Geometry of two integrable systems: discrete functions Z^c via circle patterns, conservation laws and linear congruences

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Contents

Chapter 1

Introduction

This thesis consists of two parts, devoted to geometrical properties of discrete and smooth integrable systems. The property of being integrable is used only in Part I to derive the defining equations for discrete versions of holomorphic maps z^c. The underling geometry is that of circle patterns, the governing equations are discrete versions of Painlevé and Riccati equations.

Part II deals with integrable conservation law systems, the geometric objects behind them are projective line congruences, Veronese variety and Dupin isoparametric hypersurfaces.

In the detailed Introduction all the definitions are given, the posed problems discussed and the main results formulated. The proofs and technicalities are provided in Parts I and II.

1.1 Discrete Z^c

The first part is devoted to a very interesting and rich object: circle patterns mimicking the holomorphic maps z^c and $\log(z)$. It has its roots in such different areas of mathematics as complex analysis, discrete geometry and the theory of integrable systems. The idea to use circles to model conformal maps stems from the property of analytic functions to map "infinitesimal circles" to "infinitesimal circles". This suggests constructing "discrete analytic function theory" on the basis of "finite" circles. It was Thurston who proposed this approach for approximating the Riemann mapping by circle packings (see [97]). Namely, one considers a finite part of a regular, say hexagonal, circle packing, which covers a given domain. Then Andreev's Theorem (see [69]) claims: one can change the radii of this finite packing, keeping the combinatorics of circle mutual tangency in such a way, that the "deformed" finite circle packing packs the unit circle. One is tempted to consider Andreevs' Theorem as a finite version of the Riemann mapping theorem.

The striking analogy between circle patterns and holomorphic maps resulted in the development of discrete analytic function theory (for a good survey see [42]). Discrete versions of uniformization theorem, maximum principle, Schwarz's lemma, rigidity properties and Dirichlet principle were established ([21, 61, 69, 84, 87]). Fast development of this rich fascinating area in recent years is caused by mutual influence and interplay of ideas and concepts from discrete geometry, complex analysis and the theory of integrable systems. Classical circle packings comprised of disjoint open disks were later generalized to circle patterns where the disks may overlap. This class opens a new page in the classical theory: it turned out that these circle patterns are governed by discrete integrable equation (the stationary Hirota equation [100]), thus providing one with the whole machinery of the integrable system theory [28]. Schramm's circle patterns can be also retrieved by imposing a certain symmetry (or degeneracy) condition on Clifford lattice studied in [67] in the framework of Schwarzien Kadomtsev-Petviashvili hierarchy.

Different underlying combinatorics were considered: Schramm introduced square grid circle patterns, generalized by Bobenko and Hoffmann to hexagonal patterns with constant intersection angles in [25], hexagonal circle patterns with constant multi-ratios were studied by Bobenko, Hoffman and Suris in [23].

A very natural theme in this theory is the difficult and subtle question on convergence of properly normalized sequences of circles patterns to their smooth counterparts. It was settled by Rodin and Sullivan [85] for general circle packings, He and Schramm [62] showed that the convergence is C^∞ for hexagonal packings, the uniform convergence for square grid circle patterns was established by Schramm [87].

On the other hand not very many explicit examples of analogs of standard holomorphic functions are known: for circle packings with the hexagonal combinatorics the only explicitly described examples are Doyle spirals [20], which are discrete analogues of exponential maps [36], and conformally symmetric packings, which are analogues of a quotient of Airy functions [22]. For patterns with overlapping circles more examples are constructed: discrete versions of $\exp(z)$, $\operatorname{erf}(z)$ ([87]), z^c, $\log(z)$ ([2]) are constructed for patterns with underlying combinatorics of the square grid; z^c, $\log(z)$ are also described for hexagonal patterns with both multi-ratio ([23]) and constant angle ([25]) properties.

Discrete z^c is not only a very interesting example in discrete conformal geometry. It has mysterious relationships to other fields. It is constructed via some discrete isomonodromic problem and is governed by discrete Painlevé II equation (see [77]), thus giving geometrical interpretation thereof. Its linearization defines Green's function on critical graphs (see [29]) found in [66] in the theory of Dirac operator. It seems to be an important tool for investigation of more general circle patterns and discrete minimal surfaces (see [26] for a brief survey and [24, 35] for more details).

• Schramm's circle patterns

The original Schramm's definition for square grid circle patterns is as follows.

Definition 1 *[87] Let G be a subgraph of the 1-skeleton of the cell complex with vertices $\mathbb{Z}+i\mathbb{Z} = \mathbb{Z}^2$. A square grid circle pattern for G is an indexed collection of circles on the complex plane*

$$\{C_z : z \in V(G),\ C_z \subset \mathbb{C}\}$$

that satisfy:
1) if $z, z+i \in V(G)$ then the intersection angle of C_z and C_{z+i} is $\pi/2$,
2) if $z, z+1 \in V(G)$ then the intersection angle of C_z and C_{z+1} is $\pi/2$,
3) if $z, z+1+i \in V(G)$ (or $z, z-1+i \in V(G)$) then the disks, defined by C_z and C_{z+1+i} (C_z and C_{z-1+i} respectively) are tangent and disjoint,
4) if $z, z_1, z_2 \in V(G)$, $|z_1 - z_2| = \sqrt{2}$, $|z - z_1| = |z - z_2| = 1$ (i.e. C_{z_1},C_{z_2} are tangent and C_z intersects C_{z_1} and C_{z_2}) and $z_2 = z + i(z_1 - z)$ (i.e. z_2 is one step counterclockwise from z_1), then the circular order of the triplet of points $C_z \cap C_{z_1} - C_{z_2}$,$C_{z_1} \cap C_{z_2}$,$C_z \cap C_{z_2} - C_{z_1}$ agrees with the orientation of C_z.

To visualize the analogy between Schramm's circle patterns and conformal maps, consider regular pattern composed of unit circles and suppose that the radii are being deformed in a way to preserve the orthogonality of neighboring circles and the tangency of half-neighboring ones. Discrete maps taking circle centers of the unit circles of the standard regular pattern to the circle centers of the deformed pattern mimic classical holomorphic functions, the deformed radii being analogous to $|f'(z)|$ (see Fig. 1.1). Now let us consider the lattice composed of the centers of circles of Schramm's pattern and of circle intersection points. It is straightforward to check that the elementary quadrilaterals of this new refined lattice are of kite form and Möbius equivalent to squares. This property can be reformulated in terms of the cross-ratios of the vertices.

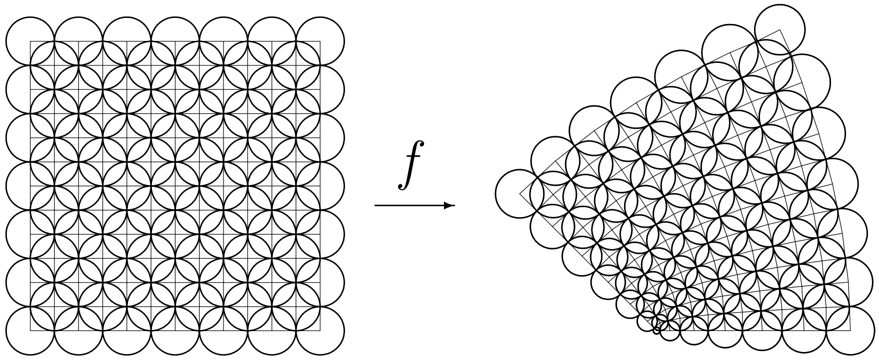

Figure 1.1: Schramm's circle patterns as discrete conformal map. Shown is the discrete version of the holomorphic mapping $z^{3/2}$.

Definition 2 *A map $f : \mathbb{Z}^2 \to \mathbb{R}^2 = \mathbb{C}$ is called a discrete conformal map if all its elementary quadrilaterals are conformal squares, i.e., their cross-ratios are equal to -1:*

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) := \frac{(f_{n,m} - f_{n+1,m})(f_{n+1,m+1} - f_{n,m+1})}{(f_{n+1,m} - f_{n+1,m+1})(f_{n,m+1} - f_{n,m})} = -1. \tag{1.1}$$

Thus Schramm's circle patterns is a special case of discrete conformal mapping. The definition above was introduced in [27] in the framework of discrete integrable geometry, originally without any relation to circle patterns. For some examples see also [63].

This definition is motivated by the following properties:
1) it is Möbius invariant,
2) a smooth map $f : D \subset \mathbb{C} \to \mathbb{C}$ is conformal (holomorphic or antiholomorphic) if and only if

$$\lim_{\epsilon \to 0} q(f(x, y), f(x + \epsilon, y)f(x + \epsilon, y + \epsilon)f(x, y + \epsilon)) = -1$$

for all $(x, y) \in D$.

In other words equation (1.1) for complex f is a discrete analog of Cauchy-Riemann equations. The essential difference to the smooth case is that most of the solutions to (1.1) have a behavior, which is far from that of the usual holomorphic maps: namely, interiors of neighboring elementary quadrilaterals can intersect. This is illustrated by the following naive method to construct a discrete analogue of the function $f(z) = z^c$: let us start with $f_{n,0} = n^c,\ n \geq 0$, $f_{0,m} = (im)^c,\ m \geq 0$, and then compute $f_{n,m}$ for each $n, m > 0$ using equation (1.1). The result is the left lattice in Fig. 1.2. Therefore it would be more consistent to define discrete conformal map as an cross-ratio preserving *immersion* on the vertices of cell decomposition of $\mathbb{C}$. To avoid confusion we will follow the already established terminology.

Definition 3 *A discrete conformal map $f_{n,m}$ is called an immersion if interiors of adjacent elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ are disjoint.*

We also will be interested in more subtle global properties of discrete conformal maps.

Definition 4 *A discrete conformal map $f_{n,m}$ is called embedded if interiors of different elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ do not intersect.*

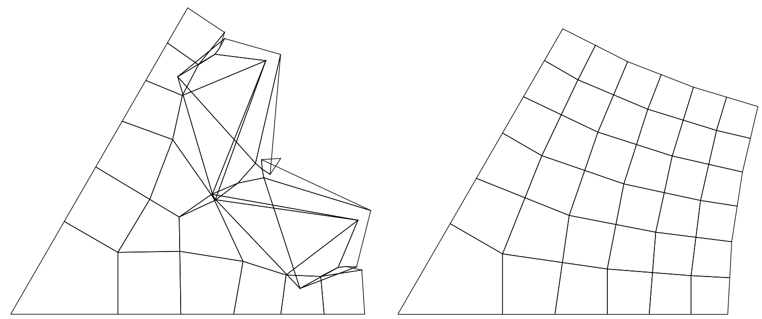

Figure 1.2: Two discrete conformal maps with close initial data for $n = 0$ and $m = 0$. The second lattice describes a discrete version of the holomorphic mapping $z^{2/3}$.

To illustrate the difference between the immersed and embedded discrete conformal maps $f : \mathbb{Z}^2_+ \to \mathbb{C}$, where $\mathbb{Z}^2_+ = \{(n,m) \in \mathbb{Z}^2 : n, m \geq 0\}$ let us imagine that the elementary quadrilaterals of the map are made of elastic inextensible material and are glued along the corresponding edges to form a surface with border. If this surface is immersed then it is locally flat. Being dropped down it will not have folds. At first sight it seems to be sufficient to guarantee embeddedness, provided $f_{n,0} \to \infty$ and $f_{0,m} \to \infty$ as $n \to \infty$. But a surface with such properties still may have some limit curve with self-intersections giving overlapping quadrilaterals. Hypothetical example of such a surface is shown in Fig. 1.3.

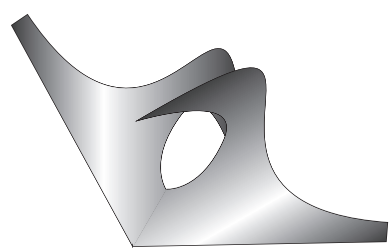

Figure 1.3: Surface glued of quadrilaterals of immersed but non-embedded discrete map.

To construct discrete analogue of z^c, which is the right lattice presented in Fig. 1.2, a more complicated approach is needed. The crucial step is supplementing equation (1.1) with the following non-autonomous constraint:

$$cf_{n,m} = 2n\frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})} + 2m\frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}. \qquad (1.2)$$

This constraint, as well as its compatibility with (1.1), is derived from some monodromy problem in Chapter 2. For $c = 1$ it appeared in [78].

Let us assume $0 < c < 2$. Motivated by the asymptotics of constraint (1.2) at $n, m \to \infty$ and the properties

$$z^c(\mathbb{R}_+) \in \mathbb{R}_+, \quad z^c(i\mathbb{R}_+) \in e^{c\pi i/2}\mathbb{R}_+$$

of the holomorphic mapping z^c we use the following definition [28] of the "discrete" z^c.

Definition 5 *The discrete conformal map* $Z^c : \mathbb{Z}^2_+ \to \mathbb{C}$, $0 < c < 2$ *is the solution of (1.1),(1.2) with the initial conditions*

$$Z^c(0,0) = 0, \quad Z^c(1,0) = 1, \quad Z^c(0,1) = e^{c\pi i/2}. \tag{1.3}$$

Obviously, $Z^c(n,0) \in \mathbb{R}_+$ and $Z^c(0,m) \in e^{c\pi i/2}(\mathbb{R}_+)$ for each $n, m \in \mathbb{N}$.

Given initial data $f_{0,0} = 0$, $f_{1,0} = 1$, $f_{0,1} = e^{i\alpha}$ with $\alpha \in \mathbb{R}$, constraint (1.2) allows one to compute $f_{n,0}$ and $f_{0,m}$ for all $n, m \geq 1$. Now using equation (1.1) one can successively compute $f_{n,m}$ for each $n, m \in \mathbb{N}$. It turned out that all edges at the vertex $f_{n,m}$ with $n+m = 0 \pmod 2$ are of the same length

$$|f_{n\pm1,m} - f_{n,m}| = |f_{n,m\pm1} - f_{n,m}| \tag{1.4}$$

and all angles between the neighboring edges at the vertex $f_{n,m}$ with $n + m = 1 \pmod 2$ are equal to $\pi/2$. Thus for any $n, m :\ n+m = 0 \pmod 2$ the points $f_{n+1,m}$, $f_{n,m+1}$, $f_{n-1,m}$, $f_{n,m-1}$ lie on the circle with the center $f_{n,m}$. All such circles form a circle pattern of Schramm type (see [87]), i.e. the circles of neighboring quadrilaterals intersect orthogonally and the circles of half-neighboring quadrilaterals with common vertex are tangent. Consider the sublattice $\{n, m :\ n + m = 0 \pmod 2\}$ and denote by $\mathbb{V}$ its quadrant

$$\mathbb{V} = \{z = N + iM :\ N, M \in \mathbb{Z}^2, M \geq |N|\}, \tag{1.5}$$

where

$$N = (n-m)/2, \quad M = (n+m)/2.$$

We will use complex labels $z = N + iM$ for this sublattice. Denote by $C(z)$ the circle of radius

$$R(z) = |f_{n,m} - f_{n\pm1,m}| = |f_{n,m} - f_{n,m\pm1}|$$

with the center at $f_{N+M,M-N} = f_{n,m}$.

The proof of the geometrical and asymptotic properties of discrete Z^c, formulated below, is based on the analysis of equations for $R(z)$. For instance, a square grid circle pattern is immersed if and only if the corresponding radius function $R(z)$ satisfies the following equation

$$R(z)^2 = \frac{\left(\frac{1}{R(z+1)} + \frac{1}{R(z+i)} + \frac{1}{R(z-1)} + \frac{1}{R(z-i)}\right) R(z+1)R(z+i)R(z-1)R(z-i)}{R(z+1) + R(z+i) + R(z-1) + R(z-i)}, \tag{1.6}$$

which is a discrete analogue of the equation $\Delta \log(R) = 0$. (Recall that $R(z)$ is an analog of $|f'(z)|$ and for an analytic function f the equation $\Delta \log(|f'(z)|) = 0$ holds.)

• **Circle patterns with prescribed intersection angles**

It turns out that Schramm's circle patterns is a special case of a more general scheme, giving circle patterns with more flexible combinatorics. The simplest generalization is square grid circle patterns with prescribed intersection angles giving Schramm's patterns as a special case.

Definition 6 *Let* G *be a subgraph of the 1-skeleton of the cell complex with vertices* $\mathbb{Z}+i\mathbb{Z} = \mathbb{Z}^2$ *and* $0 < \alpha < \pi$. *A square grid circle pattern for* G *with intersection angles* α *is an indexed collection of circles on the complex plane*

$$\{C_z : z \in V(G),\ C_z \subset \mathbb{C}\}$$

that satisfy:
1) if $z, z+i \in V(G)$ *then the intersection angle of* C_z *and* C_{z+i} *is* α ,

2) if $z, z+1 \in V(G)$ then the intersection angle of C_z and C_{z+1} is $\pi - \alpha$,
3) if $z, z+1+i \in V(G)$ (or $z, z-1+i \in V(G)$) then the disks, defined by C_z and C_{z+1+i} (C_z and C_{z-1+i} respectively) are tangent and disjoint,
4) if $z, z_1, z_2 \in V(G)$, $|z_1 - z_2| = \sqrt{2}$, $|z - z_1| = |z - z_2| = 1$ (i.e. C_{z_1},C_{z_2} are tangent and C_z intersects C_{z_1} and C_{z_2}) and $z_2 = z + i(z_1 - z)$ (i.e. z_2 is one step counterclockwise from z_1), then the circular order of the triplet of points $C_z \cap C_{z_1} - C_{z_2}$,$C_{z_1} \cap C_{z_2}$,$C_z \cap C_{z_2} - C_{z_1}$ agrees with the orientation of C_z.

The intersection angle is the angle at the corner of the disc intersection domain (Fig. 1.4). In what follows we call circle patterns with $\alpha = \pi/2$ *orthogonal*.

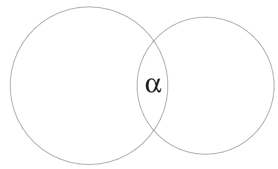

Figure 1.4: Circles intersection angle.

As above let us refine the lattice by circle intersection points. Then all the elementary quadrilaterals are conformal rhombi, i.e. equation (1.1) in Definition 2 generalizes to the following equation:

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) = e^{-2i\alpha}, \tag{1.7}$$

and the definitions of discrete conformal map and that of discrete Z^c are modified in a straightforward way.

Definition 7 *A map $f : \mathbb{Z}^2 \to \mathbb{R}^2 = \mathbb{C}$ is called a discrete conformal map if it satisfies equation (1.7).*

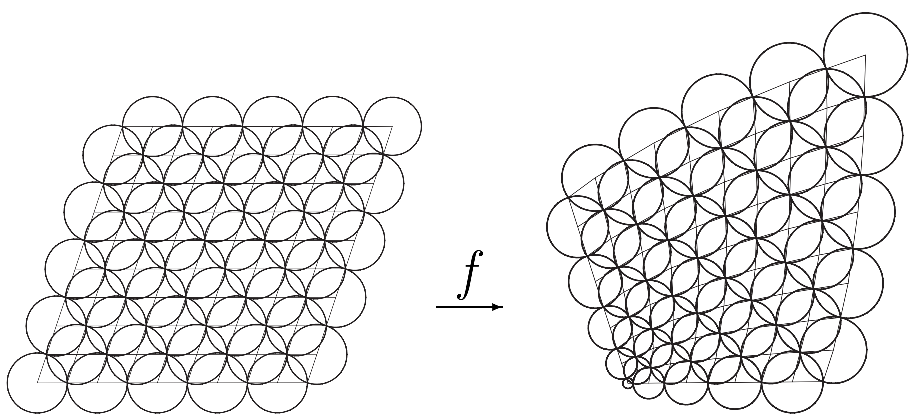

Figure 1.5: Schramm's type circle patterns with prescribed intersection angles as a discrete conformal map. The discrete version of the holomorphic mapping $z^{3/2}$. The case $\tan\alpha = 3$.

Definition 8 *The discrete conformal map $Z^c : \mathbb{Z}^2_+ \to \mathbb{C}$, $0 < c < 2$ is the solution of (1.7),(1.2) with the initial conditions*

$$Z^c(0,0) = 0, \quad Z^c(1,0) = 1, \quad Z^c(0,1) = e^{ci\alpha}. \tag{1.8}$$

• Hexagonal circle patterns

Equation (1.7) allows one also to define discrete Z^c with hexagonal combinatorics. The crucial step is to extend cross-ration equation (1.7) on $\mathbb{Z}^3$, to generalize constraint (1.2) and then restrict solutions on some regular sublattice of $\mathbb{Z}^3$ equations for the radii of the studied circle patterns in the whole Q-sublattice with even $k+l+m$. Hexagonal combinatorics are obtained on a sub-lattice of $\mathbb{Z}^3$ as follows: consider the subset

$$H = \{(k,l,m) \in \mathbb{Z}^3 : \ |k+l+m| \leq 1\}$$

and join by edges those vertices of H whose (k,l,m)-labels differ by 1 only in one component. The obtained quadrilateral lattice QL has two types of vertices: for $k+l+m=0$ the corresponding vertices have 6 adjacent edges, while the vertices with $k+l+m=\pm 1$ have only 3. Suppose that the vertices with 6 neighbors correspond to centers of circles in the complex plane $\mathbb{C}$ and the vertices with 3 neighbors correspond to intersection points of circles with the centers in neighboring vertices. Thus we obtain a circle pattern with hexagonal combinatorics.

Circle patterns where the intersection angles are constant for each of 3 types of quadrilateral faces (see Fig.1.6) were introduced in [25]. A special case of such circle patterns mimicking

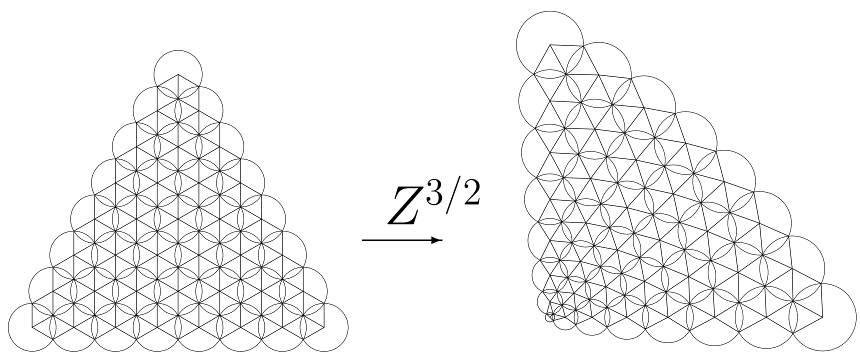

Figure 1.6: Hexagonal circle patterns as a discrete conformal map.

holomorphic map z^c and $\log(z)$ is given by the restriction to an H-sublattice of a special isomonodromic solution of some *integrable system* on the lattice $\mathbb{Z}^3$. Equations for the field variable $f : \mathbb{Z}^3 \to \mathbb{C}$ of this system are:

$$\begin{aligned} q(f_{k,l,m}, f_{k,l+1,m}, f_{k-1,l+1,m}, f_{k-1,l,m}) &= e^{-2i\alpha_1}, \\ q(f_{k,l,m}, f_{k,l,m-1}, f_{k,l+1,m-1}, f_{k,l+1,m}) &= e^{-2i\alpha_2}, \\ q(f_{k,l,m}, f_{k+1,l,m}, f_{k+1,l,m-1}, f_{k,l,m-1}) &= e^{-2i\alpha_3}, \end{aligned} \tag{1.9}$$

where $\alpha_i > 0$ satisfy $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. Equations (1.9) mean that the cross-ratios of images of faces of elementary cubes are constant for each type of faces, while the restriction $\alpha_1+\alpha_2+\alpha_3 = \pi$ ensures their compatibility.

The isomonodromic problem for this system (see section 2.2 for the details, where we present the necessary results from [25]) specifies the non-autonomous constraint

$$\begin{aligned} cf_{k,l,m} = 2k\frac{(f_{k+1,l,m}-f_{k,l,m})(f_{k,l,m}-f_{k-1,l,m})}{f_{k+1,l,m}-f_{k-1,l,m}} + \\ 2l\frac{(f_{k,l+1,m}-f_{k,l,m})(f_{k,l,m}-f_{k,l-1,m})}{f_{k,l+1,m}-f_{k,l-1,m}} + \end{aligned} \tag{1.10}$$

$$2m\frac{(f_{k,l,m+1}-f_{k,l,m})(f_{k,l,m}-f_{k,l,m-1})}{f_{k,l,m+1}-f_{k,l,m-1}}.$$

Constraint (1.10) is compatible with (1.9) (see [25]). Compatibility is understood as a solvability of some Cauchy problem. Obviously that implies compatibility of (1.7) and (1.2) for $\mathbb{Z}^2$-lattice.

In particular, a solution to (1.9),(1.10) in the subset

$$Q = \{(k,l,m) \in \mathbb{Z}^3 |\ k \geq 0,\ l \geq 0,\ m \leq 0\} \tag{1.11}$$

is uniquely determined by its values

$$f_{1,0,0},\ f_{0,1,0},\ f_{0,0,-1}.$$

Indeed, the constraint (1.10) gives $f_{0,0,0} = 0$ and defines f along the coordinate axis $(n,0,0)$, $(0,n,0)$, $(0,0,-n)$. Then all other $f_{k,l,m}$ with $(k,l,m) \in Q$ are calculated through the cross-ratios (1.9).

Proposition 1 *[25] The solution $f : Q \to \mathbb{C}$ of the system (1.9),(1.10) with the initial data*

$$f_{1,0,0} = 1,\ f_{0,1,0} = e^{i\phi},\ f_{0,0,-1} = e^{i\psi} \tag{1.12}$$

determines a circle pattern. For all $(k,l,m) \in Q$ with even $k+l+m$ the points $f_{k\pm1,l,m}$, $f_{k,l\pm1,m}$, $f_{k,l,m\pm1}$ lie on a circle with the center $f_{k,l,m}$, i.e. all elementary quadrilaterals of the image of Q are of kite form.

Moreover, equations (1.9) ensure that for the points $z_{k,l,m}$ with $k+l+m = \pm1$, where 3 circles meet, the intersection angles are α_i or $\pi - \alpha_i$, $i = 1,2,3$ (see Fig.1.6 where the isotropic case $\alpha_i = \pi/3$ of regular and $Z^{3/2}$-pattern is shown).

According to Proposition 1, the discrete map $z_{k,l,m}$, restricted to H, defines a circle pattern with circle centers $z_{k,l,m}$ for $k+l+m = 0$, each circle intersecting 6 neighboring circles. At each intersection points three circles meet.

However, for most initial data $\phi, \psi \in \mathbb{R}$, the behavior of the obtained circle pattern is quite irregular: interiors of different elementary quadrilaterals intersect. Define $Q_H = Q \cap H$.

Definition 9 *[25] The hexagonal circle patterns Z^c, $0 < c < 2$ with intersection angles $\alpha_1, \alpha_2, \alpha_3$, $\alpha_i > 0$, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ is the solution $Z^c : Q \to \mathbb{C}$ of (1.9) subject to (1.10) and with the initial data*

$$Z^c_{1,0,0} = 1,\ Z^c_{0,1,0} = e^{ic(\alpha_2+\alpha_3)},\ Z^c_{0,0,-1} = e^{ic\alpha_3} \tag{1.13}$$

restricted to Q_H.

Definition 10 *A discrete map $f : Q_H \to \mathbb{C}$ is called an immersion if interiors of adjacent elementary quadrilaterals are disjoint.*

It is interesting to note that the square grid circle pattern Z^c can be obtained from hexagonal one by limit procedure $\alpha_3 \to +0$ and by $\alpha_1 \to \pi - \alpha_2$. These limit circle patterns still can be defined by (1.9), (1.10) by imposing the self-similarity condition that $f_{k,l,m}$ depends only on l and $k - m$.

• **Circle patterns of discrete** z^2 **and** $\log(z)$

Definition 8 was given for $0 < c < 2$. For $c < 0$ or $c > 2$ the radius $R(1+i) = c/(2-c)$ of the corresponding circle pattern found as a solution to equations for $R(z)$ becomes negative and some elementary quadrilaterals around $f_{0,0}$ intersect. But for $c = 2$, one can renormalize the initial values of f so that the corresponding map remains an immersion. Let us consider orthogonal Z^c, with $0 < c < 2$, and make the following renormalization for the corresponding radii: $R \to \frac{2-c}{c} R$. Then as $c \to 2-0$ we have

$$R(0) = \frac{2-c}{c} \to +0, \quad R(1+i) = 1, \quad R(i) = \frac{2-c}{c} \tan \frac{c\pi}{4} \to \frac{2}{\pi}.$$

Definition 11 *([2]) Orthogonal* $Z^2 : \mathbb{Z}_+^2 \to \mathbb{R}^2 = \mathbb{C}$ *is the solution of (1.7), (1.2) with* $c = 2$ *and the initial conditions*

$$Z^2(0,0) = Z^2(1,0) = Z^2(0,1) = 0,$$

$$Z^2(2,0) = 1, \quad Z^2(0,2) = -1, \quad Z^2(1,1) = i\frac{2}{\pi}.$$

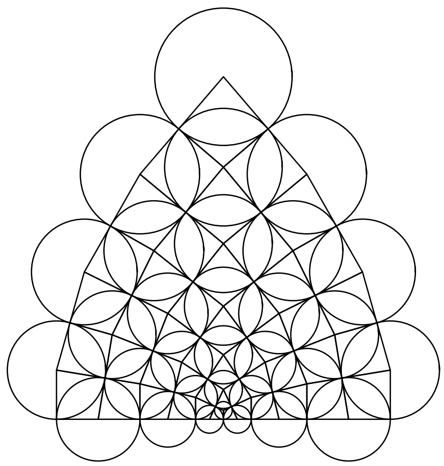

Figure 1.7: Discrete Z^2.

In this definition, equations (1.7),(1.2) are understood to be regularized through multiplication by their denominators. Note that for the radii on the border one has $R(N+iN) = N$, i.e. R is linear in agreement with $\left|\frac{d}{dz}(z^2)\right| = |z|$. If $R(z)$ is a solution to (1.6) and therefore defines some immersed circle patterns, then $\tilde{R}(z) = \frac{1}{R(z)}$ also solves (1.6). This reflects the fact that for any discrete conformal map f there is *dual discrete conformal map* f^* defined by (see [28])

$$f^*_{n+1,m} - f^*_{n,m} = -\frac{1}{f_{n+1,m} - f_{n,m}}, \quad f^*_{n,m+1} - f^*_{n,m} = \frac{1}{f_{n,m+1} - f_{n,m}}.$$

The smooth limit of the duality is

$$\frac{df^*(z)}{dz}\frac{df(z)}{dz} = -1.$$

The dual of $f(z) = z^2$ is, up to a constant, $f^*(z) = \log z$. Motivated by this observation, we define the discrete logarithm as the discrete map dual to Z^2, i.e. the map corresponding to the circle pattern with radii

$$R_{\text{Log}}(z) = \frac{1}{R_{Z^2}(z)},$$

where R_{Z^2} are the radii of the circles for Z^2. Here one has $R_{\mathrm{Log}}(0) = \infty$, i.e. the corresponding circle is a straight line. The corresponding constraint (1.2) can be also derived as a limit. Indeed, consider the map $g = \frac{2-c}{c}Z^c - \frac{2-c}{c}$. This map satisfies (1.7) and the constraint

$$c\left(g_{n,m} + \frac{2-c}{c}\right) = 2n\frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} + 2m\frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}.$$

Keeping in mind the limit procedure used to determine Z^2, it is natural to define the discrete analogue of $\log(z)$ as the limit of g as $c \to +0$. The corresponding constraint becomes

$$1 = n\frac{(g_{n+1,m} - g_{n,m})(g_{n,m} - g_{n-1,m})}{(g_{n+1,m} - g_{n-1,m})} + m\frac{(g_{n,m+1} - g_{n,m})(g_{n,m} - g_{n,m-1})}{(g_{n,m+1} - g_{n,m-1})}. \tag{1.14}$$

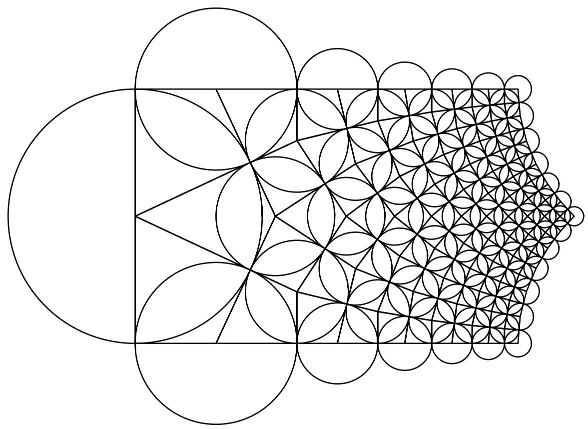

Figure 1.8: Discrete Log.

Definition 12 *([2])* Log *is the map* $\mathrm{Log}: \mathbb{Z}^2_+ \to \mathbb{R}^2 = \bar{\mathbb{C}}$ *satisfying (1.7) and (1.14) with the initial conditions*

$$\mathrm{Log}(0,0) = \infty,\ \mathrm{Log}(1,0) = 0,\ \mathrm{Log}(0,1) = \mathrm{i}\pi,$$

$$\mathrm{Log}(2,0) = 1,\ \mathrm{Log}(0,2) = 1 + \mathrm{i}\pi,\ \mathrm{Log}(1,1) = \mathrm{i}\frac{\pi}{2}.$$

It is interesting that the circle patterns for discrete z^2 and $\log(z)$ were originally guessed by Schramm and Kenyon without any connection to the theory of integrable systems and isomonodromy problem. Moreover, it was not proved that they are immersed.

The definitions of discrete z^2 and $\log(z)$ for hexagonal combinatorics and for square grid combinatorics with prescribed intersection angles are given in Section 6. The idea again is the re-normalization of initial data.

• **Asymptotic behavior of discrete z^c and** $\log(z)$

An effective approach to the description of circle patterns is given by the theory of integrable systems (see [23, 25, 28]). For example, Schramm's circle patterns are governed by a difference equation which is the stationary Hirota equation (see [87]). This approach proved to be especially useful for the construction of discrete z^c and $\log(z)$ in [2, 23, 25, 28] with the aid of some isomonodromy problem. Another connection with the theory of discrete integrable equations was revealed in [2, 3, 4]: embedded circle patterns are described by special solutions of discrete Painlevé II and discrete Riccati equations, thus giving geometrical interpretation thereof.

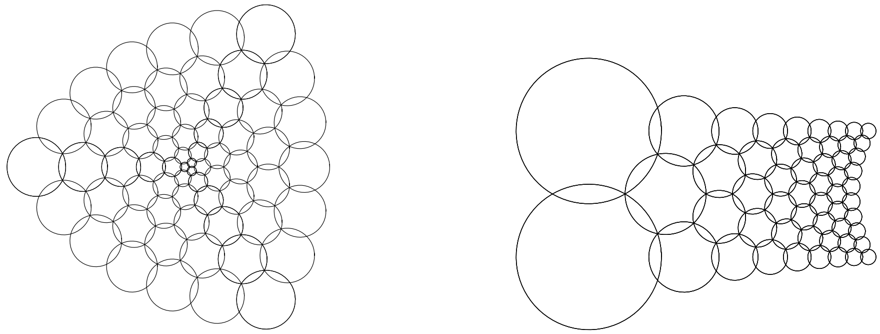

Figure 1.9: Hexagonal Z^2 and Log.

The main tool to establish all the above mentioned properties of $R(z)$ was a special case of discrete Painlevé-II equation:

$$(n+1)(x_n^2-1)\left(\frac{x_{n+1}+x_n/\varepsilon}{\varepsilon+x_nx_{n+1}}\right)-n(1-x_n^2/\varepsilon^2)\left(\frac{x_{n-1}+\varepsilon x_n}{\varepsilon+x_{n-1}x_n}\right)=cx_n\frac{\varepsilon^2-1}{2\varepsilon^2}.$$

The embedded Z^c corresponds to the unitary solution $x_n = e^{i\beta_n}$ of this equation with $x_0 = e^{ic\alpha/2}$ $0 < \beta_n < \alpha$. Here β_n is define by $f_{n,n+1} - f_{n,n} = e^{2i\beta_n}(f_{n+1,n} - f_{n,n})$.

As discrete Painlevé was used to prove the existence of discrete Z^c with regular behaviour, discrete Riccati equation is the tool to find the corresponding inital conditions for equations for $R(z)$. For $R(z)$ to be positive it is necessary that some discrete Riccati equation has positive solution. This leads to assymptotical analysis of solutions to *discrete Riccati equations*.

As the Riccati differential equation possesses the Panlevé property we are tempted to conclude that circle patterns Z^c and Log are described by discrete equations with Painlevé property though there is no satisfactory generalization thereof to discrete equations.

One of the principal questions in modelling standard holomorhic maps by infinite circle patterns is the asymptotics of their discrete analogs. The asymptotical behavior of discrete z^c and $\log(z)$ are exactly that of their smooth counterparts. Bearing in mind the highly non-trivial origin of the governing discrete equations this fact is really astonishing. The tool to prove the asymptotics is again some discrete Painleve equation, whose special reduction is the equation above.

For smooth Painlevé equations similar asymptotic problems have been studied in the frames of the isomonodromic deformation method [41, 56, 64]. In particular, connection formulas were derived. These formulas describe the asymptotics of solutions for $n \to \infty$ as a function of initial conditions. Some discrete Painlevé equations were studied in this framework in [55]). The geometric origin of our equations permits us to prove asymptotics by studying linearized equations and using the found geometric properties of the solutions without using the heavy isomonodromic technique. Moreover, isomonodromic methods seem to be insufficient for our purposes since we need to control x_n for finite n's as well.

• Main results

Now we give the plan of the Part I and formulate the main results. In Chapter 2 the defining equations (1.7),(1.2) are derived from some isomonodromic problem. In Chapter 3 the equations for radius function $R(z)$ are derived and the geometric properties of the circle patterns Z^c and Log are reformulated in terms of the solutions. In Chapter 4 we study discrete Riccati equations and prove that the above given initial conditions for Z^c and Log must be necessarily satisfied for immersed solutions. It is interesting that it was possible to find the general solution of this

Riccati equation and to express it through the hypergeometric function. In Chapter 5 we prove the existence of immersed hexagonal Z^c and of embedded square grid Z^c, which follows from the existence of the special separatrix solution of the corresponding discrete Painlevé equation. Chapter 6 extends these results for Z^2 and Log. Thus, the geometric properties are formulated as follows.

Theorem 1 *Square grid Z^c with $0 < c \le 2$ and* Log *are embedded. Hexagonal Z^c with $0 < c \le 2$ and* Log *are immersions.*

In Chapter 7 we prove the following asymptotical results.

Theorem 2 *The radius function $R(N+iM)$ of the orthogonal square grid Z^c satisfies:*

$$R(N_0 + iM) \simeq K(c)M^{c-1} \quad \text{as} \quad M \to \infty,$$

with constant $K(c)$ independent of N_0. For the orthogonal square grid Log *the corresponding asymptotics is*

$$R(N_0 + iM) \simeq K(0)M^{-1} \quad \text{as} \quad M \to \infty.$$

Moreover,

$$Z^c(n_0 + n, m_0 + n) \simeq e^{c\pi i/4}K(c)n^c \quad \text{as} \quad n \to \infty.$$

Finally, in Chapter 8 we discuss the uniqueness of discrete Z^c, some other examples and generalizations.

1.2 Geometry of integrable conservation laws

The second Part is devoted to geometric properties of integrable systems of hydrodynamic type, namely to conservation law systems. Hyperbolic systems of conservation laws

$$u^i_t = f^i(u)_x = v^i_j(u)u^j_x, \quad v^i_j = \frac{\partial f^i}{\partial u^j}, \quad i = 1, ..., n, \tag{1.15}$$

naturally arise in a variety of physical applications and are known to possess a rich mathematical and geometric structure (see [44, 65, 68, 91, 99]). It was observed that many constructions of the theory of systems of conservation laws (1.15) are parallel to that of the projective theory of congruences.

• Hyperbolic systems of conservation laws and congruences of lines

The correspondence proposed in [7] (see also [9] and [10]) associates with any system (1.15) an n-parameter family of lines

$$y^i = u^i \, y^0 - f^i(u), \quad i = 1, ..., n \tag{1.16}$$

in $(n+1)$-dimensional projective space $\mathbb{P}^{n+1}$ with affine coordinates $y^0, ..., y^n$. (Here y^0 is not one of homogeneous coordinates!) In the case $n = 2$ we obtain a two-parameter family or a congruence of lines in $\mathbb{P}^3$. In the 19th century the theory of congruences was one of the most popular chapters of classical differential geometry (see, e.g., [54]). We keep the name"congruence" for any n-parameter family of lines (1.16) in $\mathbb{P}^{n+1}$.

It turns out that the basic concepts of the theory of systems of conservation laws, such as shock and rarefaction curves, Riemann invariants, reciprocal transformations, linearly degenerate systems and systems of Temple class [95] acquire a clear and simple projective interpretation when reformulated in the language of the theory of congruences. For instance, this correspondence enabled the classification of systems of Temple class to be reduced to a much simpler geometric problem of the classification of congruences with either planar or conical developable surfaces. In particular, the results of [95] became intuitive geometric statements about families of lines in projective space. Another application of the proposed correspondence was the construction of the Laplace and Lévy transformations of hydrodynamic type systems in Riemann invariants [45, 47], which, on the geometric level, have been a subject of extensive research in projective differential geometry.

Remark. It should be emphasized that the correspondence between systems (1.15) and congruences in $\mathbb{P}^{n+1}$ is not one-to-one: "degenerate" congruences are to be excluded. Indeed, let

$$y^i = g^i(u)y^0 - f^i(u), \quad u = (u^1, ..., u^n) \tag{1.17}$$

be an arbitrary n-parameter family of lines in $\mathbb{P}^{n+1}$; notice that $g^i(u)$, as well as $f^i(u)$, may happen to be not functionally independent. Associated with such a congruence is a system of conservation laws

$$g^i(u)_t = f^i(u)_x, \quad i = 1, ..., n, \tag{1.18}$$

which, for functionally dependent $g^i(u)$, is not in Cauchy normal form. System (1.18) can be transformed to the Cauchy normal form provided the characteristic polynomial

$$\det\left(\lambda\frac{\partial g^i}{\partial u^j} - \frac{\partial f^i}{\partial u^j}\right) \tag{1.19}$$

is not identically zero, which is equivalent to the requirement that the lines (1.17) do not belong to a hypersurface in $\mathbb{P}^{n+1}$. Hypersurfaces in $\mathbb{P}^{n+1}$ carrying n-parameter families of lines are interesting in their own. For $n = 2$ these are planes. In the case $n = 3$ these are either one-parameter families of planes or three-dimensional quadrics [89]. For $n = 4$ among obvious examples are two-parameter families of planes or one-parameter families of three-dimensional quadrics. (See [86, 98] for the classification results.) In what follows we consider nondegenerate *hyperbolic* congruences only, which means that the characteristic polynomial (1.19) is not zero identically and its roots are real and pairwise distinct. Any such congruence can be locally parametrized in the form (1.16).

• Rarefaction curves

Let $\lambda^i(u)$ be the eigenvalues of the matrix v^i_j of system (1.15), assumed real and pairwise distinct. Let $\xi^i(u)$ be the corresponding eigenvectors: $v\ \xi^i = \lambda^i\ \xi^i$. Rarefaction curves, defined as the integral curves of the eigenvectors ξ_i, play a crucial role in the theory of hydrodynamic type systems. Thus, there are n families of rarefaction curves, and for any point in u-space there is exactly one rarefaction curve from each family passing through it. Due to the correspondence (1.16), a curve in $u-$space defines a ruled surface, i.e., a one-parameter family of lines in $\mathbb{P}^{n+1}$. In [7] (see also [9]) the following important property was established.

Theorem 3 *[7, 9] Ruled surfaces defined by rarefaction curves of the i-th family are developable, i.e., their rectilinear generators are tangential to a curve. This curve can be parametrized in the form*

$$y^0 = \lambda^i, \quad y^1 = u^1\lambda^i - f^1(u), \ ..., \ y^n = u^n\lambda^i - f^n(u), \tag{1.20}$$

where u varies along the rarefaction curve.

The curve (1.20) constitutes a singular locus of the developable surface called its *cuspidal edge*.

• Focal hypersurfaces

The focal hypersurface $M_i \subset \mathbb{P}^{n+1}$ is a collection of cuspidal edges corresponding to rarefaction curves of the i-th family. Therefore parametric equations of M_i coincide with (1.20), where u is now allowed to take all possible values. By the construction each line of the congruence (1.16) is tangential to M_i. The idea of focal hypersurfaces is obviously borrowed from optics: thinking of the congruence lines as the rays of light, one can intuitively imagine focal hypersurfaces as the locus in $\mathbb{P}^{n+1}$, where the light concentrates (the German literature uses the more suggestive term ‘Brennflächen’, i.e. ‘burning surfaces’). Since the system of conservation laws (1.15) is strictly hyperbolic, there are precisely n developable surfaces passing through a line of the congruence (1.16), and each line is tangential to n focal hypersurfaces.

• **Shock curves**

Shock curves play a fundamental role in the theory of weak solutions of systems (1.15). A shock curve with the vertex at u_0 is the set of points in u-space such that

$$\sigma(u^i - u_0^i) + f^i(u) - f^i(u_0) = 0, \quad i = 1, ..., n, \tag{1.21}$$

for some function $\sigma(u, u_0)$. For any u on the shock curve the discontinuous function

$$\begin{aligned} u(x,t) &= u_0, \quad x \leq \sigma t, \\ u(x,t) &= u, \quad x \geq \sigma t, \end{aligned}$$

is a weak solution of (1.15). Notice that (1.21) implies that the lines l_u and l_{u_0}, corresponding to the points u and u_0, intersect in $\mathbb{P}^{n+1}$. This implies that the shock curve with the vertex at u_0 defines a special ruled surface of the congruence (1.16) consisting of the lines of the congruence intersecting l_{u_0} ([7, 9]). Lax showed that a shock curve with the vertex at a generic point u_0 splits into n branches, the i-th branch being C^2-tangent of the associated rarefaction curve of the i-th family passing through u_0. As pointed out by a number of authors, there are situations when shock curves coincide with their associated rarefaction curves. Systems with coinciding shock and rarefaction curves were studied by Temple [95]. His main theorem can be formulated as follows.

Theorem 4 *[95] Rarefaction curves of the i-th family coincide with the associated branches of the shock curve if and only if either*
1) every rarefaction curve of the i-th family is a straight line in the $u-$space
or
2) the characteristic speed λ^i is constant along rarefaction curves of the i-th family,

$$L_i(\lambda^i) = 0,$$

where $L_i = \xi_i^k \frac{\partial}{\partial u^k}$ is the Lie derivative in the direction of ξ_i.

Both these condition have a very natural geometric interpretation.

Theorem 5 *[7, 9] Rarefaction curves of the ith family are straight lines if and only if the associated developable surfaces are planar, that is, their cuspidal edges are plane curves.*

Theorem 6 *[7, 9] The characteristic speed λ^i is linearly degenerate if and only if the associated developable surfaces are conical, that is, their generators meet at a point. The corresponding focal hypersurface M_i degenerates into a submanifold of codimension two.*

Recall that systems satisfying condition 2) are known as *linearly degenerate*.
This theorem introduces the following two natural classes of systems.

Systems with linear rarefaction curves, geometrically characterized by the planarity of cusp-edges (or, equivalently, planarity of developable surfaces) of the associated congruence (1.15) (see [7, 9]).

Linearly degenerate systems, characterized by condition 2) being satisfied for all $i = 1, ..., n$. Geometrically this condition means that developable surfaces of the congruence (1.15) are conical, and therefore all focal hypersurfaces M_i degenerate into submanifolds of codimension two

([7, 9]). This geometric result allows one to write down a general implicit formula for the fluxes $f_i(u)$ of linearly degenerate systems. (These formulae were previously known for $n=2$ only). As demonstrated in [7], (see also [9] and [10]) the properties formulated above provide an intuitive geometric proof of Temple's Theorem.

• **T-systems**

In the second Part we investigate and classify systems of conservation laws which simultaneously satisfy both conditions of Temple's Theorem:

(a) The rarefaction curves of system (1.15)) are straight lines in coordinates $u^1, ..., u^n$.

(b) The eigenvalues λ^i are linearly degenerate, i.e. constant along rarefaction curves of the i-th family.

