# On formal local cohomology, colocalization and endomorphism ring of top local cohomology modules

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# Dedication

**To Ferdowsi the Great, Iranian Poet.**

**To My parents, Ali and Simin.**

**To my wife, Sanaz.**

# Acknowledgment

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# Contents

# Chapter 1

# Introduction

The main objects of study in this thesis are local cohomology modules. We write $H^i_{\mathfrak{a}}(M)$ for the $i$th local cohomology of a module $M$ with respect to some ideal $\mathfrak{a}$. We refer the reader to see [Br-Sh], [Gr2], [Sch4], [Hun2] and [Eis] as suitable sources to study local cohomology and related subjects. Let us first introduce the subject and main problems. After this we will present some known related results and finally we will give a summary of the results obtained in this work.

Local cohomology was introduced by Grothendieck [Gr], in the early 1960s, in part to answer the following conjecture of Pierre Samuel:

**Conjecture 1.0.1.** *Let $R$ be a Noetherian local ring and $\widehat{R}$ its completion with respect to the maximal ideal. If $\widehat{R}$ is a complete intersection and for each prime ideal $P$ of $R$ of height $\leq 3$, $R_P$ is a UFD, then $R$ is a UFD.*

Among many other attributes, local cohomology allows one to answer many seemingly difficult questions. A good example of such a problem, where local cohomology provides a partial answer, is the question of how many generators ideals have up to radical. In general, if $\mathfrak{b}$ is an ideal of a ring $R$, the radical of $\mathfrak{b}$ is the ideal

$$\operatorname{Rad}\mathfrak{b} = \{x \in R : x^m \in \mathfrak{b} \text{ for some } m\}.$$

We say an ideal $\mathfrak{b}$ is generated up to radical by $n$ elements if there exist $x_1, ..., x_n \in \mathfrak{b}$ such that $\operatorname{Rad}(\mathfrak{b}) = \operatorname{Rad}(x_1, ..., x_n)$. For example, the ideal $\mathfrak{b} \subseteq k[x, y]$ generated by $x^2, xy, y^2$ is generated up to radical by the two elements $x, y$. Recall that the radical of an ideal $\mathfrak{a}$ is the intersection of all primes ideals which contain $\mathfrak{a}$.

Given an ideal $\mathfrak{a}$ what is the least number of elements needed to generate it up to radical? A particular example of this problem is the following: let $R = k[x, y, u, v]$ be a polynomial ring in four variables over the field $k$. Consider the ideal $\mathfrak{a} = (xu, xv, yu, yv)$. This ideal is its own radical, i.e. $\mathfrak{a} = \text{Rad}(\mathfrak{a})$. The four given generators of $\mathfrak{a}$ are minimal. On the other hand, it can be generated up to radical, by the three elements $xu, yv, xv + yu$. This holds since $(xv)^2 = xv(xv + yu) - (xu)(yv) \in (xu, yv, xv + yu)$. Are there two elements which generate it up to radical? Could there even be one element which generates $\mathfrak{a}$ up to radical?

The answer to the last question is no, there cannot be just one element generating the ideal $\mathfrak{a}$ up to radical, due to an obstruction first proved by Krull, namely the height of the ideal. Krull's famous height theorem states:

**Theorem 1.0.2.** *(Generalized principal ideal Theorem) Let $R$ be a Noetherian ring and $\mathfrak{a} = (x_1, ..., x_n)$ be an ideal generated by $n$ elements. If $\mathfrak{p}$ is a minimal prime over $\mathfrak{a}$, then the height of $\mathfrak{p}$ is at most $n$. In particular, if an ideal $\mathfrak{a}$ is generated up to radical by $n$ elements, then the height of $\mathfrak{a}$ is at most $n$.*

In the example we are considering, the height of $\mathfrak{a}$ is two as it is the intersection of the two height two ideals $(x, y)$ and $(u, v)$. Krull's height theorem implies that two is the smallest number of polynomials which could generate $\mathfrak{a}$ up to radical. This still begs the question, are there two polynomials $F, G \in \mathfrak{a}$ such that $\text{Rad}(F, G) = \mathfrak{a}$?

Trying to find two such polynomials $F, G$ by some type of random search would be hard, if not impossible. Of course if there are no such polynomials, no search would find them, but even if two such polynomials do exist, it is likely no random search would find them. The problem is that these polynomials would normally be extremely special, so that writing down general polynomials in $\mathfrak{a}$ would not work. Instead, we would like to find, in some cohomology theory, an obstruction to being generated up to radical by two elements. Local cohomology provides such an obstruction. To a ring $R$ and ideal $\mathfrak{b}$, we will associate for $i \geq 0$ modules $H^i_{\mathfrak{b}}(R)$ with the properties that

(1) $H^i_{\mathfrak{b}}(R) = H^i_{\text{Rad}(\mathfrak{b})}(R)$, and

(2) if $\mathfrak{b}$ is generated by $k$-elements, then $H^i_{\mathfrak{b}}(R) = 0$ for all $i > k$.

Finally, for $\mathfrak{a} = (xu, xv, yu, yv)$, we will prove that $H^3_{\mathfrak{a}}(R) \neq 0$, and therefore $\mathfrak{a}$ cannot be generated up to radical by two elements.

Item (2), is one of the most powerful tools in local cohomology. In a view of above notes, we would like to extend the description to the above question to this question that how many equations it takes to define an algebraic set X set-theoretically over an algebraically closed field. Of course $X$ can be defined by $n$ equations if and only if there is an ideal $\mathfrak{c}$ with $n$ generators, having the same radical as $I(X)$, the ideal of $X$. Since the local cohomology $H^i_{\mathfrak{a}}(M)$ depends only on the radical of $\mathfrak{a}$, we would have $H^i_{I(X)}(M) = H^i_{\mathfrak{c}}(M) = 0$ for all $i > n$ and all modules $M$. See [Schm-Vog] and [St-Vog] for some examples where this technique is used, and [Lyu] for a recent survey including many pointers to the literature.

For an $R$-module $M$ and an ideal $\mathfrak{a}$, consider the family of local cohomology modules $\{H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$ . For every $n$ there is a natural homomorphism $H^i_{\mathfrak{m}}(M/\mathfrak{a}^{n+1}M) \to H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ such that the family forms a projective system. The projective limit $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of $M$ with respect to $\mathfrak{a}$. Formal local cohomolgy modules used by Peskine and Szpiro in [Pes-Szp] when $R$ is a regular ring. Recently Schenzel [Sch] has defined formal local cohomology modules for a local ring $(R, \mathfrak{m})$ and a finitely generated $R$-module $M$. For more information see chapter three.

## 1.1 Objectives and conclusions

In the sequel, we are going to introduce the considered problems and results in this work:

• In Chapter 2 we introduce the definitions and notations will be used throughout this work.

At first we give the definition of local cohomology modules in conjunction with some of their properties. Next the concept of colocalization which is introduced by A. Richardson will be considered. Richardson's definition has a great advantage in contrast to the previous definitions, i.e. it preserves Artinian modules through the colocalization.

• Important problems concerning local cohomology modules are vanishing, finiteness and Artinianness results (e.g. [Hun]). Not so much is known about the Artinianness of formal local cohomology modules. In [Asgh-Divan] Asgharzadeh and Divani-Aazar have investigated some properties of these kind of modules. For instance they showed that $\mathfrak{F}_{\mathfrak{a}}^{\dim M}(M)$ is Artinian ( [Asgh-Divan, Lemma 2.2]), but $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is not Artinian in general, at $i = \operatorname{fgrade}(\mathfrak{a}, M)$ and $i = \dim M/\mathfrak{a}M$ where they are the first respectively last non-zero amount of formal local cohomology modules (cf. [Asgh-Divan, Theorem 2.7]). We pursue this line to find out conditions for Artinianness of formal local cohomology modules. As a main result in section 3.2 we have following Theorem:

**Theorem 1.1.1.** *(cf. Theorem 3.2.4) Let $(R, \mathfrak{m})$ be a local ring and $M$ be a finitely generated $R$-module. For given integers $i$ and $t > 0$, the following statements are equivalent:*

*(1)* $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}_{\mathfrak{a}}^{i}(M)) \subseteq V(\mathfrak{m}\widehat{R})$ *for all $i < t$;*

*(2)* $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ *is Artinian for all $i < t$;*

*(3)* $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}_{\mathfrak{a}}^{i}(M)) \subseteq V(\mathfrak{a}\widehat{R})$ *for all $i < t$.*

*(4)* $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{i}(M)))$ *for all $i < t$;*

*Suppose that $t \leq \operatorname{depth} M$, then the above conditions are equivalent to*

*(5)* $\mathfrak{F}_{\mathfrak{a}}^{i}(M) = 0$ *for all $i < t$;*

*where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of $R$.*

This Theorem can be considered as the dual to the Faltings' finiteness Theorem (cf. [Br-Sh, Theorem 9.1.2]) for formal local cohomology modules.

• For an $R$-module $M$, $\operatorname{Cosupp}_R(M) \subseteq V(\operatorname{Ann}_R M)$, for definition of cosupport of a module, see chapter three. When $M$ is representable, then $\operatorname{Cosupp}_R(M) = V(\operatorname{Ann}_R M)$ (cf. Theorem 3.3.2). Of a particular interest is to see when the cosupport of formal local cohomology module is a closed subset of $\operatorname{Spec} R$ in the Zariski topology. More precisely in order to show that $\operatorname{Cosupp}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ being closed, it is enough to show that $\operatorname{Coass}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ is finite (cf. Lemma 3.4.5), so it has encouraged us to consider the $\operatorname{Coass}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ extensively.

• Of a particular interest are the first non-vanishing (resp. the last non-vanishing) cohomological degree of the local cohomology modules $H_{\mathfrak{a}}^{i}(M)$, known as the grade $\operatorname{grade}(\mathfrak{a}, M)$ (resp. cohomological dimension $\operatorname{cd}(\mathfrak{a}, M)$). It is a well-known fact that

$$\operatorname{grade}(\mathfrak{a}, M) \leq \operatorname{cd}(\mathfrak{a}, M) \leq \dim M.$$

In the case of $\mathfrak{a} = \mathfrak{m}$ it follows that $\operatorname{cd}(\mathfrak{m}, M) = \dim M$. While for an arbitrary ideal $\mathfrak{a} \subset R$ the Hartshorne-Lichtenbaum Vanishing Theorem says that

$$H^d_{\mathfrak{a}}(M) = 0 \Longleftrightarrow \dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} > 0 \text{ for all } \mathfrak{p} \in \operatorname{Ass}_{\widehat{R}} \widehat{M} \text{ such that } \dim \widehat{R}/\mathfrak{p} = d,$$

$d = \dim M$ (see [Hart] and [Br-Sh]). Here $\widehat{M}$ resp. $\widehat{R}$ denotes the completion of $M$ resp. $R$. When $H^d_{\mathfrak{a}}(M) \neq 0$ one of the most important views concerning this is to express $H^d_{\mathfrak{a}}(M)$ via $H^d_{\mathfrak{m}}(M)$. More precisely the kernel of the natural epimorphism $H^{\dim M}_{\mathfrak{m}}(M) \to H^{\dim M}_{\mathfrak{a}}(M)$ has been calculated explicitly.

**Theorem 1.1.2.** *(cf. Theorem 4.1.5 and Corollary 4.1.7) Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let $M$ be a finitely generated $R$-module and $d = \dim M$. Then there is a natural isomorphism*

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/Q_{\mathfrak{a}\widehat{R}}(\widehat{M})) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/P_{\mathfrak{a}}(\widehat{M})\widehat{M}),$$

*where $Q_{\mathfrak{a}\widehat{R}}(\widehat{M})$ is a certain submodule of $\widehat{M}$ (cf. 4.1.3) and $P_{\mathfrak{a}}(\widehat{M}) \subseteq \widehat{R}$ is the ideal as defined in 4.1.6.*

The above results lead us to establish some properties of $\operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$. First of all a brief about endomorphism rings could be instrumental for understanding the content.

One can often translate properties of an object into properties of its endomorphism ring. For instance, a module is indecomposable if and only if its endomorphism ring does not contain any non-trivial idempotents (cf. [Jacob]). Note that a module $M$ is decomposable if $M = M_1 \oplus M_2$ where $M_i \neq 0$ for $i = 1, 2$ are submodules of $M$. Otherwise $M$ is indecomposable.

Not so much is known about the ring $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ and its relation to a given ring $R$. In Theorem 4.2.2, for a local ring $(R, \mathfrak{m})$ and its $\mathfrak{m}$-adic completion $\widehat{R}$, we show that the map

$$\Phi : \widehat{R} \to \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$$

is an isomorphism if and only if $Q_{\mathfrak{a}\widehat{R}}(\widehat{R}) = 0$ and $\widehat{R}/Q_{\mathfrak{a}\widehat{R}}(\widehat{R})$ satisfies Serre's condition $S_2$ (for more details see section two of chapter 4). Furthermore we show that $\operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ is a finitely generated $\widehat{R}$-module and it is a commutative semi-local Noetherian ring (cf. Theorem 4.2.2(3),(4)).

- In Chapter 5, we give some connectedness Theorems. Let $R$ be a commutative ring. The spectrum of $R$, denoted by $\operatorname{Spec}(R)$, is the topological space consisting of all prime ideals of $R$, with topology defined by the closed sets $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$, for each ideal $\mathfrak{a}$ of $R$. This topology is called the Zariski topology. Clearly if $R$ is nonzero, then $\operatorname{Spec} R$ is non-empty. $\operatorname{Spec} R$ enjoys very nice properties. For instance it is compact and moreover it is irreducible if and only if its nilradical is a prime ideal (a topological space $X$ is irreducible if it cannot be written as a union of two closed proper subsets $A, B$ of $X$). However $\operatorname{Spec}(R)$ is not a connected space in general. It is known that for a local ring $R$, $\operatorname{Spec} R$ is connected. More generally $\operatorname{Spec} R$ is disconnected if and only if $R$ contains a non-trivial idempotents element. The concept of a topological space being connected in codimension $k$ ($\in \mathbb{N} \cup \{0\}$) was made precise by Hartshorne [Hart2]. For definitions and more details see also chapter 5.

Next we recall a definition given by Hochster and Huneke (see [Hoch-Hun, (3.4)]).

**Definition 1.1.3.** *Let $(R, \mathfrak{m})$ denote a local ring. We denote by $\mathbb{G}(R)$ the undirected graph whose vertices are primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R = \dim R/\mathfrak{p}$, and two distinct vertices $\mathfrak{p}, \mathfrak{q}$ are joined by an edge if and only if $(\mathfrak{p}, \mathfrak{q})$ is an ideal of height one.*

We extend a classical result of Hochster-Huneke to an arbitrary ideal $\mathfrak{a}$ of $R$ as follows:

**Theorem 1.1.4.** *(cf. Theorem 5.2.5) Let $(R, \mathfrak{m})$ denote a complete local ring and $d = \dim R$. For an ideal $\mathfrak{a} \subset R$ the following conditions are equivalent:*

*(1) $H^d_{\mathfrak{a}}(R)$ is indecomposable.*

*(2) $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), E(R/\mathfrak{m}))$ is indecomposable.*

*(3) The endomorphism ring of $H^d_{\mathfrak{a}}(R)$ is a local ring.*

*(4) The graph $\mathbb{G}(R/Q_{\mathfrak{a}}(R))$ is connected,*

for the definition of $Q_{\mathfrak{a}}(R)$, see 4.1.3.

• In chapter 6 at first we give some results on the attached prime ideals of local cohomology via colocalization. Next we give a short simple proof to the Sharp's asymptotic prime divisor. Let $R$ be a commutative ring and $\mathfrak{a}$ an ideal of $R$. For every Artinian $R$-module $A$, $\operatorname{Att}(0:_A \mathfrak{a}^n)$ and $\operatorname{Att}(0:_A \mathfrak{a}^n/0:_A \mathfrak{a}^{n-1})$ are ultimately constant and $At(\mathfrak{a}, A)$ and $Bt(\mathfrak{a}, A)$ denote their ultimate constant values (cf. [Sh2]). In [Sh1], Sharp showed that

$$At(\mathfrak{a}, A) \setminus Bt(\mathfrak{a}, A) \subseteq \operatorname{Att}_R(A)$$

for every Artinian module $A$, by generalization of Heinzer-Lantz Theorem. Schenzel [Sch2] has given an alternative proof for mentioned Theorem in case that for a local ring $(R, \mathfrak{m})$, if $\mathfrak{m} \in At_R(\mathfrak{a}, A) \setminus Bt_R(\mathfrak{a}, A)$, then $\mathfrak{m} \in \operatorname{Att}_R A \cap V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of prime ideals of $R$ containing ideal $\mathfrak{a}$. Then we give a short simple proof for Sharp's Theorem using the concept of colocalization introduced by Richardson [Rich], (cf. Theorem 6.0.8).

**Note on references**: Some of the materials in this Thesis have been submitted elsewhere. Some of the results have been appeared in [E].

# Chapter 2

# Preliminaries

In this chapter, we give a brief summary of subjects that are used throughout this thesis and provide proofs for the lesser-known results. For a more in-depth treatment of the subject, we introduce suitable references in each section.

## 2.1 Definitions and basic properties of Local cohomology

In this section we present a quick review of local cohomology. For omitted proofs and more details we refer the reader to [Br-Sh].

Let $M$ be an $R$-module and $\mathfrak{a} \subset R$ be an ideal, set

$$\Gamma_{\mathfrak{a}}(M) = \{x \in M : \mathfrak{a}^n x = 0 \text{ for some } n \geq 0\},$$

simply it implies the following equality:

$$\Gamma_{\mathfrak{a}}(M) = \bigcup_n (0 :_M \mathfrak{a}^n).$$

$\Gamma_{\mathfrak{a}}$ is a covariant $R$-linear functor which is left exact and additive. For $i \in \mathbb{N}_0$, the $i$-th right derived functor of $\Gamma_{\mathfrak{a}}$ is denoted by $H^i_{\mathfrak{a}}$ and will be referred to as the $i$-th local cohomology functor with respect to $\mathfrak{a}$. In other words, if $\mathfrak{I}^\bullet$ is an injective resolution of $M$, then $H^i_{\mathfrak{a}}(M) = H^i(\Gamma_{\mathfrak{a}}(\mathfrak{I}^\bullet))$ for all $i \geq 0$. As an alternative definition for local cohomology module one can use the following:

$$H^i_{\mathfrak{a}}(M) = \varinjlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

To compute $H^i_{\mathfrak{a}}(M)$ one can also use the Čech complex. If $\mathfrak{a} = (a_1, ..., a_n)$, then $H^i_{\mathfrak{a}}(M)$ is the $i$-th cohomology of the complex

$$0 \to M \to \oplus M_{a_i} \to \oplus_{i<j} M_{a_i a_j} \to ... \to M_{a_1 \dots a_n} \to 0.$$

It is noteworthy to mention that if $\mathfrak{b}$ is another ideal with the same radical as $\mathfrak{a}$, then $H^i_{\mathfrak{b}}(M) = H^i_{\mathfrak{a}}(M)$ for all $i$ and for all $R$-module $M$.

Let $\mathfrak{a}$ be an ideal of $R$, an $R$-module $M$ is called $\mathfrak{a}$-torsion-free when $\Gamma_{\mathfrak{a}}(M) = 0$ and $M$ is $\mathfrak{a}$-torsion when $\Gamma_{\mathfrak{a}}(M) = M$.

**Lemma 2.1.1.** *( [Br-Sh, Lemma 2.1.1 ]) Let $\mathfrak{a} \subset R$ be an ideal and $M$ an $R$-module. Assume that $M$ is finitely generated. Then $M$ is $\mathfrak{a}$-torsion-free if and only if $\mathfrak{a}$ contains a non-zerodivisor on $M$.*

**Proof.** Let $r \in \mathfrak{a}$ be a non-zerodivisor on $M$ and Assume that $m \in \Gamma_{\mathfrak{a}}(M)$ be an arbitrary element. So there exists an integer $n$ such that $\mathfrak{a}^n m = 0$. Then it follows that is $r^n m = 0$, from which we deduce that $m = 0$. □

**Lemma 2.1.2.** *( [Br-Sh, Corollary 2.1.7]) Let $\mathfrak{a} \subset R$ be an ideal and $M$ an $R$-module.*

*(1) Let $M$ be an $\mathfrak{a}$-torsion $R$-module. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$.*

*(2) For each $R$-module $N$ and for all $i > 0$*

$$H^i_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(N)) = 0 \quad \text{and} \quad H^i_{\mathfrak{a}}(N) \cong H^i_{\mathfrak{a}}(N/\Gamma_{\mathfrak{a}}(N)).$$

One of the most useful properties of local cohomology is the following Theorem:

**Theorem 2.1.3.** *Let $f : R \to S$ be a ring homomorphism of Noetherian rings, $\mathfrak{a}$ an ideal of $R$ and $i \in \mathbb{Z}$.*

*(1) (**Independence Theorem**, [Br-Sh, Theorem 4.2.1]) Let $M$ be an $S$-module. Then $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}S}(M)$ as $S$-modules where the first local cohomology is considered over the ring $R$.*

*(2) (**Flat base change Theorem**, [Br-Sh, Theorem 4.3.2]) Assume that $f$ is a flat homomorphism and $M$ an $R$-module. Then there is an isomorphism*

$$H^i_{\mathfrak{a}}(M) \otimes_R S \cong H^i_{\mathfrak{a}S}(M \otimes_R S).$$

Note that the homomorphisms $R \to \widehat{R}$ and $R \to R_{\mathfrak{p}}$ are flat, so in the light of Theorem 2.3 one can see that

$$H^i_{\mathfrak{a}}(M) \otimes_R \widehat{R} \cong H^i_{\mathfrak{a}\widehat{R}}(M \otimes_R \widehat{R})$$

and

$$H^i_{\mathfrak{a}}(M) \otimes_R R_{\mathfrak{p}} \cong H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M \otimes_R R_{\mathfrak{p}}).$$

### 2.1.1 Ideal transforms

We denote the covariant, $R$-linear functor $\varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n, .)$ by $D_{\mathfrak{a}}$ which is called the $\mathfrak{a}$-transform functor. For each $R$-module $M$,

$$D_{\mathfrak{a}}(M) = \varinjlim_{n\in\mathbb{N}} \operatorname{Hom}_R(\mathfrak{a}^n, M).$$

There are some important connections between the $\mathfrak{a}$-transform functor and local cohomology functors. Below we state one of such connections will be used in this work:

**Theorem 2.1.4.** *( [Br-Sh, Theorem 2.2.4]) For each R-module M, the sequence*

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow D_{\mathfrak{a}}(M) \longrightarrow H^1_{\mathfrak{a}}(M) \longrightarrow 0$$

*is exact.*

Let $\mathfrak{a} = aR$ be a principal ideal, then $\mathfrak{a}$-transform functor can be state explicitly by localization, i.e. $D_{Ra}(M) \cong M_a$ (cf. [Br-Sh, Theorem 2.2.16]).

### 2.1.2 Vanishing Theorems

**Theorem 2.1.5.** ***(Grothendieck's Vanishing Theorem)*** *( [Br-Sh, Theorem 6.1.2]) Let M be an R-module. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i >$ dim M.*

**Theorem 2.1.6.** ***(The Non-Vanishing Theorem)*** *( [Br-Sh, Theorem 6.1.4]) Assume that $(R, \mathfrak{m})$ is local, and let M be a non-zero finitely generated R-module of dimension n. Then $H^n_{\mathfrak{m}}(M) \neq 0$.*

When $(R, \mathfrak{m})$ is a local ring and the nonzero finitely generated $R$-module $M$ has dimension $n$, then $H^n_{\mathfrak{m}}(M) \neq 0$, so that in view of Grothendieck's Vanishing Theorem, $n = \dim M$ is the greatest integer $i$ for which $H^i_{\mathfrak{m}}(M) \neq 0$

### 2.1.3 Artinian local cohomology modules

The following theorem is a useful tool to see when local cohomology is Artinian:

**Theorem 2.1.7.** *( [Mel, Theorem 1.3]) Assume that $M$ is an $\mathfrak{a}$-torsion $R$-module for which $(0 :_M \mathfrak{a})$ is Artinian. Then $M$ is Artinian.*

Immediately one can exploit the above theorem to prove next Theorem on Artinianness of local cohomology modules:

**Theorem 2.1.8.** *( [Br-Sh, Theorem 7.1.3 and 7.1.6]) Assume that $(R, \mathfrak{m})$ is local and let $M$ be a finitely generated $R$-module. Then*

*(1) the $R$-module $H^i_{\mathfrak{m}}(M)$ is Artinian for all $i \in \mathbb{N}_0$.*

*(2) the $R$-module $H^{\dim M}_{\mathfrak{a}}(M)$ is Artinian.*

**Proof.**

(1) We prove it by induction on $i$. Obviously $H^0_{\mathfrak{m}}(M)$ is of finite length, thus Artinian, since it is a finite module with support in $\{\mathfrak{m}\}$. Assume we have shown the conclusion for $i-1$, where $i \geq 1$. By replacing $M$ by $M/\Gamma_{\mathfrak{a}}(M)$, we may assume that there is an $M$-regular element $a$ in $\mathfrak{m}$. The exact sequence

$$0 \to M \xrightarrow{a} M \to M' \to 0$$

where $M' = M/aM$, yields an exact sequence

$$H^{i-1}_{\mathfrak{m}}(M') \to H^i_{\mathfrak{m}}(M) \xrightarrow{a} H^i_{\mathfrak{m}}(M).$$

By hypothesis $H^{i-1}_{\mathfrak{m}}(M')$ is Artinian, so $0 :_{H^i_{\mathfrak{m}}(M)} a$, the kernel of multiplication by the element $a$ on $H^i_{\mathfrak{m}}(M)$, is Artinian. In addition, any element in $H^i_{\mathfrak{m}}(M)$ is annihilated by a power of $a$, since $a \in \mathfrak{m}$. It follows from Theorem 2.1.7 that $H^i_{\mathfrak{m}}(M)$ is Artinian.

