approximated up to degree three and thus the value function up to degree four. In total, 3623 coefficients are calculated, while only 758 of them are non-zero. The simulation results (Fig. 2.7-2.9) show an increasing performance using higher-order approximations. The total cost arising from the control error and the control effort is reduced by nearly 11 % when* $\kappa^{[1;2]}$ *is compared with* $\kappa^{[1]}$ *and more then 12 % comparing* $\kappa^{[1;3]}$ *and* $\kappa^{[1]}$*. The third-order approximation gives the best result in the overall cost as well as the control performance. The altitude is reduced to zero without any overshoot, while the target value in the x-direction is faster than with the other control laws, and the drift in y- direction is smoothly compensated and reduced. Higher-order approximations do not always have to lead to better results. Sometimes odd degrees are better than even ones or the other way around.*

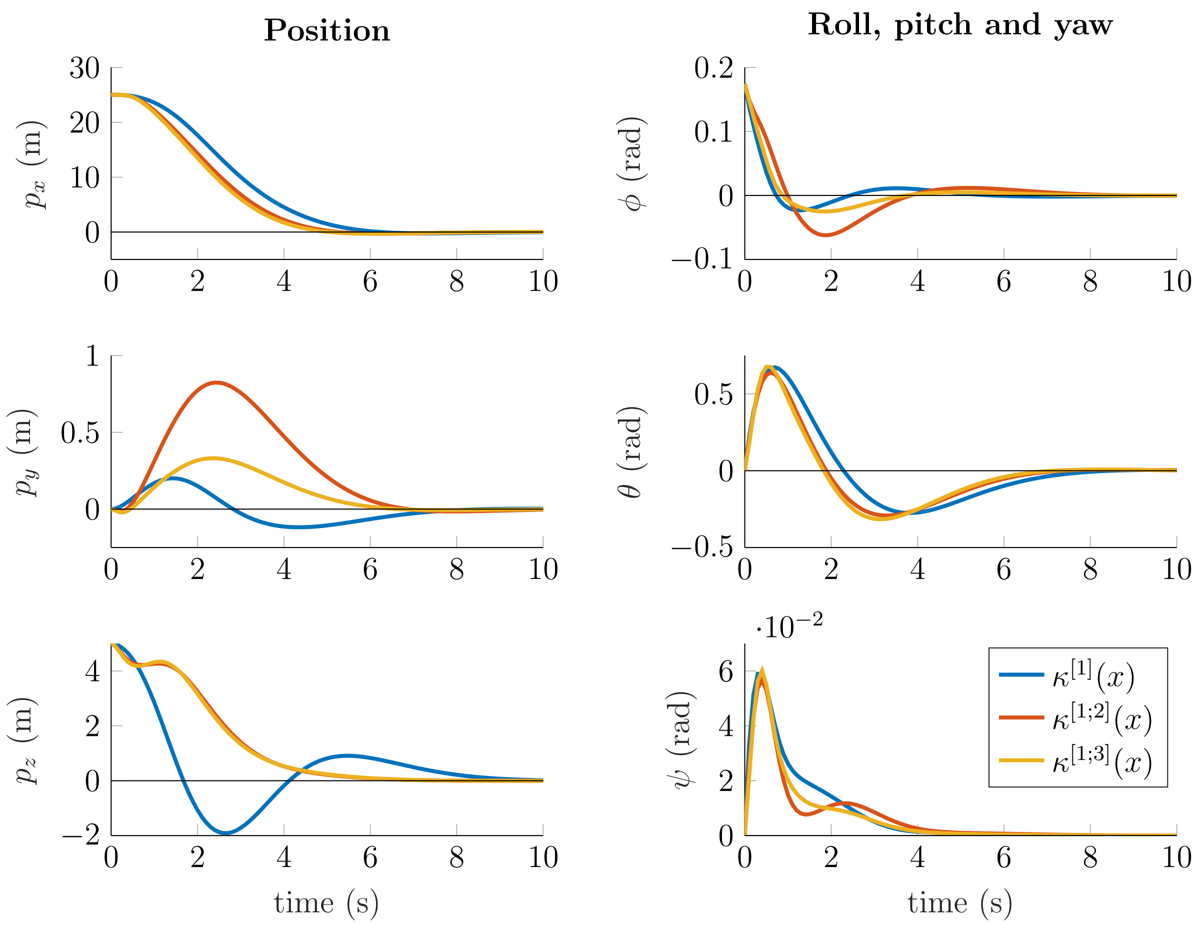


**Figure 2.7:** Propagation of six out of twelve states (12 states model)

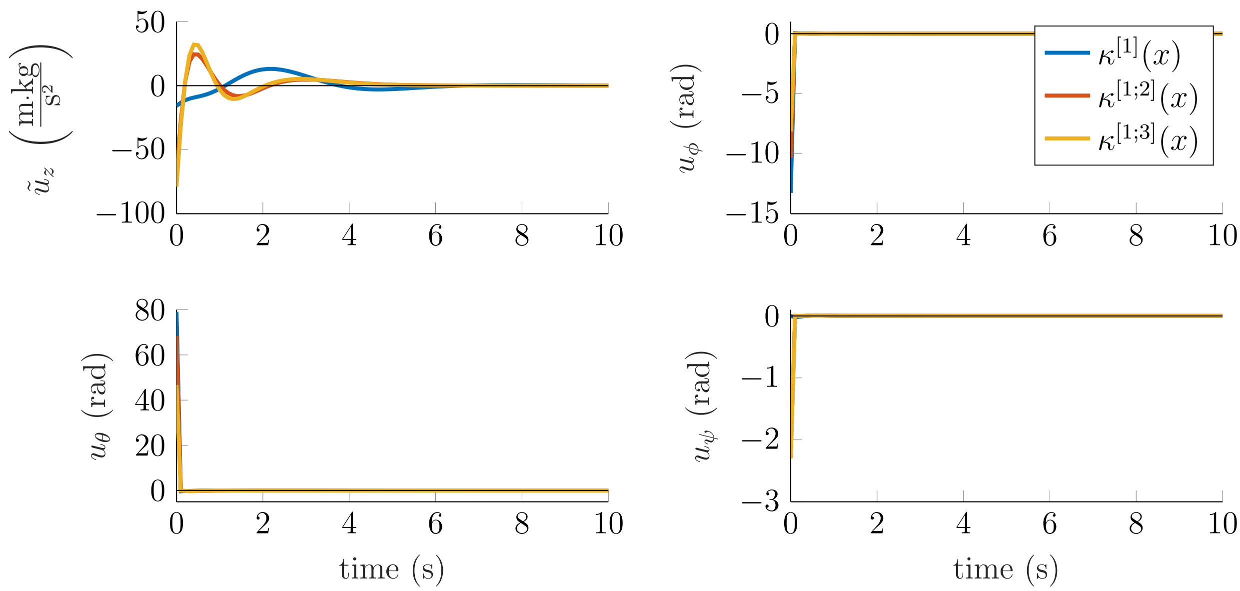


**Figure 2.8:** Control input for the thrust and the moments about the axis (12 states model)

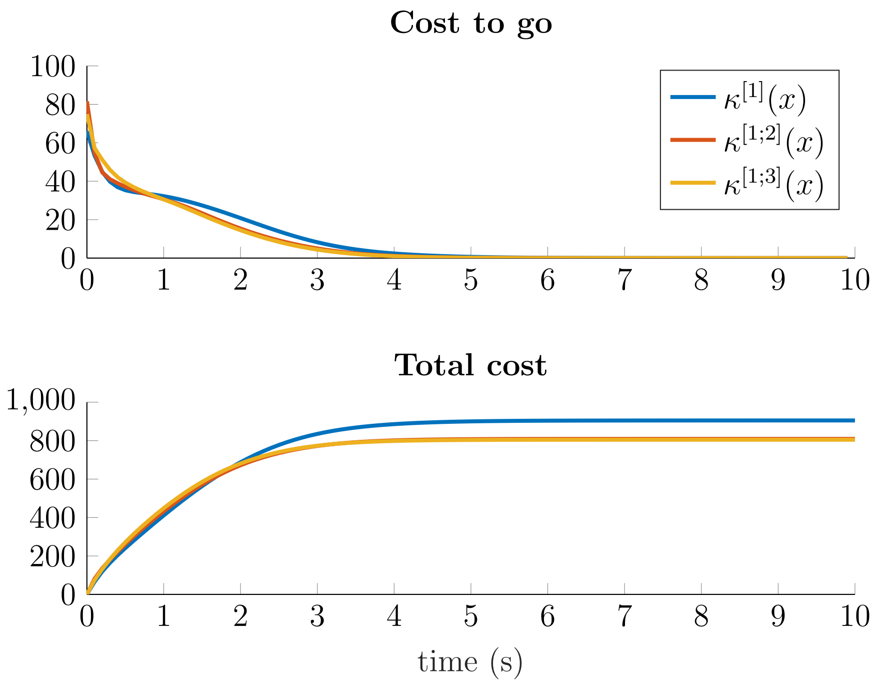


**Figure 2.9:** Resulting cost (12 states model)

## 2.2 Convergence proof for continuous-time systems

As mentioned before, even though every degree of the power series of the optimal input and the value function can be calculated, the convergence resp. the existence of the power series is not necessarily given. Therefore in this section, the convergence proof given by E. G. Al'brekht [3] is recapitulated in detail and also generalized. The proof only covers a special class of optimal control problems and not the general case, e. g. Al'brekht does not consider multiple input variables and uses $\ell_{xx} = I_{n_x}$, $\ell_{xu} = 0_{n_x \times 1}$, and $\ell_{uu} = 1$ in the cost function.
The main idea behind the proof is to first upper bound each degree of the value function, using the system dynamics and the cost function. Doing so a dominating converging power series is constructed to guarantee the convergence. Since in the special case, the control law can be stated in terms of the value function and constant matrices, its convergence is evident.

Throughout this section, an optimal control problem of the following form is considered.

$$\begin{aligned} \pi\big(x(0)\big) &= \min_{u(.)} \int_0^\infty \ell\big(x(\tau), u(\tau)\big)\,\mathrm{d}\tau \\ \text{s.t. } \dot{x} &= f(x) + Gu \end{aligned} \tag{OCP}$$

In this case, the system dynamics is linear regarding the input variables $u \in \mathbb{R}^{n_u}$, while the cost function is given by the following quadratic form:

$$\ell(x,u) = \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + x^{\mathrm{T}}\ell_{xu}u + \frac{1}{2}u^{\mathrm{T}}\ell_{uu}u. \tag{$\ell$}$$

Thus the cost does not contain terms of order higher than two, while the function $f\colon \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is considered to be analytic with $f(0) = 0$ and

$$f(x) = Fx + \sum_{k=2}^{\infty} f^{[k]}(x). \tag{$f$}$$

**Theorem 2** (Local convergence of $\pi(x)$ and $\kappa(x)$)**.**
*Let $\kappa$ be the optimal control input of (OCP) and $\pi$ the corresponding optimal cost function. If $f$ and $\ell$ respect the conditions (I)-(III) from Section 2.1, then $\kappa$ and $\pi$ can be expressed in terms of the system states $x$ and they are locally analytic.*

*Proof.* The proof follows the lines of [3] with some minor generalizations. Note that in principle the input does not have to be 1-dimensional and instead of $\ell_{xx} = I_{n_x}$, $\ell_{xu} = 0_{n_x \times n_u}$, and $\ell_{uu} = I_{n_u}$ arbitrary matrices that fulfill (I)-(III) are allowed.
The optimal input and the value function are denoted by

$$\kappa(x) = Kx + \sum_{k=2}^{\infty} \kappa^{[k]}(x) \tag{$\kappa$}$$

$$\text{and } \pi(x) = \frac{1}{2}x^{\mathrm{T}}\pi_{xx}x + \sum_{k=3}^{\infty} \pi^{[k]}(x). \tag{$\pi$}$$

The constants $\kappa_0$ and $\pi_0$ as well as the 1-by-$n_x$ matrix $\pi_x$ are neglected since they are zero anyway. The corresponding Hamilton-Jacobi-Bellman equation of the system is becomes

$$0 = \nabla_x\pi(x) \cdot \big(f(x) + G\kappa(x)\big) + \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + x^{\mathrm{T}}\ell_{xu}\kappa(x) + \frac{1}{2}\kappa(x)^{\mathrm{T}}\ell_{uu}\kappa(x) \tag{HJBE-1}$$

and the first-order optimality condition by

$$0 = \nabla_x\pi(x) \cdot G + x^{\mathrm{T}}\ell_{xu} + \kappa^{\mathrm{T}}(x) \cdot \ell_{uu}. \tag{HJBE-2}$$

At first, simplified defining equations for $\kappa^{[k]}(x)$ and $\pi^{[k]}(x)$ ($k \geq 1$) have to be established. For $K$ and $\pi_{xx}$, one obtains ($K$) and ($\pi_{xx}$) from Section 2.1.

$$K = -\ell_{uu}^{-1} \cdot (\ell_{xu}^{\mathrm{T}} + G^{\mathrm{T}}\pi_{xx}) \tag{$K$}$$

$$0 = \pi_{xx}F + F^{\mathrm{T}}\pi_{xx} + \ell_{xx} - (\ell_{xu} + \pi_{xx}G) \cdot \ell_{uu}^{-1} \cdot (\ell_{xu}^{\mathrm{T}} + G^{\mathrm{T}}\pi_{xx}) \tag{$\pi_{xx}$}$$

From (HJBE-2) one can easily derive

$$\kappa^{[k]}(x) = -\ell_{uu}^{-1} \cdot G^{\mathrm{T}} \cdot \nabla_x\pi^{[k+1]}(x)^{\mathrm{T}} \tag{$\kappa^{[k]}$}$$

for $k \geq 2$. Using (HJBE-1), $\pi^{[k+1]}(x)$ is defined by the equation

$$
\begin{aligned}
0 = &\sum_{i=1}^{k} \nabla_x \pi^{[i+1]}(x) \cdot \left( f^{[k-i+1]}(x) + G\kappa^{[k-i+1]}(x) \right) \\
&+ x^{\mathrm{T}} \ell_{xu} \cdot \kappa^{[k]}(x) + \frac{1}{2} \sum_{i=1}^{k} \kappa^{[i]}(x)^{\mathrm{T}} \cdot \ell_{uu} \cdot \kappa^{[k-i+1]}(x).
\end{aligned}
$$

Using $(K)$ to cancel out all terms, which depend on $\kappa^{[k]}$, and substituting $(\kappa^{[k]})$, one derives the following identity.

$$
\begin{aligned}
\nabla_x \pi^{[k+1]}(x) \cdot (F + GK) \cdot x = &- \sum_{i=1}^{k-1} \nabla_x \pi^{[i+1]}(x) \cdot f^{[k-i+1]}(x) \\
&+ \frac{1}{2} \sum_{i=2}^{k-1} \nabla_x \pi^{[i+1]}(x) \cdot G \ell_{uu}^{-1} G^{\mathrm{T}} \cdot \nabla_x \pi^{[k-i+2]}(x)^{\mathrm{T}}
\end{aligned}
\qquad (\pi^{[k+1]})
$$

To prove the convergence of the power series $(\pi)$, it is desired to find a series $\left(C_k^{\pi}\right)_{k \geq 2}$ such that

$$
\sum_{k=2}^{\infty} C_k^{\pi} \cdot r^k < \infty \quad \text{and} \quad \left| \pi^{[k]}(x) \right| \leq C_k^{\pi} \cdot r^k
$$

for sufficiently small $r = ||x||$.
Since $f(.)$ is analytic at least in a neighborhood of the origin, Remark 16 $(c)$ implies the existence of a series $\left(C_k^{f}\right)_{k \geq 1}$ with

$$
\sum_{k=1}^{\infty} C_k^{f} \cdot r^k < \infty \quad \text{and} \quad \left| f^{[k]}(x) \right| \leq C_k^{f} \cdot r^k
$$

for sufficiently small $r$. In the following, the inequality (A.5) from Theorem 10 will be applied to the different degrees of $\pi$.

$$
\left| \nabla_x \pi^{[k]}(x) \right| \leq k \cdot C_k^{\pi} \cdot r^{k-1}
$$

The second degree of the value function can be easily upper bounded using the spectral norm of $\pi_{xx}$, which is its largest eigenvalue. To keep the notation simple, this eigenvalue divided by two will be called $C_2^{\pi}$.

$$
\left| \pi^{[2]}(x) \right| = \left| \frac{1}{2} x^{\mathrm{T}} \pi_{xx} x \right| \leq C_2^{\pi} \cdot r^2
$$

Going to the next degree, one finds from $(\pi^{[k+1]})$ an equation for $\nabla_x \pi^{[3]}(x)$.

$$
\left. \frac{\mathrm{d}\pi^{[3]}}{\mathrm{d}t}(x) \right|_{\dot{x} = Fx + GKx} = \nabla_x \pi^{[3]}(x) \cdot (F + GK) \cdot x = -x^{\mathrm{T}} \pi_{xx} \cdot f^{[2]}(x)
$$

Via integration it follows that

$$\left|\pi^{[3]}(x)\right|_{\dot{x}=Fx+GKx} = \left|\int_0^\infty -x^{\mathrm{T}}\pi_{xx}\cdot f^{[2]}(x)\,\mathrm{d}t\right| \leq \int_0^\infty \left|x^{\mathrm{T}}\pi_{xx}\cdot f^{[2]}(x)\right|\mathrm{d}t \leq 2C_2^f C_2^\pi \int_0^\infty r^3(t)\,\mathrm{d}t.$$

Since the linear system is stable, even exponentially stable, it is possible to find an upper bound for the states and, therefore, also $r = ||x||$.

$$r(t) \leq r_0 \cdot \mathrm{e}^{-\alpha\cdot t}\ , \quad r_0 = ||x(0)||,\ \alpha > 0$$

Thus after solving the integral, $C_3^\pi$ is found.

$$\left|\pi^{[3]}(x)\right|_{\dot{x}=Fx+GKx} \leq \frac{2C_2^f C_2^\pi}{3\alpha}\cdot r_0^3 = C_3^\pi \cdot r_0^3$$

Theorem 10 implies

$$\left|\nabla_x\pi^{[3]}(x)\right| \leq 3\cdot C_3^\pi \cdot r_0^2 = \frac{2C_2^f C_2^\pi}{\alpha}\cdot r_0^2.$$

Before discussing the general case, $C_4^\pi$ will be calculated to gain more insight how those constants are found. Thus $(\pi^{[k+1]})$ is integrated.

$$\begin{aligned}
\left|\pi^{[4]}(x)\right|_{\dot{x}=Fx+GKx} &= \Bigg|\int_0^\infty -x^{\mathrm{T}}\pi_{xx}\cdot f^{[3]}(x) - \nabla_x\pi^{[3]}(x)\cdot f^{[2]}(x) \\
&\qquad + \frac{1}{2}\nabla_x\pi^{[3]}(x)\cdot G\ell_{uu}^{-1}G^{\mathrm{T}}\cdot\nabla_x\pi^{[3]}(x)^{\mathrm{T}}\,\mathrm{d}t\Bigg| \\
&\leq \int_0^\infty 2C_3^f C_2^\pi\cdot r^4(t) + 3C_2^f C_3^\pi \cdot r^4(t) + \frac{9}{2}C_G^2\cdot C_{uu}\cdot (C_3^\pi)^2\cdot r^4(t)\,\mathrm{d}t \\
&\leq \frac{1}{4\alpha}\cdot\left(2C_3^f C_2^\pi + 3C_2^f C_3^\pi + \frac{9}{2}C_G^2 C_{uu}\cdot(C_3^\pi)^2\right)\cdot r_0^4 =: C_4^\pi\cdot r_0^4
\end{aligned}$$

Here $C_{uu}$ resp. $C_G$ denotes the spectral norm of $\ell_{uu}^{-1}$ resp. $G$.

The general case $k \geq 2$ can be handled in a similar way.

$$
\begin{aligned}
&\left|\pi^{[k+1]}(x)\right|_{\dot{x}=Fx+GKx} \\
&\leq \int_0^\infty \sum_{i=1}^{k-1} (i+1) \cdot C_{i+1}^{\pi} \cdot r^i(t) \cdot C_{k-i+1}^{f} \cdot r^{k-i+1}(t)\,\mathrm{d}t \\
&\quad + \int_0^\infty \sum_{i=2}^{k-1} \frac{1}{2}(i+1) \cdot C_{i+1}^{\pi} \cdot r^i(t) \cdot C_G^2 C_{uu} \cdot (k-i+2) \cdot C_{k-i+2}^{\pi} \cdot r^{k-i+1}(t)\,\mathrm{d}t \\
&\leq \frac{\sum\limits_{i=1}^{k-1} (i+1) \cdot C_{i+1}^{\pi} C_{k-i+1}^{f} + \sum\limits_{i=2}^{k-1} \frac{1}{2}(i+1) \cdot (k-i+2) \cdot C_G^2 C_{uu} \cdot C_{i+1}^{\pi} C_{k-i+2}^{\pi}}{(k+1) \cdot \alpha} \cdot r_0^{k+1} \\
&=: C_{k+1}^{\pi} \cdot r_0^{k+1}
\end{aligned}
$$

Given these coefficients, a converging domination series for $\left(C_k^{\pi}\right)_{k \geq 2}$ would also imply the desired convergence. To do so, another equation is introduced, and it will be shown that its solution is, in fact, a series with the desired properties.

$$
C_G^2 C_{uu} \cdot \left(\frac{\mathrm{d}\gamma}{\mathrm{d}r}(r)\right)^2 + \left(\sum_{i=2}^{\infty} C_i^f \cdot q^{i-1} - a\right) \cdot r \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}r}(r) + b \cdot r^2 = 0 \tag{2.1}
$$

The constant numbers $a$ and $b$ will be determined during the following calculation, while $q \in \mathbb{R}_{\geq 0}$ can be seen as a vector norm similar to $r$. Clearly, (2.1) is only well-defined where $f$ is analytic. One may observe that $C_1^f$ is not a part of the power series, so it does not seem to have any influence on the solution. But later calculation shows that, in fact, $a$ and $b$ depend on $F$.
Finding (2.1) and a solution of it is the key element of this proof. All the steps that have been done before are not surprising and also unavoidable. Clearly, (2.1) admits two solutions. Those solutions are of the form $\gamma(r) = \frac{1}{2} g(q) \cdot r^2$, where $g(.)$ is a solution of the quadratic equation

$$
C_G^2 C_{uu} \cdot g^2(q) + \left(\sum_{i=2}^{\infty} C_i^f \cdot q^{i-1} - a\right) \cdot g(q) + b = 0. \tag{2.2}
$$

If $q$ is sufficiently small and

$$
0 < b < \frac{a^2}{4 C_G^2 C_{uu}}, \tag{2.3}
$$

two solutions

$$
g_{1/2}(q) = \frac{a - \sum\limits_{i=2}^{\infty} C_i^f \cdot q^{i-1} \pm \sqrt{\left(a - \sum\limits_{i=2}^{\infty} C_i^f \cdot q^{i-1}\right)^2 - 4b \cdot C_G^2 C_{uu}}}{2 C_G^2 C_{uu}} \tag{2.4}
$$

are found. From those two solutions, $g_2(q)$ is taken and from now on just denoted with $g(q)$. It will be seen later why $g_1(q)$ can not be the desired solution. First note that the solution $g(.)$ is analytic in the domain containing the origin where $\sum_{i=2}^{\infty} C_i^f \cdot q^{i-1} < \infty$. Furthermore, it can be locally expanded into a power series

$$g(q) = \sum_{i=2}^{\infty} C_i^g \cdot q^{i-2} < \infty. \tag{2.5}$$

Analog to the calculation in Section 2.1, this series can be found degree-wise. Starting with the constant part from (2.5) and (2.2) or (2.4), which defines $C_2^g$.

$$0 = C_G^2 C_{uu} \cdot (C_2^g)^2 - a \cdot C_2^g + b \quad \Leftrightarrow \quad C_2^g = \frac{a - \sqrt{a^2 - 4b \cdot C_G^2 \cdot C_{uu}}}{2C_G^2 C_{uu}} > 0$$

Taking the linear part from (2.2) and leaving out $q$, since it is arbitrary, leads to $C_3^g$.

$$\begin{aligned} 0 &= 2C_G^2 C_{uu} \cdot C_2^g C_3^g + C_2^f C_2^g - a \cdot C_3^g \\ \Rightarrow C_3^g &= \frac{C_2^f C_2^g}{a - 2C_G^2 C_{uu} \cdot C_2^g} \end{aligned}$$

$a$ and $b$ have to be chosen such that

$$a - 2C_G^2 C_{uu} \cdot C_2^g = \sqrt{a^2 - 4b \cdot C_G^2 C_{uu}}$$

is strictly greater than zero. This is the case if $b$ (still depending on $a$) is defined as

$$b = \frac{a^2 - \alpha^2}{4C_G^2 \cdot C_{uu}},$$

which ensures

$$a^2 - 4b \cdot C_G^2 C_{uu} = \alpha^2 > 0.$$

Obviously, this choice of $b$ also fulfills (2.3), if $a > \alpha$. Using $\alpha$ here involves also the linearized system and, therefore, $F + GK$. Combining all formulas and definitions gives

$$C_3^g = \frac{1}{\alpha} \cdot C_2^f \cdot C_2^g > 0$$

and for $a \geq \alpha + 2C_G^2 C_{uu} \cdot C_2^\pi$ also $C_2^g = \dfrac{a - \alpha}{2C_G^2 C_{uu}} \geq C_2^\pi$.
If $g_1(q)$ would have been used, then $C_k^g$ $(k \geq 3)$ would be negative, which can not be an upper bound for $C_k^\pi$.

More general, the $C_k^g$ are obtained by taking the $(k-2)$-th degree in equation (2.2).

$$
\begin{aligned}
0 &= C_G^2 C_{uu} \cdot \sum_{i=2}^{k} C_i^g \cdot C_{k-i+2}^g + \sum_{i=2}^{k-1} C_i^f \cdot C_{k-i+1}^g - a \cdot C_k^g. \\
\Rightarrow \alpha \cdot C_k^g &= (a - 2C_G^2 C_{uu} \cdot C_2^g) \cdot C_k^g = C_G^2 C_{uu} \cdot \sum_{i=3}^{k-1} C_i^g \cdot C_{k-i+2}^g + \sum_{i=2}^{k-1} C_i^f \cdot C_{k-i+1}^g \\
\Rightarrow C_k^g &= \frac{1}{\alpha} \cdot \left( C_G^2 C_{uu} \cdot \sum_{i=3}^{k-1} C_i^g \cdot C_{k-i+2}^g + \sum_{i=2}^{k-1} C_i^f \cdot C_{k-i+1}^g \right) > 0
\end{aligned}
$$

Finally, this can be compared to

$$
C_k^\pi = \frac{\sum_{i=2}^{k-1} (k-i+1) \cdot C_i^f C_{k-i+1}^\pi + C_G^2 C_{uu} \cdot \sum_{i=3}^{k-1} \frac{1}{2} i \cdot (k-i+2) \cdot C_i^\pi C_{k-i+2}^\pi}{k \cdot \alpha}
$$

and it is clear that

$$
C_k^\pi \leq C_k^g \cdot \frac{\max\limits_{i \in \{3,\ldots,k-1\}} i \cdot (k-i+2)}{2k} \leq C_k^g \cdot \frac{(k+2)^2}{8k},
$$

since $C_2^\pi \leq C_2^g$. This makes the convergence of $(\pi)$ is clear since

$$
\lim_{k\to\infty} \left( \frac{(k+2)^2}{8k} \right)^{1/k} \searrow 1
$$

and

$$
|\pi(x)| \leq \sum_{k=2}^{\infty} C_k^\pi \cdot r^k \leq \sum_{k=2}^{\infty} \frac{(k+2)^2}{8k} \cdot C_k^g \cdot r^k < \infty
$$

Since $\kappa(x)$ is a product of locally analytic functions it is also locally analytic with the same area of convergence. □

## 2.3 Area of convergence

In Section 2.2, the local convergence of the power series $(\kappa)$ and $(\pi)$ has been established. Both series are the solution of an optimal control problem (OCP). A minimal area of convergence resp. an inner approximation of the domain where the solution of (OCP) can be found via Al'brekht's Method is not provided yet. However, the proof of Theorem 2 offers a possibility to find a criterion that characterizes the domain. With the same notation as in Section 2.2 it is assumed that $f(.)$ is analytical in a neighborhood $\mathcal{D}$ of the origin. Recapitulating that the dominating series for $|\pi(x)|$ is given by

$$g(r) = \frac{a - \sum\limits_{i=2}^{\infty} C_i^f \cdot r^{i-1} - \sqrt{\left(a - \sum\limits_{i=2}^{\infty} C_i^f \cdot r^{i-1}\right)^2 - 4b \cdot C_G^2 \cdot C_{uu}}}{2C_G^2 \cdot C_{uu}}.$$

$g(r)$ is locally analytic as a combination of analytical functions and exists for all $r$ such that $f(.)$ is analytic in $\mathcal{D}_r := \{x \in \mathcal{D} : ||x|| \leq r\}$ and

$$0 \leq \left(a - \sum_{i=2}^{\infty} C_i^f \cdot q^{i-1}\right)^2 - 4b \cdot C_G^2 \cdot C_{uu}.$$

Since the constants $a$ and $b$ were chosen as

$$a \geq \alpha + 2C_G^2 C_{uu} \cdot C_2^{\pi} \quad \text{and} \quad b = \frac{a^2 - \alpha^2}{4C_G^2 \cdot C_{uu}},$$

this inequality is equivalent to

$$0 < a^2 - \alpha^2 \leq \left(a - \sum_{i=2}^{\infty} C_i^f \cdot r^{i-1}\right)^2.$$

Taking the square root on both sides leads to

$$0 < \sqrt{a^2 - \alpha^2} \leq a - \sum_{i=2}^{\infty} C_i^f \cdot r^{i-1}$$

and thus

$$\sum_{i=2}^{\infty} C_i^f \cdot r^{i-1} \leq a - \sqrt{a^2 - \alpha^2}.$$

The right-hand side of this inequality is decreasing in $a$. Therefore it reaches its maximum for $a = \alpha + 2C_G^2 C_{uu} \cdot C_2^{\pi}$.

$$\begin{aligned} &\sum_{i=2}^{\infty} C_i^f \cdot r^{i-1} \leq \alpha + 2C_G^2 C_{uu} \cdot C_2^{\pi} - \sqrt{(\alpha + 2C_G^2 C_{uu} \cdot C_2^{\pi})^2 - \alpha^2} \\ \Leftrightarrow &\sum_{i=2}^{\infty} C_i^f \cdot r^{i} \leq \left(\alpha + 2C_G^2 C_{uu} \cdot C_2^{\pi} - \sqrt{(\alpha + 2C_G^2 C_{uu} \cdot C_2^{\pi})^2 - \alpha^2}\right) \cdot r =: \beta \cdot r \end{aligned} \tag{2.6}$$

Remember, $C_G$,$C_{uu}$, and $C_2^{\pi}$ denote the spectral norm of $G$, $\ell_{uu}^{-1}$ resp. $\frac{1}{2}\pi_{xx}$, and $\alpha$ is such that $r(t) \leq r_0 \cdot \mathrm{e}^{-\alpha t}$. A good value for $\alpha$ can for example be found by

$$\begin{aligned} &- \max\left\{\Re(\lambda) : \lambda \text{ eigenvalue of } F + GK\right\} \\ \text{resp. } &\min\left\{-\Re(\lambda) : \lambda \text{ eigenvalue of } F + GK\right\}. \end{aligned}$$

Since $\alpha$ depends on $F$, the area of convergence also depends on the linear part of the system dynamics $f(.)$, whereas the left-hand side of (2.6) depends on higher degrees. The constant $\beta$ depends on $\alpha$, $C_G$, $C_{uu}$, and $C_2^\pi$. $\alpha$ should be chosen maximal since it increases $\beta$, which can be seen by calculating the following derivative.

$$\frac{\partial \beta}{\partial \alpha}(\alpha, C_G, C_{uu}, C_2^\pi) = 1 - \frac{\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi - \alpha}{\sqrt{(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi)^2 - \alpha^2}}$$

$\frac{\partial \beta}{\partial \alpha}$ is positive as $\alpha$, $C_G$, $C_{uu}$, and $C_2^\pi$ are also positive, which implies that

$$\frac{\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi - \alpha}{\sqrt{(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi)^2 - \alpha^2}} < 1.$$

This is shown by the following calculation.

$$\begin{aligned}
(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi)^2 - 2\alpha \cdot (\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi) + \alpha^2 &< (\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi)^2 - \alpha^2 \\
\Leftrightarrow -2\alpha \cdot (\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi) + 2\alpha^2 &< 0 \\
\Leftrightarrow -(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi) + \alpha &< 0 \\
\Leftrightarrow \alpha + 2C_G^2 C_{uu} \cdot C_2^\pi &> 0
\end{aligned}$$

**Corollary 2** (Area of convergence of $\pi(x)$ and $\kappa(x)$)**.**
*The area of convergence can be bound by inequality (2.6). More precisely, the power series ($\pi$) and ($\kappa$) from Section 2.1 converge and exist for all $x$ such that*

$$\sum_{i=2}^{\infty} C_i^f \cdot ||x||^i \leq \left(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi - \sqrt{(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi)^2 - \alpha^2}\right) \cdot ||x||. \tag{2.7}$$

**Remark 1.** *If the system dynamics is linear and the cost function is quadratic, then the solution of the optimal control problem (OCP) from Section 2.1 can be expressed as*

$$\pi(x) = \frac{1}{2} x^{\mathrm{T}} \pi_{xx} x \quad \text{and} \quad \kappa(x) = Kx.$$

*Both clearly exists everywhere in $\mathbb{R}^{n_x}$. The same result is also provided by (2.7) since $C_i^f = 0$ for all $i \geq 2$.*

Throughout this chapter, Al'brekht's Method in continuous-time has been discussed. A convergence proof for the power series of the value function and the control law that considers more general systems and cost functions than the original proof given in [3] has been given. In addition, a new possibility to characterize the area of convergence was investigated. Furthermore, the effectiveness of the power series approach has been shown using two quadcopter examples.

# 3 Approximated Optimal Control with Variable Parameters

In contrast to the existing method that solves fixed nonlinear optimal control problems, parameter-dependent dynamics are of interest in this chapter. Instead of solving an OCP for a particular choice of model parameters, Al'brekht's Method is extended to allow systems with parametric right-hand sides. It can be used, for example, if the specifications of a plant are changed, but the model stays the same, or in case of unmanned aerial vehicles, if their load changes. More importantly, it might be used for changing environments (e. g. wind, temperature) or measurable disturbances. Having a parametric offline calculated control law enables to react fast to such changes. The parameters have to be known or should be estimated in a reasonable time. The control scheme is updated and might also include parameter estimation (Fig. 3.1).

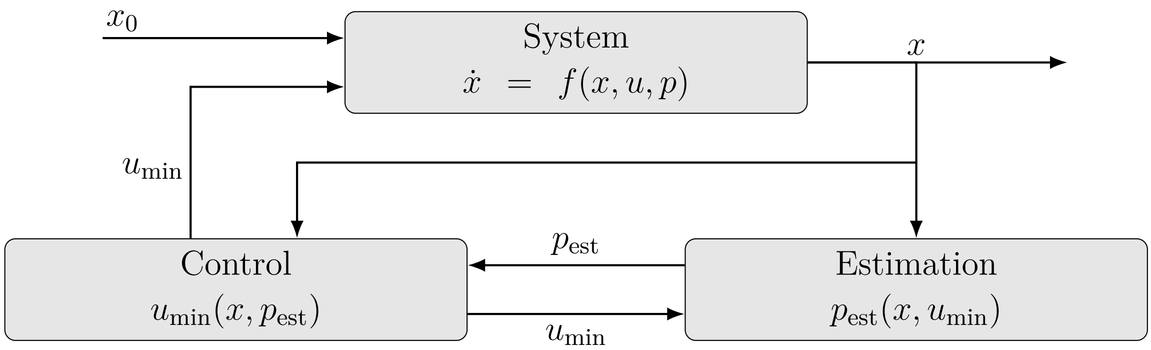


**Figure 3.1:** Overall control scheme including parameter estimation and parameter-dependent controller

Even though the control input $u_{\min}$ changes with the estimated parameters $p_{\text{est}}$ when the states $x$ are fixed, the proposed method does not do active probing, as in the concept of [25, 99]. In contrast to other methods which require an active redesign of the controller, e. g. [81], the parameters are directly integrated in the explicit controller.

There are only little explicit control laws that depend and adjust to model parameters. Axehill et. al. [5, 6] e. g. worked on parametric explicit suboptimal control, where the constraints are parameter dependent. Aguilar and Krener [1, 51, 55] introduced an external system with known dynamics to allow for changing parameters. This system has a direct effect on the states and can be interpreted as time-varying parameters or an explicit time-dependence. The setup in these three papers looks more general, but the dynamics of the exosystem need to be known beforehand and affect the design of the controller. This is not the case for the extension, which is explained in the next section.

## 3.1 Parameter-dependent continuous-time systems

In this section, Al'brekht's Method will be extended to include parameters $p$ in the dynamical system. Therefore the control law $\kappa$ and the value function $\pi$ depend on the parameters. The basic calculation steps are the same as for the non-parametric case in Section 2.1. At first, only multiplicative parameters are considered. Hence the LQR conditions towards the linear part of the system dynamics $f$ and the second-order part of the cost function $\ell$ remain unchanged. Additionally, the origin remains a fixed point for the system. Later in Remark 3, it will be discussed how to handle additive parameters as well as parameter-dependent cost functions.
During the whole calculation, it is assumed that the parameters are time-independent or that the time dependence is comparably slow to the normal system dynamics. Otherwise, as discussed in (18) $(a)$, the time derivative $\dot{p}$ has to be known, and the parameters actually similar to the states, as mentioned before [1, 51, 55].
The optimal cost and input do not contain parts that solely depend on the parameters, i. e. the control action and the cost at the origin remains zero. Furthermore, if $p$ is vanishing, the nominal case is obtained.
At the end of this section, local stability is proven and the effectiveness of the proposed method is underlined via two quadcopter examples in the case of multiplicative parameters and a bioreactor example in the case of additive parameters.

The considered parameter-dependent optimal control problem states as the following.

$$\begin{aligned} \pi\big(x(0), p\big) &= \min_{u(.)} \int_0^\infty \ell\big(x(\tau), u(\tau)\big)\,\mathrm{d}\tau \\ \text{s.t. } \dot{x} &= f(x, u, p) \end{aligned} \tag{OCP}$$

Here the system dynamics and $\pi$ depend on the parameters $p$. To apply Al'brekht's Method again, the assumption that $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ and $\ell\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$ are analytic is made. The condition $f(0,0) = 0$ from Section 2.1 has to be generalized, i. e. it needs to hold for every choice of the parameters:

$$\forall p \in \mathbb{R}^{n_p}: \; f(0,0,p) = 0. \tag{$f_0$}$$

Since $f(0,0,.)\colon \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ is analytic, one observes that the complete power series has to vanish and, therefore, it holds:

$$\forall k \in \mathbb{N}, p \in \mathbb{R}^{n_p} : f^{[k]}(0,0,p) = 0$$

The right-hand side of the dynamics and the cost function can be written using power series expansions.

$$f(x,u,p) = \sum_{k=1}^{\infty} f^{[k]}(x,u,p) = Fx + Gu + f^{[2]}(x,u,p) + \dots \tag{$f$}$$

$$\ell(x,u) = \sum_{k=2}^{\infty} \ell^{[k]}(x,u) = \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + x^{\mathrm{T}}\ell_{xu}u + \frac{1}{2}u^{\mathrm{T}}\ell_{uu}u + \ell^{[3]}(x,u) + \dots \tag{$\ell$}$$

The matrices $F, \ell_{xx} \in \mathbb{R}^{n_x \times n_x}$, $G, \ell_{xu} \in \mathbb{R}^{n_x \times n_u}$, and $\ell_{uu} \in \mathbb{R}^{n_u \times n_u}$ need to fulfill the same conditions as in Section 2.1.

(I) The second-order part of the cost function is convex in $(x,u)$ and strictly convex in $u$, i.e.

$$\begin{pmatrix} \ell_{xx} & \ell_{xu} \\ \ell_{xu}^{\mathrm{T}} & \ell_{uu} \end{pmatrix} \succeq 0 \quad \text{and} \quad \ell_{uu} \succ 0.$$

(II) The linearized system resp. the pair $(F,G)$ is stabilizable.

(III) The pair $(F, \ell_{xx})$ is detectable.

