Set-Valued Convex Analysis

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Contents

1 Introduction

The main topic of this thesis is convex analysis for set-valued functions. The main achievement is the perception that every important formula from the scalar theory has a set-valued counterpart. We will prove a set-valued version of the Fenchel-Moreau-Theorem, weak and strong duality, a set-valued max-formula for the directional derivative and provide a full calculus for the conjugate, directional derivative and the subdifferential of set-valued functions. The results rely on an appropriate choice of an lattice ordered subset of the power set of an quasi-ordered linear space as image space for set-valued functions.

"Closed convex sets in a locally convex topological vector space may be described in a dual way: they are identical with the intersection of the closed half-spaces which contain them. ... Closed convex functions in a locally convex topological vector space also permit a dual description...: they are pointwise least upper bounds of the affine functions which do not exceed them in size. Such a duality permits one to establish a connection between a convex function and the dual object — the conjugate function." [59]

"The theorems of convex analysis relate the operations of conjugation... and taking a subdifferential with algebraic, set-theoretic and ordering operations over convex sets and functions. Other subjects of study include all possible dual relations between... functions and their conjugates..." [58]

In the following, we will introduce an approach to set-valued convex analysis which is based on constructions presented in [23] and in total accordance with the well-known scalar theory as presented in [62]. There are two fundamental questions arising, when dealing with set-valued functions mapping one locally convex topological vector space X into the power set $\mathcal{P}(Z)$ of another locally convex topological vector space Z. The first is the question of an appropriate dual space, the other that of the difference between two sets.

In most approaches known to the author, the dual space has been chosen to be $\mathcal{L}(X, Z)$, the set of linear continuous operators mapping X into Z, compare [7, 48, 49, 57, 60, 63] and others. Once the dual problem is considered as a set-valued, rather then a vector-valued problem, a change of image spaces from infimum-oriented to supremum-oriented sets can be observed, compare [39, 38, 48, 54], which is due to the difference in the formulation of the dual problem. This is natural, as long as the difference between two sets is defined as the algebraic difference. Moreover, most frequently additional structural assumptions have to be made on the image space. One frequently used assumption is that the topological interior of the ordering cone $C \subseteq Z$ is nonempty, compare [4, 12, 24, 53, 56].

In our approach, we will only assume X and Z to be rich enough to allow the classic separation theorems in $X \times Z$ and, moreover, Z to be quasi ordered by a closed convex cone $\{0\} \subsetneq C \subsetneq Z$. The dual variables will be pairs $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ and by introducing an order theoretically motivated difference between subsets of Z, we are able to translate virtually every concept of the scalar theory to the set-valued theory of convex functions, such as conjugate and biconjugate functions, the directional derivative or the subdifferential. The main contribution of this thesis is, that we are able to supply a full calculus for the conjugate, directional derivative and the subdifferential of set-valued functions. Along the way, we are able to prove weak and, under an additional constrained assumption, strong duality.

One characteristic of the presented results is that each set-valued result has an equivalent representation through a family of basically scalar results. Notice, however, that we are not scalarizing the original problem in the ordinary way but rather represent it through a family of scalarizations without loss of information on the original problem. The vector-valued situation can be rediscovered in the results by replacing a vector-valued function by a set-valued extension. Thus, the presented results generalize known results in vector-valued convex analysis to more general image spaces. Apart from the various applications of set-valued analysis, such as in multicriteria-optimization, the author believes that the almost one-to-one correspondence between the scalar theory and the presented set-valued theory is in itself highly interesting.

The present text focuses on central concepts (conjugate, subdifferential, directional derivative, tangent and normal cone for epigraphs) for set-valued convex functions and their mutual relationships (max-formula, biconjugation, subdifferential calculus etc.). This leaves e.g. application to set and vector optimization problems for future research. While most of the fundamental theorems of convex analysis are presented in their set-valued form, it has not been the intention of the author to go into greater detail about more specialized results under additional assumptions. Also, most of the text is restricted to the analysis of convex functions. Though it presents no greater problem to generalize to more general functions, the author believes that the basic structure is more obvious this way.

The text is structured as follows. In the second chapter, we will collect some basic facts about order relations, conlinear spaces and functions mapping one conlinear space into another. We will introduce a difference operation on quasi-ordered conlinear spaces which is, to our knowledge, new. We then turn to a special case of a conlinear, quasi-ordered space, the power set $\mathcal{P}(Z)$ of a locally convex topological vector space Z, quasi-ordered by a closed convex cone C. We will discuss special classes of set-valued functions and specify the associated image spaces. In the end of this chapter, we will introduce a slightly altered version of the previously introduced difference operator on the power set $\mathcal{P}(Z)$. This is the first of many places, where the representation of a set-valued expression through a family of basically scalar expressions will occur.

The third chapter is dedicated to the scalarized representation of set-valued functions. This chapter supplies us with a strong tool in proving most of the subsequent statements. In fact, the set-valued theory presented in this thesis can either be derived from the scalar theory, which is the approach chosen in most parts of this thesis, or independent from this and including the scalar results a special case. The second approach has been chosen in some proofs, but is exploited to a greater extend in [23].

The fourth chapter introduces a convex conjugate and biconjugate of Fenchel-Rockafellar type. After discussing some basic results of these functions, we will prove a number of basic duality results, such as a sum- and chain-rule, as well as weak and, under additional assumptions, strong duality and a sandwich theorem. The results in this chapter are closely related to those presented in [23]. Exploiting the full potency of the difference of sets presented earlier, we are able to translate the conjugates presented there into convex functions in accordance to the classic Fenchel-Rockafellar conjugate.

In the fifth chapter the directional derivative of set-valued functions is defined and its basic properties will be discussed.

The sixth chapter is dedicated to the subdifferential of a convex set-valued function. In fact, this chapter contains the most notable difference between the classic scalar theory and our theory. While the subdifferential of a proper convex scalar function $\varphi : X \to \mathrm{I\!R} \cup \{+\infty\}$ at a point $x_0 \in \operatorname{dom} \varphi$ can be equally described via the directional derivative of φ at x_0 or the conjugate $\varphi^* : X^* \to \mathrm{I\!R} \cup \{+\infty\}$, these two definitions yield to different concepts of a subdifferential in our case. While at first glance this seems to be a shortcoming of our approach, the reason for this difference has to be sought in the scalar theory, as we neither assume $x_0 \in \operatorname{dom} F$, nor do we assume properness for the set-valued function. In the non-pathological cases, we will prove that both concepts of the subdifferential coincide in our theory as well, while the equality does not hold even in the scalar case when extending the definition of the subdifferential to non-proper functions $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$.

The seventh chapter is concerned with the tangent and normal cones of certain extensions of a convex set-valued function. In accordance to the scalar results, a connection between the tangent cone and the directional derivative on the one hand and the normal cone and the subdifferential on the other hand is proven. Note that we do not, in general, use the tangent or normal cone of the epigraph of F at some element $(x, z) \in X \times Z$, but rather stick to the family of cones associated with the family of scalarizations introduced earlier.

The Appendix consists of two parts, the first one dedicated to illustrating the presented theory on some standard examples, the second summarizing some known facts of scalar convex analysis as well as extending the classic definitions therein to the case of non-proper functions.

2 Basic Framework

The power set of a real linear space Z has been shown to be a conlinear space as introduced in [22]. Also, if Z is quasi ordered by a non trivial convex cone $C \subseteq Z$, then this order relation can be extended to an order relation on $\mathcal{P}(Z)$. A more detailed discussion of this ordering can be found in [35] and also in [22, 23]. As pointed out in the references, there are in fact two canonical extensions to an order relation on the power set, one being appropriate when dealing with concavity, the other when dealing with convexity. As we are only interested in convex functions in the following, we will refrain from introducing both extensions.

After introducing the basic notions, we will introduce a difference operation on quasi-ordered conlinear spaces in subsection 2.2.2, which is, to our knowledge new. This difference operation will prove to be of great importance in the following chapters, as through it we will be able to define concepts such as conjugate functions, the directional derivative or the subdifferential of a function $F : X \to \mathcal{P}(Z)$ (in a point $x_0 \in X$) in total accordance to the well-known scalar definitions. In fact, in the special case of the conlinear space $\mathrm{I\!R} \cup \{\pm\infty\}$, the new difference is an extension of the classic $''-''$ as will be illustrated in subsection 8.2. In sequence, we will introduce the basic notations for functions mapping one conlinear space into another and then turn to the special case of set-valued functions, which will be the focus of the rest of this thesis.

In subsection 2.3, we will introduce certain subsets of the power set of a locally convex space as the image spaces of convex and closed set-valued functions and identify these spaces as ordered conlinear spaces and then turn our attention to certain, "almost linear" set-valued functions. The set of these conlinear functions will serve as a replacement for the topological dual space later on.

Finally in subsection 2.4, we will specify the difference operation on conlinear spaces to the special case of the power set of a locally convex separable and quasi-ordered space.

2.1 Order structures

2.1.1 Definition. *[16] Let Y be a nonvoid set and $\leq$ a binary relation in Y. Then $\leq$ is called*

a) reflexive*, if for all $y \in Y$ it holds $y \leq y$,*

b) transitive*, if $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in Y$,*

c) antisymmetric*, if $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in Y$.*

If $\leq$ is reflexive and transitive, then $\leq$ is called a quasi-order *and the couple $(Y, \leq)$ is called a* quasi-ordered set*. If $\leq$ is reflexive, transitive and antisymmetric, then $\leq$ is called a* partial order *and the couple $(Y, \leq)$ is called a* partially ordered set*.*

If $(Y, \leq)$ is quasi-ordered (partially ordered), then for any subset M of Y the couple $(M, \leq)$ is quasi-ordered (partially ordered). For $x, y \in Y$ with $x \leq y$ we also write $y \geq x$.

Having a quasi-ordered set $(Y, \leq)$, it is a standard procedure to define equivalence classes, denoted by

$$[y] := \{x \in Y |\; x \leq y,\, y \leq x\}\,.$$

The set of all equivalence classes together with the relation

$$[y_1] \leq [y_2] \quad \Leftrightarrow y_1 \leq y_2$$

is a partially ordered set.

A subset $M \subseteq Y$ of Y is called *bounded from above (below)*, if in Y exists an element y such, that for all $m \in M$ it holds $m \leq y$ $(y \leq m)$. In this case, y is called an upper (lower) bound of M. If for an upper (lower) bound y of $M \subseteq Y$ holds $y \leq \tilde{y}$ $(\tilde{y} \leq y)$ for all upper (lower) bounds of M, then y is called *infimum (supremum)* of M, denoted by $\inf M$ $(\sup M)$. The infimum and supremum of Y are, if they exist, unique. A quasi-ordered set $(Y, \leq)$ is called *order complete*, if every nonvoid subset $M \subseteq Y$ has an infimum and supremum in Y. If $(Y, \leq)$ is order complete and ordered, that the infimum and supremum of each subset of Y exists and is unique. By definition, we set $\inf \emptyset = \sup Y$ and $\sup \emptyset = \inf Y$.

Let $(Y, \leq)$ be an order complete set, $\{y_\lambda\}_{\lambda \in \Lambda} \subseteq Y$ a net in Y. We define

$$\liminf_{\lambda \to 0} y_\lambda = \sup_{\rho \in \Lambda} \inf_{\lambda \leq \rho} y_\lambda$$

and

$$\limsup_{\lambda \to 0} A_\lambda = \inf_{\rho \in \Lambda} \sup_{\lambda \leq \rho} y_\lambda.$$

Let $(Y, \leq)$ be a quasi-ordered set, $\mathcal{P}(Y)$ the set of all subsets of Y, including the empty set and Y. We extend the order relation $\leq$ to a quasi-order $\preccurlyeq$ on $\mathcal{P}(Y)$ by

$$M \preccurlyeq N \quad :\Leftrightarrow \quad \forall n \in N : \exists m \in M : \; m \leq n.$$

If $M = \{m\}$ and $N = \{n\}$, then $M \preccurlyeq N$ holds if and only if $m \leq n$. If $N \subseteq M$, then it holds $M \preccurlyeq N$. Thus, $\emptyset$ is the largest element of $\mathcal{P}(Z)$ with respect to $\preccurlyeq$ and Y is the smallest element of $\mathcal{P}(Z)$ with respect to $\preccurlyeq$.

2.1.2 Example. *Let Y be a nonvoid set endowed with the order relation $=$. The extension $\preccurlyeq$ of $=$ equals $\supseteq$ and $(\mathcal{P}(Y), \supseteq)$is a partially ordered, order complete set. The infimum and supremum of a subset $\mathcal{A} \subseteq \mathcal{P}(Y)$ with respect to $\supseteq$are*

$$\inf{}_{\supseteq} \mathcal{A} = \bigcup_{A \in \mathcal{A}} A,$$
$$\sup{}_{\supseteq} \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

For any $A \in \mathcal{P}(Y)$ it holds $\emptyset \subseteq A \subseteq Y$, therefore the smallest element of $\mathcal{P}(Y)$ with respect to $\supseteq$ is Y, the greatest $\emptyset$.

The set of all equivalence classes in $\mathcal{P}(Y)$ with respect to $\preccurlyeq$ together with the relation

$$[M] \preccurlyeq [N] \quad \Leftrightarrow \quad M \preccurlyeq N$$

is partially ordered and order complete. This set can be identified with a subset of $(\mathcal{P}(Y), \supseteq)$.

2.2 Ordered conlinear spaces

2.2.1 Basic notations

2.2.1 Definition (Conlinear Space)**.** *[22] A nonvoid set Y together with an addition "$+$" and a multiplication with nonnegative reals "$\cdot$" is said to be a* real conlinear space $(Y, +, \cdot)$, *if* *a)* $(Y,+)$ *is a commutative monoid, that is*

$$\begin{aligned} \forall x,y,z\in Y: \quad x+(y+z) &= (x+y)+z, \\ \forall x,y\in Y: \quad x+y &= y+x, \\ \exists 0\in Y: \forall x\in Y: \quad 0+x &= x, \end{aligned}$$

b) $\cdot(t,y)=ty$ *for every* $t\geq 0$ *and* $y\in Y$ *satisfies the following conditions.*

$$\begin{aligned} \forall s,t\geq 0, y\in Y: \quad s(ty) &= (st)y, \\ \forall y\in Y: \quad 1y &= y, \\ \forall y\in Y: \quad 0y &= 0, \\ \forall t\geq 0, y_1,y_2\in Y: \quad t(y_1+y_2) &= ty_1+ty_2. \end{aligned}$$

If $X\subseteq Y$ *is closed under addition and multiplication positive reals, then* $(X,+,\cdot)$ *is called a conlinear subspace of* $(Y,+,\cdot)$.

Note that no multiplication with negative real numbers is defined. Also, the second distributive law $(s+t)y=sy+ty$ does not have to be valid even for $s,t\geq 0$. As a consequence, the conlinear structure is stable under passing to the power set of Y, $\mathcal{P}(Y)$ when the addition and multiplication are defined adequately. Here, as throughout the text, $\mathcal{P}(Z)$ denotes the set of all subsets of Y, including the empty set $\emptyset$ and Y itself.

2.2.2 Definition. *Let* $(Y,+,\cdot)$ *be a conlinear space. The Minkowski sum of two subsets* $A,B\in\mathcal{P}(Z)$ *and the product of a subset* $A\in\mathcal{P}(Z)$ *with a real number are defined as follows.*

$$\begin{aligned} \forall A,B\in\mathcal{P}(Y): \quad A+B &:= \{a+b \mid a\in A,\, b\in B\} \\ \forall t\in\mathbb{R}\setminus\{0\}, A\in\mathcal{P}(Y): \quad tA &:= \{ta \mid a\in A\} \\ \forall A\in\mathcal{P}(Y): \quad 0A &:= \{0\} \end{aligned}$$

For any set $A\in\mathcal{P}(Z)$ it holds $A+\emptyset=\emptyset+A=\emptyset$ and $0\emptyset=\{0\}$. We will abbreviate $A+\{z\}$ by $A+z$, $A+\{-z\}$ by $A-z$ and $A-B$ for $A+(-1)B$ for $z\in Z$ and $A,B\in\mathcal{P}(Z)$.

2.2.3 Proposition. *[22] Let* $(Y,+,\cdot)$ *be a conlinear space, then* $(\mathcal{P}(Y),+,\cdot)$ *supplied with the multiplication with nonnegative real numbers defined in 2.2.2 is a conlinear space.*

2.2.4 Definition (ordered conlinear spaces)**.** *[22] Let* $(Y,+,\cdot)$ *be a conlinear space and* $\leq$ *a quasi-order on* Y *satisfying*

a) For all $x,y,z\in Y$, $x\leq y$ *implies* $x+z\leq y+z$.

b) For $x,y\in Y$ *and* $0\leq t$, $x\leq y$ *implies* $tx\leq ty$.

Then $(Y,+,\cdot,\leq)$ *is called a* quasi-ordered conlinear space*. If* $\leq$ *is a partial order, then* $(Y,+,\cdot,\leq)$ *is called an* ordered conlinear space*.*

If $(Y,+,\cdot,\leq)$ is an ordered conlinear space, then $(\mathcal{P}(Y),+,\cdot,\supseteq)$ and $(\mathcal{P}(Y),+,\cdot,\subseteq)$ with the Minkowski sum and the multiplication defined in 2.2.2 are order complete, ordered conlinear spaces.

2.2.5 Proposition. *Let $(Y, +, \cdot, \leq)$ be an order complete quasi-ordered conlinear space, $M, N \subseteq Y$ subsets of Y and $t \geq 0$.*

a) It holds $t \inf M = \inf tM$ and $t \sup M = \sup tM$.

b) If $M \subseteq N$ then $\inf M \geq \inf N$ and $\sup M \leq \sup N$.

c) It holds $\inf(M + N) \geq \inf M + \inf N$ and $\sup(M + N) \leq \sup M + \sup N$

PROOF.

a) First, let $t = 0$, then $tM = \{0\}$ and $0 = t \inf M = \inf \{0\}$. If $t > 0$ and $M \neq \emptyset$, then

$$\forall x \in M : \quad t \in M \leq tx,$$

so $t \inf M \leq \inf tM$ and equally $\inf M \leq \frac{1}{t} \inf tM \leq \inf M$. As $(Y, +, \cdot, \leq)$ is quasi ordered and order complete, this proves the first statement. If $M = \emptyset$ and $t > 0$, then $t \inf M = \inf tM = \sup Y$. By the same arguments, $t \sup M = \sup tM$ holds.

b) Let $M \neq \emptyset$. As $\inf N \leq n$ holds for all $n \in N$, $\inf N \leq m$ holds for all $m \in M \subseteq N$ and thus $\inf N \leq \inf M$. If $M = \emptyset$, then $\inf M = \sup Y$ and the inequality is immediate. The same argumentation proves $\sup M \leq \sup N$.

c) If $M = \emptyset$, then $M + N = \emptyset$ and $\inf M + \inf N = \inf(M + N) = \sup Y$. Now let $M, N \neq \emptyset$, then it holds

$$\forall m \in M, \, n \in N : \quad \inf M + \inf N \leq m + n,$$

hence $\inf M + \inf N \leq \inf(M + N)$. Likewise the inequality $\sup(M + N) \leq \sup M + \sup N$ can be shown. ■

As pointed out in [38], the last two inequalities are in general no equalities.

A subset $A \subseteq Y$ is called *convex*, if

$$\forall t \in (0, 1) : \quad (tA + (1 - t)A) \subseteq A.$$

The *convex hull* of $A \subseteq Y$ is a convex set defined by

$$\operatorname{co} A := \bigcap_{\substack{A \subseteq B, \\ B \subseteq Y \text{is convex}}} B$$

and again is a convex subset of Y.

A subset $A \subseteq Y$ is called a *cone* iff for all $t > 0$ it holds $tA = A$. The *conical hull* of $A \subseteq Y$ is a cone defined by

$$\operatorname{cone} A := \{ta | \; t > 0, a \in A\}.$$

A cone $A \subseteq Y$ is a convex subset of Y if and only if $A + A \subseteq A$.

2.2.2 The inf–difference in order complete conlinear spaces

The classic difference operation $''-''$ on $\mathbb{R}$ has two basic interpretations. Firstly, and most commonly used is the algebraic character, that is $''-''$ serves as an inverse operator to $+$. On the other hand, an order-theoretic interpretation can be given, as

$$a - b = \inf \{c \in \mathbb{R} |\ b + c \geq a\}$$

for all $a, b \in \mathbb{R}$. Especially, $-a = 0 - a$ is the inverse element of $a \in \mathbb{R}$. One characteristic of conlinear spaces Y is that in general no inverse element of $a \in Y$ exists. Thus it is not possible to define $a - b$ by $a + (-b)$. However, the order-theoretic view provides a possibility to define an order-difference $\vartriangleleft$ on an quasi-ordered conlinear space Y. Obviously, the operator depends on the specific order relation in use. To avoid confusion, we will use the sign $0 \vartriangleleft a$, when referring to the order theoretic difference of 0 and $a \in Y$, while $-a$ denotes the inverse element of $a \in Y$ or $-1a$, if this element exists.

To our knowledge, the generalization to an operator on quasi-ordered conlinear spaces has not been done before. In subsection 2.4 and subsection 8.2, we will go into more details about the properties of the difference when the space under consideration is the power set of a quasi-ordered locally convex separable space or the set of the extended real numbers.

Notice that in fact the order-theoretic interpretation could also be stated as

$$a - b = \sup \{c \in Y |\ b + c \leq a\}$$

for all $a, b \in Y$. This interpretation proves appropriate when dealing with concavity rather than convexity, as in that case also the addition even on the extended real numbers is appropriately defined as the sup-addition rather then the inf-addition, compare [49] or [23].

2.2.6 Definition (inf-difference in conlinear spaces)**.** *In an order complete, quasi-ordered conlinear space* $(Y, +, \cdot, \leq)$ *we define the operation* $\vartriangleleft$ *by*

$$\forall x, y \in Y : \quad x \vartriangleleft y := \inf \{z \in Y |\ y + z \geq x\}$$

2.2.7 Lemma. *Let* $(Y, +, \cdot, \leq)$ *be an order complete, quasi-ordered conlinear space with largest element* $+\infty$ *and smallest element* $-\infty$*. Let* $0 \leq t \in \mathbb{R}$ *and* $a, b, x, y \in Y$*. If* $(+\infty) + (-\infty) = +\infty$*, then it holds*

a) $a \vartriangleleft (+\infty) = -\infty$, *and*

$(-\infty) \vartriangleleft a = -\infty$,

b) $t(a \vartriangleleft b) = ta \vartriangleleft tb$,

c) if $a \leq b$*, then* $a \vartriangleleft x \leq b \vartriangleleft x$ *and* $x \vartriangleleft b \leq x \vartriangleleft a$,

d) $a \vartriangleleft b \leq (a \vartriangleleft x) + (x \vartriangleleft b)$

e) $(a + x) \vartriangleleft (x + b) \leq a \vartriangleleft b$

f) $(a + x) \vartriangleleft (b + y) \leq (a \vartriangleleft b) + (x \vartriangleleft y)$.

Proof.

a) By definition, $a \lhd b = \inf\{s \in Y \mid b + s \geq a\}$. Replacing a by $-\infty$ or b by $+\infty$, the result is immediate.

b) If $a = -\infty$ or $b = +\infty$, then $t(a \lhd b) = t(-\infty)$, equality holds. Let $a \neq -\infty$ and $b \neq +\infty$. From 2.2.5 it holds

$$\begin{aligned} \forall t > 0: \quad t(a \lhd b) &= \inf\{tx \mid b + x \geq a\} \\ &= \inf\{x \mid tb + x \geq ta\} \\ &= (ta \lhd tb). \end{aligned}$$

If $t = 0$, then $0(a \lhd b) = 0$ and $(0a \lhd 0b) = 0$.

c) Let $a \leq b$, then $x + y \geq b$ implies $x + y \geq a$ and $a + y \geq x$ implies $b + y \geq x$. Hence,

$$a \lhd x \leq b \lhd x$$

and

$$x \lhd b \leq x \lhd a$$

holds true by 2.2.5.

d) Let $a \leq x + s$, $x \leq b + t$, then $x + s \leq b + t + s$ holds by 2.1.1 and $a \leq b + t + s$ by 2.2.4. By 2.2.5,

$$a \lhd b \leq (a \lhd x) + (x \lhd b)$$

is proven.

e) Let $b + s \geq a$, then $b + s + x \geq a + x$. Again, the proof goes by 2.2.5.

f) Let $b + s \geq a$, $y + t \geq x$ then $b + y + s + t \geq a + x$. Again, the inequality holds by 2.2.5. ■

2.2.3 Functions mapping into order complete conlinear spaces

Let $(X, +, \cdot)$ be a conlinear space and $(Y, +, \cdot, \leq)$ be an order complete, quasi-ordered conlinear space with largest element $+\infty$ and smallest element $-\infty$. The set $\mathcal{F} := \{f : X \to Y\}$ supplied with the point-wise addition and multiplication

$$\begin{aligned} \forall f_1, f_2 \in \mathcal{F},\, x \in X: \quad (f_1 + f_2)(x) &= f_1(x) + f_2(x), \\ \forall t > 0, f \in \mathcal{F},\, x \in X: \quad (tf)(x) &= t(f(x)), \\ \forall f \in \mathcal{F},\, x \in X: \quad (0f)(x) &= 0 \end{aligned}$$

is a conlinear space with neutral element $f \equiv 0$. Supplied with the point-wise order

$$f_1 \leq f_2 \quad \Leftrightarrow \forall x \in X: \; f_1(x) \leq f_2(x),$$

The quadruple $(\mathcal{F}, +, \cdot, \leq)$ is an order complete, quasi-ordered conlinear space with largest element $f \equiv +\infty$ and smallest element $f \equiv -\infty$. If $\leq$ is a partial order on Y, then $(\mathcal{F}, +, \cdot, \leq)$ is an ordered conlinear space. The inf-difference on $(\mathcal{F}, +, \cdot, \leq)$ is the pointwise inf-difference,

$$\forall x \in X: \quad (f_1 \lhd f_2)(x) = f_1(x) \lhd f_2(x).$$

Let $(Z, +, \cdot)$ be another conlinear space, $g : Z \to Y$ a function from Z into Y and $A : X \to Z$ a function satisfying

$$\begin{aligned} \forall x_1, x_2 \in X: \quad A(x_1 + x_2) &= A(x_1) + A(x_2), \\ \forall x \in X, t > 0: \quad A(tx) &= tA(x), \\ A(0) &= 0. \end{aligned}$$

Then we define

$$\begin{aligned} \forall x \in X: \quad gA(x) &:= g(Ax), \\ \forall z \in Z: \quad (Af)(z) &= \inf_{Ax=z} f(x). \end{aligned}$$

The *infimal convolution* of $f_1, f_2 : X \to Y$ is defined as $(f_1 \Box f_2) : X \to Y$ with

$$(f_1 \Box f_2)(x) := \inf_{x_1 + x_2 = x} \left(f_1(x_1) + f_2(x_2) \right).$$

2.2.8 Definition. *The* domain *of a function $f : X \to Y$ is defined by*

$$\operatorname{dom} f := \{x \in X |\ f(x) \neq +\infty\}.$$

A function $f : X \to Y$ is proper*, if $\operatorname{dom} f \neq \emptyset$ and $f(x) \neq -\infty$ for all $x \in X$.*

2.2.9 Definition. *A function $f : X \to Y$ is*

a) convex*, if*

$$\forall t \in (0,1), x_1, x_2 \in X: \quad f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2),$$

b) positively homogeneous*, if*

$$\forall t > 0, x \in X: \quad f(tx) = tf(x),$$

c) subadditive*, if*

$$\forall x_1, x_2 \in X: \quad f(x_1 + x_2) \leq f(x_1) + f(x_2)$$

and additive*, if*

$$\forall x_1, x_2 \in X: \quad f(x_1 + x_2) = f(x_1) + f(x_2),$$

d) sublinear*, if f is positively homogeneous and subadditive.*

2.2.10 Example. *The* indicator function $I_M : X \to Y$ *of $M \subseteq X$ is defined as*

$$I_M(x) := \begin{cases} 0, & \text{if } x \in M; \\ +\infty, & \text{else.} \end{cases}$$

The function $I_M : X \to Y$ is convex if and only if $M \subseteq X$ is a convex subset of X, positively homogeneous, if and only if $M \subseteq X$ is a cone, proper if and only if $\emptyset \neq M \neq X$ and $0 \neq -\infty$ holds for $0, -\infty \in Y$.

2.3 Set-valued functions

In this chapter, we introduce a class of functions mapping a linear space X, into the power set of a linear space Z, quasi-ordered by a convex cone $C \subsetneq Z$ with $\{0\} \subsetneq C$. When introducing topological properties of the functions, we assume X and Z to be supplied with a locally convex separated topology. The topological duals of X and Z will be denoted by X^* and Z^*, respectively. We will define the epigraph of a set-valued function as a subset of $X \times Z$. The power set $\mathcal{P}(Z)$ endowed with the Minkowski sum, a multiplication with positive real numbers and the order relation $\supseteq$ is an order complete ordered conlinear space with greatest element $\emptyset$ and smallest element Z. The convexity notion we will choose coincides with the so called cone-convexity or C-convexity, compare [8, 14, 37] and others. By definition, we call a function $F : X \to \mathcal{P}(Z)$ closed, iff its epigraph is closed. The images of convex and closed functions have certain characteristic properties, through which we identify conlinear subspaces of $(\mathcal{P}(Z), +, \cdot, \supseteq)$ as the adequate image spaces for our further investigations.

In subsection 2.3.2, we will introduce a set of functions with "almost linear" structure, the conlinear functions mapping X into $\mathcal{P}(Z)$, and in sequence the conaffine functions, which are our counterpart to the scalar affine functions. The set of the conlinear functions will serve as a dual space later on. In fact, for fixed $z^* \in C^* \setminus \{0\}$, the set $\left\{S_{(x^*,z^*)} | \, x^* \in X^*\right\}$ can be identified with X^*. Thus, instead of speaking of one dual space, we could as well speak of a family of dual spaces, each equivalent to X^*. This structure, namely a "scalar" family representing one set-valued singleton without loss of information will appear throughout this thesis in various forms.

At this point, we would like to mention that one major difference of our approach to most of the known approaches to set-valued convex analysis lies in the fact, that we do not use the set $\mathcal{L}(X, Z)$ as dual space, compare for example [6, 31, 37, 56]. The functions $T \in \mathcal{L}(X, Z)$ will be represented by a family of conlinear functions, namely $\left\{S_{(-T^*z^*,z^*)} : X \to \mathcal{P}(Z) | \; z^* \in C^* \setminus \{0\}\right\}$. Thus, any result about the classic dual space $\mathcal{L}(X, Z)$ can be deduced from our theory as well. Related approaches to ours have been presented in [3, 4, 23] and [39].

2.3.1 Set-valued proper, convex and closed functions

From now on, we will assume X and Y to be real linear spaces containing at least two elements and $C \subseteq Z$ to be a convex cone with $0 \in C$ and $C \neq Z$. The cone C generates a quasi-order $\leq_C$ on Z by means of $z_1 \leq_C z_2$ if and only if $z_2 \in z_1 + C$ for $z_1, z_2 \in Z$. Again, $(\mathcal{P}(Z), +, \cdot, \supseteq)$is an order complete ordered conlinear space. The extension of $\leq_C$ to an order relation on $\mathcal{P}(Z)$ is defined by

$$A \preccurlyeq_C B \;\Leftrightarrow\; B \subseteq A + C,$$

For a more detailed discussion of this order relation, see [22, 23, 34]. The order relation $\preccurlyeq_{\{0\}}$ is equal to the relation $\supseteq$. For $\preccurlyeq_Z$ it holds

$$\forall A \neq \emptyset, \forall B \in \mathcal{P}(Z) : A \preccurlyeq_Z B$$
$$\{B \in \mathcal{P}(Z) | \; \emptyset \preccurlyeq_Z B\} = \{\emptyset\}.$$

Two elements $A, B \in \mathcal{P}(Z)$ are equivalent with respect to $\preccurlyeq_C$ if $A \subseteq B + C$ and $B \subseteq A + C$, therefore we can identify the subset

$$\mathcal{P}_C(Z) := \{A \in \mathcal{P}(Z) | \; A = A + C\}$$

of $\mathcal{P}(Z)$ with the set of equivalence classes with respect to $\preccurlyeq_C$. It holds $A \preccurlyeq_C B$ for $A, B \in \mathcal{P}_C(Z)$ if and only if $A \supseteq B$. Modifying the multiplication with 0 by $0A = C$ for $A \in \mathcal{P}_C(Z)$, $(\mathcal{P}_C(Z), +, \cdot, \supseteq)$ with the modified scalar multiplication is an order complete, ordered conlinear subspace of $(\mathcal{P}(Z), +, \cdot, \supseteq)$.

2.3.1 Definition. *A set-valued function $F : X \to \mathcal{P}_C(Z)$ is* proper *if and only if*

$$\begin{aligned} \operatorname{dom} F = \quad & \{x \in X |\ F(x) \neq \emptyset\} \neq \emptyset, \\ \forall x \in X : \quad & F(x) \neq Z. \end{aligned}$$

*If additionally $(F(x) - C) \setminus F(x) \neq \emptyset$ holds for all $x \in \operatorname{dom} F$, then F is called C-*proper.

The definition of proper set-valued functions is a special case of 2.2.8, while the definition of C-proper functions cannot, in general be derived from the former definition. If C is generating, then a function $F : X \to \mathcal{P}_C(Z)$ is proper if and only if it is C-proper.

The images of a convex function $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$ have certain properties which allows us to work with even more specialized subspaces. Recall that $F : X \to (\mathcal{P}(Z), +, \preccurlyeq_C)$ is convex if

$$\forall t \in (0,1),\ \forall x_1, x_2 \in X : \quad F(tx_1 + (1-t)x_2) \preccurlyeq_C tF(x_1) + (1-t)F(x_2)$$

or equivalently if the *epigraph* of F,

$$\operatorname{epi} F := \{(x, z) \in X \times Z |\ F(x) \preccurlyeq_C \{z\}\}$$

is convex.

If a function $F : X \to (\mathcal{P}(Z), +, \preccurlyeq_C)$ is convex, then $F(x) + C \in \mathcal{P}_C(Z)$ is convex for all $x \in X$, not necessarily $F(x)$ itself. Therefore, we identify $F : X \to (\mathcal{P}(Z), +, \preccurlyeq_C)$ with $\tilde{F} : X \to \mathcal{P}_C(Z)$ defined by $\tilde{F}(x) = F(x) + C$. Thus, a convex set-valued function maps into the set of with respect to C lower convex subsets of Z,

$$Q_C(Z) := \{A \subseteq Z |\ A = \operatorname{co}(A + C)\}.$$

Note that $(Q_C(Z), +, \cdot, \supseteq)$ is an order complete, ordered conlinear subspaces of $(\mathcal{P}_C(Z), +, \cdot \supseteq)$. Moreover, the second distributive law

$$\forall A \in Q_C(Z),\ \forall t_1, t_2 \geq 0 : \quad t_1 A + t_2 A = (t_1 + t_2) A$$

holds and $(Q_C(Z), +, \cdot \preccurlyeq_C)$ is an ordered cone in the sense of [32].

The *convex hull of a function* $F : X \to (\mathcal{P}(Z), +, \preccurlyeq_C)$ is (uniquely) defined by

$$z \in (\operatorname{co} F)(x) \ :\Leftrightarrow (x, z) \in \operatorname{co} \operatorname{epi} F$$

It holds $\operatorname{epi} \operatorname{co} F = \operatorname{co} \operatorname{epi} F$ and it is obvious that $\operatorname{co} F$ is a convex function mapping X into the space $(Q_C(Z), +, \supseteq)$.

From now on, X and Z will be assumed to be supplied with a separated locally convex topology. The topological dual spaces of X and Z will be denoted by X^* and Z^*. There exists a neighborhood base of $0 \in X$, $\mathcal{U}_X$, consisting of convex balanced absorbing open sets only. In the following, let $\mathcal{U}_X$, $\mathcal{U}_Z$ always denote such a neighborhood base of $0 \in X$ and $0 \in Z$, respectively. The value of a linear continuous function $x^* \in X^*$ at $x \in X$ will be denoted by $x^*(x)$. Analogously, $z^*(z)$ will denote the value of $z^* \in Z^*$ at $z \in Z$. We set

$$H(z^*) := \{z \in Z |\ z^*(z) \le 0\}$$

for any $z^* \in Z^*$. If $z^* \ne 0$, the set $H(z^*)$ is a closed half space.

The order cone $C \subseteq Z$ is considered to be closed, the (negative) *dual cone* of C is denoted by

$$C^* := \{z^* \in Z^* |\ \forall c \in C:\ z^*(c) \le 0\}.$$

Obviously, $0 \in C^*$ and the set $C^* \setminus \{0\}$ is not empty, as $C \subsetneq Z$ holds. Moreover, as C is closed and convex,

$$C = \{z \in Z |\ \forall z^* \in C^*:\ z^*(z) \le 0\} = \bigcap_{z^* \in C^* \setminus \{0\}} \{z \in Z |\ z^*(z) \le 0\}$$

holds true. If $z^* \in C^* \setminus \{0\}$, then $H(z^*) \supseteq C$ holds true.

2.3.2 Definition. *A function $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$ is said to be* closed *iff* $\operatorname{epi} F$ *is closed with respect to the product topology on $X \times Z$.*

2.3.3 Lemma. *If $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$ is closed, then $F(x) + C$ is closed, possibly empty, for each $x \in X$.*

Proof. It holds $z \in \operatorname{cl}(F(x)+C)$ if and only if for any $V \in \mathcal{U}_Z$ it holds $(\{z\}+V) \cap (F(x)+C) \ne \emptyset$. This implies that for any $U \times V \in \mathcal{U}_X \times \mathcal{U}_Z$ it holds $(\{(x,z)\} + U \times V) \cap \operatorname{cl}\operatorname{epi} F \ne \emptyset$, hence $(x,z) \in \operatorname{epi} F$. ■

Thus, a closed function $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$ maps into the set of all lower closed subsets of Z, defined as

$$\mathcal{P}^t_C(Z) := \{A \in \mathcal{P}(Z) |\ A = \operatorname{cl}(A + C)\}.$$

The Minkowski sum of two closed sets is not automatically closed. Redefining the addition for elements $A, B \in \mathcal{P}^t_C(Z)$ by

$$A \oplus B := \operatorname{cl}(A + B)$$

and the multiplication with $0 \in \mathbb{R}$ by $0A = C$, $(\mathcal{P}^t_C(Z), \oplus, \cdot, \supseteq)$ is again an order complete, ordered conlinear subspace of $(\mathcal{P}^{(}Z), +, \cdot, \supseteq)$.

The *closed hull* of a function $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$ is (uniquely) defined by

$$z \in (\operatorname{cl} F)(x) \ :\Leftrightarrow (x,z) \in \operatorname{cl}\operatorname{epi} F.$$

It holds $\operatorname{epi}(\operatorname{cl} F) = \operatorname{cl}\operatorname{epi} F$ and it is obvious that $\operatorname{cl} F$ is a closed function mapping X into the space $(\mathcal{P}^t_C(Z), +, \cdot, \supseteq)$.

2.3.4 Lemma. *Let $F_1, F_2 : X \to \mathcal{P}(Z)$, F_2 closed and*

$$\forall x \in X:\ F_2(x) \supseteq F_1(x),$$

then it holds

$$\forall x \in X:\ F_2(x) \supseteq (\operatorname{cl} F_1)(x).$$

PROOF. The epigraph of F_2 is a closed set with $\operatorname{epi} F_2 \supseteq \operatorname{cl} \operatorname{epi} F_1$, which proves the statement. ■

The values of a closed convex function $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$ satisfy $F(x) = \operatorname{cl} \operatorname{co}(F(x)+C)$, thus we obtain

$$\mathcal{Q}_C^t(Z) := \{A \in \mathcal{P}(Z) |\ A = \operatorname{cl} \operatorname{co}(A + C)\},$$

the set of all lower closed convex subsets of Z. Again, $(\mathcal{Q}_C^t(Z), \oplus\cdot, \supseteq)$ is an order complete, ordered conlinear subspace of $(\mathcal{P}(Z), +, \cdot, \supseteq)$.

The closed convex hull of a function $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$ is defined via

$$z \in (\operatorname{cl} \operatorname{co} F)(x) \ :\Leftrightarrow (x, z) \in \operatorname{cl} \operatorname{co} \operatorname{epi} F.$$

The function $\operatorname{cl} \operatorname{co} F$ is a closed convex function with $\operatorname{epi}(\operatorname{cl} \operatorname{co} F) = \operatorname{cl} \operatorname{co} \operatorname{epi} F$ mapping X to $(\mathcal{Q}_C^t(Z), \oplus \supseteq)$.

Notice that even for a single-valued function $F : X \to (\mathcal{P}(Z), +, \cdot, \preccurlyeq_C)$, the operations cl, co and $\operatorname{cl} \operatorname{co}$ produce mappings into $\mathcal{P}_C^t(Z)$, $Q_C(Z)$ and $Q_C^t(Z)$, respectively.

For a subset $\mathcal{A} \subseteq \mathcal{Q}_C^t(Z)$ of $\mathcal{Q}_C^t(Z)$, the infimum $\inf_C \mathcal{A} \in \mathcal{Q}_C^t(Z)$ (supremum $\sup_C \mathcal{A} \in \mathcal{Q}_C^t(Z)$) of $\mathcal{A}$ is the greatest (smallest) lower (upper) bound of $\mathcal{A}$, that is

$$\inf{}_C \mathcal{A} = \operatorname{cl} \operatorname{co} \bigcup_{A \in \mathcal{A}} A$$

and

$$\sup{}_C \mathcal{A} = \bigcap_{A \in \mathcal{A}} A.$$

The largest element of $\mathcal{Q}_C^t(Z)$ is $\emptyset$, the smallest Z.

The limes inferior of a function $F : X \to \mathcal{Q}_C^t(Z)$ in $x \in X$ is defined via

$$\liminf_{y \to 0} F(x + y) := \sup_{U \in \mathcal{U}_X} \inf_{y \in U \setminus \{0\}} F(x + y),$$

therefore,

$$\liminf_{y \to 0} F(x + y) := \bigcap_{U \in \mathcal{U}_X} \operatorname{cl} \operatorname{co} \bigcup_{y \in U \setminus \{0\}} F(x + y).$$

2.3.5 Corollary. *For any convex function $F : X \to \mathcal{Q}_C^t(Z)$ it holds*

$$(\operatorname{cl} F)(x) = \bigcap_{U \in \mathcal{U}_X} \operatorname{cl} \bigcup_{y \in U} F(x + y).$$

The function F is closed if and only if $\liminf_{y \to 0} F(x + y) \subseteq F(x)$ holds for all $x \in X$.

PROOF. By definition we have

$$\operatorname{cl} \operatorname{epi} F = \{(x, z) |\ \forall U \in \mathcal{U}_X, V \in U_Z : (\{(x, z)\} + U \times V) \cap \operatorname{epi} F \neq \emptyset\}.$$

This can be rewritten as

$$\operatorname{cl} \operatorname{epi} F = \left\{(x, z) |\ z \in \bigcap_{U \in \mathcal{U}_X} \operatorname{cl} \bigcup_{y \in U} F(x + y)\right\}$$

obtaining $(\operatorname{cl} F)(x) = \liminf_{y \to 0} F(x + y) \cup F(x)$.

On the other hand, if $\liminf_{y\to 0} F(x+y) \subseteq F(x)$, then $\operatorname{cl}(F(x)) = (\operatorname{cl} F)(x)$ by the above formula. As F maps into $\mathcal{Q}_C^t(Z)$, we have by definition $\operatorname{cl}(F(x)) = F(x)$ and so we obtained the desired result. ■

If F is not closed at $x_0 \in X$, i.e. $\liminf_{y\to 0} F(x_0+y) \supsetneq F(x)$, then $\liminf_{y\to 0} F(x_0+y) = (\operatorname{cl} F)(x_0)$. Otherwise, if $\liminf_{y\to 0} F(x_0+y) \subseteq F(x_0)$, then $F(x) = (\operatorname{cl} F)(x)$ holds true.

2.3.6 Remark. *For a set-valued function $F : X \to (Q_C(Z), +, \supseteq)$, the epigraph and the* graph *of F*

$$\operatorname{graph} F := \{(x,z) |\ z \in F(x)\}$$

are identical, as

$$F(x) \preccurlyeq_C \{z\} \ \Leftrightarrow z \in F(x) = F(x) + C.$$

If $F : X \to \mathcal{P}(Z)$ is a closed convex function, then F and the extension $F_C : X \to \mathcal{Q}_C^t(Z)$ of F, defined by

$$\forall x \in X : \quad F_C(x) := \operatorname{cl}(F(x) + C)$$

have the same epigraph. This does not hold for more general set-valued functions. However, as C is considered to be closed and convex, a vector-valued function $f : X \to Z \cup \{+\infty\}$ and its set-valued extension $F_C : X \to \mathcal{Q}_C^t(Z)$ defined by

$$\forall x \in X : \quad F_C(x) := f(x) + C,$$

with $+\infty + C = \emptyset$ have the same epigraph. Therefore, it is more convenient to use the notion $\operatorname{epi} F$ *in the following, rather than* $\operatorname{graph} F$.

2.3.2 Set-valued conlinear and conaffine functions

For $(x^*, z^*) \in X^* \times Z^* \setminus \{0\}$, we define $S_{(x^*,z^*)} : X \to \mathcal{P}(Z)$ by

$$\forall x \in X : \quad S_{(x^*,z^*)}(x) = \{z \in Z |\ -z^*(z) \geq x^*(x)\}.$$

In fact, $S_{(x^*,z^*)} : X \to (Q^t_{H(z^*)}(Z), +, \cdot, \preccurlyeq_{H(z^*)})$ holds. Moreover, $S_{(x^*,z^*)} : X \to \mathcal{Q}_C^t(Z)$ is true if and only if $z^* \in C^* \setminus \{0\}$. Functions of the type of $S_{(x^*,z^*)} : X \to \mathcal{P}(Z)$ with $z^* \in C^* \setminus \{0\}$ are called *conlinear*, which is motivated by the following properties.

2.3.7 Proposition. *Let $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$, then*

a) $S_{(x^*,z^*)} : X \to \mathcal{Q}_C^t(Z)$ and $\operatorname{dom} S_{(x^*,z^*)} = X$.

b) $S_{(x^*,z^*)} : X \to \mathcal{Q}_C^t(Z)$ is proper. If additionally $z^* \in C^* \setminus -C^*$, then $S_{(x^*,z^*)}$ is C-proper.

c) For each $x \in X$, $x^* \in X^*$ it holds $S_{(0,z^*)}(x) = S_{(x^*,z^*)}(0) = H(z^*)$.

d) $S_{(x^*,z^*)} : X \to \mathcal{Q}_C^t(Z)$ is closed, positively homogenous and additive. In particular, $S_{(x^*,z^*)}(x) + S_{(x^*,z^*)}(-x) = H(z^*)$.

e) If $z \in Z$ is chosen such, that $-z^*(z) = 1$, then

$$\forall x \in X : \quad S_{(x^*,z^*)}(x) = x^*(x)z + H(z^*).$$

Proof. Direct calculation. ■

2.3.8 Proposition. *[23] A conlinear function $S_{(x^*,z^*)}$ is proper if and only if $z^* \in C^* \setminus \{0\}$ holds, it is C-proper if and only if $z^* \in C^* \setminus -C^*$.*

For any $z^* \in Z^* \setminus \{0\}$, the set $\left\{S_{(x^*,z^*)} | \; x^* \in X^*\right\}$ supplied with the pointwise addition and multiplication with positive reals

$$\begin{aligned}(S_{(x_1^*,z^*)} + S_{(x_2^*,z^*)})(x) &= S_{(x_1^*+x_2,z^*)}(x),\\ (tS_{(x^*,z^*)})(x) &= S_{(tx^*,z^*)}(x)\end{aligned}$$

for all $x \in X$, $x_1^*, x_2^*, x^* \in X^*$, $t \in \mathbb{R}$ and the neutral element $S_{(0,z^*)} \equiv H(z^*)$ is isomorph to X^*, see [23].

For $t > 0$ it holds

$$\forall x \in X: \quad S_{(tx^*,z^*)}(x) = S_{(x^*,\frac{1}{t}z^*)}(x) = tS_{(x^*,z^*)}(x)$$

and

$$\forall x \in X: \quad S_{(-tx^*,z^*)}(x) = \{0\} \lhd_{z^*} S_{(x^*,\frac{1}{t}z^*)}(x) = \{0\} \lhd_{z^*} tS_{(x^*,z^*)}(x).$$

2.3.9 Proposition. *[23] Let $T \in \mathcal{L}(X, Z)$ be a linear continuous operator. Then*

$$\forall x \in X: \quad T(x) \in S_{(x^*,z^*)}(x) \;\Leftrightarrow\; x^* = -T^*(z^*).$$

It holds $S_{(-T^*z^*,z^*)}(x) = Tx + H(z^*)$ for all $x \in X$.

From 2.3.9, it can be derived, that a linear continuous operator $T \in \mathcal{L}(X, Z)$ can be represented without loss of information by the family

$$\left\{S_{(-T^*z^*,z^*)} | \; z^* \in Z^*\right\}.$$

As we are only dealing with functions mapping into $\mathcal{Q}_C^t(Z)$, it is without loss of generality when we represent $T \in \mathcal{L}(X, Z)$ by the set of all $S_{(-T^*z^*,z^*)}$ with $z^* \in C^* \setminus \{0\}$. It is easy to prove that for a function $F : X \to \mathcal{Q}_C^t(Z)$ it holds

$$\operatorname{epi} T = \{(x, z) \in X \times Z | \; T(x) \le z\} \supseteq \operatorname{epi} F$$

if and only if

$$\forall z^* \in C^* \setminus \{0\}, \forall x \in X: \quad S_{(-T^*z^*,z^*)}(x) \supseteq F(x).$$

Thus, $T(x) \le z$ holds for all $x \in X$ and $z \in F(x)$ if and only if $S_{(-T^*z^*,z^*)}$ is a conlinear minorant of F for all $z^* \in C^* \setminus \{0\}$.

A function $F : X \to \mathcal{Q}_C^t(Z)$ is called *conaffine*, if there is $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ and $z \in Z$ such, that

$$\forall x \in X: \quad F(x) = S_{(x^*,z^*)}(x) + z.$$

2.3.10 Proposition. *[23] Let $F : X \to \mathcal{Q}_C^t(Z)$ be a function. The following statements are equivalent:*

a) F is the pointwise supremum of its C-proper (proper, but not C-proper) conaffine minorants, *b) F is closed, convex and C-proper (proper, but not C-proper) or $F \equiv \emptyset$ or $F \equiv Z$.*

Especially, any conaffine function is proper. It is C-proper, if $z^* \in C^* \setminus -C^*$ and proper, but not C-proper, if $z^* \in C^* \cap C^*$. A function $F : X \to \mathcal{Q}^t_C(Z)$ has a conaffine minorant if and only if $\mathrm{cl}\,\mathrm{co}\, F : X \to \mathcal{Q}^t_C(Z)$ is proper.

2.3.11 Example. *The* set-valued support function $\Sigma_{M^*} : X \to \mathcal{Q}^t_C(Z)$ *of $M^* \subseteq X^* \times C^* \setminus \{0\}$ is defined as*

$$\forall x \in X : \; \Sigma_{M^*}(x) := \bigcap_{(x^*,z^*)\in M^*} S_{(x^*,z^*)}(x).$$

It holds $\Sigma_{M^*} \equiv Z$, if $M^* = \emptyset$. If $z^* \in C^* \cap -C^*$ holds true for all $(x^*, z^*) \in M^*$, then it holds $\Sigma_{M^*}(x) = \Sigma_{M^*}(x) - C$ for all $x \in X$, therefore $\Sigma_{M^*}(x)$ is not C-proper in this case.

2.4 The z^*-difference

In the following we will introduce a family of difference operators of the power set of a locally convex separable space Z, quasi-ordered by a nontrivial closed convex cone C. Each such difference operator $\lhd_{z^*}$ with $z^* \in C^* \setminus \{0\}$ happens to be the difference operator introduced in subsection 2.2.2 on the quasi-ordered conlinear space $(\mathcal{P}(Z), \oplus, \cdot, \preccurlyeq_{H(z^*)})$. The difference of two sets $A, B \in \mathcal{P}(Z)$ coincides with the difference of the z^*-hulls of A and B defined as $\mathrm{cl}\,(A + H(z^*))$ and $\mathrm{cl}\,(B + H(z^*))$, respectively.

