PROPOSITIONES NONNVLLÆ AD THEORIAM ÆSTIMATIONIS ERRORVM IN TRIANGVLIS PLANIS ET SPHÆRICIS PERTINENTES

QVAS
RECTORE VNIVERSITATIS EBERHARDINÆ CAROLINÆ
MAGNIFICENTISSIMO
SERENISSIMO DVCE ET DOMINO
DOMINO
CAROLO
DVCE WIRTEMBERGIÆ ET TECCIÆ REGNANTE
REL. REL.

PRÆSIDE
VIRO EXCELLENTISSIMO ATQVE AMPLISSIMO
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VNIVERSITATIS ET COLLEGII ILLVSTRIS PROF. PHYSICES
ET MATHESEOS PVBL. ORD.
PRÆCEPTORE AC PATRONO SVO PIE DEVENERANDO
PRO RITE CONSEQVENDIS MAGISTERII PHILOSOPHICI HONORIBVS
DIE AVG. MDCCLXXXIII.
PVBLICE AD DISPVTANDVM PROPONIT
AVCTOR
IOANNES WILHELMVS CAMERER
OHNASTETTENSIS
MAGISTERII PHILOSOPHICI CANDIDATVS IN ILLVSTRI
STIPENDIO THEOLOGICO.

TVBINGAE *TYPIS FVESIANIS.*

§. 1.

Mathesis quum jure suo præ cæteris disciplinis eam laudem habeat, quod exactissima certitudine, summoque rigore præcepta sua demonstret: tamen, utprimum ad ipsa corpora, eaque, quæ in illis observantur, dimetienda applicatur; idem illi accidit, quod reliquis humanis artibus fere omnibus, ut nempe ad veritatem magis minusve prope accedat, raro eam ipsam assequatur. Neque id mirum: dum enim theoretice circa magnitudinem rerum versatur, data omnia ut exactissime vera assumit, atque exinde certis argumentis ad ea, quæ incognita adhuc erant, concludit, quorum ita veram indubiamque magnitudinem necessario deprehendit. Sic exempli gratia rigidissime demonstratur; si in duobus triangulis tria latera sint exacte æqualia, fore & tres angulos, atque ipsa etiam triangula exacte æqualia. At alia longe res est, cumprimum idem practice considerandum venit. Tum enim partim ob sensuum imbecillitatem, partim ob instrumentorum, optimorum etiam, imperfectiones, necessario fit, ut data jam non summum rigorem habeant; atque ita ea etiam, quæ inde concluduntur, eundem jam nunquam, aut raro certe obtinere possint.

Sic melioris certe notæ debent esse instrumenta, quorum ope millesimam adhuc digiti partem distinguere possimus: adeoque in priori exemplo id tantum scire poterimus, latera duorum triangulorum a se invicem ne millesima quidem digiti parte discrepare; plane non discrepare ope dimensionis nunquam certi erimus; & proinde saltim concludere poterimus, angulos etiam, & tota triangula haud posse multum a se invicem differre.

§. 2.

At habet tamen etiam hac in re Mathesis proprias suas virtutes. Quum enim propter eas, quas diximus difficultates, errores plane evitare non liceat: saltim determinat, qua ratione alii ab aliis pendeant; quem exactitudinis gradum habere debeant instrumenta, si errores ex eorum imperfectionibus enati certum aliquem limitem non transcendere debeant; quem situm eligere debeamus, ut sit ad evitandos errores

 quam

quam maxime commodus; quos contra casus evitare debeamus, quod facillime ex iis errores satis notabiles oriri posse prævideamus. Et quum pleraque, quæ dimetimur, ope triangulorum planorum aut sphæricorum determinare soleamus; etiam errores, qui in triangulis obtinent, eorumque inter se nexus præcipue a nobis considerari merentur. Nempe quum in triangulis planis & sphæricis ex tribus quibuscunque elementis, inter quæ in planis triangulis unum semper latus datum esse debet, tria reliqua trianguli elementa, aliaque plura, quæ inde pendent, invenire possimus; manifestum est, si vel minimus error in unum, aut duo, aut omnia tria data irrepserit, necessario etiam ea omnia, quæ quæruntur, simul variatum iri; nisi forte errores isti ita comparati sint, ut se mutuo tollant. Quantam vero talis datorum variatio vim habeat ad reliqua omnia simul varianda, id deinde accuratius debet determinari. Atque hanc errorum theoriam adornarunt magistri in rebus mathematicis peritissimi, *Cotesius*, *Wolfius*, *Marinonius*, *Bouguerus*, *Kæstnerus*, *Lambertus*, aliique. Duas hac in re vias ingressi sunt viri doctissimi: alii enim, *Cotesio* præeunte, ope figurarum elicere studuerunt, quemnam inter se nexum habeant errores in triangulis occurrentes: alii calculo uti maluerunt. Et illorum quidem consilium eo inprimis se commendat, quod ita tota res oculo præsens sistitur, atque in casibus simplicioribus mira sæpe facilitate expeditur. At, si plures simul errores locum habeant; figura & demonstratio, uti præclare monet *Lambertus*, (Beyträge zur practischen Geometrie §. 335) fiunt intricatiores; quum contra in calculo casus etiam magis compositi parum difficultatis habeant. Accedit, quod, eodem observante, in consideratione figuræ ii semper casus distingui debent, quibus trianguli elementa commisso aliquo errore crescunt, ab iis, quibus decrescunt; quem laborem calculus signorum + — ope facillime absolvit.

§. 3.

Quamvis autem insignia sint virorum, quos dixi, hac in re merita; superesse tamen adhuc nonnulla videntur, in quibus vires meas exercere possim. Et triangula quidem plana quod attinet: id modo determinatum esse video, quantum, mutato uno, aut duobus, aut tribus omnibus datis, mutentur simul reliqua latera vel anguli. At datis tribus trianguli elementis sufficientibus, plura adhuc alia determinata sunt, quæ omnia, si data mutentur, pariter variari necesse est. Inprimis in dimensione agrorum, omnique adeo Geometria practica sæpissime postulatur, ut area trianguli, ex tribus quibuscunque ejus elementis, dummodo unum inter ea latus datum sit, determinetur. Haud inutile igitur fore putavi, si investigarem; quantam variationem subeat area triangularis, si data vel tantillum mutentur. Qua in re ita versabor, ut primum quidem eos casus simpliciores persequar, quibus unum tantum trianguli elementum variatur; ac figurarum ope eruam, quid talis variatio in varianda area triangulari efficere possit: tum autem ope calculi formulas generales quæram, e quibus speciales pro singulis diversis casibus sponte fluunt. Deinde in triangulis sphæricis ii tantum casus simpliciores, quibus unum modo datorum variationem aliquam patitur, adhuc geometrice, vel calculo determinati fuerunt; ii autem, quibus plura simul variantur, nondum generaliter investigati sunt. Hic itaque calculum adhibere, & formulas generales afferre constitui; è quibus deinde etiam pro iis casibus, quibus unus aut duo saltim errores occurrunt, formulæ speciales facile deducentur.

Solent in hac disquisitione errores ut infinite parvi assumi; quod nempe ii tantum hic considerantur, qui oculi, aut instrumentorum vitio oriuntur, qui plerumque admodum exigui sunt. (vid. Lambert. lib. cit. §. 334) Si itaque inposterum de erroribus minimis, aut variationibus minimis uniuscujuscunque, aut plurium trianguli elementorum sermo erit; id ita intelligendum erit: formulam, quæ ita invenitur, à veritate eo propius absuturam, quo minor sit error commissus.

§. 4.