The optimal cost function $\pi\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}$ is assumed to be analytic in $(x,p)$ and can be, similar as in the non-parametric case, written as

$$\begin{aligned} \pi(x,p) &= \sum_{k=1}^{\infty} \pi^{[k]}(x,p) \\ &= \pi_x x + \pi_p p + \frac{1}{2}x^{\mathrm{T}}\pi_{xx}x + x^{\mathrm{T}}\pi_{xp}p + \frac{1}{2}p^{\mathrm{T}}\pi_{pp}p + \pi^{[3]}(x,p) + \dots, \end{aligned} \tag{$\pi$}$$

where the constant part is omitted. This procedure has been justified before. For the same reason it is not necessary to consider the constant in the control law $u_{\min} := \kappa\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_u}$ with

$$\kappa(x,p) = \sum_{k=1}^{\infty} \kappa^{[k]}(x,p) = Kx + Lp + \kappa^{[2]}(x,p) + \dots \tag{$\kappa$}$$

The matrices $\pi_x \in \mathbb{R}^{1 \times n_x}$, $\pi_p \in \mathbb{R}^{1 \times n_p}$, $\pi_{xx} \in \mathbb{R}^{n_x \times n_x}$, $\pi_{xp} \in \mathbb{R}^{n_x \times n_p}$, $\pi_{pp} \in \mathbb{R}^{n_p \times n_p}$, $K \in \mathbb{R}^{n_u \times n_x}$, and $L \in \mathbb{R}^{n_u \times n_p}$ remain to be determined, if possible.
Similar to the non-parametric case, the Hamilton-Jacobi-Bellman equation (see Remark 18 ($c$)) is used as optimality criterion.

$$0 = \nabla_x \pi(x,p) \cdot f\big(x, \kappa(x,p), p\big) + \ell\big(x, \kappa(x,p)\big) \tag{HJBE-1}$$

$$0 = \nabla_x \pi(x,p) \cdot \nabla_u f\big(x, \kappa(x,p), p\big) + \nabla_u \ell\big(x, \kappa(x,p)\big) \tag{HJBE-2}$$

Starting with degree zero in both equations leads to

$$(\text{HJBE-1})^{[0]} : \ 0 = 0$$

and

$$(\text{HJBE-2})^{[0]} : \ 0 = \pi_x \cdot G. \tag{$\pi_x$-1}$$

These two equations are the same as in the non-parametric case. The first degree of (HJBE-1) though gives the first small difference.

$$\begin{aligned}(\text{HJBE-1})^{[1]} : \ 0 &= \pi_x \cdot \Big(Fx + G(Kx + Lp)\Big) = \pi_x \cdot Fx \\ \Rightarrow 0 &= \pi_x \cdot F \end{aligned} \tag{$\pi_x$-2}$$

Combining condition (II) and the Hautus Lemma 3 with ($\pi_x$-1) and ($\pi_x$-2) shows $\pi_x$ has to be 0. $\pi_p$ can not be obtained since it is not contained in any equation. The same holds for $\pi_{pp}$, $\pi^{[3]}(0,p)$, and so forth. Thus $\pi(x,p)$ can not be calculated completely, but it makes sense to set the part that can not be found to 0 since no cost should arise at the origin. In other words, the optimal cost is 0 if the initial value is at the desired point. Furthermore, the control input $\kappa(0,p)$ needs to be zero, which implies $L = 0$, $\kappa^{[2]}(0,p) = 0$, and so forth. A more rigorous justification will be done throughout the calculation. This implication can also only be done under the assumption ($f_0$).
The next higher degree in both equations (HJBE-1) and (HJBE-2) is investigated in the following.

$$(\text{HJBE-2})^{[1]} : \ 0 = \Big(x^{\mathrm{T}}\pi_{xx} + p^{\mathrm{T}}\pi_{xp}^{\mathrm{T}}\Big) \cdot G + x^{\mathrm{T}}\ell_{xu} + (Kx + Lp)^{\mathrm{T}} \cdot \ell_{uu}$$

Since the states $x$ and the parameters $p$ are independent, this equation can be separated into two, and formulas for $K$ and $L$ are obtained.

$$K = -\ell_{uu}^{-1} \cdot \Big(G^{\mathrm{T}}\pi_{xx} + \ell_{xu}^{\mathrm{T}}\Big) \tag{$K$}$$

$$L = -\ell_{uu}^{-1} \cdot G^{\mathrm{T}}\pi_{xp} \tag{$L$-1}$$

Of course, $K$ as well as $\pi_{xx}$ have to be the same as in the non-parametric case.

$$\begin{aligned}(\text{HJBE-1})^{[2]} : \ 0 = &\Big(x^{\mathrm{T}}\pi_{xx} + p^{\mathrm{T}}\pi_{xp}^{\mathrm{T}}\Big) \cdot \Big(Fx + G \cdot (Kx + Lp)\Big) + \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x \\ &+ x^{\mathrm{T}}\ell_{xu} \cdot (Kx + Lp) + \frac{1}{2}(Kx + Lp)^{\mathrm{T}} \cdot \ell_{uu} \cdot (Kx + Lp)\end{aligned} \tag{3.1}$$

Because of the independence of polynomials that are quadratic in the states, quadratic in the parameters, and polynomials that contain one state and one parameter, this equation is separated into three. The first one contains all terms that only contain states and are homogeneous of degree two. Again the state vector $x$ is omitted.

$$0 = \pi_{xx} \cdot (F + GK) + (F + GK)^{\mathrm{T}} \cdot \pi_{xx} + \ell_{xx} + \ell_{xu} K + K^{\mathrm{T}} \ell_{xu}^{\mathrm{T}} + K^{\mathrm{T}} \ell_{uu} K$$

Replacing $K$ leads to the continuous algebraic Riccati equation, as in the non-parametric case.

$$0 = \pi_{xx} F + F^{\mathrm{T}} \pi_{xx} + \ell_{xx} - \left(\pi_{xx} G + \ell_{xu}\right) \cdot \ell_{uu}^{-1} \cdot \left(G^{\mathrm{T}} \pi_{xx} + \ell_{xu}^{\mathrm{T}}\right) \tag{$\pi_{xx}$}$$

Thus $\pi_{xx}$ is already fixed and only $\pi_{xp}$ has to be found. Taking all mixed terms from (3.1), the first condition towards $\pi_{xp}$ is derived.

$$0 = \pi_{xx} \cdot GL + (F + GK)^{\mathrm{T}} \cdot \pi_{xp} + \ell_{xu} L + K^{\mathrm{T}} \ell_{uu} L$$

Substituting $K$ and $L$ drastically simplifies this equality and leads to

$$0 = F^{\mathrm{T}} \pi_{xp} - (\pi_{xx} G + \ell_{xu}) \cdot \ell_{uu}^{-1} \cdot G^{\mathrm{T}} \pi_{xp}. \tag{3.2}$$

One can already foresee that $\pi_{xp}$ will be 0 if $\begin{pmatrix} F & G \end{pmatrix}$ has rank $n_x$, which is the case due to (II). The third equation, which is obtained from (3.1), when only terms that are quadratic in the parameters $p$ are taken, provides a further condition towards $\pi_{xp}$.

$$0 = \pi_{xp}^{\mathrm{T}} \cdot GL + L^{\mathrm{T}} G^{\mathrm{T}} \cdot \pi_{xp} + L^{\mathrm{T}} \ell_{uu} L \overset{(L\text{-}1)}{=} -\pi_{xp}^{\mathrm{T}} G \cdot \ell_{uu}^{-1} \cdot G^{\mathrm{T}} \pi_{xp}$$

Since $\ell_{uu}^{-1}$ is obviously positive definite, it follows

$$0 = G^{\mathrm{T}} \pi_{xp}. \tag{$\pi_{xp}$-1}$$

$$(3.2) \Rightarrow 0 = F^{\mathrm{T}} \pi_{xp} \tag{$\pi_{xp}$-2}$$

($\pi_{xp}$-1), together with ($\pi_{xp}$-2), leads to $\pi_{xp} = 0$ if $\begin{pmatrix} F & G \end{pmatrix}$ has full rank as expected. Going back to the formula ($L$-1) immediately gives

$$L = 0. \tag{$L$-2}$$

As mentioned before, $\pi_{pp}$ is not a part of any equation and thus can not be found.
As a next step, the second degree of the control law $\kappa$ and the third degree of the value function $\pi$ will be derived. Afterward, the general case $\kappa^{[k]}$, $\pi^{[k+1]}$ for $k \geq 2$ is outlined. At first, the second degree of the first-order condition (HJBE-2) is studied.

$$\begin{aligned}(\text{HJBE-2})^{[2]}:\ 0 = {} & \nabla_x \pi^{[3]}(x,p) \cdot G + x^{\mathrm{T}} \pi_{xx} \cdot \nabla_u f^{[2]}(x, Kx, p) \\ & + \nabla_u \ell^{[3]}(x, Kx) + \kappa^{[2]}(x,p)^{\mathrm{T}} \cdot \ell_{uu}\end{aligned}$$

Since $\ell_{uu}$ is positive definite, it is invertible and an explicit formula for $\kappa^{[2]}(x,p)$ can be given.

$$\kappa^{[2]}(x,p) = -\ell_{uu}^{-1} \cdot \Big( G^{\mathrm{T}} \cdot \nabla_x \pi^{[3]}(x,p)^{\mathrm{T}} + \nabla_u f^{[2]}(x,Kx,p)^{\mathrm{T}} \cdot \pi_{xx} x + \nabla_u \ell^{[3]}(x,Kx)^{\mathrm{T}} \Big) \tag{$\kappa^{[2]}$}$$

To keep the formulas shorter, these equations are not split up according to the states, parameters, and mixed terms anymore. To derive $\kappa^{[2]}(x,p)$, first $\pi^{[3]}(x,p)$ has to be found.

$$(\text{HJBE-1})^{[3]}: \; 0 = \nabla_x \pi^{[3]}(x,p) \cdot (F+GK) \cdot x + x^{\mathrm{T}} \pi_{xx} \cdot \Big( f^{[2]}(x,Kx,p) + G\kappa^{[2]}(x,p) \Big) + \ell^{[3]}(x,Kx) + \Big( x^{\mathrm{T}} \ell_{xu} + x^{\mathrm{T}} K^{\mathrm{T}} \ell_{uu} \Big) \cdot \kappa^{[2]}(x,p)$$

Collecting all terms that depend on $\kappa^{[2]}(x,p)$ and using $(K)$ implies the following simplification.

$$0 = \nabla_x \pi^{[3]}(x,p) \cdot (F+GK) \cdot x + \ell^{[3]}(x,Kx) + x^{\mathrm{T}} \pi_{xx} \cdot f^{[2]}(x,Kx,p) \tag{$\pi^{[3]}$}$$

$(\pi^{[3]})$ is a partial differential equation in terms of the states, but it is linear in terms of the coefficients that define the polynomial $\pi^{[3]}(x,p)$. Those coefficients are the actual unknowns. Using Corollary 8 and the fact that $F+GK$ is stable, one obtains the solvability of the linear equation system. Having $\pi^{[3]}$, the calculation of $\kappa^{[2]}$ is straight forward. Since all the known parts from equation $(\pi^{[3]})$ are constant or linear in the parameters $p$, Corollary 8 also implies $\nabla_p \pi^{[3]}(0,p) = 0$ and $\kappa^{[2]}(0,p) = 0$.
Having this knowledge, the general case $k \geq 2$ is investigated.

$$(\text{HJBE-2})^{[k]}: \; 0 = \sum_{i=1}^{k} \nabla_x \pi^{[i+1]}(x,p) \cdot \Big[ \nabla_u f\big(x, \kappa(x,p), p\big) \Big]^{[k-i]} + \Big[ \nabla_u \ell\big(x, \kappa(x,p)\big) \Big]^{[k]}$$

As before, an explicit formula for $\kappa^{[k]}(x,p)$ can be obtained.

$$\kappa^{[k]}(x,p) = -\ell_{uu}^{-1} \cdot \left( \sum_{i=1}^{k} \Big[ \nabla_u f\big(x, \kappa(x,p), p\big)^{\mathrm{T}} \Big]^{[k-i]} \cdot \nabla_x \pi^{[i+1]}(x,p)^{\mathrm{T}} + \Big[ \nabla_u \ell^{[3;k+1]}\big(x, \kappa(x,p)\big)^{\mathrm{T}} \Big]^{[k]} \right) \tag{$\kappa^{[k]}$}$$

Now taking all terms of (HJBE-1) that are homogeneous with degree $k+1$ and already splitting up the sums, one derives the following equality.

$$
\begin{aligned}
&(\text{HJBE-1})^{[k+1]}: \\
0 &= \sum_{i=1}^{k} \nabla_x \pi^{[i+1]}(x,p) \cdot \Big[f\Big(x, \kappa(x,p), p\Big)\Big]^{[k+1-i]} + \Big[\ell\Big(x, \kappa(x,p)\Big)\Big]^{[k+1]} \\
&= \nabla_x \pi^{[k+1]}(x,p) \cdot (F + GK) \cdot x + \sum_{i=2}^{k-1} \nabla_x \pi^{[i+1]}(x,p) \cdot \Big[f\Big(x, \kappa(x,p), p\Big)\Big]^{[k+1-i]} \\
&\quad + x^{\mathrm{T}} \pi_{xx} \cdot G\kappa^{[k]}(x,p) + x^{\mathrm{T}} \pi_{xx} \cdot \Big[f^{[2;k]}\Big(x, \kappa(x,p), p\Big)\Big]^{[k]} \\
&\quad + \Big(x^{\mathrm{T}} \ell_{xu} + x^{\mathrm{T}} K^{\mathrm{T}}\Big) \cdot \ell_{uu} \cdot \kappa^{[k]}(x,p) + \Big[\ell^{[3;k+1]}\Big(x, \kappa(x,p)\Big)\Big]^{[k+1]}
\end{aligned}
$$

Again $(K)$ is used to cancel out the highest order terms of the control law, which are included.

$$
\begin{aligned}
0 &= \nabla_x \pi^{[k+1]}(x,p) \cdot (F + GK) \cdot x + \sum_{i=2}^{k-1} \nabla_x \pi^{[i+1]}(x,p) \cdot \Big[f\Big(x, \kappa(x,p), p\Big)\Big]^{[k+1-i]} \\
&\quad + x^{\mathrm{T}} \pi_{xx} \cdot \Big[f^{[2;k]}\Big(x, \kappa(x,p), p\Big)\Big]^{[k]} + \Big[\ell^{[3;k+1]}\Big(x, \kappa(x,p)\Big)\Big]^{[k+1]} \qquad (\pi^{[k+1]})
\end{aligned}
$$

Thus $\kappa^{[k]}(x,p)$ is no longer part of this equation. Again Corollary 8 is used to obtain the coefficients of $\pi^{[k+1]}(x,p)$, which leads to $\kappa^{[k]}(x,p)$. The Corollary also implies $\nabla_p \pi^{[k+1]}(0,p) = 0$ and $\kappa^{[k]}(0,p) = 0$. If contrariwise, the parameters $p$ are set to zero, then $\pi(x,0)$ and $\kappa(x,0)$ are identical with the power series $(\pi)$ and $(\kappa)$ from Section 2.1. The equations that had to be solved are the same.
It has been shown that every part of the power series $(\pi)$ and $(\kappa)$ can be calculated under certain conditions, but the existence of those series is not clear at this point.

The detailed calculation that has been outlined in this section is summarized in Theorem 3.

**Theorem 3** (Determinability of $\pi$ and $\kappa$)**.**
*Consider an optimal control problem (OCP), where the function that defines the system dynamics and the cost function are analytic with power series expansions $(f)$ and $(\ell)$. Furthermore, let $(f_0)$ and the conditions (I)-(III) be fulfilled. Then each part of the power series given in $(\pi)$ and $(\kappa)$ is uniquely defined. Furthermore, for all $p$ in a neighborhood of the origin, it holds*

$$\pi(0,p) = 0, \quad \nabla_x \pi(0,p) = 0 \quad \textit{and} \quad \kappa(0,p) = 0.$$

**Remark 2** (Parametric cost functions)**.**
*It is also possible to allow parameters in the cost function $\ell\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \to \mathbb{R}$. To guarantee the degree-wise solvability of the HJBE, one, however, needs to assume*

*that*

$$\forall p \in \mathbb{R}^{n_p}:\ \ell(0,0,p) = 0 \quad \wedge \quad \begin{pmatrix}\nabla_x \ell(0,0,p) & \nabla_u \ell(0,0,p)\end{pmatrix} = \begin{pmatrix}0_{1\times n_x} & 0_{1\times n_u}\end{pmatrix}.$$

*The approach presented in [55] also considers parametric costs, but a slightly different setup. Furthermore, the parameters are part of a coordinate transformation leading to less restrictive conditions.*

$$\forall p \in \mathbb{R}^{n_p}:\ \ell(0,0,p) = 0 \quad \wedge \quad \nabla_p \ell(0,0,p) = 0$$

**Remark 3** (Additive parameters)**.**
*If the parameters do not only appear multiplicative but also additive, then Al'brekht's Method can still be applied by introducing an additional state. To do so an optimal control problem as (OCP) is considered without the condition ($f_0$). The system dynamics are given by*

$$\dot{x} = f(x,u,p) = f_0 + Fx + Gu + Jp + f^{[2]}(x,u,p) + \ldots, \tag{$f_p$}$$

*while the cost function ($\ell$) is unchanged. Here $f_0$ is an $n_x$-dimensional vector and $J \in \mathbb{R}^{n_x \times n_p}$. As previously in this section mentioned, solving this OCP via the power series approach would lead to equations, which depend on all the unknowns, and no iterative procedure can be found. Therefore an additional state $x_f$ is introduced and multiplied with $f(0,0,p)$. Doing so the condition in ($f_0$) is satisfied for the resulting OCP. The new state vector is*

$$\bar{x} = \begin{pmatrix} x \\ x_f \end{pmatrix},$$

*leading to the dynamics*

$$\dot{\bar{x}} = \begin{pmatrix} \dot{x} \\ \dot{x}_f \end{pmatrix} = \begin{pmatrix} f(0,0,p)\cdot x_f + Fx + Gu + f^{[2]}(x,u,p) + \ldots \\ -\alpha_f \cdot x_f \end{pmatrix}.$$

*Here $\alpha_f$ has to be positive to ensure the stability of $x_f$ and, therefore, the solvability of the continuous-time algebraic Riccati equation later on. Substituting the part of the right-hand side, which only depends on $p$ via its power series,*

$$f(0,0,p) = f_0 + Jp + f^{[2]}(0,0,p) + \ldots,$$

*one gains a new representation of the different polynomial degrees of the dynamics.*

$$\dot{\bar{x}} = \underbrace{\begin{pmatrix} F & f_0 \\ 0 & -\alpha_f \end{pmatrix}}_{=:\,\bar{F}} \cdot \begin{pmatrix} x \\ x_f \end{pmatrix} + \underbrace{\begin{pmatrix} G \\ 0 \end{pmatrix}}_{=:\,\bar{G}} \cdot u + \underbrace{\begin{pmatrix} f^{[2]}(x,u,p) - f^{[2]}(0,0,p) + Jp\cdot x_f \\ 0 \end{pmatrix}}_{=:\,\bar{f}^{[2]}(\bar{x},u,p)} + \ldots \tag{$\bar{f}$}$$

*Using the additional state, the cost function and its defining matrices are also rewritten, while the cost stays actually the same.*

$$\bar{\ell}(\bar{x},u) = \bar{x}^{\mathrm{T}} \underbrace{\begin{pmatrix} \ell_{xx} & 0 \\ 0 & 0 \end{pmatrix}}_{=:\, \ell_{\bar{x}\bar{x}}} \bar{x} + \bar{x}^{\mathrm{T}} \underbrace{\begin{pmatrix} \ell_{xu} \\ 0 \end{pmatrix}}_{=:\, \ell_{\bar{x}u}} u + \frac{1}{2} u^{\mathrm{T}} \ell_{uu} u + \ell^{[3]}(x,u) + \ldots = \ell(x,u) \qquad (\bar{\ell})$$

*The value function and the control law are now stated as power series in terms of $(\bar{x}, p)$. During the application of the control, $x_f$ is always set to $1$. In this case, the first $n_x$ rows of the system dynamics ($\bar{f}$) are the same as in ($f_p$).*

$$\bar{\pi}(\bar{x},p) = \frac{1}{2}\bar{x}^T \underbrace{\begin{pmatrix} \pi_{xx} & \pi_{xx_f} \\ \pi_{xx_f}^{\mathrm{T}} & \pi_{x_f x_f} \end{pmatrix}}_{=:\, \pi_{\bar{x}\bar{x}}} \bar{x} + \bar{\pi}^{[3]}(\bar{x},p) + \ldots \qquad (\bar{\pi})$$

$$\bar{\kappa}(\bar{x},p) = \underbrace{\begin{pmatrix} K & K_f \end{pmatrix}}_{=:\, \bar{K}} \cdot \bar{x} + \bar{\kappa}^{[2]}(\bar{x},p) + \ldots \qquad (\bar{\kappa})$$

*Doing so, all assumptions of the nominal case are fulfilled and the Hamilton-Jacobi-Bellman equation and its derivative with respect to the input $u$ are solved degree-wise, as shown at the beginning of this section. The matrices $\pi_{xx} \in \mathbb{R}^{n_x \times n_x}$ and $K \in \mathbb{R}^{n_u \times n_x}$ are found as in the nominal case, see ($\pi_{xx}$) and ($K$). The other parts $\pi_{xx_f} \in \mathbb{R}^{n_x}$ and $\pi_{x_f x_f} \in \mathbb{R}$ of the second order of the value function can be calculated uniquely since a positive $\alpha_f$ implies the stabilizability of $(\bar{F}, \bar{G})$ and the detectability of $(\bar{F}, \ell_{\bar{x}\bar{x}})$. The vector $K_f \in \mathbb{R}^{n_u}$ is given by*

$$K_f = -\ell_{uu}^{-1} \cdot G^{\mathrm{T}} \pi_{xx_f}. \qquad (K_f)$$

*Higher orders are solved accordingly, which will not be outlined here.*
*Including a constant part $f_0$ in the dynamics also generalizes the theory shown in Section 2.1. Allowing additive parameters and not only multiplicative ones leads to a whole new class of applications, e. g. constant drift and offset terms, encountered in robotics or biological and chemical systems.*

## Local stability

It remains to examine the stability of the closed-loop for the parametrized explicit control law, which can be summarised in the following corollary.

**Corollary 3** (Local stability).
*Under the requirements of Theorem 3 local stability is achieved for sufficiently small parameters $p$, if the power series ($\pi$) and ($\kappa$) are converging.*

*Proof.* For a fixed parameter vector $p \in \mathbb{R}^{n_p}$, a local Lyapunov function candidate is given by $\tilde{\pi}(x) := \pi(x,p)$. Analogously to the value function, the control law is defined as $\tilde{u}_{\min}(x) := \tilde{\kappa}(x) := \kappa(x,p)$. Both functions are the solution of

$$\pi\big(x(0)\big) = \min_{u(.)} \int_0^\infty \ell\big(x(\tau), u(\tau)\big)\,\mathrm{d}\tau$$
$$\text{s.t. } \dot{x} = \tilde{f}(x,u),$$

where $\tilde{f}(x,u) := f(x,u,p)$. If $p$ is sufficiently small, $\tilde{f}^{[1]}(.,.)$ inherits the properties of $f^{[1]}(.,.,.)$. Therefore $\tilde{\pi}(.)$ and $\tilde{\kappa}(.)$ can be written as

$$\tilde{\pi}(x) = \frac{1}{2} x^{\mathrm{T}} \tilde{\pi}_{xx} x + o(||x||^3),$$
$$\tilde{\kappa}(x) = \tilde{K}x + o(||x||^2).$$

Thus there exists an $\varepsilon > 0$ such that

$$\tilde{\pi}(x) \geq 0$$

for all $x \in B_\varepsilon(0)$. Equality only holds for vanishing $x$ due to the positive definiteness of $\tilde{\pi}_{xx}$. Therefore, $\tilde{\pi}(.)$ is locally positive definite. Using the Hamilton-Jacobi-Bellman equation (HJBE-1) from Section 2.1, it is seen that $\dot{\tilde{\pi}}\big(x(t)\big)$ is locally negative definite.

$$\begin{aligned}\dot{\tilde{\pi}}\big(x(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\pi}\big(x(t)\big) = \nabla_x \tilde{\pi}\big(x(t)\big) \cdot \tilde{f}\big(x(t), \tilde{\kappa}(x(t))\big) \overset{\text{(HJBE-1)}}{=} -\ell\big(x(t), \tilde{\kappa}(x(t))\big) \\ &= -\frac{1}{2}x(t)^{\mathrm{T}}\ell_{xx}x(t) - x(t)^{\mathrm{T}}\ell_{xu}\tilde{K}x(t) - \frac{1}{2}x(t)^{\mathrm{T}}\tilde{K}^{\mathrm{T}}\ell_{uu}\tilde{K}x(t) + o\big(||x(t)||^3\big)\end{aligned}$$

Since $\ell^{[2]}(x, \tilde{K}x)$ is positive definite, there exists an $\varepsilon > 0$ such that

$$\dot{\tilde{\pi}}\big(x(t)\big) \leq 0$$

for all $x(t) \in B_\varepsilon(0)$. Again equality holds only for vanishing $x(t)$. Thus $\tilde{\pi}(x) = \pi(x,p)$ is a local Lyapunov function for

$$\dot{x} = \tilde{f}\big(x, \tilde{\kappa}(x)\big) = f\big(x, \kappa(x,p), p\big)$$

and fixed $p \in \mathbb{R}^{n_p}$. If $p \in \mathbb{R}^{n_p}$ varies but stays in $\overline{B_\delta(0)}$ for a sufficiently small $\delta > 0$, then $x \mapsto \pi(x,p)$ is used as a local Lyapunov function. In the previous case, $x \mapsto \pi(x,p)$ was a local Lyapunov function for $x \in B_\varepsilon(0)$. Clearly, $\varepsilon$ depends on $p$. Since $p \mapsto \pi(x,p)$ is $C^\infty$, one can see that $p \mapsto \varepsilon(p)$ is continuous and thus takes its minimum value in the compact set $\overline{B_\delta(0)}$. This minimum is denoted with $\varepsilon_{\min}$ and must be greater than zero. Else there would be a parameter $p$ that contradicts the first case. Therefore $x \mapsto \pi(x,p)$ is a Lyapunov function for $x \in B_{\varepsilon_{\min}}(0)$. □

**Remark 4.** *The local stability result is asymptotic in the sense that is holds for the exact control law. It does not consider the case that the power series of the control law is truncated at a certain degree.*

## Simulation evaluation

The two quadcopter examples from Section 2.1 are revisited. As explained in Remark 3, additive parameters and constants in the power series of the system dynamics can now also be handled. Thus another example from the field of biochemistry is also investigated.

**Example 3** (Quadcopter: 10 states, 6 parameters)**.**
*The system dynamics considered in Example 1 are extended with variable parameters* $p = \begin{pmatrix}\omega_{xx} & \omega_{xy} & \omega_{xz} & \omega_{yx} & \omega_{yy} & \omega_{yz}\end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^6$ *in the following way.*

$$\begin{aligned}
\dot{p}_x &= v_x & \dot{v}_x &= g \cdot \tan\left(\frac{\phi}{\mathrm{rad}}\right) + \omega_{xx} \cdot v_x + \omega_{xy} \cdot v_y + \omega_{xz} \cdot v_z \\
\dot{p}_y &= v_y & \dot{v}_y &= g \cdot \tan\left(\frac{\theta}{\mathrm{rad}}\right) + \omega_{yx} \cdot v_x + \omega_{yy} \cdot v_y + \omega_{yz} \cdot v_z \\
\dot{p}_z &= v_z & \dot{v}_z &= -g + k_t \cdot u_z \\
\dot{\phi} &= -d_1 \cdot \phi + v_\phi & \dot{v}_\phi &= -d_0 \cdot \phi + n_0 \cdot u_\phi \\
\dot{\theta} &= -d_1 \cdot \theta + v_\theta & \dot{v}_\theta &= -d_0 \cdot \theta + n_0 \cdot u_\theta,
\end{aligned}$$

*The added terms can be seen as state-depended uncertainties, as usually considered. Those are usually used for aircraft and quadcopter models, see [45, 75, 96]. The parameters allow, for example, to describe wind disturbing the system in x and y-direction. The effect on the altitude is usually smaller and neglected here. If the parameters p can be estimated via perhaps additional sensors or the movement of the quadcopter, then the control scheme from Example 1 can be updated with another block, see Fig. 3.1. The estimation is not part of this work and will be treated as a black box. Using the same notation, cost function, and transformation of the input as before, Al'brekht's Method with included parameters is applicable. The power series of the control law was calculated up to degree three. Thus finding* 180 *non-zero out of a total of* 2904 *coefficients, while the value function is approximated up to degree four with a total of* 4828 *coefficients from which* 345 *are non-zero, see Table 7.1. The initial values were again* $p_x = 25\,\mathrm{m}$, $p_z = 5\,\mathrm{m}$, *and* $\theta = 0.1745\,\mathrm{rad}$ *(*$\approx 10°$*), while the time-dependent parameters are chosen as*

$$\omega_{xx} = \omega_{yx} = \frac{1}{5} \cdot \sin(t)\,\mathrm{s}^{-1}, \; \omega_{xy} = \omega_{yy} = \frac{1}{5} \cdot \cos(t)\,\mathrm{s}^{-1} \text{ and } \omega_{xz} = \omega_{yz} = 0\,\mathrm{s}^{-1}.$$

*As seen in Fig. 3.3, higher-order approximations lead to a better compensation of the wind effects. It can also be seen that the controllers (*$\kappa^{[1]}(x)$, $\kappa^{[1;3]}(x)$*) obtained via*

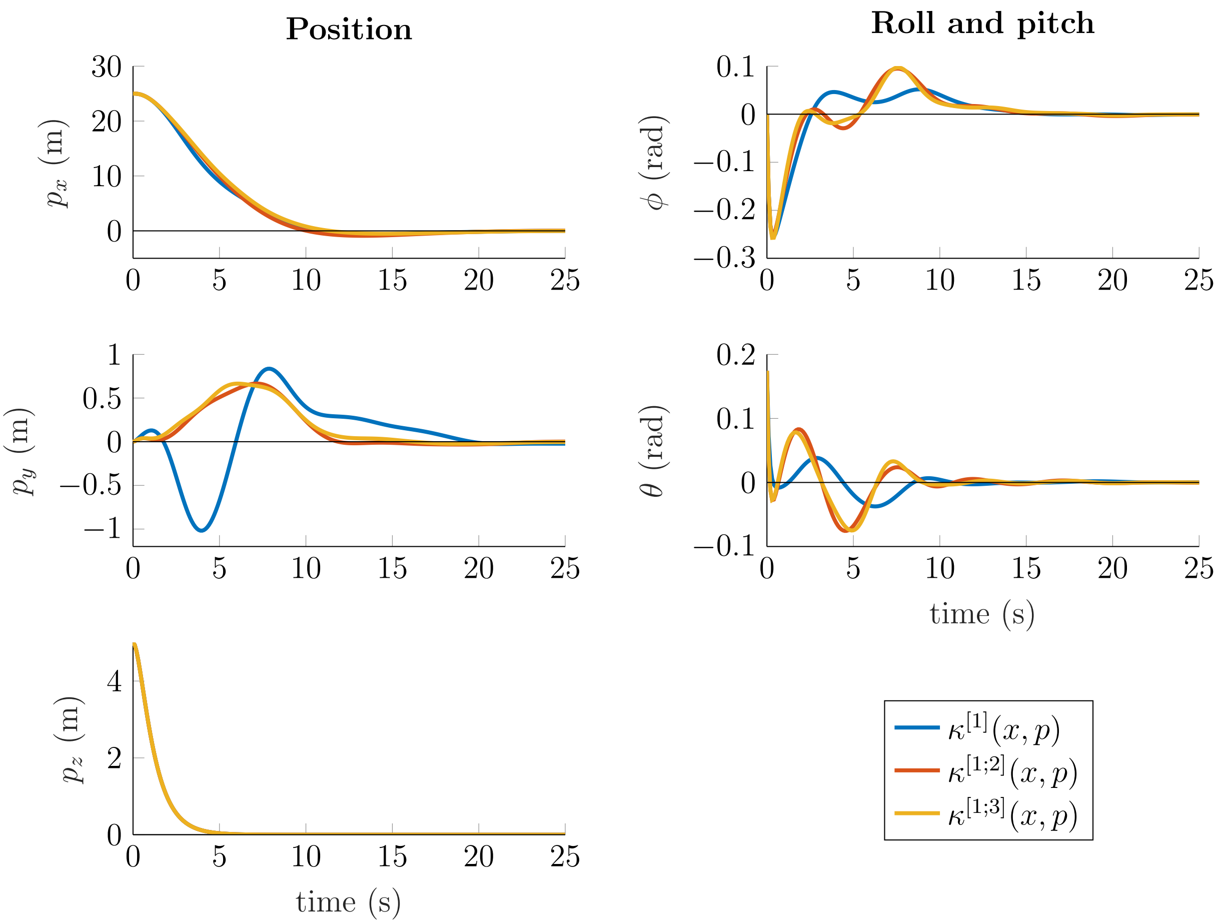


**Figure 3.2:** Propagation of five out of ten states (10 states model with parameters)

*Al'brekht's original method do not compensate the changing wind at all. The first-order approximation $\kappa^{[1]}(x,p)$ actually does not depend on the parameters and is equal to $\kappa^{[1]}(x)$. All controllers lead to the same behavior in the altitude, see Fig. 3.2. This is due to the linearity of the involved ODE's. Thus the control input (Fig. 3.4) for the altitude is the same for all approximations. The other control inputs are less smooth and show a very strong control action for a short time. The magnitude of the input depends on the initial values. Such strong control actions are not desired in reality. Furthermore, the alleviation of the wind effect causes a much higher cost (Fig. 3.5) due to the large input values for the angular accelerations.*

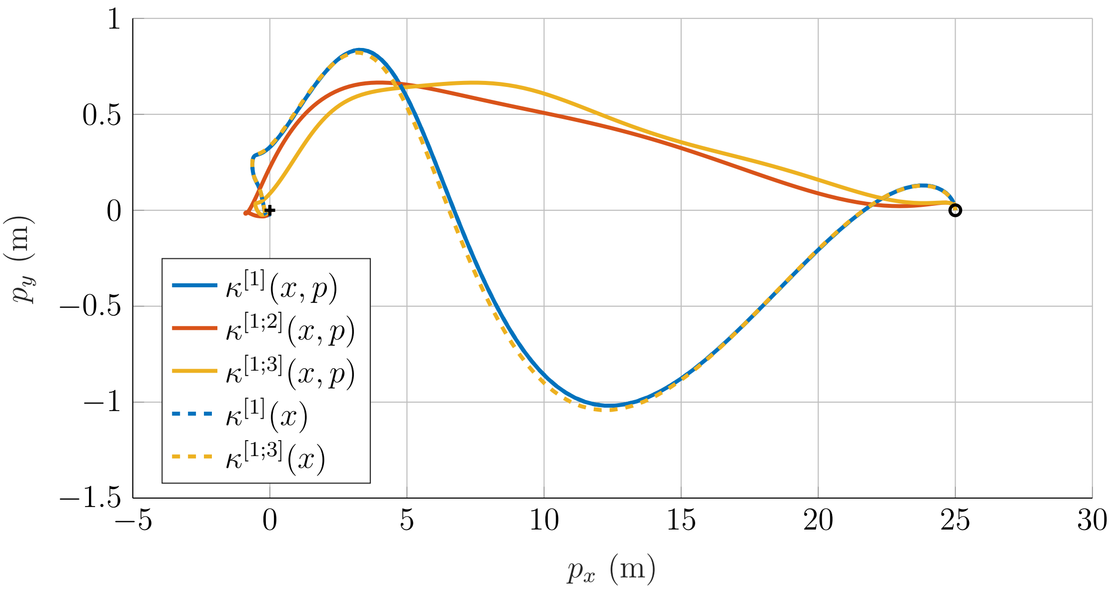


**Figure 3.3:** Position of the quadcopter in the $x$-$y$-plane (10 states model with parameters)

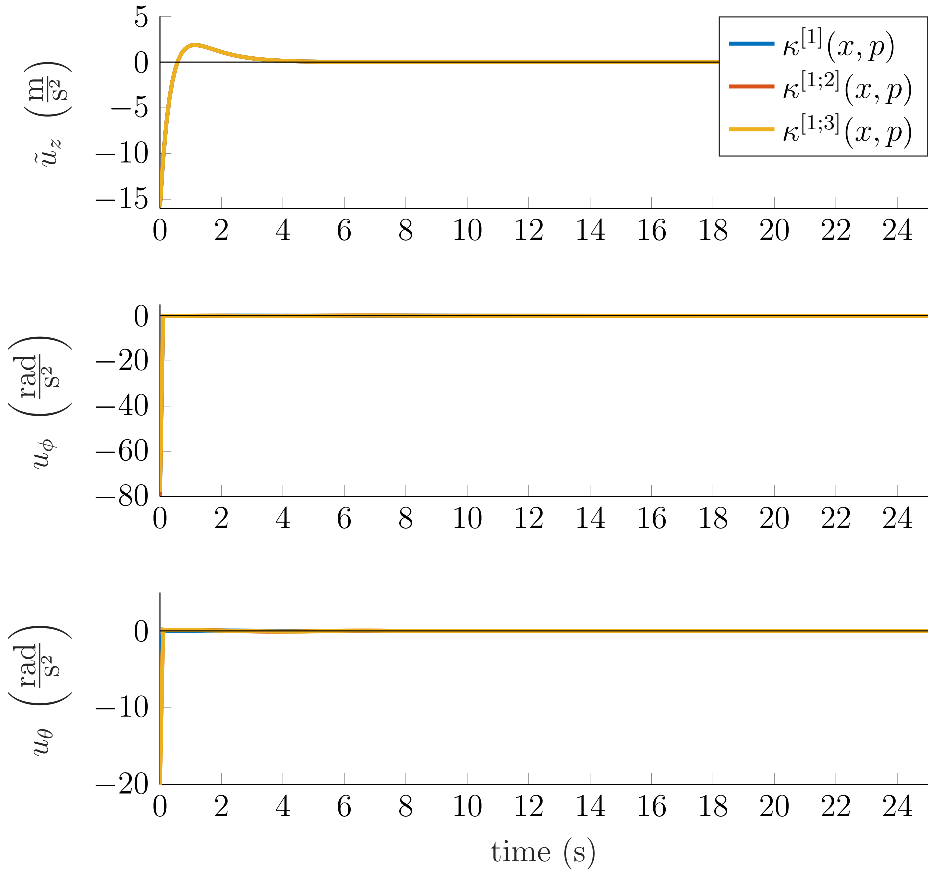


**Figure 3.4:** Control input for the vertical and angular accelerations (10 states model with parameters)

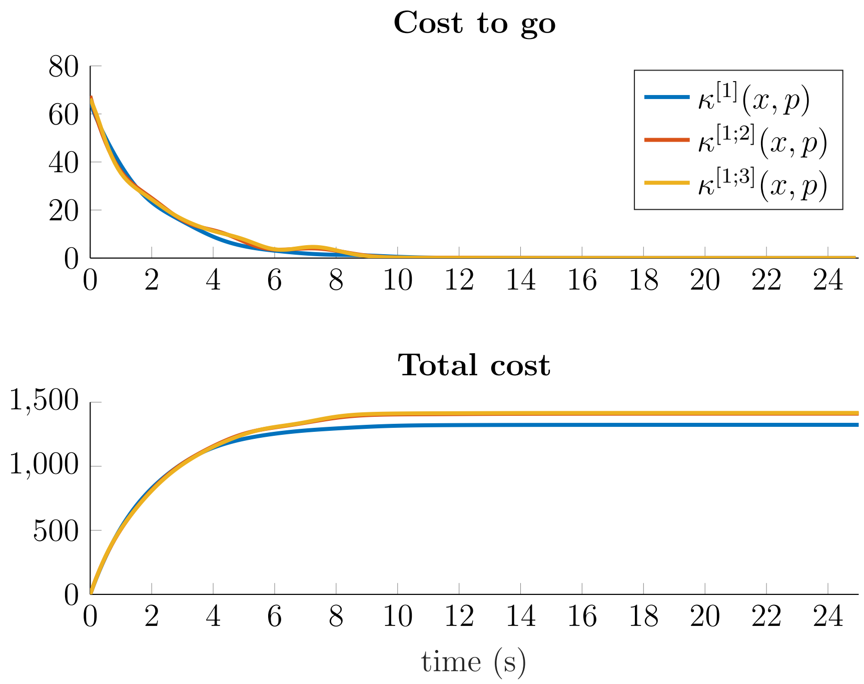


**Figure 3.5:** Resulting cost (10 states model with parameters)

**Example 4** (Quadcopter: 12 states, 6 parameters)**.**
*The ODE's from Example 2 are also extended to contain parameters. This is done in the same way as in the previous example. The velocities in* $x$ *and* $y$*-direction are given by*

$$\dot{v}_x = \frac{v_\psi \cdot v_y - v_\theta \cdot v_z}{\mathrm{rad}} - g \cdot \sin(\theta) + \omega_{xx} \cdot v_x + \omega_{xy} \cdot v_y + \omega_{xz} \cdot v_z$$
$$\dot{v}_y = \frac{v_\phi \cdot v_z - v_\psi \cdot v_x}{\mathrm{rad}} + g \cdot \sin(\phi)\cos(\theta) + \omega_{yx} \cdot v_x + \omega_{yy} \cdot v_y + \omega_{yz} \cdot v_z.$$

*The parameters representing the wind are set to*

$$\omega_{xx} = \omega_{yx} = \frac{1}{5} \cdot \sin(t)\,\mathrm{s}^{-1},\ \omega_{xy} = \omega_{yy} = \frac{1}{5} \cdot \cos(t)\,\mathrm{s}^{-1} \text{ and } \omega_{xz} = \omega_{yz} = 0\,\mathrm{s}^{-1}.$$

*Notice that this corresponds to rather strong wind. In [39] parameters with values in* $[0\,\mathrm{s}^{-1}, 0.01\,\mathrm{s}^{-1}]$ *are used. Here bigger values can be used due to the small velocities. The initial values are taken from Example 2. The control law has been approximated up to degree 3, and thus the value function up to degree 4. In total,* 12 612 *coefficients had to be found, see Table 7.1.* 1915 *of those are non-zero respectively relevant for the control. The first-order approximation is worse at handling or rejecting the wind disturbance (Fig. 3.6 and 3.7). Looking at the position and the angles of the quadcopter (Fig. 3.6), it can be seen that higher-order approximations achieve the control goal faster. Furthermore, they do not show an overshoot in the altitude* $p_z$ *like* $\kappa^{[1]}(x,p)$*. Comparing the performance of the controllers obtained via the extended method with the ones from Example AM-conti-example-12-states, it is obvious that the parametric*

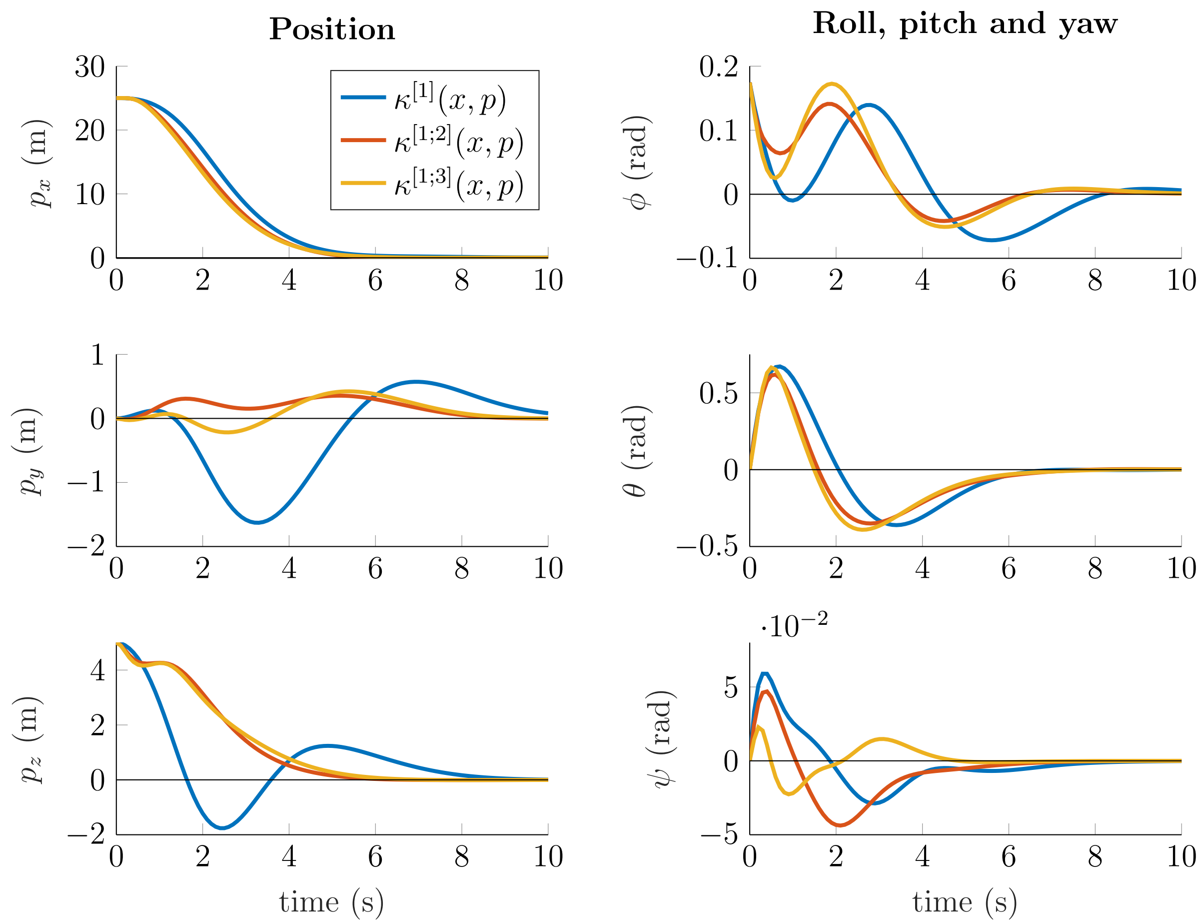


**Figure 3.6:** Propagation of six out of twelve states (12 states model with parameters)

*control laws can handle the disturbance more efficiently. The drift in y-direction is successfully reduced. The cost to reach the origin (Fig. 3.9) is reduced by 5 %, which is only half compared to the non-parametric case. The control input (Fig. 3.8) shows very strong actions at the beginning for higher-order approximations as in Example 3. Again this is mainly caused by the initial values and an adaption to the wind.*

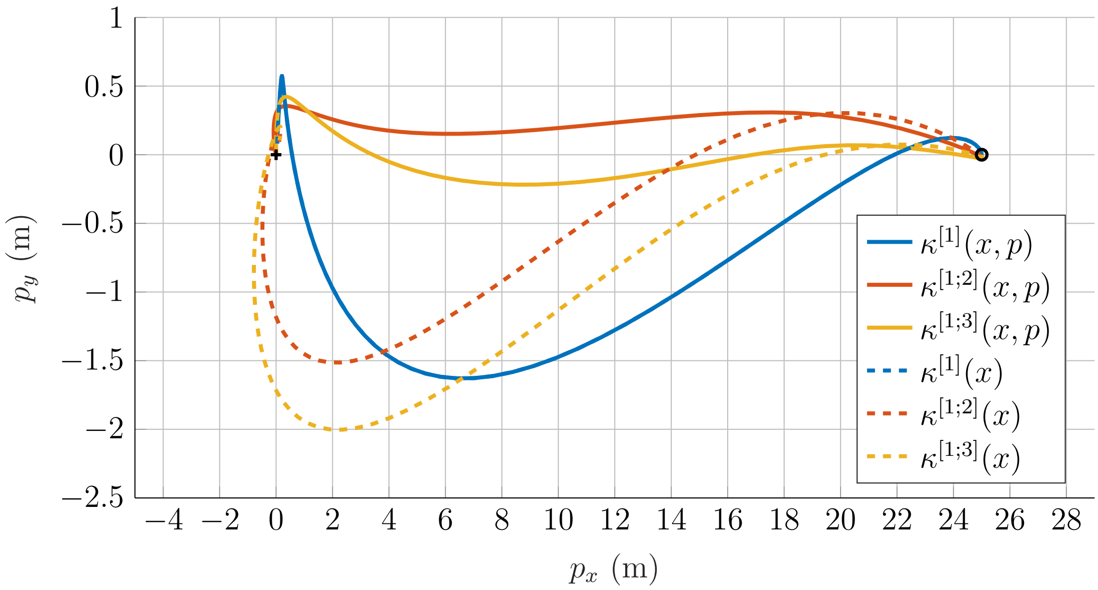


**Figure 3.7:** Position of the quadcopter in the $x$-$y$-plane (12 states model with parameters)

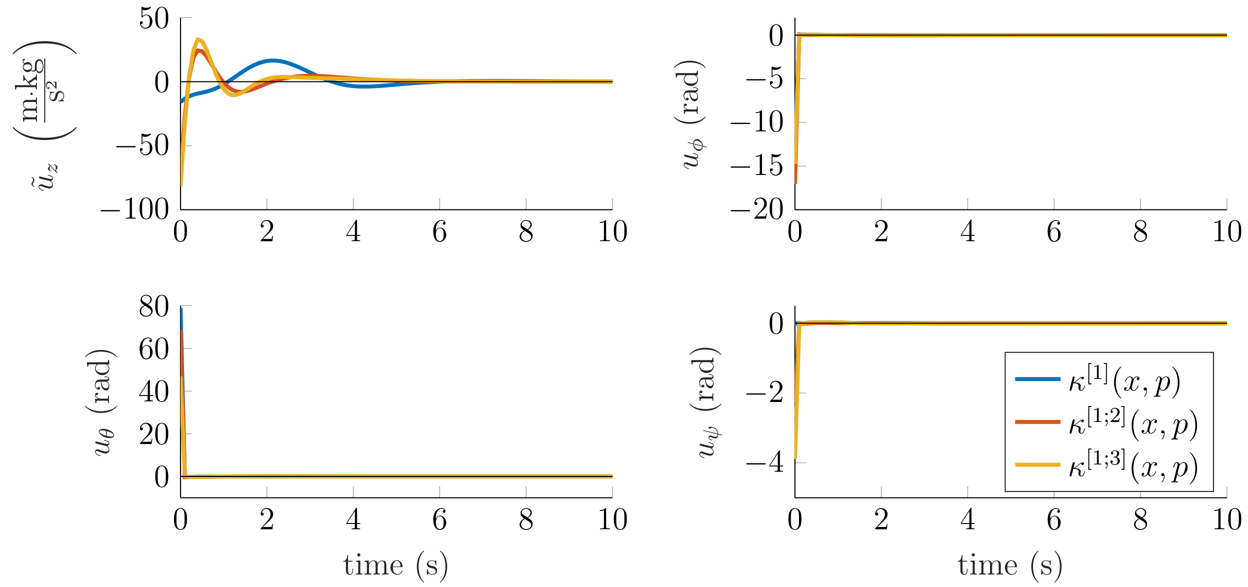


**Figure 3.8:** Control input for the thrust and the moments about the axis (12 states model with parameters)

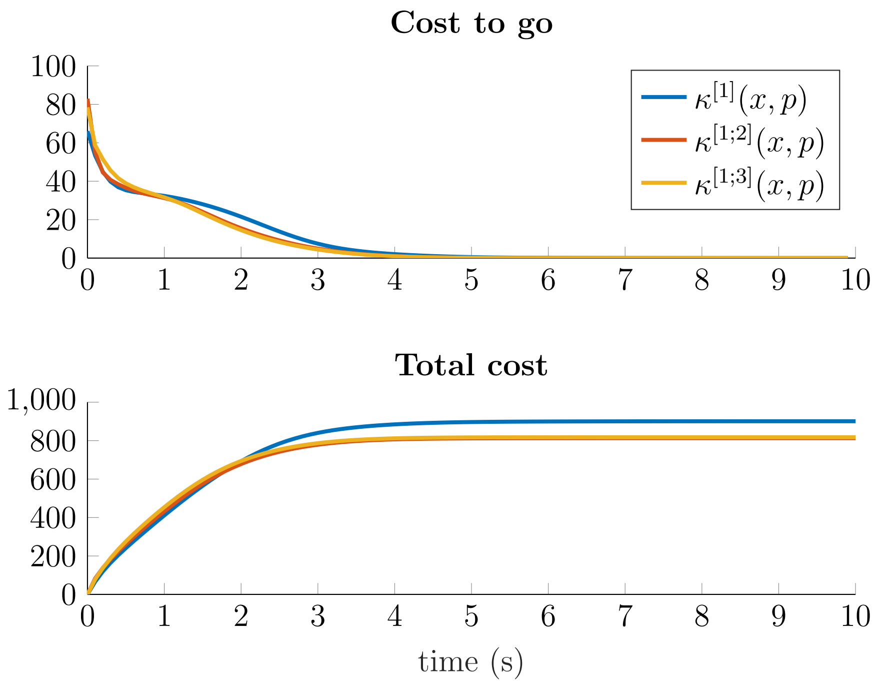


**Figure 3.9:** Resulting cost (12 states model with parameters)

**Example 5** (Bioreactor rhodospirillum rubrum, 3 states, 3 parameters)**.**
*Aiming to evaluate Al'brekht's Method in case of additive parameters an example from the field of biochemistry is investigated. The model*

$$\begin{aligned}
\dot{x}_b &= \left(v_b \cdot \frac{\Delta CRP_*^5}{\Delta CRP_*^5 + k^5} + v_{b,min}\right) \cdot \left(\frac{x_s}{x_s + k_s} + \frac{x_f}{x_f + k_f}\right) \cdot x_b - D \cdot x_b \\
\dot{x}_s &= -\frac{1}{Y_{bs}} \cdot \frac{x_s}{x_s + k_s} \cdot x_b + D \cdot (F_s - x_s) \\
\dot{x}_f &= -\frac{1}{Y_{bf}} \cdot \frac{x_f}{x_f + k_f} \cdot x_b + D \cdot (F_f - x_f)
\end{aligned}$$