Systems (1.15) satisfying both these conditions will be called **T-systems** for short. Systems of this type naturally arise in the theory of equations of associativity of 2D topological field theory [43]. In view of the results formulated above, developable surfaces of the corresponding congruence (1.16) must be planar and conical simultaneously, and therefore are planar pencils of lines. The corresponding focal hypersurfaces M_i degenerate into n submanifolds of codimension 2. In all examples discussed further the focal submanifolds M_i are glued together to form an algebraic variety $V^{n-1} \subset \mathbb{P}^{n+1}$ of codimension 2, so that the lines of the congruence (1.16) can be characterized as n-secants of V^{n-1}. We propose a complete description of T-systems for $n=3$, which is a generalization to the smooth setting of the classical result of Castelnuovo [37] classifying linear congruences in $\mathbb{P}^4$ in the algebraic-geometrical case. For $n=2$ the reader can easily derive it by elementary geometric considerations.

• **Riemann invariants**

It may happen that the $(n-1)$-dimensional submanifold M_i degenerates into a linear subspace of codimension 2. This is closely related to the property for system (1.15) to possess *Riemann invariants.*

Definition. *The Riemann invariant for ith characteristic speed λ^i is a function $R(u)$ such that*

$$R_t = \lambda^i R_x$$

by virtue of (1.15).

If system (1.15) possesses a Riemann invariant for each characteristic speed one can find new dependent variables $R^i(u)$ such that equations (1.15) take the diagonal form

$$R^i_t = \lambda^i R^i_x, \quad i = 1, ..., n$$

(no summation). As shown in [10], the existence of Riemann invariants implies that the focal nets (i.e., nets cut out by developable surfaces on each of the focal submanifolds M_i) are conjugate and holonomic. Congruences of this type and their transformations have been a subject of extensive research in projective differential geometry. Among others, the Laplace and Lévy transformations play fundamental roles. Being translated into the language of systems of conservation laws, these constructions lead to nontrivial transformations of semi-Hamiltonian systems of hydrodynamic type, which were investigated recently in [45] and [47].

• **Reciprocal transformations**

Let $B(u)dx + A(u)dt$ and $N(u)dx + M(u)dt$ be two conservation laws of system (1.15), understood as one-forms closed by virtue of (1.15). In the new independent variables X, T defined by

$$dX = B(u)dx + A(u)dt, \quad dT = N(u)dx + M(u)dt, \tag{1.22}$$

system (1.15) takes the form

$$U^i_T = F^i(U)_X, \quad i = 1, ..., n,$$

where

$$U^i = \frac{u^i M - f^i N}{BM - AN}, \quad F^i = \frac{f^i B - u^i A}{BM - AN},$$

or, if one prefers to work with old field variables,

$$u^i_T = V^i_j(u) u^j_X, \quad i = 1, ..., n,$$

where $V = (Bv - AE)(ME - Nv)^{-1}, \ E = id$. The new characteristic speeds Λ^k are

$$\Lambda^k = \frac{\lambda^k B - A}{M - \lambda^k N}. \tag{1.23}$$

Transformations (1.22) are called *reciprocal*. Reciprocal transformations are known to preserve linear degeneracy (see [46]). Moreover, particular reciprocal transformations (1.22), with both integrals being linear combinations of the "canonical" integrals $u^i dx + f^i dt$ of system (1.15),

$$\begin{aligned} dX &= (\alpha_i u^i + \alpha) dx + (\alpha_i f^i + \tilde{\alpha}) dt, \\ dT &= (\beta_i u^i + \beta) dx + (\beta_i f^i + \tilde{\beta}) dt, \end{aligned} \tag{1.24}$$

(here $\alpha_i, \alpha, \tilde{\alpha}, \beta_i, \beta, \tilde{\beta}$ are arbitrary constants), are known to preserve the class of T-systems ([7, 9]).

Furthermore, affine transformations

$$U^i = C^i_j u^j + D^i, \quad C^i_j = const, \ D^i = const, \ \det \mathrm{C}^\mathrm{i}_\mathrm{j} \neq 0, \tag{1.25}$$

obviously transform T-systems to T-systems.

Theorem 7 *[7, 9] The transformation group generated by reciprocal transformations (1.24) and affine transformations (1.25) is isomorphic to the group of projective transformations of* $\mathbb{P}^{n+1}$.

Thus, the classification of systems of conservation laws up to transformations (1.24) and (1.25) is equivalent to the classification of the corresponding congruences up to projective equivalence. This observation was the main reason for introducing the geometric correspondence between conservation law systems and line congruences.

One can readily establish that for $n = 2$ the congruences corresponding to T-systems are linear (that is, defined by two linear equations in Plücker coordinates) and consist of all lines intersecting two fixed skew lines in $\mathbb{P}^3$. Since any two linear congruences in $\mathbb{P}^3$ are projectively equivalent, there exists essentially a unique two-component T-system.

Example 1 Consider the wave equation

$$f_{tt} - f_{xx} = 0. \tag{1.26}$$

Introducing the variables $a = f_{xx}, \ b = f_{xt}$, we readily rewrite (1.26) as a linear two-component system of conservation laws

$$a_t = b_x, \qquad b_t = a_x, \tag{1.27}$$

which is obviously a T-system (any linear system of conservation laws is a T-system since its eigenvalues and eigenvectors are constant). The corresponding congruence (1.16)

$$y^1 = a y^0 - b, \qquad y^2 = b y^0 - a \tag{1.28}$$

consists of all lines intersecting the two skew lines $y^0 = 1, \ y^1 = -y^2$ and $y^0 = -1, \ y^1 = y^2$.

Example 2 Consider the Monge-Ampère equation

$$f_{xt}^2 - f_{xx} f_{tt} = 1. \tag{1.29}$$

Introducing the variables $a = f_{xx},\ b = f_{xt}$ (see [73]), we readily rewrite (1.29) as a two-component system of conservation laws

$$a_t = b_x, \qquad b_t = \left(\frac{b^2 - 1}{a}\right)_x, \tag{1.30}$$

which proves to be a T-system. The corresponding congruence

$$y^1 = ay^0 - b, \qquad y^2 = by^0 - \frac{b^2 - 1}{a} \tag{1.31}$$

consists of all lines intersecting the two skew lines $y^1 = 1$, $y^0 = y^2$ and $y^1 = -1$, $y^0 = -y^2$.

Since congruences (1.28) and (1.31) are projectively equivalent, the corresponding systems (1.27) and (1.30) are reciprocally related, thus providing a linearization of the nonlinear Monge-Ampère equation (1.29) (which, of course, is not a new result).

The main result of Chapter 9 is the classification of three-component T-systems or, in geometric language, congruences in $\mathbb{P}^4$ whose developable surfaces are planar pencils of lines. The example that motivated this research comes from the theory of equations of associativity of two-dimensional topological field theory.

Example 3 Let us consider the Monge-Ampère type equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}, \tag{1.32}$$

known as the WDVV or the associativity equation, which was thoroughly investigated by Dubrovin in [43]. Introducing the variables

$$a = f_{xxx},\ b = f_{xxt},\ c = f_{xtt}, \tag{1.33}$$

we readily rewrite (1.32) as a three-component system of conservation laws [72]

$$a_t = b_x, \qquad b_t = c_x, \qquad c_t = (b^2 - ac)_x \tag{1.34}$$

which was observed to be a T-system in [7] (also [9]). The corresponding congruence in $\mathbb{P}^4$

$$y^1 = ay^0 - b, \qquad y^2 = by^0 - c, \qquad y^3 = cy^0 - b^2 + ac \tag{1.35}$$

coincides with the set of trisecant lines of the Veronese variety projected from $\mathbb{P}^5$ into $\mathbb{P}^4$ (see section 9.1). The projected Veronese variety is the focal variety of the congruence (1.35). As follows from the classification result presented below, this example is generic.

We prove that congruences in $\mathbb{P}^4$, whose developable surfaces are planar pencils of lines are necessarily linear (that is, defined by three linear equations in the Plücker coordinates). In the parametrisation (1.16) the Plücker coordinates of a congruence in $\mathbb{P}^4$ are

$$u^1,\ u^2,\ u^3, \qquad f^1,\ f^2,\ f^3, \qquad u^1f^2 - u^2f^1,\ u^1f^3 - u^3f^1,\ u^2f^3 - u^3f^2.$$

Linear congruences are characterized by three linear relations among them

$$\alpha + \alpha_i u^i + \beta_i f^i + \alpha_{ij}(u^i f^j - u^j f^i) = 0,$$

where $\alpha, \ \alpha_i, \ \beta_i, \ \alpha_{ij}$ are arbitrary constants. Solving these equations for $f^1, \ f^2, \ f^3$, we arrive at the general formula for the fluxes of three-component T-systems. Notice that the congruence (1.35) is linear.

Unlike the case of $\mathbb{P}^3$, the proof of the linearity of these congruences in $\mathbb{P}^4$ requires a long computation bringing a certain exterior differential system into involutive form (notice that the linearity does not necessarily hold in $\mathbb{P}^5$ as simple examples from section 9.4 show). Once the linearity is established, one can make use of the results of Castelnuovo [37] who classified linear congruences in $\mathbb{P}^4$. He found six projectively different types, thus providing a list of six three-component T-systems which are not reciprocally related. Below we list them as scalar third-order Monge-Ampère type equations. They assume the form (1.15) in the variables $a = f_{xxx}, \ b = f_{xxt}, \ c = f_{xtt}$. As systems of conservation laws they differ, in particular, by a number of Riemann invariants they possess. Recall that the existence of a Riemann invariant implies the reducibility of the focal variety of the corresponding congruence: if a T-system possesses k Riemann invariants, the focal variety contains k linear subspaces of codimension two.

Theorem 8 *Any strictly hyperbolic T-system of 3 conservation laws can be reduced by a reciprocal transformation to one from the following list.*

I. *T-systems, which possess no Riemann invariants:*

$$f_{xxx}f_{ttt} - f_{xxt}f_{ttx} = 1 \tag{1.36}$$

and

$$f_{xxt}^2 + f_{xtt}^2 - f_{xxx}f_{xtt} - f_{ttt}f_{xxt} = 1. \tag{1.37}$$

The focal varieties of the corresponding congruences are non-singular projections of the Veronesé variety into $\mathbb{P}^4$. The congruences consist of the trisecant lines of these projections. (Notice that there are two essentially different projections which are not equivalent over the reals.)

II *T-systems, which possess one Riemann invariant:*

$$f_{xxx}f_{ttt} - f_{xxt}f_{ttx} = 0 \tag{1.38}$$

and

$$f_{xxt}^2 + f_{xtt}^2 - f_{xxx}f_{xtt} - f_{ttt}f_{xxt} = 0. \tag{1.39}$$

The corresponding focal varieties are reducible and consist of a cubic scroll and a plane which intersects the cubic scroll along its directrix. (Notice that equations (1.36) and (1.38) are related to (1.37) and (1.39) by a complex change of variables $x \to (x+t)/\sqrt{2}, \ t \to i(x-t)/\sqrt{2}$.)

III *T-system with two Riemann invariants:*

$$f_{xtt}^2 - f_{xxt}f_{ttt} = 1. \tag{1.40}$$

(This reduces to the Monge-Ampère equation (1.29) for $\tilde{f} = f_t$.) The corresponding focal variety consists of a two-dimensional quadric and two planes which intersect the quadric along rectilinear generators of different families.

IV *T-system with three Riemann invariants:*

$$f_{ttt} - f_{xxt} = 0. \tag{1.41}$$

The corresponding focal variety consists of three planes.

We discuss the geometry of these examples in more detail in Chapter 9.

Remark. Equation (1.36) was discussed by Dubrovin in [43]. As shown in [49], after the transformation $\tilde{x} = t,\ \tilde{t} = f_{xx},\ \tilde{f}_{\tilde{x}\tilde{x}} = -f_{xt},\ \tilde{f}_{\tilde{x}\tilde{t}} = x,\ \tilde{f}_{\tilde{t}\tilde{t}} = f_{tt}$ equation (1.36) takes the form (1.32): $\tilde{f}_{\tilde{t}\tilde{t}\tilde{t}} = \tilde{f}^2_{\tilde{x}\tilde{x}\tilde{t}} - \tilde{f}_{\tilde{x}\tilde{x}\tilde{x}}\tilde{f}_{\tilde{x}\tilde{t}\tilde{t}}$. (Notice that this is not a contact transformation.) Geometrically equations (1.32) and (1.36) correspond to projectively equivalent congruences. Equation (1.38) was discussed before in [50] and [92]. The classification of third order equations of Monge-Amperé type was given in [8].

• Reducible systems of conservation laws

Note that all the 3-component T-systems are reducible to a single third order (generically nonlinear) differential equations. These systems, which have been a subject of research in [8] and [30], are investigated and characterized geometrically in section 9.5. They can be defined as systems transformable to an n-th order scalar PDE

$$f\left(\frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial x^{n-1}\partial t}, \dots, \frac{\partial^n u}{\partial t^n}\right) = 0$$

by a transformation of type (1.33). In geometric language, the reducibility of system (1.15) implies that the associated congruence (1.16) belongs to the intersection of linear complexes of rank 4.

• Isoparametric hypersurfaces and linear congruences

On T-systems of 4 conservation laws we impose the following condition

$$\frac{(\lambda^1 - \lambda^2)(\lambda^3 - \lambda^4)}{(\lambda^2 - \lambda^3)(\lambda^4 - \lambda^1)} = -1, \tag{1.42}$$

i.e., characteristic speeds λ^i form a *harmonic quadruplet*. This condition comes from the theory of integrable systems and is necessary for the integrability. Consideration is restricted to nondiagonalizable systems without Riemann invariants. For T-systems condition (1.42) is very restrictive: it turns out that up to reciprocal transformations there are only 2 such systems over the reals, the corresponding congruences being linear. The geometry of the focal surfaces can be described in terms of isoparametric hypersurfaces in Euclidean (pseudo-Euclidean) space. Namely the Cartan isoparametric hypersurface $M^4 \subset \mathbb{S}^5 \subset \mathbb{E}^6$, represented as the intersection of the unit sphere with the zero level $P = 0$ of a fourth order homogeneous polynomial, is a non-singular 4-dimensional hypersurface. Through each point $m \in M^4$ passes a unique great circle $\mathbb{S}^1(m)$ in $\mathbb{S}^5$ that is orthogonal to M^4. Thus a 4-parameter family of such circles is obtained. Each great normal circle intersects the focal surface, consisting of two components, at four points forming a harmonic quadruplet on $\mathbb{S}^1(m)$. Regarding affine coordinates in $\mathbb{E}^6$ as homogeneous coordinates in $\mathbb{P}^5$, one arrives at a 4-parameter family of lines $l(m) \subset \mathbb{P}^5$, each line being defined by the 2-dimensional plane of the great circle $\mathbb{S}^1(m)$. This congruence and its pseudo-Euclidean counterpart are the normal forms of the classification obtained in Chapter 10 (see [13]).

• Geometry of solutions for $n = 3$

Consider the Plücker image of congruence (1.35), which is a three-dimensional submanifold M^3 of the Grassmanian $\mathbb{G}(1, 4) \subset \mathbb{P}^9$. Since congruence (1.35) is linear, M^3 is an intersection of $\mathbb{G}(1, 4)$ with $\mathbb{P}^6$. Moreover, M^3 is covered by a two-parameter family of lines (images of planar pencils) so that there are three lines passing through each point of M^3. Let M^2 be a surface in M^3, $p \in M^2$ be a point, and T_pM^2 a tangent plane to M^2 at p. Intersecting T_pM^2 with the three planes spanned by each pair of the three lines of M^3 passing through p, we obtain three characteristic directions in T_pM^2. The integral trajectories thereof foliate M^2 by three 1-parameter families of curves. Therefore, there is a characteristic 3-web invariantly defined on each surface M^2 in M^3. One can show that solutions of the system (1.32) are those M^2 for which the characteristic 3-web is hexagonal (has zero curvature). We refer to [18] for the necessary definitions and to [51, 53] for the proof of above statement.

• Implicit ODEs with hexagonal web of solutions

The hexagonal 3-web described above is the image of the characteristic web on solutions in (t, x)-plane understood as in the theory of PDE. It is determined by some implicit ordinary differential equation. This ODE for quasi-linear nonlinear PDEs depends on the particular solution. In Chapter 11 we study implicit ODEs with hexagonal web of solutions. (And we will use x, y instead of t, x to be consistent with the notations in the theory of ODE.) For regular points, where all curves intersects transversally, there is no local invariants. Thus non trivial information is encoded in singularities of solution webs of implicit characteristic equations.

Consider an implicit ordinary differential equation

$$F(x, y, p) = 0 \tag{1.43}$$

with a smooth or real analytic F. This ODE defines a surface M:

$$M := \{(x, y, p) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : \ F(x, y, p) = 0\}, \tag{1.44}$$

where (x, y, p) are coordinates in the jet space $J^1(\mathbb{R}, \mathbb{R})$ with $p = \frac{dy}{dx}$. Generically the condition $\mathrm{grad}(F)|_{F(x,y,p)=0} \neq 0$ holds true for any point $m = (x, y, p) \in M$, i.e. M is smooth. If the projection $\pi : M \to \mathbb{R}^2$, $(x, y, p) \mapsto (x, y)$ is a local diffeomorphism at a point $m \in M$ then this point is called *regular*. In some neighborhood of the projection $\pi(m)$ of a regular point m equation (1.43) can be solved for p thus defining an explicit ODE.

If the projection π is not a local diffeomorphism at m, then the point m is called a *singular* point of implicit ODE (1.43). The set of all singular points is called the *criminant* of equation (1.43) or the *apparent contour* of the surface M and will be denoted by C:

$$C := \{(x, y, p) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : F(x, y, p) = F_p(x, y, z) = 0\},$$

where the low subscript denotes a partial derivative: $F_p = \frac{\partial F}{\partial p}$.

Studying of generic singular points of implicit ODEs was initiated by Thom in [96]. Due to Whitney's Theorem such points are folds and cusps of the projection π. Local normal forms for generic singularities were conjectured by Dara in [39] and for a generic fold point were established by Davydov in [40]. The classification list for a generic fold point of the projection π is exhausted by a well folded saddle point, a well folded node point, a well folded focus point and a *regular* singular point, where the contact plane is transverse to the criminant. Cusp points were studied by Dara [39], Bruce [32] and Hayakawa, Ishikawa, Izumiya, Yamaguchi [60].
Usually the following *regularity condition* is imposed at each point of the criminant:

$$\mathrm{rank}((x, y, p) \mapsto (F, F_p)) = 2. \tag{1.45}$$

This regularity condition implies that the criminant is a smooth curve. At each point m outside the criminant C the contact plane $dy - pdx = 0$ cuts the tangent plane T_mM along a line thus giving a direction field, which takes the form:

$$\tau = [F_p : pF_p : -(F_x + pF_y)], \tag{1.46}$$

in the coordinates (x, y, p). This direction field is called the *characteristic field* of M. The projection $\pi(\gamma)$ of an integral curve $\gamma \subset M$ of the characteristic field τ is called a *solution* of ODE (1.43). If $F_{pp} \neq 0$ at a point $m \in C$ on the criminant then (1.43) reduces locally to an ODE quadratic in p. Such equations were the subject of intensive study. See, for example, [33, 34, 38, 59].

Suppose equation (1.43) has a triple root p_0 at (x_0, y_0) then the equation $F = 0$ can be written locally as a cubic equation

$$p^3 + a(x,y)p^2 + b(x,y)p + c(x,y) = 0, \tag{1.47}$$

as follows from the Division Theorem. Thus, if in a domain $U \subset \mathbb{R}^2$ outside the discriminant curve $\Delta := \pi(C)$ this cubic equation has 3 real roots p_1, p_2, p_3, we have 3-web formed by solutions of (1.43). A generic 3-web has a nontrivial invariant. In the differential-geometric context this invariant is the curvature form of the web. Therefore any general local classification of cubic implicit ODEs (1.47) necessarily has functional moduli (cf. [40]). Moreover, this invariant is topological in nature hence even the topological classification will have functional moduli if no restriction is imposed on the class of ODE's. (See also [74] and [75], where web structure was used for studying geometric properties of differential equations.)

In Chapter 11 we consider cubic ODEs (1.47) with a hexagonal web of solutions. Equations of this type describe, for example, webs of characteristics on solutions of *integrable* systems of three PDEs of hydrodynamic type, considered in Chapter 9 (see also [52]), or characteristic web of WDVV associativity equation (see Example 5 below).

Definition 13 *Let $U \subset \mathbb{R}^2$ be the open set, where (1.47) has 3 real roots p_1, p_2, p_3 and suppose $U \neq \emptyset$. We say that (1.47) has a hexagonal 3-web of solutions if for the projection $\pi(m)$ of each regular point $m \in M$ with $\pi(m) \in U$ there is a local diffeomorphism at $\pi(m)$ mapping the solutions of (1.47) to three families of parallel lines.*

The case of hexagonal web of solutions is also the most symmetric, i.e. the Lie symmetry pseudogroup of (1.43) at a regular point has the largest possible dimension 3. The list of normal forms turnes out to be finite provided regularity condition (1.45) is satisfied. These forms are given by the following examples.

Example 4 The classical Graf and Sauer theorem [58] states that a 3-web of straight lines is hexagonal iff the web lines are tangents to an algebraic curve of class 3, i.e. the dual curve is cubic. This implies immediately that the following cubic Clairaut equation has a hexagonal 3-web of solutions:

$$p^3 + px - y = 0.$$

The solutions are the lines $p = const$ enveloping a semicubic parabola. (See Fig. 1.10) Note that the contact plane is tangent to M along the criminant, i.e. the criminant is a *Legendrian* curve.

Example 5 Consider an associativity equation

$$u_{xxx} = u_{xyy}^2 - u_{xxy}u_{yyy},$$

describing 3-dimensional Frobenius manifolds (see [43] and Chapter 9). Each of its solutions $u(x, y)$ defines a characteristic web in the plane, which is hexagonal as was shown in [8]. Characteristics are integral curves of the vector field

$$\partial_x - \lambda(x,y)\partial_y,$$

where λ satisfy the characteristic equation

$$\lambda^3 + u_{yyy}\lambda^2 - 2u_{xyy}\lambda + u_{xxy} = 0.$$

For the solution $u = \frac{x^2y^2}{4} + \frac{x^5}{60}$ the characteristic equation becomes

$$p^3 + 2xp + y = 0$$

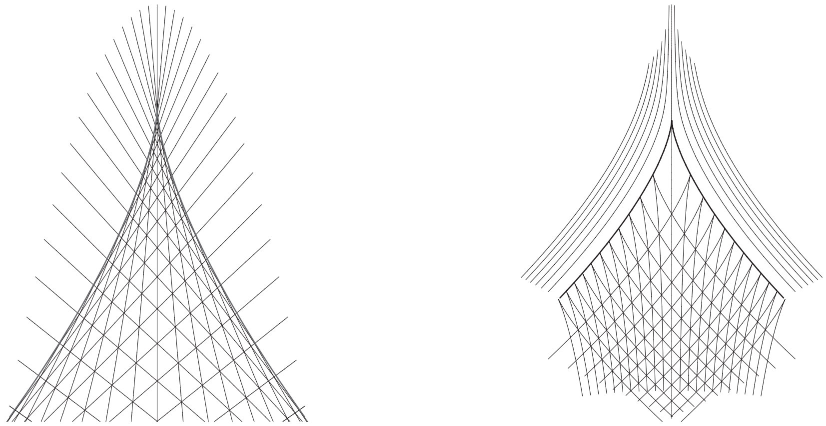

Figure 1.10: Solutions of $p^3 + px - y = 0$ (left) and $p^3 + 2xp + y = 0$ (right) with horizontal y-axis.

after the substitution $x \to -x$, $y \to -y$, $\lambda \to -p$. The criminant of this ODE is not Legendrian and its solutions have ordinary cusps on the discriminant (see Fig. 1.10). The discriminant is also a solution. In the analytic setting the above two normal forms were conjectured by Nakai in [76].

We find also local normal forms at points, where the projection π has a fold, i.e. the cubic ODE factors into a quadratic and a linear term.

Example 6 Suppose the criminant of a quadratic ODE is Legendrian, then this ODE is locally equivalent to

$$p^2 = y. \tag{1.48}$$

Solutions of this ODE together with the lines $dx = 0$ form a hexagonal 3-web (see Fig. 1.11). In fact, both the lines $dy = 0$ or the parabolas $2dy - xdx = 0$ also supplement the 2-web of solutions of (1.48) to a hexagonal 3-web, but the surfaces of the corresponding cubic equations $p(p^2 - y) = 0$ and $(2p - x)(p^2 - y) = 0$ are not smooth at $m = (0, 0, 0)$. If we agree to consider a quadratic equation as a cubic with one root at infinity, then equation (1.48) is the third normal form in our list. The following coordinate change $y = \tilde{y} + \frac{\tilde{x}^2}{4}$, $x = \tilde{x}$ straightens the solutions, transforming ODE (1.48) to a quadratic Clairaut equation

$$\tilde{p}^2 + \tilde{p}\tilde{x} - \tilde{y} = 0.$$

As the the lines $dx = 0$ are preserved this example is also a special case of the Graf and Sauer Theorem.

Example 7 Suppose the criminant of a quadratic ODE is not Legendrian, then this ODE is locally equivalent to

$$p^2 = x. \tag{1.49}$$

Solutions of this ODE together with the lines $dx = 0$ form a hexagonal web (See Fig. 1.11). The lines $dy = 0$ also complete the 2-web of solutions of (1.49) to a hexagonal 3-web, but again the surface M of the corresponding cubic equation $p(p^2 - x) = 0$ is not smooth at $m = (0, 0, 0)$.

Example 8 For completeness let us mention the case of a regular point of an implicit cubic ODE. If its 3-web of solutions is hexagonal, then it can be mapped to the web of 3 families of parallel lines $dx = 0$, $dy = 0$ and $dx + dy = 0$. This gives

$$p(p + 1) = 0.$$

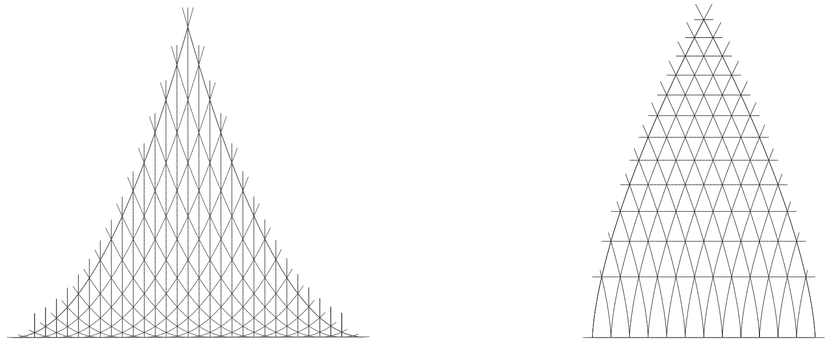

Figure 1.11: Solutions of $p^2 = y$ and the lines $dx = 0$ (left). Solutions of $p^2 = x$ and the lines $dx = 0$ with horizontal y-axis (right).

Now we can formulate our classification theorem.

Theorem 9 *Suppose functions a, b, c are real analytic and the following conditions hold for an implicit cubic ODE*

$$F(x, y, p) := p^3 + a(x, y)p^2 + b(x, y)p + c(x, y) = 0$$

at a point $m = (x_0, y_0, p_0) \in M := \{(x, y, p) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : \ F(x, y, p) = 0\}$:
1) this equation has a hexagonal 3-web of solutions,
2) $dF|_m \neq 0$,
3) $\operatorname{rank}((x, y, p) \mapsto (F, F_p))|_m = 2$ if m lies on the criminant C.
Then this ODE is equivalent to one of the following five forms with respect to some local real analytic isomorphism:

$$\begin{array}{lll} i) & p^3 + 2xp + y = 0, & \text{if } p_0 \text{ is triple and the criminant is transverse to the} \\ & & \quad \text{contact field in a punctured neighborhood of } m, \\ ii) & p^3 + px - y = 0, & \text{if } p_0 \text{ is triple and the criminant is Legendrian,} \\ iii) & p^2 = y, & \text{if } p_0 \text{ is double and the criminant is Legendrian,} \\ iv) & p^2 = x, & \text{if } p_0 \text{ is double and the criminant is transverse} \\ & & \text{to the contact plane at } m, \\ v) & p(p+1) = 0, & \text{if the roots are pairwise distinct at } \pi(m) = (x_0, y_0). \end{array} \tag{1.50}$$

If the functions a, b, c are smooth and conditions 1),2),3) are satisfied, then there is a diffeomorphism of a neighborhood of the point (x_0, y_0) onto a neighborhood of the point $(0, 0)$ reducing the above cubic ODE either to one of the four equations ii)-v) or to an equation that coincides with i) within the domain, where i) has three real roots.

The main difficulty in proving the above classification theorem brings the case of irreducible cubic ODE. The idea is to lift its 3-web of solutions to M and then to the plane $E : \ p_1 + p_2 + p_3 = 0$ in space of roots of the cubic equation $p^3 + A(x, y)p + B(x, y) = 0$. (Note that the general case reduces to this cubic.) Then this 3-web at the plane E has $\mathbb{D}_3$-symmetry permuting the roots. Using the regularity condition we construct a $\mathbb{D}_3$-equivariant diffeomorphism "upstairs", matching the web to that of a corresponding normal form. Due to the $\mathbb{D}_3$-symmetry the constructed diffeomorphism is lowerable to some diffeomorphism "downstairs", i.e. to a point transformation in the plane of solutions. Most of the claims and the proofs below are given for the smooth case and for some neighborhood of the projection $\pi(m)$ of $m \in M$, if it is not stated explicitly. In section 11.3 we discuss how to get rid of the annoying stipulation in Theorem 9 for the smooth case i) by replacing Definition 13 with a less geometric one.

Part I

Discrete Z^c

Chapter 2

Discrete Z^c via a monodromy problem

2.1 Square grid combinatorics

In this section we derive equation (1.1) as the compatibility condition (see [2]) of the Lax pair

$$\Psi_{n+1,m} = U_{n,m}\Psi_{n,m} \quad \Psi_{n,m+1} = V_{n,m}\Psi_{n,m} \tag{2.1}$$

found by Nijhoff and Capel [78]:

$$U_{n,m} = \begin{pmatrix} 1 & -u_{n,m} \\ \frac{\lambda}{u_{n,m}} & 1 \end{pmatrix} \quad V_{n,m} = \begin{pmatrix} 1 & -v_{n,m} \\ -\frac{\lambda}{v_{n,m}} & 1 \end{pmatrix}, \tag{2.2}$$

where λ is the spectral parameter and

$$u_{n,m} = f_{n+1,m} - f_{n,m}, \quad v_{n,m} = f_{n,m+1} - f_{n,m}.$$

Whereas equation (1.1) is invariant with respect to fractional linear transformations $f_{n,m} \to (pf_{n,m}+q)/(rf_{n,m}+s)$, the constraint (1.2) is not. By applying a fractional linear transformation and shifts of n and m, (1.2) is generalized to the following form:

$$\beta f_{n,m}^2 + \gamma f_{n,m} + \delta = 2(n-\phi)\frac{(f_{n+1,m} - f_{n,m})(f_{n,m} - f_{n-1,m})}{(f_{n+1,m} - f_{n-1,m})} +$$

$$2(m-\psi)\frac{(f_{n,m+1} - f_{n,m})(f_{n,m} - f_{n,m-1})}{(f_{n,m+1} - f_{n,m-1})}, \tag{2.3}$$

where $\beta, \gamma, \delta, \phi, \psi$ are arbitrary constants.

Theorem 10 *$f : \mathbb{Z}^2 \to \mathbb{C}$ is a solution to the system (1.1, 2.3) if and only if there exists a solution $\Psi_{n,m}$ to (2.1, 2.2) satisfying the following differential equation in λ:*

$$\frac{d}{d\lambda}\Psi_{n,m} = A_{n,m}\Psi_{n,m}, \quad A_{n,m} = -\frac{B_{n,m}}{1+\lambda} + \frac{C_{n,m}}{1-\lambda} + \frac{D_{n,m}}{\lambda}, \tag{2.4}$$

with $\lambda-$independent matrices $B_{n,m},\ C_{n,m},\ D_{n,m}$. The matrices $B_{n,m},\ C_{n,m},\ D_{n,m}$ in (2.4) *are of the following structure:*

$$
\begin{array}{rcl}
B_{n,m} & = & -\dfrac{n-\phi}{u_{n,m}+u_{n-1,m}}\begin{pmatrix} u_{n,m} & u_{n,m}u_{n-1,m} \\ 1 & u_{n-1,m}\end{pmatrix} - \dfrac{\phi}{2}I \\
C_{n,m} & = & -\dfrac{m-\psi}{v_{n,m}+v_{n,m-1}}\begin{pmatrix} v_{n,m} & v_{n,m}v_{n,m-1} \\ 1 & v_{n,m-1}\end{pmatrix} - \dfrac{\psi}{2}I \\
D_{n,m} & = & \begin{pmatrix} -\frac{\gamma}{4}-\frac{\beta}{2}f_{n,m} & -\frac{\beta}{2}f_{n,m}^2-\frac{\gamma}{2}f_{n,m}-\frac{\delta}{2} \\ -\frac{\beta}{2} & \frac{\gamma}{4}+\frac{\beta}{2}f_{n,m}\end{pmatrix}.
\end{array}
$$

The constraint (2.3) is compatible with (1.1).

Proof: Compatibility. Direct but rather long computation shows that if the constraint (2.3) holds for 3 vertices of an elementary quadrilateral it holds for the fourth vertex. A map $f : \mathbb{Z}^2 \to \mathbb{C}$ satisfying equation (1.1) and the constraint (2.3) is uniquely determined by its values at four vertices, for example, $f_{n_0,m_0}, f_{n_0,m_0\pm1}, f_{n_0+1,m_0}$. Indeed starting with these data and consequently applying (2.3) and (1.1) one determines $f_{n,m_0}, f_{n,m_0\pm1}$ for all n. Now, applying (2.3) we get the values $f_{n,m_0\pm2}$, $\forall n$. Note that, due to the observation above, equation (1.1) is automatically satisfied for all obtained elementary quadrilaterals. Proceeding further as above one determines $f_{n,m_0\pm3}$, $f_{n,m_0\pm4}, \ldots$ and thus $f_{n,m}$ for all n, m.

Necessity. Now let $f_{n,m}$ be a solution to the system (1.1),(2.3). Define $\Psi_{0,0}(\lambda)$ as a nontrivial solution of linear equation (2.4) with $A(\lambda)$ given by Theorem 10. Equations (2.1) determine $\Psi_{n,m}(\lambda)$ for any n, m. By direct computation, one can check that the compatibility conditions of (2.4) and (2.1)

$$
\begin{array}{rcl}
U_{n,m+1}V_{n,m} & = & V_{n+1,m}U_{n,m}, \\
\dfrac{d}{d\lambda}U_{n,m} & = & A_{n+1,m}U_{n,m} - U_{n,m}A_{n,m}, \\
\dfrac{d}{d\lambda}V_{n,m} & = & A_{n,m+1}V_{n,m} - V_{n,m}A_{n,m},
\end{array}
\tag{2.5}
$$

are equivalent to (1.1, 2.3).

Sufficiency. Conversely, let $\Psi_{n,m}(\lambda)$ satisfy (2.4) and (2.1) with some $\lambda-$independent matrices $B_{n,m}$, $C_{n,m}$, $D_{n,m}$. Note that the identity

$$\det \Psi_{n,m}(\lambda) = (1+\lambda)^n(1-\lambda)^m \det \Psi_{0,0}(\lambda)$$

for determinants implies

$$\mathrm{tr}A_{n,m}(\lambda) = \frac{n}{1+\lambda} - \frac{m}{1-\lambda} + a(\lambda), \tag{2.6}$$

where $a(\lambda)$ is independent of n and m. (Thus, up to the term $D_{n,m}/\lambda$, equation (2.4) is the simplest one possible.) From (2.6) it follows that $\mathrm{tr}\, B_{n,m} = -n$, $\mathrm{tr}\, C_{n,m} = -m$. Equations

(2.5) are equivalent to equations for their principal parts at $\lambda = 0$, $\lambda = -1$, $\lambda = 1$, $\lambda = \infty$:

$$D_{n+1,m} \begin{pmatrix} 1 & -u_{n,m} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -u_{n,m} \\ 0 & 1 \end{pmatrix} D_{n,m}, \tag{2.7}$$

$$D_{n,m+1} \begin{pmatrix} 1 & -v_{n,m} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -v_{n,m} \\ 0 & 1 \end{pmatrix} D_{n,m}, \tag{2.8}$$

$$B_{n+1,m} \begin{pmatrix} 1 & -u_{n,m} \\ -\frac{1}{u_{n,m}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -u_{n,m} \\ -\frac{1}{u_{n,m}} & 1 \end{pmatrix} B_{n,m}, \tag{2.9}$$

$$B_{n,m+1} \begin{pmatrix} 1 & -v_{n,m} \\ \frac{1}{v_{n,m}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -v \\ \frac{1}{v} & 1 \end{pmatrix} B_{n,m}, \tag{2.10}$$

$$C_{n+1,m} \begin{pmatrix} 1 & -u_{n,m} \\ \frac{1}{u_{n,m}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -u_{n,m} \\ \frac{1}{u_{n,m}} & 1 \end{pmatrix} C_{n,m}, \tag{2.11}$$

$$C_{n,m+1} \begin{pmatrix} 1 & -v_{n,m} \\ -\frac{1}{v_{n,m}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -v_{n,m} \\ -\frac{1}{v_{n,m}} & 1 \end{pmatrix} C_{n,m}, \tag{2.12}$$

$$(D_{n+1,m} - B_{n+1,m} - C_{n+1,m})E_{2,1} - E_{2,1}(D_{n,m} - B_{n,m} - C_{n,m}) = E_{2,1}, \tag{2.13}$$

$$(D_{n,m+1} - B_{n,m+1} - C_{n,m+1})E_{2,1} - E_{2,1}(D_{n,m} - B_{n,m} - C_{n,m}) = E_{2,1}. \tag{2.14}$$

Here

$$E_{2,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.15}$$

From (2.9, 2.10) and $\operatorname{tr} B_{n,m} = -n$, it follows that

$$B_{n,m} = -\frac{n-\phi}{u_{n,m} + u_{n-1,m}} \begin{pmatrix} u_{n,m} & u_{n,m}u_{n-1,m} \\ 1 & u_{n-1,m} \end{pmatrix} - \frac{\phi}{2} I.$$

Similarly, (2.11, 2.12) and $\operatorname{tr} C_{n,m} = -m$ imply

$$C_{n,m} = -\frac{m-\psi}{v_{n,m} + v_{n,m-1}} \begin{pmatrix} v_{n,m} & v_{n,m}v_{n,m-1} \\ 1 & v_{n,m-1} \end{pmatrix} - \frac{\psi}{2} I.$$

Here, ϕ and ψ are constants independent of n, m. The function $a(\lambda)$ in (2.6), independent of n and m, can be normalized to vanish identically, i.e. $\operatorname{tr} D_{n,m} = 0$. Substitution of

$$D = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

into equations (2.7, 2.8) yields

$$c_{n+1,m} = c_{n,m}, \quad c_{n,m+1} = c_{n,m}, \tag{2.16}$$

$$a_{n+1,m} = a_{n,m} - u_{n,m}c_{n,m}, \quad a_{n,m+1} = a_{n,m} - v_{n,m}c_{n,m}, \tag{2.17}$$

$$b_{n+1,m} = b_{n,m} + u_{n,m}(a_{n,m} + a_{n+1,m}), \quad b_{n,m+1} = b_{n,m} + v_{n,m}(a_{n,m} + a_{n,m+1}). \tag{2.18}$$

Thus c is a constant independent of n, m. Equations (2.17) can be easily integrated

$$a_{n,m} = -cf_{n,m} + \theta$$

where θ is independent of n, m (recall that $u_{n,m} = f_{n+1,m} - f_{n,m}$, $v_{n,m} = f_{n,m+1} - f_{n,m}$). Substituting this expression into (2.18) and integrating we get

$$b_{n,m} = -cf_{n,m}^2 + 2\theta f_{n,m} + \mu,$$

for some constant μ. Now (2.13) and (2.14) imply

$$b_{n,m} = -\frac{n-\phi}{u_{n,m}+u_{n-1,m}} u_{n,m}u_{n-1,m} - \frac{m-\psi}{v_{n,m}+v_{n,m-1}} v_{n,m}v_{n,m-1},$$

which is equivalent to constraint (2.3) after identifying $c = \frac{\beta}{2}$, $\theta = -\frac{\gamma}{4}$, $\mu = -\frac{\delta}{2}$. $\square$

Further, we will deal with the special case in (2.3) where $\beta = \delta = \phi = \psi = 0$, leading to the discrete Z^γ. Constraint (1.2) and the corresponding monodromy problem were obtained in [77] for the case $\gamma = 1$, and generalized to the case of arbitrary γ in [28]. One can derive equation (1.7) in the same way as above. We get it as a consequence of the results form [25] (see next section).

2.2 Hexagonal combinatorics

Equations (1.9) have the Lax representation [25]:

$$\begin{aligned}
\Psi_{k+1,l,m}(\lambda) &= L^{(1)}(e,\lambda)\Psi_{k,l,m}(\lambda), \\
\Psi_{k,l+1,m}(\lambda) &= L^{(2)}(e,\lambda)\Psi_{k,l,m}(\lambda), \\
\Psi_{k,l,m+1}(\lambda) &= L^{(3)}(e,\lambda)\Psi_{k,l,m}(\lambda),
\end{aligned} \tag{2.19}$$

where λ is the spectral parameter and $\Psi(\lambda) : \mathbb{Z}^3 \to \mathrm{GL}(2,\mathbb{C})$ is the wave function. The matrices $L^{(n)}$ are defined on the edges $e = (p_{out}, p_{in})$ of $\mathbb{Z}^3$ connecting two neighboring vertices and oriented in the direction of increasing $k+l+m$:

$$L^{(n)}(e,\lambda) = \begin{pmatrix} 1 & f_{in} - f_{out} \\ \lambda \frac{\Delta_n}{f_{in}-f_{out}} & 1 \end{pmatrix}, \tag{2.20}$$

with parameters Δ_n fixed for each type of edges. The zero-curvature condition on the faces of elementary cubes of $\mathbb{Z}^3$ is equivalent to equations (1.9) with $\Delta_n = e^{i\delta_n}$ for properly chosen δ_n. Indeed, each elementary quadrilateral of $\mathbb{Z}^3$ has two consecutive positively oriented pairs of edges e_1, e_2 and e_3, e_4. Then the compatibility condition

$$L^{(n_1)}(e_2)L^{(n_2)}(e_1) = L^{(n_2)}(e_4)L^{(n_1)}(e_3)$$

is exactly one of the equations (1.9). This Lax representation is a generalization of the one found in [78] for the square lattice.

A solution $f : \mathbb{Z}^3 \to \mathbb{C}$ of equations (1.9) is called *isomonodromic* if there exists a wave function $\Psi(\lambda) : \mathbb{Z}^3 \to \mathrm{GL}(2,\mathbb{C})$ satisfying (2.19) and the following linear differential equation in λ:

$$\frac{d}{d\lambda}\Psi_{k,l,m}(\lambda) = A_{k,l,m}(\lambda)\Psi_{k,l,m}(\lambda), \tag{2.21}$$

where $A_{k,l,m}(\lambda)$ are some 2×2 matrices meromorphic in λ with the order and position of their poles being independent of k, l, m. Isomonodromic solutions are important in many applications. In particular, for the first time the isomonodromy method was used to solve a discrete equation appearing in quantum gravity [55].

The simplest non-trivial isomonodromic solutions satisfy the constraint:

$$\begin{aligned}
bf_{k,l,m}^2 + cf_{k,l,m} + d = {} & 2(k-a_1)\frac{(f_{k+1,l,m}-f_{k,l,m})(f_{k,l,m}-f_{k-1,l,m})}{f_{k+1,l,m}-f_{k-1,l,m}} + \\
& 2(l-a_2)\frac{(f_{k,l+1,m}-f_{k,l,m})(f_{k,l,m}-f_{k,l-1,m})}{f_{k,l+1,m}-f_{k,l-1,m}} + \\
& 2(m-a_3)\frac{(f_{k,l,m+1}-f_{k,l,m})(f_{k,l,m}-f_{k,l,m-1})}{f_{k,l,m+1}-f_{k,l,m-1}}.
\end{aligned} \tag{2.22}$$

Theorem 11 *[25] Let $f : \mathbb{Z}^3 \to \mathbb{C}$ be an isomonodromic solution to (1.9) with the matrix $A_{k,l,m}$ in (2.21) of the form*

$$A_{k,l,m}(\lambda) = \frac{C_{k,l,m}}{\lambda} + \sum_{n=1}^{3} \frac{B^{(n)}_{k,l,m}}{\lambda - \frac{1}{\Delta_n}} \tag{2.23}$$

with λ-independent matrices $C_{k,l,m}$, $B^{(n)}_{k,l,m}$ and normalized by tr $A_{0,0,0}(\lambda) = 0$*. Then these matrices have the following form:*

$$C_{k,l,m} = \frac{1}{2}\begin{pmatrix} -bf_{k,l,m} - c/2 & bf^2_{k,l,m} + cf_{k,l,m} + d \\ b & bf_{k,l,m} + c/2 \end{pmatrix}$$

$$B^{(1)}_{k,l,m} = \frac{k - a_1}{f_{k+1,l,m} - f_{k-1,l,m}} \begin{pmatrix} f_{k+1,l,m} - f_{k,l,m} & (f_{k+1,l,m} - f_{k,l,m})(f_{k,l,m} - f_{k-1,l,m}) \\ 1 & f_{k,l,m} - f_{k-1,l,m} \end{pmatrix} + \frac{a_1}{2} I$$

$$B^{(2)}_{k,l,m} = \frac{l - a_2}{f_{k,l+1,m} - f_{k,l-1,m}} \begin{pmatrix} f_{k,l+1,m} - f_{k,l,m} & (f_{k,l+1,m} - f_{k,l,m})(f_{k,l,m} - f_{k,l-1,m}) \\ 1 & f_{k,l,m} - f_{k,l-1,m} \end{pmatrix} + \frac{a_2}{2} I$$

$$B^{(3)}_{k,l,m} = \frac{m - a_3}{f_{k,l,m+1} - f_{k,l,m-1}} \begin{pmatrix} f_{k,l,m+1} - f_{k,l,m} & (f_{k,l,m+1} - f_{k,l,m})(f_{k,l,m} - f_{k,l,m-1}) \\ 1 & f_{k,l,m} - f_{k,l,m-1} \end{pmatrix} + \frac{a_3}{2} I$$

and $f_{k,l,m}$ satisfies (2.22).

Conversely, any solution $f : \mathbb{Z}^3 \to \mathbb{C}$ to the system (1.9),(2.22) is isomonodromic with $A_{k,l,m}(\lambda)$ given by the formulas above.

The special case $b = a_1 = a_2 = a_3 = 0$ with shift $z \to z - d/c$ implies (1.10).

Chapter 3

Circle patterns

3.1 Schramm's square grid circle patterns.

In this section we show that Z^c of Definition 8 is a special case of circle patterns with the combinatorics of the square grid with prescribed angle intersection. Here more general initial conditions are considered:

$$f_{1,0} = 1, \;\; f_{0,1} = e^{i\beta} \tag{3.1}$$

with real β.

In what follows we say that the triangle (f_1, f_2, f_3) has *positive (negative) orientation* if

$$\frac{f_3 - f_1}{f_2 - f_1} = \left| \frac{f_3 - f_1}{f_2 - f_1} \right| e^{i\phi} \;\; \text{with } 0 \le \phi \le \pi \quad (-\pi < \phi < 0).$$

Lemma 1 *Let $q(f_1, f_2, f_3, f_4) = e^{-2i\alpha}$, $0 < \alpha < \pi$.*

- *If $|f_1 - f_2| = |f_1 - f_4|$ and the triangle (f_1, f_2, f_4) has positive orientation then $|f_3 - f_2| = |f_3 - f_4|$ and the angle between $[f_1, f_2]$ and $[f_2, f_3]$ is $(\pi - \alpha)$.*
- *If $|f_1 - f_2| = |f_1 - f_4|$ and the triangle (f_1, f_2, f_4) has negative orientation then $|f_3 - f_2| = |f_3 - f_4|$ and the angle between $[f_1, f_2]$ and $[f_2, f_3]$ is α.*
- *If the angle between $[f_1, f_2]$ and $[f_1, f_4]$ is α and the triangle (f_1, f_2, f_4) has positive orientation then $|f_3 - f_2| = |f_1 - f_2|$ and $|f_3 - f_4| = |f_4 - f_1|$.*
- *If the angle between $[f_1, f_2]$ and $[f_1, f_4]$ is $(\pi - \alpha)$ and the triangle (f_1, f_2, f_4) has negative orientation then $|f_3 - f_2| = |f_1 - f_2|$ and $|f_3 - f_4| = |f_4 - f_1|$.*

Proof: straightforward. $\square$

Proposition 2 *All the elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ for the solution of (1.7),(1.2) with initial (3.1) are of kite form: all edges at the vertex $f_{n,m}$ with $n + m = 0 \pmod 2$ are of the same length. Moreover, each elementary quadrilateral has one of the forms enumerated in Lemma 1.*

Proof: Given initial $f_{0,1}$ and $f_{1,0}$ constraint (1.2) allows one to compute $f_{n,0}$ and $f_{0,m}$ for all $n, m \ge 1$. Induction gives the following equidistant property:

$$f_{2n,0} - f_{2n-1,0} = f_{2n+1,0} - f_{2n,0}, \quad f_{0,2m} - f_{0,2m-1} = f_{0,2m+1} - f_{0,2m} \tag{3.2}$$

for every $n \ge 1$, $m \ge 1$. Now using (1.7) one can successively compute $f_{n,m}$ for each $n, m \in \mathbb{N}$. Lemma 1 completes the proof by induction. $\square$

Corollary 1 *The circumscribed circles of the quadrilaterals* $(f_{n-1,m},\ f_{n,m-1},\ f_{n+1,m},\ f_{n,m+1})$ *with* $n+m=0 \pmod 2$ *form a circle pattern of Schramm type (see [87]) with prescribed intersection angles (see Fig. 3.1).*

In fact, Proposition 2 implies that for $n+m=0 \pmod 2$ the points $f_{n\pm1,m}$,$f_{n,m\pm1}$ lie on the circle with the center at $f_{n,m}$. For the most β (namely for $\beta \neq \alpha$) this discrete conformal map is not an immersion.