Theor. I. Fig. 1. Permanente trianguli rectilinei ABC uno quovis latere AB, cum angulo ipsi adjacente B; erit CD variatio minima lateris reliqui adjacentis BC, ad hoc ipsum latus BC, ut ACD variatio minima areæ triangularis, ad ABC ipsam aream triangularem.

Quum enim triangula ACD, & ABC eandem habeant altitudinem; erunt inter se, uti bases, i. e. uti CD ad BC.

Notandum hic, fore hanc propoſitionem adhuc veram, quamcunque variatio lateris BC magnitudinem habuerit; dum in demonſtratione nihil occurrit, quod ſupponeret, eſſe variationem minimam: at ita poſui ad ſervandam uniformitatem cum ſequentibus propoſitionibus, quæ tantum de variationibus minimis valent.

§. 5.

Theor. II. Iisdem manentibus, erit ACD variatio minima areæ triangularis ad dimidium rectangulum contentum ſub latere AC angulo permanenti oppoſito, & DE variatione minima ejusdem lateris, ut tangens anguli ACB lateri permanenti oppoſiti ad radium.
Eſt enim

$$\triangle ACD : \tfrac{1}{2} AC.DE = \begin{cases} CE : DE \text{ ob } AC = AD \text{ eo propius, quo minor eſt } DE \\ \text{tang } ACB : \text{ſin. tot. eo propius, quo minor eſt angulus } CAD. \end{cases}$$

§. 6.

Theor. III. Iisdem manentibus erit FG menſura variationis minimæ anguli utriusque reliqui ad duplum radii ſui AF, ut variatio minima areæ triangularis ad quadratum lateris AC angulo permanenti oppoſiti. Nam

$$FG : AF = \begin{cases} CE : AC \\ CE.AC : AC^2 \end{cases}$$

Et, quum ſit $\frac{CE.AC}{2} = \triangle ACD$, erit

$$FG : 2AF = \triangle ACD : AC^2$$

§. 7.

Theor. IV. Manentibus uno quovis latere BC, & angulo ipſi oppoſito BAC = BDC; erit FG variatio minima anguli utriusvis reliqui ABC ad duplum radii, ut variatio minima areæ triangularis ad exceſſum, quo quadratum lateris AB, angulos inter permanentem, & variatum interjacentis, ſuperat quadratum reliqui lateris AC angulo variato oppoſiti.

Fig. 2.

Sit enim BF = Cf = radio tabulari, erit FG = fg. Jam manifeſtum eſt, trianguli ABC incrementum æquale eſſe △DOB—△AOC. Eſt vero △DOB = ½ BD.EO, &, quum,

ſi

si variatio assumatur minima, sit $BD = AB$, $EO = AI = \frac{AB.FG}{\text{sin. tot.}}$:
erit $\triangle DOB = \frac{AB^2.FG}{2.\text{ sin. tot.}}$ Eodem modo erit $\triangle AOC = \frac{AC^2.fg.}{2.\text{ sin. tot.}}$
Hinc variatio minima trianguli $= \frac{FG}{2.\text{sin.tot}} . (AB^2 - AC^2)$
vel $FG : 2.\text{sin. tot.} = \text{var. minim. triang.} : (AB^2 - AC^2)$

Aliter ita: $\triangle ABD : AB^2 = \begin{cases} \frac{1}{2}AB.AI : AB^2 \text{ tanto propius, quo minor est } DI, \\ AI \quad : 2AB \\ FG \quad : 2BF \end{cases}$

Pariter $\triangle ACD : AC^2 = \begin{cases} fg : 2Cf \\ FG : 2BF \end{cases}$

Itaque variatio minima $\triangle ABC : AB^2 - AC^2 = FG : 2BF$

§. 8.

Theor. V. Iisdem manentibus, erit variatio minima trianguli ad dimidium rectangulum contentum sub DI variatione minima lateris utriusvis reliqui AB, & quarta proportionali ad hoc ipsum AB, & summam atque differentiam duorum laterum AB & AC angulum permanentem comprehendentium; ut tangens supplementi anguli ACB, lateri AB oppositi, ad radium.

Nam variat. minim. triang. : $AB^2 - AC^2 = \begin{cases} FG : 2.\text{ sin. tot. } (\S. 7.) \\ AI : 2AB \end{cases}$

& $AB^2 - AC^2 : \frac{(AB^2 - AC^2).DI}{2.AB} = 2.AB : DI$

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hinc variat. minim. triang. : $\frac{1}{2}\frac{(AB^2 - AC^2)DI}{AB} = \begin{cases} AI : DI \\ \text{tang. suppl.} ACB : \text{sin.tot.} \end{cases}$

Est vero $AB : AB + AC = AB - AC : \frac{(AB^2 - AC^2)}{AB}$

§. 9.

Theor. VI. *Fig.* 3. Si trianguli rectilinei permaneant duo latera AB, & AC = AD; erit FG variatio minima anguli BAC inter duo ista latera permanentia contenti, ad tangentem ejusdem anguli, ut variatio minima areæ triangularis ad ipsam aream trianguli.

Va-

Variat. minim. triang. : $\triangle ABC = \begin{cases} DO : CK \\ AF.DO : AF.CK \end{cases}$

Sed $DO : DC = AK : AC$

$\underline{DC : FG = AC : AF}$

$DO : FG = AK : AF$

hinc $AF.DO = FG.AK$

& variat. minim. triang. : $\triangle ABC = \begin{cases} FG.AK : AF.CK \\ \frac{FG}{AF} : \frac{CK}{AK} \text{ seu } \frac{\text{tang } BAC}{\text{sin. tot.}} \\ FG : \text{tang } BAC, \text{ sumta } AF = \text{sin. tot.} \end{cases}$

§. 10.

Theor. VII. Iisdem manentibus, erit variatio minima areæ triangularis ad dimidium quadratum lateris tertii BC, in ratione composita ex ratione inversa radii ad HI variationem minimam anguli utriusvis reliqui ABC, & ex directa tangentis anguli reliqui ACB adjacentis lateri BC ad tangentem anguli BAC inter latera permanentia intercepti.

Nam $DO : CD = \begin{cases} \text{sin. tot.} : \text{sec. } BAC \\ \text{sin. } BAC : \text{tang } BAC \end{cases}$

$\underline{AB : BC = \text{sin. } ACB : \text{sin. } BAC}$

$AB.DO : BC.CD = \text{sin. } ACB : \text{tang } BAC$

$\left.\begin{matrix} CD : CE \\ BC.CD : BC.CE \end{matrix}\right\} = \begin{cases} \text{sec. } ACB : \text{sin. tot.} \\ \text{tang } ACB : \text{sin. } ACB \end{cases}$

$AB.DO : BC.CE = \text{tang } ACB : \text{tang } BAC$

$\left.\begin{matrix} CE : BC \\ BC.CE : BC^2 \end{matrix}\right\} = HI : BH$

$\left.\begin{matrix} AB.DO : \overline{BC}^2 \\ \text{variat. minim. } \triangle : \frac{1}{2}BC^2 \end{matrix}\right\} = HI.\text{tang } ACB : BH \text{ tang } BAC$

§. 11.

Theor. VIII. Iisdem manentibus, erit variatio minima areæ triangularis, ad dimidium rectangulum contentum sub latere

tere tertio BC, ejusque variatione minima DE, ut cotangens anguli BAC inter latera permanentia intercepti ad radium.