*corresponds to a bioreactor producing the bacteria rhodospirillum rubrum, see e. g. [14, 15]. The $\Delta CRP_*$ is calculated via*

$$\frac{\Delta CRP + 330\,\mathrm{mV}}{330\,\mathrm{mV}}.$$

*The state vector $x = \begin{pmatrix} x_b & x_s & x_f \end{pmatrix}^{\mathrm{T}}$ consists of the concentrations of the biomass, succinate, and fructose. The input $u = D$ stands for the dilution rate, see Fig. 3.10. There are several fixed parameters. Their meaning and values are collected in Table 3.1. The parameters $p = \begin{pmatrix} Y_{bs} & Y_{bf} & k \end{pmatrix}^{\mathrm{T}}$ are considered to be variable and their ranges are shown in Table 3.2. It is desired to stabilize the system at the concentration $x_{stat} = \begin{pmatrix} 0.8\,\mathrm{g\,L^{-1}} & 4\,\mathrm{g\,L^{-1}} & 2\,\mathrm{g\,L^{-1}} \end{pmatrix}^{\mathrm{T}}$ with a dilution rate of $D_{stat} = 1.486\,\mathrm{h^{-1}}$. Note that this is not a steady state ensuring that $f_0 = f(0,0,0)$ does not vanish. To*

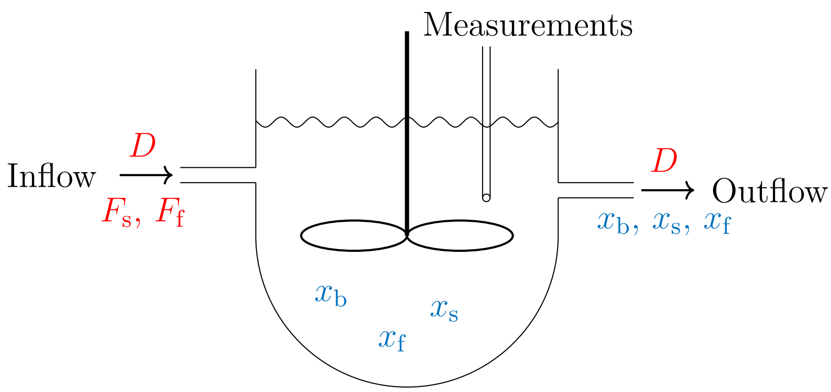


**Figure 3.10:** Scheme of a bioreactor producing rhodospirillum rubrum

| | | |
|---|---|---|
| Affinity constant for succinate consumption | $k_s$ | $0.0011\,\mathrm{g\,L^{-1}}$ |
| Affinity constant for fructose consumption | $k_f$ | $0.0011\,\mathrm{g\,L^{-1}}$ |
| Succinate concentration of the feed solution | $F_s$ | $4.85\,\mathrm{g\,L^{-1}}$ |
| Fructose concentration of the feed solution | $F_f$ | $3.04\,\mathrm{g\,L^{-1}}$ |
| CRP affinity constant of the Hill kinetic | $k$ | 0.705 |
| Coefficient for biomass growth rate | $v_b$ | $0.5\,\mathrm{h^{-1}}$ |
| Smallest growth rate | $v_{b,\min}$ | $0.5\,\mathrm{h^{-1}}$ |

**Table 3.1:** Fixed model parameters

*apply Al'brekht's Method a shift in the coordinates has to be done such that the system should be steered to the origin in the new coordinates.*

$$x_T = x - x_{stat}, \quad u_T = D_T = D - D_{stat}, \quad p_T = p - p_{ave}$$

*Here $p_{ave}$ is chosen as $\begin{pmatrix} 0.5\,\mathrm{h^{-1}} & 0.5\,\mathrm{h^{-1}} & -165\,\mathrm{mV}\end{pmatrix}^{\mathrm{T}}$, which are the mean values of the given intervals. The new system now states as*

$$\begin{aligned}
\dot{x}_{T,b} = {} & \left( v_b \cdot \frac{\Delta CRP_*^5}{\Delta CRP_*^5 + k^5} + v_{b,min} \right) \\
& \cdot \left( \frac{x_{T,s} + 4\,\mathrm{g\,L^{-1}}}{x_{T,s} + 4\,\mathrm{g\,L^{-1}} + k_s} + \frac{x_{T,f} + 2\,\mathrm{g\,L^{-1}}}{x_{T,f} + 2\,\mathrm{g\,L^{-1}} + k_f} \right) \cdot x_{T,b} + 0.8\,\mathrm{g\,L^{-1}} \\
& - (D_T + 1.486\,\mathrm{h^{-1}}) \cdot (x_{T,b} + 0.8\,\mathrm{g\,L^{-1}}) \\
\dot{x}_{T,s} = {} & -\frac{1}{Y_{T,bs} + 0.5\,\mathrm{h^{-1}}} \cdot \frac{x_{T,s} + 4\,\mathrm{g\,L^{-1}}}{x_{T,s} + 4\,\mathrm{g\,L^{-1}} + k_s} \cdot (x_{T,b} + 0.8\,\mathrm{g\,L^{-1}}) \\
& + (D_T + 1.486\,\mathrm{h^{-1}}) \cdot (F_s - x_{T,s} - 4\,\mathrm{g\,L^{-1}}) \\
\dot{x}_{T,f} = {} & -\frac{1}{Y_{T,bf} + 0.5\,\mathrm{h^{-1}}} \cdot \frac{x_{T,f} + 2\,\mathrm{g\,L^{-1}}}{x_{T,f} + 2\,\mathrm{g\,L^{-1}} + k_f} \cdot (x_{T,b} + 0.8\,\mathrm{g\,L^{-1}}) \\
& + (D_T + 1.486\,\mathrm{h^{-1}}) \cdot (F_f - x_{T,f} - 2\,\mathrm{g\,L^{-1}}),
\end{aligned}$$

| | | |
|---|---|---|
| Biomass-substrate yield coefficient of succinate | $Y_{\mathrm{bs}}$ | $[0\,\mathrm{h}^{-1}, 1\,\mathrm{h}^{-1}]$ |
| Biomass-substrate yield coefficient of fructose | $Y_{\mathrm{fs}}$ | $[0\,\mathrm{h}^{-1}, 1\,\mathrm{h}^{-1}]$ |
| Step size of culture redox potential (CRP) reduction | $\Delta\mathrm{CRP}$ | $[-330\,\mathrm{mV}, 0\,\mathrm{mV}]$ |

**Table 3.2:** Variable model parameters

*where $f_0$ calculates to $\begin{pmatrix}-0.267\,\mathrm{g\,L^{-1}\,h^{-1}} & -0.337\,\mathrm{g\,L^{-1}\,h^{-1}} & -0.0547\,\mathrm{g\,L^{-1}\,h^{-1}}\end{pmatrix}^{\mathrm{T}}$. The used cost function is again quadratic.*

$$\ell(x_T, u_T) = \frac{1}{2}||x_T||_2^2 + \frac{1}{200}u_T^2 = \frac{1}{2}||x - x_{stat}||_2^2 + \frac{1}{200}\cdot(D - D_{stat})^2$$

*The initial values are to $x_{T,b} = -0.1\,\mathrm{g\,L^{-1}}$, $x_{T,s} = -0.5\,\mathrm{g\,L^{-1}}$, and $x_{T,f} = -0.1\,\mathrm{g\,L^{-1}}$, while the time-dependent parameters are given via*

$$p_T = \begin{pmatrix}\frac{1}{25}\sin(t) & \frac{1}{25}\sin(t) & 5t\end{pmatrix}^{\mathrm{T}}.$$

*Since $f_0$ does not vanish, an additional state, which is always set to 1, is introduced. As described in Remark 3, an $\alpha_f$ needs to be chosen. In the first simulation, four choices ($\alpha_f \in \{0.01, 0.1, 1, 10\}$) are compared. Figures 3.11 and 3.12 show the results when the second-order approximation of the control law is used. The general behavior, when different $\alpha_f$ are used, does not change for higher-order approximations. Larger values for $\alpha_f$ lead to worse performance (Fig. 3.12). If $\alpha_f$ is set even bigger than 10, instability can occur. The three smaller values ($\alpha_f \in \{0.01, 0.1, 1\}$) are faster and more efficient at stabilizing the system. As pointed out before, the origin can not be reached, see Fig. 3.11. Choosing even smaller values may cause numerical problems during the calculation of the power series. The control law is nearly the same in all four cases.*
*The control law has been approximated up to degree 6 in a second simulation. The second-order control performs best (Fig. 3.14). The changing parameters are again hard to handle for the linear control law, see e. g. $x_{T,f}$ in Fig. 3.13. All others are able to stabilize the three states. The control input differs only slightly. Overall, it is shown that Al'brekht's Method can also be applied in the field of biochemistry.*

## 3.2 Convergence proof for parametric continuous-time systems

Next, the convergence proof of Theorem 2 is generalized, to include fixed but arbitrary model parameters. The basic steps of the proof remain the same and again only a special class of optimal control problems is considered. The value function is, similar to before, upper-bounded using the system dynamics and the cost function. The additional difficulty is that $\pi$ does not only increase with the norm of the states but

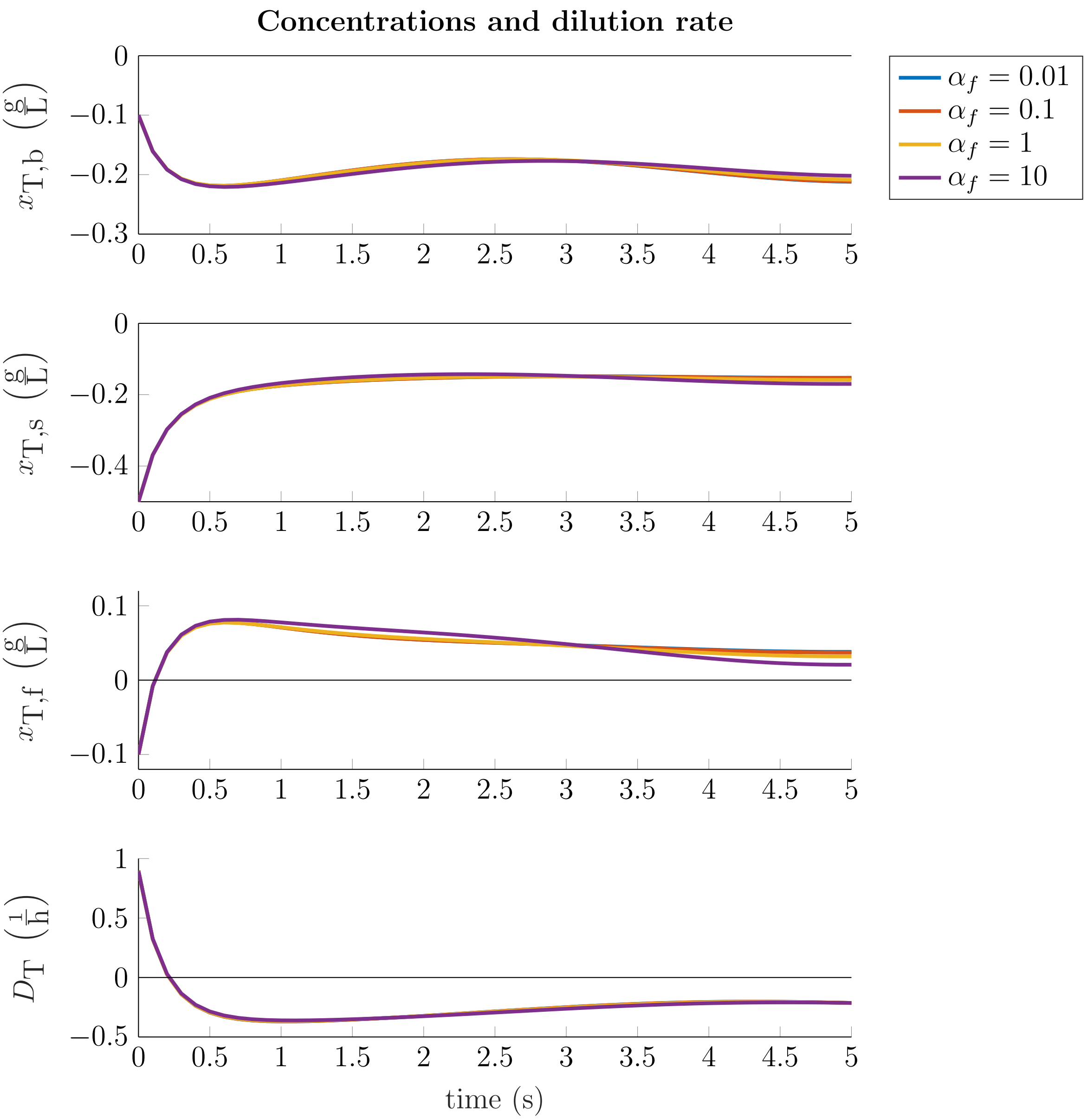


**Figure 3.11:** Propagation of the concentrations and dilution rate for different choices of $\alpha_f$ (bioreactor with parameters)

also the parameters, which are arbitrary and do not converge to zero. Therefore one has to work with $||x||$ whenever possible and with $||(x,p)||$ otherwise. Only using the later one does not lead to useful upper bounds. Other changes in the proof are just notational.

The considered optimal control problem states as the following:

$$\pi\big(x(0),p\big) = \min_{u(.)} \int_0^\infty \ell\big(x(\tau),u(\tau)\big)\,\mathrm{d}\tau \qquad \text{s.t. } \dot{x} = f(x,p) + Gu. \tag{OCP}$$

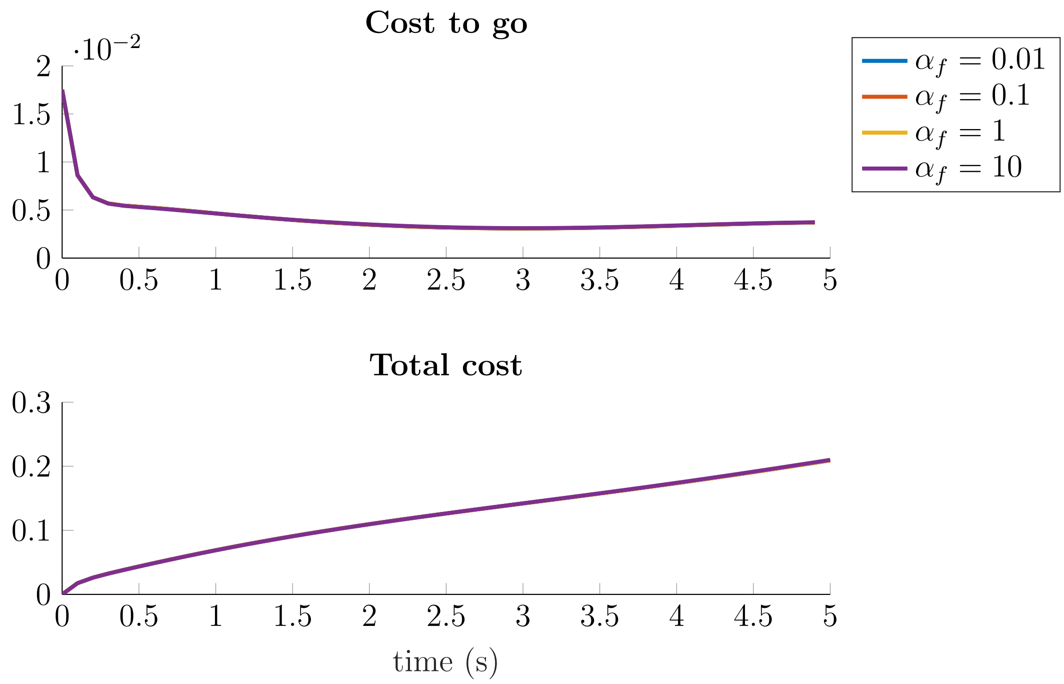


**Figure 3.12:** Resulting cost for different choices of $\alpha_f$ (bioreactor with parameters)

In this case, the system dynamics is linear in the input variables $u \in \mathbb{R}^{n_u}$, and the cost function

$$\ell(x,u) = \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + x^{\mathrm{T}}\ell_{xu}u + \frac{1}{2}u^{\mathrm{T}}\ell_{uu}u \tag{$\ell$}$$

is quadratic, thus does not contain terms of higher order. The right-hand side function $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ is considered to be analytic, and it holds $f(0,p) = 0$ for all parameters $p$. The power series expansion of the system dynamics becomes

$$f(x,p) = Fx + \sum_{k=2}^{\infty} f^{[k]}(x,p). \tag{$f$}$$

**Theorem 4** (Local convergence of $\pi(x,p)$ and $\kappa(x,p)$)**.**
*Let $\kappa$ be the optimal control input of (OCP) and $\pi$ the corresponding optimal cost function. If $F$, $G$, and $\ell$ respect the conditions (I)-(III) from Section 3.1, then $\kappa$ and $\pi$ can be expressed in terms of the system states $x$ as well as system parameters $p$ and are locally analytic.*

*Proof.* The strategy of the proof follows along the lines of Theorem 2. However, the presence of the parameters creates some additional difficulties in the determination of an upper bound of the power series.

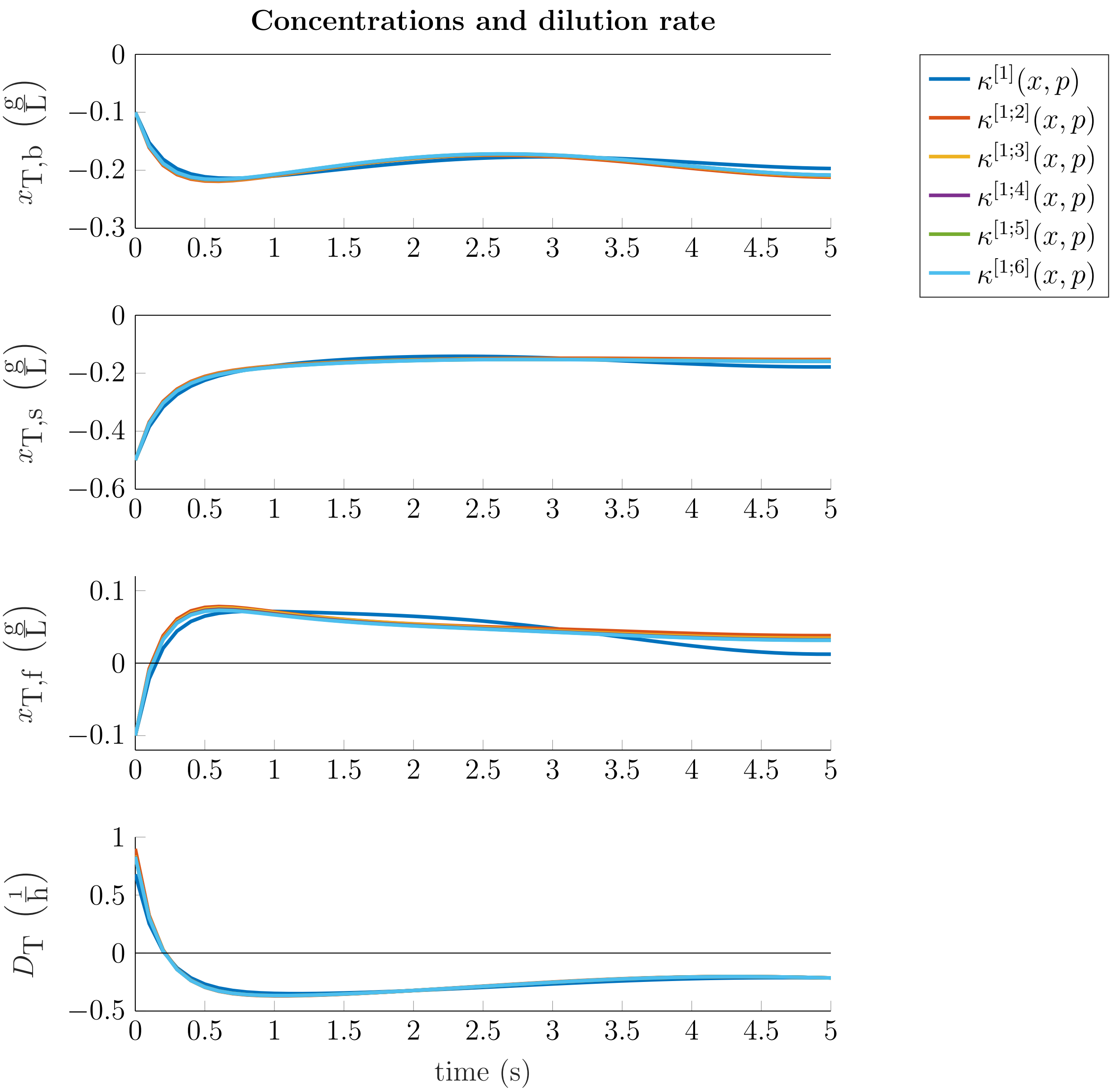


**Figure 3.13:** Propagation of the concentrations and dilution rate (bioreactor with parameters)

The optimal input and the value function are denoted by

$$\kappa(x,p) = Kx + \sum_{k=2}^{\infty} \kappa^{[k]}(x,p) \tag{$\kappa$}$$

$$\text{and } \pi(x,p) = \frac{1}{2}x^{\mathrm{T}}\pi_{xx}x + \sum_{k=3}^{\infty} \pi^{[k]}(x,p). \tag{$\pi$}$$

The constants $\kappa_0$ and $\pi_0$ as well as the 1-by-$n_x$ matrix $\pi_x$ are neglected since they are zero anyway. The calculation done in Section 3.1 resp. Theorem 3 also shows that $p \mapsto \kappa(0,p)$, $p \mapsto \pi(0,p)$ as well as $p \mapsto \nabla_x\pi(0,p)$ are zero everywhere. The

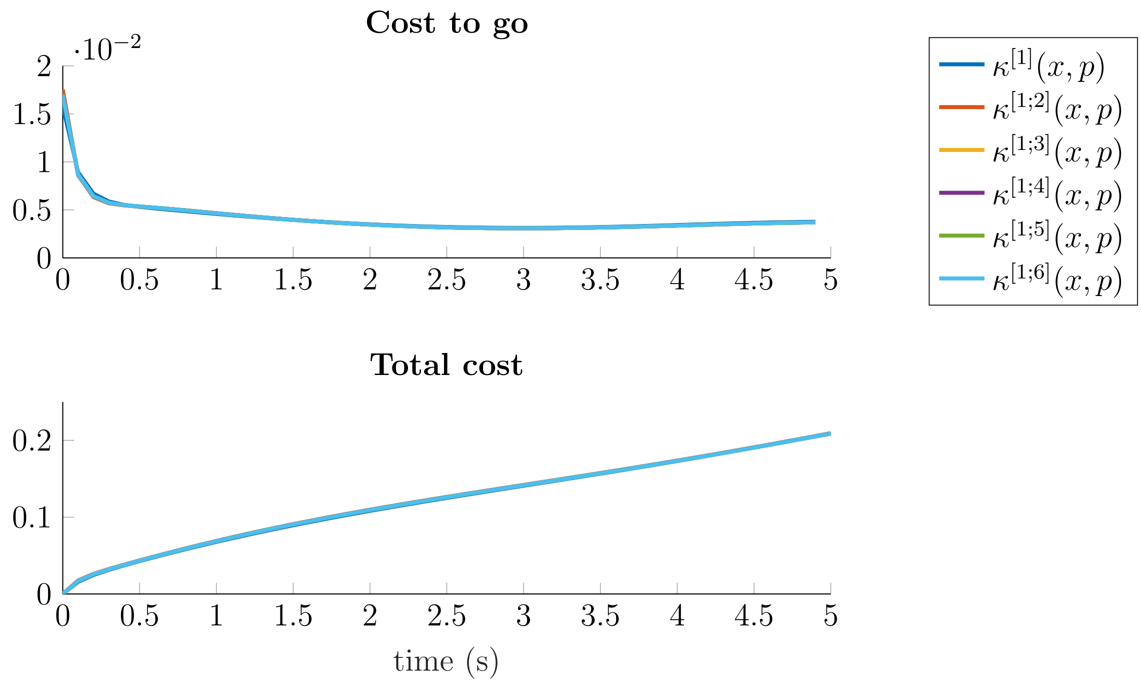


**Figure 3.14:** Resulting cost (bioreactor with parameters)

Hamilton-Jacobi-Bellman equation of the system is given by

$$\begin{aligned}0 &= \nabla_x \pi(x,p) \cdot \big(f(x,p) + G\kappa(x,p)\big) \\ &\quad + \frac{1}{2}x^{\mathrm{T}}\ell_{xx}x + x^{\mathrm{T}}\ell_{xu} \cdot \kappa(x,p) + \frac{1}{2}\kappa(x,p)^{\mathrm{T}} \cdot \ell_{uu} \cdot \kappa(x,p)\end{aligned} \tag{HJBE-1}$$

and the first-order optimality condition by

$$0 = \nabla_x \pi(x,p) \cdot G + x^{\mathrm{T}}\ell_{xu} + \kappa^{\mathrm{T}}(x,p) \cdot \ell_{uu}. \tag{HJBE-2}$$

First the defining equations for $\kappa^{[k]}(x,p)$ and $\pi^{[k+1]}(x,p)$ $(k \geq 1)$ are simplified. For $K$ and $\pi_{xx}$ one can copy $(K)$ and $(\pi_{xx})$ from Section 3.1.

$$K = -\ell_{uu}^{-1} \cdot (\ell_{xu}^{\mathrm{T}} + G^{\mathrm{T}}\pi_{xx}) \tag{$K$}$$

$$0 = \pi_{xx}F + F^{\mathrm{T}}\pi_{xx} + \ell_{xx} - (\ell_{xu} + \pi_{xx}G) \cdot \ell_{uu}^{-1} \cdot (\ell_{xu}^{\mathrm{T}} + G^{\mathrm{T}}\pi_{xx}) \tag{$\pi_{xx}$}$$

From (HJBE-2),

$$\kappa^{[k]}(x,p) = -\ell_{uu}^{-1} \cdot G^{\mathrm{T}} \cdot \nabla_x \pi^{[k+1]}(x,p)^{\mathrm{T}} \tag{$\kappa^{[k]}$}$$

is easily derived for $k \geq 2$, while (HJBE-1) shows that $\pi^{[k+1]}(x,p)$ is fixed by the equation

$$0 = \sum_{i=1}^{k} \nabla_x \pi^{[i+1]}(x,p) \cdot \left(f^{[k-i+1]}(x,p) + G\kappa^{[k-i+1]}(x,p)\right)$$
$$+ x^{\mathrm{T}} \ell_{xu} \cdot \kappa^{[k]}(x,p) + \frac{1}{2}\sum_{i=1}^{k} \kappa^{[i]}(x,p)^{\mathrm{T}} \cdot \ell_{uu} \cdot \kappa^{[k-i+1]}(x,p).$$

Using $(K)$ to cancel out all terms, which depend on $\kappa^{[k]}$, and substituting $(\kappa^{[k]})$, one derives the following identity.

$$\nabla_x \pi^{[k+1]}(x,p) \cdot (F+GK) \cdot x = -\sum_{i=1}^{k-1} \nabla_x \pi^{[i+1]}(x,p) \cdot f^{[k-i+1]}(x,p) \qquad (\pi^{[k+1]})$$
$$+ \frac{1}{2}\sum_{i=2}^{k-1} \nabla_x \pi^{[i+1]}(x,p) \cdot G\ell_{uu}^{-1} G^{\mathrm{T}} \cdot \nabla_x \pi^{[k-i+2]}(x,p)^{\mathrm{T}}$$

To prove the convergence of the power series in $(\pi)$, it is desired to find a series $\left(C_k^{\pi}\right)_{k\geq 2}$ such that

$$\sum_{k=2}^{\infty} C_k^{\pi} \cdot r^k < \infty \quad \text{and} \quad \left|\pi^{[k]}(x,p)\right| \leq C_k^{\pi} \cdot r^{k-2} \cdot r_x^2 \leq C_k^{\pi} \cdot r^k \tag{3.3}$$

for sufficiently small $r = ||\left(\begin{smallmatrix} x \\ p \end{smallmatrix}\right)||$. $r_x$ denotes the vector norm of the states $x$.
Since $f(.,.)$ is analytic at least in a neighborhood of the origin, Remark 16 $(c)$ implies the existence of a series $\left(C_k^{f}\right)_{k\geq 1}$ with

$$\sum_{k=1}^{\infty} C_k^{f} \cdot r^k < \infty \quad \text{and} \quad \left|f^{[k]}(x,p)\right| \leq C_k^{f} \cdot r^{k-1} \cdot r_x \leq C_k^{f} \cdot r^k \tag{3.4}$$

for sufficiently small $r$. In the following, inequality (A.5) from Theorem 10 will be applied to the different degrees of $\pi$.

$$\left|\nabla_x \pi^{[k]}(x,p)\right| \leq k \cdot C_k^{\pi} \cdot r^{k-2} \cdot r_x \leq k \cdot C_k^{\pi} \cdot r^{k-1}$$

The second degree of the value function can be easily over approximated using the spectral norm of $\pi_{xx}$, which is its largest eigenvalue. To keep the notation simple, this eigenvalue divided by two will be called $C_2^{\pi}$.

$$\left|\pi^{[2]}(x)\right| = \left|\frac{1}{2} x^{\mathrm{T}} \pi_{xx} x\right| \leq C_2^{\pi} \cdot r_x^2 \tag{3.5}$$

Going to the next degree, one finds from $(\pi^{[k+1]})$ an equation for $\nabla_x \pi^{[3]}(x,p)$.

$$\left.\frac{\mathrm{d}\pi^{[3]}}{\mathrm{d}t}(x,p)\right|_{\dot{x}=Fx+GKx} = \nabla_x \pi^{[3]}(x,p) \cdot (F+GK) \cdot x = -x^{\mathrm{T}} \pi_{xx} \cdot f^{[2]}(x,p)$$

Via integration it follows that

$$\begin{aligned} \left|\pi^{[3]}(x,p)\right|_{\dot{x}=Fx+GKx} &= \left|\int\limits_0^\infty -x^{\mathrm{T}}\pi_{xx}\cdot f^{[2]}(x,p)\,\mathrm{d}t\right| \leq \int\limits_0^\infty \left|x^{\mathrm{T}}\pi_{xx}\cdot f^{[2]}(x,p)\right|\mathrm{d}t \\ &\overset{(3.5),(3.4)}{\leq} 2C_2^f C_2^\pi \int\limits_0^\infty r(t)\cdot r_x^2(t)\,\mathrm{d}t. \end{aligned} \tag{3.6}$$

Since the linear system is stable, even exponential stability, it is possible to upper bound the states and, therefore, also $r_x = ||x||$.

$$r_x(t) \leq r_{x0}\cdot \mathrm{e}^{-\alpha\cdot t}\ , \quad r_{x0} = ||x(0)||,\ \alpha > 0$$

Thus $r(t)$ can also be upper bounded using $x(0)$.

$$r(t) = \left\|\begin{pmatrix} x(t) \\ p \end{pmatrix}\right\| \leq \left\|\begin{pmatrix} x(0) \\ p \end{pmatrix}\right\| =: r_0$$

$C_3^\pi$ can be found by solving the integral in (3.6).

$$\left|\pi^{[3]}(x,p)\right|_{\dot{x}=Fx+GKx} \leq \frac{2C_2^f C_2^\pi}{2\alpha}\cdot r_0\cdot r_{x0}^2 \leq \frac{3C_2^f C_2^\pi}{\alpha}\cdot r_0^3 =: C_3^\pi\cdot r_0^3$$

Theorem 10 implies

$$\left|\nabla_x\pi^{[3]}(x,p)\right| \leq 3\cdot C_3^\pi\cdot r_0\cdot r_{x0} \leq 3\cdot C_3^\pi\cdot r_0^2 = \frac{C_2^f C_2^\pi}{\alpha}\cdot r_0^2.$$

Before the general case is shown, $C_4^\pi$ will be calculated to gain more insight how those constants are obtained. To do so $(\pi^{[k+1]})$ is integrated.

$$\begin{aligned} &\left|\pi^{[4]}(x,p)\right|_{\dot{x}=Fx+GKx} \\ &= \left|\int\limits_0^\infty -x^{\mathrm{T}}\pi_{xx}\cdot f^{[3]}(x,p) - \nabla_x\pi^{[3]}(x,p)\cdot f^{[2]}(x,p)\right. \\ &\qquad \left. +\frac{1}{2}\nabla_x\pi^{[3]}(x,p)\cdot G\ell_{uu}^{-1}G^{\mathrm{T}}\cdot\nabla_x\pi^{[3]}(x,p)^{\mathrm{T}}\,\mathrm{d}t\right| \\ &\leq \int\limits_0^\infty 2C_3^f C_2^\pi\cdot r^2(t)\cdot r_x^2(t) + 3C_2^f C_3^\pi\cdot r^2(t)\cdot r_x^2(t) + \frac{9}{2}C_G^2\cdot C_{uu}\cdot (C_3^\pi)^2\cdot r^2(t)\cdot r_x^2(t)\,\mathrm{d}t \\ &\leq \frac{1}{2\alpha}\cdot\left(2C_3^f C_2^\pi + 3C_2^f C_3^\pi + \frac{9}{2}C_G^2 C_{uu}\cdot (C_3^\pi)^2\right)\cdot r_0^2\cdot r_{x0}^2 \\ &\leq \frac{1}{2\alpha}\cdot\left(2C_3^f C_2^\pi + 3C_2^f C_3^\pi + \frac{9}{2}C_G^2 C_{uu}\cdot (C_3^\pi)^2\right)\cdot r_0^4 =: C_4^\pi\cdot r_0^4 \end{aligned}$$

Here $C_{uu}$ resp. $C_G$ denotes the spectral norm of $\ell_{uu}^{-1}$ resp. $G$.

The general case $k \geq 2$ can be handled in a similar way.

$$
\begin{aligned}
&\left|\pi^{[k+1]}(x,p)\right|_{\dot{x}=Fx+GKx} \\
&\leq \int_0^\infty \sum_{i=1}^{k-1} (i+1) \cdot C_{i+1}^\pi \cdot r^{i-1}(t) r_x(t) \cdot C_{k-i+1}^f \cdot r^{k-i}(t) r_x(t)\,\mathrm{d}t \\
&\quad + \int_0^\infty \sum_{i=2}^{k-1} \frac{1}{2}(i+1) \cdot C_{i+1}^\pi \cdot r^{i-1}(t) r_x(t) \cdot C_G^2 C_{uu} \cdot (k-i+2) \cdot C_{k-i+2}^\pi \cdot r^{k-i}(t) r_x(t)\,\mathrm{d}t \\
&\leq \underbrace{\frac{\sum_{i=1}^{k-1} (i+1) \cdot C_{i+1}^\pi C_{k-i+1}^f + \sum_{i=2}^{k-1} \frac{1}{2}(i+1) \cdot (k-i+2) \cdot C_G^2 C_{uu} \cdot C_{i+1}^\pi C_{k-i+2}^\pi}{2\alpha}}_{=:\, C_{k+1}^\pi} \cdot r_0^{k-1} r_{x0}^2 \\
&\leq C_{k+1}^\pi \cdot r_0^{k+1}
\end{aligned}
$$

Having these coefficients, a converging domination series for $\left(C_k^\pi\right)_{k\geq 2}$ would also show the desired convergence. Therefore, another equation is introduced. Not surprisingly, it is the same equation as in the original proof. The goal is to show that its solution is, in fact, a series with the desired properties.

$$
C_G^2 C_{uu} \cdot \left(\frac{\mathrm{d}\gamma}{\mathrm{d}r}(r)\right)^2 + \left(\sum_{i=2}^\infty C_i^f \cdot q^{i-1} - a\right) \cdot r \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}r}(r) + b \cdot r^2 = 0 \tag{3.7}
$$

$a$ has to be chosen such that

$$
a \geq \alpha + 2C_G^2 C_{uu} \cdot C_2^\pi,
$$

while $b$ is fixed as

$$
b = \frac{a^2 - \alpha^2}{4C_G^2 \cdot C_{uu}}.
$$

$q \in \mathbb{R}_{\geq 0}$ can be seen as a vector norm like $r$. Clearly, (3.7) is only well-defined where $f(.,.)$ is analytic. As shown in the proof of Theorem 2, the solution of (3.7) is given by

$$
g(q) = \frac{a - \sum_{k=2}^\infty C_k^f \cdot q^{k-1} - \sqrt{\left(a - \sum_{k=2}^\infty C_k^f \cdot q^{k-1}\right)^2 - 4b \cdot C_G^2 C_{uu}}}{2C_G^2 C_{uu}}. \tag{3.8}
$$

$g(.)$ is analytic in a domain that contains the origin. Its power series expansion is written as

$$
g(q) = \sum_{i=2}^\infty C_k^g \cdot q^{k-2} < \infty,
$$

where the $C_i^g$ are given by

$$C_k^g = \frac{1}{\alpha} \cdot \left( C_G^2 C_{uu} \cdot \sum_{i=3}^{k-1} C_i^g \cdot C_{k-i+2}^g + \sum_{i=2}^{k-1} C_i^f \cdot C_{k-i+1}^g \right).$$

To finally show the convergence, the $C_k^g$ are compared with

$$C_k^\pi = \frac{\sum_{i=2}^{k-1} (k-i+1) \cdot C_i^f C_{k-i+1}^\pi + \sum_{i=3}^{k-1} \frac{1}{2} i \cdot (k-i+2) \cdot C_G^2 C_{uu} \cdot C_i^\pi C_{k-i+2}^\pi}{2\alpha}.$$

With $a$ chosen as outlined, $C_2^\pi$ is smaller than or equal to $C_2^g$. For $k \geq 3$ it is

$$C_k^\pi \leq C_k^g \cdot \frac{\max\limits_{i \in \{3,\ldots,k-1\}} i \cdot (k-i+2)}{2} \leq C_k^g \cdot \frac{(k+2)^2}{8}$$

and, therefore, it holds

$$|\pi(x,p)| \leq \sum_{k=2}^{\infty} C_k^\pi \cdot r^k \leq \sum_{k=2}^{\infty} \frac{(k+2)^2}{8} \cdot C_k^g \cdot r^k.$$

If the power series $g(r)$ exists resp. converges for a fixed $r$, then $\pi(x,p)$ exists and converges for all $(x,p)$ such that

$$||(x,p)|| < r.$$

That is the case since

$$\lim_{k\to\infty} \left((k+2)^2\right)^{1/k} \searrow 1.$$

Since $\kappa(x,p)$ is a product of locally analytic functions, it is also locally analytic with the same area of convergence. □

## 3.3 Area of convergence

As for the non-parametric case (see Section 2.3), it is of interest to estimate the area of convergence of the power series $(\kappa)$ and $(\pi)$ from section Section 3.2. Taking the optimal control problem (OCP) from Section 3.2 and the same calculation steps as for the non-parametric case, it is straightforward to see that convergence and, therefore, existence is achieved for all points $(x,p)$ in a neighborhood of the origin such that $f(x,p)$ is analytic and

$$\sum_{k=2}^{\infty} C_k^f \cdot ||(x,p)||^k \leq \left(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi - \sqrt{(\alpha + 2C_G^2 C_{uu} \cdot C_2^\pi)^2 - \alpha^2}\right) \cdot ||(x,p)||. \tag{3.9}$$

The constants $C_k^f$ $(k \geq 2)$ are such that

$$\left|\left|f^{[k]}(x,p)\right|\right| \leq C_k^f \cdot ||(x,p)||^k$$

and should be chosen minimal. $C_G$, $C_{uu}$, and $C_2^\pi$ denote the spectral norm of $G$, $\ell_{uu}^{-1}$, and $\frac{1}{2}\pi_{xx}$, while $\alpha$ is such that $r(t) \leq r_0 \cdot \mathrm{e}^{-\alpha t}$ as described in the proof of Theorem 4.

**Example 6.** *For the quadcopter model with 10 states from Example 3, the constants (all units are neglected)*

$$C_G = 10, \quad C_{uu} = 1, \quad C_2^\pi = 1.3286 \quad \textit{and} \quad \alpha = 0.6753$$

*are found. Using (3.9), convergence and correctness of the control law and the value function is guaranteed for all* $r = ||(x,p)|| \leq 0.003\,27$.
*For Example 4, the constants are*

$$C_G = 18.8679, \quad C_{uu} = 10, \quad C_2^\pi = 1.1457 \quad \textit{and} \quad \alpha = 0.726.$$

*Thus convergence is proven for* $r = ||(x,p)|| \leq 0.3056 \cdot 10^{-10}$.
*The results are not satisfying since they have no use in practice. Sum of squares methods, as used in [65], are likely to lead to better inner approximations but are still for away from values that are needed in reality. Definitely, there is more research needed in this field.*

In this chapter, Al'brekht's Method has been extended to allow slowly varying parameters in the system dynamics. As a result, a parametric explicit control law and value function were obtained. The proposed method has been validated using the same examples as in Chapter 2 and a bioreactor example for the case where the origin is not a steady-state. The convergence proof, local stability results, and the estimation of the domain of convergence were generalized to include the parametric case.

# 4 Including Constraints

Every good method for optimal control has to be capable of involving constraints. Including those directly in the power series approximation approach is not possible. As a possible solution, a barrier function approach is proposed. Barrier functions are often used in optimization problems since only the cost function is changed, and they can be combined with many optimal control methods, see [76]. There are several types of barrier functions. The most common are reciprocal and logarithmic barriers, see [31, 89]. Throughout this chapter, logarithmic barrier functions, compare to works of Feller and Ebenbauer [22–24] as well as Wills and Heath [98], are used. Other types could also be utilized. The only needed property is the existence of a local power series. Hunt and Krener proposed norms of even degree as penalty terms in the cost function, see [38]. However they put up restrictions since e. g. symmetry is contained. The idea is to extend the functions defining the constraints into power series. Specific logarithmic barrier functions, which tend to infinity if the boundary of the feasible region is approached, are used. The logarithm can also be expanded into a local power series, thus coinciding with Al'brekht's Method. The barrier is designed in a way that there is no change of the cost at the origin and, additionally, it will also be shown that the LQR conditions remain valid. In the end, the quadcopter examples from the previous chapter are expanded with constraints and the results are compared with the non-constraint case.

## Setup and constraints

Suppose a number $m \in \mathbb{N}$ of inequality constraints $g_i(x,u) \leq 0$ $(i \in [m])$, where $g_i \colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$. Note that if the $g_i$ are analytic in $(x,u)$, they can be expanded into power series

$$g_i(x,u) = g_{i0} + \sum_{k=1}^{\infty} g_i^{[k]}(x,u).$$

Additionally, if $(0,0)$ lies in the interior of the feasible region, the constant $g_{i0} \in \mathbb{R}$ has to be negative. Simplifying the notation, the index $i$ will be neglected in the following since all constraints will be handled equally. Straightforward solving the optimal control problem

$$
\begin{aligned}
\pi\big(x(0),p\big) &= \min_{u(.)} \int_0^\infty \ell\big(x(\tau),u(\tau)\big)\,\mathrm{d}\tau \\
\dot{x} &= f(x,u,p) \\
g(x,u) &\leq 0
\end{aligned}
\tag{OCP}
$$

with the same strategy as shown in Chapter 3 is not applicable, as the constraints define conditions for the states $x$ and input values $u$ instead of the value function $\pi$ and the optimal control law $\kappa$, which depend on $x$ and $u$. Therefore another strategy based on logarithmic barrier functions is proposed, which „hides“ the constraints in the cost function. Defining

$$
\tilde{g}(x,u) := \sum_{k=1}^{\infty} g_i^{[k]}(x,u) = g(x,u) - g_0 \tag{$\tilde{g}$}
$$

shows $\tilde{g}(x,u) \leq -g_0$ if $g(x,u) \leq 0$. Using $\tilde{g}$, it is desired to design a logarithmic barrier function $L_B\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}$, which will be added to the cost function $\ell$, such that

$$
\lim_{\tilde{g} \nearrow -g_0} L_B = \infty.
$$

If the constraints are fulfilled, then $\frac{\tilde{g}(x,u)}{g_0}$ must be greater or equal to $-1$. A suitable logarithmic barrier function is

$$
L_B(x,u) := -\log\left(1 + \frac{\tilde{g}(x,u)}{g_0}\right).
$$

Utilizing the power series expansion of the logarithm $\log(1+s)$,

$$
\log(1+s) = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{s^k}{k} = s - \frac{s^2}{2} + \frac{s^3}{3} - \frac{s^4}{4} + \dots,
$$

which converges for $s \in \mathbb{R}$ with $|s| < 1$, $L_B$ is expanded into its power series for all $(x,u)$ such that $\left|\frac{\tilde{g}(x,u)}{g_0}\right| < 1$.

$$
\begin{aligned}
L_B(x,u) &= -\log\left(1 + \frac{\tilde{g}(x,u)}{g_0}\right) = \sum_{k=1}^{\infty} L_B^{[k]}(x,u) \\
&= -\frac{\tilde{g}(x,u)}{g_0} + \frac{1}{2}\cdot\left(\frac{\tilde{g}(x,u)}{g_0}\right)^2 - \frac{1}{3}\cdot\left(\frac{\tilde{g}(x,u)}{g_0}\right)^3 + \frac{1}{4}\cdot\left(\frac{\tilde{g}(x,u)}{g_0}\right)^4 + \dots
\end{aligned}
\tag{$L_B$}
$$

The barrier function should be added to the cost function $\ell$ without changing its properties, i. e. no terms of degree 1 in $(x,u)$. Since the power series in ($\tilde{g}$) starts with linear terms, it is sufficient to remove the first term of ($L_B$). Removing the first degree also ensures local positivity of the additional cost. Multiplying $L_B$ with a penalty factor $c \in \mathbb{R}_{>0}$ to regulate the influence of the barrier function, a modified cost function is given via ($\tilde{\ell}$).

$$\tilde{\ell}(x,u) = \ell(x,u) + c \cdot L_B(x,u) + c \cdot \frac{\tilde{g}(x,u)}{g_0} \tag{$\tilde{\ell}$}$$

The optimal control problem (OCP) is therefore changed and now states as the following.

$$\begin{aligned} \pi\big(x(0),p\big) &= \min_{u(.)} \int_0^\infty \tilde{\ell}\big(x(\tau),u(\tau)\big)\,\mathrm{d}\tau \\ \dot{x} &= f(x,u,p) \end{aligned} \tag{$\widetilde{\text{OCP}}$}$$

Since for the calculation of the approximative solution $\pi$ and $\kappa$ approximations of $f$ and $\tilde{\ell}$ are used, the constraints are soft for the approximations, but the penalty cost grows to infinity while increasing the degree of approximation and approaching the boundary of the feasible region.

$$\lim_{k\to\infty} \lim_{\tilde{g}(x,u)\nearrow -g_0} \tilde{\ell}^{[2;k]}(x,u) = \infty$$

**Lemma 1** (Convexity)**.** *The changed cost function $\tilde{\ell}$ fulfills conditions (I) and (III) from Section 3.1 if the original cost function $\ell$ does.*

*Proof.* All second-order terms of $\tilde{\ell}$ are given by

$$\ell^{[2]}(x,u) + \frac{1}{2} \cdot \left(\frac{\tilde{g}^{[1]}(x,u)}{g_0}\right)^2 .$$

If $\tilde{g}^{[1]}(x,u)$ is written as $g_x \cdot x + g_u \cdot u$ for matrices $g_x \in \mathbb{R}^{1\times n_x}$ and $g_u \in \mathbb{R}^{1\times n_u}$, the following identities are obtained.

$$\tilde{\ell}_{xx} = \ell_{xx} + \frac{g_x^\mathrm{T} \cdot g_x}{g_0}, \quad \tilde{\ell}_{xu} = \ell_{xu} + \frac{g_x^\mathrm{T} \cdot g_u}{g_0}, \quad \tilde{\ell}_{uu} = \ell_{uu} + \frac{g_u^\mathrm{T} \cdot g_u}{g_0}$$

Clearly, $\tilde{\ell}_{uu}$ is positive definite. Using

$$\frac{1}{g_0} \cdot \begin{pmatrix} g_x^\mathrm{T} \cdot g_x & g_x^\mathrm{T} \cdot g_u \\ g_u^\mathrm{T} \cdot g_x & g_u^\mathrm{T} \cdot g_u \end{pmatrix} = \frac{1}{g_0} \cdot \begin{pmatrix} g_x^\mathrm{T} \\ g_u^\mathrm{T} \end{pmatrix} \cdot \begin{pmatrix} g_x & g_u \end{pmatrix} \succeq 0$$

one also has

$$\begin{pmatrix} \tilde{\ell}_{xx} & \tilde{\ell}_{xu} \\ \tilde{\ell}_{xu}^\mathrm{T} & \tilde{\ell}_{uu} \end{pmatrix} \succeq 0$$

and, therefore, condition (I) stays valid. Since $\ell_{xx}$ and $g_x^{\mathrm{T}} \cdot g_x$ are positive semidefinite the image of $\ell_{xx}$ is contained in the image of $\tilde{\ell}_{xx}$, which shows that the pair $(F, \tilde{\ell}_{xx})$ must be detectable if $(F, \ell_{xx})$ is detectable. Thus (III) also stays valid. □

**Remark 5.**

(*a*) *With the method described throughout this chapter, even non-convex constraints can be handled. For example in [45] an obstacle that has to be avoided by a quadcopter is implemented as a non-convex constraint.*

(*b*) *The proposed strategy can also be used for discrete-time systems in a similar way. Lemma 1 still stays valid.*

(*c*) *If Al'brekht's Method is used for output feedback systems, which are investigated in Chapter 5, constraints regarding the output values $y \in \mathbb{R}^{n_y}$ and the input values $u \in \mathbb{R}^{n_u}$ can be handled in the same way.*

**Remark 6** (Equality constraints)**.** *Handling equality constraints via barrier functions, in the same way, is not possible. Additionally, if one takes constraints*

$$h(x, u) = 0$$

*with $h\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^m$ $(m \in \mathbb{N})$ directly into account and assumes $h$ to be analytic, the resulting optimal control problem states conditions for the functions $\pi$ and $\kappa$ as well as the variables $(x, u)$. As mentioned before, using Al'brekht's Method in this setup does not lead to solvable equations.*

The running examples are now revisited to show possible advantages and disadvantages of the inclusion of inequality constraints.