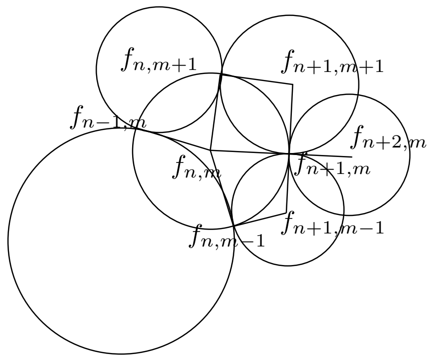

Figure 3.1: Discrete conformal maps of Schramm‘s type: kite-quadrilaterals, $n+m=0 \pmod 2$

Consider the sublattice $\{n,m:\ n+m=0 \pmod 2\}$ and denote by $\mathbb{V}$ its quadrant

$$\mathbb{V}=\{z=N+iM:\ N,M\in\mathbb{Z}^2, M\geq |N|\},$$

where

$$N=(n-m)/2,\quad M=(n+m)/2.$$

We use complex labels $z=N+iM$ for this sublattice. Denote by $C(z)$ the circle of the radius

$$R_z=|f_{n,m}-f_{n\pm1,m}|=|f_{n,m}-f_{n,m\pm1}| \tag{3.3}$$

with the center at $f_{N+M,M-N}=f_{n,m}$.

Let $\{C(z)\},\ z\in\mathbb{V}$ be a square grid circle pattern on the complex plane. Define $f_{n,m}:\mathbb{Z}^2_+\to\mathbb{C}$ as follows:
a) if $n+m=0 \pmod 2$ then $f_{n,m}$ is the center of $C(\frac{n-m}{2}+i\frac{n+m}{2})$,
b) if $n+m=1 \pmod 2$ then $f_{n,m}:=C(\frac{n-m-1}{2}+i\frac{n+m-1}{2})\cap C(\frac{n-m+1}{2}+i\frac{n+m+1}{2})=C(\frac{n-m+1}{2}+i\frac{n+m-1}{2})\cap C(\frac{n-m-1}{2}+i\frac{n+m+1}{2})$. Since all elementary quadrilaterals $(f_{n,m},\ f_{n+1,m},\ f_{n+1,m+1},\ f_{n,m+1})$ are of kite form equation (1.7) is satisfied automatically. In what follows the function $f_{n,m}$, defined as above by a) and b) is called *a discrete map corresponding to the circle pattern* $\{C(z)\}$.

Proposition 3 *Let the solution of (1.7),(1.2) with initial (3.1) be an immersion, then* $R(z)$ *defined by (3.3) satisfies the following equations:*

$$-MR_zR_{z+1}+(N+1)R_{z+1}R_{z+1+i}+(M+1)R_{z+1+i}R_{z+i}-NR_{z+i}R_z= \\ \frac{c}{2}(R_z+R_{z+1+i})(R_{z+1}+R_{z+i}) \tag{3.4}$$

for $z\in\mathbb{V}_l:=\mathbb{V}\cup\{-N+i(N-1)|N\in\mathbb{N}\}$ *and*

$$(N+M)(R_{z+i}+R_{z+1})(R_z^2-R_{z+1}R_{z-i}+\cos\alpha R_z(R_{z-i}-R_{z+1}))+ \tag{3.5}$$
$$(M-N)(R_{z-i}+R_{z+1})(R_z^2-R_{z+1}R_{z+i}+\cos\alpha R_z(R_{z+i}-R_{z+1}))=0,$$

for $z \in \mathbb{V}_{rint} := \mathbb{V} \backslash \{\pm N + iN | N \in \mathbf{N}\}$.
Conversely let $R(z) : \mathbb{V} \to \mathbb{R}_+$ satisfy (3.4) for $z \in \mathbb{V}_l$ and (3.5) for $z \in \mathbb{V}_{rint}$. Then $R(z)$ define a square grid circle patterns with intersection angles α, the corresponding discrete map $f_{n,m}$ is an immersion and satisfies (1.7),(1.2).

Proof: Circle pattern is immersed iff all triangles $(f_{n,m}, f_{n+1,m}, f_{n,m+1})$ of elementary quadrilaterals of the map $f_{n,m}$ have the same orientation (for brevity we call it orientation of quadrilaterals). Suppose that the quadrilateral $(f_{0,0}, f_{1,0}, f_{1,1}, f_{0,1})$ has positive orientation. Let the circle pattern $f_{n,m}$ be an immersion. For $n+m \equiv 1 \pmod 2$ points $f_{n,m}, f_{n-1,m+1}, f_{n-2,m}, f_{n-1,m-1}$ lie on circle with the center at $f_{n-1,m}$ and radius R_z, where $z = (n-m-1)/2 + i(n+m-1)/2$ (See the left part of Fig. 3.2). Using equation (1.7) one can compute $f_{n,m+1}$ and $f_{n,m-1}$. Lemma 1 and Proposition 2 imply that the points $f_{n-1,m}, f_{n,m}, f_{n+1,m}$ are collinear. Similarly the points $f_{n,m+1}, f_{n,m}, f_{n,m-1}$ are also collinear. Denote by R_{z+1}, R_{z+i} the radii of the circle at $f_{n,m-1}$ and $f_{n,m+1}$ respectively and by $R_{z+1+i} := R_z \frac{(f_{n+1,m} - f_{n,m})}{(f_{n,m} - f_{n-1,m})}$. Let (1.2) is satisfied at $(n-1, m)$. Then (1.2) at (n, m) is equivalent to (3.4), R_{z+1+i} being positive iff the quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ and $(f_{n,m-1}, f_{n+1,m-1}, f_{n+1,m}, f_{n,m})$ have positive orientation.

Similarly starting with (1.2) at $(n, m-1)$, where $n+m \equiv 0 \pmod 2$ (see the right part of Fig. 3.2) one can determine evolution of the cross-like figure formed by $f_{n,m-1}$, $f_{n+1,m-1}$, $f_{n,m}, f_{n-1,m-1}$, $f_{n,m-2}$ into $f_{n+1,m}$, $f_{n+2,m}$, $f_{n+1,m+1}$, $f_{n,m}$, $f_{n+1,m-1}$. Equation (1.2) at $(n+1, m)$ is equivalent to (3.4) and (3.5) at $z = (n-m)/2 + i(n+m)/2$. R_{z+1} is positive only for immersed circle pattern.

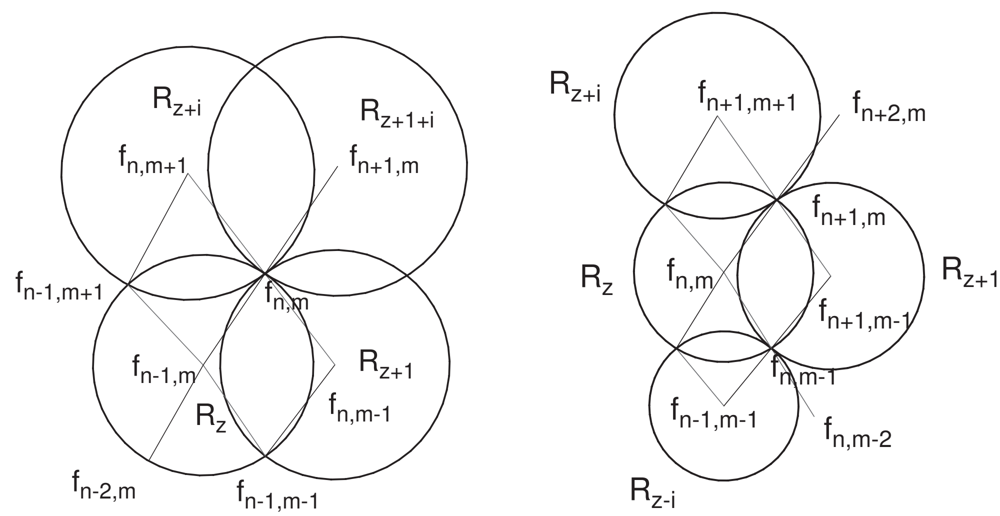

Figure 3.2: Kite-quadrilaterals of circle pattern.

Now let R_z be some positive solution to (3.4),(3.5). We rescale it so that $R_0 = 1$. This solution is completely determined by R_0, R_i. Consider solution $f_{n,m}$ of (1.7),(1.2) with initial data (3.1), where β is chosen so that the quadrilateral $(f_{0,0}, f_{1,0}, f_{1,1}, f_{0,1})$ has positive orientation and satisfies the conditions $R_0 = 1 = |f_{0,0} - f_{1,0}|$ and $R_i = |f_{1,1} - f_{1,0}|$. The map $f_{n,m}$ defines circle pattern due to Proposition 2. It is uniquely computed from these equations. To this end one have to resolve (3.4) with respect to R_{z+i+1} and use it to find $f_{n+1,m}$ from $R_{z+1+i} = \frac{R_z(f_{n+1,m} - f_{n,m})}{(f_{n,m} - f_{n-1,m}})$ and to resolve (3.5) for R_{z+i} to find $f_{n+1,m+1}$ from $R_{z+i} = R_{z+1} \frac{(f_{n+1,m+1} - f_{n+1,m})}{(f_{n+1,m} - f_{n+1,m-1})}$. Now one reverses the argument used in derivation of (3.4),(3.5) to show that f satisfies (1.7),(1.2). Moreover, since R_z is positive, at each step we get positively orientated quadrilaterals. $\square$

Remark. One easily derives from equations (3.4),(3.5) the following equation

$$\begin{gathered} R_z^2(R_{z+1}+R_{z+i}+R_{z-1}+R_{z-i})- \\ (R_{z+i}R_{z-1}R_{z-i}+R_{z+1}R_{z-1}R_{z-i}+R_{z+1}R_{z+i}R_{z-i}+R_{z+1}R_{z+i}R_{z-1})+ \\ 2R_z\cos\alpha(R_{z+1}R_{z-1}-R_{z+i}R_{z-i})=0, \end{gathered} \tag{3.6}$$

governing general square grid circle patterns with prescribed angles. For $\alpha=\pi/2$ it becomes

$$R(z)^2=\frac{\left(\frac{1}{R(z+1)}+\frac{1}{R(z+i)}+\frac{1}{R(z-1)}+\frac{1}{R(z-i)}\right)R(z+1)R(z+i)R(z-1)R(z-i)}{R(z+1)+R(z+i)+R(z-1)+R(z-i)}.$$

This equation is a discrete analogue of the equation $\Delta\log(R)=0$ in the smooth case. Similarly equation (3.5) is a discrete version of the equation $xR_y-yR_x=0$, and equation (3.4) is a discrete analogue of the equation $xR_x+yR_y=(c-1)R$.

Note that initial data (3.1) for $f_{n,m}$ imply initial data for R_z:

$$R_0=1,\quad R_i=\frac{\sin\frac{\beta}{2}}{\sin(\alpha-\frac{\beta}{2})}. \tag{3.7}$$

Theorem 12 *If for the solution R_z of (3.4),(3.5) with $c\neq 1$ and initial conditions (3.7) holds that*

$$R_z>0,\quad (c-1)(R_z^2-R_{z+1}R_{z-i}+\cos\alpha R_z(R_{z-i}-R_{z+1}))\geq 0 \tag{3.8}$$

in $\mathbb{V}_{int}$, then the corresponding discrete map is embedded.

Proof: To simplify computations we give the proof of this theorem for the case $\alpha=\frac{\pi}{2}$. For generic α it is the same with obvious modifications.

Since $R(z)>0$ the corresponding discrete map is an immersion due to Proposition 3. Consider piecewise linear curve Γ_n formed by segments $[f_{n,m},f_{n,m+1}]$ where $n>0$ and $0\leq m\leq n-1$ and the vector $\mathbf{v}_n(m)=(f_{n,m}f_{n,m+1})$ along this curve. Due to Proposition 2 this vector rotates only in vertices with $n+m=0 \pmod 2$ as m increases along the curve. The sign of the rotation angle $\theta_n(m)$, where $-\pi<\theta_n(m)<\pi$, $0<m<n$ is defined by the sign of expression $R(z)^2-R(z+i)R(z+1)$ (note that there is no rotation if this expression vanishes), where $z=(n-m)/2+i(n+m)/2$ is the label for the circle with the center in $f_{n,m}$. If $n+m=1 \pmod 2$ define $\theta_n(m)=0$.

Now the theorem hypothesis and equation (3.5) imply that the vector $\mathbf{v}_n(m)$ rotates with increasing m in the same direction for all n, and namely, clockwise for $c<1$ and counterclockwise for $c>1$.

Consider the sector $B:=\{z=re^{i\varphi}:r\geq 0,\ 0\leq\varphi\leq c\pi/4\}$. The terminal points of the curves Γ_n lie on the sector border.

Lemma 2 *For the curve Γ_n holds:*

$$\left|\sum_{m=1}^{n-1}\theta_n(m)\right|<\frac{\pi}{4}(1+|1-c|) \tag{3.9}$$

Proof of Lemma 2: Let us prove the inequality (3.9) for $1 < c < 2$ by induction for n. For $n = 1$ the inequality is obviously true since the curve Γ_1 is a segment perpendicular to $\mathbb{R}_+$. Define the angle $\alpha_n(m)$ between $i\mathbb{R}_+$ and the vector $\mathbf{v}_n(m)$ by $f_{n,m+1} - f_{n,m} = e^{i(\alpha_n(m)+\pi/2)}|f_{n,m+1} - f_{n,m}|$, where $0 \le \alpha_n(m) < 2\pi$, $0 \le m < n$. Then $\sum_{m=1}^{l} \theta_n(m) = \alpha_n(l) - \alpha_n(0) + 2\pi k_n(l)$ for some positive integer $k_n(l)$ increasing with l. Note that $\alpha_n(0) < \pi/2$, which easily follows from Propositon 2, and $\alpha_n(n-1) < (\frac{c\pi}{4} + \frac{\pi}{2}) - \frac{\pi}{2} = \frac{c\pi}{4}$ since for immersed Z^c the angle between the vector $\mathbf{v}_n(n-1)$ and $e^{ic\pi/4}\mathbb{R}_+$ is less then $\frac{\pi}{2}$. Let (3.9) holds for $n > 1$: $\left|\sum_{m=1}^{n-1} \theta_n(m)\right| = \sum_{m=1}^{n-1} \theta_n(m) < \frac{c\pi}{4}$ (all $\theta_n(m)$ are positive for $1 < c < 2$). That implies $k_n(l)$=0, since $k_n(l) = (\sum_{m=1}^{l} \theta_n(m) - \alpha_n(l) + \alpha_n(0))/2\pi \le (\sum_{m=1}^{l} \theta_n(m) + |\alpha_n(l)| + |\alpha_n(0)|)/2\pi < (c\pi/4 + c\pi/4 + \pi/2)/2\pi < 1$ and $k_n(l)$ is integer. Let $\alpha_{n+1}(l) = \alpha_n(l) + \sigma_n(l)$. All elementary quadrilaterals are of the kite form therefore $|\sigma_n(l)| < \pi/2$.

Let us prove, that $k_{n+1}(m) = 0$ for $0 \le m \le n+1$. Obviously $k_{n+1}(0) = 0$. Assume $k_{n+1}(l) = 0$ but $k_{n+1}(l+1) > 0$. The increment of l.h.s. of

$$\sum_{m=1}^{l} \theta_{n+1}(m) = \alpha_{n+1}(l) - \alpha_{n+1}(0) + 2\pi k_{n+1}(l)$$

as $l \to l+1$ is $\theta_{n+1}(l+1) < \pi$. The increment of r.h.s. is no less than $2\pi + \alpha_{n+1}(l+1) - \alpha_{n+1}(l) \ge 2\pi - \alpha_{n+1}(l) \ge 2\pi - \alpha_n(l) - |\sigma_n(l)| > 2\pi - c\pi/4 - \pi/2 > \pi$. The obtained contradiction gives $k_{n+1}(l) = 0$ and $\sum_{m=1}^{n} \theta_{n+1}(m) = \alpha_{n+1}(n) - \alpha_{n+1}(0) \le \alpha_{n+1}(n) < \frac{c\pi}{4}$. Lemma 2 is proved. $\square$

The obvious corollary of Lemma 2 is that the curve Γ_n has no self-intersection and lies in the sector B since the rotation of the vector $\mathbf{v}_n(m)$ along the curve is less then $c\pi/4 < \pi/2$. Each such curve cuts the sector B into a finite part and an infinite part. Since the curve Γ_n is convex and the borders of all elementary quadrilaterals $(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1})$ for imbedded Z^c have the positive orientation the segments of the curve Γ_{n+1} lie in the infinite part. Now the induction in n completes the proof of Theorem 12 for $1 < c < 2$. The proof for $0 < c < 1$ is similar. The differences are that $\theta_n(m)$ is not positive, the angle α is naturally defined as negative: $-2\pi < \alpha_n(m) < 0$, so that $-\pi/2 < \alpha_n(0) \le 0$ and $\frac{c\pi}{4}(2-c) < \alpha_n(n-1) < 0$. Details are left to the reader. $\square$

3.2 Hexagonal circle patterns with constant intersection angles.

In this section, like as in the previous one, we describe the hexagonal circle pattern corresponding to Z^c in terms of the radii of the circles.

Lemma 1 and Proposition 1 imply that each elementary quadrilateral of the studied circle pattern has one of the forms enumerated in Lemma 1. Proposition 1 allows us to introduce the radius function

$$r(_{K,L,M}) = |f_{k,l,m} - f_{k\pm1,l,m}| = |f_{k,l\pm1,m} - f_{k,l,m}| = |f_{k,l,m} - f_{k,l,m\pm1}|, \tag{3.10}$$

where (k,l,m) belongs to the sublattice of Q with even $k+l+m$ and (K,L,M) label this sublattice:

$$K = k - \frac{k+l+m}{2}, \quad L = l - \frac{k+l+m}{2}, \quad M = m - \frac{k+l+m}{2}. \tag{3.11}$$

The function r is defined on the sublattice

$$\tilde{Q} = \{(K,L,M) \in \mathbb{Z}^3 | L+M \le 0, \quad M+K \le 0, \quad K+L \ge 0)\}$$

corresponding to Q. Consider this function on

$$\tilde{Q}_H = \{(K,L,M) \in \mathbb{Z}^3 | K \ge 0, \quad L \ge 0, \quad M \le 0, \quad K+L+M = 0, +1\}.$$

Theorem 13 *Let the solution $f: Q_H \to \mathbb{C}$ of the system (1.9),(1.10) with initial data (1.12) be an immersion. Then function $r_{(K,L,M)}: \tilde{Q}_H \to \mathbb{R}_+$, defined by (3.10), satisfies the following equations:*

$$(r_1 + r_2)(r^2 - r_2 r_3 + r(r_3 - r_2)\cos\alpha_i)+$$
$$(r_3 + r_2)(r^2 - r_2 r_1 + r(r_1 - r_2)\cos\alpha_i) = 0 \tag{3.12}$$

on the patterns of type I and II as in Fig.3.3, with $i = 3$ and $i = 2$ respectively,

$$(L+M+1)\frac{r_4 - r_1}{r_4 + r_1} + (M+K+1)\frac{r_6 - r_3}{r_6 + r_3} + (K+L+1)\frac{r_2 - r_5}{r_2 + r_5} = c - 1 \tag{3.13}$$

on the patterns of type III and

$$r(r_1 \sin\alpha_3 + r_2 \sin\alpha_1 + r_3 \sin\alpha_2) = r_1 r_2 \sin\alpha_2 + r_2 r_3 \sin\alpha_3 + r_3 r_1 \sin\alpha_1 \tag{3.14}$$

on the patterns of type IV. Conversely, $r_{(K,L,M)}: \tilde{Q}_H \to \mathbb{R}_+$ satisfying equations (3.12),(3.13), (3.14) is the radius function of an immersed hexagonal circle pattern with constant intersection angles (i.e. corresponding to some immersed solution $f: Q_H \to \mathbb{C}$ of (1.9),(1.10)), which is determined by r uniquely.

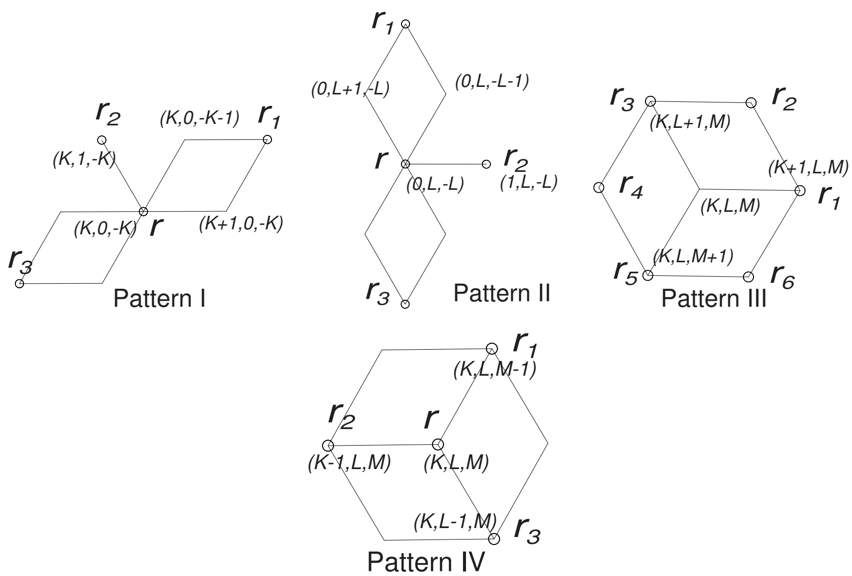

Figure 3.3: Equation patterns.

Proof: The map $f_{k,l,m}$ is an immersion iff the triangles $(f_{k,l,m},\ f_{k+1,l,m},\ f_{k,l,m-1})$, $(f_{k,l,m},\ f_{k,l,m-1},\ f_{k,l+1,m})$ and $(f_{k,l,m}, f_{k+1,l,m}, f_{k,l+1,m})$ of elementary quadrilaterals of the map $f_{k,l,m}$ have the same orientation (for brevity we call it the orientation of the quadrilaterals).

Necessity: To get equation (3.13) consider the configuration of two star-like figures with centers at $f_{k,l,m}$ with $k + l + m = 1 \pmod 2$ and at $f_{k+1,l,m}$, connected by five edges in the k-direction as shown on the left part of Fig.3.4. Let $r_i,\ i = 1, ..., 6$ be the radii of the circles with the centers at the vertices neighboring $f_{k,l,m}$ as in Fig.3.4. As follows from Lemma 1, the vertices $f_{k,l,m}$, $f_{k+1,l,m}$ and $f_{k-1,l,m}$ are collinear. For immersed f the vertex $f_{k,l,m}$ lies between $f_{k+1,l,m}$ and $f_{k-1,l,m}$. Similar facts are true also for the l- and m-directions. Moreover, the orientations of elementary quadrilaterals with the vertex $f_{k,l,m}$ coincides with one of the standard lattice.

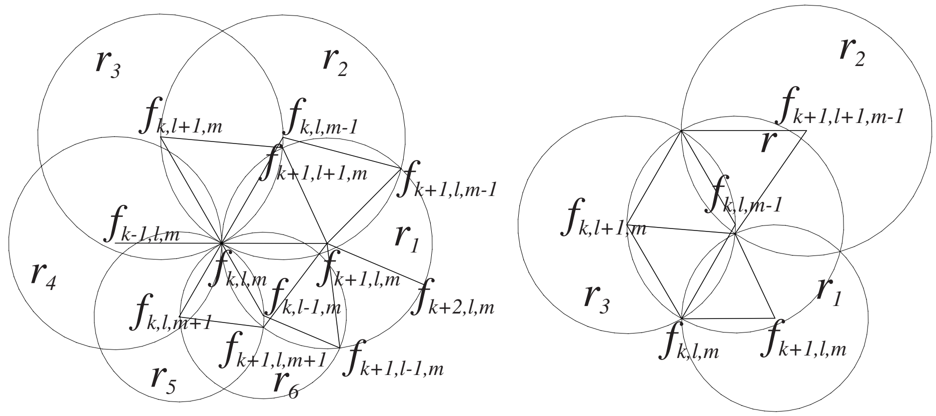

Figure 3.4: Circles.

Lemma 1 defines all angles at $f_{k,l,m}$ of these quadrilaterals. Equation (1.10) at (k,l,m) gives $f_{k,l,m}$:

$$f_{k,l,m} = \frac{2e^{is}}{c}\left(k\frac{r_1 r_4}{r_1 + r_4} + l\frac{r_3 r_6}{r_3 + r_6}e^{i(\alpha_2+\alpha_3)} + m\frac{r_2 r_5}{r_2 + r_5}e^{i(\alpha_1+\alpha_2+2\alpha_3)}\right),$$

where $e^{is} = (f_{k+1,l,m} - f_{k,l,m})/r_1$. Lemma 1 allows one to compute $f_{k+1,l,m-1}$, $f_{k+1,l+1,m}$, $f_{k+1,l,m+1}$ and $f_{k+1,l-1,m}$ using the form of quadrilaterals (they are shown in Fig. 3.4). Now equation (1.10) at $(k+1,l,m)$ defines $f_{k+2,l,m}$. Condition $|f_{k+2,l,m} - f_{k+1,l,m}| = r_1$ with the labels (3.11) yields equation (3.13).

For $l = 0$ values $f_{k+1,0,m}$, $f_{k+2,0,m}$, $f_{k+1,0,m-1}$ and the equation for the cross-ratio with α_3 give the radius r with the center at $f_{k+2,0,m-1}$. Note that for $l = 0$ the term with r_6 and r_5 drops out of equation (3.13). Using this equation and the permutation $R \to r_1$, $r_1 \to r$, $r_2 \to r_2$, $r_5 \to r_3$, one gets equation (3.12) with $i = 3$. The equation for pattern II is derived similarly.

To derive (3.14) consider the figure on the right part of Fig.3.4 where $k+l+m = 1 \pmod 2$ and r_1, r_2, r_3 and r are the radii of the circles with the centers at $f_{k+1,l,m}$, $f_{k+1,l+1,m-1}$, $f_{k,l+1,m}$ and $f_{k,l,m-1}$, respectively. Elementary geometrical considerations and Lemma 1 applied to the forms of the shown quadrilaterals give equation (3.14).

Remark. Equation (3.14) is derived for $r = r(_{K,L,M})$, $r_1 = r(_{K,L,M-1})$, $r_2 = r(_{K-1,L,M})$, $r_3 = r(_{K,L-1,M+1})$. However it holds true also for $r_1 = r(_{K,L,M+1})$, $r_2 = r(_{K+1,L,M})$, $r_3 = r(_{K,L+1,M+1})$ since it gives the radius of the circle through the three intersection points of the circles with radii r_1, r_2, r_3 intersecting at prescribed angles as shown in the right part of Fig.3.4. Later, we refer to this equation also for this pattern.

Sufficiency: Now let $r(_{K,L,M}) : \tilde{Q}_H \to \mathbb{R}_+$ be some positive solution to (3.12),(3.13),(3.14). We can re-scale it so that $r(0,0,0) = 1$. Starting with $r(1,0,-1)$ and $r(0,1,-1)$ one can compute r everywhere in $\tilde{Q}_H$: r in a "black" vertex (see Fig.3.5) is computed from (3.13). (Note that only r at "circled" vertices is used: so to compute $r_{1,1,-1}$ one needs $r(1,0,-1)$ and $r(0,1,-1)$.) The function r in "white" vertices on the border $\partial\tilde{Q}_H = \{(K,0,-K)|K \in \mathbf{N}\} \cup \{(0,L,-L)|L \in \mathbf{N}\}$ is given by (3.12). Finally, r in "white" vertices in $Q_H^{int} = Q_H \setminus \partial\tilde{Q}_H$ is computed from (3.14). In Fig.3.5 labels show the order of computing r.

Lemma 3 *Any solution $r(_{K,L,M}) : \tilde{Q}_H \to \mathbb{R}$ to (3.12),(3.13),(3.14) with $0 \le c \le 2$, which is positive for inner vertices of $\tilde{Q}_H$ defines some $f_{k,l,m}$ satisfying (1.9) in Q. Moreover, all the triangles $(f_{k,l,m}, f_{k+1,l,m}, f_{k,l,m-1})$, $(f_{k,l,m}, f_{k,l,m-1}, f_{k,l+1,m})$ and $(f_{k,l,m}, f_{k+1,l,m}, f_{k,l+1,m})$ have positive orientation.*

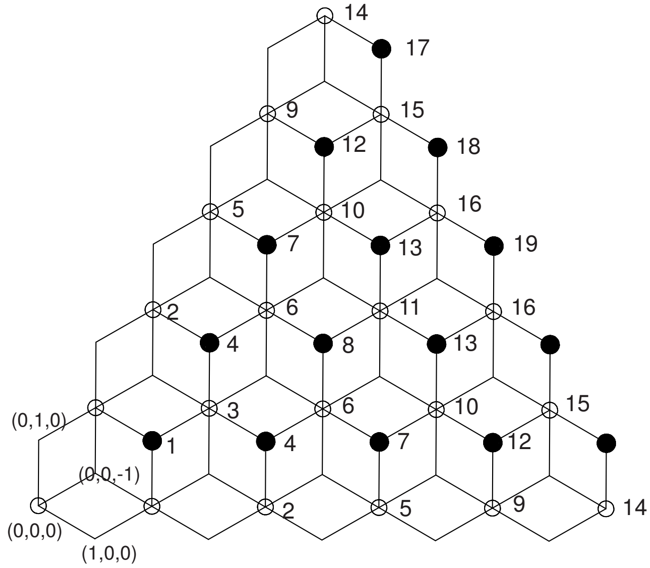

Figure 3.5: Computing r in $\tilde{Q}_H$.

Proof of the lemma: One can place the circles with radii $r(_{K,L,M})$ into the complex plane $\mathbb{C}$ in the way prescribed by the hexagonal combinatorics and the intersection angles. Taking the circle centers and the intersection points of neighboring circles, one recovers $f_{k,l,m}$ for $k+l+m=0,\pm 1$ up to translation and rotation. This procedure is an analog of analytical continuation of holomorphic function. Reversing the arguments used in the derivation of (3.12),(3.13),(3.14), one observes from the forms of the quadrilaterals that equations (1.9) are satisfied. Now using (1.9), one recovers z in the whole Q. Equation (3.14) ensures that the radii r remain positive, which implies the positive orientations of the triangles $(f_{k,l,m}, f_{k+1,l,m}, f_{k,l,m-1})$ and $(f_{k,l,m}, f_{k,l,m-1}, f_{k,l+1,m})$, $(f_{k,l,m}, f_{k+1,l,m}, f_{k,l+1,m})$. The Lemma is proved.

Consider a solution $z : Q \to \mathbb{C}$ of the system (1.9),(1.10) with initial data (1.12), where ϕ and ψ are chosen so that the triangles $(f_{0,0,0)},\ f_{1,0,0},\ f_{0,0,-1})$ and $(f_{0,0,0)},\ f_{0,0,-1},\ f_{0,1,0})$ have positive orientations and satisfy conditions $r(1,0,-1) = |f_{1,0,-1} - f_{1,0,0}|$ and $r(0,1,-1) = |f_{0,1,-1} - f_{0,0,-1}|$. The map $f_{k,l,m}$ defines circle pattern due to Proposition 1 and coincides with the map defined by Lemma 3 due to the uniqueness of the solution. $\square$

Since the cross-ratio equations and the constraint are compatible, the equations for the radii are also compatible. Starting with $r(0,0,0)$, $r(1,0,-1)$ and $r(0,1,-1)$, one can compute $r(_{K,L,M})$ everywhere in $\tilde{Q}$.

Lemma 4 *Let a solution $r(_{K,L,M}) : \tilde{Q} \to \mathbb{R}$ of (3.12),(3.13),(3.14) be positive in the planes given by equations $K+M=0$ and $L+M=0$ then it is positive everywhere in $\tilde{Q}$.*

Proof: As follows from equation (3.14), r is positive for positive r_i, $i=1,2,3$. As r at $(K,K,-K)$, $(K+1,K,-K-1)$ and $(K,K+1,-K-1)$ is positive, r at $(K,K,K-1)$ is also positive. Now starting from r at $(K,K,-K-1)$ and having $r>0$ at $(N,K+1,-K-1)$ and $(N,K,-K)$, one obtains positive r at $(N,K,-K-1)$ for $0 \le N < K$ by the same reason. Similarly, r at $(K,N,-K-1)$ is positive. Thus from positive r at the planes $K+M=0$ and $L+M=0$, we get positive r at the planes $K+M=-1$ and $L+M=-1$. Induction completes the proof. $\square$

Lemma 5 *Let a solution $r_{(K,L,M)} : \tilde{Q} \to \mathbb{R}$ of (3.12),(3.13),(3.14) be positive in the lines parameterized by n as $(n, 0, -n)$ and $(0, n, -n)$. Then it is positive in the border planes of $\tilde{Q}$ specified by $K + M = 0$ and $L + M = 0$.*

Proof: We prove this lemma for $K + M = 0$. For the other border plane the proof is proved similar. Equation (3.13) for $(K, L, -K - 1)$ gives

$$r_2 = r_5 \frac{(2L + c)r_1 + (2K + c)r_4}{(2K + 2 - c)r_1 + (2L + 2 - c)r_4} \tag{3.15}$$

therefore r_2 is positive provided r_1, r_5 and r_4 are positive. For $K = L$ it reads as

$$r_2 = r_5 \frac{(2K + c)}{(2K + 2 - c)}. \tag{3.16}$$

It allows us to compute recursively r at $(K, K, -K)$ starting with r at $(0, 0, 0)$. Obviously, $r > 0$ for $(K, K, -K)$ if $r > 0$ at $(0, 0, 0)$. This property together with the condition $r > 0$ at $(n, 0, -n)$ imply the conclusion of the lemma since equation (3.4) gives r everywhere in the border plane of $\tilde{Q}$ specified by $K + M = 0$. $\square$

Lemmas 4 and 5 imply that the hexagonal circle pattern Z^c is an immersion if $r > 0$ at $(N, 0, -N)$ and $(0, N, -N)$.

Chapter 4

Discrete Riccati equation and hypergeometric functions

In this chapter we show that the initial condition for radius function is uniquely determined for the solution to be positive.

Let r_n and R_n be radii of the circles with the centers at $f_{2n,0}$, $f_{2n+1,1}$ respectively (see Fig.4.1).

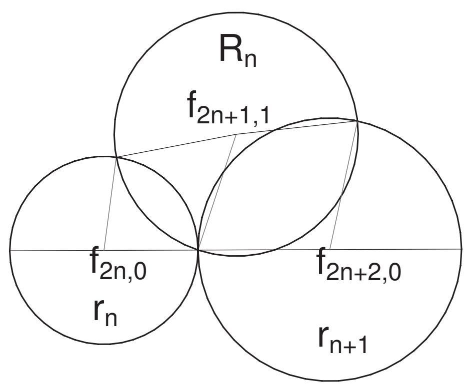

Figure 4.1: Circles on the border.

Constraint (1.2) and property (3.2) gives

$$r_{n+1} = \frac{2n+c}{2(n+1)-c} r_n.$$

From elementary geometric considerations one gets

$$R_{n+1} = \frac{r_{n+1} - R_n \cos\alpha}{R_n - r_{n+1}\cos\alpha} r_{n+1}$$

Define

$$p_n = \frac{R_n}{r_n}, \quad g_n(c) = \frac{2n+c}{2(n+1)-c}$$

and denote $t = \cos\alpha$ for brevity. Now the equation for radii $R,\ r$ takes the form:

$$p_{n+1} = \frac{g_n(c) - tp_n}{p_n - tg_n(c)}. \tag{4.1}$$

Remark. Equation (4.1) is a discrete version of *Riccati* equation. This title is motivated by the following properties:

- cross-ratio of each four-tuple of its solutions is constant since p_{n+1} is Möbius transform of p_n,
- general solution is expressed in terms of solution of some linear equation (see below this linearisation).

Below we call (4.1) d-Riccati equation.

Theorem 14 *Solution of discrete Riccati equation (4.1) with $\alpha \neq \pi/2$ is positive for $n \geq 0$ iff*

$$p_0 = \frac{\sin\frac{c\alpha}{2}}{\sin\frac{(2-c)\alpha}{2}} \tag{4.2}$$

The proof is based on the closed form of the general solution of d-Riccati linearisation. It is linearised by the standard Ansatz

$$p_n = \frac{y_{n+1}}{y_n} + t g_n(c) \tag{4.3}$$

which transforms it into

$$y_{n+2} + t(g_{n+1}(c)+1)y_{n+1} + (t^2-1)g_n(c)y_n = 0 \tag{4.4}$$

One can guess that there is only one initial value p_0 giving positive d-Riccati solution from the following consideration: $g_n(c) \to 1$ as $n \to \infty$, and the general solution of equation (4.4) with limit values of coefficients is $y_n = c_1(-1)^n(1+t)^n + c_2(1-t)^n$. So $p_n = \frac{y_{n+1}}{y_n} + tg_n(c) \to -1$ for $c_1 \neq 0$. However c_1, c_2 defines only asymptotics of a solution. To relate it to initial values one needs some kind of *connection formulas*. Fortunately it is possible to find the general solution to (4.4).

Proposition 4 *The general solution to (4.4) is*

$$y_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1-\frac{c}{2})}\left(c_1\lambda_1^{n+1-c/2}F\left(\frac{3-c}{2}, \frac{c-1}{2}, \frac{1}{2}-n, z_1\right) + \right. \tag{4.5}$$

$$\left. + c_2\lambda_2^{n+1-c/2}F\left(\frac{3-c}{2}, \frac{c-1}{2}, \frac{1}{2}-n, z_2\right)\right)$$

where $\lambda_1 = -t-1$, $\lambda_2 = 1-t$, $z_1 = (t-1)/2$, $z_2 = -(1+t)/2$ and F stands for the hypergeometric function.

Proof: Solutions were found by slightly modified *symbolic method* (see [31] for method description). Substitution

$$y_n = u_x\lambda^x, \quad x = n+1-c/2 \tag{4.6}$$

transforms (4.4) into

$$\lambda^2(x+1)xu_{x+2} + 2t(x+\frac{c+1}{2})xu_{x+1} + (t^2-1)(x+c-1)(x+1)u_x = 0. \tag{4.7}$$

We are looking for solution in the form

$$u_x = \sum_{m=-\infty}^{\infty} a_m v_{x,m} \tag{4.8}$$

where $v_{x,m}$ satisfies

$$(x+m)v_{x,m} = v_{x,m+1}, \quad xv_{x+1,m} = v_{x,m+1}. \tag{4.9}$$

Remark. Note that the label m in (4.8) is running by step 1 but is not necessary integer therefore $v_{x,m}$ is a straightforward generalization of $x^{(m)} = (x+m-1)(x+m-2)...(x+1)x$ playing the role of x^m in the calculus of finite differences. General solution to (4.9) is expressed in terms of Γ-function:

$$v_{x,m} = c\frac{\Gamma(x+m)}{\Gamma(x)} \tag{4.10}$$

Stirling formula [17] for large x

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x}+O\left(\frac{1}{x^2}\right)\right) \tag{4.11}$$

gives the asymptotics for $v_{x,m}$:

$$v_{x,m} \simeq cx^m \quad \text{for} \quad x \to \infty. \tag{4.12}$$

Substituting (4.8) into (4.7), using of (4.9) and collecting similar terms one gets the following equation for the coefficients:

$$(\lambda^2+2t\lambda+t^2-1)a_{m-2} + 2\left(\frac{1+c}{2}-m\right)(t\lambda+t^2-1)a_{m-1}+$$

$$(t^2-1)(1-m)(c-1-m)a_m = 0 \tag{4.13}$$

Choosing $\lambda_1 = -t-1$ or $\lambda_2 = 1-t$ kills the term with a_{m-2}. To make series (4.8) convergent we can use the freedom in m to truncate (4.8) on one side. The choice $m \in \mathbb{Z}$ or $m \in c+\mathbb{Z}$ leads to divergent series. For $m \in \frac{c+1}{2}+\mathbb{Z}$ equation (4.13) gives $a_{\frac{c+1}{2}+k} = 0$ for all non-negative integer k and

$$a_{\frac{c+1}{2}-k-1} = \frac{1-t^2}{t\lambda+t^2-1}\frac{(k-\frac{c-1}{2})(k-1+\frac{c-1}{2})}{2k}a_{\frac{c+1}{2}-k} \tag{4.14}$$

where $\lambda = \lambda_1, \lambda_2$. Substitution of solution of this recurrent relation in terms of the Γ-functions and (4.10) yields

$$y_x = \lambda^x\sum_{k=1}^{\infty}\left(\frac{1-t^2}{2(t\lambda+t^2-1)}\right)^k\frac{\Gamma(k-\frac{c-1}{2})\Gamma(k-1+\frac{c-1}{2})\Gamma(x+\frac{c+1}{2}-k)}{\Gamma(k)\Gamma(x)} \tag{4.15}$$

Lemma 6 *For both $\lambda = -t-1, 1-t$ series (4.15) converges for all x.*

Proof of Lemma 6: Since $z = \frac{1-t^2}{2(t\lambda+t^2-1)} = (t-1)/2, -(1+t)/2$ for λ_1, λ_2 respectively and $t = \cos\alpha < 1$ the convergence of (4.15) depends on the behavior of

$$\frac{\Gamma(k-\frac{c-1}{2})\Gamma(k-1+\frac{c-1}{2})\Gamma(x+\frac{c+1}{2}-k)}{\Gamma(k)}.$$

Stirling formula (4.11) ensures that this expression is bounded by $ck^{\phi(x,c)}$ for some c an $\phi(x,c)$ which gives convergence.

Series (4.15) is expressed in terms of hypergeometric functions:

$$y_x = \lambda^x\frac{\Gamma(x+\frac{c-1}{2})\Gamma(1-\frac{c-1}{2})\Gamma(\frac{c-1}{2})}{\Gamma(x)}F(1-\frac{c-1}{2}, \frac{c-1}{2}, 1-\left(x+\frac{c-1}{2}\right), z)$$

where

$$F(1-\frac{c-1}{2},\frac{c-1}{2},1-\left(x+\frac{c-1}{2}\right),z)=1+z\frac{(1-\frac{c-1}{2})(\frac{c-1}{2})}{(1-(x+\frac{c-1}{2}))}+...+ \tag{4.16}$$

$$z^k\frac{\left[(1-\frac{c-1}{2})(2-\frac{c-1}{2})...(k-\frac{c-1}{2})\right]\left[(\frac{c-1}{2})(1+\frac{c-1}{2})...(k-1\frac{c-1}{2})\right]}{(1-(x\frac{c-1}{2}))...(k-(x+\frac{c-1}{2}))}+...$$

Here the standard designation $F(a,b,c,z)$ for hypergeometric function as a holomorphic at $z=0$ solution for equation

$$z(1-z)F_{zz}+[c-(a+b+1)z]F_z-abF=0 \tag{4.17}$$

is used. □

Now we can complete the proof of Proposition 4. Due to linearity the general solution of (4.4) is given by superposition of any two linear independent solutions. As was shown each summond in (4.5) satisfies the equation (4.4). To finish the proof of Proposition 4 one has to show that the particular solutions with $c_1=0,\ c_2\neq 0$ and $c_1\neq 0,\ c_2=0$ are linearly independent, which follows from the following Lemma.

Lemma 7 *As $n\to\infty$ solution(4.5) has the asymtotics*

$$y_n\simeq (n+1-c/2)^{\frac{c-1}{2}}(c_1\lambda_1^{n+1-c/2}+c_2\lambda_2^{n+1-c/2}) \tag{4.18}$$

Proof: For $n\to\infty$ series representation (4.16) gives $F(\frac{3-c}{2},\frac{c-1}{2},\frac{1}{2}-n,z_1)\simeq 1$. Stirling formula (4.11) defines asymptotics of the factor $\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1-\frac{c}{2})}$ (compare with (4.12)). □

Proof of Theorem 14: For positive p_n it is necessary that $c_1=0$: it follows from asymptotics (4.18) substituted into (4.3). Let us define

$$s(z)=1+z\frac{(1-\frac{c-1}{2})(\frac{c-1}{2})}{\frac{1}{2}}+...+z^k\frac{(k-\frac{c-1}{2})...(1-\frac{c-1}{2})(\frac{c-1}{2})(k-1+\frac{c-1}{2})}{k!(k-\frac{1}{2})...\frac{1}{2}}... \tag{4.19}$$

It is the hypergeometric function $F(\frac{3-c}{2},\frac{c-1}{2},\frac{1}{2}-n,z)$ with $n=0$. A straightforward manipulation with series shows that

$$p_0=1+\frac{2(c-1)}{2-c}z+\frac{4z(z-1)}{2-c}\frac{s'(z)}{s(z)} \tag{4.20}$$

where $z=\frac{1+t}{2}$. Note that p_0 as a function of z satisies some ordinary differential equation of first order since $\frac{s'(z)}{s(z)}$ satisfies Riccati equation obtained by reduction of (4.17). Computation shows that $\frac{\sin\frac{c\alpha}{2}}{\sin\frac{(2-c)\alpha}{2}}$ satisfies the same ODE. Since both expression (4.20) and (4.2) are equal to 1 for $z=0$ they coincide everywhere. □

Proposition 5 *If there exists immersed $f_{n,m}$ satisfying (1.7),(1.2),(3.1) it is defined by initial data (1.8).*

Proof: For $\alpha\neq\pi/2$ the claim follows from Theorem 14. For the case $\alpha=\pi/2$, any solution for (4.1) with $p_0>0$ is positive. Nevertheless x_0 is in this case also unique and is specified by (4.2). For orthogonal square grid circle patterns (i.e. $\alpha=\pi/2$) the analysis is as follows. One has

$$R(\pm(N+1)+i(N+1))=\frac{2N+c}{2(N+1)-c}R(\pm N+iN), \tag{4.21}$$

$$R(N-1+iN)R(N+i(N+1))=R^2(N+iN). \tag{4.22}$$

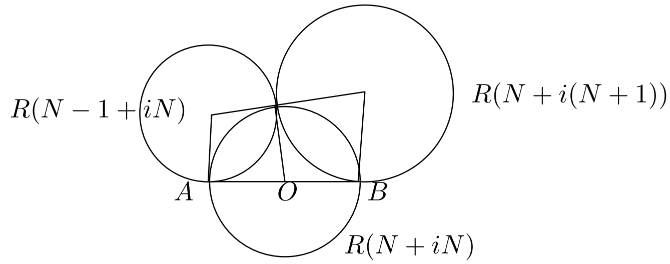

Figure 4.2: Points $f_{n,0}$ are collinear.