If $\mathrm{cl}\,(A + H(z^*)) \neq \emptyset$ or Z, then $\mathrm{cl}\,(A + H(z^*)$ can be identified with a real number $t_A \in \mathrm{I\!R}$ via $\mathrm{cl}\,(A + H(z^*)) = z_A + H(z^*)$ with $z_A \in Z$ and $t_A = -z^*(z_A)$ and it holds $\{0\} \lhd_{z^*} A = \{z \in Z |\; -z^*(z) \geq -t_A\}$. On the other hand, $\{0\} \lhd_{z^*} \emptyset = \{z \in Z |\; -z^*(z) \geq -\infty\} = Z$ and $\{0\} \lhd_{z^*} Z = \{z \in Z |\; -z^*(z) \geq +\infty\} = \emptyset$. Identifying $\emptyset \in \mathcal{P}(Z)$ with $+\infty \in \mathrm{I\!R} \cup \{\pm\infty\}$ and $Z \in \mathcal{P}(Z)$ with $-\infty \in \mathrm{I\!R} \cup \{\pm\infty\}$, this indicates a close relationship between the z^*-difference on $\mathcal{P}(Z)$ and the order-difference $\lhd$ on $\mathrm{I\!R} \cup \{\pm\infty\}$, which will be discussed in detail in subsection 8.2. The mentioned connection will be of virtual importance for the theory of vector-valued convex functions as presented in this thesis.

In [20], a difference operation for subsets of the set $\mathrm{I\!R}^n$ has been discussed, called the Pontryagin-difference (P-difference) in reference to [46]. The same construction is used in [47, 55], while in [18, 19] a slightly different approach is used, which is introduced in [13] and generalized in [27]. The P-difference of two sets $A, B \in \mathcal{Q}^t_C(Z)$ coincides with the intersection of all z^*-differences of A and B and equals $A \lhd B$, using the definition of the inf-difference given in 2.2.6. The advantage of the z^*-difference is, that, contrary to the inf-difference, $A \lhd_{z^*} B \neq \emptyset$ holds for all $A, B \in \mathcal{Q}^t_C(Z) \setminus \{\emptyset, Z\}$ and $\{0\} \lhd_{z^*} A$ is the inverse of A with respect to the addition in $Q^t_{H(z^*)}(Z)$, whenever $A \neq \emptyset$, and $\mathrm{cl}\,(A + H(z^*)) \neq Z$ holds.

2.4.1 Definition. *Let $A, B \in \mathcal{P}(Z)$, then the difference $A \lhd_{z^*} B$ with respect to $z^* \in Z^* \setminus \{0\}$ is defined by*

$$A \lhd_{z^*} B := \{z \in Z |\; B + z \subseteq \mathrm{cl}\,(A + H(z^*))\}.$$

Notice that for $t > 0$ it holds $(A \lhd_{tz^*} B) = (A \lhd_{z^*} B)$.

2.4.2 Lemma. *For $A, B \in \mathcal{P}(Z)$ and $z^* \in Z^* \setminus \{0\}$ it holds*

$$A \lhd_{z^*} B = \left\{ z \in Z |\; -z^*(z) \geq \left[\inf_{a\in A} (-z^*(a)) \lhd \inf_{b\in B} (-z^*(b)) \right] \right\}$$

PROOF. Let $A, B \in \mathcal{P}(Z)$. The set $\mathrm{cl}\,(A + H(z^*))$ is convex consists of exactly those elements $z \in Z$ with $-z^*(z) \geq \inf\limits_{a\in A}(-z^*(a))$. Therefore it holds

$$\begin{aligned} A \lhd_{z^*} B &= \left\{ z \in Z |\, \forall b \in B:\ (-z^*(b) - z^*(z)) \geq \inf_{a\in A}(-z^*(a)) \right\} \\ &= \left\{ z \in Z |\, (\inf_{b\in B}(-z^*(b)) - z^*(z)) \geq \inf_{a\in A}(-z^*(a)) \right\}. \end{aligned}$$

By definition,

$$\left(\inf_{a\in A}(-z^*(a)) \lhd \inf_{b\in B}(-z^*(b)) \right) = \inf_{t\in\mathrm{I\!R}} \left(\inf_{b\in B}(-z^*(b)) + t \geq \inf_{a\in A}(-z^*(a)) \right)$$

holds, see 8.2.1. Therefore, the assertion is proven. ■

2.4.3 Remark. *From 2.4.2 it is immediate, that*

$$A \lhd_{z^*} B = (\mathrm{cl}\,(A + H(z^*)) \lhd_{z^*} \mathrm{cl}\,(B + H(z^*))) \in Q^t_{H(z^*)}(Z)$$

holds for all $A, B \subseteq Z$ and

$$\inf_{z\in(A\lhd_{z^*}B)}(-z^*(z)) = \inf_{a\in A}(-z^*(a)) \lhd \inf_{b\in B}(-z^*(b)).$$

If $z^* \in C^* \setminus \{0\}$, then $C \subseteq H(z^*)$ and $(A \lhd_{z^*} B) \in \mathcal{Q}^t_C(Z)$.

In fact $\lhd_{z^*}$ is the inf-substraction on $(Q^t_{H(z^*)}(Z), +, \cdot, \supseteq)$ as defined in 2.2.6. If $z^* \in C^* \setminus \{0\}$, then $(Q^t_{H(z^*)}(Z), +, \cdot, \supseteq)$ is a conlinear subspace of $(\mathcal{Q}^t_C(Z), +, \cdot, \preccurlyeq_C)$. The operation $\lhd_{z^*}$ is not, however the inf-substraction on $\mathcal{Q}^t_C(Z)$, as this would be given by

$$\begin{aligned} \forall A, B \in \mathcal{Q}^t_C(Z):\quad A \lhd B &= \mathrm{cl\,co} \bigcup_{M\in\mathcal{Q}^t_C(Z)} (B + M \subseteq A) \\ &= \{ z \in Z |\, B + z \subseteq A \}. \end{aligned}$$

It holds

$$\forall A, B \in \mathcal{Q}^t_C(Z),\, \forall z^* \in C^* \setminus \{0\}:\quad A \lhd B \subseteq A \lhd_{z^*} B.$$

2.4.4 Proposition. *For $z^* \in C^* \setminus \{0\}$, denote by $\preccurlyeq_{z^*}$ the order relation defined by $H(z^*)$. The relation $\preccurlyeq_{z^*}$ is a quasi-order on $\mathcal{Q}^t_C(Z)$. If $A \supseteq B$ holds for $A, B \in \mathcal{Q}^t_C(Z)$, then $A \preccurlyeq_{z^*} B$ holds true. For any $A \in \mathcal{Q}^t_C(Z)$ it holds*

$$\inf{}_{z^*} \{A\} = (A \lhd_{z^*} \{0\}) \in \mathcal{Q}^t_C(Z)$$

and for a non empty set $\mathcal{A} \subseteq \mathcal{Q}^t_C(Z)$ it holds

$$\inf{}_{z^*} \mathcal{A} = \mathrm{cl} \bigcup_{A\in\mathcal{A}} (A \lhd_{z^*} \{0\}) \in \mathcal{Q}^t_C(Z).$$

The quadruple $(\mathcal{Q}^t_C(Z), +, \cdot, \preccurlyeq_{z^*})$ is an order complete quasi-ordered conlinear space.

PROOF. For $A, B \in \mathcal{Q}^t_C(Z)$ it holds

$$A \preccurlyeq_{z^*} B \quad\Leftrightarrow\quad B \subseteq \mathrm{cl}\,(A + H(z^*)).$$

It is easily checked that $\preccurlyeq_{z^*}$ is reflexive and transitive. Obviously, $A \preccurlyeq_{z^*} B$ holds if $A \supseteq B$. The set $(A \vartriangleleft_{z^*} \{0\}) \in \mathcal{Q}^t_C(Z)$ is a lower bound of $\{A\}$. Moreover, if $B \preccurlyeq_{z^*} A$ holds for $B \in \mathcal{Q}^t_C(Z)$, then $B \supseteq (A \vartriangleleft_{z^*} \{0\})$ holds, as $B = \operatorname{cl} B$ and therefore $(A \vartriangleleft_{z^*} \{0\}) = \inf_{\preccurlyeq_{z^*}} \{A\}$. For a nonvoid set $\mathcal{A} \subseteq \mathcal{Q}^t_C(Z)$ it holds

$$\forall A \in \mathcal{A}: \quad \operatorname{cl} \bigcup_{A \in \mathcal{A}} (A \vartriangleleft_{z^*} \{0\}) \preccurlyeq_{z^*} A.$$

If for $B \in \mathcal{Q}^t_C(Z)$

$$\forall A \in \mathcal{A}: \quad B \supseteq A + H(z^*)$$

holds, then

$$B \supseteq \operatorname{cl} \bigcup_{A \in \mathcal{A}} (A \vartriangleleft_{z^*} \{0\})$$

holds, as $B = \operatorname{cl} B$ holds true. Therefore, $(\mathcal{Q}^t_C(Z), +, \cdot, \preccurlyeq_{z^*})$ is order complete and the assertion is proven. ■

2.4.5 Definition. *For a subset $A \subseteq Z$ and $z^* \in Z^*$, the z^*-hull of A is defined by* $\operatorname{cl}(A + H(z^*))$.

It holds $\operatorname{cl}(\emptyset + H(z^*)) = \emptyset$ for all $z^* \in Z^*$. If $A \neq \emptyset$, then $\operatorname{cl}(A + H(0)) = Z$ and for all $z^* \in Z^* \setminus \{0\}$ the set $\operatorname{cl}(A + H(z^*))$ is a closed half-space or equal to Z.

2.4.6 Definition. *Let $z^* \in Z^*$. A set $A \subseteq Z$ is called z^*-proper, if $\emptyset \neq \operatorname{cl}(A + H(z^*)) \neq Z$.*

2.4.7 Remark. *Let $A \subseteq Z$ and $z^* \in C^* \setminus \{0\}$, then $(A \vartriangleleft_{z^*} \{0\}) = \operatorname{cl}(A + H(z^*))$ holds and*

$$A \oplus (\{0\} \vartriangleleft_{z^*} A) = \begin{cases} H(z^*), & \text{if } A \text{ is } z^*\text{-proper;} \\ \emptyset, & \text{else.} \end{cases}$$

If A is z^*-proper, then $(A \vartriangleleft_{z^*} A) = H(z^*)$ holds. Recall that $H(z^*)$ is the neutral element in $Q^t_{H(z^*)}(Z)$. A set A is z^*-proper for at least one $z^* \in C^* \setminus \{0\}$ if and only if $\emptyset \neq \operatorname{cl}(A + C) \neq Z$.

Obviously no subset $A \subseteq Z$ is 0-proper as $H(0^*) = Z$ and either $A = \emptyset$ and $\operatorname{cl}(A + H(0)) = \emptyset$, or $\operatorname{cl}(A + H(0)) = Z$. Also, $A \subsetneq Z$ is by no means sufficient for A being z^*-proper for all $z^* \in Z^* \setminus \{0\}$.

2.4.8 Remark. *Denoting $(Y, +, \cdot, \leq) = (Q^t_{H(z^*)}(Z), +, \cdot, \supseteq)$, then with 2.4.3 and 2.2.7 the following properties hold for any given $A, B \in \mathcal{P}(Z)$ and $z^* \in Z^* \setminus \{0\}$.*

a) For $t > 0$ it holds $tA \vartriangleleft_{z^*} tB = t(A \vartriangleleft_{z^*} B)$.

b) If $A \subseteq B \oplus H(z^*)$, then $(A \vartriangleleft_{z^*} D) \subseteq (B \vartriangleleft_{z^*} D)$ and $(D \vartriangleleft_{z^*} B) \subseteq (D \vartriangleleft_{z^*} A)$.

c) It holds $(A \vartriangleleft_{z^*} D) \oplus (D \vartriangleleft_{z^*} B) \subseteq A \vartriangleleft_{z^*} B$.

d) It holds $(A \vartriangleleft_{z^*} B) \subseteq (A \oplus D) \vartriangleleft_{z^*} (D \oplus B)$.

e) It holds $(A \vartriangleleft_{z^*} B) \oplus (D \vartriangleleft_{z^*} E) \subseteq (A \oplus D) \vartriangleleft_{z^*} (B \oplus E)$.

It is easy to check that

$$A \lhd_{z^*} \emptyset = Z \lhd_{z^*} A = Z$$

holds for all $z^* \in Z^*$ and $A \in \mathcal{P}(Z)$. Moreover, if A is z^*-proper, then

$$A \lhd_{z^*} Z = \emptyset \lhd_{z^*} A = \emptyset.$$

2.4.9 Proposition. *Let $z^* \in Z^* \setminus \{0\}$ and A, B, D and $E \in \mathcal{P}(Z)$*

a) It holds

$$\begin{aligned} \forall s,t>0: \quad A \lhd_{z^*} (tB \oplus sB) &= A \lhd_{z^*} (s+t)B \\ (tA \oplus sA) \lhd_{z^*} B &= (t+s)A \lhd_{z^*} B. \end{aligned}$$

b) If A and $B \in \mathcal{P}(Z)$ are z^*-proper, then it holds

$$(A \lhd_{z^*} B) \oplus (D \lhd_{z^*} E) = (A \oplus D) \lhd_{z^*} (B \oplus E).$$

c) If D and either A or B is z^*-proper, then it holds

$$\begin{aligned} A \lhd_{z^*} B &= (A \lhd_{z^*} D) \oplus (D \lhd_{z^*} B) \\ &= (A \oplus D) \lhd_{z^*} (D \oplus B). \end{aligned}$$

Proof.

a) If A and $B \in \mathcal{P}(Z)$ are z^*-proper, then it holds

$$(A \lhd_{z^*} B) \oplus (D \lhd_{z^*} E) = (A \oplus D) \lhd_{z^*} (B \oplus E).$$

As pointed out in 2.4.3, the sets A and B can be identified with their z^*-hulls. Moreover, $tA \oplus sA = \operatorname{cl}(tA + sA) = (t+s)A$ holds true as by assumption A is z^*-proper and thus $\operatorname{cl}((tA \oplus sA) + H(z^*)) = \operatorname{cl}((t+s)A + H(z^*))$ holds true.

b) As A and B are assumed to be z^*-proper, it holds $\emptyset \neq (A \lhd_{z^*} B) \neq Z$. Without loss of generality, assume that A, B, D and $E \in Q^t_{H(z^*)}(Z)$ holds true. If $(D \lhd_{z^*} E) = \emptyset$, then it holds $D \neq E$ and either $D = \emptyset$ or $E = Z$. In this case,

$$(A \lhd_{z^*} B) \oplus (D \lhd_{z^*} E) = (A \oplus D) \lhd_{z^*} (B \oplus E) = \emptyset.$$

If $(D \lhd_{z^*} E) = Z$, then either $D = Z$ or $E = \emptyset$. In this case,

$$(A \lhd_{z^*} B) \oplus (D \lhd_{z^*} E) = (A \oplus D) \lhd_{z^*} (B \oplus E) = Z.$$

If $(D \lhd_{z^*} E)$ is a z^*-proper set, then both D and E are z^*-proper sets. We identify A with a real number $a \in \mathbb{R}$ by

$$A = \{z \in Z| \ -z^*(z) \geq a\}$$

and define $b, d, e \in \mathbb{R}$ likewise. It holds

$$\begin{aligned} (A \lhd_{z^*} B) \oplus (D \lhd_{z^*} E) &= \{z \in Z| \ -z^*(z) \geq (a-b) + (d-e)\} \\ &= \{z \in Z| \ -z^*(z) \geq (a+d) - (b+e)\} \\ &= (A \oplus D) \lhd_{z^*} (B \oplus E) = Z. \end{aligned}$$

c) If D and either A or B is z^*-proper, then by b) it holds

$$(A \lhd_{z^*} D) \oplus (D \lhd_{z^*} B) = (A \oplus D) \lhd_{z^*} (D \oplus B).$$

If $\operatorname{cl}(A + H(z^*)) = Z$ or $B = \emptyset$, then

$$A \lhd_{z^*} B = (A \lhd_{z^*} D) \oplus (D \lhd_{z^*} B) = (A \oplus D) \lhd_{z^*} (D \oplus B) = Z.$$

If $A = \emptyset$ or $\operatorname{cl}(B + H(z^*)) = Z$, then

$$A \lhd_{z^*} B = (A \lhd_{z^*} D) \oplus (D \lhd_{z^*} B) = (A \oplus D) \lhd_{z^*} (D \oplus B) = \emptyset,$$

as either A or B is assumed to be z^*-proper. ■

2.4.10 Remark. *If $S_{(x^*,z^*)} : X \to \mathcal{Q}_C^t(Z)$ is a conlinear function with $x^* \in X^*$, $z^* \in C^* \setminus \{0\}$, then*

$$S_{(x^*,z^*)}(x_1 - x_2) = S_{(x^*,z^*)}(x_1) \lhd_{z^*} S_{(x^*,z^*)}(x_2),$$

which is another argument for the "almost linearity" of these functions.

For functions $F_1, F_2 : X \to \mathcal{Q}_C^t(Z)$, the difference (with respect to $z^* \in Z^* \setminus \{0\}$), $(F_1 \lhd_{z^*} F_2)$ is defined as the pointwise difference,

$$\forall x \in X : \quad (F_1 \lhd_{z^*} F_2)(x) = F_1(x) \lhd_{z^*} F_2(x).$$

Again, $\lhd_{z^*}$ is the inf-difference on the conlinear space $\left(\left\{F : X \to Q_{H(z^*)}^t(Z)\right\}, +, \cdot, \supseteq\right)$ for all $z^* \in Z^* \setminus \{0\}$. It holds

$$\forall F_1, F_2 : X \to \mathcal{Q}_C^t(Z), \forall z^* \in Z^* \setminus \{0\} : (F_1 \lhd_{z^*} F_2) : X \to Q_{H(z^*)}^t(Z)$$

and $(F_1 \lhd_{z^*} F_2) : X \to \mathcal{Q}_C^t(Z)$ if $z^* \in C^* \setminus \{0\}$. Notice that $(F_1 \lhd_{z^*} F_2) \equiv Z$, if $\operatorname{cl}(F_1(x) + H(z^*)) = Z$ for all $x \in \operatorname{dom} F_2$. Moreover, if $F : X \to \mathcal{Q}_C^t(Z)$ and $z^* \in Z^* \setminus \{0\}$, then by 2.4.2 it holds

$$\forall x \in X : \quad (F \lhd_{z^*} \{0\})(x) = \left\{ z \in Z | \; -z^*(z) \geq \inf_{z \in F(x)} (-z^*(z)) \right\}.$$

3 Scalarization

In this chapter, we will show that each set-valued convex function $F : X \to \mathcal{Q}_C^t(Z)$ can be equivalently described by a family of scalar functions $\left\{\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\} \mid z^* \in C^* \setminus \{0\}\right\}$. These scalar functions are convex (subadditive, positively homogeneous), if $F : X \to \mathcal{Q}_C^t(Z)$ is convex (subadditive, positively homogeneous). The set-valued function is proper (C-proper), if and only if at least one scalarization $\varphi_{(F,z^*)} : X \to \mathrm{I\!R}\cup\{\pm\infty\}$ with $z^* \in C^* \setminus \{0\}$ $(z^* \in C^* \setminus -C^*)$ is proper. Topological properties turn out to be somewhat more difficult as in general the scalarizations of closed set-valued functions are not closed.

It will turn out later, that also the conjugate, directional derivative or the subdifferential of a set-valued function are fully described by the set of the conjugates, directional derivatives or the subdifferentials of these scalarizations. Therefore, this chapter provides us with a strong tool allowing us to derive the set-valued theory of convex analysis from the well-known scalar theory.

It has been pointed out in [23], that scalarization, in the way we apply it, is not ”as in many references about vector-optimization problems (for example [29], [41]),... just a useful tool to find real-valued substitutes for vector-valued problems, but another way of representing a set-valued theory”.

It will be shown at another place that it is not necessary to apply the scalar theory to obtain the theory presented in this thesis. This approach has its beauty in the fact that the scalar convex analysis can be derived as a special case of the new set-valued theory as well as the fact that it sheds some new light to the fundamental (algebraic) structures needed to deal with convexity. On the other hand, the approach chosen in most part of the present work stresses the fact, that the set-valued theory can be completely derived from the scalar theory. Thus, the scalar and set-valued theory are in fact of equal power and one can be derived from the other.

Throughout this chapter, X, Y and Z will be locally convex, separable spaces with the dual spaces X^*, Y^* and Z^*. As before, $C \subseteq Z$ is a closed convex cone with $\{0\} \subsetneq C \subsetneq Z$ and C^* the negative dual of C.

3.1 Definition and basic results

3.1.1 Proposition. *For a function $F : X \to \mathcal{Q}_C^t(Z)$ and $z^* \in C^* \setminus \{0\}$ it holds*

$$\forall x \in X : \quad F(x) \subseteq (F \lhd_{z^*} \{0\})(x)$$

and $\operatorname{dom} F = \operatorname{dom}(F \lhd_{z^*} \{0\})$. *Moreover, it holds*

$$\forall x \in X : \quad F(x) = \bigcap_{z^* \in C^* \setminus \{0\}} (F \lhd_{z^*} \{0\})(x).$$

Proof. For all $z^* \in C^* \setminus \{0\}$, $x \in X$ it holds $(F \lhd_{z^*} \{0\})(x) = \operatorname{cl}(F(x) + H(z^*))$, thus the first two assertions are immediate. For $x_0 \in \operatorname{dom} F$, the set $F(x)$ is convex, closed and $F(x) = F(x) + C$. By 2.4.2,

$$\forall z^* \in C^* \setminus \{0\}, \forall x \in X : \quad (F \lhd_{z^*} \{0\})(x) = \left\{\bar{z} \in Z \mid -z^*(\bar{z}) \geq \inf_{z \in F(x)} (-z^*(z))\right\}$$

holds true. If $F(x_0) = Z$, then

$$\forall z^* \in C^* \setminus \{0\} : \quad \inf_{z \in F(x_0)} (-z^*(z)) = -\infty,$$
$$(F \lhd_{z^*} \{0\})(x_0) = Z.$$

Otherwise, let $z_0 \notin F(x_0)$, then by a separation argument it exists $z^* \in C^* \setminus \{0\}$ such, that

$$-z^*(z_0) < \inf_{z \in F(x)} (-z^*(z))$$

holds true and thus $z_0 \notin (F \lhd_{z^*} \{0\})(x_0)$. ■

3.1.2 Definition. *With a function $F : X \to \mathcal{Q}^t_C(Z)$ and an element $z^* \in Z^* \setminus \{0\}$, associate the function $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$\varphi_{(F,z^*)}(x) := \inf \{-z^*(z) |\ z \in F(x)\}.$$

In fact, for a fixed $x \in X$, then function $z^* \mapsto \varphi_{(F,z^*)}(x)$ is the negative support function of $F(x)$,

$$\forall x \in X, z^* \in Z^* : \quad \varphi_{(F,z^*)}(x) = -\sigma(z^* | F(x)).$$

3.1.3 Example. *If $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$ holds, then $\varphi_{(S_{(x^*,z^*)},tz^*)}(x) = tx^*(x)$ holds for all $t > 0$ and all $x \in X$, while $\varphi_{(S_{(x^*,z^*)},\bar{z}^*)} \equiv -\infty$ holds for all $\bar{z}^* \in C^* \setminus \operatorname{cl}\operatorname{cone}\{z^*\}$.*

3.1.4 Remark. *From 2.4.2, we know that $A \lhd_{z^*} \{0\} = \left\{ z \in Z |\ -z^*(z) \geq \inf_{z \in A} (-z^*(z)) \right\}$ holds for any $A \in \mathcal{P}(Z)$ and $z^* \in Z^* \setminus \{0\}$. Thus, for a function $F : X \to \mathcal{Q}^t_C(Z)$ and $z^* \in C^* \setminus \{0\}$ it holds*

$$\forall x \in X : \quad (F \lhd_{z^*} \{0\})(x) = \left\{ z \in Z |\ -z^*(z) \geq \varphi_{(F,z^*)}(x) \right\},$$
$$\varphi_{(F,z^*)}(x) = \varphi_{(F \lhd_{z^*} \{0\}, z^*)}(x).$$

Moreover, $\operatorname{dom} \varphi_{(F,z^*)} = \operatorname{dom} F$ holds true by 3.1.1 and for $x \in \operatorname{dom} F$ it holds $\varphi_{(F,z^*)}(x) = -\infty$ if and only if $F(x) \lhd_{z^*} \{0\} = Z$.

If $z^* \notin \operatorname{cone}\{\bar{z}^*\}$, then $(F(x) \lhd_{z^*} \{0\}) \lhd_{\bar{z}^*} \{0\} = Z$ holds for all $x \in \operatorname{dom} F$, thus

$$\forall x \in \operatorname{dom} F : \quad \varphi_{(F \lhd_{z^*} \{0\}, \bar{z}^*)}(x) = -\infty.$$

3.1.5 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function.*

a) For all $z^* \in Z^* \setminus \{0\}$ it holds $\operatorname{dom} F = \operatorname{dom} \varphi_{(F,z^*)}$ and $\varphi_{(F,z^*)}(x) \in \mathbb{R}$ holds if and only if $F(x)$ is a z^*-proper set.

b) For all $z^* \in Z^* \setminus \{0\}$, $t > 0$ we have

$$\varphi_{(F,tz^*)}(x) = \varphi_{(tF,z^*)}(x) = t\varphi_{(F,z^*)}(x)$$

and for $z^*_1, z^*_2 \in Z^* \setminus \{0\}$ we have

$$\varphi_{(F,z^*_1+z^*_2)}(x) \geq \varphi_{(F,z^*_1)}(x) + \varphi_{(F,z^*_2)}(x)$$

c) If $z^* \notin (C^* \setminus \{0\})$, then $\varphi_{(F,z^*)}(x) = -\infty$ for every $x \in \operatorname{dom} F$.

PROOF.

a) This is 3.1.4.

b) As for $t >$ and $z^* \in Z^* \setminus \{0\}$ it holds

$$\forall x \in X : \quad (F \triangleleft_{z^*} \{0\})(x) = (F \triangleleft_{tz^*} \{0\})(x),$$

the first statement is proven by direct calculation from the formula

$$\forall x \in X : \quad (F \triangleleft_{z^*} \{0\})(x) = \left\{ z \in Z | \ -z^*(z) \geq \varphi_{(F,z^*)}(x) \right\}.$$

The second statement can be seen by easy calculation from the definition of $\varphi_{(F,z^*)}(x)$.

$$\begin{aligned}\forall x \in X : \quad \varphi_{(F,z_1^*+z_2^*)} &= \inf_{z\in F(x)} ((z_1^* + z_2^*)(z)) \\ &\geq \inf_{z\in F(x)} (z_1^*(z)) + \inf_{z\in F(x)} (z_2^*(z)) \\ &= \varphi_{(F,z_1^*)}(x) + \varphi_{(F,z_2^*)}(x).\end{aligned}$$

c) If $z^* \notin C^* \setminus \{0\}$, then $\mathrm{cl}\,(F(x) + H(z^*)) = Z$ holds for all $x \in \mathrm{dom}\, F$, therefore $\varphi_{(F,z^*)}(x) = -\infty$ holds for all $x \in \mathrm{dom}\, F$. ■

For a function $F : X \to \mathcal{Q}_C^t(Z)$ and $z^* \in C^* \setminus \{0\}$ it holds

$$(F \triangleleft_{z^*} \{0\})(x) = \left\{ z \in Z | \ -z^*(z) \geq \varphi_{(F,z^*)}(x) \right\},$$

thus

$$\begin{aligned}\mathrm{epi}\,(F \triangleleft_{z^*} \{0\}) &= \left\{ (x,z) \in X \times Z | \ \exists (x,t) \in \mathrm{epi}\,\varphi_{(F,z^*)} \ : \ -z^*(z) = t \right\} \\ \mathrm{epi}\,\varphi_{(F,z^*)} &= \{ (x,t) \in X \times \mathbb{R} | \ \exists (x,z) \in \mathrm{epi}\,(F \triangleleft_{z^*} \{0\}) : \ -z^*(z) = t \}.\end{aligned}$$

Moreover, as $F(x) \subseteq (F \triangleleft_{z^*} \{0\})(x)$ holds for all $x \in X$, it holds

$$\begin{aligned}\mathrm{epi}\, F &\subseteq \left\{ (x,z) \in X \times Z | \ \exists (x,t) \in \mathrm{epi}\,\varphi_{(F,z^*)} \ : \ -z^*(z) = t \right\} \\ \mathrm{epi}\,\varphi_{(F,z^*)} &\supseteq \{ (x,t) \in X \times \mathbb{R} | \ \exists (x,z) \in \mathrm{epi}\, F : \ -z^*(z) = t \}.\end{aligned}$$

The following example will show that in general equation holds in neither inclusion.

3.1.6 Example. *Let $C = \mathbb{R}^2_+$ and $F : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ be a function defined by*

$$\forall x \in \mathbb{R} : \quad F(x) := \left\{ (t, \frac{1}{t}) | \ t > 0 \right\} + C.$$

Let $z^* = (0,-1)$, then for all $(x,z) \in \mathrm{epi}\, F$ it holds $-z^*(z) > \varphi_{(F,z^*)}(x) = 0$. Therefore,

$$\mathrm{epi}\,\varphi_{(F,z^*)} \supseteq \{ (x,r) \in X \times \mathbb{R} | \ \exists (x,z) \in \mathrm{epi}\, F : \ -z^*(z) = r \}$$

$$\mathrm{epi}\, F \subseteq \left\{ (x,z) \in X \times Z | \ \exists (x,r) \in \mathrm{epi}\,\varphi_{(F,z^*)} : \ -z^*(z) = r \right\}$$

but equality does not hold in neither inclusion.

3.1.7 Theorem. *For a function $F : X \to \mathcal{Q}_C^t(Z)$ it holds*

$$\forall x \in X: \quad F(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z | \; -z^*(z) \geq \varphi_{(F,z^*)}(x) \right\}.$$

Proof. By 3.1.1, it holds

$$\forall x \in X: \quad F(x) = \bigcap_{z^* \in C^* \setminus \{0\}} (F \lhd_{z^*} \{0\})(x)$$

and

$$\forall x \in X: \quad (F \lhd_{z^*} \{0\})(x) = \left\{ z \in Z | \; -z^*(z) \geq \varphi_{(F,z^*)}(x) \right\}.$$

■

An immediate result from 3.1.7 is, that it holds

$$\operatorname{epi} F = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ (x,z) \in X \times Z | \; (x, -z^*(z)) \in \operatorname{epi} \varphi_{(F,z^*)} \right\}.$$

3.1.8 Remark. *If $t > 0$, then $(F(x) \lhd_{tz^*} \{0\}) = (F(x) \lhd_{z^*} \{0\})$. Therefore, if $C^* \setminus \{0\}^* = \operatorname{cone} B^*$ with $B^* \subseteq Z^*$, then*

$$\forall x \in X: \quad F(x) = \bigcap_{z^* \in B^*} \left\{ z \in Z | \; -z^*(z) \geq \varphi_{(F,z^*)}(x) \right\}$$

holds true.

3.1.9 Proposition. *If $F : X \to \mathcal{Q}_C^t(Z)$ is a function, $x \in X$, $z \in Z$ and $z^* \in C^* \setminus \{0\}$, then the following statements are equivalent*

$$\begin{aligned} -z^*(z_0) &= \varphi_{(F,z^*)}(x) \\ F(x) \lhd_{z^*} \{0\} &= z_0 + H(z^*) \\ F(x) \lhd_{z^*} \{z_0\} &= H(z^*). \end{aligned}$$

Proof. It holds $-z^*(z) = \varphi_{(F,z^*)}(x)$ if and only if

$$F(x) \lhd_{z^*} \{0\} = \{ z \in Z | \; -z^*(z) \geq -z^*(z_0) \} = z_0 + H(z^*).$$

By 2.4.2,

$$F(x) \lhd_{z^*} \{z_0\} = \left\{ z \in Z | \; -z^*(z) \geq \varphi_{(F,z^*)}(x) - (-z^*(z_0)) \right\}$$

holds, proving the equivalence. ■

3.1.10 Proposition (Scalarization of vector-valued functions)**.** *a) For a function $F : X \to \mathcal{Q}_C^t(Z)$ it holds $F(x) = z + C$ if and only if*

$$\forall z^* \in C^* \setminus \{0\}: \quad -z^*(z) = \varphi_{(F,z^*)}(x).$$

b) Let $f : X \to Z \cup \{\pm\infty\}$ be a vector-valued function and $F_C : X \to \mathcal{Q}_C^t(Z)$ its set-valued extension defined as $F(x) = f(x) + C$ for all $x \in X$, then

$$\forall z^* \in C^* \setminus \{0\}: \quad \varphi_{(F,z^*)}(x) = \begin{cases} -z^*(f(x)), & \text{if } x \in \operatorname{dom} F\ ; \\ +\infty, & \text{else.} \end{cases}$$

For $z_1^*, z_2^* \in C^* \setminus \{0\}$ it holds

$$\forall x \in X: \quad \varphi_{(F,z_1^*+z_2^*)}(x) = \varphi_{(F,z_1^*)}(x) + \varphi_{(F,z_2^*)}(x).$$

Proof.

a) It holds

$$z + C = \bigcap_{z^* \in C^* \setminus \{0\}} \{y \in Z | \; -z^*(y) \geq -z^*(z)\} ,$$

so if

$$\forall z^* \in C^* \setminus \{0\} : \quad -z^*(z) = \varphi_{(F,z^*)}(x)$$

holds, then $F(x) = \{z\} + C$ is true.

On the other hand let $F(x) = \{z\} + C$, then

$$\forall z^* \in C^* \setminus \{0\} : \quad F(x) \lhd_{z^*} \{0\} = z_0 + H(z^*)$$

and therefore by above the statement holds true.

b) This is obvious from a). ■

The equation

$$\forall x \in X : \quad \varphi_{(F,z_1^*+z_2^*)}(x) = \varphi_{(F,z_1^*)}(x) + \varphi_{(F,z_2^*)}(x)$$

is not true in more general cases of $F : X \to \mathcal{Q}_C^t(Z)$, as it is essential in 3.1.10, that the infimum in the definition of $\varphi_{(F,z^*)}(x)$ is attained in the same element $f(x)$ for all $z^* \in C^* \setminus \{0\}$.

3.2 Algebraic properties

3.2.1 Proposition. *Let $F, F_1, F_2 : X \to \mathcal{Q}_C^t(Z)$ be three functions, $z^* \in C^* \setminus \{0\}$ and $t > 0$.*

a) It holds

$$\forall x \in X : \quad \varphi_{(tF,z^*)}(x) = t\varphi_{(F,z^*)}(x).$$

b) It holds

$$\forall x \in X : \quad \varphi_{(F_1 \oplus F_2,z^*)}(x) = \varphi_{(F_1,z^*)}(x) + \varphi_{(F_2,z^*)}(x).$$

c) It holds

$$\forall x \in X : \quad \varphi_{(F_1 \lhd_{z^*} F_2,z^*)}(x) = \varphi_{(F_1,z^*)}(x) \lhd \varphi_{(F_2,z^*)}(x)$$

d) It holds

$$\forall x \in X : \quad \varphi_{(F_1 \Box F_2,z^*)}(x) = \left(\varphi_{(F_1,z^*)} \Box \varphi_{(F_2,z^*)}\right)(x)$$

e) If $A \in \mathcal{L}(X,Y)$ and $G : Y \to Z$, then it holds

$$\forall y \in Y : \quad \varphi_{(AF,z^*)}(y) = A\varphi_{(F,z^*)}(y)$$
$$\forall x \in X : \quad \varphi_{(GA,z^*)}(x) = \varphi_{(G,z^*)}(Ax).$$

Proof.

a) By definition, $\varphi_{(tF,z^*)}(x) = \inf\limits_{z \in tF(x)} (-z^*(z))$, so the statement is immediate.

b) By 2.4.4,

$$\forall A \in \mathcal{Q}_C^t(Z): \quad \inf{}_{z^*}\{A\} = (A \triangleleft_{z^*} \{0\}) \in \mathcal{Q}_C^t(Z)$$

holds and by 2.2.5, $\inf(A \oplus B) \leq \inf A \oplus \inf B$ holds for all $A, B \in \mathcal{Q}_C^t(Z)$. Thus by 3.1.4 it holds

$$\forall x \in X: \quad \varphi_{(F_1 \oplus F_2, z^*)}(x) \leq \varphi_{(F_1,z^*)}(x) + \varphi_{(F_2,z^*)}(x).$$

It holds

$$\forall A, B \in \mathcal{Q}_C^t(Z): \quad (A \triangleleft_{z^*} \{0\}) \oplus (B \triangleleft_{z^*} \{0\}) \subseteq (A \oplus B) \triangleleft_{z^*} (\{0\} \oplus \{0\}) = (A \oplus B) \triangleleft_{z^*} \{0\}$$

by 2.4.8 and hence

$$\forall x \in X: \quad \varphi_{(F_1,z^*)}(x) + \varphi_{(F_2,z^*)}(x) \geq \varphi_{(F_1 \oplus F_2, z^*)}(x)$$

holds true.

c) It holds

$$\varphi_{(F_1 \triangleleft_{z^*} F_2), z^*}(x) = \inf\{-z^*(z) | \ z \in (F_1(x) \triangleleft_{z^*} F_2(x))\}$$

And by 2.4.2 it holds

$$(F_1(x) \triangleleft_{z^*} F_2(x)) = \left\{z \in Z | \ -z^*(z) \geq \varphi_{(F_1,z^*)}(x) \triangleleft \varphi_{(F_2,z^*)}(x)\right\},$$

which is the desired result.

d) By definition,

$$\forall x \in X: \quad (F_1 \Box F_2)(x) := \operatorname{cl}\operatorname{co} \bigcup_{x_1 + x_2 = x} (F_1(x_1) + F_2(x_2))$$

holds. As X is a linear space,

$$\forall x \in X: \quad (F_1 \Box F_2)(x) := \operatorname{cl}\operatorname{co} \bigcup_{\bar{x} \in X} (F_1(\bar{x}) + F_2(x - \bar{x}))$$

holds. As $-z^* \in Z^*$ holds,

$$\forall x \in X: \quad \varphi_{(F_1 \Box F_2, z^*)}(x) = \inf_{z \in \bigcup\limits_{\bar{x} \in X} (F_1(\bar{x}) + F_2(x - \bar{x}))} (-z^*(z))$$

and therefore

$$\begin{aligned}\forall x \in X: \quad \varphi_{(F_1 \Box F_2, z^*)}(x) &= \inf_{\bar{x} \in X}\Big(\inf_{\substack{z_1 \in F_1(\bar{x}),\\ z_2 \in F_2(x - \bar{x})}} (-z^*(z_1 + z_2))\Big)\\ &= \inf_{\bar{x} \in X}\Big(\varphi_{(F_1,z^*)}(\bar{x}) + \varphi_{(F_2,z^*)}(x - \bar{x})\Big)\\ &= (\varphi_{(F_1,z^*)} \Box \varphi_{(F_2,z^*)})(x)\end{aligned}$$

e) It holds

$$\varphi_{(AF,z^*)}(y) = \inf\left\{-z^*(z)|\; z \in \operatorname{cl}\operatorname{co} \bigcup_{Ax=y} F(x)\right\},$$

hence

$$\begin{aligned}\varphi_{(AF,z^*)}(y) &= \inf\{\inf\{-z^*(z)|\; z \in F(x)\} \mid Ax = y\} \\ &= (A\varphi_{(F,z^*)})(y).\end{aligned}$$

For the second equation note that

$$\varphi_{(GA,z^*)}(x) = \inf\{-z^*(z)|\; z \in G(Ax)\},$$

hence

$$\varphi_{(GA,z^*)}(x) = \varphi_{(G,z^*)}A(x).$$

■

3.2.2 Lemma. *Let $F, G : X \to \mathcal{Q}^t_C(Z)$ be two functions. It holds*

$$\forall x \in X : \quad F(x) \supseteq G(x)$$

if and only if

$$\forall x \in X : \quad \forall z^* \in C^* \setminus \{0\} : \quad \varphi_{(F,z^*)}(x) \le \varphi_{(G,z^*)}(x).$$

Proof. If $F(x) \supseteq G(x)$ holds for all $x \in X$, then by 3.1.2 it holds $\varphi_{(F,z^*)}(x) \le \varphi_{(G,z^*)}(x)$ for all $x \in X$ and $z^* \in C^* \setminus \{0\}$. On the other hand, if $\varphi_{(F,z^*)}(x) \le \varphi_{(G,z^*)}(x)$ holds for all $x \in X$ and $z^* \in C^* \setminus \{0\}$, then by 3.1.7 it holds $F(x) \supseteq G(x)$ for all $x \in X$. ■

3.2.3 Lemma. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a function, then*

a) F is convex if and only if for all $z^* \in C^* \setminus \{0\}$ the function $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is convex.

b) It holds

$$\forall x \in X, \forall z^* \in C^* \setminus \{0\},: \quad (\operatorname{co}\varphi_{(F,z^*)})(x) = \varphi_{(\operatorname{co}F),z^*}(x).$$

Proof.

a) By definition 2.2.9, F is convex if and only if for each $t \in (0,1)$ and for $x_1, x_2 \in X$ it holds

$$F(tx_1 + (1-t)x_2) \supseteq tF(x_1) + (1-t)F(x_2).$$

Therefore by 3.1.2 and 3.2.1, $\varphi_{(F,z^*)}(tx_1 + (1-t)x_2) \le t\varphi_{(F,z^*)}(x_1) + (1-t)\varphi_{(F,z^*)}(x_2)$ holds for all $z^* \in C^* \setminus \{0\}$.

If the function $\varphi_{(F,z^*)}$ is convex for all $z^* \in C^* \setminus \{0\}$, then

$$\forall x_1, x_2 \in X, \forall t \in (0,1) : \quad \varphi_{(F,z^*)}(tx_1 + (1-t)x_2) \le t\varphi_{(F,z^*)}(x_1) + (1-t)\varphi_{(F,z^*)}(x_2).$$

Therefore, $F(tx_1 + (1-t)x_2) \supseteq tF(x_1) + (1-tF(x_2)$ holds by 3.1.7, so F is convex.

b) As $\varphi_{(\operatorname{co} F, z^*)}$ is a convex minorant of $\varphi_{(F,z^*)}$, we know that $\varphi_{(\operatorname{co} F, z^*)} \le \operatorname{co} \varphi_{(F,z^*)}$.

The epigraph of $\operatorname{co} F$ is equal to $\operatorname{co} \operatorname{epi} F$ and it holds

$$\operatorname{epi} F = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ (x,z) \in X \times Z |\ (x, -z^*(z)) \in \operatorname{epi} \varphi_{(F,z^*)} \right\}.$$

Thus,

$$\begin{aligned}
\operatorname{epi} \operatorname{co} F &= \operatorname{co} \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ (x,z) \in X \times Z |\ (x, -z^*(z)) \in \operatorname{epi} \varphi_{(F,z^*)} \right\} \\
&\subseteq \bigcap_{z^* \in C^* \setminus \{0\}} \operatorname{co} \left\{ (x,z) \in X \times Z |\ (x, -z^*(z)) \in \operatorname{epi} \varphi_{(F,z^*)} \right\} \\
&= \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ (x,z) \in X \times Z |\ (x, -z^*(z)) \in \operatorname{co} \operatorname{epi} \varphi_{(F,z^*)} \right\} \\
&= \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ (x,z) \in X \times Z |\ (x, -z^*(z)) \in \operatorname{epi} \operatorname{co} \varphi_{(F,z^*)} \right\}.
\end{aligned}$$

The function $G : X \to \mathcal{Q}_C^t(Z)$ defined by

$$\operatorname{epi} G = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ (x,z) \in X \times Z |\ (x, -z^*(z)) \in \operatorname{epi} \operatorname{co} \varphi_{(F,z^*)} \right\}$$

is convex and it holds

$$\begin{aligned}
\forall x \in X, \forall z^* \in C^* \setminus \{0\} : \quad \varphi_{(\operatorname{co} F, z^*)}(x) &= \inf_{(x,z) \in \operatorname{co} \operatorname{epi} F} (-z^*(z)) \\
&\ge \inf_{(x,z) \in \operatorname{epi} G} (-z^*(z)) \\
&\ge \operatorname{co} \varphi_{(F,z^*)}(x).
\end{aligned}$$

■

3.2.4 Remark. *By similar proofs one can show that a function $F : X \to \mathcal{Q}_C^t(Z)$ is positively homogenous, subadditive or sublinear if and only if for all $z^* \in C^* \setminus \{0\}$ the function $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is positively homogeneous, subadditive or sublinear.*

3.2.5 Lemma. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a function and $x \in \operatorname{dom} F$.*

a) It holds $F(x) \ne Z$ if and only if there is a $z^* \in C^* \setminus \{0\}$ such that $\varphi_{(F,z^*)}(x) \in \mathrm{I\!R}$.

b) It holds $F(x) \ne F(x) - C$ if and only if there is a $z^* \in C^* \setminus -C^*$ such that $\varphi_{(F,z^*)}(x) \in \mathrm{I\!R}$.

c) It holds $F(x) \ne Z$ and $F(x) = F(x) - C$ if and only if there is a $z^* \in (C^* \cap -C^*) \setminus \{0\}$ such that $\varphi_{(F,z^*)}(x) \in \mathrm{I\!R}$ and for all $z^* \in C^* \setminus C^*$ it holds $\varphi_{(F,z^*)}(x) = -\infty$.

Proof.

a) By 3.1.7, it holds

$$F(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z |\ -z^*(z) \ge \varphi_{(F,z^*)}(x) \right\},$$

thus, $F(x) = Z$ holds if and only if

$$\forall z^* \in C^* \setminus \{0\} : \quad \varphi_{(F,z^*)}(x) = -\infty$$

b) Let $z_0 \in F(x)$ and $c \in C$ such, that $z_0 - c \notin F(x)$. By a separation argument it exists $z^* \in Z^* \setminus \{0\}$ such, that $-z^*(z_0 - c) < \varphi_{(F,z^*)}(x)$. As z_0 was chosen as an element of $F(x)$, $z^* \in C^* \setminus -C^*$ holds true.

c) The third assertion is a combination of the first two. ■

In general, the scalarizations of a proper convex set-valued function are not proper, as the following examples shows.

3.2.6 Example. *a) Let $C = \operatorname{cl}\operatorname{cone}\{(0,1)\} \subseteq \mathbb{R}^2$ and $F : X \to \mathcal{P}(\mathbb{R}^2)$ be a function defined by*

$$\forall x \in X: \quad F(x) := \left\{(t,t^2) | \, t \in \mathbb{R}\right\} + C$$

and $z^* = (-1,0) \in C^* \cap -C^*$ and $z_n^* = (-1, -\frac{1}{n})$ for all $n \in \mathbb{N}$. The function F is C-proper and $\operatorname{core}\operatorname{dom} F \neq \emptyset$ and φ_{z^*} is identically $-\infty$ while $\varphi_{z_n^*}$ is identically $-\frac{3}{4}n$ for all $n \in \mathbb{N}$.

b) Let $C = \operatorname{cl}\operatorname{cone}\{(1,1)\} \subseteq \mathbb{R}^2$ and $F : X \to \mathcal{P}(\mathbb{R}^2)$ be a function defined by

$$\forall x \in X: \quad F(x) = R^2_+.$$

The function F is convex and C-proper and for all $z^* \in C^* \setminus R^2_-$ it holds $\varphi_{z^*} \equiv -\infty$.

3.2.7 Definition. *A function $F : X \to \mathcal{P}(Z)$ is called z^*-proper for $z^* \in C^* \setminus \{0\}$, if $(F \lhd_{z^*} \{0\}) : X \to \mathcal{Q}^t_C(Z)$ is proper.*

A function $F : X \to \mathcal{Q}^t_C(Z)$ is z^*-proper if and only if $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is proper.

3.2.8 Lemma. *If $F : X \to \mathcal{Q}^t_C(Z)$ is z^*-proper with $z^* \in C^* \setminus \{0\}$ ($z^* \in C^* \setminus -C^*$), then F is proper (C-proper).*

Proof. Let F be z^*-proper with $z^* \in C^* \setminus \{0\}$, then $\operatorname{dom} F = \operatorname{dom} \varphi_{(F,z^*)} \neq \emptyset$. Moreover,

$$\forall x \in X: \quad F(x) \subseteq (F \lhd_{z^*} \{0\})(x) \neq Z$$

holds, thus F is proper. If additionally $z^* \in C^* \setminus -C^*$ holds, then

$$\forall x \in \operatorname{dom} F: \quad -\infty = \inf_{z \in F(x) - C} (-z^*(z)) < \varphi_{(F,z^*)}(x),$$

thus $F(x) \neq F(x) - C$ holds for all $x \in \operatorname{dom} F$ and therefore F is C-proper. ■

3.2.9 Lemma. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a convex function and $x_0 \in \operatorname{core}\operatorname{dom} F$, then F is proper (C-proper) if and only if it exists $z^* \in C^* \setminus \{0\}$ ($z^* \in C^* \setminus -C^*$) such that $F(x_0)$ is a z^*-proper set.*

Proof. If it exists $z^* \in C^* \setminus \{0\}$ ($z^* \in C^* \setminus -C^*$) such that F is z^*-proper, then by 3.2.8 F is proper (C-proper).

Let $F : X \to \mathcal{Q}^t_C(Z)$ be proper. By 3.2.3, $\varphi_{(F,z^*)}$ is convex for all $z^* \in C^* \setminus \{0\}$. By 3.2.5 there is $z_0^* \in C^* \setminus \{0\}$ such that $\varphi_{(F,z_0^*)}(x_0) \in \mathbb{R}$. If $\varphi_{(F,z_0^*)}(x) = -\infty$ for some $x \in X$, then by 8.3.1 it holds

$$\forall x \in \operatorname{core}\operatorname{dom} F: \quad \varphi_{(F,z_0^*)}(x) = -\infty,$$

a contradiction. The proof for F being C-proper goes along parallel arguments with $z_0^* \in C^* \setminus -C^*$. ■

3.2.10 Theorem. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a convex and proper function and $x_0 \in \operatorname{core}\operatorname{dom} F$, then*

$$F(x_0) = \bigcap_{\substack{z^* \in C^* \setminus \{0\}, \\ \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z |\ (-z^*, z) \geq \varphi_{(F,z^*)}(x_0) \right\}.$$

If F is additionally $C-$proper, then

$$F(x_0) = \bigcap_{\substack{z^* \in C^* \setminus -C^*, \\ \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z |\ (-z^*, z) \geq \varphi_{(F,z^*)}(x_0) \right\}.$$

PROOF. By 3.1.5,

$$\forall z^* \in C^* \setminus \{0\} : \quad \operatorname{dom} \varphi_{(F,z^*)} = \operatorname{dom} F$$

and therefore $x_0 \in \operatorname{core}\operatorname{dom} \varphi_{(F,z^*)}$ for all $z^* \in C^* \setminus \{0\}$. By 3.2.3, each scalarization $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is convex and by 3.1.7 it holds

$$F(x_0) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z |\ (-z^*, z) \geq \varphi_{(F,z^*)}(x_0) \right\}.$$

By 3.2.9, the set

$$\begin{aligned} & \left\{ z^* \in C^* \setminus \{0\} \ : \ \varphi_{(F,z^*)}(x_0) \in \mathrm{I\!R} \right\} \\ = & \left\{ z^* \in C^* \setminus \{0\} \ : \ \varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\} \text{ is proper} \right\} \end{aligned}$$

is nonvoid. On the other hand if $\varphi_{(F,z_0^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is not proper for $z_0^* \in C^* \setminus \{0\}$, then

$$\forall x \in \operatorname{core}\operatorname{dom} \varphi_{(F,z^*)} : \quad \varphi_{(F,z_0^*)}(x) = -\infty.$$

Therefore, the first assertion holds true

If F is C-proper, then exists $z_0^* \in C^* \setminus -C^*$ such that $\varphi_{(F,z_0^*)}$ is proper, compare 3.2.9. In particular, $\varphi_{(F,z_0^*)}(x_0) \in \mathrm{I\!R}$ because of 3.2.5. Let $z_0 \notin F(x_0)$. As $F(x_0) = \operatorname{cl}\operatorname{co}(F(x_0) + C)$, by a separation argument there is $z^* \in C^* \setminus \{0\}$ and $\varepsilon > 0$ such, that $-z^*(z_0) < \varphi_{(F,z^*)}(x_0) - 2\varepsilon$. Choosing $t > 0$ small enough it holds

$$\begin{aligned} -\varepsilon < -t z_0^*(z_0) < \varepsilon, \\ -\varepsilon < \varphi_{(F,t z_0^*)}(x_0) < \varepsilon. \end{aligned}$$

Furthermore, $(z^* + t z_0^*) \in C^* \setminus -C^*$ and

$$-(z^* + t z_0^*)(z_0) < \varphi_{(F,z^*)}(x_0) - \varepsilon < \varphi_{(F,z^*)}(x_0) + \varphi_{(F,t z_0^*)}(x_0) \leq \varphi_{(F,z^* + t z_0^*)}(x_0).$$

Therefore, the second assertion is proven. ■

The result of 3.2.10 does not hold for $x \notin \operatorname{core}\operatorname{dom} F$ in general, as the following example shows.