**Example 7** (Quadcopter: 10 states, 6 parameters, 11 constraints)**.**
*Taking Example 3 from Section 3.1 with the additional the state and input constraints*

$$p_x \leq 4\,\mathrm{m}, \quad |\phi| \leq \frac{\pi}{4}\,\mathrm{rad}, \quad |\theta| \leq \frac{\pi}{4}\,\mathrm{rad}, \quad |u_\phi| \leq \frac{\pi}{9}\,\mathrm{rad/s}^2, \quad |u_\theta| \leq \frac{\pi}{9}\,\mathrm{rad/s}^2,$$
$$-\frac{g}{k_t} \leq \tilde{u}_z \leq 2g - \frac{g}{k_t}, \quad (\Leftrightarrow u_z \in [0, 2g]),$$

*which can also be found in [48], shows a 90 % reduction of the input $u_\phi$ and $u_\theta$, see Fig. 4.3. Thus fulfilling the constraints, which is not the case when the constraints are not taken into account. The input $\tilde{u}_z$ is the same for all shown control laws since the corresponding subsystem is linear and there is no danger of violating its constraints for the given initial value. The initial values that were used are $p_x = 1$ and $p_z = 1$, while the other states are set to zero. Since the optimal control law is approximated up to a certain degree, the constraints are „soft“ and do not necessarily need to be obeyed for other initial values. In fact, if the quadcopter is initialized with a large tilt, the*

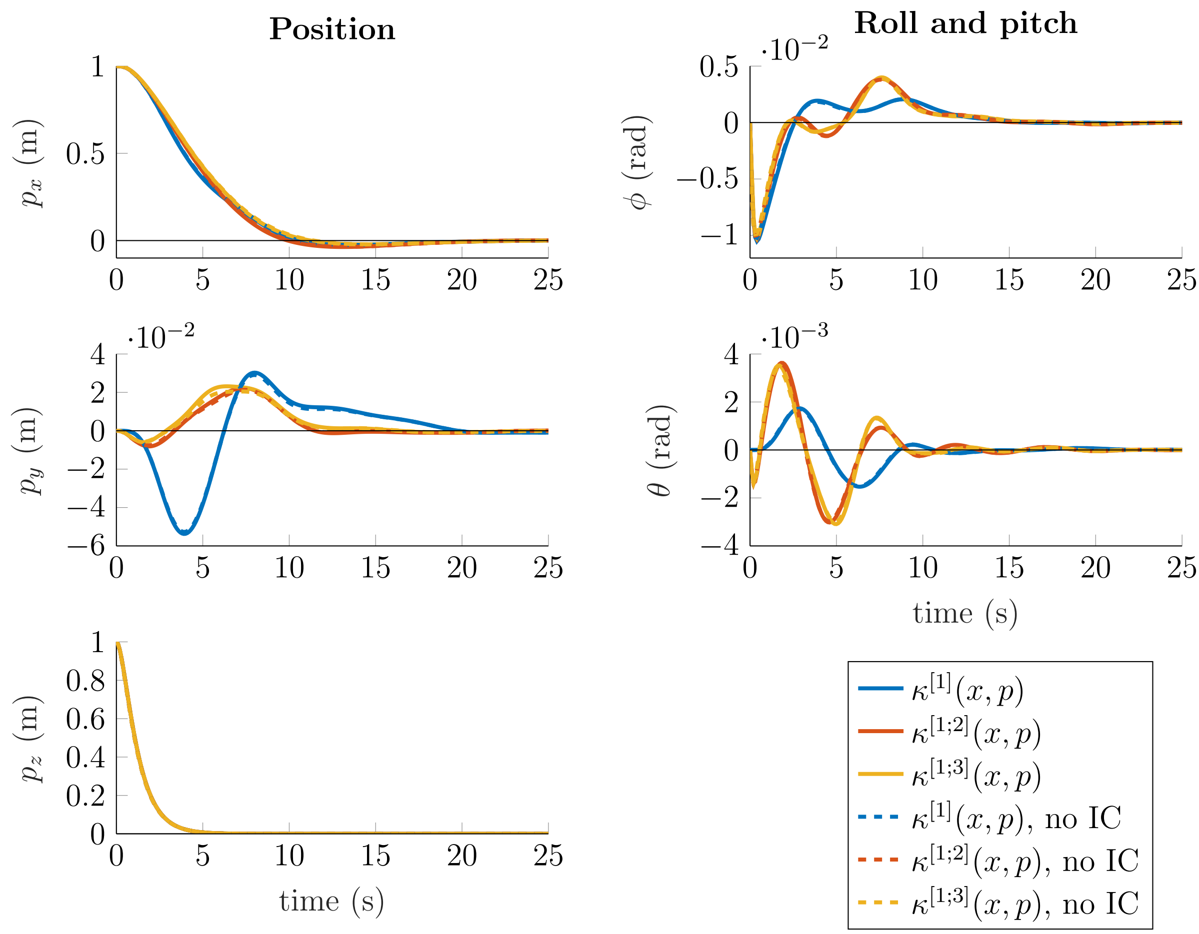


**Figure 4.1:** Propagation of five out of ten states (10 states model with parameters and constraints)

*controller has to react with a large control action to achieve stability. The behavior of the states nearly does not change in the considered scenario, see Fig. 4.1. The state constraint $p_x \leq 4\,\mathrm{m}$ only has a minor effect. The much smaller control action of the controllers considering the constraints leads to less alleviation of the wind disturbance, see Fig. 4.2. The drift in the y-direction is therefore slightly increased. Overall higher-order approximations remain to perform better at controlling the system.*

*In [45] non-convex constraints have been used for the same model. It could be shown that the quadcopter can avoid obstacles using the outlined method. Higher-order approximations also increased the safety.*

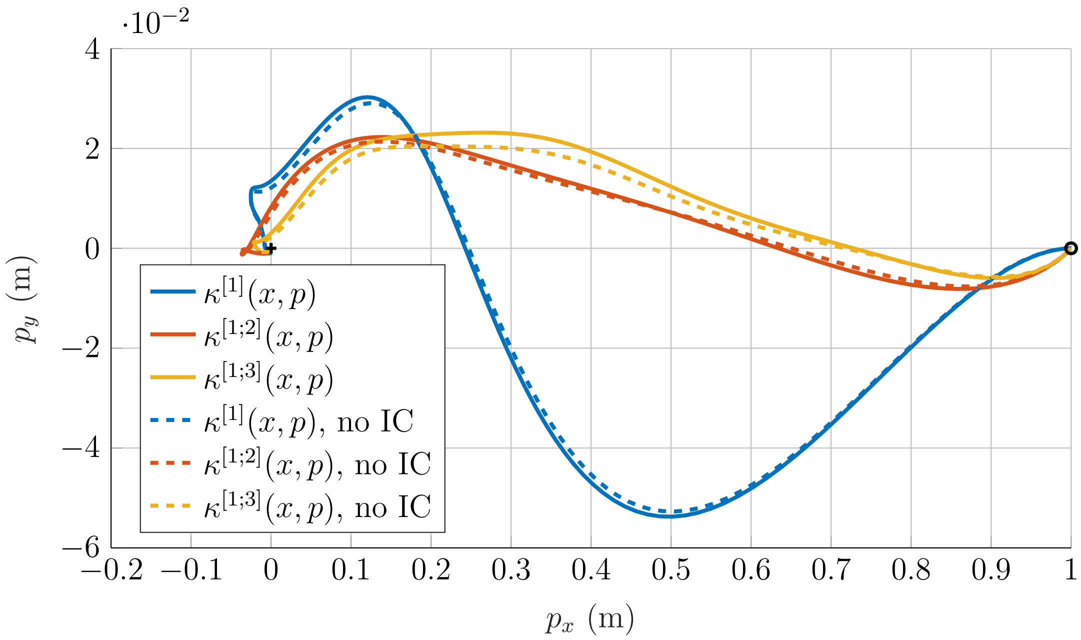


**Figure 4.2:** Position of the quadcopter in the $x$-$y$-plane (10 states model with parameters and constraints)

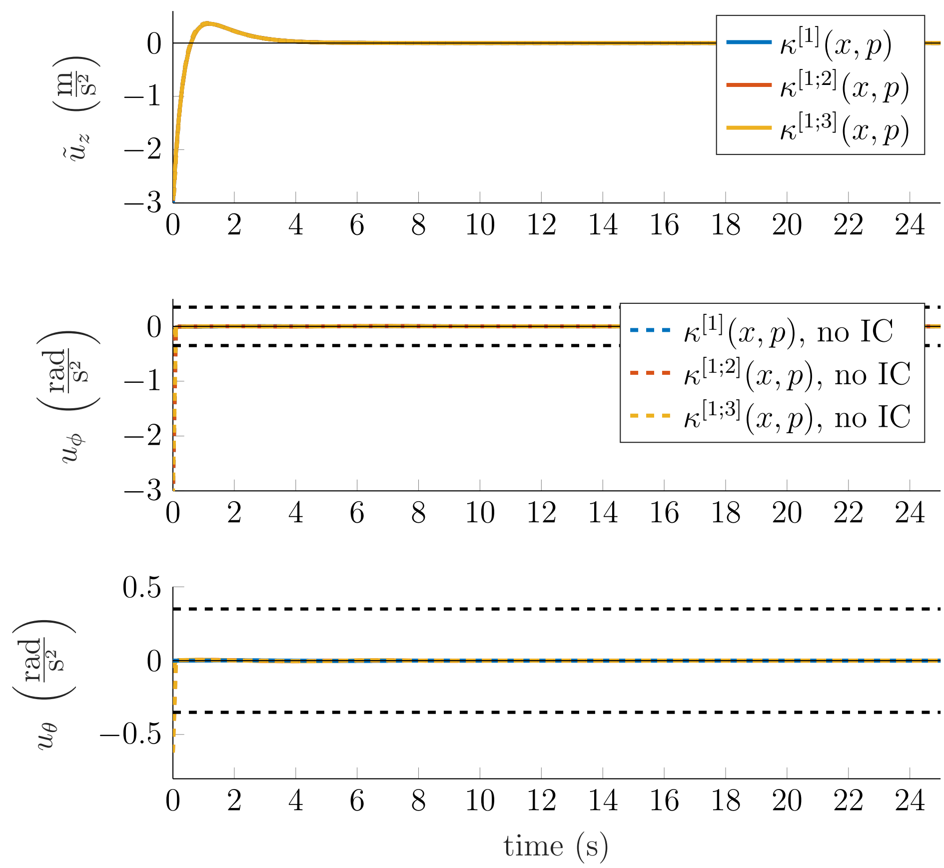


**Figure 4.3:** Control input for the vertical and angular accelerations (10 states model with parameters and constraints)

**Example 8** (Quadcopter: 12 states, 6 parameters, 8 constraints)**.**
*The dynamics and parameters from Example 4 are revisited. Furthermore, the following input constraints are taken into account.*

$$\begin{aligned}
|u_\phi| \leq \frac{\pi}{9}\,\mathrm{rad/s^2}, \quad |u_\theta| \leq \frac{\pi}{9}\,\mathrm{rad/s^2}, \quad |u_\psi| \leq \frac{\pi}{9}\,\mathrm{rad/s^2},\\
-10.8\,\mathrm{kg}\cdot\mathrm{m/s^2} \leq \tilde{u}_z \leq 18.5\,\mathrm{kg}\cdot\mathrm{m/s^2}
\end{aligned}$$

*The boundaries for the moments are the same as in Example 7. For the thrust, some virtual constraints, which were found in previous works [40], are used. The initial values are $p_x = 1$ and $p_z = 1$ (other states are zero), and the choice of the parameters stays the same as in Example 4. Similar to the previous example in this section, the control input $u_\phi$, $u_\theta$, and $u_\psi$ is reduced by around 90 %, see Fig. 4.6. Note that only the constraints which are violated by the controllers from Chapter 3 are shown in the picture. The controllers that take the constraints into account fulfill all of them given the mentioned initial values. The states are nearly unchanged (Fig. 4.4), while the drift caused by the parameters representing the wind is slightly increased (Fig. 4.5). The overall cost also increases but not significantly due to the smaller control action and therefore slower convergence to the target point, see Fig. 4.7.*

Throughout this chapter, it was shown how the power series approach can take inequality constraint into account without reducing its applicability. The effectiveness was investigated using the running examples. Furthermore, the results were compared to the unconstrained case. Unfortunately, convergence can not be guaranteed so far since Theorem 4 from Chapter 3 only allows quadratic cost functions.

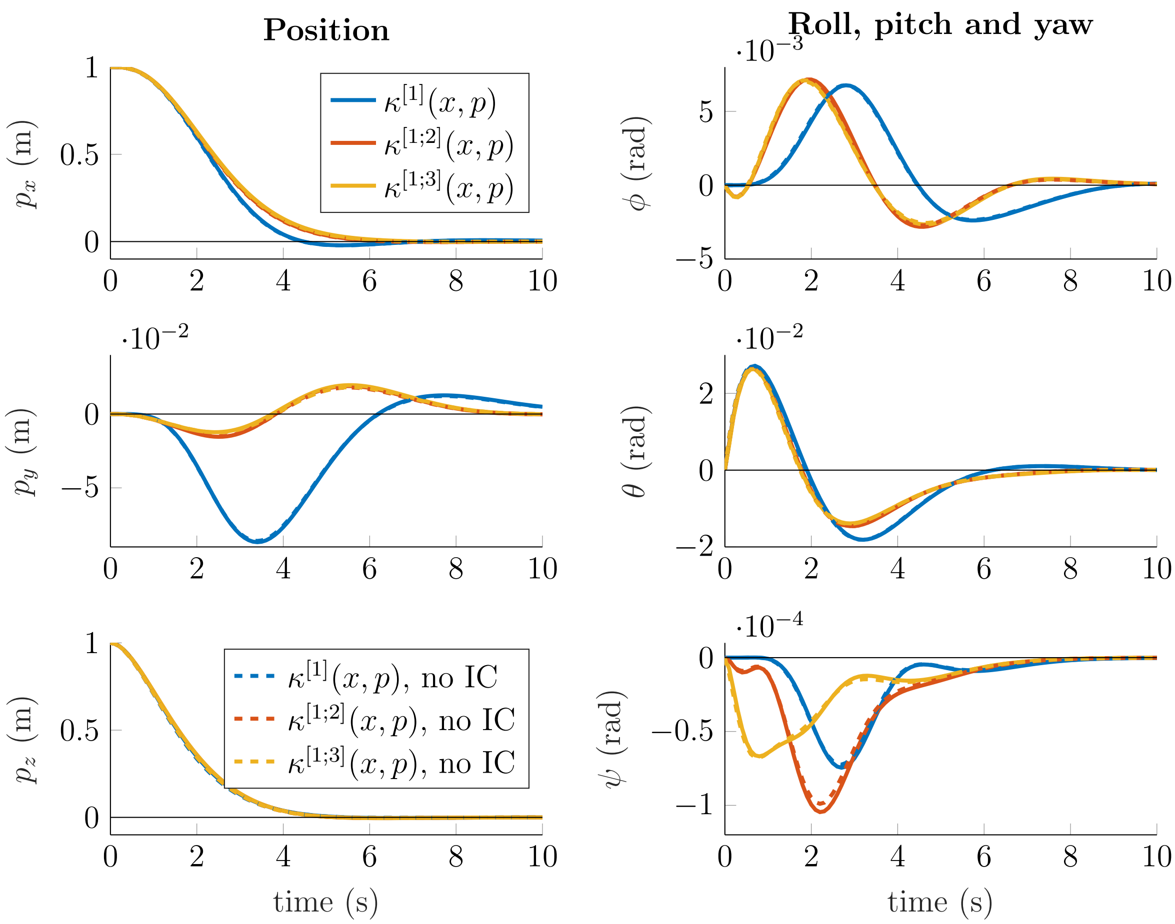


**Figure 4.4:** Propagation of six out of twelve states (12 states model with parameters and constraints)

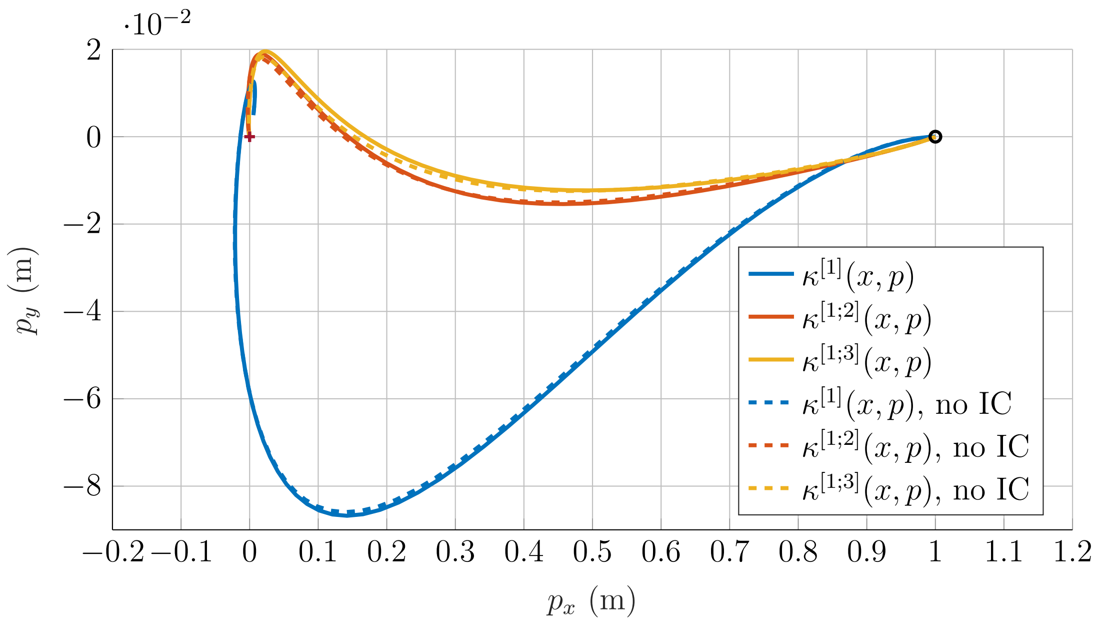


**Figure 4.5:** Position of the quadcopter in the $x$-$y$-plane (12 states model with parameters and constraints)

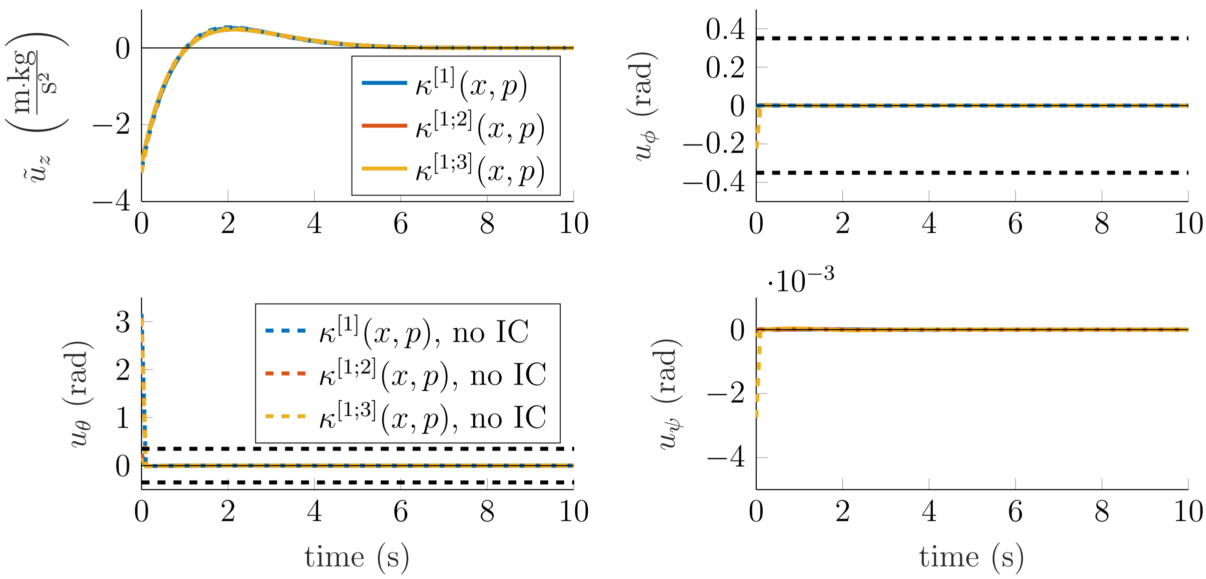


**Figure 4.6:** Control input for the thrust and the moments about the axis (12 states model with parameters and constraints)

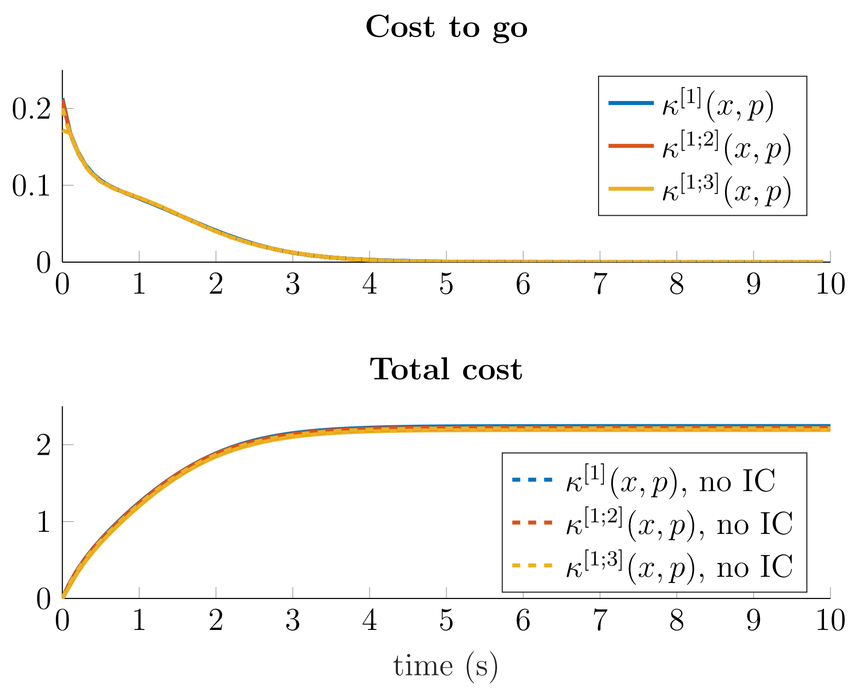


**Figure 4.7:** Resulting cost (12 states model with parameters and constraints)

# 5 Output-Feedback

In this chapter, the output-feedback case, sometimes also called output control, is considered. For a general introduction to this topic see e. g. [47, 92]. The states are now not directly accessible for control, see Fig. 5.1. While in principle, one could use a state estimator to recover the states, sometimes it is not desirable or challenging. Examples are systems with large state dimensions, where not all states are needed for the feedback, which might lead to ill-conditioned inverse problems. Throughout this chapter, it is the desire to find controllers that directly map from the output $y$ to the controls $u_{\text{min}}$ including the known or estimated parameters $p$, see Fig. 5.1.

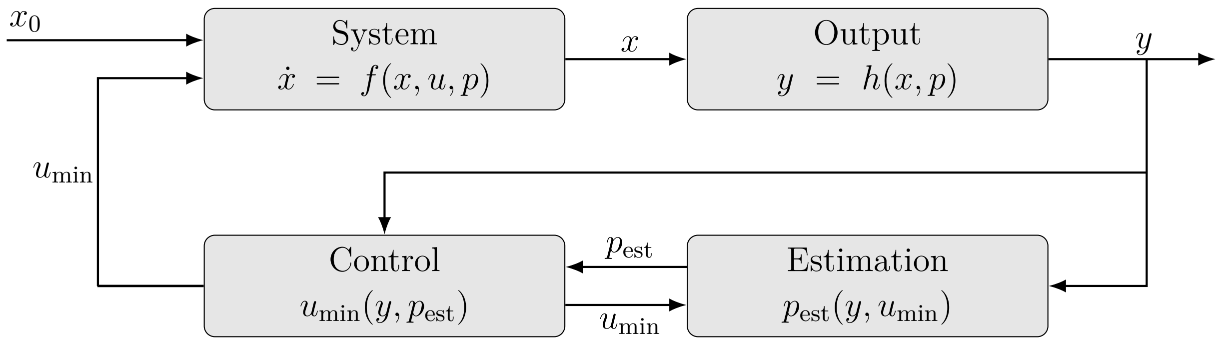


**Figure 5.1:** Control scheme for output control with parameter estimation

Thus it is the task to find an optimal control law in terms of only the measurement information $y$ and the known or estimated parameters $p_{\text{est}}$, which stabilizes the output at the origin, see [43, 44]. The existing works of other authors [1, 41, 55] consider output regulation, using Al'brekht's power series approach, but use a slightly different setup. They aim to steer the output to a submanifold of the state space and keep it on the manifold defined by the output function $h$. However, they assume that all states are accessible resp. measurable. Often, not the full state can be measured, and estimating it, might involve knowing exact equations for all states, which sometimes is challenging, e. g. in biotechnology applications (Example 11).

## 5.1 Output-feedback in continuous time

The construction of the output-feedback is first investigated in continuous time and thereafter in Chapter 6 in discrete time. Additionally to the parametric case, which has been investigated in Chapter 3, an analytic output function is mapping the states to the output $y$. At first, multiplicative parameters in the output are considered.

Remark 8 then generalizes the result to additive (and multiplicative) parameters. Additionally to the pure parametric case, it is assumed that on a submanifold of the state space, the linearisation of the output function $h$ has a local inverse. Thus linear output dynamics can be found. This dynamical system is not unique since it depends on the choice of the submanifold. Furthermore, the linear output system and the second-order of the cost function need to fulfill LQR conditions. Under these conditions, a similar but involved procedure as used in Sections 2.1 and 3.1 is derived. The calculation is only valid within the corresponding submanifold. Thus the control law and the optimal cost function also depend on the particular choice of the mapping between output and state space and are therefore not unique.
The approach allows to establish local stability, see Corollary 4, and its performance is underlined using extended examples from the previous chapters. The effect of different choices of the submanifold is discussed in Example 9.

## Setup and power series solution

The optimal control problem that is considered throughout this section is stated in (OCP).

$$\begin{aligned} \pi\Big(y(0),p\Big) &= \min_{u(.)} \int_0^\infty \ell\Big(y(\tau),u(\tau)\Big)\,\mathrm{d}\tau \\ \text{s.t. } \dot{x} &= f(x,u,p) \\ y &= h(x,p) \end{aligned} \tag{OCP}$$

Note that the cost function is stated in terms of $y$ and $u$. The objective is to control the system via the input $u \in \mathbb{R}^{n_u}$ only using the measurements $y \in \mathbb{R}^{n_y}$ as well as the parameters $p \in \mathbb{R}^{n_p}$. It is assumed that the dimension of the states should be at least the one of the output. The system dynamics $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ and the output function $h\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_y}$ are required to be analytic functions with power series expansions

$$f(x,u,p) = \sum_{k=1}^{\infty} f^{[k]}(x,u,p) = Fx + Gu + f^{[2]}(x,u,p) + \dots \tag{$f$}$$

$$\text{and } h(x,p) = \sum_{k=1}^{\infty} h^{[k]}(x,p) = Hx + h^{[2]}(x,p) + \dots, \tag{$h$}$$

where the parameters only appear multiplicative to ensure unique solvability.

$$\forall p \in \mathbb{R}^{n_p} : f(0,0,p) = 0 \tag{$f_0$}$$

$$h(0,p) = 0 \tag{$h_0$}$$

Since $p \mapsto f(0,0,p)$ and $p \mapsto h(0,p)$ are also analytic but remain constant at 0, the complete power series need to vanish.

$$\begin{aligned} \forall k \in \mathbb{N}, p \in \mathbb{R}^{n_p} : f^{[k]}(0,0,p) &= 0 \\ h^{[k]}(0,p) &= 0 \end{aligned}$$

The cost function $\ell \colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}$ depends solely on the input values $u$ and the output vector $y$ instead of the states $x$. $\ell(.,.)$ is also required to be analytic in $(y,u)$. Its power series representation is given by

$$\ell(y,u) = \sum_{k=2}^{\infty} \ell^{[k]}(y,p) = \frac{1}{2} y^{\mathrm{T}} \ell_{yy} y + y^{\mathrm{T}} \ell_{yu} u + \frac{1}{2} u^{\mathrm{T}} \ell_{uu} u + \ell^{[3]}(y,u) + \dots \tag{$\ell$}$$

There is no linear part in the cost function and no constants in the system dynamics as well as the output function. Further restrictions towards the matrices $F \in \mathbb{R}^{n_x \times n_x}$, $G \in \mathbb{R}^{n_x \times n_u}$, $H \in \mathbb{R}^{n_y \times n_x}$, $\ell_{yy} \in \mathbb{R}^{n_y \times n_y}$, $\ell_{yu} \in \mathbb{R}^{n_y \times n_u}$, and $\ell_{uu} \in \mathbb{R}^{n_u \times n_u}$ are needed, as outlined in the following derivations.
Similar to the derivations in Chapter 2 and 3, the value function $\pi \colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_p} \to \mathbb{R}$ and the control law $u_{\min} := \kappa \colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_u}$ are assumed to be analytic. Both functions depend on the output vector $y$ and the parameters $p$.

$$\begin{aligned} \pi(y,p) &= \sum_{k=1}^{\infty} \pi^{[k]}(y,p) \\ &= \pi_y y + \pi_p p + \frac{1}{2} y^{\mathrm{T}} \pi_{yy} y + y^{\mathrm{T}} \pi_{yp} p + \frac{1}{2} p^{\mathrm{T}} \pi_{pp} p + \pi^{[3]}(y,p) + \dots \end{aligned} \tag{$\pi$}$$

$$\kappa(y,p) = \sum_{k=1}^{\infty} \kappa^{[k]}(y,p) = Ky + Lp + \kappa^{[2]}(y,p) + \dots \tag{$\kappa$}$$

Note that $\pi(0,0)$ and $\kappa(0,0)$ need to vanish since there is no control needed at the origin $(y,p) = (0,0)$, and no cost arises.
To obtain the power series solution, the Hamilton-Jacobi-Bellman equation is used. Its derivation in case of output control can be found in the Appendix A Remark 18 and leads to:

$$0 = \nabla_y \pi(y,p) \cdot \nabla_x h(x,p) \cdot f\big(x, \kappa(y,p), p\big) + \ell\big(y, \kappa(y,p)\big), \tag{HJBE-1}$$

$$0 = \nabla_y \pi(y,p) \cdot \nabla_x h(x,p) \cdot \nabla_u f\big(x, \kappa(y,p), p\big) + \nabla_u \ell\big(y, \kappa(y,p)\big). \tag{HJBE-2}$$

Similar to the derivations in Chapter 2 and 3, the two equations are separated degree-wise after substituting all functions by their power series. The constant part of (HJBE-1) does not contain any information, but (HJBE-2) leads to the first condition for $\pi_y \in \mathbb{R}^{1 \times n_y}$.

$$(\text{HJBE-2})^{[0]} : \quad 0 = \pi_y \cdot H \cdot G \tag{$\pi_y$-1}$$

A second and final condition is gained from the linear part of (HJBE-1).

$$\begin{aligned}
(\text{HJBE-1})^{[1]}: \quad 0 &= \pi_y \cdot H \cdot \Big(Fx + G \cdot (Ky + Lp)\Big) \overset{(\pi_y\text{-}1)}{=} \pi_y \cdot H \cdot Fx \\
\Rightarrow 0 &= \pi_y \cdot H \cdot F
\end{aligned} \tag{$\pi_y$-2}$$

Combining (HJBE-1) and (HJBE-2) leads to $\pi_y = 0$ if

(I) $\operatorname{rank}(HF\ HG) = n_y$.

As before, $\pi_p \in \mathbb{R}^{1\times n_p}$ is needed. With $p = 0$ one must obtain the solution of the non-parametric case. This implies that $p \mapsto \pi(0,p)$ and $p \mapsto \kappa(0,p)$ must vanish and, therefore,

$$\forall k \in \mathbb{N},\, p \in \mathbb{R}^{n_p} : \pi^{[k]}(0,p) = 0 \text{ and } \kappa^{[k]}(0,p) = 0.$$

Taking all terms that are linear with respect to the states $x$, output values $y$, and parameters $p$ from (HJBE-2) representations of $K \in \mathbb{R}^{n_u\times n_y}$ and $L \in \mathbb{R}^{n_u\times n_p}$ are obtained.

$$(\text{HJBE-2})^{[1]}: \quad 0 = \Big(y^{\mathrm{T}}\pi_{yy} + p^{\mathrm{T}}\pi_{yp}^{\mathrm{T}}\Big) \cdot HG + y^{\mathrm{T}}\ell_{yu} + \Big(y^{\mathrm{T}}K^{\mathrm{T}} + p^{\mathrm{T}}L^{\mathrm{T}}\Big) \cdot \ell_{uu}$$

The vectors $y$ and $p$ can be chosen arbitrarily in a domain that contains the origin. Therefore, $(\text{HJBE-2})^{[1]}$ contains two equations and $K$ and $L$ can be stated independently.

$$K = -\ell_{uu}^{-1} \cdot \Big(\ell_{yu}^{\mathrm{T}} + G^{\mathrm{T}}H^{\mathrm{T}}\pi_{yy}\Big) \tag{$K$}$$

$$L = -\ell_{uu}^{-1} \cdot G^{\mathrm{T}}H^{\mathrm{T}}\pi_{yp} \tag{$L$-1}$$

To find those matrices the quadratic part of (HJBE-1) is investigated.

$$\begin{aligned}
(\text{HJBE-1})^{[2]}: \quad 0 = {}& \Big(y^{\mathrm{T}}\pi_{yy} + p^{\mathrm{T}}\pi_{yp}^{\mathrm{T}}\Big) \cdot H \cdot \Big(\underline{Fx} + G \cdot (Ky + Lp)\Big) + \frac{1}{2}y^{\mathrm{T}}\ell_{yy}y \\
&+ y^{\mathrm{T}}\ell_{yu} \cdot (Ky + Lp) + \frac{1}{2}\Big(y^{\mathrm{T}}K^{\mathrm{T}} + p^{\mathrm{T}}L^{\mathrm{T}}\Big) \cdot \ell_{uu} \cdot (Ky + Lp)
\end{aligned} \tag{5.1}$$

(5.1) does not only depend on $y$ and $p$ but also on $x$. But the output does indeed depend on the states and those two can not be separated. One could replace $y$ with $Hx$ but $\pi$ and $\kappa$ are expressed in terms of the output vector $y$. So it makes more sense to substitute $x$ in terms of $y$. Therefore $H \in \mathbb{R}^{n_y\times n_x}$ is required to have full rank, which limits the class of systems that the method can be applied to. Thus a right inverse $\tilde{H} \in \mathbb{R}^{n_x\times n_y}$ of $H$ exists resp.

(II) $H \cdot \tilde{H} = I_{n_y}$.

To keep the homogeneity of degree two in all terms of (5.1), only the linear part of the output $y$, i. e. $y = Hx$, is taken. Using condition (I) it is

$$x = \tilde{H} y \tag{5.2}$$

in an $n_y$-dimensional subspace of $\mathbb{R}^{n_x}$. Since $x \mapsto HFx$ maps to $\mathbb{R}^{n_y}$ it contains the same information as $y \mapsto HF\tilde{H}y$. Now (5.1) is separated in three equations, starting with the part that is quadratic in terms of $y$.

$$\begin{aligned} 0 &= \pi_{yy} \cdot H \cdot \left(F\tilde{H} + GK\right) + (F\tilde{H} + GK)^{\mathrm{T}} \cdot H^{\mathrm{T}} \cdot \pi_{yy} \\ &\quad + \ell_{yy} + \ell_{yu}K + K^{\mathrm{T}}\ell_{yu} + K^{\mathrm{T}}\ell_{uu}K \\ &\overset{(K)}{=} \pi_{yy}HF\tilde{H} + \tilde{H}^{\mathrm{T}}F^{\mathrm{T}}H^{\mathrm{T}}\pi_{yy} + \ell_{yy} - \left(\ell_{yu} + \pi_{yy}HG\right) \cdot \ell_{uu}^{-1} \cdot \left(\ell_{yu}^{\mathrm{T}} + G^{\mathrm{T}}H^{\mathrm{T}}\pi_{yy}\right) \end{aligned}$$

Defining

$$\tilde{F} := HF\tilde{H} \quad \text{and} \quad \tilde{G} := HG \tag{$\tilde{F}|\tilde{G}$}$$

a continuous time algebraic Riccati equation is obtained.

$$0 = \pi_{yy}\tilde{F} + \tilde{F}^{\mathrm{T}}\pi_{yy} + \ell_{yy} - (\ell_{yu} + \pi_{yy}\tilde{G}) \cdot \ell_{uu}^{-1} \cdot (\ell_{yu}^{\mathrm{T}} + \tilde{G}^{\mathrm{T}}\pi_{yy}) \tag{$\pi_{yy}$}$$

This equation emits a positive definite solution $\pi_{yy} \in \mathbb{R}^{n_y \times n_y}$ if the following conditions hold.

(III) The second order of the cost function is convex with respect to $(y, u)$ and strictly convex with respect to $u$.

$$\begin{pmatrix} \ell_{yy} & \ell_{yu} \\ \ell_{yu}^{\mathrm{T}} & \ell_{uu} \end{pmatrix} \succeq 0, \quad \ell_{uu} \succ 0$$

(IV) The pair $\left(\tilde{F}, \tilde{G}\right)$ is stabilizable.

(V) The pair $(\tilde{F}, \ell_{yy})$ is detectable.

Since $\pi_{yy}$ is the solution of the Riccati equation, it is clear that all eigenvalues of $\tilde{F} + \tilde{G}K$ have a negative real part. As $\tilde{F} + \tilde{G}K$ describes the linear part of the output dynamics, it is stable.

$$\dot{y} = H\dot{x} = H \cdot (Fx + GKy) \overset{(5.2)}{=} (\tilde{F} + \tilde{G}K) \cdot y \tag{LinOut}$$

Going back to (5.1), the parts that are linear in terms of $y$ and linear in terms of $p$ are collected.

$$0 = \pi_{yy}\tilde{G}L + \left(\tilde{F}^{\mathrm{T}} + K^{\mathrm{T}}\tilde{G}^{\mathrm{T}}\right) \cdot \pi_{yp} + \ell_{yu}L + K^{\mathrm{T}}\ell_{uu}L$$

Applying ($L$-1) leads to the first condition towards $\pi_{yp} \in \mathbb{R}^{n_y \times n_p}$.

$$0 = \tilde{F}^{\mathrm{T}} \pi_{yp} - \left(\pi_{yy} \tilde{G} + \ell_{yu}\right) \cdot \ell_{uu}^{-1} \cdot \tilde{G}^{\mathrm{T}} \pi_{xp} \tag{5.3}$$

A second condition is obtained by taking all terms that are quadratic with respect to the parameters $p$.

$$0 = \pi_{yp}^{\mathrm{T}} \tilde{G} L + L^{\mathrm{T}} \tilde{G}^{\mathrm{T}} \pi_{yp} + L^{\mathrm{T}} \ell_{uu} L$$

Using the formula for $L$ leads to

$$0 = -\pi_{xp}^{\mathrm{T}} \tilde{G} \cdot \ell_{uu}^{-1} \cdot \tilde{G}^{\mathrm{T}} \pi_{yp}.$$

Since $\ell_{uu}^{-1}$ has full rank, this equation can only be true for

$$0 = \tilde{G}^{\mathrm{T}} \pi_{yp}. \tag{$\pi_{yp}$-1}$$

This information can be used to simplify (5.3) further.

$$0 = \tilde{F}^{\mathrm{T}} \pi_{yp} \tag{$\pi_{yp}$-2}$$

Taking ($\pi_{yp}$-1) and ($\pi_{yp}$-2), together with (I), the vanishing of the mixed term becomes clear.

$$\pi_{yp} = 0 \tag{$\pi_{yp}$}$$

As outlined before, $\pi_{pp}$ is also vanishing.
To find $\kappa^{[2]}(y,p)$ and $\pi^{[3]}(y,p)$, (HJBE-2)$^{[2]}$ and (HJBE-1)$^{[3]}$ are investigated.

$$\begin{aligned}
\text{(HJBE-2)}^{[2]}: \quad 0 = {} & y^{\mathrm{T}} \pi_{yy} \cdot \nabla_x h^{[2]}(x,p) \cdot G + y^{\mathrm{T}} \pi_{yy} H \cdot \nabla_u f^{[2]}(x, Ky, p) \\
& + \nabla_y \pi^{[3]}(y,p) \cdot \tilde{G} + \nabla_u \ell^{[3]}(y, Ky) \\
& + \left(\kappa^{[2]}(y,p) + K \cdot h^{[2]}(x,p)\right)^{\mathrm{T}} \cdot \ell_{uu}
\end{aligned}$$

Here $y$ is first replaced by $h(x,p)$ to cover the complete dynamics, and afterward, $x$ is substituted by $\tilde{H}y$. Thus a formula for $\kappa^{[2]}(y,p)$ is given as follows.

$$\begin{aligned}
\kappa^{[2]}(y,p) = -\ell_{uu}^{-1} \cdot \Big( & G^{\mathrm{T}} \cdot \nabla_x h^{[2]}(\tilde{H}y, p)^{\mathrm{T}} \cdot \pi_{yy} y + \nabla_u f^{[2]}(\tilde{H}y, Ky, p)^{\mathrm{T}} \cdot H^{\mathrm{T}} \pi_{yy} y \\
& + \tilde{G}^{\mathrm{T}} \cdot \nabla_y \pi^{[3]}(y,p)^{\mathrm{T}} + \nabla_u \ell^{[3]}(y, Ky)^{\mathrm{T}} \Big) - K \cdot h^{[2]}(\tilde{H}y, p)
\end{aligned} \tag{$\kappa^{[2]}$}$$

$\kappa^{[2]}$ depends on $\pi^{[3]}$, which is defined by (HJBE-1)$^{[3]}$.

$$
\begin{aligned}
\text{(HJBE-1)}^{[3]}: \quad 0 &= y^{\mathrm{T}}\pi_{yy} \cdot \Big(\nabla_x h^{[2]}(\tilde{H}y,p) \cdot (F\tilde{H}y + GKy) + H \cdot f^{[2]}(\tilde{H}y, Ky, p)\Big) \\
&+ y^{\mathrm{T}}\pi_{yy} \cdot \tilde{G}\kappa^{[2]}(y,p) + \nabla_y \pi^{[3]}(y,p) \cdot H \cdot (F\tilde{H}y + GKy) \\
&+ \ell^{[3]}(y, Ky) + y^{\mathrm{T}}\ell_{yu}\kappa^{[2]}(y,p) + y^{\mathrm{T}}K^{\mathrm{T}}\ell_{uu}\kappa^{[2]}(y,p)
\end{aligned}
$$

Using $(K)$ for simplification leads to a partial differential equation for $\pi^{[3]}(y,p)$.

$$
\begin{aligned}
0 &= \Big(\nabla_y \pi^{[3]}(y,p) + h^{[2]}(\tilde{H}y,p)^{\mathrm{T}} \cdot \pi_{yy}\Big) \cdot (\tilde{F} + \tilde{G}K) \cdot y + \ell^{[3]}(y, Ky) \\
&+ y^{\mathrm{T}}\pi_{yy} \cdot \Big(\nabla_x h^{[2]}(\tilde{H}y,p) \cdot (F\tilde{H}y + GKy) + H \cdot f^{[2]}(\tilde{H}y, Ky, p)\Big)
\end{aligned} \tag{$\pi^{[3]}$}
$$

($\pi^{[3]}$) has the structure as the equality in (7.5) from Corollary 8 in the next chapter. Furthermore, $\tilde{F} + \tilde{G}K$ is stable. Thus Corollary 8 guarantees a unique solution for the coefficients of $\pi^{[3]}$.
Based on the previous derivations the general case $\kappa^{[k]}(y,p)$ and $\pi^{[k+1]}(y,p)$ for $k \geq 2$ is considered.

$$
\begin{aligned}
\text{(HJBE-2)}^{[k]}: \quad 0 &= \nabla_y \pi\Big(h(\tilde{H}y,p),p\Big) \cdot \nabla_x h(\tilde{H}y,p) \cdot \nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)^{[k]} \\
&+ \Big[\nabla_u \ell\Big(h(\tilde{H}y,p), \kappa\Big(h(\tilde{H}y,p),p\Big)\Big)\Big]^{[k]} \\
&= \nabla_y \pi\Big(h(\tilde{H}y,p),p\Big) \cdot \nabla_x h(\tilde{H}y,p) \cdot \nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)^{[k]} \\
&+ \Big[\nabla_u \ell^{[3;k+1]}\Big(h(\tilde{H}y,p), \kappa\Big(h(\tilde{H}y,p),p\Big)\Big)\Big]^{[k]} + \kappa^{[k]}(y,p)^{\mathrm{T}} \cdot \ell_{uu} \\
&+ \Big(\Big[\kappa^{[1;k-1]}\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k]}\Big)^{\mathrm{T}} \cdot \ell_{uu}
\end{aligned}
$$

$\kappa^{[k]}(y,p)$ can be isolated, and a formula is given by ($\kappa^{[k]}$).

$$
\begin{aligned}
\kappa^{[k]}(y,p) = &-\ell_{uu}^{-1} \cdot \Big[\Big(\nabla_y \pi\Big(h(\tilde{H}y,p),p\Big) \cdot \nabla_x h(\tilde{H}y,p) \cdot \nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)\Big)^{\mathrm{T}}\Big]^{[k]} \\
&- \ell_{uu}^{-1} \cdot \Big[\nabla_u \ell^{[3;k+1]}\Big(h(\tilde{H}y,p), \kappa\Big(h(\tilde{H}y,p),p\Big)\Big)^{\mathrm{T}}\Big]^{[k]} \\
&- \Big(\Big[\kappa^{[1;k-1]}\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k]}\Big)
\end{aligned} \tag{$\kappa^{[k]}$}
$$

The only unknown is $\pi^{[k+1]}(y,p)$. It is given via (HJBE-1)$^{[k+1]}$.

$$
\begin{aligned}
0 &= \Big[\nabla_y \pi\Big(h(\tilde{H}y,p),p\Big) \cdot \nabla_x h(\tilde{H}y,p) \cdot f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)\Big]^{[k+1]} \\
&\quad + \Big[\ell\Big(h(\tilde{H}y,p), \kappa\Big(h(\tilde{H}y,p),p\Big)\Big)\Big]^{[k+1]} \\
&= \Big[\nabla_y \pi\Big(h(\tilde{H}y,p),p\Big) \cdot \nabla_x h(\tilde{H}y,p) \cdot f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)\Big]^{[k+1]} \\
&\quad + y^{\mathrm{T}} \pi_{yy} \tilde{G} \cdot \kappa^{[k]}(y,p) \\
&\quad + \Big[\ell^{[3;k+1]}\Big(h(\tilde{H}y,p), \kappa\Big(h(\tilde{H}y,p),p\Big)\Big)\Big]^{[k+1]} + y^{\mathrm{T}} \ell_{yu} \cdot \kappa^{[k]}(y,p) \\
&\quad + \sum_{i=2}^{k} h^{[i]}(\tilde{H}y,p)^{\mathrm{T}} \cdot \ell_{yu} \cdot \Big[\kappa\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k+1-i]} \\
&\quad + y^{\mathrm{T}} K^{\mathrm{T}} \cdot \ell_{uu} \cdot \kappa^{[k]}(y,p) \\
&\quad + \frac{1}{2} \sum_{i=2}^{k-1} \Big(\Big[\kappa\Big(h(\tilde{H}y,p),p\Big)\Big]^{[i]}\Big)^{\mathrm{T}} \cdot \ell_{uu} \cdot \Big[\kappa\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k+1-i]}
\end{aligned}
$$

$(K)$ is used for simplification, and $\nabla_y \pi^{[k+1]}(y,p)$ is separated.

$$
\begin{aligned}
0 &= \nabla_y \pi^{[k+1]}(y,p) \cdot (\tilde{F} + \tilde{G}K) \cdot y \\
&\quad + \Big[\nabla_y \pi^{[2;k]}\Big(h(\tilde{H}y,p),p\Big) \cdot \nabla_x h(\tilde{H}y,p) \cdot f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)\Big]^{[k+1]} \\
&\quad + \Big[\ell^{[3;k+1]}\Big(h(\tilde{H}y,p), \kappa\Big(h(\tilde{H}y,p),p\Big)\Big)\Big]^{[k+1]} \\
&\quad + \sum_{i=2}^{k} h^{[i]}(\tilde{H}y,p)^{\mathrm{T}} \cdot \ell_{yu} \cdot \Big[\kappa\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k+1-i]} \\
&\quad + \frac{1}{2} \sum_{i=2}^{k-1} \Big(\Big[\kappa\Big(h(\tilde{H}y,p),p\Big)\Big]^{[i]}\Big)^{\mathrm{T}} \cdot \ell_{uu} \cdot \Big[\kappa\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k+1-i]}
\end{aligned}
\tag{$\pi^{[k+1]}$}
$$

Corollary 8 from Section 7.2 ensures the unique solvability of $(\pi^{[k+1]})$ since all eigenvalues of $\tilde{F} + \tilde{G}K$ must have a negative real part. Thus $\kappa^{[k]}$ and $\pi^{[k+1]}$ can be found under the made assumptions.