Eſt enim
$$\begin{aligned} DO : CD &= \text{coſin}\ BAC : \text{ſin. tot.} \\ CD : DE &= \text{ſin. tot.} : \text{ſin.}\ ACB \end{aligned}$$

$$\left.\begin{matrix} DO : DE \\ AB.DO : AB.DE \end{matrix}\right\} = \text{coſin.}\ BAC : \text{ſin.}\ ACB$$

$$\left.\begin{matrix} AB : BC \\ AB.DE : BC.DE \end{matrix}\right\} = \text{ſin.}\ ACB : \text{ſin.}\ BAC$$

$$\left.\begin{matrix} AB.DO : BC.DE \\ \text{variat. minim.}\ \triangle : \tfrac{1}{2} BC.DE \end{matrix}\right\} = \left\{\begin{matrix} \text{coſin}\ BAC : \text{ſin.}\ ABC \\ \text{cotang}\ BAC : \text{ſin. tot.} \end{matrix}\right.$$

§. 12.

Theor. IX. *Fig. 4.* Si in triangulo rectilineo permaneant duo anguli (adeoque etiam tertius); erit variatio minima lateris cujuscunque ad dimidium ejusdem lateris, ut variatio minima areæ ad ipſam aream.

Nempe $\triangle ADF : \triangle ABC = AD^2 : AB^2$

$$BCDF : \triangle ABC = \left\{\begin{matrix} \left\{\begin{matrix} AD^2 - AB^2 \\ (AD + AB)\ BD \\ 2.AB\ .\ BD \end{matrix}\right\} : AB^2 \\ 2.\ BD \quad : \quad AB \end{matrix}\right.$$

§. 13.

His propoſitionibus omnes caſus poſſibiles, dum duo ſemper datorum invariata manent, comprehendi, facile patet. Permanebunt enim vel latus quodcunque cum angulo ipſi contiguo, & variabitur

1. latus alterum angulo permanenti adjacens Theor. I.
2. latus angulo permanenti oppoſitum II.
3. alteruter, ac proinde uterque reliquus angulus III.

vel latus quodcunque cum angulo ipſi oppoſito, & variabitur

1. uterque reliquus angulus IV.
2. alterutrum reliquum latus V.

vel duo quæcunque latera, & variabitur

1. angulus interceptus VI.
2. alteruter reliquus angulus VII.
3. latus tertium VIII.

vel duo, adeoque tres omnes anguli, & variabitur

latus quodcunque IX.

Cæterum probe hic attendendum eſt, num creſcente vel decreſcente aliquo datorum area ſimul creſcat, an decreſcat. Id vero ex conſideratione figuræ diſcendum. Regulæ enim hic allatæ tantum docent, quantam variationem ſubeat area trianguli, dum unum datorum minimam quandam variationem patitur; utrum iſta areæ variatione creſcat triangulum, an decreſcat, non determinant.

§. 14.

Jam vero ad eos progredimur caſus, quibus plura data ſimul variantur, in quibus propter eas, quas diximus, cauſſas, calculo utemur. Area trianguli determinari poteſt vel datis duobus lateribus cum angulo incluſo; vel duobus lateribus cum angulo alterutri laterum oppoſito; vel duobus angulis, & latere interjacente; vel duobus angulis, & latere alterutri angulorum oppoſito; vel tribus lateribus. Habebimus igitur quinque, pro his diverſis caſibus, diverſas formulas, ad quas exprimendas denotet A aream trianguli; P, Q, R, latera; p, q, r angulos iſtis lateribus oppoſitos.

§. 15.

Datis igitur duobus lateribus P, Q cum angulo incluſo r, erit

$$A = \frac{P.Q.\sin. r.}{2}$$

$$lA = lP + lQ + l\sin. r - l2$$

$$\frac{dA}{A} = \frac{dP}{P} + \frac{dQ}{Q} + \left\{ \begin{array}{l} \frac{\text{cofin } r}{\text{fin } r} dr \\ \frac{dr}{\text{tang} r} \end{array} \right.$$

§. 16.

Si data ſint duo latera P, Q, cum angulo q alterutri oppoſito; erit $2A = P.Q.\sin r = P.Q.\sin(p+q)$. Sin $(p+q)$ autem $=$ $\sin p \cosin q + \sin q \cosin p$. At $\sin p : \sin q = P : Q$ vel $\sin p = \frac{P.\sin q}{Q}$. Coſin $p = \pm\sqrt{\left(1 - \frac{P^2 \sin q^2}{Q^2}\right)}$ prout angulus p acutus, vel obtuſus, vel coſin $p = \pm\frac{1}{Q}\sqrt{(Q^2 - P^2 \sin q^2)}$

Hinc $2A = P^2 \sin q . \cosin q \pm P \sin q \sqrt{(Q^2 - P^2 \sin q^2)}$

$$2\,dA = 2P \sin q \cosin q\, dP + P^2 \cosin q^2\, dq - P^2 \sin q^2\, dq$$
$$\pm\sqrt{(Q^2 - P^2 \sin q^2)} \sin q .\, dP \pm \sqrt{(Q^2 - P^2 \sin q^2)}\, P \cosin q\, dq$$
$$+ \frac{P.Q \sin q\, dQ - P^2 \sin q^3\, dP - P^3 \sin q^2 \cosin q\, dq}{\pm\sqrt{(Q^2 - P^2 \sin q^2)}}.$$

$$2dA = dP\left(2P \sin q \cosin q \pm \sin q \sqrt{(Q^2 - P^2 \sin q^2)} - \frac{P^2 \sin q^3}{\pm\sqrt{(Q^2 - P^2 \sin q^2)}}\right)$$
$$+ dq\left(P^2 \cos q^2 - P^2 \sin q^2 \pm \sqrt{(Q^2 - P^2 \sin q^2)} P \cosin q - \frac{P^3 \sin q^2 \cos q}{\pm\sqrt{(Q^2 - P^2 \sin q^2)}}\right)$$
$$+ dQ . \frac{P Q \sin q}{\pm\sqrt{(Q^2 - P^2 \sin q^2)}}$$

§. 17.

Formula hæc ſatis complicata concinnior redditur, ſi præter elementa trianguli, quæ immediate data ſupponuntur, alia ab ipſis pendentia in eam introducantur.

Nempe cum ſit $P \sin q = Q \sin p$; & $\pm\sqrt{\overline{Q^2 - P^2 \sin q}^2}$ $= Q \cosin p$ (§. 16): ſit:

$$2P \sin q \cosin q \pm \sin q \sqrt{(Q^2 - P^2 \sin q^2)} - \frac{P^2 \sin q^3}{\pm\sqrt{(Q^2 - P^2 \sin q^2)}}$$
$$= 2P \sin q \cosin q + Q \sin q \cosin p - \frac{P^2 \sin q^3}{Q . \cos p}$$
$$= \frac{\sin q}{Q . \cosin p}\left(2 P Q \cosin p \cosin q + Q^2 \cosin p^2 - P^2 \sin q^2\right)$$

$$= \frac{\sin p}{P \,\mathrm{cosin}\, p} \left(2 PQ \,\mathrm{cosin}\, p \,\mathrm{cosin}\, q + Q^2 - Q^2 \sin p^2 - PQ \sin p \sin q\right)$$

$$= \frac{\mathrm{tang}\, p}{P} \left(Q^2 + 2 P Q \,\mathrm{cosin}\, p \,\mathrm{cosin}\, q - 2 PQ \sin p \sin q\right)$$

$$= \frac{\mathrm{tang}\, p}{P} \left(Q^2 - 2 PQ \,\mathrm{cosin}\, r\right) = \mathrm{tang}\, p \left(\frac{R^2 - P^2}{P}\right)$$

Hinc membrum primum formulæ generalis §. 16. ita exprimi potest:

$$d P. \frac{\mathrm{tang}\, p}{P} (R^2 - P^2)$$

§. 18.