3.2.11 Example. *Let $C = \operatorname{cl}\operatorname{cone}\{(0,1)\} \subseteq \mathrm{I\!R}^2$ and $F : \mathrm{I\!R} \to \mathrm{I\!R}^2$ a function defined by*

$$F(x) := \begin{cases} H((0,-1)), & \text{if } x > 0; \\ C, & \text{if } x = 0; \\ \emptyset, & \text{else.} \end{cases}$$

Obviously $\operatorname{core}\operatorname{dom} F \neq \emptyset$ *holds and F is convex. $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is proper if and only if $z^* \in \operatorname{cone}\{(0,-1)\}$ but*

$$F(0) \subsetneq \bigcap_{t>0} \left\{ z \in \mathrm{I\!R}^2 |\ -t(0,-1)(z) \geq \varphi_{(F,t(0,-1))}(0) \right\} = H(z^*).$$

3.3 Topological properties

It will turn out later on, that the richest theory is, as in the scalar case, that of proper closed convex functions. Therefore, this subsection comprises the most important tools for our further investigations. It will turn out that the correspondence between the topological properties of a function $F : X \to \mathcal{Q}^t_C(Z)$ and those of its scalarizations $\varphi_{(F,z^*)}$ is not as immediate as it was the case for the algebraic properties. Still, the results presented in the following will be sufficient to develop a theory of set-valued convex functions in a one-to-one correspondence to the scalar theory.

3.3.1 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function and $z^* \in C^* \setminus \{0\}$.*

a) It holds $\operatorname{cl}\varphi_{(F,z^*)}(x) = \varphi_{(\operatorname{cl}(F \lhd_{z^*} \{0\}),z^*)}(x)$ for all $x \in X$.

b) The function $(F \lhd_{z^*} \{0\}) : X \to \mathcal{Q}^t_C(Z)$ is closed if and only if $\varphi_{(F,z^*)}$ is closed.

c) If $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is closed for all $z^* \in C^* \setminus \{0\}$, then F is closed.

Proof.

a) By definition,

$$\forall x \in X : \quad \operatorname{cl}\varphi_{(F,z^*)}(x) = \sup_{U \in \mathcal{U}_X} \inf_{\bar{x} \in U} \varphi_{(F,z^*)}(x + \bar{x})$$

and

$$\operatorname{epi}(F \lhd_{z^*} \{0\}) = \Big\{(x,z) \in X \times Z |\ -z^*(z) \geq \varphi_{(F,z^*)}(x)\Big\}.$$

It holds

$$\begin{aligned}
& -z^*(z) \geq \operatorname{cl}\varphi_{(F,z^*)}(x) \\
\Leftrightarrow\ & \forall U \in \mathcal{U}_X,\ \forall \varepsilon > 0,\ \exists \bar{x} \in U : \quad -z^*(z) + \varepsilon \geq \varphi_{(F,z^*)}(x + \bar{x}) \\
\Leftrightarrow\ & \forall U \in \mathcal{U}_X,\ \forall V \in \mathcal{U}_Z,\ \exists \bar{x} \in U,\ \exists \bar{z} \in V : \quad -z^*(z + \bar{z}) \geq \varphi_{(F,z^*)}(x + \bar{x}) \\
\Leftrightarrow\ & \forall U \in \mathcal{U}_X,\ \forall V \in \mathcal{U}_Z,\ \exists \bar{x} \in U,\ \exists \bar{z} \in V : \quad (x + \bar{x}, z + \bar{z}) \in \operatorname{epi}(F \lhd_{z^*} \{0\}) \\
\Leftrightarrow\ & (x,z) \in \operatorname{cl}\operatorname{epi}(F \lhd_{z^*} \{0\}).
\end{aligned}$$

Thus,

$$\operatorname{epi}\operatorname{cl}(F \lhd_{z^*} \{0\}) = \Big\{(x,z) \in X \times Z |\ -z^*(z) \geq \operatorname{cl}\varphi_{(F,z^*)}(x)\Big\}$$

and $\varphi_{(\operatorname{cl}(F \lhd_{z^*} \{0\}),z^*)}(x) = \operatorname{cl}\varphi_{(F,z^*)}(x)$ holds for all $x \in X$.

b) The function $(F \lhd_{z^*} \{0\}) : X \to \mathcal{Q}^t_C(Z)$ is closed if and only if $\operatorname{epi}(F \lhd_{z^*} \{0\})$ is closed. In this case,

$$\begin{aligned}
\forall x \in X : \quad \varphi_{(F,z^*)}(x) &= \inf_{(x,z) \in \operatorname{epi}(F \lhd_{z^*} \{0\})} (-z^*(z)) \\
&= \inf_{(x,z) \in \operatorname{cl}\operatorname{epi}(F \lhd_{z^*} \{0\})} (-z^*(z)) \\
&= \operatorname{cl}\varphi_{(F,z^*)}(x).
\end{aligned}$$

On the other hand, if $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is closed, then $\operatorname{epi}(F \lhd_{z^*} \{0\})$ is closed and therefore $(F \lhd_{z^*} \{0\})$ is closed.

c) If $\varphi_{(F,z^*)}$ is closed for all $z^* \in C^* \setminus \{0\}$, then $\operatorname{epi}(F \lhd_{z^*} \{0\})$ is closed for all $z^* \in C^* \setminus \{0\}$. By 3.1.7 or by 3.1.1,

$$\operatorname{epi} F = \bigcap_{z^* \in C^* \setminus \{0\}} \operatorname{epi}(F \lhd_{z^*} \{0\})$$

and thus $\operatorname{epi} F$ is closed, therefore, $F(x) = (\operatorname{cl} F)(x)$ holds for all $x \in X$. ■

A function $F : X \to \mathcal{Q}_C^t(Z)$ being closed does not mean that all $\varphi_{(F,z^*)}$ are necessarily closed.

3.3.2 Example. *The set-valued function $F : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ in 3.1.6 is proper, closed and convex, as $\operatorname{dom} F \neq \emptyset$ and $\operatorname{epi} F$ is a closed, convex set. With $z^* = (0, -1)$, the scalarization $\varphi_{(F,z^*)} : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is not closed at 0, as $\varphi_{(F,z^*)}(0) = +\infty$ and $\operatorname{cl} \varphi_{(F,z^*)}(0) = 0$.*

For a function $F : X \to \mathcal{Q}_C^t(Z)$ and $z^* \in C^* \setminus \{0\}$, it holds $\operatorname{dom} \operatorname{cl} \varphi_{(F,z^*)} \supseteq \operatorname{dom} \varphi_{(\operatorname{cl} F,z^*)} = \operatorname{dom} \operatorname{cl} F$. As can be seen in 3.3.2, the opposite inclusion does not hold.

3.3.3 Definition. *A function $F : X \to \mathcal{Q}_C^t(Z)$ is called z^*-closed for $z^* \in C^* \setminus \{0\}$, if $(F \lhd_{z^*} \{0\}) : X \to \mathcal{Q}_C^t(Z)$ is a closed function.*

From 3.3.1 we know that a function $F : X \to \mathcal{Q}_C^t(Z)$ is closed, if it is z^*-closed for all $z^* \in C^* \setminus \{0\}$. Moreover, F is z^*-closed if and only if $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is closed.

The property 3.3.1c) (F is z^*-closed for all $z^* \in C^* \setminus \{0\}$) is called C-upper hemicontinuity in [45]. It is shown there that if all images of $F : X \to \mathcal{Q}_C^t(Z)$ can be represented as $B(x) + C$, where $B(x) \subseteq Z$ is a bounded set for all $x \in X$, then $F : X \to \mathcal{Q}_C^t(Z)$ is closed if and only if it is z^*-closed for all $z^* \in C^* \setminus \{0\}$.

3.3.4 Lemma. *For a given function $F : X \to \mathcal{Q}_C^t(Z)$ it holds*

$$\forall z^* \in C^* \setminus \{0\} : \quad \forall x \in X : \quad \operatorname{cl} \varphi_{(\operatorname{cl} F,z^*)}(x) = (\operatorname{cl} \varphi_{(F,z^*)})(x) \leq \varphi_{(\operatorname{cl} F,z^*)}(x).$$

Proof. It holds

$$\forall z^* \in C^* \setminus \{0\} : \quad \operatorname{epi} F \subseteq \operatorname{epi}(F \lhd_{z^*} \{0\})$$

and thus

$$\forall z^* \in C^* \setminus \{0\} : \quad \operatorname{cl} \operatorname{epi} F \subseteq \operatorname{cl} \operatorname{epi}(F \lhd_{z^*} \{0\}).$$

Hence by 3.3.1, $\operatorname{cl} \varphi_{(F,z^*)}(x) \leq \varphi_{(\operatorname{cl} F,z^*)}(x)$ holds for all $x \in X$. From this one can see that $\operatorname{cl} \varphi_{(F,z^*)}(x) \leq \operatorname{cl} \varphi_{(\operatorname{cl} F,z^*)}(x)$ holds for all $x \in X$, as $\operatorname{cl} \varphi_{(\operatorname{cl} F,z^*)}$ is the greatest closed minorant of $\varphi_{(\operatorname{cl} F,z^*)}$. On the other hand, $\varphi_{(\operatorname{cl} F,z^*)}(x) \leq \varphi_{(F,z^*)}(x)$ holds for all $x \in X$ and thus $\operatorname{cl} \varphi_{(F,z^*)}(x) \geq \operatorname{cl} \varphi_{(\operatorname{cl} F,z^*)}(x)$. ■

3.3.5 Proposition. *If $F : X \to \mathcal{Q}_C^t(Z)$ is convex, then*

$$\forall x \in X : \quad (\operatorname{cl} F)(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z \mid -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$

PROOF. The inclusion "$\subseteq$" holds by 3.3.4. As F is convex, the set $\operatorname{cl} \bigcup_{y \in U} F(x+y)$ is convex for any $U \in \mathcal{U}_X$.

It holds $z_0 \notin (\operatorname{cl} F)(x_0)$ if and only if $(x_0, z_0) \notin \operatorname{cl} \operatorname{epi} F$. Therefore there are $U_0 \in \mathcal{U}_X, V_0 \in \mathcal{U}_Z$ such that

$$(x_0, z_0) \notin \operatorname{epi} F + (U_0 \times V_0)$$

an so $z_0 \notin \operatorname{cl} \bigcup_{y \in U_0} F(x+y)$.

If $\operatorname{cl} \bigcup_{y \in U_0} F(x+y) = \emptyset$ for some $U_0 \in \mathcal{U}_X$, then

$$\forall z^* \in C^* \setminus \{0\}, \, y \in U_0 : \quad \varphi_{(F,z^*)}(x+y) = \operatorname{cl} \varphi_{(F,z^*)}(x) = +\infty$$

and therefore

$$\emptyset = (\operatorname{cl} F)(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$

Now let $\operatorname{cl} \bigcup_{y \in U_0} F(x+y) \neq \emptyset$. By a separation argument we get $z^* \in C^* \setminus \{0\}, \, \alpha \in \mathbb{R}$ such that

$$-z^*(z_0) < \alpha \leq \inf \left\{ -z^*(z) | \; z \in \operatorname{cl} \bigcup_{y \in U_0} F(x+y) \right\} = \inf_{y \in U_0} \varphi_{(F,z^*)}(x+y).$$

As we have $\inf \left\{ \varphi_{(F,z^*)}(x+y) \,|\, y \in U_0 \right\} \leq \left(\operatorname{cl} \varphi_{(F,z^*)} \right)(x)$, it follows that

$$z_0 \notin \left\{ z \in Z | \; -z^*(z) \geq \left(\operatorname{cl} \varphi_{(F,z^*)} \right)(x) \right\}.$$

■

Notice that the formula in 3.3.5 is stated only for convex functions.

3.3.6 Proposition. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$, $x_0 \in \operatorname{core} \operatorname{dom} F$. Then either $\operatorname{cl} F$ is z^*-proper and $\varphi_{(F,z^*)}(x_0) = \operatorname{cl} \varphi_{(F,z^*)}(x_0)$, or $\operatorname{cl} \varphi_{(F,z^*)}(x) = -\infty$ for all $x \in \operatorname{dom} F$.*

PROOF. The function $\varphi_{(F,z^*)}$ is convex for every $z^* \in C^* \setminus \{0\}$, so by 8.3.1, $\operatorname{cl} \varphi_{(F,z^*)}(x_0) = \varphi_{(F,z^*)}(x_0)$ or $\operatorname{cl} \varphi_{(F,z^*)}(x) = -\infty$ for all $x \in \operatorname{dom} F$. If $\operatorname{cl} \varphi_{(F,z^*)}(x_0) = \varphi_{(F,z^*)}(x_0) > -\infty$, then by 3.3.4 $\operatorname{cl} F$ is z^*-proper. ■

3.3.7 Proposition. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex function, $(x_0, z_0) \in \operatorname{int} \operatorname{epi} F$. If F is z^*-proper for $z^* \in C^* \setminus \{0\}$, then $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is continuous at x_0. If F is not z^*-proper, then $\varphi_{(F,z^*)}(x) = -\infty$ holds for all $x \in \operatorname{int} \operatorname{dom} F$.*

PROOF. It holds $(x_0, z_0) \in \operatorname{int} \operatorname{epi} F$ if and only if there is $U \in \mathcal{U}_X$ and $V \in \mathcal{U}_Z$ such that $((x_0, z_0) + U \times V) \subseteq \operatorname{epi} F$. Therefore for all $z^* \in C^* \setminus \{0\}$ exists $\varepsilon > 0$ such that $\{(x_0, -z^*(z_0))\} + U \times (-\varepsilon, \varepsilon) \subseteq \operatorname{epi} \varphi_{(F,z^*)}$, so for all $z^* \in C^* \setminus \{0\}$ it holds $(x_0, -z^*(z_0)) \in \operatorname{int} \operatorname{epi} \varphi_{(F,z^*)}$. If F is z^*-proper, then $\varphi_{(F,z^*)}$ is proper and $\varphi_{(F,z^*)}(x_0) \in \mathbb{R}$. By 8.3.3 it holds $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is continuous. If F is not z^*-proper, then $\varphi_{(F,z^*)}(\bar{x}) = -\infty$ holds for some $\bar{x} \in X$. Thus by 8.3.3, $\varphi_{(F,z^*)}(x) = -\infty$ holds for all $x \in \operatorname{int} \operatorname{dom} \varphi_{(F,z^*)} = \operatorname{int} \operatorname{dom} F$, ■

3.3.8 Corollary. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a sublinear function, $(0, z_0) \in \operatorname{int} \operatorname{epi} F$. Then for all $x \in X$ there exists $z \in Z$ such, that $(x, z) \in \operatorname{int} \operatorname{epi} F$.*

Proof. It holds $(0, z_0) \in \operatorname{int} \operatorname{epi} F$ if and only it exists $U \in \mathcal{U}_X$ and $V \in \mathcal{U}_Z$ such, that

$$\forall (x,z) \in U \times V : \quad (x, z + z_0) \in \operatorname{int} \operatorname{epi} F.$$

Moreover $\operatorname{dom} F = \operatorname{cone} U = X$ and

$$\forall t > 0,\, \forall x \in U : \quad F(tx) = tF(x)$$

and $\operatorname{int} \operatorname{epi} F$ is a convex cone, thus

$$\forall t > 0,\, \forall x \in U : (tx, t(z + z_0)) \in \operatorname{int} \operatorname{epi} F.$$

■

The property $(x, z) \in \operatorname{int} \operatorname{epi} F$ will prove to be an assumption strong enough to state various strong duality results for convex functions. The same result can be achieved by the following assumptions.

3.3.9 Proposition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $x_0 \in \operatorname{dom} F$ and one of the two following conditions holds.*

a) The function F is C-continuous in x_0 in the sense of [45], that is

$$\forall U \in \mathcal{U}_Z : \exists V \in \mathcal{U}_X : \forall x \in V : F(x_0) \subseteq F(x_0 + x) + U,\; F(x_0 + x) \subseteq F(x_0) + U.$$

b) The function F is continuous in x_0 in the sense of [21] Definition 2.5.1., that is if $D \subseteq Z$ is an open set, then

$$F(x_0) \subseteq D \;\Rightarrow\; \exists U_X \in \mathcal{U}_X :\; \forall x \in U_X :\; F(x_0 + x) \subseteq D,$$
$$F(x_0) \cap D \neq \emptyset \;\Rightarrow\; \exists U_X \in \mathcal{U}_X :\; \forall x \in U_X :\; F(x_0 + x) \cap D \neq \emptyset.$$

If F is z^*-proper for $z^* \in C^* \setminus \{0\}$, then $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is continuous at x_0. If F is not z^*-proper, then $\varphi_{(F,z^*)}(x) = -\infty$ holds for all $x \in \operatorname{int} \operatorname{dom} F$.

Proof.

a) From

$$\forall U \in \mathcal{U}_Z : \exists V \in \mathcal{U}_X : \forall x \in V : F(x_0) \subseteq F(x_0 + x) + U,\; F(x_0 + x) \subseteq F(x_0) + U,$$

it holds for all $z^* \in C^* \setminus \{0\}$, that

$$\forall \varepsilon > 0 : \exists V \in \mathcal{U}_X : \forall x \in V : \varphi_{(F,z^*)}(x_0 + x) - \varepsilon \leq \varphi_{(F,z^*)}(x_0) \leq \varphi_{(F,z^*)}(x_0 + x) + \varepsilon,$$

hence each scalarization of F is continuous at $x_0 \in \operatorname{dom} \varphi_{(F,z^*)}$ or $\varphi_{(F,z^*)}(x_0) = -\infty$. In the latter case, $\varphi_{(F,z^*)}(x) = -\infty$ holds for all $x \in \operatorname{int} \operatorname{dom} F$ as $\varphi_{(F,z^*)}$ is convex.

b) Let $z_0 \in F(x_0)$ hold, then

$$\forall U_Z \in \mathcal{U}_Z :\; \exists U_X \in \mathcal{U}_X : \forall x \in U_X : \quad F(x_0 + x) \cap (z_0 + U_Z) \neq \emptyset,$$

therefore, for all $z^* \in C^* \setminus \{0\}$ it holds

$$\forall \varepsilon > 0 :\; \exists U_X \in \mathcal{U}_X : \forall x \in U_X : \quad \varphi_{(F,z^*)}(x_0 + x) \leq -z^*(z_0) + \varepsilon \leq \varphi_{(F,z^*)}(x_0) + \varepsilon.$$

On the other hand, $F(x_0) \subseteq F(x_0) + U_Z$ holds for all $U_Z \in \mathcal{U}_Z$ and thus

$$\forall U_Z \in \mathcal{U}_Z : \exists U_X \in \mathcal{U}_X : \forall x \in U_X : \quad F(x_0 + x) \subseteq (F(x_0) + U_Z).$$

Therefore, for all $z^* \in C^* \setminus \{0\}$ it holds

$$\forall \varepsilon > 0 : \exists U_X \in \mathcal{U}_X : \forall x \in U_X : \quad \varphi_{(F,z^*)}(x_0 + x) \geq \varphi_{(F,z^*)}(x_0) - \varepsilon.$$

Hence, each scalarization of F is continuous at $x_0 \in \operatorname{dom} \varphi_{(F,z^*)}$ or $\varphi_{(F,z^*)}(x_0) = -\infty$. In the latter case, $\varphi_{(F,z^*)}(x) = -\infty$ holds for all $x \in \operatorname{int} \operatorname{dom} F$ as $\varphi_{(F,z^*)}$ is convex. ■

3.3.10 Remark. *a) If $(x_0, z_0) \in \operatorname{int} \operatorname{epi} F$ holds, then $z_0 \in \operatorname{int} F(x_0)$ is true, while this cannot be derived from the assertions in 3.3.9.*

b) From the proof of 3.3.9 it can be derived that $F(x_0) = (\operatorname{cl} F)(x_0)$, if either

$$\forall U \in \mathcal{U}_Z : \exists V \in \mathcal{U}_X : \forall x \in V : F(x_0 + x) \subseteq F(x_0) + U \tag{3.3.1}$$

or for all open sets $D \subseteq Z$ holds

$$F(x_0) \subseteq D \;\Rightarrow\; \exists U_X \in \mathcal{U}_X : \; \forall x \in U_X : \; F(x_0 + x) \subseteq D. \tag{3.3.2}$$

If either (3.3.1) *or* (3.3.2) *holds, then F is z^*-closed in x_0 for all $z^* \in C^* \setminus \{0\}$.*

3.3.11 Theorem. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a closed convex function.*

a) If F is proper, then

$$\forall x \in X : \quad F(x) = \bigcap_{\substack{z^* \in C^* \setminus \{0\}, \\ \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$

b) If F is C-proper, then

$$\forall x \in X : \quad F(x) = \bigcap_{\substack{z^* \in C^* \setminus -C^*, \\ \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$

Proof.

a) The equation
$$F(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$
holds by 3.3.5. By 3.2.3, $\varphi_{(F,z^*)}$ is convex for all $z^* \in C^* \setminus \{0\}$. Suppose there is no $z^* \in C^* \setminus \{0\}$ such, that $\operatorname{cl} \varphi_{(F,z^*)}$ is proper, then

$$F(x) = \begin{cases} Z, & \text{for } x \in \operatorname{dom} F; \\ \emptyset, & \text{else.} \end{cases}$$

holds by 8.3.1. This is a contradiction, as F is proper, therefore,

$$\exists z_0^* \in C^* \setminus \{0\} : \quad \operatorname{cl} \varphi_{(F,z_0^*)} \text{ is proper}$$

It holds $z_0 \notin F(x)$ if and only if $z_0 \notin \bigcap\limits_{U \in \mathcal{U}_X} \operatorname{cl} \bigcup\limits_{y \in U} F(x+y)$.

If $\operatorname{cl} \bigcup\limits_{y \in U_0} F(x+y) = \emptyset$ for some $U_0 \in \mathcal{U}_X$, then

$$\forall z^* \in C^* \setminus \{0\}, \, y \in U_0 : \quad \varphi_{(F,z^*)}(x+y) = \operatorname{cl} \varphi_{(F,z^*)}(x) = +\infty$$

and therefore

$$\emptyset = F(x) = \bigcap_{\substack{z^* \in C^* \setminus \{0\}, \\ \operatorname{cl} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$

Now let $\operatorname{cl} \bigcup\limits_{y \in U_0} F(x+y) \neq \emptyset$ and $z_0 \notin \operatorname{cl} \bigcup\limits_{y \in U_0} F(x+y)$. By a separation argument we get $z^* \in C^* \setminus \{0\}$, $\alpha \in \mathbb{R}$ such that

$$-z^*(z_0) < \alpha \leq \inf \left\{ -z^*(z) | \; z \in \operatorname{cl} \bigcup_{y \in U_0} F(x+y) \right\} \leq \operatorname{cl} \varphi_{(F,z^*)}(x).$$

Especially, $\operatorname{cl} \varphi_{(F,z^*)}$ is proper and

$$z_0 \notin \left\{ z \in Z | \; -z^*(z) \geq \left(\operatorname{cl} \varphi_{(F,z^*)} \right)(x) \right\}.$$

Therefore, the statement is proven.

b) Let F be C-proper. From the above we know that

$$\forall x \in X : \quad F(x) \subseteq \bigcap_{\substack{z^* \in C^* \setminus -C^*, \\ \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$

First, let $x \in \operatorname{dom} F$, then

$$\exists z \in (F(x) - C) \setminus F(x). \tag{3.3.3}$$

and there is $z^* \in C^* \setminus \{0\}$ such, that $\operatorname{cl} \varphi_{(F,z^*)}$ is proper and $-z^*(z) < \varphi_{(F,z^*)}(x)$. Suppose $z^* \in C^* \cap -C^*$, then by (3.3.3) it holds $-z^*(z) \geq \varphi_{(F,z^*)}(x) \geq \operatorname{cl} \varphi_{(F,z^*)}(x)$, which is a contradiction, so

$$\exists z_0^* \in C^* \setminus -C^* : \quad \operatorname{cl} \varphi_{(F,z_0^*)} \text{ is proper.} \tag{3.3.4}$$

If $\operatorname{cl} \bigcup\limits_{y \in U_0} F(x+y) = \emptyset$ for some $U_0 \in \mathcal{U}_X$, then

$$\forall z^* \in C^* \setminus \{0\}, \, y \in U_0 : \quad \varphi_{(F,z^*)}(x+y) = \operatorname{cl} \varphi_{(F,z^*)}(x) = +\infty,$$

hence

$$\emptyset = F(x) = \bigcap_{\substack{z^* \in C^* \setminus -C^*, \\ \operatorname{cl} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(F,z^*)}(x) \right\}.$$

Now let $\operatorname{cl} \bigcup\limits_{y \in U_0} F(x+y) \neq \emptyset$ and $z_0 \notin \operatorname{cl} \bigcup\limits_{y \in U_0} F(x+y)$. As before, by a separation theorem there exist $z^* \in C^* \setminus \{0\}$, $\alpha \in \mathbb{R}$ such that

$$-z^*(z_0) < \alpha < \operatorname{cl} \varphi_{(F,z^*)}(x).$$

For any $t > 0$ it holds $tz_0^* + z^* \in C^* \setminus -C^*$.

If $-(tz_0^* + z^*)(z_0) < \operatorname{cl} \varphi_{(F,tz_0^*+z^*)}(x)$ holds for some $t > 0$, then we are finished. Therefore suppose that

$$\begin{aligned}\forall t > 0: \quad -tz_0^*(z_0) - z^*(z_0) &\geq \operatorname{cl} \varphi_{(F,tz_0^*+z^*)}(x) \\ &\geq \operatorname{cl}\left(\operatorname{cl} \varphi_{(F,tz_0^*)}(x) + \operatorname{cl} \varphi_{(F,z^*)}(x)\right) \\ &> \alpha + t \operatorname{cl} \varphi_{(F,z_0^*)}(x) \\ &> -z^*(z_0) + t \operatorname{cl} \varphi_{(F,z_0^*)}(x).\end{aligned}$$

Then

$$\forall t > 0: \quad t(-z_0^*(z_0) - \operatorname{cl} \varphi_{(F,z_0^*)}(x)) > \alpha + z^*(z_0) > 0$$

which is a contradiction with t small enough, as $\varphi_{(F,z_0^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is proper.

Therefore, the statement is proven. ∎

3.3.12 Corollary. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a function with $\operatorname{dom} F \neq \emptyset$. If $(\operatorname{cl} \operatorname{co} F) : X \to \mathcal{Q}_C^t(Z)$ is proper, then*

$$\forall x \in X: \quad (\operatorname{cl} \operatorname{co} F)(x) = \bigcap_{\substack{z^* \in C^* \setminus \{0\}, \\ \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z \mid -z^*(z) \geq \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)}(x) \right\}.$$

If $\operatorname{cl} \operatorname{co} F : X \to \mathcal{Q}_C^t(Z)$ is C-proper, then

$$\forall x \in X: \quad (\operatorname{cl} \operatorname{co} F)(x) = \bigcap_{\substack{z^* \in C^* \setminus -C^*, \\ \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z \mid -z^*(z) \geq \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)}(x) \right\}.$$

Proof. For each $z^* \in C^* \setminus \{0\}$ it holds

$$\forall x \in X: \quad \operatorname{cl} \varphi_{(\operatorname{cl} \operatorname{co} F,z^*)}(x) = \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)}(x).$$

Thus, both assertions hold by 3.3.11. ∎

3.3.13 Corollary. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a function. The closed convex hull $\operatorname{cl} \operatorname{co} F$ of F is proper if and only if*

$$\exists z^* \in C^* \setminus \{0\}: \quad \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ is proper.}$$

The function $\operatorname{cl} \operatorname{co} F$ is C-proper if and only if

$$\exists z^* \in C^* \setminus -C^*: \quad \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ is proper.}$$

Proof. Let $\operatorname{cl} \operatorname{co} F$ be proper, from

$$\forall x \in \operatorname{dom} F: \quad (\operatorname{cl} \operatorname{co} F)(x) = \bigcap_{\substack{z^* \in C^* \setminus \{0\}, \\ \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)} \text{ proper}}} \left\{ z \in Z \mid -z^*(z) \geq \operatorname{cl} \operatorname{co} \varphi_{(F,z^*)}(x) \right\}$$

we deduce the existence of at least one $z^* \in C^* \setminus \{0\}$ with $\operatorname{cl} \operatorname{co} \varphi_{(F,z^*)}$ proper. On the other hand let $\operatorname{cl} \operatorname{co} \varphi_{(F,z_0^*)}$ be proper for $z_0^* \in C^* \setminus \{0\}$, then $\operatorname{dom} F \neq \emptyset$ and

$$\forall x \in \operatorname{dom} F: \quad (\operatorname{cl} \operatorname{co} F)(x) \subseteq \left\{ z \in Z \mid -z^*(z) \geq \varphi_{(F,z^*)}(x) \right\} \subsetneq Z,$$

so $\operatorname{cl}\operatorname{co} F$ is proper. ■

For the scalar functions $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ it is well-known, that $\operatorname{cl}\operatorname{co} \varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is proper if and only if $\operatorname{dom} \varphi_{(F,z^*)} \neq \emptyset$ and it exists $x^* \in X^*$ and $t \in \mathbb{R}$ such, that $x^*(x) - t \leq \varphi_{(F,z^*)}(x)$ holds for all $x \in X$. Setting $-z^*(e) = 1$ with $e \in Z$, it holds

$$S_{(x^*,z^*)}(x) - te = \{z \in Z | \; -z^*(z) \geq x^*(x) - t\}$$

for all $x \in X$. Thus, $\operatorname{cl}\operatorname{co} F$ is proper (C-proper) if and only if there is a (C-proper) conaffine minorant of F and $\operatorname{dom} F \neq \emptyset$.

4 Conjugation

In the case of a scalar function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$, the conjugate mapping of φ is defined as

$$\varphi^*(x^*) := \sup_{x \in X} (x^*(x) - \varphi(x)) \tag{4.0.5}$$

for all $x^* \in X^*$. The biconjugate is defined as

$$\varphi^{**}(x) := \sup_{x \in X} (x^*(x) - \varphi^*(x^*)) \tag{4.0.6}$$

for all $x \in X$, compare [62] for both formulas.

For vector-valued functions, the conjugate of $f : X \to Z$ has been defined for example in [9, 17, 36, 63] as

$$f^*(T) := \sup_{x \in X} (T(x) - f(x)),$$

using $\mathcal{L}(X, Z)$ as the set of dual variables and understanding the supremum in the sense of the vector-order. This approach requires the image space to be order-complete. Moreover, in general the supremum of a set can be far away from the original set itself. Therefore, another approach defines a set-valued conjugate of a vector-valued function, compare [40, 41, 54, 57], as "... for a vector problem, its dual ... is a problem whose objective function is set-valued whatever the objective of the primal problem be."([41], p.57). In these approaches, the dual variables are also linear continuous operators $T \in \mathcal{L}(X, Z)$ and for many results the assumption $\operatorname{int} C \neq \emptyset$ is necessary. In [48, 54, 39] it can be observed how the difference in the definition of set-valued conjugate of a function in fact causes a change of image-spaces from infimum-oriented sets to supremum-oriented sets. This has been avoided in [23], but at the cost of the convexity of the conjugate function. Our definition in fact is a variation of the one given in [23]. By exploiting the possibilities of the z^*-difference introduced in subsection 2.4 we are able to define a convex set-valued conjugate of a set-valued function $F : X \to \mathcal{Q}_C^t(Z)$, the conjugate mapping $X^* \times C^* \setminus \{0\}$ into $\mathcal{Q}_C^t(Z)$. Likewise, the definition of the biconjugate will be inspired by (4.0.6). In sequence, we will prove a Fenchel-Moreau-Theorem, a sum- and chain rule, weak duality and, under an additional constrained assumption strong duality and a sandwich theorem.

Throughout this chapter, X, Y and Z are assumed to be locally convex separable spaces with the corresponding dual spaces X^*, Y^* and Z^* and Z is quasi-ordered by a closed convex cone $C \subsetneq Z$ with $\{0\} \subsetneq C$.

4.1 Definition and basic results

Let $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ and $-z \in Z$ generate an affine minorant of $F : X \to \mathcal{Q}_C^t(Z)$, that is

$$\forall x \in X : \quad F(x) + z \subseteq S_{(x^*,z^*)}(x).$$

Then

$$\bigcap_{x \in X} \left(S_{(x^*,z^*)}(x) \lhd_{z^*} F(x) \right) \supseteq z + H(z^*).$$

Note that both sides of the inclusion are nonempty elements of $Q_{H(z^*)}^t(Z)$. Hence, $S_{(x^*,z^*)} - z : X \to \mathcal{Q}_C^t(Z)$ is an affine minorant of $F : X \to \mathcal{Q}_C^t(Z)$ if and only if

$$z \in \bigcap_{x \in X} \left(S_{(x^*,z^*)}(x) \lhd_{z^*} F(x) \right).$$

4.1.1 Definition. *Let $F : X \to \mathcal{P}(Z)$ be a function.*

a) The conjugate $F^* : X^* \times C^* \setminus \{0\} \to \mathcal{Q}_C^t(Z)$ of F is defined by

$$F^*(x^*, z^*) := \bigcap_{x \in X} \Big(S_{(x^*,z^*)}(x) \lhd_{z^*} F(x) \Big).$$

for all $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$.

b) The convex biconjugate $F^{**} : X \to \mathcal{Q}_C^t(Z)$ is defined by

$$F^{**}(x) := \bigcap_{(x^*,z^*) \in X^* \times C^* \setminus \{0\}} \Big(S_{(x^*,z^*)}(x) \lhd_{z^*} F^*(x^*, z^*) \Big).$$

for all $x \in X$.

For convenience, we will abbreviate the conjugate and the biconjugate of the scalarizations to $\varphi^*_{(F,z^*)} : X^* \to \mathrm{I\!R} \cup \{\pm\infty\}$ and $\varphi^{**}_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ for all $z^* \in C^* \setminus \{0\}$.

4.1.2 Lemma. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a function. It holds*

$$\forall x^* \in X^*, z^* \in C^* \setminus \{0\} : \quad F^*(x^*, z^*) = \Big\{ z \in Z | \ -z^*(z) \geq \varphi^*_{(F,z^*)}(x^*) \Big\}$$

and

$$\forall x^* \in X^*, z^* \in C^* \setminus \{0\} : \quad \varphi^*_{(F,z^*)}(x^*) = \inf \{ -z^*(z) | \ z \in F^*(x^*, z^*) \}.$$

Proof. By 3.2.1c and 3.1.3 it holds $\varphi_{((S_{(x^*,z^*)} \lhd_{z^*} F), z^*)}(x) = x^*(x) \lhd \varphi_{(F,z^*)}(x)$, so

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\} : \quad F^*(x^*, z^*) = \Big\{ z \in Z | \ -z^*(z) \geq \varphi^*_{(F,z^*)}(x^*) \Big\} \tag{4.1.1}$$

holds, as by definition 8.3.7 $\varphi^*_{(F,z^*)}(x^*) = \sup\limits_{x \in X}(x^*(x) \lhd \varphi_{(F,z^*)}(x))$ holds true. Moreover, from (4.1.1) it is immediate that

$$\begin{aligned} \forall x^* \in X^*, z^* \in C^* \setminus \{0\} : \quad & \inf \{ -z^*(z) | \ z \in F^*(x^*, z^*) \} \\ & = \inf \Big\{ -z^*(z) | \ -z^*(z) \geq \varphi^*_{(F,z^*)}(x^*) \Big\} \\ & = \varphi^*_{(F,z^*)}(x^*). \end{aligned}$$

■

4.1.3 Proposition. *Let $F : X \to \mathcal{Q}_C^t(Z)$, $z_0^* \in C^* \setminus \{0\}$ and $z^* \in C^* \setminus \mathrm{cl\,cone}\, \{z_0^*\}$.*

a) For all $x^* \in X^*$ it holds

$$\inf \{ -z^*(z) | \ z \in F^*(x^*, z_0^*) \} = \begin{cases} +\infty, & \text{if } F^*(x^*, z_0^*) = \emptyset; \\ -\infty, & \text{else.} \end{cases}$$

If $z^* = t z_0^*$ holds for $t > 0$, then $\inf \{ -z^*(z) | \ z \in F^*(x^*, z_0^*) \} = t \varphi^*_{(F,z_0^*)}(x^*)$ holds for all $x^* \in X^*$.

b) For the conjugate of the z^*-hull of F it holds

$$(F \lhd_{z^*} \{0\})^*(x^*, z_0^*) = \begin{cases} F^*(x^*, z_0^*), & \text{if } z^* \in \mathrm{cone}\, \{z_0^*\}; \\ Z, & \text{if } \mathrm{dom}\, F = \emptyset; \\ \emptyset, & \text{else.} \end{cases}$$

Proof.

a) By 4.1.1 and 4.1.2,

$$F^*(x^*, z_0^*) = \bigcap_{x\in X} \left\{z \in Z|\; F(x) + z \subseteq S_{(x^*,z_0^*)}(x)\right\}$$
$$= \left\{z \in Z|\; -z_0^*(z) \geq \varphi^*_{(F,z_0^*)}(x^*)\right\} \in Q^t_{H(z_0^*)}(Z)$$

holds. If $F^*(x^*, z_0^*) = \emptyset$, then

$$\forall z^* \in C^* \setminus \{0\} : \quad \inf \{-z^*(z)|\; z \in F^*(x^*, z_0^*)\} = +\infty.$$

If $\emptyset \neq F^*(x^*, z_0^*) \subseteq (S_{(x^*,z^*)}(x) \lhd_{z^*} F(x)) \in Q^t_{H(z_0^*)}(Z)$ holds, then

$$\forall z^* \in C^* \setminus \{0\} : \quad \inf \{-z^*(z)|\; z \in F^*(x^*, z_0^*)\}$$
$$= \inf \left\{-z^*(z)|\; -z_0^*(z) \geq \varphi^*_{(F,z_0^*)}(x^*)\right\} = \begin{cases} t\varphi^*_{(F,z_0^*)}(x^*), & \text{if } z^* = tz_0^*,\ t > 0; \\ -\infty, & \text{else.} \end{cases}$$

b) By 2.4.3 it holds

$$\forall x \in X : \quad (F \lhd_{z^*} \{0\})(x) = F(x) \oplus H(z^*).$$

For all $z^* \in C^* \setminus \{0\}$, $t > 0$ it holds $H(tz^*) = H(z^*)$ and thus

$$\forall x \in X : \quad (F \lhd_{z^*} \{0\})(x) = (F \lhd_{tz^*} \{0\})(x).$$

Thus for $x^* \in X^*$, $z^*, z_0^* \in C^* \setminus \{0\}$ and $t > 0$ it holds

$$(F \lhd_{z^*} \{0\})^*(x^*, z_0^*) = (F \lhd_{tz^*} \{0\})(x^*, z_0^*).$$

Again by 2.4.3 it holds

$$(F \lhd_{z^*} \{0\})^*(x^*, z_0^*) = \bigcap_{x\in X} \left\{z \in Z|\; (F(x) \oplus H(z^*) \oplus H(z_0^*)) + z \subseteq S_{(x^*,z_0^*)}(x)\right\}$$
$$= \begin{cases} F^*(x^*, z_0^*), & \text{if } z^* \in \operatorname{cone}\{z_0^*\}; \\ Z, & \text{if } \operatorname{dom} F = \emptyset; \\ \emptyset, & \text{else.} \end{cases}$$

■

4.1.4 Remark. *By the formula*

$$\forall x^* \in X^*,\ \forall z^* \in C^* \setminus \{0\},\ \forall t > 0 : \quad (F \lhd_{tz^*} \{0\})^*(x^*, z^*) = F^*(x^*, z^*),$$

it is tempting to try to define a conjugate of the z^*-hull of $F : X \to \mathcal{Q}^t_C(Z)$, mapping X^* to $Q^t_{H(z^*)}(Z)$ by

$$\forall x^* \in X^*,\ \forall z^* \in C^* \setminus \{0\} : \quad (F \lhd_{z^*} \{0\})^*(x^*) = F^*(x^*, z^*).$$

This mapping would not be well defined, as $(F \lhd_{z^*} \{0\})(x) = (F \lhd_{tz^*} \{0\})(x)$ holds for all $t > 0$ and $x \in X$, but in general

$$(F^*(x^*, z^*) \neq F^*(x^*, tz^*).$$

The same problem arises if the conjugate of $F \lhd_{z^*} \{0\}$ is defined as a mapping from $X^* \times \mathbb{R}_+$ to $Q^t_{H(z^*)}(Z)$ by

$$\forall x^* \in X^*,\ \forall z^* \in C^* \setminus \{0\},\ \forall t > 0 : \quad (F \lhd_{z^*} \{0\})^*(x^*, t) = F^*(x^*, tz^*).$$

4.1.5 Lemma. *Let $F : X \to \mathcal{P}(Z)$ be a function and $z^* \in C^* \setminus \{0\}$. It holds*

$$\forall x \in X : \quad F^{**}(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z | \ - z^*(z) \geq \varphi^{**}_{(F,z^*)}(x) \right\}$$

and $\varphi^{**}_{(F,z^*)}(x^*) \leq \varphi_{(F^{**},z^*)}(x)$.

Proof. By 2.4.3, 3.1.3 and 4.1.2 it holds

$$(S_{(x^*,z^*)}(x) \lhd_{z^*} F^*(x^*, z^*)) = \left\{ z \in Z | \ - z^*(z) \geq (x^*(x) \lhd \varphi^*_{(F,z^*)}(x^*)) \right\}$$

for all $x^* \in X^*$ and all $z^* \in C^* \setminus \{0\}$. Therefore, by 8.3.7

$$\begin{aligned} \forall x \in X : \quad F^{**}(x) &= \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z | \ \forall x^* \in X^* : -z^*(z) \geq (x^*(x) \lhd \varphi^*_{(F,z^*)}(x^*)) \right\} \\ &= \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z | \ - z^*(z) \geq \varphi^{**}_{(F,z^*)}(x) \right\}. \end{aligned}$$

From this formula, $\inf\limits_{z \in F^{**}(x)} -z^*(z) \geq \varphi^{**}_{(F,z^*)}(x)$ follows, hence $\varphi^{**}_{(F,z^*)}(x) \leq \varphi_{(F^{**},z^*)}(x)$. ■

The functions $\varphi^{**}_{(F,z^*)}(x)$ and $\varphi_{(F^{**},z^*)}(x)$ are not necessarily equal as the following example shows.

4.1.6 Example. *Let $F : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2), C = \mathbb{R}^2_+$ be a function defined by*

$$\forall x > 0 : \quad F(x) := \left\{ (\frac{1}{x}, 0) \right\} + C$$

and $F(x) = \emptyset$ for $x \leq 0$. Let $z^* = (0, -1) \in C^* \setminus -C^*$, then $F = F^{**}$ and $\varphi_{(F,z^*)}(0) = +\infty$ while $\varphi^{**}_{(F,z^*)}(0) = 0$.

A function $F : X \to \mathcal{P}(Z)$ has the same conjugate (and therefore biconjugate) as the function $\bar{F} : X \to \mathcal{Q}^t_C(Z)$ defined by $\bar{F}(x) := \operatorname{cl} \operatorname{co} (F(x) + C)$. Therefore it is no restriction to start with functions mapping into $\mathcal{Q}^t_C(Z)$.

One is tempted to define

$$\tilde{F}^*(x^*) = \bigcap_{z^* \in C^* \setminus \{0\}} F^*(x^*, z^*),$$

even more so as it holds

$$\tilde{F}^*(x^*) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z | \ - z^*(z) \geq \varphi^*_{(F,z^*)}(x^*) \right\}.$$

However, this definition would leave us with a poorer theory concerning the conjugate of a set-valued function.

4.1.7 Example. *Let $F : X \to Q^t_{\operatorname{cone}\{(1,0)\}}(\mathbb{R}^2)$ be a function defined by $F \equiv \mathbb{R}^2_+$ and $z^*_0 = (-1, 1) \in C^* \setminus \{0\}$. Then*

$$(F \lhd_{z^*_0} \{0\}) \equiv Z$$

and hence $\tilde{F}^*(x^*) \subseteq F^*(x^*, z^*_0) = \emptyset$ for all $x^* \in X^*$. On the other hand $F^*(x^*, z^*) = H(z^*)$ for all $(x^*, z^*) \in X^* \times (R^2_+)^* \setminus \{0\}$.

In [23], the negative set-valued conjugate has been defined, avoiding the substraction of functions in the definition of F^* and F^{**}. For a function $F : X \to \mathcal{Q}^t_C(Z)$, the negative conjugate and the biconjugate are defined by

$$
\begin{aligned}
-F^*(x^*, z^*) &:= \operatorname{cl} \bigcup_{x\in X} \Big(F(x) + S_{(x^*,z^*)}(-x)\Big)\\
\tilde{F}^{**}(x) &:= \bigcap_{(x^*,z^*)\in X^*\times C^*\setminus\{0\}} \Big(-F^*(x^*, z^*) + S_{(x^*,z^*)}(x)\Big).
\end{aligned}
$$

for all $x^* \in X^*$, $z^* \in C^* \setminus \{0\}$ and $x \in X$. It turns out that $F^*(x^*, z^*) = (\{0\} \lhd_{z^*} -F^*(x^*, z^*))$, while $F^{**}(x) = \tilde{F}^{**}(x)$ holds for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ and $x \in X$.

4.1.8 Proposition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function, $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ and $x \in X$, then it holds*

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\} : \quad F^*(x^*, z^*) = (\{0\} \lhd_{z^*} -F^*(x^*, z^*))$$

and

$$\forall x \in X : \quad \tilde{F}^{**}(x) = F^{**}(x).$$

Proof. It holds

$$-F^*(x^*, z^*) = \operatorname{cl} \bigcup_{x\in X} \operatorname{cl} \Big(F(x) + S_{(x^*,z^*)}(-x)\Big)$$

and by 3.2.1 b) and 3.1.3

$$\operatorname{cl}\left(F(x) + S_{(x^*,z^*)}(-x)\right) = \Big\{z \in Z |\ -z^*(z) \geq \varphi_{(F,z^*)}(x) + x^*(-x)\Big\}$$

for every $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$. Thus,

$$-F^*(x^*, z^*) = \Big\{z \in Z |\ -z^*(z) \geq \inf_{x\in X} (\varphi_{(F,z^*)}(x) + x^*(-x))\Big\}$$

holds for all $x^* \in X^*$, $z^* \in C^* \setminus \{0\}$. By 2.4.2,

$$\{0\} \lhd_{z^*} -F^*(x^*, z^*) = \Big\{z \in Z |\ -z^*(z) + \inf_{x\in X} (\varphi_{(F,z^*)}(x) + x^*(-x)) \geq 0\Big\}$$

holds and therefore

$$
\begin{aligned}
\{0\} \lhd_{z^*} -F^*(x^*, z^*) &= \bigcap_{x\in X} \Big\{z \in Z |\ -z^*(z) + \varphi_{(F,z^*)}(x) + x^*(-x) \geq 0\Big\}\\
&= \bigcap_{x\in X} \Big\{z \in Z |\ -z^*(z) + \varphi_{(F,z^*)}(x) \geq x^*(x)\Big\}\\
&= \bigcap_{x\in X} \Big\{z \in Z |\ -z^*(z) \geq (x^*(x) \lhd \varphi_{(F,z^*)}(x))\Big\}\\
&= \Big\{z \in Z |\ -z^*(z) \geq \varphi^*_{(F,z^*)}(x^*)\Big\}\\
&= F^*(x^*, z^*).
\end{aligned}
$$

For all $x \in X$ it holds

$$F^{**}(x) = \Big\{z \in Z |\ \forall (x^*, z^*) \in X^* \times C^* \setminus \{0\} : -z^*(z) \geq (x^*(x) \lhd \varphi^*_{(F,z^*)}(x^*))\Big\}$$

As x^* maps into $\mathbb{R}$, it holds

$$\forall x \in X: \quad (x^*(x) \lhd \varphi^*_{(F,z^*)}(x^*)) = x^*(x) + \inf_{y \in X} (\varphi_{(F,z^*)}(x) + x^*(-y))$$

and thus it holds $F^{**}(x) = \tilde{F}^{**}(x)$ for all $x \in X$. ■

From now on, the conjugate and biconjugate of a function $F : X \to \mathcal{Q}^t_C(Z)$ will be defined as in 4.1.1.

4.1.9 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function.*

a) The function $F^*_{z^*} : X^* \to \mathcal{Q}^t_C(Z)$ defined as $F^*_{z^*}(x^*) = F^*(x^*, z^*)$ for all $x^* \in X^*$ is convex for all $z^* \in C^* \setminus \{0\}$.

b) The function $F^{**} : X \to \mathcal{Q}^t_C(Z)$ is convex and closed.

c) For all $x \in X$ it holds $F^{**}(x) \supseteq (\operatorname{cl} \operatorname{co} F)(x)$.

d) The Young-Fenchel inequality holds:

$$\forall x \in X, (x^*, z^*) \in X^* \times (C^* \setminus \{0\}) : \ F^*(x^*, z^*) \subseteq \Big(S_{(x^*,z^*)}(x) \lhd_{z^*} F(x)\Big)$$

Moreover, $(S_{(x^*,z^*)} - z) : X \to \mathcal{Q}^t_C(Z)$ with $x^* \in X^*$, $z^* \in C^* \setminus \{0\}$ and $z \in Z$ is a conaffine minorant of F if and only if $z \in F^*(x^*, z^*)$ and

$$\forall x \in X: \quad F^*(x^*, z^*) \oplus F(x) \subseteq S_{(x^*,z^*)}(x)$$

e) If for all $x \in X$ it holds $F_1(x) \supseteq F_2(x)$, then

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\} : \quad F^*_1(x^*, z^*) \subseteq F^*_2(x^*, z^*)$$

and

$$\forall x \in X: \quad F^{**}_1(x) \supseteq F^{**}_2(x).$$

f) For all $(x^*, z^*) \in X^* \times (C^* \setminus \{0\})$ it holds $F^*(x^*, z^*) = (\operatorname{cl} \operatorname{co} F)^*(x^*, z^*)$.

g) For all $x \in X$, $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ it holds $(F^{**})^*(x^*, z^*) = F^*(x^*, z^*)$.

Proof.

a) The function $\varphi^*_{(F,z^*)}(x^*)$ is convex, 3.1.5, therefore $F^*_{z^*} : X^* \to \mathcal{Q}^t_C(Z)$ is convex.

b) The biconjugate $\varphi^{**}_{(F,z^*)}$ is convex and closed for any $z^* \in C^* \setminus \{0\}$ and by 4.1.5 it holds $\varphi^{**}_{(F,z^*)} \le \varphi_{(F^{**},z^*)}$, so $\varphi^{**}_{(F,z^*)} \le \operatorname{cl} \operatorname{co} \varphi_{(F^{**},z^*)}$ holds true. By 3.2.3 and 3.3.5 it holds

$$\forall x \in X: \ (\operatorname{cl} \operatorname{co} F^{**})(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \Big\{ z \in Z | \ -z^*(z) \ge \operatorname{cl} \operatorname{co} \varphi_{(F^{**},z^*)}(x) \Big\} \subseteq F^{**}(x).$$

Therefore, F^{**} is convex and closed.

c) It holds $\varphi^{**}_{(F,z^*)} \le \varphi_{(F,z^*)}$, therefore

$$\forall x \in X: \ F^{**}(x) \supseteq \bigcap_{z^* \in C^* \setminus \{0\}} \Big\{ z \in Z | \ -z^*(z) \ge \varphi_{(F,z^*)}(x) \Big\} = F(x).$$

As F^{**} is closed and convex, $F^{**}(x) \supseteq (\operatorname{cl} \operatorname{co} F)(x)$ holds for all $x \in X$.

d) The first inequality holds by definition. The function $S_{(x^*,z^*)} - z$ is a conaffine minorant of F if and only if $-z^*(z) \geq \varphi^*_{(F,z^*)}(x^*)$, that is $z \in F^*(x^*, z^*)$. Also, $S_{(x^*,z^*)} - z$ is a conaffine minorant of F if and only if $S_{(x^*,z^*)}$ is a conlinear minorant of $F + z$, thus the second inclusion holds.

e) If for all $x \in X$ it holds $F_1(x) \supseteq F_2(x)$, then for all $z^* \in C^* \setminus \{0\}$ it holds $\varphi^*_{(F_1,z^*)}(x^*) \geq \varphi^*_{(F_2,z^*)}(x^*)$ for all $x^* \in X^*$ and $\varphi^{**}_{(F_1,z^*)}(x) \leq \varphi^{**}_{(F_2,z^*)}(x)$ for all $x \in X$, giving the desired result.

f) It holds

$$\forall z^* \in C^* \setminus \{0\}\,,\ x^* \in X^* : \quad \varphi^*_{(F,z^*)}(x^*) = (\operatorname{cl}\operatorname{co}\varphi_{(F,z^*)})^*(x^*).$$

Moreover,

$$\begin{aligned} \forall z^* \in C^* \setminus \{0\}\,,\ x \in X : \quad \operatorname{cl}\operatorname{co}\varphi_{(F,z^*)}(x) &\leq \varphi_{(\operatorname{cl}\operatorname{co}F,z^*)}(x), \\ \varphi_{(\operatorname{cl}\operatorname{co}F,z^*)}(x) &\leq \varphi_{(F,z^*)}(x) \end{aligned}$$

holds by 3.2.3 and 3.3.4. It holds

$$\forall z^* \in C^* \setminus \{0\}\,,\ x^* \in X^* : \quad \varphi^*_{(\operatorname{cl}\operatorname{co}F,z^*)}(x^*) = \varphi^*_{(F,z^*)}(x^*).$$

Combined with 4.1.2, this is the desired result.

g) It holds

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\} : \quad (\varphi^{**}_{(F,z^*)})^*(x^*) = \varphi^*_{(F,z^*)}(x^*)$$

and therefore $\varphi^*_{(F^{**},z^*)}(x^*) = \varphi^*_{(F,z^*)}(x^*)$. Applying 4.1.2, the claim is proven.