**Remark 7.** *The conditions (II), (IV), and (V) do not define the right inverse matrix $\tilde{H} \in \mathbb{R}^{n_x \times n_y}$ uniquely. Furthermore, the substitution $x = \tilde{H}y$ can only be done for $x$ in an $n_y$-dimensional subspace of $\mathbb{R}^{n_x}$, which is defined by $\tilde{H}$. Thus the optimal feedback $\kappa(y,p)$ and value function $\pi(y,p)$ also depend on $\tilde{H}$ and are therefore not unique. The existence of $\tilde{H}$ will be addressed in Lemma 2 and Corollary 6. How one finds a suitable $\tilde{H}$ will be discussed in Chapter 7.*

**Theorem 5** (Determinability of $\pi$ and $\kappa$)**.**
*Consider the optimal control problem (OCP), where all functions are analytic with power series expansions ($f$), ($h$), and ($\ell$). Furthermore, let ($f_0$), ($h_0$), and the condi-*

*tions (II)-(V) be fulfilled. Then each part of the power series given in ($\pi$) and ($\kappa$) is uniquely defined. In addition, for all p in a neighborhood of the origin, it holds*

$$\pi(0,p) = 0, \quad \nabla_y \pi(0,p) = 0 \quad \textit{and} \quad \kappa(0,p) = 0.$$

*Proof.* It is sufficient to gain condition (I) out of (IV). In this case, the claim follows from the derivations itself. (IV) and the Hautus Lemma 3 imply

$$\operatorname{rank}\begin{pmatrix}\tilde{F} & \tilde{G}\end{pmatrix} = \begin{pmatrix}HF\tilde{H} & HG\end{pmatrix} = n_y.$$

The columns of $HF\tilde{H}$ are linear combinations of the columns of $HF$. Thus $\begin{pmatrix}HF & HG\end{pmatrix}$ must have had already the full rank, which is stated in (I). □

**Remark 8** (Additive parameters)**.**
*Similar to the state-feedback case, see Remark 3, additive parameters and constants in the dynamics and the output function can be handled using additional states and outputs. Considered is an optimal control problem as in (OCP) without the conditions ($f_0$) and ($h_0$). Therefore the system dynamics and the output function state as*

$$\dot{x} = f(x,u,p) = f_0 + Fx + Gu + Jp + f^{[2]}(x,u,p) + \dots, \tag{$f_p$}$$
$$y = h(x,p) = h_0 + Hx + Mp + h^{[2]}(x,p) + \dots, \tag{$h_p$}$$

*while the cost function in ($\ell$) stays unchanged. $f_0$ and $h_0$ are vectors in $\mathbb{R}^{n_x}$ resp. $\mathbb{R}^{n_y}$, whereas $J \in \mathbb{R}^{n_x \times n_p}$ and $M \in \mathbb{R}^{n_y \times n_p}$. As already known, with this setup, the OCP can not be solved iterative using Al'brekht's Method. Thus additional state and output variables are introduced to rewrite ($f_p$) and ($h_p$) such that ($f_0$) and ($h_0$) are fulfilled. As in Remark 3, a state $x_f$ is introduced, which will be multiplied with $f(0,0,p)$. Similar to that, another state $x_h$ is multiplied with $h(0,p)$. Since both variables need to be part of the control law and the value function, they need to appear in the output. Thus $y_f = x_f$ and $y_h = x_h$ are introduced. The new state and output vector are*

$$\bar{x} = \begin{pmatrix} x \\ x_f \\ x_h \end{pmatrix} \quad \textit{resp.} \quad \bar{y} = \begin{pmatrix} y \\ y_f \\ y_h \end{pmatrix},$$

*while the dynamics is given by*

$$\dot{\bar{x}} = \begin{pmatrix} \dot{x} \\ \dot{x}_f \\ \dot{\bar{x}}_h \end{pmatrix} = \begin{pmatrix} f(0,0,p)\cdot x_f + Fx + Gu + f^{[2]}(x,u,p) - f^{[2]}(0,0,p) + \dots \\ -\alpha_f \cdot x_f \\ -\alpha_h \cdot x_h \end{pmatrix}.$$

*As in the state-feedback case in Section 3.1, $\alpha_f$ and $\alpha_h$ have to be positive to ensure the stability of $x_f$ resp. $x_h$ and, therefore, the solvability of the CARE later on. The* *extended output function is shown in the following equation.*

$$\bar{y} = \begin{pmatrix} y \\ y_f \\ y_h \end{pmatrix} = \begin{pmatrix} h(0,p) \cdot x_h + Hx + h^{[2]}(x,p) - h^{[2]}(0,p) + \dots \\ x_f \\ x_h \end{pmatrix}$$

*Replacing* $f(0,0,p)$ *and* $h(0,p)$ *with their power series,*

$$\begin{aligned} f(0,0,p) &= f_0 + Jp + f^{[2]}(0,0,p) + \dots, \\ h(0,p) &= h_0 + Mp + h^{[2]}(0,p) + \dots \end{aligned}$$

*leads to a new representation of the different polynomial degrees of the dynamics and the output function.*

$$\begin{aligned} \dot{\bar{x}} = & \underbrace{\begin{pmatrix} F & f_0 & 0 \\ 0 & -\alpha_f & 0 \\ 0 & 0 & -\alpha_h \end{pmatrix}}_{=: \bar{F}} \cdot \begin{pmatrix} x \\ x_f \\ x_h \end{pmatrix} + \underbrace{\begin{pmatrix} G \\ 0 \\ 0 \end{pmatrix}}_{=: \bar{G}} \cdot u \\ & + \underbrace{\begin{pmatrix} f^{[2]}(x,u,p) - f^{[2]}(0,0,p) + Jp \cdot x_f \\ 0 \\ 0 \end{pmatrix}}_{=: \bar{f}^{[2]}(\bar{x},u,p)} + \dots, \end{aligned} \tag{$\bar{f}$}$$

$$\bar{y} = \underbrace{\begin{pmatrix} H & 0 & h_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=: \bar{H}} \cdot \bar{x} + \underbrace{\begin{pmatrix} h^{[2]}(x,p) - h^{[2]}(0,p) + Mp \cdot x_h \\ 0 \\ 0 \end{pmatrix}}_{=: h^{[2]}(\bar{x},p)} + \dots \tag{$\bar{h}$}$$

*Having two additional output variables,* $y_f$ *and* $y_h$*, the cost function is rewritten as the following.*

$$\bar{\ell}(\bar{y},u) = \bar{y}^{\mathrm{T}} \underbrace{\begin{pmatrix} \ell_{yy} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=: \ell_{\bar{y}\bar{y}}} \bar{y} + \bar{y}^{\mathrm{T}} \underbrace{\begin{pmatrix} \ell_{yu} \\ 0 \\ 0 \end{pmatrix}}_{=: \ell_{\bar{y}u}} u + \frac{1}{2} u^{\mathrm{T}} \ell_{uu} u + \ell^{[3]}(\bar{y},u) + \dots = \ell(y,u) \tag{$\bar{\ell}$}$$

*It remains to find/define a matrix* $\bar{\tilde{H}} \in \mathbb{R}^{(n_x+2)\times(n_y+2)}$ *with the properties (II), (IV), and (V). One may choose*

$$\bar{\tilde{H}} = \begin{pmatrix} \tilde{H} & 0 & -\tilde{H} \cdot h_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{$\bar{\tilde{H}}$}$$

*where $\tilde{H}$ is such that (II), (IV), and (V) are fulfilled using $F$ and $G$. With this particular choice, one has*

$$\bar{H} \cdot \tilde{\bar{H}} = I_{n_y+2}$$

*and, therefore, condition (II) is satisfied. Furthermore, it is*

$$\tilde{\bar{F}} = \bar{H} \cdot \bar{F} \cdot \tilde{\bar{H}} = \begin{pmatrix} \tilde{F} & H \cdot f_0 & -(\alpha_h + \tilde{F}) \cdot h_0 \\ 0 & -\alpha_f & 0 \\ 0 & 0 & -\alpha_h \end{pmatrix} \quad \text{and} \quad \tilde{\bar{G}} = \bar{H} \cdot \bar{G} = \begin{pmatrix} \tilde{G} \\ 0 \\ 0 \end{pmatrix},$$

*where $\tilde{F} = H \cdot F \cdot \tilde{H}$ and $\tilde{G} = H \cdot G$. Choosing $\alpha_f$ and $\alpha_h$ positive ensures the properties (IV) and (V) using $\tilde{\bar{H}}$ as in ($\tilde{\bar{H}}$).*
*The value function and the control law can now be stated as power series in terms of the new output vector $\bar{y}$ and the parameters $p$. As done for the state-feedback case, the new variables $y_f$ and $y_h$ are always set to 1, when the control law is applied. In this case, the first $n_x$ rows of the system dynamics ($\bar{f}$) are the same as in ($f_p$), while the first $n_y$ rows of the output function ($\bar{h}$) are the same as in ($h_p$).*

$$\bar{\pi}(\bar{y}, p) = \frac{1}{2}\bar{y}^{\mathrm{T}} \underbrace{\begin{pmatrix} \pi_{yy} & \pi_{yy_f} & \pi_{yy_h} \\ \pi_{yy_f}^{\mathrm{T}} & \pi_{y_f y_f} & \pi_{y_f y_h} \\ \pi_{yy_h}^{\mathrm{T}} & \pi_{y_f y_h}^{\mathrm{T}} & \pi_{y_h y_h} \end{pmatrix}}_{=:\, \pi_{\bar{y}\bar{y}}} \bar{y} + \bar{\pi}^{[3]}(\bar{y}, p) + \ldots \tag{$\bar{\pi}$}$$

$$\bar{\kappa}(\bar{y}, p) = \underbrace{\begin{pmatrix} K & K_f & K_h \end{pmatrix}}_{=:\, \bar{K}} \cdot \bar{y} + \bar{\kappa}^{[2]}(\bar{y}, p) + \ldots \tag{$\bar{\kappa}$}$$

*With these definitions, all assumptions are fulfilled, and the Hamilton-Jacobi-Bellman equation and its derivative with respect to the input $u$ are solved degree-wise, as shown at the beginning of this section. The matrices $\pi_{yy} \in \mathbb{R}^{n_y \times n_y}$ and $K \in \mathbb{R}^{n_u \times n_y}$ are found as in the nominal case, see ($\pi_{yy}$) and ($K$). The other parts of the second order of the value function can be calculated uniquely because all conditions are holding. The vectors $K_f \in \mathbb{R}^{n_u}$ and $K_h \in \mathbb{R}^{n_u}$ are given by*

$$K_f = -\ell_{uu}^{-1} \cdot \tilde{G}^{\mathrm{T}} \pi_{yy_f}. \tag{$K_f$}$$

$$\text{and } K_h = -\ell_{uu}^{-1} \cdot \tilde{G}^{\mathrm{T}} \pi_{yy_h}. \tag{$K_h$}$$

*Higher orders are solved accordingly. Their additional dependencies are not further investigated here.*

**Corollary 4** (Local stability)**.**
*Under the requirements of Theorem 5, local stability of the output is achieved for sufficiently small parameters $p$, if the power series ($\pi$) and ($\kappa$) are converging.*

*Proof.* For a fixed parameter vector $p \in \mathbb{R}^{n_p}$, a local Lyapunov function candidate is given by $\tilde{\pi}(y) := \pi(y,p)$. Analogously to the value function, the control law is defined as $\tilde{u}_{\min}(y) := \tilde{\kappa}(y) := \kappa(y,p)$. Both functions are the solution of

$$\begin{aligned} \tilde{\pi}\big(y(0)\big) &= \min_{u(.)} \int_0^\infty \ell\big(y(\tau), u(\tau)\big)\,\mathrm{d}\tau \\ \text{s.t. } \dot{x} &= \tilde{f}(x,u), \\ y &= \tilde{h}(x), \end{aligned}$$

where $\tilde{f}(x,u) := f(x,u,p)$ and $\tilde{h}(x) = h(x,p)$. If $p$ is sufficiently small, $\tilde{f}^{[1]}(.,.)$ resp. $\tilde{h}(.)$ inherits the properties of $f^{[1]}(.,.,.)$ resp. $h(.,.)$. Therefore, $\tilde{\pi}(.)$ and $\tilde{\kappa}(.)$ can be written as

$$\begin{aligned} \tilde{\pi}(y) &= \frac{1}{2} y^{\mathrm{T}} \tilde{\pi}_{yy} y + o(||y||^3), \\ \tilde{\kappa}(y) &= \tilde{K} y + o(||y||^2). \end{aligned}$$

Thus there exists an $\varepsilon > 0$ such that

$$\tilde{\pi}(y) \geq 0$$

for all $y \in B_\varepsilon(0)$. Equality only holds for vanishing $y$ since $\tilde{\pi}_{yy}$ must be positive definite. Therefore, $\tilde{\pi}(.)$ is locally positive definite. Using the Hamilton-Jacobi-Bellman equation,

$$0 = \nabla_y \tilde{\pi}(y) \cdot \nabla_x \tilde{h}(x) \cdot f\big(x, \tilde{\kappa}(y)\big) + \ell\big(y, \tilde{\kappa}(y)\big), \tag{HJBE}$$

it is seen that $\dot{\pi}\big(y(t)\big)$ is locally negative definite.

$$\begin{aligned} \dot{\tilde{\pi}}\big(y(t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\pi}\big(y(t)\big) = \nabla_y \tilde{\pi}\big(y(t)\big) \cdot \nabla_x \tilde{h}\big(x(t)\big) \cdot \tilde{f}\big(x(t), \tilde{\kappa}(y(t))\big) \overset{\text{(HJBE)}}{=} -\ell\big(y(t), \tilde{\kappa}(y(t))\big) \\ &= -\frac{1}{2} y(t)^{\mathrm{T}} \ell_{yy} y(t) - y(t)^{\mathrm{T}} \ell_{yu} \tilde{K} y(t) - \frac{1}{2} y(t)^{\mathrm{T}} \tilde{K}^{\mathrm{T}} \ell_{uu} \tilde{K} y(t) + o\big(||y(t)||^3\big) \end{aligned}$$

Since $\ell^{[2]}(y, \tilde{K}y)$ is positive definite, there exists an $\varepsilon > 0$ such that

$$\dot{\tilde{\pi}}\big(y(t)\big) \leq 0$$

for all $y(t) \in B_\varepsilon(0)$. Again equality holds only for $y(t) = 0$. Thus $\tilde{\pi}(y) = \pi(y,p)$ is a local Lyapunov function for

$$\dot{y} = \nabla_x \tilde{h}(x) \cdot \tilde{f}\big(x, \tilde{\kappa}(y)\big) = \nabla_x h(x,p) \cdot f\big(x, \kappa(y,p), p\big)$$

and fixed $p \in \mathbb{R}^{n_p}$. If $p \in \mathbb{R}^{n_p}$ varies but stays in $\overline{B_\delta(0)}$ for a sufficiently small $\delta > 0$, then $y \mapsto \pi(y,p)$ is used as local Lyapunov function. In the previous case, $y \mapsto \pi(y,p)$ was a local Lyapunov function for $y \in B_\varepsilon(0)$. Clearly, $\varepsilon$ depends on $p$.

Since $p \mapsto \pi(y,p)$ is $C^\infty$, one can see that $p \mapsto \varepsilon(p)$ is continuous and thus takes its minimum value in the compact set $\overline{B_\delta(0)}$. This minimum is denoted with $\varepsilon_{\min}$ and must be greater than zero. Else there would be a parameter $p$ that contradicts the first case. Therefore, $y \mapsto \pi(y,p)$ is a Lyapunov function for $y \in B_{\varepsilon_{\min}}(0)$. $\square$

**Remark 9.** *If in the optimal output-feedback problem (OCP) the dimension $n_y$ of the output is larger than the state dimension $n_x$, then the maximal rank that the matrix $H \in \mathbb{R}^{n_y \times n_x}$ could have is $n_x$ and (5.1) is not solvable, since $H$ does not have a right inverse. $\pi_{yy}$ can not be obtained. Calculating $H^{\mathrm{T}}\pi_{yy}H$ is possible if the rank of $H$ is $n_x$. But in this case, all states would be observable, and there is no need for output-feedback.*

## Simulations and evaluation

The quadcopter and the bioreactor examples from Chapter 3 are used again to show the capabilities of the extension of Al'brekht's Method. In the first example, different choices of $\tilde{H}$ are considered. In the last example, the behavior of a nonlinear output function with additive parameters is investigated.

**Example 9** (Quadcopter: 10 states, 6 parameters, 6 output variables)**.**
*The system dynamics that have been used in Example 3 are extended by an output function. There are no inequalities included. It is assumed that*

$$y = \begin{pmatrix} p_z & v_x & v_y & v_z & v_\phi & v_\theta \end{pmatrix}^{\mathrm{T}}$$

*can be measured. In other words, all velocities and the altitude can be obtained. In reality, this model relates to a quadcopter, which lost GPS and sensors signals to reconstruct the roll and pitch angles. In this case, the output-feedback solution can be used as a fallback controller as done in [43]. Figure 5.2 shows the control scheme. If the full state information is available, a nominal controller is used. As soon as a fault is detected, the quadcopter switches to the fallback, i. e. the output-feedback, controller which e. g. reduces the drift due to the wind and lands safely.*

*The cost function*

$$\ell(y, u_T) = \frac{1}{2} y^{\mathrm{T}} \ell_{yy} y + \frac{1}{2} u_T^{\mathrm{T}} \ell_{uu} u_T,$$

*where* $\ell_{yy} = \operatorname{diag}(1,1,1,1,100,100)$ *and* $\ell_{uu} = \frac{1}{10} \cdot I_3$ *is used. It remains to choose a*

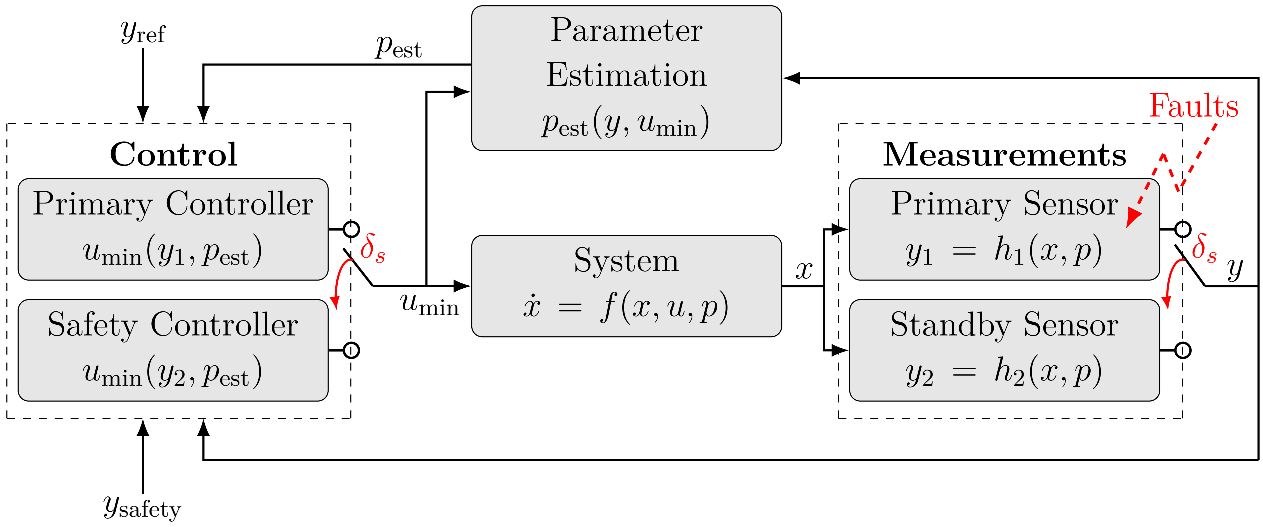


**Figure 5.2:** Fallback control scheme

*suitable matrix $\tilde{H}$. To this end, two different choices of $\tilde{H}$ are investigated.*

$$\tilde{H}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{H}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \underline{1} & 0 & \underline{1} & 0 & \underline{1} & 0 \\ 0 & \underline{1} & 0 & \underline{1} & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

*The not underlined entries are forced by the condition $H \cdot \tilde{H} = I_6$. The other entries are chosen to satisfy the remaining conditions. Clearly, $\tilde{H}_1$ is a simpler choice and thus also leads to a simpler polynomial representation of the control law.*
*Fig. 5.3 to 5.6 show the closed-loop simulation starting from initial condition $x_0 = \begin{pmatrix} 0 & 0 & 25 & 5 & 0 & 0 & 0 & 0.1745 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$. While the same time-dependent parameters*

$$\omega_{xx} = \frac{1}{5}, \ \omega_{xy} = \omega_{xz} = 0$$
$$\omega_{yx} = -\frac{1}{5} \cdot t, \ \omega_{yy} = \omega_{yz} = 0$$

*are used, completely different behaviours are obtained. The control law has been approximated up to degree four, and the value function up to degree five. Therefore, in principle 11 632 coefficients had to be determined. Due to the weak nonlinearities of*

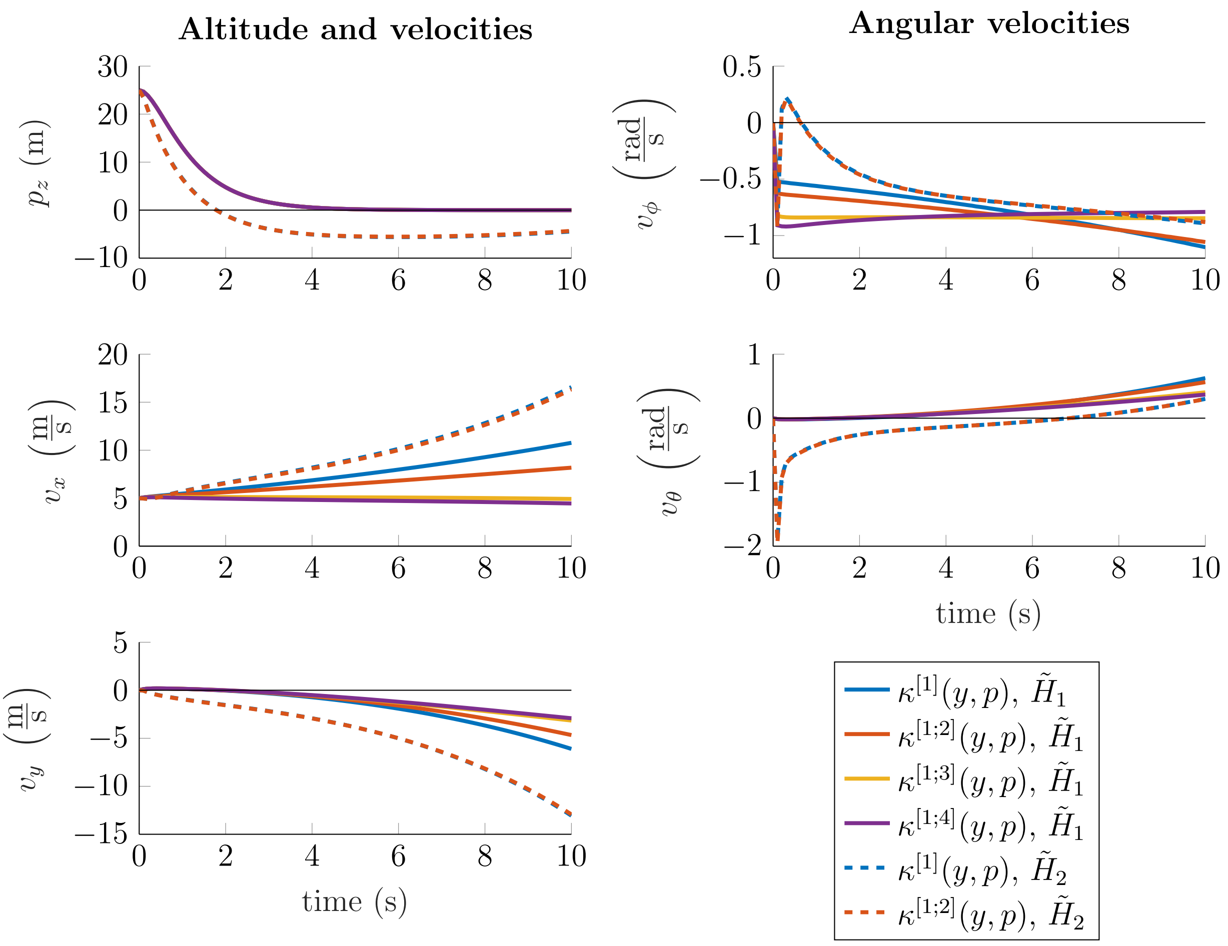


**Figure 5.3:** Propagation of five out of six output variables (10 states model with parameters, output-feedback)

*the system only* 697 *coefficients are non-zero. As one can see in Figure 5.4, the drift of the quadcopter is reduced when the approximation degree is increased. The performance of the controllers based on* $\tilde{H}_2$ *is worse than all controllers based on* $\tilde{H}_1$*. Not only is the drifting distance is twice as much but they also fail to efficiently stabilize the altitude* $p_z$*, see Fig. 5.3. For the control laws based on* $\tilde{H}_1$*, one can see an increasing magnitude of the control input (Fig. 5.5), especially for* $u_\theta$*. This behavior is a reaction to the increasing wind. Since the drift is reduced the overall cost is also decreased using better approximations (Fig. 5.6).*

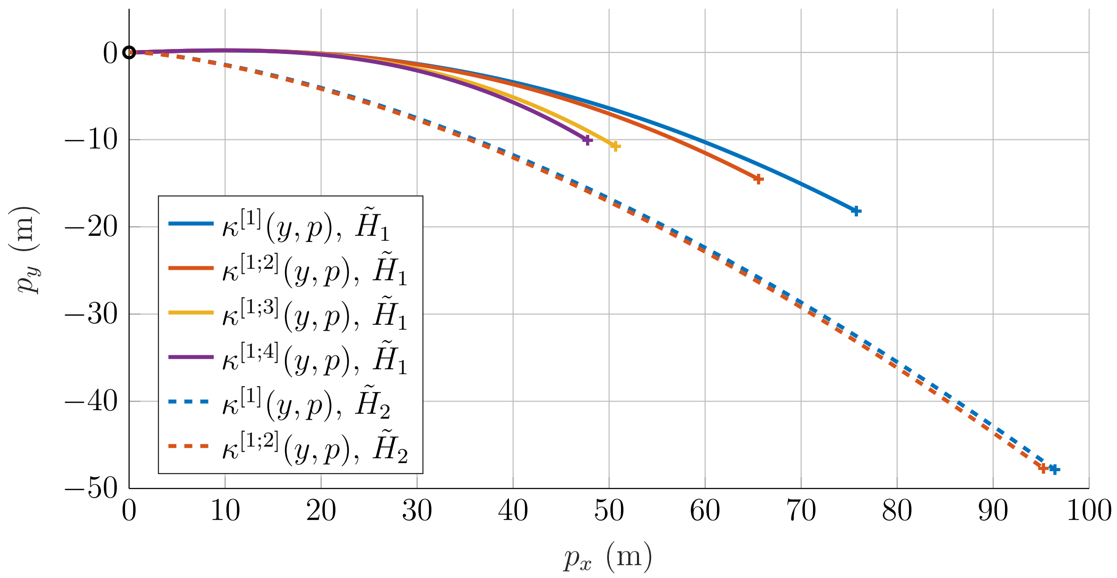


**Figure 5.4:** Position of the quadcopter in the $x$-$y$-plane (10 states model with parameters, output-feedback)

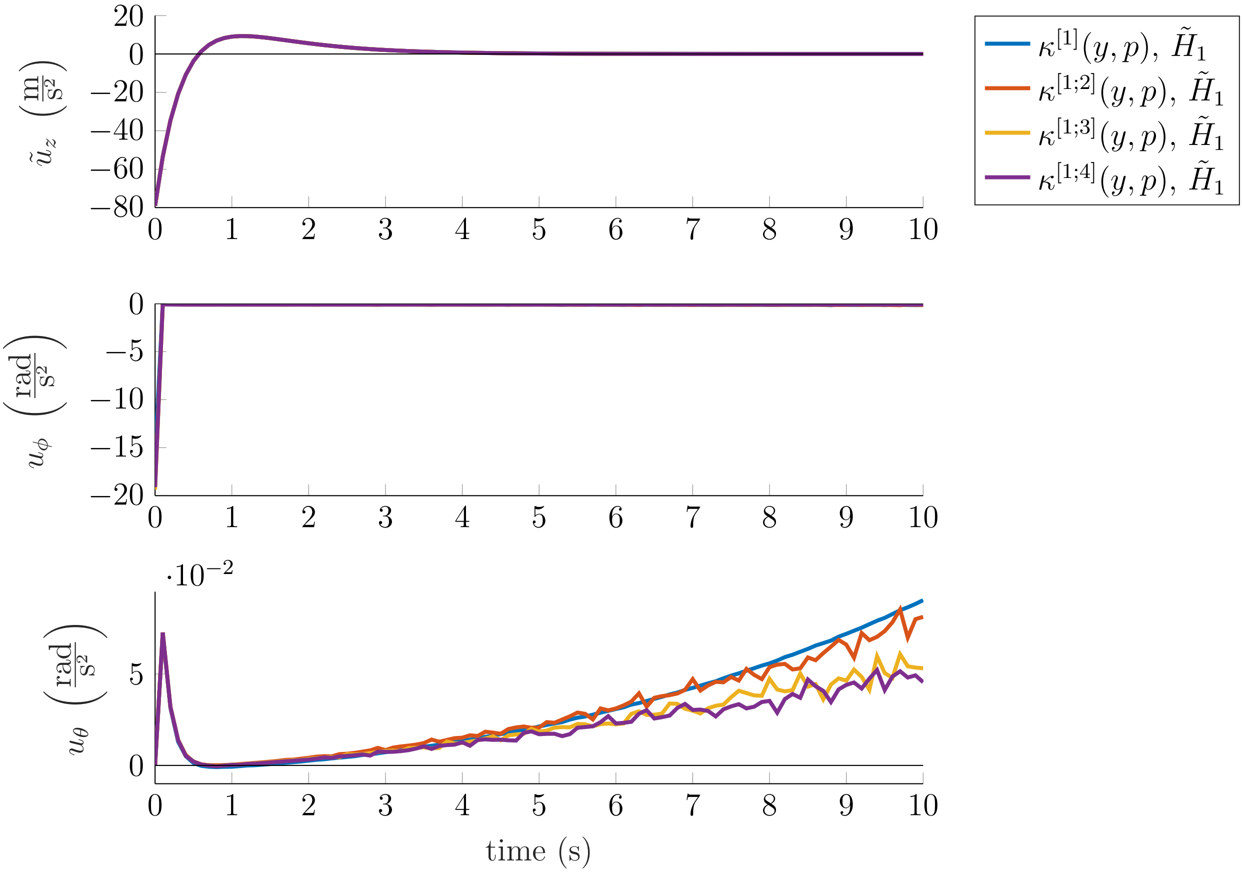


**Figure 5.5:** Control input for the vertical and angular accelerations (10 states model with parameters, output-feedback)

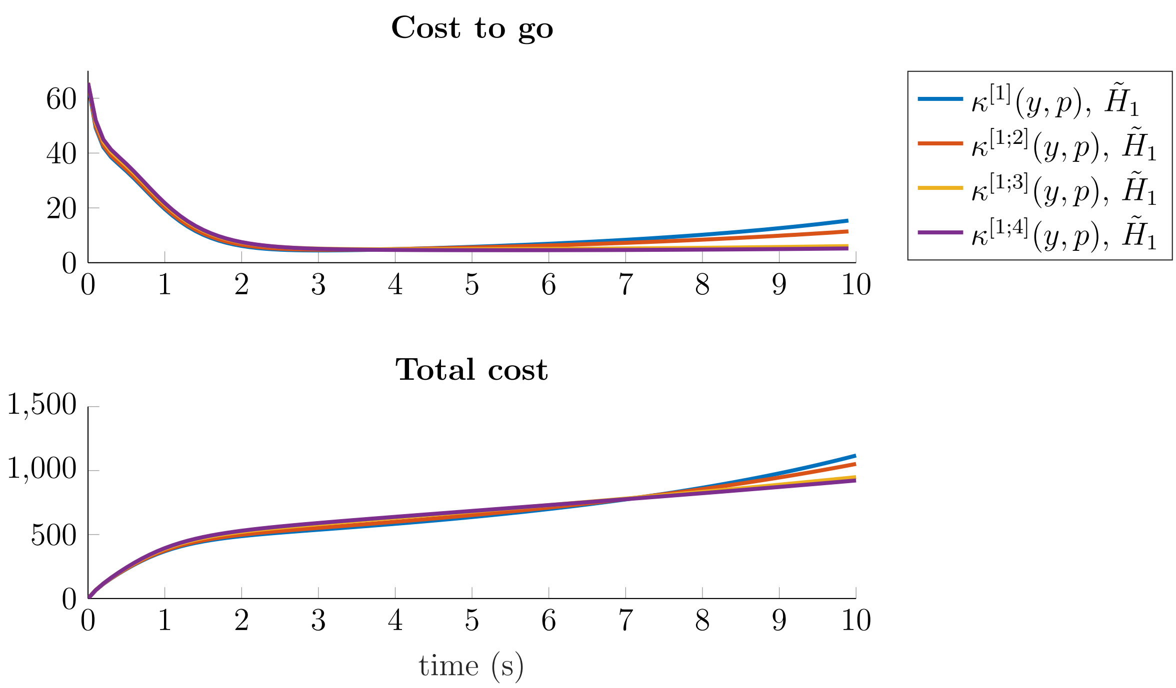


**Figure 5.6:** Resulting cost (10 states model with parameters, output-feedback)

**Example 10** (Quadcopter: 12 states, 6 parameters, 7 output variables)**.**
*For the quadcopter model with 12 states, the output is assumed to contain the seven measurements*

$$y = \begin{pmatrix} p_z & v_x & v_y & v_z & v_\phi & v_\theta & v_\psi \end{pmatrix}^{\mathrm{T}}.$$

*The needed right inverse matrix is chosen as*

$$\tilde{H} = \begin{pmatrix} 0 & 0 & \underline{1} & 0 & \underline{1} & 0 & 0 \\ 0 & 0 & 0 & \underline{1} & 0 & \underline{1} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \underline{1} & 0 & 0 & 0 & \underline{1} & 0 & 0 \\ 0 & \underline{1} & 0 & 0 & 0 & \underline{1} & 0 \\ 0 & 0 & \underline{1} & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

*where the not underlined entries are given via the condition $H \cdot \tilde{H} = I_7$. The remaining* *entries guarantee the conditions (IV) and (V). The parameters are chosen as*

$$\omega_{xx} = \frac{1}{5},\ \omega_{xy} = \omega_{xz} = 0$$
$$\omega_{yx} = -\frac{1}{50} \cdot t,\ \omega_{yy} = \omega_{yz} = 0,$$

*while the cost function is given by*

$$\ell(y, u_T) = \frac{1}{2} y^{\mathrm{T}} \ell_{yy} y + \frac{1}{2} u_T^{\mathrm{T}} \ell_{uu} u_T,$$

*where* $\ell_{yy} = \mathrm{diag}(1, 1, 1, 1, 100, 100, 100)$ *and* $\ell_{uu} = \frac{1}{10} \cdot I_4$ *are used. The initial values are* $p_z = 1\,\mathrm{m}$ *and* $v_x = 0.2\,\mathrm{m\,s^{-1}}$*, while the other states are set to zero. In the considered scenario, the wind becomes stronger over time, which then leads to higher velocities in* $x$*- and* $y$*-direction (Fig. 5.7) and less stable behavior. The velocities can not be stabilized at zero. When the wind becomes very strong after* $6\,\mathrm{s}$ *the altitude is also affected. If one is only interested in the drift of the quadcopter then* $\kappa^{[1;2]}$ *would be the best choice for our scenario, see Fig. 5.8. The other two control laws result in a bigger distance to the origin. The results can differ with different initial values or parameters. The applied control action at the start of the simulation shows the same disadvantage of the power series approach as in the previous examples. The values can be unreasonably large, see Fig. 5.9. According to the total cost (Fig. 5.10), the third-order approximation leads to the worst performance due to the cost induced by the velocity in* $x$*-direction.*

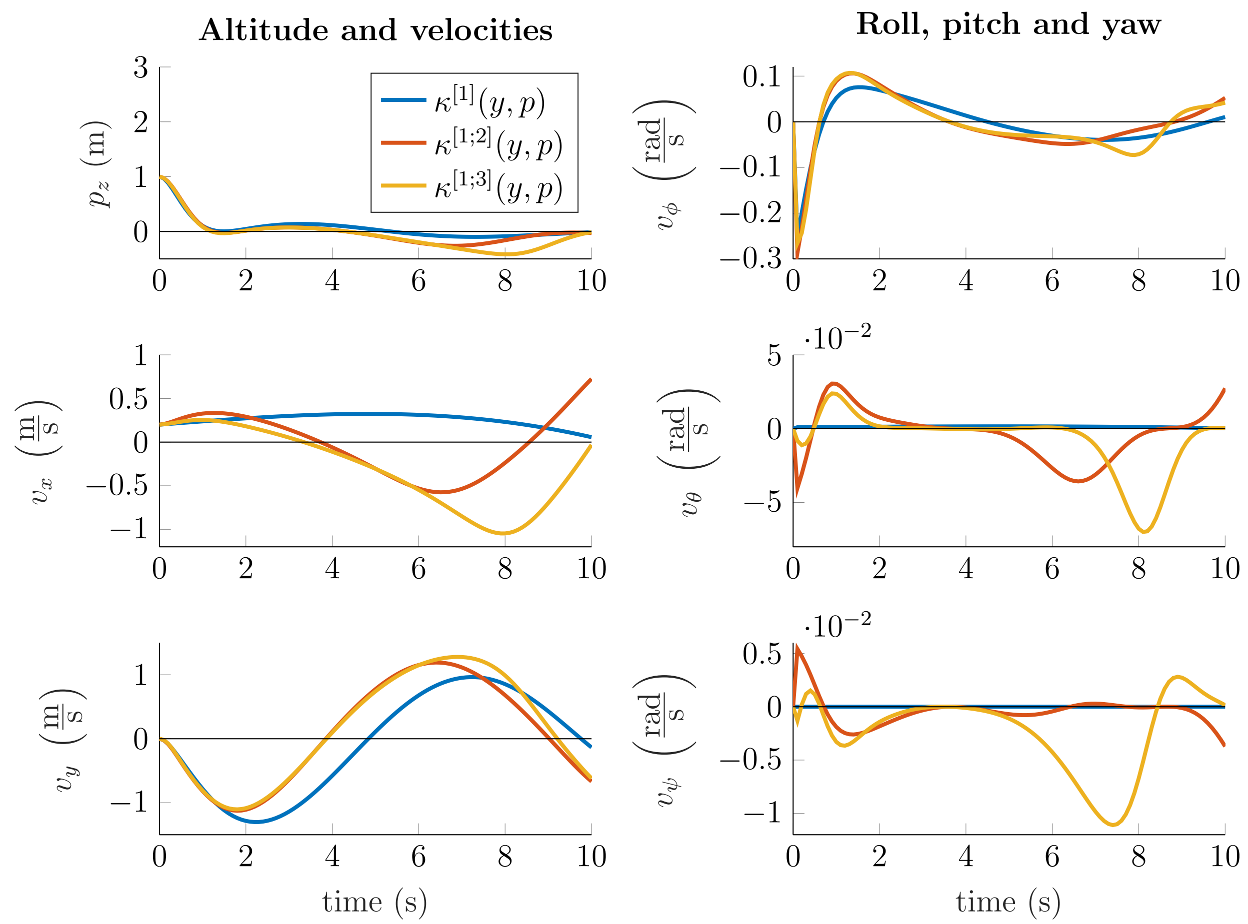


**Figure 5.7:** Propagation of six out of seven output variables (12 states model with parameters, output-feedback)

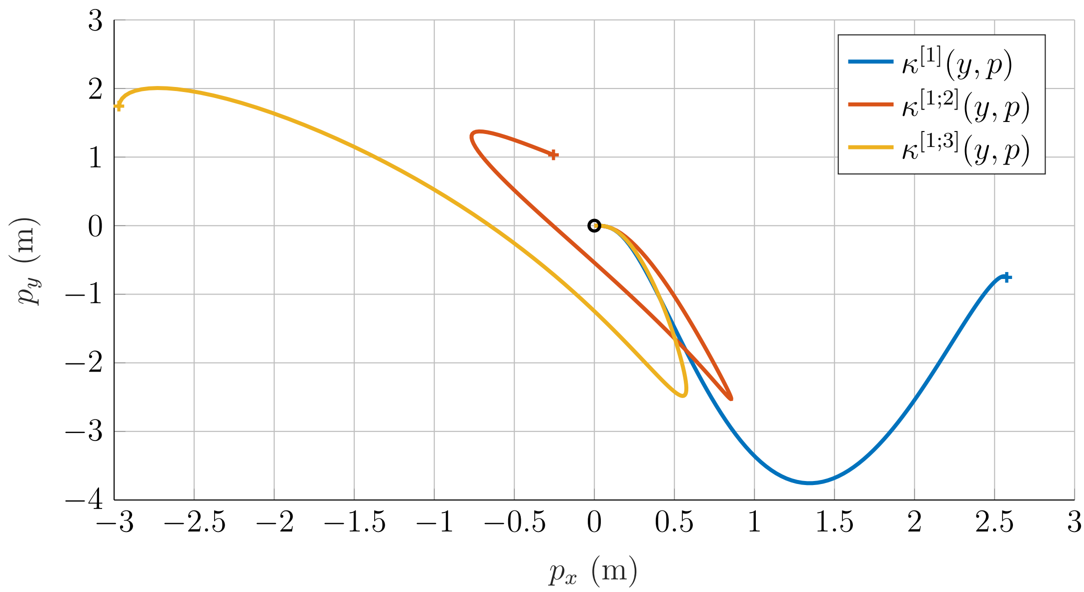


**Figure 5.8:** Position of the quadcopter in the $x$-$y$-plane (12 states model with parameters, output-feedback)

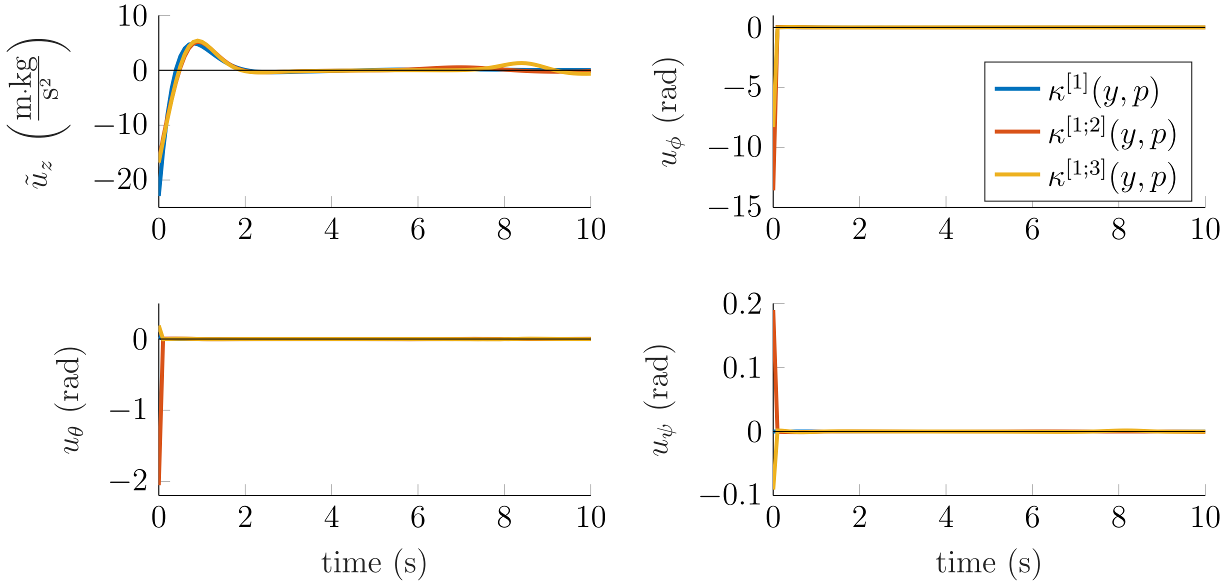


**Figure 5.9:** Control input for the thrust and the moments about the axis (12 states model with parameters, output-feedback)