These equations again make it possible to find $R(N+iN)$ and $R(N+i(N+1))$ in a closed form. From the initial condition (1.8) we have

$$R(0) = 1, \quad R(i) = \tan\frac{c\pi}{4}. \tag{4.23}$$

Equation (4.21) implies

$$R(\pm N + iN) = \frac{c(2+c)...(2(N-1)+c)}{(2-c)(4-c)...(2N-c)}. \tag{4.24}$$

We now show that substituting the asymptotics of $R(z)$ for immersed $f_{n,m}$, one necessarily gets $R(i) = \tan\frac{c\pi}{4}$.
Indeed, formula (4.24) yields the following representation in terms of the Γ- function:

$$R(N+iN) = H(c)\frac{\Gamma(N+c/2)}{\Gamma(N+1-c/2)},$$

where

$$H(c) = \frac{c\Gamma(1-c/2)}{2\Gamma(1+c/2)}. \tag{4.25}$$

From the Stirling formula (4.11) one obtains

$$R(N+iN) = H(c)N^{c-1}\left(1 + O\left(\frac{1}{N}\right)\right). \tag{4.26}$$

Now let $R(i) = a\tan\frac{c\pi}{4}$ where a is a positive constant. Equation (4.22) (again equivalent to the fact that the centers of all the circles $C(N+iN)$ lie on a straight line) yields

$$R(N+i(N+1)) = \left(a\tan\frac{c\pi}{4}\right)^{(-1)^N}\left(\frac{(2(N-1)+c)(2(N-3)+c)(2(N-5)+c)...}{(2N-c)(2(N-2)-c)(2(N-4)-c)...}\right)^2.$$

Using the product representation for $\tan x$,

$$\tan x = \frac{\sin x}{\cos x} = \frac{x\left(1-\frac{x^2}{\pi^2}\right)...\left(1-\frac{x^2}{(k\pi)^2}\right)...}{\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{(3\pi)^2}\right)...\left(1-\frac{4x^2}{((2k-1)\pi)^2}\right)...}$$

one arrives at

$$R(N+i(N+1)) = a^{(-1)^N}H(c)N^{(c-1)}\left(1+O\left(\frac{1}{N}\right)\right). \tag{4.27}$$

Solving equation (3.5) with respect to $R^2(z)$ we get

$$R^2(z) = G(N, M, R(z+i), R(z+1), R(z-i)) :=$$

$$\frac{R(z+i)R(z+1)R(z-i)+R^2(z+1)\left(\frac{M+N}{2M}R(z-i)+\frac{M-N}{2M}R(z+i)\right)}{R(z+1)+\frac{M+N}{2M}R(z+i)+\frac{M-N}{2M}R(z-i)}. \tag{4.28}$$

For $z \in \mathbf{V}$, $R(z+i) \geq 0$, $R(z+1) \geq 0$, $R(z-i) \geq 0$, the function G is monotonic:

$$\frac{\partial G}{\partial R(z+i)} \geq 0,\ \frac{\partial G}{\partial R(z+1)} \geq 0,\ \frac{\partial G}{\partial R(z-i)} \geq 0.$$

Thus, any positive solution $R(z)$ with $z \in \mathbb{V}$ must satisfy

$$R^2(z) \geq G(N, M, 0, R(z+1), R(z-i)).$$

Substituting asymptotics (4.26) and (4.27) of R into this inequality and taking the limit $K \to \infty$, for $N = 2K$, we get $a^2 \geq 1$. Similarly, for $N = 2K+1$ one obtains $\frac{1}{a^2} \geq 1$, which implies $a = 1$. □

Proposition 6 *If there is an immersed hexagonal Z^c it satisfies initial data 1.13*

Proof: Let r_n and R_n be the radii of the circles of the circle pattern defined by $z_{k,l,m}$ with the centers at $z_{2n,0,0}$ and $z_{2n+1,0,-1}$ respectively. Again one has

$$r_{n+1} = \frac{2n+c}{2(n+1)-c} r_n$$

and

$$R_{n+1} = \frac{r_{n+1} - R_n \cos\alpha}{R_n - r_{n+1}\cos\alpha} r_{n+1}$$

Now the claim follows from Theorem 14 and Proposition 5. □

Chapter 5

Discrete Painlevé equations

Let R_z be a solution of (3.4) and (3.5) with initial condition (4.23). For $z \in \mathbb{V}_{int}$ define $P_{N,M} = P_z = \frac{R_{z+1}}{R_{z-i}}$, $Q_{N,M} = Q_z = \frac{R_z}{R_{z-i}}$. Then (3.5) and (3.4) are rewritten as follows

$$Q_{N,M+1} = \frac{(N-M)Q_{N,M}(1+P_{N,M})(Q_{N,M}-P_{N,M}\cos\alpha)-(M+N)P_{N,M}S_{N,M}}{Q_{N,M}[(M+N)S_{N,M}-(M-N)(1+P_{N,M})(P_{N,M}-Q_{N,M}\cos\alpha)]}, \tag{5.1}$$

$$P_{N,M+1} = \frac{(2M+c)P_{N,M}+(2N+c)Q_{N,M}Q_{N,M+1}}{(2(N+1)-c)P_{N,M}+(2(M+1)-c)Q_{N,M}Q_{N,M+1}}, \tag{5.2}$$

where

$$S_{N,M} = Q^2_{N,M} - P_{N,M} + Q_{N,M}(1-P_{N,M})\cos\alpha.$$

Property (3.8) for (5.1),(5.2) reads as

$$(c-1)S_{N,M} \geq 0,\ Q_{N,M} > 0,\ P_{N,M} > 0. \tag{5.3}$$

Equations (5.1),(5.2) can be considered as a dynamical system for variable M.

Theorem 15 *There exists such $a > 0$ that (3.8) holds for the solution R_z of (3.4),(3.5) with initial conditions*

$$R_0 = 1, \qquad R_i = a. \tag{5.4}$$

Proof: Due to the following Lemma it is sufficient to prove (3.8) only for $0 < c < 1$.

Lemma 8 *If R_z is a solution of (3.4),(3.5) for c then $1/R_z$ is a solution of (3.4),(3.5) for $\tilde{c} = 2 - c$.*

Lemma is proved by straightforward computation.

Let $0 < c < 1$ and $(P_{N,M}, Q_{N,M})$ correspond to the solution of (3.4),(3.5) with initial conditions (5.4). Define real function $F(P)$ on $\mathbb{R}_+$ implicitly by $F^2 - P + F(1-P)\cos\alpha = 0$ for $0 \leq P \leq 1$ and by $F(P) \equiv 1$ for $1 \leq P$.

Designate

$$D_u := \{(P,Q) : P > 0, Q > F(P)\}, \quad D_d := \{(P,Q) : Q < 0\},$$

$$D_0 := \{(P,Q) : P > 0, 0 \leq Q \leq F(P)\}, \quad D_f := \{(P,Q) : P \leq 0, Q \geq 0\}$$

as in Fig. 5.1. Now define the infinite sequences $\{q_n\}, \{p_n\},\ n \in \mathbb{N}$ as follows:

$$\{q_n(a)\} := \{Q_{0,1}, Q_{0,2}, Q_{1,2}, Q_{0,3}, Q_{1,3}, Q_{2,3}, ..., Q_{0,M}, Q_{1,M}, ..., Q_{M-1,M}, ...\},$$

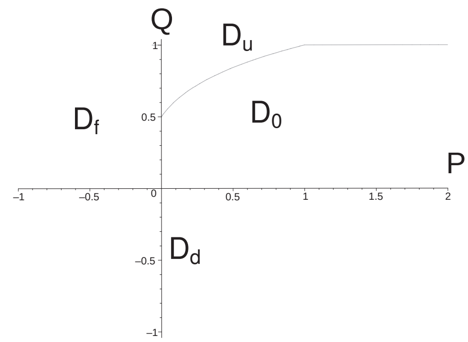

Figure 5.1: The case $\cos\alpha = -1/2$.

$$\{p_n(a)\} := \{P_{0,1}, P_{0,2}, P_{1,2}, P_{0,3}, P_{1,3}, P_{2,3}, ..., P_{0,M}, P_{1,M}, ..., P_{M-1,M}, ...\}.$$

and the sets

$$A_u(n) := \{a \in \mathbb{R}_+ : (p_n(a), q_n(a)) \in D_u, (p_k(a), q_k(a)) \in D_0 \ \forall \ 0 < k < n\},$$

$$A_d(n) := \{a \in \mathbb{R}_+ : (p_n(a), q_n(a)) \in D_d, (p_k(a), q_k(a)) \in D_0 \ \forall \ 0 < k < n\}.$$

$A_u(n)$ and $A_d(n)$ are open sets since the denominators of (5.1),(5.2) do not vanish in D_0. Moreover, direct computation shows that $A_u(1) \neq \emptyset$ and $A_d(2) \neq \emptyset$, therefore the sets

$$A_u := \cup A_u(k), \quad A_d := \cup A_d(k)$$

are not empty. Finally, define

$$A_0 := \{a \in \mathbb{R}_+ : (p_n(a), q_n(a)) \in D_0, \ \forall \ n \in \mathbb{N}\}.$$

Note that A_0, A_u, A_d are mutually disjoint and the sequences $\{p_n\}, \{q_n\}$ is so constructed that

$$\mathbb{R}_+ = A_0 \cup A_u \cup A_d. \tag{5.5}$$

Indeed $(P_{N,M}, Q_{N,M})$ can not jump from D_0 into D_f in one step $M \to M+1$ since $P_{N,M+1}$ is positive for positive $P_{N,M}, Q_{N,M}, Q_{N,M+1}$. Relation (5.5) would be impossible for $A_0 = \emptyset$, since the connected set $\mathbb{R}_+$ can not be covered by two open disjoint nonempty subsets A_u and A_d, therefore $A_0 \neq \emptyset$. $\square$

Now we are ready to prove the following geometrical property of square grid circle patterns Z^c.

Theorem 16 *The square grid discrete map Z^c with $0 < c < 2$ is embedded.*

Proof: Theorems 15 and 12 ensure that there is imbedded Z^c for each $a \in A_0$. Proposition 5 implies that the set A_0 consists of only one element, namely, $A_0 = \{\sin\frac{\gamma\alpha}{2} / \sin\frac{(2-\gamma)\alpha}{2}\}$. $\square$

Proposition 7 *For $N = 0$ system (5.1),(5.2) for $Q_{N,M}, P_{N,M}$ reduces to the special case of discrete Painlevé equation:*

$$(n+1)(x_n^2 - 1)\left(\frac{x_{n+1} + x_n/\varepsilon}{\varepsilon + x_n x_{n+1}}\right) - n(1 - x_n^2/\varepsilon^2)\left(\frac{x_{n-1} + \varepsilon x_n}{\varepsilon + x_{n-1}x_n}\right) = c x_n \frac{\varepsilon^2 - 1}{2\varepsilon^2}, \tag{5.6}$$

where $\varepsilon = e^{i\alpha}$. Namely, the map $f : \mathbb{Z}^2_+ \to \mathbb{C}$ satisfying (1.7) and (1.2) with initial data $f_{0,0} = 0$, $f_{0,1} = 1$, $f_{0,1} = e^{i\beta}$ is an immersion if and only if the solution x_n of the equation (5.6) with $x_0 = e^{i\beta/2}$, is of the form $x_n = e^{i\beta_n}$, where $\beta_n \in (0, \alpha)$.

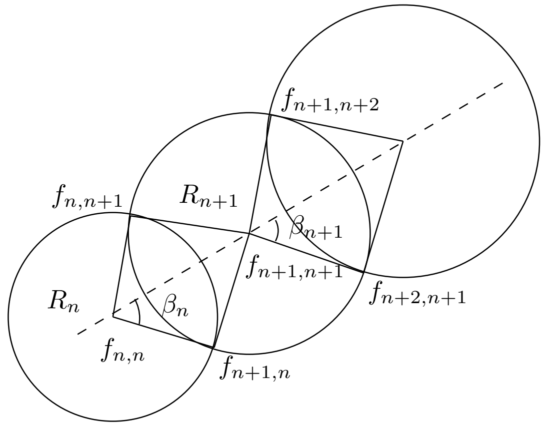

Figure 5.2: Diagonal circles

Proof: To simplify computations let us prove the claim for $\alpha = \pi/2$. Define $R_n := R(in) > 0$, and define $\beta_n \in (0, \pi/2)$ through $f_{n,n+1} - f_{n,n} = e^{2i\beta_n}(f_{n+1,n} - f_{n,n})$. By symmetry, all the points $f_{n,n}$ lie on the diagonal $\arg f_{n,n} = \beta/2$.

Taking into account that all elementary quadrilaterals are of the kite form, one obtains

$$f_{n+2,n+1} = e^{i\beta/2}(g_{n+1} + R_{n+1}e^{-i\beta_{n+1}}), \quad f_{n+1,n+2} = e^{i\beta/2}(g_{n+1} + R_{n+1}e^{i\beta_{n+1}}),$$

$$f_{n+1,n} = e^{i\beta/2}(g_{n+1} - iR_{n+1}e^{-i\beta_n}), \quad f_{n,n+1} = e^{i\beta/2}(g_{n+1} + iR_{n+1}e^{i\beta_n}),$$

and

$$R_{n+1} = R_n \tan \beta_n, \tag{5.7}$$

where $g_{n+1} = |f_{n+1,n+1}|$ (see Fig. 5.2). Now the constraint (1.2) for $(n+1, n+1)$ is equivalent to

$$cg_{n+1} = 2(n+1)R_{n+1}\left(\frac{e^{i\beta_n} + ie^{i\beta_{n+1}}}{i + e^{i(\beta_n+\beta_{n+1})}}\right).$$

Similarly,

$$cg_n = 2nR_n\left(\frac{e^{i\beta_{n-1}} + ie^{i\beta_n}}{i + e^{i(\beta_{n-1}+\beta_n)}}\right).$$

Putting these expressions into the equality

$$g_{n+1} = g_n + e^{-i\beta_n}(R_n + iR_{n+1})$$

and using (5.7) one obtains (5.6) with $x_n = e^{i\beta_n}$. This proves the necessity part.

Now let us suppose that there is a solution $x_n = e^{i\beta_n}$ of (5.6) with $\beta_n \in (0, \pi/2)$. This solution determines a sequence of orthogonal circles along the diagonal $e^{i\beta/2}\mathbb{R}_+$, and thus the points $f_{n,n},\ f_{n\pm1,n},\ f_{n,n\pm1}$, for $n \geq 1$. Now equation (1.1) determines $f_{n,m}$ in $\mathbb{Z}^2_+$. Since $\beta_n \in (0, \pi/2)$, the interiors of the quadrilaterals $(f_{n,n}, f_{n+1,n}, f_{n+1,n+1}, f_{n,n+1})$ on the diagonal, and of the quadrilaterals $(f_{n,n-1}, f_{n+1,n-1}, f_{n+1,n}, f_{n,n})$ are disjoint. That means that we have positive solution $R(z)$ of (3.4),(3.5) for $z = iM,\ z = 1 + iM,\ N \in \mathbb{N}$. Given $R(iM) > 0$, equation (3.4) determines $R(z)$ for all $z \in \mathbb{V}$. Moreover, $R(z)$ is positive, as follows from equation (3.4) by induction. (Compare with Lemma 5.) $\square$

Remark. Equation (5.6) is a special case of *discrete Painlevé equation* that has appeared in the literature in a completely different context. Namely, it is related to the following discrete

Painlevé equation

$$\frac{2\zeta_{n+1}}{1-X_{n+1}X_n}+\frac{2\zeta_n}{1-X_nX_{n-1}}=\mu+\nu+\zeta_{n+1}+\zeta_n+$$

$$\frac{(\mu-\nu)(r^2-1)X_n+r(1-X_n^2)[\frac{1}{2}(\zeta_n+\zeta_{n+1})+(-1)^n(\zeta_n-\zeta_{n+1}-2m)]}{(r+X_n)(1+rX_n)},$$

which was considered in [79], and is called the generalized d-PII equation. The corresponding transformation for $\alpha=\pi/2$ is

$$X=\frac{(1+i)(x-i)}{\sqrt{2}(x+1)}$$

with $\zeta_n=n$, $r=-\sqrt{2}$, $\mu=0$, $(\zeta_n-\zeta_{n+1}-2m)=0$, $\gamma=(2\nu-\zeta_n+\zeta_{n+1})$. For a more general reduction of cross-ratio equation see [77].

Now consider hexagonal Z^c. We prove the existence of an initial value $f_{0,0,-1}$ such that $r(n,0,-n)>0$, $\forall n\in\mathbb{N}$. (We have already shown that this value, if there is any, is unique and is $f_{0,0,-1}=e^{c\alpha_3}$.)

Proposition 8 *Suppose equation (5.6), where $\varepsilon=e^{i\alpha_3}$, has a unitary solution $x_n=e^{i\beta_n}$ in the sector $0<\beta_n<\alpha_3$. Then $r(n,0,-n)$, $n\geq 0$ is positive.*

Proof: The proof follows from Proposition 7, as hexagonal Z^c defines also square grid Z^c for constant intersection angles α_3. For $f_{1,0,0}=1$ and unitary $f_{1,0,-1}$, the equations for the cross-ratio with α_3 and (1.10) again reduce to (5.6) with unitary $x_n^2=(f_{n,0,-n-1}-f_{n,0,n})/(f_{n+1,0,-n}-f_{n,0,n})$. Note that for $n=0$ the term with x_{-1} drops out of (5.6); therefore the solution for $n>0$ is determined by x_0 only. The condition $0<\beta_n<\alpha_3$ means that all triangles $(z_{n,0,-n},z_{n+1,0,n},z_{n,0,-n-1})$ have positive orientation. Hence $r(n,0,-n)$ are all positive. □

Below we omit the index of α so that $\varepsilon=e^{i\alpha}$.

Theorem 17 *A unitary solution $x_n=e^{i\beta_n}$ to (5.6) exists in the sector $0<\beta_n<\alpha$.*

Proof: Equation (5.6) allows us to represent x_{n+1} as a function of n,x_{n-1} and x_n in a recurrent form: $x_{n+1}=\Phi(n,x_{n-1},x_n)$. Say, for orthogonal Z^c

$$x_{n+1}=\Phi(n,x_{n-1},x_n):=$$

$$-x_{n-1}\frac{nx_n^{-2}+i(\gamma-1)x_{n-1}^{-1}x_n^{-1}+(\gamma-1)+i(2n+1)x_{n-1}^{-1}x_n+(n+1)x_n^2}{nx_n^2-i(\gamma-1)x_{n-1}x_n+(\gamma-1)-i(2n+1)x_{n-1}x_n^{-1}+(n+1)x_n^{-2}}. \tag{5.8}$$

Obviously, this equation possesses unitary solutions.

$\Phi(n,u,v)$ maps the torus $T^2=S^1\times S^1=\{(u,v)\in\mathbb{C}:\ |u|=|v|=1\}$ into S^1 and has the following properties:

- For all $n\in\mathbb{N}$ it is a continuous map on $A_I\times\bar{A}_I$ where $A_I=\{e^{i\beta}:\beta\in(0,\alpha)\}$ and $\bar{A}_I$ is the closure of A_I . Values of Φ on the border of $A_I\times\bar{A}_I$ are defined by continuity: $\Phi(n,u,\varepsilon)=-1$, $\Phi(n,u,1)=-\varepsilon$.
- For $(u,v)\in A_I\times A_I$ one has $\Phi(n,u,v)\in A_I\cup A_{II}\cup A_{IV}$, where $A_{II}=\{e^{i\beta}:\beta\in(\alpha,\pi]\}$ and $A_{IV}=\{e^{i\beta}:\beta\in[\alpha-\pi,0)\}$. That means that x cannot jump in one step from A_I into $A_{III}=\{e^{i\beta}:\beta\in(-\pi,\alpha-\pi)\}$.

Let $x_0=e^{i\beta_0}$. Note that although (5.6) is a difference equation of the second order its solution x_n for $n\geq 0$ is determined by its value x_0. Then $x_n=x_n(\beta_0)$. Define $S_n=\{\beta_0:\ x_k(\beta_0)\in\bar{A}_I\ \forall\ 0\leq k\leq n\}$. Then S_n is a closed set since Φ is continuous on $A_I\times\bar{A}_I$. As a closed subset of a segment it is a collection of disjoint segments S_n^l.

Lemma 9 *There exists sequence* $\{S_n^{l(n)}\}$ *such that:*

- $S_n^{l(n)}$ *is mapped by* $x_n(\beta_0)$ *onto* $\bar{A}_I$,
- $S_{n+1}^{l(n+1)} \subset S_n^{l(n)}$.

The lemma is proved by induction. For $n = 0$ it is trivial. Suppose it holds for n. As $S_n^{l(n)}$ is mapped by $x_n(\beta_0)$ onto $\bar{A}_I$, continuity considerations and

$$\Phi(n,u,\varepsilon) = -1, \quad \Phi(n,u,1) = -\varepsilon \tag{5.9}$$

imply: $x_{n+1}(\beta_0)$ maps $S_n^{l(n)}$ onto $A_I \cup A_{II} \cup A_{IV}$ and at least one of the segments $S_{n+1}^l \subset S_n^{l(n)}$ is mapped into $\bar{A}_I$. This proves the lemma. □

Since the segments of $\{S_n^{l(n)}\}$ constructed in lemma 9 are nonempty, there exists $\bar{\beta}_0 \in S_n$ for all $n \geq 0$. For this $\bar{\beta}_0$, the value $x_n(\bar{\beta}_0)$ is not on the border of $\bar{A}_0$ since then $x_{n+1}(\beta_0)$ would jump out of $\bar{A}_I$. □

Finally we can prove the following geometrical property of hexagonal Z^c.

Theorem 18 *The hexagonal* Z^c *with constant intersection angles and* $0 < c < 2$ *is an immersion.*

Proof: Theorem 17 and Proposition 8 imply that there exits an immersed hexagonal Z^c. Now Proposition 6 gives the initial data for that immersion. □

Remark. Condition (3.8) does not imply that β_n is monotonous. For orthogonal circle pattern the angle β_m is related to radii by $\tan\beta_m = R(i(m+1))/R(im)$, in terms of radii the condition (3.8) is reformulated as

$$(c-1)(R(i(m+1)) - R(i(m-1))) \geq 0.$$

Chapter 6

Circle patterns Z^2 and Log

• Square grid circle patterns Z^2 and Log

The definitions of discrete z^2 and $\log(z)$ for orthogonal square grid circle patterns were given in the Introduction. In this section we define them also for hexagonal and square grid circle patterns with prescribed intersection angles. Let us again consider square grid Z^c, with $0 < c < 2$, and make the following renormalization for the corresponding radii: $R \to \frac{2-c}{c}R$. Then as c approaches 2 from below, i.e. $c \to 2-0$, from (3.7) with $\beta = c\alpha$ we have

$$R(0) = \frac{2-c}{c} \to +0, \quad R(1+i) = 1, \quad R(i) = \frac{2-c}{c}\frac{\sin(c\alpha/2)}{\sin(\alpha - c\alpha/2)} \to \frac{\sin\alpha}{\alpha}.$$

Definition 14 *$Z^2 : \mathbb{Z}^2_+ \to \mathbb{R}^2 = \mathbb{C}$ is the solution of (1.7), (1.2) with $c = 2$ and the initial conditions*

$$Z^2(0,0) = Z^2(1,0) = Z^2(0,1) = 0, \quad Z^2(2,0) = 1,$$
$$Z^2(0,2) = e^{2i\alpha}, \quad Z^2(1,1) = \frac{\sin\alpha}{\alpha}e^{i\alpha}.$$

In this definition equations (1.7) and (1.2) are again understood to be regularized through multiplication by their denominators. To define discrete $\log(z)$ we use again the obvious symmetry $R \to \frac{1}{R}$ of equations for radius function R.

Proposition 9 *Let $R(z)$ be a solution of the system (3.4),(3.5) for some c. Then $\tilde{R}(z) = \frac{1}{R(z)}$ is a solution of (3.4),(3.5) with $\tilde{c} = 2 - c$.*

This proposition reflects the fact that for any discrete conformal map f there is *dual discrete conformal map* f^* defined by (see [28])

$$f^*_{n+1,m} - f^*_{n,m} = -\frac{1}{f_{n+1,m} - f_{n,m}}, \quad f^*_{n,m+1} - f^*_{n,m} = \frac{1}{f_{n,m+1} - f_{n,m}}.$$

Obviously this transformation preserves the kite form of elementary quadrilaterals and therefore is well-defined for Schramm's circle patterns. The smooth limit of the duality is

$$(f^*)' = -\frac{1}{f'}.$$

The dual of $f(z) = z^2$ is, up to a constant, $f^*(z) = \log(z)$. Motivated by this observation, we define the discrete logarithm as the discrete map dual to Z^2, i.e. the map corresponding to the circle pattern with radii

$$R_{\mathrm{Log}}(z) = 1/R_{Z^2}(z),$$

where R_{Z^2} are the radii of the circles for Z^2. Here one has $R_{\mathrm{Log}}(0) = \infty$, i.e. the corresponding circle is a straight line. The corresponding constraint (1.2) becomes (1.14). As shown in the Introduction it can be also derived as a limit.

Definition 15 Log : $\mathbb{Z}_+^2 \to \mathbb{R}^2 = \overline{\mathbb{C}}$ *is the map satisfying (1.7) and (1.14) with the initial conditions*

$$\mathrm{Log}(0,0) = \infty,\ \mathrm{Log}(1,0) = 0,\ \mathrm{Log}(0,1) = 2\mathrm{i}\alpha,$$
$$\mathrm{Log}(2,0) = 1,\ \mathrm{Log}(0,2) = 1 + 2\mathrm{i}\alpha,\ \mathrm{Log}(1,1) = \frac{\alpha}{\sin\alpha}\mathrm{e}^{\mathrm{i}\alpha}.$$

Proposition 10 *Square grid discrete maps Z^2 and* Log *are immersions.*

Proof: To simplify the calculations assume $\alpha = \pi/2$. Consider the discrete conformal map $\frac{2-c}{c}Z^c$ with $0 < c < 2$. The corresponding solution x_n of (5.6) is a continuous function of c. So there is a limit as $c \to 2-0$, of this solution with $x_n \in A_I$, $x_0 = i$, and $x_1 = \frac{-1+i\pi/2}{1+i\pi/2} \in A_I$. The solution x_n of (5.6) with the property $x_n \in A_I$ satisfies $x_n \neq 1$, $x_n \neq i$ for $n > 0$ (see (5.9)). Now, reasoning as in the proof of Proposition 7, we get that Z^2 is an immersion. The only difference is that $R(0) = 0$. The circle $C(0)$ lies on the border of $\mathbb{V}$, so Schramm's result (see [87]) claiming that corresponding circle pattern is immersed is true. Log corresponds to the dual circle pattern, with $R_{\mathrm{Log}}(z) = 1/R_{Z^2}(z)$, which implies that Log is also an immersion. □

Theorem 19 *Discrete conformal maps Z^2 and* Log *are embedded.*

Proof: The circle radii for Z^2 and Log are subject to equations (3.4),(3.5) with $c = 2$ and $c = 0$ respectively. For these values of c Theorem 12 is true: the proof is the same since Z^2 and Log are immersed. Due to Lemma 8 it is suffices to prove the property (3.8) only for Z^2. Consider the discrete conformal map $\frac{2-c}{c}Z^c$ with $0 < c < 2$. The corresponding solution $R(z)$ of (3.4),(3.5) is a continuous function of c. So there is a limit as $c \to 2-0$, of this solution with the property (3.8), which is violated only for $z = 0$ since $R(0) = 0$. □

• **Hexagonal circle patterns Z^2 and** Log

For hexagonal case we can also define discrete z^2 and $\log(z)$ as discrete conformal maps satisfying the corresponding constraint and initial conditions. To avoid the renormalization of initial conditions we rather define discrete z^2 and $\log(z)$ by their radius functions. Formula (3.16) with $c = 2$ gives also infinite $r(1,1,-1)$. The way around this difficulty is again renormalization $z \to (2-c)z/c$ and a limit procedure $c \to 2-0$, which leads to the re-normalization of initial data (see [25]). As follows from (4.2), this renormalization implies:

$$r(0,0,0) = 0,\ r(1,1,-1) = 1,$$
$$r(1,0,-1) = \frac{\sin\alpha_3}{\alpha_3},\ r(0,1,-1) = \frac{\sin\alpha_2}{\alpha_2}. \tag{6.1}$$

Proposition 11 *The solution to (3.12),(3.13),(3.14) with* $c = 2$ *and initial data (6.1) is positive.*

Proof: This follows from Lemmas 4 and 5 since Theorem 17 is true also for the case $c = 2$. Indeed, solution x_n is a continuous function of c. Therefore it has a limit value as $c \to 2-0$ and it lies in the sector A_I. □

Lemma 3 implies that there exists a hexagonal circle pattern with radius function r.

Definition 16 *The hexagonal circle pattern Z^2 has a radius function specified by Proposition 11.*

Equations (3.12),(3.13),(3.14) have the symmetry

$$r \to \frac{1}{r}, \; c \to 2 - c, \tag{6.2}$$

which is again the *duality transformation* (see [28]).

Definition 17 *[25] The hexagonal circle pattern* Log *is a circle pattern dual to* Z^2.

Discrete Z^2 and Log are shown in Fig. 1.9.

Theorem 20 *The hexagonal circle patterns* Z^2 *and* Log *are immersions.*

Proof: For Z^2 this follows from Proposition 11. Hence the values of $1/r$, where r is radius function for Z^2, are positive except for $r(0,0,0) = \infty$. Lemma 3 completes the proof. $\square$

Chapter 7

Asymptotics of Z^c and Log

Here we restrict our discussions to orthogonal square grid Z^c. Equation (4.26) gives the asymptotical behavior of the radius function on the border of $\mathbb{V}$. Due to constraint (1.2) this asymptotic gives also asymptotical behavior of $f_{n,0}$ and $f_{0,m}$. Taking further terms from the Stirling formula (4.11), one gets the following asymptotics for Z^c

$$Z^{\gamma}_{n,k} = \frac{2H(c)}{c}\left(\frac{n+ik}{2}\right)^c\left(1+O\left(\frac{1}{n^2}\right)\right), \quad n\to\infty, \quad k=0,1, \tag{7.1}$$

having a proper smooth limit. Here the constant $H(c)$ is given by (4.25).

Conjecture 1 *The discrete conformal map Z^c has the following asymptotic behavior*

$$Z^c_{n,m} = \frac{2H(c)}{c}\left(\frac{n+im}{2}\right)^c\left(1+o\left(\frac{1}{\sqrt{n^2+m^2}}\right)\right).$$

Due to representation (2.4) the discrete conformal map Z^c can be studied by the isomonodromic deformation method. One can probably prove the above Conjecture by applying a technique similar to the one used in [55] for an equation, which is continuous in the first and discrete in the second variable.

Another result on behavior of discrete Z^c at infinity follows from its geometrical properties:

$$\lim_{n\to\infty} Z^{\gamma}_{n,m} = \infty, \quad \lim_{m\to\infty} Z^{\gamma}_{n,m} = \infty.$$

In fact, since the terminal points of the curves Γ_n (see Theorem 12 for the definition of the curve) lie on the sector border the proof easily follows from convexity of the curves Γ_n, inequality (3.9) and asymptotics of (4.26).

We will prove the asymptotics of $R(z)$ (and therefore for Z^c) for $\mathrm{Im}(z)\to\infty$. Let us formulate the following statement.

Proposition 12 *The radius function $R(z)$ of orthogonal square grid Z^c satisfy the following equations*

$$R(z)^2 = \frac{\left(\frac{1}{R(z+1)}+\frac{1}{R(z+i)}+\frac{1}{R(z-1)}+\frac{1}{R(z-i)}\right)R(z+1)R(z+i)R(z-1)R(z-i)}{R(z+1)+R(z+i)+R(z-1)+R(z-i)} \tag{7.2}$$

$$
\begin{aligned}
&(N+M)(R(z)^2-R(z+1)R(z-i))(R(z+i)+R(z+1))\\
&+(M-N)(R(z)^2-R(z+i)R(z+1))(R(z+1)+R(z-i))=0,
\end{aligned}
\tag{7.3}
$$

$$
\begin{aligned}
&(N+M)(R(z)^2-R(z+i)R(z-1))(R(z-1)+R(z-i))\\
&+(M-N)(R(z)^2-R(z-1)R(z-i))(R(z+i)+R(z-1))=0,
\end{aligned}
\tag{7.4}
$$

$$
\begin{aligned}
&(N+M)(R(z)^2-R(z+i)R(z-1))(R(z+1)+R(z+i))\\
&+(N-M)(R(z)^2-R(z+1)R(z+i))(R(z+i)+R(z-1))=0.
\end{aligned}
\tag{7.5}
$$

Proof: Equations (7.3) is just (3.5) for $\alpha=\pi/2$. Further one derives equations (7.2), (7.4), (7.5) from (3.5) and (3.4) by direct computation. □

To treat Z^c and Log on equal footing we agree that the case $c=0$ in equations corresponds to discrete $\log(z)$. For the edges of unit squares with the vertices in $\mathbb{V}$ defined by (1.5) we introduce X and Y via the radius ratios:

$$
\frac{1+X_{N,M}}{1-X_{N,M}}=\frac{R_{N+1,M}}{R_{N,M}},\qquad \frac{1+Y_{N,M}}{1-Y_{N,M}}=\frac{R_{N,M}}{R_{N,M-1}}.
\tag{7.6}
$$

In these variables equations (3.5) and (3.4) read as:

$$
(M-N)\frac{X_{N,M}+Y_{N,M+1}}{1-X_{N,M}Y_{N,M+1}}+(M+N)\frac{X_{N,M}-Y_{N,M}}{1+X_{N,M}Y_{N,M}}=0,
\tag{7.7}
$$

$$
(M-N)\frac{Y_{N,M+1}-X_{N,M}}{1-X_{N,M}Y_{N,M+1}}+(M+N+1)\frac{X_{N,M+1}+Y_{N,M+1}}{1+X_{N,M+1}Y_{N,M+1}}=c-1.
\tag{7.8}
$$

Moreover, X and Y satisfy

$$
\frac{X_{N,M}+Y_{N,M+1}}{1-X_{N,M}Y_{N,M+1}}=\frac{X_{N-1,M}+Y_{N,M}}{1-X_{N-1,M}Y_{N,M}},
\tag{7.9}
$$

$$
\frac{X_{N,M}+Y_{N+1,M+1}}{1+X_{N,M}Y_{N+1,M+1}}=\frac{X_{N,M+1}+Y_{N,M+1}}{1+X_{N,M+1}Y_{N,M+1}},
\tag{7.10}
$$

where (7.9) is equivalent to (7.2) and (7.10) is the compatibility condition.

Conditions (3.8), which hold for discrete z^c and $\log(z)$, turn out to be so restrictive for the corresponding solutions $X_{N,M},Y_{N,M}$ of (7.7), (7.8), that they allow one to compute their asymptotic behavior. For definiteness we consider the case $c>1$.

Lemma 10 *For the solution $X_{N,M},Y_{N,M}$ of (7.7),(7.8) corresponding to Z^c in $\mathbb{V}$ with $c>1$ holds true:*

$$
-\frac{c-1}{M-N}\le X_{N,M}\le\frac{c-1}{M+N},\qquad 0\le Y_{N,M+1}\le\frac{c-1}{M+N}+\frac{2(c-1)}{M-N}.
\tag{7.11}
$$

Proof: Note that for the studied solutions $R\ge 0$ and therefore $-1\le X_{N,M}\le 1$ and $-1\le Y_{N,M}\le 1$. Let us denote for brevity $X_{N,M}$ by X, $X_{N,M+1}$ by $\bar{X}$, $Y_{N,M}$ by Y, $Y_{N,M+1}$ by $\bar{Y}$, and $R_{N,M}$ by R_1, $R_{N+1,M}$ by R_2, $R_{N+1,M+1}$ by R_3, $R_{N,M+1}$ by R_4, $R_{N,M-1}$ by R_5. Then (3.8) together with (3.5),(7.4),(7.5) imply

$$
R_1^2\le R_2R_4,\quad R_1^2\ge R_2R_5,\quad R_3^2\ge R_2R_4,\quad R_4^2\ge R_1R_3,\quad R_2^2\le R_1R_3.
\tag{7.12}
$$

Rewriting the first inequality as $\frac{R_1}{R_2}\le\frac{R_4}{R_1}$ and taking into account $1+X\ge 0$ we have

$$
X+\bar{Y}\ge 0.
\tag{7.13}
$$

Similarly the second inequality of (7.12) infers $Y - X \geq 0$ or after shifting

$$\bar{Y} \geq \bar{X}. \tag{7.14}$$

Combining the first and the third inequalities of (7.12) one gets $R_3 \geq R_1$ or

$$\bar{X} + \bar{Y} \geq 0, \tag{7.15}$$

which is equivalent to

$$X + \bar{Y}_{N+1,M+1} \geq 0 \tag{7.16}$$

due to (7.10). Similarly the forth and the fifth imply $R_4 \geq R_2$ or

$$\bar{Y} \geq X. \tag{7.17}$$

Comparing (7.15) with (7.14) one gets $\bar{Y} \geq 0$. Rewriting (3.4) as

$$\frac{R_3}{R_1} = \frac{(2N+c)R_4 + (2M+c)R_2}{(2(M+1)-c)R_4 + (2(N+1)-c)R_2}$$

and taking into account $\frac{R_3}{R_1} \geq 1$ and $c \leq 2$ we can estimate $\frac{R_2}{R_4}$ in $\mathbb{V}$:

$$\frac{R_2}{R_4} \geq \frac{1 - \frac{c-1}{M-N}}{1 + \frac{c-1}{M-N}}$$

which reads as

$$\bar{Y} - X \leq (1 - \bar{Y}X)\epsilon_{N,M}, \quad \epsilon_{N,M} = \frac{c-1}{M-N}.$$

As $(1 - \bar{Y}X) \leq 2$ we have

$$\bar{Y} - X \leq 2\epsilon_{N,M}. \tag{7.18}$$

Similarly we can solve (3.4) with respect to $\frac{R_4}{R_2}$:

$$\frac{R_4}{R_2} = \frac{(2M+c)R_1 + (-2(N+1)+c)R_3}{(-2N-c)R_1 + (2(M+1)-c)R_3}.$$

Note that for $M > N$ holds $(-2N-c)R_1 + (2(M+1)-c)R_3 \geq 0$ as $R_3 \geq R_1$, and $c \leq 2$. This together with $R_4 \geq R_2$ and $(1 - \bar{Y}\bar{X}) \leq 2$ gives after some calculations

$$\bar{X} + \bar{Y} \leq 2\delta_{N,M}, \tag{7.19}$$

with $\delta_{N,M} = \frac{c-1}{M+N+1}$. Now inequalities (7.13),(7.17),(7.18) yield

$$-\epsilon_{N,M} \leq X$$

and (7.14),(7.15),(7.19) imply

$$\bar{X} \leq \delta_{N,M},$$

which gives the first inequality of (7.11) after shifting backwards from $M+1$ to M . Using (7.18) we easily get the second one. $\square$

The estimations obtained allow one to find asymptotic behavior for the radius-function in $M-$direction.

Theorem 21 *For the solution $R_{N,M}$ of (3.5),(3.4) corresponding to discrete Z^c and* Log *in $\mathbb{V}$ holds true:*

$$R_{N_0,M} \simeq K(c)M^{c-1} \quad \text{as} \quad M \to \infty, \tag{7.20}$$

with constant $K(c)$ independent of N_0.

Proof: Because of the duality

$$R_{z^{2-c}} = \frac{1}{R_{z^c}}$$

it is enough to consider the case $c > 1$. Let us introduce $n = M - N_0$, $x_n = X_{N_0,M}$, $y_n = Y_{N_0,M}$. Then for large n Lemma 10 allows one to rewrite equations (7.7),(7.8) as

$$\begin{array}{l} x_{n+1} = 5x_n - 2y_n + \frac{c-1}{n} + U_{N_0}(n), \\ y_{n+1} = -2x_n + y_n + V_{N_0}(n), \end{array} \tag{7.21}$$

where $U_{N_0}(n)$ and $V_{N_0}(n)$ are defined by discrete Z^c and satisfy

$$U_{N_0}(n) < C_1 \frac{1}{n^2}, \quad V_{N_0}(n) < C_2 \frac{1}{n^2}, \tag{7.22}$$

for natural n and some constants C_1, C_2 (depending on N_0). Thus the solution (x_n, y_n), corresponding to discrete Z^c is a special solution of linear non-homogeneous system (7.21) having the order $\frac{1}{n}$ for large n. The eigenvalues of the system matrix are positive numbers $\lambda_1 = 3 - 2\sqrt{2} < 1$ and $\lambda_2 = 3 + 2\sqrt{2} > 1$. In the diagonal form system (7.21) takes the form:

$$\varphi_{n+1} = A\varphi_n + s_n + r_n \tag{7.23}$$

with

$$\varphi_n = \begin{pmatrix} 1, & 1 \\ 1+\sqrt{2}, & 1-\sqrt{2} \end{pmatrix}^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad s_n = \begin{pmatrix} 2-\sqrt{2} \\ 2+\sqrt{2} \end{pmatrix} \frac{c-1}{4n},$$

and $|r_n| \le \frac{G}{n^2}$ for some G. Looking for the solution in the form

$$\varphi_n = A^n c_n$$

we have the following recurrent formula for c_n:

$$c_{n+1} = A^{-n}(s_n + r_n) + c_n.$$

Integrating one gets

$$c_{n+1} = c_1 + A^{-2}(s_1 + r_1) + A^{-3}(s_2 + r_2) + ... + A^{-1-n}(s_n + r_n).$$

For components of $\varphi_n = (a_n, b_n)^T$ this implies

$$\begin{array}{l} a_n = \lambda_1^n \left(a_1 + \frac{(2-\sqrt{2})(c-1)}{4} \left(\frac{1}{\lambda_1^2} + ... + \frac{1}{(n-1)\lambda_1^n} \right) + \left(\frac{G_1}{\lambda_1^2} + ... + \frac{G_{n-1}}{(n-1)^2 \lambda_1^n} \right) \right) = \\ = a_1 \lambda_1^n + \frac{(2-\sqrt{2})(c-1)}{4} \left(\frac{1}{n-1} + \frac{\lambda_1}{(n-2)} + ... + \lambda_1^{n-2} \right) + \left(\frac{G_1}{\lambda_1^2} + \frac{G_2}{2^2 \lambda_1^3} + ... + \frac{G_{n-1}}{(n-1)^2 \lambda_1^n} \right) \end{array}$$

with some bounded sequence G_n: $|G_n| \le G$. The first sum, corresponding to s_n, is estimated as follows:

$$\begin{array}{l} \frac{1}{n-1} \left(1 + \frac{n-1}{n-2}\lambda_1 + \frac{n-1}{n-3}\lambda_1^2 ... + (n-1)\lambda_1^{n-2} \right) = \\ = \frac{1}{n-1} \left(1 + \lambda_1 + \lambda_1^2 ... + \lambda_1^{n-2} \right) + \frac{\lambda_1}{n-1} \left(\frac{1}{n-2} + \frac{2}{n-3}\lambda_1 ... + (n-2)\lambda_1^{n-3} \right) = \\ = \frac{1}{n-1} \frac{1-\lambda_1^{n-1}}{1-\lambda_1} + \frac{\lambda_1}{(n-1)(n-2)} \left(1 + \frac{2(n-2)}{n-3}\lambda_1 + \frac{3(n-2)}{n-4}\lambda_1^2 ... + (n-2)^2 \lambda_1^{n-3} \right) = \\ = \frac{1}{n-1} \frac{1-\lambda_1^{n-1}}{1-\lambda_1} + \frac{\lambda_1}{(n-1)(n-2)} F_1(n, \lambda_1) \end{array}$$

where

$$F_1(n,\lambda_1) < (1 + 2\times 3\lambda_1 + 3\times 4\lambda_1^2 + ...) = \frac{2}{(1-\lambda_1)^3} - 1,$$

as $|\lambda_1| < 1$ and $(n-2)/(n-k) < k$ for $k \le n-1$. The second sum is estimated by

$$\begin{aligned}
&\tfrac{G}{(n-1)^2}\left(1 + \tfrac{(n-1)^2}{(n-2)^2}\lambda_1 + \tfrac{(n-1)^2}{(n-3)^2}\lambda_1^2 + ... + \tfrac{(n-1)^2}{(n-k)^2}\lambda_1^{k-1} + ... + (n-1)^2\lambda_1^{n-2}\right) = \\
&\tfrac{G}{(n-1)^2}(1 + \lambda_1 + \lambda_1^2 + ... + \lambda_1^{n-2} + \\
&\tfrac{(n-1)^2-(n-2)^2}{(n-2)^2}\lambda_1 + ... + \tfrac{(n-1)^2-(n-k)^2}{(n-k)^2}\lambda_1^{k-1} + ... + ((n-1)^2-1)\lambda_1^{n-2}) = \\
&= \tfrac{G}{(n-1)^2}\tfrac{1-\lambda_1^{n-1}}{1-\lambda_1} + \\
&\tfrac{G}{(n-1)^2}\left(\tfrac{(2(n-2)+1}{(n-2)^2}\lambda_1 + ... + \tfrac{(2(n-k)(k-1)+(k-1)^2}{(n-k)^2}\lambda_1^{k-1} + ... + (2(n-2) + (n-2)^2)\lambda_1^{n-2}\right) \le \\
&\tfrac{G}{(n-1)^2}\left(\tfrac{1}{1-\lambda_1} + 2\lambda_1(1 + 2\lambda_1 + 3\lambda_1^2 + ...) + \lambda_1(1 + 2^2\lambda_1 + 3^2\lambda_1^2 + ...)\right) = \tfrac{G}{(n-1)^2}F_2(\lambda_1).
\end{aligned}$$

Summing up we conclude:

$$a_n = \frac{(2-\sqrt{2})(c-1)}{4(1-\lambda_1)}\frac{1}{n} + O\left(\frac{1}{n^2}\right). \tag{7.24}$$

For the second component b_n one has

$$b_n = b_1\lambda_2^n + \frac{(2+\sqrt{2})(c-1)}{4}\lambda_2^n\left(\frac{1}{\lambda_2^2} + \frac{1}{2\lambda_2^3} + ... + \frac{1}{(n-1)\lambda_2^n}\right) + \lambda_2^n\left(\frac{H_1}{\lambda_2^2} + \frac{H_2}{2^2\lambda_2^3} + ... + \frac{H_{n-1}}{(n-1)^2\lambda_2^n}\right)$$

with some bounded sequence H_n: $|H_n| \le H$. The first sum in the previous formula is estimated as

$$\begin{aligned}
&\lambda_2^{n-1}\textstyle\int_0^{\frac{1}{\lambda_2}}(1 + x + x^2 ... + x^{n-2})dx = \lambda_2^{n-1}\int_0^{\frac{1}{\lambda_2}}\frac{1-x^{n-1}}{1-x}dx = \\
&\lambda_2^{n-1}\left(\textstyle\int_0^{\frac{1}{\lambda_2}}\frac{1}{1-x}dx - \int_0^{\frac{1}{\lambda_2}}(x^{n-1} + x^n + ...)dx\right) = \\
&\lambda_2^{n-1}\left(-\ln(1-x)\mid_0^{\frac{1}{\lambda_2}} - \tfrac{1}{\lambda_2^n}\left(\tfrac{1}{n} + \tfrac{1}{(n+1)\lambda_2} + \tfrac{1}{((n+2)\lambda_2^2}...\right)\right) = \\
&\lambda_2^{n-1}\ln\tfrac{\lambda_2}{\lambda_2-1} - \tfrac{1}{n\lambda_2}\left(1 + \tfrac{n}{(n+1)\lambda_2} + \tfrac{n}{((n+2)\lambda_2^2}...\right) = \\
&\lambda_2^{n-1}\ln\tfrac{\lambda_2}{\lambda_2-1} - \tfrac{1}{n\lambda_2}\left(\left(1 + \tfrac{1}{\lambda_2} + \tfrac{1}{\lambda_2^2} + ...\right) - \tfrac{1}{\lambda_2}\left(\tfrac{1}{(n+1)} + \tfrac{2}{(n+2)\lambda_2} + ... + \tfrac{k}{(n+k)\lambda_2^{k-1}} + ...\right)\right) = \\
&\lambda_2^n\tfrac{1}{\lambda_2}\ln\tfrac{\lambda_2}{\lambda_2-1} - \tfrac{1}{n(\lambda_2-1)} + \tfrac{S_n(\lambda_2)}{n(n+1)\lambda_2^2},
\end{aligned}$$

Where

$$\begin{aligned}
S_n(\lambda_2) = &\left(1 + \tfrac{2(n+1)}{(n+2)\lambda_2} + \tfrac{3(n+1)}{(n+3)\lambda_2^2} + ... + \tfrac{k(n+1)}{(n+k)\lambda_2^{k-1}} + ...\right) < \\
&\left(1 + \tfrac{2}{\lambda_2} + \tfrac{3}{\lambda_2^2} + ... + \tfrac{k}{\lambda_2^{k-1}} + ...\right) = \tfrac{\lambda_2^2}{(\lambda_2-1)^2}.
\end{aligned}$$

For the second sum in the formula for b_n one has

$$\begin{aligned}
&\tfrac{1}{\lambda_2^2}\left(H_1 + \tfrac{H_2}{2^2\lambda_2} + ... + \tfrac{H_{n-1}}{(n-1)^2\lambda_2^{n-2}}\right) = \\
&F_3(\lambda_2) - \tfrac{1}{\lambda_2^2}\left(\tfrac{H_n}{n^2\lambda_2^{n-1}} + \tfrac{H_{n+1}}{(n+1)^2\lambda_2^n} + ... + \tfrac{H_{n+k}}{(n+k)^2\lambda_2^{n+k-1}} + ...\right)
\end{aligned}$$

where

$$F_3(\lambda_2) = \tfrac{1}{\lambda_2^2}\left(H_1 + \tfrac{H_2}{2^2\lambda_2} + ... + \tfrac{H_{k-1}}{(k-1)^2\lambda_2^{k-2}} + ...\right)$$

and the second sum is estimated from above as

$$\begin{aligned}
&\tfrac{H}{\lambda_2^{n+1}n^2}\left(1 + \tfrac{n^2}{(n+1)^2\lambda_2} + \tfrac{n^2}{(n+2)^2\lambda_2^2} + ... + \tfrac{n^2}{(n+k)^2\lambda_2^k} + ...\right) < \\
&\tfrac{H}{\lambda_2^{n+1}n^2}\left(1 + \tfrac{1}{\lambda_2} + \tfrac{1}{\lambda_2^2} + \tfrac{1}{\lambda_2^2} + ...\right) = \tfrac{H}{\lambda_2^n n^2(\lambda_2-1)}.
\end{aligned}$$

Finally

$$b_n = \lambda_2^n \left(b_1 + \frac{(2+\sqrt{2})(c-1)}{4\lambda_2} \ln \frac{\lambda_2}{\lambda_2 - 1} + F_3(\lambda_2) \right) - \frac{(2+\sqrt{2})(c-1)}{4(\lambda_2 - 1)} \frac{1}{n} + O\left(\frac{1}{n^2}\right). \quad (7.25)$$

As $b_n \to 0$ with $n \to \infty$ and $\lambda_2 > 1$ one deduces that the coefficient by λ_2^n vanishes. For original variables x_n, y_n the found asymptotics (7.24),(7.25) have especially simple form:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{c-1}{2n} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + O\left(\frac{1}{n^2}\right). \quad (7.26)$$

Asymptotic (7.26), the second equation (7.6) and

$$R_{N_0, N_0+n} = R_{N_0, N_0} \prod_{k=1}^{n} \frac{2k + (c-1)}{2k - (c-1)}$$

imply (7.20). The independence of $K(c)$ on N_0 easily follows from the first equation (7.6) and $x_n \to 0$. □

The found asymptotics implies

$$\tan \alpha_n \simeq \left(1 + \frac{1}{n}\right)^{c-1} \quad \text{as} \quad n \to \infty \quad (7.27)$$

for the corresponding solution $u_n = e^{i\alpha_n}$ of (5.6). Further, equation (1.2) allows one to "integrate" asymptototics (7.20) to get

$$Z^c(n_0 + n, m_0 + n) \simeq e^{c\pi i/4} K(c) n^c \quad \text{as} \quad n \to \infty. \quad (7.28)$$

Thus the circles of Z^c not only cover the whole infinite sector with the angle $c\pi/2$ but the circle centers and intersection points mimic smooth map $z \to z^c$ also asymptotically. Morover, $R(z)$, being analogous to $|f'(z)|$ of the corresponding smooth map, has the "right" asymptotics as well.