Pariter

$$P^2 \,\mathrm{cosin}\, q^2 - P^2 \sin q^2 \pm P \,\mathrm{cosin}\, q \sqrt{(Q^2 - P^2 \sin q^2)} - \frac{P^3 \sin q^2 \,\mathrm{cosin}\, q}{\pm\sqrt{(Q^2 - P^2 \sin q^2)}}$$

$$= P \,\mathrm{cosin}\, q \left(P \,\mathrm{cosin}\, q - \frac{P^2 \sin q^2}{Q \,\mathrm{cosin}\, p}\right) - P^2 \sin q^2 + PQ \,\mathrm{cosin}\, p \,\mathrm{cosin}\, q$$

$$= \frac{P \,\mathrm{cosin}\, q}{Q \,\mathrm{cosin}\, p} (PQ \,\mathrm{cosin}\, p \,\mathrm{cosin}\, q - PQ \sin p \sin q) + PQ \,\mathrm{cosin}\, p \,\mathrm{cosin}\, q - PQ \sin p \sin q$$

$$= - PQ \left(\frac{P \,\mathrm{cosin}\, q}{Q \,\mathrm{cosin}\, p} + 1\right) \mathrm{cosin}\, r$$

$$= - \frac{PQ}{\sin p \sin q} \cdot \frac{\sin p}{\mathrm{cosin}\, p} \left(\frac{P \sin q \,\mathrm{cosin}\, q}{Q} + \mathrm{cosin}\, p \sin q\right) \mathrm{cosin}\, r$$

$$= - \frac{R^2}{\sin r^2} \cdot \mathrm{tang}\, p \,(\sin p \,\mathrm{cosin}\, q + \mathrm{cosin}\, p \sin q)\, \mathrm{cosin}\, r$$

$$= - R^2 . \frac{\mathrm{tang}\, p \,.\, \mathrm{cosin}\, r}{\sin r}$$

$$= - R^2 . \frac{\mathrm{tang}\, p}{\mathrm{tang}\, r}$$

Hinc membrum secundum formulæ generalis §. 16. erit $= - dq. R^2 \frac{\mathrm{tang}\, p}{\mathrm{tang}\, r}$.

§. 19.

Denique $\frac{P . Q \sin q}{\pm \sqrt{(Q^2 - P^2 \sin q^2)}} = \frac{P \sin q}{\mathrm{cosin}\, p} = \frac{Q \sin p}{\mathrm{cosin}\, p} = Q \,\mathrm{tang}\, p.$

Vel membrum tertium §. 16. $= dQ . Q \,\mathrm{tang}\, p$.

Inde

Inde ex §§. 17. 18. 19. jam hanc formulam habemus:

$$2\,dA = dP.\frac{\tan p}{P}(R^2 - P^2) - dq\frac{R^2 \tan p}{\tan r} + dQ.Q \tan p.$$

seu $$2\,dA = \left(\frac{R^2 - P^2}{P} dP + Q dQ - \frac{R^2\,dq}{\tan r}\right) \tan p$$

§. 20.

Si dati sint duo anguli r & p cum latere interjacente Q, erit $A = \frac{PQ \sin r}{2}$

Sed $P = \frac{Q.\sin p}{(\sin p+r)}$. Hinc $A = \frac{Q^2 \sin p.\sin r}{2.\sin(p+r)}$.

$$lA = 2lQ + l\sin p + l\sin r - l2 - l\sin(p+r)$$

$$\frac{dA}{A} = \frac{2dQ}{Q} + \frac{dp.\cos p}{\sin p} + \frac{dr.\cos r}{\sin r} - \frac{d(p+r)\cos(p+r)}{\sin(p+r)}$$

vel $$\frac{dA}{A} = \frac{2dQ}{Q} + \frac{dp.\sin r}{\sin p.\sin(p+r)} + \frac{dr \sin p}{\sin r.\sin(p+r)}$$

§. 21.

Quum sit $\left.\begin{matrix}\sin p\\ \sin r\end{matrix}\right\} : \sin(p+r) = \left.\begin{matrix}P\\ R\end{matrix}\right\} : Q$

erit $\frac{dA}{A} = \frac{2dQ}{Q} + \frac{dp.R}{Q.\sin p} + \frac{dr.P}{Q.\sin r}$ vel, multiplicando per $\frac{PQ\sin r}{2} = A$

$$dA = P\sin r\,dQ + \frac{R^2}{2}dp + \frac{P^2}{2}dr$$

§. 22.

Si dati sint duo anguli r, & q cum latere alterutri angulorum opposito, Q.

erit $A = \frac{PQ\sin r}{2}$. Sed $P = \frac{Q\sin(r+q)}{\sin q}$. Hinc $A = \frac{Q^2 \sin r \sin(r+q)}{2\sin q}$

$$lA = 2lQ + l\sin r + l\sin(r+q) - l2 - l\sin q$$

$$\frac{dA}{A} = \frac{2dQ}{Q} + \frac{\cos r}{\sin r}dr + \frac{\cos(r+q)}{\sin(r+q)}d(r+q) - \frac{\cos q}{\sin q}dq$$

$$\frac{dA}{A} = \frac{2\,dQ}{Q} + dr\,\frac{(\mathrm{sin}\,(r+q)\,\mathrm{cosin}\,r + \mathrm{cosin}\,(r+q)\,\mathrm{sin}\,r)}{\mathrm{sin}\,r\,.\,\mathrm{sin}\,(r+q)}$$

$$+\,dq\,\frac{(\mathrm{sin}\,q\,\mathrm{cosin}\,(r+q) - \mathrm{cosin}\,q\,\mathrm{sin}\,(r+q))}{\mathrm{sin}\,q\,.\,\mathrm{sin}\,(r+q)}$$

$$\frac{dA}{A} = \frac{2\,dQ}{Q} + dr\,\frac{\mathrm{sin}\,(2r+q)}{\mathrm{sin}\,r\,.\,\mathrm{sin}\,(r+q)} - \frac{dq\,\mathrm{sin}\,r}{\mathrm{sin}\,q\,.\,\mathrm{sin}\,(r+q)}$$

§. 23.

$\frac{\mathrm{Sin}\,(2r+q)}{\mathrm{sin}\,r\,.\,\mathrm{sin}\,(r+q)}$ seu (§. 22.) $\frac{\mathrm{cosin}\,r}{\mathrm{sin}\,r} + \frac{\mathrm{cosin}\,(r+q)}{\mathrm{sin}\,(r+q)} = \frac{\mathrm{cosin}\,r}{\mathrm{sin}\,r} - \frac{\mathrm{cosin}\,p}{\mathrm{sin}\,p}$

$= \frac{P\,\mathrm{cosin}\,r - R\,\mathrm{cosin}\,p}{R\,\mathrm{sin}\,p}$ quia $P\,\mathrm{sin}\,r = R\,\mathrm{sin}\,p$

Sed $2PQ\,\mathrm{cosin}\,r = P^2 + Q^2 - R^2 = 2Q^2 - (Q^2 + R^2 - P^2) = 2Q^2 - 2QR\,\mathrm{cos}.\,p$

proinde $P\,\mathrm{cosin}\,r = Q - R\,\mathrm{cosin}\,p$

& $\frac{\mathrm{sin}\,(2r+q)}{\mathrm{sin}\,r\,.\,\mathrm{sin}\,(r+q)} = \frac{Q - 2R\,\mathrm{cosin}\,p}{R\,\mathrm{sin}\,p}$.