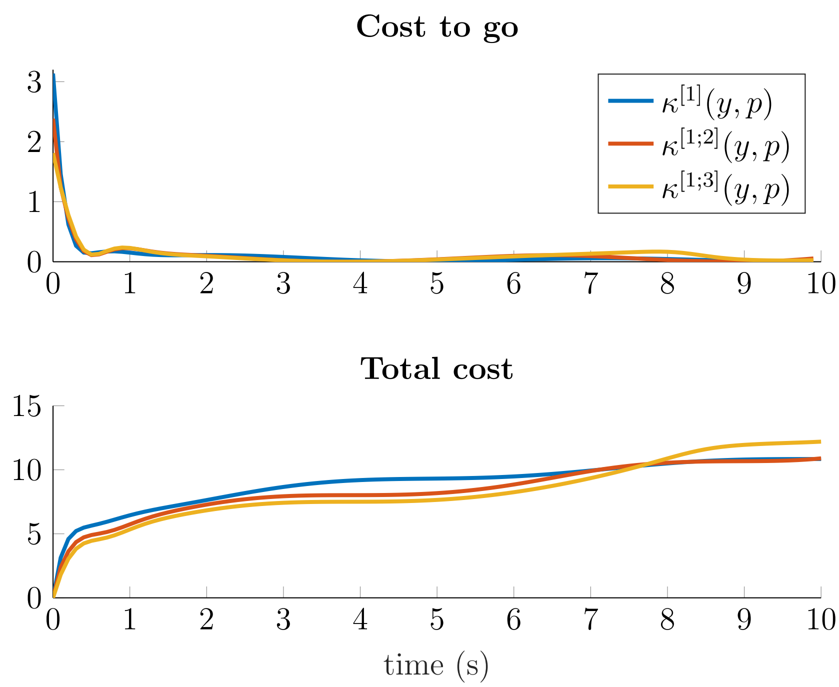


**Figure 5.10:** Resulting cost (12 states model with parameters, output-feedback)

**Example 11** (Bioreactor rhodospirillum rubrum, 3 states, 2 parameters, 1 output variable).
*Example 5 from Section 3.1 is revisited. The parameters $p$ are set to $p_{ave}$ and are now fixed. Instead of having the full state information, it is assumed that only the biomass can be measured. The measurement information is given via*

$$y_{T,b} = h(x_T, p_{out}) = p_{const} + (1 + p_{lin}) \cdot x_{T,b}.$$

*The output function contains two parameters $p_{out} = \begin{pmatrix} p_{const} & p_{lin} \end{pmatrix}^{\mathrm{T}}$ indicating measurement noise. The matrix $\tilde{H} \in \mathbb{R}^{3\times 1}$ is chosen as $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$. $\alpha_f$ and $\alpha_h$ are set to $0.01$. The cost function is*

$$\ell(y_{T,b}, u_T) = \frac{1}{2} y_{T,b}^2 + \frac{1}{200} u_T^2$$

*and the initial values $x_{T,b} = -0.1\,\mathrm{g\,L^{-1}}$, $x_{T,s} = -0.5\,\mathrm{g\,L^{-1}}$, and $x_{T,f} = -0.1\,\mathrm{g\,L^{-1}}$. The parameters are set to $p_{const} = 1 + \frac{1}{50}\sin(t)\,\mathrm{g\,L^{-1}}$ and $p_{lin} = \frac{1}{50}\sin(t)$, representing a permanent offset and minor noise. The measurement noise is not estimated in this scenario. Therefore, $p_{out}$ is set to zero inside the controller, and higher-order approximations lose one of their advantages. Not surprisingly, all control laws are able to keep the system output (Fig. 5.11) stable. All three states are also stabilized but can not reach the desired points. The cost shows a phenomenon that is often observed using Al'brekht's Method. In this case, the approximations up to odd degrees show a better performance than the even ones. Applying $\kappa^{[1;5]}$ produces the least cost (Fig. 5.12) and is, therefore, throughout this scenario, slightly cheaper than $\kappa^{[1;3]}$ and $\kappa^{[1]}$. Nevertheless, this may change when different parameters or initial values are used.*

## 5.2 Convergence proof for continuous-time output-feedback control

During this section, the convergence proofs, which have been done in Section 2.2 for the nominal case and in Section 3.2 for the parametric case, will be generalized for the output-feedback case with fixed but arbitrary model parameters. To prove the convergence, the optimal cost function is upper bounded via the system dynamics, the cost function, and the linear part of the output function. The main obstacle one has to overcome is to work in the output space and only allowing states in the submanifold of the state space, which is defined by $\tilde{H}$. Once the upper bounds are found, a dominating converging series is constructed using the same idea as in the proofs of Theorems 2.2 and 3.2. At several instances, there appears an additional factor representing the spectral norm of the matrix defining the linear part of the output function. Thus the result looks more complex, but the idea remains the same.

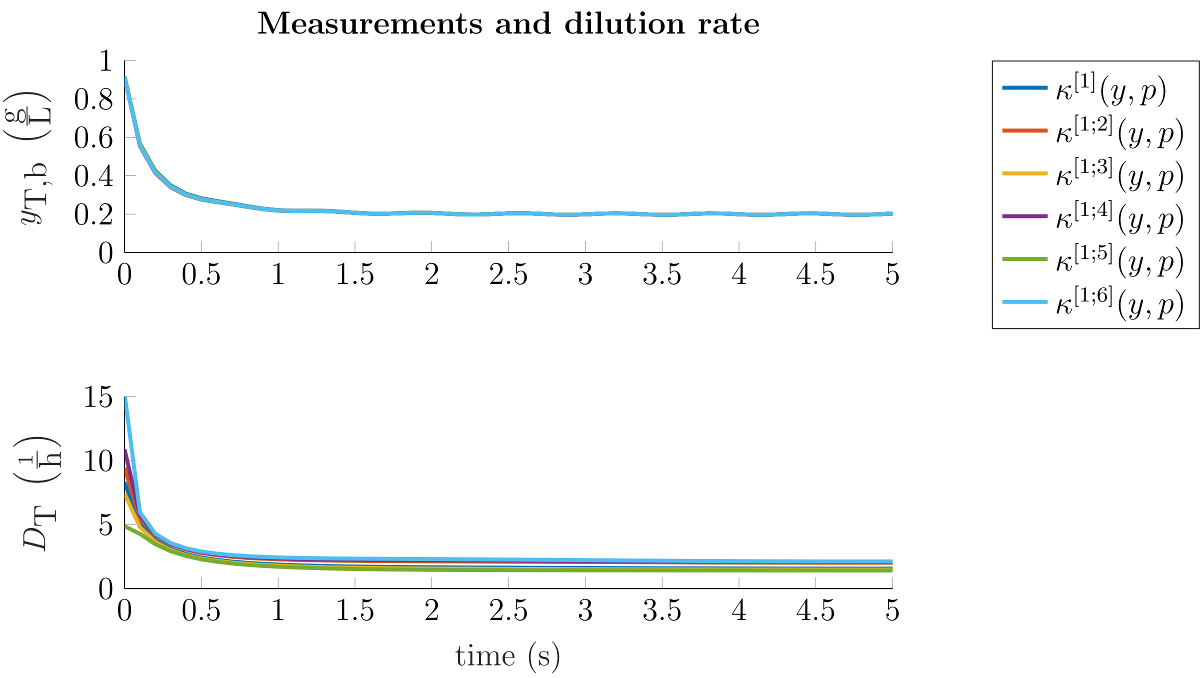


**Figure 5.11:** Measurement of the biomass concentration and dilution rate (bioreactor with parameters, output-feedback)

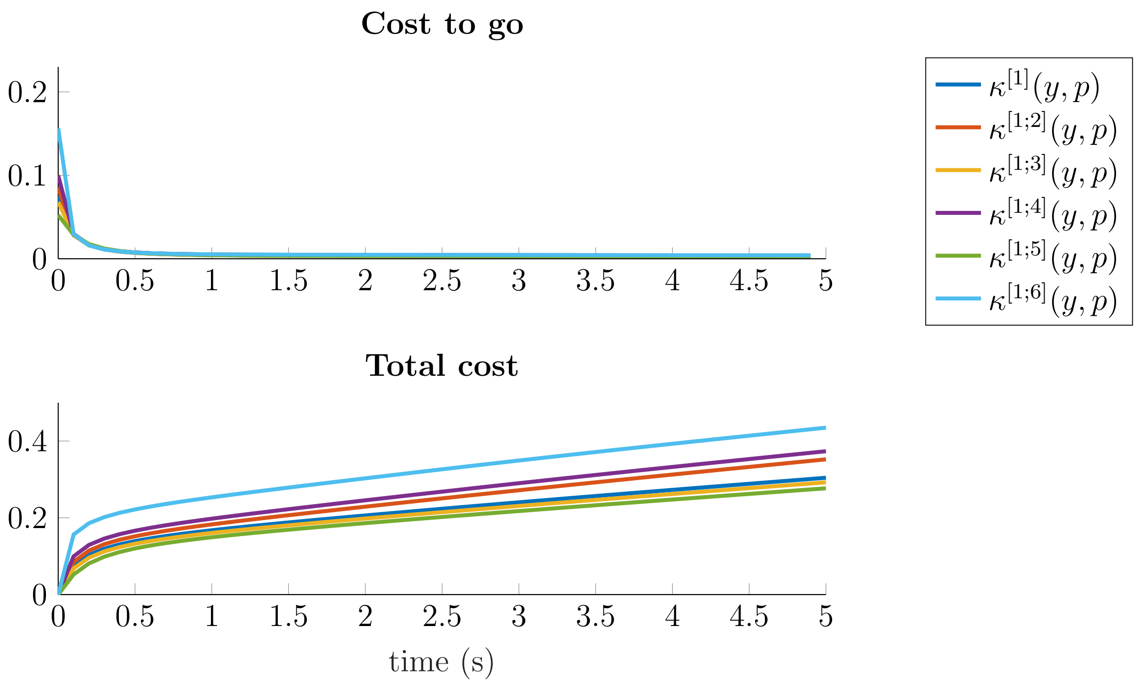


**Figure 5.12:** Resulting cost (bioreactor with parameters, output-feedback)

The considered optimal control problem states as the following.

$$\begin{aligned} \pi\big(y(0),p\big) &= \min_{u(.)} \int_0^\infty \ell\big(y(\tau),u(\tau)\big)\,\mathrm{d}\tau \\ \text{s.t. } \dot{x} &= f(x,p) + Gu \\ y &= Hx \end{aligned} \tag{OCP}$$

In this case, the system dynamics is linear with respect to the input variables $u \in \mathbb{R}^{n_u}$, the output $y \in \mathbb{R}^{n_y}$ is linear with respect to the states $x \in \mathbb{R}^{n_x}$, and the cost function

$$\ell(y,u) = \frac{1}{2}y^{\mathrm{T}}\ell_{yy}y + y^{\mathrm{T}}\ell_{yu}u + \frac{1}{2}u^{\mathrm{T}}\ell_{uu}u \tag{$\ell$}$$

is quadratic, thus does not contain terms of higher order. The function $f\colon \mathbb{R}^{n_x}\times\mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ is considered to be analytic, and it holds $f(0,p) = 0$ for all parameters $p \in \mathbb{R}^{n_p}$. The power series expansion of the system dynamics is shown in ($f$), and the output dimension $n_y$ is considered to be smaller or equal to the state space dimension $n_x$.

$$f(x,p) = Fx + \sum_{k=2}^{\infty} f^{[k]}(x,p) \tag{$f$}$$

**Theorem 6** (Local convergence of $\pi(y,p)$ and $\kappa(y,p)$)**.**
*Let $\kappa$ be the optimal control input of (OCP) and $\pi$ the corresponding optimal cost function. If $F$, $G$, $H$, and $\ell$ respect the conditions (II)-(V) from Section 5.1, then $\kappa$ and $\pi$ can be expressed in terms of the output variables $y$, as well as system parameters $p$ and are locally analytic.*

*Proof.* The proof follows along the lines of Theorem 4 and Theorem 2. Nevertheless, some adjustments have to be made since the calculation is done in the output space instead of state space. The optimal input and the value function are denoted by

$$\kappa(y,p) = Ky + \sum_{k=2}^{\infty} \kappa^{[k]}(y,p) \tag{$\kappa$}$$

$$\text{and } \pi(y,p) = \frac{1}{2}y^{\mathrm{T}}\pi_{yy}y + \sum_{k=3}^{\infty} \pi^{[k]}(y,p). \tag{$\pi$}$$

The constants $\kappa_0 = \kappa(0,0)$ and $\pi_0 = \pi(0,0)$, as well as the 1-by-$n_y$ matrix $\pi_y$, are neglected since they are zero anyway. The derivations in Section 5.1 resp. Theorem 5 show that $p \mapsto \kappa(0,p)$, $p \mapsto \pi(0,p)$ as well as $p \mapsto \nabla_y\pi(0,p)$ are zero everywhere. The Hamilton-Jacobi-Bellman equation, which corresponds to the optimal control problem

(OCP), states as

$$
\begin{aligned}
0 &= \nabla_y \pi(y,p) \cdot H \cdot \Big(f(x,p) + G\kappa(y,p)\Big) \\
&\quad + \frac{1}{2} y^{\mathrm{T}} \ell_{yy} y + y^{\mathrm{T}} \ell_{yu} \cdot \kappa(y,p) + \frac{1}{2} \kappa(y,p)^{\mathrm{T}} \cdot \ell_{uu} \cdot \kappa(y,p),
\end{aligned}
\tag{HJBE-1}
$$

and the first-order optimality condition is given as

$$
0 = \nabla_y \pi(y,p) \cdot HG + y^{\mathrm{T}} \ell_{yu} + \kappa^{\mathrm{T}}(y,p) \cdot \ell_{uu}. \tag{HJBE-2}
$$

First simplified defining equations for $\kappa^{[k]}(y,p)$ and $\pi^{[k+1]}(y,p)$ ($k \geq 1$) need to be obtained. For $K$ and $\pi_{yy}$ one can copy ($K$) and ($\pi_{yy}$) from Section 5.1.

$$
K = -\ell_{uu}^{-1} \cdot (\ell_{yu}^{\mathrm{T}} + \tilde{G}^{\mathrm{T}} \pi_{yy}) \tag{$K$}
$$

$$
0 = \pi_{yy} \tilde{F} + \tilde{F}^{\mathrm{T}} \pi_{yy} + \ell_{yy} - (\ell_{yu} + \pi_{yy} \tilde{G}) \cdot \ell_{uu}^{-1} \cdot (\ell_{yu}^{\mathrm{T}} + \tilde{G}^{\mathrm{T}} \pi_{yy}) \tag{$\pi_{yy}$}
$$

Here $\tilde{F} \in \mathbb{R}^{n_y \times n_y}$ is, like in ($\tilde{F}|\tilde{G}$) from Section 5.1, defined as $HF\tilde{H}$, while $\tilde{G}$ is given by $HG$, and $\tilde{H}$ is a right inverse of $H$ such that $H \cdot \tilde{H} = I_{n_y}$ as required in condition (II) in Section 5.1. From (HJBE-2),

$$
\kappa^{[k]}(y,p) = -\ell_{uu}^{-1} \cdot \tilde{G}^{\mathrm{T}} \cdot \nabla_y \pi^{[k+1]}(y,p)^{\mathrm{T}} \tag{$\kappa^{[k]}$}
$$

is easily derived for $k \geq 2$, while (HJBE-1) shows that $\pi^{[k+1]}(y,p)$ is fixed via the equation

$$
\begin{aligned}
0 &= \sum_{i=1}^{k} \nabla_y \pi^{[i+1]}(y,p) \cdot H \cdot \Big(f^{[k-i+1]}(\tilde{H}y,p) + G\kappa^{[k-i+1]}(y,p)\Big) \\
&\quad + y^{\mathrm{T}} \ell_{yu} \cdot \kappa^{[k]}(y,p) + \frac{1}{2} \sum_{i=1}^{k} \kappa^{[i]}(y,p)^{\mathrm{T}} \cdot \ell_{uu} \cdot \kappa^{[k-i+1]}(y,p),
\end{aligned}
$$

where $x$ has been replaced by $\tilde{H}y$. This is possible in an $n_y$-dimensional subspace of $\mathbb{R}^{n_x}$. Using ($K$) to cancel out all terms depending on $\kappa^{[k]}$ and substituting ($\kappa^{[k]}$), one derives the following identity.

$$
\begin{aligned}
\nabla_y \pi^{[k+1]}(y,p) \cdot (\tilde{F} + \tilde{G}K) \cdot y &= -\sum_{i=1}^{k-1} \nabla_y \pi^{[i+1]}(y,p) \cdot H \cdot f^{[k-i+1]}(\tilde{H}y,p) \\
&\quad + \frac{1}{2} \sum_{i=2}^{k-1} \nabla_y \pi^{[i+1]}(y,p) \cdot \tilde{G} \ell_{uu}^{-1} \tilde{G}^{\mathrm{T}} \cdot \nabla_y \pi^{[k-i+2]}(y,p)^{\mathrm{T}}
\end{aligned}
\tag{$\pi^{[k+1]}$}
$$

To prove the convergence of the power series in ($\pi$), it is desired to find a series $\left(C_k^{\pi}\right)_{k \geq 2}$ such that

$$
\sum_{k=2}^{\infty} C_k^{\pi} \cdot r^k < \infty \quad \text{and} \quad \left|\pi^{[k]}(y,p)\right| \leq C_k^{\pi} \cdot r^{k-2} \cdot r_y^2 \leq C_k^{\pi} \cdot r^k \tag{5.4}
$$

for sufficiently small $r = \left|\left| \binom{y}{p} \right|\right|$. $r_y$ denotes the norm or the output vector $y$. Since $f(.,.)$ is analytic at least in a neighborhood of the origin, Remark 16 $(c)$ implies the existence of a series $\left(C_k^f\right)_{k\geq 1}$ with

$$\sum_{k=1}^{\infty} C_k^f \cdot r^k < \infty \quad \text{and} \quad \left|f^{[k]}(\tilde{H}y,p)\right| \leq C_k^f \cdot r^{k-1} \cdot r_y \leq C_k^f \cdot r^k \tag{5.5}$$

for sufficiently small $r$. In the following, the inequality (A.5) from Theorem 10 is applied to the different degrees of $\pi$.

$$\left|\nabla_y \pi^{[k]}(y,p)\right| \leq k \cdot C_k^\pi \cdot r^{k-2} \cdot r_y \leq k \cdot C_k^\pi \cdot r^{k-1}$$

The second degree of the value function can be easily over approximated using the spectral norm of $\pi_{yy}$, which is its largest eigenvalue. This eigenvalue divided by two will be called $C_2^\pi$.

$$\left|\pi^{[2]}(y)\right| = \left|\frac{1}{2} y^{\mathrm{T}} \pi_{yy} y\right| \leq C_2^\pi \cdot r_y^2 \tag{5.6}$$

Shifting to the next degree, one finds from $(\pi^{[k+1]})$ an equation for $\nabla_y \pi^{[3]}(y,p)$.

$$\left.\frac{\mathrm{d}\pi^{[3]}}{\mathrm{d}t}(y,p)\right|_{\dot{y}=\tilde{F}y+\tilde{G}Ky} = \nabla_y \pi^{[3]}(y,p) \cdot (\tilde{F}+\tilde{G}K) \cdot y = -y^{\mathrm{T}} \pi_{yy} \cdot H \cdot f^{[2]}(\tilde{H}y,p)$$

Via integration it follows

$$\begin{aligned} \left|\pi^{[3]}(y,p)\right|_{\dot{y}=\tilde{F}y+\tilde{G}Ky} &= \left|\int_0^\infty -y^{\mathrm{T}} \pi_{yy} \cdot H \cdot f^{[2]}(\tilde{H}y,p)\,\mathrm{d}t\right| \leq \int_0^\infty \left|y^{\mathrm{T}} \pi_{yy} \cdot H \cdot f^{[2]}(y,p)\right| \mathrm{d}t \\ &\overset{(5.6),(5.5)}{\leq} 2C_2^f C_H C_2^\pi \int_0^\infty r(t) \cdot r_y^2(t)\,\mathrm{d}t, \end{aligned} \tag{5.7}$$

where $C_H$ is the spectral norm of $H$. Since the linear part of the output system is stable, even exponential stability, it is possible to upper bound the output and, therefore, also $r_y = ||y||$.

$$r_y(t) \leq r_{y0} \cdot \mathrm{e}^{-\alpha \cdot t}\ , \quad r_{y0} = ||y(0)||,\ \alpha > 0$$

Thus $r(t)$ can also be upper bounded using $y(0)$.

$$r(t) = \left|\left|\begin{pmatrix} y(t) \\ p \end{pmatrix}\right|\right| \leq \left|\left|\begin{pmatrix} y(0) \\ p \end{pmatrix}\right|\right| =: r_0$$

Solving the integral in (5.7), $C_3^\pi$ can be found.

$$\left|\pi^{[3]}(y,p)\right|_{\dot{y}=\tilde{F}y+\tilde{G}Ky} \le \frac{2C_2^f C_H C_2^\pi}{2\alpha}\cdot r_0\cdot r_{y0}^2 \le \frac{3C_2^f C_H C_2^\pi}{\alpha}\cdot r_0^3 =: C_3^\pi\cdot r_0^3$$

Theorem 10 implies

$$\left|\nabla_y\pi^{[3]}(y,p)\right| \le 3\cdot C_3^\pi\cdot r_0\cdot r_{y0} \le 3\cdot C_3^\pi\cdot r_0^2 = \frac{C_2^f C_H C_2^\pi}{\alpha}\cdot r_0^2.$$

Before the general case is shown, $C_4^\pi$ will be calculated to gain more inside how those constants are found. Thus $(\pi^{[k+1]})$ is integrated again.

$$\begin{aligned}
&\left|\pi^{[4]}(y,p)\right|_{\dot{y}=\tilde{F}y+\tilde{G}Ky}\\
&= \Bigg|\int_0^\infty -y^{\mathrm{T}}\pi_{yy}\cdot H\cdot f^{[3]}(y,p) - \nabla_y\pi^{[3]}(y,p)\cdot H\cdot f^{[2]}(y,p)\\
&\qquad + \frac{1}{2}\nabla_y\pi^{[3]}(y,p)\cdot\tilde{G}\ell_{uu}^{-1}\tilde{G}^{\mathrm{T}}\cdot\nabla_y\pi^{[3]}(y,p)^{\mathrm{T}}\,\mathrm{d}t\Bigg|\\
&\le \int_0^\infty 2C_3^f C_H C_2^\pi\cdot r^2(t)r_y^2(t) + 3C_2^f C_H C_3^\pi\cdot r^2(t)r_y^2(t) + \frac{9}{2}C_G^2 C_H^2 C_{uu}\cdot(C_3^\pi)^2\cdot r^2(t)r_y^2(t)\,\mathrm{d}t\\
&\le \frac{1}{2\alpha}\cdot\left(2C_3^f C_H C_2^\pi + 3C_2^f C_H C_3^\pi + \frac{9}{2}C_G^2 C_H^2 C_{uu}\cdot(C_3^\pi)^2\right)\cdot r_0^2\cdot r_{y0}^2\\
&\le \frac{1}{2\alpha}\cdot\left(2C_3^f C_H C_2^\pi + 3C_2^f C_H C_3^\pi + \frac{9}{2}C_G^2 C_H^2 C_{uu}\cdot(C_3^\pi)^2\right)\cdot r_0^4 =: C_4^\pi\cdot r_0^4
\end{aligned}$$

$C_{uu}$ resp. $C_G$ denotes the spectral norm of $\ell_{uu}^{-1}$ resp. $G$.
The general case $k\ge 2$ can now be handled in the same way.

$$\begin{aligned}
&\left|\pi^{[k+1]}(y,p)\right|_{\dot{y}=\tilde{F}y+\tilde{G}Ky}\\
&\le \int_0^\infty \sum_{i=1}^{k-1}(i+1)\cdot C_{i+1}^\pi\cdot r^{i-1}(t)r_y(t)\cdot C_H\cdot C_{k-i+1}^f\cdot r^{k-i}(t)r_y(t)\,\mathrm{d}t\\
&\quad + \int_0^\infty \sum_{i=2}^{k-1}\frac{1}{2}(i+1)\cdot C_{i+1}^\pi r^{i-1}(t)r_y(t)\cdot C_G^2C_H^2C_{uu}\cdot(k-i+2)\cdot C_{k-i+2}^\pi r^{k-i}(t)r_y(t)\,\mathrm{d}t\\
&\le \underbrace{\frac{\sum_{i=1}^{k-1}(i+1)\cdot C_{i+1}^\pi C_H C_{k-i+1}^f + \sum_{i=2}^{k-1}\frac{1}{2}(i+1)(k-i+2)\cdot C_G^2C_H^2C_{uu}C_{i+1}^\pi C_{k-i+2}^\pi}{2\alpha}}_{=:\,C_{k+1}^\pi}\cdot r_0^{k-1}r_{y0}^2\\
&\le C_{k+1}^\pi\cdot r_0^{k+1}
\end{aligned}$$

Knowing these coefficients, a converging domination series for $\left(C_k^\pi\right)_{k\ge 2}$ would show the desired convergence. Therefore, another equation is introduced, and it will be shown that its solution is, in fact, a series with the desired properties.

$$C_G^2 C_H^2 C_{uu} \cdot \left(\frac{\mathrm{d}\gamma}{\mathrm{d}r}(r)\right)^2 + \left(\sum_{i=2}^{\infty} C_i^f \cdot q^{i-1} - a\right) \cdot C_H \cdot r \cdot \frac{\mathrm{d}\gamma}{\mathrm{d}r}(r) + b \cdot r^2 = 0 \tag{5.8}$$

The constant numbers $a$ and $b$ will be determined during the following calculation, while $q \in \mathbb{R}_{\geq 0}$ can also be seen as a norm of a vector like $r$. Clearly, (5.8) is only well-defined where $f(.,.)$ is analytic. The constant $C_1^f$ influences the solution of (5.8) via $a$ and $b$, which will depend on $F$.
Clearly, (5.8) admits two solutions, which are given in the form $\gamma(r) = \frac{1}{2} g(q) \cdot r^2$, where $g(.)$ is a solution of the quadratic equation

$$C_G^2 C_H^2 C_{uu} \cdot g^2(q) + \left(\sum_{i=2}^{\infty} C_i^f \cdot q^{i-1} - a\right) \cdot C_H \cdot g(q) + b = 0. \tag{5.9}$$

If $q$ is sufficiently small and

$$0 < b < \frac{a^2}{4 C_G^2 C_{uu}} \tag{5.10}$$

two solutions

$$g_{1/2}(q) = \frac{a - \sum\limits_{i=2}^{\infty} C_i^f \cdot q^{i-1} \pm \sqrt{\left(a - \sum\limits_{i=2}^{\infty} C_i^f \cdot q^{i-1}\right)^2 - 4b \cdot C_G^2 C_{uu}}}{2 C_G^2 C_H C_{uu}} \tag{5.11}$$

are found. From those two solutions, $g_2(q)$ is taken and from now on just denoted with $g(q)$. It will be seen later why $g_1(q)$ can not be the correct one. The solution $g(.)$ is analytic in the domain where $\sum\limits_{i=2}^{\infty} C_i^f \cdot q^{i-1} < \infty$ and $q$ are sufficiently small. Furthermore, it can be locally expanded into a power series

$$g(q) = \sum_{i=2}^{\infty} C_i^g \cdot q^{i-2} < \infty. \tag{5.12}$$

This series can also be found degree-wise. Starting with the constant part from (5.12) and (5.9) or (5.11), which defines $C_2^g$.

$$0 = C_G^2 C_H^2 C_{uu} \cdot (C_2^g)^2 - a \cdot C_H \cdot C_2^g + b \quad \Leftrightarrow \quad C_2^g = \frac{a - \sqrt{a^2 - 4b \cdot C_G^2 \cdot C_{uu}}}{2 C_G^2 C_H C_{uu}} > 0$$

Taking the linear part from (5.9) and leaving out $q$, since it is arbitrary, leads to $C_3^g$.

$$
\begin{aligned}
0 &= 2C_G^2C_H^2C_{uu}\cdot C_2^gC_HC_3^g + C_2^fC_HC_2^g - a\cdot C_H\cdot C_3^g\\
\Rightarrow C_3^g &= \frac{C_2^fC_2^g}{a-2C_G^2C_HC_{uu}\cdot C_2^g}
\end{aligned}
$$

$a$ and $b$ have to be chosen such that

$$
a-2C_G^2C_HC_{uu}\cdot C_2^g = \sqrt{a^2-4b\cdot C_G^2C_{uu}}
$$

is strictly greater than zero. This is the case if $b$ (still depending on $a$) is defined as

$$
b=\frac{a^2-\alpha^2}{4C_G^2\cdot C_{uu}},
$$

which ensures

$$
a^2-4b\cdot C_G^2C_{uu}=\alpha^2>0.
$$

Obviously, this choice of $b$ also fulfills (2.3), if $a>\alpha$. Using $\alpha$ here involves also the linearized system and, therefore, $F+GK$. Putting all formulas and definitions together gives

$$
C_3^g=\frac{1}{\alpha}\cdot C_2^f\cdot C_2^g>0
$$

and for $a\geq\alpha+2C_G^2C_{uu}\cdot C_2^\pi$ also $C_2^g=\dfrac{a-\alpha}{2C_G^2C_{uu}}\geq C_2^\pi$.
If $g_1(q)$ would have been used, then $C_k^g$ $(k\geq 3)$ would be negative, which can not be an upper bound for $C_k^\pi$.
More general, the $C_k^g$ are derived by taking the $(k-2)$-th degree in equation (5.9).

$$
\begin{aligned}
0 &= C_G^2C_H^2C_{uu}\cdot\sum_{i=2}^{k}C_i^g\cdot C_{k-i+2}^g + C_H\cdot\sum_{i=2}^{k-1}C_i^f\cdot C_{k-i+1}^g - a\cdot C_H\cdot C_k^g.\\
\Rightarrow \alpha\cdot C_k^g &= (a-2C_G^2C_HC_{uu}\cdot C_2^g)\cdot C_k^g = C_G^2C_HC_{uu}\cdot\sum_{i=3}^{k-1}C_i^g\cdot C_{k-i+2}^g+\sum_{i=2}^{k-1}C_i^f\cdot C_{k-i+1}^g\\
\Rightarrow C_k^g &= \frac{1}{\alpha}\cdot\left(C_G^2C_HC_{uu}\cdot\sum_{i=3}^{k-1}C_i^g\cdot C_{k-i+2}^g+\sum_{i=2}^{k-1}C_i^f\cdot C_{k-i+1}^g\right)>0
\end{aligned}
$$

Finally, this result can be compared to

$$
C_k^\pi=\frac{C_H\cdot\sum_{i=2}^{k-1}(k-i+1)\cdot C_i^fC_{k-i+1}^\pi + C_G^2C_H^2C_{uu}\cdot\sum_{i=3}^{k-1}\frac{1}{2}i\cdot(k-i+2)\cdot C_i^\pi C_{k-i+2}^\pi}{2\alpha},
$$

and it is seen that for $k \geq 3$

$$C_k^\pi \leq C_k^g \cdot C_H \cdot \frac{\max\limits_{i \in \{3,\ldots,k-1\}} i \cdot (k-i+2)}{2} \leq C_k^g \cdot C_H \cdot \frac{(k+2)^2}{8}$$

and, therefore, it holds

$$|\pi(y,p)| \leq \sum_{k=2}^{\infty} C_k^\pi \cdot r^k \leq C_H \cdot \sum_{k=2}^{\infty} \frac{(k+2)^2}{8} \cdot C_k^g \cdot r^k.$$

If the power series $g(r)$ exists resp. converges for a fixed $r$, then $\pi(y,p)$ exists and converges for all $(y,p)$ such that

$$||(y,p)|| < r.$$

That is the case because of

$$\lim_{k \to \infty} \left((k+2)^2\right)^{1/k} \searrow 1.$$

$\kappa(y,p)$ can be seen as a product of locally analytic functions. Thus it is also locally analytic with the same area of convergence. □

## 5.3 Area of convergence

Similar to the results of Section 2.3 and 3.3, the convergence area and thus the area of existence of the solution

$$\pi(y,p) = \frac{1}{2} y^{\mathrm{T}} \pi_{yy} y + \sum_{k=3}^{\infty} \pi^{[k]}(y,p), \quad \kappa(y,p) = Ky + \sum_{k=2}^{\infty} \kappa^{[k]}(y,p)$$

of the continuous-time optimal output-feedback problem

$$\pi\big(y(0),p\big) = \min_{u(.)} \int_0^\infty \frac{1}{2} y^{\mathrm{T}}(\tau) \cdot \ell_{yy} \cdot y(\tau) + y^{\mathrm{T}}(\tau) \cdot \ell_{yu} \cdot u(\tau) + \frac{1}{2} u^{\mathrm{T}}(\tau) \cdot \ell_{uu} \cdot u(\tau) \, \mathrm{d}\tau$$

$$\text{s.t. } \dot{x} = f(x,p) + Gu$$

$$y = Hx,$$

as given in Section 5.2, can be estimated. If $C_G$, $C_H$, $C_{uu}$, and $C_2^\pi$ are the spectral norms of $G$, $H$, $\ell_{uu}^{-1}$, and $\frac{1}{2}\pi_{yy}$, and $\alpha$ the maximal value such that for the solution of the initial value problem

$$\dot{y} = (\tilde{F} + \tilde{G}K) \cdot y, \quad y(0) = y_0 \in \mathbb{R}^{n_y}$$

$||y(t)|| \leq ||y_0|| \cdot \mathrm{e}^{-\alpha t}$ holds for $\tilde{F}$ and $\tilde{G}$ as in $(\tilde{F}|\tilde{G})$ in Section 5.1 and $C_k^f$ $(k \geq 2)$ such that

$$\left\|f^{[k]}(\tilde{H}y, p)\right\| \leq C_k^f \cdot ||(y,p)||.$$

Then convergence and existence of $\pi$ and $\kappa$ are given for all $(y, p)$ in a neighborhood of the origin such that $f$ is analytic and (5.13) holds.

$$\sum_{k=2}^{\infty} C_k^f \cdot ||(y,p)||^k \leq \left(\alpha + 2C_G^2 C_{uu} C_2^\pi - \sqrt{(\alpha + 2C_G^2 C_{uu} C_2^\pi)^2 - \alpha^2}\right) \cdot ||(y,p)|| \quad (5.13)$$

**Remark 10.** *If the system dynamics $f$ is linear with respect to the states, it is clear that the value function and the optimal control law can be expressed as*

$$\pi(y) = \frac{1}{2} y^{\mathrm{T}} \pi_{yy} y \quad \text{and} \quad \kappa(y) = Ky.$$

*Both exist everywhere in $\mathbb{R}^{n_y}$. The same result is also found via (5.13), since $C_k^f = 0$ for all $k \geq 2$.*

**Example 12.** *For the quadcopter model (10 states, 6 output variables) from Example 9, the constants (all units are again neglected)*

$$C_G = 10, \quad C_{uu} = 1, \quad C_2^\pi = 1.4896 \quad \text{and} \quad \alpha = 0.8136$$

*are found for $\tilde{H}_1$. For $\tilde{H}_2$, one obtains $C_2^\pi = 2.116\,56$ and $\alpha = 0.9312$. Using 5.13, both choices lead to convergence for all $r = ||(y,p)|| \leq 0.007\,55$, which is more than twice the radius of the state-feedback case. The reason are fewer variables in the power series and less conservative estimations.*
*For Example 10, the constants are*

$$C_G = 18.8679, \quad C_{uu} = 10, \quad C_2^\pi = 1.8477 \quad \text{and} \quad \alpha = 1.0101.$$

*Thus convergence is given for $r = ||(y,p)|| \leq 0.43 \cdot 10^{-10}$, which is slightly bigger compared to the state feedback case.*
*As mentioned in Example 6, these results are of no use in practice, and other methods should be investigated.*

# 6 The Discrete-Time Case

Analogous to the continuous-time output-feedback, which has been worked on in Section 5.1, results for the discrete-time case can be derived. The solution procedure is the same as in continuous-time. As optimality criteria, Bellmans equation and its derivative with respect to the inputs are used. To guarantee a solution, again LQR conditions and a matrix $\tilde{H}$ as in Section 5.1 are required. An existence resp. convergence proof for the control law and the value function is, however, not provided. The main challenge in deriving such results is the more complicated dependence of both power series on each other. This dependence is also not simplified if a linear output function, quadratic cost, and a linear input regarding the system dynamics are considered. To the best knowledge of the author, a convergence proof in discrete-time does not exist so far. This issue will be tackled in future research. Nevertheless, the local stability of the closed-loop system using approximated control laws can be guaranteed.

For the discrete-time case the following optimal control problem is considered.

$$\begin{aligned} \pi\Big(y(0), p\Big) &= \min_{u(.)} \sum_{n=0}^{\infty} \ell\Big(y(n), u(n)\Big) \\ \text{s.t. } x^+(n) &= x(n+1) = f\Big(x(n), u(n), p\Big), \\ y(n) &= h\Big(x(n), p\Big) \quad n \in \mathbb{N}_0 \end{aligned} \tag{OCP}$$

Therein $x \in \mathbb{R}^{n_x}$ are the states, $u \in \mathbb{R}^{n_u}$ are input variables, $p \in \mathbb{R}^{n_p}$ are the parameters, and $y \in \mathbb{R}^{n_y}$ the output variables. The system dynamics $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ and the output function $h\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_y}$ are assumed to be analytic with power series expansions

$$f(x,u,p) = \sum_{k=1}^{\infty} f^{[k]}(x,u,p) = Fx + Gu + f^{[2]}(x,u,p) + \dots \tag{$f$}$$

$$\text{and } h(x,p) = \sum_{k=1}^{\infty} h^{[k]}(x,p) = Hx + h^{[2]}(x,p) + \dots, \tag{$h$}$$

where $F \in \mathbb{R}^{n_x \times n_x}$ and $H \in \mathbb{R}^{n_y \times n_x}$. The parameters appear multiplicative to ensure unique solvability.

$$\forall p \in \mathbb{R}^{n_p} : f(0,0,p) = 0 \tag{$f_0$}$$

$$h(0,p) = 0 \tag{$h_0$}$$

Since the maps $p \mapsto f(0,0,p)$ and $p \mapsto h(0,p)$ are also analytic but constant at 0, the power series need to vanish completely.

$$\begin{aligned} \forall k \in \mathbb{N}, p \in \mathbb{R}^{n_p} : f^{[k]}(0,0,p) &= 0 \\ h^{[k]}(0,p) &= 0 \end{aligned}$$

The cost function $\ell \colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}$ is stated in terms of the input variables and the output variables instead of the system states. $\ell$ also needs to be analytic.

$$\ell(y,u) = \sum_{k=2}^{\infty} \ell^{[k]}(y,u) = \frac{1}{2} y^{\mathrm{T}} \ell_{yy} y + y^{\mathrm{T}} \ell_{yu} u + \frac{1}{2} u^{\mathrm{T}} \ell_{uu} u + \ell^{[3]}(y,u) + \dots \tag{$\ell$}$$

The matrices $\ell_{yy} \in \mathbb{R}^{n_y \times n_y}$, $\ell_{yu} \in \mathbb{R}^{n_y \times n_u}$, and $\ell_{uu} \in \mathbb{R}^{n_u \times n_u}$ must together with $F$ and $H$ fulfill further conditions, which are similar to the continuous case and will be shown later throughout the calculation.
The optimal cost function $\pi \colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_p} \to \mathbb{R}$ is assumed to be analytic with a power series expansion of the form

$$\begin{aligned} \pi(y,p) &= \sum_{k=1}^{\infty} \pi^{[k]}(y,p) \\ &= \pi_y y + \pi_p p + \frac{1}{2} y^{\mathrm{T}} \pi_{yy} y + y^{\mathrm{T}} \pi_{yp} + \frac{1}{2} p^{\mathrm{T}} \pi_{pp} p + \pi^{[3]}(y,p) + \dots, \end{aligned} \tag{$\pi$}$$

while the cost-minimizing control law $\kappa \colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_u}$ can be written as follows.

$$\kappa(y,p) = \sum_{k=1}^{\infty} \kappa^{[k]}(y,p) = Ky + Lp + \kappa^{[2]}(y,p) + \dots \tag{$\kappa$}$$

Both series depend on parameters and output variables since the full state information may not be available and can not be used. From ($f_0$) and ($h_0$), it is clear that the maps $p \mapsto \pi(0,p)$ and $p \mapsto \kappa(0,p)$ are zero everywhere. Thus $\pi_p \in \mathbb{R}^{1 \times n_p}$, $\pi_{pp} \in \mathbb{R}^{n_p \times n_p}$, $\pi^{[3]}(0,p)$, ... and $L \in \mathbb{R}^{n_u \times n_p}$, $\kappa^{[2]}(0,p)$, ... are zero. Furthermore, $\pi_{yp} \in \mathbb{R}^{n_y \times n_p}$, $\nabla_y \pi^{[3]}(0,p)$, ... must also vanish.
Bellman's principle of optimality is given in terms of $(y,p)$,

$$\pi(y,p) = \pi(y^+,p) + \ell\big(y, \kappa(y,p)\big),$$

where $y^+(n) = y(n+1)$ for $n \in \mathbb{N}_0$. $y^+$ can be replaced via

$$y^+ = h(x^+,p) = h\Big(f\big(x, \kappa(y,p), p\big), p\Big).$$

Thus the Bellman equation states as

$$\pi(y,p) = \pi\Big(h\Big(f\big(x,\kappa(y,p),p\big),p\Big),p\Big) + \ell\big(y,\kappa(y,p)\big) \tag{BDP-1}$$

and the first-order optimality criterion becomes

$$0 = \nabla_y \pi\Big(h\Big(f\Big(x, \kappa(y,p), p\Big), p\Big), p\Big) \cdot \nabla_x h\Big(f\Big(x, \kappa(y,p), p\Big), p\Big) \cdot \nabla_u f\Big(x, \kappa(y,p), p\Big) + \nabla_u \ell\Big(y, \kappa(y,p)\Big). \tag{BDP-2}$$

Substituting all power series into (BDP-1) and (BDP-2) leads to a separate equation for each degree. Degree zero of (BDP-1) does not contain any useful information, while the constant part of (BDP-2) leads to

$$0 = \pi_y \cdot H \cdot G. \tag{$\pi_y$-1}$$

$\pi_y$ is not completely fixed in equation ($\pi_y$-1). Thus (BDP-1)[1] is investigated next.

$$\begin{aligned} \pi_y y + \pi_p p &= \pi_y \cdot H \cdot \Big(Fx + G(Ky + Lp)\Big) + \pi_p p \\ &\overset{(\pi_y\text{-}1)}{=} \pi_y \cdot H \cdot Fx + \pi_p p \\ \Rightarrow 0 &= \pi_y \cdot H \cdot (F - I_{n_x}) \end{aligned} \tag{$\pi_y$-2}$$

Combining ($\pi_y$-1) and ($\pi_y$-2) shows that $\pi_y$ has to vanish if

(I) $\begin{pmatrix} H \cdot (F - I_{n_x}) & HG \end{pmatrix}$ has full rank.

$\pi_p$ can not be obtained here but as outlined before it has to vanish. (BDP-2)[1] contains information about $K \in \mathbb{R}^{n_u \times n_y}$ and $L \in \mathbb{R}^{n_u \times n_p}$.

$$\begin{aligned} 0 = {} & \Big(\Big(\underline{Fx} + G(Ky + Lp)\Big)^{\mathrm{T}} \cdot H^{\mathrm{T}} \pi_{yy} + p^{\mathrm{T}} \pi_{yp}^{\mathrm{T}}\Big) \cdot HG \\ & + y^{\mathrm{T}} \cdot \ell_{yu} + (Ky + Lp)^{\mathrm{T}} \cdot \ell_{uu} \end{aligned} \tag{6.1}$$

As for the continuous case, $x$ is replaced in terms of $y$. Thus $H$ has to have full rank:

(II) $\exists\, \tilde{H} \in \mathbb{R}^{n_x \times n_y} : \; H \cdot \tilde{H} = I_{n_y}$.