■

Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function, then $\operatorname{cl}\operatorname{co}F : X \to \mathcal{Q}^t_C(Z)$ is proper if and only if there is a conaffine minorant $S_{(x^*,z^*)} - z : X \to \mathcal{Q}^t_C(Z)$ of F with $(x^*, z^*) \in C^* \setminus \{0\}$ and $z \in Z$. Such a minorant exists on the other hand if and only if $F^*(x^*, z^*) \supseteq z + H(z^*)$, so

$$z \in (F^*(x^*, z^*) \lhd_{z^*} \{0\}) = F^*(x^*, z^*)$$

and $(x^*, z^*) \in \operatorname{dom} F^*$.

4.1.10 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function.*

a) It holds $\operatorname{dom} F = \emptyset$ *if and only if* $F^* \equiv Z$*, if and only if* $F^*(x^*_0, z^*_0) = Z$ *for some* $(x^*_0, z^*_0) \in X^* \times C^* \setminus \{0\}$.

b) It holds $\operatorname{dom} F^*(\cdot, z^*) = \emptyset$ *if and only if* $(\operatorname{cl}\operatorname{co}(F \lhd_{z^*} \{0\}))(x_0) = Z$ *for some* $x_0 \in X$.

c) There is $x_0 \in X$ *with* $(\operatorname{cl}\operatorname{co}F)(x_0) = Z$ *if and only if* $\operatorname{dom} F^* = \emptyset$.

d) $\operatorname{cl}\operatorname{co}F : X \to \mathcal{Q}^t_C(Z)$ *is proper if and only if* $F^* : X^* \times C^* \setminus \{0\} \to \mathcal{Q}^t_C(Z)$ *is proper.*

e) $\operatorname{cl}\operatorname{co}F : X \to \mathcal{Q}^t_C(Z)$ *is proper but not C-proper if and only if* $\operatorname{cl}\operatorname{co}F : X \to \mathcal{Q}^t_C(Z)$ *is proper and* $\operatorname{dom} F^* \subseteq X^* \times (C^* \setminus -C^*)$.

Proof.

a) It holds $\operatorname{dom} F = \emptyset$ if and only if $\operatorname{dom} \varphi_{(F,z^*)} = \emptyset$ holds for all (for one) $z^* \in C^* \setminus \{0\}$. This is equivalent to $\varphi^*_{(F,z^*)} \equiv -\infty$ for all (for one) $z^* \in C^* \setminus \{0\}$.

b) It holds $F^*(x^*, z^*) = \emptyset$ if and only if $\varphi^*_{(F,z^*)}(x^*) = +\infty$. This is equivalent to $\operatorname{cl} \operatorname{co} \varphi_{(F,z^*)}(x_0) = -\infty$, which is true if and only if $(\operatorname{cl} \operatorname{co}(F \triangleleft_{z^*} \{0\}))(x_0) = Z$.

c) It holds $\operatorname{dom} F^* = \emptyset$ if and only if $F : X \to \mathcal{Q}^t_C(Z)$ has no conaffine minorant. This is the case if and only if $(\operatorname{cl} \operatorname{co} F)(x_0) = Z$.

d) Direct conclusion of the above.

e) The function $\operatorname{cl} \operatorname{co} F$ is not C-proper if and only if F has no C-proper conaffine minorant, that is $F^*(x^*, z^*) = \emptyset$ for all $z^* \in C^* \cap -C^*$. ■

Recall that in general $(\operatorname{cl} \operatorname{co}(F \triangleleft_{z^*} \{0\}))(x) \supsetneq (\operatorname{cl} \operatorname{co} F \triangleleft_{z^*} \{0\})(x)$, hence $\operatorname{cl} \operatorname{co} F : X \to \mathcal{Q}^t_C(Z)$ is z^*-proper if $F^*(x^*, z^*) \neq \emptyset$.

4.1.11 Definition. *The function*

$$\Sigma(\cdot|\operatorname{epi} F) : (X^* \times C^* \setminus \{0\}) \times C^* \setminus \{0\} \to \mathcal{Q}^t_C(Z),$$

defined by

$$\Sigma(((x^*, z_1^*), z_2^*)|\operatorname{epi} F) := \bigcap_{(x,z) \in \operatorname{epi} F} S_{((x^*,z_1^*),z_2^*)}(x, z)$$

for all $x^* \in X^*$ *and* $z_1^*, z_2^* \in C^* \setminus \{0\}$. *is called the set-valued support function of* epi F, *mapping* $(X^* \times C^* \setminus \{0\}) \times C^* \setminus \{0\}$ *into* $\mathcal{Q}^t_C(Z)$.

The definition of $\Sigma(\cdot|\operatorname{epi} F) : (X^* \times C^* \setminus \{0\}) \times C^* \setminus \{0\} \to \mathcal{Q}^t_C(Z)$ is in analogy to the scalar support function of the epigraph of a function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$, which is defined by

$$\sigma((x^*, t)|\operatorname{epi} \varphi) = \sup_{(x,r) \in \operatorname{epi} \varphi} (x^*, t)(x, r)$$

for all $x^* \in X^*$ and $t \in \mathbb{R}$.

For all $x^* \in X^*$ and $z_1^*, z_2^* \in C^* \setminus \{0\}$ and $(x, z) \in X \times Z$ it holds

$$S_{((x^*,z_1^*),z_2^*)}(x, z) = \{y \in Z |\ -z_2^*(y) \geq x^*(x) + z_1^*(z)\} \in Q^t_{H(z_2^*)}(Z).$$

Thus, $S_{((x^*,z_1^*),z_2^*)}(x, z) \subseteq H(z_2^*)$ holds if and only if $z \in S_{(x^*,z_1^*)}(x)$. Moreover, $S_{((x^*,z_1^*),z_2^*)}(x, z) \supseteq \bar{z} + H(z_2^*)$ holds if and only if $-z_2^*(\bar{z}) \geq x^*(x) + z_1^*(z)$.

4.1.12 Proposition. *Let* $F : X \to \mathcal{Q}^t_C(Z)$ *be a function,* $x^* \in X^*$ *and* $z_1^*, z_2^* \in C^* \setminus \{0\}$. *It holds* $\Sigma(((x^*, z_1^*), z_2^*)|\operatorname{epi} F) \in Q^t_{H(z_2^*)}$ *and with* $-z_2^*(e) = 1$ *it holds*

$$\Sigma(((x^*, z_1^*), z_2^*)|\operatorname{epi} F) = \begin{cases} \emptyset, & \text{if } F^*(x^*, z_1^*) = \emptyset; \\ Z, & \text{if } F^*(x^*, z_1^*) = Z; \\ \varphi^*_{(F,z_1^*)}(x^*)e + H(z_2^*), & \text{else.} \end{cases}$$

Proof. By definition, it holds

$$\begin{aligned} F^*(x^*, z_1^*) &= \left\{\bar{z} \in Z |\ \operatorname{epi} S_{(x^*,z_1^*)} \supseteq \operatorname{epi}(F + \bar{z})\right\} \\ &= \{\bar{z} \in Z |\ \forall (x, z) \in \operatorname{epi} F : \ x^*(x) + z_1^*(z) \leq -z_1^*(\bar{z})\}. \end{aligned}$$

It holds $F^*(x^*, z_1^*) = Z$ if and only if $\operatorname{epi} F = \emptyset$. In this case it holds $\Sigma(\cdot|\operatorname{epi} F) \equiv Z$.

If $F^*(x^*, z_1^*) = \emptyset$ holds, then for all $t \in \mathbb{R}$ there is $(x, z) \in \operatorname{epi} F$ such, that $x^*(x) + z_1^*(z) > t$, thus $\Sigma(((x^*, z_1^*), \cdot)|\operatorname{epi} F) \equiv \emptyset$.

The set $F^*(x^*, z_1^*)$ is neither empty nor Z if and only if $\varphi^*_{(x^*, z_1^*)} \in \mathbb{R}$. In this case,

$$\sup\{x^*(x) + z_1^*(z) |\ (x, z) \in \operatorname{epi} F\} = \varphi^*_{(F, z_1^*)}(x^*) \in \mathbb{R}$$

and with $-z_2^*(e) = 1$ it holds

$$\Sigma(((x^*, z_1^*), z_2^*)|\operatorname{epi} F) \supseteq \varphi^*_{(F, z_1^*)}(x^*)e + H(z^*).$$

For $\bar{z} \in \Sigma(((x^*, z_1^*), z_2^*)|\operatorname{epi} F) \setminus (\varphi^*_{(F, z_1^*)}(x^*)e + H(z^*))$ it holds $-z^*(\bar{z}) < \varphi^*_{(F, z_1^*)}$ and

$$\forall (x, z) \in \operatorname{epi} F: \quad x^*(x) + z_1^*(z) \le -z_2^*(\bar{z}),$$

a contradiction. ■

The image of $\Sigma(((x^*, z_1^*), z_2^*)|\operatorname{epi} F)$ is a translation of $F^*(x^*, z_1^*)$ into the space $Q^t_{H(z_2^*)}(Z)$.

4.1.13 Proposition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function, $x^* \in X^*$ and $z_1^*, z_2^* \in C^* \setminus \{0\}$, then*

$$\bigcap_{(x,z) \in \operatorname{epi} F} S_{(x^*, z_1^*, z_2^*)}(x, z) = I^*_{\operatorname{epi} F}(x^*, z_1^*, z_2^*),$$

$$F^*(x^*, z^*) = I^*_{\operatorname{epi} F}(x^*, z^*, z^*).$$

Proof. For all $x^* \in X^*$ and $z_1^*, z_2^* \in C^* \setminus \{0\}$ it holds

$$\begin{aligned} I^*_{\operatorname{epi} F}((x^*, z_1^*), z_2^*) &= \bigcap_{(x,z) \in X \times Z} \left[S_{((x^*, z_1^*), z_2^*)}(x, z) \lhd_{z^*} I_{\operatorname{epi} F}(x, z)\right] \\ &= \bigcap_{(x,z) \in \operatorname{epi} F} S_{((x^*, z_1^*), z_2^*)}(x, z). \end{aligned}$$

If $-z^*(e) = 1$ holds for $z^* \in C^* \setminus \{0\}$, then $F^*(x^*, z^*) = \varphi^*_{(F, z^*)}(x^*)e + H(z^*)$ holds by 4.1.2, if $F^*(x^*, z^*)$ is a proper set. Thus, the claim holds by 4.1.12. ■

4.1.14 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function.*

a) $\operatorname{dom} F = \emptyset$ *if and only if* $\operatorname{dom} F^{**} = \emptyset$.

b) There is x_0 with $(\operatorname{cl}\operatorname{co} F)(x_0) = Z$ if and only if $F^{**} \equiv Z$.

c) $\operatorname{cl}\operatorname{co} F$ *is proper if and only if $F^{**} : X \to \mathcal{Q}^t_C(Z)$ is proper.*

Proof.

a) For all $z^* \in C^* \setminus \{0\}$ it holds $\operatorname{dom} F = \operatorname{dom} \varphi_{(F, z^*)}$. Therefore, $\operatorname{dom} F = \emptyset$ is equivalent to

$$\forall z^* \in C^* \setminus \{0\}\, x \in X: \ \varphi^{**}_{(F, z^*)}(x) = +\infty,$$

therefore it is equivalent to $\operatorname{dom} F^{**} = \emptyset$.

b) $(\operatorname{cl}\operatorname{co} F)(x_0) = Z$ if and only if for all $z^* \in C^* \setminus \{0\}$ it holds $\operatorname{cl}\operatorname{co} \varphi_{(F, z^*)} = -\infty$. This is equivalent to

$$\forall z^* \in C^* \setminus \{0\}, x \in X: \ \varphi^{**}_{(F, z^*)}(x) = -\infty,$$

which again is equivalent to F^{**} being identically Z.

c) From 4.1.9, $F^{**}(x) \supseteq (\operatorname{cl}\operatorname{co} F)(x)$. With the above, this is the statement. ■

4.1.15 Theorem (Fenchel-Moreau-Theorem)**.** *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function.*

a) It holds

$$\forall x \in X : \quad F^{**}(x) \supseteq (\operatorname{cl}\operatorname{co} F)(x).$$

b) It holds

$$\forall x \in X : \quad F^{**}(x) = (\operatorname{cl}\operatorname{co} F)(x)$$

if and only if $\operatorname{cl}\operatorname{co} F$ is proper or identically Z or $\emptyset$.

c) If F is convex and $x_0 \in \operatorname{dom} F$ such that $F(x_0) = (\operatorname{cl} F)(x_0)$, then $F^{**}(x_0) = F(x_0)$. If additionally $F(x_0) \neq Z$, then $F^{**} = \operatorname{cl} F$ and F^{**} is proper.

Proof.

a) See 4.1.9.

b) By 4.1.14, $F^{**} = \operatorname{cl}\operatorname{co} F$ if $\operatorname{cl}\operatorname{co} F$ is identically $\emptyset$ or Z. Furthermore, F^{**} is proper if and only if $\operatorname{cl}\operatorname{co} F$ is proper. Let $\operatorname{cl}\operatorname{co} F$ be proper. In this case, by 3.3.11 it holds

$$\begin{aligned}
\forall x \in X : \; (\operatorname{cl}\operatorname{co} F)(x) &= \bigcap_{\substack{z^* \in C^* \setminus \{0\}, \\ \operatorname{cl}\operatorname{co} \varphi_{(F,z^*)} \text{ is proper}}} \left\{ z \in Z |\; z^*(z) \geq \operatorname{cl}\operatorname{co} \varphi_{(F,z^*)}(x) \right\} \\
&= \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z |\; z^*(z) \geq \varphi^{**}_{(F,z^*)}(x) \right\} \\
&= F^{**}(x).
\end{aligned}$$

If for $x_0 \in X$ it holds $(\operatorname{cl}\operatorname{co} F)(x_0) = Z$, then $\varphi^{**}_{(F,z^*)}$ is identically $-\infty$, therefore $F^{**} \equiv Z$ and $(\operatorname{cl}\operatorname{co} F)(x) = F^{**}(x)$ holds for all $x \in X$ if and only if $F \equiv Z$.

c) If $F(x_0) = (\operatorname{cl} F)(x_0) = Z$, then F^{**} is identically Z, especially $F^{**}(x_0) = F(x_0) = Z$. Let $(\operatorname{cl} F)(x_0) \neq Z$. Then, by 3.3.5 it exists $z^* \in C^* \setminus \{0\}$ such that $\operatorname{cl} \varphi_{(F,z^*)}(x_0) \in \mathbb{R}$. Therefore, $\operatorname{cl} \varphi_{(F,z^*)}$ is proper and $\operatorname{cl} F$ is proper. Thus, $(\operatorname{cl} F)(x) = F^{**}(x)$ holds for all $x \in X$, especially $F^{**}(x_0) = F(x_0) \neq Z$ and F^{**} is proper. ■

4.1.16 Remark. *If $x \in \operatorname{dom} F$ or $\operatorname{dom} F = \emptyset$, then $\varphi^{**}_{(F,z^*)}(x) = \operatorname{cl} \varphi_{(F^{**},z^*)}(x)$.*

Proof. In general, $\varphi^{**}_{(F,z^*)}(x) \leq \operatorname{cl} \varphi_{(F^{**},z^*)}(x)$ and $\operatorname{cl} \varphi_{(F^{**},z^*)}(x) \leq \operatorname{cl}\operatorname{co} \varphi_{(F,z^*)}(x)$ holds for all $x \in X$. If $x \in \operatorname{dom} F$ or $\operatorname{dom} F = \emptyset$, then $\varphi^{**}_{(F,z^*)}(x) = \operatorname{cl}\operatorname{co} \varphi_{(F,z^*)}(x)$, therefore, $\varphi^{**}_{(F,z^*)}(x) = \operatorname{cl} \varphi_{(F^{**},z^*)}(x)$ holds true. ■

4.2 Calculus

4.2.1 Proposition. *Let $G : Y \to Q^t_C(Z)$, $t > 0$ and $A : X \to Y$ be a linear homeomorphism, $y_0 \in Y$, $z_0 \in Z$ and $(x_0^*, z_0^*) \in X^* \times C^* \setminus \{0\}$ and $F : X \to Q^t_C(Z)$ is defined by*

$$F(x) = tG(Ax + y_0) + S_{(x_0^*, z_0^*)}(x) + z_0$$

for all $x \in X$.

a) If $\operatorname{dom} G = \emptyset$ holds, then $F^* \equiv Z$.

b) If $\operatorname{dom} G \neq \emptyset$ *and* $z^* \notin \operatorname{cone}\{z_0^*\}$ *holds, then* $F^*(x^*, z^*) = \emptyset$.

c) If $z^* = sz_0^*$ *holds for* $s > 0$*, then*

$$\begin{aligned} F^*(x^*, sz_0^*) &= tG^*(\frac{1}{t}A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*) + S_{(A^{-1*}(x_0^* - \frac{1}{s}x^*), z_0^*)}(y_0) - z_0 \\ &= tG^*(\frac{1}{t}A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*) \lhd_{z^*} \Big(S_{(A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(y_0) + z_0\Big) \end{aligned}$$

holds for all $x^* \in X^*$.

Proof.

a) If $\operatorname{dom} G = \emptyset$, then $\operatorname{dom} F = \emptyset$ and thus by 4.1.10 it holds $F^* \equiv Z$.

b) If $\operatorname{dom} G \neq \emptyset$ and $z^* \notin \operatorname{cone}\{z_0^*\}$ holds, then $\operatorname{dom} F \neq \emptyset$ and for all $x \in \operatorname{dom} F$ it holds $F(x) \lhd_{z^*} \{0\} = Z$. Thus by 4.1.10 it holds $F^*(x^*, z^*) = \emptyset$.

c) By definition,

$$F^*(x^*, z^*) = \bigcap_{x \in X} \Big(S_{(x^*, z^*)}(x) \lhd_{z^*} F(x)\Big)$$

holds. If $s > 0$, then

$$\begin{aligned} S_{(x^*, sz^*)}(x) \lhd_{z^*} F(x) &= \Big\{z \in Z |\ tG(Ax + y_0) + S_{(x_0^*, z_0^*)}(x) + z_0 + z \subseteq S_{(x^*, sz_0^*)}(x)\Big\} \\ &= \Big\{z \in Z |\ tG(Ax + y_0) + S_{(x_0^*, z_0^*)}(x) + z \subseteq S_{(x^*, sz_0^*)}(x)\Big\} - z_0 \end{aligned}$$

holds. Recall that for $s > 0$ and $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ it holds $S_{(x^*, sz^*)}(x) = S_{(\frac{1}{s}x^*, z^*)}(x)$ for all $x \in X$. Moreover,

$$\begin{aligned} &S_{(x_0^*, z_0^*)}(x) \in Q^t_{H(z^*)}(Z) \setminus \{\emptyset, Z\} \\ &S_{(x_0^*, z_0^*)}(x) + S_{(-x_0^*, z_0^*)}(x) = H(z^*) \end{aligned}$$

holds and thus for $y = Ax + y_0$ it holds

$$\begin{aligned} S_{(x^*, sz^*)}(x) \lhd_{z^*} F(x) &= \Big\{z \in Z |\ tG(Ax + y_0) + z \subseteq S_{(\frac{1}{s}x^* - x_0^*, z_0^*)}(x)\Big\} - z_0 \\ &= t\Big\{z \in Z |\ G(Ax + y_0) + z \subseteq S_{(\frac{1}{t}(\frac{1}{s}x^* - x_0^*), z_0^*)}(x)\Big\} - z_0 \\ &= t\Big\{z \in Z |\ G(y) + z \subseteq S_{(\frac{1}{t}A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(y - y_0)\Big\} - z_0 \\ &= t\Big\{z \in Z |\ G(y) + z \subseteq S_{(\frac{1}{t}A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(y)\Big\} \\ &\quad + S_{(A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(-y_0) - z_0. \end{aligned}$$

Thus it holds

$$\begin{aligned} F^*(x^*, sz_0^*) &= tG^*((\frac{1}{t}A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)) + S_{(A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(-y_0) - z_0 \\ &= tG^*(\frac{1}{t}A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*) + S_{(A^{-1*}(x_0^* - \frac{1}{s}x^*), z_0^*)}(y_0) - z_0. \end{aligned}$$

Also,

$$S_{(A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(-y_0) - z_0 = \{0\} \lhd_{z^*} \Big(S_{(A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(y_0) + z_0\Big) \in Q^t_{H(z_0^*)}(Z) \setminus \{\emptyset, Z\}$$

holds and thus

$$F^*(x^*, sz_0^*) = tG^*(\frac{1}{t}A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*) \lhd_{z^*} \Big(S_{(A^{-1*}(\frac{1}{s}x^* - x_0^*), z_0^*)}(y_0) + z_0\Big).$$

■

Under the assumptions of 4.2.1, $\operatorname{dom} F^* \subseteq \left\{S_{(x^*,z_0^*)} |\ x^* \in X^*\right\}$ holds if $\operatorname{dom} G \neq \emptyset$.

The claim in 4.2.1c) is in accordance to the scalar result which can be found in [26], Proposition 3.3.3.4.

4.2.2 Proposition. *Let $F_i : X \to \mathcal{Q}_C^t(Z)$ for $n \geq 2$ and $i = 1, ..., n$.*

a) If $F(x) = \inf\limits_{i=1,..,n} F_i(x) = \operatorname{cl}\operatorname{co} \bigcup\limits_{i=1,...,n} F_i(x)$ holds for all $x \in X$, then

$$F^*(x^*, z^*) = \bigcap_{i=1,...,n} F_i^*(x^*, z^*)$$

holds for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$.

b) If $F(x) = \sup\limits_{i=1,..,n} F_i(x) = \bigcap\limits_{i=1,...,n} F_i(x)$ holds for all $x \in X$, then

$$F^*(x^*, z^*) \supseteq \operatorname{co}\,(\inf_{i=1,...,n} F_i^*)(x^*, z^*)$$

holds for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$.

c) Let $F(x) = \sup\limits_{i=1,..,n} F_i(x) = \bigcap\limits_{i=1,...,n} F_i(x)$ hold for all $x \in X$. If for all $i = 1, ..., n$ the functions F_i are convex and $\operatorname{dom} F_i = X$ and for all except possibly one F_i one of the assumptions in 3.3.7 and 3.3.9 holds for all $x \in X$, then

$$F^*(x^*, z^*) = \operatorname{co}\,(\inf_{i=1,...,n} F_i^*)(x^*, z^*)$$

holds for all $x^* \in X^*$. Moreover for every $x^* \in X^*$ with $(x^*, z^*) \in \operatorname{dom} F^*$ there exist $(x_i^*, z^*) \in \operatorname{dom} F_i^*$, $i = 1, ..., n$ and nonnegative numbers t_i, $i = 1, ..., n$ adding up to 1 such that $t_1 x_1^* + ... + t_n x_n^* = x^*$ and

$$F^*(x^*, z^*) = t_1 F_1^*(x_1^*, z^*) + ... + t_n F^*(x_n^*, z^*).$$

Proof.

a) For all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ it holds

$$F^*(x^*, z^*) = \bigcap_{x \in X} \Big(S_{(x^*,z^*)}(x) \lhd_{z^*} \operatorname{cl}\operatorname{co} \bigcup_{i=1,...,n} F_i(x)\Big).$$

By 2.4.3,

$$\begin{aligned}&S_{(x^*,z^*)}(x) \lhd_{z^*} \operatorname{cl}\operatorname{co} \bigcup_{i=1,...,n} F_i(x)\\ =&S_{(x^*,z^*)}(x) \lhd_{z^*} \bigcup_{i=1,...,n} F_i(x),\end{aligned}$$

thus

$$\begin{aligned}F^*(x^*, z^*) &= \bigcap_{x \in X} \left\{z \in Z |\ \bigcup_{i=1,...,n} F_i(x) + z \subseteq S_{(x^*,z^*)}(x)\right\}\\ &= \bigcap_{i=1,...,n} \bigcap_{x \in X} \left\{z \in Z |\ F_i(x) + z \subseteq S_{(x^*,z^*)}(x)\right\}\\ &= \bigcap_{i=1,...,n} F_i^*(x^*, z^*)\end{aligned}$$

b) For all $i = 1, ..., n$ it holds

$$(S_{(x^*,z^*)}(x) \lhd_{z^*} F_i(x)) \subseteq (S_{(x^*,z^*)}(x) \lhd_{z^*} \bigcap_{i=1,...,n} F_i(x))$$

for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$ and $x \in X$, thus the claim holds.

c) If $F^*(x^*, z^*) = \emptyset$, then we are done, so without loss of generality we can assume that $F^*(x^*, z^*) \in Q^t_{H(z^*)}(Z) \setminus \{\emptyset\}$.
If $F_1(x) \lhd_{z^*} \{0\} = Z$ holds for some $x \in X$, then $F_1^*(\cdot, z^*) \equiv \emptyset$ and

$$\bigcap_{i=1,...,n} F_i(x) = \bigcap_{i=2,...,n} F_i(x),$$
$$\mathrm{cl} \bigcup_{i=1,...n} F_i^*(x^*, z^*) = \mathrm{cl} \bigcup_{i=2,...n} F_i^*(x^*, z^*),$$

so without loss of generality we can assume that F_i is z^*-proper for all $i = 1, ..., n$ and all $z^* \in C^* \setminus \{0\}$. Thus, for every $z^* \in C^* \setminus \{0\}$ the scalarizations of all F_i are finite everywhere and all except for possibly one are continuous. Applying [26], 3.3.4,Theorem 2 it holds

$$\varphi^*_{(F,z^*)}(x^*) = \mathrm{co}\,(\inf_{i=1,...,n} \varphi^*_{(F_i,z^*)})(x^*).$$

and for every $x^* \in X^*$ with $x^* \in \mathrm{dom}\,\varphi^*_{(F,z^*)}$ there exist $x_i^* \in \mathrm{dom}\,\varphi^*_{(F_i,z^*)}$, $i = 1, ..., n$ and nonnegative numbers t_i, $i = 1, ..., n$ adding up to 1 such that $t_1 x_1^* + ... + t_n x_n^* = x^*$ and

$$\varphi^*_{(F,z^*)}(x^*) = t_1 \varphi^*_{(F_1,z^*)}(x^*) + ... + t_n \varphi^*_{(F_n,z^*)}(x^*).$$

Thus,

$$\begin{aligned}
F^*(x^*, z^*) &= \left\{z \in Z | -z^*(z) \geq \varphi^*_{(F,z^*)}(x^*)\right\} \\
&= \left\{z \in Z | -z^*(z) \geq t_1 \varphi^*_{(F_1,z^*)}(x^*) + ... + t_n \varphi^*_{(F_n,z^*)}(x^*)\right\} \\
&= t_1 \left\{z \in Z | -z^*(z) \geq \varphi^*_{(F_1,z^*)}(x^*)\right\} + ... + t_n \left\{z \in Z | -z^*(z) \geq \varphi^*_{(F_n,z^*)}(x^*)\right\} \\
&= t_1 F_1^*(x^*, z^*) + ... + t_n F_n^*(x^*, z^*) \\
&\subseteq \mathrm{co}\,(F_1^* + ... + F_n^*)(x^*, z^*)
\end{aligned}$$

holds, proving the claim. ■

Notice that the result in 4.2.2c) in fact is slightly more general than the known scaler result as we do not assume properness for the functions $F_i : X \to \mathcal{Q}^t_C(Z)$.

4.2.3 Definition. *For $F_1, F_2 : X^* \times C^* \setminus \{0\} \to \mathcal{Q}^t_C(Z)$, the infimal convolution of F_1^* and F_2^* in $(x^*, z^*) \in X \times C^* \setminus \{0\}$ is defined by*

$$(F_1^* \Box F_2^*)(x^*, z^*) := \mathrm{cl} \bigcup_{x^* \in X^*} (F_1^*(\bar{x}^*, z^*) + (F_2)^*(x^* - \bar{x}^*, z^*)).$$

Notice that in fact the infimal convolution $(F_1^* \Box F_2^*)$ is an operation defined on the set $\{G : X^* \to \mathcal{Q}^t_C(Z)\}$, thus identifying $F^*(x^*, z^*) = F^*_{z^*}(x^*)$ for all $z^* \in C^* \setminus \{0\}$ and all $x^* \in X^*$, this is the ordinary definition of the infimal convolution, introduced in subsection 2.2.3.

4.2.4 Proposition. *Let $F_i : X \to \mathcal{Q}^t_C(Z)$ for $n \geq 2$ and $i = 1, ..., n$.*

a) If $F(x) = (F_1 \square ... \square F_n)(x)$ holds for all $x \in X$ and $\operatorname{dom} F_1 + .. + \operatorname{dom} F_n \neq \emptyset$, then it holds

$$F^*(x^*, z^*) = (F_1^* + .. + F_n^*)(x^*, z^*)$$

for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$.

b) If $F(x) = (F_1 + ... + F_n)(x)$ holds for all $x \in X$, then it holds

$$F^*(x^*, z^*) \supseteq (F_1^* \square ... \square F_n^*)(x^*, z^*)$$

for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$.

c) If $F(x) = (F_1 + ... + F_n)(x)$ holds for all $x \in X$ and for all $i = 1, ..., n$ the functions F_i are convex and z_0^*-proper for $z_0 \in C^* \setminus \{0\}$ and for all except possibly one F_i one of the assumptions in 3.3.7 and 3.3.9 holds in $x_0 \in \bigcap\limits_{i=1,...n} \operatorname{dom} F_i$, then

$$F^*(x^*, z_0^*) = (F_1^* \square ... \square F_n^*)(x^*, z_0^*)$$

holds for all $x^* \in X^*$. Moreover for every $x^* \in X^*$ with $(x^*, z_0^*) \in \operatorname{dom} F^*$ there exist $(x_i^*, z_0^*) \in \operatorname{dom} F_i^*$, $i = 1, ..., n$ such that $x_1^* + ... + x_n^* = x^*$ and

$$F^*(x^*, z_0^*) = F_1^*(x_1^*, z_0^*) + ... + F^*(x_n^*, z_0^*).$$

Proof.

a) Without loss of generality, assume $n = 2$. By definition,

$$(F_1 \square F_2)(x) = \operatorname{cl} \operatorname{co} \bigcup_{y \in X} (F_1(y) + F_2(x - y))$$

holds for all $x \in X$. Thus by 2.4.8e) and 2.4.3 it holds

$$\begin{aligned}
(F_1 \square F_2)^*(x^*, z^*) &= \bigcap_{x \in X} \Big(S_{(x^*, z^*)}(x) \lhd_{z^*} \bigcup_{y \in X} (F_1(y) + F_2(x - y)) \Big) \\
&= \bigcap_{x \in X} \left\{ z \in Z | \bigcup_{y \in X} (F_1(y) + F_2(x - y) + z) \subseteq S_{(x^*, z^*)}(x) \right\} \\
&= \bigcap_{x, y \in X} \Big(S_{(x^*, z^*)}(x) \lhd_{z^*} (F_1(y) + F_2(x - y)) \Big)
\end{aligned}$$

By assumption it exists $x_0, y_0 \in X$ such, that $y_0 \in \operatorname{dom} F_1$ and $x_0 - y_0 \in \operatorname{dom} F_2$. If $y \notin \operatorname{dom} F_1$ or $x - y \notin \operatorname{dom} F_2$, then

$$S_{(x^*, z^*)}(x) \lhd_{z^*} (F_1(y) + F_2(x - y)) = Z.$$

Therefore it holds

$$(F_1 \square F_2)^*(x^*, z^*) = \bigcap_{\substack{y \in \operatorname{dom} F_1, \\ x \in \operatorname{dom} F_2 + y}} S_{(x^*, z^*)}(x) \lhd_{z^*} (F_1(y) + F_2(x - y)).$$

If for $y \in \operatorname{dom} F_1$ and $x \in \operatorname{dom} F_2 + y$ it holds $F_1(y) \lhd_{z^*} \{0\} = Z$ or $F(x-y) \lhd_{z^*} \{0\} = Z$, then $(F_1(y) + F(x-y)) \lhd_{z^*} \{0\} = Z$ and

$$S_{(x^*,z^*)}(x) \lhd_{z^*} (F_1(y) + F(x-y)) = (S_{(x^*,z^*)}(y) \lhd_{z^*} F_1(y)) + (S_{(x^*,z^*)}(y) \lhd_{z^*} F(x-y)) = \emptyset.$$

for all $x^* \in X^*$. In this case, $(F_1 \Box F_2)^*(\cdot, z^*) = F_1^*(\cdot, z^*) + F_2^*(\cdot, z^*) \equiv \emptyset$. From now on, let F_1 and F_2 be z^*-proper functions and $y \in \operatorname{dom} F_1$ and $x \in \operatorname{dom} F_2 + y$. By 2.4.8e) it holds

$$S_{(x^*,z^*)}(x) \lhd_{z^*} (F_1(y) + F(x-y)) = (S_{(x^*,z^*)}(y) \lhd_{z^*} F_1(y)) + (S_{(x^*,z^*)}(y) \lhd_{z^*} F(x-y))$$

for all $x^* \in X^*$. Thus,

$$\begin{aligned}(F_1 \Box F_2)^*(x^*, z^*) &= \bigcap_{x,y \in X} (S_{(x^*,z^*)}(y) \lhd_{z^*} F_1(y)) + (S_{(x^*,z^*)}(x-y) \lhd_{z^*} F_2(x-y)) \\ &= F_1^*(x^*, z^*) + F_2^*(x^*, z^*).\end{aligned}$$

b) Without loss of generality, assume $n = 2$. By 2.4.8e) it holds

$$\begin{aligned}(F_1 + F_2)^*(x^*, z^*) &= \bigcap_{x \in X} \Big(S_{(x^*,z^*)}(x) \lhd_{z^*} (F_1(x) + F_2(x)) \Big) \\ &= \bigcap_{x \in X} \Big((S_{(\bar{x}^*,z^*)}(x) + S_{(x^*-\bar{x}^*,z^*)}(x)) \lhd_{z^*} (F_1(x) + F_2(x)) \Big) \\ &\supseteq \bigcap_{x \in X} \Big((S_{(\bar{x}^*,z^*)}(x) \lhd_{z^*} F_1(x)) + (S_{(x^*-\bar{x}^*,z^*)}(x) \lhd_{z^*} +F_2(x)) \Big) \\ &\supseteq F_1^*(\bar{x}^*, z^*) + F_2^*(x^* - \bar{x}^*, z^*).\end{aligned}$$

for all $x^*, \bar{x}^* \in X^*$ and all $z^* \in C^* \setminus \{0\}$.

c) By assumption, the scalarizations $\varphi_{(F_i,z_0^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ are convex and proper and for all but possibly one $i \in \{1, ..., n\}$ there is $t_i \in \mathbb{R}$ such, that $(x_0, t_i) \in \operatorname{int} \operatorname{epi} \varphi_{(F_i,z_0^*)}$. Thus, all but possibly one scalarization $\varphi_{(F_i,z_0^*)}$ are continuous at x_0. By the scalar sum rule as found in [26], 3.3.4, Theorem 1 and by 3.2.1 d)it holds

$$\varphi^*_{(F,z_0^*)}(x^*) = (\varphi^*_{(F_1,z_0^*)} \Box ... \Box \varphi^*_{(F_n,z_0^*)})(x^*)$$

holds for all $x^* \in X^*$. Moreover for every $x^* \in X^*$ with $x^* \in \operatorname{dom} \varphi^*_{(F,z_0^*)}$ there exist $x_i^* \in \operatorname{dom} \varphi^*_{(F_i,z_0^*)}$, $i = 1, ..., n$ such that $x_1^* + ... + x_n^* = x^*$ and

$$\varphi^*_{(F,z_0^*)}(x^*) = \varphi^*_{(F_1,z_0^*)}(x_1^*) + ... + \varphi^*_{(F_n,z_0^*)}(x_n^*).$$

Thus by 4.1.2 it holds

$$F^*(x^*, z_0^*) = F_1^*(x_1^*, z_0^*) + ... + F_n^*(x_n^*, z_0^*) \subseteq (F_1^* \Box ... F_n^*)(x^*, z_0^*)$$

And thus the equation is proven. ∎

4.2.5 Definition. *If $A : X \to Y$ is a linear continuous operator, $G : Y \to \mathcal{Q}_C^t(Z)$ and $F \to \mathcal{Q}_C^t(Z)$, then the functions $A^*G^* : X^* \times C^* \setminus \{0\} \to \mathcal{Q}_C^t(Z)$ and $F^*A^* : Y^* \times C^* \setminus \{0\} \to \mathcal{Q}_C^t(Z)$ are defined by*

$$(A^*G^*)(x^*, z^*) := \operatorname{cl} \bigcup_{A^*y^*=x^*} G^*(y^*, z^*)$$

for all $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$ and

$$F^*A^*(y^*, z^*) := F^*(A^*y^*, z^*)$$

for all $y^* \in Y^*$ and $z^* \in C^* \setminus \{0\}$.

Identifying $F^*(x^*, z^*) = F^*_{z^*}(x^*)$ for all $z^* \in C^* \setminus \{0\}$ and all $x^* \in X^*$, this is the ordinary definition of $A^*G^*_{z^*} : X^* \to Q^t_{H(z^*)}(Z)$ and $F^*_{z^*}A^* : Y^* \to \mathcal{Q}^t_C(Z)\,^t_{H(z^*)}(Z)$ introduced in subsection 2.2.3.

4.2.6 Proposition. *Let $A : X \to Y$ be a linear continuous operator, $G : Y \to \mathcal{Q}^t_C(Z)$ and $F \to \mathcal{Q}^t_C(Z)$.*

a) It holds

$$(AF)^*(y^*, z^*) = F^*A^*(y^*, z^*)$$

for all $(y^*, z^*) \in Y^* \times C^* \setminus \{0\}$.

b) It holds

$$(GA)^*(x^*, z^*) \supseteq A^*G^*(x^*, z^*)$$

for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$.

c) If G is convex and one of the assumptions in 3.3.7 and 3.3.9 holds in Ax_0, $x_0 \in X$, then

$$(GA)^*(x^*, z^*) = A^*G^*(x^*, z^*)$$

holds for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$. Moreover for every $(x^*, z^*) \in \operatorname{dom}(GA)^*$ there exist $(y^*, z^*) \in \operatorname{dom} Y^* \times C^* \setminus \{0\}$, such that $A^*y^* = x^*$ holds and

$$(GA)^*(x^*, z^*) = G^*(y^*, z^*).$$

Proof.

a) For all $(y^*, z^*) \in Y^* \times C^* \setminus \{0\}$ it holds

$$S_{(y^*,z^*)}(y) \lhd_{z^*} \operatorname{cl} \bigcup_{Ax=y} F(x) = \bigcap_{Ax=y} S_{(A^*y^*,z^*)}(x) \lhd_{z^*} F(x).$$

Thus the claim is immediate.

b) For all $(y^*, z^*) \in Y^* \times C^* \setminus \{0\}$ it holds

$$\begin{aligned}\bigcap_{y\in Y} \Big(S_{(y^*,z^*)}(y) \lhd_{z^*} G(y)\Big) &\subseteq \bigcap_{x\in X} \Big(S_{(y^*,z^*)}(Ax) \lhd_{z^*} GA(x)\Big)\\ &= \bigcap_{x\in X} \Big(S_{(A^*y^*,z^*)}(x) \lhd_{z^*} GA(x)\Big)\end{aligned}$$

And thus the inclusion is proven.

c) By 3.2.1e) and 4.1.2 it holds

$$(GA)^*(x^*, z^*) = \Big\{z \in Z| \ -z^*(z) \geq (\varphi_{(G,z^*)}A)^*(x^*)\Big\}$$

for all $z^* \in C^* \setminus \{0\}$ and $x^* \in X^*$. By assumption, $\varphi_{(G,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is either continuous at $Ax_0 \in \operatorname{dom} G$ or $\varphi_{(G,z^*)}(x) = -\infty$ holds for all $y \in \operatorname{dom} G$. In the latter case, $\operatorname{dom} \varphi^*_{(GA,z^*)} = \emptyset$, as $Ax_0 \in \operatorname{dom} G$. Thus, in this case $(GA)^*(\cdot, z^*) \equiv \emptyset$, proving the statement. If $\varphi_{(G,z^*)}(Ax_0) \in \mathbb{R}$, then

$$(\varphi_{(G,z^*)}A)^*(x^*) = A^*\varphi^*_{(G,z^*)}(x^*)$$

holds and for all $x^* \in \operatorname{dom}(\varphi_{(G,z^*)}A)^*$ there exists $y^* \in Y^*$ with $A^*y^* = x^*$ such, that

$$(\varphi_{(G,z^*)}A)^*(x^*) = \varphi^*_{(G,z^*)}(y^*),$$

compare [26], 3.3.4 Theorem 3. Thus by 4.1.2, for all $(x^*, z^*) \in \operatorname{dom}(GA)^*$ there exists $y^* \in Y^*$ with $A^*y^* = x^*$ such, that

$$(GA)^*(x^*, z^*) = G^*(y^*, z^*) \subseteq A^*G^*(x^*, z^*),$$

proving the statement. ∎

The result of 4.2.6c) is stronger then the well-known scalar result as it can be found in [26], as we do not need any properness assumption on $G : X \to \mathcal{Q}^t_C(Z)$.

4.2.7 Remark. *For $A \in \mathcal{L}(X, Y)$, $x_0^* \in X^*$, $y_0^* \in Y^*$ and $z_0^* \in C^* \setminus \{0\}$ it holds*

$$(AS_{(x_0^*,z_0^*)})^*(y^*, z^*) = I_{\operatorname{cone}\{(x_0^*,z_0^*)\}}(A^*y^*, z^*) + H(z_0^*)$$

and

$$(S_{(y_0^*,z_0^*)}A)^*(x^*, z^*) = I_{\operatorname{cone}\{(Ay_0^*,z_0^*)\}}(x^*, z^*) + H(z_0^*).$$

4.2.8 Proposition (Chain-rule)**.** *Let $F : X \to \mathcal{Q}^t_C(Z)$ and $G : Y \to \mathcal{Q}^t_C(Z)$ and $A : X \to Y$ a continuous linear operator. It holds*

$$(F + GA)^*(x^*, z^*) \supseteq F^*(x^* + A^*y^*, z^*) + G^*(-y^*, z^*)$$

for all $x^* \in X^*$, $y^* \in Y^*$ and $z^* \in C^* \setminus \{0\}$. If additionally F and G are convex and z^*-proper and it exists $x_0 \in X$ such, that $x_0 \in \operatorname{dom} F$ and $(Ax_0, z_0) \in \operatorname{int}\operatorname{epi} G$ for some $z_0 \in Z$, then equality holds and for all $x^* \in X^*$ it exists $y_0^* \in Y^*$ such, that

$$(F + GA)^*(x^*, z^*) = F^*(x^* - A^*y_0^*, z^*) + G^*(y_0^*, z^*) \neq Z$$

Proof. The first inclusion is immediate from 4.2.4b) and 4.2.6b). Under the additional assumptions,

$$\exists y^* \in Y^* : (F + GA)^*(x^*, z^*) = F^*(x^* - A^*y_0^*, z^*) + G^*(y_0^*, z^*)$$

holds with 4.2.4c) and 4.2.6c) for all $x^* \in X^*$. As $\operatorname{dom}(F + GA) \neq \emptyset$ is assumed, $(F + GA)^*(x^*, z^*) \neq Z$ holds for all $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$. ∎

With the assumption in 4.2.8 it holds

$$\begin{aligned}(F + GA)^*(0, z^*) &= \bigcap_{x\in X} (\{0\} \lhd_{z^*} (F(x) + G(Ax))\\ &= \{0\} \lhd_{z^*} \operatorname{cl}\operatorname{co} \bigcup_{x\in X} (F(x) + G(Ax))\\ &\supseteq \operatorname{cl} \bigcup_{y^*\in Y^*} (F^*(A^*y^*, z^*) + G^*(-y^*, z^*)).\end{aligned}$$

With this we are now able to state a Fenchel-Rockafellar type duality theorem.

4.2.9 Theorem (Fenchel-Rockafellar-Duality). *Let $F : X \to \mathcal{Q}^t_C(Z)$ and $G : Y \to \mathcal{Q}^t_C(Z)$ and $A : X \to Y$ a continuous linear operator. Denote*

$$P := \operatorname{cl}\operatorname{co} \bigcup_{x\in X} (F(x) + GA(x))$$

and

$$D := \bigcap_{(y^*,z^*)\in Y^*\times(C^*\setminus\{0\})} \left[\{0\} \lhd_{z^*} (F^*(A^*y^*, z^*) + G^*(-y^*, z^*))\right].$$

a) It holds $D \supseteq P$.

b) If additionally F and G are convex and z^*-proper and $x_0 \in \operatorname{dom} F$ and $(Ax_0, z) \in \operatorname{int}\operatorname{epi} G$ for some $z \in Z$, then

$$P \lhd_{z^*} \{0\} = D \lhd_{z^*} \{0\} \neq \emptyset$$

and there is $y^* \in Y^*$ such that

$$(P \lhd_{z^*} \{0\}) = (D \lhd_{z^*} \{0\}) = \{0\} \lhd_{z^*} (F^*(A^*y^*, z^*) + G^*(-y^*, z^*)).$$

c) If F and G are convex and z^*-proper for all $z^* \in C^* \setminus \{0\}$ and $x_0 \in \operatorname{dom} F$ and $(Ax_0, z) \in \operatorname{int}\operatorname{epi} G$ for some $z \in Z$, then

$$P = D \neq \emptyset$$

and to every $z^* \in C^* \setminus \{0\}$ there is $y^*_{z^*} \in Y^*$ such that

$$P = D = \bigcap_{z^*\in C^*\setminus\{0\}} \Big(\{0\} \lhd_{z^*} (F^*(A^*y^*_{z^*}, z^*) + G^*(-y^*_{z^*}, z^*))\Big).$$

Proof.

a) For all $z^* \in C^* \setminus \{0\}$ it holds

$$\{0\} \lhd_{z^*} P = (F + GA)^*(0, z^*)$$

and

$$\{0\} \lhd_{z^*} D = \{0\} \lhd_{z^*} \operatorname{cl} \bigcup_{(y^*,z^*)\in Y^*\times(C^*\setminus\{0\})} (F^*(A^*y^*, z^*) + G^*(-y^*, z^*)).$$

Therefore $\{0\} \lhd_{z^*} D \subseteq \{0\} \lhd_{z^*} P$ holds for all $z^* \in C^* \setminus \{0\}$ by 4.2.8 and thus

$$\forall z^* \in C^* \setminus \{0\} : \quad P \lhd_{z^*} \{0\} \subseteq D \lhd_{z^*} \{0\}.$$

As

$$P = \bigcap_{z^*\in C^*\setminus\{0\}} (P \lhd_{z^*} \{0\}),$$
$$D = \bigcap_{z^*\in C^*\setminus\{0\}} (D \lhd_{z^*} \{0\})$$

holds, the inclusion is proven.

b) Under the additional assumptions by 4.2.8 it holds

$$\exists y_0^* \in Y^* : \quad (F + GA)^*(0, z^*) = (F^*(A^* y_0^*, z^*) + G^*(-y_0^*, z^*)) \neq Z$$

and

$$(F^*(A^* y_0^*, z^*) + G^*(-y_0^*, z^*)) = \operatorname{cl} \bigcup_{y^* \in Y^*} (F^*(A^* y^*, z^*) + G^*(-y^*, z^*)).$$

The statement holds as

$$\begin{aligned} P \lhd_{z^*} \{0\} &= \{0\} \lhd_{z^*} (F + GA)^*(0, z^*), \\ D \lhd_{z^*} \{0\} &= \{0\} \lhd_{z^*} (F^*(A^* y^*, z^*) + G^*(-y^*, z^*)) \neq \emptyset. \end{aligned}$$

c) As the assumptions of b) hold for all $z^* \in C^* \setminus \{0\}$, the result is easily derived from the previous result as

$$\begin{aligned} P &= \bigcap_{z^* \in C^* \setminus \{0\}} (P \lhd_{z^*} \{0\}), \\ D &= \bigcap_{z^* \in C^* \setminus \{0\}} (D \lhd_{z^*} \{0\}) \end{aligned}$$

holds. ■

4.2.10 Corollary (Sandwich-Theorem)**.** *Let $F : X \to \mathcal{Q}_C^t(Z)$ and $G : Y \to \mathcal{Q}_C^t(Z)$ be convex and z^*-proper and $A : X \to Y$ a continuous linear operator. If it exists $x_0 \in \operatorname{dom} F$ such that $(Ax_0, z) \in \operatorname{int} \operatorname{epi} G$ for some $z \in Z$ and for every $x \in X$ it holds $F(x) \subseteq \{0\} \lhd_{z^*} G(Ax)$, then there exist $y^* \in Y^*$ and $z_0 \in Z$ such that*

$$\forall x \in X : \ F(x) \subseteq S_{(A^* y^*, z^*)}(x) - z_0 \subseteq \{0\} \lhd_{z^*} G(Ax),$$

and

$$\begin{aligned} &z_0 \in F^*(Ay^*, z^*), \\ &z_0 \in \{0\} \lhd_{z^*} G^*(-y^*, z^*) \end{aligned}$$

holds. If additionally $F(x_0) \lhd_{z^*} \{0\} = \{0\} \lhd_{z^*} G(Ax_0)$, then z_0 can be chosen such that

$$\begin{aligned} F^*(Ay^*, z^*) &= z_0 + H(z^*), \\ G^*(-y^*, z^*) &= -z_0 + H(z^*). \end{aligned}$$

Proof.