(2) By induction on $d := \dim M$. If $d = 0$, then $M$ is of finite length, and so is its submodule $\Gamma_{\mathfrak{a}}(M)$. So assume $n \geq 1$ and put $\overline{M} = M/\Gamma_{\mathfrak{a}}(M)$. Then $H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{a}}(\overline{M})$ and $\dim M \geq \dim \overline{M}$. If $\overline{M} = 0$ or $d > \dim \overline{M}$, then $H^d_{\mathfrak{a}}(\overline{M}) = 0$, so we may assume that $\mathfrak{a}$ contains an $M$-regular element $a$. Putting $M' = M/aM$, we have $\dim M' = d - 1$ and the exact sequence

$$0 \to M \xrightarrow{a} M \to M' \to 0$$

yields an exact sequence

$$H_{\mathfrak{a}}^{d-1}(M') \to H_{\mathfrak{a}}^{d}(M) \xrightarrow{a} H_{\mathfrak{a}}^{d}(M).$$

By the induction hypothesis, $H_{\mathfrak{a}}^{d-1}(M')$ is Artinian, so $0 :_{H_{\mathfrak{a}}^{d}(M)} a$, is Artinian. Since $\bigcup_{n\geq 1}(0 :_{H_{\mathfrak{a}}^{d}(M)} a^n) = H_{\mathfrak{a}}^{d}(M)$. Now $H_{\mathfrak{a}}^{d}(M)$ is Artinian by Theorem 2.1.7.

□

Each Artinian $R$-module has a natural structure as an (Artinian) $\widehat{R}$-module. In fact for an Artinian $R$-module $M$ there is an $\widehat{R}$-isomorphism $\psi : M \otimes_R \widehat{R} \to M$ for which

$$\psi(\textstyle\sum_{i=1}^{u} x_i \otimes \widehat{a}_i) = \sum_{i=1}^{u} \widehat{a}_i x_i$$

(for $x_1, ..., x_u \in M$ and $\widehat{a}_1, ..., \widehat{a}_u \in \widehat{R}$). Because each element of $M$ is annihilated by some power of $\mathfrak{m}$ (cf. [Sh-Vam, 3.21] and [Mat, 3.4(1)]). Now let $x \in M, \widehat{a} \in \widehat{R}$ and $(\mathfrak{a}_n)_{n\geq 1}$ be a Cauchy sequence of elements of $R$ which converges to $\widehat{a}$ in $\widehat{R}$. Then the values of the sequence $(\mathfrak{a}_n x)_{n\geq 1}$ of elements of $M$ are ultimately constant. It is straightforward to check that $M$ may be given the structure of an $\widehat{R}$-module in such a way that $\widehat{a}x$ is equal to the ultimate constant value of the sequence $(\mathfrak{a}_n x)_{n\geq 1}$. It follows that a subset of $M$ is an $R$-submodule if and only if it is an $\widehat{R}$-submodule (cf. [Sh3, lemma 2.1]).

## 2.2 Canonical module

In this section we present a quick review of canonical modules. The notion of a canonical module of a (Noetherian) local ring is due to Grothendieck [Gr2]. In the sequel we define canonical modules via local cohomology modules.

**Theorem 2.2.1.** *(Grothendieck) Suppose that the local ring $(R, \mathfrak{m})$ is the factor ring of a Gorenstein ring $(S, \mathfrak{n})$ with $r = \dim S$. Then there are functorial isomorphisms*

$$H_{\mathfrak{m}}^{i}(M) \cong \mathrm{Hom}_R(\mathrm{Ext}_S^{r-i}(M, S), E), i \in \mathbb{Z},$$

*for any finitely generated $R$-module $M$, where $E$ denotes the injective hull of the residue field. (see also [Sch4, Theorem 1.8] for an alternative proof)*

In the situation of Theorem 2.2.1 we introduce a few abbreviations. For $i \in \mathbb{Z}$ put

$$K_M^i = \mathrm{Ext}_S^{r-i}(M, S).$$

Moreover for $i = \dim M$ we often write $K_M$ instead of $K_M^{\dim M}$ . The module $K_M$ is called the canonical module of $M$. In the case of $M = R$ it coincides with the classical definition of the canonical module of $R$ (cf. [Herz-Kunz]). By the Matlis duality and by Theorem 2.2.1 the modules $K_M^i$ do not depend up to isomorphisms on the presentation of the Gorenstein ring $S$. Clearly $K_M^i = 0$ for all $i > \dim M$ and $i < 0$.

For a finitely generated $R$-module $M$ say it satisfies Serre's condition $S_k$, $k \in \mathbb{N}$, provided

$$\operatorname{depth} M_{\mathfrak{p}} \geq \min\{k, \dim M_{\mathfrak{p}}\} \text{ for all } \mathfrak{p} \in \operatorname{Supp} M.$$

Note that $M$ satisfies $S_1$ if and only if it is unmixed. $M$ is a Cohen-Macaulay module if and only if it satisfies $S_k$ for all $k \in \mathbb{N}$.

**Theorem 2.2.2.** *(cf. [Sch4, Theorem 1.14]) Let $M$ denote a finitely generated, equidimensional $R$-module with $d = \dim M$, where $R$ is a factor ring of a Gorenstein ring. Then for an integer $k \geq 1$ the following statements are equivalent:*

*(1) $M$ satisfies condition $S_k$.*

*(2) The natural map $\tau_M : M \to K_{K_M}$ is bijective (resp. injective for $k = 1$) and $H_{\mathfrak{m}}^i(K_M) = 0$ for all $d - k + 2 \leq i < d$.*

It turns out that for a module $M$ satisfying $S_2$ the natural map $\tau_M : M \to K_{K_M}$ is an isomorphism.

**Corollary 2.2.3.** *(cf. [Sch4, Corollary 1.15]) With the notation of Theorem 2.2.2, suppose that the $R$-module $M$ satisfies the condition $S_2$. For an integer $k \geq 2$ the following conditions are equivalent:*

*(1) $K_M$ satisfies condition $S_k$.*

*(2) $H_{\mathfrak{m}}^i(M) = 0$ for all $d - k + 2 \leq i < d$.*

**Remark 2.2.4.** *By the previous result the canonical module $K_M$ of $M$ is a Cohen-Macaulay module provided $M$ is a Cohen-Macaulay module.*

**Lemma 2.2.5.** *Let $M$ denote a finitely generated $R$-module of dimension $d$. Then*

*(1) $K_M$ satisfies $S_2$ ( [Sch4, Lemma 1.9(e)]).*

*(2) $\operatorname{Ann}_R K_M = (\operatorname{Ann}_R M)_d$ ( [Sch3, Proposition 3.4]).*

*(3) $\operatorname{Ass}_R K_M = (\operatorname{Ass}_R M)_d$ ( [Gr2, Proposition 6.6]).*

Note that for an ideal $\mathfrak{a} \subset R$ with $\dim R/\mathfrak{a} = d$ we will denote by $\mathfrak{a}_d$ the intersection of those primary components in a minimal reduced primary decomposition of $\mathfrak{a}$ which are of dimension $d$. If $Z \subset \operatorname{Spec} R$ and $d \in N$, then we put $Z_d = \{\mathfrak{p} \in Z : \dim R/\mathfrak{p} = d\}$.

In the end we express some significant facts on canonical modules could be useful in the next chapters. We refer the reader to [Aoya], [Aoya-Goto] and [Hoch-Hun].

**Theorem 2.2.6.** *Let $R$ be a local ring of dimension $d$ and with canonical module $K_R$. Let $h$ be the natural map from $R$ to $\operatorname{Hom}_R(K_R, K_R)$. Then*

*(1) $\operatorname{Hom}_R(K_R, K_R)$ is a semi-local ring which is finitely generated as an $R$-module.*

*(2) $\operatorname{Hom}_R(K_R, K_R)$ is a commutative ring.*

*(3) The map $h$ is an isomorphism if and only if $R$ is $S_2$ if and only if $\widehat{R}$ is $S_2$.*

## 2.3 Colocalization

For a given commutative ring $R$ and a multiplicative closed subset $S \subset R$, the functor $S^{-1}(-) = S^{-1}R \otimes -$ is the well-known localization functor. It is known that for a Noetherian $R$-module $M$, $S^{-1}(M)$ is a Noetherian $S^{-1}R$-module. For an Artinian module $N$, $S^{-1}(N)$ is sometimes zero. It is a natural question whether there exists a functor $S_{-1}(-)$ (which is called colocalization functor) from the category of $R$-modules to the category of $S^{-1}R$-modules to be well-behavior by Artinian modules.

Recently, A. S. Richardson [Rich] has proposed the definition for colocalization fulfilled the expected properties. In particular, $S_{-1}(-)$ preserves secondary representations and attached primes (the duals of primary decompositions and associated primes; cf. [Mac] and [Br-Sh, Section 7.2]) and the colocalization of an

Artinian $R$-module is an Artinian $S^{-1}R$-module (cf. Theorem 2.3.4). This colocalization functor should define a sensible cosupport (see Definition 3.3.1). In particular, the cosupport of a nonzero module should be nonempty, the cosupport of an Artinian module $N$ should be $V(\operatorname{Ann} N)$, and the cosupport of a finitely generated module should, like the ordinary support of an Artinian module, consist solely of maximal ideals (cf. Theorem 3.3.2).

Melkersson and Schenzel [Mel-Sch] defined the colocalization functor as $\operatorname{Hom}_R(S^{-1}R, -)$ where this definition works well when restricted to the class of Artinian modules, with the exception that it almost never takes an Artinian module to an Artinian module. However, this definition does not work at all well for non-Artinian modules. For example, if $S$ is a multiplicative closed set of integers which includes a nonunit, then $\operatorname{Hom}(S^{-1}\mathbb{Z}, \mathbb{Z}) = 0$, which says that the cosupport of $\mathbb{Z}$, under this definition, is empty, which is definitely not what we want.

Throughout this Thesis we use the concept of colocalization due to Richardson.

**Definition 2.3.1.** *Let $B$ be a commutative ring. Let $E_B$ be the injective hull of $\oplus B/\mathfrak{m}$, the sum running over all maximal ideals $\mathfrak{m}$ of $B$, and let $D_B$ be the functor $\operatorname{Hom}(-, E_B)$.*

This module $E_B$ is the minimal injective cogenerator of the category of $B$-modules; that is, it is the smallest injective module with the property that, for every module $M$ and nonzero $x \in M$, there is a homomorphism $\varphi : M \to E_B$ with $\varphi(x) \neq 0$.

Let $R$ be a commutative ring and $S$ a multiplicative closed subset of $R$.

**Definition 2.3.2.** *For any $R$-module $M$, the co-localization of $M$ relative to $S$ is the $S^{-1}R$-module $S_{-1}M = D_{S^{-1}R}(S^{-1}D_A(M))$. If $S = R \setminus \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} R$, we write ${}^{\mathfrak{p}}M$ for $S_{-1}M$.*

It follows from the definition $S_{-1}(-)$ is exact and additive functor, as it is a composition of exact, additive functors.

As mentioned above Richardson's definition of colocalization preserves Artinian modules through the colocalization. In order to prove this claim we need the following result due to Ooishi (cf. [Ooish, Theorem 1.6]). By the completion of a semi- local ring $B$ with maximal ideals $\mathfrak{m}_1, ..., \mathfrak{m}_r$, we mean the direct sum of completion $\widehat{B_i}$ of local rings $B_i = B_{\mathfrak{m}_i}$, $i = 1, ..., r$ (cf. [Nag]).

**Theorem 2.3.3.** *Assume $B$ is semi-local and Noetherian.*

*(1) If $M$ is a finitely generated $B$-module, then $D_B(M)$ is Artinian.*

*(2) If $M$ is an Artinian $B$-module then $D_B(M)$ is finitely generated over the completion of $B$.*

*(3) If $B$ is complete and $M$ is either finitely generated or Artinian, then $M \cong D_B^2(M)$.*

Now by definition of colocalization and Theorem 2.3.3 we have:

**Theorem 2.3.4.** *(cf. [Rich, Theorem 2.3]) Suppose $R$ is semi-local and complete. If $S^{-1}R$ is also semi-local, but not necessarily complete, then $S_{-1}(-)$ takes Artinian $R$-modules to Artinian $S^{-1}R$-modules.*

Vanishing and non-vanishing of $S_{-1}(-)$ is appeared in the next lemma:

**Lemma 2.3.5.** *(cf. [Rich, Lemma 2.1]) Let $M$ be an $R$-module.*

*(1) If $sM = 0$ for some $s \in S$ then $S_{-1}M = 0$.*

*(2) If $\bigcap_{s \in S} sM \neq 0$, then $S_{-1}M \neq 0$.*

## 2.4 Attached primes

The theory of attached prime ideals and secondary representation of a module has been developed by I.G. MacDonald in [Mac] which is in a certain sense dual to the theory of associated prime ideals and primary decompositions. This theory was applied to the theory of local cohomology by him and R.Y. Sharp (cf. [Mac] and [Sch3]). In the sequel we express a brief review of some facts which are used in the further chapters. we refer the reader to see [Mats] for more information.

A non-zero $R$-module $S$ is called secondary when for each $r \in R$, either $rS = S$ or there exists $n \in \mathbb{N}$ such that $r^n S = 0$. When this is the case, $\mathfrak{p} = \mathrm{Rad}(0 :_R S)$ is a prime ideal of $R$ and $S$ is called $\mathfrak{p}$-secondary $R$-module. Furthermore a secondary representation of an $R$ module $M$ is an expression for $M = S_1 + S_2 + ... + S_t$, $t \in \mathbb{N}$ as a sum of finitely many secondary submodules of $M$. One may assume that the $\mathfrak{p}_i = \mathrm{Rad}(0 : S_i), i = 1, 2, ..., t$ are all distinct and by omitting redundant summands, that the representation is minimal. Then the set of prime ideals $\{\mathfrak{p}_1, ..., \mathfrak{p}_t\}$ does not depend on the representation and it is called the set of attached prime ideals of $M$, and denoted by $\mathrm{Att}(M)$. As a result note that Artinian modules are representable moreover an Artinian module $A$ is nonzero if and only if $\mathrm{Att}\, A \neq \emptyset$. For more information see also [Br-Sh].

The set of associated primes of $\mathrm{Hom}(M, N)$ where $M$ is a finitely generated and $N$ any module over a Noetherian ring $R$ is useful to investigate the associated primes of Matlis dual of finite modules (cf. [Bour]):

**Proposition 2.4.1.** *Let $R$ be a Noetherian ring, $M$ a finitely generated $R$-module and $N$ an arbitrary $R$-module. Then* $\mathrm{Ass}_R \mathrm{Hom}(M, N) = \mathrm{Ass}_R N \cap \mathrm{Supp}_R M$.

Its dual to Artinian modules has been proved in [Mel-Sch, Proposition 5.2]:

**Proposition 2.4.2.** *Let $R$ be a commutative ring, $A$ an Artinian $R$-module and $N$ a finitely presented $R$-module. Then* $\mathrm{Att}_R A \otimes_R N = \mathrm{Att}_R A \cap \mathrm{Supp}_R N$.

Next proposition shows the relation between $\mathrm{Ass}\, M$ and $\mathrm{Att}\, D(M)$ where $M$ is a Noetherian $R$-module:

**Proposition 2.4.3.** *(cf. [Br-Sh, 10.2.20]) Let $(R, \mathfrak{m})$ be a local ring and $M$ be a Noetherian $R$-module. Then $D_R(M)$ is an Artinian $R$-module and* $\mathrm{Att}_R D(M) = \mathrm{Ass}_R M$.

In the next Proposition we see an Artinian analogue of the well-known fact that if $N$ is a Noetherian $R$-module and $r \in R$, then $r$ is a non-zerodivisor on $N$ if and only if $r$ lies outside all the associated prime ideals of $N$.

**Proposition 2.4.4.** *(cf. [Br-Sh, Proposition 7.2.11]) Let $A$ be an Artinian $R$-module and $r \in R$. Then*

*(1) $rA = A$ if and only if $r \in R \setminus \cup_{\mathfrak{p} \in \mathrm{Att}\, A} \mathfrak{p}$; and*

*(2) $\mathrm{Rad}(0 :_R A) = \cap_{\mathfrak{p} \in \mathrm{Att}\, A} \mathfrak{p}$.*

**Proof.** Clearly we may assume that $A \neq 0$, since $\mathrm{Att}\, 0 = \emptyset$. Let $A = S_1 + S_2 + ... + S_n$ with $S_i$ $\mathfrak{p}_i$-secondary $(1 \leq i \leq n)$ be a minimal secondary representation of $M$.

(1) Suppose that $r \in R \setminus \cup_{\mathfrak{p} \in \mathrm{Att}\, A} \mathfrak{p}$; then $rS_i = S_i$ for all $i = 1, ..., n$ and so $rA = A$. On the other hand if $r \in \mathfrak{p}_j$ for some $j$ with $1 \leq j \leq n$, then $r^h S_j = 0$ for a sufficiently large integer $h$, and so

$$r^h A = r^h S_1 + r^h S_2 + ... + r^h S_n \subseteq \textstyle\sum_{i=1, i \neq j}^{n} S_i \subset A.$$

(2) In order to prove just note that

$$\operatorname{Rad}(0 :_R A) = \cap_{i=1}^{n} \operatorname{Rad}(0 :_R S_i) = \cap_{i=1}^{n} \mathfrak{p}_i.$$

□

# Chapter 3

# Results on formal local cohomology

In this chapter we deal with the formal local cohomolgy modules which is used by Peskine and Szpiro in [Pes-Szp] when R is a regular ring. Recently Schenzel [Sch] has defined formal local cohomology modules for a local ring $(R, \mathfrak{m})$ and a finitely generated $R$-module $M$. Not so much is known about their properties. Recently there were some attempts in order to investigate Artinianness properties of formal local cohomology. For instance it has been shown that $\mathfrak{F}_{\mathfrak{a}}^{\dim M}(M)$ is Artinian but $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is not Artinian in general, for $i = \operatorname{fgrade}(\mathfrak{a}, M)$ and $i = \dim M/\mathfrak{a}M$ where they are the first respectively last non-zero amount of formal local cohomology modules (cf. [Asgh-Divan, 2.2 and 2.7]) or see Lemma 3.2.1 and Theorem 3.2.2 below. We pursue this line to find out conditions for Artinianness of formal local cohomology modules. Let $i < t$, $t > 0$ be two integers, we give some equivalent conditions for Artinianness of $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ for all $i < t$ (cf. Theorem 3.2.4). In fact among the other conditions we show that Artinianness of all $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ for all $i < t$ where $t$ itself is $\leq \operatorname{depth} M$, implies vanishing of all those $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$.

Of a particular interest are the closed subsets of $\operatorname{Spec} R$ in the Zariski topology. We consider the cosupport of $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ to see when it is a closed subset of $\operatorname{Spec} R$. For an Artinian module $M$, it is known that $\operatorname{Cosupp} M = V(\operatorname{Ann} M)$. More precisely in order to show that $\operatorname{Cosupp}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ being closed, it is enough to show that $\operatorname{Coass}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ is finite (cf. Lemma 3.4.5), so it has encouraged us to consider the $\operatorname{Coass}(\mathfrak{F}_{\mathfrak{a}}^{i}(M))$ extensively.

## 3.1 Formal Local Cohomology

Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring, $\mathfrak{a}$ an ideal of $R$ and let $M$ be an $R$-module. Consider the family of local cohomology modules $\{H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)\}_{n\in\mathbb{N}}$. For every $n$ there is a natural homomorphism $H^i_{\mathfrak{m}}(M/\mathfrak{a}^{n+1}M) \to H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ such that the family forms a projective system. The projective limit $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of $M$ with respect to $\mathfrak{a}$. Formal local cohomolgy modules used by Peskine and Szpiro in [Pes-Szp] when $R$ is a regular ring.

Let $\mathbf{x} = \{x_1, ..., x_r\}$ denote a system of elements such that $\mathfrak{m} = \operatorname{Rad}(\mathbf{x})$. In [Sch, Proposition 3.2], Schenzel has proved the following isomorphisms to give a new aspect of formal local cohomology modules via cohomology of inverse limit of projective systems $\{\check{C}_{\mathbf{x}} \otimes M/\mathfrak{a}^n M\}$:

$$\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \cong H^i(\varprojlim_n(\check{C}_{\mathbf{x}} \otimes M/\mathfrak{a}^n M))$$

where $\check{C}_{\mathbf{x}}$ denotes the Čech complex of $R$ with respect to $\mathbf{x}$.

Formal local cohomology is well-behaved under completion:

**Proposition 3.1.1.** *[Sch, Proposition 3.3] Let M be a finitely generated R-module. Then* $\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$, $i \in \mathbb{Z}$, *has a natural structure as an* $\widehat{R}$*-module and there are isomorphisms*

$$\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \cong \varprojlim_n H^i_{\widehat{\mathfrak{m}}}(\widehat{M}/\mathfrak{a}^n \widehat{M})$$

*for all* $i \in \mathbb{Z}$.

**Proof.** Let $N$ be a finitely generated $R$-module. Then it is known that $H^i_{\mathfrak{m}}(N)$, $i \in \mathbb{Z}$, is an Artinian $R$-module (cf. Theorem 2.1.8). Because of the flatness of $\widehat{R}$ over $R$ there are $R$-isomorphisms $H^i_{\mathfrak{m}}(N) \cong H^i_{\widehat{\mathfrak{m}}}(\widehat{N})$ for all $i \in \mathbb{Z}$. Now take $N = M/\mathfrak{a}^n M$ and pass to the projective limit. Then this proves the claim. □

By the result of T. Kawasaki (cf. [Kawas]) $R$ possesses a dualizing complex $D^\bullet_R$ if and only if $R$ is the factor ring of a Gorenstein ring. By dualizing complex we mean a bounded complex of injective $R$-modules whose cohomology modules $H^i(D^\bullet_R), i \in \mathbb{Z}$, are finitely generated $R$-module (cf. [Hart3] or [Sch4]). In the light of Proposition 3.1.1 and Cohen's structure Theorem we can assume the existence of a dualizing complex in order to consider the formal local cohomology. Using this view formal local cohomology could be express in terms of a certain local cohomology of the dualizing complex (cf. [Sch, Theorem 3.5]):

$$\begin{aligned}\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) &\cong \operatorname{Hom}_R(H^{-i}_{\mathfrak{a}}(\operatorname{Hom}_R(M, D^{\bullet}_R)), E)\\ &\cong \operatorname{Hom}_R(H^{-i}(\check{C}_{\mathbf{x}} \otimes \operatorname{Hom}_R(M, D^{\bullet}_R)), E),\end{aligned}$$

for all $i \in \mathbb{Z}$, where $M$ be a finitely generated $R$-module, $E = E_R(R/\mathfrak{m})$ denotes the injective hull of the residue field $k$ and $\mathfrak{a} = \operatorname{Rad}(\mathbf{x})$. As a consequence of above explanations there is a following description of formal local cohomology as a Matils dual of a certain generalized local cohomology introduced by Herzog [Herzog] in the case local ring $(R, \mathfrak{m})$ is a factor ring of a local Gorenstein ring $S$:

$$\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \cong \operatorname{Hom}_R(H^{t-i}_{\mathfrak{a}S}(M, S), E), \ \ i \in \mathbb{Z},$$

where $M$ is considered as an $S$-module, $\dim S = t$ and $E = E(R/\mathfrak{m})$ is as above. (cf. [Sch, Remark 3.6]).

One of the notable properties of formal local cohomology modules is that

$$\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \cong (\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M))^{\mathfrak{a}},$$

i.e. they are $\mathfrak{a}$-adically complete for a finitely generated module $M$ (cf. [Sch, Theorem 3.9]). From this isomorphism one can deduce that $\bigcap_{t \geq 1}(\mathfrak{a}^t \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)) = 0$.