Chapter 8

Remarks on discrete Z^c

8.1 Discrete Z^c is unique

Discrete map Z^c is defined via constraint (1.2), which is an isomonodromy condition for some linear equation. This seems to be rather far-fetched approach. It would be more naturally to define Z^c via circle patterns in a pure geometrical way: square grid Z^c, $0 < c < 2$ with prescribed intersection angles α is an embedded infinite Schramm's type square grid circle pattern with prescribed intersection angles, the circles $C(z)$ being labeled by

$$\mathbb{V} = \{z = N + iM : \ N, M \in \mathbb{Z}^2, M \geq |N|\},$$

satisfying the following conditions:
1) the circles cover the infinite sector with the angle $c\alpha$,
2) the centers of the border circles $C(N+iN)$ and $C(-N+iN)$ lie on the borders of this sector.

Conjecture 2 *Up to re-scaling there is unique square grid Z^c with prescribed intersection angles.*

A naive method to look for so defined orthogonal circle pattern Z^c is to start with some equidistant $f_{n,0} \in \mathbb{R}$, $f_{0,m} \in i^c\mathbb{R}$:

$$|f_{2n,0} - f_{2n\pm1,0}| = |f_{0,2n} - f_{0,2n\pm1}|$$

and then compute $f_{n,m}$ for any $n, m > 0$ using equation (1.7). Such $f_{n,m}$ determines some circle pattern. But so determined map has a not very nice behavior (see an example in Fig. 8.1).

For $c = 2/k, \ k \in \mathbb{N}$ the proof of Theorem 2 easily follows from the rigidity results obtained in [61]. (See an example of such Z^c at Fig. 8.2.)

Infinite embedded circle patterns define some (infinite) convex ideal polyhedron in $\mathbb{H}^3$ and the group generated by inversions in its faces. The known results (see for example [93]) imply the conjecture claim for rational c, but seem to be inapplicable for irrational as the group, generated by reflections in the circles and the sector borders, is not discrete any more. Unfortunately, the rigidity results for finite polyhedra [83] does not seem to be carried over to infinite case by induction.

For othogonal Z^c Conjecture 2 was recently proved by Bücking in her PhD [35] thesis by adapting result from the theory of random walks.

One can also consider solutions of (1.7) subjected to (1.2) where n and m are not integer. It turned out that there exist initial data, so that the corresponding solution define immersed circle patterns. (The equations for radii are the same and therefore compatible. Existence of positive solution is provable by the same arguments for the corresponding discrete Painlevé equation.) It is natural to call such circle patterns discrete map $z \to (z + z_0)^c$. In this case it can not be defined pure geometrically as the centers of border circles are not collinear.

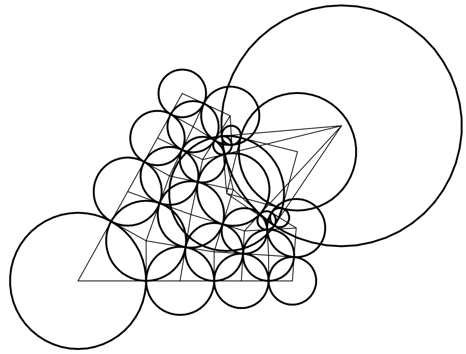

Figure 8.1: Non-immersed discrete conformal map

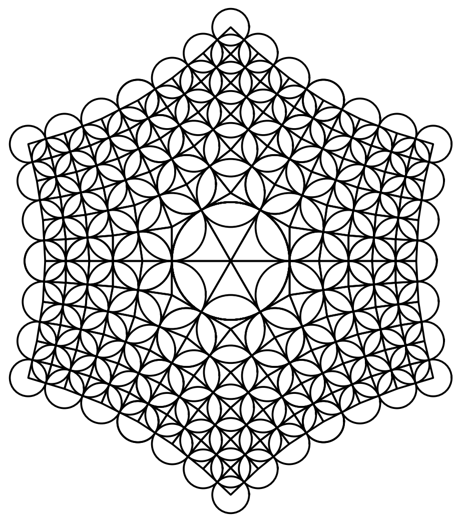

Figure 8.2: Schramm's circle pattern corresponding to $Z^{2/3}$

8.2 Discrete maps Z^c with $c \notin [0, 2]$

Starting with $Z^c,\ c \in [0, 2]$ one can easily define Z^c for arbitrary c by applying some simple transformations of discrete conformal maps and Schramm's circle patterns. Denote by S_c the Schramm's circle pattern associated to $Z^c,\ c \in (0, 2]$. Applying the inversion of the complex plane $z \mapsto \tau(z) = 1/z$ to S_c one obtains a circle pattern τS_c, which is also of Schramm's type. It is natural to define the discrete conformal map $Z^{-c},\ c \in (0, 2]$, through the centers and intersection points of circles of τS_c. On the other hand, constructing the dual Schramm's circle pattern (see Proposition 9) for Z^{-c} we arrive at a natural definition of Z^{2+c}. Intertwining the inversion and the dualization described above, one constructs circle patterns corresponding to Z^c for any c. To define immersed Z^c one should discard some points (and some circles) near $(n, m) = (0, 0)$ from the definition domain.

To give a precise description of the corresponding discrete conformal maps in terms of the constraint (1.2) and initial data for arbitrarily large c a more detailed consideration is required. To any Schramm circle pattern S there corresponds a one-parameter family of discrete conformal maps described in [28]. Take an arbitrary point $P_\infty \in \mathbb{C} \cup \infty$. Reflect it through all the circles of

S. The resulting extended lattice is a discrete conformal map and is called a *central extension* of S. As a special case, choosing $P_\infty = \infty$, one obtains the centers of the circles, and thus, the discrete conformal map considered here. Composing the discrete map $Z^c : \mathbb{Z}^2{}_+ \to \mathbb{C}$ with the inversion $\tau(z) = 1/z$ of the complex plane one obtains the discrete conformal map $g(n,m) = \tau(Z^c(n,m))$ satisfying the constraint (1.2) with the parameter $c_g = -c$. This map is the central extension of τS_c corresponding to $P_\infty = 0$. Let us define Z^{-c} as the central extension of τS_c corresponding to $P_\infty = \infty$. The map Z^{-c} defined in this way also satisfies the constraint (1.2) due to the following

Lemma 11 *Let S be a Schramm's circle pattern and $f^\infty : \mathbb{Z}^2_+ \to \mathbb{C}$ and $f^0 : \mathbb{Z}^2_+ \to \mathbb{C}$ be its two central extensions corresponding to $P_\infty = \infty$ and $P_\infty = 0$, respectively. Then f^∞ satisfies (1.2) if and only if f^0 satisfies (1.2).*

Proof: If f^∞ (or f^0) satisfies (1.2), then $f^\infty_{n,0}$ (respectively $f^0_{n,0}$) lie on a straight line, and so do $f^\infty_{0,m}$ (respectively $f^0_{0,m}$). A straightforward computation shows that $f^\infty_{n,0}$ and $f^0_{n,0}$ satisfy (1.2) simultaneously, and the same statement holds for $f^\infty_{0,m}$ and $f^0_{0,m}$. Since (1.7) is compatible with (1.2) f^0 (respectively f^∞) satisfy (1.2) for any $n, m \geq 0$. $\square$

Let us now describe Z^K for $K \in \mathbb{N}$ as a special solutions of (1.7),(1.2).

Definition 18 *$Z^K : \mathbb{Z}^2_+ \to \mathbb{R}^2 = \mathbb{C}$, where $K \in \mathbb{N}$, is the solution of (1.7, 1.2) with $c = K$ and the initial conditions*

$$Z^K(n,m) = 0 \text{ for } n+m \leq K-1, \quad (n,m) \in \mathbb{Z}^2_+. \tag{8.1}$$

$$Z^K(K,0) = 1, \tag{8.2}$$

$$Z^K(K-1,1) = i\frac{2^{K-1}\Gamma^2(K/2)}{\pi\Gamma(K)}. \tag{8.3}$$

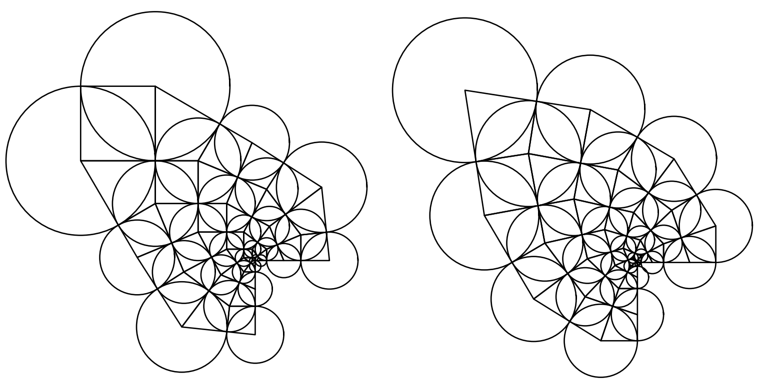

Figure 8.3: Discrete Z^3.

The initial condition (8.1) corresponds to the identity

$$\frac{d^k z^K}{dz^k}(z=0) = 0, \qquad k < K,$$

in the smooth case. For odd $K = 2N+1$, condition (8.3) reads

$$Z^{2N+1}(2N,1) = i\frac{(2N-1)!!}{(2N)!!},$$

and follows from constraint (1.2). For even $K = 2N$, any value of $Z^K(K-1,1)$ is compatible with (1.2). In this case formula (8.3) can be derived from the asymptotics

$$\lim_{N\to\infty} \frac{R(N+iN)}{R(N+i(N+1))} = 1$$

and reads

$$Z^{2N}(2N-1,1) = i\frac{2}{\pi}\frac{(2N-2)!!}{(2N-1)!!}.$$

We conjecture that so defined Z^K are immersed.

Note that for odd integer $K = 2N+1$, discrete Z^{2N+1} in Definition 18 is slightly different from the one previously discussed in this section. Indeed, by intertwining the dualization and the inversion (as described above) one can define two different versions of Z^{2N+1}. One is obtained from the circle pattern corresponding to discrete $Z(n,m) = n+im$ with centers in $n+im,\ n+m = 0 \pmod 2$. The second one comes from Definition 18 and is obtained by the same procedure from $Z(n,m) = n+im$, but in this case the centers of the circles of the pattern are chosen in $n+im,\ n+m = 1 \pmod 2$. These two versions of Z^3 are presented in Figure 8.3. The left figure shows Z^3 obtaied through Definition 18. Note that this map is immersed, in contrast to the right lattice of Figure 8.3 which has overlapping quadrilaterals at the origin (see Figure 8.4). All the asymptotic results obtained so far can be carried over on this case as

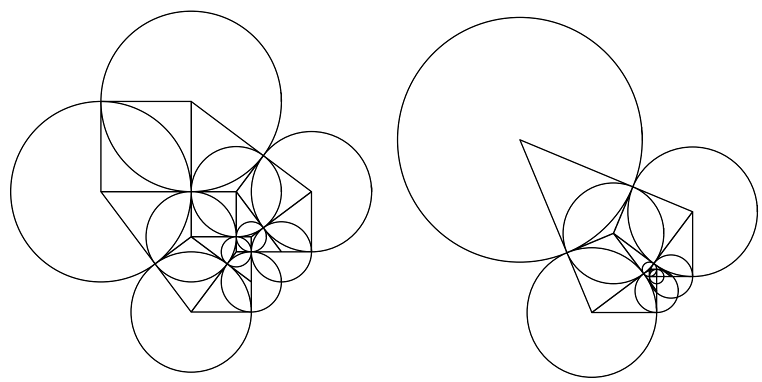

Figure 8.4: Detail view of two versions of discrete Z^3.

well as on hexagonal Z^c with constant intersection angles defined in [25] since the governing equations are essentially the same ([5]).

8.3 Circle patterns with quasi-regular combinatorics.

One can deregularize the prescribed combinatorics by a projection of $\mathbb{Z}^n$ into a plane as follows (see [94]). Consider $\mathbb{Z}^n_+ \subset \mathbb{R}^n$. For each coordinate vector $\mathbf{e}_i = (e_i^1, ..., e_i^n)$, where $e_i^j = \delta_i^j$ define a unit vector ξ_i in $\mathbb{C} = \mathbb{R}^2$ so that for any pair of indices i, j, vectors ξ_i, ξ_j form a basis in $\mathbb{R}^2$. Let $\Omega \in \mathbb{R}^n$ be some 2-dimensional simply connected cell complex with vertices in $\mathbb{Z}^n_+$. Choose some $x_0 \in \Omega$. Define the map $P : \Omega \to \mathbb{C}$ by the following conditions:

- $P(x_0) = P_0$,
- if x, y are vertices of Ω and $y = x + \mathbf{e}_i$ then $P(y) = P(x) + \xi_i$.

It is easy to see that P is correctly defined and unique.
We call Ω a *projectable* cell complex if its image $\omega = P(\Omega)$ is embedded, i.e. intersections of images of different cells of Ω do not have inner parts. Using projectable cell complexes one can obtain combinatorics of Penrose tilings.

It is natural to define "discrete conformal map on ω" as a discrete complex immersion function f on vertices of ω preserving the cross-ratios of the ω-cells. The argument of f can be labeled by the vertices x of Ω. Hence for any cell of Ω, constructed on $\mathbf{e}_k, \mathbf{e}_j$, the function f satisfies the following equation for the cross-ratios:

$$q(f_x, f_{x+\mathbf{e_k}}, f_{x+\mathbf{e_k}+\mathbf{e_j}}, f_{x+\mathbf{e_j}}) = e^{-2i\alpha_{k,j}}, \tag{8.4}$$

where $\alpha_{k,j}$ is the angle between ξ_k and ξ_j, taken positively if (ξ_k, ξ_j) has positive orientation and taken negatively otherwise.

Now suppose that f is a solution to (8.4) defined on the whole $\mathbb{Z}_+^n$. Equation (8.4) is compatible with the constraint

$$cf_x = \sum_{s=1}^{n} 2x_s \frac{(f_{x+\mathbf{e_s}} - f_x)(f_x - f_{x-\mathbf{e_s}})}{f_{x+\mathbf{e_s}} - f_{x-\mathbf{e_s}}}. \tag{8.5}$$

This constraint could be derived from some discrete isomonodromy problem (for $n = 3$ see [25]) which ensures the compatibility.

Now we can define discrete $Z^c : \omega \to \mathbb{C}$ for projectable Ω as solution to (8.4),(8.5) restricted on Ω. Initial conditions for this solution are of the form (1.8) so that the restrictions of f on each two-dimensional coordinate lattices is an immersion defining circle pattern with prescribed intersection angles. This definition naturally generalizes the definition of discrete square grid and hexagonal Z^c considered above. For the latter one chooses $\Omega = \{(k, l, m) : k + l + m = 0, \pm 1\}$.

Conjecture 3 *The discrete $z^c : \omega \to \mathbb{C}$ is an immersion.*

8.4 Square grid circle patterns Erf

For square grid combinatorics and $\alpha = \pi/2$, Schramm [87] constructed circle pattern mimicking holomorphic function $\mathrm{erf}(z) = (2/\pi) \int e^{-z^2} dz$ by giving the radius function explicitly. Namely, let n, m label the circle centers so that the pairs of circles $C(N, M)$, $C(N+1, M)$ and $C(N, M)$, $C(N, M + 1)$ are orthogonal and the pairs $C(N, M)$, $C(N + 1, M + 1)$ and $C(N, M + 1)$, $C(N + 1, M)$ are tangent. Then

$$R(N + iM) = e^{NM} \tag{8.6}$$

satisfies the equation for a radius function:

$$R^2(r_1 + r_2 + r_3 + r_4) - (r_2 r_3 r_4 + r_1 r_3 r_4 + r_1 r_2 r_4 + r_1 r_2 r_3) = 0, \tag{8.7}$$

where $R = R(N + iM)$, $r_1 = R(N + 1 + iM)$, $r_2 = R(N + i(M + 1))$, $r_3 = R(N - 1 + iM)$, $r_4 = R(N + i(M - 1))$. For square grid circle patterns with intersection angles α for $c(n, m)$, $C(N + 1, M)$ and $\pi - \alpha$ for $C(N, M)$, $C(N, M + 1)$ governing equation (8.7) becomes

$$R^2(r_1 + r_2 + r_3 + r_4) - (r_2 r_3 r_4 + r_1 r_3 r_4 + r_1 r_2 r_4 + r_1 r_2 r_3) + 2R \cos \alpha (r_1 r_3 - r_2 r_4) = 0.$$

It is easy to see that it has the same solution (8.6) and therefore it defines a square grid circle pattern, which is a discrete Erf. A hexagonal analog of Erf is not known.

Schramm [87] showed that the above circle pattern actually is a discrete analog of $\mathrm{erf}(\sqrt{i}z)$ but an exact analog of $\mathrm{erf}(z)$ does not exist. The obstacle is purely combinatorial. There is a hope that combinatorics of projectable cells can give more examples of discrete analogs of classical functions.

8.5 Discrete Z^c and Log without circles

Further generalizations of discrete Z^c and Log are possible. One can relax the unitary condition for cross-ratios and consider solutions to

$$q(f_{n,m}, f_{n+1,m}, f_{n+1,m+1}, f_{n,m+1}) = \kappa^2 e^{-2i\alpha} \tag{8.8}$$

subjected to the same constraint (1.2) with the initial data

$$f_{1,0} = 1, \; f_{0,1} = \frac{e^{ic\alpha}}{\kappa}. \tag{8.9}$$

This solution is a discrete analog of Z^c defined on the vertices of regular parallelogram lattice (see Fig. 8.5). However, thus obtained mappings are deprived of geometrical flavor as they do

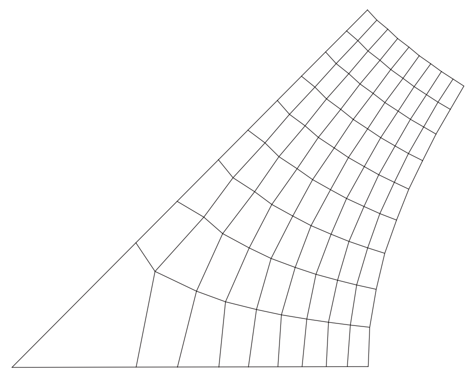

Figure 8.5: Discrete $Z^{1/2}$, $\kappa = 2$, $\alpha = \pi/2$.

not define circle patterns.

Part II

Integrable conservation law systems

Chapter 9

Integrable systems of three conservation laws and linear congruences

9.1 Geometry of focal varieties

To start discussing geometry of the examples mentioned in the Introduction, let us prove the translation of the property of having Riemann invariants into geometric language of line congruences. It is convenient to introduce the expansions

$$[L_i, L_j] = c^k_{ij} L_k, \quad c^k_{ij} = -c^k_{ji}. \tag{9.1}$$

(Recall that L_i is Lie derivative in the flow of i-th eigenvector.) There is a simple criterion for the existence of Riemann invariants in terms of the coefficients c^i_{jk} defined by (9.1).

Proposition 13 *The characteristic speed λ^i possesses a Riemann invariant if and only if $c^i_{jk} = 0$ for any $(j,k) \neq i$.*

Theorem 22 *If the characteristic speed λ^i of a T-system (1.15) possesses a Riemann invariant, then the corresponding focal submanifold M_i is a linear subspace of codimension 2.*

Proof: Let us consider, for definiteness, the focal hypersurface M_1 with the radius vector

$$\mathbf{r}_1 = (\lambda^1, u^1\lambda^1 - f^1, ..., u^n\lambda^1 - f^n),$$

corresponding to the characteristic speed λ^1. We will need the following relations between the densities u and the fluxes f of conservation laws of system (1.15):

$$L_i(f) = \lambda^i L_i(u) \text{ for any } i = 1, ..., n, \tag{9.2}$$

$$L_iL_j(u) = \frac{L_j(\lambda^i)}{\lambda^j - \lambda^i} L_i(u) + \frac{L_i(\lambda^j)}{\lambda^i - \lambda^j} L_j(u) + \frac{\lambda^j - \lambda^k}{\lambda^i - \lambda^j} c^k_{ij} L_k(u), \ i \neq j, \tag{9.3}$$

(see e.g. [91], [99]). In particular, $f = f^s$ and $u = u^s$ satisfy (9.2) and (9.3) for any $s = 1, ..., n$. Introducing $\mathbf{l} = (1,\ u^1, ..., u^n)$ and applying $L_1, ..., L_n$ to the radius vector $\mathbf{r}_1$, one readily obtains $L_1(\mathbf{r}_1) = L_1(\lambda^1)\mathbf{l} = 0$ as $L_1(\lambda^1) = 0$ by linear degeneracy. Thus, the condition $L_1(\mathbf{r}_1) = 0$ implies that M_1 is independent of R^1, where R^1 is the Riemann invariant corresponding to λ^1. Since

$$L_i(\mathbf{r}_1) = L_i(\lambda^1)\mathbf{l} + (\lambda^1 - \lambda^i) L_i(\mathbf{l}),$$

the tangent space TM_1 is spanned by $n-1$ vectors $L_i(\lambda^1)\mathbf{l}+(\lambda^1-\lambda^i)L_i(\mathbf{l}),\ (i\neq 1)$. This tangent space belongs to the hyperplane H_1 spanned by n vectors $\mathbf{l}, L_i(\mathbf{l}),\ (i\neq 1)$ which depends only on the variable R^1 since $L_k(H_1)\in H_1$ for any $k\neq 1$. This follows from the relations

$$L_k(\mathbf{l})\in H_1, \quad L_k^2(\mathbf{l})=p_kL_k(\mathbf{l})\in H_1, \quad L_kL_jH_1\in H_1 \ \ (k,j\neq 1).$$

Here $L_k^2(\mathbf{l})=p_kL_k(\mathbf{l})$ due to the linearity of rarefaction curves, and $L_kL_jH_1\in H_1$ by virtue of (9.3) and the condition $c^1_{kj}=0$ (Proposition 13). On the other hand, M_1 (and hence TM_1) does not depend on R^1 due to linear degeneracy. Consequently, the tangent space of M_1 is the intersection of any two hyperplanes H_1 which correspond to the two different values of R^1. Therefore, it is stationary and coincides with the focal submanifold M_1. □

• Veronesé variety

Let us first recall some of the well-known properties of the Veronesé variety $V^2\subset\mathbb{P}^5$ realising $\mathbb{P}^5$ as the space of 3×3 symmetric matrices $Z^{ij},\ i,j=0,1,2$. Veronesé variety V^2 is a subvariety of matrices of rank 1

$$Z=\left(\begin{array}{ccc} Z^{00} & Z^{01} & Z^{02}\\ Z^{10} & Z^{11} & Z^{12}\\ Z^{20} & Z^{21} & Z^{22}\end{array}\right).$$

It can be viewed as the canonical embedding $F:\mathbb{P}^2\to V^2\subset\mathbb{P}^5$ defined by

$$Z^{ij}=X^iX^j,\ i=0,1,2, \tag{9.4}$$

where $[X^0:X^1:X^2]$ are homogeneous coordinates in $\mathbb{P}^2$. Veronesé variety coincides with the singular locus of the *cubic symmetroid* defined by the equation

$$\det Z^{ij}=0,$$

which is also the bisecant variety $S(V^2)$ of V^2 consisting of symmetric matrices of rank two. Under the embedding (9.4) each line in $\mathbb{P}^2$ is mapped onto a conic on V^2, therefore, Veronesé variety carries a 2-parameter family of conics. The projective automorphism group of V^2 coincides with the natural action of PSL_3 on $\mathbb{P}^5$

$$Z\to g^TZg, \quad g\in PSL_3, \tag{9.5}$$

which obviously preserves V^2.

Below we discuss in some more detail the geometry of congruences associated with the equations (1.32), (1.36) – (1.41).

• Equations without Riemann invariants

In this subsection we discuss equations (1.32), (1.36) and (1.37). Rewritten as systems of conservation laws, they do not possess Riemann invariants, so that the corresponding focal varieties will be irreducible. We explicitly demonstrate that they concide with different non-singular projections of the Veronesé variety.

Equation (1.32). The focal variety of the corresponding congruence (1.35) is defined by (1.20)

$$y^0=\lambda, \quad y^1=a\lambda-b, \quad y^2=b\lambda-c, \quad y^3=c\lambda-b^2+ac, \tag{9.6}$$

where λ satisfies the characteristic equation

$$\lambda^3+a\lambda^2-2b\lambda+c=0. \tag{9.7}$$

One can verify that the three focal surfaces (9.6) corresponding to the three different values of λ are, in fact, "glued" together to form the algebraic variety defined in this affine chart by a system of seven cubics

$$
\begin{aligned}
&(y^0)^3 + y^0y^1 - y^2 = 0, \quad (y^2)^2 + y^3(y^0)^2 = 0, \quad y^1y^2y^3 + y^0(y^3)^2 - (y^2)^3 = 0, \\
&y^2(y^0)^2 + y^1y^2 + y^0y^3 = 0, \quad (y^3)^2 - y^1(y^2)^2 + y^0y^2y^3 + y^3(y^1)^2 = 0, \\
&y^0y^1y^3 - y^0(y^2)^2 - y^2y^3 = 0, \quad y^0y^2 + y^1(y^0)^2 + (y^1)^2 + y^3 = 0.
\end{aligned}
\tag{9.8}
$$

Variety (9.8) coincides with the projection of the Veronesé variety $V^2 \subset \mathbb{P}^5$

$$
y^0 = \frac{Z^{02}}{Z^{22}}, \quad y^1 = \frac{Z^{12} - Z^{00}}{Z^{22}}, \quad y^2 = \frac{Z^{01}}{Z^{22}}, \quad y^3 = -\frac{Z^{11}}{Z^{22}}
$$

from the point

$$
\begin{pmatrix} Z^{00} & 0 & 0 \\ 0 & 0 & Z^{00} \\ 0 & Z^{00} & 0 \end{pmatrix}
\tag{9.9}
$$

into $\mathbb{P}^4$. Notice that this point does not belong to the bisecant variety $S(V^2)$ and hence the projection is non-singular. Here we list some of the main properties of this projection which are, of course, well-known.

1. The manifold of trisecant lines of the focal variety (9.8), which we denote by $M^3 \subset \mathbb{G}(1,4)$, is three-dimensional.

2. For each point p on the focal variety the set of trisecants passing through p forms a planar pencil with the vertex p.

3. The intersection of the abovementioned planar pencil with the focal variety consists of the point p and a conic. Let us demonstrate this by a direct calculation. Since $(1, \lambda, \lambda^2)^T$ is the eigenvector of the system (1.34) corresponding to the eigenvalue λ, the rarefaction curve passing through p is given parametrically by

$$
(a,\ b,\ c) + s(1,\ \lambda,\ \lambda^2) = (a + s,\ b + s\lambda,\ c + s\lambda^2),
$$

s being the parameter (recall that rarefaction curves are lines). The corresponding one-parameter pencil of lines

$$
\begin{aligned}
y^1 &= (a + s)y^0 - (b + s\lambda), \\
y^2 &= (b + s\lambda)y^0 - (c + s\lambda^2), \\
y^3 &= (c + s\lambda^2)y^0 - (b + s\lambda)^2 + (a + s)(c + s\lambda^2)
\end{aligned}
$$

belongs to the plane with parametric equations

$$
\begin{aligned}
y^0 &= X, \\
y^1 &= aX - b + Y, \\
y^2 &= bX - c + \lambda Y, \\
y^3 &= cX - b^2 + ac + \lambda^2 Y.
\end{aligned}
$$

It can be readily verified that the intersection of this plane with the focal variety consists of the point $X = \lambda,\ Y = 0$ and the parabola $Y + X^2 + (a + \lambda)X + \lambda^2 + a\lambda - 2b = 0$.

Remark. The submanifold $M^3 \subset \mathbb{G}(1,4)$ can be equivalently described as the image of the mapping $\mathbb{P}^4 \to \mathbb{P}^6$ defined by the system of cubics (9.8): this mapping blows down the lines of the congruence (1.35), so that the image is indeed three-dimensional.

Equation (1.36). Rewritten as a system of conservation laws

$$
a_t = b_x, \quad b_t = c_x, \quad c_t = ((1 + bc)/a)_x,
\tag{9.10}
$$

this equation is associated with the congruence

$$y^1 = ay^0 - b, \quad y^2 = by^0 - c, \quad y^3 = cy^0 - (1 + bc)/a \tag{9.11}$$

the focal variety of which is defined by (1.20)

$$y^0 = \lambda, \quad y^1 = a\lambda - b, \quad y^2 = b\lambda - c, \quad y^3 = c\lambda - (1 + bc)/a \tag{9.12}$$

where λ satisfies the characteristic equation

$$\lambda^3 - \frac{b}{a}\lambda^2 - \frac{c}{a}\lambda + \frac{1 + bc}{a^2} = 0. \tag{9.13}$$

One can verify that the three focal surfaces (9.12) corresponding to the three different values of λ are glued together to form the algebraic variety defined in this affine chart by a system of cubics

$$\begin{aligned}
&1 + y^0(y^1)^2 + y^1y^2 = 0, \quad y^1(y^0)^2 - y^3 = 0, \quad (y^0)^3 + y^0y^2y^3 + (y^3)^2 = 0, \\
&y^0 + y^0y^1y^2 + y^1y^3 = 0, \quad y^0y^3 - y^2(y^0)^2 + y^1(y^3)^2 - y^3(y^2)^2 = 0, \\
&(y^0)^2 + y^0y^1y^3 + y^2y^3 = 0, \quad y^0y^1 + (y^1)^2y^3 - y^2 - y^1(y^2)^2 = 0.
\end{aligned} \tag{9.14}$$

This algebraic variety is the projection of the Veronesé variety

$$y^0 = -\frac{Z^{02}}{Z^{12}}, \quad y^1 = -\frac{Z^{11}}{Z^{12}}, \quad y^2 = \frac{Z^{22} - Z^{01}}{Z^{12}}, \quad y^3 = -\frac{Z^{00}}{Z^{12}}$$

from the point

$$\begin{pmatrix} 0 & Z^{01} & 0 \\ Z^{01} & 0 & 0 \\ 0 & 0 & Z^{01} \end{pmatrix} \tag{9.15}$$

into $\mathbb{P}^4$. Notice that the two points (9.9) and (9.15) are equivalent under the action of the group (9.5) preserving the Veronesé variety (indeed, both matrices have the same Lorentzian signature). Hence, both projections and the corresponding congruences of trisecants are projectively equivalent. To be explicit, the projective transformation

$$y^0 = -\frac{1}{Y^1}, \ y^1 = \frac{Y^2}{Y^1}, \ y^2 = \frac{Y^0}{Y^1}, \ y^3 = -\frac{Y^3}{Y^1} \tag{9.16}$$

identifies the systems of cubics (9.8) and (9.14). Applying this transformation to the congruence (1.35) and introducing the new parameters $A = -1/c, \ B = b/c, \ C = a - b^2/c$, we readily rewrite (1.35) in the form

$$Y^1 = AY^0 - B, \quad Y^2 = BY^0 - C, \quad Y^3 = CY^0 - (1 + BC)/A$$

which coincides with (9.11). This gives geometric explanation of the transformation between equations (1.32) and (1.36) mentioned in the Introduction. On the level of systems of conservation laws (1.34) and (9.10), this transformation is a reciprocal equivalence.

Equation (1.37). Rewritten as a system of conservation laws

$$a_t = b_x, \quad b_t = c_x, \quad c_t = ((c^2 + b^2 - ac - 1)/b)_x, \tag{9.17}$$

this equation is associated with the congruence

$$y^1 = ay^0 - b, \quad y^2 = by^0 - c, \quad y^3 = cy^0 - ((c^2 + b^2 - ac - 1)/b), \tag{9.18}$$

whose focal surfaces are glued together to form the algebraic variety that is the projection of the Veronesé variety:

$$y^0 = -\frac{Z^{02}}{Z^{12}}, \quad y^1 = \frac{Z^{00} - Z^{22}}{Z^{12}}, \quad y^2 = \frac{Z^{01}}{Z^{12}}, \quad y^3 = \frac{Z^{11} - Z^{22}}{Z^{12}},$$

from the point

$$\begin{pmatrix} Z^{00} & 0 & 0 \\ 0 & Z^{00} & 0 \\ 0 & 0 & Z^{00} \end{pmatrix} \tag{9.19}$$

into $\mathbb{P}^4$. Notice that this point is not equivalent (over the reals) to the points (9.9) and (9.15) under the action of the group (9.5) (indeed, the signature of (9.19) is Euclidean). Hence, the congruence (9.18) is not projectively equivalent to any of the congruences (1.35) or (9.11). The corresponding systems of conservation laws are not reciprocally related.

We point out that the Veronesé variety $V^2 \subset \mathbb{P}^5$, being the intersection of quadrics, does not possess trisecant lines. Trisecants appear only after we project V^2 into $\mathbb{P}^4$. Indeed, let P_0 be a point in $\mathbb{P}^5$ not on the bisecant variety $S(V^2)$. Viewed as a 3×3 symmetric matrix, P_0 defines a non-degenerate conic in $\mathbb{P}^2$

$$\sum_{i,j=0}^{2} (P_0^{-1})^{ij} X^i X^j = 0 \tag{9.20}$$

where $[X^0 : X^1 : X^2]$ are homogeneous coordinates. If a plane passes through P_0 and cuts V^2 in three points, then pre-images of these points under the embedding (9.4) are pairwise conjugate with respect to the conic (9.20). Conversely, P_0 lies in the plane spanned by the images under (9.4) of any three points in $\mathbb{P}^2$ that are pairwise conjugate with respect to (9.20). Thus, there is a three-parameter family of planes passing through P_0 and cutting V^2 in three points. Projecting this family from the point P_0 into $\mathbb{P}^4$, we arrive at the congruence of lines in $\mathbb{P}^4$. By the construction, its lines are trisecants of the projection $\pi_{P_0}(V^2)$, which is the focal surface of our congruence. To see that developable surfaces of the congruence are planar pencils of lines, we consider a line L in $\mathbb{P}^2$ defined by equation $L_0X^0 + L_1X^1 + L_2X^2 = 0$. Under the embedding (9.4), this line corresponds to a conic on V^2 lying in the so called *conisecant plane* of V^2. In matrix form equations of this plane are $LZ = 0$. The three-dimensional subspace Λ spanned by P_0 and the conisecant plane consists of all Z such that the vectors LZ and LP_0 are collinear. In addition to the conic in the conisecant plane, Λ intersects V^2 in the point P_0^L whose pre-image in $\mathbb{P}^2$ under (9.4) has homogeneous coordinates P_0L^T. Consider now the one-parameter family of planes in $\mathbb{P}^5$ lying in Λ and passing through the line joining P_0 and P_0^L. Clearly, each of these planes intersects V^2 in three points, and the projection of this one-parameter family of planes into $\mathbb{P}^4$ will be a planar pencil of lines. This gives developable surfaces of our congruence.

• Equations with one Riemann invariant

Now let us discuss equations (1.38) and (1.39). Since both equations possess only one Riemann invariant, the corresponding focal varieties will be reducible, consisting of a cubic scroll and a plane intersecting the cubic scroll along its directrix.

Equation (1.38) can be rewritten as a system of conservation laws

$$\begin{aligned} a_t &= b_x, \\ b_t &= c_x, \\ c_t &= (bc/a)_x, \end{aligned} \tag{9.21}$$

the characteristic speeds of which are $\lambda^1 = b/a$ and $\lambda^2, \lambda^3 = \pm\sqrt{c/a}$. The only Riemann invariant $R^1 = c/a$ corresponds to λ^1. The focal surface corresponding to λ^1 is the plane

$$y^1 = y^3 = 0, \tag{9.22}$$

while the focal surfaces corresponding to λ^2 and λ^3 are glued together to form the cubic scroll defined by a system of quadrics

$$y^0y^1 + y^2 = 0, \quad y^0y^2 + y^3 = 0, \quad y^1y^3 - (y^2)^2 = 0. \tag{9.23}$$

The plane (9.22) intersects the cubic scroll along its directix

$$y^1 = y^2 = y^3 = 0. \tag{9.24}$$

The cubic scroll (9.23) can be obtained by projecting the Veronesé variety

$$y^0 = \frac{Z^{02}}{Z^{12}}, \quad y^1 = \frac{Z^{11}}{Z^{12}}, \quad y^2 = -\frac{Z^{01}}{Z^{12}}, \quad y^3 = \frac{Z^{00}}{Z^{12}},$$

from the point

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z^{22} \end{pmatrix}.$$

Notice that the center of this projection lies on the Veronesé variety. The directrix (9.24) is the image of the tangent plane $Z^{00} = Z^{01} = Z^{11} = 0$ to the Veronesé variety in the centre of projection, and the plane (9.22) is the projection of the three-dimensional linear subspace in $\mathbb{P}^5$ spanned by the tangent plane and the point

$$\begin{pmatrix} 0 & Z^{01} & 0 \\ Z^{01} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.25}$$

on the bisecant variety. Thus, the focal variety of our congruence is reducible and consists of the plane (9.22) and the cubic scroll (9.23). Like in the case of systems without Riemann invariants, holds true

1. the manifold of trisecants of the focal variety is three-dimensional, and
2. for a fixed point p on the focal variety the set of trisecants passing through p forms a planar pencil with the vertex p. If p belongs to the plane (9.22), the corresponding planar pencil cuts the focal variety in the point p and a conic. If p belongs to the cubic scroll, it cuts the focal variety in the point p and a pair of lines.

Equation (1.39) can be rewritten as a system of conservation laws

$$\begin{aligned} a_t &= b_x, \\ b_t &= c_x, \\ c_t &= ((c^2 + b^2 - ac)/b)_x, \end{aligned} \tag{9.26}$$

the characteristic speeds of which are $\lambda^1 = c/b$ and $\lambda^2, \lambda^3 = (c - a \pm \sqrt{4b^2 + (c-a)^2})/2b$. The only Riemann invariant $R^1 = (c-a)/b$ corresponds to λ^1. The focal surface corresponding to λ^1 is the plane

$$y^1 = y^3, \quad y^2 = 0, \tag{9.27}$$

while the focal surfaces corresponding to λ^2 and λ^3 are glued together to form the cubic scroll defined by a system of quadrics

$$y^0y^3 + y^2 = 0, \quad y^0y^2 + y^1 = 0, \quad y^1y^3 - (y^2)^2 = 0. \tag{9.28}$$

The plane (9.27) intersects the cubic scroll (9.28) along its directrix

$$y^1 = y^2 = y^3 = 0. \tag{9.29}$$

The cubic scroll (9.28) can be obtained by projecting V^2

$$y^0 = \frac{Z^{02}}{Z^{12}}, \quad y^1 = \frac{Z^{00}}{Z^{12}}, \quad y^2 = -\frac{Z^{01}}{Z^{12}}, \quad y^3 = \frac{Z^{11}}{Z^{12}},$$

from the point

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z^{22} \end{pmatrix}$$

on V^2. The directrix (9.29) is the image of the tangent plane $Z^{00} = Z^{01} = Z^{11} = 0$ to V^2 in the centre of projection, and the plane (9.27) is the image of the three-dimensional linear subspace spanned by the tangent plane and the point

$$\begin{pmatrix} Z^{00} & 0 & 0 \\ 0 & Z^{00} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.30}$$

on the bisecant variety.

A coordinate-free construction of the congruences discussed above can be described as follows. Take a point $P_0 \in S(V^2)$ which is represented by a symmetric matrix of rank two. Then there is a nonzero vector $X \in \mathbb{P}^2$ such that $P_0X = 0$. Consider the tangent plane to V^2 at the point $F(X)$ (recall that F is the canonocal embedding of Veronesé variety). The projection of V^2 into $\mathbb{P}^4$ from the point $F(x)$ is a cubic scroll. The projection of the tangent plane is the directrix. The projection of the three-space spanned by the tangent plane and P_0 is the plane intersecting the cubic scroll along its directrix.

Although the last two examples look pretty similar, they are not projectively equivalent. Indeed, the points (9.25) and (9.30) have different signatures.

• Equations with two Riemann invariants.

Here we discuss equation (1.40). Due to the existence of two Riemann invariants, the corresponding focal variety will be reducible consisting of two planes and a two-dimensional quadric.

Equation (1.40) can be rewritten as a system of conservation laws

$$\begin{aligned} a_t &= b_x, \\ b_t &= c_x, \\ c_t &= ((c^2 - 1)/b)_x \end{aligned} \tag{9.31}$$

with the characteristic speeds $\lambda^1 = 0$ and $\lambda^2, \lambda^3 = (c \mp 1)/b$. The system has two Riemann invariants $(c \pm 1)/b$ corresponding to λ^2 and λ^3, respectively. The focal surfaces of the associated congruence corresponding to λ^2 and λ^3 are the planes

$$\begin{aligned} y^2 &= \mp 1, \\ y^0 &= \mp y^3, \end{aligned} \tag{9.32}$$

while the third focal surface, corresponding to λ^1, is the quadric

$$y^0 = 0, \quad y^1y^3 - (y^2)^2 + 1 = 0. \tag{9.33}$$

The planes (9.32) intersect the quadric (9.33) along the rectilinear generators

$$y^0 = 0, \; y^2 = \mp 1, \; y^3 = 0$$

which belong to different families and meet at infinity.

One can describe this congruence in a coordinate-free form as follows. Consider a quadric Q in a hyperplane $\Lambda \subset \mathbb{P}^4$. Choose a point $p \in Q$ and draw two rectilinear generators l_1, l_2 of Q through p. Choose two planes π_1 and π_2 which are not in Λ such that $l_i \subset \pi_i$ and $\pi_1 \cap \pi_2 = p$. The union of $\pi_1,\ \pi_2$ and Q is the focal variety in question. Its trisecants define a congruence in $\mathbb{P}^4$.

• **Equations with three Riemann invariants.**

As follows from Theorem 22, focal varieties of congruences corresponding to diagonalizable n-component T-systems are collections of n linear subspaces of codimension two in $\mathbb{P}^{n+1}$. For $n = 4$ we have 3 planes in $\mathbb{P}^4$. To ensure the nondegeneracy, we require that the points of their pairwise intersections are distinct.

Equation (1.41) can be rewritten as a linear system of conservation laws

$$\begin{aligned} a_t &= b_x, \\ b_t &= c_x, \\ c_t &= b_x \end{aligned} \tag{9.34}$$

with the characteristic speeds $\lambda^1 = 0,\ \lambda^2 = 1,\ \lambda^3 = -1$. Being linear, this system has 3 Riemann invariants. The focal surfaces of the associated conruence are the planes

$$y^1 = y^3,\ y^0 = 0 \ \text{ for } \ \lambda^1 = 0,$$

$$y^3 = -y^2,\ y^0 = 1 \ \text{ for } \ \lambda^2 = 1$$

and

$$y^3 = y^2,\ y^0 = -1 \ \text{ for } \ \lambda^1 = 0,$$

respectively.

9.2 Linear congruences.

Definition 19 *The congruence (1.16) is called linear (or general linear) if its Plücker coordinates*

$$u^i,\ f^i,\ u^i f^j - u^j f^i$$

satisfy n linear equations of the form

$$\alpha + \alpha_i u^i + \beta_i f^i + \alpha_{ij}(u^i f^j - u^j f^i) = 0 \tag{9.35}$$

where $\alpha,\ \alpha_i,\ \beta_i,\ \alpha_{ij}$ are arbitrary constants.

Notice that equations (9.35), being linear in f, define f^i as explicit functions of u. We emphasize that all examples discussed above belong to this class.

Proposition 14 *To each linear congruence corresponds a T-system of conservation laws.*

Proof: Let q be any fixed point in P^{n+1}. For the lines passing through q we have $f^i = u^i q^0 - q^i$, which, upon the substitution into (9.35), implies a linear system for u. In general, this system possesses a unique solution, so that there exists a unique line of our congruence passing through q (such congruences are said to be of order one). The focal variety V (also called the jump locus) consists of those q for which the corresponding linear system is not uniquely solvable for u. One can easily see that V has codimension at least two, and in the case it equals two, the developable surfaces are planar pencils of lines. Moreover, the intersection of any of these planes with the focal variety V consists of a point and a plane curve of the order $n - 1$. Thus, there is a T-system associated with any linear congruence in $\mathbb{P}^{n+1}$. □

Theorem 23 *A congruence, corresponding to a three-component T-system is linear.*

Proof is given in the next section.

In the case $n = 4$ the geometry of focal varieties of general linear congruences, known as the Palatini scrolls, was investigated in [82] (see also [70] and [81] for further properties of the Palatini scrolls). (It is a great pleasure to thank F. Zak for providing these references.)

There are at least two different ways one could approach the classification of three-component T-systems or, equivalently, the line congruences in $\mathbb{P}^4$, whose developable surfaces are planar pencils of lines. The first way is to establish their linearity. In the parametrization (1.16) this means that the three-dimensional surface with the radius-vector $(u, f, u \wedge f)$ representing our congruence in the Grassmanian $\mathbb{G}r(1, \mathbb{P}^4)$ lies in a linear subspace of codimension three. After the linearity is established, the results of Castelnuovo [37], who demonstrated that the corresponding focal varieties are projections into $\mathbb{P}^4$ of the Veronesé variety in $\mathbb{P}^5$, complete the classification and imply Theorem 8.

Another way makes use of the Theorem of Segre [90] saying that a surface in projective space carrying a two-parameter family of plane curves (not lines) is either a cone or the surface of Veronesé V^2 or its projection into $\mathbb{P}^4$. Moreover, the corresponding plane curves are conics. This theorem is intimately related to our problem. Indeed, let M_1, M_2, M_3 be three focal surfaces of our congruence in $\mathbb{P}^4$. Take a point $p \in M_1$ and consider the planar pencil of lines passing through it. This plane intersects M_2 and M_3 in the curves γ_2 and γ_3, respectively. Varying p, we conclude that both M_2 and M_3 contain two-parameter families of plane curves and hence are projections of the Veronesé variety (the case of a cone can be easily ruled out). Moreover, the curves γ_2 and γ_3 are conics. To show that both M_2 and M_3 are actually parts of one and the same Veronesé variety, it is sufficient to demonstrate that γ_2 amd γ_3 are parts of one and the same plane conic. This can be done as follows: intersect γ_2 and γ_3 by a line passing through p and construct the tangent lines to γ_2 and γ_3 through the points of intersection. These lines meet in a point q lying in the same plane. Doing this for all lines of the pencil with vertex p we arrive at the curve q, which clearly must be a line (called the polar of p) in case γ_2 and γ_3 are parts of one and the same conic.