As both F and GA are z^*-proper, it holds

$$\begin{aligned} \emptyset \neq F(x) &\subseteq \{0\} \lhd_{z^*} G(Ax) \neq Z, \\ H(z^*) &\supseteq F(x) + G(Ax) \neq \emptyset \end{aligned}$$

for all $x \in \operatorname{dom} F \cap \operatorname{dom} GA$. Moreover,

$$\operatorname{cl} \bigcup_{x \in X} (F(x) + G(Ax)) = \operatorname{cl} \bigcup_{x \in \operatorname{dom} F \cap \operatorname{dom} GA} (F(x) + G(Ax)).$$

Applying 4.2.9 there exists $y^* \in Y^*$ such, that

$$H(z^*) \supseteq (\bigcup_{x\in X}(F(x)+G(Ax))) \lhd_{z^*} \{0\} = \{0\} \lhd_{z^*} \Big(F^*(A^*y^*, z^*) + G^*(-y^*)\Big) \neq \emptyset.$$

Thus by 2.4.9 it holds

$$Z \neq F^*(A^*y^*, z^*) \supseteq \{0\} \lhd_{z^*} G^*(-y^*) \neq \emptyset.$$

Therefore, for all $z \in \{0\} \lhd_{z^*} G^*(-y^*, z^*)$ it holds

$$\begin{aligned} F(x) &\subseteq S_{(y^*,z^*)}(Ax) - z, \\ G(Ax) &\subseteq S_{(-y^*,z^*)}(Ax) + z. \end{aligned}$$

As by 4.1.9 $S_{(x^*,z^*)} - z$ is a conaffine minorant of F if and only if $z \in F^*(x^*, z^*)$, the inclusion $G(Ax) \subseteq S_{(-y^*,z^*)}(Ax) + z$ can be transformed into

$$\begin{aligned} \{0\} \lhd_{z^*} G(Ax) &\supseteq (\{0\} \lhd_{z^*} S_{(y^*,z^*)}(Ax)) - z \\ &= S_{(A^*y^*,z^*)}(x) - z \end{aligned}$$

proving the claimed inclusions. If additionally $F(x_0) \lhd_{z^*} \{0\} = \{0\} \lhd_{z^*} G(Ax_0)$, then holds

$$H(z^*) = \{0\} \lhd_{z^*} \Big(F^*(A^*y^*, z^*) + G^*(-y^*)\Big) \neq \emptyset$$

and thus

$$Z \neq F^*(A^*y^*, z^*) = \{0\} \lhd_{z^*} G^*(-y^*) \neq \emptyset$$

and therefore there exists $z_0 \in Z$ such that

$$z_0 + H(z^*) = F^*(A^*y^*, z^*) = \{0\} \lhd_{z^*} G^*(-y^*)$$

and

$$F(x) \subseteq S_{(y^*,z^*)}(Ax) - z_0 \subseteq \{0\} \lhd_{z^*} G(Ax)$$

holds by the same calculations as above. ∎

5 Directional Derivative

In [10, 26, 50], the directional derivative of a convex function $f : X \to \mathrm{I\!R} \cup \{+\infty\}$ at $x_0 \in \operatorname{dom} f$ is defined as

$$f'(x_0, x) = \lim_{t \downarrow 0} \frac{1}{t}(f(x_0 + tx) - f(x_0)) \tag{5.0.1}$$

for $x \in X$, when the limit exists in $\mathrm{I\!R} \cup \{\pm\infty\}$. It is well-known that for convex proper functions and $x_0 \in \operatorname{dom} f$ it holds

$$f'(x_0, x) = \inf_{t>0} \frac{1}{t}(f(x_0 + tx) - f(x_0))$$

as the difference quotient $\frac{1}{t}(f(x_0 + tx) - f(x_0))$ does not increase, as $t > 0$ converges towards 0. Moreover, if $\operatorname{cl}(f'(x_0, \cdot))(0) = 0$ holds, then

$$\exists s \in \mathrm{I\!R},\ s \geq f(x_0) : \quad \operatorname{cl} \operatorname{epi}(f'(x_0, \cdot)) = T_{\operatorname{epi} f}(x_0, s) \tag{5.0.2}$$

holds, where $T_{\operatorname{epi} f}(x_0, s) := \operatorname{cl}\{t(\operatorname{epi} f - (x_0, s)) |\ t > 0\}$ denotes the tangent cone of $\operatorname{epi} f$ at $(x_0, s) \in \operatorname{epi} f$. Of course, as $f(x_0) \in \mathrm{I\!R}$, it has to holds $s = f(x_0)$.

If $\operatorname{cl} \operatorname{epi}(f'(x_0, \cdot))(0) = 0$, then

$$f'(x_0, x) = \sup \{x^*(x) |\ x^* \leq f'(x_0, \cdot)\}$$

holds for all $x \in X$, compare [62]. If an additional constrained assumption holds, then the max-formula

$$f'(x_0, x) = \max \{x^*(x) |\ x^* \leq f'(x_0, \cdot)\}$$

holds for all $x \in X$, compare [10].

Replacing the classic difference $''-''$ by the inf-difference $''\vartriangleleft''$, we succeed to prove that each of the mentioned properties of the directional derivative, but (5.0.2), holds in the more general case of $f : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ with $x_0 \in X$, $f(x_0)$ not necessarily finite. The formula (5.0.2) can only be achieved, if $x_0 \in \operatorname{dom} f$ holds. If $f(x_0) = -\infty$, then

$$\forall s \in \mathrm{I\!R} : \quad \operatorname{cl} \operatorname{epi}(f'(x_0, \cdot)) = T_{\operatorname{epi} f}(x_0, s) = (\operatorname{dom} f - x_0) \times \mathrm{I\!R}.$$

The formulas (5.0.1) and (5.0.2) gave rise to various definitions of a directional derivative of vector-valued and set-valued functions, compare for example [15, 44, 65, 7, 14, 31, 28, 5] and the references therein. Closest related to our approach appears to be the definition of the derivative or epiderivative in [1]. The most popular approach seems to be that of the contingent epiderivative, introduced in [28], compare also [51, 52] and related concepts. As the aim in those works is to establish a subdifferential which is a subset of $\mathcal{L}(X, Z)$, the contingent epiderivative is vector-valued. We will return to discuss this approach later on in section 7.

For the present, we restrict our further investigations on the case of convex set-valued functions $F : X \to \mathcal{Q}^t_C(Z)$ and define a family of set-valued directional derivatives of F at $x_0 \in X$ parallel to (5.0.1), namely

$$F'_{z^*}(x_0, x) = \inf_{t>0} \frac{1}{t}(F(x_0 + tx) \vartriangleleft_{z^*} F(x_0))$$

for every $x \in X$ and $z^* \in C^* \setminus \{0\}$. It will be shown in subsection 7.1, that a formula related to (5.0.2) holds for all $x_0 \in \operatorname{dom} F$, that is

$$\forall x_0 \in \operatorname{dom} F : \; \exists z_0 \in F(x_0) \lhd_{z^*} \{0\} : \quad \operatorname{cl} \operatorname{epi} (F'_{z^*}(x_0, \cdot)) = T_{\operatorname{epi} (F \lhd_{z^*} \{0\})}(x_0, z_0).$$

It is notable, that again we do not assume that the function $F : X \to \mathcal{Q}^t_C(Z)$ or its z^*-hull $F \lhd_{z^*} \{0\} : X \to Q^t_{H(z^*)}(Z)$ is proper, neither do we assume that $x_0 \in \operatorname{dom} F$ holds. Each directional derivative will prove to be sublinear and the difference quotient

$$G(t) = \frac{1}{t}(F(x_0 + tx) \lhd_{z^*} F(x_0))$$

proves to be non-increasing as $t > 0$ converges towards 0, that is for $0 < t_1 < t_2$ it holds $G(t_1) \supseteq G(t_2)$.

We will prove that if $(\operatorname{cl} (F'_{z^*}(x_0, \cdot)))(0) = H(z^*)$, then $(\operatorname{cl} (F'_{z^*}(x_0, \cdot)))$ is the pointwise supremum of the conlinear minorants of $\operatorname{cl} (F'_{z^*}(x_0, \cdot))$ and under an additional constrained assumption we will provide a set-valued max-formula.

In subsection 5.2, we will summarize some calculus rules for the directional derivatives, including a sum- and a chain-rule.

The results achieved in this chapter very naturally lead to one possible definition of the subdifferential, or rather a family of subdifferentials of a convex set-valued function mapping into $\mathcal{Q}^t_C(Z)$, which will be discussed in detail in the following section 6. In fact this approach will turn out to coincide in all but the "pathological" cases with the subdifferentials defined via the conjugate of $F : X \to \mathcal{Q}^t_C(Z)$.

Throughout this chapter, X, Y and Z are assumed to be locally convex separable spaces with the corresponding dual spaces X^*, Y^* and Z^* and Z is quasi-ordered by a closed convex cone $C \subsetneq Z$ with $\{0\} \subsetneq C$.

5.1 Definition and basic results

5.1.1 Definition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function. The directional derivative $F'_{z^*}(x_0, \cdot) : X \to \mathcal{P}(Z)$ in $x_0 \in X$ with respect to $z^* \in C^* \setminus \{0\}$ is defined by*

$$F'_{z^*}(x_0, x) := \operatorname{cl} \bigcup_{t>0} \frac{1}{t} \left(F(x_0 + tx) \lhd_{z^*} F(x_0) \right).$$

Especially, if $F : X \to \mathcal{Q}^t_C(Z)$ is a convex function and $x_0 \in X$, then

$$F'_{z^*}(x_0, 0) = \begin{cases} F(x_0) \lhd_{z^*} \{0\}, & \text{if } F(x_0) \neq \emptyset; \\ Z, & \text{else.} \end{cases}$$

holds and therefore $F'_{z^*}(x_0, 0) = Z$ if $F(x_0)$ is not a z^*-proper set, that is if $(F(x_0) \lhd_{z^*} \{0\}) \in \{\emptyset, Z\}$, compare 2.4.6.

5.1.2 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $x_0 \in X$ and $z^* \in C^* \setminus \{0\}$. It holds*

$$F'_{z^*}(x_0, \cdot) : X \to Q^t_{H(z^*)}(Z)$$

with

$$\forall x \in X : \quad F'_{z^*}(x_0, x) = \left\{ z \in Z | \; -z^*(z) \geq \varphi'_{(F,z^*)}(x_0, x) \right\}.$$

Proof. By definition,

$$\begin{aligned}\forall x \in X: \quad F'_{z^*}(x_0,x) &= \left\{z \in Z \mid \forall t>0: \, -z^*(z) \geq \frac{1}{t}\left(\varphi_{(F,z^*)}\left(x_0+tx\right) \triangleleft \varphi_{(F,z^*)}\left(x_0\right)\right)\right\} \\ &= \left\{z \in Z \mid \, -z^*(z) \geq \inf_{t>0} \frac{1}{t}\left(\varphi_{(F,z^*)}\left(x_0+tx\right) \triangleleft \varphi_{(F,z^*)}\left(x_0\right)\right)\right\} \\ &= \left\{z \in Z \mid \, -z^*(z) \geq \varphi'_{(F,z^*)}(x_0,x)\right\} \in Q^t_{H(z^*)}(Z).\end{aligned}$$

■

5.1.3 Lemma. *Let $F: X \to \mathcal{Q}^t_C(Z)$ be a convex function, $x \in X$ and $s > 0$.*

a) It holds $\varphi_{(F'_{z^*}(x_0,\cdot),sz^*)}(x) = s\varphi'_{(F,z^*)}(x_0,x)$.
If $z_0^* \in C^* \setminus \operatorname{cl}\operatorname{cone}\{z^*\}$, then $\varphi_{(F'_{z^*}(x_0,\cdot),z_0^*)}(x) = -\infty$, if $x \in \operatorname{dom} F'_{z^*}(x_0,\cdot)$ and $+\infty$, else.

b) It holds

$$\forall s > 0: \; F'_{sz^*}(x_0,x) = F'_{z^*}(x_0,x).$$

Proof.

a) From 5.1.2, it holds $\varphi_{(F'_{z^*}(x_0,\cdot),z^*)}(x) = \varphi'_{(F,z^*)}(x_0,x)$ and $\varphi_{(F'_{z^*}(x_0,\cdot),sz^*)}(x) = s\varphi'_{(F,z^*)}(x_0,x)$. If $z_0^* \in C^* \setminus (\operatorname{cone}\{z^*\} \cup \{0\}$, then $\varphi_{(F'_{z^*}(x_0,\cdot),z_0^*)}(x) = -\infty$ holds if $x \in \operatorname{dom} F'_{z^*}(x_0,\cdot)$, as $F'_{z^*}: X \to Q^t_{H(z^*)}(Z)$. If $F'_{z^*}(x_0,x) = \emptyset$, then

$$\varphi_{(F'_{z^*}(x_0,\cdot),z_0^*)}(x) = \inf\{-z_0^*(z) \mid z \in \emptyset\} = +\infty.$$

b) Let $s > 0$, then

$$\begin{aligned}\forall x \in X: \; F'_{sz^*}(x_0,x) &= \left\{z \in Z \mid \, -sz^*(z) \geq \varphi'_{(F,sz^*)}(x_0,x) = s\varphi'_{(F,z^*)}(x_0,x)\right\} \\ &= F'_{z^*}(x_0,x).\end{aligned}$$

■

5.1.4 Proposition. *For a convex function $F: X \to \mathcal{Q}^t_C(Z)$ and $z^* \in C^* \setminus \{0\}$ it holds*

$$\forall x_0, x \in X: \quad F'_{z^*}(x_0,x) = (F \triangleleft_{z^*} \{0\})'_{z^*}(x_0,x) = (F'_{z^*}(x_0,\cdot) \triangleleft_{z^*} \{0\})(x).$$

Moreover, $F'_{z^*}(x_0,\cdot)$ is closed if and only if it is z^*-closed, proper if and only if it is z^*-proper, C-proper, if and only if it is proper and $z^* \in C^* \setminus -C^*$.

Proof. As

$$\forall x \in X: \quad F'_{z^*}(x_0,x) = \left\{z \in Z \mid \, -z^*(z) \geq \varphi'_{(F,z^*)}(x_0,x)\right\}$$

holds by 5.1.2, the equality of $F'_{z^*}(x_0,x)$ and $(F'_{z^*}(x_0,\cdot) \triangleleft_{z^*} \{0\})(x)$ for all $z^* \in C^* \setminus \{0\}$ and all $x_0, x \in X$ is immediate with 3.1.4. With 2.4.3 it holds

$$(F(x_0+sx) \triangleleft_{z^*} F(x_0)) = \Big((F \triangleleft_{z^*} \{0\})(x_0+sx) \triangleleft_{z^*} (F \triangleleft_{z^*} \{0\})(x_0)\Big),$$

thus $F'_{z^*}(x_0,x) = (F \triangleleft_{z^*} \{0\})'_{z^*}(x_0,x)$ holds for all $z^* \in C^* \setminus \{0\}$ and all $x_0, x \in X$. As $F'_{z^*}(x_0,\cdot)$ maps into $Q^t_{H(z^*)}(Z)$, $F'_{z^*}(x_0,\cdot)$ is proper if and only if it is z^*-proper, C-proper if and only if it is proper and $z^* \in C^* \setminus -C^*$. Applying 3.3.1, $F'_{z^*}(x_0,\cdot)$ is closed if and only if it is z^*-closed. ■

5.1.5 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$.*

a) If $F(x_0) = \emptyset$, then $F'_{z^*}(x,x) = Z$ for all $x \in X$.

b) If $(F(x_0) \lhd_{z^*} \{0\}) = Z$, then

$$F'_{z^*}(x_0, x) = \begin{cases} Z, & \text{if } \exists t > 0 : \; x_0 + tx \in \operatorname{dom} F; \\ \emptyset, & \text{else.} \end{cases}$$

In this case, $F'_{z^*}(x_0, 0) = Z$.

c) If $(F(x_0 + s_0 x) \lhd_{z^*} \{0\}) = Z$ for some $s_0 > 0$, then it holds $F'_{z^*}(x_0, x) = Z$.

Proof.

a) If $F(x_0) = \emptyset$, then

$$\forall x \in X, \, \forall t > 0 : \quad \frac{1}{t}\Big(F(x_0 + tx) \lhd_{z^*} F(x_0)\Big) = Z$$

holds an thus $F'_{z^*}(x_0, \cdot) \equiv Z$.

b) If $F(x_0) \lhd_{z^*} \{0\} = Z$ and $(x_0 + tx) \in \operatorname{dom} F$, then for all $s \in (0,t)$ it holds $F(x_0 + sx) \lhd_{z^*} \{0\} = Z$ and thus

$$\forall s \in (0,t) : \quad \frac{1}{s}\Big(F(x_0 + sx) \lhd_{z^*} F(x_0)\Big) = Z,$$

therefore $F'_{z^*}(x_0, x) = Z$.

c) If $(F(x_0 + s_0 x) \lhd_{z^*} \{0\}) = Z$, then $\varphi_{(F,z^*)}(x_0 + s_0 x) = -\infty$. If $x_0 \in \operatorname{dom} F$, then it holds $\varphi_{(F,z^*)}(x_0 + sx) = -\infty$ for all $s \in (0, s_0)$ and by 2.4.3 it holds

$$F'_{z^*}(x_0, x) = \operatorname{cl} \bigcup_{s>0} \frac{1}{s} \Big\{ z \in Z | \; -z^*(z) \geq (\varphi_{(F,z^*)}(x_0 + sx) \lhd \varphi_{(F,z^*)}(x_0)) \Big\} = Z.$$

If $x_0 \notin \operatorname{dom} F$, then we can apply a), completing the proof. ■

5.1.6 Corollary. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a convex function and $z^* \in C^* \setminus \{0\}$, then $F'_{z^*}(x_0, \cdot) : X \to Q^t_{H(z^*)}(Z)$ is positively homogeneous. If additionally $x_0 \in \operatorname{dom} F$ and $F(x_0) \lhd_{z^*} \{0\} \neq Z$ holds, then $F'_{z^*}(x_0, 0) = H(z^*)$.*

Proof. Let $s > 0$, then

$$\forall t > 0 : \quad \frac{1}{t}\Big(F(x_0 + tsx) \lhd_{z^*} F(x_0)\Big) = s\frac{1}{ts}\Big(F(x_0 + tsx) \lhd_{z^*} F(x_0)\Big) \in Q^t_{H(z^*)}(Z)$$

and thus the function $F'_{z^*}(x_0, \cdot)_X \to Q^t_{H(z^*)}(Z)$ is positively homogeneous. If $F(x_0)$ is a z^*-proper set, then $F(x_0) \lhd_{z^*} F(x_0) = H(z^*)$, proving the statement. ■

5.1.7 Proposition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function and $z^* \in C^* \setminus \{0\}$.*

a) If $F(x_0)$ is z^*-proper, then $F'_{z^*}(x_0, 0) = H(z^*)$ and $F'_{z^*}(x_0, 0) = Z$, else.

b) If $x_0 \in \operatorname{dom} F$, then $\operatorname{dom} F'_{z^*}(x_0, \cdot) = \{0\} \cup \operatorname{cone}(\operatorname{dom} F + \{-x_0\})$.

c) The function $F'_{z^*}(x_0, \cdot) : X \to Q^t_{H(z^*)}(Z)$ is sublinear.

d) If $F(x_0)$ is z^*-proper, then

$$\forall x \in X : \quad \{0\} \lhd_{z^*} F'_{z^*}(x_0, x) \supseteq F'_{z^*}(x_0, -x).$$

e) For $z_0^*, z^* \in C^* \setminus \{0\}$ it holds

$$(F'_{z_0^*}(x_0, \cdot))^*(x^*, z^*) = \begin{cases} H(z^*), & \text{if } S_{(x^*,z^*)} \preccurlyeq F'_{z_0^*}(x_0, \cdot)\ ; \\ \emptyset, & \text{else.} \end{cases}$$

Especially, $(F'_{z_0^*}(x_0, \cdot))^*(x^*, z^*) = \emptyset$, if $z_0^* \notin \operatorname{cone}\{z^*\}$.

f) It holds

$$(F'_{z^*}(x_0, \cdot))^{**}(x) = \bigcap_{S_{(x^*,z^*)} \preccurlyeq F'_{z^*}(x_0, \cdot)} (S_{(x^*,z^*)}(x)).$$

PROOF.

a) For all $x_0 \in X$ it holds

$$F'_{z^*}(x_0, 0) = F(x_0) \lhd_{z^*} F(x_0) = \begin{cases} H(z^*), & \text{if } F(x_0) \text{ is a } z^*\text{-proper set;} \\ Z, & \text{else.} \end{cases}$$

b) It holds $0 \in \operatorname{dom} F'_{z^*}(x_0, \cdot)$. If $x \neq 0$ and $t > 0$, then

$$\frac{1}{t}(F(x_0 + tx) \lhd_{z^*} F(x_0)) \neq \emptyset \ \Leftrightarrow\ x \in \frac{1}{t}(\operatorname{dom} F - x_0).$$

Thus, $\operatorname{dom} F'_{z^*}(x_0, \cdot) = \{0\} \cup \operatorname{cone}(\operatorname{dom} F - x_0)$ holds, as

$$F'_{z^*}(x_0, x) \neq \emptyset \ \Leftrightarrow\ \exists t > 0 :\ x \in \frac{1}{t}(\operatorname{dom} F - x_0).$$

c) Define $G : X \to Q^t_{H(z^*)}(Z)$ by

$$G(x) = F(x_0 + x) \lhd_{z^*} F(x_0)$$

for all $x \in X$. It holds

$$\operatorname{epi} G = \{(x, z) \in X \times Z |\ F(x_0) + z \subseteq F(x_0 + x)\}.$$

If $t \in (0, 1)$ and $(x_1, z_1), (x_2, z_2) \in \operatorname{epi} G$, then $F(x_0) \subseteq tF(x_0) + (1 - t)F(x_0)$ and

$$\begin{aligned} F(x_0) + (tz_1 + (1 - t)z_2) &\subseteq t(F(x_0) + z_1) + (1 - t)(F(x_0) + z_2) \\ &\subseteq tF(x_0 + x_1) + (1 - t)F(x_0 + x_2) \\ &\subseteq F(x_0 + tx_1 + (1 - t)x_2), \end{aligned}$$

as F is convex. Thus, $\operatorname{epi} G$ is a convex set and G a convex function. If $0 < s_1 < s_2$ holds, then there is $t \in (0, 1)$ such, that $s_1 = ts_2 + (1 - t)0$, therefore

$$\begin{aligned} G(s_1 x) &\supseteq tG(s_1 x) + (1 - t)G(0) \\ \frac{1}{s_1} G(s_1 x) &\supseteq \frac{1}{s_2} G(s_1 x) + \frac{1 - t}{ts_2} G(0) \\ &= \begin{cases} \frac{1}{s_2} G(s_1 x), & \text{if } F(x_0) \text{ is a } z^*\text{-proper set;} \\ \frac{1}{s_2} G(s_1 x) + Z, & \text{else.} \end{cases} \end{aligned}$$

Thus, the difference quotient $\frac{1}{t}G(tx)$ is, not increasing when $t > 0$ is decreasing.

First, let $s > 0$ and $x \in X$. Then $\frac{1}{t}G(tsx) = s\frac{1}{ts}G(tsx)$ and hence $F'_{z^*}(x_0, \cdot)$ is positively homogeneous as

$$F'_{z^*}(x_0, sx) = s \operatorname{cl} \bigcup_{ts>0} \frac{1}{ts}G(tsx) = sF'_{z^*}(x_0, x).$$

Involving 2.4.8e) and the facts that F is a convex function and $F(x_0)$ is a convex set it can be seen that

$$F'_{z^*}(x_0, x_1 + x_2) \supseteq \operatorname{cl} \bigcup_{t>0} \frac{1}{t}(G(tx_1) + G(tx_2)).$$

For $0 < s < t$ it holds

$$G(tx_1) + G(sx_2) \subseteq G(tx_1) + G(tx_2),$$

thus $F'_{z^*}(x_0, \cdot)$ is subadditive.

d) As $F'_{z^*}(x_0, \cdot)$ is sublinear, it holds

$$F'_{z^*}(x_0, x) + F'_{z^*}(x_0, -x) \subseteq F'_{z^*}(x_0, 0).$$

If $F(x_0)$ is z^*-proper, then $F'_{z^*}(x_0, 0) = H(z^*)$. Thus,

$$\begin{aligned} F'_{z^*}(x_0, -x) &\subseteq \{z \in Z |\ F'_{z^*}(x_0, x) + z \subseteq H(z^*)\} \\ &= \{0\} \lhd_{z^*} F'_{z^*}(x_0, x). \end{aligned}$$

e) By definition,

$$(F'_{z_0^*}(x_0, \cdot))^*(x^*, z^*) = \bigcap_{x \in X} (S_{(x^*,z^*)}(x) \lhd_{z^*} F'_{z_0^*}(x_0, x))$$

holds. Especially,

$$(F'_{z_0^*}(x_0, \cdot))^*(x^*, z^*) \subseteq (S_{(x^*,z^*)}(0) \lhd_{z^*} F'_{z_0^*}(x_0, 0)) \tag{5.1.1}$$

holds and $F'_{z_0^*}(x_0, 0) \in \{Z, H(z_0^*)\}$. Thus $(F'_{z_0^*}(x_0, \cdot))^*(x^*, z^*) = \emptyset$ if $z_0^* \notin \operatorname{cone}\{z^*\}$. By 5.1.3 it holds $F'_{tz_0^*}(x_0, x) = F'_{z_0^*}(x_0, x)$ for all $x \in X$ and $t > 0$, thus without loss we can assume $z^* = z_0^*$ from now on.

If $F'_{z_0^*}(x_0, 0) = Z$, then $F'_{z_0^*}(x_0, \cdot)$ has no conlinear minorant and by (5.1.1) it holds

$$(F'_{z^*}(x_0, \cdot))^*(x^*, z^*) = \emptyset.$$

From now on suppose $F'_{z^*}(x_0, 0) = H(z^*)$ By (5.1.1) it holds $(F'_{z^*}(x_0, \cdot))^*(x^*, z^*) \subseteq H(z^*)$.

If $S_{(x^*,z^*)}(x) \supseteq F'_{z^*}(x_0, x)$ holds for all $x \in X$, then

$$(S_{(x^*,z^*)}(x) \lhd_{z^*} F'_{z^*}(x_0, x)) \supseteq H(z^*)$$

and thus

$$(F'_{z^*}(x_0, \cdot))^*(x^*, z^*) = H(z^*).$$

Both $S_{(x^*,z^*)}$ and $F'_{z^*}(x_0,\cdot)$ map into $Q^t_{H(z^*)}(Z)$. If $S_{(x^*,z^*)}$ is not a conlinear minorant of $F'_{z^*}(x_0,\cdot)$, then there is $x \in X$ such, that $F'_{z^*}(x_0,x) \supset S_{(x^*,z^*)}(x)$ holds and it exists $s > 0$ such that

$$(S_{(x^*,z^*)}(x) \lhd_{z^*} F'_{z^*}(x_0,x)) \subseteq se + H(z^*)$$

with $z^*(e) = 1$. As both functions are positively homogeneous, we can apply 2.4.8a to prove

$$\begin{aligned}
(F'_{z^*}(x_0,\cdot))^*(x^*,z^*) &\subseteq \bigcap_{t>0} (S_{(x^*,z^*)}(tx) \lhd_{z^*} F'_{z^*}(x_0,tx)) \\
&\subseteq \bigcap_{t>0} (tS_{(x^*,z^*)}(x) \lhd_{z^*} tF'_{z^*}(x_0,x)) \\
&\subseteq \bigcap_{t>0} t(S_{(x^*,z^*)}(x) \lhd_{z^*} F'_{z^*}(x_0,x)) \\
&\subseteq \bigcap_{t>0} tse + H(z^*) \\
&= \emptyset.
\end{aligned}$$

f) By definition,

$$(F'_{z^*}(x_0,\cdot)^{**}(x) = \bigcap_{(x^*,z^)\in X^*\times C^*\setminus\{0\}} \Big(S_{(x^*,z^*)}(x) \lhd_{z^*} (F'_{z^*}(x_0,\cdot)^*(x^*,z^*)\Big)$$

and therefore

$$\begin{aligned}
(F'_{z^*}(x_0,\cdot)^{**}(x) &= \bigcap_{S_{(x^*,z^*)}\preccurlyeq F'_{z^*}(x_0,\cdot)} \Big(S_{(x^*,z^*)}(x) \lhd_{z^*} H(z^*)\Big) \\
&= \bigcap_{S_{(x^*,z^*)}\preccurlyeq F'_{z^*}(x_0,\cdot)} S_{(x^*,z^*)}(x).
\end{aligned}$$

■

The function $(F'_{z^*}(x_0,\cdot))^* : (X^* \times C^* \setminus \{0\}) \to Q^t_{H(z^*)}(Z)$ can be understood as the indicator function of the set $\Big\{(x^*,z^*) |\ S_{(x^*,z^*)} \preccurlyeq F'_{z^*}(x_0,\cdot)\Big\}$ mapping into $Q^t_{H(z^*)}(Z)$, a subset of $\mathcal{Q}^t_C(Z)$. Recall that the neutral element in $Q^t_{H(z^*)}(Z)$ is $H(z^*) \supseteq C$, while C is the neutral Element in $\mathcal{Q}^t_C(Z)$.

Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function. From 5.1.7 it is immediate, that if $F(x_0)$ is a z^*-proper set and $F'_{z^*}(x_0,x) = Z$ holds, then $F'_{z^*}(x_0,-x) = \emptyset$. Moreover, $\operatorname{dom} F'_{z^*}(x_0,\cdot) = X$ if $x_0 \in \operatorname{core}\operatorname{dom} F$.

5.1.8 Theorem. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a convex function and $x_0 \in \operatorname{core}\operatorname{dom} F$, then $\operatorname{dom} F'_{z^*}(x_0,\cdot) = X$ and the following are equivalent.*

a) The set $F(x_0)$ is z^*-proper.

b) The function F is z^*-proper.

c) The directional derivative $F'_{z^*}(x_0,\cdot)$ is proper.

PROOF. As $\operatorname{dom} F'_{z^*}(x_0,\cdot) = \operatorname{cone}(\operatorname{dom} F + x_0)$ holds, the first result is immediate. The set $F(x_0)$ is z^*-proper if and only if $\varphi_{(F,z^*)}(x_0) \in \mathbb{R}$. As F is convex and $x_0 \in \operatorname{core}\operatorname{dom} F = \operatorname{core}\operatorname{dom}\varphi_{(F,z^*)}$ holds by assumption, $\varphi_{(F,z^*)}$ is convex and $\varphi_{(F,z^*)}(x_0) \in \mathbb{R}$ holds if and only if $\varphi_{(F,z^*)}$ is a proper function. This is equivalent to F being z^*-proper, thus the first equivalence is proven. If $F'_{z^*}(x_0,\cdot)$ is proper, then $F'_{z^*}(x_0,0) = H(z^*)$ and thus $F(x_0)$ is z^*-proper. On the other hand, if $F(x_0)$ is z^*-proper, then $F'_{z^*}(x_0,0) = H(z^*)$ and

$$\forall \{0\} \lhd_{z^*} F'_{z^*}(x_0,x) \supseteq F'_{z^*}(x_0,-x).$$

If $F'_{z^*}(x_0,x) = Z$, then $F'_{z^*}(x_0,-x) = \emptyset$, a contradiction, as $\operatorname{dom} F'_{z^*}(x_0,\cdot) = X$. ■

5.1.9 Proposition. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a convex function and $x_0 \in \operatorname{dom} F$, then there exists $z_0 \in F(x_0) \lhd_{z^*} \{0\}$ such, that*

$$\operatorname{epi} F'_{z^*}(x_0,\cdot) \supseteq \operatorname{cone}(\operatorname{epi} F - (x_0,z_0)) \cup (\{0\} \times H(z^*)).$$

PROOF. If $F(x_0) \lhd_{z^*} \{0\} = Z$, then by 5.1.5

$$\begin{aligned} \operatorname{epi} F'_{z^*}(x_0,\cdot) &= \Big(\operatorname{cone}(\operatorname{dom} F - x_0) \times Z\Big) \cup \Big(\{0\} \times Z\Big) \\ &\supseteq \operatorname{cone}(\operatorname{epi} F - (x_0,z)) \cup (\{0\} \times H(z^*)) \end{aligned}$$

holds for all $z \in F(x_0) \lhd_{z^*} \{0\}$. If $F(x_0) \lhd_{z^*} \{0\} \neq Z$, then it exists $z_0 \in Z$ such, that $F(x_0) \lhd_{z^*} \{0\} = z_0 + H(z^*)$ and

$$\forall x \in X : \quad F'_{z^*}(x_0,x) \supseteq F(x_0+x) - z_0 + H(z^*)$$

and $F'_{z^*}(x_0,0) = H(z^*)$. Thus,

$$\operatorname{epi} F'_{z^*}(x_0,\cdot) \supseteq \operatorname{cone}(\operatorname{epi} F - (x_0,z_0)) \cup (\{0\} \times H(z^*))$$

holds, as $F'_{z^*}(x_0,\cdot)$ is a sublinear map from X into $Q^t_{H(z^*)}(Z)$. ■

5.1.10 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$ and $s > 0$.*

a) It holds $\varphi_{(\operatorname{cl} F'_{z^*}(x_0,\cdot),sz^*)}(x) = s \operatorname{cl} \varphi'_{(F,z^*)}(x_0,x)$.
If $z_0^* \in C^* \setminus \operatorname{cl}\operatorname{cone}\{z^*\}$, then $\varphi_{(\operatorname{cl} F'_{z^*}(x_0,\cdot),z_0^*)}(x) = -\infty$.

b) It holds
$$(\operatorname{cl} F'_{z^*}(x_0,\cdot))(x) = \Big\{z \in Z |\ -z^*(z) \geq \operatorname{cl} \varphi'_{(F,z^*)}(x_0,x)\Big\}.$$

PROOF.

a) By 3.1.5 and 3.3.1 it holds

$$\varphi_{(\operatorname{cl} F'_{z^*}(x_0,\cdot),sz^*)}(x) = s \operatorname{cl} \varphi_{(F'_{z^*}(x_0,\cdot),z^*)}(x).$$

By 5.1.3, it holds

$$\varphi_{(\operatorname{cl} F'_{z^*}(x_0,\cdot),sz^*)}(x) = s \operatorname{cl} \varphi'_{(F,z^*)}(x_0,x)$$

and the second claim.

b) By 3.3.5, the claim is immediate. ■

5.1.11 Lemma. *Let $z^* \in C^* \setminus \{0\}$ and $F : X \to \mathcal{Q}_C^t(Z)$ be a convex z^*-proper function. If $F'_{z^*}(x_0, 0) = (\operatorname{cl} F'_{z^*}(x_0, \cdot))(0) = H(z^*)$ holds, then*

$$\forall x \in X : \ (\operatorname{cl} F'_{z^*}(x_0, \cdot))(x) = \bigcap_{S_{(x^*,z^*)} \preccurlyeq F'_{z^*}(x_0,\cdot)} S_{(x^*,z^*)}(x).$$

Proof. By 5.1.10 it holds

$$\forall x \in X : \ (\operatorname{cl} F'_{z^*}(x_0, \cdot))(x) = \left\{ z \in Z | \ -z^*(z) \geq \operatorname{cl} \varphi'_{(F,z^*)}(x_0, x) \right\}.$$

If $F'_{z^*}(x_0, 0) = (\operatorname{cl} F'_{z^*}(x_0, \cdot))(0) = H(z^*)$ holds, then $\varphi'_{(F,z^*)}(x_0, 0) = \operatorname{cl} \varphi'_{(F,z^*)}(x_0, 0) = 0$, thus by 8.3.14 it holds

$$\forall x \in X : \ \operatorname{cl} \varphi'_{(F,z^*)}(x_0, x) = \sup_{x^* \leq \varphi'_{(F,z^*)}(x_0,\cdot)} x^*(x).$$

■

5.1.12 Lemma. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex function and $x_0 \in \operatorname{dom} F$, then $\operatorname{cl} F'_{z^*}(x_0, \cdot) : X \to \mathcal{Q}_C^t(Z)$ is proper if and only if $F'_{z^*}(x_0, 0) = (\operatorname{cl} F'_{z^*}(x_0, \cdot))(0) = H(z^*)$.*

Proof. Again, with

$$(\operatorname{cl} F'_{z^*}(x_0, \cdot))(x) = \left\{ z \in Z | \ -z^*(z) \geq \operatorname{cl} \varphi'_{(F,z^*)}(x_0, x) \right\}.$$

and the scalar result 8.3.14. ■

5.1.13 Proposition. *If $F : X \to \mathcal{Q}_C^t(Z)$ is a convex function and $(x_0, \bar{z}) \in \operatorname{int} \operatorname{epi} F$, then for all $x \in X$ it exists $z \in Z$ such, that $(x, z) \in \operatorname{int} \operatorname{epi} F'_{z^*}(x_0, \cdot)$. If additionally $F(x_0) \lhd_{z^*} \{0\} \neq Z$, then $F'_{z^*}(x_0, \cdot)$ is proper.*

Proof. By 5.1.9, it exists $z_0 \in F(x_0) \lhd_{z^*} \{0\}$ such, that

$$\operatorname{epi} F'_{z^*}(x_0, \cdot) \supseteq \operatorname{cone} (\operatorname{epi} F - (x_0, z_0)) \cup \{0\} \times H(z^*).$$

Thus, if $(x_0, \bar{z}) \in \operatorname{int} \operatorname{epi} F$ holds, then $(0, \bar{z} - z_0) \in \operatorname{int} \operatorname{epi} F'_{z^*}(x_0, \cdot)$. The function $F'_{z^*}(x_0, \cdot)$ is sublinear, thus by 3.3.8 for all $x \in X$ it exists $z \in Z$ such, that $(x, z) \in \operatorname{int} \operatorname{epi} F'_{z^*}(x_0, \cdot)$. The remaining claim is 5.1.8, as $x_0 \in \operatorname{int} \operatorname{dom} F = \operatorname{core} \operatorname{dom} F$ is assumed. ■

5.1.14 Proposition. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex proper function and $x_0 \in \operatorname{dom} F$. If F is z^*-proper for $z^* \in C^* \setminus \{0\}$ and $(x_0, z_0) \in \operatorname{int} \operatorname{epi} F$, then*

$$\forall x \in X : \ \exists S_{(x^*,z^*)} \preccurlyeq F'_{z^*}(x_0, \cdot) : \ F'_{z^*}(x_0, x) = S_{(x^*,z^*)}(x).$$

Proof. By 3.3.7, the scalarization $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is proper and continuous in $x_0 \in \operatorname{int} \operatorname{dom} \varphi_{(F,z^*)}$. Thus, by 8.3.15 it holds

$$\forall x \in X : \quad \varphi'_{(F,z^*)}(x_0, x) = \operatorname{cl} (\varphi'_{(F,z^*)}(x_0, \cdot))(x) \in \mathbb{R}.$$

Thus by 5.1.2 and 3.3.1

$$\forall x \in X : \quad F'_{z^*}(x_0, x) = \operatorname{cl} (F'_{z^*}(x_0, \cdot))(x)$$

and each image $F'_{z^*}(x_0, x)$ is a z^*-proper set. Moreover, by 8.3.15 it holds

$$\forall x \in X : \exists x^* \leq \varphi'_{(F,z^*)}(x_0, \cdot) : \quad \varphi'_{(F,z^*)}(x_0, x) = x_0^*(x).$$

It holds $x^* \leq \varphi'_{(F,z^*)}(x_0, \cdot)$ if and only if $S_{(x^*,z^*)} \preccurlyeq F'_{z^*}(x_0, \cdot)$ and $\varphi'_{(F,z^*)}(x_0, x) = x_0^*(x)$ holds if and only if $S_{(x^*,z^*)}(x) = F'_{z^*}(x_0, x)$, thus the claim is proven. ■

In fact, the assumption of $(x_0, z_0) \in \operatorname{int} \operatorname{epi} F$ could be replaced by either of the conditions introduced in 3.3.9. The next theorem will provide another version of the set-valued max-Formula introduced in 5.1.14.

5.1.15 Theorem (The Max-Formula)**.** *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex proper (C-proper) function and $x_0 \in \operatorname{dom} F$. If $(x_0, z_0) \in \operatorname{int} \operatorname{epi} F$, then it exists $z^* \in C^* \setminus \{0\}$ ($z^* \in C^* \setminus -C^*$) such that F is z^*-proper and*

$$\forall x \in X : \exists z \in Z : \quad (x, z) \in \operatorname{int} \operatorname{epi} (F'_{z^*}(x_0, \cdot)).$$

Moreover, $F'_{z^*}(x_0, \cdot)$ proper, $\operatorname{dom} F'_{z^*}(x_0, \cdot) = X$ and

$$\forall x \in X : \exists S_{(x^*,z^*)} \preccurlyeq F'_{z^*}(x_0, \cdot) : \quad F'_{z^*}(x_0, x) = S_{(x^*,z^*)}(x)$$

Proof. A function is proper (C-proper) if and only if it is z^*-proper for at least one $z^* \in C^* \setminus \{0\}$ ($z^* \in C^* \setminus -C^*$). Thus, by 5.1.13 $F'_{z^*}(x_0, \cdot)$ proper and $\operatorname{dom} F'_{z^*}(x_0, \cdot) = X$ and

$$\forall x \in X : \exists z \in Z : \quad (x, z) \in \operatorname{int} \operatorname{epi} (F'_{z^*}(x_0, \cdot)).$$

By 5.1.14, the final formula holds true. ■

5.2 Calculus

5.2.1 Proposition. *Let $G : Y \to Q^t_C(Z)$ be a convex function and $t > 0$, $A \in \mathcal{L}(X, Y)$, $y_0 \in Y$, $z_0 \in Z$, $(x_0^*, z_0^*) \in X^* \times C^* \setminus \{0\}$ and $z^* \in C^* \setminus \{0\}$. The function $F : X \to Q^t_C(Z)$ is defined by*

$$F(x) = tG(Ax + y_0) + S_{(x_0^*,z_0^*)}(x) + z_0$$

for all $x \in X$.

a) If $x_0 \in \operatorname{dom} F$ and $z^* \notin \operatorname{cone} \{z_0^*\}$ holds, then

$$F'_{z^*}(x_0, x) = \begin{cases} Z, & \text{if } x \in \operatorname{cone} (\operatorname{dom} F - x_0); \\ \emptyset, & \text{else.} \end{cases}$$

b) If $z^* \in \operatorname{cone} \{z_0^*\}$ holds, then

$$F'_{z^*}(x_0, x) = tG'_{z^*}(Ax_0 + y_0, Ax) + S_{(x_0^*,z_0^*)}(x).$$

Proof.

a) If $x_0 \in \operatorname{dom} F$ and $z^* \notin \operatorname{cone} \{z_0^*\}$ holds, then $F(x_0) \lhd_{z^*} \{0\} = Z$. Thus the claim holds by 5.1.5.

b) If $z^* \in \operatorname{cone}\{z_0^*\}$ holds, then

$$F'_{z^*}(x_0,x) = F'_{z_0^*}(x_0,x)$$

holds for all $x \in X$ by 5.1.3. For all $s > 0$ it holds

$$\begin{aligned}
&\frac{1}{s}F(x_0+sx) \lhd_{z^*} F(x_0)\\
=&\frac{1}{s}\Big((tG(Ax_0+y_0+sAx)+S_{(x_0^*,z^*)}(x_0+sx)+z_0) \lhd_{z^*} (tG(Ax_0+y_0)+S_{(x_0^*,z^*)}(x_0)+z_0)\Big)\\
=&\frac{1}{s}\Big((tG(Ax_0+y_0+sAx)+S_{(x_0^*,z^*)}(x_0+sx)) \lhd_{z^*} (tG(Ax_0+y_0)+S_{(x_0^*,z^*)}(x_0))\Big).
\end{aligned}$$

Recall that for $s,t>0$ and $(x^*,z^*) \in X^* \times C^* \setminus \{0\}$ it holds $tS_{(x^*,z^*)}(sx) = sS_{(tx^*,z^*)}(x)$ for all $x \in X$. Moreover it holds

$$\begin{aligned}
\forall x \in X: \quad & S_{(x_0^*,z_0^*)}(x) \in Q^t_{H(z^*)}(Z) \setminus \{\emptyset, Z\},\\
& S_{(x_0^*,z_0^*)}(x) + S_{(-x_0^*,z_0^*)}(x) = H(z^*),
\end{aligned}$$

so

$$\begin{aligned}
\frac{1}{s}F(x_0+sx) \lhd_{z^*} F(x) &= \frac{1}{s}\Big((tG(Ax_0+y_0+sAx)+sS_{(x_0^*,z^*)}(x)) \lhd_{z^*} tG(Ax_0+y_0)\Big)\\
&= t\frac{1}{s}\Big((G(Ax_0+y_0+sAx)+sS_{(\frac{1}{t}x_0^*,z^*)}(x)) \lhd_{z^*} G(Ax_0+y_0)\Big)\\
&= t\frac{1}{s}\Big(G(Ax_0+y_0+sAx) \lhd_{z^*} G(Ax_0+y_0)\Big) + S_{(x_0^*,z^*)}(x)
\end{aligned}$$

holds and the claim is proven as

$$\begin{aligned}
F'_{z^*}(x_0,x) &= \operatorname{cl}\bigcup_{s>0}\frac{1}{s}F(x_0+sx) \lhd_{z^*} F(x)\\
&= t\operatorname{cl}\bigcup_{s>0}\frac{1}{s}\Big(G(Ax_0+y_0+sAx) \lhd_{z^*} G(Ax_0+y_0)\Big) + S_{(x_0^*,z^*)}(x).
\end{aligned}$$

■

Recall that $F'_{z^*}(x_0,\cdot) \equiv Z$ holds, if $x_0 \notin \operatorname{dom} F$.

5.2.2 Proposition. *Let $F_i : X \to \mathcal{Q}^t_C(Z)$ for $n \geq 2$ and $i = 1,...,n$ be convex functions and $z^* \in C^* \setminus \{0\}$.*

a) If $F(x) = (\operatorname{co}(\inf_{i=1,\dots,n} F_i))(x) = (\operatorname{co}(\operatorname{cl}\operatorname{co}\bigcup_{i=1,\dots,n} F_i))(x)$ holds for all $x \in X$ and $F_j(x_0) \lhd_{z^*} \{0\} = F(x_0) \lhd_{z^*} \{0\}$ for $j = 1,...,m \leq n$ and $x_0 \in X$, then

$$F'_{z^*}(x_0,x) \supseteq \bigcup_{j=1,\dots,m} F'_{jz^*}(x_0,x)$$

holds for all $x \in X$. Especially, if $F_k(x_0) = F(x_0)$ holds for $k = 1,...,l \leq n$, then

$$F'_{\bar{z}^*}(x_0,x) \supseteq \bigcup_{j=1,\dots,m} F'_{j\bar{z}^*}(x_0,x)$$

holds for all $x \in X$ and $\bar{z}^* \in C^* \setminus \{0\}$.

b) If $F(x) = \sup\limits_{i=1,..,n} F_i(x) = \bigcap\limits_{i=1,...,n} F_i(x)$ holds for all $x \in X$ and $F_j(x_0) \triangleleft_{z^*} \{0\} = F(x_0) \triangleleft_{z^*} \{0\}$ for $j = 1, ..., m \leq n$ and $x_0 \in X$, then

$$F'_{z^*}(x_0, x) \subseteq \bigcap_{j=1,...,m} F'_{jz^*}(x_0, x)$$

holds for all $x \in X$. Especially, if $F_k(x_0) = F(x_0)$ holds for $k = 1, ..., l \leq n$, then

$$F'_{\bar{z}^*}(x_0, x) \subseteq \bigcap_{j=1,...,m} F'_{j\bar{z}^*}(x_0, x)$$

holds for all $x \in X$ and $\bar{z}^* \in C^* \setminus \{0\}$.

Proof.

a) If $F(x) = \operatorname{cl}\operatorname{co} \bigcup\limits_{i=1,...,n} F_i(x)$ holds for all $x \in X$ and $F_j(x_0) \triangleleft_{z^*} \{0\} = F(x_0) \triangleleft_{z^*} \{0\}$ for $j = 1, ..., m \leq n$ and $x_0 \in X$, then

$$F(x_0 + x) \triangleleft_{z^*} F(x_0) \supseteq F_j(x_0 + x) \triangleleft_{z^*} F_j(x_0)$$

holds for all $x \in X$. The difference quotient

$$\frac{1}{t}\Big(F_j(x_0 + tx) \triangleleft_{z^*} F_j(x_0)\Big)$$

is non-increasing, as $t > 0$ converges to 0. Thus,

$$\operatorname{cl} \bigcup_{t>0} \frac{1}{t}(F(x_0 + tx) \triangleleft_{z^*} F(x_0)) \supseteq \operatorname{cl} \bigcup_{t>0} \frac{1}{t}(F_j(x_0 + tx) \triangleleft_{z^*} F_j(x_0))$$

holds and therefore $F'_{z^*}(x_0, x) \supseteq F'_{jz^*}(x_0, x)$ holds for all $x \in X$ and $j = 1, ...m$. If $F(x_0) = F_k(x_0)$ then especially $F(x_0) \triangleleft_{\bar{z}^*} \{0\} = F_k(x_0) \triangleleft_{\bar{z}^*} \{0\}$ holds for all $\bar{z}^* \in C^* \setminus \{0\}$, thus the second inclusion is a special case of the first.

b) If $F(x) = \sup\limits_{i=1,..,n} F_i(x) = \bigcap\limits_{i=1,...,n} F_i(x)$ holds for all $x \in X$ and $F_j(x_0) \triangleleft_{z^*} \{0\} = F(x_0) \triangleleft_{z^*} \{0\}$ for $j = 1, ..., m \leq n$ and $x_0 \in X$, then $F : X \to \mathcal{Q}^t_C(Z)$ is convex and

$$F(x_0 + x) \triangleleft_{z^*} F(x_0) \subseteq F_j(x_0 + x) \triangleleft_{z^*} F_j(x_0)$$

holds for all $x \in X$. Thus $F'_{z^*}(x_0, x) \subseteq F'_{jz^*}(x_0, x)$ holds for all $x \in X$ and $j = 1, ...m$. If $F(x_0) = F_k(x_0)$ then especially $F(x_0) \triangleleft_{\bar{z}^*} \{0\} = F_k(x_0) \triangleleft_{\bar{z}^*} \{0\}$ holds for all $\bar{z}^* \in C^* \setminus \{0\}$, thus the second inclusion is a special case of the first.

■

5.2.3 Proposition. *Let $F_i : X \to \mathcal{Q}^t_C(Z)$ be convex functions for $n \geq 2$ and $i = 1, ..., n$ and $z^* \in C^* \setminus \{0\}$.*

a) If $F(x) = (F_1 \square ... \square F_n)(x)$ holds for all $x \in X$ and $F_1(x_1).. + F_n(x_n) = F(x_0)$ holds for $x_i \in X$ with $x_1 + ... + x_n = x_0$ and $F(x_0)$ is a z^*-proper set, then it holds

$$F'_{z^*}(x_0, x) = (F'_{1z^*}(x_1, \cdot) \square ... \square F'_{nz^*}(x_n, \cdot))(x)$$

for all $x \in X$.

b) If $F(x) = (F_1 + ... + F_n)(x)$ holds for all $x \in X$, then it holds

$$F'_{z^*}(x_0, x) \supseteq F'_{1z^*}(x_0, x) + ... + F'_{nz^*}(x_0, x)$$

for all $x_0, x \in X$.

c) If $F(x) = (F_1 + ... + F_n)(x)$ holds for all $x \in X$ and $\operatorname{dom} F_i = X$ and F_i is z^*-proper for all but possibly one $i \in \{1, ..., n\}$, then

$$F'_{z^*}(x_0, x) = F'_{1z^*}(x_0, x) + ... + F'_{nz^*}(x_0, x)$$

for all $x_0, x \in X$.

Proof.

a) Without loss of generality we assume $n = 2$. If $F(x) = (F_1 \square F_2)(x)$ holds for all $x \in X$ and $\emptyset \neq F_1(x_1) + F_2(x_2) = F(x_0) \neq Z$ holds for $x_i \in X$ with $x_1 + x_2 = x_0$, then it holds

$$\begin{aligned} F'_{z^*}(x_0, \bar{x}) &= \operatorname{cl} \bigcup_{t>0} \operatorname{cl} \bigcup_{\bar{x}_1 + \bar{x}_2 = \bar{x}} \frac{1}{t} \Big((F_1(x_1 + t\bar{x}_1) + F_2(x_2 + \bar{x}_2)) \lhd_{z^*} (F_1(x_1) + F_2(x_2)) \Big) \\ &= \operatorname{cl} \bigcup_{\bar{x}_1 + \bar{x}_2 = \bar{x}} \operatorname{cl} \bigcup_{t>0} \frac{1}{t} \Big((F_1(x_1 + t\bar{x}_1) \lhd_{z^*} F_1(x_1)) + (F_2(x_2 + \bar{x}_2) \lhd_{z^*} F_2(x_2)) \Big) \end{aligned}$$

As the difference quotient $\frac{1}{t}(F_i(x_i + t\bar{x}_i) \lhd_{z^*} F_i(x_i))$ is not increasing as t converges towards 0, it holds

$$\begin{aligned} F'_{z^*}(x_0, \bar{x}) &= \operatorname{cl} \bigcup_{\bar{x}_1 + \bar{x}_2 = \bar{x}} (F'_{1z^*}(x_1, \bar{x}_1) + F'_{2z^*}(x_2, \bar{x}_2)) \\ &= (F'_{1z^*}(x_1, \cdot) \square F'_{2z^*}(x_2, \cdot))(\bar{x}). \end{aligned}$$

b) Again we assume $n = 2$. If $F(x) = (F_1 + F_2)(x)$ holds for all $x \in X$, then by 2.4.8 it holds

$$F(x_0 + x) \lhd_{z^*} F(x_0) \supseteq (F(x_0 + x) \lhd_{z^*} F_1(x_0)) + (F_2(x_0 + x) \lhd_{z^*} F_2(x_0))$$

for all $x_0, x \in X$. As the difference quotient $\frac{1}{t}(F_i(x_i + t\bar{x}_i) \lhd_{z^*} F_i(x_i))$ is not increasing as t converges towards 0, it holds

$$F'_{z^*}(x_0, x) \supseteq (F'_{1z^*}(x_0, x) + F'_{2z^*}(x_0, x)).$$

c) We assume $n = 2$ and the additional assumptions are fulfilled by $F_1 : X \to \mathcal{Q}^t_C(Z)$. Then $F_1(x) \notin \{\emptyset, Z\}$ holds for all $x \in X$ and hence

$$F_1(x) + (\{0\} \lhd_{z^*} F_1(x)) = H(z^*).$$

Therefore,

$$\begin{aligned} &F(x_0 + x) \lhd_{z^*} F(x_0) \\ &\quad = \{z \in Z |\ (F_1(x_0) + F_2(x_0)) + z \subseteq (F_1(x_0 + x) + F_2(x_0 + x))\} \\ &\quad = \{z \in Z |\ (F_2(x_0)) + z \subseteq F_2(x_0 + x))\} + (F_1(x_0 + x) + (\{0\} \lhd_{z^*} F_1(x_0)) \\ &\quad = (F_2(x_0 + x) \lhd_{z^*} F_2(x_0)) + (F_1(x_0 + x) \lhd_{z^*} F_1(x_0)). \end{aligned}$$

Again, as the difference quotients are decreasing as $t > 0$ converges towards 0, the claim is proven.