Hinc $dr\,\frac{\mathrm{sin}\,(2r+q)}{\mathrm{sin}\,r\,.\,\mathrm{sin}\,(r+q)} = dr\,\frac{(Q - 2R\,\mathrm{cosin}\,p)}{R\,\mathrm{sin}\,p}$, $\frac{dq\,.\,\mathrm{sin}\,r}{\mathrm{sin}\,q\,\mathrm{sin}\,(r+q)} = dq\,.\,\frac{R}{Q\,\mathrm{sin}\,p}$.

Hinc formula §. 22. jam in hanc transformatur:

$\frac{dA}{A} = \frac{2\,dQ}{Q} + dr\,\frac{(Q - 2R\,\mathrm{cosin}\,p)}{R\,\mathrm{sin}\,p} - dq\,\frac{R}{Q\,\mathrm{sin}\,p}$ &, multiplicando per $QR\,\mathrm{sin}\,p = 2A$

$$2\,dA = 2R\,\mathrm{sin}\,p\,dQ + dr\,(Q^2 - 2QR\,\mathrm{cosin}\,p) - dq\,R^2$$

At $Q^2 - 2QR\,\mathrm{cosin}\,p = P^2 - R^2$

Hinc $2\,dA = 2R\,\mathrm{sin}\,p\,dQ + dr\,(P^2 - R^2) - dq\,R^2$

§. 24.

Si data sint tria latera (fig. 3.) $AC = P$, $BC = Q$, $AB = R$: erit ex Element. II, 12 & 13.

$$\pm 2AB\,.\,AK = AB^2 + AC^2 - BC^2$$

$$AK = \frac{AB^2 + AC^2 - BC^2}{\pm 2AB}$$

CK

$$CK^2 = AC^2 - AK^2 = AC^2 - \frac{(AB^2 + AC^2 - BC^2)^2}{4\,AB^2}$$

$$= \frac{4\,AB^2\,AC^2 - (AB^2 + AC^2 - BC^2)^2}{4\,AB^2}$$

Area trianguli $= A = \frac{CK.AB}{2}$. $\quad A^2 = \frac{CK^2.AB^2}{4}$

$$= \frac{4\,AB^2\,AC^2 - (AB^2 + AC^2 - BC^2)^2}{16}$$

$$16\,A^2 = 4\,R^2\,P^2 - (R^2 + P^2 - Q^2)^2$$

$$dA = \frac{(R^2+Q^2-P^2)\,PdP + (P^2+R^2-Q^2)\,QdQ + (P^2+Q^2-R^2)\,RdR}{8\,A}$$

§. 25.

Quum sit $R^2 + Q^2 - P^2 = 2\,QR \operatorname{cosin} p$

$P^2 + R^2 - Q^2 = 2\,PR \operatorname{cosin} q$

$P^2 + Q^2 - R^2 = 2\,PQ \operatorname{cosin} r$ erit

$$2\,A\,dA = \frac{dP}{2}.\,PQR \operatorname{cosin} p + \frac{dQ}{2}\,PQR \operatorname{cosin} q + \frac{dR}{2}\,PQR \operatorname{cosin} r.$$

& dividendo per $A = \frac{QR \sin p}{2} = \frac{PR \sin q}{2} = \frac{PQ \sin r}{2}$

$$2\,dA = dP.\,P \operatorname{cotang} p + dQ.\,Q \operatorname{cotang} q + dR.\,R \operatorname{cotang} R.$$

§. 26.

Ex his formulis omnes casus facile solvuntur; dum nempe, si non omnia tria data mutentur, eorum, quæ manent, differentialia = o ponuntur. Sic, si unum tantum datorum varietur; reliqua duo invariata maneant; horum differentialibus = o positis consequentur regulæ superioribus theorematibus demonstratæ. Pariter inde deducentur regulæ pro iis casibus, ubi duo quævis datorum variantur, unum manet invariatum; quas hic apponere, superfluum videtur. Cæterum eædem formulæ adhuc ostendunt: quantam variationem subeat unum quodvis trianguli elementum, dum area trianguli, & duo ejus alia quæcunque elementa minimas aliquas mutationes patiuntur; docent etiam: quem inter se nexum habere debeant errores datorum, si requiratur ut area trianguli non mutetur; & vice versa.

§. 27.

Ad triangula sphærica nunc pergo. Sunt apud *Cotesium* (de æstimatione errorum in mixta Mathesi per variationes partium trianguli plani & sphærici) decem & orto theoremata geometrice demonstrata, quæ omnes eos casus complectuntur, quibus unum tantum trianguli sphærici elementum variatur. Analogias Cotesianas symbolice expressas, adjunctis pluribus aliis, absque demonstrationibus (quarum curiosos ad *Cotesium* remittit) in Mem. de l'Acad. des sciences de Paris, & postea in Leçons d'Astronomie exhibuit de la Caille; multisque exemplis illustravit. Eædem, Cotesiana methodo demonstratæ, pariter symbolice expressæ inveniuntur apud de *la Lande*, (Astronomie T. III. L. XXIII. §. 3746 sqq.) *Brackenhofferum*, (Sphæricorum formulare Part. II. Sect. IV.) *Scherfferum* (Institutiones Geometriæ sphæricæ cap. I. art. IX). *Kæstnerus* denique, (astronomische Abhandlungen. Erste Sammlung IIte Abhandl. 2tes Cap. S. 95 sqq.) & *Klügelius*, (analytische Trigonometrie 8tes Cap. S. 221 sqq.) præeunte in solvendo quodam problemate particulari huc pertinente *Eulero* in Comment. Petropol. Tom. VIII. eosdem casus speciales calculo subjecerunt. Generales formulas nemo, quod sciam, exposuit; quæ tamen & ipsæ, cum plura data errori obnoxia sint, applicationem nanciscuntur; & breviter inde deducendis regulis casuum specialium inserviunt. *Lamberti* igitur secutus exemplum, quo in tractandis triangulorum planorum variationibus præivit (Anmerkungen und Zusæze zur practischen Geometrie §. 336 sqq.); rationes, quas invicem habent variationes quorumcunque quatuor elementorum trianguli sphærici, investigabo. Comprehendere eas licebit quatuor formulis generalibus; cum quatuor quæcunque trianguli sphærici elementa ita a se invicem pendeant, ut ex tribus determinari possit quartum. Erunt nempe hæc quatuor trianguli elementa a se invicem pendentia aut tria latera cum uno angulo; aut tres anguli cum uno latere; aut duo anguli, cum duobus lateribus, quorum unum alterutri horum angulorum opponitur, alterum illis interjacet; aut duo anguli cum duobus lateribus, quæ ipsis opponuntur. Formulas, quibus ipsorummet talium quatuor elementorum rationes assignantur, ex Trigonometria sphærica cognitas supponam. Litteræ P, Q, R iterum latera; p, q, r angulos ipsis oppositos significabunt.

§. 28.

§. 28.

