FORMULAE

DE SERIERUM REVERSIONE

DEMONSTRATIO VNIVERSALIS

SIGNIS LOCALIBUS

COMBINATORIO-ANALYTICORUM

VICARIIS

EXHIBITA

DISSERTATIO ACADEMICA

AUCTORE

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LIPSIAE

LITTERIS SOMMERIIS.

Quae sunt communes problematibus circa Series quam plurimis, praesertim his, quae Methodo, generalissima illa quidem et latissime patente, Leibnitiana tractantur, per coefficientes fictos, qui assumuntur tanquam dati, difficultates: has ipsas pariter, quae de *Serierum Reversione* praecipit, vti vulgo instituitur, analysis, experta est longe grauissimas, vbi Infinitinomiorum, quae postulantur, Dignitates, et coefficientium fictorum, sequentium per antecedentium omnium recursum, determinatio, molestiam plane taediosam creant, et ad substitutiones tantum non insuperabiles deducunt. Quibus incommodis remedium attulit paratissimum certissimumque *Excellentissimus* HINDENBVRGIVS, Praeceptor atque Patronus plurimum mihi venerandus, exposita *noua Methodo combinatorio-analytica*, quam libris duobus: *Infinitinomii Dignitatum Exponentis indeterminati Historia, Leges ac Formulae — Gottingae* 1779. 4to, et *Noui Systematis Permutationum, Combinationum ac Variationum primae lineae — Lipsiae* 1781, 4to, luculenter pertractauit, eoque ipso, qui fructus ex *Combinatoria Arte* ad *Analysin* redundent longe vberrimi, exemplis quam plurimis et maxime perspicuis ostendit. Huius igitur methodi ope, quae ipsius est prae ceteris praestantia, *serierum*, a factorum polynomicorum quotcunque complicatione oriundarum, *Genesis* et *Terminus generalis*, *Infinitinomiorum* autem, pari facilitate atque Binomiorum, *Dignitates* exhiberi, coefficientes quaesiti singuli, ex ordine et extra ordinem, a praecedentibus independenter, in terminis statim simplicissimis, per seriei non fictae sed datae coefficientes atque exponentes, deriuari, ad alia problemata, per *Methodum* inprimis *Potentiarum*, accommodari, substitutionum autem molestissima diuerticula penitus evitari possunt.

His praesidiis et adminiculis adiutus ESCHENBACHIVS, *Vir Clarissimus*, peculiari libello: *De serierum Reuersione, formulis analytico-combinatoriis exhibita — Lipsiae* 1789 edito, problema hoc complicatissimum ingeniose tractauit, sic, vt formulam, terminis *combinatorio-analyticis* constantem exhiberet, qua seriei quaesitae coefficiens indeterminatus, extra ordinem, et ab antecedentibus independenter, datis solummodo seriei propositae coefficientibus et exponentibus, determinatur. Demonstratio *Formulae Eschenbachianae* (ita enim ab auctore eam appellare non dubitaui) cum neque in ipsius Auctoris libello

 tradita

tradita appareat, neque etiam ita ſit comparata, vt in oculos ſtatim incurrat, operae pretium me facturum eſſe arbitratus ſum, ſi opportunitate ſcribendi oblata, quae *Eſchenbachius* deſiderantibus in medio reliquit, ſpecimem qualiscunque diligentiae editurus, tractanda mihi ſumſerim, *duplicem demonſtrationem*, quam diu de hac re cogitans tandem inveni, hunc in finem propoſiturus.

Erunt fortaſſe, qui mihi obiiciant, meam formulae *Eſchenbachianae* demonſtrationem eſſe ſuperfluam. Nam *de la Grangium*, *Virum Celeb.* iam tradidiſſe formulam (a), *Fiſcheri* quidem *V. Cl.* iudicio (ex cuius libro (b) nuperrime edito primam eius accepi notitiam) *generaliſſime*, *maximoque*, *cuius Analyſis capax ſit*, *rigore* demonſtratam, ex qua formula *Eſchenbachiana* derivari poſſit. Sciendum autem eſt, demonſtrationem formulae *de la Grangii*, contra, quam contendit *Fiſcherus*, non ſolum non generalem eſſe, (valet enim tantummodo ſi m *numerus* eſt *integer poſitiuus*), ſed demonſtrationis etiam rigorem dubiis adhuc obnoxium reperiri. Ex *Identitate* enim *valorum* ſerierum huiusmodi