∎

5.2.4 Proposition. *Let $A : X \to Y$ be a linear continuous operator and $G : Y \to \mathcal{Q}^t_C(Z)$ and $F \to \mathcal{Q}^t_C(Z)$ convex functions and $z^* \in C^* \setminus \{0\}$.*

a) It holds

$$(GA)'_{z^*}(x_0, x) = G'_{z^*}(Ax_0, Ax)$$

for all $x_0, x \in X$.

b) If $AF(Ax_0) = F(x_0)$ holds for $x_0 \in X$, then

$$(AF)'_{z^*}(Ax_0, Ax) \supseteq F'_{z^*}(x_0, x)$$

holds for all $x \in X$.

Proof.

a) This is immediate from 5.2.1.

b) If $AF(Ax) = F(Ax)$, then

$$AF(Ax_0 + Ax) \lhd_{z^*} AF(Ax_0) \supseteq F(x_0 + x) \lhd_{z^*} F(x_0)$$

holds for all $x \in X$. The rest is immediate, as F is assumed to be convex. ■

6 Subdifferential

In classic scalar convex analysis, the subdifferential of a proper convex function $f : X \to \mathrm{I\!R} \cup \{+\infty\}$ at $x_0 \in \operatorname{dom} f$ can be defined equivalently as the set of those elements $x^* \in X^*$ majorized by the directional derivative $f'(x_0, \cdot)$ of f in x_0,

$$\forall x \in X : \quad x^*(x - x_0) \le f(x) - f(x_0) \tag{6.0.1}$$

or those, for which the Young-Fenchel inequality holds for x_0 with equality,

$$\forall x \in X : \quad x^*(x_0) - f(x_0) \ge x^*(x) - f(x), \tag{6.0.2}$$

compare [10, 50, 62, 64]. These two concepts are no longer equivalent even for extended real functions, when $f : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is not proper. However, they coincide for convex functions, whenever $f(x_0) \in \mathrm{I\!R} \cup \{+\infty\}$ and the domain of f is not empty.

The distinction can be found implicitly in [63], where the subdifferential of a vector-valued function $f : X \to Z$ is defined by the an inequality of the type of (6.0.1) and it is pointed out, that $T \in \mathcal{L}(X, Z)$ is a subgradient of f at $x_0 \in \operatorname{dom} f$ if and only if

$$f^c(T) := \sup_{x \in X} (T(x) - f(x))$$

exists and $T(x_0) - f(x_0) = f^c(T)$.

The relation between both concepts of subdifferentials will hold valid in the concept of subdifferentials of convex set-valued functions as presented in the following. We will distinguish between the subdifferential of a convex function $F : X \to \mathcal{Q}^t_C(Z)$ in $x_0 \in X$ obtained via the directional derivative, $\partial F(x_0)$ and the extended subdifferential of a convex function $F : X \to \mathcal{Q}^t_C(Z)$ in $x_0 \in X$ obtained via the conjugate, $\partial_{ext} F(x_0)$, both consisting of conlinear set-valued functions $S_{(x^*,z^*)} : X \to \mathcal{Q}^t_C(Z)$ with $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$. Both kinds of subdifferentials can be represented by a family of (extended) z^*-subdifferentials in X^*, denoted accordingly by $\partial_{z^*} F(x_0)$ and $\partial_{ext,z^*} F(x_0)$. While the z^*-subdifferential and extended z^*-subdifferential coincide, whenever $F : X \to \mathcal{Q}^t_C(Z)$ is a z^*-proper function, the subdifferential and extended subdifferential of F coincide if and only if F is z^*-proper for all $z^* \in C^* \setminus \{0\}$. The latter situation occurs naturally, if F majorizes a vector-valued function $f : X \to Z$ or at least $F(x_0)$ and $-F(x_0)$ are minorized by elements $z_+, z_- \in Z$. This situation has been exploited in various approaches in the literature, compare [6, 12, 37, 61].

In subsection 7.2 we will point out that the z^*-subdifferential of a convex function at $x_0 \in X$ can be equally defined via the normal cone of $\operatorname{epi}(F \lhd_{z^*} \{0\})$ at an element (x_0, z_0) with $z_0 \in F(x_0) \lhd_{z^*} \{0\}$. Closely related to this are approaches introduced in [1] or [42, 43], differing from our concept mainly by the choice of the normal cone. Also, the concept of our simple z^*-subdifferential $\partial_{z*} F(x_0)$ appears as a scalarized characterization of the subdifferential in [56].

In [4], the subdifferential of a function is defined via a scalarized version of the Young-Fenchel equality. The Azimov-subdifferential of F in (x_0, z_0) is closely related to the extended z^*-subdifferential of F at x_0 in our approach.

Whenever the subdifferential of a vector- or set-valued function is understood to be a subset of $\mathcal{L}(X, Z)$ as in [60], there are basically two different types of relation to our approach. The fist appears, whenever the defining inequality is formulated via the inclusion in $T(x - x_0) + C$, as in [6, 60] and others. In this case, $T \in \mathcal{L}(X, Z)$ is a vector-valued subgradient of $F : X \to \mathcal{Q}^t_C(Z)$ if and only if $S_{(-T^*z^*,z^*)} \in \partial G(x_0)$ (or $\in \partial_{ext} G(x_0)$) holds for all $z^* \in C^* \setminus \{0\}$ for a certain minorant $G : X \to \mathcal{Q}_C^t(Z)$ of F. On the other hand, if empty intersection with $T(x - x_0) - \operatorname{int} C$ as in [12, 48, 54] in the defining inequality is used, then $T \in \mathcal{L}(X, Z)$ is a vector-valued subgradient if and only if $S_{(-T^* z_0^*, z_0^*)} \in \partial G(x_0)$ (or $\in \partial_{ext} G(x_0)$) holds for at least one $z_0^* \in C^* \setminus \{0\}$ for a specified minorant $G : X \to \mathcal{Q}_C^t(Z)$ of F.

Exemplarily, we will illuminate the connection between our subdifferential and the approach chosen in [60] and [65] in the final part of this chapter.

Throughout this chapter, X, Y and Z are assumed to be locally convex separable spaces with the corresponding dual spaces X^*, Y^* and Z^* and Z is quasi-ordered by a closed convex cone $C \subsetneq Z$ with $\{0\} \subsetneq C$.

6.1 Definition and basic results

6.1.1 Definition. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex function*

a) The subdifferential of F in $x_0 \in X$ is defined as

$$\partial F(x_0) := \left\{ S_{(x^*,z^*)} |\; x^* \in X^*, z^* \in C^* \setminus \{0\}, \forall x \in X : S_{(x^*,z^*)}(x) \supseteq F'_{z^*}(x_0, x) \right\}.$$

The subdifferential of F in $x_0 \in X$ with respect to $z^* \in C^* \setminus \{0\}$ is defined as

$$\partial_{z^*} F(x_0) := \left\{ x^* \in X^* |\; S_{(x^*,z^*)} \in \partial F(x_0) \right\}.$$

b) The extended subdifferential of F in x_0 is defined as

$$\partial_{ext} F(x_0) := \left\{ S_{(x^*,z^*)} |\; x^* \in X^*, z^* \in C^* \setminus \{0\}, S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) \subseteq F^*(x^*, z^*) \right\}.$$

The extended subdifferential of F in $x_0 \in X$ with respect to $z^* \in C^* \setminus \{0\}$ is defined as

$$\partial_{z^*,ext} F(x_0) := \left\{ x^* \in X^* |\; S_{(x^*,z^*)} \in \partial_{ext} F(x_0) \right\}.$$

Recall that for all $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$ it holds $S_{(x^*,z^*)}(x) = \{z \in Z |\; -z^*(z) \geq x^*(x)\} \in Q_{H(z^*)}^t(Z)$ for all $x \in X$ or equally, with $e \in Z$ such that $-z^*(e) = 1$, then $S_{(x^*,z^*)}(x) = x^*(x)e + H(z^*)$ holds for all $x \in X$.

Also, for $x^* \in X^*$, $z^* \in C^* \setminus \{0\}$ and $t > 0$ it holds

$$S_{(tx^*,tz^*)}(x) = S_{(x^*,z^*)}(x)$$

for all $x \in X$. Thus, $t\partial_{z^*} F(x) = \partial_{tz^*} F(x)$ holds for all $t > 0$ and $z^* \in C^* \setminus \{0\}$. Moreover, $S_{(x^*,z^*)} \in \partial F(x)$ holds if and only if $S_{(tx^*,tz^*)} \in \partial F(x)$ holds for all $t > 0$.

6.1.2 Corollary. *For a function convex $F : X \to \mathcal{Q}_C^t(Z)$, $x \in X$ and $z^* \in C^* \setminus \{0\}$ it holds*

$$\partial_{z^*} F(x) = \left\{ x^* \in X^* |\; \forall x \in X : \, x^*(x) \leq \varphi'_{(F,z^*)}(x_0, x) \right\}$$

and

$$\partial_{z^*,ext} F(x) = \left\{ x^* \in X^* |\; \forall x \in X : \, x^*(x_0) \lhd \varphi_{(F,z^*)}(x_0) \leq \varphi^*_{(F,z^*)}(x^*) \right\}.$$

Proof. It holds

$$\inf\{-z^*(z)|\ z \in F(x^*, z^*)\} = \varphi^*_{(F,z^*)}(x^*),$$
$$\inf\{-z^*(z)|\ z \in F'_{z^*}(x_0, x)\} = \varphi'_{(F,z^*)}(x_0, x)$$

and

$$\inf\left\{-z^*(z)|\ z \in S_{(x^*,z^*)}(x)\right\} = x^*(x).$$

Thus, the result is immediate with 2.4.2. ■

6.1.3 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $x_0 \in X$ and $z^* \in C^* \setminus \{0\}$. If $\operatorname{dom} F = \emptyset$ or $F(x_0) \lhd_{z^*} \{0\} = Z$, then $\partial_{z^*}(x_0) = \emptyset$ and $\partial_{z^*,ext} F(x_0) = X^*$. Otherwise it holds $\partial_{z^*} F(x_0) = \partial_{z^*,ext} F(x_0)$.*

Proof. If $\operatorname{dom} F = \emptyset$, then $F^* \equiv Z$ and $F'_{z^*}(x_0, \cdot) \equiv Z$ for all $z^* \in C^* \setminus \{0\}$, thus $\partial_{z^*,ext} F(x) = X^*$ and $\partial_{z^*} F(x) = \emptyset$ holds for all $z^* \in C^* \setminus \{0\}$. If $F(x_0) \lhd_{z^*} \{0\} = Z$, then $S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) = \emptyset$ holds for all $x^* \in X^*$, thus $\partial_{z^*,ext} F(x) = X^*$. Moreover, $F'_{z^*}(x_0, 0) = Z$ holds and thus $\partial F(x_0) = \emptyset$. From now on suppose that $F(x_0)$ is a z^*-proper set. It holds $F(x_0) \lhd_{z^*} \{0\} \notin \{\emptyset, Z\}$ and by standard argumentation

$$\forall x \in X: \quad S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) \subseteq S_{(x^*,z^*)}(x) \lhd_{z^*} F(x)$$

is equivalent to

$$\forall x \in X: \quad S_{(x^*,z^*)}(x) \supseteq \frac{1}{t}(F(x_0 + tx) \lhd_{z^*} F(x)).$$

Therefore, $\partial_{z^*} F(x_0) = \partial_{z^*,ext} F(x_0)$ holds. ■

6.1.4 Proposition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$. If $\operatorname{dom} F \neq \emptyset$ and $F(x_0) \lhd_{z^*} \{0\} \neq Z$ holds, then the following statements are equivalent:*

a)

$$x^* \in \partial_{z^*} F(x_0),$$

b)

$$\forall x \in X:\ S_{(x^*,z^*)}(x - x_0) \supseteq F(x) \lhd_{z^*} F(x_0),$$

c)

$$\forall x \in X:\ F(x_0) \oplus S_{(x^*,z^*)}(x - x_0) \supseteq F(x).$$

Proof. It holds $x^* \in \partial_{z^*} F(x_0)$ if and only if for all $t > 0$ and all $x \in X$ holds

$$S_{(x^*,z^*)}(tx) \supseteq F(x_0 + tx) \lhd_{z^*} F(x_0).$$

With $\bar{x} = x_0 + tx$ this is equivalent to

$$\forall x \in X:\ S_{(x^*,z^*)}(\bar{x} - x_0) \supseteq F(\bar{x}) \lhd_{z^*} F(x_0).$$

If $F(x_0) = \emptyset$ and $x \in \operatorname{dom} F$, then $\partial_{z^*} F(x_0) = \emptyset$ and equally

$$\forall x^*: \quad \emptyset = F(x_0) \oplus S_{(x^*,z^*)}(x - x_0) \not\supseteq F(x).$$

On the other hand if $F(x_0) \lhd_{z^*} \{0\} \notin \{\emptyset, Z\}$, then

$$F(x_0) \oplus (\{0\} \lhd_{z^*} F(x_0)) = H(z^*)$$

and thus $x^* \in \partial_{z^*} F(x_0)$ holds if and only if

$$\forall x \in X: \ S_{x^*,z^*}(\bar{x} - x_0) \oplus F(x_0) \supseteq F(\bar{x}).$$

■

6.1.5 Remark. *If $F : X \to \mathcal{Q}_C^t(Z)$ is a convex function, $x_0 \in \operatorname{dom} F$ and $F(x_0) \lhd_{z^*} \{0\} \neq Z$, then for all $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$ it exists $z_0 \in Z$ such that $S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) = z_0 + H(z^*)$. Moreover, it holds $x^* \in \partial_{z^*,ext} F(x_0)$ if and only if $S_{(x^*,z^*)} - z_0$ is a conaffine minorant of F. This again is true if and only if $z_0 \in F^*(x^*, z^*)$.*

6.1.6 Corollary. *If $F : X \to \mathcal{Q}_C^t(Z)$ is a convex and z_0^*-proper function for $z_0^* \in C^* \setminus \{0\}$, then*

$$0 \in \partial_{z_0^*} F(x_0) \quad \Leftrightarrow \quad F(x_0) \lhd_{z_0^*} \{0\} = \operatorname{cl} \bigcup_{x \in X} (F(x) \lhd_{z_0^*} \{0\}).$$

and

$$F(x_0) = \operatorname{cl} \bigcup_{x \in X} F(x) \quad \Leftrightarrow \quad \forall z^* \in C^* \setminus \{0\} : S_{(0,z^*)} \in \partial_{ext} F(x_0).$$

Proof. It holds $S_{(0,z^*)}(x - x_0) = H(z^*)$ for all $x \in X$, so if $F : X \to \mathcal{Q}_C^t(Z)$ is a convex and z_0^*-proper function for $z_0^* \in C^* \setminus \{0\}$, then by 6.1.4 it holds $0 \in \partial_{z_0^*} F(x_0)$ if and only if

$$\forall x \in X: \ F(x_0) \lhd_{z_0^*} \{0\} \supseteq F(x).$$

As $F(x_0) \lhd_{z_0^*} \{0\} \in Q_{H(z_0^*)}^t(Z)$, the first claim is proven.

As by assumption F is z_0^*-proper, it holds $\operatorname{dom} F \neq \emptyset$ and either it holds $F(x_0) \lhd_{z^*} \{0\} = Z$, or $\partial_{z^*} F(x_0) = \partial_{z^*,ext} F(x_0)$, therefore the second claim holds. ■

6.1.7 Lemma. *Let $F_1, F_2 : X \to \mathcal{Q}_C^t(Z)$ be two convex functions, $x_0 \in X$ and $z^* \in C^* \setminus \{0\}$. If $F_1(x) \supseteq F_2(x)$ holds for all $x \in X$ and $F_1(x_0) \lhd_{z^*} \{0\} = F_2(x_0) \lhd_{z^*} \{0\}$, then*

$$\partial_{z^*} F_1(x_0) \subseteq \partial_{z^*} F_2(x_0),$$
$$\partial_{z^*,ext} F_1(x_0) \subseteq \partial_{z^*,ext} F_2(x_0).$$

Especially, if $F_1(x_0) = F_2(x_0)$ holds, then

$$\partial F_1(x_0) \subseteq \partial F_2(x_0),$$
$$\partial_{ext} F_1(x_0) \subseteq \partial_{ext} F_2(x_0).$$

Proof. As $F_1(x_0) \lhd_{z^*} \{0\} = F_2(x_0) \lhd_{z^*} \{0\}$ holds, it can be proven by direct calculation that $F'_{1z^*}(x_0, x) \supseteq F'_{2z^*}(x_0, x)$ holds for all $x \in X$, thus the first inclusion holds. By assumption, $F_1^*(x^*, z^*) \subseteq F_2^*(x^*, z^*)$ holds for all $x^* \in X^*$, compare 4.1.9, thus the second inclusion holds. If $F_1(x_0) = F_2(x_0)$ holds, then $F_1(x_0) \lhd_{z^*} \{0\} = F_2(x_0) \lhd_{z^*} \{0\}$ holds for all $z^* \in C^* \setminus \{0\}$. ■

6.1.8 Proposition. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$.*

a) If $\partial_{z^*,ext} F(x_0) \neq \emptyset$*, then*

$$\begin{aligned} F(x_0) \lhd_{z^*} \{0\} &= (\mathrm{cl}\,(F \lhd_{z^*} \{0\}))(x_0), \\ \partial_{z^*,ext} F(x_0) &= \partial_{z^*,ext} (\mathrm{cl}\,\mathrm{co}\, F)(x_0), \\ \partial_{z^*} F(x_0) &= \partial_{z^*} (\mathrm{cl}\,\mathrm{co}\, F)(x_0) \end{aligned}$$

holds.

b) If $\partial_{z^*,ext} F(x_0) \neq \emptyset$ *holds for all* $z^* \in C^* \setminus \{0\}$*, then*

$$\begin{aligned} F(x_0) &= (\mathrm{cl}\, F)(x_0), \\ \partial_{ext} F(x_0) &= \partial_{ext} (\mathrm{cl}\,\mathrm{co}\, F)(x_0), \\ \partial F(x_0) &= \partial (\mathrm{cl}\,\mathrm{co}\, F)(x_0) \end{aligned}$$

holds.

Proof.

a) It holds $x^* \in \partial_{z^* ext} F(x_0)$ if and only if

$$S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) \subseteq F^*(x^*, z^*)$$

and $F^*(x^*, z^*) = (\mathrm{cl}\,(F \lhd_{z^*} \{0\}))^*(x^*, z^*)$ holds for all $x^* \in X^*$. If

$$F(x_0) \lhd_{z^*} \{0\} \subsetneq (\mathrm{cl}\,(F \lhd_{z^*} \{0\}))(x_0)$$

holds, then

$$S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) \supsetneq S_{(x^*,z^*)}(x_0) \lhd_{z^*} (\mathrm{cl}\,(F \lhd_{z^*} \{0\}))(x_0) \supseteq F^*(x^*, z^*),$$

a contradiction. The second equation is immediate from the definition 6.1.1.

As either $\mathrm{dom}\, F = \emptyset$, $F(x_0) \lhd_{z^*} \{0\} = Z$ or $\partial_{z^*} F(x_0) = \partial_{z^*,ext} F(x_0)$ holds by compare 6.1.3 and the same holds for $(\mathrm{cl}\,(F \lhd_{z^*} \{0\}))$, the last equation is immediate.

b) If $\partial_{z^*,ext} F(x_0) \neq \emptyset$ holds for all $z^* \in C^* \setminus \{0\}$, then

$$\begin{aligned} F(x_0) \lhd_{z^*} \{0\} &= (\mathrm{cl}\,(F \lhd_{z^*} \{0\}))(x_0), \\ \partial_{z^*,ext} F(x_0) &= \partial_{z^*,ext} (\mathrm{cl}\, F)(x_0), \\ \partial_{z^*} F(x_0) &= \partial_{z^*} (\mathrm{cl}\, F)(x_0) \end{aligned}$$

holds for all $z^* \in C^* \setminus \{0\}$. By 6.1.1, the rest is immediate. ■

6.1.9 Corollary. *Let* $F : X \to \mathcal{Q}^t_C(Z)$ *be a convex function,* $x \in X$ *and* $z^* \in C^* \setminus \{0\}$*. If*

$$\partial_{z^*} F(x) \neq \emptyset,$$

then F *is* z^**-proper,* $x \in \mathrm{dom}\, F$ *and* $F(x_0) \lhd_{z^*} \{0\} = (\mathrm{cl}\,(\mathrm{co}\, F \lhd_{z^*} \{0\})(x_0)$ *If*

$$\forall z^* \in C^* \setminus \{0\} : \quad \partial_{z^*} F(x) \neq \emptyset,$$

then $F(x) = (\mathrm{cl}\,\mathrm{co}\, F)(x)$ *and* $\partial F(x) = \partial (\mathrm{cl}\,\mathrm{co}\, F)(x)$*.*

Proof. By 6.1.3, $x \in \operatorname{dom} F$ holds and F is z^*-proper, as $\partial_{z^*} F(x) \neq \emptyset$. The rest is immediate by 6.1.8. ■

Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex function, $x \in X$. It is easy to see that for every $z^* \in C^* \backslash \{0\}$. the set $\partial_{z^*} F(x)$ is convex, possibly empty. In general, the set

$$\left\{(x^*, z^*) \in X^* \times C^* \setminus \{0\} \mid S_{(x^*,z^*)} \in \partial F(x)\right\}$$

is not convex, as the following example shows.

6.1.10 Example. *Let $F : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2), C = \mathbb{R}^2_+$ be a function defined by*

$$F(0) := \left\{(t, \frac{1}{t}) | \ t > 0\right\} + C$$

$$F(1) := \left\{(t, \frac{1}{2t}) | \ t > 0\right\} + C$$

and for $0 < s < 1$ $F(s) := sF(1) + (1-s)F(0)$ and $F(x) := \emptyset$, else. Then $S_{(0,(-2,0))}, S_{(0,(0,-2))} \in \partial F(0)$ but $S_{(0,(-1,-1))} \in \operatorname{co} \partial F(0) \setminus \partial F(0)$

Proof.

Let $z^* \in \{(-2,0),(0,-2)\}$, then

$$\varphi'_{(F,z^*)}(0,x) = \begin{cases} +\infty, & \text{if } x < 0; \\ 0, & \text{else.} \end{cases}$$

and therefore $S_{(0,z^*)} \in \partial F(0)$. On the other hand, let $z^* = (-1,-1)$. Then

$$\varphi_{(F,z^*)}(x) = \begin{cases} 2 + x(\sqrt{2} - 2), & \text{if } 0 \le x \le 1; \\ +\infty, & \text{else.} \end{cases}$$

Therefore,

$$\varphi'_{(F,z^*)}(0,1) = \sqrt{2} - 2$$

and $S_{(0,(-1,-1))} \notin \partial F(0)$. ■

6.1.11 Lemma. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a sublinear function, $z^* \in C^* \setminus \{0\}$.*

a) The set $\partial_{z^*} F(0)$ is not empty if and only if

$$(\operatorname{cl}(F \lhd_{z^*} \{0\}))(0) = F(0) \lhd_{z^*} \{0\} = H(z^*).$$

In this case,

$$\partial_{z^*} F(0) = \left\{x^* \in X^* | \ \forall x \in X : \ S_{(x^*,z^*)}(x) \supseteq F(x)\right\},$$

$$\forall x \in X : \quad \partial_{z^*} F(x) = \left\{x^* \in \partial_{z^*} F(0) | \ S_{(x^*,z^*)}(x) = F(x) \lhd_{z^*} \{0\}\right\}$$

holds and

$$\forall x \in X : \quad (\operatorname{cl}(F \lhd_{z^*} \{0\}))(x) = \bigcap_{x^* \in \partial_{z^*} F(0)} S_{(x^*,z^*)}(x).$$

b) If $\partial_{z^*} F(0)$ is not empty for all $z^* \in C^* \setminus \{0\}$, then

$$\forall x \in X : \quad (\operatorname{cl} F) = \bigcap_{\substack{x^* \in \partial_{z^*} F(0),\\ z^* \in C^* \setminus \{0\}}} S_{(x^*.z^*)}(x).$$

Proof.

a) If $\partial_{z^*} F(0) \neq \emptyset$, then by 6.1.9 and 6.1.3

$$(\operatorname{cl}(F \lhd_{z^*} \{0\}))(0) = F(0) \lhd_{z^*} \{0\} = H(z^*)$$

as $F(0) \lhd_{z^*} \{0\} \in \{\emptyset, H(z^*), Z\}$ holds.

The scalarization $\varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\}$ is sublinear and proper and $\varphi_{(F,z^*)}(0) = \operatorname{cl} \varphi_{(F,z^*)}(0)$. Thus,

$$\varphi^*_{(F,z^*)}(x^*) = \begin{cases} 0, & \text{if } \forall x \in X : x^*(x) \leq \varphi_{(F,z^*)}(x); \\ +\infty, & \text{else.} \end{cases}$$

And $\varphi^*_{(F,z^*)}(x_0^*) \neq +\infty$ holds for at least one $x^* \in X^*$.

By 6.1.2, and 4.1.2

$$\begin{aligned} \partial_{z^*} F(0) &= \{x^* \in X^* | \, F^*(x^*, z^*) \neq \emptyset\} \\ &= \left\{x^* \in X^* | \, \forall x \in X : S_{(x^*,z^*)}(x) \supseteq F(x)\right\} \end{aligned}$$

holds and by 6.1.1

$$\partial_{z^*} F(x) = \left\{x^* \in X^* | \, \forall x \in X : S_{(x^*,z^*)}(x) \supseteq F(x), \, S_{(x^*,z^*)}(x) = F(x) \lhd_{z^*} \{0\}\right\}$$

holds for all $x \in X$.

Obviously,

$$\forall x \in X : \quad (\operatorname{cl}(F \lhd_{z^*} \{0\}))(x) \subseteq \bigcap_{x^* \in \partial_{z^*} F(0)} S_{(x^*,z^*)}(x)$$

holds. On the other hand, suppose

$$(x_0, z_0) \in \bigcap_{x^* \in \partial_{z^*} F(0)} \operatorname{epi} S_{(x^*,z^*)} \setminus \operatorname{cl} \operatorname{epi}(F \lhd_{z^*} \{0\})$$

Notice that

$$\partial_{z^*} F(0) = \{x^* \in X^* | \, \sup\{x^*(x) + z^*(z) | \, (x, z) \in \operatorname{epi} F\} = 0\}$$

Then by a separation argument it exists $(x_0^*, z_0^*) \in X^* \times Z^* \setminus \{(0,0)\}$ such, that

$$x_0^*(x_0) + z_0^*(z_0^*) > \sup\{x_0^*(x) + z_0^*(z) | \, (x, z) \in \operatorname{cl} \operatorname{epi}(F \lhd_{z^*} \{0\})\} = 0$$

as $(0,0) \in \operatorname{epi}(F \lhd_{z^*} \{0\})$ and F is sublinear. Suppose $z_0^* = 0$, then

$$\forall x^* \in \partial_{z^*} F(0) : \quad \forall x \in X : S_{(x^* + x_0^*, z^*)}(x) \supseteq F(x),$$

a contradiction. Thus without loss of generality it holds $z_0^* = z^*$ and it holds

$$\forall x^* \in \partial_{z^*} F(0) : \quad \forall x \in X : S_{(x^* + x_0^*, z^*)}(x) \supseteq F(x),$$

a contradiction.

b) If $\partial_{z^*} F(0)$ is not empty for all $z^* \in C^* \setminus \{0\}$, then

$$\forall z^* \in C^* \setminus \{0\}, \forall x \in X: \quad (\mathrm{cl}\,(F \lhd_{z^*} \{0\}))(x) = \bigcap_{x^* \in \partial_{z^*} F(0)} S_{(x^*,z^*)}(x)$$

holds and

$$(\mathrm{cl}\, F)(x) = \bigcap_{z^* \in C^* \setminus \{0\}} (\mathrm{cl}\,(F \lhd_{z^*} \{0\}))(x).$$

■

6.1.12 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function and $x_0 \in X$, then*

$$\partial_{z^*} F(x_0) = \partial_{z^*} F'_{z^*}(x_0, 0),$$
$$\partial F(x_0) = \bigcup_{z^* \in C^* \setminus \{0\}} \partial F'_{z^*}(x_0, 0).$$

Proof. As the functions $S_{(x^*,z^*)} : X \to Q^t_{H(z^*)}(Z)$ are closed, it holds

$$\partial_{z^*} F(x_0) = \left\{ x^* \in X^* |\ \forall x \in X : \ S_{(x^*,z^*)}(x) \supseteq (\mathrm{cl}\,(F'_{z^*}(x_0, \cdot)))(x) \right\}.$$

If $(\mathrm{cl}\,(F'_{z^*}(x_0,\cdot)))(0) = F'_{z^*}(x_0,0)$ holds, then we can apply 6.1.11, proving the result. Otherwise $(\mathrm{cl}\,(F'_{z^*}(x_0,\cdot)))(0) = Z$ and thus $\partial_{z^*} F(x_0) = \emptyset$. Again, by 6.1.11 this is the desired result. ■

6.1.13 Corollary. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function and $x_0 \in \mathrm{dom}\, F$. For $z^* \in C^* \setminus \{0\}$ the following statements are equivalent.*

a) $\partial_{z^*} F(x_0) \neq \emptyset$,

b) $(\mathrm{cl}\, F'_{z^*}(x_0, \cdot))(0) = F'_{z^*}(x_0, 0) = H(z^*)$.

If one of these statement is true, then $\mathrm{cl}\, F'_{z^*}(x_0, \cdot)$ *is proper and*

$$\forall x \in X \quad (\mathrm{cl}\, F'_{z^*}(x_0, \cdot))(x) = \bigcap_{S_{(x^*,z^*)} \in \partial F(x_0)} S_{(x^*,z^*)}(x). \tag{6.1.1}$$

Proof. The equivalence holds by 6.1.12 and 6.1.11.

If $(\mathrm{cl}\, F'_{z^*}(x_0, \cdot))(0) = F'_{z^*}(x_0, 0) = H(z^*)$, then by 6.1.11

$$\forall x \in X \quad (\mathrm{cl}\, F'_{z^*}(x_0, \cdot))(x) = \bigcap_{S_{(x^*,z^*)} \in \partial F(x_0)} S_{(x^*,z^*)}(x) \tag{6.1.2}$$

holds and hence $F'_{z^*}(x_0, \cdot)$ is proper. ■

6.1.14 Theorem (Max-Formula)**.** *If $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function and $(x_0, z) \in$* $\mathrm{int\,epi}\, F$*, then*

$$\forall x \in X : \ \exists z \in Z : \quad (x, z) \in \mathrm{int\,epi}\,(F'_{z^*}(x_0, \cdot)).$$

If additionally F is z^*-proper, then $F'_{z^*}(x_0, \cdot) : X \to Q^t_{H(z^*)}(Z)$ is proper with $\mathrm{dom}\, F'_{z^*}(x_0, \cdot) = X$ *and*

$$\forall x \in X : \quad \exists S_{(x^*,z^*)} \in \partial F(x_0) : \ F'_{z^*}(x_0, x) = S_{(x^*,z^*)}(x).$$

Proof. By 5.1.13 it holds

$$\forall x \in X : \exists z \in Z : \quad (x,z) \in \operatorname{int}\operatorname{epi}\left(F'_{z^*}(x_0,\cdot)\right)$$

and $F'_{z^*}(x_0,\cdot) : X \to Q^t_{H(z^*)}(Z)$ is proper with $\operatorname{dom} F'_{z^*}(x_0,\cdot) = X$ if F is z^*-proper. By 5.1.15, the last equation holds true. ■

6.1.15 Definition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a function and $z^* \in C^* \setminus \{0\}$. Then $F^*_{z^*} : X^* \to Q^t_{H(z^*)}(Z)$ is defined by*

$$F^*_{z^*}(x^*) := F^*(x^*, z^*)$$

for all $x^* \in X^*$. The subdifferential of $F^*_{z^*}$ in $x^*_0 \in X^*$ is defined as

$$\partial F^*_{z^*}(x^*_0) := \left\{ x \in X | \ \forall x^* \in X^* : \ S_{(x^*-x^*_0,z^*)}(x) \supseteq F^*(x^*,z^*) \lhd_{z^*} F^*(x^*_0,z^*) \right\}.$$

The extended subdifferential of $F^*_{z^*}$ in $x^*_0 \in X^*$ is defined as

$$\partial_{ext} F^*_{z^*}(x^*_0) := \left\{ x \in X | \ \forall x^* \in X^* : \ S_{(x^*_0,z^*)}(x) \lhd_{z^*} F^*(x^*_0,z^*) \subseteq S_{(x^*,z^*)}(x) \lhd_{z^*} F^*(x^*,z^*) \right\}.$$

As by 4.1.9 the function $F^*_{z^*}$ is convex, it holds

$$\forall x^* \in X^* : \quad (F^*_{z^*})'_{z^*}(x^*_0, x^*) = \operatorname{cl} \bigcup_{t>0} \frac{1}{t}\Big(F^*_{z^*}(x^*_0 + t x^*) \lhd_{\bar{z}^*} F^*_{z^*}(x^*_0) \Big).$$

For convenience, we will abbreviate $(F^*_{z^*})'_{z^*}(x^*_0, x^*) = (F^*_{z^*})'(x^*_0, x^*)$.

6.1.16 Lemma. *For $F : X \to \mathcal{Q}^t_C(Z)$ and $z^* \in C^* \setminus \{0\}$ the function $F^*_{z^*} : X^* \to Q^t_{H(z^*)}(Z)$ is convex. The directional derivative of $F^*_{z^*} : X^* \to Q^t_{H(z^*)}(Z)$ with respect to $\bar{z}^* \in C^* \setminus \{0\}$ is*

$$(F^*_{z^*})'_{\bar{z}^*}(x^*_0, x^*) = (F^*_{z^*})'_{z^*}(x^*_0, x^*) \lhd_{\bar{z}^*} \{0\}.$$

Proof. If $x^*_0 \notin \operatorname{dom} F^*_{z^*}$, then

$$\forall \bar{z}^* \in C^* \setminus \{0\}, \forall x^* \in X^* : \quad (F^*_{z^*})'_{\bar{z}^*}(x^*_0, x^*) = Z.$$

Else, if $x^*_0 \in \operatorname{dom} F^*_{z^*}$, then for all $x^* \in X^*$ it holds

$$F^*_{z^*}(x^*) \lhd_{\bar{z}^*} F^*_{z^*}(x^*_0) = \Big(F^*_{z^*}(x^*) \lhd_{z^*} F^*_{z^*}(x^*_0) \Big) \lhd_{\bar{z}^*} \{0\},$$

proving the statement. ■

6.1.17 Lemma. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a function and $z^* \in C^* \setminus \{0\}$, then it holds*

$$\partial F^*_{z^*}(x^*_0) = \left\{ x \in X | \ \forall x^* \in X^* : \ S_{(x^*,z^*)}(x) \supseteq (F^*_{z^*})'(x^*_0, x^*) \right\},$$
$$\partial_{ext} F^*_{z^*}(x^*_0) = \left\{ x \in X | \ \forall x^* \in X^* : \ S_{(x^*_0,z^*)}(x) \lhd_{z^*} F^*(x^*_0, z^*) \subseteq F^{**}(x) \lhd_{z^*} \{0\} \right\}.$$

PROOF. It holds $x \in \partial F^*_{z^*}(x_0^*)$ if and only if

$$\forall x^* \in X^* : \; S_{(x^*-x_0^*,z^*)}(x) \supseteq F^*(x^*,z^*) \triangleleft_{z^*} F^*(x_0^*,z^*).$$

in this case, for $t > 0$ and $\bar{x}^* = tx^* + x_0^*$ it holds

$$\forall x^* \in X^* : \; S_{(x^*,z^*)}(x) \supseteq \frac{1}{t}\Big(F^*(x^*,z^*) \triangleleft_{z^*} F^*(x_0^*,z^*)\Big),$$

proving the first statement. The second statement is immediate, as

$$F^{**}(x) \triangleleft_{z^*} \{0\} = \bigcap_{x^* \in X^*} (S_{(x^*,\bar{z}^*)} \triangleleft_{z^*} F^*(x^*,z^*)).$$

■

6.1.18 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ is a function and $z^* \in C^* \setminus \{0\}$. If $\operatorname{dom} F = \emptyset$ or $(\operatorname{cl}\operatorname{co} F)(x_0) \triangleleft_{z^*} \{0\} = Z$ for $x_0 \in X$, then*

$$\partial F^*_{z^*}(x_0^*) = \emptyset,$$
$$\partial_{ext} F^*_{z^*}(x_0^*) = X.$$

If $(\operatorname{cl}\operatorname{co} F) : X \to \mathcal{Q}^t_C(Z)$ is a z^*-proper function, then

$$\partial F^*_{z^*}(x^*) = \partial_{ext} F^*_{z^*}(x^*)$$

holds for all $x^* \in X^*$.

PROOF. If it holds $\operatorname{dom} F = \emptyset$, then $F^*_{z^*} \equiv Z$ holds for all $z^* \in C^* \setminus \{0\}$, thus

$$(F^*_{z^*})'(x_0^*,\cdot) \equiv Z,$$
$$F^{**} \equiv \emptyset,$$

while

$$S_{(x_0^*,z^*)}(x) \triangleleft_{z^*} F^*(x_0^*,z^*) = \emptyset.$$

On the other hand let $(\operatorname{cl}\operatorname{co} F)(x_0) \triangleleft_{z^*} \{0\} = Z$, then $F^*_{z^*} \equiv \emptyset$ holds for all $z^* \in C^* \setminus \{0\}$, thus

$$(F^*_{z^*})'(x_0^*,\cdot) \equiv Z,$$
$$F^{**} \equiv Z,$$

Thus, by 6.1.17 the desired statement holds if $(\operatorname{cl}\operatorname{co} F)$ is not z^*-proper. Next, let $(\operatorname{cl}\operatorname{co} F) : X \to \mathcal{Q}^t_C(Z)$ be z^*-proper, then $F^{**}(x) \triangleleft_{z^*} \{0\} = (\operatorname{cl}\operatorname{co} F)(x) \triangleleft_{z^*} \{0\}$ holds for all $x \in X$ and thus F^{**} is z^*-proper. In this case, $F^*_{z^*} : X \to \mathcal{Q}^t_C(Z)$ is z^*-proper and it holds

$$\forall x^* \in X^* : \quad S_{(x^*-x_0^*,z^*)}(x_0) \supseteq F^*(x^*,z^*) \triangleleft_{z^*} F^*(x_0^*,z^*)$$

if and only if

$$\forall x^* \in X^* : \quad S_{(x_0^*,z^*)}(x_0) \triangleleft_{z^*} F^*(x_0^*,z^*) \supseteq S_{(x^*,z^*)}(x_0) \triangleleft_{z^*} F^*(x^*,z^*).$$

Thus, the claim is proven. ■

6.1.19 Proposition. *Let* $F : X \to \mathcal{Q}_C^t(Z)$ *be a function and* $z^* \in C^* \setminus \{0\}$. *For* $x_0^* \in X^*$ *it holds*

$$\partial F_{z^*}^*(x_0^*) = \{x_0 \in X |\, x_0^* \in \partial_{z^*} F^{**}(x_0)\}\,,$$
$$\partial_{ext} F_{z^*}^*(x_0^*) = \{x_0 \in X |\, x_0^* \in \partial_{z^*,ext} F^{**}(x_0)\}\,.$$

If additionally $(\operatorname{cl}\operatorname{co} F) : X \to \mathcal{Q}_C^t(Z)$ *is a* z^**-proper function, then*

$$\begin{aligned}\partial F_{z^*}^*(x_0^*) &= \partial_{ext} F_{z^*}^*(x_0^*)\\ &= \{x_0 \in X |\, x_0^* \in \partial_{z^*}(\operatorname{cl}\operatorname{co} F)(x_0)\}\\ &= \{x_0 \in X |\, x_0^* \in \partial_{z^*,ext}(\operatorname{cl}\operatorname{co} F)(x_0)\}\,.\end{aligned}$$

Proof. By 4.1.9, $F^{**} : X \to \mathcal{Q}_C^t(Z)$ is a convex function and $(F^{**})^*(x^*, z^*) = F^*(x^*, z^*)$ holds for all $x^* \in X^*$ and $z^* \in C^* \setminus \{0\}$. Moreover, F^{**} is either z^*-proper, or $\operatorname{dom} F^{**} = \emptyset$ or $F^{**} \lhd_{z^*} \{0\} \equiv Z$. If $\operatorname{dom} F^{**} = \emptyset$, then $F_{z^*}^* \equiv Z$ and

$$\begin{aligned}\forall x \in X, \forall x^* \in X^* : \quad & x \in \partial_{ext} F_{z^*}^*(x^*),\, x^* \in \partial_{z^*,ext} F^{**}(x),\\ & \partial F_{z^*}^*(x^*) = \emptyset,\, \partial_{z^*} F^{**}(x) = \emptyset.\end{aligned}$$

Equally, if $F^{**} \lhd_{z^*} \{0\} \equiv Z$ holds, then $\operatorname{dom} F_{z^*}^* = \emptyset$ and

$$\begin{aligned}\forall x \in X, \forall x^* \in X^* : \quad & x \in \partial_{ext} F_{z^*}^*(x^*),\, x^* \in \partial_{z^*,ext} F^{**}(x),\\ & \partial F_{z^*}^*(x^*) = \emptyset,\, \partial_{z^*} F^{**}(x) = \emptyset.\end{aligned}$$

From now on, suppose that F^{**} is z^*-proper. Then $F_{z^*}^* : X^* \to Q_{H(z^*)}^t(Z)$ is a proper function, especially, $F_{z^*}^*$ is a z^*-proper convex function. Thus, by 6.1.18

$$\forall x_0^* \in X^* : \quad \partial_{ext} F_{z^*}^*(x_0^*) = \partial F_{z^*}^*(x_0^*)$$

holds. Moreover,

$$\forall x_0 \in X : \quad \partial_{z^*} F^{**}(x_0) = \partial_{z^*,ext} F^{**}(x_0)$$

holds. If $x_0 \in \partial F_{z^*}^*(x_0^*)$ holds, then

$$S_{(x_0^*,z^*)} \lhd_{z^*} F^{**}(x_0) \subseteq (F^{**})^*(x_0^*, z^*),$$

thus $x_0^* \in \partial_{z^*} F^{**}(x_0)$. The same inclusion provides

$$S_{(x_0^*,z^*)} \lhd_{z^*} F^*(x_0^*, z^*) \subseteq F^{**}(x_0),$$

if $x_0^* \in \partial_{z^*} F^{**}(x_0)$, so in this case $x_0 \in \partial F_{z^*}^*(x_0^*)$.

If $(\operatorname{cl}\operatorname{co} F) : X \to \mathcal{Q}_C^t(Z)$ is a z^*-proper function, then $(\operatorname{cl}\operatorname{co} F)(x) = F^{**}(x)$ holds for all $x \in X$, proving the statement. ■

6.1.20 Theorem. *Let* $F : X \to \mathcal{Q}_C^t(Z)$ *be a convex* z^**-proper function for* $z^* \in C^* \setminus \{0\}$. *If*

$$\forall x \in X : \quad F(x) \lhd_{z^*} \{0\} = (\operatorname{cl} F)(x) \lhd_{z^*} \{0\}$$

holds, then

$$\partial F_{z^*}^*(x^*) = \partial_{ext} F_{z^*}^*(x^*)$$

holds for all $x^* \in X^*$ *and*

$$\partial F_{z^*}^*(x_0^*) = \{x_0 \in X |\, x_0^* \in \partial_{z^*} F(x_0)\}\,.$$

PROOF. If $F : X \to \mathcal{Q}_C^t(Z)$ is z^*-proper and

$$\forall x \in X : \quad F(x) \lhd_{z^*} \{0\} = (\mathrm{cl}\,\mathrm{co}\, F)(x) \lhd_{z^*} \{0\}$$

holds, then $(\mathrm{cl}\,\mathrm{co}\, F) : X \to \mathcal{Q}_C^t(Z)$ is z^*-proper and by 6.1.19

$$\partial F^*_{z^*}(x^*) = \partial_{ext}\, F^*_{z^*}(x^*)$$

holds for all $x^* \in X^*$ and

$$\partial F^*_{z^*}(x_0^*) = \{x_0 \in X |\; x_0^* \in \partial_{z^*}(\mathrm{cl}\,\mathrm{co}\, F)(x_0)\}\,.$$

Moreover,

$$\partial_{z^*} F(x_0) = \{x_0 \in X |\; x_0^* \in \partial_{z^*}(\mathrm{cl}\, F)(x_0)\}$$

holds for all $x_0 \in X$, proving the statement. ■

6.1.21 Corollary. *If $F : X \to \mathcal{Q}_C^t(Z)$ is a closed convex function which is z^*-proper for all $z^* \in C^* \setminus \{0\}$, then*

$$\forall x \in X : \quad \partial F(x) = \left\{ S_{(x^*,z^*)} |\; x \in \partial F^*_{z^*}(x^*) \right\}.$$

PROOF. As

$$\forall x \in X : \quad \partial F(x) = \bigcup_{z^* \in C^* \setminus \{0\}} \left\{ S_{(x^*,z^*)} |\; x^* \in \partial_{z^*} F(x) \right\}$$

holds and by 6.1.20

$$\forall x \in X : \quad \partial_{z^*} F(x) = \{x^* \in X^* |\; x \in \partial_{z^*} F(x)\}\,,$$

the claim is immediate. ■

6.2 Calculus

Recall that it holds

$$\begin{aligned} \partial_{z^*} F(x_0) :&= \left\{ x^* \in X^* |\; S_{(x^*,z^*)} \in \partial F(x_0) \right\} \\ &= \left\{ x^* \in X^* |\; S_{(x^*,z^*)} \in \partial (F \lhd_{z^*} \{0\})(x_0) \right\}. \end{aligned}$$

Moreover it holds

$$\partial F(x_0) = \bigcup_{z^* \in C^* \setminus \{0\}} \left\{ S_{(x^*,z^*)} |\; x^* \in \partial_{z^*} F(x_0) \right\}.$$

Thus without loss of generality we can assume $F(x) = (F \lhd_{z^*} \{0\})(x)$ for all $x \in X$ when dealing with the set $\partial_{z^*} F(x_0)$.

If $\mathrm{dom}\, F = \emptyset$ or $F(x_0) \lhd_{z^*} \{0\} = Z$ holds, then by 6.1.4 it holds

$$\forall x^* \in X^* : \quad S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) \subseteq F^*(x^*, z^*) = Z \tag{6.2.1}$$

and $\partial_{z^*} F(x_0) = \emptyset$. In any other case (that is $\mathrm{dom}\, F \neq \emptyset$ and $F(x_0) \lhd_{z^*} \{0\} \neq Z$) it holds

$$\partial_{z^*} F(x_0) = \left\{ x^* \in X^* |\; S_{(x^*,z^*)}(x_0) \lhd_{z^*} F(x_0) \subseteq F^*(x^*, z^*) \right\}. \tag{6.2.2}$$

6.2.1 Proposition. *Let $G : Y \to Q^t_C(Z)$ be a convex function and $t > 0$, $y_0 \in Y$, $z_0 \in Z$ and $(x^*_0, z^*_0) \in X^* \times C^* \setminus \{0\}$ and $A : X \to Y$ a linear homeomorphism. The function $F : X \to Q^t_C(Z)$ is defined by*

$$F(x) = tG(Ax + y_0) + S_{(x^*_0,z^*_0)}(x) + z_0$$

for all $x \in X$.

a) For all $x_0 \in X$ it holds

$$\partial F(x_0) = \partial(F \lhd_{z^*_0} \{0\})(x_0).$$

Especially, $S_{(x^*,z^*)} \in \partial F(x_0)$ holds if and only if

$$(x^*, z^*) \in \operatorname{cone}\Big(\partial_{z^*_0} F(x_0) \times \{z^*_0\}\Big).$$

b) It holds

$$\partial_{z^*_0} F(x_0) = tA^* \partial_{z^*_0} G(Ax_0 + y_0) + x^*$$

and

$$\partial F(x_0) = \Big\{S_{(tA^*y^*+x^*_0,z^*)} |\; S_{(y^*,z^*_0)} \in \partial G(Ax_0 + y_0)\Big\}.$$

Proof.

a) Notice that $F(x) = (F \lhd_{z^*_0} \{0\})(x)$ holds for all $x \in X$. Moreover, $S_{(tx^*,tz^*)}(x) = S_{(x^*,z^*)}(x)$ holds for all $t > 0$ and $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$. If $x_0 \notin \operatorname{dom} F$, then $\partial F(x_0) = \emptyset$. From now on, assume that $x_0 \in \operatorname{dom} F$. It holds $S_{(x^*,z^*)} \in \partial F(x_0)$ if and only if

$$\forall x \in X : \quad S_{(x^*,z^*)}(x) \supseteq F'_{z^*}(x_0, x) \in Q^t_{H(z^*)}(Z).$$

By 5.2.1, $F'_{z^*}(x_0, 0) = Z$ if $z^* \notin \operatorname{cone}\{z^*_0\}$, thus the claim is proven.

b) The set $F(x_0)$ is z^*_0-improper if and only if the set $G(Ax_0 + y)$ is z^*_0-improper. In this case

$$\partial_{z^*_0} F(x_0) = tA^* \partial_{z^*_0} G(Ax_0 + y_0) + x^* = \emptyset.$$

Therefore let us assume that $F(x_0)$ is a z^*_0-proper set. By 4.2.1 it holds

$$F^*(x^*, z^*_0) = \Big(tG^*(\frac{1}{t}A^{-1*}(x^* - x^*_0), z^*_0) \lhd_{z^*} S_{(x^*-x^*_0,z^*_0)}(A^{-1}y_0)\Big) - z_0.$$

Thus by ((6.2.2)) it holds

$$\begin{aligned}
&\partial_{z^*_0} F(x_0)\\
&\quad= \Big\{x^* \in X^* |\; \Big(S_{(x^*,z^*_0)}(x_0) \lhd_{z^*} F(x_0)\Big) \subseteq F^*(x^*, z^*_0)\Big\}\\
&\quad= \Big\{x^* \in X^* |\; \Big(S_{(x^*-x^*_0,z^*_0)}(x_0 + A^{-1}y_0) \lhd_{z^*} tG(Ax_0 + y_0)\Big) \subseteq tG^*(\frac{1}{t}A^{-1*}(x^* - x^*_0), z^*_0)\Big\}\\
&\quad= \Big\{x^* \in X^* |\; \Big(S_{(\frac{1}{t}A^{-1*}(x^*-x^*_0),z^*_0)}(Ax_0 + y_0) \lhd_{z^*} G(Ax_0 + y_0)\Big) \subseteq G^*(\frac{1}{t}A^{-1*}(x^* - x^*_0), z^*_0)\Big\}\\
&\quad= tA^* \Big\{y^* \in Y^* |\; \Big(S_{(y^*,z^*_0)}(Ax_0 + y_0) \lhd_{z^*} G(Ax_0 + y_0)\Big) \subseteq G^*(y^*, z^*_0)\Big\} + x^*_0\\
&\quad= tA^* \partial_{z^*_0} G(Ax_0 + y_0) + x^*_0.
\end{aligned}$$

■

6.2.2 Proposition. *Let $F_i : X \to \mathcal{Q}^t_C(Z)$ for $n \geq 2$ and $i = 1, ..., n$ be convex functions.*

a) If $F(x) = (\operatorname{co}(\inf_{i=1,\ldots,n} F_i))(x) = (\operatorname{co}(\operatorname{cl}\operatorname{co} \bigcup_{i=1,\ldots,n} F_i))(x)$ holds for all $x \in X$ and $F_j(x_0) \lhd_{z^*} \{0\} = F(x_0) \lhd_{z^*} \{0\}$ for $j = 1, ..., m \leq n$ and $x_0 \in X$, then

$$\partial_{z^*} F(x_0) \subseteq \bigcap_{j=1,\ldots,m} \partial_{z^*} F_j(x_0).$$

Especially, if $F_k(x_0) = F(x_0)$ holds for $k = 1, ..., l \leq n$, then

$$\partial F(x_0) \subseteq \bigcap_{k=1,\ldots,l} \partial F_k(x_0).$$

b) If $F(x) = \sup_{i=1,..,n} F_i(x) = \bigcap_{i=1,\ldots,n} F_i(x)$ holds for all $x \in X$ and $F_j(x_0) \lhd_{z^*} \{0\} = F(x_0) \lhd_{z^*} \{0\}$ for $j = 1, ..., m \leq n$ and $x_0 \in X$, then

$$\partial_{z^*} F(x_0) \supseteq \operatorname{co} \bigcup_{j=1,\ldots,m} \partial_{z^*} F_j(x_0).$$

Especially, if $F_k(x_0) = F(x_0)$ holds for $k = 1, ..., l \leq n$, then

$$\partial F(x_0) \supseteq \operatorname{co} \bigcup_{k=1,\ldots,m} \partial F_k(x_0).$$

Proof.

a) By 5.2.2, $F'_{z^*}(x_0, x) \supseteq F'_{jz^*}(x_0, x)$ holds for all $x \in X$, if $F_j(x_0) \lhd_{z^*} \{0\} = F(x_0) \lhd_{z^*} \{0\}$. Thus,

$$\begin{aligned} \partial_{z^*} F(x_0) &= \left\{x^* \in X^* \mid \forall x \in X : S_{(x^*,z^*)}(x) \supseteq F'_{z^*}(x_0, x)\right\} \\ &\subseteq \partial_{z^*} F_j(x_0). \end{aligned}$$

If $F_k(x_0) = F(x_0)$, then for all $z^* \in C^* \setminus \{0\}$ above inclusion holds true.

b) By 5.2.2, $F'_{z^*}(x_0, x) \subseteq F'_{jz^*}(x_0, x)$ holds for all $x \in X$, if $F_j(x_0) \lhd_{z^*} \{0\} = F(x_0) \lhd_{z^*} \{0\}$. Thus,

$$\begin{aligned} \partial_{z^*} F(x_0) &= \left\{x^* \in X^* \mid \forall x \in X : S_{(x^*,z^*)}(x) \supseteq F'_{z^*}(x_0, x)\right\} \\ &\supseteq \partial_{z^*} F_j(x_0). \end{aligned}$$

If $F_k(x_0) = F(x_0)$, then for all $z^* \in C^* \setminus \{0\}$ above inclusion holds true.