Unfortunately, both proofs require differential identities, which do not immediately follow from the geometric data given. Thus, it proves necessary to directly investigate the exterior differential system governing three-component T-systems, transforming it into the involutive form. Once it has been done, both properties mentioned above reduce to simple calculation.

9.3 Structure equations

• Exterior representation

Studying nondiagonalizable systems (1.15) it is convenient to use the following exterior notation: let $l^i = (l^i_1(u), l^i_2(u), ..., l^i_n(u))$ be left eigenvectors of the matrix $v^i_j(u)$ corresponding to the eigenvalues $\lambda^i(u)$, i.e. $l^i_j v^j_k = \lambda^i l^i_k$. With the eigenforms $\omega^i = l^i_j du^j$ the system (1.15) is rewritten in the following exterior form:

$$\omega^i \wedge (dx + \lambda^i dt) = 0, \qquad i = 1, ..., n. \tag{9.36}$$

The differentiation of ω^i and λ^i gives the structure equations

$$d\omega^i = \tilde{c}^i_{jk} \omega^j \wedge \omega^k \quad (\tilde{c}^i_{jk} = -\tilde{c}^i_{kj}), \quad d\lambda^i = \lambda^i_j \omega^j, \tag{9.37}$$

that contain all the necessary information about the system under study. If ω^i are normalized in such a way that $\omega^i(\xi_j) = \delta^i_j$ then the structure coefficients $\tilde{c}^i_{jk}$ are related to the structure coefficients c^i_{jk} of (9.1) by $\tilde{c}^i_{jk} = -c^i_{jk}$.

• **Structure equations for nondiagonalizable systems.**

The next three theorems claim that for nondiagonalizable systems the structure equations for ω^i take surprisingly simple forms. We give the detailed proof only for the first theorem and less detailed plans of proof for the others. Notice that ω^i are defined up to nonzero normalization $\omega^i \to p^i\omega^i, \ p^i \neq 0$.

Theorem 24 *The eigenforms of a 3×3 T-system without Riemann invariants can be normalized so that the structure equations take the form*

$$d\omega^1 = \omega^2\wedge\omega^3, \quad d\omega^2 = \epsilon\omega^3\wedge\omega^1, \quad d\omega^3 = \omega^1\wedge\omega^2, \ \text{where } \epsilon = \pm 1. \tag{9.38}$$

Proof: For a system without Riemann invariants the forms ω^i can be normalized in such a way, that the structure equations assume the form:

$$\begin{array}{c} d\omega^1 = a\omega^1\wedge\omega^2 + b\omega^1\wedge\omega^3 + \omega^2\wedge\omega^3, \\ d\omega^2 = p\omega^2\wedge\omega^1 + q\omega^2\wedge\omega^3 + \epsilon\omega^3\wedge\omega^1, \\ d\omega^3 = r\omega^3\wedge\omega^1 + s\omega^3\wedge\omega^2 + \omega^1\wedge\omega^2, \ \text{where } \epsilon = \pm 1. \end{array} \tag{9.39}$$

Below we assume that $\epsilon = 1$, since the complex normalization

$$\omega^1 \to i\omega^1, \ \omega^2 \to \omega^2, \ \omega^3 \to i\omega^3$$

reduces the case $\epsilon = -1$ to the case $\epsilon = 1$, which allows to treat both cases on the equal footing. Since the systems under consideration are strictly hyperbolic the eigenforms ω^i constitute a basis so that the differential of any function $u(w)$ can be decomposed by ω^i: $du = u_i\omega^i$, where for brevity we use the notation $u_i = L_i(u)$. Let $u^i(w)$ be the densities of conservation laws, which provide the variables rectifying rarefaction curves. Differentiating the relations $du = u_i\omega^i$, $df = \lambda^i u_i\omega^i$ (compare with (9.2)) and equating to zero the coefficients in $\omega^i\wedge\omega^j$, one obtains

$$\begin{array}{l} u_{12} = u_2\frac{\lambda^2_1}{\lambda^1-\lambda^2} + u_1\frac{\lambda^1_2}{\lambda^2-\lambda^1} + u_1 a + u_3\frac{\lambda^3-\lambda^2}{\lambda^1-\lambda^2}, \\ u_{21} = u_2\frac{\lambda^2_1}{\lambda^1-\lambda^2} + u_1\frac{\lambda^1_2}{\lambda^2-\lambda^1} + u_2 p + u_3\frac{\lambda^3-\lambda^1}{\lambda^1-\lambda^2}, \\ u_{31} = u_3\frac{\lambda^3_1}{\lambda^1-\lambda^3} + u_1\frac{\lambda^1_3}{\lambda^3-\lambda^1} + u_3 r + u_2\frac{\lambda^1-\lambda^2}{\lambda^1-\lambda^3}, \\ u_{13} = u_3\frac{\lambda^3_1}{\lambda^1-\lambda^3} + u_1\frac{\lambda^1_3}{\lambda^3-\lambda^1} + u_1 b + u_2\frac{\lambda^3-\lambda^2}{\lambda^1-\lambda^3}, \\ u_{32} = u_3\frac{\lambda^3_2}{\lambda^2-\lambda^3} + u_2\frac{\lambda^2_3}{\lambda^3-\lambda^2} + u_3 s + u_1\frac{\lambda^1-\lambda^2}{\lambda^2-\lambda^3}, \\ u_{23} = u_3\frac{\lambda^3_2}{\lambda^2-\lambda^3} + u_2\frac{\lambda^2_3}{\lambda^3-\lambda^2} + u_2 q + u_1\frac{\lambda^1-\lambda^3}{\lambda^2-\lambda^3}, \end{array}$$

where u_{ij} are defined as $du_i = u_{ij}\omega^j$. Let $\vec{u} = (u^1, u^2, u^3)$ be functionally independent densities of three conservation laws; then $\vec{u_i}$ is the i-th right-hand eigenvector in these variables. Since the rarefaction curves are straight lines there exist scalars A, B, C such that

$$\vec{u_{11}} = A\,\vec{u_1}, \quad \vec{u_{22}} = B\,\vec{u_2}, \quad \vec{u_{33}} = C\,\vec{u_3}\ .$$

Taking into account the known expressions for $\vec{u_{ij}}$ when $i \neq j$, one obtains the following expressions for $d\,\vec{u_i}$:

$$\begin{array}{l} d\,\vec{u_1} = A\vec{u_1}\,\omega^1 + \left(\vec{u_1}\,\left(a + \frac{\lambda^1_2}{\lambda^2-\lambda^1}\right) + \frac{\vec{u_2}\,\lambda^2_1}{\lambda^1-\lambda^2} + \frac{\vec{u_3}\,(\lambda^3-\lambda^2)}{\lambda^1-\lambda^2}\right)\omega^2 \\ \quad + \left(\vec{u_1}\,\left(b + \frac{\lambda^1_3}{\lambda^3-\lambda^1}\right) + \frac{\vec{u_2}\,(\lambda^3-\lambda^2)}{\lambda^1-\lambda^3} + \frac{\vec{u_3}\,\lambda^3_1}{\lambda^1-\lambda^3}\right)\omega^3 \\ d\,\vec{u_2} = \left(\vec{u_2}\,\left(p + \frac{\lambda^2_1}{\lambda^1-\lambda^2}\right) + \frac{\vec{u_1}\,\lambda^1_2}{\lambda^2-\lambda^1} + \frac{\vec{u_3}\,(\lambda^3-\lambda^1)}{\lambda^1-\lambda^2}\right)\omega^1 + B\vec{u_2}\,\omega^2 \\ \quad + \left(\vec{u_2}\,\left(q + \frac{\lambda^2_3}{\lambda^3-\lambda^2}\right) + \frac{\vec{u_1}\,(\lambda^1-\lambda^3)}{\lambda^2-\lambda^3} + \frac{\vec{u_3}\,\lambda^3_2}{\lambda^2-\lambda^3}\right)\omega^3 \end{array} \tag{9.40}$$

$$d\,\vec{u_3}=\left(\vec{u_3}\,\left(r+\tfrac{\lambda_1^3}{\lambda^1-\lambda^3}\right)+\tfrac{\vec{u_1}\,\lambda_3^1}{\lambda^3-\lambda^1}+\tfrac{\vec{u_2}\,(\lambda^1-\lambda^2)}{\lambda^1-\lambda^3}\right)\omega^1$$
$$+\left(\vec{u_3}\,\left(s+\tfrac{\lambda_2^3}{\lambda^2-\lambda^3}\right)+\tfrac{\vec{u_1}\,(\lambda^1-\lambda^2)}{\lambda^2-\lambda^3}+\tfrac{\vec{u_2}\,\lambda_3^2}{\lambda^3-\lambda^2}\right)\omega^2+C\vec{u_3}\,\omega^3$$

where $\vec{u_i}=(u_i^1,u_i^2,u_i^3)$.

Differentiating these equations and equating to zero the coefficients in $\omega^i\wedge\omega^j$, one obtains 9 equations linear with respect to $\vec{u_j}$. Since the required u^i are functionally independent, these equations split with respect to $\vec{u_j}$. As a result one gets 27 equations for derivatives of λ^k, of the coefficients A,B,C, and of the coefficients of the structure equations a,b,p,q,r,s. The coefficients in $\vec{u_3}\,\omega^1\wedge\omega^2$ and in $\vec{u_2}\,\omega^3\wedge\omega^1$ of the differentiation of the first eaquation of (9.40) allows to find A and λ_1^1, the coefficients in $\vec{u_3}\,\omega^1\wedge\omega^2$ and in $\vec{u_1}\,\omega^2\wedge\omega^3$ of the differentiation of the second equation of (9.40) gives B and λ_2^2, and, finally, the coefficients in $\vec{u_1}\,\omega^2\wedge\omega^3$ and $\vec{u_2}\,\omega^3\wedge\omega^1$ of the differentiation of the third equation of (9.40) gives C and λ_3^3:

$$\lambda_1^1=\frac{(2\,p-2\,r)\left(\lambda^1-\lambda^3\right)\left(\lambda^2-\lambda^1\right)}{\lambda^2-\lambda^3},\tag{9.41}$$

$$\lambda_2^2=\frac{(2\,a-2\,s)\left(\lambda^2-\lambda^3\right)\left(\lambda^1-\lambda^2\right)}{\lambda^1-\lambda^3},\tag{9.42}$$

$$\lambda_3^3=\frac{(2\,q-2\,b)\left(\lambda^2-\lambda^3\right)\left(\lambda^1-\lambda^3\right)}{\lambda^1-\lambda^2},\tag{9.43}$$

$$A=\frac{(r-p)\left(\lambda^2-2\,\lambda^1+\lambda^3\right)}{\lambda^2-\lambda^3}+\frac{2\,\lambda_1^3\,\left(\lambda^2-\lambda^1\right)}{(\lambda^1-\lambda^3)\,(\lambda^2-\lambda^3)}+\frac{2\,\lambda_1^2\,\left(\lambda^3-\lambda^1\right)}{(\lambda^2-\lambda^3)\,(\lambda^2-\lambda^1)},\tag{9.44}$$

$$B=\frac{(s-a)\left(\lambda^1-2\,\lambda^2+\lambda^3\right)}{\lambda^1-\lambda^3}+\frac{2\,\lambda_2^3\,\left(\lambda^1-\lambda^2\right)}{(\lambda^1-\lambda^3)\,(\lambda^2-\lambda^3)}+\frac{2\,\lambda_2^1\,\left(\lambda^3-\lambda^2\right)}{(\lambda^1-\lambda^3)\,(\lambda^1-\lambda^2)},\tag{9.45}$$

$$C=\frac{(q-b)\left(\lambda^1-2\,\lambda^3+\lambda^2\right)}{\lambda^1-\lambda^2}+\frac{2\,\lambda_3^1\,\left(\lambda^2-\lambda^3\right)}{(\lambda^1-\lambda^3)\,(\lambda^1-\lambda^2)}+\frac{2\,\lambda_3^2\,\left(\lambda^3-\lambda^1\right)}{(\lambda^2-\lambda^3)\,(\lambda^1-\lambda^2)}.\tag{9.46}$$

For linear degenerate systems the first three of these equations imply

$$r=p,\;\; s=a,\;\; q=b.\tag{9.47}$$

For eigenvalues of linear degenerate systems and their first derivatives holds

$$\begin{array}{l} d\lambda^1=\lambda_2^1\omega^2+\lambda_3^1\omega^3\\ d\lambda^2=\lambda_1^2\omega^1+\lambda_3^2\omega^3\\ d\lambda^3=\lambda_1^3\omega^1+\lambda_2^3\omega^2 \end{array}\tag{9.48}$$

$$\begin{array}{l} d\lambda_2^1=(p\lambda_2^1-\lambda_3^1)\omega^1+\lambda_{22}^1\omega^2+\lambda_{23}^1\omega^3\\ d\lambda_3^1=(r\lambda_3^1+\lambda_2^1)\omega^1+(\lambda_{23}^1-q\lambda_2^1+s\lambda_3^1)\omega^2+\lambda_{33}^1\omega^3\\ d\lambda_1^2=\lambda_{11}^2\omega^1+(a\lambda_1^2+\lambda_3^2)\omega^2+(\lambda_{31}^2+b\lambda_1^2-r\lambda_3^2)\omega^3\\ d\lambda_3^2=\lambda_{31}^2\omega^1+(s\lambda_3^2-\lambda_1^2)\omega^2+\lambda_{33}^2\omega^3\\ d\lambda_1^3=\lambda_{11}^3\omega^1+\lambda_{12}^3\omega^2+(b\lambda_1^3-\lambda_2^3)\omega^3\\ d\lambda_2^3=(\lambda_{12}^3-a\lambda_1^3+p\lambda_2^3)\omega^1+\lambda_{22}^3\omega^2+(q\lambda_2^3+\lambda_1^3)\omega^3 \end{array}\tag{9.49}$$

Taking into account (9.47) and substituting the expressins for A,B,C into relations (9.40), one obtains a closed system with respect to $\vec{h}_1$, $\vec{h}_2$, $\vec{h}_3$. Differentiating this system and spliting with respect to $\vec{u_i}\,\omega^j\wedge\omega^k$ one gets 27 equations for the second derivatives of λ^i and first derivatives of a,b,p. These equations allow to find all second derivatives of λ^i. Morover, from these equation it followes that $a=b=p=0$.

Indeed, the coefficients in $\vec{u_1}\ \omega^1 \wedge \omega^2$, $\vec{u_2}\ \omega^1 \wedge \omega^2$, $\vec{u_1}\ \omega^2 \wedge \omega^3$, $\vec{u_2}\ \omega^2 \wedge \omega^3$, $\vec{u_3}\ \omega^3 \wedge \omega^1$ of the derivative of the first equation (9.40) define λ^3_{12}, λ^2_{11}, λ^1_{23}, λ^2_{31}, λ^3_{11} respectively. (We do not write down these intermediate expression for nonzero a, b, p.) Substituting these derivatives into the coefficients in $\vec{u_3}\ \omega^2 \wedge \omega^3$ and $\vec{u_1}\ \omega^3 \wedge \omega^1$ one gets the equations

$$\begin{aligned} pa - 5b - a_1 = 0 \\ bp + 5a - b_1 = 0 \end{aligned}$$

which determines a_1 and b_1. It is remarkable that these equations for structure coefficients do not include the eigenvalues and their derivatives.

Similarly, differentiation of the second equation (9.40) defines λ^1_{22}, λ^3_{22}, from the coefficients in $\vec{u_1}\ \omega^1 \wedge \omega^2$, $\vec{u_3}\ \omega^2 \wedge \omega^3$ respectively and gives the equations

$$\begin{aligned} &pa + 5b - p_2 = 0 \\ &ab - 5p - b_2 = 0 \\ &bp + 3a - p_3 = 0 \\ &ab - 2a_3 + b_2 - p = 0 \end{aligned}$$

for p_2, a_3, b_2, p_3 from the coefficients in $\vec{u_2}\ \omega^1 \wedge \omega^2$, $\vec{u_1}\ \omega^3 \wedge \omega^1$, $\vec{u_2}\ \omega^3 \wedge \omega^1$, $\vec{u_2}\ \omega^2 \wedge \omega^3$ respectively.

Finally, differentiation of the third equation (9.40) defines λ^2_{33}, λ^1_{33}, from the coefficients in $\vec{u_2}\ \omega^2 \wedge \omega^3$, $\vec{u_1}\ \omega^3 \wedge \omega^1$ respectively and the equations $b = 0$, $p = 0$, $a = 0$ from the coefficients in $\vec{u_3}\ \omega^1 \wedge \omega^2$, $\vec{u_3}\ \omega^2 \wedge \omega^3$, $\vec{u_3}\ \omega^3 \wedge \omega^1$ respectively.

For $a = b = p = 0$ not only system (9.40) but also the system (9.49) is in involution. The second derivatives λ^i_{jk} and λ^i_{jj} can be obtained by cyclic permutation from

$$\begin{aligned} \lambda^3_{12} = \lambda^3_1\lambda^3_2\left(\frac{1}{\lambda^3-\lambda^1} + \frac{1}{\lambda^3-\lambda^2}\right) + \lambda^3_2\lambda^2_1\left(\frac{1}{\lambda^1-\lambda^2} + \frac{1}{\lambda^2-\lambda^3}\right) - \lambda^3_1\lambda^1_2\left(\frac{1}{\lambda^3-\lambda^1} + \frac{1}{\lambda^1-\lambda^2}\right) + \\ +\lambda^2_3\left(\frac{\lambda^1-\lambda^3}{\lambda^1-\lambda^2}\right)^2 - \lambda^1_3\left(\frac{\lambda^2-\lambda^3}{\lambda^2-\lambda^1}\right)^2 \end{aligned} \tag{9.50}$$

$$\lambda^2_{11} = -2\lambda^2_1\lambda^3_1\left(\frac{1}{\lambda^2-\lambda^3} + \frac{1}{\lambda^3-\lambda^1}\right) + 2\frac{(\lambda^2_1)^2}{\lambda^2-\lambda^3} + 2\frac{(\lambda^2-\lambda^3)(\lambda^2-\lambda^1)}{\lambda^3-\lambda^1} \tag{9.51}$$

□

Differential form ω^i is proportional to the total differential of a certain function $\omega^i = p^i dR^i$ if and only if $d\omega^i \wedge \omega^i = 0$ (this is a special case of the Frobenius theorem). Recall that the function R^i is called Riemann invariant.

Theorem 25 *The eigenforms of a 3×3 T-system with one Riemann invariant can be normalized so that the structural equations take the form*

$$d\omega^1 = \epsilon\omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^3 \wedge \omega^1, \quad \omega^3 = dR^3, \text{ where } \epsilon = \pm 1. \tag{9.52}$$

Proof: Let the system have only one Riemann invariant R^3: $dR^3 = \omega^3$. Then the structure equations can be normalized as follows:

$$\begin{aligned} d\omega^1 &= a\omega^1 \wedge \omega^2 + b\omega^1 \wedge dR^3 + c\omega^2 \wedge dR^3, \\ d\omega^2 &= p\omega^2 \wedge \omega^1 + q\omega^2 \wedge dR^3 + dR^3 \wedge \omega^1. \end{aligned} \tag{9.53}$$

As in the case without Riemann invariants, differentiation $d(d\vec{f}) = 0$ allows to find $\vec{u}_{ij} = \vec{U}_{ij}(\lambda^l, \lambda^l_m, \vec{u}_n, a, b, c, p, q)$, $\quad i \neq j, \quad l, m, n = 1, 2, 3$. Now the linear degeneracy $\lambda^i_i = 0, \quad i = 1, ..., 3$ and the compatibility conditions for

$$\begin{aligned} d\vec{u}_1 &= A_1\vec{u}_1\omega^1 + \vec{U}_{12}\omega^2 + \vec{U}_{13}\omega^3, \\ d\vec{u}_2 &= \vec{U}_{21}\omega^1 + A_2\vec{u}_2\omega^2 + \vec{U}_{23}\omega^3, \\ d\vec{u}_3 &= \vec{U}_{31}\omega^1 + \vec{U}_{32}\omega^2 + A_3\vec{u}_3\omega^3 \end{aligned} \tag{9.54}$$

imply

$$\begin{gathered} a_1 = p_2, \quad q_1 = b_1, \quad q_2 = b_2, \quad a_3 - b_2 = pc + aq, \\ b_1 - p_3 = a - pb, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 2c(q-b). \end{gathered} \tag{9.55}$$

The first five formulae (9.55) are equivalent to $d(p\,\omega^1 + a\,\omega^2 + b\,dR^3) = 0$ and $d((q-b)\,dR^3) = 0$. Define the functions ϕ and ψ by

$$-\frac{d\phi}{\phi} = p\,\omega^1 + a\,\omega^2 + b\,\omega^3, \quad \frac{d\psi}{\psi} = (q-b)dR^3.$$

It is clear that ψ depends only on R^3. Renormalize the forms and introduce the new Riemann invariant $\widehat{R}^3$ as follows:

$$\widehat{\omega}^1 = \frac{\omega^1}{\phi}, \quad \widehat{\omega}^2 = \frac{\psi}{\phi}\omega^2, \quad d\widehat{R}^3 = \psi(R^3)dR^3.$$

Thus renormalized forms satisfy

$$d\widehat{\omega}^1 = \frac{c}{\psi^2}\widehat{\omega}^2 \wedge \widehat{\omega}^3, \quad d\widehat{\omega}^2 = d\widehat{R}^3 \wedge \widehat{\omega}^1.$$

The last three equations (9.55) give $d(c/\psi^2) = 0$, so c/ψ^2 is constant. One can always choose ψ to guarantee $c/\psi^2 = \pm 1$. With structure equation (9.52) system (9.54) is involutive. □

Theorem 26 *The eigenforms of a 3×3 T-system with two Riemann invariants can be normalized so that the structure equations take the form*

$$d\omega^1 = \omega^2 \wedge \omega^3, \quad \omega^2 = dR^2, \quad \omega^3 = dR^3. \tag{9.56}$$

Proof: One can normalize ω^1 so that

$$d\omega^1 = a\,\omega^1 \wedge dR^2 + b\,\omega^1 \wedge dR^3 + dR^2 \wedge dR^3. \tag{9.57}$$

Now the compatibility conditions for (9.54) with $\lambda^i_i = 0, \quad i = 1, ..., 3$ imply

$$a_1 = a_3 = b_1 = b_2 = 0,$$

which means that a is a function only of R^2 and b depends only on R^3.
Define $d\widehat{R}^2 = exp\left(\int a(R^2)dR^2\right)dR^2, \quad d\widehat{R}^3 = exp\left(\int b(R^3)dR^3\right)dR^3$,
$\widehat{\omega}^1 = exp\left(\int a(R^2)dR^2 + \int b(R^3)dR^3\right)\omega^1$. The so renormalized form and so redifined Riemann invariants satisfy $d\widehat{\omega}^1 = d\widehat{R}^2 \wedge d\widehat{R}^3$. As before, with structure equation (9.56) system (9.54) is involutive. □

Remark. Equation (1.36) has structure equations (9.38) with $\epsilon = -1$, for equation (1.37) $\epsilon = 1$. Equations (1.38) and (1.39) have structure equations (9.52) with $\epsilon = -1$ and with $\epsilon = 1$ respectively.

• Proof of Theorem 23

In the parametrization (1.16) linearity of the congruence means that the three-dimensional surface with the radius-vector $\mathbf{q} = (1, \mathbf{u}, \mathbf{f}, \mathbf{u} \wedge \mathbf{f})$ representing the congruence in the Grassmanian $G(1,4)$, where $\mathbf{u} = (u^1, u^2, u^3)$, $\mathbf{f} = (f^1, f^2, f^3)$, lies in a linear subspace of codimension 3. The osculating space of this surface is spanned by

$$\mathbf{q}_i = L_i(\mathbf{q}), \quad \mathbf{q}_{ii} = L_i^2(\mathbf{q}), \quad \mathbf{q}_{ij} = L_j L_i(\mathbf{q}).$$

Conditions $\lambda^i_i = 0$, $\mathbf{u}_{ii} = p_i \mathbf{u}_i$ and $\mathbf{f}_i = \lambda^i \mathbf{u}_i$ imply

$$\mathbf{u}_i = (0, \mathbf{u}_i, \lambda^i \mathbf{u}_i, \mathbf{u}_i \wedge (\mathbf{f} - \lambda^i \mathbf{u}))$$

and

$$\mathbf{q}_{ii} = p_i \mathbf{q}_i \equiv 0 \text{ mod}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3).$$

Using the first equation of (9.40) one has

$$\mathbf{q}_{12} = L_2(\mathbf{q}_1) =$$

$$\frac{\lambda_1^2}{\lambda^1 - \lambda^2}\mathbf{q}_2 + \lambda_1^2(0,0,\mathbf{u}_2,\mathbf{u}\wedge\mathbf{u}_2) + \frac{\lambda_2^1}{\lambda^2-\lambda^1}\mathbf{q}_1 + \frac{(\lambda^3-\lambda^2)(\lambda^1-\lambda^3)}{\lambda^1-\lambda^2}(\mathbf{q}_3 + (0,0,\mathbf{u}_3,\mathbf{u}\wedge\mathbf{u}_3))+$$

$$\lambda_2^1(0,0,\mathbf{u}_1,\mathbf{u}\wedge\mathbf{u}_1) + (\lambda^2-\lambda^1)(0,0,0,\mathbf{u}_1\wedge\mathbf{u}_2) \equiv \lambda_1^2(0,0,\mathbf{u}_2,\mathbf{u}\wedge\mathbf{u}_2)+$$

$$\frac{(\lambda^3-\lambda^2)(\lambda^1-\lambda^3)}{\lambda^1-\lambda^2}(0,0,\mathbf{u}_3,\mathbf{u}\wedge\mathbf{u}_3) + \lambda_2^1(0,0,\mathbf{u}_1,\mathbf{u}\wedge\mathbf{u}_1)+$$

$$(\lambda^2-\lambda^1)(0,0,0,\mathbf{u}_1\wedge\mathbf{u}_2) \text{ mod}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3).$$

Deenote r.h.s. by $\tilde{\mathbf{q}}_{12}$ and define $\tilde{\mathbf{q}}_{23}$ and $\tilde{\mathbf{q}}_{31}$ in a similar way. Thus the osculating space is spanned by 6 vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \tilde{\mathbf{q}}_{12}, \tilde{\mathbf{q}}_{23}, \tilde{\mathbf{q}}_{31}$. Using formula (9.50) and these ones obtained from (9.50) by cyclic permutation one gets by direct computation that $L_3(\tilde{\mathbf{q}}_{12}) \equiv 0 \text{ mod}(\tilde{\mathbf{q}}_{12}, \tilde{\mathbf{q}}_{23}, \tilde{\mathbf{q}}_{31})$. This relation along with $\mathbf{q}_{ii} \equiv 0 \text{ mod}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ implies that the osculating space is stationary so the three-dimensional surface representing the congruence lies in 6 dimensional linear subspace, which has codimension 3. For the cases with Riemann invariants the proof is essentially the same. □

9.4 Completely exeptional Monge-Ampére type equations.

Another important class of examples of T-systems is provided by completely exceptional Monge-Ampére equations studied in [30]. Equations of this type are defined as follows. Introduce the Hankel matrix

$$\begin{vmatrix} \frac{\partial^{2m}u}{\partial x^{2m}} & \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t} & \frac{\partial^{2m}u}{\partial x^{2m-2}\partial t^2} & \dots & \frac{\partial^{2m}u}{\partial x^m \partial t^m} \\ \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t} & \frac{\partial^{2m}u}{\partial x^{2m-2}\partial t^2} & \frac{\partial^{2m}u}{\partial x^{2m-3}\partial t^3} & \dots & \frac{\partial^{2m}u}{\partial x^{m-1}\partial t^{m+1}} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial^{2m}u}{\partial x^m \partial t^m} & \frac{\partial^{2m}u}{\partial x^{m-1}\partial t^{m+1}} & \frac{\partial^{2m}u}{\partial x^{m-2}\partial t^{m+2}} & \dots & \frac{\partial^{2m}u}{\partial t^{2m}} \end{vmatrix} \tag{9.58}$$

and denote by $M_{J,K}(u)$ its minor of order l whose rows and columns are encoded in the multi-indices $J = (j_1, ..., j_l)$ and $K = (k_1, ..., k_l)$, respectively. PDE's in question are defined by linear combinations of these minors:

$$\sum A^{J,K} M_{J,K} = 0, \tag{9.59}$$

where the summation is over all possible l, J, K, and $A^{J,K}$ are arbitrary constants. Any such equation can be rewritten as

$$\frac{\partial^{2m}u}{\partial t^{2m}} = f\left(\frac{\partial^{2m}u}{\partial x^{2m}}, \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t}, ..., \frac{\partial^{2m}u}{\partial x^1 \partial t^{2m-1}}\right),$$

and after the introduction of $a^1 = \frac{\partial^{2m}u}{\partial x^{2m}}$, $a^2 = \frac{\partial^{2m}u}{\partial x^{2m-1}\partial t}$, ..., $a^{2m} = \frac{\partial^{2m}u}{\partial x^1\partial t^{2m-1}}$, assumes the conservative form

$$a^1_t = a^2_x, \quad a^2_t = a^3_x, \quad ..., \quad a^{2m}_t = f(a^1, a^2, ..., a^{2m})_x. \tag{9.60}$$

One can show that this is always a T-system (in fact, its linear degeneracy was demonstrated in [30]), and the corresponding congruence (1.16) has the following properties:

– its developable surfaces are planar pencils of lines,
– its focal variety has codimension at least 2,
– each developable surface intersects the focal variety in a point, which is the vertex of the pencil, and a plane curve of degree $n-1$.

To obtain systems of this type for odd n, one should consider equations (9.59) which are independent of $\frac{\partial^{2m}u}{\partial t^{2m}}$. Introducing $v = \frac{\partial u}{\partial x}$ and rewriting the resulting equation for v (which is of order $2m-1$) as a system of conservation laws, one arrives at congruence (1.16) with the properties as formulated above. When $n \geq 4$ these congruences are not necessarily linear. In this case the focal varieties will necessarily be singular, as follows from [70].

9.5 Reducible systems and linear complexes

Let us consider a PDE of the form

$$\frac{\partial^n u}{\partial t^n} = f\left(\frac{\partial^n u}{\partial x^n}, \frac{\partial^n u}{\partial x^{n-1}\partial t}, ..., \frac{\partial^n u}{\partial x \partial t^{n-1}}\right).$$

Introducing the variables $a^1 = \frac{\partial^n u}{\partial x^n},\ a^2 = \frac{\partial^n u}{\partial x^{n-1}\partial t},\ ..., a^n = \frac{\partial^n u}{\partial x \partial t^{n-1}}$, we can rewrite it as a system of conservation laws

$$a^1_t = a^2_x, \quad a^2_t = a^3_x,\ ...,\ a^n_t = f(a^1, a^2, ..., a^n)_x. \tag{9.61}$$

Definition 20 *A system of conservation laws is said to be reducible if it can be cast into the form (9.61) by an appropriate reciprocal transformation (1.24) together with an affine change of dependent variables.*

Notice that all examples discussed so far are reducible by construction. There exists a simple geometric criterion for a system of conservation laws to be reducible.

Definition 21 *A linear complex in $\mathbb{P}^{n+1}$ is a family of lines, whose Plücker coordinates P^{ij} are subject to a linear constraint $A_{ji}P^{ij} = 0$, where $A_{ij} = -A_{ji} = const$.*

Recall that to the line in $\mathbb{P}^{n+1}$ passing through the points with homogeneous coordinates $X = [X^0 : X^1 : ... : X^{n+1}]$ and $Y = [Y^0 : Y^1 : ... : Y^{n+1}]$ there corresponds a point in the Grassmanian $\mathbb{G}(1, n+1)$ with Plücker coordinates $P^{ij} = X^iY^j - X^jY^i,\ \ i,j = 0, ..., n+1$. If one considers Plücker coordinates as an $(n+2)\times(n+2)$ skew-symmetric matrix P of rank 2, any linear constraint can be rewritten in the form $\mathrm{tr}AP = 0$, where A is a skew-symmetric matrix. Intersection of $n-1$ linear complexes is given by $n-1$ linear equations

$$\mathrm{tr}A^\alpha P = 0,\ \ \alpha = 1, ..., n-1, \tag{9.62}$$

where the matrices A^α are linearly independent.

Remark. Linear congruence is determined by n such equations. The focal variety of linear congruence is the determinantal variety

$$M = \{X \in \mathbb{P}^{n+1} : \mathrm{rk}\{A^\alpha_{ij}X^i\} < n\}, \tag{9.63}$$

$\alpha = 0, ..., n-1,\ i,j = 0, ..., n+1$. The lines of the congruence are n-secants of the focal variety M.

Define the map $A(\mu)$ by

$$\mathbb{CP}^{n-2} \ni (\mu_1 : \mu_2 : ...\mu_{n-1}) \to A(\mu) = \sum_\alpha \mu_\alpha A^\alpha.$$

Theorem 27 *System of conservation laws (1.15) is reducible iff the corresponding congruence (1.16) lies in the intersection of n-1 linear complexes (9.62) such that*
1) $\mathrm{rank} A(\mu) = 4$ *for all* $\mu \in \mathbb{CP}^{n-2}$*,*
2) there exists an n*-dimensional linear subspace* $L \subset V^{n+2}$*, which is Lagrangian with respect to all skew-symmetric scalar products* $\{X, Y\}_\mu := X^T A(\mu) Y$*.*

Recall that the subspace of a linear space with a skew-symmetric scalar product $\{,\}$ is called Lagrangian if $\{X, Y\} = 0$ for all $X, Y \in L$. In parametrization (1.16), the Plücker coordinates of a congruence in $\mathbb{P}^{n+1}$ are

$$1, \quad u, \quad f, \quad u \wedge f,$$

so that the corresponding matrix P is

$$P = \begin{pmatrix} 0 & 1 & u^1 & \dots & u^n \\ -1 & 0 & f^1 & \dots & f^n \\ -u^1 & -f^1 & & & \\ \vdots & \vdots & & u^i f^j - u^j f^i & \\ -u^n & -f^n & & & \end{pmatrix}.$$

Equations (9.61) imply that the basis of matrices A^α can be choosen in the form

$$A^\alpha = \begin{pmatrix} 0 & D^\alpha \\ -(D^\alpha)^T & 0 \end{pmatrix}$$

where D^α are $2 \times n$ matrices with only two nonzero entries:

$$D^1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad ..., \quad D^{n-1} = \begin{pmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & -1 & 0 \end{pmatrix}.$$

Notice that the condition $\mathrm{tr} A^\alpha P = 0$ is equivalent to $f^\alpha = u^{\alpha+1}$. It is easy to see that both conditions 1) and 2) are fulfilled. To prove that these conditions are also sufficient is a bit more difficult. Let us restrict our consideration to the cases $n = 2$, $n = 3$ and give a full proof of the theorem in the end of this section.

In the case $n = 2$ conditions 1) and 2) imply that the 4×4 matrix A determining the linear complex in question, is nondegenerate. Any such matrix can be transformed into the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

with the corresponding congruence in $\mathbb{P}^3$

$$Y^1 = aY^0 - bY^{n+1}, \quad Y^2 = bY^0 - f(a, b)Y^{n+1},$$

giving rise to a reducible 2-component system.

Now let the congruence in $\mathbb{P}^4$ associated with a 3-component system lie in the intersection of 2 general linear complexes. If the matrix $A(\mu)$ has rank 4 for any $\mu \in \mathbb{CP}^2$, the kernel $\xi(\mu)$ of $\{,\}_\mu$ is one-dimensional. So the map $\xi(\mu) : \mathbb{CP}^1 \ni \mu \to \mathbb{P}^4$ is correctly defined, the image of $\mathbb{P}^1$ being a plane conic (see [37]). Condition 1) guarantees that this conic is nondegenerate. As follows from the results of Castelnuovo [37], the intersection of 2 general linear complexes in P^3 is projectively equivalent to

$$y^1 = ay^0 - b, \quad y^2 = by^0 - c, \quad y^3 = cy^0 - f. \tag{9.64}$$

Congruences belonging to the complex (9.64) are specified by one extra relation of the form $f = f(a, b, c)$, thus giving rise to reducible systems.

Remark. The Lagrangian subspace L in Theorem 27 can be described constructively as a linear span of $\xi(\mu)$, the $\xi(\mu) \in V^{n+2}$ being the (n-2)-dimensional kernel of $\{X, Y\}_\mu$, so that the criterion given by Theorem 27 is effective. One has to construct the linear span L of $\xi(\mu)$ and verify two conditions: $\mathrm{dim}L = n$ and $\{X, Y\}_\mu = 0$ for all $X, Y \in L,\ \mu \in \mathbb{CP}^{n-2}$.

We emphasize that for $n = 2, 3$ condition 2) follows from condition 1).
We need the following Lemma to prove Theorem 27.

Lemma 12 *Let $B(\mu)$ be $2 \times m$ matrix, whose entries are linear forms in $[\mu_1 : \mu_2 : ... : \mu_m] \in \mathbb{CP}^{m-1}$, i.e., $b_{ij}(\mu) = b^k_{ij}\mu_k$. Then there exists $\tilde{\mu} \in \mathbb{CP}^{m-1}$ such that* $\mathrm{rank}B(\tilde{\mu}) = 1$.

Proof: Equation $\det|\nu_1 b^i_{1k} - \nu_2 b^i_{2k}| = 0$ has at least one root $[\tilde{\nu}_1 : \tilde{\nu}_2] \in \mathbb{CP}^1$. Therefore, the system of linear equations $\sum_i (\tilde{\nu}_1 b^i_{1k} - \tilde{\nu}_2 b^i_{2k})\mu_i = 0$ has a nontrivial solution $\tilde{\mu} = (\mu_1(\tilde{\nu}), ..., \mu_m(\tilde{\nu}))$.
□

• Proof of Theorem 27

We only need to prove that conditions 1) and 2) are sufficient (the necessity follows from the discussion above. Let $L \subset V^{n+2}$ be an n-dimensional subspace which is Lagrangian for all $\{,\}_\mu$. Choose a basis $e_1, e_2, ..., e_{n+2}$ in V^{n+2} such that the last n vectors $e_3, ..., e_n$ constitute a basis of L. In this basis,

$$A^\alpha = \begin{pmatrix} C^\alpha & B^\alpha \\ -(B^\alpha)^T & 0 \end{pmatrix} \tag{9.65}$$

where B^α is a $2 \times n$ matrix and C^α is a skew-symmetric 2×2 matrix.

Consider $n - 1$ linear equations for $\xi \in L$:

$$\{e_1, \xi\}_\alpha = 0, \tag{9.66}$$

where $\{,\}_\alpha$ is the skew-symmetric scalar product defined by A^α. There exists a nonzero solution of this system. Choose this solution to be the basis vector e_3. In this basis $A^\alpha_{1,3} = 0$. There must exist α for which $A^\alpha_{2,3} \neq 0$. Otherwise Lemma 12 implies that there is μ such that $\mathrm{rank}B(\mu) = 1$, which means $\mathrm{rank}A(\mu) = 2$. Choose the matrix A^α for which $A^\alpha_{2,3} \neq 0$ to be A^1. Normalize A^1 so that $A^1_{2,3} = -1$ and for $\alpha = 2, ..., n-1$ replace A^α by $A^\alpha + A^\alpha_{2,3}A^1$. Now all matrices A^α with $\alpha = 2, ..., n-1$ have zero first column in B^α. Applying the same procedure to the linear span of $\{e_4, ..., e_{n+2}\}$ with $\alpha = 2, ..., n-1$, one ends up with B^2 of the form

$$\begin{pmatrix} 0 & 0 & ... \\ 0 & -1 & ... \end{pmatrix}$$

and A^α for $\alpha = 3, ..., n-1$ having two first zero columns in B^α. After n-1 steps, matrices A^α will have the following form: $\alpha - 1$ first columns of B^α are zero and the αth column is $(0, -1)^T$. In particular, B^{n-1} is of the form

$$\begin{pmatrix} 0 & ... & 0 & 0 & b^{n-1} \\ 0 & ... & 0 & -1 & c \end{pmatrix}$$

with $b^{n-1} \neq 0$. Note that up to now only $e_3, ..., e_{n+1}$ were fixed. Replacing e_{n+2} by $e_{n+2} + ce_{n+1}$, one gets $\tilde{c} = 0$ and $\tilde{b}_{n-1} = b_{n-1} \neq 0$.

After replacing A^{n-2} by $A^{n-2} - \frac{A^{n-2}_{1,n+2}}{b_{n-1}}A^{n-1}$, it takes the form

$$\begin{pmatrix} 0 & ... & 0 & 0 & b^{n-2} & 0 \\ 0 & ... & 0 & -1 & c & r \end{pmatrix}$$

Replacing $e_{n+2} \to e_{n+2} + re_n$, $e_{n+1} \to e_{n+1} + ce_n$ does not change B^{n-1} and kills c and r. Repeating this "backward" procedure one transforms all matrices A^α to the following form: all columns of B^α are zero except for αth, which is $(0,-1)^T$, and $(\alpha+1)$st, which is $(b^\alpha, 0)^T$ with $b^\alpha \neq 0$.

The above transformations of V^{n+2} induce reciprocal transformations of the associated system of conservation laws. As a result, the r.h.s. of the system assume the form

$$f^1 = b^1 u^2 + c^1, \quad f^2 = b^2 u^3 + c^2, ..., f^{n-1} = b^{n-1} u^{n-2} + c^{n-1},$$

where the constants c^α are the nonzero elements of the corresponding 2×2 matrices C^α in (9.65). Finally, the renormalization $u^i \to d^i u^i$ with $d^i = \prod_i^{n-1} b^k$ completes the proof, since the constants c^k do not effect (1.15). $\square$

Chapter 10

Integrable systems of four conservation laws and isoparametric hypersurfaces

10.1 Isoparametric hypersurfaces and linear congruences in $\mathbb{P}^5$

In this chapter we describe an explicit geometric construction of the linear congruence corresponding to a four-component T-system without Riemann invariants and harmonic cross-ratio of characteristic speeds in more detail. It is associated with the Cartan isoparametric hypersurface $M^4 \subset \mathbb{S}^5 \subset \mathbb{E}^6$, which can be represented as the intersection of the unit sphere

$$(U^1)^2 + (U^2)^2 + (U^3)^2 + (U^4)^2 + (U^5)^2 + (U^6)^2 = 1 \tag{10.1}$$

with the zero level $P = 0$ of the fourth order polynomial

$$\begin{aligned} P(U) = -((U^1)^2 + (U^2)^2 + (U^3)^2 + (U^4)^2 + (U^5)^2 + (U^6)^2)^2 + \\ 2((U^1)^2 + (U^2)^2 + (U^3)^2 - (U^4)^2 - (U^5)^2 - (U^6)^2)^2 + 8(U^1U^4 + U^2U^5 + U^3U^6)^2. \end{aligned} \tag{10.2}$$

Since M^4 is a non-singular 4-dimensional hypersurface, with each point $m \in M^4$ one can associate a unique great circle $\mathbb{S}^1(m)$ in $\mathbb{S}^5$ which is orthogonal to M^4, so that a 4-parameter family of such circles is obtained. Each great normal circle intersects the "focal" surfaces $M_\pm \subset \mathbb{S}^5$, determined by the equations

$$P(U) = \pm 1, \tag{10.3}$$

at four points forming a harmonic quadruplet on $\mathbb{S}^1(M)$. The focal surfaces $M_\pm \subset \mathbb{S}^5$ are 3-dimensional, M_- being the Stiefel manifold

$$|p|^2 = |q|^2, \quad (p, q) = 0, \tag{10.4}$$

here $p = (U^1, U^2, U^3)^T$, $q = (U^4, U^5, U^6)^T$, whereas M_+ is the cubic scroll

$$p \wedge q = 0. \tag{10.5}$$

Regarding U^i, $i = 1, ..., 6$, as homogeneous coordinates in $\mathbb{P}^5$, one arrives at a 4-parameter family of lines $l(m) \subset \mathbb{P}^5$, each line being defined by the 2-dimensional plane of the great circle

$\mathbb{S}^1(m)$. Moreover, equations (10.4) and (10.5), without the original restriction $|p|^2 + |q|^2 = 1$, specify two components of the focal variety F^3 of the congruence $\{l(m)|m \in M^4\}$, the cross-ratios of intersection points $\{l(m) \cap F^3\}$ being equal to -1. Since the focal surface F^3 is 3-dimensional, the corresponding system (1.15) is linearly degenerate. In the affine chart $U^6 \neq 0$ with coordinates $y^i = U^i/U^6$, $i = 1, ..., 5$, this congruence can be parametrized by the parameters a, b, c, d as follows:

$$\begin{aligned} y^1 &= ay^3 + c, \\ y^2 &= by^3 + d, \\ y^4 &= cy^3 + \tfrac{abd+c^3+cd^2-cb^2-c}{ac+bd}, \\ y^5 &= dy^3 + \tfrac{bac+d^3+dc^2-da^2-d}{ac+bd}. \end{aligned} \tag{10.6}$$

One can check by direct computation that the congruence under consideration is linear. In the coordinates $X^0 = U^1,\ X^1 = U^2,\ X^2 = U^3,\ X^3 = U^4,\ X^4 = U^5,\ X^5 = U^6$ the matrices A^α have block forms,

$$A^0 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},\ A^i = \begin{pmatrix} G^i & 0 \\ 0 & G^i \end{pmatrix},\ i = 1, 2, 3, \tag{10.7}$$

where $E = \mathrm{diag}\{1, 1, 1\}$, and the matrices G^i are defined as follows:

$$G^1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ G^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ G^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since the congruence (10.6) is linear, the corresponding system (1.15) has rectilinear rarefaction curves. Its focal variety does not content linear subspaces of codimension 2, therefore, the system does not possess Riemann invariants.

Developable surfaces of the congruence (which are planar pencils of lines) intersects the focal variety at a point, which is the vertex of the pencil, and a reducible plane cubic, which is a union of a straight line and a conic. The line and the point are conjugate with respect to the conic and lie on the same component of the focal variety.

Remark. Condition (1.42) is easily verified: if the characteristic polynomial of (1.15) is

$$a_4(u)\lambda^4 + 4a_3(u)\lambda^3 + 6a_2(u)\lambda^2 + 4a_1(u)\lambda + a_0(u),$$

then (1.42) is equivalent to

$$\det \begin{vmatrix} a_4 & a_3 & a_2 \\ a_3 & a_2 & a_1 \\ a_2 & a_1 & a_0 \end{vmatrix} = 0, \tag{10.8}$$

see, e.g., [80].

Remark. Consider a pencil of lines and a cubic curve in $\mathbb{P}^2$. Each line cuts the cubic at 3 points. If the vertex of the pencil and these 3 points form an harmonic quadruplet for each line of the pencil, then the cubic must necessarily be reducible (a union of a straight line and a conic), moreover, the line and the vertex are conjugate with respect to the conic. This fact seems to be classical, however, we are unable to provide the reader with an exact reference.

Proposition 15 *The system (1.15) corresponding to the congruence (10.6) is integrable.*

Proof: In the homogeneous coordinates U^i, the lines $\{l(m) :\ m \in M^4\}$ of the congruence can be parametrized by s_1, s_2 as follows:

$$\vec{U} = \vec{\mathbf{r}}(m)s_1 + \vec{\mathbf{n}}(m)s_2,$$

where $\vec{\mathbf{r}}(m)$ is the position vector of the Cartan isoparametric hypersurface $M^4 \subset \mathbb{S}^5 \subset \mathbb{E}^6$, and $\vec{\mathbf{n}}(m) \in T_m\mathbb{S}^5$ is the unit normal. The reparametrization of this congruence in terms of U^3, U^6 shows that the corresponding system (1.15) is a reciprocal transform of the following system on M^4 (see [48]),

$$\vec{\mathbf{r}}_t = \vec{\mathbf{n}}_x, \tag{10.9}$$

which has the exterior representation

$$\begin{aligned} \Omega^1 \wedge (dx + \lambda^1 dt) &= 0, \\ \Omega^2 \wedge (dx + \lambda^2 dt) &= 0, \\ (\Omega^3 + \phi) \wedge (dx + \lambda^3 dt) &= 0, \\ (\Omega^1 - \phi) \wedge (dx + \lambda^4 dt) &= 0; \end{aligned} \tag{10.10}$$

here Ω^i satisfy the $SO(3)$ Maurer-Cartan equations, $d\phi = 0$ and all λ^i are constant. This system is equivalent to the system describing a resonant 4-wave interaction and, therefore, is integrable (see [48] for details). □

Remark. General linear congruences in $\mathbb{P}^4$ are obtained from the Cartan isoparametric hypersurface $M^3 \subset \mathbb{S}^4 \subset \mathbb{E}^5$ by similar construction. The essential difference is that the "focal" submanifolds $M_\pm$ are antipodal in $\mathbb{S}^4$ so that the focal variety of the congruence is irreducible.