Keeping only quadratic terms, $x$ is replaced by $\tilde{H}y$. This substitution can be done in an $n_y$-dimensional subspace of $\mathbb{R}^{n_x}$ and makes it possible to calculate $\kappa$ and $\pi$. Thus (6.1) leads to

$$\begin{aligned} 0 = {} & \Big(\Big(F\tilde{H}y + G(Ky + Lp)\Big)^{\mathrm{T}} \cdot H^{\mathrm{T}} \pi_{yy} + p^{\mathrm{T}} \pi_{yp}^{\mathrm{T}}\Big) \cdot HG \\ & + y^{\mathrm{T}} \cdot \ell_{yu} + (Ky + Lp)^{\mathrm{T}} \cdot \ell_{uu} \end{aligned} \tag{6.2}$$

Similar to the continuous-time case, this equation can be separated in two since $y$ and $p$ are independent. Starting with the equation that is defined by the output variables to find $K$ one has

$$0 = (F\tilde{H} + GK)^{\mathrm{T}} \cdot H^{\mathrm{T}} \pi_{yy} HG + \ell_{yu} + K^{\mathrm{T}} \ell_{uu}$$

resp.

$$K = -\Big(G^{\mathrm{T}} H^{\mathrm{T}} \pi_{yy} H G + \ell_{uu}\Big)^{-1} \cdot \Big(\ell_{yu}^{\mathrm{T}} + G^{\mathrm{T}} H^{\mathrm{T}} \pi_{yy} H F \tilde{H}\Big).$$

Defining

$$\tilde{F} := HF\tilde{H} \quad \text{and} \quad \tilde{G} := HG \tag{$\tilde{F}|\tilde{G}$}$$

the matrix $K$ is simply given by

$$K = -\Big(\tilde{G}^{\mathrm{T}} \pi_{yy} \tilde{G} + \ell_{uu}\Big)^{-1} \cdot \Big(\ell_{yu}^{\mathrm{T}} + \tilde{G}^{\mathrm{T}} \pi_{yy} \tilde{F}\Big). \tag{$K$}$$

Going back to (6.2) and only take terms that are linear with respect to the parameters $p$ shows

$$0 = L^{\mathrm{T}} \tilde{G}^{\mathrm{T}} \pi_{yy} \tilde{G} + \pi_{yp}^{\mathrm{T}} \tilde{G} + L^{\mathrm{T}} \ell_{uu}$$

and, therefore, leads to a formula for $L$, which is actually 0 as mentioned above. But it is outlined in more detail in the following.

$$L = -\Big(\tilde{G}^{\mathrm{T}} \pi_{yy} \tilde{G} + \ell_{uu}\Big)^{-1} \cdot \tilde{G}^{\mathrm{T}} \pi_{yp} \tag{$L$-1}$$

The next step is to collect all terms of degree two from (BDP-1).

$$\begin{aligned}
&\frac{1}{2} y^{\mathrm{T}} \pi_{yy} y + y^{\mathrm{T}} \pi_{yp} p + \frac{1}{2} p^{\mathrm{T}} \pi_{pp} p \\
&= \frac{1}{2}\Big(Fx + G(Ky + Lp)\Big)^{\mathrm{T}} \cdot H^{\mathrm{T}} \pi_{yy} H \cdot \Big(Fx + G(Ky + Lp)\Big) \\
&\quad + \Big(Fx + G(Ky + Lp)\Big)^{\mathrm{T}} \cdot H^{\mathrm{T}} \pi_{yp} p + \frac{1}{2} p^{\mathrm{T}} \pi_{pp} p + \frac{1}{2} y^{\mathrm{T}} \ell_{yy} y \\
&\quad + y^{\mathrm{T}} \ell_{yu} \cdot (Ky + Lp) + \frac{1}{2}(Ky + Lp)^{\mathrm{T}} \cdot \ell_{uu} \cdot (Ky + Lp)
\end{aligned}$$

$x$ is again replaced by $\tilde{H}y$. The term that contains $\pi_{pp} \in \mathbb{R}^{n_p \times n_p}$ is canceled out of the equation and can not be obtained here. As outlined before it has to vanish.

$$\begin{aligned}
\frac{1}{2} y^{\mathrm{T}} \pi_{yy} y + y^{\mathrm{T}} \pi_{yp} p &= \frac{1}{2}\Big(\tilde{F}y + \tilde{G}(Ky + Lp)\Big)^{\mathrm{T}} \cdot \pi_{yy} \cdot \Big(\tilde{F}y + \tilde{G}(Ky + Lp)\Big) \\
&\quad + \Big(\tilde{F}y + \tilde{G}(Ky + Lp)\Big)^{\mathrm{T}} \cdot \pi_{yp} p + \frac{1}{2} y^{\mathrm{T}} \ell_{yy} y \\
&\quad + y^{\mathrm{T}} \ell_{yu} \cdot (Ky + Lp) + \frac{1}{2}(Ky + Lp)^{\mathrm{T}} \cdot \ell_{uu} \cdot (Ky + Lp)
\end{aligned} \tag{6.3}$$

Setting $p$ to zero and removing $y$, since it can be chosen arbitrary and independent from $p$, leads to the following matrix equation.

$$\pi_{yy} = (\tilde{F} + \tilde{G}K)^{\mathrm{T}} \cdot \pi_{yy} \cdot (\tilde{F} + \tilde{G}K) + \ell_{yy} + \ell_{yu} K + K^{\mathrm{T}} \ell_{yu}^{\mathrm{T}} + K^{\mathrm{T}} \ell_{uu} K$$

Substitution of $(K)$ gives the well known discrete time algebraic Riccati equation.

$$\pi_{yy} = \tilde{F}^{\mathrm{T}}\pi_{yy}\tilde{F} + \ell_{yy} - \left(\ell_{yu}^{\mathrm{T}} + \tilde{F}^{\mathrm{T}}\pi_{yy}\tilde{G}\right) \cdot \left(\tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G} + \ell_{uu}\right)^{-1} \cdot \left(\ell_{yu} + \tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{F}\right) \quad (\pi_{yy})$$

$(\pi_{yy})$ has a solution $\pi_{yy} \succ 0$ if

(III) the second order of the cost function is convex in $(y, u)$ and strictly convex in $u$ resp.

$$\begin{pmatrix} \ell_{yy} & \ell_{yu} \\ \ell_{yu}^{\mathrm{T}} & \ell_{uu} \end{pmatrix} \succeq 0 \quad \text{and} \quad \ell_{uu} \succ 0,$$

(IV) the pair $(\tilde{F},\ \tilde{G})$ is stabilizable and

(V) the pair $(\tilde{F}, \ell_{yy})$ is detectable.

Having this solved, one investigates all terms of (6.3) that contain $y$ and $p$.

$$\begin{aligned} \pi_{yp} &= (\tilde{F} + \tilde{G}K)^{\mathrm{T}} \cdot \pi_{yy} \cdot \tilde{G}L + (\tilde{F} + \tilde{G}K)^{\mathrm{T}} \cdot \pi_{yp} + \ell_{yu}L + K^{\mathrm{T}}\ell_{uu}L \\ &= \tilde{F}^{\mathrm{T}}\pi_{yy}\tilde{G}L + K^{\mathrm{T}}\tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G}L + \tilde{F}^{\mathrm{T}}\pi_{yp} + K^{\mathrm{T}}\tilde{G}^{\mathrm{T}}\pi_{yp} + \ell_{yu}L + K^{\mathrm{T}}\ell_{uu}L \end{aligned} \quad (6.4)$$

Considering (6.3) and replacing $y$ with 0 leads to another condition for $\pi_{yp}$.

$$0 = L^{\mathrm{T}}\tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G}L + L^{\mathrm{T}}\tilde{G}^{\mathrm{T}}\pi_{yp} + \pi_{yp}^{\mathrm{T}}\tilde{G}L + L^{\mathrm{T}}\ell_{uu}L$$

Substituting $L$ via $(L\text{-}1)$ simplifies this condition to

$$0 = -\pi_{yp}^{\mathrm{T}}\tilde{G} \cdot \left(\tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G} + \ell_{uu}\right)^{-1} \cdot \tilde{G}^{\mathrm{T}}\pi_{yp},$$

where one can see that

$$0 = \tilde{G}^{\mathrm{T}}\pi_{yp}, \qquad (\pi_{yp}\text{-}1)$$

since $\tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G} + \ell_{uu}$ is positive definite. Therefore, $L$ must vanish.

$$L = 0 \qquad (L\text{-}2)$$

Using $(\pi_{yp}\text{-}1)$ in (6.4) yields

$$0 = (\tilde{F}^{\mathrm{T}} - I_{n_y}) \cdot \pi_{yp}, \qquad (\pi_{yp}\text{-}2)$$

which together with $(\pi_{yp}\text{-}1)$ shows

$$\pi_{yp} = 0. \qquad (\pi_{yp})$$

To obtain $\kappa^{[2]}(y,p)$ the second order of (BDP-2) must be investigated.

$$\begin{aligned}
0 &= \nabla_y \pi^{[3]}\Big(H(Fx+GKy),p\Big)\cdot \tilde{G} + h^{[2]}(Fx+GKy,p)^{\mathrm{T}}\cdot \pi_{yy}\tilde{G} \\
&\quad + f^{[2]}(x,Ky,p)^{\mathrm{T}}\cdot H^{\mathrm{T}}\pi_{yy}\tilde{G} + \Big(\kappa^{[2]}(y,p)+K\cdot h^{[2]}(x,p)\Big)^{\mathrm{T}}\cdot \tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G} \\
&\quad + (Fx+GKy)^{\mathrm{T}}\cdot H^{\mathrm{T}}\pi_{yy}\cdot \nabla_x h^{[2]}(Fx+GKy,p)\cdot G \\
&\quad + (Fx+GKy)^{\mathrm{T}}\cdot H^{\mathrm{T}}\pi_{yy}\cdot H\cdot \nabla_u f(x,Ky,p) \\
&\quad + \nabla_u \ell^{[3]}(y,Ky) + h^{[2]}(x,p)^{\mathrm{T}}\cdot \ell_{yu} + \Big(\kappa^{[2]}(y,p)+K\cdot h^{[2]}(x,p)\Big)^{\mathrm{T}}\cdot \ell_{uu}
\end{aligned}$$

Again $x$ is replaced by $\tilde{H}y$ to remove one dependent variable. Thus a formula for the second degree of the control law can be given.

$$\begin{aligned}
\kappa^{[2]}(y,p) &= -\Big(\tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G}+\ell_{uu}\Big)^{-1}\cdot \Bigg(\tilde{G}^{\mathrm{T}}\nabla_y\pi^{[3]}(\tilde{F}y+\tilde{G}Ky,p)^{\mathrm{T}} \\
&\quad + \tilde{G}^{\mathrm{T}}\pi_{yy}\cdot \Big(h^{[2]}(F\tilde{H}y+GKy,p)+H\cdot f^{[2]}(\tilde{H}y,Ky,p)+\tilde{G}K\cdot h^{[2]}(\tilde{H}y,p)\Big) \\
&\quad + G^{\mathrm{T}}\cdot \nabla_x h^{[2]}(F\tilde{H}y+GKy,p)^{\mathrm{T}}\cdot \pi_{yy}\cdot (\tilde{F}y+\tilde{G}Ky) \\
&\quad + \nabla_u f(\tilde{H}y,Ky,p)^{\mathrm{T}}\cdot H^{\mathrm{T}}\cdot \pi_{yy}\cdot(\tilde{F}y+\tilde{G}Ky) + \nabla_u\ell^{[3]}(y,Ky)^{\mathrm{T}} \\
&\quad + \ell_{yu}^{\mathrm{T}}\cdot h^{[2]}(\tilde{H}y,p)\Bigg) - K\cdot h^{[2]}(\tilde{H}y,p)
\end{aligned} \tag{$\kappa^{[2]}$}$$

Except for $\nabla_y\pi^{[3]}$, everything is known in this formula. $\pi^{[3]}$ is found if the third degree of (BDP-1) is solved.

$$\begin{aligned}
&\pi^{[3]}(y,p) + y^{\mathrm{T}}\pi_{yy}\cdot h^{[2]}(x,p) \\
&= \pi^{[3]}\Big(H(Fx+GKy),p\Big) + (Fx+GKy)^{\mathrm{T}}\cdot H^{\mathrm{T}}\pi_{yy}\cdot h^{[2]}(Fx+GKy,p) \\
&\quad + (Fx+GKy)^{\mathrm{T}}\cdot H^{\mathrm{T}}\pi_{yy}H\cdot f^{[2]}(x,Ky,p) \\
&\quad + (Fx+GKy)^{\mathrm{T}}\cdot H^{\mathrm{T}}\pi_{yy}HG\cdot \Big(\kappa^{[2]}(y,p)+K\cdot h^{[2]}(x,p)\Big) \\
&\quad + \ell^{[3]}(y,Ky) + x^{\mathrm{T}}H^{\mathrm{T}}\ell_{yu}\cdot \Big(\kappa^{[2]}(y,p)+K\cdot h^{[2]}(x,p)\Big) \\
&\quad + h^{[2]}(x,p)^{\mathrm{T}}\cdot \ell_{yu}\cdot Ky + y^{\mathrm{T}}K^{\mathrm{T}}\cdot \ell_{uu}\cdot \Big(\kappa^{[2]}(y,p)+K\cdot h^{[2]}(x,p)\Big)
\end{aligned}$$

For simplification, $(K)$ is applied to remove all terms that contain $\kappa^{[2]}$. Furthermore, $x=\tilde{H}y$ is used.

$$\begin{aligned}
&\pi^{[3]}(y,p) + y^{\mathrm{T}}\pi_{yy}\cdot h^{[2]}(\tilde{H}y,p) \\
&= \pi^{[3]}\Big((\tilde{F}+\tilde{G}K)y,p\Big) + y^{\mathrm{T}}(\tilde{F}+\tilde{G}K)^{\mathrm{T}}\cdot \pi_{yy}\cdot h^{[2]}(F\tilde{H}y+\tilde{G}Ky,p) \\
&\quad + y^{\mathrm{T}}(\tilde{F}+\tilde{G}K)^{\mathrm{T}}\cdot \pi_{yy}H\cdot \Big(f^{[2]}(\tilde{H}y,Ky,p)+GK\cdot h^{[2]}(\tilde{H}y,p)\Big) \\
&\quad + y^{\mathrm{T}}(\tilde{F}+\tilde{G}K)^{\mathrm{T}}\cdot \pi_{yy}\tilde{G}K\cdot h^{[2]}(\tilde{H}y,p) + \ell^{[3]}(y,Ky) \\
&\quad + y^{\mathrm{T}}\ell_{yu}K\cdot h^{[2]}(\tilde{H}y,p) + h^{[2]}(\tilde{H}y,p)^{\mathrm{T}}\cdot \ell_{yu}\cdot Ky + y^{\mathrm{T}}K^{\mathrm{T}}\cdot \ell_{uu}K\cdot h^{[2]}(\tilde{H}y,p)
\end{aligned} \tag{$\pi^{[3]}$}$$

Since $\tilde{F} + \tilde{G}K$ must be stable as $K$ is the solution of an LQR, $(\pi^{[3]})$ has the shape of equation (7.3) from Corollary 7. Thus the coefficients of $\pi^{[3]}$ can be found and are unique. Having this $\kappa^{[2]}(y, p)$ is also found.
Next, the general case $\pi^{[k+1]}$ and $\kappa^{[k]}$ for $k \geq 2$ is tackled. The steps are exactly the same as for the calculation of $\pi^{[3]}$ and $\kappa^{[2]}$ but the formulas are more complicated. First, all $k$-th order terms of (BDP-2) will be collected. $y$ is replaced with the power series of the output function, while $x$, afterward, is written using $\tilde{H}y$.

$$
\begin{aligned}
0 &= \Big[\nabla_y \pi\Big(h\Big(f\Big(x, \kappa(h(\tilde{H}y, p), p), p\Big), p\Big), p\Big) \cdot \nabla_x h\Big(f\Big(\tilde{H}y, \kappa(h(\tilde{H}y, p), p), p\Big), p\Big) \\
&\qquad \cdot \nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y, p), p\Big), p\Big)\Big]^{[k]} + \Big[\nabla_u \ell\Big(h(\tilde{H}y, p), \kappa\Big(h(\tilde{H}y, p), p\Big)\Big)\Big]^{[k]} \\
&= \Big[\nabla_y \pi^{[3;k+1]}\Big(h\Big(f\Big(\tilde{H}y, \kappa(h(\tilde{H}y, p), p), p\Big), p\Big), p\Big) \cdot \nabla_x h\Big(f\Big(\tilde{H}y, \kappa(h(\tilde{H}y, p), p), p\Big), p\Big) \\
&\qquad \cdot \nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y, p), p\Big), p\Big)\Big]^{[k]} \\
&\quad + \Big[\Big(h\Big(f\Big(\tilde{H}y, \kappa^{[1;k-1]}(h(\tilde{H}y, p), p), p\Big), p\Big), p\Big) \cdot \pi_{yy} \cdot \nabla_x h\Big(f\Big(\tilde{H}y, \kappa(h(\tilde{H}y, p), p), p\Big), p\Big) \\
&\qquad \cdot \nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y, p), p\Big), p\Big)\Big]^{[k]} \\
&\quad + \Big[\nabla_u \ell^{[3;k+1]}\Big(h(\tilde{H}y, p), \kappa\Big(h(\tilde{H}y, p), p\Big)\Big)\Big]^{[k]} + h^{[k]}(\tilde{H}y, p)^{\mathrm{T}} \cdot \ell_{yu} \\
&\quad + \Big(\Big[\kappa^{[1;k-1]}\Big(h(\tilde{H}y, p), p\Big)\Big]^{[k]}\Big)^{\mathrm{T}} \cdot \ell_{uu} + \kappa^{[k]}(y, p)^{\mathrm{T}} \cdot \Big(\ell_{uu} + \tilde{G}^{\mathrm{T}} \pi_{yy} \tilde{G}\Big)
\end{aligned}
$$

$\kappa^{[k]}(y, p)$ can be separated and only depends on one unknown, which is $\nabla_y \pi^{[k+1]}$.

$$
\begin{aligned}
\kappa^{[k]}(y,p) = -\Big(\ell_{uu} + \tilde{G}^{\mathrm{T}}\pi_{yy}\tilde{G}\Big)^{-1} \cdot \Bigg( & \Big[\nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)^{\mathrm{T}} \\
& \cdot \nabla_x h\Big(f\Big(\tilde{H}y, \kappa(h(\tilde{H}y,p),p),p\Big),p\Big)^{\mathrm{T}} \\
& \cdot \nabla_y \pi^{[3;k+1]}\Big(h\Big(f\Big(\tilde{H}y, \kappa(h(\tilde{H}y,p),p),p\Big),p\Big),p\Big)^{\mathrm{T}}\Big]^{[k]} \\
& + \Big[\nabla_u f\Big(\tilde{H}y, \kappa\Big(h(\tilde{H}y,p),p\Big),p\Big)^{\mathrm{T}} \\
& \cdot \nabla_x h\Big(f\Big(\tilde{H}y, \kappa(h(\tilde{H}y,p),p),p\Big),p\Big)^{\mathrm{T}} \cdot \pi_{yy} \\
& \cdot \Big(h\Big(f\Big(\tilde{H}y, \kappa^{[1;k-1]}(h(\tilde{H}y,p),p),p\Big),p\Big),p\Big)^{\mathrm{T}}\Big]^{[k]} \\
& + \Big[\nabla_u \ell^{[3;k+1]}\Big(h(\tilde{H}y,p), \kappa\Big(h(\tilde{H}y,p),p\Big)\Big)^{\mathrm{T}}\Big]^{[k]} \\
& + \ell_{yu} \cdot h^{[k]}(\tilde{H}y,p)\Bigg) \\
- \Big[\kappa^{[1;k-1]}\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k]} &
\end{aligned} \tag{$\kappa^{[k]}$}
$$

$\pi^{[k+1]}(y,p)$ is uniquely defined via the equation that is obtained by collecting all terms from (BDP-1) that are homogeneous with degree $k+1$.

$$
\begin{aligned}
\Big[\pi\Big(h(\tilde{H}y,p),p\Big)\Big]^{[k+1]} &= \Big[\pi\Big(h\Big(f(\tilde{H}y, \kappa(h(\tilde{H}y,p),p),p),p\Big),p\Big)\Big]^{[k+1]} \\
&\quad + \Big[\ell\Big(h(\tilde{H}y,p), \kappa(h(\tilde{H}y,p),p)\Big)\Big]^{[k+1]} \\
&= \Big[\pi^{[3;k+1]}\Big(h\Big(f(\tilde{H}y, \kappa(h(\tilde{H}y,p),p),p),p\Big),p\Big)\Big]^{[k+1]} \\
&\quad + \Big[\ell^{[3;k+1]}\Big(h(\tilde{H}y,p), \kappa(h(\tilde{H}y,p),p)\Big)\Big]^{[k+1]} \\
&\quad + \Big[\pi^{[2]}\Big(h\Big(f(\tilde{H}y, \kappa^{[1;k-1]}(h(\tilde{H}y,p),p),p),p\Big),p\Big)\Big]^{[k+1]} \\
&\quad + y^{\mathrm{T}}(\tilde{F} + \tilde{G}K)^{\mathrm{T}} \cdot \pi_{yy}\tilde{G} \cdot \kappa^{[k]}(y,p) \\
&\quad + \Big[\ell^{[2]}\Big(h(\tilde{H}y,p), \kappa^{[1;k-1]}(h(\tilde{H}y,p),p)\Big)\Big]^{[k+1]} \\
&\quad + y^{\mathrm{T}} \cdot \ell_{yu} \cdot \kappa^{[k]}(y,p) + y^{\mathrm{T}}K^{\mathrm{T}} \cdot \ell_{uu} \cdot \kappa^{[k]}(y,p)
\end{aligned}
$$

$(K)$ is again used for simplification and removes all terms that contain $\kappa^{[k]}$ from the equation.

Thus the only unknown is $\pi^{[k+1]}$.

$$\begin{aligned}\pi^{[k+1]}(y,p) &= \pi^{[k+1]}\Big((\tilde F+\tilde G K)y,p\Big) - \Big[\pi^{[2;k]}\Big(h(\tilde H y,p),p\Big)\Big]^{[k+1]}\\ &+ \Big[\pi^{[3;k]}\Big(h\Big(f(\tilde H y,\kappa(h(\tilde H y,p),p),p),p\Big),p\Big)\Big]^{[k+1]}\\ &+ \Big[\ell^{[3;k+1]}\Big(h(\tilde H y,p),\kappa(h(\tilde H y,p),p)\Big)\Big]^{[k+1]}\\ &+ \Big[\pi^{[2]}\Big(h\Big(f(\tilde H y,\kappa^{[1;k-1]}(h(\tilde H y,p),p),p),p\Big),p\Big)\Big]^{[k+1]}\\ &+ \Big[\ell^{[2]}\Big(h(\tilde H y,p),\kappa^{[1;k-1]}(h(\tilde H y,p),p)\Big)\Big]^{[k+1]}\end{aligned} \tag{$\pi^{[k+1]}$}$$

Applying Corollary 7 ensures the unique solvability of this equation. Thus $\kappa^{[k]}$ is also known and all parts of the power series ($\pi$) and ($\kappa$) can be found. One should keep in mind that the existence resp. the convergence of both series is not guaranteed and needs to be evaluated separately.

**Remark 11.** *As for the continuous case, the matrix $\tilde H$ is not unique, and thus the optimal feedback $\kappa(y,p)$ and value function $\pi(y,p)$ also depend on $\tilde H$. How a suitable $\tilde H$ can be found will be shown in Lemma 2 in the following chapter.*

**Theorem 7** (Determinability of $\pi$ and $\kappa$)**.**
*Consider a discrete-time optimal control problem (OCP), where $f(x,u,p)$ and $\ell(y,u)$ are analytic functions with power series expansions ($f$) and ($\ell$). If additionally, the conditions (II)-(V), as well as ($f_0$), and ($h_0$) are fulfilled, then each part of the power series ($\pi$) and ($\kappa$) is uniquely defined. Furthermore, for all $p$ in a neighborhood of the origin, it holds*

$$\pi(0,p)=0,\quad \nabla_y\pi(0,p)=0 \quad \textit{and} \quad \kappa(0,p)=0.$$

*Proof.* Similar to the proof of Theorem 7, it suffices to show condition (I).
(IV) and Lemma 3 give

$$\operatorname{rank}\begin{pmatrix}\tilde F - I_{n_y} & \tilde G\end{pmatrix} = \begin{pmatrix}H\cdot(F-I_{n_x})\cdot\tilde H & HG\end{pmatrix} = n_y.$$

The columns of $H\cdot(F-I_{n_x})\cdot\tilde H$ are linear combinations of the columns of $H\cdot(F-I_{n_x})$. Thus $\begin{pmatrix}H\cdot(F-I_{n_x}) & HG\end{pmatrix}$ must have full rank and (I) is shown. □

**Remark 12** (Additive parameters)**.**
*Using the procedure of Remark 8, additive parameters resp. not vanishing functions $p\mapsto f(0,0,p)$ and $p\mapsto h(0,p)$ are also handled in this setup. One can copy the procedure, but use the linear dynamics*

$$\begin{aligned} x_f^+ &= (1-\alpha_f)\cdot x_f\\ \textit{and } x_h^+ &= (1-\alpha_h)\cdot x_h.\end{aligned}$$

*Only the matrices $\bar{F}$ and $\tilde{\bar{F}}$ have to be stated/calculated again. All other matrices and functions follow the same definitions.*

$$\bar{F} = \begin{pmatrix} F & f_0 & 0 \\ 0 & 1-\alpha_f & 0 \\ 0 & 1-\alpha_h & 0 \end{pmatrix}, \qquad \tilde{\bar{F}} = \begin{pmatrix} \tilde{F} & H \cdot f_0 & (1-\alpha_h-\tilde{F}) \cdot h_0 \\ 0 & 1-\alpha_f & 0 \\ 0 & 0 & 1-\alpha_h \end{pmatrix}$$

**Corollary 5** (Local stability)**.**
*If the power series ($\pi$) and ($\kappa$) are converging and under the requirements of Theorem 7, local stability of the output is achieved for sufficiently small parameters $p$.*

*Proof.* Fix a parameter vector $p \in \mathbb{R}^{n_p}$. Then $\tilde{\pi}(y) := \pi(y,p)$ is used as a local Lyapunov function candidate. The control law is defined as $\tilde{u}_{\min}(y) := \tilde{\kappa}(y) := \kappa(y,p)$. Both functions are the solution of

$$\begin{aligned} \tilde{\pi}\big(y(0)\big) &= \min_{u(.)} \sum_{n=0}^{\infty} \ell\big(y(n), u(n)\big) \\ \text{s.t. } x^+ &= \tilde{f}(x,u), \\ y &= \tilde{h}(x), \end{aligned}$$

where $\tilde{f}(x,u) := f(x,u,p)$ and $\tilde{h}(x) := h(x,p)$. If $p$ is sufficiently small, then $\tilde{f}^{[1]}(.,.)$ resp. $\tilde{h}^{[1]}(.,.)$ inherits the properties of $f^{[1]}(.,.,.)$ resp. $h^{[1]}(.,.,.)$. Therefore, $\tilde{\pi}(.)$ and $\tilde{\kappa}(.)$ can be written as

$$\begin{aligned} \tilde{\pi}(y) &= \frac{1}{2} y^{\mathrm{T}} \tilde{\pi}_{yy} y + o(||y||^3), \\ \tilde{\kappa}(y) &= \tilde{K} y + o(||y||^2). \end{aligned}$$

Thus there exists an $\varepsilon > 0$ such that

$$\tilde{\pi}(y) \geq 0$$

for all $y \in B_\varepsilon(0)$. Due to the positive definiteness of $\tilde{\pi}_{yy}$, equality only holds for vanishing $y$. Therefore $\tilde{\pi}(.)$ is locally positive definite. Using Bellman's equation,

$$\tilde{\pi}(y) = \tilde{\pi}\Big(\tilde{h}(\tilde{f}(x,\kappa(y)))\Big) + \ell\big(y,\kappa(y)\big),$$

it is seen that $\tilde{\pi}\Big(\tilde{h}\big(\tilde{f}(x,\tilde{\kappa}(y))\big)\Big) - \tilde{\pi}(y)$ is locally negative definite since (III) is holding. Thus $\tilde{\pi}(y) = \pi(y,p)$ is a local Lyapunov function for

$$y^+ = \tilde{h}\Big(\tilde{f}(x,\tilde{\kappa}(y))\Big) = h\Big(f(x,\kappa(y,p),p),p\Big),$$

fixed $p \in \mathbb{R}^{n_p}$ and $y \in B_\varepsilon(0)$. If $p \in \mathbb{R}^{n_p}$ varies but stays in $\overline{B_\delta(0)}$ for a sufficiently small $\delta > 0$, then $y \mapsto \pi(y,p)$ is used as local Lyapunov function. In the previous case, $y \mapsto \pi(y,p)$ was a local Lyapunov function for $y \in B_\varepsilon(0)$. Obviously, $\varepsilon$ depends on $p$, but $p \mapsto \pi(y, p)$ is $C^\infty$ and, therefore, $p \mapsto \varepsilon(p)$ is at least continuous and takes its minimum value in the compact set $\overline{B_\delta(0)}$. Denote this minimum with $\varepsilon_{\min}$. $\varepsilon_{\min}$ must be greater than zero else there would be a parameter $p$ that contradicts the first case. Therefore, $y \mapsto \pi(y, p)$ is a Lyapunov function for $y \in B_{\varepsilon_{\min}}(0)$. $\square$

**Remark 13.** *If, in the discrete-time optimal control problem (OCP), the output dimension $n_y$ is larger than the state dimension $n_x$, then the same problem as addressed in Remark 9 arises. The maximal rank that the matrix $H \in \mathbb{R}^{n_y \times n_x}$ could have is $n_x$, and (6.3) is not solvable since $H$ does not have a right inverse, and $\pi_{yy}$ can not be obtained. Calculating $H^{\mathrm{T}}\pi_{yy}H$ is possible if the rank of $H$ is $n_x$. But in this case, all states would be observable, and there is no need for output-feedback.*

**Remark 14.** *Throughout this section, Al'brekht's Method has been shown for parametric output-feedback control, which is the most general case considered in this work. Setting $y = h(x, p) = x$ leads to the discrete-time pendant of Section 3.1. If additionally the parameters p are set to zero, the discrete-time version of the original method is obtained. Both cases were not discussed separately since the procedure is the same as shown here. Nevertheless, the continuous-time method was shown in all three variants to make it easier for the reader to understand all the details.*

In this chapter, it has been outlined how Al'brekht's power series approach can be used for parametric output-feedback. The additional requirements make it less applicable. However, the convergence proof has been extended, and local stability was also proven. In the next chapter, the discrete-time counterpart of Al'brekht's Method is investigated considering the output-feedback case and thus also the nominal and the parametric case.

# 7 Toolbox for Obtaining Approximated Solutions

All presented results and methods, namely, the classical non-parametric case (Chapter 2), the parametric case (Chapter 3), the constraint case (Chapter 4), the output-feedback case (Chapter 5), the discrete-time version (Chapter 6), and the case where additive parameters are present in the system dynamics (Remark 3) resp. the output function (Remarks 8 and 12) have been implemented. The implementation of the „Solver for Al'brekht's Method" or short SAM has been done in C++ using two special toolboxes, namely, GiNaC [95] and CLN [30]. GiNaC is an abbreviation for GiNaC is Not a CAS, where CAS stands for Computer Algebra System. It is used for symbolic calculation and especially for symbolic differentiation. It allows to perform the calculations much faster than, e. g. MATLAB. GiNaC also utilizes the CLN library for efficient numeric calculation with arbitrarily high precision. Both toolboxes can be downloaded from (`https://www.ginac.de/Download.html`) resp. (`https://www.ginac.de/CLN/`).

SAM reads the optimal control problems with infinite horizon from a text file. It then checks for consistency of the input and decides which version of Al'brekht's Method has to be used. The output is then written into three files. The first one, a -*.txt*-file, only contains the different parts of the power series of the control law $\kappa^{[k]}(x/y, p)$ and the optimal cost function $\pi^{[k+1]}(x/y, p)$ for $k \in [d]$, where $d \in \mathbb{N}$ is the user-chosen degree of approximation. The matrices $K$ and $\pi_{xx}$ resp. $\pi_{yy}$ are also given explicitly. The next file is an *.m*-file, which states the complete OCP in MATLAB syntax. The different approximations of the control law are applied to the system dynamics and the cost function. The results are saved in separated variables. After running this file, all created variables are saved for further purposes in a *.mat*-file. Using the last file, which is used with MATLAB, the user can automatically simulate all the controllers. The initial values, the time horizon, and time-dependent parameters can be adjusted. Plots of the state and output (if present) propagation as well as the change of the parameters, the controls, the cost to go, and the total cost are provided. All files are tested in *MATLAB 2017a* and higher. Older versions might require minor adjustments.

A similar MATLAB toolbox, containing the cases of Al'brekht's Method that can be found in the literature, the so-called Nonlinear Systems Toolbox [52] exists. This toolbox is accessibly via MATLAB but is also based on C++ code. It can also consider other partial differential equations such as e. g. the Francis-Byrnes-Isidori PDE, the Kazantzis-Kravaris PDE and is not only focusing on the Hamilton-Jacobi-Bellman and the Bellman equation. Nevertheless, for this thesis, an own solver was needed,

since the considered setups for Al'brekht's Method in the parametric case and the output feedback, as well as some solution approaches (e. g. handling constraints) are different.
The remainder of this chapter is organized as follows. In Section 7.1, the existence of the matrix $\tilde{H} \in \mathbb{R}^{n_y \times n_x}$ is investigated as this is central for the output-feedback case. The proofs also show how its calculation is implemented in SAM. Section 7.2 deals with the calculation of the coefficients of the different degrees of the power series of $\pi$ and $\kappa$. The proofs therein show how linear equation systems are derived from the partial derivative (continuous time) resp. recursive (discrete time) equations. Tables stating the calculation time for the examples shown in the Chapters 2 to 5 and the number of coefficients, that had to be calculated, are provided.

## 7.1 Calculation of $\tilde{H}$

The matrix $\tilde{H} \in \mathbb{R}^{n_y \times n_x}$, which was first mentioned in Section 5.1, can be calculated via SAM. Due to the non-uniqueness of $\tilde{H}$, SAM only selects one possible solution out of in general infinitely many. If SAM does not find a suitable $\tilde{H}$, then it does not necessarily mean that there exists non, since at the current state of the art the calculation contains some trial and error. This will become clear in the following derivation, in which conditions for the existence of $\tilde{H}$ are derived.
The matrix $\tilde{H}$ needs to fulfill the following conditions:

(I) First, it has to be a right inverse of a matrix $H \in \mathbb{R}^{n_y \times n_x}$, which has full rank,

$$H \cdot \tilde{H} = I_{n_y},$$

(II) the pair $\left(\tilde{F}, \tilde{G}\right) = \left(HF\tilde{H}, HG\right)$ has to be controllable for $(F, G)$ controllable, and

(III) the pair $\left(\tilde{F}, \ell_{yy}\right)$ has to be observable.

These conditions now may raise the question, whether such a matrix exists at all and under which conditions can the existence be guaranteed. The following lemma provides a first step towards the existence and calculation of $\tilde{H}$.

**Lemma 2** (Existence of $\tilde{H}$).
*If*

*(I')* $H \in \mathbb{R}^{n_y \times n_x}$ *has full rank,*

*(II')* $(HG \;\; HFG \;\; \dots \;\; HF^{n_x-1}G)$ *has rank* $n_y$*, and*

*(III')* *the pair* $(F, H^{\mathrm{T}}\ell_{yy}H)$ *is observable,*

*then there exists* $\tilde{H} \in \mathbb{R}^{n_x \times n_y}$ *such that*

*(I")* $H \cdot \tilde{H} = I_{n_y}$,

*(II")* $(\tilde{F}, \tilde{G}) = (HF\tilde{H}, HG)$ *is controllable, and*

*(III")* $(\tilde{F}, \ell_{yy})$ *is observable.*

*Proof.* $H$ has rank $n_y$ and it is $n_y \leq n_x$. So it is clear that there exists a right inverse $\tilde{H}$. Thus (I") is obvious. To show (II"), one needs to find $\tilde{H}$ such that

$$\operatorname{rank}\begin{pmatrix} HG & HF\tilde{H}HG & \dots (HF\tilde{H})^{n_x-1}HG \end{pmatrix} = n_y.$$

Adding unit vectors as rows to $H$, a quadratic and invertible matrix can be obtained. Thus there exist $n_x - n_y$ vectors $v_1, \dots, v_{n_x-n_y} \in \mathbb{R}^{n_x}$ such that

$$\begin{pmatrix} H \\ e_{i_1}^{\mathrm{T}} \\ \vdots \\ e_{i_{n_x-n_y}}^{\mathrm{T}} \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{H} & v_1 & \dots & v_{n_x-n_y} \end{pmatrix}$$

for some $i_1, \dots, i_{n_x-n_y} \in [n_x]$. Since $H \cdot v_k = 0$ for all $k \in [n_x - n_y]$, the vectors $v_k$ can be added to the columns of $\tilde{H}$ without violating (I"). Having this, $\tilde{H}$ can be chosen such that

$$\operatorname{rank}(HF) = \operatorname{rank}(HF\tilde{H}) = \operatorname{rank}(\tilde{F})$$

and

$$\operatorname{rank}(HG \; HF\tilde{H} \cdot HG \; \dots (HF\tilde{H})^{n_y-1} \cdot HG) = n_y$$

as well as

$$\operatorname{rank}\begin{pmatrix} \ell_{yy} \\ \ell_{yy} \cdot HF\tilde{H} \\ \vdots \\ \ell_{yy} \cdot (HF\tilde{H})^{n_y-1} \end{pmatrix} = n_y,$$

which shows (II") and (III"). □

**Remark 15.** *Condition (II') in Lemma 2 states that the outputs $y$ but not all the states $x$ have to be controllable. Condition (III') basically states the observability of the states $x$.*

Motivated by the proof, SAM uses linear combinations of the vectors $v_1, \dots, v_{n_x-n_y}$ and adds it to the columns of an $\tilde{H}$ gained via condition (I'). At the current state of the art, this procedure is still based on trial and error and may not find a suitable matrix even though the existence is clear.
In the subsequent corollary, the result from Lemma 2 is generalized.

**Corollary 6** (Existence of $\tilde{H}$)**.**
*The assumptions of Lemma 2 can be weakened to the following.*

*(I*)* $H \in \mathbb{R}^{n_y \times n_x}$ *has full rank,*
*(II*) the non-controllable outputs are stable and*
*(III*) the pair* $(F, H^{\mathrm{T}}\ell_{yy}H)$ *is detectable.*

*Having this the existence of* $\tilde{H} \in \mathbb{R}^{n_x \times n_y}$ *such that*

*(I**)* $H \cdot \tilde{H} = I_{n_y}$,
*(II**)* $(\tilde{F}, \tilde{G}) = (HF\tilde{H}, HG)$ *is stabilizable and*
*(III**)* $(\tilde{F}, \ell_{yy})$ *is detectable.*

*is still guaranteed.*

*Proof.* Again condition (I*) directly implies the existence of $\tilde{H} \in \mathbb{R}^{n_x \times n_y}$ fulfilling (I**). For the other conditions, the system states are first split up into four groups using the Kalman decomposition [70]. To do so a coordinate transformation $z := Tx$ with $T \in \mathrm{GL}(n_x; \mathbb{R})$ is applied.

$$\dot{z} = TFT^{-1} \cdot z + TG \cdot u =: \bar{F}z + \bar{G}u$$
$$y = HT^{-1} \cdot z =: \bar{H}z$$

Now $z$ is written as $\begin{pmatrix} z_{\mathrm{c,o}} & z_{\mathrm{c,no}} & z_{\mathrm{nc,o}} & z_{\mathrm{nc,no}} \end{pmatrix}^{\mathrm{T}}$, where „c", „nc", „o", and „no" stand for controllable, non-controllable, observable, and non-observable. W.l.o.g. $y$ is assumed to be split up into its controllable and non-controllable resp. observable and non-observable parts and is written as $\begin{pmatrix} y_{\mathrm{c,o}} & y_{\mathrm{c,no}} & y_{\mathrm{nc,o}} & y_{\mathrm{nc,no}} \end{pmatrix}^{\mathrm{T}}$. This can be achieved using another coordinate transformation, which just switches the rows of $H$. Therefore the output matrix $H$ is written in a similar way, $H = \begin{pmatrix} H_{\mathrm{c,o}} & H_{\mathrm{c,no}} & H_{\mathrm{nc,o}} & H_{\mathrm{nc,no}} \end{pmatrix}^{\mathrm{T}}$ and condition (II*) implies

$$\mathrm{rank}\begin{pmatrix} H_{\mathrm{c}}G & H_{\mathrm{c}}FG & \ldots & H_{\mathrm{c}}F^{n_x-1}G \end{pmatrix} = \mathrm{rank}(H_{\mathrm{c}}).$$

So one obtains condition (II') from Lemma 2 for the controllable outputs. Because of (II*) the non-controllable outputs are stable, which then leads to (II**).
Furthermore, the matrix $H^{\mathrm{T}}\ell_{yy}H$ can be split into rows belonging to observable and non-observable outputs. The first part and $\bar{F}$ are then used for condition (III') in Lemma 2, while the others have to be stable due to (III*). Thus Lemma 2 can be applied to achieve (II**) and (III**) at the same time. □

## 7.2 Coefficients of polynomials

Efficient calculation of the coefficients of the power series is of core importance to obtain the explicit control law. Using the results and methods shown in the previous chapters, a procedure to calculate the coefficients of the power series expansions of the value function and the control law has been implemented. Before going into detail, the number of coefficients of a polynomial, which is homogeneous with degree $d \in \mathbb{N}$ and depends on $n \in \mathbb{N}$ variables, has to be found. This number can be calculated recursively. Therefore let $N(n, d)$ denote the number of those coefficients. If $n \geq 2$ and $d \geq 2$, then $N(n, d)$ is given via

$$\frac{(n+d-1)!}{d! \cdot (n-1)!} = N(n-1, d) + N(n, d-1) \text{ resp. } \binom{n+d-1}{d},$$

while $N(1, d) = 1$ and $N(n, 1) = n$. The following table shows the results for $N(.,.)$. Numbers greater than 100 000 are neglected to compress the table. The numbers, which are relevant for the examples that are discussed throughout this work, are underlined.

| $n \backslash d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 3 | 6 | 10 | 15 | 21 | 28 | 36 | 45 | 55 | 66 |
| 4 | 4 | 10 | 20 | 35 | 56 | 84 | 120 | 165 | 220 | 286 |
| 5 | 5 | 15 | 35 | 70 | 126 | 210 | $\underline{330}$ | 495 | 715 | 1001 |
| 6 | 6 | 21 | 56 | 126 | 252 | 462 | $\underline{792}$ | 1287 | 2002 | 3003 |
| 7 | 7 | 28 | 84 | 210 | 462 | $\underline{924}$ | $\underline{1716}$ | 3003 | 5005 | 8008 |
| 8 | 8 | 36 | 120 | 330 | 792 | 1716 | 3432 | 6435 | 11 440 | 19 448 |
| 9 | 9 | 45 | 165 | 495 | 1287 | 3003 | 6335 | 12 870 | 34 310 | 43 758 |
| 10 | 10 | 55 | 220 | 715 | $\underline{2002}$ | 5005 | 11 440 | 24 310 | 48 620 | 92 378 |
| 11 | 11 | 66 | 286 | 1001 | 3003 | 8008 | 19 448 | 43 758 | 92 378 | × |
| 12 | 12 | 78 | 364 | 1365 | $\underline{4368}$ | 12 376 | 31 824 | 75 582 | × | × |
| 13 | 13 | 91 | 455 | $\underline{1820}$ | 6188 | 18 564 | 50 388 | × | × | × |
| 14 | 14 | 105 | 560 | 2380 | 8568 | 27 132 | 77 520 | × | × | × |
| 15 | 15 | 120 | 680 | 3060 | 11 628 | 38 760 | × | × | × | × |
| 16 | 16 | 136 | 816 | $\underline{3876}$ | 15 504 | 54 264 | × | × | × | × |
| 17 | 17 | 153 | 969 | 4845 | 20 349 | 74 613 | × | × | × | × |
| 18 | 18 | 171 | 1140 | $\underline{5985}$ | 26 334 | × | × | × | × | × |

**Table 7.1:** Number of coefficients of polynomials with degree $d$ in $n$ variables

One can see that the complexity increases with the degree $d$ but even more with the number of variables $n$. The calculation time increases accordingly and is shown in Table 7.2 for the examples of the previous chapters. The calculation has been carried out on a Dell Optiplex 780 SFF Desktop-PC (Core 2 Quad Q8400, 2,66 GHz) with 4038 MB RAM. It is not visible from the table, but definitely worth mentioning that the calculation time also depends on the „difficulty“ of the nonlinearities. Thus including constraints increases the calculation time even though the number of variables and the degree stay the same, see Table 7.2.

| Model | specification\Degree | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Quadcopter | $n_p = 0$ | 11.0 s | 349 s | 127 min | - | - |
| $n_x = 10$ | $n_p = 6$ | 309 s | 662 min | - | - | - |
| | $n_p = 6,\ n_c = 11$ | 312 s | 646 min | - | - | - |
| | $n_p = 6,\ n_c = 6$ | 313 s | 646 min | - | - | - |
| $\tilde{H}_1$ | $n_y = 6$ | 0.5 s | 5.1 s | 40 s | 251 s | 22 min |
| $\tilde{H}_2$ | $n_y = 6$ | 0.6 s | 6.7 s | 53 s | 340 s | 29 min |
| $\tilde{H}_1$ | $n_y = 6,\ n_p = 6$ | 21.6 s | 22 min | 13.5 h | - | - |
| $\tilde{H}_2$ | $n_y = 6,\ n_p = 6$ | 29.7 s | 31 min | 19.7 h | - | - |
| $\tilde{H}_1$ | $n_y = 6,\ n_p = 6,\ n_c = 6$ | 22.1 s | 25 min | - | - | - |
| $\tilde{H}_2$ | $n_y = 6,\ n_p = 6,\ n_c = 6$ | 30.4 s | 34 min | - | - | - |
| Quadcopter | $n_p = 0$ | 44 s | 37 min | - | - | - |
| $n_x = 12$ | $n_p = 6$ | 15 min | 39.0 h | - | - | - |
| | $n_p = 6,\ n_c = 8$ | 15 min | 41 h | - | - | - |
| | $n_p = 6,\ n_c = 15$ | 15 min | 41 h | - | - | - |
| | $n_y = 7$ | 1.4 s | 20 s | 198 s | 26 min | - |
| $\tilde{H}$ given | $n_y = 7,\ n_p = 6$ | 51 s | 61 min | - | - | - |
| $\tilde{H}$ not given | $n_y = 7,\ n_p = 6$ | 73 s | 87 min | - | - | - |
| $\tilde{H}$ given | $n_y = 7,\ n_p = 6,\ n_c = 8$ | 53 s | 69 min | - | - | - |
| Bioreactor | $n_p = 3$ | 0.79 s | 12.3 s | 133 s | 18 min | 8.3 h |
| $n_x = 3$ | $n_y = 1,\ n_p = 2$ | 0.45 s | 1.5 s | 12 s | 80 s | 436 s |

**Table 7.2:** Calculation time for two quadcopter models and a bioreactor example with different settings regarding the parameters, measurements and constraints

The next two theorems and corollaries state the existence and calculability of $\pi^{[k+1]}(x/y,p)$ and $\kappa^{[k]}(x/y,p)$ ($k \geq 2$) first in discrete and then in continuous time. It can be seen via comparing formula ($\pi^{[k+1]}$) from Chapter 6 with (7.3) resp. formula ($\pi^{[k+1]}$) from Section 2.1 with (7.4) and so forth. From the proofs, one can also see how the linear equations for the coefficients are derived within SAM.