Si proposita sint tria latera P, Q, R cum angulo quocunque q: erit

$\sin P . \sin R \operatorname{cosin} q = \operatorname{cosin} Q - \operatorname{cosin} R . \operatorname{cosin} P.$

Hinc $dP . \operatorname{cosin} P \sin R \operatorname{cos} q + dR \sin P \operatorname{cosin} R \operatorname{cosin} q - dq \sin P \sin R \sin q$

$= - dQ \sin Q + dR \sin R \operatorname{cosin} P + dP \sin P \operatorname{cosin} R$

$dQ . \sin Q - dP(\sin P \operatorname{cos} R - \operatorname{cos} P \sin R \operatorname{cos} q) - dR(\sin R \operatorname{cos} P - \operatorname{cos} R \sin P \operatorname{cos} q)$

$= dq \sin P \sin R \sin q$

$$dQ . \sin Q - dP\left(\sin P \operatorname{cosin} R - \frac{\operatorname{cosin} P \operatorname{cosin} Q - \operatorname{cosin} P^2 \operatorname{cosin} R}{\sin P}\right)$$

$$- dR\left(\sin R \operatorname{cos} P - \frac{\operatorname{cos} R \operatorname{cos} Q - \operatorname{cos} R^2 \operatorname{cos} P}{\sin R}\right) = dq \sin P \sin R \sin q.$$

$$dQ . \sin Q - dP\left(\frac{\operatorname{cosin} R - \operatorname{cosin} P \operatorname{cosin} Q}{\sin P}\right) - dR\left(\frac{\operatorname{cos} P - \operatorname{cos} R \operatorname{cos} Q}{\sin R}\right)$$

$= dq \sin P \sin R \sin q.$

§. 29.

Fiet hæc formula simplicior, introductis aliis trianguli elementis.

Nam $\operatorname{cosin} R - \operatorname{cosin} P \operatorname{cosin} Q = \sin P \sin Q \operatorname{cosin} r$

& $\operatorname{cosin} P - \operatorname{cosin} R \operatorname{cosin} Q = \sin Q \sin R \operatorname{cosin} p$

Hinc $dQ \sin Q - dP \sin Q \operatorname{cosin} r - dR \sin Q \operatorname{cosin} p = dq \sin P \sin R \sin q$

$$dQ - dP \operatorname{cosin} r - dR \operatorname{cosin} p = \frac{dq \sin P \sin R \sin q}{\sin Q}$$

Sed $\sin Q : \sin q = \begin{cases} \sin R : \sin r \\ \sin P : \sin p \end{cases}$

hinc $dQ - dP \operatorname{cosin} r - dR \operatorname{cosin} p = dq \begin{cases} \sin R \sin p \\ \sin P \sin r \end{cases}$

§. 30.

Si propositi sint tres anguli p, q, r cum latere quocunque Q: est

$\sin p . \sin r . \operatorname{cosin} Q = \operatorname{cosin} q + \operatorname{cosin} r \operatorname{cosin} p.$

Hinc $dp . \operatorname{cosin} p \sin r \operatorname{cosin} Q + dr \sin p \operatorname{cosin} r \operatorname{cos} Q - dQ \sin p \sin r \sin Q$

$= - dq \sin q - dr \sin r \operatorname{cosin} p - dp \operatorname{cosin} r \sin p$

$$dq \sin q + dp(\sin p \,\text{cosin}\, r + \text{cosin}\, p \sin r \,\text{cosin}\, Q) + dr(\sin r \,\text{cosin}\, p + \sin p \,\text{cosin}\, r \,\text{cosin}\, Q) = dQ \sin p \sin r \sin Q.$$

$$dq \sin q + dp\left(\sin p \,\text{cosin}\, r + \frac{\text{cosin}\, p \,\text{cosin}\, q + \text{cosin}\, p^2 \,\text{cosin}\, r}{\sin p}\right)$$

$$+ dr\left(\sin r \,\text{cosin}\, p + \frac{\text{cosin}\, r \,\text{cos.}\, q + \text{cosin}\, r^2 \,\text{cos.}\, p}{\sin r}\right) = dQ \sin p \sin r \sin Q$$

$$dq \sin q + dp\left(\frac{\text{cosin}\, r + \text{cosin}\, p \,\text{cosin}\, q}{\sin p}\right) + dr\left(\frac{\text{cosin}\, p + \text{cosin}\, r \,\text{cosin}\, q}{\sin r}\right)$$

$$= dQ \sin p \sin r \sin Q.$$

§. 31.

Quia $\text{cosin}\, r + \text{cosin}\, p \,\text{cosin}\, q = \sin p \sin q \,\text{cosin}\, R$

$\text{cosin}\, p + \text{cosin}\, r \,\text{cosin}\, q = \sin r \sin q \,\text{cosin}\, P$

erit $dq \sin q + dp \sin q \,\text{cosin}\, R + dr \sin q \,\text{cosin}\, P = dQ \sin p \sin r \sin Q$

$$dq + dp \,\text{cos.}R + dr \,\text{cos.}P = dQ.\frac{\sin p \sin r \sin Q}{\sin q} = dQ \begin{cases} \sin p \sin R \\ \sin r \sin P \end{cases}$$

§. 32.

Si propositi sint duo anguli p, q cum duobus lateribus Q, R, quorum unum alterutri horum angulorum opponitur, alterum illis interjacet: est

$$\sin R \,\text{cotang}\, Q = \text{cosin}\, p \,\text{cosin}\, R + \sin p \,\text{cotang}\, q$$

$$dR \,\text{cosin}\, R \,\text{cotang}\, Q - dQ\frac{\sin R}{\sin Q^2} = -dp \sin p \,\text{cosin}\, R - dR \,\text{cosin}\, p \sin R$$

$$+ dp \,\text{cosin}\, p \,\text{cotang}\, q - dq\frac{\sin p}{\sin q^2}$$

$$dp(\sin p \,\text{cosin}\, R - \text{cosin}\, p \,\text{cotang}\, q) + dq\,\frac{\sin p}{\sin q^2}$$

$$= dQ.\frac{\sin R}{\sin Q^2} - dR\,(\text{cosin}\, p \sin R + \text{cosin}\, R \,\text{cotang}\, Q)$$

$$dp\left(\sin p \,\text{cos.}R - \frac{\text{cos.}\,p \sin R \,\text{cot.}\,Q - \text{cos.}\,p^2 \,\text{cos.}\,R}{\sin p}\right) + dq\left(\frac{\sin p}{\sin q}\right)^2 \frac{1}{\sin p}$$

$$= dQ$$

$$=dQ\frac{\sin R}{\sin Q^2}-dR\left(\frac{\sin R^2 \operatorname{cotang} Q-\sin R \sin p \operatorname{cotang} q}{\operatorname{cosin} R}+\operatorname{cos.} R \operatorname{cot.} Q\right)$$

$$dp\left(\frac{\operatorname{cosin} R-\operatorname{cosin} p \sin R \operatorname{cotang} Q}{\sin p}\right)+dq\left(\frac{\sin P}{\sin Q}\right)^2\frac{1}{\sin p}$$

$$=dQ\frac{\sin R}{\sin Q^2}-dR\left(\frac{\operatorname{cotang} Q-\sin R \sin p \operatorname{cotang} q}{\operatorname{cosin} R}\right)$$

$$dp \sin Q\left(\frac{\operatorname{cosin} R \sin Q-\operatorname{cosin} p \sin R \operatorname{cosin} Q}{\sin p}\right)+dq\frac{\sin P^2}{\sin p}$$

$$=dQ \sin R-dR \sin Q\left(\frac{\operatorname{cosin} Q-\sin R \sin p \operatorname{cotang} q \sin Q}{\operatorname{cosin} R}\right)$$

$$dp \sin R \sin Q\left(\frac{\operatorname{cotang} R \sin Q-\operatorname{cosin} p \operatorname{cosin} Q}{\sin p}\right)+dq\frac{\sin P^2}{\sin p}$$

$$=dQ \sin R-dR \sin Q\left(\frac{\operatorname{cosin} Q-\sin R \sin p \operatorname{cotang} q \sin Q}{\operatorname{cosin} R}\right)$$

§. 33.