We may also consider the following Remark in (cf. [Hel-St, Remark 3.1]):

**Remark 3.1.2.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring, $\mathfrak{a}$ an ideal of $R$ and $M$ an $R$-module such that* $\operatorname{Supp}_R(M) \subseteq V(\mathfrak{a})$. *Then the natural map*

$$D(M) \to D(M)^{\mathfrak{a}}$$

*is an isomorphism. In particular, $\bigcap_{l \in \mathbb{N}} \mathfrak{a}^l D(M) = 0$, where $D(M) = \operatorname{Hom}_R(M, E(R/\mathfrak{m}))$.*

**Proof.** We have to show that the canonical map

$$D(M) \longrightarrow \varprojlim_{l \in \mathbb{N}}(D(M)/\mathfrak{a}^l D(M))$$

is bijective; but one has

$$\begin{aligned}D(M) &= D(\Gamma_{\mathfrak{a}}(M))\\ &= D(\varinjlim_{l \in \mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^l, M))\\ &= \varprojlim_{l \in \mathbb{N}} D(\operatorname{Hom}_R(R/\mathfrak{a}^l, M))\\ &= \varprojlim_{l \in \mathbb{N}} D(M)/\mathfrak{a}^l D(M)\end{aligned}$$

where the last equality follows by Hom $-\otimes$-adjointness. Now it is easy to see that this is the canonical isomorphism $D(M) \overset{\cong}{\to} D(M)^{\mathfrak{a}}$. $\square$

In the sequel we consider the behaviour of formal cohomology with short exact sequences of $R$-modules.

**Theorem 3.1.3.** *(cf. [Sch, Theorem 3.11]) Let $(R, \mathfrak{m})$ denote a local ring. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ denote a short exact sequence of finitely generated $R$-modules. For an ideal $\mathfrak{a}$ of $R$ there is a long exact sequence*

$$... \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(A) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(B) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(C) \longrightarrow \mathfrak{F}^{i+1}_{\mathfrak{a}}(A) \longrightarrow ....$$

Let $M$ be a finitely generated $R$-module. For an $R$-submodule $N$ of $M$ denote by $N :_M \langle \mathfrak{m} \rangle$ the ultimate constant $R$-module $N :_M \mathfrak{m}^n$, $n$ large.

Let $0 = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M} Z(\mathfrak{p})$ denote a minimal primary decomposition of 0 in $M$. Moreover, let $\mathfrak{a}$ denote an ideal of $R$. Then define

$$T_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{Ass}_R M : \dim R/(\mathfrak{a}, \mathfrak{p}) = 0\}.$$

Furthermore, put

$$u_M(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} M \setminus T_{\mathfrak{a}}(M)} Z(\mathfrak{p}).$$

Now it will be shown that $u_M(\mathfrak{a})$ plays an important role in order to understand the 0-th formal cohomology module.

**Lemma 3.1.4.** *(cf. [Sch, Lemma 4.1])With the previous notation we have:*

*(1)* $\bigcap_{n \geq 1} (a^n M :_M < \mathfrak{m} >) = u_M(\mathfrak{a})$.

*(2)* $\operatorname{Ass}_R(u_M(\mathfrak{a})) = T_{\mathfrak{a}}(M)$.

*(3)* $\varprojlim_n H^0_{\mathfrak{m}}(M/\mathfrak{a}^n M) \cong u_{\widehat{M}}(\mathfrak{a}\widehat{R})$.

**Sketch of the proof:** In order to prove (1) and (2) it is enough to consider

$$\bigcap_{n \geq 1} (a^n M :_M < \mathfrak{m} >) = \bigcap_{\mathfrak{p} \in \operatorname{Supp}(M/\mathfrak{a}^n M) \setminus V(\mathfrak{m})} \ker(M \to M_{\mathfrak{p}})$$

(cf. [Sch3, Lemma 2.1]) and note that $\ker(M \to M_{\mathfrak{p}}) = Z(\mathfrak{p})$.

For (3) note that by Proposition 3.1.1 we may assume that $R$ respectively $M$ are complete ring respectively module. Consider the short exact sequence

$$0 \to \{\mathfrak{a}^n M\}_{n \in \mathbb{N}} \to \{\mathfrak{a}^n M :_M \langle \mathfrak{m} \rangle\}_{n \in \mathbb{N}} \to \{H^0_{\mathfrak{m}}(M/\mathfrak{a}^n M)\}_{n \in \mathbb{N}} \to 0,$$

where $H^0_{\mathfrak{m}}(M/\mathfrak{a}^n M) = \mathfrak{a}^n M :_M \langle \mathfrak{m} \rangle / \mathfrak{a}^n M$.

By passing to the projective limit it provides an injection

$$0 \to \bigcap_{n\geq 1}(\mathfrak{a}^n M :_M \langle \mathfrak{m} \rangle) \stackrel{\varphi}{\to} \varprojlim_n H^0_{\mathfrak{m}}(M/\mathfrak{a}^n M).$$

In order to prove that $\varphi$ is surjective, use the fact that $M$ as an $\mathfrak{m}$-adically complete module is also $\mathfrak{a}$-adically complete, see [Sch, Lemma 4.1] for the details. □

For a finitely generated module $M$ the largest non-vanishing value of $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is known. To be more precise consider next Theorem:

**Theorem 3.1.5.** *(cf. [Sch, Theorem 4.5]) Let $M$ be a finitely generated $R$-module. Then*

$$\dim_R M/\mathfrak{a}M = \sup\{i \in \mathbb{Z} : \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \neq 0\}.$$

**Sketch of the proof:** By virtue of Grothendieck's vanishing Theorem 2.1.5

$$\dim_R M/\mathfrak{a}M \geq \sup\{i \in \mathbb{Z} : \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \neq 0\}.$$

Consider the short exact sequence

$$0 \to \mathfrak{a}^n M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^n M \to 0,$$

it yields the epimorphism

$$H^d_{\mathfrak{m}}(M/\mathfrak{a}^{n+1}M) \to H^d_{\mathfrak{m}}(M/\mathfrak{a}^n M) \to 0,$$

of nonzero $R$-modules for all $n \in \mathbb{N}$(cf. Theorem 2.1.6) where $d := \dim_R M/\mathfrak{a}M$. Hence the inverse limit $\varprojlim_n H^d_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is non-zero. □

The infimum for the non-vanishing of formal local cohomology is called the formal grade. Let $M$ be a finitely generated $R$-module, then it is defined as

$$\operatorname{fgrade}(\mathfrak{a}, M) = \inf\{i \in \mathbb{Z} : \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \neq 0\}.$$

For more information see [Sch].

The Mayer-Vietoris sequence in local cohomology is an important tool for connectedness phenomenons (cf. chapter 5). Here is the analogue of it for formal local cohomology:

**Theorem 3.1.6.** *(cf. [Sch, Theorem 5.1]) Let $\mathfrak{a}$, $\mathfrak{b}$ two ideals of a local ring $(R, \mathfrak{m})$. For a finitely generated R-module M there is the long exact sequence*

$$\ldots \to \varprojlim_n H^i_{\mathfrak{m}}(M/(\mathfrak{a} \cap \mathfrak{b})^n M) \to \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \oplus \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{b}^n M) \to \ldots.$$

$$\varprojlim_n H^i_{\mathfrak{m}}(M/(\mathfrak{a}, \mathfrak{b})^n M) \to \ldots,$$

*where $i \in \mathbb{Z}$.*

The long exact sequence relates the $\mathfrak{a}$-formal cohomology to the $(\mathfrak{a}, xR)$-formal cohomology for any element $x \in \mathfrak{m}$. To be more precise:

**Theorem 3.1.7.** *(cf. [Sch, Theorem 3.15]) Let $x \in \mathfrak{m}$ denote an element of $(R, \mathfrak{m})$. For an ideal $\mathfrak{a}$ and a finitely generated R-module M there is the long exact sequence*

$$\ldots \to \operatorname{Hom}(R_x, \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)) \to \varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \to$$

$$\varprojlim_n H^i_{\mathfrak{m}}(M/(\mathfrak{a}, x)^n M) \to \ldots,$$

*where $i \in \mathbb{Z}$.*

## 3.2 On Artinianness results

Important problems concerning local cohomology modules are vanishing, finiteness and Artinianness results. In the present section we study the vanishing and Artinianness conditions of formal local cohomology modules as our main result. Not so much is known about the mentioned properties. In [Asgh-Divan] Asgharzadeh and Divani-Aazar have investigated some properties of formal local cohomology modules. For instance they showed the following lemma. From now on for simplicity we use the notation $\mathfrak{F}^i_{\mathfrak{a}}(M)$ for $\varprojlim_n H^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$.

**Lemma 3.2.1.** *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d. Then $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian.*

**Proof.** It was proved by induction on the number of generators of ideal $\mathfrak{a}$ in [Asgh-Divan, Lemma 2.2]). Here we give an alternative proof:

By Independence Theorem we may assume that $\operatorname{Ann} M = 0$ and so $d = \dim R$. As $H^d_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ is right exact ($n \in \mathbb{N}$) we have

$$\begin{aligned} H^d_{\mathfrak{m}}(M/\mathfrak{a}^n M) &\cong H^d_{\mathfrak{m}}(R) \otimes_R M/\mathfrak{a}^n M \\ &\cong H^d_{\mathfrak{m}}(M) \otimes_R R/\mathfrak{a}^n \\ &\cong H^d_{\mathfrak{m}}(M)/\mathfrak{a}^n H^d_{\mathfrak{m}}(M). \end{aligned}$$

Since $H^d_{\mathfrak{m}}(M)$ is an Artinian module so there exists an integer $n_0$ such that for all integer $m \geq n_0$ we have $\mathfrak{a}^m H^d_{\mathfrak{m}}(M) = \mathfrak{a}^{n_0} H^d_{\mathfrak{m}}(M)$. Then one can see that

$$\mathfrak{F}^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{m}}(M)/\mathfrak{a}^{n_0} H^d_{\mathfrak{m}}(M),$$

which is an Artinian module. □

By virtue of the proof of Lemma 3.2.1, we may consider $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R(H^d_{\mathfrak{m}}(M)) \cap V(\mathfrak{a})$.

Next they showed that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is not Artinian in general, at $i = \operatorname{fgrade}(\mathfrak{a}, M)$ and $i = \dim M/\mathfrak{a}M$. To be more precise the following result holds.

**Theorem 3.2.2.** *(cf. [Asgh-Divan, Theorem 2.7]) Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$.*

*(1) If $M$ is a finitely generated $R$-module such that $f := \operatorname{fgrade}(\mathfrak{a}, M) < \operatorname{depth} M$, then $\mathfrak{F}^f_{\mathfrak{a}}(M)$ is not Artinian.*

*(2) If $R$ is Cohen-Macaulay and $\operatorname{ht} \mathfrak{a} > 0$, then $\mathfrak{F}^{dimR/a}_{\mathfrak{a}}(R)$is not Artinian.*

We pursue this line to find out conditions for Artinianness of formal local cohomology modules.

**Lemma 3.2.3.** *Let $(R, \mathfrak{m})$ be a complete local ring and $M$ a finitely generated $R$-module. Then*

$$\operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R \mathfrak{F}^0_{\mathfrak{a}}(M)} V(\mathfrak{p}).$$

*Moreover* $\operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) \cap V(\mathfrak{a}) \subseteq V(\mathfrak{m})$ .

**Proof.** We only prove the first part because the second part is a consequence of it. As $\mathfrak{F}^0_{\mathfrak{a}}(M)$ is a finitely generated $R$-module (Lemma 3.1.4), then in order to prove the claim it is enough to consider $0 :_R \mathfrak{F}^0_{\mathfrak{a}}(M)$. Let $0 = \cap_{\mathfrak{p} \in \operatorname{Ass} M} Z(\mathfrak{p})$ denote a minimal primary decomposition of 0 in $M$. By virtue of Lemma 3.1.4(2) $\mathfrak{F}^0_{\mathfrak{a}}(M) = \cap_{\mathfrak{p} \in \operatorname{Ass} M \setminus T_{\mathfrak{a}}(M)} Z(\mathfrak{p})$. Now the proof is clear. To this end note that $\operatorname{Ass}_R \mathfrak{F}^0_{\mathfrak{a}}(M) = T_{\mathfrak{a}}(M)$. □

**Theorem 3.2.4.** *Let $(R, \mathfrak{m})$ be a local ring and $M$ be a finitely generated $R$-module. For given integers $i$ and $t > 0$, the following statements are equivalent:*

*(1)* $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{m}\widehat{R})$ *for all $i < t$;*

*(2)* $\mathfrak{F}^i_{\mathfrak{a}}(M)$ *is Artinian for all $i < t$;*

*(3)* $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a}\widehat{R})$ *for all $i < t$.*

*(4)* $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}_R(\mathfrak{F}^i_{\mathfrak{a}}(M)))$ *for all $i < t$;*

*Suppose that $t \leq$ depth $M$, then the above conditions are equivalent to*

*(5)* $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ *for all $i < t$;*

*where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of $R$.*

**Proof.**
$(1) \Rightarrow (3)$ and $(2) \Rightarrow (1)$ are obvious.

$(3) \Rightarrow (2)$ : We argue by induction on $t$. By passing to the completion, we may assume that $R$ is complete (cf. Proposition 3.1.1).

Let $t = 1$, then $i = 0$. As $\mathfrak{F}^0_{\mathfrak{a}}(M)$ is a finitely generated submodule of $M$ and since by assumption $\operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$, then by lemma (3.2.3)

$$\operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) = \operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) \cap V(\mathfrak{a}) \subseteq V(\mathfrak{m}).$$

Hence $\mathfrak{F}^0_{\mathfrak{a}}(M)$ is Artinian.

Now let $t > 1$, put $\overline{M} = M/H^0_{\mathfrak{a}}(M)$. From the exact sequence

$$0 \longrightarrow H^0_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0$$

we get the long exact sequence

$$... \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(H^0_{\mathfrak{a}}(M)) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(\overline{M}) \longrightarrow \mathfrak{F}^{i+1}_{\mathfrak{a}}(H^0_{\mathfrak{a}}(M)) \longrightarrow ....$$

As $\mathfrak{F}^i_{\mathfrak{a}}(H^0_{\mathfrak{a}}(M)) = H^i_{\mathfrak{m}}(H^0_{\mathfrak{a}}(M))$ is an Artinian $R$-module for every $j \in \mathbb{Z}$ (cf. Theorem 2.1.8), then one can see that $\operatorname{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(\overline{M})) \subseteq V(\mathfrak{a})$ for all $i < t$. Hence it is enough to show that $\mathfrak{F}^i_{\mathfrak{a}}(\overline{M})$ is Artinian, so we may assume that $H^0_{\mathfrak{a}}(M) = 0$. Hence there exists an $M$-regular element $x$ in $\mathfrak{a}$ such that from the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM = \widetilde{M} \longrightarrow 0$$

we deduce the long exact sequence

$$\dots \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M) \xrightarrow{x} \mathfrak{F}^i_{\mathfrak{a}}(M) \longrightarrow \mathfrak{F}^i_{\mathfrak{a}}(\widetilde{M}) \longrightarrow \mathfrak{F}^{i+1}_{\mathfrak{a}}(M) \longrightarrow \dots. \qquad (*)$$

As for all $i < t$, $\operatorname{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$ , it implies that $\operatorname{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(\widetilde{M})) \subseteq V(\mathfrak{a})$ for all $i < t-1$. Hence by induction hypothesis $\mathfrak{F}^i_{\mathfrak{a}}(\widetilde{M})$ is Artinian for all $i < t-1$. Therefore in the view of $(*)$, $(0 :_{\mathfrak{F}^i_{\mathfrak{a}}(M)} x)$ is Artinian for all $i < t$ .

On the other hand since $\operatorname{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$ for all $i < t$ , one can see that

$$\mathfrak{F}^i_{\mathfrak{a}}(M) = \bigcup (0 :_{\mathfrak{F}^i_{\mathfrak{a}}(M)} \mathfrak{a}^{\alpha}) \subseteq \bigcup (0 :_{\mathfrak{F}^i_{\mathfrak{a}}(M)} x^{\alpha}) \subseteq \mathfrak{F}^i_{\mathfrak{a}}(M)$$

so $\mathfrak{F}^i_{\mathfrak{a}}(M) = \bigcup (0 :_{\mathfrak{F}^i_{\mathfrak{a}}(M)} x^{\alpha})$ . Therefore by Theorem 2.1.7, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ will be Artinian.

(2) $\Rightarrow$ (4) : Since $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-adically complete for every $i \in \mathbb{Z}$ , then $\bigcap_n \mathfrak{a}^n \mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. Moreover for all $i < t$, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian. Hence there is an integer $u$ such that $\mathfrak{a}^u \mathfrak{F}^i_{\mathfrak{a}}(M) = 0$.

(4) $\Rightarrow$ (3) Without loss of generality we may assume that $R$ is complete. Let $\mathfrak{a}^n(\mathfrak{F}^i_{\mathfrak{a}}(M)) = 0$ for some integer $n$ and $\mathfrak{p} \in \operatorname{Supp}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \setminus V(\mathfrak{a})$. Then one can write

$$\mathfrak{F}^i_{\mathfrak{a}}(M) = \mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{a}^n(\mathfrak{F}^i_{\mathfrak{a}}(M)).$$

Now apply $- \otimes R_{\mathfrak{p}}$ to the both sides of the above equality to get $\mathfrak{F}^i_{\mathfrak{a}}(M)_{\mathfrak{p}} = 0$ which is a contradiction.

(1) $\Rightarrow$ (5) : By passing to the completion we may assume that $R$ is complete. We use induction on $t$. Let $t = 1, i = 0$.

As $\operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{m})$, so $\mathfrak{F}^0_{\mathfrak{a}}(M)$ must be zero. Otherwise since

$$\emptyset \neq \operatorname{Ass}(\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq \operatorname{Supp}(\mathfrak{F}^0_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{m})$$

so

$$\mathfrak{m} \in \operatorname{Ass}(\mathfrak{F}^0_{\mathfrak{a}}(M)) = \{\mathfrak{p} \in \operatorname{Ass} M; \dim(R/\mathfrak{a} + \mathfrak{p}) = 0\},$$

this is contradiction to depth $M > 0$.

Now let depth $M \geq t > 1$. Thus there exists $x \in \mathfrak{m}$ that is an $M$-regular element. Consider the short exact sequence

$$0 \to M \xrightarrow{x^l} M \to M/x^l M = \bar{M} \to 0$$

for every $l$. So we have the following long exact sequence

$$... \to \mathfrak{F}_{\mathfrak{a}}^{i-1}(\bar{M}) \to \mathfrak{F}_{\mathfrak{a}}^{i}(M) \xrightarrow{x^l} \mathfrak{F}_{\mathfrak{a}}^{i}(M) \to \mathfrak{F}_{\mathfrak{a}}^{i}(\bar{M}) \to ...$$

for every $l$.

As $\operatorname{depth} \bar{M} = \operatorname{depth} M - 1 > 0$ and for all $i < t-1$, $\operatorname{Supp}(\mathfrak{F}_{\mathfrak{a}}^{i}(\bar{M})) \subseteq V(\mathfrak{m})$. Then by induction hypothesis $\mathfrak{F}_{\mathfrak{a}}^{i}(\bar{M}) = 0$ for all $i < t-1$. So for every $l$, $(0 :_{\mathfrak{F}_{\mathfrak{a}}^{t-1}(M)} x^l)$ is a homomorphic image of $\mathfrak{F}_{\mathfrak{a}}^{t-2}(\bar{M})$. Hence $(0 :_{\mathfrak{F}_{\mathfrak{a}}^{t-1}(M)} x^l) = 0$ for every $l$.

Take into account that by assumption $\operatorname{Supp}(\mathfrak{F}_{\mathfrak{a}}^{i}(M)) \subseteq V(\mathfrak{m})$ for every $i < t$, hence $\mathfrak{F}_{\mathfrak{a}}^{t-1}(M) = \cup(0 :_{\mathfrak{F}_{\mathfrak{a}}^{t-1}(M)} x^l) = 0$ . This completes the proof. □

One can see that Theorem 3.2.4 can be considered as the dual to the Faltings' finiteness Theorem (cf. [Br-Sh, Theorem 9.1.2]).

The following example can be instrumental for understanding Theorem 3.2.4.

**Example 3.2.5.** *(cf. [Sch, Example 5.2]) Let k be a field. Let $R = k[|x_1, x_2, x_3, x_4|]$ denote the formal power series ring in four variables over k. Put $\mathfrak{a} = (x_1, x_2)R$, $\mathfrak{b} = (x_3, x_4)R$ and $\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b}$. Then the Mayer-Vietoris sequence provides the isomorphism $R \cong \mathfrak{F}_{\mathfrak{c}}^{1}(R)$. To this end note that $(\mathfrak{a}, \mathfrak{b})$ is the maximal ideal of the complete local ring R. Therefore* $\operatorname{Supp} \mathfrak{F}_{\mathfrak{c}}^{1}(R) = \operatorname{Spec} R$ *and clearly R is not Artinian, here $i < t = 2$.*

## 3.3 Cosupport

In this section we examine the cosupport of formal local cohomology. Yassemi in [Yas] has defined the $\operatorname{CoSupp}_R M$ as the set of prime ideals $\mathfrak{q}$ such that there exists a cocyclic homomorphic image $L$ of $M$ with $\mathfrak{p} \supseteq \operatorname{Ann}(L)$ (an $R$-module $L$ is cocyclic if $L$ is a submodule of $E(R/\mathfrak{m})$ for some $\mathfrak{m} \in \max(R)$). His definition is equivalent with Melkerson-Schenzel's definition for Artinian $R$-modules (cf. [Yas]). Melkerson-Schenzel's definition of colocalization does not preserve Artinian $R$-module to Artinian $S^{-1}R$-module through colocalization for a multiplicative closed subset of $R$ (cf. [Mel-Sch]). We use the concept of cosupport has introduced by A. Richardson (cf. 2.3.2).

**Definition 3.3.1.** *For any R-module M, the co-support of M is* $\operatorname{CoSupp} M = \{\mathfrak{p} \in \operatorname{Spec} R : {}^{\mathfrak{p}}M \neq 0\}$.

For brevity we often write $\operatorname{CoSupp} M$ for $\operatorname{CoSupp}_R M$ when there is no ambiguity about the ring $R$.

Following Theorem makes the cosupport of a module more clear:

**Theorem 3.3.2.** *(cf. [Rich, Theorem 2.7]) Let $R$ be a ring and $M$ an $R$-module.*

*(1)* $\operatorname{CoSupp} M = \operatorname{Supp} D_R(M)$*, where $D_R$ is defined in (2.3.1).*

*(2)* $\operatorname{CoSupp} M = \emptyset$ *if and only if $M = 0$.*

*(3)* $\operatorname{CoSupp} M \subseteq V(\operatorname{Ann} M)$*.*

*(4) If $M$ is representable, then* $\operatorname{CoSupp} M = \{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supset \mathfrak{q}$ *for some* $\mathfrak{q} \in \operatorname{Att} M\} = V(\operatorname{Ann} M)$*.*

*(5) If $0 \to M' \to M \to M'' \to 0$ is exact, then* $\operatorname{CoSupp} M = \operatorname{CoSupp} M' \cup \operatorname{CoSupp} M''$*.*

*(6) If $M$ is finitely generated then* $\operatorname{CoSupp} M = V(\operatorname{Ann} M) \cap \max(R)$*.*

**Sketch of the proof:**

(1) It is clear by definition.

(2) It follows by (1).

(3) Use Lemma 2.3.5 to prove.

(4) Let $M$ be representable so $M = \sum_{i=1}^{n} N_i$, $\operatorname{Rad}(\operatorname{Ann}_R N_i) = \mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Spec} R$ for $i \in \{1, ..., n\}$ and $n \in \mathbb{N}$. Then $0 :_R M \subseteq \mathfrak{p}$ if and only if $0 :_R N_j \subseteq \mathfrak{p}$ for some $j \in \{1, ..., n\}$. It proves the second equality. In order to prove the first equality it is enough to show that $V(\operatorname{Ann} M) \subseteq \operatorname{CoSupp} M$ which follows by Theorem 6.0.8.

(5) Follows from the exactness of colocalization.

(6) since $M$ is finitely generated, we have $D_R(M)_{\mathfrak{p}} \cong \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (E_R)_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec} R$, so $\operatorname{Supp} D_R(M) \subseteq \operatorname{Supp} M \cap \operatorname{Supp} E_A = V(\operatorname{Ann} M) \cap \max(R)$. On the other hand, if $\mathfrak{m}$ is maximal, then $(E_R)_{\mathfrak{m}} \cong E_{R_{\mathfrak{m}}}$, so

$$\begin{aligned} {}^{\mathfrak{m}}M &\cong \operatorname{Hom}_{R_{\mathfrak{m}}}(\operatorname{Hom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, E_{R_{\mathfrak{m}}}), E_{R_{\mathfrak{m}}}) \\ &\cong M_{\mathfrak{m}} \otimes \operatorname{Hom}_{R_{\mathfrak{m}}}(E_{R_{\mathfrak{m}}}, E_{R_{\mathfrak{m}}}) \\ &\cong \widehat{M_{\mathfrak{m}}}, \end{aligned}$$

which is nonzero if and only if $\mathfrak{m} \in V(\operatorname{Ann} M)$.