**Theorem 8.**
*Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ with all eigenvalues inside the unit disc. Furthermore, let $q(x)$ be a polynomial with degree of homogeneity $d \in \mathbb{N}$, i. e. for all $\lambda \in \mathbb{R}$ it holds*

$$q(\lambda x) = \lambda^d \cdot q(x).$$

*Then there exists a polynomial $p(x)$ with degree of homogeneity $d \in \mathbb{N}$ such that*

$$p(x) - p(Ax) = q(x). \tag{7.1}$$

*A similar result and proof can be found in [29].*

*Proof.* The polynomials $p(x)$ and $q(x)$ can be written in the form

$$\sum_{\substack{(i_1,\dots,i_d)\in[n]^d \\ i_j \le i_{j+1},\ j\in[n-1]}} c_{i_1,\dots,i_d} \cdot x_{i_1} \cdot \ldots \cdot x_{i_d} = \begin{pmatrix} c_{1,\dots,1} & \dots & c_{n,\dots,n} \end{pmatrix} \cdot \begin{pmatrix} x_1^d \\ x_1^{d-1} \cdot x_2 \\ \vdots \\ x_n^d \end{pmatrix}, \tag{7.2}$$

where $c_i \in \mathbb{R}$ is replaced by $c^p_{i_1,\dots,i_d}$ resp. $c^q_{i_1,\dots,i_d}$.
To show the existence of $p(.)$, the coefficients $c^p_{i_1,\dots,i_d}$ will be calculated. To do so, two cases are distinguished.
<u>Case 1:</u> $A$ is a diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{R}$.

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_d \end{pmatrix} \quad \Rightarrow \quad Ax = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_d x_d \end{pmatrix}$$

Thus, in this case, $p(Ax)$ is simply given by

$$\begin{aligned} p(Ax) &= \sum_{\substack{(i_1,\dots,i_d)\in[n]^d \\ i_j \le i_{j+1},\ j\in[n-1]}} c^p_{i_1,\dots,i_d} \cdot \lambda_{i_1} x_{i_1} \cdot \ldots \cdot \lambda_{i_d} x_{i_d} \\ &= \begin{pmatrix} c^p_{1,\dots,1} & \dots & c^p_{n,\dots,n} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1^d & 0 & \dots & 0 \\ 0 & \lambda_1^{d-1} \cdot \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n^d \end{pmatrix} \cdot \begin{pmatrix} x_1^d \\ x_1^{d-1} \cdot x_2 \\ \vdots \\ x_n^d \end{pmatrix}, \end{aligned}$$

and it is easy to calculate the left-hand side of (7.1).

$$p(x) - p(Ax) = \begin{pmatrix} c^p_{1,\dots,1} & \dots & c^p_{n,\dots,n} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1-\lambda_1^d & 0 & \dots & 0 \\ 0 & 1-\lambda_1^{d-1}\cdot\lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1-\lambda_n^d \end{pmatrix}}_{=:B} \cdot \begin{pmatrix} x_1^d \\ x_1^{d-1}\cdot x_2 \\ \vdots \\ x_n^d \end{pmatrix}$$

$$= \begin{pmatrix} c^q_{1,\dots,1} & \dots & c^q_{n,\dots,n} \end{pmatrix} \cdot \begin{pmatrix} x_1^d \\ x_1^{d-1}\cdot x_2 \\ \vdots \\ x_n^d \end{pmatrix}$$

Since $\lambda_i \in (-1,1)$ for all $i \in [n]$, it is clear that $1-\lambda_{i_1}\cdot\ldots\cdot\lambda_{i_d} \in (0,2)$. Leaving out the variables $x$, since they are arbitrary and the equation has to hold in a neighborhood of the origin, leads to

$$\begin{pmatrix} c^p_{1,\dots,1} & \dots & c^p_{n,\dots,n} \end{pmatrix} = \begin{pmatrix} c^q_{1,\dots,1} & \dots & c^q_{n,\dots,n} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{1-\lambda_1^d} & 0 & \dots & 0 \\ 0 & \frac{1}{1-\lambda_1^{d-1}\cdot\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{1-\lambda_n^d} \end{pmatrix},$$

and, therefore, $p(.)$ is determined and unique.
Case 2: $A$ is not a diagonal matrix.
Let $A_{\mathrm{JNF}} \in \mathbb{C}^{n\times n}$ be the Jordan normal form of $A$ and $S \in \mathrm{GL}(n;\mathbb{C})$ the associated transformation matrix.

$$A = S^{-1} \cdot A_{\mathrm{JNF}} \cdot S$$

Define $z := Sx$ and $\tilde{p}(z) := p(S^{-1}z)$ as well as $\tilde{q}(z) := q(S^{-1}z)$.

$$p(x) - p(Ax) = \tilde{p}(z) - \tilde{p}(A_{\mathrm{JNF}}z) = \tilde{q}(z)$$

If $\tilde{c}^p_{i_1,\dots,i_d} \in \mathbb{C}$ and $\tilde{c}^q_{i_1,\dots,i_d} \in \mathbb{R}$ $\left((i_1,\dots,i_d) \in [n]^d\right)$ are the coefficients of $\tilde{p}(.)$ resp. $\tilde{q}(.)$, then with a similar calculation as in case 1 one obtains

$$\begin{pmatrix} \tilde{c}^p_{1,\dots,1} & \dots & \tilde{c}^p_{n,\dots,n} \end{pmatrix} = \begin{pmatrix} \tilde{c}^q_{1,\dots,1} & \dots & \tilde{c}^q_{n,\dots,n} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{1-\lambda_1^d} & * & * & * \\ 0 & \frac{1}{1-\lambda_1^{d-1}\cdot\lambda_2} & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \frac{1}{1-\lambda_n^d} \end{pmatrix}.$$

The later matrix is well defined since all eigenvalues lie inside the complex unit disc, hence $1 - \lambda_{i_1} \cdot \lambda_{i_d} \neq 0$. Thus $\tilde{p}(.)$ resp. $p(.)$ is also well defined in this case. □

This result can be generalized to also include polynomials depending on parameters, which was needed in formula ($\pi^{[k+1]}$) in Chapter 6. Here the parameters are denoted with $y$ since $p$ is used as a polynomial.

**Corollary 7.**
*Let $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$, and $A \in \mathbb{R}^{n_x \times n_x}$ with all eigenvalues inside the unit disc. Furthermore, let $q(x,y)$ be a polynomial with degree of homogeneity $d \in \mathbb{N}$ in the variables $(x,y)$ such that $q(0,y)$ equals zero. Then there exists a polynomial $p(x,y)$ with degree of homogeneity $d \in \mathbb{N}$ such that*

$$p(x,y) - p(Ax,y) = q(x,y). \tag{7.3}$$

*Proof.* (7.3) can be rewritten as

$$p\begin{pmatrix} x \\ y \end{pmatrix} - p\left( \underbrace{\begin{pmatrix} A & 0_{n_x \times n_y} \\ 0_{n_y \times n_x} & I_{n_y} \end{pmatrix}}_{=: \tilde{A}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right) = q\begin{pmatrix} x \\ y \end{pmatrix}.$$

Now Theorem 8 is applied although $n_y$ eigenvalues of $\tilde{A}$ are 1 and not inside the unit disc. The result stays nearly the same. One only needs to take care of the part of $B$, that is not invertible. But this part defines the coefficients that belong to $p(0,y)$ resp. $q(0,y)$, hence those are set to zero. The same is true for the second part of the proof. □

The continuous versions of the last theorem and corollary are given next. In Corollary 8, $y$ again plays the role of the parameters.

**Theorem 9.**
*Let $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ such that all its eigenvalues have a negative real part. Furthermore, let $q(x)$ be a polynomial with degree of homogeneity $d \in \mathbb{N}$. Then there*

*exists a polynomial $p(x)$ with degree of homogeneity $d \in \mathbb{N}$ such that*

$$\nabla_x p(x) \cdot Ax = q(x). \tag{7.4}$$

*Proof.* The polynomials $p(x)$ and $q(x)$ can be written in the form of (7.2), where $c_i \in \mathbb{R}$ is replaced by $c^p_{i_1,\dots,i_d}$ resp. $c^q_{i_1,\dots,i_d}$.
To show the existence of $p(.)$, the coefficients $c^p_{i_1,\dots,i_d}$ will be calculated. To do so, two cases are distinguished.
Case 1: $A$ is a diagonal matrix with only negative eigenvalues $\lambda_1, \dots, \lambda_d$.

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_d \end{pmatrix} \quad \Rightarrow \quad Ax = \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_d x_d \end{pmatrix}$$

Therefore, the left-hand side of (7.4) is given by

$$\begin{aligned}
&\begin{pmatrix} c^p_{1,\dots,1} & \dots & c^p_{n,\dots,n} \end{pmatrix} \cdot \nabla_x \begin{pmatrix} x_1^d \\ x_1^{d-1} \cdot x_2 \\ \vdots \\ x_n^d \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_d x_d \end{pmatrix} \\
&= \begin{pmatrix} c^p_{1,\dots,1} & \dots & c^p_{n,\dots,n} \end{pmatrix} \cdot \begin{pmatrix} d & 0 & \dots & 0 \\ d-1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & d \end{pmatrix} \cdot \begin{pmatrix} x_1^{d-1} & 0 & \dots & 0 \\ x_1^{d-1} \cdot x_2 & x_1^d & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & x_n^{d-1} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_d x_d \end{pmatrix} \\
&= \begin{pmatrix} c^p_{1,\dots,1} & \dots & c^p_{n,\dots,n} \end{pmatrix} \cdot \begin{pmatrix} d \cdot \lambda_1 & 0 & \dots & 0 \\ (d-1) \cdot \lambda_1 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & d \cdot \lambda_n \end{pmatrix} \cdot \begin{pmatrix} x_1^d \\ x_1^{d-1} \cdot x_2 \\ \vdots \\ x_n^d \end{pmatrix}.
\end{aligned}$$

Re substituting this result into (7.4) and leaving out the variables $x$ leads to

$$\begin{pmatrix} c^p_{1,\dots,1} & \dots & c^p_{n,\dots,n} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} d \cdot \lambda_1 & 0 & \dots & 0 \\ (d-1) \cdot \lambda_1 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & d \cdot \lambda_n \end{pmatrix}}_{=:B} = \begin{pmatrix} c^q_{1,\dots,1} & \dots & c^q_{n,\dots,n} \end{pmatrix}.$$

Since the eigenvalues $\lambda_1, \dots, \lambda_n$ are negative, the matrix $B$ is invertible, and the coefficients of $p(.)$ can be obtained and are unique.
Case 2: $A$ is not a diagonal matrix.
Let $A_{\text{JNF}} \in \mathbb{C}^{n \times n}$ be the Jordan normal form of $A$ and $S \in \text{GL}(n; \mathbb{C})$ the associated

transformation matrix.

$$A = S^{-1} \cdot A_{\mathrm{JNF}} \cdot S$$

Here a Jordan normal form with ones below the main diagonal is chosen, wherein the proof of Theorem 8 the ones are above the main diagonal.
Define $z := Sx$ and $\tilde{p}(z) := p(S^{-1}z)$ as well as $\tilde{q}(z) := q(S^{-1}z)$.

$$\nabla_x p(x) \cdot Ax = \nabla_z \tilde{p}(z) \cdot SAS^{-1}z = \nabla_z \tilde{p}(z) \cdot A_{\mathrm{JNF}} z = \tilde{q}(z)$$

If $\tilde{c}^p_{i_1,\ldots,i_d} \in \mathbb{C}$ and $\tilde{c}^q_{i_1,\ldots,i_d} \in \mathbb{R}$ $\left((i_1,\ldots,i_d) \in [n]^d\right)$ are the coefficients of $\tilde{p}(.)$ resp. $\tilde{q}(.)$, then with a similar calculation as in case 1 leads to

$$\begin{pmatrix} \tilde{c}^p_{1,\ldots,1} & \ldots & \tilde{c}^p_{n,\ldots,n} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} d \cdot \lambda_1 & 0 & \ldots & 0 \\ * & \lambda_2 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & * & d \cdot \lambda_n \end{pmatrix}}_{=:C} = \begin{pmatrix} \tilde{c}^q_{1,\ldots,1} & \ldots & \tilde{c}^q_{n,\ldots,n} \end{pmatrix}$$

and again the matrix $C$ is invertible and the coefficients of $\tilde{p}(.)$ resp. $p(.)$ can be obtained and are unique. □

**Corollary 8.**
*Let $x \in \mathbb{R}^{n_y}$, $y \in \mathbb{R}^{n_y}$, and $A \in \mathbb{R}^{n_x \times n_x}$ such that all its eigenvalues have a negative real part. Furthermore, let $q(x,y)$ be a polynomial with degree of homogeneity $d \in \mathbb{N}$ in the variables $(x,y)$ such that $q(0,y)$ equals zero. Then there exists a polynomial $p(x,y)$ with degree of homogeneity $d \in \mathbb{N}$ such that*

$$\nabla_x p(x,y) \cdot Ax = q(x,y). \tag{7.5}$$

*Proof.* Equation (7.5) can be rewritten as

$$\begin{pmatrix} \nabla_x p(x,y) & \nabla_y p(x,y) \end{pmatrix} \cdot \underbrace{\begin{pmatrix} A & 0_{n_x \times n_y} \\ 0_{n_y \times n_x} & 0_{n_y \times n_y} \end{pmatrix}}_{=:\, \tilde{A}} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q(x,y) \\ 0 \end{pmatrix}.$$

Now Theorem 9 is used even though $n_y$ eigenvalues of $\tilde{A}$ are zero. The part of the matrices $B$ or $C$ that are not invertible belongs to the part that contains only the $y$ variables. The corresponding coefficients can be set to zeros since they belong to $p(0,y)$ resp. $q(0,y)$. □

The calculations performed in SAM are based on the proofs and derivations that has been outlined in this chapter. This allows to efficiently obtain the feedback laws for the regular, parametric and output-feedback case in continuous and discrete time.

# 8 Conclusion and Outlook

This work considers the explicit solution of continuous and discrete-time optimal control problems via power series expansion. Several specific extensions of Al'brekht's Method have been developed, including parameters dependent solutions and the structured inclusion of constraints via barrier functions. Furthermore, a novel extension to the output-feedback case has been proposed. Detailed convergence proofs, as well as stability proofs, have been developed. Additionally, a new approach for the derivation of an inner approximation of the area of convergence has been investigated. One of the open problems is the expansion of the proofs to a more general class of nonlinear systems and non-quadratic cost functions. The estimation of the region of convergence also leaves room for further investigation.

The parametric explicit solution enables to broaden the application area significantly. Parametric uncertainties or disturbances can be handled effectively, thus opening the door for a wide field of applications. The proposed procedure of dealing with the constraints allows non-convex and non-symmetric constraints without the need of finding a suitable penalty function.

Extending the power series approach to do optimal control, using only the measurement information, has not been done before. It, therefore, leads to a new field of applications and is an advancement in the theory of explicit nonlinear optimal control. The non-uniqueness of the right inverse matrix $\tilde{H}$ is a drawback, which needs to be investigated further. A „simpler" choice leads to simpler representations of the power series of the control law but not necessarily to a better control performance.

The effectiveness of the proposed methods was shown, using realistic examples from the fields of unmanned aerial vehicles as well as the control of bioreactors. This was enabled by the toolbox SAM, which is capable of calculating the power series expansions.

Future research will focus on generalizing the convergence proofs to a broader class of optimal control problems. Furthermore, the power series approach can be easily combined with, for example, path following. Estimating the region where stability can be guaranteed should be pursued further. Parametric constraints, e. g. in case of moving obstacles, are also worthy of being investigated. The software SAM is under further development and constantly improved and expanded as the repertoire of the different versions of Al'brekht's approach grows.

# A Mathematical Preliminaries

**Definition 1** (Lyapunov function).

(a) *Let*

$$\dot{x} = f(x)$$

*with $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an $n$-dimensional ($n \in \mathbb{N}$) autonomous continuous-time system. A function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ is called a* ***Lyapunov function*** *if*

*(I) $V(x)$ is positive definite and*

*(II) $\nabla_x V(x) \cdot f(x)$ is negative definite.*

*$V$ is called a* ***local Lyapunov function****, if both conditions hold locally, [63].*

(b) *Let*

$$x^+ = f(x)$$

*with $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be an $n$-dimensional ($n \in \mathbb{N}$) autonomous discrete-time system. A function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ is called a* ***Lyapunov function*** *if*

*(III) $V(x)$ is positive definite and*

*(IV) $V\big(f(x)\big) - V(x)$ is negative definite.*

*$V$ is called a* ***local Lyapunov function****, if both conditions hold locally, [11, 29].*

**Lemma 3** (Hautus Lemma).
*Consider a linear continuous or discrete-time system with matrices $F \in \mathbb{R}^{n_x \times n_x}$ and $G \in \mathbb{R}^{n_x \times n_u}$. Furthermore, let the system be stabilizable. Then for each $\lambda \in \mathbb{C}$, it holds*

$$\operatorname{rank}\begin{pmatrix} F - \lambda \cdot I_{n_x} & G \end{pmatrix} = n_x.$$

*In particular, one has $\operatorname{rank}\begin{pmatrix} F & G \end{pmatrix} = n_x$ and $\operatorname{rank}\begin{pmatrix} F - I_{n_x} & G \end{pmatrix} = n_x$.*

*Proof.* The proof can be found, for example, in [91] and [102]. □

## Analytic functions

**Definition 2** (Analytic functions).
*Let $f\colon D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ and $n \in \mathbb{N}$. $f$ is called* ***analytic*** *or* ***locally analytic*** *in*

*the domain $D$ if the power series*

$$f(x) = \sum_{|\alpha|\geq 0} \frac{D^\alpha f(x_0)}{\alpha!} \cdot (x-x_0)^\alpha = f(x_0) + \nabla_x f(x_0)\cdot(x-x_0) + \dots \tag{A.1}$$

*is convergent for every $x_0 \in D$ and $x$ in a neighborhood of $x_0$.*
*Hereby $\alpha \in \mathbb{N}_0^n = \mathbb{N}_0 \times \ldots \times \mathbb{N}_0$ is a multi-index and*

$$|\alpha| = \sum_{i=1}^n \alpha_i, \quad \alpha! = \prod_{i=1}^n \alpha_i.$$

*Therefore, the derivative $D^\alpha$ is defined by*

$$D^\alpha f(x_0) := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}(x_0),$$

*and $(x-x_0)^k$ is given via*

$$(x-x_0)^\alpha = \prod_{i=1}^n (x_i - x_{0,i})^{\alpha_i}.$$

*$f$ is called **globally analytic** if $D = \mathbb{R}^n$.*

**Remark 16.** (*a*) *For an analytic function $f$, the power series (A.1) is equivalent to its Taylor series.*

(*b*) *If the function $f$ is multidimensional, i. e. $f\colon D \to \mathbb{R}^m$ ($m \in \mathbb{N}$), then Definition 2 can be used component-wise.*

(*c*) *Equivalent to the definition, a function $f\colon D \to \mathbb{R}$ is analytic if $f \in C^\infty(D;\mathbb{R})$ and it is*

$$f(x) = \sum_{k=0}^\infty f^{[k]}(x) = \sum_{k=0}^\infty \sum_{|\alpha|=k} \frac{D^\alpha f(x_0)}{\alpha!}\cdot(x-x_0)^\alpha$$

*such that for every compact set $K \subseteq D$ with $x_0 \in K$ there exists a constant $C \in \mathbb{R}_{>0}$ with*

$$|f^{[k]}(x)| = \sum_{|\alpha|=k} \frac{D^\alpha f(x_0)}{\alpha!}\cdot(x-x_0)^\alpha \leq C^k \cdot ||x-x_0||^k \tag{A.2}$$

*for each $x \in K$. This result can be gleaned in [50].*

## Matrix norms and appraisal of derivatives of polynomials

**Definition 3** (Spectral norm)**.**
*Let $A$ be an $n \times m$-dimensional matrix with real entries. Then the **spectral norm** of*

*A is defined as*

$$||A|| = ||A||_2 := \sqrt{\lambda_{max}(A^{\mathrm{T}} \cdot A)},$$

*where $\lambda_{max}(A^{\mathrm{T}} \cdot A)$ is the largest eigenvalue of $A^{\mathrm{T}} \cdot A$.*

**Remark 17** (Spectral and Frobenius norm inequality)**.**
*The spectral norm is upper bounded by the Frobenius norm, see [33].*

$$||A||_2 \leq \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |A_{i,j}|^2} =: ||A||_F \tag{A.3}$$

**Theorem 10** (Appraisal of derivatives of polynomials)**.**
*Let $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^n$ $(n_x, n_p, n \in \mathbb{N})$ be an analytic function with power series*

$$f(x,p) = \sum_{k=1}^{\infty} f^{[k]}(x,p), \tag{A.4}$$

*where $\left(\begin{smallmatrix} x \\ p \end{smallmatrix}\right) \in \mathbb{R}^{n_x+n_p}$. If for $k \in \mathbb{N}$,*

$$\left\| f^{[k]}(x,p) \right\| \leq C_k \cdot \left\| \begin{pmatrix} x \\ p \end{pmatrix} \right\|^k$$

*holds one also has*

$$\left\| \nabla_x f^{[k]}(x,p) \right\| \leq k \cdot C_k \cdot \left\| \begin{pmatrix} x \\ p \end{pmatrix} \right\|^{k-1} \tag{A.5}$$

*with the same constant $C_k \in \mathbb{R}_{>0}$.*

*Proof.* Each component $j \in [n]$ of the $k$-th degree of the power series (A.4) can be written as

$$f_j^{[k]}(x,p) = \sum_{|m| = \left|\left(\begin{smallmatrix} m_x \\ m_p \end{smallmatrix}\right)\right| = k} c_{j,m} \cdot x^{m_x} \cdot p^{m_p}, \tag{A.6}$$

where $m = \left(\begin{smallmatrix} m_x \\ m_p \end{smallmatrix}\right) \in \mathbb{N}_0^{n_x+n_p}$ is a multi-index such that

$$|m| = \sum_{i=1}^{n_x+n_p} m_i.$$

The scalar polynomials $x^{m_x}$ respectively $p^{m_p}$ are given by

$$x^{m_x} = \prod_{i=1}^{n_x} x_i^{m_{x,i}} \quad \text{and} \quad p^{m_p} = \prod_{i=1}^{n_p} p_i^{m_{p,i}}.$$

Now set $\left(\begin{smallmatrix} m_x \\ m_p \end{smallmatrix}\right) = r \cdot \xi$ with $r \in \mathbb{R}_{>0}$ and $\xi \in \mathbb{S}^{n_x+n_p-1}$. For arbitrary $j \in [n]$ and $k \in \mathbb{N}$, the expression in (A.6) can be bounded as follows.

$$\left|f_j^{[k]}(x,p)\right| \leq r^k \cdot \underbrace{\sum_{|m|=k} |c_{j,m}|}_{=:C_{j,k}} \cdot \underbrace{|\xi|^k}_{=1} = C_{j,k} \cdot r^k$$

Thus, for the vector that contains the $k$-th degree of $f$ one has

$$\left\|f^{[k]}(x,p)\right\| \leq \underbrace{\sqrt{\sum_{j=1}^{n} C_{j,k}^2}}_{=:C_k} \cdot r^k = C_k \cdot r^k.$$

Now taking the derivative with respect to $x_i$ for $i \in [n_x]$ gives

$$\nabla_{x_i} f_j^{[k]}(x,p) = \sum_{|m|=k} c_{j,m} \cdot m_{x,i} \cdot x^{m_x - e_i}, \tag{A.7}$$

where $e_i$ is the $i$-th unit vector in $\mathbb{R}^{n_x}$ and $m_{x,i}$ is the $i$-th component of $m_x$. The derivative in (A.7) can be bounded in the same way as (A.6).

$$\left\|\nabla_{x_i} f_j^{[k]}(x,p)\right\| \leq r^{k-1} \cdot m_{x,i} \cdot \underbrace{\sum_{|m|=k} |c_{j,m}|}_{=C_{j,k}} \cdot \underbrace{|\xi|^{k-1}}_{=1} \leq k \cdot C_{j,k} \cdot r^{k-1} \tag{A.8}$$

This finally leads to an upper bound of the norm of the matrix

$$\nabla_x f^{[k]}(x,p) = \begin{pmatrix} \nabla_{x_1} f_1^{[k]}(x,p) & \dots & \nabla_{x_{n_x}} f_1^{[k]}(x,p) \\ \vdots & \ddots & \vdots \\ \nabla_{x_1} f_n^{[k]}(x,p) & \dots & \nabla_{x_{n_x}} f_n^{[k]}(x,p) \end{pmatrix}.$$

$$\begin{aligned} \left\|\nabla_x f^{[k]}(x,p)\right\| &\overset{(A.3)}{\leq} \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n_x} \left|\nabla_{x_i} f_j^{[k]}(x,p)\right|^2} \overset{(A.8)}{\leq} r^{k-1} \cdot \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n_x} m_{x,i}^2 \cdot C_{j,k}^2} \\ &= r^{k-1} \cdot \underbrace{\sqrt{\sum_{j=1}^{n} C_{j,k}^2}}_{=C_k} \cdot \sqrt{\sum_{i=1}^{n_x} m_{x,i}^2} \leq \sum_{i=1}^{n_x} m_{x,i} \cdot C_k \cdot r^{k-1} \\ &\leq k \cdot C_k \cdot r^{k-1} \end{aligned}$$

□

# Derivation of the Hamilton-Jacobi-Bellman equation

To derive the **Hamilton-Jacobi-Bellman equation**, the following optimal control problem is considered.

$$
\begin{aligned}
\pi\big(y(0), p(0)\big) &= \min_{u(.)} \int_0^{\infty} \ell\big(y(\tau), u(\tau)\big)\,\mathrm{d}\tau \\
\text{s.t. } \dot{x} &= f(x, u, p) \\
y &= h(x, p)
\end{aligned} \tag{OCP}
$$

Hereby $x\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$, $y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$, $u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_u}$, and $p\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_p}$ represent the system states, output variables, input variables, and system parameters/uncertainties. $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$ represents the system dynamics, $h\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_y}$ the output function, and $\ell\colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}$ the cost function. $\pi\colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_p} \to \mathbb{R}$ is the optimal cost function resp. value function. Using Bellman's principle of optimality, the optimal cost function can be for any $t \geq 0$ and $\Delta t \geq 0$ recursively written as

$$
\pi\big(y(t), p(t)\big) = \pi\big(y(t+\Delta t), p(t+\Delta t)\big) + \min_{u(.)} \int_t^{t+\Delta t} \ell\big(y((\tau), u(\tau)\big)\,\mathrm{d}\tau. \tag{A.9}
$$

If the parameter function is known at any time and also differentiable, then a first-order approximation of $\pi$ can be given.

$$
\begin{aligned}
\pi\big(y(t+\Delta t), p(t+\Delta t)\big) = \pi\big(y(t), p(t)\big) &+ \nabla_y \pi\big(y(t), p(t)\big) \cdot \nabla_x h\big(x(t), p(t)\big) \cdot \dot{x}(t) \cdot \Delta t \\
&+ \nabla_y \pi\big(y(t), p(t)\big) \cdot \nabla_p h\big(x(t), p(t)\big) \cdot \dot{p}(t) \cdot \Delta t \\
&+ \nabla_p \pi\big(y(t), p(t)\big) \cdot \dot{p}(t) \cdot \Delta t + o(\Delta t)
\end{aligned} \tag{A.10}
$$

Here $o(\Delta t)$ is the set of all functions $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$
\lim_{\Delta t \to 0} \left| \frac{g(\Delta t)}{\Delta t} \right| = 0,
$$

which is known as the **little-o notation**, which is part of the **Bachmann-Landau notation**. Combining (A.9) and (A.10) yields:

$$
\begin{aligned}
0 = \min_{u(.)} \Bigg\{ &\nabla_y \pi\big(y(t), p(t)\big) \cdot \nabla_x h\big(x(t), p(t)\big) \cdot f\big(x(t), u(t), p(t)\big) \cdot \Delta t \\
&+ \nabla_y \pi\big(y(t), p(t)\big) \cdot \nabla_p h\big(x(t), p(t)\big) \cdot \dot{p}(t) \cdot \Delta t \\
&+ \nabla_p \pi\big(y(t), p(t)\big) \cdot \dot{p}(t) \cdot \Delta t + \int_t^{t+\Delta t} \ell\big(y(\tau), u(\tau)\big)\,\mathrm{d}\tau + o(\Delta t) \Bigg\}
\end{aligned}
$$

Dividing by $\Delta t$ and taking $\lim\limits_{\Delta t \to 0}$ leads to the following minimization problem.

$$
\begin{aligned}
0 = \min_{u(.)} \Bigg\{ &\nabla_y \pi(y,p) \cdot \nabla_x h(x,p) \cdot f(x,u,p) + \nabla_y \pi(y,p) \cdot \nabla_p h(x,p) \cdot \dot{p} \\
&+ \nabla_p \pi(y,p) \cdot \dot{p} + \ell(y,u) \Bigg\}
\end{aligned}
\tag{A.11}
$$

To see that one may use the following identity.

$$
\lim_{\Delta t \to 0} \left( \frac{1}{\Delta t} \int_t^{t+\Delta t} \ell\big(y(\tau), u(\tau)\big) \,\mathrm{d}\tau + \frac{o(\Delta t)}{\Delta t} \right) = \ell\big(y(t), u(t)\big)
$$

The minimizing argument of (A.11) will be called $u_{\min}$ or $\kappa$, and, similarly to $\pi$, it depends on the output variables $y$ and the parameters $p$. Therefore, equation (A.11) can be written as

$$
\begin{aligned}
0 = \nabla_y \pi(y,p) \cdot \Big( &\nabla_x h(x,p) \cdot f\big(x, \kappa(y,p), p\big) + \nabla_p h(x,p) \cdot \dot{p} \Big) \\
&+ \nabla_p \pi(y,p) \cdot \dot{p} + \ell\big(y, \kappa(y,p)\big).
\end{aligned}
\tag{HJBE-1}
$$

The second equation is now given by the first-order optimality condition.

$$
0 = \nabla_y \pi(y,p) \cdot \nabla_x h(x,p) \cdot \nabla_u f\big(x, \kappa(y,p), p\big) + \nabla_u \ell\big(y, \kappa(y,p)\big) \tag{HJBE-2}
$$

**Remark 18.** (*a*) *In the previous calculation, it is observed that the parameters $p$ actually behave like the states $x$. Which means the time-derivative $\dot{p}$ has to be known. Since this is usually not realistic, one may assume time-independent parameters, which simplifies the Hamilton-Jacobi-Bellman equation (HJBE-1) and its derivative with respect to the input (HJBE-2).*

$$
0 = \nabla_y \pi(y,p) \cdot \nabla_x h(x,p) \cdot f\big(x, \kappa(y,p), p\big) + \ell\big(y, \kappa(y,p)\big) \tag{HJBE-1'}
$$
$$
0 = \nabla_y \pi(y,p) \cdot \nabla_x h(x,p) \cdot \nabla_u f\big(x, \kappa(y,p), p\big) + \nabla_u \ell\big(y, \kappa(y,p)\big) \tag{HJBE-2'}
$$

(*b*) *In this setup, the output function $h$ does not depend on the control input $u$, else the time derivative $\dot{u} = \frac{\mathrm{d}}{\mathrm{d}t}\kappa$ would be a part of the HJBE. The solvability of the resulting equations is still unclear if Al'brekht's Method is used.*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \kappa\big(x(t), p(t)\big) = &\nabla_x \kappa\big(x(t), p(t)\big) \cdot f\Big(x(t), \kappa\big(x(t), p(t)\big), p(t)\Big) \\
&+ \nabla_p \kappa\big(x(t), p(t)\big) \cdot \dot{p}(t)
\end{aligned}
$$

(*c*) *If $h(x,p) = x$, i. e. the output variables are identical with the states ($y = x$), then the HJBE (HJBE-1') and its input-derivative (HJBE-2') can be simplified* *further.*

$$0 = \nabla_x \pi(x,p) \cdot f\Big(x, \kappa(x,p), p\Big) + \ell\Big(x, \kappa(x,p)\Big) \tag{HJBE-1”}$$

$$0 = \nabla_x \pi(x,p) \cdot \nabla_u f\Big(x, \kappa(x,p), p\Big) + \nabla_u \ell\Big(x, \kappa(x,p)\Big) \tag{HJBE-2”}$$

($d$) *If additionally to* ($c$) *the parameters* $p$ *are set to* 0 *or are just removed, then (HJBE-1”) and (HJBE-2”) result in the well known nominal HJBE and its derivative.*

$$0 = \nabla_x \pi(x) \cdot f\Big(x, \kappa(x)\Big) + \ell\Big(x, \kappa(x)\Big) \tag{HJBE-1”’}$$

$$0 = \nabla_x \pi(x) \cdot \nabla_u f\Big(x, \kappa(x)\Big) + \nabla_u \ell\Big(x, \kappa(x)\Big) \tag{HJBE-2”’}$$

## Pontryagin’s Minimum Principle for parametric output-feedback

In this section, a comparison between the minimum principle introduced by Lev Semyonovich Pontryagin (1908-1988) (see [12, 79]) and the Hamilton-Jacobi-Bellman equations (William Rowan Hamilton (1805-1865), Carl Gustav Jacob Jacobi (1804-1851), Richard Ernest Bellman (1920-1984)) will be done. To do so, an optimal control problem

$$\min_{u(.)} \int_0^\infty \ell\Big(y(\tau), u(\tau)\Big)\, \mathrm{d}\tau \tag{OCP}$$

$$\dot{x} = f(x,u,p), \tag{$f$}$$

$$y = h(x,p), \tag{$h$}$$

where the system dynamics $f\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_x}$, the output function $h\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_y}$, and the cost function $\ell\colon \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \to \mathbb{R}$ are at least once continuously differentiable and the parameters $p \in \mathbb{R}^{n_p}$ are fixed. The control target is $y = 0$ resp. $x = 0$, if all states are observable. Following the steps of Pontryagin, at first, the Hamiltonian function $\mathcal{H}\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_\lambda} \times \mathbb{R}^{n_\mu} \to \mathbb{R}$ is defined via

$$\mathcal{H}(x,y,u,p,\lambda,\mu) = \ell(y,u) + \lambda^{\mathrm{T}} \cdot f(x,u,p) + \mu^{\mathrm{T}} \cdot y \tag{$\mathcal{H}$}$$

and a total cost function $J\colon \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_p} \times \mathbb{R}^{n_\lambda} \times \mathbb{R}^{n_\mu} \to \mathbb{R}$ as integral over the cost $\ell$.

$$J(x,y,u,p,\lambda,\mu) = \int_0^\infty \ell\Big(y(\tau), u(\tau)\Big)\, \mathrm{d}\tau \tag{$J$}$$

$\lambda\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$ is called the co-state, whereas $\mu\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$ can be seen as a co-output. Both could also be called Lagrange multipliers.

Using the Hamiltonian, the total cost can be reformulated and, therefore, its explicit dependence on $(x, p, \lambda, \mu)$ becomes clear.

$$J(x, y, u, p, \lambda, \mu) = \int_0^\infty \ell(y, u)\,\mathrm{d}\tau \underset{(f),(h)}{\overset{(\mathcal{H})}{=}} \int_0^\infty \mathcal{H}(x, y, u, p, \lambda, \mu) - \lambda^\mathrm{T} \cdot \dot{x} - \mu^\mathrm{T} \cdot h(x, p)\,\mathrm{d}\tau$$

Now let $x\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$, $y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$, $u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_u}$, $\lambda\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$, and $\mu\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$ be an optimal solution of (OCP) for a fixed $p \in \mathbb{R}^{n_p}$ under the conditions $(f)$ and $(h)$.
Take another point close to the optimal solution above:
$x + \Delta x\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$, $y + \Delta y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$, $u + \Delta u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_u}$, $\lambda + \Delta\lambda\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$, $\mu + \Delta\mu\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$
Therefore, one also has functions $\Delta x\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$, $\Delta y\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$, $\Delta u\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_u}$, $\Delta\lambda\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_x}$, and $\Delta\mu\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_y}$. Furthermore, the initial point for the states should be the same for all trajectories ($\Delta x(0) = 0$). The resulting total cost is given by

$$\begin{aligned}
&J(x + \Delta x, y + \Delta y, u + \Delta u, p, \lambda + \Delta\lambda, \mu + \Delta\mu)\\
&= \int_0^\infty \mathcal{H}(x + \Delta x, y + \Delta y, u + \Delta u, p, \lambda + \Delta\lambda, \mu + \Delta\mu)\\
&\qquad - (\lambda + \Delta\lambda)^\mathrm{T} \cdot (x \dot{+} \Delta x) - (\mu + \Delta\mu)^\mathrm{T} \cdot h(x + \Delta x, p)\,\mathrm{d}\tau.
\end{aligned}$$

To find criteria for an optimal solution, the first variation of $J$ is needed.

$$\begin{aligned}
&J(x + \Delta x, y + \Delta y, u + \Delta u, p, \lambda + \Delta\lambda, \mu + \Delta\mu) - J(x, y, u, p, \lambda, \mu)\\
&= \int_0^\infty \nabla_x \mathcal{H}(x, y, u, p, \lambda, \mu) \cdot \Delta x + \nabla_y \mathcal{H}(x, y, u, p, \lambda, \mu) \cdot \Delta y + \nabla_u \mathcal{H}(x, y, u, p, \lambda, \mu) \cdot \Delta u\\
&\quad + \nabla_\lambda \mathcal{H}(x, y, u, p, \lambda, \mu) \cdot \Delta\lambda + \nabla_\mu \mathcal{H}(x, y, u, p, \lambda, \mu) \cdot \Delta\mu - (\lambda + \Delta\lambda)^\mathrm{T} \cdot (x \dot{+} \Delta x)\\
&\quad + \lambda^\mathrm{T} \cdot \dot{x} - h(x, p)^\mathrm{T} \cdot \Delta\mu - \mu^\mathrm{T} \cdot \nabla_x h(x, p) \cdot \Delta x\,\mathrm{d}\tau + o(\Delta x, \Delta y, \Delta u, \Delta\lambda, \Delta\mu)
\end{aligned}$$

The sum under the integral can be simplified using the following identities:

$$-(\lambda + \Delta\lambda)^\mathrm{T} \cdot (x \dot{+} \Delta x) + \lambda^\mathrm{T} \cdot \dot{x} = -\lambda^\mathrm{T} \cdot \dot{\Delta x} - \Delta\lambda^\mathrm{T} \cdot \dot{\Delta x} - \Delta\lambda^\mathrm{T} \cdot \dot{x}$$

Since it is hard to evaluate $\lambda^\mathrm{T} \cdot \dot{\Delta x}$, it is replaced using partial integration.

$$-\int_0^\infty \lambda^\mathrm{T} \cdot \dot{\Delta x}\,\mathrm{d}\tau = -\left[\lambda^\mathrm{T} \cdot \Delta x\right]_{\tau=0}^\infty + \int_0^\infty \dot{\lambda}^\mathrm{T} \cdot \Delta x\,\mathrm{d}\tau = \int_0^\infty \dot{\lambda}^\mathrm{T} \cdot \Delta x\,\mathrm{d}\tau$$

Remember, the control objective is to steer the system output to the origin (resp. $x = 0$, if all states are observable), and the initial states are fixed. Thus

$$\lim_{t \to \infty} \Delta x(t) = 0 \quad \text{and} \quad \Delta x(0) = 0$$

are holding. $\Delta\lambda^{\mathrm{T}} \cdot \dot{\Delta x}$ can be neglected since it belongs to the class $o(\Delta x, \Delta y, \Delta u, \Delta\lambda, \Delta\mu)$. Now let $\Delta x$, $\Delta y$, $\Delta u$, $\Delta\lambda$, and $\Delta\mu$ go to 0 but independently. Then the following identities are obtained:

$$\begin{aligned}
-\dot{\lambda}^{\mathrm{T}} &= \nabla_x \mathcal{H}(x, y, u, p, \lambda, \mu) - \mu^{\mathrm{T}} \cdot \nabla_x h(x, p) \\
0 &= \nabla_y \mathcal{H}(x, y, u, p, \lambda, \mu) \\
0 &= \nabla_u \mathcal{H}(x, y, u, p, \lambda, \mu) \\
\dot{x}^{\mathrm{T}} &= \nabla_\lambda \mathcal{H}(x, y, u, p, \lambda, \mu) \\
h(x, p)^{\mathrm{T}} &= \nabla_\mu \mathcal{H}(x, y, u, p, \lambda, \mu)
\end{aligned}$$

Re-substituting the Hamiltonian function ($\mathcal{H}$) and setting $\lambda$, $\mu$, and $\kappa := u$ dependent on $y$ and $p$ gives

$$\begin{aligned}
-\dot{\lambda}^{\mathrm{T}}(y, p) &= \lambda^{\mathrm{T}}(y, p) \cdot \nabla_x f\big(x, \kappa(y, p), p\big) - \mu^{\mathrm{T}}(y, p) \cdot \nabla_x h(x, p), && \text{(PMP-1)} \\
0 &= \nabla_y \ell\big(y, \kappa(y, p)\big) + \mu^{\mathrm{T}}(y, p), && \text{(PMP-2)} \\
0 &= \nabla_u \ell\big(y, \kappa(y, p)\big) + \lambda^{\mathrm{T}}(y, p) \cdot \nabla_u f\big(x, \kappa(y, p), p\big), && \text{(PMP-3)} \\
\dot{x} &= f\big(x, \kappa(y, p), p\big), && \text{(PMP-4)} \\
h(x, p) &= y, && \text{(PMP-5)}
\end{aligned}$$

where (PMP-1) and (PMP-2) together lead to

$$-\dot{\lambda}^{\mathrm{T}}(y, p) = \lambda^{\mathrm{T}}(y, p) \cdot \nabla_x f\big(x, \kappa(y, p), p\big) + \nabla_y \ell\big(y, \kappa(y, p)\big) \cdot \nabla_x h(x, p). \quad \text{(PMP-1*)}$$

**Remark 19.** (*a*) *In the setup above, the parameters $p$ do not depend on the time and the output function $h$ does not depend on the control input $u$ to be consistent with the setup used for the Hamilton-Jacobi-Bellman equations, see (OCP) in the previous section and Remark 18* (*a*) *and* (*b*)*.*

(*b*) *If input to state control is considered ($y = h(x, p) = x$), then the equations (PMP-1)-(PMP-5) are reduced.*

$$\begin{aligned}
-\dot{\lambda}^{\mathrm{T}}(x, p) &= \lambda^{\mathrm{T}}(x, p) \cdot \nabla_x f\big(x, \kappa(x, p), p\big) + \nabla_x \ell\big(x, \kappa(x, p)\big) && \text{(PMP-1')} \\
0 &= \nabla_u \ell\big(x, \kappa(x, p)\big) + \lambda^{\mathrm{T}}(x, p) \cdot \nabla_u f\big(x, \kappa(x, p), p\big) && \text{(PMP-2')} \\
\dot{x} &= f\big(x, \kappa(x, p), p\big) && \text{(PMP-3')}
\end{aligned}$$

(*c*) *If, in addition to part* (*b*)*, there are no parameters present, the (PMP-1')-(PMP-*

*3') result in the well-known equations obtained via the classic version of Pontryagin's Minimum Principle.*

$$-\dot{\lambda}^{\mathrm{T}}(x) = \lambda^{\mathrm{T}}(x) \cdot \nabla_x f\big(x, \kappa(x)\big) + \nabla_x \ell\big(x, \kappa(x)\big) \tag{PMP-1”}$$

$$0 = \nabla_u \ell\big(x, \kappa(x)\big) + \lambda^{\mathrm{T}}(x) \cdot \nabla_u f\big(x, \kappa(x)\big) \tag{PMP-2”}$$

$$\dot{x} = f\big(x, \kappa(x)\big) \tag{PMP-3”}$$

(*d*) *The left-hand sides of (PMP-1*), (PMP-1',) and (PMP-1”) can be replaced utilizing*

$$\begin{aligned}
\dot{\lambda}(y,p) &= \frac{\mathrm{d}\lambda}{\mathrm{d}t}(y,p) = \nabla_y \lambda(y,p) \cdot \nabla_x h(x,p) \cdot f\big(x, \kappa(y,p), p\big), \\
\dot{\lambda}(x,p) &= \frac{\mathrm{d}\lambda}{\mathrm{d}t}(x,p) = \nabla_x \lambda(x,p) \cdot f\big(x, \kappa(x,p), p\big), \\
\dot{\lambda}(x) &= \frac{\mathrm{d}\lambda}{\mathrm{d}t}(x) = \nabla_x \lambda(x) \cdot f\big(x, \kappa(x)\big)
\end{aligned}$$

*and, therefore, the equations gained via PMP do not contain any explicit time-derivatives.*

The equality of the optimality conditions gained from PMP and HJB are seen if the PMP equations are rewritten and integrated. For example, (PMP-1*) can be seen as a Jacobian.

$$\begin{aligned}
0 &= \lambda^{\mathrm{T}}(y,p) \cdot \nabla_x f(x,u,p) + \nabla_y \ell(y,u) \cdot \nabla_x h(x,p) \\
&\quad + f(x,u,p)^{\mathrm{T}} \cdot \nabla_x h(x,p)^{\mathrm{T}} \cdot \nabla_y \lambda(y,p)^{\mathrm{T}} \\
&= \Big[\nabla_x\Big(\lambda^{\mathrm{T}}(y,p) \cdot f(x,u,p) + \ell(y,u)\Big)\Big]_{u=\kappa(y,p)}
\end{aligned}$$

After integration, one obtains that

$$\lambda^{\mathrm{T}}(y,p) \cdot f\big(x, \kappa(y,p), p\big) + \ell\big(y, \kappa(y,p)\big)$$

must be constant. Now using the setup from Section 5.1, $0 = y = h(x,p)$ implies $x = 0$ and $\kappa(y,p) = 0$. Hence it is

$$\lambda^{\mathrm{T}}(y,p) \cdot f\big(x, \kappa(y,p), p\big) + \ell\big(y, \kappa(y,p)\big) = 0.$$

Furthermore, replacing $\lambda^{\mathrm{T}}(y,p)$ with

$$\nabla_y \pi(y,p) \cdot \nabla_x h(x,p)$$

results in the Hamilton-Jacobi-Bellman equation (HJBE-1). The same substitution shows the equality of (PMP-3) and (HJBE-2).

**Remark 20.** *In case of state-feedback* $(y = x)$, $\lambda^{\mathrm{T}}(x, p)$ *is replaced by* $\nabla_x \pi(x, p)$. *Thus the equations (PMP-1') (after integration with respect to x) and (PMP-2') become the Hamilton-Jacobi-Bellman equations (HJBE-2"), (HJBE-2")*
*from Remark 18. The same holds for the non-parametric case.*

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