$\blacksquare$

6.2.3 Proposition. *Let $F_i : X \to \mathcal{Q}^t_C(Z)$ be convex functions for $n \geq 2$ and $i = 1, ..., n$.*

a) If $F(x) = (F_1 \square ... \square F_n)(x)$ holds for all $x \in X$ and $\operatorname{dom} F_1 + .. + \operatorname{dom} F_n \neq \emptyset$, then it holds

$$\partial F(x) \supseteq \bigcap_{i=1,\ldots n} \partial F_i(x_i)$$

for all $x \in X$ with $x = x_1 + ... + x_n$.

b) If $F(x) = (F_1 + ... + F_n)(x)$ holds for all $x \in X$, then it holds

$$\partial_{z^*}F(x_0) \subseteq \partial_{z^*}F_1(x_0) + ... + \partial_{z^*}F_n(x_0),$$
$$\partial F(x_0) \subseteq \left\{ S_{(x^*,z^*)} |\; x^* \in \partial_{z^*}F_1(x_0) + ... + \partial_{z^*}F_n(x_0) \right\}$$

for all $x_0 \in X$.

c) If $F(x) = (F_1 + ... + F_n)(x)$ holds for all $x \in X$ and for all $i = 1, ..., n$ the functions F_i are convex and z^*-proper and for all except possibly one F_i it holds

$$\exists z \in Z : \quad (x_0, z) \in \operatorname{int} \operatorname{epi} F,$$

for all $x_0 \in \bigcap_{i=1,...,n} \operatorname{dom} F_i$, then

$$\partial_{z^*}F(x) = \partial_{z^*}F_1(x) + ... + \partial_{z^*}F_n(x)$$

holds for all $x \in X$. If additionally the functions F_i are z^*-proper for all $z^* \in C^* \setminus \{0\}$, then

$$\partial F(x) = \left\{ S_{(x^*,z^*)} |\; x^* \in \partial_{z^*}F_1(x) + ... + \partial_{z^*}F_n(x) \right\}$$

holds for all $x \in X$.

Proof.

a) Let $n = 2$ and $x^* \in \partial_{z^*}F_1(\bar{x}) \cap \partial_{z^*}F_2(x - \bar{x})$. Then $F_1(\bar{x})$ and $F_2(x - \bar{x})$ are z^*-proper sets and by 2.4.9 it holds

$$S_{(x^*,z^*)}(x) \lhd_{z^*} (F(\bar{x}) + F_2(x - \bar{x})) \subseteq F^*(x^*, z^*),$$

hence $x^* \in \partial_{z^*}F(x)$.

b) By 5.2.3, $F'_{z^*}(x_0, x) \subseteq F'_{1z^*}(x_0, x) + ... + F'_{nz^*}(x_0, x)$ holds for all $x \in X$. Thus if

$$x^* = x_1^* + ... + x_n^*,$$
$$\forall i = 1, ..., n, \forall x \in X : \quad S_{(x_i^*,z^*)}(x) \supseteq F'_{iz^*}(x_0, x)$$

holds, then

$$\forall x \in X : \quad S_{(x^*,z^*)}(x) \supseteq F'_{z^*}(x_0, x)$$

and hence the claim is proven.

c) Without loss of generality we assume $n = 2$ and $(x_0, z) \in \operatorname{int} \operatorname{epi} F_1$ for $x_0 \in \operatorname{dom} F_2$ and $z \in Z$. If $\partial F(x) = \emptyset$ for $x \in X$, then we are finished, as $\partial F(x) \supseteq \partial F_1(x) + \partial_2 F(x)$. Now suppose $S_{(x^*,z^*)} \in \partial F(x)$, then $x \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2 = \operatorname{dom} F$ and by 5.1.13 it holds

$$\forall \bar{x} \in X : \exists z \in Z : \quad (\bar{x}, z) \in \operatorname{int} \operatorname{epi} F'_{z^*}(x, \cdot)$$

And for all $\bar{x} \in X$ it holds $F'_{z^*}(x_0, x) \neq Z$. If $F_{2z^*}(x, \cdot)$ is proper we can apply 4.2.4c) and 5.1.7e) to achieve

$$\begin{aligned} \partial_{z^*}F(x) &= \{x^* \in X^* |\; (x^*, z^*) \in \operatorname{dom} F'_{z^*}(x, \cdot)^*\} \\ &= \{x^* \in X^* |\; (x^*, z^*) \in \operatorname{dom} F'_{1z^*}(x, \cdot)^* + \operatorname{dom} F'_{2z^*}(x, \cdot)^*\} \\ &= \partial_{z^*}F_1(x) + \partial_{z^*}F_2(x). \end{aligned}$$

As $x \in \operatorname{dom} F_2$ holds, $\operatorname{dom} F'_{2z^*}(x,\cdot) \neq \emptyset$. Therefore, suppose $F'_{2z^*}(x,y) = Z$ for $y \in X$. Then, by 5.2.3 it holds

$$F'_{z^*}(x,y) \supseteq F'_{1z^*}(x,y) + F'_{2z^*}(x,y) = Z$$

and thus $\partial_{z^*} F(x) = \emptyset$, a contradiction. ∎

6.2.4 Proposition. *Let $A : X \to Y$ be a linear continuous operator and $G : Y \to \mathcal{Q}^t_C(Z)$ and $F \to \mathcal{Q}^t_C(Z)$ convex functions.*

a) It holds

$$\partial_{z^*}(AF)(Ax) \supseteq A^* \partial_{z^*} F(Ax)$$

for all $x \in X$ and all $z^* \in C^* \setminus \{0\}$. If $AF(Ax) \lhd_{z^*} \{0\} = F(Ax) \lhd_{z^*} \{0\}$, then equality holds.

b) It holds

$$A^* \partial_{z^*} G(Ax) \subseteq \partial_{z^*} GA(x)$$

for all $x \in X$ and all $z^* \in C^* \setminus \{0\}$.

c) If for G one of the assumptions in 3.3.7 and 3.3.9 holds in Ax_0, $x_0 \in X$, then

$$A^* \partial_{z^*} G(Ax) = \partial_{z^*} GA(x)$$

holds for all $x \in X$ and all $z^* \in C^* \setminus \{0\}$. Especially,

$$\partial GA(x) = \left\{ S_{(A^*y^*,z^*)} |\ S_{(y^*,z^*)} \in \partial G(Ax) \right\}$$

Proof.

a) By 4.2.6 it holds $(AF)^*(y^*,z^*) = F^*(A^*y^*,z^*)$. In general it holds

$$S_{(y^*,z^*)}(Ax) \lhd_{z^*} AF(Ax) \subseteq S_{(A^*y^*,z^*)}(x) \lhd_{z^*} F(Ax),$$

thus by (6.2.1) it holds $\partial_{z^*}(AF)(Ax) \supseteq A^* \partial_{z^*}(F)(Ax)$. If $AF(Ax) \lhd_{z^*} \{0\} = F(Ax) \lhd_{z^*} \{0\}$, then

$$S_{(y^*,z^*)}(Ax) \lhd_{z^*} AF(Ax) = S_{(A^*y^*,z^*)}(x) \lhd_{z^*} F(Ax)$$

and $y^* \in \partial_{z^*}(AF)(Ax)$ holds if and only if $A^* \in A^* \partial_{z^*} F(Ax)$.

b) It holds $G^*(y^*,z^*) \subseteq A^* G^*(A^*y^*,z^*) \subseteq (GA)^*(A^*y^*,z^*)$ and thus

$$\begin{aligned} A^* \partial_{z^*} G(Ax) &\subseteq \left\{ A^*y^* \in X^* |\ S_{(A^*y^*,z^*)}(x) \lhd_{z^*} GA(x) \subseteq (GA)^*(A^*y^*,z^*) \right\} \\ &\subseteq \partial_{z^*} GA(x). \end{aligned}$$

c) Under the additional assumptions by 4.2.6, for every $(x^*,z^*) \in X^* \times C^* \setminus \{0\}$ with $(GA)^*(x^*,z^*) \neq \emptyset$ there exists $y^* \in Y^*$ such, that

$$A^*y^* = x^*,\ (GA)^*(x^*,z^*) = G^*(y^*,z^*).$$

If $G^*(x^*,z^*) = \emptyset$, then $S_{(x^*,z^*)} \notin \partial(GA)(x)$. Otherwise,

$$\begin{aligned} A^* \partial_{z^*} G(Ax) &= \left\{ A^*y^* \in X^* |\ S_{(A^*y^*,z^*)}(x) \lhd_{z^*} GA(x) \subseteq (GA)^*(A^*y^*,z^*) \right\} \\ &= \partial_{z^*} GA(x) \end{aligned}$$

holds and thus the claim holds. ∎

6.3 Comparison to vector-valued convex analysis

From now on, we consider a convex function f mapping a nonempty set $\operatorname{dom} f \subseteq X$ to Z and define $f(x) = +\infty$ for $x \notin \operatorname{dom} f$. Furthermore it will be assumed that $x_0 \in \operatorname{int} \operatorname{dom} f$, f is continuous at x_0 regarded as a mapping from X to Z.

The set-valued extension of f is defined as $F : X \to \mathcal{Q}^t_C(Z)$ with

$$\forall x \in X : \quad F(x) := f(x) + C$$

with $\{+\infty\} + C = \emptyset$. The epigraph of f is defined as

$$\operatorname{epi} f := \{(x, z) \in X \times Z |\ z \in f(x) + C\} .$$

Notice that in fact $\operatorname{epi} F = \operatorname{epi} f$.

For $f : X \to Z \cup \{+\infty\}$ and $z^* \in Z^*$, the function $z^* f : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is defined by

$$\forall x \in X : \ z^* f(x) = z^*(f(x))$$

with $z^*(+\infty) := +\infty$ and it holds $\varphi_{(F,z^*)}(x) = (-z^* f)(x)$ for ll $x \in X$.

As by assumption $\operatorname{dom} f = \operatorname{dom} F$ is nonempty and for every $x \in X$ it holds $f(x) + C \neq Z$, the set-valued extension F of f is z^*-proper for all $z^* \in C^* \setminus \{0\}$.

6.3.1 Lemma. *For every $z^* \in C^* \setminus \{0\}$ the function $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is continuous at $x_0 \in \operatorname{int} \operatorname{dom} f$ and $F(x_0) = (\operatorname{cl} F)(x_0)$.*

Proof. As $\varphi_{(F,z^*)}(x) = -z^*(f(x))$ holds for all $x \in \operatorname{dom} f$, the function $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{+\infty\}$ is proper and $x_0 \in \operatorname{int} \operatorname{dom} \varphi_{(F,z^*)}$. As f is continuous at x_0, it holds

$$\forall V \in \mathcal{U}_Z : \exists U \in \mathcal{U}_X : \ \forall x \in U : \ f(x_0 + x) \in f(x_0) + V.$$

From this it follows that

$$\forall \varepsilon > 0 : \exists U \in \mathcal{U}_X : \ \forall x \in U : \ -z^*(f(x_0)) - \varepsilon \le -z^*(f(x_0 + x)) \le -z^*(f(x_0)) + \varepsilon.$$

Therefore the function $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is continuous at x_0 for all $z^* \in C^* \setminus \{0\}$ and by 3.3.5 it holds that $F : X \to \mathcal{Q}^t_C(Z)$ is closed at x_0. ■

The subdifferential of f at $x_0 \in \operatorname{int} \operatorname{dom} f$ is defined as

$$\partial f(x_0) := \{T \in \mathcal{L}(X, Z) |\ \forall x \in X : \ T(x - x_0) \le f(x) - f(x_0)\} . \tag{6.3.1}$$

6.3.2 Theorem. *Let $T \in \mathcal{L}(X, Z)$ be a linear continuous operator. Then $T \in \partial f(x_0)$ if and only if*

$$\forall z^* \in C^* \setminus \{0\} : \ S_{(-T^* z^*, z^*)} \in \partial F(x_0).$$

Proof. By 6.1.4, the statement

$$\forall z^* \in C^* \setminus \{0\} : \ S_{(-T^* z^*, z^*)} \in \partial F(x_0)$$

is true if and only if

$$\forall z^* \in C^* \setminus \{0\} , \, x \in X : \ -z^*(Tx) \le -z^*(f(x_0 + x) - f(x_0)).$$

This is true if and only if

$$\forall x \in X, \ f(x_0 + x) - f(x_0) \in \{Tx\} + C,$$

which is by definition $T \in \partial f(x_0)$. ■

Under the assumptions of 6.3.2, $-z^*(\partial f(x_0)) \subseteq \partial \varphi_{(F,z^*)}(x_0)$ holds true.

6.3.3 Proposition. *If for all* $z^* \in C^* \setminus \{0\}$ *the set* $\partial_{z^*} F(x_0) = \left\{ x^* \in X^* | \, S_{(x^*, z^*)} \in \partial F(x_0) \right\}$ *is single valued and* $\operatorname{int} C^* \neq \emptyset$*, then there exists* $-T^* \in \mathcal{L}(Z^*, X^*_\sigma)$ *such, that*

$$\partial F(x_0) = \left\{ S_{(-T^* z^*, z^*)} | \, z^* \in C^* \setminus \{0\} \right\}$$

and $\partial f(x_0) = \{T\}$.

Proof. It is to prove that the function $-t^* : C^* \setminus \{0\} \to X^*$ defined by

$$\forall z^* \in C^* \setminus \{0\} : \quad -t^*(z^*) := x^*, \; \partial \varphi_{(F,z^*)}(x_0) = \{x^*\}$$

can be uniquely extended to a mapping $-T^* \in \mathcal{L}(Z^*, X^*_\sigma)$. The equation

$$\partial F(x_0) = \left\{ S_{(-T^* z^*, z^*)} | \, z^* \in C^* \setminus \{0\} \right\}$$

is then a immediate conclusion of 6.1.2 and $\partial f(x_0) = \{T\}$ holds by 6.3.2.

Define $-t^*(0) = 0$. As for $t > 0$ holds $S_{(tx^*, tz^*)} = S_{(x^*, z^*)}$ for all $(x^*, z^*) \in X^* \times C^* \setminus \{0\}$, $-t^*$ is obviously positively homogenous.

As $\varphi_{(F, z_1^* + z_2^*)}(x) = \varphi_{(F, z_1^*)}(x) + \varphi_{(F, z_2^*)}(x)$ for all $x \in X$, $z_1^*, z_2^* \in C^* \setminus \{0\}$ holds under the given assumptions and $f(x_0) \in Z$, $-t^*$ is additive.

As $\operatorname{int} C^* \neq \emptyset$ has been presumed, each $z^* \in Z^*$ can be represented as $z^* = z_1^* - z_2^*$, with $z_1^*, z_2^* \in C^*$ and $-t^*$ can be uniquely extended to a linear mapping $-T^* : Z^* \to X^*$ defined by

$$-T^*(z^*) = -t^*(z_1^*) - (-t^*)(z_2^*).$$

Now let $U \subseteq X^*$ be defined through $x \in X \setminus \{0\}$ with

$$U := \{x^* \in X^* | \, -1 < x^*(x) < 1\}.$$

Now choose $0 < t$ small enough for $x_0 \pm tx \in \operatorname{dom} f$ and define $V \subseteq Z^*$ by

$$V := \{z^* \in Z^* | \, -t < z^*(f(x_0 + tx) + f(x_0)), z^*(f(x_0 - tx) - f(x_0)) < t\}.$$

Thus,

$$\begin{aligned} \forall z^* \in V \cap C^* \setminus \{0\} : \qquad & -T^* z^*(tx) \leq \varphi'_{(F,z^*)}(x_0, tx) \leq -z^*(f(x_0 + tx) + f(x_0)) < t, \\ & -T^* z^*(-tx) \leq \varphi'_{(F,z^*)}(x_0, -tx) \leq -z^*(f(x_0 - tx) + f(x_0)) < t, \end{aligned}$$

that is $-T^*(V \cap C^* \setminus \{0\}) \subseteq U$. Since $\operatorname{int} C^* \neq \emptyset$, there is an open set $W \subseteq Z^*$ and $z_0^* \in C^* \setminus \{0\}$ such, that

$$\{z_0^*\} + W \subseteq C^* \setminus \{0\} \cap V$$

and we get

$$-T^*(\{z_0^*\} + W) \subseteq U.$$

Therefore, $-T^* \in \mathcal{L}(Z^*, X^*_\sigma)$. ■

Under the assumptions of 6.3.3, $F'_{z^*}(x_0, x) = T(x) + H(z^*)$ holds for all $z^* \in C^* \setminus \{0\}$ and $x \in X$. The above proof is based on the idea of the proof of Proposition 2.5. in [65].

6.3.4 Corollary. *[65] Let* X *be a reflexive Banach space,* $\operatorname{int} C^*_\tau \neq \emptyset$ *and all order intervals* $[z_1, z_2] \subseteq Z$ *are relatively compact in* Z_σ*. As always,* $f : X \to Z \cup \{+\infty\}$ *is convex and* f *is continuous at* $x_0 \in \operatorname{int} \operatorname{dom} f$ *and* $F : X \to \mathcal{Q}^t_C(Z)$ *is the set-valued extension on* f*. Then*

$$\forall z^* \in C^* \setminus \{0\} : \quad -z^* \partial f(x_0) = \partial \varphi_{(F,z^*)}(x_0).$$

6.3.5 Theorem. *Under the assumptions of 6.3.4,* $\partial f(x_0)$ *is single valued if and only if*

$$\partial F(x_0) = \left\{ S_{(-T^* z^*, z^*)} | \, z^* \in C^* \setminus \{0\} \right\}.$$

Proof. Immediate from 6.3.3 and 6.3.4. ■

7 Tangent Cone and Normal Cone

The results in this chapter will provide an alternative description of the z^*-directional derivatives $F'_{z^*}(x_0,\cdot)$ and the z^*-subdifferentials $\partial_{z^*}F(x_0)$ of a convex function $F : X \to \mathcal{Q}^t_C(Z)$. In 7.1.4 it will be proven that for any $x_0 \in \operatorname{dom} F$ there exists $z_0 \in F(x_0) \lhd_{z^*} \{0\}$ such, that

$$\operatorname{cl}(\operatorname{epi} F'_{z^*}(x_0,\cdot)) = T_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0,z_0)$$

and in 7.2.4, that $x^* \in \partial_{z^*}F(x_0)$ holds if and only if

$$\exists z_0 \in F(x_0) \lhd_{z^*} \{0\} : \quad (x^*,z^*) \in N_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0,z_0).$$

This far, we are in accordance to the well-known scalar results, compare [10, 50]. It turns out that the tangent cone of $\operatorname{epi} F$ at $(x_0,z_0) \in \operatorname{epi} F$ in general is a subset of the epigraph of the supremum of the z^*-directional derivatives of F at x_0. Likewise, the normal cone of $\operatorname{epi} F$ at $(x_0,z_0) \in \operatorname{epi} F$ coincides with the set

$$\left\{(x^*,z^*) \in X^* \times C^* \setminus \{0\} \mid S_{(x^*,z^*)} \in \partial F(x_0)\right\}$$

only in the special case when $F(x_0) = z_0 + C$ holds.

As it has be shown in [10], the tangent cone and the contingent cone (for definition, see [2]) of $\operatorname{epi} F$ at $(x_0,z_0) \in \operatorname{epi} F$ coincide when F is a convex function and thus $\operatorname{epi} F$ is a convex set. The statement in [10] is stated in the finite dimensional case, while the proof does not make use of the finite dimension, thus it holds in the more general case as well. Thus, the z^*-directional derivatives are minorants of the (set-valued extension of the) contingent epiderivative $DF(x_0,z_0) : X \to Z$ of F in $(x_0,z_0) \in \operatorname{epi} F$ defined in [31], when it exists, and $DF(x_0,z_0)$ will turn out to be the pointwise supremum of the z^*-directional derivatives, if it exists. The same holds true for the derivative and epiderivative of F in $(x_0,z_0) \in \operatorname{epi} F$, as defined in [1], as there the tangent cone is already used as an initial concept. In those concepts, an element $(x_0,z_0) \in \operatorname{epi} F$ is fixed and the derivatives are defined with respect to this point. It is notable that in our approach, in general $z_0 \in F(x_0)$ does not hold and moreover is not fixed in advance.

Throughout this chapter, X and Z are assumed to be locally convex separable spaces with the corresponding dual spaces X^* and Z^*, and Z is quasi-ordered by a closed convex cone $C \subsetneq Z$ with $\{0\} \subsetneq C$.

7.1 Tangent cone

7.1.1 Definition. *[1] Let Y be a locally convex separable space, $M \subseteq Y$ a convex set and $y_0 \in M$. The tangent cone of M in y_0 is defined by*

$$T_M(y_0) := \operatorname{cl}\operatorname{cone}(M - y_0).$$

7.1.2 Corollary. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$. For $(x_0,z_0) \in$* epi $(F \lhd_{z^*} \{0\})$*, the tangent cone of* epi $(F \lhd_{z^*} \{0\})$ *in (x_0,z_0) is given by*

$$T_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0,z_0) := \operatorname{cl} \bigcup_{t>0} \frac{1}{t}\left(\operatorname{epi}\,(F \lhd_{z^*} \{0\}) - (x_0,z_0)\right).$$

Proof. Setting $Y = X \times Z$, $M = \operatorname{epi}\,(F \lhd_{z^*} \{0\})$ and $y_0 = (x_0,z_0) \in \operatorname{epi}\,(F \lhd_{z^*} \{0\})$, the result is immediate from 7.2.1. ∎

7.1.3 Lemma. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$ and $(x_0, z_0) \in$ $\mathrm{epi}\,(F \lhd_{z^*} \{0\})$.*

a) If $z_0 + H(z^*) \subsetneq (F(x_0) \lhd_{z^*} \{0\})$, then

$$T_{\mathrm{epi}\,(F \lhd_{z^*} \{0\})}(x_0, z_0) = \mathrm{cone}\,(\mathrm{dom}\, F - x_0) \times Z.$$

b) If $z_0 + H(z^*) = F(x_0) \lhd_{z^*} \{0\}$, then

$$T_{\mathrm{epi}\,(F \lhd_{z^*} \{0\})}(x_0, z_0) = \mathrm{cl}\,\mathrm{epi}\, F'_{z^*}(x_0, \cdot)\,.$$

Proof.

a) If $z_0 + H(z^*) \subsetneq (F(x_0) \lhd_{z^*} \{0\})$, then $z_0 \in \mathrm{int}\, F(x_0) \lhd_{z^*} \{0\}$ holds and the statement is immediate with 7.1.1.

b) If $z_0 + H(z^*) = F(x_0) \lhd_{z^*} \{0\}$, then

$$\mathrm{epi}\,(F'_{z^*}(x_0, \cdot) = \bigcup_{t>0} \frac{1}{t}\left(\mathrm{epi}\ (F \lhd_{z^*} \{0\}) - (x_0, z_0)\right),$$

proving the claim. ■

7.1.4 Theorem. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a convex function, $x_0 \in \mathrm{dom}\, F$ and $z^* \in C^* \setminus \{0\}$, then there is $z_0 \in F(x_0) \lhd_{z^*} \{0\}$ such, that*

$$T_{\mathrm{epi}\,(F \lhd_{z^*} \{0\})}(x_0, z_0) = \mathrm{cl}\,\mathrm{epi}\, F'_{z^*}(x_0, \cdot)\,.$$

If additionally $(\mathrm{cl}\,(F'_{z^*}(x_0, \cdot)))(0) = H(z^*)$ holds, then

$$T_{\mathrm{epi}\,(F \lhd_{z^*} \{0\})}(x_0, z_0) = \bigcap_{x^* \in \partial_{z^*} F(x_0)} \mathrm{epi}\, S_{(x^*, z^*)}.$$

Proof. The first equation is proven in 5.1.9, the second in 5.1.11. ■

Even if $F(x_0)$ is a z^*-proper set, in general there is no $z_0 \in F(x_0)$ such that $z_0 + H(z^*) = F(x_0) \lhd_{z^*} \{0\}$ as the following example will show. Therefore the assumption of $(x_0, z_0) \in \mathrm{epi}\ (F \lhd_{z^*} \{0\})$ in 7.1.3 is notably weaker than $z_0 \in F(x_0)$.

7.1.5 Example. *Let $F : X \to Q^t_{\mathbb{R}^2_+}\left(\mathbb{R}^2\right)$ defined by $F \equiv \left\{\left(t, \frac{1}{t}\right) \mid t > 0\right\} + \mathbb{R}^2_+$ and $z^* = (-1, 0)$, then $F \lhd_{z^*} \{0\} \equiv (0,0) + H(z^*)$ holds true, while*

$$\forall (x,z) \in \mathrm{epi}\, F : \quad z + H(z^*) \subsetneq F(x) \lhd_{z^*} \{0\}\,.$$

Thus, $T_{\mathrm{epi}\,(F \lhd_{z^*} \{0\})}(x_0, z_0) = X \times Z$ for all $(x_0, z_0) \in \mathrm{epi}\, F$ while

$$\forall x \in \mathrm{dom}\, F : \quad T_{(F, z^*)}(x, (0,0)) = X \times H(z^*) = \mathrm{cl}\,\mathrm{epi}\,\left(F'_{z^*}(x_0, \cdot)\right).$$

As $\mathrm{epi}\, F \subseteq \mathrm{epi}\,(F \lhd_{z^*} \{0\}$ holds for all $z^* \in C^* \setminus \{0\}$, it can be seen that

$$T_{\mathrm{epi}\, F}(x_0, z_0) \subseteq \bigcap_{z^* \in C^* \setminus \{0\}} T_{(F, z^*)}(x_0, z_0)$$

holds true. The inclusion is in general not an equality, as the following example shows.

7.1.6 Example. *[33], p. 199. Let $F : X \to \mathcal{P}\left(l^1\right)$, $C := \left\{z = (z_n)_{n\in\mathbb{N}} \in l^1 \mid \forall n \in \mathbb{N} : z_n \geq 0\right\}$ and $F \equiv C$. The dual cone is $C^* = \left\{z^* = (z_n^*)_{n\in\mathbb{N}} \in l^\infty \mid \forall n \in \mathbb{N} : z_n^* \leq 0\right\}$. For all $z^* \in C^* \setminus \{0\}$ it holds $\varphi_{(F,z^*)} \equiv 0$. Take $z_0 \in C$ with*

$$\forall n \in \mathbb{N} : \quad z_{0,n} = \left(\frac{1}{2}\right)^{n-1},$$

then

$$\forall z^* \in C^* \setminus \{0\} : \quad z_0 + H\left(z^*\right) \subsetneq \left(F\left(0\right) \lhd_{z^*} \{0\}\right)$$

and $T_{\operatorname{epi}\left(F \lhd_{z^*} \{0\}\right)}\left(0, z_0\right) = X \times l^1$ while

$$T_{\operatorname{epi} F}\left(0, z_0\right) = X \times \operatorname{cl}\left\{z \in l^1 \mid \exists t > 0 \forall n \in \mathbb{N} : z_n \geq -t\left(\frac{1}{2}\right)^{n-1}\right\}$$

and for all $x \in X$ and for $z := \left(-\left(\frac{3}{4}\right)^{n-1}\right)_{n\in\mathbb{N}}$ it holds $(x, z) \notin T_{\operatorname{epi} F}\left(0, z_0\right)$.

Proof. As

$$\operatorname{epi} F - (0, z_0) = X \times \left\{z \in l^1 \mid \forall n \in \mathbb{N} : z_n \geq -\left(\frac{1}{2}\right)^{n-1}\right\}$$

holds, $(x, z) \in T_{\operatorname{epi} F}\left(0, z_0\right)$ holds if and only if

$$\begin{aligned} \forall \varepsilon > 0 : \exists x_\varepsilon \in l^1, t > 0 : \quad & \sum_{n\in\mathbb{N}} \left|x_{\varepsilon,n}\right| < \varepsilon, \\ \forall n \in \mathbb{N} : \quad & -\left(\frac{3}{4}\right)^{n-1} + x_{\varepsilon,n} \geq -t\left(\frac{1}{2}\right)^{n-1}. \end{aligned}$$

Thus,

$$\forall n \in \mathbb{N} : \quad x_{\varepsilon,n} \geq \left(\left(\frac{3}{2}\right)^{n-1} - t\right)\left(\frac{1}{2}\right)^{n-1},$$

a contradiction to $x_\varepsilon \in l^1$. ∎

7.1.7 Corollary. *Let $F : X \to \mathcal{Q}_C^t(Z)$ be a convex function, $x_0 \in X$. If $F\left(x_0\right) = z_0 + C$, then*

$$T_{\operatorname{epi} F}\left(x_0, z_0\right) = \bigcap_{z^* \in C^* \setminus \{0\}} T_{(F,z^*)}\left(x_0, z_0\right).$$

If additionally $(\operatorname{cl}(F'_{z_0^*}(x_0, \cdot)))(0) = H(z_0^*)$ holds for at least one $z_0^* \in C^* \setminus \{0\}$, then

$$T_{\operatorname{epi} F}\left(x_0, z_0\right) = \bigcap_{S_{(x^*,z^*)} \in \partial F(x_0)} \operatorname{epi} S_{(x^*,z^*)}.$$

Proof. By assumption it holds $F\left(x_0\right) = z_0 + C$, so $F(x_0) \lhd_{z^*} \{0\} = z_0 + H(z^*)$ holds for all $z^* \in C^* \setminus \{0\}$ and thus

$$\forall x \in X, \forall z^* \in C^* \setminus \{0\} : \quad F(x) \lhd_{z^*} F(x_0) = F(x) \lhd_{z^*} \{z_0\}.$$

Therefore,

$$\forall x \in X : \quad F(x) - z_0 = \bigcap_{z^* \in C^* \setminus \{0\}} \left(F(x) \lhd_{z^*} \{z_0\}\right)$$

holds and therefore

$$\operatorname{epi} F - (x_0, z_0) = \bigcap_{z^* \in C^* \setminus \{0\}} (\operatorname{epi} (F \lhd_{z^*} \{0\}) - (x_0, z_0)).$$

This proves

$$T_{\operatorname{epi} F} (x_0, z_0) = \bigcap_{z^* \in C^* \setminus \{0\}} T_{(F,z^*)} (x_0, z_0) .$$

If additionally $(\operatorname{cl} (F'_{z_0^*}(x_0, \cdot)))(0) = H(z_0^*)$ holds for at least one $z_0^* \in C^* \setminus \{0\}$, then

$$T_{\operatorname{epi} (F \lhd_{z_0^*} \{0\})} (x_0, z_0) = \bigcap_{x^* \in \partial_{z_0^*} F(x_0)} \operatorname{epi} S_{(x^*, z_0^*)}$$

holds by 7.1.4. By 5.1.12,

$$T_{\operatorname{epi} (F \lhd_{z^*} \{0\})}(x_0, z_0) = \operatorname{cl} \operatorname{cone} (\operatorname{dom} F - x_0) \times Z$$

if and only if $(\operatorname{cl} (F'_{z_0^*}(x_0, \cdot)))(0) = Z$. In this case, $\partial_{z^*} F(x_0) = \emptyset$ and

$$T_{\operatorname{epi} (F \lhd_{z_0^*} \{0\})} (x_0, z_0) \cap T_{\operatorname{epi} (F \lhd_{z^*} \{0\})} (x_0, z_0) = \bigcap_{x^* \in \partial_{z_0^*} F(x_0)} \operatorname{epi} S_{(x^*, z_0^*)} \cap \bigcap_{x^* \in \partial_{z^*} F(x_0)} \operatorname{epi} S_{(x^*, z^*)},$$

thus the statement is proven. ■

7.1.8 Remark. *In [31], the contingent epiderivative of a set-valued function $F : X \to \mathcal{P}(Z)$ at $(x_0, z_0) \in \operatorname{epi} F$ is defined as a single-valued function $DF(x_0, z_0) : X \to Z$ with*

$$\operatorname{epi} (DF(x_0, z_0)) = T_{\operatorname{epi} F}(x_0, z_0).$$

As we are only interested in (C-)convex functions, the contingent cone in the original definition coincides with the tangent cone of $\operatorname{epi} F$ in $(x_0, z_0) \in \operatorname{epi} F$, compare [10]. Moreover, let $F_C : X \to \mathcal{Q}^t_C(Z)$ be defined by

$$\forall x \in X : \quad F_C(x) := \operatorname{cl} (F(x) + C).$$

Then for all $(x_0, z_0) \in \operatorname{epi} F$ it holds

$$T_{\operatorname{epi} F_C}(x_0, z_0) = T_{\operatorname{epi} F}(x_0, z_0)$$

and thus $DF(x_0, z_0) = DF_C(x_0, z_0)$ holds for all $(x_0, z_0) \in \operatorname{epi} F$. Thus without loss of generality suppose that $F(x) = F_C(x)$ holds for all $x \in X$. If $DF(x_0, z_0)$ exists, then especially $DF(x_0, z_0)(0) \in Z$ holds. By definition, the set

$$DF(x_0, z_0)(0) + C = \{\bar{z} \in Z |\ (0, \bar{z}) \in \operatorname{epi} DF(x_0, z_0)\}$$

is given by

$$\{\bar{z} \in Z |\ (0, \bar{z}) \in \operatorname{cl} \operatorname{cone} (\operatorname{epi} F - (x_0, z_0))\} \supseteq \operatorname{cl} \{t(z - z_0) \in Z |\ z \in F(x_0),\ t > 0\} .$$

Therefore,

$$\forall z \in F(x_0) : \quad \forall z^* \in C^* \setminus \{0\} : \quad \inf_{t>0} -tz^*(z - z_0) \geq 0.$$

Thus, $\varphi_{(F,z^*)}(x_0) \geq -z^*(z_0)$ *holds for all* $z^* \in C^* \setminus \{0\}$ *and therefore* $F(x_0) \subseteq z_0 + C$ *holds true. As* $z_0 \in F(x_0)$ *holds by assumption and* $F(x_0) \in \mathcal{Q}_C^t(Z)$ *, it holds* $F(x_0) = z_0 + C$*.*

Likewise, if $DF(x_0, z_0)$ *exists, then for all* $x \in X$ *it holds*

$$\forall z^* \in C^* \setminus \{0\} : \quad \inf \{-z^*(z) |\, z \in (\operatorname{cl} F'_{z^*}(x_0, \cdot))(x)\} \geq -z^*(DF(x_0, z_0)(x))$$

by 7.1.7 and therefore

$$\begin{aligned} \forall z^* \in C^* \setminus \{0\} : \quad (\operatorname{cl} F'_{z^*}(x_0, \cdot))(x) &= DF(x_0, z_0)(x) + H(z^*), \\ DF(x_0, z_0)(x) + C &= \bigcap_{z^* \in C^* \setminus \{0\}} (\operatorname{cl} F'_{z^*}(x_0, \cdot))(x) \end{aligned}$$

holds for all $x \in X$*, as* $T_{\operatorname{epi} F}(x_0, z_0) \subseteq T_{\operatorname{epi}(F \lhd_{z^*} \{0\})}(x_0, z_0)$ *holds for all* $z^* \in C^* \setminus \{0\}$ *.*

7.1.9 Corollary. *Let* $F : X \to \mathcal{Q}_C^t(Z)$ *be a convex function,* $z_0 \in F(x_0)$*. If* $F(x_0) = z_0 + C$*, then*

$$\partial F(x_0) = \left\{ S_{(x^*, z^*)} |\, \operatorname{epi} S_{(x^*, z^*)} \supseteq T_{\operatorname{epi} F}(x_0, z_0) \right\}.$$

Proof. The set $\partial F(x_0)$ is nonempty if and only if $(\operatorname{cl}(F'_{z_0^*}(x_0, \cdot)))(0) = H(z_0^*)$ holds for at least one $z_0^* \in C^* \setminus \{0\}$. In this case, the statement is proven by 7.1.7. On the other hand, let $\partial F(x_0) = \emptyset$ holds. Then

$$\forall z^* \in C^* \setminus \{0\} : \quad (\operatorname{cl}(F'_{z^*}(x_0, \cdot)))(0) = Z$$

holds and as $T_{\operatorname{epi} F}(x_0, z_0) = \bigcap_{z^* \in C^* \setminus \{0\}} T_{\operatorname{epi}(F \lhd_{z^*} \{0\})}(x_0, z_0)$ holds by 7.1.7,

$$\left\{ S_{(x^*, z^*)} |\, \operatorname{epi} S_{(x^*, z^*)} \supseteq T_{\operatorname{epi} F}(x_0, z_0) \right\} = \emptyset$$

holds true. ■

The case of $F(x_0) = z_0 + C$ occurs naturally when $F(x) := f(x) + C$ for all $x \in X$, where $f : X \to Z$ is a vector-valued function.

7.2 Normal cone

7.2.1 Definition. *[1] Let* Y *be a locally convex separable space,* $M \subseteq Y$ *a convex set and* $y_0 \in M$*. The normal cone of* M *in* y_0 *is defined by*

$$N_M(y_0) := \{y^* \in Y^* |\, \forall y \in M : y^*(y - y_0) \leq 0\}.$$

7.2.2 Corollary. *Let* $F : X \to \mathcal{Q}_C^t(Z)$ *be a convex function,* $z^* \in C^* \setminus \{0\}$ *. For* $(x_0, z_0) \in$ epi $(F \lhd_{z^*} \{0\})$*, the normal cone of* epi $(F \lhd_{z^*} \{0\})$ *in* (x_0, z_0) *is defined by*

$$N_{\operatorname{epi}(F \lhd_{z^*} \{0\})}(x_0, z_0) := \{(x^*, \bar{z}^*) \in X^* \times Z^* |\, \forall (x, z) \in \operatorname{epi}(F \lhd_{z^*} \{0\}) : x^*(x - x_0) + \bar{z}^*(z - z_0) \leq 0\}.$$

Proof. Setting $Y = X \times Z$, $M = \operatorname{epi}(F \lhd_{z^*} \{0\})$ and $y_0 = (x_0, z_0) \in \operatorname{epi}(F \lhd_{z^*} \{0\})$, the result is immediate from 7.2.1. ■

7.2.3 Remark. *If* $F : X \to \mathcal{Q}_C^t(Z)$ *is a convex function,* $z^* \in C^* \setminus \{0\}$ *and* $(x_0, z_0) \in$ epi $(F \lhd_{z^*} \{0\})$*, then*

$$\begin{aligned} N_{\operatorname{epi}(F \lhd_{z^*} \{0\})}(x_0, z_0) &= (T_{\operatorname{epi}(F \lhd_{z^*} \{0\})}(x_0, z_0))^* \\ &= \left\{ (x^*, \bar{z}^*) \in X^* \times Z^* |\, T_{\operatorname{epi}(F \lhd_{z^*} \{0\})}(x_0, z_0) \subseteq \operatorname{epi} S_{(x^*, \bar{z}^*)} \right\} \\ &\subseteq X^* \times (\operatorname{cone}\{z_0^*\} \cup \{0\}) \end{aligned}$$

holds true.

7.2.4 Theorem. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$. If $x_0 \in \operatorname{dom} F$, then it exists $z_0 \in F(x_0) \lhd_{z^*} \{0\}$ such, that*

$$N_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0) = \operatorname{cone}\{(x^*, z^*)|\; x^* \in \partial_{z^*}F(x_0)\} \cup \{(x^*, 0)|\; \forall x \in \operatorname{dom} F : x^*(x) \le 0\}.$$

Moreover, $x^* \in \partial_{z^*}F(x_0)$ holds if and only if

$$\exists z_0 \in F(x_0) \lhd_{z^*} \{0\} : \quad (x^*, z^*) \in N_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0).$$

Proof. The first result is immediate from 7.2.3 and 7.1.4. The second holds true as $x^* \in \partial_{z^*}F(x_0)$ holds if and only if $S_{(x^*,z^*)}$ is a minorant of $F'_{z^*}(x_0, \cdot)$. Applying the first result, the statement is proven. ■

7.2.5 Lemma. *If $F : X \to \mathcal{Q}^t_C(Z)$ is a convex function and $(x_0, z_0) \in \operatorname{epi} F$, then it holds*

$$N_{\operatorname{epi} F}(x_0, z_0) \supseteq \bigcup_{z^* \in C^* \setminus \{0\}} N_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0).$$

Proof. It holds

$$N_{\operatorname{epi} F}(x_0, z_0) = (T_{\operatorname{epi} F}(x_0, z_0))^*$$

and

$$\begin{aligned}
\bigcup_{z^* \in C^* \setminus \{0\}} N_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0) &= \bigcup_{z^* \in C^* \setminus \{0\}} (T_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0))^* \\
&\subseteq \Big(\bigcap_{z^* \in C^* \setminus \{0\}} T_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0)\Big)^* \\
&= (T_{\operatorname{epi} F}(x_0, z_0))^* \\
&= N_{\operatorname{epi} F}(x_0, z_0).
\end{aligned}$$

■

Example 7.1.6 supplies an example where the inclusion is real.

7.2.6 Proposition. *Let $F : X \to \mathcal{Q}^t_C(Z)$ be a convex function, $z^* \in C^* \setminus \{0\}$. If $F(x_0) = z_0 + C$, then*

$$N_{\operatorname{epi} F}(x_0, z_0) = \bigcup_{z^* \in C^* \setminus \{0\}} N_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0).$$

Proof. By 7.1.9,

$$\partial F(x_0) = \left\{S_{(x^*,z^*)}|\; z^* \in C^* \setminus \{0\},\, (x^*, z^*) \in N_{\operatorname{epi}(F \lhd_{z^*}\{0\})}(x_0, z_0)\right\}.$$

Moreover,

$$\begin{aligned}
N_{\operatorname{epi} F}(x_0, z_0) = &\left\{(x^*, z^*) \in X^* \times C^* \setminus \{0\} \mid S_{(x^*,z^*)} \in \partial F(x_0)\right\} \\
&\cup \{(x^*, 0) \in X^* \times Z^* | \forall x \in \operatorname{dom} F :\; x^*(x) \le 0\} \\
= &\bigcup_{z^* \in C^* \setminus \{0\}} N_{\operatorname{epi}(F \lhd_{z^*}\{0\}}(x_0, z_0).
\end{aligned}$$

■

8 Appendix

8.1 Examples

In this chapter, we will present some special functions in order to illustrate the theory presented in the main part of this work.

8.1.1 Example (Conlinear functions)**.** *Let* $(x_0^*, z_0^*) \in X^* \times C^* \setminus \{0\}$ *and* $F = S_{(x_0^*, z_0^*)} : X \to \mathcal{Q}_C^t(Z)$ *. Then*

a) Let $z^* \in C^* \setminus \{0\}$ *, then*

$$\forall x \in X: \quad : \varphi_{(F,z^*)}(x) = \begin{cases} t x_0^*(x), & \textit{if } z^* = t z_0^* \textit{ with } t > 0; \\ -\infty, & \textit{else.} \end{cases}$$

Moreover, $\left\{ z^* \in C^* \setminus \{0\} \mid \varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\} \textit{ is proper} \right\} = \operatorname{cone} \{z_0^*\}$.

b) The function $F : X \to \mathcal{Q}_C^t(Z)$ *is conlinear, proper and closed. If* $z_0^* \in C^* \setminus -C^*$*, then* F *is* C*-proper*

c) It holds $\operatorname{dom} F = X$ *and* $\operatorname{epi} F$ *is a closed half space in* $X \times Z$*.*

d) Let $z^* \in C^* \setminus \{0\}$ *, then*

$$\forall x_0, x \in X: \quad : F'_{z^*}(x_0, x) = \begin{cases} tF(x), & \textit{if } z^* = t z_0^* \textit{ with } t > 0; \\ Z, & \textit{else.} \end{cases}$$

e) Let $x^* \in X^*$*,* $z^* \in C^* \setminus \{0\}$ *, then it holds*

$$F^*(x^*, z^*) = \begin{cases} H(z_0^*), & \textit{if } (x^*, z^*) \in \operatorname{cone} \{(x_0^*, z_0^*)\}; \\ \emptyset, & \textit{else.} \end{cases}$$

f) It holds

$$\forall x \in X: \quad F^{**}(x) = F(x)$$

g) It holds

$$\forall x \in X: \quad \partial F(x) = \left\{ S_{(x_0^*, z_0^*)} \right\}$$

and

$$\forall x \in X: \quad \partial_{ext} F(x) = \left\{ S_{(x_0^*, z_0^*)} \right\} \cup \left\{ S_{(x^*, z^*)} \mid z^* \in C^* \setminus \operatorname{cone} \{z_0^*\} \right\}.$$

PROOF.

a) For $z^* \in C^* \setminus \{0\}$, the function $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is defined by

$$\forall x \in X: \quad \varphi_{(F,z^*)}(x) := \inf_{z \in F(x)} \{-z^*(z)\}$$

and $F(x) = \{z \in Z \mid -z_0^*(z) \geq x_0^*(x)\}$ for all $x \in X$, thus for $z^* \in C^* \setminus \{0\}$ it holds

$$\forall x \in X: \quad : \varphi_{(F,z^*)}(x) = \begin{cases} t x_0^*(x), & \text{if } z^* = t z_0^* \text{ with } t > 0; \\ -\infty, & \text{else.} \end{cases}$$

Therefore, $\left\{ z^* \in C^* \setminus \{0\} \mid \varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\} \text{ is proper} \right\} = \operatorname{cone} \{z_0^*\}$.

b) As

$$\forall x \in X: \quad F(x) = (F \lhd_{z_0^*} \{0\})(x)$$

holds by 3.3.1, $F : X \to \mathcal{Q}_C^t(Z)$ is closed and by proper by 3.2.5. Also by 3.2.5, F is C-proper, if and only if $z_0^* \in C^* \setminus -C^*$.

c) It holds $\operatorname{dom} F = \operatorname{dom} \varphi_{(F,z^*)} = X$ for all $z^* \in C^* \setminus \{0\}$ and

$$\operatorname{epi} F = \{(x,z) | \; -z_0^*(z) \geq x_0^*(x)\},$$

a closed half space in $X \times Z$.

d) The directional derivative of $F : X \to \mathcal{Q}_C^t(Z)$ in $x_0 \in X$ with respect to $z^* \in C^* \setminus \{0\}$ is defined by

$$\forall x \in X: \quad F'_{z^*}(x_0, x) = \left\{z \in Z | \; -z^*(z) \geq \varphi'_{(F,z^*)}(x_0, x)\right\}$$

thus

$$\forall x \in X: \quad F'_{z^*}(x_0, x) = \begin{cases} F(x), & \text{if } z^* = t z_0^* \text{ with } t > 0; \\ Z, & \text{else.} \end{cases}$$

e) If $(x^*, z^*) \in \operatorname{cone}\{(x^*, z^*)\}$ holds, then

$$\forall x \in X: \quad S_{(x^*,z^*)}(x) = S_{(x_0^*,z_0^*)}(x).$$

The conjugate of $F : X \to \mathcal{Q}_C^t(Z)$ is given by

$$\forall x^* \in X^*, \; z^* \in C^* \setminus \{0\}: \quad F^*(x^*, z^*) = \left\{z \in Z | \; -z^*(z) \geq \varphi^*_{(F,z^*)}(x^*)\right\}$$

and

$$\varphi^*_{(F,z^*)}(x^*) = \iota_{\operatorname{cone}\{(x_0^*,z_0^*)\}}(x^*, z^*),$$

so

$$\forall x^* \in X^*, \; z^* \in C^* \setminus \{0\}: \quad F^*(x^*, z^*) = \left(I_{\operatorname{cone}\{(x_0^*,z_0^*)\}} \lhd_{z^*} \{0\}\right)(x^*, z^*).$$

f) As $F : X \to \mathcal{Q}_C^t(Z)$ is closed and proper, $F = F^{**} : X \to \mathcal{Q}_C^t(Z)$ holds by 4.1.15.

g) By 6.1.2 , the subdifferential of $F : X \to \mathcal{Q}_C^t(Z)$ in $x \in X$ is given by

$$\partial F(x) = \left\{S_{(x^*,z^*)} \; x^* \in \partial\varphi_{(F,z^*)}(x)\right\},$$

thus $\partial F(x) = \left\{S_{(x_0^*,z_0^*)}\right\}$. On the other hand, by 6.1.4 it holds

$$\partial_{ext} F(x) = \partial F(x) \cup \left\{S_{(x^*,z^*)} \; z^* \in C^* \setminus (\operatorname{cone}\{z_0^*\} \cup \{0\})\right\}.$$

■

8.1.2 Example (Conaffine functions)**.** *Let* $(x_0^*, z_0^*) \in X^* \times C^* \setminus \{0\}$, $z_0 \in Z$ *and*

$$\forall x \in X: \quad F(x) = S_{(x_0^*,z_0^*)}(x) + \{z_0\}.$$

a) It holds

$$\forall x \in X, z^* \in C^* \setminus \{0\}: \quad : \varphi_{(F,z^*)}(x) = \begin{cases} t(x_0^*(x) - z_0^*(z_0)), & \textit{if } z^* = t z_0^* \textit{ with } t > 0; \\ -\infty, & \textit{else.} \end{cases}$$

Moreover, $\left\{z^* \in C^* \setminus \{0\} \,|\, \varphi_{(F,z^*)} : X \to \mathrm{IR} \cup \{\pm\infty\} \textit{ is proper}\right\} = \operatorname{cone}\{z_0^*\}$.

b) The function $F : X \to \mathcal{Q}^t_C(Z)$ is conaffine and closed and proper. If $z_0^* \in C^* \setminus -C^*$, then F is C-proper.

c) It holds $\operatorname{dom} F = X$ and $\operatorname{epi} F$ is a shifted closed half space of $X \times Z$.

d) The directional derivative of F in x_0 with respect to $z^* \in C^* \setminus \{0\}$ is given by

$$\forall x_0, x \in X, z^* \in C^* \setminus \{0\} : \quad : F'_{z^*}(x_0, x) = \begin{cases} tS_{(x_0^*, z_0^*)}(x), & \textit{if } z^* = tz_0^* \textit{ with } t > 0; \\ Z, & \textit{else.} \end{cases}$$

e) The conjugate $F^* : X^* \times C^* \setminus \{0\} \to \mathcal{Q}^t_C(Z)$ of F is

$$\forall x^* \in X^*,\ z^* \in C^* \setminus \{0\} : \quad F^*(x^*, z^*) = \begin{cases} -z_0 + H(z_0^*), & \textit{if } (x^*, z^*) \in \operatorname{cone}\{(x_0^*, z_0^*)\}; \\ \emptyset, & \textit{else.} \end{cases}$$

f) For all $x \in X$ it holds $F^{**}(x) = F(x)$.