□

It is known that for every $R$-module $M$ and every integer $i$ we have $\operatorname{Supp} H^i_{\mathfrak{a}}(M) \subseteq V(\mathfrak{a})$. It is natural to ask whether this is true for formal local cohomology. We give an affirmative answer to the above question in the case $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian while $(R, \mathfrak{m})$ is a local ring and $M$ a finitely generated $R$-module. At first, we should make some preparations.

**Proposition 3.3.3.** *(cf. [Rich, Proposition 2.5]) Let $M$ and $N$ be $R$-modules with $M$ finitely generated, and let $i$ be any integer.*

*(1)* $S_{-1} \operatorname{Tor}^R_i(M, N) \cong \operatorname{Tor}^{S^{-1}R}_i(S^{-1}M, S_{-1}N)$.

*(2)* $S_{-1} \operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S_{-1}N)$.

**Lemma 3.3.4.** *Let $R$ be a ring and $M$, $N$ be $R$-modules. Then the following statements are true:*

*(1)* $\operatorname{CoSupp}(M)$ *is stable under specialization, i.e.*

$$\mathfrak{p} \in \operatorname{Cosupp}(M), \mathfrak{p} \subseteq \mathfrak{q} \Rightarrow \mathfrak{q} \in \operatorname{Cosupp}(M).$$

*(2) Let $M$ be a finite module, then* $\operatorname{CoSupp}(M \otimes_R N) \subseteq \operatorname{Supp} M \cap \operatorname{CoSupp} N$.

**Proof.**

(1) Let $\mathfrak{p} \in \operatorname{Cosupp}(M)$, then by definition $D_{R_{\mathfrak{p}}}(D_R(M)_{\mathfrak{p}})$ is nonzero and so is $D_R(M)_{\mathfrak{p}}$. As $0 \neq D_R(M)_{\mathfrak{p}} = (D_R(M)_{\mathfrak{q}})_{\mathfrak{p}R_{\mathfrak{q}}}$, then $D_R(M)_{\mathfrak{q}} \neq 0$. It implies that ${}^{\mathfrak{q}}M \neq 0$.

(2) Let $\mathfrak{p} \in \operatorname{CoSupp}(M \otimes_R N)$, then $0 \neq {}^{\mathfrak{p}}(M \otimes_R N) = M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} {}^{\mathfrak{p}}N$, by (3.3.3). So $M_{\mathfrak{p}} \neq 0$ and ${}^{\mathfrak{p}}N \neq 0$. Hence $\mathfrak{p} \in \operatorname{Supp} M \cap \operatorname{CoSupp} N$.

□

Next Lemma plays a significant role to lead us to the desired result.

**Lemma 3.3.5.** *Let $\mathfrak{a}$ be an ideal of a ring $R$. Let $N$ be an Artinian $R$-module with* $\operatorname{Att}_R(N) \subseteq V(\mathfrak{a})$. *Then* $\operatorname{CoSupp} N \subseteq V(\mathfrak{a})$.

**Proof.** As $N$ is Artinian so the descending chain

$$\mathfrak{a}N \supseteq \mathfrak{a}^2N \supseteq ... \supseteq \mathfrak{a}^nN \supseteq ...$$

of submodules of $N$ is stable, i.e. there exists an integer $k$ that $\mathfrak{a}^kN = \mathfrak{a}^{k+1}N$. As $\mathrm{Att}_R(N/\mathfrak{a}^kN) = \mathrm{Att}_R(N) \cap V(\mathfrak{a})$ (cf. Proposition 2.4.1) and $\mathrm{CoSupp}(N/\mathfrak{a}^kN) \subseteq V(\mathfrak{a})$ by virtue of lemma 3.3.4, then by passing to $N/\mathfrak{a}^kN$ we may assume that $\mathfrak{a}^kN = 0$.

Let $\mathfrak{p} \in \mathrm{CoSupp}\, N$, then ${}^{\mathfrak{p}}N \neq 0$ so by Lemma 2.3.5, for every $s \in S = R \setminus \mathfrak{p}$, $sN \neq 0$. On the other hand as $\bigcap_n \mathfrak{a}^nN = \mathfrak{a}^kN = 0$, hence for every $s \in S$, $sN \nsubseteq \mathfrak{a}^tN$. Then for all $s \in S$, $s \notin \mathfrak{a}^t$. It means that $\mathfrak{p} \in V(\mathfrak{a})$. □

**Corollary 3.3.6.** *Let $i \in \mathbb{Z}$. Let $(R, \mathfrak{m})$ be a local ring and $M$ be a finitely generated $R$-module. Assume that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is an Artinian $R$-module, then* $\mathrm{CoSupp}\, \mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq V(\mathfrak{a})$.

**Proof.** As $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian and $\mathfrak{a}$-adically complete so there exists an integer $k$ such that $\bigcap_{n\geq 1} \mathfrak{a}^n\mathfrak{F}^i_{\mathfrak{a}}(M) = \mathfrak{a}^k\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$. Hence Proposition 2.4.4(2) implies that $\mathrm{Att}\, \mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq V(\mathfrak{a})$, so it follows that $\mathrm{CoSupp}\, \mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq V(\mathfrak{a})$ by Lemma 3.3.5. □

**Remark 3.3.7.** *Converse of corollary (3.3.6) is not true in general. Let $R = k[|x|]$ denote the formal power series ring over a field $k$. Put $\mathfrak{a} = (x)R$. Then*

$$\mathrm{CoSupp}\, \mathfrak{F}^0_{\mathfrak{a}}(R) = \mathrm{Supp}\, D_R(D_R(H^1_{\mathfrak{a}}(R))) = \mathrm{Supp}\, H^1_{\mathfrak{a}}(R) \subseteq V(\mathfrak{a})$$

*but $\mathfrak{F}^0_{\mathfrak{a}}(R)$ is not Artinian.*

We now turn our attention to prove Theorem (3.3.10). For this reason we give some preliminary lemmas:

**Lemma 3.3.8.** *Let $(R, \mathfrak{m})$ be a local ring and $M$ be a finitely generated $R$-module. Then*

$$\mathfrak{F}^c_{\mathfrak{a}}(M) \cong \mathfrak{F}^c_{\mathfrak{a}}(R) \otimes_R M,$$

*where $c := \dim R/\mathfrak{a}$.*

**Proof.** At first note that by definition of inverse limit, $\mathfrak{F}^j_{\mathfrak{a}}(-)$ preserves finite direct sum, for every $j \in \mathbb{Z}$. Furthermore $\mathfrak{F}^c_{\mathfrak{a}}(-)$ is a right exact functor (cf. Theorem 3.1.5). Hence by Watts' Theorem ( [Rot, Theorem 3.33]) the claim is proved. □

Lemma 3.3.8 declares that $\mathfrak{F}^c_{\mathfrak{a}}(R) = 0$ if and only if $\mathfrak{F}^c_{\mathfrak{a}}(M) = 0$ for all finitely generated $R$-module $M$.

We utilize the useful consequence of Gruson's Theorem (e.g., [Vas, Corollary 4.3]) allows us to reduce to the case $M = R$ when considering the cosupport of top formal local cohomology modules.

**Lemma 3.3.9.** *Let $M$ be a finite faithful $R$-module and $N$ an arbitrary $R$-module. Then $M \otimes_R N = 0$ if and only if $N = 0$.*

**Proof.** We cite a proof is appeared in a note due to B. Johnson. Suppose $M \otimes_R N = 0$. It suffices to show that $N \otimes_R L = 0$ for any $R$-module $L$. Let $\lambda(L)$ denote the length of the shortest filtration of $L$ such that the factor modules of the filtration are homomorphic images of direct sums of copies of $M$ (This filtration exists by Grusson's Theorem, cf. [Vas, Theorem 4.1]). If $\lambda(L) = 1$ then $L$ is the homomorphic image of $\oplus_n M$ for some $n$. Since $M \otimes_R N = 0$, certainly $\oplus_n M \otimes_R N = 0$ and hence $M \otimes_R L = 0$. Suppose $\lambda(L) > 1$. Then there exists a short exact sequence $0 \to L^{'} \to L \to L^{''} \to 0$ such that $\lambda(L^{'})$ and $\lambda(L^{''})$ are less than $\lambda(L)$ By induction, $N \otimes_R L^{''} = 0 = N \otimes_R L^{'}$. By the right exactness of tensor products, we see that $N \otimes_R L = 0$. $\square$

Next Theorem is the analogue for formal local cohomology of the result due to Huneke-Katz-Marley in [Hun-Kat-Mar, Proposition 2.1]:

**Theorem 3.3.10.** *Let $(R, \mathfrak{m})$ be a local ring. Let $M$ be a finitely generated $R$-module. Then*

*(1)* $\operatorname{CoSupp}(\mathfrak{F}^c_{\mathfrak{a}}(M)) = \operatorname{CoSupp}(\mathfrak{F}^c_{\mathfrak{a}}(R/J))$,

*(2)* $\operatorname{Supp}(\mathfrak{F}^c_{\mathfrak{a}}(M)) = \operatorname{Supp}(\mathfrak{F}^c_{\mathfrak{a}}(R/J))$,

*where $J := \operatorname{Ann}_R M$ and $c := \dim R/\mathfrak{a}$.*

**Proof.** (1): Since for every $i \in \mathbb{Z}$, $\mathfrak{F}^i_{\mathfrak{a}}(M) \cong \mathfrak{F}^i_{\mathfrak{a}(R/J)}(M)$, by Independence Theorem (cf. 2.1.3(1)), we may replace $R$ by $R/J$ to assume that $M$ is faithful. Note that for $\dim R/(\mathfrak{a}, J) < c$, there is nothing to prove because, $\mathfrak{F}^c_{\mathfrak{a}}(M) = 0$.

In the view of lemma 3.3.8 and Proposition 3.3.3, for every $\mathfrak{p} \in \operatorname{Spec} R$

$$^{\mathfrak{p}}\mathfrak{F}^c_{\mathfrak{a}}(M) \cong M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} {}^{\mathfrak{p}}\mathfrak{F}^c_{\mathfrak{a}}(R).$$

As $M_{\mathfrak{p}}$ is faithful $R_{\mathfrak{p}}$-module, (3.3.9) implies that $M_{\mathfrak{p}} \otimes_R {}^{\mathfrak{p}}\mathfrak{F}^c_{\mathfrak{a}}(R) = 0$ if and only if ${}^{\mathfrak{p}}\mathfrak{F}^c_{\mathfrak{a}}(R) = 0$, which completes the proof.

(2): To prove we use the localization instead of colocalization in the proof of (1). □

## 3.4 Coassociated primes

There have been three earlier attempts to dualize the theory of associated primes. The first one was made by I.G. Macdonald in [Mac] by defining the set $\mathrm{Att}(M)$ of attached prime ideals of an A-module M. The theory of attached primes is particularly well-behaved when M has a secondary representation (which is the dual notion to primary decomposition). However, in general this theory is not completely satisfactory.

Next, L. Chambless [Cham], H. Zöschinger [Z2] and S. Yassemi [Yas] defined the set $\mathrm{Coass}_R(M)$ of coassociated prime ideals of an $R$-module $M$. Yassemi's definition of coassociated primes (below) is equivalent with Macdonald's definition when $M$ has secondary representation, and that this is equivalent with Chambless and Zöschinger's definitions (cf. [Yas] for details).

**Definition 3.4.1.** *(1) For any maximal ideal $\mathfrak{m}$ of $R$ we define a duality functor $D_{\mathfrak{m}}(-) = \mathrm{Hom}(-, E(R/\mathfrak{m}))$ where $E(R/\mathfrak{m})$ is the injective envelope of $R/\mathfrak{m}$.*

*(2) We say that an R-module L is cocyclic if L is a submodule of $E(R/\mathfrak{m})$ for some $\mathfrak{m} \in \max(R)$. In other words $L \subseteq D_{\mathfrak{m}}(R)$ for some $\mathfrak{m} \in \max(R)$.*

*(3) Let M be an R-module. A prime ideal $\mathfrak{p}$ of R is called a coassociated prime of M if there exists a cocyclic homomorphic image L of M such that $\mathfrak{p} = \mathrm{Ann}(L)$. The set of coassociated prime ideals of M is denoted by $\mathrm{Coass}_R(M)$*

For brevity we often write $\mathrm{Coass}(M)$ for $\mathrm{Coass}_R(M)$ when there is no ambiguity about the ring $R$.

Below we collect some facts on coassociated primes, for more details see [Yas].

**Theorem 3.4.2.** *Let $M, M', M''$ be R-modules.*

*(1) $\mathfrak{p} \in \mathrm{Coass}(M)$ if and only if there exists $\mathfrak{m} \in \max(R) \cap V(\mathfrak{p})$ such that $\mathfrak{p} \in \mathrm{Ass}(D_{\mathfrak{m}}(M))$.*

(2) *If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence, then* $\operatorname{Coass}(M'') \subseteq \operatorname{Coass}(M) \subseteq \operatorname{Coass}(M') \cup \operatorname{Coass}(M'')$.

(3) *If $M$ is a finite $R$-module and $N$ is any $R$-module, then*

$$\operatorname{Coass}(M \otimes N) = \operatorname{Supp} M \cap \operatorname{Coass}(N).$$

(4) *If $M$ is an Artinian $R$-module, then* $\operatorname{Coass}(M)$ *is finite.*

**Proof.** see [Yas, 1.7, 1.10, 1.21, 1.22]. □

Next we show the relation between coassociated primes of a module and Richardson's cosupport.

**Lemma 3.4.3.** *Let $(R, \mathfrak{m})$ be a local ring and $M$ an $R$-module. Then the following statements are true:*

(1) $\operatorname{Coass}(M) \subseteq \operatorname{CoSupp}(M)$.

(2) *Every minimal element of* $\operatorname{CoSupp}(M)$ *belongs to* $\operatorname{Coass}(M)$.

(3) *For any Noetherian $\widehat{R}$-module $M$,* $\operatorname{Coass}(M) = \operatorname{CoSupp}(M) \subseteq \{\mathfrak{m}\}$, *where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of $R$.*

**Proof.**

(1) Let $\mathfrak{p} \in \operatorname{Coass}(M)$. Then $\mathfrak{p} \in \operatorname{Ass} D_R(M)$ so $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}} D_R(M)_\mathfrak{p}$ which implies that $0 \neq \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, D_R(M)_\mathfrak{p})$, so it remains nonzero by applying $\operatorname{Hom}_{R_\mathfrak{p}}(-, E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$. It follows by definition that $\mathfrak{p} \in \operatorname{CoSupp}(M)$.

(2) Let $\mathfrak{p} \in \min \operatorname{CoSupp}(M)$, then $\mathfrak{p} \in \min \operatorname{Supp} D_R(M)$ (cf. 3.3.2). Hence $\mathfrak{p} \in \min \operatorname{Ass} D_R(M)$ and it follows that $\mathfrak{p} \in \min \operatorname{Coass}(M)$.

(3) It is clear that $\operatorname{Coass}(M) = \emptyset$ if and only if $M = 0$ if and only if $\operatorname{CoSupp}(M) = \emptyset$. In the case $\operatorname{Coass}(M)$ is non-empty the claim follows by (1) and (2). □

It should be noted that $\operatorname{Supp}(\mathfrak{F}^i_\mathfrak{a}(M))$ is closed when $\operatorname{Ass}(\mathfrak{F}^i_\mathfrak{a}(M))$ is finite. In fact for a local Gorenstein ring $(R, \mathfrak{m})$, $\operatorname{Ass}(\mathfrak{F}^i_\mathfrak{a}(R)) = \operatorname{Ass} D_R(H^{\dim R - i}_\mathfrak{a}(R))$ where it was discussed extensively in [Hel]. Take into account that it is not finite in general.

**Remark 3.4.4.** *( [Asgh-Divan, Remark 2.8(vi)]) Let* $(R, \mathfrak{m})$ *be complete Gorenstein and equicharacteristic ring with* $\dim R > 2$*. Let* $\mathfrak{p}$ *be a prime ideal of* $R$ *of height* $2$ *and take* $x \in \mathfrak{m} \setminus \mathfrak{p}$*. Then by [Hel, Corollary 2.6],* $\operatorname{Ass}_R(\mathfrak{F}^{\dim R-1}_{(x)}(R)) = \operatorname{Spec} R \setminus V((x))$*. Since* $\operatorname{ht} \mathfrak{p} = 2$*, there are infinitely many prime ideals of* $R$ *which are contained in* $\mathfrak{p}$*, and so* $\operatorname{Ass}_R(\mathfrak{F}^{\dim R-1}_{(x)}(R))$ *is infinite.*

Our motivation to consider the $\operatorname{Coass}(\mathfrak{F}^i_{\mathfrak{a}}(M))$ arises from the following Lemma. Of a particular interest are the closed subsets of $\operatorname{Spec} R$ in the Zariski topology. We consider to the cosupport of $\mathfrak{F}^i_{\mathfrak{a}}(M)$ to see when it is a closed subset of $\operatorname{Spec} R$. For an Artinian module $N$, it is known that $\operatorname{Cosupp} N = V(\operatorname{Ann} N)$ (cf. 3.3.2). More precisely in order to show that $\operatorname{Cosupp}(\mathfrak{F}^i_{\mathfrak{a}}(M))$ being closed, it is enough to show that $\operatorname{Coass}(\mathfrak{F}^i_{\mathfrak{a}}(M))$ is finite, so it has encouraged us to consider the $\operatorname{Coass}(\mathfrak{F}^i_{\mathfrak{a}}(M))$.

**Lemma 3.4.5.** *Let* $(R, \mathfrak{m})$ *be a local ring and* $M$ *be an* $R$*-module. The set of minimal primes in* $\operatorname{CoSupp}(M)$ *is finite if and only if* $\operatorname{CoSupp}(M)$ *is a closed subset of* $\operatorname{Spec} R$*.*

**Proof.** Let $\operatorname{CoSupp}(M) = V(\mathfrak{b})$ for some ideal $\mathfrak{b}$ of $R$. As $R$ is Noetherian then so is $R/\mathfrak{b}$. It turns out that the set of minimal elements of $\operatorname{CoSupp}(M)$ is finite.

For the reverse direction, let $\mathfrak{p}_1, ..., \mathfrak{p}_t$ be the minimal prime ideals of $\operatorname{CoSupp}(M)$. Put $\mathfrak{q} := \cap_i \mathfrak{p}_i$. We claim that $\operatorname{CoSupp} M = V(\mathfrak{q})$.

It is clear that $\operatorname{CoSupp}(M) \subseteq V(\mathfrak{q})$. For the opposite direction assume that there is a prime ideal $Q \supset \mathfrak{q}$. Then $Q \supset \mathfrak{p}_j$, for some $1 \leq j \leq t$ so the proof follows by lemma 3.3.4(1). $\square$

Take into account that when $R$ is a complete local Gorenstein ring and $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is assumed to be either Noetherian or Artinian module, then

$$\operatorname{CoSupp}(\mathfrak{F}^i_{\mathfrak{a}}(M)) = \operatorname{Supp} H^{\dim R-i}_{\mathfrak{a}}(M, R).$$

As we have seen in Theorem 3.2.2, for a Cohen-Macaulay ring $R$ with $\operatorname{ht} \mathfrak{a} > 0$, $\mathfrak{F}^{\dim R/\mathfrak{a}}_{\mathfrak{a}}(M)$ is not Artinian. Moreover $\mathfrak{F}^{\dim M/\mathfrak{a}M}_{\mathfrak{a}}(M)$ is not finitely generated for $\dim M/\mathfrak{a}M > 0$ (cf. [Asgh-Divan, Theorem 2.6 (ii)]). Below we give an alternative proof.

**Theorem 3.4.6.** *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* $M$ *a finitely generated* $R$*-module. Assume that* $\dim M/\mathfrak{a}M > 0$*. Then* $\mathfrak{F}^{\dim M/\mathfrak{a}M}_{\mathfrak{a}}(M)$ *is not a finitely generated* $R$*-module.*

**Proof.** Put $c := \dim M/\mathfrak{a}M$. In the contrary assume that $\mathfrak{F}^c_{\mathfrak{a}}(M)$ is a finitely generated $R$-module. Let $x \in \mathfrak{m}$ be a parameter element of $M/\mathfrak{a}M$. Hence Theorem 3.1.7 implies the following long exact sequence

$$... \to \operatorname{Hom}(R_x, \mathfrak{F}^c_{\mathfrak{a}}(M)) \to \mathfrak{F}^c_{\mathfrak{a}}(M) \to \mathfrak{F}^c_{(\mathfrak{a},x)}(M) \to ...,$$

where $i \in \mathbb{Z}$. As $\dim M/(\mathfrak{a}, x)M < \dim M/\mathfrak{a}M$, then $\mathfrak{F}^c_{(\mathfrak{a},x)}(M) = 0$. Now let $f \in \operatorname{Hom}(R_x, \mathfrak{F}^c_{\mathfrak{a}}(M))$. Fix an arbitrary integer $n$, so

$$f(1/x^n) = x^m f(1/x^{m+n}) \in x^m \mathfrak{F}^c_{\mathfrak{a}}(M),$$

for every integer $m$. It implies that $f(1/x^n) \in \bigcap_m x^m \mathfrak{F}^c_{\mathfrak{a}}(M) = 0$ by Krull's Theorem, hence $f = 0$. Now it follows that $\mathfrak{F}^c_{\mathfrak{a}}(M) = 0$, which is a contradiction, see 3.1.5. □

Now we examine the set of coassociated primes of top formal local cohomology to show that by some assumptions on $R$, it could be finite.

**Proposition 3.4.7.** *Let $\mathfrak{a}$ be an ideal of a complete Gorenstein local ring $(R, \mathfrak{m})$ and $c := \dim R/\mathfrak{a}$. Let $M$ be a finitely generated $R$-module. Then*

$$\operatorname{Coass}_R(\mathfrak{F}^c_{\mathfrak{a}}(M)) = \operatorname{Supp}_R M \cap \operatorname{Ass}_R(H^{\operatorname{ht}\mathfrak{a}}_{\mathfrak{a}}(R)).$$

**Proof.**

$$\begin{aligned} \operatorname{Coass}_R(\mathfrak{F}^c_{\mathfrak{a}}(M)) &= \operatorname{Coass}_R(\mathfrak{F}^c_{\mathfrak{a}}(R) \otimes_R M) \\ &= \operatorname{Supp}_R M \cap \operatorname{Coass}_R(\mathfrak{F}^c_{\mathfrak{a}}(R)) \\ &= \operatorname{Supp}_R M \cap \operatorname{Ass}_R(H^{\operatorname{ht}\mathfrak{a}}_{\mathfrak{a}}(R)), \end{aligned}$$

where the first equality is clear by lemma 3.3.8, the second equality follows by 3.4.2(3). □

It should be noted that by hypotheses in Proposition 3.4.7, $\operatorname{ht}\mathfrak{a} = \operatorname{grade}_R \mathfrak{a}$ (by $\operatorname{grade}_R \mathfrak{a}$ we mean the common length of maximal $R$-sequences in $\mathfrak{a}$) and it is well-known that $\operatorname{Ass}_R(H^{\operatorname{grade}_R \mathfrak{a}}_{\mathfrak{a}}(R))$ is finite, e.g see [Mar, Proposition 1.1] or [Hel2, Theorem 1].

**Corollary 3.4.8.** *Keep the notations and hypotheses in Proposition 3.4.7,*

$$\mathfrak{F}^c_{\mathfrak{a}}(M) = 0 \quad \textit{if and only if} \quad \operatorname{Supp}_R M \cap \operatorname{Ass}_R(H^{\operatorname{ht}\mathfrak{a}}_{\mathfrak{a}}(R)) = \emptyset.$$

**Proof.** It follows by the fact that $\mathfrak{F}^c_{\mathfrak{a}}(M) = 0$ if and only if $\operatorname{Coass}_R(\mathfrak{F}^c_{\mathfrak{a}}(M)) = \emptyset$. $\square$

In the light of Lemma 3.1.4, one can see that for a local ring $R$, $\operatorname{Coass}_R(\mathfrak{F}^0_{\mathfrak{a}}(R))$ is not the same with $\operatorname{Coass}_{\widehat{R}}(\mathfrak{F}^0_{\mathfrak{a}}(R))$.