Quia etiam $\operatorname{cotang} R \sin Q=\operatorname{cosin} p \operatorname{cosin} Q+\sin p \operatorname{cotang} r$

adeoque $\frac{\operatorname{cotang} R \sin Q-\operatorname{cosin} p \operatorname{cosin} Q}{\sin p}=\operatorname{cotang} r$

& $\sin R \sin Q \sin p \operatorname{cotang} q=\frac{\sin R \sin Q \sin p \operatorname{cos.} q}{\sin q}=\sin P \sin R \operatorname{cos.} q$

proinde

$$\frac{\operatorname{cos.} Q-\sin R \sin Q \sin p \operatorname{cot.} q}{\operatorname{cosin} R}=\frac{\operatorname{cos.} Q-\sin P \sin R \operatorname{cos.} q}{\operatorname{cosin} R}=\operatorname{cos.} P \text{ (§.28.)}$$

erit $dp \sin R \sin Q \operatorname{cotang} r+dq\frac{\sin P^2}{\sin p}=dQ \sin R-dR \sin Q \operatorname{cos.} P$

$$dp \operatorname{cotang} r+dq\frac{\sin p}{\sin q \sin r}=\frac{dQ}{\sin Q}-dR\frac{\operatorname{cosin} P}{\sin R}$$

§. 34.

Si propoſiti ſint duo anguli q, r cum duobus lateribus ipſis oppoſitis Q, R: eſt

$$\text{ſin } R \ \text{ſin } q = \text{ſin } Q \ \text{ſin } r$$

$$l \ \text{ſin } R + l \ \text{ſin } q = l \ \text{ſin } Q + l \ \text{ſin } r$$

$$dR \frac{\text{coſin } R}{\text{ſin } R} + dq \frac{\text{coſin } q}{\text{ſin } q} = dQ \frac{\text{coſin } Q}{\text{ſin } q} + dr \frac{\text{coſin } r}{\text{ſin } r}$$

$$\frac{dR}{\text{tang } R} + \frac{dq}{\text{tang } q} = \frac{dQ}{\text{tang } Q} + \frac{dr}{\text{tang } r}$$

§. 35.

Liceat nunc uſum harum propoſitionum in triangulis planis & ſphæricis paucis adhuc facilioribus exemplis illuſtrare. Rectæ indefinitæ AB alia indefinita AC ſub angulo recto exacte inſiſtat. Abſciſſa in recta AB, à puncto A inde, baſi AD (quod abſque errore peragi poſſe ſupponemus); & angulo ad alterum ejus extremum D applicato, ope inſtrumenti, quod locum errori exiguo relinquat: conſtruendum ſit triangulum rectangulum areæ datæ. Quæritur magnitudo baſis AD, & anguli ei ad D applicandi; qua fiat, ut idem in angulo D conſtruendo admiſſus error exiguus in aream trianguli variandam quam minime influat. Quum latus AD, & alter angulus adjacens A exacte dari ponantur; erit ex Theoremate tertio §. 6.

Fig. 5.

$$\text{variatio minima areæ} = \frac{DE^2 \times \text{variat. minim. ang. D.}}{2. \ \text{ſin. tot.}}$$

Erit igitur cæteris paribus error in area trianguli eo minor, quo minus eſt latus DE. Inveniendum itaque erit inter omnia triangula rectangula, quæ datam aream habent, id cujus hypotenuſa eſt minima. Sit area trianguli $= a^2$; unus cathetus $= x$: erit alter $= \frac{2a^2}{x}$; & quadratum hypotenuſæ $= V = x^2 + \frac{4a^4}{x^2}$: proinde $\frac{dV}{dx} = 2x - \frac{8a^4}{x^3}$ Ex quo $=$ o poſito fit $x^4 = 4a^4$; & $x = a\sqrt{2}$ Hinc alter cathetus $= \frac{2a^2}{x}$ fit $= \frac{2a^2}{a\sqrt{2}} = a\sqrt{2}$. Ergo triangulum æquicrurum, & angulus D semirectus sit, oportet. Et cum $\frac{d^2V}{dx^2} = 2 + \frac{24a^4}{x^4} = + 8$; reapse minimum est, quod ita invenitur. Error itaque, in aream trianguli ex errore anguli D redundans, minimus erit; si AD sumatur $= a\sqrt{2}$, & ad punctum D construatur angulus semirectus. Hoc ipso casu ostendit Cotesius, etiam lateris AC errorem fore minimum: quo nostrum enunciatum confirmatur.

Si vero super data basi AD ope duorum angulorum quorumcunque, ejus extremis applicandorum, construendum sit triangulum areæ datæ: erit ex §. 21. variatio ex erroribus horum angulorum in aream redundans $= dA = dp\frac{R^2}{2} + dr\frac{P^2}{2}$. Vel, si eundem in utroque angulo errorem committi ponamus; $dA = dp\left(\frac{R^2+P^2}{2}\right)$. Quo minor itaque fuerit summa quadratorum laterum, angulis p & r oppositorum; eo minor erit error in area trianguli. Quæstio igitur huc redit, ut determinetur: quodnam inter omnia triangula, quæ habent aream datam a^2, & basin datam b, ita comparatum sit, ut V summa quadratorum duorum reliquorum laterum sit minima. In hoc triangulo perpendiculum ex vertice anguli oppositi in basin demissum erit $= \frac{2a^2}{b}$. Idem perpendiculum basin secabit in duas partes: quarum si una sit x; erit altera $b - x$. Quadratum vero ejus lateris trianguli, quod adjacet segmento baseos x, erit $= x^2 + \frac{4a}{b^2}$; & quadratum alterius lateris, quod adjacet segmento $b - x$, erit $= b^2 - 2bx + x^2 + \frac{4a^4}{b^2}$: quorum quadratorum summa $V = 2x^2 + \frac{8a^4}{b^2} - 2bx + b^2$. Vnde $\frac{dV}{dx} = 4x - 2b$. Ex quo $= 0$ posito fit $x = \frac{1}{2}b$; proinde triangulum æquicrurum. Et cum $\frac{d^2V}{dx^2} = +4$; reipsa V, summa quadratorum crurum trianguli, ac pro-

proinde variatio areæ illius minima fit. Si errores angulorum p & r æquales quidem, ſed oppoſiti ponantur; ita nimirum, ut ſit $dp = -dr$; foret $dA = dp\left(\frac{R^2 - P^2}{2}\right)$. Hic igitur inquirendum eſſet; quodnam inter omnia triangula, quæ habent aream datam a^2, & baſin b, ita comparatum ſit, ut differentia quadratorum duorum reliquorum laterum ſit minima. Manentibus omnibus, ut antea, eſſet hæc differentia quadratorum $= b^2 - 2bx$: unde patet, tum hanc expreſſionem minimam eſſe, aut potius plane evaneſcere, ubi ſit $2bx = b^2$; ſeu $x = \frac{1}{2}b$. Foret igitur etiam hoc caſu triangulum æquicrurum aptiſſimum ad efficiendum, ut area trianguli ab erroribus angulorum p & r minimam patiatur variationem.