g) The subdifferential of F in $x \in X$ is

$$\forall x \in X : \quad \partial F(x) = \left\{ S_{(x_0^*, z_0^*)} \right\}$$

and

$$\forall x \in X : \quad \partial_{ext} F(x) = \left\{ S_{(x_0^*, z_0^*)} \right\} \cup \left\{ S_{(x^*, z^*)} | \, z^* \in C^* \setminus (\operatorname{cone}\{z_0^*\} \cup \{0\}) \right\}.$$

Proof. Let $G : X \to \mathcal{Q}^t_C(Z)$ be defined as $G(x) = S_{(x_0^*, z_0^*)}(x)$ for all $x \in X$, then

$$\forall z^* \in C^* \setminus \{0\}, x \in X : \quad \varphi_{(F,z^*)}(x) = \varphi_{(G,z^*)}(x) - z^*(z_0).$$

a) By 8.1.1 for $z^* \in C^* \setminus \{0\}$ it holds $\left\{ z^* \in C^* \setminus \{0\} \,|\, \varphi_{(F,z^*)} : X \to \mathbb{R} \cup \{\pm\infty\} \text{ is proper} \right\} = \operatorname{cone}\{z_0^*\}$ and

$$\forall x \in X : \quad : \varphi_{(F,z^*)}(x) = \begin{cases} t(x_0^*(x) - z_0^*(z_0)), & \text{if } z^* = tz_0^* \text{ with } t > 0; \\ -\infty, & \text{else.} \end{cases}$$

b) As

$$\forall x \in X : \quad F(x) = (F \lhd_{z_0^*} \{0\})(x)$$

holds by 3.3.1, $F : X \to \mathcal{Q}^t_C(Z)$ is closed and by proper by 3.2.5. Also by 3.2.5, F is C-proper, if and only if $z_0^* \in C^* \setminus -C^*$.

c) It holds $\operatorname{dom} F = \operatorname{dom} \varphi_{(F,z^*)} = X$ for all $z^* \in C^* \setminus \{0\}$ and

$$\operatorname{epi} F = \{(x, z) | \, -z_0^*(z) \geq x_0^*(x) + z^*(z_0)\},$$

a shifted closed half space in $X \times Z$.

d) The directional derivative of $F : X \to \mathcal{Q}^t_C(Z)$ in $x_0 \in X$ with respect to $z^* \in C^* \setminus \{0\}$ is defined by

$$\forall x \in X : \quad F'_{z^*}(x_0, x) = \left\{ z \in Z | \, -z^*(z) \geq \varphi'_{(F,z^*)}(x_0, x) \right\}$$

and $\varphi'_{(F,z^*)}(x_0, x) = \varphi'_{(G,z^*)}(x_0, x)$ for all $x_0, x \in X$, thus

$$\forall x \in X : \quad F'_{z^*}(x_0, x) = \begin{cases} S_{(x_0^*, z_0^*)}(x), & \text{if } z^* = tz_0^* \text{ with } t > 0; \\ Z, & \text{else.} \end{cases}$$

e) The conjugate of $F : X \to \mathcal{Q}_C^t(Z)$ is given by

$$\forall x^* \in X^*,\, z^* \in C^* \setminus \{0\} : \quad F^*(x^*, z^*) = \left\{ z \in Z | \; -z^*(z) \geq \varphi^*_{(F,z^*)}(x^*) \right\}$$

and

$$\varphi^*_{(F,z^*)}(x^*) = \iota_{\operatorname{cone}\left\{(x_0^*, z_0^*)\right\}}(x^*, z^*) + z^*(z_0),$$

so

$$\forall x^* \in X^*,\, z^* \in C^* \setminus \{0\} : \quad F^*(x^*, z^*) = \left(I_{\operatorname{cone}\left\{(x_0^*, z_0^*)\right\}} \lhd_{z^*} \{0\} \right)(x^*, z^*) - z_0.$$

f) As $F : X \to \mathcal{Q}_C^t(Z)$ is closed and proper, $F = F^{**} : X \to \mathcal{Q}_C^t(Z)$ holds by 4.1.15.

g) By 6.1.2 , the subdifferential of $F : X \to \mathcal{Q}_C^t(Z)$ in $x \in X$ is given by

$$\partial F(x) = \left\{ S_{(x^*,z^*)} \; x^* \in \partial \varphi_{(F,z^*)}\,(x) \right\},$$

thus $\partial F(x) = \left\{ S_{(x_0^*, z_0^*)} \right\}$. On the other hand, by 6.1.4 it holds

$$\partial_{ext} F(x) = \partial F(x) \left\{ S_{(x^*,z^*)} \; z^* \in C^* \setminus (\operatorname{cone}\,\{z_0^*\} \cup \{0\}) \right\}.$$

■

8.1.3 Example (Sublinear functions and the set-valued support function)**.** *Let $\emptyset \neq M^* \subseteq X^* \times C^* \setminus \{0\}$, then the support-function $\Sigma(\cdot|M^*) : X \to \mathcal{Q}_C^t(Z)$ of M^* is defined by*

$$\forall x \in X : \quad \Sigma(x|M^*) := \bigcap_{(x^*,z^*) \in M^*} S_{(x^*,z^*)}(x).$$

The function $\Sigma(\cdot|M^*) : X \to \mathcal{Q}_C^t(Z)$ is proper, closed and sublinear. It holds $C \subseteq \Sigma(0|M^*)$ and

$$\operatorname{cl}\operatorname{co}(\operatorname{cone} M^*) = \left\{ (x^*, z^*) | \; \forall x \in X : \; S_{(x^*,z^*)}(x) \supseteq \Sigma(x|M^*) \right\} \subseteq X^* \times C^*.$$

Let $P : X \to \mathcal{Q}_C^t(Z)$ be a proper, sublinear closed function, then

$$\forall x \in X : \quad P(x) = \Sigma(x|M_P^*)$$

with

$$M_P^* := \left\{ (x^*, z^*) | \; \forall x \in X : \; S_{(x^*,z^*)}(x) \supseteq P(x) \right\} \subseteq X^* \times C^*.$$

The set M_P^* is a nonempty, closed, convex cone.

a) Let $M^*_{(P,z^*)} = \{x^* \in X^* | \, (x^*, z^*) \in M_P^*\}$ for all $z^* \in C^* \setminus \{0\}$, then

$$\forall z^* \in C^* \setminus \{0\},\, x \in X : \quad \operatorname{cl}\varphi_{(P,z^*)}(x) = \sigma(x|M^*_{(P,z^*)})$$

where $\sigma(\cdot|M^*_{(P,z^*)}) : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ denotes the scalar support function. and

$$\left\{ z^* \in C^* \setminus \{0\} \; : \; \operatorname{cl}\varphi_{(F,z^*)} \; : X \to \mathrm{I\!R} \cup \{\pm\infty\} \;\; \textit{is proper} \right\} = \left\{ z^* \in C^* \setminus \{0\} \,|\, M^*_{(P,z^*)} \neq \emptyset \right\}.$$

b) The sets $\operatorname{dom} P$ and $\operatorname{epi} P$ are convex cones, P is $C-$proper, if and only if there exists $(x_0^*, z_0^*) \in M_P^*$ with $z_0^* \in C^* \setminus -C^*$.

c) Let $z^* \in C^* \setminus \{0\}$, then it holds

$$\forall x_0, x \in X: \; P'_{z^*}(x_0, x) \supseteq \begin{cases} P(x) \lhd_{z^*} \{0\}, & \textit{if } x_0 \in \operatorname{dom} P \textit{ and } M_{(P,z^*)} \neq \emptyset; \\ Z, & \textit{else.} \end{cases}$$

d) The conjugate of P is given by

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\}: \quad P^*(x^*, z^*) = \begin{cases} H(z^*), & \textit{if } (x^*, z^*) \in M_P^*; \\ \emptyset, & \textit{else.} \end{cases}$$

and $P^{**}(x) = P(x)$ for all $x \in X$.

e) The subdifferential of P in $x \in X$ is given by

$$\forall x \in X: \quad \partial P(x) = \left\{ S_{(x^*,z^*)} | \; (x^*, z^*) \in M_P^* \setminus (X^* \times \{0\}): \; S_{(x^*,z^*)}(x) = (P \lhd_{z^*} \{0\})(x) \right\}.$$

and

$$\partial_{ext} P(x) = \partial P(x) \cup \left\{ S_{(x^*,z^*)} | \; (x^*, z^*) \in ((X^* \times C^* \setminus \{0\}) \setminus M_P^*) \right\}.$$

For $x = 0$ it holds $\partial P(0) = M_P^* \setminus (X^* \times \{0\})$.

Proof. As

$$\operatorname{epi}(\Sigma(\cdot|M^*) = \bigcap_{(x^*,z^*) \in M^*} \{z \in Z | \; x^*(x) + z^*(z) \leq 0\}$$

and $\emptyset \neq M^* \subseteq X^* \times C^*$ hold, $C \subseteq \Sigma(0, |M^*)$ and $\operatorname{epi}(\Sigma(\cdot|M^*)$ is a closed convex cone. The set

$$\bar{M}^* := \left\{ (x^*, z^*) \in X^* \times C^* | \; \forall x \in X: \; S_{(x^*,z^*)}(x) \supseteq \Sigma(x|M^*) \right\}$$

is identical to the set

$$\{(x^*, z^*)| \; \forall x \in X: \; \{z \in Z | \; x^*(x) + z^*(z) \leq 0\} \supseteq \{z \in Z | \; \forall (x^*, z^*) \in M^*: \; x^*(x) + z^*(z) \leq 0\}\}.$$

As $S_{(x^*,z^*)}(x) = S_{(tx^*,tz^*)}(x)$ holds for all $t > 0$, $\bar{M}^* = \operatorname{cone} \bar{M}^*$. Obviously, $\bar{M}^* \subseteq (X^* \times C^*)$. If $(x^*, z^*) \in (X^* \times C^*) \setminus \operatorname{cl} \operatorname{co}(\operatorname{cone} M^*)$, then by a separation argument there exists $(0, 0) \neq (x, z) \in X \times Z$ and $\alpha \in \mathbb{R}$ such, that

$$x^*(x) + z^*(z) > \alpha \geq \sigma((x, z)|M^*).$$

Taking $(x_0, z_0) \in X \times Z$ such, that $x^*(x_0) + z^*(z_0) = -\alpha$, then

$$x^*(x + x_0) + z^*(z + z_0) > 0 \geq \sigma((x + x_0, z + z_0)|M^*),$$

and therefore $(x^*, z^*) \notin \bar{M}^*$. Thus,

$$\bar{M}^* = \operatorname{cl} \operatorname{co}(\operatorname{cone} M^*).$$

A proper, sublinear closed function $P : X \to \mathcal{Q}_C^t(Z)$ is nonempty at 0, as by 3.3.11 it holds

$$\forall x \in X: \quad P(0) = \bigcap_{\substack{z^* \in C^* \setminus \{0\}, \\ \operatorname{cl} \varphi_{(P,z^*)} \text{ proper}}} \left\{ z \in Z | \; -z^*(z) \geq \operatorname{cl} \varphi_{(P,z^*)}(0) \right\}$$

and the functions $\operatorname{cl}\varphi_{(P,z^*)} : X \to \mathrm{IR} \cup \{\pm\infty\}$ are sublinear. Therefore, if $\operatorname{cl}\varphi_{(P,z^*)}$ is proper, then $\operatorname{cl}\varphi_{(P,z^*)}(0) = 0$, so $C \subseteq P(0)$. By 8.3.2, for a proper sublinear and closed function $\operatorname{cl}\varphi_{(P,z^*)} : X \to \mathrm{IR} \cup \{\pm\infty\}$ it holds

$$\forall x \in X : \quad \operatorname{cl}\varphi_{(P,z^*)}(x) = \sup\left\{x^*(x) |\, \forall y \in X : \, x^*(y) \leq \varphi_{(P,z^*)}(y)\right\}.$$

The function $x^* \in X^*$ is a minorant of $\varphi_{(P,z^*)}$ if and only if $S_{(x^*,z^*)}$ is a minorant of P and moreover

$$\forall x \in X : \quad \left\{z \in Z | \, - z^*(z) \geq \operatorname{cl}\varphi_{(P,z^*)}(x)\right\} = \bigcap_{x^* \leq \varphi_{(P,z^*)}} S_{(x^*,z^*)}(x).$$

Thus

$$\forall x \in X : \quad P(x) = \Sigma(x|M_P^*)$$

with

$$M_P^* := \left\{(x^*, z^*) |\, \forall x \in X : \, S_{(x^*,z^*)}(x) \supseteq P(x)\right\} \subseteq X^* \times C^*$$

holds. As P is proper, M_P^* in nonempty. By the same arguments as used for $\bar{M}^* = \operatorname{cl}\operatorname{co}(\operatorname{cone} M^*)$, $M_P^* = \operatorname{cl}\operatorname{co}(\operatorname{cone} M_P^*)$ is proven .

a) This has already been shown above.

b) The first statement is immediate.

   $P : X \to \mathcal{Q}_C^t(Z)$ is C-proper if and only if there is a C-proper affine minorant $S_{(x^*,z^*)} + z$ of P. As P is sublinear, $z = 0$ must hold. The function $S_{(x^*,z^*)} : X \to \mathcal{Q}_C^t(Z)$ is C-proper if and only if $z^* \in C^* \setminus -C^*$ and it is a minorant of P if and only if $(x^*, z^*) \in M_P^*$.

c) If $x_0 \notin \operatorname{dom} P$ or $M_{(P,z^*)} = \emptyset$, then $\varphi'_{(F,z^*)}(x_0, \cdot) \equiv -\infty$, thus

$$\forall x \in X : \quad F'_{z^*}(x_0, x) = \left\{z \in Z | \, - z^*(z) \geq \varphi'_{(F,z^*)}(x_0, x)\right\} = Z.$$

   Otherwise,

$$\begin{aligned}
\forall x \in X : \quad F'_{z^*}(x_0, x) &= \operatorname{cl} \bigcup_{t>0} \frac{1}{t}\left(P(x_0 + tx) \lhd_{z^*} P(x_0)\right) \\
&\supseteq \operatorname{cl} \bigcup_{t>0} \frac{1}{t}\left(P(x_0) + tP(x) \lhd_{z^*} P(x_0)\right) \\
&\supseteq \operatorname{cl} \bigcup_{t>0} \left(P(x_0) \lhd_{z^*} P(x_0)\right) + \left(P(x) \lhd_{z^*} \{0\}\right) \\
&\supseteq \operatorname{cl} \bigcup_{t>0} \left(P(x) \lhd_{z^*} \{0\}\right).
\end{aligned}$$

d) For all $z^* \in C^* \setminus \{0\}$ it holds $\varphi^*_{(P,z^*)}(x^*) = \iota_{M^*_{(P,z^*)}}(x^*, z^*)$. The first statement follows from

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\} : \quad P^*(x^*, z^*) = \left\{z \in Z | \, - z^*(z) \geq \varphi^*_{(P,z^*)}(x^*)\right\}.$$

   As P is closed, convex, proper, $P^{**}(x) = P(x)$ holds for all $x \in X$.

e) It holds
$$\forall x \in X: \quad \partial F(x) = \left\{ S_{(x^*,z^*)} |\, x^* \in \partial\varphi_{(F,z^*)}(x) \right\}.$$
The set $\partial\varphi_{(F,z^*)}(x)$ in nonempty if and only if $\varphi_{(F,z^*)}(0) = \operatorname{cl}\varphi_{(F,z^*)}(0) = 0$, if and only if $M^*_{(P,z^*)} \neq \emptyset$. In this case it holds by 8.3.13
$$\begin{aligned} & \partial F(0) = && \left\{ S_{(x^*,z^*)} |\, x^* \in M^*_{(P,z^*)} \right\} \\ \forall x \in X: \quad & \partial F(x) = && \left\{ S_{(x^*,z^*)} |\, x^* \in M^*_{(P,z^*)},\, S_{(x^*,z^*)}(x) = P(x) \right\} \end{aligned}$$
and
$$\partial_{ext} F(x) = \partial F(x) \cup \left\{ S_{(x^*,z^*)} |\, M^*_{(P,z^*)} = \emptyset \right\}.$$
■

8.1.4 Example (Indicator function)**.** *Let $M \subseteq X$ and $F = I_M : X \to \mathcal{Q}^t_C(Z)$ defined by*
$$\forall x \in X: \quad I_M(x) := \begin{cases} C, & \text{if } x \in M; \\ \emptyset, & \text{else.} \end{cases}$$

a) For all $z^* \in C^* \setminus \{0\}$ it holds $\varphi_{(F,z^*)}(x) = \iota_M(x)$ with $\iota_M : X \to \mathrm{I\!R} \cup \{+\infty\}$ denoting the scalar indicator function of M.

b) It holds $\operatorname{dom} F = M$ *and* $\operatorname{epi} F = M \times C$

c) The function $I_M : X \to \mathcal{Q}^t_C(Z)$ is convex (closed) if and only if $M \subseteq X$ is convex (closed), C-proper if and only if $M \neq \emptyset$.

d) The conjugate of $F : X \to \mathcal{Q}^t_C(Z)$ is the support function of M,
$$\forall x^* \in X^*,\, z^* \in C^* \setminus \{0\}: \quad F^*(x^*, z^*) = \bigcap_{x \in M} S_{(x^*,z^*)}(x).$$

e) For all $x \in X$ it holds $F^{**}(x) = (\operatorname{cl}\operatorname{co} F)(x) = I_{\operatorname{cl}\operatorname{co} M}(x)$

f) If M is a convex subset of X and $x_0 \in M$, then
$$F'_{z^*}(x_0, x) = \left(I_{\operatorname{cone}(M - x_0) \cup \{0\}} \lhd_{z^*} \{0\} \right)(x)$$

g) If M is a convex subset of X and $x_0 \in M$, then
$$\partial F(x_0) = \left\{ S_{(x^*,z^*)} |\, \forall x \in M: \, x^*(x - x_0) \le 0 \right\}$$
and $\partial_{ext} F(x_0) = \partial F(x_0)$.

Proof.

a) It holds
$$\forall z^* \in C^* \setminus \{0\}, x \in X: \; \varphi_{(F,z^*)}(x) = \inf -z^*(z) |\, z \in F(x).$$
Therefore, for all $z^* \in C^* \setminus \{0\}$ it holds $\varphi_{(F,z^*)} = \iota_M(x)$.

b) Direct calculation

c) The convex (closed) hull of $F : X \to \mathcal{Q}^t_C(Z)$ is defined via the convex (closed) hull of the epigraph of F, thus

$$\forall x \in X : \quad \operatorname{co} F(x) = \{z \in Z |\ (x, z) \in \operatorname{co}\operatorname{epi} F\} = \{z \in Z |\ (x, z) \in \operatorname{co} M \times C\}.$$

The indicator function is C-proper if and only if $\varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is proper for some $z^* \in C^* \setminus -C^*$, thus if and only if $M \neq \emptyset$.

d) By definition,

$$F^*(x^*, z^*) = \bigcap_{x \in X} \left(S_{(x^*,z^*)}(x) \lhd_{z^*} F(x) \right),$$

so

$$F^*(x^*, z^*) = \bigcap_{x \in M} S_{(x^*,z^*)}(x).$$

e) It holds $\operatorname{cl}\operatorname{co}\operatorname{epi} F = (\operatorname{cl}\operatorname{co} M) \times C$, therefore, for all $x \in X$, $(\operatorname{cl}\operatorname{co} F)(x) = I_{\operatorname{cl}\operatorname{co} M}(x)$. By 4.1.5 it holds

$$\forall x \in X : \ F^{**}(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \left\{ z \in Z |\ -z^*(z) \geq \varphi^{**}_{(F,z^*)}(x) \right\}$$

As

$$\varphi^{**}_{(F,z^*)}(x) = \sup \{x^*(x) - \sigma(x^* | \operatorname{cl}\operatorname{co} M)\} = \iota_{\operatorname{cl}\operatorname{co} M}(x)$$

for all $x \in X$,

$$\forall x \in X : \ F^{**}(x) = \bigcap_{z^* \in C^* + \setminus \{0\}} \{z \in Z |\ -z^*(z) \geq \iota_{\operatorname{cl}\operatorname{co} M}(x)\}$$

which is $F^{**} = I_{\operatorname{cl}\operatorname{co} M}$.

f) By 5.1.2,

$$F'_{z^*}(x_0, x) = \left\{ z \in Z |\ -z^*(z) \geq \varphi'_{(F,z^*)}(x_0, x) \right\}.$$

For all $x \in X$ it holds $\varphi'_{(F,z^*)}(x_0, x) = \iota_{\operatorname{cone}(M + \{-x_0\})}(x)$, therefore

$$F'_{z^*}(x_0, x) = I_{\operatorname{cone}(M + \{-x_0\})}(x) + H(z^*).$$

g) By 6.1.2,

$$\partial F(x_0) = \left\{ S_{(x^*,z^*)} |\ \forall x \in X : \ x^*(x) \leq \varphi'_{(F,z^*)}(x_0, x) \right\}$$

Each scalarization $\varphi_{(F,z^*)}$ is equal to the scalar indicator function of M, so

$$\partial F(x_0) = \left\{ S_{(x^*,z^*)} |\ \forall x \in M : \ x^*(x - x_0) \leq 0 \right\}.$$

$$\partial F(x_0) = \partial ext\, F(x_0) = \left\{ S_{(x^*,z^*)} |\ \forall x \in M : \ S_{(x^*,z^*)}(x - x_0) \supseteq H(z^*) \right\}$$

■

8.1.5 Example (Scalar functions)**.** *Let $f : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ be a function and $C = \mathrm{I\!R}_+$. The set-valued extension $F : X \to \mathcal{P}(\mathrm{I\!R})$ of f is defined by*

$$F(x) := \begin{cases} f(x) + C, & \text{if } f(x) \in \mathrm{I\!R}; \\ \mathrm{I\!R}, & \text{if } f(x) = -\infty; \\ \emptyset, & \text{if } f(x) = +\infty. \end{cases}$$

The set $C^* \setminus \{0\}$ is the set $\{t \in \mathrm{I\!R} |\ t < 0\}$.

a) If $t \in C^* \setminus \{0\}$ *, then*

$$\forall x \in X: \quad \varphi_{(F,t)}(x) = -t\varphi_{(F,-1)}(x) = -tf(x).$$

Especially, $f(x) = \varphi_{(F,-1)}(x)$ *for all* $x \in X$*. Moreover, if* f *is proper, then*

$$\left\{t \in C^* \setminus \{0\} \,:\, \varphi_{(F,t)} : X \to \mathrm{I\!R} \cup \{\pm\infty\} \text{ is proper}\right\} = C^* \setminus \{0\}.$$

b) It holds $\operatorname{dom} F = \operatorname{dom} f$ *and* $\operatorname{epi} F = \operatorname{epi} f$*.*

c) The function F *is convex (*$C-$*proper, subadditive, positively homogeneous, closed), if and only if* f *is convex (proper, subadditive, positively homogeneous, closed).*

d) If f *is convex, then the directional derivative of* F *in* $x_0 \in \operatorname{dom} f$ *with respect to* $t \in C^* \setminus \{0\}$ *is*

$$\begin{aligned}\forall x_0, x \in X:\ F_t'(x_0,x) &= \{r \in \mathrm{I\!R} |\ -t(z) \geq (-tf)'(x_0,x) = -tf'(x_0,x)\} \\ &= f'(x_0,x) + C.\end{aligned}$$

If $f(x_0) = +\infty$*, then* $F_t'(x_0,x) = \mathrm{I\!R}$ *for all* $x \in X$*.*

e) The conjugate of F *is*

$$\begin{aligned}\forall (x^*,t) \in X^* \times C^* \setminus \{0\}: \quad F^*(x^*,t) &= F^*(-\frac{1}{t}x^*, -1) \\ &= \left\{r \in \mathrm{I\!R} |\ r \geq f^*(-\frac{1}{t}x^*)\right\} \\ &= f^*(-\frac{1}{t}x^*) + C.\end{aligned}$$

f) For the biconjugate of F *it holds*

$$\begin{aligned}\forall x \in X: \quad F^{**}(x) &= \bigcap_{t \in C^* \setminus \{0\}} \{r \in \mathrm{I\!R} |\ -tr \geq (-tf)^{**}(x) = -tf^{**}(x)\} \\ &= f^{**}(x) + C.\end{aligned}$$

g) If f *is convex and* $x_0 \in X$*, then* $x^* \in \partial f(x_0)$ *(*$x^* \in \partial_{ext} f(x_0)$*) if and only if*

$$S_{(x^*,-1)} \in \partial F(x_0)\ (S_{(x^*,-1)} \in \partial_{ext} F(x_0)).$$

8.1.6 Example (Extended vector-valued functions)**.** *Let* $f : X \to Z \cup \{+\infty\}$ *be a convex proper function,* $F(x) = f(x) + C$*.*

a) Let $z^* \in C^* \setminus \{0\}$ *, then*

$$\forall x \in X: \quad \varphi_{(F,z^*)}(x) = -z^*(f(x))$$

and

$$\left\{z^* \in C^* \setminus \{0\} \,:\, \varphi_{(F,z^*)} : X \to \mathrm{I\!R} \cup \{\pm\infty\} \text{ is proper}\right\} = C^* \setminus \{0\}.$$

b) It holds $\operatorname{dom} F = \operatorname{dom} f$ *and* $\operatorname{epi} F = \operatorname{epi} f$*.*

c) The function F *is convex and proper,* $C-$*proper if* $C^* \setminus -C^* \neq \emptyset$*. Moreover,* F *is subadditive (positively homogeneous), if and only if* f *is subadditive (positively homogeneous).*

d) If f is convex, then the directional derivative of F in $x_0 \in \operatorname{dom} f$ with respect to $z^* \in C^* \setminus \{0\}$ is given by

$$\forall z^* \in C^* \setminus \{0\}, x_0, x \in X: \ F'_{z^*}(x_0, x) = \{z \in Z| \ -z^*(z) \geq (-z^* f)'(x_0, x)\}.$$

If $f(x_0) = +\infty$, then $F'_{z^*}(x_,x) = Z$ for all $x \in X$.

e) The conjugate of F is defined by

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\}: \quad F^*(x^*, z^*) = \{z \in Z| \ -z^*(z) \geq (-z^* f)^*(x^*)\}.$$

f) For the biconjugate of F it holds

$$\forall x \in X: \quad F^{**}(x) = \bigcap_{z^* \in C^* \setminus \{0\}} \{z \in Z| \ -z^*(z) \geq (-z^* f)^{**}(x)\}.$$

g) If f is convex, $x_0 \in \operatorname{int} \operatorname{dom} f$ and $T \in \mathcal{L}(X, Z)$, then $T \in \partial f(x_0)$ if and only if

$$\forall z^* \in C^* \setminus \{0\}: \ S_{(-T^* z^*, z^*)} \in \partial F(x_0) = \partial_{ext} F(x_0).$$

Proof.

a) By definition,

$$\forall x \in X: \quad \varphi_{(F,z^*)}(x) = \inf \{-z^*(z)| \ z \in f(x) + C\} = -z^*(f(x)).$$

b) The effective domain of F is

$$\operatorname{dom} F = \{x \in X| \ f(x) + C \neq \emptyset\} = \operatorname{dom} f,$$

and for the epigraph it holds

$$\operatorname{epi} F = \{(x, z) \in X \times Z| \ z \in (f(x) + C)\} = \operatorname{epi} f,$$

c) As

$$\forall z^* \in C^* \setminus \{0\}: \quad \varphi_{(F,z^*)}: X \to \mathrm{I\!R} \cup \{\pm\infty\} \text{ is proper and convex,}$$

by 3.2.5 and 3.2.3 F is convex and C-proper if $C^* \setminus -C^* \neq \emptyset$. Equally, F is positively homogeneous or subadditive if and only if for all $z^* \in C^* \setminus \{0\}$ the scalarization $\varphi_{(F,z^*)}$ is, which is equivalent to $f: X \to Z$ being positively homogeneous or subadditive.

d) As $\varphi_{(F,z^*)}(x) = -z^*(f(x))$ holds for all $z^* \in C^* \setminus \{0\}$, $x \in X$, this is immediate with 5.1.2.

e) This holds by 4.1.2.

f) This is 4.1.5

g) This is 6.3.2. ■

8.1.7 Example (Vector Norm). *In [30], a function $f: X \to C$ is called a vector norm if*

VN1) It holds $f(x) = 0$ if and only if $x = 0$.

VN2) For all $t \in \mathbb{R}$, $x \in X$ it holds $f(tx) = |t| \cdot f(x)$.

VN3) For all $x_1, x_2 \in X$ it holds $f(x_1) + f(x_2) \in f(x_1 + x_2) + C$.

Let $f : X \to C$ be a vector norm defined everywhere. For $F : X \to \mathcal{Q}_C^t(Z)$ defined by

$$\forall x \in X : \; F(x) := f(x) + C$$

and $z^* \in C^* \setminus \{0\}$, it holds

VN1') The scalarization $\varphi_{(F,z^*)}(x) = -z^*(f(x))$ is a semi-norm, $\varphi_{(F,z^*)}(x) = 0$ if and only if $f(x) \in H(z^*) \cap -H(z^*)$.

VN2') It holds

$$\{x \in X|\; F(x) = C\} = \{x \in X|\; f(x) \in C \cap -C\}.$$

VN3') For all $t \in \mathbb{R}$, $x \in X$ it holds $F(tx) = |t| \cdot F(x)$.

VN4') For all $x_1, x_2 \in X$ it holds $F(x_1) + F(x_2) \subseteq F(x_1 + x_2)$.

Let

$$U_{z^*} := \{x \in X|\; -z^* f(x) \leq 1\},$$

then for $G : \left\{S_{(x^*,z^*)}|\; (x^*, z^*) \in X^* \times C^* \setminus \{0\}\right\} \to \mathcal{Q}_C^t(Z)$ defined by

$$\forall (x^*, z^*) \in X^* \times C^* \setminus \{0\} : \; G(S_{(x^*,z^*)}) := \bigcap_{x \in U_{z^*}} S_{(x^*,z^*)}(x)$$

it holds

VN1 For all $z^* \in C^* \setminus \{0\}$, the function $\varphi_{(G,z^*)}(x^*) = \sup\limits_{x \in U_{z^*}} x^*(x)$ is a norm.*

VN2 It holds $G(S_{(x^*,z^*)}) = H(z^*)$ if and only if $x^* = 0$.*

VN3 For all $t \in \mathbb{R}$, $x \in X$ it holds $G(S_{(tx^*,z^*)}) = |t| \cdot G(S_{(x^*,z^*)})$.*

VN4 For all $S_{(x_1^*,z^*)}, S_{(x_2^*,z^*)} \in X$ it holds $G(S_{(x_1^*+x_2^*,z^*)}) \subseteq G(S_{(x_1^*,z^*)}) + G(S_{(x_2^*,z^*)})$.*

It holds

a) It holds

$$F^*(x^*, z^*) = I_{\left\{S_{(x^*,z^*)}|\; \varphi_{(G,z^*)}(x^*) \leq 1\right\}}(S_{(x^*,z^*)}) + H(z^*).$$

b) If $z^* f(x) = 0$, then

$$\partial F(x) = \left\{S_{(x^*,z^*)}|\; \varphi_{(G,z^*)}(x^*) \leq 1,\; x^*(x) = 0\right\}.$$

and else

$$\partial F(x) = \left\{S_{(x^*,z^*)}|\; \varphi_{(G,z^*)}(x^*) = 1\right\}.$$

Proof.

a) $\varphi^*(F,z^*)(x^*) = \sup x \in X(x^*(x) \lhd \varphi_{(F,z^*)}(x)) = \iota_{x^* \in X^* |\, x^* \leq \varphi_{(F,z^*)}}(x^*)$

If $x^* \leq \varphi_{(F,z^*)}$, then for $x \in U_{z^*}$ it holds $x^*(x) \leq 1$. On the other hand, if $x^*(x) \leq 1$ holds for all $x \in U_{z^*}$, then $x^* \leq \varphi_{(F,z^*)}$ holds. Therefore,

$$\varphi^*(F,z^*)(x^*) = \iota_{\{x^* \in X^* |\, \forall x \in U_{z^*} : x^*(x) \leq 1\}}(x^*).$$

Therefore,

$$F^*(x^*,z^*) = I_{\left\{S_{(x^*,z^*)} |\, \varphi_{(G,z^*)}(x^*) \leq 1\right\}}(S_{(x^*,z^*)}) + H(z^*).$$

b)

$$\forall x \in X :\ \partial\varphi_{(F,z^*)}(x) = \left\{x^* \in X^* |\, \varphi_{(F,z^*)}(x) - x^*(x) \leq -\varphi_{(F,z^*)}(x^*)\right\}.$$

Therefore,

$$\forall x \in X :\ \partial\varphi_{(F,z^*)}(x) \subseteq \{x^* \in X^* |\, \forall x \in U_{z^*} :\ x^*(x) \leq 1\}.$$

Let $\varphi_{(F,z^*)}(x) = t \neq 0$, then $\frac{1}{|t|}x \in U_{z^*}$ and

$$\partial\varphi_{(F,z^*)}(x) = \left\{x^* \in X^* |\, x^*(\frac{1}{|t|}x) \geq 1\right\},$$

so

$$\partial\varphi_{(F,z^*)}(x) = \left\{S_{(x^*,z^*)} |\, \varphi_{(G,z^*)}(x^*) = 1\right\}$$

Let $\varphi_{(F,z^*)}(x) = 0$, $x^* \in \{x^* \in X^* |\, \forall x \in U_{z^*} :\ x^*(x) \leq 1\}$. Then $\varphi_{(F,z^*)}(x) - x^*(x) \leq -\varphi^*_{(F,z^*)}(x^*)$ if and only if $x^*(x) \geq 0$. On the other hand,

$$\partial\varphi_{(F,z^*)}(x) = \left\{x^* \in X^* |\, \forall \bar{x} \in X :\ \varphi_{(F,z^*)}(\bar{x}) - x^*(\bar{x}) \geq \varphi_{(F,z^*)}(x) - x^*(x)\right\},$$

therefore $x^*(x) \leq 0$, so

$$\partial F(x) = \left\{S_{(x^*,z^*)} |\, \varphi_{(G,z^*)}(x^*) \leq 1,\ x^*(x) = 0\right\}.$$

■

8.2 Arithmetic in the extended real numbers

Viewing the extended real numbers as an extension of the ordered linear space $(\mathbb{R}, +, \cdot, \leq)$, it is necessary to extend the addition, multiplication with positive real numbers, especially 0 and the difference defined on $\mathbb{R}$ to operators defined on $\mathbb{R} \cup \{\pm\infty\}$. Again, the algebraic interpretation of the difference will not be obtained for the extended definition of the difference operator.

An investigation of concavity rather then convexity would require a change from the inf-addition and -difference to the sup-addition and -difference. In fact, this can be achieved by the "multiplication with -1", causing a change of spaces.

The set $\mathbb{R}$ is ordered by the relation $\leq$ generated by the cone $\mathbb{R}_+ \cup \{0\} = \{t \in \mathbb{R} |\ t \geq 0\}$. By defining $(-\infty) + A = \mathbb{R}$ and $(+\infty) + A = \emptyset$ for any nonvoid subset $A \subseteq \mathbb{R}$, the order relation extends to $\mathbb{R} \cup \{\pm\infty\}$ by $r \leq s$ if and only if $s \in r + \mathbb{R}_+ \cup \{0\}$ for $r, s \in \mathbb{R} \cup \{\pm\infty\}$.

We will extend the operation $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to an operation on $\mathbb{R} \cup \{\pm\infty\}$ by means of

$$\forall r, s \in \mathbb{R} \cup \{\pm\infty\} : \quad r + s := \inf \{x + y |\ x, y \in \mathbb{R}, x \geq r, y \geq s\} .$$

In particular, $(+\infty) + r = +\infty$ for all $r \in \mathbb{R} \cup \{\pm\infty\}$.

Moreover, $\mathbb{R} \cup \{\pm\infty\}$ is supplied with a multiplication with positive real numbers by means of

$$\begin{aligned} 0 \cdot r &= 0, \\ t \cdot r &= \inf \{tx \in \mathbb{R} |\ x \geq r\} \end{aligned}$$

for all $t > 0$ and $r \in \mathbb{R} \cup \{\pm\infty\}$.

The set $\mathbb{R} \cup \{\pm\infty\}$ supplied with extended addition and the multiplication with positive numbers along with the order $\leq$ is an order complete, ordered conlinear space.

Notice that in general no inverse element $-x$ exists for $x \in \mathbb{R} \cup \{\pm\infty\}$. Anyway, for $s, t \geq 0$ it holds $(s + t)x = sx + tx$ for all $x \in \mathbb{R} \cup \{\pm\infty\}$. Therefore, $(\mathbb{R} \cup \{\pm\infty\}, +, \cdot, \leq)$ is an ordered cone in the sense of [32].

Analog to the addition, the substraction $- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be extended to an operation on $\mathbb{R} \cup \{\pm\infty\}$ which coincides with the inf-difference on $(\mathbb{R} \cup \{\pm\infty\}, +, \cdot, \leq)$.

8.2.1 Proposition. *The inf-difference* $\lhd : \mathbb{R} \cup \{\pm\infty\} \times \mathbb{R} \cup \{\pm\infty\} \to \mathbb{R} \cup \{\pm\infty\}$ *is given by*

$$\forall r, s \in \mathbb{R} \cup \{\pm\infty\} : \quad r \lhd s = \inf \{t \in \mathbb{R} |\ s + t \geq r\} .$$

If $r, s \in \mathbb{R}$ *holds, then* $(r \lhd s) = r - s$.

Proof. By definition,

$$\begin{aligned} r \lhd s &= \inf \{t \in \mathbb{R} |\ s + t \geq r\} \\ &= \begin{cases} +\infty, & \text{if } \{t \in \mathbb{R} |\ s + t \geq r\} = \emptyset; \\ -\infty, & \text{if } \{t \in \mathbb{R} |\ s + t \geq r\} = \mathbb{R}; \\ r - s, & \text{else.} \end{cases} \qquad = \inf \{t \in \mathbb{R} \cup \{\pm\infty\} \mid s + t \geq r\} \end{aligned}$$

holds for all $r, s \in \mathbb{R} \cup \{\pm\infty\}$. ■

In particular, $(-\infty) \lhd r = -\infty$ and $r \lhd (+\infty) = -\infty$ for all $r \in \mathbb{R} \cup \{\pm\infty\}$.

8.2.2 Lemma. *Let* $t \geq 0$, $a, b, x, y \in \mathbb{R} \cup \{\pm\infty\}$. *It holds* *a)*

$$\begin{aligned} a \lhd (+\infty) &= -\infty, \\ (-\infty) \lhd a &= -\infty, \end{aligned}$$

b)

$$t(a \lhd b) = ta \lhd tb,$$

c) if $a \leq b$, then

$$a \lhd x \leq b \lhd x$$

and

$$x \lhd b \leq x \lhd a,$$

d)

$$a \lhd b \leq (a \lhd x) + (x \lhd b)$$

e)

$$(a + x) \lhd (x + b) \leq a \lhd b$$

f)

$$(a + x) \lhd (b + y) \leq (a \lhd b) + (x \lhd y)$$

PROOF. By 2.2.7, it is only left to prove

$$\begin{aligned} \forall a \in \mathbb{R} \cup \{\pm\infty\} : \quad a \lhd (+\infty) &= -\infty, \\ (-\infty) \lhd a &= -\infty. \end{aligned}$$

By definition, $(a \lhd b) = \inf \{x \in \mathbb{R} | \, b + x \geq a\}$. As

$$\begin{aligned} \forall x \in \mathbb{R}, b \in \mathbb{R} \cup \{\pm\infty\} : \quad b + x &\geq -\infty \\ +\infty + x &= +\infty \end{aligned}$$

holds, the result is immediate. ■

Obviously, in the last three inequalities equality holds if $a, b, x, y \in \mathbb{R}$. Otherwise though, equality is not true in general. An easy result is

$$a \lhd a = \begin{cases} 0, & \text{if } a \in \mathbb{R}; \\ -\infty, & \text{else.} \end{cases}$$

and $0 \lhd a = -a$ with $-(\pm\infty) := (\mp\infty)$.

8.2.3 Example. *a) Let $a, b = 0$ and $x = +\infty$, then $a \lhd b = 0$ while*

$$\begin{aligned} (a \lhd x) + (x \lhd b) &= +\infty \\ (a + x) \lhd (x + b) &= -\infty. \end{aligned}$$

b) Let $a, x, y = +\infty$, $b = 0$, then

$$\begin{aligned} (a \lhd b) + (x \lhd y) &= +\infty \\ (a + x) \lhd (b + y) &= -\infty. \end{aligned}$$

If $r \leq s$ holds for $r, s \in \mathbb{R} \cup \{\pm\infty\}$ we have

$$(0 \lhd r) \geq (0 \lhd s)$$

We will not define a multiplication with negative numbers on $\mathbb{R} \cup \{\pm\infty\}$, instead we will make use of the expression $0 \lhd r$. For $r \in \mathbb{R}$ it holds $0 \lhd r = -r$, while $0 \lhd (\pm\infty) = (\mp\infty)$ and therefore $s + (0 \lhd s) \geq 0$ for $s \in \mathbb{R} \cup \{\pm\infty\}$.

It is important to remember that $+$ and $\lhd$ are not inverse operators in general. The advantage of making use of these operators is that we no longer have to restrict ourselves from terms like $(+\infty) - (-\infty)$ as both addition and substraction are defined on the whole space $\mathbb{R} \cup \{\pm\infty\}$.

8.3 Scalar Convex Analysis

In the following, we will summarize some well-known facts from the theory of scalar convex analysis as presented in [10, 25, 50, 62, 64]. As it is common, and also necessary, to reduce definitions such as that of the directional derivative or the subdifferential of a scalar function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ to the case of $|\varphi(x_0)| \neq +\infty$ when using the classic difference $''-''$, we will also include some extended definitions and discuss the new special cases occurring when using the inf-difference instead. It turns out that the extended definitions coincide with the classic ones everywhere but in "pathological" cases. The proves of each such statement can be easily done by applying the definition of $\lhd$. Throughout this chapter, X will be a locally convex separable space with the dual space X^*.

8.3.1 Basic facts and definitions

Let X be a locally convex separable space.

For a function $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$, the effective domain of φ is defined as

$$\operatorname{dom} \varphi := \{x \in X |\ \varphi(x) \neq +\infty\}.$$

The epigraph of φ is defined as

$$\operatorname{epi} \varphi = \{(x, t) \in X \times \mathbb{R} : |\ t \geq \varphi(x)\}.$$

8.3.1 Lemma. *[62] Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ be a convex function*

a) If $x_0 \in \operatorname{core} \operatorname{dom} \varphi$ with $\varphi(x) \in \mathbb{R}$, then φ is proper.

b) If φ is closed then either φ is proper or $\varphi(x) = -\infty$ for all $x \in \operatorname{dom} \varphi$.

c) If $x_0 \in \operatorname{core} \operatorname{dom} \varphi$, then either $\operatorname{cl} \varphi(x_0) = -\infty$ or $\operatorname{cl} \varphi(x) = \varphi(x)$ holds for all $x \in \operatorname{dom} \varphi$.

8.3.2 Lemma. *If $\operatorname{cl} \varphi$ is sublinear and proper, then $\operatorname{cl} \varphi(0) = 0$ and*

$$\operatorname{cl} \varphi(x) = \sup \{x^*(x) | x^* \leq \varphi\}.$$

Proof. As $\operatorname{cl} \varphi$ is assumed to be sublinear and proper, it holds $\operatorname{cl} \varphi(0) \in \{+\infty, 0\}$. As moreover $\operatorname{cl} \varphi$ is proper, there exists $x_0 \in \operatorname{dom} \operatorname{cl} \varphi$ and it holds $\operatorname{cl} \varphi(0) \leq \lim_{t \downarrow 0} \operatorname{cl} \varphi(t x_0) = 0$, thus $\operatorname{cl} \varphi(0) \neq +\infty$. By [11], every closed proper convex function is the pointwise supremum of its affine minorants. As $\operatorname{cl} \varphi(0) = 0$, the statement is proven. ■

8.3.3 Lemma. *[26] If $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is convex and $(x_0, s) \in \operatorname{int}\operatorname{epi}\varphi$ for some $s \in \mathrm{I\!R}$, then φ is continuous at $x_0 \in \operatorname{dom}\varphi$ or $\varphi(x) = -\infty$ for all $x \in \operatorname{dom}\varphi$.*

8.3.4 Definition. *[26] Let Y be another locally convex space and $A : X \to Y$ is a linear homeomorphism. If $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$, then*

$$\forall y \in Y : \quad A\varphi(y) = \inf_{Ax=y} \varphi(x).$$

If $\gamma : Y \to \mathrm{I\!R} \cup \{\pm\infty\}$, then

$$\forall x \in X : \quad \gamma A(y) = \gamma(Ax).$$

8.3.5 Definition. *[11] Let $M \subseteq X$ and $M^* \subseteq X^*$. the support function of M is defined by*

$$\forall x^* \in X^* : \quad \sigma(x^*|M) = \sup\{x^*(x)|\ x \in M\}$$

and likewise the support function of M^* is defined by

$$\forall x \in X : \quad \sigma(x|M^*) = \sup\{x^*(x)|\ x^* \in M^*\}$$

8.3.6 Definition. *[11] The indicator function of $M \subseteq X$ is defined by*

$$\forall x \in X : \quad \iota_M(x) = \sup\{x^*(x)|\ x \in M\} \begin{cases} 0, & \text{if } x \in M; \\ +\infty, & \text{else.} \end{cases}$$

We use the notion $\iota_M : X \to \mathrm{I\!R} \cup \{+\infty\}$ for the scalar indicator function in order to distinguish it from the set-valued indicator function $I_M : X \to \mathcal{Q}_C^t(Z)$ in the main part of this thesis.

8.3.2 Conjugation

8.3.7 Definition. *[11] For a function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$, the convex conjugate $\varphi^* : X^* \to \mathrm{I\!R} \cup \{\pm\infty\}$ is defined by*

$$\varphi^*(x^*) := \sup_{x \in X}(x^*(x) \lhd \varphi(x)),$$

the biconjugate $\varphi^{**} : X \to \mathrm{I\!R} \cup \{\pm\infty\}$

$$\varphi^{**}(x) := \sup_{x^* \in X^*}(x^*(x) \lhd \varphi^*(x^*)).$$

Obviously, the conjugate and the biconjugate are classically defined with $-$ instead of $\lhd$. As $x^* : X \to \mathrm{I\!R}$ has only real values, the extended substraction makes no difference in the definition.

8.3.3 Directional derivative and subdifferential

8.3.8 Definition. *The directional derivative of a convex function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is defined by*

$$\varphi'(x_0, x) = \lim_{s \downarrow 0} \frac{1}{s}\left(\varphi(x_0 + sx) \lhd \varphi(x_0)\right)$$

when the limit exists in $\mathrm{I\!R} \cup \{\pm\infty\}$.

If $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is a proper function and $x_0 \in \operatorname{dom}\varphi$, then our directional derivative coincides with the classic one for every $x \in X$ as found in [26] and others. If $x_0 \notin \operatorname{dom}\varphi$, then $\varphi'(x_0, x) = -\infty$ holds for all $x \in X$. If $\varphi(x_0) = -\infty$, then for all $s > 0$ and $x \in X$ it holds

$$\varphi(x_0 + sx) \lhd \varphi(x_0) = \begin{cases} -\infty, & \text{if } \varphi(x_0 + sx) = -\infty\ ; \\ +\infty, & \text{else.} \end{cases}$$

8.3.9 Lemma. *If $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is a convex function, then the directional derivative in $x_0 \in X$ exists for all $x \in X$ and it holds*

$$\varphi'(x_0, x) = \inf_{t>0} \frac{1}{t} \left(\varphi(x_0 + tx) \lhd \varphi(x_0)\right).$$

PROOF. If φ is proper and $x_0 \in \operatorname{dom}\varphi$, then this is the classic case as found in [26]. Otherwise, if $x_0 \notin \operatorname{dom}\varphi$, then $\varphi'(x_0, x) = -\infty$ holds for all $x \in X$. If $\varphi(x_0) = -\infty$, then either $x \notin \operatorname{cone}(\operatorname{dom}\varphi - x_0)$ and $\varphi'(x_0, x) = +\infty$, or

$$\exists s > 0 : \quad \varphi(x_0 + sx) - \infty$$

and thus $\varphi'(x_0, x) = -\infty$. ■

8.3.10 Remark. *Let $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ be a convex function, $\varphi(x_0) = -\infty$.*

a) Take $x \notin \operatorname{dom}\varphi'(x_0, \cdot)$. Then $\varphi'(x_0, x) = +\infty$, but $\varphi'(x_0, 0) = -\infty \neq 0 = 0(\varphi'(x_0, x))$. Therefore, the directional derivative is not positively homogeneous under the definition including 0, found in [21].

b) For any $x \in X$, the directional derivative $\varphi'(x_0, \cdot)$ can take values in $\{\pm\infty\}$ at $-x, x \in X$. Therefore the inequality

$$0 \lhd \varphi'(x_0, x) \leq \varphi'(x_0, -x)$$

in general does not hold if $\varphi(x_0) = -\infty$.

8.3.11 Definition. *The subdifferential of a convex function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is defined by*

$$\partial\varphi(x_0) := \{x^* \in X^* |\ \forall x \in X :\ x^*(x) \leq \varphi'(x_0, x)\}.$$

The extended subdifferential of a function $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ is defined by

$$\partial_{ext}\,\varphi(x_0) := \{x^* \in X^* |\ x^*(x_0) \lhd \varphi(x_0) \geq \varphi^*(x^*)\}.$$

8.3.12 Lemma. *Let $\varphi : X \to \mathrm{I\!R} \cup \{\pm\infty\}$ be a convex function and $x_0 \in X$. If $\varphi(x_0) = -\infty$ or $\operatorname{dom}\varphi = \emptyset$, then*

$$\partial\varphi(x_0) = \emptyset,$$
$$\partial_{ext}\,\varphi(x_0) = X^*.$$

If φ is proper, then

$$\partial\varphi(x_0) = \partial_{ext}\,\varphi(x_0).$$

Proof. If $\varphi(x_0) = -\infty$ or $\operatorname{dom}\varphi = \emptyset$, then

$$\varphi'(x_0, \cdot) \equiv -\infty,$$
$$\varphi^* \equiv \begin{cases} -\infty, & \text{if } \operatorname{dom}\varphi = \emptyset; \\ +\infty, & \text{if } \varphi(x_0) = -\infty. \end{cases}$$

Thus, $\partial\varphi(x_0) = \emptyset$ and $\partial_{ext}\varphi(x_0) = X^*$. If $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is proper, then the equality is well-known, compare [62]. ■

The statement in 8.3.12 shows, that in fact our definitions are very easy extensions of the classic definition subdifferential, where $\partial\varphi(x_0)$ is only defined for proper functions with $x_0 \in \operatorname{dom}\varphi$, see [62].

8.3.13 Lemma. *[62]Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ be a sublinear function, then $\partial\varphi(0) \neq \emptyset$ if and only if $\operatorname{cl}\varphi(0) = \varphi(0) = 0$. In this case, φ is proper and*

$$\partial\varphi(0) = \{x^* \in X^* | \, \forall x \in X : \, x^*(x) \leq \varphi(x)\}$$

and

$$\begin{aligned} \forall x \in X : \, \partial\varphi(x) &= \{x^* \in \partial\varphi(0) | \, x^*(x) = \varphi(x)\} \\ \operatorname{cl}\varphi(x) &= \sup_{x^* \in \partial\varphi(0)} x^*(x). \end{aligned}$$

8.3.14 Lemma. *[62] Let φ be a convex function, then $\partial\varphi(x_0) \neq \emptyset$ if and only if $\varphi'(x_0, 0) = \operatorname{cl}\varphi'(x_0, 0) = 0$. In this case, φ is proper and it holds*

$$\operatorname{cl}\varphi'(x_0, x) = \sup\{x^*(x) | x^* \in \partial\varphi(x_0)\}.$$

8.3.15 Lemma (Max-Formula). *[62] Let φ be a convex proper function and $x_0 \in \operatorname{dom}\varphi$. If φ is continuous at x_0, then $\partial\varphi(x_0) \neq \emptyset$, $\varphi'(x_0, \cdot)$ is continuous and finite and*

$$\forall x \in X : \, \exists x_0^* \in \partial\varphi(x_0) : \, \varphi'(x_0, x) = x_0^*(x).$$

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