**Remark 3.4.9.** *(1) (cf. [Z, Beispiel 2.4]) Let $(R, \mathfrak{m})$ be a local ring, then*

$$\operatorname{Coass}_R \widehat{R} = \{\mathfrak{m}\} \cup \{\mathfrak{p} \in \operatorname{Spec} R : R/\mathfrak{p} \text{ is not complete}\}.$$

*(2) $\operatorname{Coass}_{\widehat{R}} \mathfrak{F}^0_{\mathfrak{a}}(R)$ is finite, as $\mathfrak{F}^0_{\mathfrak{a}}(R)$ is a finitely generated $\widehat{R}$-module (cf. Lemma 3.1.4) but $\operatorname{Coass}_R \mathfrak{F}^0_{\mathfrak{a}}(R)$ is not finite in general. The example in [Asgh-Divan, Remark 2.8(iii)] shows it more clear. let $T := \mathbb{Q}[X,Y]_{(X,Y)}$ and $\mathfrak{a} := (X,Y)T$. Then $\mathfrak{F}^0_{\mathfrak{a}}(T) = \widehat{T} = \mathbb{Q}[[X,Y]]$. For each integer $n$, let $\mathfrak{p}_n := (X - nY)T$. Then it is easy to see that $T/\mathfrak{p}_n \cong \mathbb{Q}[Y]_{(Y)}$, and so it is not a complete local ring. By (1), $\operatorname{Coass}_T \mathfrak{F}^0_{\mathfrak{a}}(T) = \{\mathfrak{a}\} \cup \{\mathfrak{p} \in \operatorname{Spec} T : T/\mathfrak{p} \text{ is not complete }\}$. Hence $\operatorname{Coass}_T(\mathfrak{F}^0_{\mathfrak{a}}(T))$ is not finite.*

**Proposition 3.4.10.** *Let $i \in \mathbb{Z}$. Let $\mathfrak{a} \subset R$ be an ideal of a ring $R$. If $\operatorname{Coass}_R \mathfrak{F}^i_{\mathfrak{a}}(R)$ is finite, then so is $\operatorname{Coass}_R \mathfrak{F}^i_{\mathfrak{a}}(R/H^0_{\mathfrak{a}}(R))$. In the case $\mathfrak{F}^i_{\mathfrak{a}}(R/H^0_{\mathfrak{a}}(R))$ is Artinian, the converse can be true.*

**Proof.** Consider the exact sequence

$$0 \to H^0_{\mathfrak{a}}(R) \to R \to R/H^0_{\mathfrak{a}}(R) = \overline{R} \to 0.$$

It provides the following long exact sequence

$$\ldots \to \mathfrak{F}^i_{\mathfrak{a}}(H^0_{\mathfrak{a}}(R)) \xrightarrow{\psi} \mathfrak{F}^i_{\mathfrak{a}}(R) \xrightarrow{\varphi} \mathfrak{F}^i_{\mathfrak{a}}(\overline{R}) \to \mathfrak{F}^{i+1}_{\mathfrak{a}}(H^0_{\mathfrak{a}}(R)) \to \ldots, \qquad (*)$$

for every $i$.

As $\mathfrak{F}^i_{\mathfrak{a}}(H^0_{\mathfrak{a}}(R)) = H^i_{\mathfrak{m}}(H^0_{\mathfrak{a}}(R))$ is Artinian, it follows that $\operatorname{Coass}(\mathfrak{F}^i_{\mathfrak{a}}(H^0_{\mathfrak{a}}(R)))$ is finite.

By virtue of $(*)$, we get the following short exact sequence

$$0 \to U \to \mathfrak{F}^i_{\mathfrak{a}}(\overline{R}) \to U' \to 0,$$

where $U = \operatorname{coker} \psi$ and $U' = \operatorname{coker} \varphi$. It implies that $\operatorname{Coass} \mathfrak{F}^i_{\mathfrak{a}}(\overline{R})$ is finite. To this end consider $\operatorname{Coass}(U)$ is finite, by assumption and 3.4.2(2). Furthermore $\operatorname{Coass}(U')$ is finite as $\mathfrak{F}^{i+1}_{\mathfrak{a}}(H^0_{\mathfrak{a}}(R))$ is Artinian. $\square$

Now we are going to give more information on the last non-vanishing formal local cohomology module.

**Theorem 3.4.11.** *Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$. Let $\mathfrak{F}^d_{\mathfrak{a}}(R) = 0$. Then:*

*(1) If $\mathfrak{p} \in \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)$, then it implies that $\dim(R/(\mathfrak{a}, \mathfrak{p})) = d - 1$.*

*(2) $\operatorname{Assh}(R) \cap \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R) \subseteq \{\mathfrak{p} \in \operatorname{Spec} R : \dim R/\mathfrak{p} = d, \operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) \neq \mathfrak{m}\}$.*

*(3) If $\operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R) \subseteq \operatorname{Assh}(R)$, then $\{\mathfrak{p} \in \operatorname{Spec} R : \dim(R/(\mathfrak{a}, \mathfrak{p})) = d - 1\} \subseteq \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)$.*

**Proof.**

(1) Let $\mathfrak{p} \in \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)$. As $\mathfrak{F}^d_{\mathfrak{a}}(R) = 0$, by Theorem 3.1.5

$$\dim R/(\mathfrak{a}, \mathfrak{p}) \leq \dim R/\mathfrak{a} \leq d - 1.$$

On the other hand $\mathfrak{p} \in \operatorname{Coass}(R/\mathfrak{p} \otimes_R \mathfrak{F}^{d-1}_{\mathfrak{a}}(R))$, because

$$\operatorname{Coass}(R/\mathfrak{p} \otimes_R \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)) = \operatorname{Supp} R/\mathfrak{p} \cap \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R).$$

It yields with the similar argument to lemma 3.3.8 that $0 \neq R/\mathfrak{p} \otimes_R \mathfrak{F}^{d-1}_{\mathfrak{a}}(R) = \mathfrak{F}^{d-1}_{\mathfrak{a}}(R/\mathfrak{p})$. So we have $\dim R/(\mathfrak{a}, \mathfrak{p}) \geq d - 1$. It completes the proof.

(2) Let $\mathfrak{p} \in \operatorname{Assh}(R) \cap \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)$. Then similar to (1), $\mathfrak{F}^{d-1}_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$ and moreover $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) \neq \mathfrak{m}$. To this end note that if $\operatorname{Rad}(\mathfrak{a} + \mathfrak{p}) = \mathfrak{m}$, then $\mathfrak{F}^{d-1}_{\mathfrak{a}}(R/\mathfrak{p}) = 0$ by Grothendieck's vanishing Theorem.

(3) Let $\mathfrak{p} \in \operatorname{Spec} R$ and $\dim(R/(\mathfrak{a}, \mathfrak{p})) = d - 1$. Then it follows that $\emptyset \neq \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R/\mathfrak{p}) = \operatorname{Supp}(R/\mathfrak{p}) \cap \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)$. Let $\mathfrak{q} \in \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)$ then, $\mathfrak{q} \supseteq \mathfrak{p}$, but by assumption $\mathfrak{q}$ is minimal so we deduce that $\mathfrak{q} = \mathfrak{p}$. □

**Remark 3.4.12.** *The inclusion in Theorem 3.4.11(2) is not an equality in general. For example Let $R = k[[x, y, z]]$ denote the formal power series ring in three variables over a field k. Let $\mathfrak{a} = (x, y)$ be an ideal of R which is of dimension one and put $\mathfrak{p} = 0$. It is clear that $\mathfrak{F}^{3-1}_{\mathfrak{a}}(R) = 0 = \mathfrak{F}^3_{\mathfrak{a}}(R)$, that is $\operatorname{Coass} \mathfrak{F}^{3-1}_{\mathfrak{a}}(R) = \emptyset$.*

**Lemma 3.4.13.** *Let $(R, \mathfrak{m})$ be a local complete ring and $\mathfrak{a}$ an ideal of R. Let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$. Then $\mathfrak{q} \in \operatorname{Coass}_R(\widehat{R_{\mathfrak{p}}})$ implies that $\mathfrak{q} \subseteq \mathfrak{p}$.*

**Proof.** The proof is straightforward. Let $\mathfrak{q} \in \mathrm{Coass}_R(\widehat{R_{\mathfrak{p}}})$, then

$$0 \neq \mathrm{Hom}_R(R/\mathfrak{q}, \mathrm{Hom}_R(\widehat{R_{\mathfrak{p}}}, E_R(R/\mathfrak{m}))) = \mathrm{Hom}_R(R/\mathfrak{q} \otimes_R \widehat{R_{\mathfrak{p}}}, E_R(R/\mathfrak{m})).$$

It yields that

$$0 \neq R/\mathfrak{q} \otimes_R \widehat{R_{\mathfrak{p}}} = R/\mathfrak{q} \otimes_R R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \widehat{R_{\mathfrak{p}}}.$$

It is clear that $R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}} \neq 0$ and so $\mathfrak{q}$ must be contained in $\mathfrak{p}$. □

Next result shows that for a one dimensional ideal $\mathfrak{a}$ of a complete local ring $R$ of dimension $d$, $\mathrm{Cosupp}\, \mathfrak{F}_{\mathfrak{a}}^{d-1}(R)$ is closed.

**Theorem 3.4.14.** *Let $(R, \mathfrak{m})$ be a local complete ring of dimension $d$. Let $\mathfrak{a}$ be an ideal of dimension one. Then*

$$\mathfrak{F}_{\mathfrak{a}}^{d-1}(R) = 0, \text{ when } d > 2,$$

*in particular* $\mathrm{Coass}_R\, \mathfrak{F}_{\mathfrak{a}}^{d-1}(R) = \emptyset$.

$$\mathrm{Coass}_R\, \mathfrak{F}_{\mathfrak{a}}^{d-1}(R) \subseteq \{\mathfrak{m}\}, \text{ when } d = 1$$

*and in the case $d = 2$*

$$\mathrm{Coass}_R\, \mathfrak{F}_{\mathfrak{a}}^{d-1}(R) = \textstyle\bigcup_{i=1}^{r} \mathrm{Coass}_R(\widehat{R_{\mathfrak{p}_i}}) = \{\mathfrak{p}_1, ..., \mathfrak{p}_r\} \cup (\bigcup_{j=1}^{s}\{\mathfrak{q}_j : R_{\mathfrak{p}_i}/\mathfrak{q}_j R_{\mathfrak{p}_i} \text{ is not complete }\}),$$

*where $\mathfrak{p}_1, ...\mathfrak{p}_r$ are minimal prime ideals of $\mathfrak{a}$ and $\mathfrak{q}_1, ...\mathfrak{q}_s$ are minimal prime ideals of $R$ with $\mathfrak{q}_j \subseteq \mathfrak{p}_i$ for $i \in \{1, ..., r\}$.*

*In particular* $\mathrm{Cosupp}\, \mathfrak{F}_{\mathfrak{a}}^{d-1}(R)$ *is closed for all $d > 0$.*

**Proof.** For $d > 2$ and $d = 1$, the claim is clear.

Let $d = 2$. Suppose that $\mathfrak{p}_1, ..., \mathfrak{p}_r$ be the minimal prime ideals of $\mathfrak{a}$. Put $S = \bigcap_{i=1}^{r}(R \setminus \mathfrak{p}_i)$, choose $y \in \mathfrak{m} \setminus \bigcup_{i=1}^{r} \mathfrak{p}_i$. By Theorem 2.1.4, for any $n \in \mathbb{N}$ we have

$$0 \to H_{\mathfrak{m}}^0(R/\mathfrak{a}^n) \to R/\mathfrak{a}^n \to D_{(y)}(R/\mathfrak{a}^n) \to H_{\mathfrak{m}}^1(R/\mathfrak{a}^n) \to 0,$$

where $D_{(y)}(R/\mathfrak{a}^n)$ is the $(y)$-transform functor. One can see that $D_{(y)}(R/\mathfrak{a}^n) \cong R_S/\mathfrak{a}^n R_S$, so we get the following exact sequence

$$0 \to H_{\mathfrak{m}}^0(R/\mathfrak{a}^n) \to R/\mathfrak{a}^n \to R_S/\mathfrak{a}^n R_S \to H_{\mathfrak{m}}^1(R/\mathfrak{a}^n) \to 0.$$

Furthermore $R_S/\mathfrak{a}^n R_S \cong \oplus_{i=1}^{r} R_{\mathfrak{p}_i}/\mathfrak{a}^n R_{\mathfrak{p}_i}$. All the modules satisfying the Mittag-Leffler condition so by applying inverse limits we get

$$0 \to R/\mathfrak{F}_{\mathfrak{a}}^0(R) \to \oplus_{i=1}^{r} \widehat{R_{\mathfrak{p}_i}} \to \mathfrak{F}_{\mathfrak{a}}^1(R) \to 0.$$

It yields that $\operatorname{Coass}_R(\mathfrak{F}^1_{\mathfrak{a}}(R)) \subseteq \bigcup_{i=1}^r \operatorname{Coass}_R(\widehat{R_{\mathfrak{p}_i}}) \subseteq \operatorname{Coass}_R(\mathfrak{F}^1_{\mathfrak{a}}(R)) \cup \{\mathfrak{m}\}$. In the view of lemma 3.4.13, $\operatorname{Coass}_R(\mathfrak{F}^1_{\mathfrak{a}}(R)) = \bigcup_{i=1}^r \operatorname{Coass}_R(\widehat{R_{\mathfrak{p}_i}})$. Now the claim is proved by Remark 3.4.9(1). To this end note that $\operatorname{Coass}_R(\widehat{R_{\mathfrak{p}_i}}) = \operatorname{Coass}_{R_{\mathfrak{p}_i}}(\widehat{R_{\mathfrak{p}_i}}) \cap R$ for every $i \in \{1, ..., r\}$. □

**Remark 3.4.15.** *Keep the notations and hypotheses in Theorem 3.4.14 and let $M$ be a finitely generated $R$-module. As $R$ is complete so by Cohen's structure Theorem, there exists a Gorenstein local ring $(S, \mathfrak{n})$ where $R$ is a homomorphic image of $S$ and* $\dim R = \dim S$. *Then by virtue of 3.3.8 we have*

$$\operatorname{Ass}_R H^1_{\mathfrak{a}S}(M, S) \subseteq \operatorname{Coass} \mathfrak{F}^{d-1}_{\mathfrak{a}}(R)$$

*is finite.*

# Chapter 4

# Top local cohomology modules

In this chapter we consider $H_{\mathfrak{a}}^{\dim M}(M)$; the last possible non-vanishing local cohomology module. It is known by the Grothendieck's vanishing Theorem (cf. 2.1.5) that $H_{\mathfrak{a}}^{i}(M) = 0$ for all $i > \dim M$.

Let $(R, \mathfrak{m})$ be a local ring and $M$ be a finitely generated $R$-module. Then there is the long exact sequence

$$H_{\mathfrak{m}}^{i}(M) \to H_{\mathfrak{a}}^{i}(M) \to \varinjlim_{n} \operatorname{Ext}_{R}^{i}(\mathfrak{m}^{n}/\mathfrak{a}^{n}, M) \to \ldots$$

relating the local cohomology of $M$ with respect to $\mathfrak{a}$ and $\mathfrak{m}$ resp. It follows by Hartshorne's result, see [Hart, p. 417], that $\varinjlim_{n} \operatorname{Ext}_{R}^{\dim M}(\mathfrak{m}^{n}/\mathfrak{a}^{n}, M) = 0$. Therefore $H_{\mathfrak{a}}^{\dim M}(M)$ is - as an epimorphic image of $H_{\mathfrak{m}}^{\dim M}(M)$ - an Artinian $R$-module.

The kernel of the natural epimorphism $H_{\mathfrak{m}}^{\dim M}(M) \to H_{\mathfrak{a}}^{\dim M}(M)$ was calculated in [Divan-Sch], but here we calculate it more precisely in the first section. Furthermore it yields a new equivalent statement to Hartshorne-Lichtenbaum vanishing Theorem.

The above results lead us to establish some properties of $\operatorname{Hom}_{\widehat{R}}(H_{\mathfrak{a}}^{d}(R), H_{\mathfrak{a}}^{d}(R))$ in section two. First of all a brief about endomorphism rings could be instrumental for understanding the content.

Let $G$ be an abelian group. An endomorphism of $G$ is a group homomorphism from $G$ to itself. The set $\operatorname{End} G$ of endomorphisms of $G$ is a ring where the addition is defined point-wise and the multiplication is given by composition: Given $f, g \in \operatorname{End} G$, the sum $f + g$ is the function defined by $(f+g)(x) = f(x) + g(x)$ and the product $fg$ is the function defined by $(fg)(x) = f(g(x))$. Let $R = \operatorname{End} G$.

Then $G$ is a left $R$-module where the scalar multiplication is just function evaluation that is, given $f \in R$ and $x \in G$, the scalar product $fx$ is just $f(x)$.

If $k$ is a field and we consider the $k$-vector space $k^n$, then the endomorphism ring of $k^n$ (which consists of all $k$-linear maps from $k^n$ to $k^n$) is naturally identified with the ring of $n \times n$ matrices with entries in $k$ which is not commutative in general.

One can often translate properties of an object into properties of its endomorphism ring. For instance, a module is indecomposable if and only if its endomorphism ring does not contain any non-trivial idempotents (cf. [Jacob]). Note that a module $M$ is decomposable if $M = M_1 \oplus M_2$ where $M_i \neq 0$ for $i = 1, 2$ are submodules of $M$. Otherwise $M$ is indecomposable. It follows that if $\operatorname{End} M$ for a module $M \neq 0$ is local, then $M$ is indecomposable.

Not so much is known about the ring $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ and its relation to a given ring $R$. In Theorem 4.2.2, for a local ring $(R, \mathfrak{m})$ and its $\mathfrak{m}$-adic completion $\widehat{R}$, we show that in some cases the map

$$\Phi : \widehat{R} \to \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$$

could be an isomorphism. Furthermore we show that $\operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ is a commutative semi-local Noetherian ring which is a finitely generated $\widehat{R}$-module.

## 4.1 Ideas around Hartshorne-Lichtenbaum vanishing Theorem

Let $(R, \mathfrak{m})$ be a commutative, Noetherian local ring (with identity) of dimension $d$, and let $\mathfrak{a}$ be a proper ideal of $R$. It is well known that, for an $R$-module $M$, the local cohomology modules $H^i_{\mathfrak{a}}(M)$ vanish for all $i > d$, while $H^d_{\mathfrak{a}}(M) \cong M \otimes_R H^d_{\mathfrak{a}}(R)$. These results accord some importance to $H^d_{\mathfrak{a}}(R)$ and a sufficient condition for its vanishing is given by the following theorem, which was first proved by R. Hartshorne.

**Theorem 4.1.1.** *( [Hart, 3.1] and also [Pes-Szp, III,3.1]) If, for every minimal prime ideal $\mathfrak{q}$ of $\widehat{R}$ of dimension $d$, we have $\dim(\widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{q}) \geq 1$, then $H^d_{\mathfrak{a}}(R) = 0$.*

In both Hartshorne's proof [Hart, Theorem 3.1] and the proof of C. Peskine and L. Szpiro [Pes-Szp, III,3.1], an important ingredient is an analysis of a situation in which $R$ is complete: Hartshorne reduces to a case where $R$ is a complete, normal, local domain; Peskine and Szpiro work with a complete Gorenstein local ring to obtain the result. Both proofs use Chevalley's Theorem [Zar-Sam, VIII, Sec.5, Theorem 13] for a complete local ring to compare topologies defined in terms of symbolic prime powers with ideal-adic topologies. For more information cf. [Call-Sh]. After them F. W. Call and R.Y. Sharp [Call-Sh] used symbolic prime powers rather differently in order to analyse the case when $R$ is Gorenstein by consideration of properties of a minimal injective resolution for $R$. As further references for the proof of Hartshorne-Lichtenbaum Theorem see [Sch3] and [Divan-Sch] used canonical modules.

In the following we bring a proof of the Hartshorne-Lichtenbaum Vanishing Theorem was appeared in [Br-Sh]. For a new proof we refer the reader to Hellus' Habilitation [Hel]. When $R$ is a $d$-dimensional complete local domain, the statement simplifies: $H^d_{\mathfrak{a}}(M) = 0$ for every $R$-module $M$ if $\dim R/\mathfrak{a} > 0$.

**Theorem 4.1.2.** ***(Hartshorne-Lichtenbaum Vanishing Theorem)*** *Suppose that $(R, \mathfrak{m})$ is local of dimension $d$ and also that $\mathfrak{a}$ is proper. Then the following statements are equivalent:*

*(1)* $H^d_{\mathfrak{a}}(R) = 0$;

*(2) For each (necessarily minimal) prime ideal $\mathfrak{p}$ of $\widehat{R}$, satisfying* $\dim \widehat{R}/\mathfrak{p} = d$, *we have* $\dim \widehat{R}/(\mathfrak{a}\widehat{R} + \mathfrak{p}) > 0$.

**Proof.** (1) $\Rightarrow$ (2): Assume that $H^d_{\mathfrak{a}}(R) = 0$ and that there exists a prime ideal $\mathfrak{p}$ of $\widehat{R}$ such that $\dim \widehat{R}/\mathfrak{p} = d$ but $\dim \widehat{R}/(\mathfrak{a}\widehat{R} + \mathfrak{p}) = 0$. Since the natural ring homomorphism $R \to \widehat{R}$ is flat, it follows from the flat base change Theorem (cf. 2.1.3(2)) that there is an $\widehat{R}$-isomorphism $H^d_{\mathfrak{a}}(R) \otimes_R \widehat{R} \cong H^d_{\mathfrak{a}\widehat{R}}(\widehat{R})$, and so $H^d_{\mathfrak{a}\widehat{R}}(\widehat{R}) = 0$.

Now $\mathfrak{m}\widehat{R}$ is the maximal ideal of the local ring $\widehat{R}$, and our assumptions mean that $(\widehat{R}/\mathfrak{p}, \mathfrak{m}\widehat{R}/\mathfrak{p})$ is a $d$-dimensional local ring and $(\mathfrak{a}\widehat{R} + \mathfrak{p})/\mathfrak{p}$ is an $(\mathfrak{m}\widehat{R}/\mathfrak{p})$-primary ideal of this ring. It therefore follows from Theorem 2.1.6 that $H^d_{(\mathfrak{a}\widehat{R}+\mathfrak{p})/\mathfrak{p}}(\widehat{R}/\mathfrak{p}) \neq 0$. We now deduce from the Independence Theorem that $H^d_{\mathfrak{a}\widehat{R}}(\widehat{R}/\mathfrak{p}) \neq 0$. Therefore we have $H^d_{\mathfrak{a}\widehat{R}}(\widehat{R}) \neq 0$, and this is a contradiction.

(2) $\Rightarrow$ (1): Suppose $H^d_{\mathfrak{a}}(R) \neq 0$. Then $H^d_{\mathfrak{a}\widehat{R}}(\widehat{R}) \neq 0$. Simply we may reduce to the case $(R, \mathfrak{m})$ is a local Gorenstein domain such that $\mathfrak{p} \in \operatorname{Spec} R$ and $H^d_{\mathfrak{p}}(R) \neq 0$

with $\dim R/\mathfrak{p} = 1$. This is impossible as

$$H^d_{\mathfrak{p}}(R) = \varinjlim_n \operatorname{Ext}^d_R(R/\mathfrak{p}^{(j)}, R) = 0,$$

as $\operatorname{depth} R/\mathfrak{p}^{(j)} > 0$ for every $j \in \mathbb{N}$. □

In order to extend the equivalent relations to Hartshorne-Lichtenbaum vanishing Theorem we show the relation between $H^d_{\mathfrak{a}}(M)$ and $H^d_{\mathfrak{m}}(M)$, where $\dim M = d$.

By a primary submodule we mean a proper submodule $N$ of a module $M$ such that whenever $r \in R$, $m \in M \setminus N$ and $rm \in N$, then there exists a positive integer $n$ such that $r^n M \subseteq N$. $N$ is called $\mathfrak{p}$-primary where $\mathfrak{p}$ is the prime ideal $\mathfrak{p} = \operatorname{Rad}(N :_R M)$.

For an $R$-module $M$ let $0 = \cap_{i=1}^n Q_i(M)$ denote a minimal primary decomposition of the zero submodule of $M$. That is $Q_i(M), i = 1, \ldots, n$, is a $\mathfrak{p}_i$ primary submodule of $M$. Clearly $\operatorname{Ass}_R M = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

**Definition 4.1.3.** *Let $\mathfrak{a} \subset R$ denote an ideal of $R$. We define two disjoint subsets $U, V$ of* $\operatorname{Ass}_R M$ *related to* $\mathfrak{a}$

*(a)* $U = \{\mathfrak{p} \in \operatorname{Ass}_R M | \dim R/\mathfrak{p} = d \text{ and } \dim R/\mathfrak{a} + \mathfrak{p} = 0\}$.

*(b)* $V = \{\mathfrak{p} \in \operatorname{Ass}_R M | \dim R/\mathfrak{p} < d \text{ or } \dim R/\mathfrak{p} = d \text{ and } \dim R/\mathfrak{a} + \mathfrak{p} > 0\}$.