Remark. The proof of Proposition 15 implies that the system (1.15) corresponding to the congruence (10.6) has the exterior representation (10.10), however, λ^i are no longer constant.

Introducing potentials u, v by

$$u_{xx} = a, \ u_{xt} = -c, \ v_{xx} = b, \ v_{xt} = -d,$$

one can rewrite this hydrodynamic type system as a pair of two second order equations

$$u_{xt} = -\det \begin{pmatrix} u_{tt} & u_{xt} & v_{xt} \\ u_{xt} & u_{xx} & v_{xx} \\ v_{xx} & -v_{xt} & u_{xt} \end{pmatrix}, \quad v_{xt} = -\det \begin{pmatrix} v_{tt} & v_{xt} & u_{xt} \\ v_{xt} & v_{xx} & u_{xx} \\ u_{xx} & -u_{xt} & v_{xt} \end{pmatrix}.$$

Recall that three-component T-systems can be cast into the form

$$a_t = b_x, \quad b_t = c_x, \quad c_t = f(a, b, c)_x \tag{10.11}$$

by an appropriate reciprocal transformation (1.24) combined with an affine change of dependent variables. System (9.61) can be rewritten as a single third order PDE

$$\frac{\partial^3 u}{\partial t^3} = f\left(\frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial x^2 \partial t}, \frac{\partial^3 u}{\partial x \partial t^2}\right)$$

after the substitution $a = \frac{\partial^3 u}{\partial x^3}, \ b = \frac{\partial^3 u}{\partial x^2 \partial t}, c = \frac{\partial^3 u}{\partial x \partial t^2}$. According to Theorem 27 the analogous change of variables for the four-component system (1.15) corresponding to the congruence (10.6) does not exist.

10.2 Symmetry properties

Consider a linear congruence in $\mathbb{P}^{n+1}$ specified by a collection of n skew-symmetric matrices A^α as in (9.62). By abuse of notation we use the same symbol A also for the following map:

$$\mathbb{P}^{n-1} \ni (\mu_0 : \mu_1 : ...\mu_{n-1}) \to A(\mu) = \sum_\alpha \mu_\alpha A^\alpha.$$

For even n, to each μ such that

$$\mathrm{Pf}(A(\mu)) \equiv \sqrt{\mathrm{det} A(\mu)} = 0 \tag{10.12}$$

there corresponds a line $l(\mu) \in M \in \mathbb{P}^{n+1}$ that belongs to the kernel of $A(\mu)$. Conversely, there exists a map $f : X \in M \to \mu \in V_A$, where $V_A \in \mathbb{P}^{n-1}$ is defined by (10.12). Hence, this map defines the structure of a $\mathbb{P}^1$-bundle over V_A on the jump locus (9.63).

For any $G \in GL(n+2, R)$, the Pfaffian of $A(\mu)$ transforms as

$$\mathrm{Pf}(G^T A(\mu) G) = |\mathrm{det} G| \mathrm{Pf}(A(\mu)).$$

Hence, all projectively equivalent congruences have the same variety $V_A \subset \mathbb{P}^{n-1}$ defined by (10.12). Notice that the matrices A^α chosen as in (10.7) form a basis of the Lie algebra $so(3) \times so(2)$ represented in $so(6)$:

$$[A^0, A^i] = 0,\ i = 1,2,3,\ \ [A^1, A^2] = A^3,\ [A^2, A^3] = A^1,\ [A^3, A^1] = A^2.$$

Therefore, the corresponding Lie group $SO(3) \times SO(2) \subset SO(6)$ leaves the congruence under consideration invariant:

$$G^T A(\mu) G^T = \mu_\alpha G^{-1} A^\alpha G = \mu_\alpha C^\alpha_\beta A^\beta = \tilde{\mu}_\beta A^\beta = A(\tilde{\mu}),$$

where C^α_β is the adjoint representation of $SO(3) \times SO(2)$, and

$$\tilde{\mu}_\beta = \mu_\alpha C^\alpha_\beta. \tag{10.13}$$

On the other hand,

$$\mathrm{Pf}(G^T A(\mu) G) = |\mathrm{det} G| \mathrm{Pf}(A(\mu)) = \mathrm{Pf}(A(\mu)) = \mathrm{Pf}(A(\tilde{\mu})),$$

so that (10.13) gives a symmetry of V_A. Since the kernel of the adjoint representation is $SO(2)$, this symmetry group is $SO(3)$. The Pfaffian of the congruence is factorized as

$$\mathrm{Pf}(A(\mu)) = \mu_0(\mu_0^2 - \mu_1^2 - \mu_2^2 - \mu_3^2)$$

so that the cubic surface V_A degenerates into a union of the plane

$$\mu_0 = 0 \tag{10.14}$$

and the quadric

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = \mu_0^2. \tag{10.15}$$

The cubic scroll (10.5) and the intersection of quadrics (10.4) are $\mathbb{P}^1$-bundles over the plane (10.14) and the quadric (10.15), respectively. Considering (10.14) as an infinite plane in $\mathbb{P}^3$, one can look at transformations (10.13) as the $SO(3)$ symmetry of the quadric (10.15).

Remark. The proposed interpretation of the $SO(3) \times SO(2)$ symmetry of the congruence as the orthogonal group, is only valid under a special choice of the basis (10.7). The general projective transformation destroys it, but retains the symmetry group. Therefore, any congruence projectively equivalent to (10.6) has a symmetry group isomorphic to $SO(3) \times SO(2)$. In terms of the system (1.15), this symmetry is interpreted as an *autoreciprocal* transformation, i.e., as a reciprocal transformation which, after being combined with a local change of field variables, leaves the equation (1.15) invariant.

Remark. This symmetry can also be read off the focal varieties (10.4) and (10.5). Indeed, if one represents the coordinates U in the form of a 3×2 matrix $r = (p, q)$, then (10.4) and (10.5) become manifestly invariant under the linear transformations

$$r \to g_1 r g_2, \quad g_1 \in SO(3), \quad g_2 \in SO(2) \tag{10.16}$$

Here g_1 is a simultaneous rotation of p and q, A^i being infinitesimal generators of such rotations. The transformation g_2 can be interpreted as a rotation in the 2-dimensional plane spanned by p and q, represented by the matrix A^0 in the basis (10.7).

10.3 Isoparametric hypersurfaces in a pseudoeuclidean space and linear congruences in $\mathbb{P}^5$

The congruence (10.6) has a pseudoeuclidean counterpart. One can start with the same focal varieties (10.4) and (10.5), where now (p, q) is a scalar product of the signature (2,1) defined by the matrix

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and consider a linear congruence formed by its four-secants. The basis of linear constraints A^α can be chosen as

$$A^0 = \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix} \tag{10.17}$$

and A^1, A^2, A^3 as in (10.7). For this congruence the Pfaffian reads:

$$\mathrm{Pf}(A(\mu)) = \mu_0(\mu_0^2 + 2\mu_1\mu_3 + \mu_2^2).$$

In the affine chart $U^2 \neq 0$ with coordinates $y^i = U^i/U^2,\ i = 1, 3, 4, 5, 6$, this congruence is parametrized by a, b, c, d as follows:

$$\begin{aligned} y^1 &= ay^5 + \tfrac{2a^2b-acd-a+bc^2}{ad+bc}, \\ y^3 &= by^5 + \tfrac{2ab^2-bcd-b+ad^2}{ad+bc}, \\ y^4 &= cy^5 + a, \\ y^5 &= dy^5 + b. \end{aligned} \tag{10.18}$$

Remark. This congruence has a symmetry group isomorphic to $SO(2,1)\times SO(2)$, which is also the symmetry group of the focal variety. This symmetry is interpreted similarly to (10.16), with the only difference that $g_1 \in SO(2,1)$, whereas $g_2 \in SO(2)$ leaves (10.4) invariant regardless of the signature.

With the potentials u, v defined by

$$u_{xx} = -c,\ u_{xt} = a,\ v_{xx} = -d,\ v_{xt} = b,$$

the corresponding system (1.15) takes the form:

$$u_{xt} = \det\begin{pmatrix} u_{tt} & u_{xt} & v_{xt} \\ u_{xt} & u_{xx} & v_{xx} \\ -u_{xx} & u_{xt} & -v_{xt} \end{pmatrix}, \quad v_{xt} = \det\begin{pmatrix} v_{tt} & v_{xt} & u_{xt} \\ v_{xt} & v_{xx} & u_{xx} \\ -v_{xx} & v_{xt} & -u_{xt} \end{pmatrix}.$$

10.4 Classification

In this section we formulate the main result of classification of four-component non-diagonalizable T-systems having harmonic cross-ratio of characteristic speeds. This result is differential-geometric, as we do not assume the linearity of the corresponding congruence.

Theorem 28 *Let a congruence in $\mathbb{P}^5$ has the following properties:*

- *its Plücker image in $\mathbb{G}(1,5)$ is connected,*
- *its focal variety has codimension 2 and does not contain linear subspaces of codimension 2,*

- *its developable surfaces are planar pencils of lines, transversal to the focal varieties,*
- *each line cuts the focal surface at four points forming an harmonic quadruplet, each pair of these points not coinciding identically.*

Then this congruence is projectively equivalent over the reals to either (10.6) or (10.18).

The most difficult part of the theorem is the following lemma.

Lemma 13 *If (1.15) is the system corresponding to a congruence with the properties as in Theorem 28, then its eigenforms can be normalized so that the exterior representation takes the form (10.10) where Ω^i satisfy the Maurer-Cartan equations of the $SO(3)$ or $SO(2,1)$ groups.*

Proof: The proof requires a long computation bringing a certain exterior differential system into involutive form. To derive and analyze these equations we used computer algebra system MAPLE 7. We do not present all intermediate formulas as they are extremely awkward. Let us only sketch the proof and final formulas.

Since the system under study does not possess Riemann invariants, at least one of the coefficients $\tilde{c}^i_{jk}$ with $j,k \neq i$ in (9.37) does not vanish for each $i = 1, ..., 4$. By an appropriate normalization $\omega^i \to r^i\omega^i,\ r^i \neq 0$, one can make them constant. (We allow r^i to be complex to treat all cases on equal footing). Thus, there are $3^4 = 81$ possibilities to consider. As the equation (1.42) is invariant under the permutations $\{(1,2,3,4), (2,1,3,4), (1,2,4,3), (2,1,4,3), (3,4,1,2), (4,3,2,1), (4,3,1,2), (3,4,2,1)\}$, only 15 of these possibilities are essentially different. They are presented in the following table, where i, j, k are the indices of non-zero $\tilde{c}^i_{jk}$.

Different cases															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
i	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k	j,k
1	2,3	2,3	3,4	3,4	3,4	2,3	2,3	3,4	2,3	2,3	2,3	2,3	2,3	3,4	2,3
2	4,1	1,3	3,1	3,4	3,1	1,3	4,1	3,1	4,1	1,3	1,3	4,1	4,1	3,1	4,1
3	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	1,2	4,1	4,1	4,1	4,1	2,4	2,4
4	1,2	1,2	1,2	1,2	3,1	3,1	3,1	3,2	3,2	1,2	1,3	1,3	2,3	1,2	3,1

The eigenforms ω^i constitute a basis, therefore the differential of any function u can be decomposed as $du = u_i\omega^i$, where, for brevity, we use the notation $u_i = L_i(u)$. Let $\vec{u} = (u^1, u^2, u^3, u^4)$ be the densities of conservation laws for which rarefaction curves are linear. Then there exist scalars p_i such that

$$\vec{u}_{ii} = p_i\, \vec{u}_i.$$

Differentiating the relations $du^j = u^j_i\omega^i$, $df^j = \lambda^i u^j_i\omega^i$ and equating to zero coefficients at $\omega^i \wedge \omega^j$, one obtains all mixed second derivatives $u^j_{ik},\ i \neq k$. Thus, one has

$$d\,\vec{u}_i = \vec{U}_{ij}\,(\lambda^k, \lambda^k_l, p_n, \tilde{c}^s_{qr}, \vec{u}_m)\omega^j. \tag{10.19}$$

Note that F_{ij} are linear in $\vec{u}_m$. Differentiating these equations and equating to zero coefficients at $\omega^i \wedge \omega^j$, one obtains $4 \times C^2_4 = 24$ equations which are linear with respect to $\vec{u}_j$. Since u^i are functionally independent, these equations split with respect to $\vec{u}_j$. As a result, one gets $24 \times 4 = 96$ differential equations for λ^k, p_i and the structure coefficients $\tilde{c}^s_{qr}$. The analysis of these equations along with (1.42), the linear degeneracy conditions

$$\lambda^i_i = 0, \quad i = 1, ..., 4, \tag{10.20}$$

and the equations $d(d(\omega^i)) = 0$, $d(d(\lambda^i)) = 0$, shows that the structure equations for the system under study are

$$\begin{aligned} d\omega^1 &= \tfrac{1}{2}\omega^2 \wedge \omega^3 + \tfrac{1}{2}\omega^2 \wedge \omega^4, \\ d\omega^2 &= \tfrac{1}{2}\omega^3 \wedge \omega^1 + \tfrac{1}{2}\omega^4 \wedge \omega^1, \\ d\omega^3 &= \omega^1 \wedge \omega^2, \quad d\omega^4 = \omega^1 \wedge \omega^2. \end{aligned} \tag{10.21}$$

These equations imply that $\Omega^1 = \omega^1$, $\Omega^2 = \omega^2$, $\Omega^3 = (\omega^3 + \omega^4)/2$ satisfy the structure equations of the $SO(3)$-group, and $d\phi = d(\omega^3 - \omega^4)/2 = 0$. The complex normalization

$$\omega^1 = i\Omega^1, \quad \omega^2 = \Omega^2, \quad \omega^3 = i\Omega^3 + \phi, \quad \omega^4 = i\Omega^4 - \phi$$

gives the structure equations

$$\begin{aligned} d\omega^1 &= \tfrac{1}{2}\omega^2 \wedge \omega^3 + \tfrac{1}{2}\omega^2 \wedge \omega^4, \\ d\omega^2 &= -\tfrac{1}{2}\omega^3 \wedge \omega^1 - \tfrac{1}{2}\omega^4 \wedge \omega^1, \\ d\omega^3 &= \omega^1 \wedge \omega^2, \quad d\omega^4 = \omega^1 \wedge \omega^2 \end{aligned} \tag{10.22}$$

for the pseudo-Euclidean counterpart. □

Remark. Given the structure equations (10.21), one can compute p_i,

$$\begin{aligned} p_1 &= \tfrac{2\lambda^2_1(\lambda^3-\lambda^1)}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)} + \tfrac{2\lambda^3_1(\lambda^2-\lambda^1)}{(\lambda^3-\lambda^1)(\lambda^3-\lambda^2)}, \\ p_2 &= \tfrac{2\lambda^1_2(\lambda^3-\lambda^2)}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)} + \tfrac{2\lambda^3_2(\lambda^1-\lambda^2)}{(\lambda^3-\lambda^1)(\lambda^3-\lambda^2)}, \\ p_3 &= \tfrac{2\lambda^1_3(\lambda^2-\lambda^3)}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)} + \tfrac{2\lambda^2_3(\lambda^1-\lambda^3)}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}, \\ p_4 &= \tfrac{2\lambda^2_4(\lambda^1-\lambda^3)}{(\lambda^2-\lambda^3)(\lambda^1-\lambda^2)} + \tfrac{2\lambda^1_4(\lambda^2-\lambda^3)}{(\lambda^1-\lambda^2)(\lambda^3-\lambda^1)}, \end{aligned}$$

as well as all second derivatives of λ^1, λ^2 and λ^3, as functions $L^i_{jk}(\lambda^m, \lambda^n_l)$ of λ^1, λ^2, λ^3 and first derivatives thereof. These satisfy the above mentioned 96 equations along with $d(d(\lambda^i)) = 0$, equations (10.20), (1.42) and their differentials. Moreover, the system

$$\begin{aligned} d\lambda^1 &= \lambda^1_2\omega^2 + \lambda^1_3\omega^3 + \lambda^1_4\omega^4 \\ d\lambda^2 &= \lambda^2_1\omega^1 + \lambda^2_3\omega^3 + \lambda^2_4\omega^4 \\ d\lambda^3 &= \lambda^3_1\omega^1 + \lambda^3_2\omega^2 + \left(2\lambda^2_4\tfrac{(\lambda^3-\lambda^1)^2}{(\lambda^2-\lambda^1)^2} + 2\lambda^1_4\tfrac{(\lambda^2-\lambda^3)^2}{(\lambda^2-\lambda^1)^2}\right)\omega^4, \end{aligned} \tag{10.23}$$

for λ^1, λ^2, λ^3 (here λ^3_4 is found from (1.42) and $\lambda^4_4 = 0$) along with the equations

$$d\lambda^i_j = L^i_{jk}(\lambda^m, \lambda^n_l)\omega^k \tag{10.24}$$

for the 8 first derivatives λ^1_2, λ^1_3, λ^1_4, λ^2_1, λ^2_3, λ^2_4, λ^3_1, λ^3_2, turns out to be in involution. For the case $SO(3) \times SO(2)$ the exact formulas for L^i_{jk} are given below.

• Proof of Theorem 28

Let λ^i and ω^i be characteristic speeds and the eigenforms of the system corresponding to (10.6). Then there exist 4 conservation laws with functionally independent densities u^i rectifying rarefaction curves. One can consider u^i as local coordinates parametrizing the congruence. Let $\tilde\lambda^i$ and $\tilde\omega^i$ be the characteristic speeds and the eigenforms of the system corresponding to a congruence G which satisfies the hypothesis of the theorem. According to lemma 13, the structure equations for $\tilde\omega^i$ are either as for (10.6) or (10.18). Suppose they are as for (10.6). (The proof for the case (10.18) is the same). Then one can take $\tilde\omega = \omega$ and consider u^i as local coordinates for the congruence G as well. Then λ^i and $\tilde\lambda^i$ satisfy the same system (10.23), (10.24). We show that there exist such constants α_i, α, $\tilde\alpha$, β_i, β, $\tilde\beta$ that, locally,

$$\tilde\lambda^k = \Lambda^k \equiv \frac{\lambda^k(\alpha_i u^i + \alpha) - (\alpha_i f^i + \tilde\alpha)}{(\beta_i f^i + \tilde\beta) - \lambda^k(\beta_i u^i + \beta)} \equiv \frac{\lambda^k B - A}{M - \lambda^k N}, \quad i = 1, 2, 3. \tag{10.25}$$

To this end it suffices to find the constants which satisfy (10.25) and $\tilde{\lambda}^i_j = \Lambda^i_j$ only at one point $\vec{u}_0$ since system (10.23), (10.24) is completely integrable as a Pfaffian system. Let us fix some $\vec{u}_0$. Then (10.25) defines A, B, M, N at $\vec{u}_0$ up to a common factor. Moreover, $BM - AN \neq 0$ since λ^i are distinct (as well as $\tilde{\lambda}^i$). A direct computation yields

$$\Lambda^1_2 = \lambda^1_2 \frac{(BM - AN)}{(M - \lambda^1 N)^2} + \frac{(\lambda^1 - \lambda^2)}{(M - \lambda^1 N)^2}\{B_2(M - \lambda^1 N) + N_2(\lambda^1 B - A)\}, \tag{10.26}$$

where we have used $f_i = \lambda^i u_i$. Similarly,

$$\Lambda^3_2 = \lambda^3_2 \frac{(BM - AN)}{(M - \lambda^3 N)^2} + \frac{(\lambda^3 - \lambda^2)}{(M - \lambda^3 N)^2}\{B_2(M - \lambda^3 N) + N_2(\lambda^3 B - A)\}. \tag{10.27}$$

The linear system

$$\tilde{\lambda}^1_2 = \Lambda^1_2, \ \tilde{\lambda}^3_2 = \Lambda^3_2$$

defines B_2, N_2 uniquely since the determinant of this system is

$$\tfrac{(\lambda^1-\lambda^2)(\lambda^3-\lambda^2)(\lambda^1-\lambda^3)(AN-MB)}{(M-\lambda^1 N)^2(M-\lambda^3 N)^2} \neq 0.$$

Similarly, the equations

$$\tilde{\lambda}^2_1 = \Lambda^2_1, \ \tilde{\lambda}^3_1 = \Lambda^3_1$$

give B_1, N_1. Finally, B_3, N_3 and B_4, N_4 are determined from

$$\tilde{\lambda}^1_3 = \Lambda^1_3, \ \tilde{\lambda}^2_3 = \Lambda^2_3$$

and

$$\tilde{\lambda}^1_4 = \Lambda^1_4, \ \tilde{\lambda}^2_4 = \Lambda^2_4,$$

respectively. Thus obtained A, B, N, M and B_i, N_i define α_i, α, $\tilde{\alpha}$, β_i, β, $\tilde{\beta}$, which means that the system corresponding to G is a reciprocal transform of the system corresponding to (10.6). Thus, we have also proved that u^i are not just local coordinates for G, but also the "rectifying" densities of conservation laws of the corresponding system (1.15). Since the congruences under consideration define connected manifolds in $\mathbb{G}(1,5)$, this local equivalence is extended globally.□

Finally, we present formulas for $L^i_{jk}(\lambda^m, \lambda^n_l)$ of (10.24). As λ^4 is found from (1.42) and λ^3_4 is obtained from $\lambda^4_4 = 0$, the permutation symmetry used in Lemma 13 is lost. Therefore, we give all necessary expressions without referring to index permutations.

$$\lambda^1_{21}=-\lambda^1_3-\lambda^1_4,\quad \lambda^1_{22}=\frac{2\lambda^1_2\lambda^3_2(\lambda^1-\lambda^2)}{(\lambda^2-\lambda^3)(\lambda^1-\lambda^3)}+\frac{2(\lambda^1_2)^2}{\lambda^1-\lambda^3},\quad \lambda^1_{31}=\tfrac{1}{2}\lambda^1_2,\quad \lambda^1_{32}=\lambda^1_{23},\quad \lambda^1_{41}=\tfrac{1}{2}\lambda^1_2,$$

$$\lambda^1_{23}=\frac{(2\lambda^1-\lambda^2-\lambda^3)\lambda^1_2\lambda^1_3}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}+\frac{(\lambda^1-\lambda^3)\lambda^1_2\lambda^2_3}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^2-\lambda^1)\lambda^1_3\lambda^3_2}{(\lambda^2-\lambda^3)(\lambda^3-\lambda^1)}-\frac{(\lambda^1-\lambda^3)^2\lambda^2_1}{2(\lambda^2-\lambda^3)^2}+\frac{(\lambda^1-\lambda^2)^2\lambda^3_1}{2(\lambda^2-\lambda^3)^2},$$

$$\lambda^1_{24}=\frac{(2\lambda^1-\lambda^2-\lambda^3)\lambda^1_2\lambda^1_4}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}+\frac{(\lambda^3-\lambda^1)\lambda^1_2\lambda^2_4}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^2-\lambda^1)\lambda^1_4\lambda^3_2}{(\lambda^2-\lambda^3)(\lambda^3-\lambda^1)}+\frac{(\lambda^1-\lambda^3)^2\lambda^2_1}{2(\lambda^2-\lambda^3)^2}-\frac{(\lambda^1-\lambda^2)^2\lambda^3_1}{2(\lambda^2-\lambda^3)^2},$$
$$\lambda^1_{33}=\frac{2(\lambda^1_3)^2}{\lambda^1-\lambda^2}+\frac{2(\lambda^1-\lambda^3)\lambda^1_3\lambda^2_3}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}{2(\lambda^2-\lambda^3)},\quad \lambda^1_{42}=\lambda^1_{24},\quad \lambda^1_{43}=\lambda^1_{34},\quad \lambda^2_{12}=\lambda^2_3+\lambda^2_4,$$

$$\lambda^1_{34}=\frac{(\lambda^1-\lambda^3)\lambda^1_4\lambda^2_3}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^3-\lambda^1)\lambda^1_3\lambda^2_4}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{2(\lambda^3-\lambda^2)\lambda^1_3\lambda^1_4}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)},\quad \lambda^2_{13}=\lambda^2_{31},\quad \lambda^2_{32}=-\tfrac{1}{2}\lambda^2_1,$$

$$\lambda^1_{44}=\frac{2(\lambda^3-\lambda^2)(\lambda^1_4)^2}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{2(\lambda^3-\lambda^1)\lambda^1_4\lambda^2_4}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}{2(\lambda^3-\lambda^2)},\quad \lambda^2_{41}=\lambda^2_{14},\quad \lambda^2_{42}=-\tfrac{1}{2}\lambda^2_1,$$

$$\lambda^2_{11}=\frac{2(\lambda^2_1)^2}{\lambda^2-\lambda^3}+\frac{2(\lambda^1-\lambda^2)\lambda^2_1\lambda^3_1}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^3)},\quad \lambda^2_{33}=\frac{2(\lambda^3-\lambda^1)(\lambda^2_3)^2}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}+\frac{2(\lambda^2-\lambda^3)\lambda^1_3\lambda^2_3}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}+\frac{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}{2(\lambda^1-\lambda^3)},$$
$$\lambda^2_{14}=\frac{(2\lambda^2-\lambda^1-\lambda^3)\lambda^2_1\lambda^2_4}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^3-\lambda^2)\lambda^1_4\lambda^2_1}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}+\frac{(\lambda^2-\lambda^1)\lambda^2_4\lambda^3_1}{(\lambda^3-\lambda^2)(\lambda^3-\lambda^1)}-\frac{(\lambda^2-\lambda^3)^2\lambda^1_2}{2(\lambda^3-\lambda^1)^2}+\frac{(\lambda^1-\lambda^2)^2\lambda^3_2}{2(\lambda^1-\lambda^3)^2},$$

$$\lambda^2_{31}=\frac{(2\lambda^2-\lambda^1-\lambda^3)\lambda^2_1\lambda^2_3}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^2-\lambda^3)\lambda^1_3\lambda^2_1}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}+\frac{(\lambda^2-\lambda^1)\lambda^2_3\lambda^3_1}{(\lambda^3-\lambda^2)(\lambda^3-\lambda^1)}-\frac{(\lambda^2-\lambda^1)^2\lambda^3_2}{2(\lambda^3-\lambda^1)^2}+\frac{(\lambda^2-\lambda^3)^2\lambda^1_2}{2(\lambda^1-\lambda^3)^2},$$

$$\lambda^2_{34}=\frac{2(\lambda^3-\lambda^1)\lambda^2_3\lambda^2_4}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}+\frac{(\lambda^3-\lambda^2)\lambda^2_3\lambda^1_4}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}+\frac{(\lambda^2-\lambda^3)\lambda^1_3\lambda^2_4}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)},\quad \lambda^2_{43}=\lambda^2_{34},\quad \lambda^3_{13}=-\tfrac{1}{2}\lambda^3_2,\quad \lambda^3_{23}=\tfrac{1}{2}\lambda^3_1,$$

$$\lambda^2_{44}=\frac{2(\lambda^2_4)^2}{\lambda^2-\lambda^1}+\frac{2(\lambda^3-\lambda^2)\lambda^1_4\lambda^2_4}{(\lambda^2-\lambda^1)(\lambda^3-\lambda^1)}+\frac{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}{2(\lambda^3-\lambda^1)},\quad \lambda^3_{21}=\lambda^3_{12}-2\left(\frac{(\lambda^3-\lambda^1)^2\lambda^2_4}{(\lambda^2-\lambda^1)^2}+\frac{(\lambda^2-\lambda^3)^2\lambda^1_4}{(\lambda^2-\lambda^1)^2}\right),$$

$$\lambda^3_{11}=\frac{2(\lambda^3_1)^2}{\lambda^3-\lambda^2}+\frac{2(\lambda^3-\lambda^1)\lambda^2_1\lambda^3_1}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}+\frac{(\lambda^3-\lambda^1)(\lambda^3-\lambda^2)}{\lambda^2-\lambda^1},\quad \lambda^3_{22}=\frac{2(\lambda^3_2)^2}{\lambda^3-\lambda^1}+\frac{2(\lambda^3-\lambda^2)\lambda^1_2\lambda^3_2}{(\lambda^2-\lambda^1)(\lambda^3-\lambda^1)}+\frac{(\lambda^3-\lambda^1)(\lambda^2-\lambda^3)}{(\lambda^2-\lambda^1)},$$

$$\lambda^3_{12}=\frac{(2\lambda^3-\lambda^1-\lambda^2)\lambda^3_1\lambda^3_2}{(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)}+\frac{(\lambda^3-\lambda^2)\lambda^1_2\lambda^3_1}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)}+\frac{(\lambda^3-\lambda^1)\lambda^3_2\lambda^2_1}{(\lambda^2-\lambda^1)(\lambda^2-\lambda^3)}-\frac{(\lambda^2-\lambda^3)^2\lambda^1_3}{(\lambda^2-\lambda^1)^2}+\frac{(\lambda^2-\lambda^3)^2\lambda^1_4}{(\lambda^1-\lambda^2)^2}+$$

$$+\frac{(\lambda^3-\lambda^1)^2\lambda^2_3}{(\lambda^1-\lambda^2)^2}+\frac{(\lambda^1-\lambda^3)^2\lambda^2_4}{(\lambda^1-\lambda^2)^2}$$

$$\lambda^3_{14}=\frac{2(2\lambda^3-\lambda^1-\lambda^2)(\lambda^1-\lambda^3)\lambda^3_1\lambda^2_4}{(\lambda^1-\lambda^2)^2(\lambda^2-\lambda^3)}+\frac{2(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)\lambda^1_4\lambda^2_1}{(\lambda^1-\lambda^2)^3}+\frac{4(\lambda^3-\lambda^2)\lambda^1_4\lambda^3_1}{(\lambda^1-\lambda^2)^2}+\frac{2(\lambda^3-\lambda^1)^3\lambda^2_1\lambda^2_4}{(\lambda^2-\lambda^3)(\lambda^2-\lambda^1)^3}+\frac{\lambda^3_2}{2},$$

$$\lambda^3_{24}=\frac{2(2\lambda^3-\lambda^1-\lambda^2)(\lambda^3-\lambda^2)\lambda^3_2\lambda^1_4}{(\lambda^1-\lambda^2)^2(\lambda^3-\lambda^1)}+\frac{2(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)\lambda^1_2\lambda^2_4}{(\lambda^2-\lambda^1)^3}+\frac{4(\lambda^3-\lambda^1)\lambda^2_4\lambda^3_2}{(\lambda^1-\lambda^2)^2}+\frac{2(\lambda^3-\lambda^2)^3\lambda^1_2\lambda^1_4}{(\lambda^3-\lambda^1)(\lambda^2-\lambda^1)^3}-\frac{\lambda^3_1}{2}.$$

10.5 Concluding remarks on integrable systems of conservation laws

The obtained results suggest two conjectures on the structure of congruences in $\mathbb{P}^{n+1}$ whose developable surfaces are planar pencils of lines.

Conjecture 4 *The focal varieties of such congruences are algebraic (possibly, reducible and singular).*

Conjecture 5 *The intersection of the focal variety with a developable surface (which is a planar pencil of lines) consists of a point (the vertex of the pencil) and a plane curve of degree $n-1$.*

For $n = 2$ this is obvious. For $n = 3$ it follows from the results presented above. Both conjectures are true for general linear congruences in $\mathbb{P}^{n+1}$ (see Section 9.2) and congruences

arising from the completely exceptional Monge-Ampère type equations (see Section 9.4). As readily follows from the discussion in these Sections focal varieties have codimension two and contain n-parameter families of plane curves (which are conics for $n = 3$). This shows that the problem in question is actually algebro-geometric.
Another obvious hypothesis concerns 4-component systems that are allowed to have Riemann invariants.

Conjecture 6 *The congruence, corresponding to a four-component T-system (1.15) with harmonic cross-ratio of characteristic speeds, is linear.*

Chapter 11

Implicit ODEs with hexagonal web of solutions

11.1 Normal forms for a fold point

In this section we establish normal forms for the case, when the projection π has a fold point at m. If cubic equation (1.47) has two coinciding roots, then the third root corresponds to a regular point of the projection π and the equation factors to a quadratic equation and a linear one. Regularity condition (1.45) for the double root p_0 implies immediately that the projection π has a fold point at m. First we find a normal form for fold points and the symmetries of this normal form. Further we look for the linear in p (i.e. explicit) equations whose solutions complete the 2-web of solutions of the quadratic normal form to a hexagonal 3-web. Finally, we bring these linear terms to some normal forms using the symmetries of the quadratic equation. We start with a Legendrian criminant, then consider non-Legendrian criminant and finally show that the case of an isolated point of tangency of the criminant and the contact plane is excluded by the regularity conditions.

• **The case of Legendrian criminant**

Proposition 16 *Consider implicit ODE (1.43) with a smooth Legendrian criminant and a smooth surface (1.44). Then characteristic field (1.46) can be smoothly extended to the criminant. Moreover, the extended characteristic field is transverse to the criminant.*

Proof: Let m be a point on the criminant. A suitable contactomorphism φ maps M to M' with the following properties:
a) the criminant of M is mapped to the line $x = y = 0$,
b) $\varphi(m) = (0, 0, 0)$,
c) the tangent plane T_mM is mapped to the plane $y = 0$.
It suffices to prove the proposition for the transformed surface $M' := \{(x, y, p) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : G(x, y, p) = 0\}$ with the Legendrian line $x = y = 0$. Condition c) allows to rewrite $G(x, y, p) = 0$ as

$$y = g(x, p),$$

while condition a) implies $g(x, p) = xf(x, p)$. Now $g_p(0, 0) = g_x(0, 0) = 0$ implies $f(0, 0) = 0$, so that $f(x, p) = pu(x, p) + xv(x, p)$ by the Hadamard lemma. As the line $x = y = 0$ is Legendrian, the form $dy - pdx$ must vanish on it:

$$\{d(xpu(x, p) + x^2v(x, p)) - pdx\}\big|_{x=0} = \{(pu(x, p)dx - pdx\}\big|_{x=0} = 0$$

or $u(0,p) \equiv 1$. Again by the Hadamard lemma one gets $u(x,p) = 1 + xw(x,p)$ and

$$y = xp + x^2 h(x,p)$$

with $h(x,p) = pw(x,p) + v(x,p)$. Now the characteristic field τ is defined by restriction of the equation $dy - pdx = 0$ to M', i.e. by $\{d(xp + x^2 h(x,p)) - pdx\}\big|_{M'} = x\{(1 + xh_p(x,p))dp + (2h(x,p) + xh_x(x,p))dx\} = 0$. This implies that the characteristic field on M' in coordinates (x,p) is generated by the vector field $(1+xh_p(x,p))\partial_x - (2h(x,p)+xh_x(x,p))\partial_p$, which is clearly smooth and transverse to the line $x = y = 0$ on M'. □

Theorem 29 *Let (1.43) be an implicit ODE such that corresponding surface (1.44) is smooth. Suppose its criminant C is a smooth Legendrian curve and the projection $\pi : (x,y,p) \to (x,y)$ has a fold singularity at $m \in C$. Then (1.43) is locally equivalent to*

$$p^2 = y \tag{11.1}$$

with respect to some diffeomorphism $\varphi : \tilde{U} \to U$, where $\tilde{U}, U \in \mathbb{R}^2$ are neighborhoods of $(0,0)$, $\pi(m)$ *and* $\varphi(0,0) = \pi(m)$.

Proof: Due to Proposition 16 the direction field τ on M is smooth. Its integral curves define a foliation $\mathcal{F}$ of a neighborhood of m. Since the projection π has a fold on the criminant, the discriminant curve $\Delta = \pi(C) \in \mathbb{R}^2$ is smooth. Let us choose new coordinates on $(U, \pi(m))$ such that the discriminant curve turns to the line $y = 0$. Then equation (1.43) is equivalent to $y = g(x,p)$. The discriminant curve is a solution therefore $g(x,p) = pf(x,p)$ by the Hadamard lemma. The criminant $y = p = 0$ is Legendrian hence $\{d(pf(x,p)) - pdx\}_{p=0} = f(x,0)dp = 0$ or $f(x,0) \equiv 0$. Now $f(x,p) = ph(x,p)$ and $y = p^2 h(x,p)$. As the projection π has a fold at $m \in M$ holds true $\partial_p^2(p^2 h(x,p))\big|_{x=p=0} = 2h(0,0) \neq 0$. Consequently characteristic field (1.46) on M with $F(x,y,p) = p^2 h(x,p) - y$ is generated by the vector field

$$(2ph(x,p) + p^2 h_p(x,p))\partial_x - (p^2 h_x(x,p) - p)\partial_p = p\{(2h(x,p) + ph_p(x,p))\partial_x + (1 - ph_x(x,p))\partial_p\}.$$

Due to $h(0,0) \neq 0$ it is transverse to the kernel of $d\pi$ on M. Hence the projection of each integral curve crossing the criminant C is smooth and tangent to the discriminant curve. Locally the equation $p^2 h(x,p) - y = 0$ can be rewritten as a quadratic equation

$$p^2 + a(x,y)p + b(x,y) = 0.$$

This easily follows from the Division Theorem since $h(0,0) \neq 0$. Moreover, as the criminant $y = p = 0$ is a Legendrian curve holds true $b(x,0) = a(x,0) = 0$. Thus one gets $a(x,y) = y\alpha(x,y)$ and $b(x,y) = y\beta(x,y)$ with $\beta(0,0) \neq 0$ since M is smooth at $m = (0,0,0)$. Consider (x,p) as local coordinates on M and define the map $i : M \to M$ by

$$i(x,p) = (x, -p - p^2 h(x,p)\alpha(x, p^2 h(x,p))).$$

This map is the involution that permutes the roots p_1, p_2 of our quadratic ODE. Let the foliation $\mathcal{F}$ on M be defined locally by $I(x,p) = const$ with $\mathrm{grad}(I)\big|_{x=p=0} \neq 0$, where I is a first integral of the characteristic field τ. Then the functions I and $J := i^*(I)$ are functionally independent as $di(\tau)$ and τ are transverse to each other. Since τ is transverse also to the criminant the partial derivative $I_x\big|_{x=p=0}$ does not vanish. Hence one can chose I so that $I(x,0) = x$ which implies $J(x,0) = x$. Let us take thus normalized functions I, J as local coordinates on M. Note that the following relation holds true

$$\pi(I,J) = \pi(J,I), \tag{11.2}$$

since $i(I,J) = (J,I)$. For the ODE $\tilde{p}^2 = \tilde{y}$ the above defined objects are as follows: $\tilde{I}(\tilde{x},\tilde{p}) = \tilde{x} - 2\tilde{p}$, $\tilde{i}(\tilde{x},\tilde{p}) = (\tilde{x}, -\tilde{p})$, $\tilde{J}(\tilde{x},\tilde{p}) = \tilde{x} + 2\tilde{p}$. Now define the diffeomorphism germ $\psi : \tilde{M}, 0 \to M, 0$

by $\psi(\tilde{I},\tilde{J}) = (\tilde{I},\tilde{J})$. By (11.2) there exists a map germ $\varphi : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ such that the following diagram commutes:

$$\begin{array}{ccc} M, 0 & \xleftarrow{\psi} & \tilde{M}, 0 \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \mathbb{R}^2, 0 & \xleftarrow{\varphi} & \mathbb{R}^2, 0 \end{array}$$

We claim that the map germ φ is the required diffeomorphism. By construction it maps solutions to solutions. To show that φ is differentiable consider the map germ $\pi \circ \psi : \tilde{M}, 0 \to \mathbb{R}^2, 0$. Applying Malgrange's Preparation Theorem to this map germ in coordinates $(\tilde{x}, \tilde{p})$ on $\tilde{M}, 0$ and (x, y) on $\mathbb{R}^2, 0$ one gets

$$x = X_0(\tilde{x}, \tilde{p}^2) + \tilde{p}X_1(\tilde{x}, \tilde{p}^2), \quad y = Y_0(\tilde{x}, \tilde{p}^2) + \tilde{p}Y_1(\tilde{x}, \tilde{p}^2),$$

where the functions X_0, X_1, Y_0, Y_1 are smooth. Since $\pi \circ \psi(\tilde{x}, \tilde{p}) = \pi \circ \psi(\tilde{x}, -\tilde{p}) = (x, y)$ holds true $X_1(\tilde{x}, \tilde{p}^2) = Y_1(\tilde{x}, \tilde{p}^2) = 0$. With $\tilde{p}^2 = \tilde{y}$ the above formulas take the form

$$x = X_0(\tilde{x}, \tilde{y}), \quad y = Y_0(\tilde{x}, \tilde{y}),$$

and therefore define a smooth map germ φ. Applying the same considerations to the map germ $\tilde{\pi} \circ \psi^{-1}$ we see that ψ has an inverse for $y \geq 0$. Therefore ψ is invertible. $\square$

Remark. Note that equation (11.1) has a nontrivial symmetry group. For example, the scaling $x \to \lambda x, \quad y \to \lambda^2 y$ leaves it invariant. Therefore the diffeomorphism φ in Theorem 29 is not unique.

Proposition 17 *Equation (11.1) has an infinite symmetry pseudogroup. Its transformations are given by*

$$\tilde{x} = F(x + 2\sqrt{y}) + F(x - 2\sqrt{y}), \quad \tilde{y} = \frac{1}{4}[F(x + 2\sqrt{y}) - F(x - 2\sqrt{y})]^2. \tag{11.3}$$

Here F is a smooth function subjected to $F'(u)|_{u=0} \neq 0$. Infinitesimal generators of this pseudogroup have the form

$$\{f(x + 2\sqrt{y}) + f(x - 2\sqrt{y})\}\partial_x + \sqrt{y}\{f(x + 2\sqrt{y}) - f(x - 2\sqrt{y})\}\partial_y, \tag{11.4}$$

where f is an arbitrary smooth function.

Proof: Consider the action of symmetry group transformation on M in coordinates (u, v), where

$$u = x - 2p, \quad v = x + 2p.$$

To preserve the foliations $\mathcal{F}$ and $i(\mathcal{F})$ determined by the direction fields τ and $di(\tau)$ it must have the following form $\bar{u} = 2F(u), \ \bar{v} = 2G(v)$. This transformation is a symmetry if it commutes with i. Note that i permutes u and v and sends ∂_p to $-\partial_p$. Hence $F = G$. Now substitutions $\bar{x} = \frac{1}{2}(F(v) + F(u))$ and $\bar{y} = \bar{p}^2 = \frac{1}{4}(F(v) + F(u))^2$ gives (11.3). Condition $F'(u)|_{u=0} \neq 0$ is equivalent to non-vanishing of the Jacobian: $\left.\frac{\partial(\tilde{x},\tilde{y})}{\partial(x,y)}\right|_{x=y=0} \neq 0$.

Now consider infinitesimal symmetries of (11.1). They are defined by operators $\xi(x, y)\partial_x + \eta(x, y)\partial_y$. These operators must be liftable to M. On M the lifted vector field must be a symmetry of foliations $\mathcal{F}$ and $i(\mathcal{F})$. In coordinates (u, v) any infinitesimal symmetry X of foliations $\mathcal{F}$ and $i(\mathcal{F})$ is easy to write down:

$$X = f(u)\partial_u + g(v)\partial_v.$$

In coordinates (x, p) on M this operator X takes the form

$$X = \frac{1}{2}(g(v) + f(u))\partial_x + \frac{1}{4}(g(v) - f(u))\partial_p.$$

X is lowerable iif $di(X) = X$. Now lowerability condition amounts to $f(u)+g(v) = f(v)+g(u)$ and $g(v) - f(u) = f(v) - g(u)$. Thus, the first equation gives $f(u) - g(u) = c = const$ and the second implies $c = 0$. Substitution $u = x - 2\sqrt{y}$, $v = x + 2\sqrt{y}$, $p = \sqrt{y}$ into X gives (11.4) for $y > 0$. The obtained transformations are well defined for $y > 0$ and are smoothly (analytically) extendable for $y \geq 0$. The possibility to extend them for $y \leq 0$ easily follows from Malgranges's Preparation Theorem: one considers the projection $\pi : M \to \mathbb{R}^2$, $(x, p) \mapsto (x, p^2)$ in coordinates x, p on M and x, y on $\mathbb{R}^2$ and observes that $F(x + 2p) + F(x - 2p)$ is an even function and $F(x + 2p) - F(x - 2p)$ is an odd function with respect to p. □

To use the above symmetries we need the following lemma.

Lemma 14 *If a smooth (analytic) function germ $f : \mathbb{R}, 0 \to \mathbb{R}, 0$ is not flat, then the vector field germ $f(u)\partial_u$ on $\mathbb{R}, 0$ is equivalent to $u^k\partial_u$ with respect to a suitable smooth (analytic) coordinate transformation $\bar{u} = F(u)$, where $k \in \mathbb{N}$ is such that $\left.\frac{d^k f(u)}{du^k}\right|_{u=0}$ is the first non-vanishing derivative at 0 .*

Proof: $F(u)$ must satisfy ODE $f(u)F'(u) = (F(u))^k$. The existence of $F(u)$ with $F'(u)|_{u=0} \neq 0$ is easily verified in both smooth and analytic cases. □

Theorem 30 *Suppose solutions of (11.1) and those of*

$$\alpha(x, y)dx + \beta(x, y)dy = 0, \tag{11.5}$$

where $\alpha(x, y), \beta(x, y)$ are non-flat functions at $(0, 0)$, form together a hexagonal 3-web. Then there is a local symmetry of (11.1) at $(0, 0)$ that maps equation (11.5) to one of the two following forms for $y \geq 0$:

$$\begin{gathered}
[(x + 2\sqrt{y})^k + (x - 2\sqrt{y})^k]dx - \tfrac{1}{\sqrt{y}}[(x + 2\sqrt{y})^k - (x - 2\sqrt{y})^k]dy = 0, \text{ or} \\
\sqrt{y}[(x + 2\sqrt{y})^k - (x - 2\sqrt{y})^k]dx - [(x + 2\sqrt{y})^k + (x - 2\sqrt{y})^k]dy = 0,
\end{gathered} \tag{11.6}$$

where k is a non-negative integer. In particular, if $\alpha(x, y), \beta(x, y)$ are non-flat functions with $(\alpha(0, 0), \beta(0, 0)) \neq (0, 0)$, then one gets three normal forms:

$$a)\ dx = 0, \quad b)\ dy = 0, \quad c)\ 2dy - xdx = 0. \tag{11.7}$$

Moreover, if equation (11.5) for $y \geq 0$ is equivalent to (11.7) item a) or (11.7) item b), then it can be reduced to (11.7) item a), respectively (11.7) item b) by a suitable symmetry of (11.1) in some neighborhood of the point $(0, 0)$

Proof: Let us introduce operators U, V of differentiation along the curves of the foliations $\mathcal{F}$ and $i(\mathcal{F})$ on M:

$$U = \partial_p + 2\partial_x, \quad V = \partial_p - 2\partial_x. \tag{11.8}$$

Then these operators commute and satisfy the following relations:

$$U(u) = 0, \quad U(v) = 4, \quad V(u) = -4, \quad V(v) = 0.$$

Consequently a direction field on M whose integral curves form a hexagonal 3-web together with $\mathcal{F}$ and $i(\mathcal{F})$ must be generated by a vector field commuting with U and V (for the detail see [18], p.17). Such a vector field has the form

$$Y = f(v)U + g(u)V.$$

The direction field generated by Y is the lift to M of the direction field induced by (11.5) iff $Y \wedge di(Y) = 0$. This gives $g = \pm f$. Projecting from M to the plane one obtains

$$\frac{1}{\sqrt{y}}[f(x + 2\sqrt{y}) - f(x - 2\sqrt{y})]\partial_x + [f(x + 2\sqrt{y}) + f(x - 2\sqrt{y})]\partial_y$$

for $g = f$ and

$$[f(x+2\sqrt{y}) + f(x-2\sqrt{y})]\partial_x + \sqrt{y}[f(x+2\sqrt{y}) - f(x-2\sqrt{y})]\partial_y,$$

for $g = -f$. Now applying symmetry (11.3) with F satisfying $f(u)F'(u) = (F(u))^k$ (see Lemma 14) we reduce (11.5) to one of the forms (11.6) for $y \geq 0$. If the found symmetry maps (11.5) to (11.7) item a), than we can construct a diffeomorphism φ such that it is the identity for $y \geq 0$ and maps integral curves of (11.5) to the lines $x = const$ as follows. As equation (11.5) is not singular at $(0,0)$ it has a smooth first integral $I(x,y)$ coinciding with x for $y \geq 0$. Then ϕ is defined by $(x,y) \mapsto (I(x,y), y)$. Similarly, for (11.7) item b) we define ϕ by $(x,y) \mapsto (x, I(x,y)$, where $I(x,y)$ is the first integral of (11.5), coinciding with y for $y \geq 0$. □

• **The case of non-Legendrian criminant**

Theorem 31 *Let (1.43) be an implicit ODE such that the corresponding surface M is smooth. Suppose its criminant C is a smooth curve, the projection $\pi : (x,y,p) \to (x,y)$ has a fold singularity at $m \in C$, and the contact plane is not tangent to M at m. Then (1.43) is locally equivalent to*

$$p^2 = x \tag{11.9}$$

with respect to some diffeomorphism $\varphi : \tilde{U} \to U$, where $\tilde{U}, U \in \mathbb{R}^2$ are neighborhoods of $(0,0)$, $\pi(m)$ and $\varphi(0,0) = \pi(m)$.