Si baſis AD data non eſſet; veruntamen, quæcunque ejus deſideretur magnitudo, exacte effici poſſet: tum ita eſſet ratiocinandum. Sit $AD = x$. Erit ſecundum ea, quæ modo vidimus, triangulum æquicrurum ſuper illa conſtruendum; ut ex erroribus angulorum p & r æqualibus minimus, qui fieri poteſt, error in aream trianguli irrepat. Et, quum hic error, ſub conditione priori loco poſita, ſit pro qualibet aſſumta baſi, uti ſumma quadratorum crurum, ſeu uti duplum quadratum alterutrius cruris trianguli: jam quæritur, quodnam inter omnia triangula æquicrura, quæ datam aream $= a^2$ habent, ſit id, cujus crura, adeoque etiam eorum quadrata ſint minima. Quum baſis ſit x, & triangulum æquicrurum: perpendiculum ex angulo ad verticem in baſin demiſſum erit $= \frac{2a^2}{x}$; & baſin ſecabit in duas partes æquales, quorum igitur unaquæque erit $= \frac{1}{2}x$. Hinc quadratum uniuscujusque cruris $= V = \frac{1}{4}x^2 + \frac{4a^4}{x^2}$. Proinde $\frac{dV}{dx} = \frac{1}{2}x - \frac{8a^4}{x^3}$. Ex quo $= 0$ poſito fit $x^4 = 16a^4$, $x = 2a$; perpendicularis in baſin demiſſa $= \frac{2a^2}{x} = a$; quadratum uniuscujusque cruris $= \frac{1}{4}.4a^2 + \frac{4a^4}{4a^2} = 2a^2$; & unumquodque crurum $= a\sqrt{2}$. Tutiſſimum igitur erit, ſumere baſin $= 2a$, & ſuper ea conſtruere triangulum æquicrurum rectangulum. Minimum enim

enim esse, quod ita obtinetur, ex eo patet: quod $\frac{d^2V}{dx^2} = \frac{1}{2} + \frac{24a^4}{x^4}$ tunc fit $= \frac{1}{2} + \frac{3}{2} = +2$.

§. 36.

In praxi Astronomica sæpe altitudo poli determinari debet ex altitudine sideris cujusdam, cujus declinatio nota est, & quod

Fig. 6. dato tempore observatur. Si P polum, V verticem observatoris, S sidus denotet: erunt PV, VS, PS complementa altitudinis poli, altitudinis sideris, & declinationis sideris. Quæritur, quæ potissimum sidera eligere debeat Astronomus; ut, si quis error in observationem altitudinis sideris irrepat, minimus tamen exinde in determinanda altitudine poli error oriatur. Erit igitur, si assumatur, tempus verum, & declinationem sideris accuratissime nota esse, ex (§. 29.) positis $VPS = q$, $VS = Q$, $PS = R$, $PV = P$, quum invariata maneant q & R, $dP = \frac{dQ}{\text{cosin } r}$: id est, error in determinanda altitudine poli, eo minor committetur, quo major est cosinus anguli r; vel, quo minorem angulum acutum, aut quo majorem obtusum facit VS cum PV: minimus ergo erit, scilicet $dP = \pm dQ$, si sidus sit in meridiano constitutum.

Si altitudo sideris fuerit exacte definita, tempus vero observationis errori obnoxium; erit $dP = -dq \sin P \tan r$: id est, si distantia temporis a meridie major justo assumta sit, invenietur altitudo æquatoris justo minor, vel altitudo poli justo major; & vice versa. Utroque casu error sub data poli elevatione eo major erit, quo magis datus verticalis distet à meridiano; maximus igitur, si fecerit verticalis angulos rectos cum meridiano: præterea eo major erit, quo major sit sinus P, seu cosinus altitudinis poli: absolute igitur maximus erit sub æquatore degentibus, ubi verticulis fecerit angulos rectos cum meridiano. Et simili modo in reliquis casibus; etiam, ubi plura data simul variantur, ratiocinandum erit.

THESES.

THESES.

I.

Est in Trigonometria ſphærica caſus, ubi datis non tribus tantum, ſed quatuor adeo trianguli elementis reliqua duo exinde colligere tamen haud licet; nempe, ſi duo anguli, adeoque etiam duo latera ipſis oppoſita ſint rectangula. Neque tamen id prohibet, quo minus dici poſſit, Trigonometriam docere ex datis tribus elementis invenire reliqua.

II.

Quæ ad notandas etiam abſente obſervatore variationes meteorologicas excogitata ſunt inſtrumenta, parum adhuc perfectionis habent; neque omnino poſſunt ejus generis inſtrumenta ſatis exacta fieri.

III.

Ea, quam Ingenhouſius invenit, antlia pnevmatica, qua aër a carbonibus abſorbetur, non poteſt inſervire omnibus iis experimentis faciundis, quæ ope vulgaris antliæ inſtitui ſolent.

IV.

Inter omnes, quæ adhucdum propoſitæ ſunt, theorias, optime pleraque Electricitatis phænomena explicare videtur Francliniana.

V.

Aptiora eſſe videntur ad avertendum fulmen ea inſtrumenta, quæ acutam cuſpidem, quam quæ globulum rotundum nubibus opponunt.

VI.

Vicinis ædificiis ejusmodi inſtrumenta, ſiquidem bene fabricata ſint, nocere poſſunt nunquam: prodeſſe ſæpius.

PEREXIMIO ATQVE PRÆSTANTISSIMO
DOMINO CANDIDATO, DISSERTATIONIS HVIVS AVCTORI
PRÆSES.

Inſigne, quod edis, **Tuorum** *in diſciplinis mathematicis profectuum, & operæ, quam illis navaſti, indefeſſæ pariter ac felicis, documentum ex animo* Tibi *gratulor: &, ut porro etiam* **Tuis** *ſtudiis cœptisque Deus benigniſſime annuat, enixe precor.*

Fig: 1.

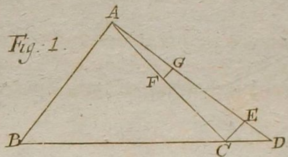


Fig: 2.

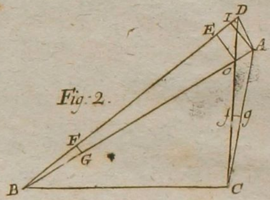


Fig: 3.

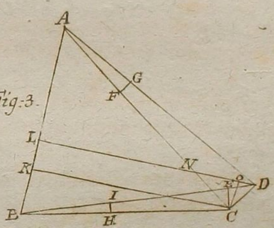


Fig: 4.

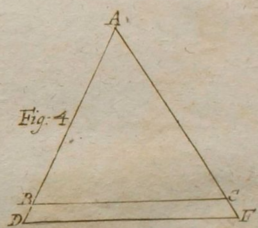
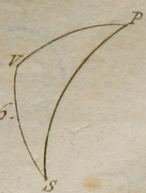


Fig: 5.

Fig: 6.



Tübingen, Diss., 1783/88

PROPOSITIONES NONNVLLÆ AD THEORIAM ÆSTIMATIONIS ERRORVM IN TRIANGVLIS PLANIS ET SPHÆRICIS PERTINENTES

QVAS
RECTORE VNIVERSITATIS EBERHARDINÆ CAROLINÆ
MAGNIFICENTISSIMO
SERENISSIMO DVCE ET DOMINO
DOMINO
CAROLO
DVCE WIRTEMBERGIÆ ET TECCIÆ REGNANTE
REL. REL.

RÆSIDE
ISSIMO ATQVE AMPLISSIMO
FRID. PFLEIDERER
LEGII ILLVSTRIS PROF. PHYSICES
HESEOS PVBL. ORD.
TRONO SVO PIE DEVENERANDO
MAGISTERII PHILOSOPHICI HONORIBVS
AVG. MDCCLXXXIII.
DISPVTANDVM PROPONIT
AVCTOR
LHELMVS CAMERER
ASTETTENSIS
PHICI CANDIDATVS IN ILLVSTRI
NDIO THEOLOGICO.

TYPIS FVESIANIS.