*Finally we define $Q_{\mathfrak{a}}(M) = \cap_{\mathfrak{p}_i \in U} Q_i(M)$. In the case $U = \emptyset$, put $Q_{\mathfrak{a}}(M) = M$.*

The following Lemma gives a better understanding of the previous definitions (see [Sch, Lemma 2.7]).

**Lemma 4.1.4.** *With the previous notation we have that*

$$\operatorname{Ass}_R Q_{\mathfrak{a}} = V,\ \operatorname{Ass}_R M/Q_{\mathfrak{a}} = U \text{ and } U \cup V = \operatorname{Ass}_R M.$$

**Proof.** Let $\operatorname{Ass}_R M = \{\mathfrak{p}_1, ..., \mathfrak{p}_n\}$ and $0 = \cap_{i=1}^n Q_i(M)$ a minimal primary decomposition. First it is clear that $\operatorname{Ass}_R M/Q_{\mathfrak{a}} = U$. Remember that $Q_{\mathfrak{a}} = \cap_{\mathfrak{p}_i \in U} Q_i(M)$ is a reduced minimal primary decomposition. By our choose of $V$ we have $V = \{\mathfrak{p} \in \operatorname{Ass}_R M | \mathfrak{p} \notin U\}$. In order to show that $\operatorname{Ass}_R Q_{\mathfrak{a}} = V$ it is enough to prove that $\operatorname{Ass}_R Q_{\mathfrak{a}} = \{\mathfrak{p} \in \operatorname{Ass}_R M | \mathfrak{p} \notin U\}$.

Let $Q'_{\mathfrak{a}}(M) = \cap_{\mathfrak{p}_i \notin U} Q_i(M)$. First note that $Q_{\mathfrak{a}} = Q_{\mathfrak{a}} + Q'_{\mathfrak{a}}(M)/Q'_{\mathfrak{a}}(M) \subseteq M/Q'_{\mathfrak{a}}(M)$. Therefore $\operatorname{Ass}_R Q_{\mathfrak{a}} \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R M | \mathfrak{p} \notin U\}$ as easily seen. Now let $\mathfrak{p} \in \{\mathfrak{p} \in \operatorname{Ass}_R M | \mathfrak{p} \notin U\}$ be a given prime ideal. Then $Q_\mathfrak{a}/Q_\mathfrak{a} \cap Z(\mathfrak{p}) \cong Q_\mathfrak{a} + Z(\mathfrak{p})/Z(\mathfrak{p})$ is a nonzero $\mathfrak{p}$-coprimary module, where $Z(\mathfrak{p})$ is a $\mathfrak{p}$-primary submodule. Since $Q_\mathfrak{a} \cap Z(\mathfrak{p})$ is part of a minimal reduced primary decomposition of 0 in $Q_\mathfrak{a}$ it follows that $\mathfrak{p} \in \operatorname{Ass}_R Q_\mathfrak{a}$, as required. The last claim is clear and follows by the definition of $U$ and $V$. $\square$

Now we are prepared in order to establish the first main result of this section. It explains in more detail the structure of $H^d_\mathfrak{a}(M), d = \dim M$.

**Theorem 4.1.5.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let $M$ be a finitely generated $R$-module and $d = \dim M$ Then there is a natural isomorphism*

$$H^d_\mathfrak{a}(M) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/Q_{\mathfrak{a}\widehat{R}}(\widehat{M})).$$

**Proof.** First note that $H^d_\mathfrak{a}(M)$ is an Artinian $R$-module. So it admits a unique $\widehat{R}$-module structure compatible with its $R$-module structure such that the natural homomorphism

$$H^d_{\mathfrak{a}\widehat{R}}(\widehat{M}) \cong H^d_\mathfrak{a}(M) \otimes_R \widehat{R} \to H^d_\mathfrak{a}(M)$$

is an isomorphism. That is, without loss of generality we may assume that $R$ is complete.

Now apply the local cohomology functor to the short exact sequence

$$0 \to Q_\mathfrak{a}(M) \to M \to M/Q_\mathfrak{a}(M) \to 0$$

it implies a natural isomorphism $H^d_\mathfrak{a}(M) \cong H^d_\mathfrak{a}(M/Q_\mathfrak{a}(M))$. To this end recall that $H^i_\mathfrak{a}(Q_\mathfrak{a}(M)) = 0$ for all $i \geq d$. The vanishing for $i = d$ follows by the Hartshorne-Lichtenbaum Vanishing Theorem because of $\operatorname{Ass}_R Q = V$, where $Q = Q_\mathfrak{a}(M)$. By the base change of local cohomology there is the isomorphism

$$H^d_\mathfrak{a}(M/Q_\mathfrak{a}(M)) \cong H^d_{\mathfrak{a}+\operatorname{Ann}_R M/Q}(M/Q).$$

In order to complete the proof it is enough to show that $\mathfrak{m} = \operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R M/Q)$. To this end consider

$$V(\mathfrak{a} + \operatorname{Ann}_R M/Q) = V(\mathfrak{a}) \cup \operatorname{Supp}_R M/Q = \cup_{\mathfrak{p}\in U} V(\mathfrak{a} + \mathfrak{p}) = \{\mathfrak{m}\},$$

as required. $\square$

In the case of $M = R$ in Theorem (4.1.5) it follows that $H^d_\mathfrak{a}(R) = H^d_{\mathfrak{m}\widehat{R}}(\widehat{R}/Q_{\mathfrak{a}\widehat{R}}(\widehat{R}))$. By the definition $Q_{\mathfrak{a}\widehat{R}}(\widehat{R})$ is equal to the intersection of all the $\mathfrak{p}$-primary components of a reduced minimal primary decomposition of the zero ideal in $\widehat{R}$ such that $\dim \widehat{R}/\mathfrak{p} = \dim R$ and $\dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} = 0$. Next we want to extend this to the case of an $R$-module $M$.

**Definition 4.1.6.** *Let $M$ denote a finitely generated module over the local ring $(R, \mathfrak{m})$. Let $\mathfrak{a} \subset R$ denote an ideal. Then define $P_{\mathfrak{a}}(M)$ as the intersection of all the primary components of $\operatorname{Ann}_R M$ such that $\dim R/\mathfrak{p} = \dim M$ and $\dim R/\mathfrak{a} + \mathfrak{p} = 0$. Clearly $P_{\mathfrak{a}}(M)$ is the pre-image of $Q_{\mathfrak{a}R/\operatorname{Ann}_R M}(R/\operatorname{Ann}_R M)$ in $R$.*

With these preparations we are able to prove the extension we have in mind.

**Corollary 4.1.7.** *Let $M$ denote a finitely generated $R$-module and $d = \dim M$. Let $I \subset R$ be an ideal. Then*

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{M}/P_{\mathfrak{a}}(\widehat{M})\widehat{M}),$$

*where $P_{\mathfrak{a}}(\widehat{M}) \subset \widehat{R}$ is the ideal as defined in Definition (4.1.6).*

**Proof.** As in the beginning of proof of Theorem (4.1.5) we may assume that $R$ is a complete local ring without loss of generality. Let $\overline{R} = R/\operatorname{Ann}_R M$. Then by base change and the right exactness there are the isomorphisms

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{a}\overline{R}}(M) \cong H^d_{\mathfrak{a}\overline{R}}(\overline{R}) \otimes_R M.$$

Now by virtue of Theorem (4.1.5) there is the isomorphism $H^d_{\mathfrak{a}\overline{R}}(\overline{R}) \cong H^d_{\mathfrak{m}}(R/P_{\mathfrak{a}}(M))$. Therefore it follows that

$$H^d_{\mathfrak{a}\overline{R}}(\overline{R}) \otimes_R M \cong H^d_{\mathfrak{m}}(R/P_{\mathfrak{a}}(M)) \otimes_R M \cong H^d_{\mathfrak{m}}(M/P_{\mathfrak{a}}(M)M),$$

which finishes the proof of the statement. □

For an Artinian $R$-module $A$, the decreasing sequence of submodules $\{\mathfrak{a}^n A\}_{n\in\mathbb{N}}$ becomes stable. Let $\langle\mathfrak{a}\rangle A$ denote the ultimative stable value of this sequence of decreasing submodules. For each Artinian $R$-module there is the theory of secondary representations; see section 2.4. In particular, for an ideal $\mathfrak{a}$ of $R$ it follows that $\langle\mathfrak{a}\rangle A = \mathfrak{a}^m A$, $m$ enough large, coincides with the sum of all $\mathfrak{p}_i$-secondary components $A_i$ of a minimal secondary representation $A = \sum_{i=1}^n A_i$ of $A$ such that $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ (where $\mathfrak{p}_i = \operatorname{Rad}(0 :_R A_i)$, $1 \leq i \leq n$). Pursuing this point of view it is shown in [Divan-Sch, Theorem 1.1] that

$$H^d_{\mathfrak{a}}(M) \cong H^d_{\mathfrak{m}}(M)/\textstyle\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_{H^d_{\mathfrak{m}}(M)} \mathfrak{a}^n).$$

**Remark 4.1.8.** *(1) Let $\mathfrak{a} \subset R$ denote an ideal. For a finitely generated R-module M there is a natural epimorphism*

$$H_{\mathfrak{m}}^d(M) \to H_{\mathfrak{a}}^d(M) \to 0, \; d = \dim M,$$

*(see [Divan-Sch]). By above explanations the kernel is described as $\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_{H_{\mathfrak{m}}^d(M)} \mathfrak{a}^n)$.*

*Let us consider the previous epimorphism as an epimorphism of $\widehat{R}$-modules. Then by Corollary (4.1.7) its kernel is equal to $P_{\mathfrak{a}}(\widehat{M})H_{\mathfrak{m}\widehat{R}}^d(\widehat{M})$, or in other words*

$$H_{\mathfrak{a}}^d(M) \cong H_{\mathfrak{m}\widehat{R}}^d(\widehat{M})/P_{\mathfrak{a}}(\widehat{M})H_{\mathfrak{m}\widehat{R}}^d(\widehat{M}).$$

*This follows easily since $H_{\mathfrak{m}\widehat{R}}^d(\widehat{M}/P_{\mathfrak{a}}(\widehat{M})\widehat{M}) \cong H_{\mathfrak{m}\widehat{R}}^d(\widehat{M}) \otimes_{\widehat{R}} \widehat{R}/P_{\mathfrak{a}}(\widehat{M})$.*

*(2) With the previous notations the following conditions are equivalent:*

*(a)* $H_{\mathfrak{a}}^d(M) = 0$.

*(b)* $H_{\mathfrak{m}\widehat{R}}^d(\widehat{M}) = P_{\mathfrak{a}}(\widehat{M})H_{\mathfrak{m}\widehat{R}}^d(\widehat{M})$.

*(c)* $H_{\mathfrak{m}}^d(M) = \sum_{n\in\mathbb{N}} < \mathfrak{m} > (0 :_{H_{\mathfrak{m}}^d(M)} \mathfrak{a}^n)$.

## 4.2 Endomorphism rings of $H_{\mathfrak{a}}^{\dim R}(R)$

In this section we consider the endomorphism rings of certain local cohomology modules $H_{\mathfrak{a}}^i(R)$. In the case of $i = \dim R$ and $\mathfrak{a} = \mathfrak{m}$ Hochster and Huneke examined the endomorphism rings of local cohomology modules(see [Hoch-Hun]) and in the case of $i = \operatorname{ht}\mathfrak{a}$ and $R$ a Gorenstein ring were studied by Schenzel (see [Sch5] and the references there). Here we continue with the case of $i = \dim R$ and an arbitrary ideal $\mathfrak{a} \subset R$.

Let $(R, \mathfrak{m})$ denote a $d$-dimensional local ring. For an ideal $\mathfrak{a} \subset R$ we investigate the endomorphism ring of $H_{\mathfrak{a}}^d(R)$. In particular, we study the natural homomorphism

$$R \to \operatorname{Hom}_R(H_{\mathfrak{a}}^d(R), H_{\mathfrak{a}}^d(R)), \quad r \mapsto m_r,$$

where $m_r$ denotes the multiplication map by $r \in R$. Since $H_{\mathfrak{a}}^d(R)$ admits the structure of an $\widehat{R}$-module (see 2.1.8) it follows that $\operatorname{Hom}_R(H_{\mathfrak{a}}^d(R), H_{\mathfrak{a}}^d(R))$ has a unique natural $\widehat{R}$-module such that the diagram

$$\begin{array}{ccc} R & \rightarrow & \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R)) \\ \downarrow & & \| \\ \widehat{R} & \rightarrow & \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R)). \end{array}$$

is commutative. That is, the map $R \rightarrow \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ factors through $\widehat{R}$. Before we study the endomorphism ring we need an auxiliary statement on the Matlis dual of $H^d_{\mathfrak{a}}(R)$.

**Lemma 4.2.1.** *Let $\mathfrak{a}$ denote an ideal in a local ring $(R, \mathfrak{m})$.*

*(1)* $T_{\mathfrak{a}}(R) = \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), E_R(R/\mathfrak{m}))$ *is a finitely generated $\widehat{R}$-module.*

*(2)* $\operatorname{Ass}_{\widehat{R}} T_{\mathfrak{a}}(R) = \{\mathfrak{p} \in \operatorname{Ass} \widehat{R} | \dim \widehat{R}/\mathfrak{p} = \dim R$ *and* $\dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} = 0\}$.

*(3)* $K_{\widehat{R}}(\widehat{R}/Q_{\mathfrak{a}}(\widehat{R})) \cong T_{\mathfrak{a}}(R)$. *In particular, It is satisfies $S_2$ situation. Furthermore when $\widehat{R}/Q_{\mathfrak{a}}(\widehat{R})$ is Cohen-Macaulay then so is $T_{\mathfrak{a}}(R)$.*

*(4)* $\operatorname{Ann}_{\widehat{R}}(H^d_{\mathfrak{a}}(R)) = Q_{\mathfrak{a}}(\widehat{R})_d$.

**Proof.**

(1) As $H^d_{\mathfrak{a}}(R)$ is an Artinian module, so by local duality $T_{\mathfrak{a}}(R)$ is a finitely generated $\widehat{R}$-module.

(2) By virtue of (1), $T_{\mathfrak{a}}(R)$ is a finitely generated $\widehat{R}$-module so by Proposition 2.4.3, $\operatorname{Ass}_{\widehat{R}} T_{\mathfrak{a}}(R) = \operatorname{Att}_{\widehat{R}} H^d_{\mathfrak{a}}(R)$.

By virtue of Corollary 4.1.7, $H^d_{\mathfrak{a}}(R) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{R}) \otimes_{\widehat{R}} \widehat{R}/P_{\mathfrak{a}}(\widehat{R})$. It follows from 2.4.2 that

$$\operatorname{Att}_{\widehat{R}} H^d_{\mathfrak{a}}(R) = \operatorname{Att}_{\widehat{R}} H^d_{\mathfrak{m}\widehat{R}}(\widehat{R}) \cap \operatorname{Supp}(\widehat{R}/P_{\mathfrak{a}}(\widehat{R})),$$

which is equal to $\{\mathfrak{p} \in \operatorname{Ass} \widehat{R} | \dim \widehat{R}/\mathfrak{p} = \dim R$ and $\dim \widehat{R}/\mathfrak{a}\widehat{R} + \mathfrak{p} = 0\}$. To this end note that $\operatorname{Att}_{\widehat{R}} H^d_{\mathfrak{m}\widehat{R}}(\widehat{R}) = \{\mathfrak{p} \in \operatorname{Ass} \widehat{R} | \dim \widehat{R}/\mathfrak{p} = \dim R\}$, cf. [Br-Sh, Theorem 7.3.2].

(3) By Hom $-\otimes$ adjointness

$$\begin{aligned} T_{\mathfrak{a}}(R) &= \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), E_R(R/\mathfrak{m})) \\ &= \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}\widehat{R}}(\widehat{R}), E_R(R/\mathfrak{m})). \end{aligned}$$

By Theorem 2.2.1 and Matlis duality it is isomorph to the following

$$\operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(K_{\widehat{R}}(\widehat{R}/Q_{\mathfrak{a}}(\widehat{R})), E_R(R/\mathfrak{m})), E_R(R/\mathfrak{m})) \cong K_{\widehat{R}}(\widehat{R}/Q_{\mathfrak{a}}(\widehat{R})).$$

Obviously by 2.2.5(1) it satisfies the $S_2$-situation. Now the last claim follows by 2.2.4.

(4) For every module $R$-module $N$, $\operatorname{Ann} N = \operatorname{Ann} D(N)$. Then it follows that $\operatorname{Ann}_{\widehat{R}}(H^d_{\mathfrak{a}}(R)) = \operatorname{Ann}_{\widehat{R}}(K_{\widehat{R}}(\widehat{R}/Q_{\mathfrak{a}}(\widehat{R}))$. By virtue of 2.2.5(2) the last one is equal to $Q_{\mathfrak{a}}(\widehat{R})_d$.

□

For an $R$-module $M$ the natural map $R \to \operatorname{Hom}_R(M, M)$ is in general neither injective nor surjective.

**Theorem 4.2.2.** *Let $\mathfrak{a}$ denote an ideal in a local ring $(R, \mathfrak{m})$. Let*

$$\Phi : \widehat{R} \to \operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$$

*the natural homomorphism. Then*

*(1)* $\ker \Phi = Q_{\mathfrak{a}\widehat{R}}(\widehat{R})_d$.

*(2)* $\Phi$ *is surjective if and only if* $\widehat{R}/Q_{\mathfrak{a}\widehat{R}}(\widehat{R})$ *satisfies* $S_2$.

*(3)* $\operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ *is a finitely generated* $\widehat{R}$*-module.*

*(4)* $\operatorname{Hom}_{\widehat{R}}(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ *is a commutative semi-local Noetherian ring.*

**Proof.** First note that as $H^d_{\mathfrak{a}}(R)$ is an Artinian $R$-module so $H^d_{\mathfrak{a}}(R) \cong H^d_{\mathfrak{a}\widehat{R}}(\widehat{R})$ (see explanations after 2.1.8). That is, without loss of generality we may assume that $R$ is a complete local ring. By virtue of Theorem (4.1.5) there is the natural isomorphism $H^d_{\mathfrak{a}}(R) \cong H^d_{\mathfrak{m}}(R/Q), Q = Q_{\mathfrak{a}}(R)$. Then

$$K_{R/Q} \cong D(H^d_{\mathfrak{m}}(R/Q)) \cong \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), E_R(R/\mathfrak{m})).$$

Because $H^d_{\mathfrak{a}}(R)$ is Artinian the Matlis' duality provides an isomorphism

$$\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R)) \cong \operatorname{Hom}_R(K_{R/Q}, K_{R/Q}).$$

Therefore the kernel of $\Phi$ equals to $\operatorname{Ann}_R K_{R/Q} = Q_d$ (cf. 2.2.5(2))which proves (1). Because the endomorphism ring of $H^d_{\mathfrak{a}}(R)$ is isomorphic to the endomorphism ring of the canonical module of $K_{R/Q}$ the results in (2), (3) and (4) are shown by 2.2.6. □

In the next we want to relate some homological properties of $T_{\mathfrak{a}}(R)$ with those of the endomorphism ring $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$ resp. $\widehat{R}/Q_{\mathfrak{a}\widehat{R}}(\widehat{R})$.

**Theorem 4.2.3.** *Let $\mathfrak{a}$ be an ideal of a complete local ring $(R, \mathfrak{m})$. For an integer $r \geq 2$ we have the following statements:*

*(1) Suppose $R/Q_{\mathfrak{a}}(R)$ has $S_2$. Then $T_{\mathfrak{a}}(R)$ satisfies the condition $S_r$ if and only if $H^i_{\mathfrak{m}}(R/Q_{\mathfrak{a}}(R)) = 0$ for $d - r + 2 \leq i < d$.*

*(2) $R/Q_{\mathfrak{a}}(R)$ satisfies the condition $S_r$ if and only if $H^i_{\mathfrak{m}}(T_{\mathfrak{a}}(R)) = 0$ for $d - r + 2 \leq i < d$ and $R/Q_{\mathfrak{a}}(R) \cong \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$.*

*In particular, if $R/Q_{\mathfrak{a}}(R)$ has $S_2$ it is a Cohen-Macaulay ring if and only if the module $T_{\mathfrak{a}}(R)$ is Cohen-Macaulay.*

**Proof.** By our conventions and definitions it follows that $T_{\mathfrak{a}}(R) \cong K_{R/Q}$, where $Q = Q_{\mathfrak{a}}(R)$, and $R/Q \cong \operatorname{Hom}_R(H^d_{\mathfrak{a}}(R), H^d_{\mathfrak{a}}(R))$. Then the statement in (1) resp. in (2) follows by virtue of 2.2.3, 2.2.2 and 4.2.2 for $M = R/Q$ resp. $M = K_{R/Q}$. □

# Chapter 5

# Connectedness

Let $R$ be a commutative ring. The spectrum of $R$ denoted by $\operatorname{Spec}(R)$, is the topological space consisting of all prime ideals of $R$ with topology defined by the closed sets $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq \mathfrak{a}\}$, for each ideal $\mathfrak{a}$ of $R$. This topology is called the Zariski topology. Clearly if $R$ is nonzero, then $\operatorname{Spec} R$ is non-empty. $\operatorname{Spec} R$ enjoys very nice properties. For instance it is compact and moreover it is irreducible if and only if its nilradical is a prime ideal (a topological space $X$ is irreducible if it cannot be written as a union of two closed proper subsets $A, B$ of $X$). However it is not a connected space in general. Recall that a topological space is connected if it cannot be written as a disjoint union of two proper closed subsets. It is known that for a local ring $R$, $\operatorname{Spec} R$ is connected. More generally $\operatorname{Spec} R$ is disconnected if and only if $R$ contains a non-trivial idempotents element. Following Remark gives an algebraic interpret of connectedness which is easily seen by definition.

**Remark 5.0.4.** *Let $I, J$ be ideals of a ring $R$. The topological space $\operatorname{Spec}(R/I) \setminus V(J)$ is disconnect whenever there are ideals $\mathfrak{a}$ and $\mathfrak{b}$ of $R$ satisfying the following conditions*

*(1) neither $\mathfrak{a}$ nor $\mathfrak{b}$ is $J$-primary,*

*(2) $\operatorname{Rad}(I) = \operatorname{Rad}(\mathfrak{a} \cap \mathfrak{b})$ and*

*(3) $\operatorname{Rad}(J) = \operatorname{Rad}(\mathfrak{a} + \mathfrak{b})$.*

The concept of a topological space being connected in codimension $k$ ($\in \mathbb{N} \cup \{0\}$) was made precise by Hartshorne [Hart2]. To be more precise we need a few more preparations. First we cite some definitions and facts related to connectedness from [Hart2].

**Definition 5.0.5.** *Let $X$ be a Noetherian topological space and $Y$ be an irreducible closed subspace of $X$. Then we define the codimension of $Y$ in $X$ to be the supremum of those integers $n$ such that there exists a sequence of closed irreducible subspaces $X_i$ of $X$,*

$$Y \subset X_0 \subset X_1 \subset ... \subset X_n \subset X.$$

*And we denote it by $codim(Y, X)$.*

Now, we can define connectedness in codimension $k$:

**Definition 5.0.6.** *Let $X$ be a Noetherian topological space, and $k \geq 0$ be an integer. If $X$ satisfies any of the following equivalent conditions*

*(1) If $Y$ is a closed subset of $X$, and $codim(Y, X) > k$, then $X \setminus Y$ is connected.*

*(2) Let $X^{'}$ and $X^{''}$ be irreducible components of $X$. Then we can find a finite sequence*

$$X^{'} = X_1, X_2, ..., X_n = X^{''}$$

*which is composed of irreducible components of $X$, such that for each $i = 1, 2, ..., n-1$, $X_i \cap X_{i+1}$ is of codimension $\leq k$ in $X$.*

*we say that $X$ is connected in codimension $k$.*

It is known that being connected in codimension $k$, for any $k$, implies being connected (cf. [Hart2]). Now we may deduce the algebraic interpret of the above definition as follows

**Definition 5.0.7.** $\operatorname{Spec}(R)$ *is connected in codimension one if* $\operatorname{Spec}(R) \setminus V(\mathfrak{a})$ *is connected, for every ideal $\mathfrak{a}$ of $R$ with* $\operatorname{ht}(\mathfrak{a}) \geq 2$ .

**Remark 5.0.8.** *A ring $R$ is connected in codimension one if and only if whenever $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $R$ such that* $\operatorname{Rad}(\mathfrak{a}) \neq 0$, $\operatorname{Rad}(\mathfrak{b}) \neq 0$ *and* $\operatorname{Rad}(\mathfrak{a} \cap \mathfrak{b}) = 0$, *then* $\operatorname{ht}(\mathfrak{a} + \mathfrak{b}) \leq 1$.