The proof is given in [15], p.27. Similar to the case of Legendrian criminant, the diffeomorphism φ is not unique.

Proposition 18 *Equation (11.9) has an infinite symmetry pseudogroup. Its transformations are given by*

$$\tilde{x} = \sqrt[3]{\frac{1}{16}[F(3y+2x\sqrt{x}) - F(3y-2x\sqrt{x})]^2}, \quad \tilde{y} = \frac{1}{6}(F(3y+2x\sqrt{x}) + F(3y-2x\sqrt{x})). \tag{11.10}$$

Here F is a smooth function subjected to $F'(u)|_{u=0} \neq 0$. Infinitesimal generators of this pseudogroup have the form

$$\frac{1}{\sqrt{x}}\{f(3y+2x\sqrt{x}) - f(3y-2x\sqrt{x})\}\partial_x + \{f(3y+2x\sqrt{x}) + f(3y-2x\sqrt{x})\}\partial_y, \tag{11.11}$$

where f is an arbitrary smooth function.

Proof: On the surface M defined by (1.44) we choose (y,p) as local coordinates. Solutions of (11.9) define foliations $\mathcal{F}$ and $i(\mathcal{F})$ by

$$u := 3y - 2p^3 = const, \quad v := 3y + 2p^3 = const,$$

where i is the involution $i : M \to M$, $(x,y,p) \to (x,y,-p)$.

To prove the finite transformation formulas observe that the symmetry group transformation $(x,y) \mapsto (\tilde{x},\tilde{y})$ lifted to M must satisfy

$$3\tilde{y} + 2\tilde{p}^3 = F(3y+2p^3), \quad 3\tilde{y} - 2\tilde{p}^3 = F(3y-2p^3).$$

In fact, to preserve the foliations $\mathcal{F}$ and $i(\mathcal{F})$ it is necessary that $3\tilde{y}+2\tilde{p}^3 = F(3y+2p^3)$, $3\tilde{y} - 2\tilde{p}^3 = G(3y-2p^3)$. On the line $p = 0$ one has $F(3y) = G(3y) = 3\tilde{y}$. This implies (11.10). The condition $F'(u)|_{u=0} \neq 0$ is equivalent to non-degeneracy of the Jacobian of the transformation.

Consider now an infinitesimal symmetry $\xi(x,y)\partial_x + \eta(x,y)\partial_y$. Lifted on M it turns to

$$X = \eta(p^2,y)\partial_y + g(y,p)\partial_p.$$

(On can write down an explicit expression for g, but we do not need it.) Since X is a symmetry of (11.9) it must satisfy

$$X(u) = 3f(u) = 3f(3y - 2p^3), \quad X(v) = 3k(v) = 3k(3y + 2p^3),$$

for some smooth functions f, k. This is equivalent to

$$\eta(p^2, y) = 2p^2 g(y, p) + f(3y - 2p^3), \quad \eta(p^2, y) = -2p^2 g(y, p) + k(3y + 2p^3).$$

Substituting $p = 0$ into the difference of the above equations

$$4p^2 g(y, p) = k(3y + 2p^3) - f(3y - 2p^3)$$

one gets $k = f$. Hence the functions η and g are well defined by

$$\eta(p^2, y) = \frac{1}{2}(f(3y + 2p^3) + f(3y - 2p^3)), \quad g(y, p) = \frac{1}{4p^2}(f(3y + 2p^3) - f(3y - 2p^3)).$$

Note that $g(y, 0) = 0$, hence X is tangent to the criminant and therefore lowerable (see [16]). Up to scaling the lowered operator X becomes (11.11). The extension of the defined transformation for $x \leq 0$ is again justified by Malgrange's Preparation Theorem. (See the detail in the proof of Theorem 17.) $\square$

Theorem 32 *Suppose the solutions of (11.9) and those of*

$$\alpha(x, y)dx + \beta(x, y)dy, \tag{11.12}$$

where $\alpha(x, y), \beta(x, y)$ are non-flat functions at $(0, 0)$, form together a hexagonal 3-web. Then there is a local symmetry of (11.9) that maps equation (11.12) to one of the two following forms for $x \geq 0$:

$$\frac{1}{\sqrt{x}}[(3y + 2x\sqrt{x})^k - (3y - 2x\sqrt{x})^k]dx - \sqrt[3]{16}[(3y + 2x\sqrt{x})^k + (3y - 2x\sqrt{x})^k]dy = 0 \text{ or}$$

$$[(3y + 2x\sqrt{x})^k + (3y - 2x\sqrt{x})^k]dx - \sqrt[3]{16}\sqrt{x}[(3y + 2x\sqrt{x})^k - (3y - 2x\sqrt{x})^k]dy = 0,$$

where k is non-negative integer. In particular, if $\alpha(x, y), \beta(x, y)$ are non-flat functions with $(\alpha(0, 0), \beta(0, 0)) \neq (0, 0)$, then equation (11.12) can be reduced to one of the following two normal forms in some neighborhood of the point $(0, 0)$:

$$a)\ dx = 0, \quad b)\ dy = 0.$$

Proof: Let us introduce operators U, V of differentiation along the curves of the foliations $\mathcal{F}$ and $i(\mathcal{F})$:

$$U = \partial_p + 2p^2\partial_y, \quad V = \partial_p - 2p^2\partial_y.$$

Then the operators $\frac{1}{p^2}U$ and $\frac{1}{p^2}V$ commute and satisfy the following relations:

$$\frac{1}{p^2}U(u) = 0, \quad \frac{1}{p^2}U(v) = 12, \quad \frac{1}{p^2}V(u) = -12, \quad \frac{1}{p^2}V(v) = 0.$$

A direction field on M, whose integral curves form a hexagonal 3-web together with $\mathcal{F}$ and $i(\mathcal{F})$, must be generated by the vector field, commuting with $\frac{1}{p^2}U$ and $\frac{1}{p^2}V$. Such a vector field has the form

$$Y = f(v)\frac{1}{p^2}U + g(u)\frac{1}{p^2}V.$$

The direction field generated by Y is the lift to M of the direction field induced by (11.12) iff $Y \wedge di(Y) = 0$. This gives $g = \pm f$ (compare with the proof of Theorem 30). Projecting from M to (x, y)-plane one obtains

$$[f(3y + 2x\sqrt{x}) + f(3y - 2x\sqrt{x})]\partial_x + \sqrt{x}[f(3y + 2x\sqrt{x}) - f(3y - 2x\sqrt{x})]\partial_y$$

for $g = f$ and

$$\frac{1}{\sqrt{x}}[f(3y + 2x\sqrt{x}) - f(3y - 2x\sqrt{x})]\partial_x + [f(3y + 2x\sqrt{x}) + f(3y - 2x\sqrt{x})]\partial_y$$

for for $g = -f$. Now applying symmetry (11.10) with F satisfying $f(u)F'(u) = (F(u))^k$ (see Lemma 14) we complete the proof. The details can be found in the proof of Theorem 30. □

Remark. Real analytic versions of Theorems 30 and 32 are true without the stipulations $y \geq 0$ and $x \geq 0$ respectively.

Remark. One can not extend the claim of Theorem 30 for $y < 0$ in the smooth case for equation (11.7) item c). Its solutions are parabolas $y = x^2/4 + C$. They cross the line $y = 0$ in two points if $C < 0$. If one smoothly deforms equation (11.7) in the domain $y < 0$ then the solution of the deformed equation starting from some point $(x_0, 0)$ with $x_0 < 0$ will not necessarily pass through $(-x_0, 0)$, i.e. this solution returns to a "wrong parabola".

Corollary 2 *If the following conditions hold for implicit ODE (1.47) at a point* $m = (x_0, y_0, p_0) \in M$*:*
1) ODE (1.43) has a hexagonal 3-web of solutions,
2) p_0 *is a double root of (1.43) at* $\pi(m) = (x_0, y_0)$*,*
3) regularity condition (1.45) is satisfied, i.e. $\mathrm{rank}((x, y, p) \mapsto (F, F_p))|_m = 2$*,*
then its criminant is either Legendrian or transverse to the contact plane field in some neighborhood of m

Proof: Denote by C_t the closed set of points on the criminant C, where the contact plane is tangent to C. Suppose m is not a point of $C \setminus C_t$ and not an interior point of C_t. Then m is a boundary point of C_t. Now Theorem 32 implies that for each point m' sufficiently close to m and such that $m' \neq m$, $m' \notin C_t$ equation (1.43) is locally equivalent to the product of the explicit ODE $dx = 0$ and quadratic equation (11.9), i.e. the solutions of the linear factor are tangent to the discriminant curve at $\pi(m')$ and therefore at $\pi(m)$. Further, Theorem 30 implies that for each point m' sufficiently close to m and such that $m' \neq m$, $m' \in C_t$ equation (1.43) is locally equivalent to the product of the explicit ODE $dx = 0$ and quadratic equation (11.1), i.e. the solutions of the quadratic factor are tangent to the discriminant curve at $\pi(m')$ and therefore at $\pi(m)$. But that means that the root p_0 is triple. Thus our assumption is false and the corollary is proved. □

Remark. As follows from the above proof the hypothesis of Corollary 2 also implies that there is no isolated points of tangency of the contact plane and the criminant.

11.2 Normal form for an ordinary cusp point

In this section we use the results of the previous one to establish normal forms for the case of a cusp singularity of the projection π on M. Regularity condition (1.45) for a triple root p_0 implies immediately that the projection π has a cusp point at $m = (x_0, y_0, p_0) \in M$. We start with Legendrian criminant, then consider non-Legendrian criminant and finally show that one can not "glue" Legendrian criminant with non-Legendrian one at the cusp point.

Lemma 15 *If the following conditions hold for implicit ODE (1.43) at a point* $m = (x_0, y_0, p_0) \in M$*:*
1) ODE (1.43) has a hexagonal 3-web of solutions,
2) p_0 *is the triple root of (1.43) at* $\pi(m) = (x_0, y_0)$*,*
3) regularity condition (1.45) is satisfied, i.e. $\operatorname{rank}((x, y, p) \mapsto (F, F_p))|_m = 2$*,*
then it is locally equivalent to

$$p^3 + A(x, y)p + B(x, y) = 0, \tag{11.13}$$

where
1) the projection π *has an ordinary cusp singularity at* $(0, 0, 0)$ *with* $A(0, 0) = B(0, 0) = 0$*,*
2) A, B *are local coordinates at* $(0, 0)$*, i.e.* $\left.\frac{\partial(A,B)}{\partial(x,y)}\right|_{x=y=0} \neq 0$
3) $B_x(0, 0) = 0$*.*

Proof: Since equation (1.43) has the triple root p_0 at $\pi(m)$ it is locally equivalent to some cubic equation (1.47). Further, the coefficient by p^2 in this cubic equation is killed by a coordinate transform of the form $y = f(\tilde{x}, \tilde{y})$, $x = \tilde{x}$, satisfying

$$3f_x(x, y) + a(x, y) = 0.$$

This transform respects the regularity condition. Thus, our implicit equation $F = 0$ becomes

$$F(x, y, p) = p^3 + A(x, y)p + B(x, y) = 0.$$

Without loss of generality it can be assumed that $(x_0, y_0) = (0, 0)$. As the equation $p_0^3 + A(0, 0)p_0 + B(0, 0) = 0$ has a triple root holds $p_0 = 0$. Therefore the functions A, B must also vanish at $(0, 0)$. Now regularity condition (1.45) at $m = (0, 0, 0) \in C$ reads as

$$\operatorname{rank}\left.\begin{pmatrix} A_x p + B_x & A_y p + By & 3p^2 + A \\ A_x & A_y & 6p \end{pmatrix}\right|_{x=y=p=0} = \left.\begin{pmatrix} B_x & By & 0 \\ A_x & A_y & 0 \end{pmatrix}\right|_{x=y=p=0} = 2.$$

Thus claims 1) and 2) are proved. Moreover, the discriminant curve

$$\Delta = \pi(\{(x, y, p) :\ p^3 + A(x, y)p + B(x, y) = 3p^2 + A(x, y) = 0\})$$

has an ordinary cusp at $\pi(m) = (x_0, y_0)$.

If the solutions of equation (11.13) form a hexagonal 3-web the curvature of this 3-web must vanish identically. This is equivalent to the following cumbersome partial differential equation for the functions A, B:

$$\begin{aligned} &(4A^3 + 27B^2)(9BA_{xx} - 2A^2A_{xy} + 6ABA_{yy} - 6AB_{xx} - 9BB_{xy} - 4A^2B_{yy}) + \\ &+108A^2BA_xB_y - 108AB^2A_xA_y + 162B^3A_y^2 + 40A^4A_yB_y - 108A^2BA_x^2 + \\ &+216A^2BB_y^2 - 36A^3B_xB_y + 108A^2BA_yB_x - 378AB^2A_yB_y - 405B^2A_xB_x + \\ &-48A^3BA_y^2 + 8A^4A_xA_y + 243B^2B_xB_y + 84A^3A_xB_x + 324ABB_x^2 = 0. \end{aligned} \tag{11.14}$$

This equation is obtained by direct lengthy but straightforward computation. (Expressions for the corresponding web curvature for a cubic ODE can be also found in [71] and [76]). As was shown above the functions A, B can be taken as local coordinates around $(0, 0)$. Then all partial derivatives of A, B with respect to x and y are smooth functions of A, B. The homogeneous part of second order of Taylor expansion of l.h.s. of (11.14) around $(0, 0)$ is

$$-405B^2A_x(0, 0)B_x(0, 0) + 243B^2B_x(0, 0)B_y(0, 0) + 324AB(B_x(0, 0))^2.$$

It must vanish. In particular, $B_x(0, 0)^2 = 0$ as the coefficient by AB. □

• **The case of Legendrian criminant**

Theorem 33 *If the following conditions hold for implicit ODE (1.43) at a point* $m = (x_0, y_0, p_0) \in M$*:*
1) ODE (1.43) has a hexagonal 3-web of solutions,
2) p_0 *is the triple root of (1.43) at* $\pi(m) = (x_0, y_0)$*,*
3) its criminant C *is a Legendrian curve with* $\operatorname{rank}((x, y, p) \mapsto (F, F_p))|_C = 2$*,*
then it is locally equivalent to the following Clairaut equation

$$P^3 + PX - Y = 0. \tag{11.15}$$

Proof: Lemma 15 reduces equation (1.43) to (11.13) with $A(0,0) = B(0,0) = B_x(0,0) = 0$ and $\left.\frac{\partial(A,B)}{\partial(x,y)}\right|_{x=y=0} \neq 0$. Thus, the tangent plane $T_m M$ to the surface M at $m = (0,0,0)$ is the plane $y = 0$ and the tangent line to the discriminant curve at $(0,0)$ is $y = 0$. (The condition $B_x(0,0) = 0$ for the case of Legendrian criminant can be obtain also by the following geometrical consideration. By Proposition 16 the characteristic field τ given by (1.46) can be smoothly extended to the criminant C. As τ is transverse to C, the projection of the integral curves of τ are smooth curves in $\mathbb{R}^2$ tangent to the discriminant curve Δ at the origin $(0,0)$. As $p_0 = 0$ this implies the claim.) With $B_x(0,0) = 0$ one gets from the regularity condition

$$A_x(0,0) \neq 0, \quad B_y(0,0) \neq 0.$$

Therefore one can choose p, A as local coordinates on the surface M at $m = (0,0,0)$ and A, B as local coordinates on $\mathbb{R}^2, 0$. In these coordinates the projection π is the Whitney map. The criminant is parameterized by p as follows

$$A = -3p^2, \quad B = 2p^3.$$

Its projection is the discriminant curve $\Delta := \{(A,B) : 27B^2 + 4A^3 = 0\}$. The set of points projected to the discriminant curve is the criminant itself and the following curve

$$D := \{(p, A) \in M : A = -\frac{3}{4}p^2\}. \tag{11.16}$$

This follows from the observation that the value $-2p$ is the third root of (11.13) at the discriminant point, where the double root is p. The curve D is tangent to C at 0, thus the characteristic field τ is transverse also to D.

Consider the following map

$$f : (\mathbb{R}^2, 0) \to (M, 0), \ (p, q) \mapsto (p, A) = (p, -q^2 - pq - p^2). \tag{11.17}$$

This map has a fold singularity on the line $L_1 := \{(p,q) : \ p + 2q = 0\}$. This line is mapped by f to the curve D since $-q^2 - pq - p^2 = -\left(q + \frac{p}{2}\right)^2 - \frac{3}{4}p^2$. Note that if $f(p,q) = (p, A)$, then $f^{-1}(p, A) = \{(p,q) \cup (p, -p-q)\}$. The pull back $\tilde{\tau}$ of the characteristic field τ by f^* from $M, 0$ to $\mathbb{R}^2, 0$ must be tangent to the kernel of df, i.e. to the vector field ∂_q, since τ is transverse to D. Moreover, the foliation $\mathcal{F}_1$ by the integral curves of $\tilde{\tau}$ is invariant with respect to the following linear involution

$$g_1 : \mathbb{R}^2 \to \mathbb{R}^2, \ (p,q) \mapsto (p, -p-q).$$

Consider also the following two linear involutions

$$g_2 : \mathbb{R}^2 \to \mathbb{R}^2, \ (p,q) \mapsto (-p-q, q), \quad g_3 : \mathbb{R}^2 \to \mathbb{R}^2, \ (p,q) \mapsto (q, p). \tag{11.18}$$

The linear maps g_1, g_2, g_3 generate the group $\mathbb{D}_3$, the symmetry group of equilateral triangle, which can be viewed as the group of linear transformations of the plane $p + q + r = 0$ in $\mathbb{R}^3$, generated by permutations of the coordinates (p, q, r) in $\mathbb{R}^3$. The orbit of a point (p, q) under this group action is the inverse image of the point $(A, B) = \pi(f(p, q)) \in \mathbb{R}^2$ under the Vieta map $V := \pi \circ f$. Therefore the three foliations $\mathcal{F}_1$, $\mathcal{F}_2 := g_2(\mathcal{F}_1)$ and $\mathcal{F}_3 := g_3(\mathcal{F}_1)$ form a hexagonal 3-web. Moreover, this 3-web is not singular at $(0, 0)$ and has the symmetry group $\mathbb{D}_3$ generated by $\{g_1, g_2, g_3\}$. Note that for Clairaut equation (11.15) the above defined three foliations are $p = const$, $p + q = const$ and $q = const$ respectively.

Now we are ready to construct the diffeomorphism φ that transforms the given ODE to normal form (11.15). Consider a domain $U, 0 \subset \mathbb{R}^2, 0$ such that the 3-web formed by the foliations $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ is regular in U. Let γ_1 be the integral curve of $\tilde{\tau}$ that passes through the origin. Pick up a point $u = (p_1, q_1) \in \gamma_1$ on this curve and draw the *Brianҫon* hexagon around 0 through u. (Let us recall the construction of the Brianҫon hexagon: one draws three curves γ_i, $i = 1, 2, 3$ of the foliations $\mathcal{F}_i$, $i = 1, 2, 3$ through the origin, picks up a point on one of this curves, say γ_1, and then goes around the origin along the foliation curves, swapping the family whenever one meets one of the fixed curves γ_i. The web is hexagonal iff one gets a closed hexagonal figure for any choice of the central "origin" point and u. See Fig. 11.1 on the left.) Let us choose u so that the following conditions hold:
1) $q_1 > 0$,
2) the Brianҫon hexagon around 0 through u is contained in U.

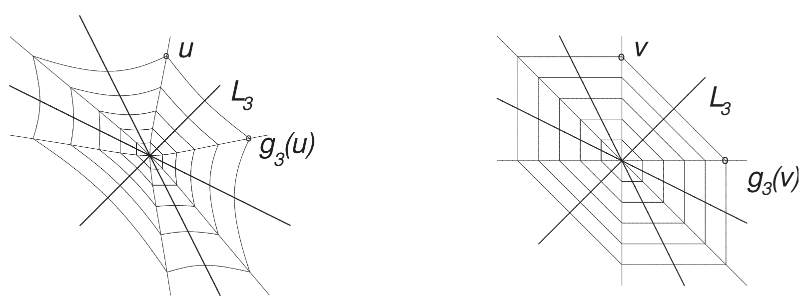

Figure 11.1: Brianҫon's hexagons and their inverse images under the diffeomorphism ψ with $u = \psi(v)$.

Then there is a unique local homeomorphism $\psi : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ such that
1) $\psi(0, 1) = u$,
2) $\psi(1, 0) = g_3(u)$,
3) it maps the foliations $p = const$, $p + q = const$ and $q = const$ to the foliations $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ respectively. (See Fig. 11.1). In fact, the points u and $g_3(u)$ lies on the same curve of the foliation $\mathcal{F}_2$ since the involution g_3 is a symmetry of $\mathcal{F}_2$. Further, there is a unique diffeomorphism, mapping the triangle $(0, 0), (0, 1), (1, 0)$ to the "triangle" $(0, 0), u, g_3(u)$ formed by the curves of the foliations $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ (see [18] p.15). This map is uniquely extended to the whole hexagon. Moreover, the constructed homeomorphism ψ is equivariant with respect to the action of $\mathbb{D}_3$ defined above. The map ψ is smooth (analytic) if the foliations $\mathcal{F}_1$,$\mathcal{F}_2$,$\mathcal{F}_3$ are smooth (analytic). Indeed, according to [18] p.155, there exists a smooth (analytic) map, taking the foliations $p = const$, $p + q = const$ and $q = const$ to $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ respectively, and this map is uniquely defined by specifying the inverse image of u. Thus, this map should coincide with the above homeomorphism ψ.

Now consider the map:

$$\pi \circ f \circ \psi : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0, \quad (P,Q) \mapsto (A,B) = \pi(f(\psi(P,Q))).$$

In coordinates it reads as

$$A = \alpha(P,Q), \;\; B = \beta(P,Q).$$

Observe that the above map is symmetric with respect to the action of $\mathbb{D}_3$. Then by the results on smooth functions, invariant with respect to finite group action, the functions α and β must depend only on the basic invariants of the above $\mathbb{D}_3$ group action (see [57]) and [88]):

$$A = \tilde{\alpha}(X,Y), \;\; B = \tilde{\beta}(X,Y),$$

where

$$X = -P^2 - PQ - Q^2, \quad Y = PQ(P+Q). \tag{11.19}$$

We claim that $\tilde{\alpha}, \tilde{\beta}$ are the components of required diffeomorphism φ. To prove that consider the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^2, 0 & \stackrel{\psi}{\longrightarrow} & \mathbb{R}^2, 0 \\ \downarrow V & & \downarrow \pi \circ f \\ \mathbb{R}^2, 0 & \stackrel{\varphi}{\longrightarrow} & \mathbb{R}^2, 0 \end{array} \tag{11.20}$$

where V is Vieta's map (11.19). Applying the same results on symmetric functions to $\theta := V \circ \psi^{-1}$ we see that the differentiable map θ is inverse to φ inside the "cusped" domain where our ODE has 3 distinct real solutions. This completes the proof in the real analytic case. For the smooth case we apply φ and for the reduced equation we consider the first integral of the direction field τ that coincides with p on the part M_3 of M that is projected to the domain with three real roots. Further, it is easy to construct through homotopy the π-lowerable diffeomorphism φ' of M that is identity on M_3 and moves the integral curves of τ to that of (11.15). The searched for diffeomorphism is $\varphi' \circ \varphi$. $\square$

• **The case of non-Legendrian criminant**

Theorem 34 *If the following conditions hold for implicit ODE (1.43) at a point* $m = (x_0, y_0, p_0) \in M$*:*
1) ODE (1.43) has a hexagonal 3-web of solutions,
2) p_0 *is the triple root of (1.43) at* $\pi(m) = (x_0, y_0)$*,*
3) the criminant C *is transverse to the contact plane field in some punctured neighborhood of* m *and* $\mathrm{rank}((x,y,p) \mapsto (F, F_p))|_C = 2$*,*
then it is locally equivalent to

$$P^3 + 2PX + Y = 0, \tag{11.21}$$

within the domain, where (11.21) has three real roots, if F *is smooth,*
and in some neighborhood of $(0,0)$*, if* F *is real analytic.*

Proof: We follow the proof scheme of Theorem 33. Namely we consider the pull-back of the form $dy - pdx$ to $\mathbb{R}^2, 0$ by the Vieta map $\pi \circ f$, where f is defined by (11.17), duplicate this pull-back form by linear involutions (11.18) and find a local diffeomorphism of $\mathbb{R}^2, 0$ matching "lifted" 3-web of our equation and that of (11.21). The difference to the previous case of Legendrian criminant is that now the web is singular; each foliation $\mathcal{F}_i$ has a saddle singular point at $f^{-1}(m)$. Therefore the classical results on hexagonal 3-web are not of much use to find the diffeomorphism "upstairs". We construct it through a homotopy of the first integrals of the corresponding foliations.

•*Differential forms of the foliations.*
By Lemma 15 equation (1.43) is equivalent to (11.13) with

$$A(0,0) = B(0,0) = B_x(0,0) = 0, \ \frac{\partial(A,B)}{\partial(x,y)}\bigg|_0 \neq 0, \ A_x(0,0) \neq 0, \ \ B_y(0,0) \neq 0.$$

Thus, the tangent plane T_mM to the surface M at $m = (0,0,0)$ is the plane $y = 0$ and the tangent line to the discriminant curve at $(0,0)$ is $y = 0$. Therefore one can choose p, A as local coordinates on the surface M, m and A, B as local coordinates on $\mathbb{R}^2, 0$. Thus, one has $x = X(A,B), \ \ y = Y(A,B)$, where $A = -p^2 - q^2 - pq$ and $p, q, r = -p-q$ are roots of (11.13). As easily follows from Theorem 32, the kernel of the pull-back form $\pi^*(dy - pdx)$ is tangent to the curve D defined by (11.16). That means that the kernel of the form

$$f^*(\pi^*(dy-pdx)) = (2p+q)(-Y_A+pX_A+qY_B-pqX_B)dp+(2q+p)(-Y_A+p(X_A+Y_B)-p^2X_b)dq$$

is tangent to the line $L_1 := \{(p,q) : \ \ p + 2q = 0\}$. Writing the above form as $\omega_1 := (2p + q)P(p,q)dp + (2q+p)Q(p,q)dq$ with suitable P, Q and passing to the coordinates (q,s), where $s := 2q + p$, one gets

$$\omega_1 = (2s - 3q)Pds + (sQ - 2(2s - 3q)P)dq.$$

Hence the tangency condition implies $P|_{s=0} = 0$. By the Hadamard lemma $P = s\tilde{P}$ hence one obtains

$$\omega_1 = s\{(2s - 3q)\tilde{P}ds + (Q - 2(2s - 3q)\tilde{P})dq\}.$$

Now consider the expression for $P = -Y_A + (s-2q)X_A + qY_B - (s-2q)qX_B$. One has $Y_A = 0$, $X_A \neq 0$ from $A_xB_y \neq 0$, $B_x = 0$. Since A is quadratic and B is cubic in p, q, the term Y_A does not have linear terms in p, q. This implies $\tilde{P}(0,0) \neq 0$. Using again the condition on L_1 one obtains

$$Q - 2(2s - 3q)\tilde{P} = s\tilde{Q}.$$

Let as normalize forms vanishing each on its own family of solutions to satisfy $\sigma_1 + \sigma_2 + \sigma_3 = 0$:

$$\sigma_1 = (q - r)(dy - pdx), \quad \sigma_2 = (r - p)(dy - qdx), \quad \sigma_3 = (p - q)(dy - rdx). \tag{11.22}$$

As shown above the pull-back of σ_1 is

$$\tilde{\sigma_1} = (2q + p)^2\{(2p + q)\tilde{P}dp + (2(2p + q)\tilde{P} + (2q + p)\tilde{Q})dq\} \tag{11.23}$$

•*Connection form.*
Following [19] consider the area form

$$\Omega := \tilde{\sigma_1} \wedge \tilde{\sigma_2} = \tilde{\sigma_2} \wedge \tilde{\sigma_3} = \tilde{\sigma_3} \wedge \tilde{\sigma_1} = (Y_AX_B - Y_BX_A)(p-q)^2(2p+q)^2(2q+p)^2dp \wedge dq$$

and the connection form

$$\gamma := h_2\tilde{\sigma_1} - h_1\tilde{\sigma_2} = h_3\tilde{\sigma_2} - h_2\tilde{\sigma_3} = h_1\tilde{\sigma_3} - h_3\tilde{\sigma_1},$$

where h_i are defined by

$$d\tilde{\sigma_i} = h_i\Omega.$$

Using (11.23) on obtains by direct calculation that

$$h_1 = \frac{R_1}{(2p+q)^2(p-q)^2},$$

where R_1 is a smooth function of p, q. Applying the cyclic permutation $p \to q, \ q \to r, \ r \to p$, one gets: $\tilde{\sigma_2} = (2p+q)^2\bar{\sigma_2}$, $h_2 = \frac{R_2}{(2q+p)^2(p-q)^2}$, where R_2 and $\bar{\sigma_2}$ are smooth. Therefore

$$\gamma = \frac{\bar{\gamma}}{(p-q)^2},$$

with a smooth form $\bar{\gamma}$. Observe that the connection form γ is symmetric with respect to the linear transformation group $\mathbb{D}_3$, generated by g_1, g_2, g_3. Therefore γ is smooth, i.e. $(p-q)^2$ divides $\bar{\gamma}$.

•*Existence of first integrals.*
As the web is hexagonal the connection form γ is closed. Therefore there exists a unique $\mathbb{D}_3$-symmetric function μ, satisfying

$$d\mu = \mu\gamma, \quad \mu(0,0) = 1.$$

Further, the forms $\mu\tilde{\sigma_i}$ are also closed. Thus, we can locally define functions u_i by

$$d(u_i) = \mu\tilde{\sigma_i}, \quad u_i(0,0) = 0,$$

satisfying the following equation, which is equivalent to the hexagonality of the web:

$$u_1 + u_2 + u_3 \equiv 0. \tag{11.24}$$

Observe that the function u_1 is skew-symmetric with respect to g_1:

$$g_1^*(u_1) = -u_1. \tag{11.25}$$

This follows from (11.22) and from the invariance of μ. Applying Hadamard's trick one estimates u_1 as follows:

$$u_1(p,q) = \int_0^1 \frac{d}{dt} u_1(tp,tq)dt = \int_0^1 \left(p\frac{\partial}{\partial p} u_1(tp,tq) + q\frac{\partial}{\partial q} u_1(tp,tq) \right) dt =$$

$$= (2q+p)^2 \int_0^1 \mu(tp,tq) \left(p(2p+q)\tilde{P}(tp,tq) + q(2(2p+q)\tilde{P}(tp,tq) + (2q+p)\tilde{Q}(tp,tq)) \right) t^3 dt.$$

Collecting similar terms, using $\tilde{P}(0,0) \neq 0$ and integrating, one has

$$u_1(p,q) = (2q+p)^3((2p+q)\hat{P}(p,q) + q\hat{Q}(p,q)), \quad \text{where} \quad \hat{P}(0,0) \neq 0. \tag{11.26}$$

•*Properties of the first integrals.*
It follows from Malgrange's Preparation Theorem that any smooth function F of (p,q) can be represented in the form

$$\begin{aligned} F(p,q) = F_0(A,B) + pF_1(A,B) + qF_2(A,B) + \\ pqF_3(A,B) + q^2F_4(A,B) + pq^2F_5(A,B), \end{aligned} \tag{11.27}$$

with smooth functions F_i. In fact, the identities $p^2 = -pq - q^2 - A$, $p^3 = -pA - B$, $p^2q = -pq^2 + B$, $q^3 = -qA - B$ imply $\langle p,q\rangle^4 \subset \langle A,B\rangle$ and $\mathcal{E}(\mathbb{R}^2)/\langle A,B\rangle = \mathbb{R}\{1,p,q,pq,q^2,pq^2\}$. (Here $\mathcal{E}(\mathbb{R}^2)$ is the local algebra of smooth function germs at $(0,0)$; $\langle p,q\rangle$ its maximal ideal, generated by the coordinate functions p and q; the maps A, B are defined as follows: $A : (p,q) \mapsto -p^2 - pq - q^2$, $B : (p,q) \mapsto p^2q + q^2p$, and $\mathbb{R}\{1,p,q,pq,q^2,pq^2\}$ is the real vector subspace of $\mathcal{E}(\mathbb{R}^2)$, spanned by $1,p,q,pq,q^2,pq^2$.) Moreover, inside the "cusped" domain with 3 real distinct solutions of our ODE the functions F_i are uniquely determined by F. For $F = u_1$ property (11.25) implies $F_4 = -F_3$, $F_2 = 2F_1 - AF_5$, $F_0 = -AF_3 - BF_5/2$. Applying $-g_2^*$ and $-g_3^*$ to u_1 one gets the other two first integrals u_2 and u_3, whose representations in form (11.27) are easily read from the representation of u_1. Now identity (11.24) implies $F_5 = 0$. Thus,

$$u_1(p,q) = (2q+p)(F_1(A,B) + pF_3(A,B)).$$

Using (11.26) one can write

$$F_1(A,B) + pF_3(A,B) = (2q+p)^2 G(p,q).$$

Representing the function G as

$$G(p,q) = G_0(A,B) + pG_1(A,B) + qG_2(A,B) + pqG_3(A,B) + q^2G_4(A,B) + pq^2G_5(A,B)$$

and substituting this representation into the above equation, one obtains

$$G(p,q) = \frac{4}{3}AG_4(A,B) + pG_1(A,B) + pqG_4(A,B) + q^2G_4(A,B). \tag{11.28}$$

•*Equivariant homotopy.*
For equation (11.21) the first integral u_1 is $u_0(p,q) := p(2q+p)^3$. Let us scale u_1 so that $\hat{P}(0,0) = \frac{1}{2}$ in (11.26). We claim that the family of functions

$$u_t(p,q) := u_0(p,q) + t(u_1(p,q) - u_0(p,q))$$

is equivariantly $\mathcal{R}$-trivial, i.e. for any $t \in [0,1]$ there is a diffeomorphism ψ_t equivariant with respect to the above defined $\mathbb{D}_3$ group action such that

$$u_t \circ \psi_t = u_0.$$

To prove this it is enough to find $\mathbb{D}_3$-equivariant vector field $\xi(p,q,t)\partial_p + \eta(p,q,t)\partial_q$ satisfying the following homotopy equation:

$$\xi(p,q,t)\frac{\partial u_t(p,q)}{\partial p} + \eta(p,q,t)\frac{\partial u_t(p,q)}{\partial q} + \frac{\partial u_t(p,q)}{\partial t} = 0, \quad \xi(0,0,t) = \eta(0,0,t) = 0.$$

A general form of a $\mathbb{D}_3$-equivariant vector field is given by

$$\xi = p\alpha(A,B) + \left(\frac{A}{3} + pq + q^2\right)\beta(A,B), \quad \eta = q\alpha(A,B) - \left(\frac{2}{3}A + q^2\right)\beta(A,B), \tag{11.29}$$

(see [16] or derive it from the representations of ξ, η in form (11.27)). Observe that the difference $(u_1(p,q) - u_0(p,q)) = \frac{\partial u_t(p,q)}{\partial t}$ also has form (11.28):

$$u_1(p,q) - u_0(p,q) = \frac{4}{3}AL(A,B) + pK(A,B) + pqL(A,B) + q^2L(A,B)$$

with $K(0,0) = 0$ due to the chosen scaling of u_1. Solving the homotopy equation yields the following expressions for α and β:

$$\alpha = \frac{6K + (6K^2 + 2A^2(K_BL - KL_B) + 9B(KL_A - K_AL) + 10AL^2)t}{M},$$

$$\beta = \frac{-12L + (6A(KL_A - K_AL) + 9B(KL_B - K_BL) + 3KL)t}{M},$$

where

$$\begin{gathered} M = -24 + [-48K - 12AK_A - 18B(K_B + 2L_A) + 8A^2L_B]t + \\ +[-24K^2 - 2A(6KK_A + 25L^2) + 3B(15K_AL - 6KK_B - 12KL_A) + \\ +2A^2(4KL_B - 10LL_A - 5K_BL) - 30ABLL_B + (27B^2 + 4A^3)(K_AL_B - K_BL_A)]t^2. \end{gathered}$$

M does not vanish at $(0,0)$ since $K(0,0) = 0$. The claim on $\mathcal{R}$-triviality of the family $u_t(p,q)$ is proved.
•*Diffeomorphism.*
We have proved that the diffeomorphism

$$\psi := \psi_1$$

maps the fibres of u_0, i.e. the curves $\{(p,q) : u_0(p,q) = const\}$ to those of u_1. Therefore, being equivariant, φ maps the foliations $\mathcal{F}_i$ of (11.21) to that of our equation (11.13). Now diagram (11.20) defines again the desired diffeomorpfism φ. $\square$

Remark. A picture of the $\mathbb{D}_3$-symmetric hexagonal 3-web, defined by the solutions of (11.21) and lifted to the plane $p+q+r=0$, is presented in Fig. 11.2 on the left. It consists of 3 foliations, one of them is shown in Fig. 11.2 in the center. On the right is the fundamental domain of $\mathbb{D}_3$-group (compare with Fig. 1.10 on the right). The flower-like form on the left suggests that the web is actually symmetric with respect to the symmetry group $\mathbb{D}_6$ of regular hexagon. In fact, it is the case since the fibers of the first integrals are permuted by the following symmetry $g_4 : \mathbb{R}^2 \to \mathbb{R}^2,\ (p,q) \mapsto (-p,-q)$.

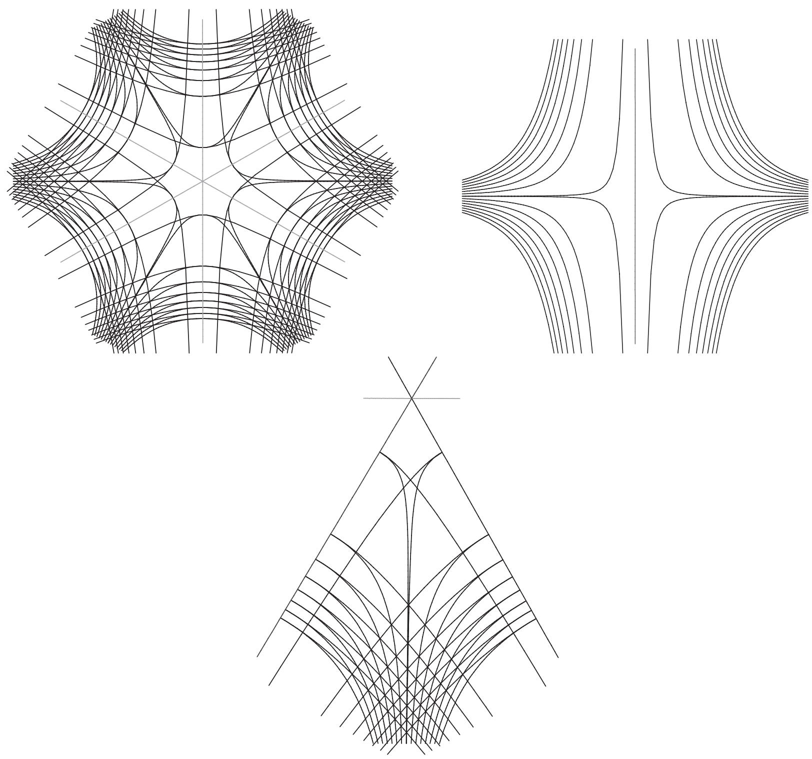

Figure 11.2: $\mathbb{D}_3$-symmetric hexagonal web of 3 foliations with saddle singularities (left), one of the foliations (center), and the fundamental domain of $\mathbb{D}_3$ (right).

11.3 Proof of the classification theorem

Now we can prove Theorem 9. If a point $m = (x_0, y_0, p_0) \in M$ is regular then our equation is locally equivalent to (1.50) case v) by definition. If p_0 is a double root then the regularity condition (1.45) implies that m is a fold point of the projection π. Futher, Corollary 2 implies that the criminant is either Legendrian or transverse to the contact plane field in some neighborhood of m. Thus by Theorems 30 and 32 the equation is locally equivalent either to (1.50) case iii) or to (1.50) case iv). Finally if p_0 is a triple root and the criminant is either Legendrian or transverse to the contact plane field in some punctured neighborhood of m then by Theorems 33 and 34 the equation is locally equivalent either to (1.50) case i) or to (1.50) case ii). To complete the proof we show that Legendrian and non-Legendrian parts of the criminant can not be glued together at a cusp point. By Lemma 15 our equation is equivalent to (11.13) with $(x_0, y_0, p_0) = (0, 0, 0)$, $B_x(0, 0) = 0$. Suppose the criminant is transverse to the contact plane field for $p > 0$ and Legemdrian for $p \leq 0$. For any point $m' \neq m$ with $p > 0$ on the curve D defined by (11.16) the direction field τ is tangent to D by theorem 32. This condition reads as

$$\left(d(A + \frac{3}{4}p^2) \wedge (dy - pdx) \right)\Big|_D = 0.$$

In coordinates p, x on M it can be rewritten as follows

$$\left((dA + \frac{3}{2}pdp) \wedge ((3p^2 + A)dp + (B_x + (A_x + B_y)p + A_y p^2)dx) \right)\Big|_D = 0.$$

Substituting $A = -\frac{3}{4}p^2$ and

$$dA = A_x dx + A_y dy = A_x dx - \frac{A_y}{B_y + A_y p}((3p^2 + A)dp + (B_x + A_x p)dx)$$

into this equation one gets

$$\frac{3}{2}p^2(A_x B_y + A_y(A_x + B_y)p + A_y^2 p^2) - p(B_x + (A_x + B_y)p + A_y p^2)(B_y + A_y p)\Big|_D = 0.$$

Parameterizing the curve D by p, expanding the above equation by Tailor's formula at $p = 0$ and equating the coefficient by p^2 to 0 one obtains

$$B_y(\frac{3}{2}A_x - (A_x + B_y))\Big|_{x=y=0} = 0,$$

which implies

$$2B_y(0, 0) - A_x(0, 0) = 0,$$

since $B_y(0, 0) \neq 0$. (We have used the Taylor formula $B_x = B_{xA}A + B_{xB}B + ... = B_{xA}(-3/4p^2) + B_{xB}(-1/4p^3) + ...$)

On the other hand, for any point $m' \neq m$ with $p \leq 0$ the contact form vanishes on the criminant:

$$dy - pdx|_C = 0.$$

In coordinates p, x on M it can be rewritten as follows

$$(B_x + (A_x + B_y)p + A_y p^2)\big|_C = 0.$$

Now the Tailor expansion at $p = 0$ for C parameterized by p gives $A_x + B_y = 0$ (B_x does not have linear in p terms). Comparing with the condition above on the non-Legendrian part one gets $A_x(0, 0) = 0$ and therefore $B_y(0, 0) = 0$, which contradicts Lemma 15. □

Remark. Unfortunately, the annoying stipulation in Theorem 9 for the smooth case (1.50) i) can not be omitted to guarantee the existence of the diffeomorphism φ reducing ODE under consideration to (1.50) i) in some neighborhood of m if one stays within the framework of geometric Definition 13. A necessary condition for that is the existence of the first integral of τ in the form $f(p, x)^2 g(p, q)^3 = \tilde{\varphi}^*(p^2(x + 3p^2/8)^3)$. Here $\tilde{\varphi}$ is the lift to M of the searched for diffeomorphism and $p^2(x + 3p^2/8)^3$ is the first integral of τ for (1.50) i). It is not hard to find a counterexample which does not have such an integral in the form $f(p, x)^2 g(p, q)^3$ with non-vanishing df, dg at $(0, 0)$. This drawback is repaired as follows. One replace definition 13 with a less geometric one.

Definition 22 *We say that implicit ODE (11.13) has a hexagonal 3-web of solutions if A, B satisfy PDE (11.14) and the domain, where (11.13) has 3 real roots p_1, p_2, p_3 is not empty.*

The proof of Theorem 34 is easily modified for the case of one real root $p_1 = p$ and two complex conjugated roots $p_{2,3} = -p/2 \pm iz$. The form σ_1 turns out to be pure imaginary but the connection form γ is real. All analytical properties being the same, one finds the diffeomorpfism φ similarly through homotopy.

11.4 Concluding remarks on implicit ODEs with hexagonal web of solutions

• **Symmetries of the normal forms.** The solutions of equation (1.50) case v) are the lines $dx = 0$, $dy = 0$ and $dx + dy = 0$. Its symmetry group is generated by the following operators:

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + y\partial_y.$$

Thus the symmetry pseudogroup of a cubic implicit ODE with hexagonal 3-web of solutions is at most 3-dimensional. In a neighborhood of the projection of a regular point $m \in M$ it is generated by the above three operators X_i in suitable coordinates. The coordinate change becomes singular on the discriminant curve and not all symmetry operators "survive" at $\pi(m) \in \Delta$. The symmetry pseudogroups of equations (1.50) case iii) and (1.50) case iv) at a fold point are generated by

$$Y_1 = x\partial_x + 2y\partial_y, \quad Y_2 = \partial_x$$

and

$$Y_1 = 2x\partial_x + 3y\partial_y, \quad Y_2 = \partial_y,$$

respectively. This easily follows from Propositions 17 and 18. Irreducible equations (1.50) case i) and (1.50) case ii) have only one-dimensional symmetry pseudogroup at $(0, 0)$:

$$Z = 2x\partial_x + 3y\partial_y.$$

• **Analytic properties.** All equations in the given normal forms are integrable in elementary functions.

• **Implicit cubic ODEs with singular surfaces M.** Suppose our cubic ODE factors out to 3 linear in p terms $p - f_i(x, y)$ such that 2 of 3 smooth surfaces $M_i := \{(x, y, p) : p = f_i(x, y)\}$ intersect transversally along a non-singular curve, the solutions of these 2 factors being transverse to the curve projection into the plane. Then one can bring these two factors to the forms $p = 0$ and $p = 2x$ respectively. The symmetry pseudogroup of the quadratic ODE $p(p-2x) = 0$ is $\tilde{y} = F(y)$, $\tilde{x} = \sqrt{F(y) - F(y - x^2)}$. If our cubic equation has a hexagonal 3-web of solutions, then its third factor is generated by the vector field $(\alpha(y-x^2)+\beta(y))\partial x+2x\beta(y)\partial_y$. As the functions α, β are arbitrary we can hope to "kill" only one of them by the above mentioned symmetry. Thus, a general classification of all cubic ODEs will have functional moduli even if one impose hexagonality condition. (Note that if the third family of solutions in this example is transverse to the first two, we have $\beta = -\alpha$ and one gets a finite classification list.)

• **Other examples.** The proof of Theorem 34 suggests the following procedure to generate cubic ODEs with a hexagonal 3-web of solutions: start with a function $F(p, q)$ written in form (11.27) with $A = p^2 + pq + q^2$, $B = pq(p + q)$, define $G(p, q) := F(q, -p - q)$, $H(p, q) := F(-p - q, p)$ and solve the following equations for F_i:

$$g_3^*(F) = \pm F, \quad F + G + H \equiv 0. \qquad (11.30)$$

This gives four of six coefficients F_i as linear combinations of the remaining two "free" functions of A, B. Then the fibers of F, G, H define a hexagonal 3-web, symmetric with respect to $\mathbb{D}_3$-group action, generated by the involutions g_1, g_2, g_3. The image of this web under Vieta map $(p,q) \mapsto (p^2+pq+q^2, pq(p+q))$ is a hexagonal 3-web of solutions of some implicit cubic ODE. For example, starting with $F = p - q$ one gets the following equation:

$$yp^3 - \frac{2}{3}x^2p^2 + xyp + \frac{1}{27}(2x^3 - 27y^2) = 0.$$

The solution 3-web of this equation is *dual* to that of (1.50) case i) (see [76]). Its surface M is not smooth at $(0,0,0)$. Note that one can also start with F such that the "free" coefficients F_i have poles at 0. This approach linearizes the problem of finding local solutions of nonlinear PDE (11.14). In the space of functions F satisfying (11.30) acts the pseudogroup of $\mathbb{D}_3$-equivariant transformations with the tangent space generated by vector fields defined by (11.29). General classification of such functions with this equivalence group seems rather unpromising since the orbit codimension quickly becomes infinite.

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