**Proof.** (cf. [Hun3]) Note that given $\mathfrak{a}$ and $\mathfrak{b}$ as above,

$$X := \operatorname{Spec}(R) \setminus V(\mathfrak{a} + \mathfrak{b}) = (V(\mathfrak{a}) \cap X) \cup (V(\mathfrak{b}) \cap X)$$

is disconnected. Thus if $R$ is connected in codimension 1, $\operatorname{ht}(\mathfrak{a}+\mathfrak{b}) \leq 1$.

On the other hand, if $R$ is not connected in codimension 1, there is an ideal $K \subseteq R$, $\operatorname{ht}(K) \geq 2$, such that $X := \operatorname{Spec}(R) \setminus V(K)$ is disconnected. Write $X = (V(\mathfrak{a}) \cap X) \cup (V(\mathfrak{b}) \cap X)$. Then $\mathfrak{a}$ and $\mathfrak{b}$ satisfy the above conditions. We now show that $\operatorname{Rad}(\mathfrak{a}+\mathfrak{b}) \supseteq K$ and hence $\operatorname{ht}(\mathfrak{a}+\mathfrak{b}) \geq \operatorname{ht}(K) \geq 2$.

Consider $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{a}+\mathfrak{b} \subseteq \mathfrak{p}$. Then $\mathfrak{p} \in V(\mathfrak{a})$ and $\mathfrak{p} \in V(\mathfrak{b})$. Thus $\mathfrak{p} \notin X$, i.e. $\mathfrak{p} \in V(K)$. Thus $K \subseteq \mathfrak{p}$ which proves $K \subseteq \operatorname{Rad}(\mathfrak{a}+\mathfrak{b})$. $\square$

Next we recall a definition given by Hochster and Huneke (see [Hoch-Hun, (3.4)]).

**Definition 5.0.9.** *Let $(R, \mathfrak{m})$ denote a local ring. We denote by $\mathbb{G}(R)$ the undirected graph whose vertices are primes $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim R = \dim R/\mathfrak{p}$, and two distinct vertices $\mathfrak{p}, \mathfrak{q}$ are joined by an edge if and only if $(\mathfrak{p}, \mathfrak{q})$ is an ideal of height one.*

Next examples make the above definition more clear (cf. [Hun3]):

**Example 5.0.10.** *(1) Let*

$$R := k[X, Y, U, V]/((X, Y) \cap (U, V)) = k[x, y, u, v].$$

*Then $R$ has two minimal primes, $\mathfrak{p} := (x, y)$ and $\mathfrak{q} := (u, v)$. Since $\operatorname{ht}(\mathfrak{p}+\mathfrak{q}) = 2$, then the graph $\mathbb{G}(R)$ consists of two vertices $\mathfrak{p}$ and $\mathfrak{q}$ that are not connected to each other.*

*(2) Let*

$$R := k[X, Y, U, V]/((X, Y) \cap (Y, U) \cap (U, V)) = k[x, y, u, v].$$

*There are three minimal primes, $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (y, u)$ and $\mathfrak{p}_3 = (u, v)$. In this case $\mathbb{G}(R)$ consists of three vertices $\mathfrak{p}_1$, $\mathfrak{p}_2$ , $\mathfrak{p}_3$ and two edges between $\mathfrak{p}_1$ , $\mathfrak{p}_2$ and $\mathfrak{p}_2$ , $\mathfrak{p}_3$.*

Connection between the above materials is appeared in Proposition 5.2.4. Our goal in this chapter is to prove some connectedness theorems for $\operatorname{Spec} R$ via endomorphism rings of top local cohomology modules.

## 5.1 Mayer-Vietoris sequence

The Mayer-Vietoris sequence has applications to connectedness properties of algebraic varieties.

**Theorem 5.1.1.** *(cf. [Br-Sh, 3.2.3]) Let $R$ be a Noetherian ring, $\mathfrak{a}, \mathfrak{b}$ be ideals of $R$ and $M$ an $R$-module. Then there exists a natural long exact sequence*

$$0 \to H^0_{\mathfrak{a}+\mathfrak{b}}(M) \to H^0_{\mathfrak{a}}(M) \oplus H^0_{\mathfrak{b}}(M) \to H^0_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \ldots \to H^i_{\mathfrak{a}+\mathfrak{b}}(M)$$
$$\to H^i_{\mathfrak{a}}(M) \oplus H^i_{\mathfrak{b}}(M) \to H^i_{\mathfrak{a}\cap\mathfrak{b}}(M) \to \ldots.$$

**Proof.** For all $n \in \mathbb{N}$ there exists a short exact sequence

$$0 \to R/(\mathfrak{a}^n \cap \mathfrak{b}^n) \to R/\mathfrak{a}^n \oplus R/\mathfrak{b}^n \to R/\mathfrak{a}^n + \mathfrak{b}^n \to 0.$$

It yields a long exact sequence of local cohomology modules by applying $\mathrm{Hom}_R(-, M)$

$$\ldots \to \mathrm{Ext}^i_R(R/\mathfrak{a}^n + \mathfrak{b}^n, M) \to \mathrm{Ext}^i_R(R/\mathfrak{a}^n \oplus R/\mathfrak{b}^n, M) \to \mathrm{Ext}^i_R(R/(\mathfrak{a}^n \cap \mathfrak{b}^n), M) \to \ldots.$$

This forms a directed system of long exact sequences. Then take direct limits to get the desired long exact sequence. To this end note that $\{\mathfrak{a}^n + \mathfrak{b}^n\}$ is cofinal with $\{(\mathfrak{a}+\mathfrak{b})^n\}$, because $\mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a}+\mathfrak{b})^n$ and $(\mathfrak{a}+\mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n$ and $\{\mathfrak{a}^n \cap \mathfrak{b}^n\}$ is cofinal with $\{(\mathfrak{a}\cap\mathfrak{b})^n\}$, because $(\mathfrak{a}\cap\mathfrak{b})^n \subseteq \mathfrak{a}^n \cap \mathfrak{b}^n$ and by the Artin-Rees Lemma, there exists $k = k(n)$ such that for all $m \geq k$

$$\mathfrak{a}^m \cap \mathfrak{b}^n = \mathfrak{a}^{m-k}(\mathfrak{a}^k \cap \mathfrak{b}^n) \subseteq \mathfrak{a}^{m-k}\mathfrak{b}^n.$$

Therefore, for $m \geq n + k$ we have

$$\mathfrak{a}^m \cap \mathfrak{b}^m \subseteq \mathfrak{a}^m \cap \mathfrak{b}^n \subseteq \mathfrak{a}^{m-k}\mathfrak{b}^n \subseteq \mathfrak{a}^n\mathfrak{b}^n \subseteq (\mathfrak{a}\cap\mathfrak{b})^n.$$

□

## 5.2 Connectedness Theorems

There are several many papers to show that local cohomology yields connectedness results, for instance see [Falt], [Falt2] , [Divan-Sch], [Hoch-Hun] and [Rung].

One of the well-known results about connectedness is the Faltings' connectedness Theorem:

**Theorem 5.2.1.** *Let* $(R, \mathfrak{m})$ *be a complete local domain. If* $\mathfrak{a}$ *is an ideal of* $R$ *with* $\operatorname{ara} \mathfrak{a} \leq \dim R - 2$*, then* $\operatorname{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}/\mathfrak{a}\}$*, the punctured spectrum of* $R/\mathfrak{a}$*, is connected.*

For an ideal $\mathfrak{a}$ of a Noetherian ring $R$

$$\operatorname{ara}(\mathfrak{a}) = \inf\{\mu(\mathfrak{b}) : \operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\mathfrak{b}), \mathfrak{b} \text{ is an ideal }\},$$

where $\mu(\mathfrak{b})$ is the minimal number of generators of the ideal $\mathfrak{b}$.

Hochster and Huneke have obtained generalizations of Faltings' connectedness Theorem. One such is [Hoch-Hun, Theorem 3.3]:

**Theorem 5.2.2.** *Let* $(R, \mathfrak{m})$ *be a complete equidimensional ring of dimension* $d$ *such that* $H^d_{\mathfrak{m}}(R)$ *is indecomposable as an* $R$*-module; equivalently, the canonical module* $K_R$ *is indecomposable.*

*If* $\mathfrak{a}$ *is an ideal of* $R$ *with* $\operatorname{ara} \mathfrak{a} \leq d - 2$*, then* $\operatorname{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}/\mathfrak{a}\}$ *is connected.*

This section is devoted to characterize the number of the maximal ideals of the endomorphism ring $\operatorname{Hom}_{\widehat{R}}(H^d_I(R), H^d_I(R)), d = \dim R$. In fact we give some equivalent statements to connectedness.

**Theorem 5.2.3.** *(cf. [Hoch-Hun, Theorem (3.6)]) Let* $(R, \mathfrak{m})$ *be a complete local equidimensional ring and* $d = \dim R$*. Then the following conditions are equivalent:*

*(1)* $H^d_{\mathfrak{m}}(R)$ *is indecomposable.*

*(2)* $K_R$*, the canonical module of* $R$ *is indecomposable.*

*(3) The ring* $\operatorname{Hom}_R(K_R, K_R)$ *is local.*

*(4) For every ideal* $J$ *of height at least two,* $\operatorname{Spec}(R) \setminus V(J)$ *is connected.*

*(5) The graph* $\mathbb{G}(R)$ *is connected.*

**Sketch of the proof:** We shall prove that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (3)$. The equivalence of (1) and (2) is clear.

$(2) \Leftrightarrow (3)$ If the $S := \operatorname{Hom}_R(K_R, K_R)$ is not local, then $K_R$ is a product of nonzero factors corresponding to the various factors rings of $S$, and this will yield a non-trivial direct sum decomposition of $K_R$ over $R$. On the other hand, if $S$ is local, it contains no idempotents other than $0, 1$, and this implies that $K_R$ is indecomposable.

(3) $\Rightarrow$ (4) To prove use Remark 5.0.4, then one can replace $\mathfrak{a}$ , $\mathfrak{b}$ by their powers and assume that $\mathfrak{a}\mathfrak{b} = 0$ but $\mathfrak{a} + \mathfrak{b}$ has height at least two. Then find a contradiction.

(4) $\Rightarrow$ (5) Suppose that one has ideals $\mathfrak{a}$ , $\mathfrak{b}$ such that $\mathfrak{a} \cap \mathfrak{b}$ is nilpotent. Then we can replace $\mathfrak{a}$ , $\mathfrak{b}$ by their radicals while only increasing $\mathfrak{a} + \mathfrak{b}$. Then each of $\mathfrak{a}$ , $\mathfrak{b}$ is a finite intersection of primes. For each minimal prime $\mathfrak{p}$ of $R$, $\mathfrak{p} \supseteq \mathfrak{a} \cap \mathfrak{b}$, and so $\mathfrak{p}$ must contain either a minimal prime of $\mathfrak{a}$ or a minimal prime of $\mathfrak{b}$. Thus, $\mathfrak{p}$ must be either a minimal prime of $\mathfrak{a}$ or a minimal prime of $\mathfrak{b}$. If we omit all non-minimal primes from the primary decomposition of $\mathfrak{a}$ (respectively, $\mathfrak{b}$) and intersect the others, we get two larger ideals whose intersection is still $\mathrm{Rad}(0)$. Thus, it is possible to give $\mathfrak{a}$ , $\mathfrak{b}$ such that $\mathrm{Rad}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{Rad}(0)$ and $\mathfrak{a} + \mathfrak{b}$ has height two if and only if one can do this with ideals $\mathfrak{a}$ , $\mathfrak{b}$ coming from a partition of the minimal primes of $R$ into two nonempty sets, with $\mathfrak{a}$ the intersection of the minimal primes in one set and $\mathfrak{b}$ the intersection of the minimal primes in the other set. If one set consists of $\{\mathfrak{p}_1, ..., \mathfrak{p}_h\}$ and the other of $\{\mathfrak{q}_1, ..., \mathfrak{q}_k\}$ we shall have $\mathfrak{a} = \cap_i \mathfrak{p}_i$, $\mathfrak{b} = \cap_j \mathfrak{q}_j$ , and $\mathfrak{a} + \mathfrak{b}$ will then have the same radical as $\cap_{i,j}(\mathfrak{p}_i + \mathfrak{q}_j)$, and will have height at least two if and only if every $\mathfrak{p}_i + \mathfrak{q}_j$ has height at least two. Thus, (4) fails if and only if the minimal primes can be partitioned into two nonempty sets such that no edge of $\mathbb{G}(R)$ joins a vertex in one set to a vertex in the other, which is precisely the condition for $\mathbb{G}(R)$ to be disconnected.

(5) $\Rightarrow$ (3) If S has two or more maximal ideals, say $\mathfrak{M}_1, ..., \mathfrak{M}_r$, where $r \geq 2$, for each $\mathfrak{M}_j$ let $\mathfrak{P}_j$ denote the set of minimal primes of $S$ contained in $\mathfrak{M}_j$ . Then $\mathfrak{P}_j$ is evidently non-empty. There is a bijection between the minimal primes of $S$ and those of $R$, so that for each $\mathfrak{P}_j$ there is a corresponding set of minimal primes $\mathfrak{Q}_j$ of $R$. To complete the argument, it will suffice to show that if $i, j$ are different then it is impossible to have an edge joining a vertex in $\mathfrak{Q}_i$ to a vertex in $\mathfrak{Q}_j$ . If there were such an edge, there would be a height one prime $\mathfrak{p}$ of $R$ containing both a minimal prime in $\mathfrak{Q}_i$ and a minimal prime in $\mathfrak{Q}_j$ . Then $R_\mathfrak{p} \cong S_\mathfrak{p}$ , and it follows that the unique prime of $S$ lying over $\mathfrak{p}$ contains both a prime of $\mathfrak{P}_i$ and a prime of $\mathfrak{P}_j$ . Let $\mathfrak{M}$ be a maximal ideal of $R$ containing $\mathfrak{p}$. Then $\mathfrak{M}$ contains both a prime of $\mathfrak{P}_i$ and a prime of $\mathfrak{P}_j$ , which is impossible: $S$ is a finite product of local rings, and each prime ideal of $S$ is therefore contained in a unique maximal ideal of $S$, forcing $\mathfrak{M}_i = \mathfrak{M} = \mathfrak{M}_j$. □

Next we are interested in the connectedness of $\mathbb{G}(R)$. That is characterized in the following statement.

**Proposition 5.2.4.** *Let $(R, \mathfrak{m})$ denote a local ring with $d = dimR$. Then the following conditions are equivalent:*

*(1) The graph $\mathbb{G}(R)$ is connected.*

*(2)* Spec $R/0_d$ *is connected in codimension one.*

*(3) For every ideal $JR/0_d$ of height at least two,* $\operatorname{Spec}(R/0_d) \setminus V(JR/0_d)$ *is connected.*

**Proof.** (1) and (3) are equivalent by Theorem 5.2.3 and by virtue of Definition 5.0.7, (2) and (3) are equivalent. □

Next we describe when the endomorphism ring of $H^d_{\mathfrak{a}}(R), d = \dim R$, is a local ring. In other words we generalized the results in 5.2.3.

**Theorem 5.2.5.** *Let $(R, \mathfrak{m})$ denote a complete local ring and $d = \dim R$. For an ideal $\mathfrak{a} \subset R$ the following conditions are equivalent:*

*(1) $H^d_{\mathfrak{a}}(R)$ is indecomposable.*

*(2) $T_{\mathfrak{a}}(R)$ is indecomposable.*

*(3) The endomorphism ring of $H^d_{\mathfrak{a}}(R)$ is a local ring.*

*(4) The graph $\mathbb{G}(R/Q_{\mathfrak{a}}(R))$ is connected.*

**Proof.** We may always assume that $Q = Q_{\mathfrak{a}}(R)$ is a proper ideal. In the case of $Q = R$ there is nothing to show. As it follows by the results in chapter four, we have the following isomorphisms

$$H^d_{\mathfrak{m}}(R/Q) \cong H^d_{\mathfrak{a}}(R), K_{R/Q} \cong T_{\mathfrak{a}}(R) \text{ and } \operatorname{End} H^d_{\mathfrak{m}}(R/Q) \cong \operatorname{End} H^d_{\mathfrak{a}}(R),$$

where End denotes the endomorphism ring. That is, we have reduced the proof of the statement to the corresponding result for $H^d_{\mathfrak{m}}(R/Q)$. Note that $d = \dim R/Q$. Then the equivalence of the conditions follows by 5.2.3. □

Now we shall describe $t$, the number of connected components of $\mathbb{G}(R/Q_{\mathfrak{a}}(R))$. A connected component of an undirected graph is a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices.

**Definition 5.2.6.** *Let $\mathfrak{a}$ be an ideal in a local ring $(R, \mathfrak{m})$. Suppose that $Q = Q_{\mathfrak{a}}(R)$ is a proper ideal. Let $\mathbb{G}_i, i = 1, \ldots, t$, denote the connected components of $\mathbb{G}(R/Q)$. Let $Q_i, i = 1, \ldots, t$, denote the intersection of all $\mathfrak{p}$-primary components of a reduced minimal primary decomposition of $Q$ such that $\mathfrak{p} \in \mathbb{G}_i$. Then $Q = \cap_{i=1}^t Q_i$ and $\mathbb{G}(R/Q_i) = \mathbb{G}_i, i = 1, \ldots, t$, is connected. Moreover, let $\mathfrak{a}_i, i = 1, \ldots, t$, denote the image of the ideal $\mathfrak{a}$ in $R/Q_i$.*

**Theorem 5.2.7.** *Let $\mathfrak{a}$ denote an ideal of a complete local ring $(R, \mathfrak{m})$ with $d = \dim R \geq 2$. Then*

$$\operatorname{End} H^d_{\mathfrak{a}}(R) \simeq \operatorname{End} H^d_{\mathfrak{a}_1}(R/Q_1) \times \ldots \times \operatorname{End} H^d_{\mathfrak{a}_t}(R/Q_t)$$

*is a semi-local ring, $\operatorname{End} H^d_{\mathfrak{a}_i}(R/Q_i), i = 1, \ldots, t$, is a local ring and therefore $t$ is equal to the number of maximal ideals of $\operatorname{End} H^d_{\mathfrak{a}}(R)$.*

*Proof.* As in the proof in Theorem 4.2.2 we have $\operatorname{End} H^d_{\mathfrak{m}}(R/Q) \simeq \operatorname{End} H^d_{\mathfrak{a}}(R)$. For an integer $1 \leq i \leq t$ we define $\tilde{Q}_i = \cap_{j=1}^i Q_j$, in particular $\tilde{Q}_t = Q$. Then there is the short exact sequence

$$0 \to R/\tilde{Q}_{i+1} \to R/\tilde{Q}_i \oplus R/Q_{i+1} \to R/(\tilde{Q}_i + Q_{i+1}) \to 0.$$

Because $\mathbb{G}_{i+1}$ and $\mathbb{G}_j$ for $j = 1, \ldots, i$, are not connected it follows by the definition that $\operatorname{ht}(\tilde{Q}_i + Q_{i+1}) \geq 2$ and therefore $\dim R/(\tilde{Q}_i + Q_{i+1}) \leq d - 2$. Whence the short exact sequence induces isomorphisms $H^d_{\mathfrak{a}}(R/\tilde{Q}_{i+1}) \simeq H^d_{\mathfrak{a}}(R/\tilde{Q}_i) \oplus H^d_{\mathfrak{a}}(R/Q_{i+1})$ and by induction

$$H^d_{\mathfrak{a}}(R/Q) \simeq \oplus_{i=1}^t H^d_{\mathfrak{a}}(R/Q_i).$$

Furthermore, because of Theorem 4.1.5 and Corollary 4.1.7 we have

$$H^d_{\mathfrak{a}}(R) \simeq H^d_{\mathfrak{m}}(R/Q) \text{ and } H^d_{\mathfrak{a}_i}(R/Q_i) \simeq H^d_{\mathfrak{a}}(R/Q_i) \simeq H^d_{\mathfrak{m}}(R/Q_i), i = 1, \ldots, t.$$

Now by Matlis duality it turns out that

$$\operatorname{End} H^d_{\mathfrak{a}}(R) \simeq \operatorname{End} K_{R/Q} \text{ and } \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R/Q_j), H^d_{\mathfrak{m}}(R/Q_i)) \simeq \operatorname{Hom}_R(K_{R/Q_i}, K_{R/Q_j})$$

for all $i, j = 1, \ldots, t$. Moreover we see that $\operatorname{Hom}_R(K_{R/Q_i}, K_{R/Q_j}) = 0$ for $i \neq j$ because

$$\operatorname{Ass}_R \operatorname{Hom}_R(K_{R/Q_i}, K_{R/Q_j}) = \operatorname{Ass}_R K_{R/Q_j} \cap \operatorname{Supp}_R R/Q_i = \emptyset$$

for all $i \neq j$ as follows by the definitions, Proposition 2.4.1 and Lemma 2.2.5(3). This implies the decomposition

$$\operatorname{End} H^d_{\mathfrak{a}}(R) \simeq \operatorname{End} H^d_{\mathfrak{a}_1}(R/Q_1) \times \ldots \times \operatorname{End} H^d_{\mathfrak{a}_t}(R/Q_t)$$

because $\operatorname{End} K_{R/Q_i} \simeq \operatorname{End} H^d_{\mathfrak{a}_i}(R/Q_i), i = 1, \ldots, t$, as follows again by Matlis duality. By Theorem 5.2.5 the endomorphism ring of $H^d_{\mathfrak{a}_i}(R/Q_i), i = 1, \ldots, t$, is a local ring. So we get the decomposition as a direct product of rings and $\operatorname{End} H^d_{\mathfrak{a}}(R)$ is a semi-local ring with $t$ as its number of maximal ideals. $\square$

# Chapter 6

# Attached primes and Sharp's asymptotic Theorem

In this chapter we study some results on attached primes of modules via colocalization (cf. 2.4). The colocalization functor preserves secondary representations and attached primes (Theorem 6.0.8 below). As an application of this one may investigate the attached primes of colocalization of local cohomology modules.

**Theorem 6.0.8.** *(cf. [Rich, Theorem 2.2]) Let $S \subseteq R$ be a multiplicatively closed subset of $R$ and $M$ be an $R$-module. Let $\mathfrak{p}$ be a prime ideal of $R$.*

*(1) If $M$ is $\mathfrak{p}$-secondary, then $S_{-1}M$ is zero, if $S \cap \mathfrak{p} \neq \emptyset$.*

*(2) If $M$ is $\mathfrak{p}$-secondary, then $S_{-1}M$ is $S^{-1}\mathfrak{p}$-secondary, if $S \cap \mathfrak{p} = \emptyset$.*

*(3) If $M$ is representable then so is $S_{-1}M$ and $\operatorname{Att} S_{-1}M = \{S^{-1}\mathfrak{p} : \mathfrak{p} \in \operatorname{Att} M$ and $S \cap \mathfrak{p} = \emptyset\}$.*

Throughout this chapter we denote by $E_R$ the injective hull of $\oplus R/\mathfrak{m}$, the sum running over all maximal ideals $\mathfrak{m}$ of $R$ and let $D_R$ be the functor $\operatorname{Hom}(-, E_R)$ (cf. Definition 2.3.1).

## 6.1 Attached primes of local cohomology

An important application of the theory of attached primes and secondary representation has been to local cohomology modules of finite $R$-module. Let $(R, \mathfrak{m})$

be a complete local ring. In the light of Theorem 2.3.4 colocalization preserves Artinian modules through colocalization. This section is based on utilize of this property.

We examine the set of attached prime ideals of last non-vanishing value of local cohomology. It is known that $H_{\mathfrak{a}}^{\dim R}(R)$ is an Artinian $R$-module (cf. Theorem 2.1.8). Hence its colocalization is an Artinian module, when $(R, \mathfrak{m})$ is a complete local ring. So the set of attached primes will be well-defined.

**Theorem 6.1.1.** *Let $(R, \mathfrak{m})$ be a complete local ring, $\mathfrak{a}$ be an ideal of $R$ and $\mathfrak{p} \in \operatorname{Spec} R$. Let $c$ be an integer such that $H_{\mathfrak{a}}^i(R) = 0$ for every $i > c$. Assume that $H_{\mathfrak{a}}^c(R)$ is Artinian. Then*

*(1)* $\operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{\mathfrak{a}}^c(R)) \subseteq \{\mathfrak{q}R_{\mathfrak{p}} : \dim R/\mathfrak{q} \geq c, \mathfrak{q} \subseteq \mathfrak{p}$ *and* $\mathfrak{q} \in \operatorname{Spec} R\}$.

*(2)* $\operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{\mathfrak{a}}^{\dim R}(R)) = \{\mathfrak{q}R_{\mathfrak{p}} : \dim R/\mathfrak{q} = \dim R, \mathfrak{q} \subseteq \mathfrak{p}, \operatorname{Rad}(\mathfrak{a} + \mathfrak{q}) = \mathfrak{m}$ *and* $\mathfrak{q} \in \operatorname{Spec} R\}$.

**Proof.**

(1) As $H_{\mathfrak{a}}^c(R)$ is an Artinian module, then by Matlis duality $H_{\mathfrak{a}}^c(R) \cong D_R(D_R(H_{\mathfrak{a}}^c(R)))$. It implies that

$$\operatorname{Att} H_{\mathfrak{a}}^c(R) = \operatorname{Att} D_R(D_R(H_{\mathfrak{a}}^c(R))) = \operatorname{Ass} D_R(H_{\mathfrak{a}}^c(R)).$$

Therefore by virtue of Theorem 6.0.8 we may have

$$\operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{\mathfrak{a}}^c(R)) = \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \operatorname{Ass}_R D_R(H_{\mathfrak{a}}^c(R)) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Since $\mathfrak{q} \in \operatorname{Ass}_R D_R(H_{\mathfrak{a}}^c(R))$, then

$$0 \neq \operatorname{Hom}_R(R/\mathfrak{q}, D_R(H_{\mathfrak{a}}^c(R))) = D_R(H_{\mathfrak{a}}^c(R) \otimes_R R/\mathfrak{q}) = D_R(H_{\mathfrak{a}}^c(R/\mathfrak{q})).$$

To this end note that the first equality follows by Hom $-\otimes$-adjointness and the second one follows by the fact that $H_{\mathfrak{a}}^c(-)$ is a right exact functor. Hence we may deduce that $H_{\mathfrak{a}}^c(R/\mathfrak{q}) \neq 0$ so $\dim R/\mathfrak{q} \geq c$.

(2) Put $d := \dim R$. If $H_{\mathfrak{a}}^d(R) = 0$ we are done. Then we assume that $H_{\mathfrak{a}}^d(R) \neq 0$.

$\subseteq$: Let $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H_{\mathfrak{a}}^d(R))$. As we have seen in part one, $H_{\mathfrak{a}}^d(R/\mathfrak{q}) \neq 0$ so $\dim R/\mathfrak{q} = d$ and by HLVT (Theorem 4.1.2) $\operatorname{Rad}(\mathfrak{a} + \mathfrak{q}) = \mathfrak{m}$.

$\supseteq$: By virtue of Theorem 6.0.8

$$\operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^d_{\mathfrak{a}}(R)) = \{\mathfrak{q}R_{\mathfrak{p}} : \mathfrak{q} \in \operatorname{Att}_R H^d_{\mathfrak{a}}(R), \mathfrak{q} \subseteq \mathfrak{p}\},$$

so it is enough to show that $\mathfrak{q} \in \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$.

As $\dim R/\mathfrak{q} = d$ and $\operatorname{Rad}(\mathfrak{a} + \mathfrak{q}) = \mathfrak{m}$, so Independence Theorem implies that $H^d_{\mathfrak{a}}(R/\mathfrak{q}) \neq 0$. Hence Proposition 2.4.2 implies that

$$\emptyset \neq \operatorname{Att}_R(H^d_{\mathfrak{a}}(R/\mathfrak{q})) = \operatorname{Att}_R(H^d_{\mathfrak{a}}(R)) \cap \operatorname{Supp}_R(R/\mathfrak{q}). \qquad (*)$$

In the contrary assume that $\mathfrak{q} \notin \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$. Then by virtue of $(*)$ there exists a prime ideal $\mathfrak{q}_0 \in \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$ such that $\mathfrak{q}_0 \supset \mathfrak{q}$ and so $\dim R/\mathfrak{q}_0 < d$. On the other $\mathfrak{q}_0 \in \operatorname{Att}_R H^d_{\mathfrak{a}}(R)$ if and only if $\mathfrak{q}_0 R_{\mathfrak{q}_0} \in \operatorname{Att}_{R_{\mathfrak{q}_0}}({}^{\mathfrak{q}_0}H^d_{\mathfrak{a}}(R))$ (cf. 6.0.8). By virtue of part one $\dim R/\mathfrak{q}_0 \geq d$ which is contradiction. Now the proof is complete.

□

**Remark 6.1.2.** *The inclusion in Theorem 6.1.1(1) is not an equality in general. Let $(R, \mathfrak{m})$ be a complete local ring of dimension $d > 0$ and $\mathfrak{p}$ be a $d$-dimensional minimal prime ideal of $R$. Assume that $H^d_{\mathfrak{a}}(R) = 0$, then $\operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^d_{\mathfrak{a}}(R)) = \emptyset$ but $\{\mathfrak{q}R_{\mathfrak{p}} : \dim R/\mathfrak{q} = d$ and $\mathfrak{q} \subseteq \mathfrak{p}\} = \{\mathfrak{p}R_{\mathfrak{p}}\}$.*

**Proposition 6.1.3.** *Let $(R, \mathfrak{m})$ be a complete local ring of dimension $d$. Let $\mathfrak{a}$ be an ideal of $R$. Assume that $H^{d-1}_{\mathfrak{a}}(R)$ is Artinian and $H^d_{\mathfrak{a}}(R) = 0$. Then*

*(1)* $\operatorname{Att}_R(H^{d-1}_{\mathfrak{a}}(R)) \subseteq \{\mathfrak{p} \in \operatorname{Spec} R : \dim R/\mathfrak{p} = d-1, \operatorname{Rad}(\mathfrak{a}+\mathfrak{p}) = \mathfrak{m}\} \cup \operatorname{Assh}(R)$.

*(2)* $\{\mathfrak{p} \in \operatorname{Spec} R : \dim R/\mathfrak{p} = d-1, \operatorname{Rad}(\mathfrak{a}+\mathfrak{p}) = \mathfrak{m}\} \subseteq \operatorname{Att}_R(H^{d-1}_{\mathfrak{a}}(R))$.

**Proof.**

(1) Let $\mathfrak{p} \in \operatorname{Att}_R H^{d-1}_{\mathfrak{a}}(R)$ so $\mathfrak{p}R_{\mathfrak{p}} \in \operatorname{Att}_{R_{\mathfrak{p}}}({}^{\mathfrak{p}}H^{d-1}_{\mathfrak{a}}(R))$, hence by Theorem 6.1.1 $\dim R/\mathfrak{p} \geq d-1$.

When $\dim R/\mathfrak{p} = d$ it follows that $\mathfrak{p} \in \operatorname{Assh}(R)$. In the case $\dim R/\mathfrak{p} = d-1$, as $\mathfrak{p} \in \operatorname{Att}_R H^{d-1}_{\mathfrak{a}}(R) = \operatorname{Ass}_R D_R(H^{d-1}_{\mathfrak{a}}(R))$ and $H^{d-1}_{\mathfrak{a}}(-)$ is a right exact functor so one can deduce that $H^{d-1}_{\mathfrak{a}}(R/\mathfrak{p}) \neq 0$. Hence by Hartshorne-Lichtenbaum vanishing Theorem there exists a prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$ of $R$ of dimension $d-1$ with $\operatorname{Rad}(\mathfrak{a}+\mathfrak{q}) = \mathfrak{m}$. Now one can see that $\mathfrak{q} = \mathfrak{p}$.

(2) Let $\dim R/\mathfrak{p} = d-1$ and $\mathrm{Rad}(\mathfrak{a}+\mathfrak{p}) = \mathfrak{m}$, then Theorem 6.1.1(2) implies that $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Att}_{R_\mathfrak{p}}({}^{\mathfrak{p}}H^{d-1}_\mathfrak{a}(R/\mathfrak{p}))$. Using 6.0.8 we deduce that $\mathfrak{p} \in \mathrm{Att}_R(H^{d-1}_\mathfrak{a}(R/\mathfrak{p}))$. Now by the epimorphism

$$H^{d-1}_\mathfrak{a}(R) \to H^{d-1}_\mathfrak{a}(R/\mathfrak{p}) \to 0$$

we see that $\mathfrak{p} \in \mathrm{Att}_R(H^{d-1}_\mathfrak{a}(R))$.

□

It is noteworthy to say that in the situation of Proposition 6.1.3 if $\mathfrak{a}$ is an ideal of dimension one, the inclusion at (1) will be an equality, see [Hel, Theorem 8.2.3].

## 6.2 Sharp's Asymptotic Theorem

Let $R$ be a commutative ring (not necessarily Noetherian) and $\mathfrak{a}$ an ideal of $R$. For every Artinian $R$-module $A$, $\mathrm{Att}_R(0 :_A \mathfrak{a}^n)$ and $\mathrm{Att}_R((0 :_A \mathfrak{a}^n)/(0 :_A \mathfrak{a}^{n-1}))$ are ultimately constant and $At_R(\mathfrak{a}, A)$ and $Bt_R(\mathfrak{a}, A)$ denote their ultimate constant values (cf. [Sh2]). Clearly $Bt_R(\mathfrak{a}, A) \subseteq At_R(\mathfrak{a}, A)$. In [Sh1], Sharp showed that

$$At(\mathfrak{a}, A) \setminus Bt(\mathfrak{a}, A) \subseteq \mathrm{Att}_R(A)$$

for every Artinian module $A$, by generalization of Heinzer-Lantz Theorem. Schenzel [Sch2] has given an alternative proof for mentioned Theorem in case that for a local ring $(R, \mathfrak{m})$, if $\mathfrak{m} \in At_R(\mathfrak{a}, A) \setminus Bt_R(\mathfrak{a}, A)$ then $\mathfrak{m} \in \mathrm{Att}_R A \cap V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of prime ideals of $R$ containing ideal $\mathfrak{a}$. In this section we give a short simple proof to Sharp's Theorem using the concept of colocalization.

At first we give some preliminary lemmas in order to prove Theorem 6.2.4 as the main result in this section.

**Lemma 6.2.1.** *Let $A$ be an Artinian $R$-module. Let $\mathfrak{p}$ be a prime ideal of $R$ and $n$ be an arbitrary integer. Then*

$${}^{\mathfrak{p}}(0 :_A \mathfrak{a}^n) = (0 :_{{}^{\mathfrak{p}}A} \mathfrak{a}^n R_\mathfrak{p}).$$

**Proof.** We prove by definition of colocalization and use of Hom $-\otimes$-adjointness as follows:

$$\begin{aligned}
{}^{\mathfrak{p}}(0 :_A \mathfrak{a}^n) &\cong D_{R_{\mathfrak{p}}}((D_R(0 :_A \mathfrak{a}^n))_{\mathfrak{p}}) \\
&\cong D_{R_{\mathfrak{p}}}((\mathrm{Hom}_R(\mathrm{Hom}_R(R/\mathfrak{a}^n, A), E_R))_{\mathfrak{p}}) \\
&\cong D_{R_{\mathfrak{p}}}((R/\mathfrak{a}^n \otimes_R D_R(A))_{\mathfrak{p}}) \\
&\cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}^n R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} D_R(A)_{\mathfrak{p}}, E(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})) \\
&\cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}^n R_{\mathfrak{p}}, D_{R_{\mathfrak{p}}}(D_R(A)_{\mathfrak{p}})) \\
&\cong \mathrm{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}^n R_{\mathfrak{p}}, {}^{\mathfrak{p}}A) \\
&\cong (0 :_{{}^{\mathfrak{p}}A} \mathfrak{a}^n R_{\mathfrak{p}}).
\end{aligned}$$

□

**Lemma 6.2.2.** *Let $A$ be an Artinian $R$-module. Let $\mathfrak{p}$ be a prime ideal of $R$ and $n$ be an arbitrary integer. Then*

$${}^{\mathfrak{p}}(0 :_A \mathfrak{a}^n / 0 :_A \mathfrak{a}^{n-1}) = (0 :_{{}^{\mathfrak{p}}A} \mathfrak{a}^n R_{\mathfrak{p}} / 0 :_{{}^{\mathfrak{p}}A} \mathfrak{a}^{n-1} R_{\mathfrak{p}}).$$

**Proof.** For an integer $n$, there is the following short exact sequence

$$0 \to 0 :_A \mathfrak{a}^{n-1} \to 0 :_A \mathfrak{a}^n \to 0 :_A \mathfrak{a}^n / 0 :_A \mathfrak{a}^{n-1} \to 0. \qquad (*)$$

As colocalization is a covariant exact functor we get

$$0 \to^{\mathfrak{p}} (0 :_A \mathfrak{a}^{n-1}) \to^{\mathfrak{p}} (0 :_A \mathfrak{a}^n) \to^{\mathfrak{p}} (0 :_A \mathfrak{a}^n / 0 :_A \mathfrak{a}^{n-1}) \to 0$$

so in the view of lemma 6.2.1, the claim is clear. □

In the case $(R, \mathfrak{m})$ is a local ring, for an Artinian $R$-module $A$ it follows by Proposition 2.4.2 that $\mathfrak{m} \notin \mathrm{Att}_R A$ if and only if $A/\mathfrak{m}A = 0$.

**Lemma 6.2.3.** *( [Sch2, lemma 3.1]) Let $\mathfrak{a}$ be an ideal of local ring $(R, \mathfrak{m})$. Suppose that $\mathfrak{m} \in At_R(\mathfrak{a}, A) \setminus Bt_R(\mathfrak{a}, A)$ for an Artinian $R$-module $A$. Then $\mathfrak{m} \in \mathrm{Att}_R A \cap V(\mathfrak{a})$.*

**Proof.** By tensoring the exact sequence $(*)$ with $R/\mathfrak{m}$, it yields the existence of an integer $m_0 \in \mathbb{N}$ such that the derived homomorphism

$$\psi_n : (0 :_A \mathfrak{a}^{n-1}) \otimes_R R/\mathfrak{m} \to (0 :_A \mathfrak{a}^n) \otimes_R R/\mathfrak{m} \qquad (**)$$

is a surjective homomorphism of non-zero and finite dimensional $R/\mathfrak{m}$-vector spaces for all $n \geq m_0$. To this end note that by assumption $\mathfrak{m} \notin Bt_R(\mathfrak{a}, A)$. By

$(**)$ there is an integer $\mathfrak{m} \in \mathbb{N}$ such that the epimorphisms $\psi_n, n \geq m$ become isomorphisms. Hence $\{(0 :_A \mathfrak{a}^n) \otimes_R R/\mathfrak{m}, \psi_n\}$ is a direct system with

$$0 \neq (0 :_A \mathfrak{a}^m) \otimes_R R/\mathfrak{m} \cong \varinjlim_n ((0 :_A \mathfrak{a}^n) \otimes_R R/\mathfrak{m}).$$

The direct limit commutes with the tensor product, i.e.

$$0 \neq (\varinjlim_n 0 :_A \mathfrak{a}^n) \otimes_R R/\mathfrak{m} \cong A \otimes_R R/\mathfrak{m} = A/\mathfrak{m}A.$$

Whence $\mathfrak{m} \in \mathrm{Att}_R A$. $\square$

**Theorem 6.2.4.** ***(Sharp's Asymptotic Theorem)*** *Let A be an Artinian R-module. Then*

$$At_R(\mathfrak{a}, A) \setminus Bt_R(\mathfrak{a}, A) \subseteq \mathrm{Att}_R A \cap V(\mathfrak{a}).$$

**Proof.** Let $\mathfrak{p} \in At_R(\mathfrak{a}, A) = \mathrm{Att}_R(0 :_A \mathfrak{a}^n)$. Then by Theorem 6.0.8 and Lemma 6.2.1

$$\begin{aligned} \mathfrak{p}R_\mathfrak{p} \in \mathrm{Att}_{R_\mathfrak{p}}(^\mathfrak{p}(0 :_A \mathfrak{a}^n)) &= \mathrm{Att}_{R_\mathfrak{p}}(0 :_{^\mathfrak{p}A} \mathfrak{a}^n R_\mathfrak{p}) \\ &= At_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p}, {}^\mathfrak{p}A). \end{aligned}$$

On the other hand by virtue of Lemma 6.2.2, $\mathfrak{p} \notin \mathrm{Att}_R(0 :_A \mathfrak{a}^n / 0 :_A \mathfrak{a}^{n-1})$ if and only if $\mathfrak{p}R_\mathfrak{p} \notin \mathrm{Att}_{R_\mathfrak{p}}(0 :_{^\mathfrak{p}A} \mathfrak{a}^n R_\mathfrak{p} / 0 :_{^\mathfrak{p}A} \mathfrak{a}^{n-1} R_\mathfrak{p}) = Bt_{R_\mathfrak{p}}(\mathfrak{a}^n R_\mathfrak{p}, {}^\mathfrak{p}A)$.

So in the view of Lemma 6.2.3, it yields that

$$\mathfrak{p}R_\mathfrak{p} \in (\mathrm{Att}_{R_\mathfrak{p}} {}^\mathfrak{p}A) \cap V(\mathfrak{a}R_\mathfrak{p}).$$

Hence $\mathfrak{p} \in (\mathrm{Att}_R A) \cap V(\mathfrak{a})$. $\square$

# Chapter 7

# Summary and further problems

The present research was devoted to a study on local cohomology modules. About the importance and motivation to work on these modules and also about their properties we refer the reader to chapter one and chapter two of this thesis.

## 7.1 Formal local cohomology

Let $(R, \mathfrak{m})$ be a local ring and $M$ be a finitely generated $R$-module. In Chapter 3, we deal with the question when formal local cohomology modules are Artinian. Our efforts led to the following result:

**Theorem 7.1.1.** *Let $(R, \mathfrak{m})$ be a local ring and $M$ be a finitely generated $R$-module. For given integers $i$ and $t > 0$, the following statements are equivalent:*

*(1)* $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{m}\widehat{R})$ *for all $i < t$;*

*(2)* $\mathfrak{F}^i_{\mathfrak{a}}(M)$ *is Artinian for all $i < t$;*

*(3)* $\operatorname{Supp}_{\widehat{R}}(\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a}\widehat{R})$ *for all $i < t$.*

*(4)* $\mathfrak{a} \subseteq \operatorname{Rad}(\operatorname{Ann}_R(\mathfrak{F}^i_{\mathfrak{a}}(M)))$ *for all $i < t$;*

*Suppose that $t \leq \operatorname{depth} M$, then the above conditions are equivalent to*

*(5)* $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ *for all $i < t$;*

Let $\varphi : R \to S$ be a flat ring homomorphism, then for any $R$-module $N$ and any $\mathfrak{q} \in \operatorname{Spec} S$

$$\operatorname{Supp}_R(N) = \{\mathfrak{q} \cap R : \mathfrak{q} \in \operatorname{Supp}_S(N \otimes_R S)\}$$

(cf. [Tou-Yas, lemma 2.1]). But according to [Asgh-Divan, Remark 2.8(vii)], the analogue of the Flat Base Change Theorem is not true in general, for formal local cohomology. It seems to be an important question to ask

• What is $\operatorname{Supp}_R(\mathfrak{F}^i_{\mathfrak{a}}(M))$?

• Formal local cohomology modules are very seldom finitely generated. It can be a natural question to find out the equivalent statements for finiteness of formal local cohomology. For instance $\mathfrak{F}^{\dim M/\mathfrak{a}M}_{\mathfrak{a}}(M)$ is not finitely generated but $\mathfrak{F}^0_{\mathfrak{a}}(M)$ is a finitely generated $\widehat{R}$-module.

Let $(R, \mathfrak{m})$ be a complete local ring then

$$\mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is finite } \Leftrightarrow \mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is finite . } (*)$$

If $\mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian ($i \in \mathbb{Z}$), then in the case $\mathfrak{a} \neq \mathfrak{m}$, as $\operatorname{Att}(\mathfrak{F}^i_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^i_{\mathfrak{a}}(M)) \subseteq V(\mathfrak{a})$, then by virtue of $(*)$, $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is not finitely generated. In this direction it is known that for an integer $t$ such that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian for all $i > t$, then $\mathfrak{F}^t_{\mathfrak{a}}(M)/\mathfrak{a}\mathfrak{F}^t_{\mathfrak{a}}(M)$ is Artinian. Consequently $\mathfrak{F}^t_{\mathfrak{a}}(M)$ is not finite provided $\mathfrak{a} \neq \mathfrak{m}$.

• One of our structural results is to find the $\operatorname{Coass}(\mathfrak{F}^{d-1}_{\mathfrak{a}}(R))$ where $(R, \mathfrak{m})$ is a complete local ring of dimension $d$ and $\mathfrak{a}$ is a one dimensional ideal of $R$. It is an open question to investigate it for an arbitrary ideal. Moreover investigation of $\operatorname{Coass}(\mathfrak{F}^{d-i}_{\mathfrak{a}}(R))$ for $i \geq 2$ could be interesting. In particular, it can be helpful to clarify some aspects of $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$, when $(R, \mathfrak{m})$ is a Gorenstein local ring.

Note that finiteness of $\operatorname{Coass}(\mathfrak{F}^i_{\mathfrak{a}}(R))$ implies that

$$\operatorname{Cosupp}(\mathfrak{F}^i_{\mathfrak{a}}(R)) = \operatorname{Supp} \operatorname{Hom}_R(\mathfrak{F}^i_{\mathfrak{a}}(R), E(R/\mathfrak{m}))$$

to be closed.

## 7.2 Top local cohomology

Ogus [Og, Corollary 2.11] in equicharacteristic 0 and Peskine and Szpiro [Pes-Szp, 5.5] in equicharacteristic $p > 0$ generalized the vanishing Theorem of Hartshorne

[Hart] for the cohomological dimension of the complement of a subvariety of projective space. Huneke and Lyubeznik [Hun-Lyu, Theorem 2.9] gave a new characteristic-free proof of it:

**Theorem 7.2.1.** *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d$ containing a field and let $\mathfrak{a}$ be an ideal of $R$. Then the following are equivalent:*

*(1)* $H^i_{\mathfrak{a}}(R) = 0$ *for* $i = d-1, d$.

*(2)* $\dim R/\mathfrak{a} \geq 2$ *and* $\operatorname{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}\}$ *is formally geometrically connected.*

Hochster and Huneke [Hoch-Hun, Theorem 3.3] generalized Faltings' connectedness Theorem as follows:

**Theorem 7.2.2.** *Let $(R, \mathfrak{m})$ be a complete equidimensional local ring of dimension $d$ such that $H^d_{\mathfrak{m}}(R)$ is an indecomposable $R$-module. Let $H^i_{\mathfrak{a}}(R) = 0$ for $i = d-1, d$, then* $\operatorname{Spec}(R/\mathfrak{a}) \setminus \{\mathfrak{m}\}$ *is connected.*

Hartshorne-Lichtenbaum vanishing Theorem gives a characterization for vanishing of $H^d_{\mathfrak{a}}(R)$. In Theorem 4.1.5, we have expressed the isomorphism $H^d_{\mathfrak{a}}(R) \cong H^d_{\mathfrak{m}\widehat{R}}(\widehat{R}/J)$ for a certain ideal $J$ of $\widehat{R}$. It translates the properties of $H^d_{\mathfrak{a}}(R)$ via $H^d_{\mathfrak{m}\widehat{R}}(\widehat{R})$, which is more known. On the other hand $H^{d-1}_{\mathfrak{a}}(R)$ is more mysterious. In Proposition 6.1.3 we gave some information about $\operatorname{Att} H^{d-1}_{\mathfrak{a}}(R)$, also some of the properties of $\operatorname{Hom}_R(H^{d-1}_{\mathfrak{a}}(R), E(R/\mathfrak{m}))$ for a one dimensional ideal $\mathfrak{a}$ have been appeared in [Hel].

Now the natural question can be as follows:

- When $H^{d-1}_{\mathfrak{a}}(R)$ is zero? What are $\operatorname{Ass} H^{d-1}_{\mathfrak{a}}(R)$ and $\operatorname{Att} H^{d-1}_{\mathfrak{a}}(R)$?

# Notation

| | |
|---|---|
| Ass | The set of associated primes |
| Att | The set of attached primes |
| Coass | The set of coassociated primes |
| Cosupp | Cosupport (of a module) |
| $\check{C}_{\mathbf{x}}$ | Čech complex |
| $D^{\bullet}$ | Dualizing complex |
| $D_R(-)$ | $\operatorname{Hom}(-, E_R)$ |
| $E_R$ | Minimal injective cogenerator of the category of $R$-modules |
| $\mathfrak{F}^i_{\mathfrak{a}}(M)$ | $i$-th formal local cohomology of a module $M$ |
| HLVT | Hartshorne-Lichtenbaum vanishing Theorem |
| $K_M$ | Canonical module of a module $M$ |
| $\max(R)$ | the set of maximal ideals of $R$ |
| $\widehat{M}$ | $\mathfrak{m}$-adic completion of a module $M$ |
| ${}^{\mathfrak{p}}M$ | Colocalization of module $M$ with respect a prime ideal $\mathfrak{p}$ |
| $\operatorname{Rad}(-)$ | Radical (of an ideal) |
| Supp | Support (of a module) |

# Declaration

Hiermit erkläre ich, dass ich diese Arbeit selbständig und ohne fremde Hilfe verfasst habe. Ich habe keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt. Die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen sind als solche kenntlich gemacht worden. Ich habe mich bisher nicht um den Doktorgrad beworben.

Halle (Saale) Majid Eghbali

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