$$a + bx^{-1} + cx^{-2} \dots \qquad \alpha + \beta x^{-1} + \gamma x^{-2} \dots$$
$$+ Ax + Bx^2 + Cx^3 \dots \quad \text{atque} \quad + \text{A}x + \text{B}x^2 + \Gamma x^3 \dots$$

neque *initium* neque *finem* habentium, non licet eodem modo, quam fieri poteſt in ſeriebus initium habentibus, coefficientes in vtraque ſerie ad eandem variabilis x poteſtatem pertinentes, aequales eſſe, tuto et directe concludere. Qua de re alio loco fuſius dicendi ſeſe offeret occaſio.

Cum libri *Fiſcheriani* mentionem iniecerim, non poſſum non breuiter hic monere, ſigna, quibus *Fiſcherus* in libro ſuo vſus eſt (*Dimenſionszeichen*) cum ſignis *Hindenburgianis Combinatoriis* (𝔞ⁿA, 𝔟ⁿB, etc.) plane convenire (c), praeter formam externam, quam, quia eorum inventionem et vſum *iniuſtiſſime* ſibi

(a) Si p radix eſt aequationis $\alpha - x + \Phi x = 0$ (vbi Φx functionem incognitae x ſignificat) formulam hanc tradidit *de la Grangius*:

$$p^m = x^m + \frac{d(x^m)\Phi x}{dx} + d\,\frac{\left[\frac{d(x^m)}{dx}(\Phi x)^2\right]}{1.2\;\; dx} + d^2\,\frac{\left[\frac{d(x^m)}{dx}(\Phi x)^3\right]}{1.2.3\;\; dx^2} + d^3\,\frac{\left[\frac{d(x^m)}{dx}(\Phi x)^4\right]}{1.2.3.4\;\; dx^3} + \text{etc.}$$

in qua, facta differentiatione, loco x, ſubſtituendum eſt α. Cf. *Nouvelle méthode pour réſoudre les Equations litterales par le moyen des ſéries. Mem. de l'Acad. Roy. des Sc. et. B. L. à Berlin* 1770. p. 251 - 326. Huius diſſertationis verſio germanica invenitur in *Michelſens 3ten Bande der Euleriſchen Einleitung in die Analyſis des Unendlichen. S.* 190 - 270.

(b) *Theorie der Dimenſionszeichen, nebſt ihrer Anwendung auf verſchiedene Materien aus der Anal. endl. Groeſſen. II. Theile* 1792. *Vorrede S. IV.*

(c) Cf. *Combinatoriſche Analytik und Theorie der Dimenſionszeichen, in Parallele geſtellt von Heinrich Auguſt Toepfer, Leipzig 1793. Seite* 52 - 61. atque Tabula III. et IIII.

ſibi vult attribui, data opera paululum immutavit. Quod, quo melius ei procederet, plagiumque ſuum, ſi fieri ullo modo poſſet, lectores harum rerum ignaros lateret, omnia, quae accuratioris libri ſui cum *Hindenburgianis* [quorum non niſi vnius (*Nov. Syſt. Perm.*) ac obiter tantum, mentionem fecit, alterius vero (*Infinit. Dignit.*) eiusque ipſius, e quo plurima hauſit, ne titulum quidem nominauit] comparationis inſtituendae occaſionem praebere potuiſſent, maximo ſtudio adhibito vitavit, eamque etiam ob cauſam *regulas* ipſas, numerorum *Diſcerptiones* (d) ac *Variationes* (e) complexionisque cuiusvis propoſitae *Numerum Permutationum* (f) inveniendi [quibus tamen ſigna ſua, quae vocat, dimenſionalia nituntur] tradere auſus non eſt, ſed exempla tantum et tabulam, ex *Hindenburgiana* penitus tranſcriptam, attulit, de *Complexionibus* autem *ſimpliciter*, *admiſſis* vel *non admiſſis repetitionibus*, earumque notis, de *Permutationibus*, *de ſignis localibus*, eorumque in Analyſi vſu longe maximo, de *diſtantiae exponentibus*, atque *Coefficientium binomialium* et *polynomialium* notis, ne verbum quidem dicere, ideoque ſuam ſignorum dimenſionalium theoriam limitibus nimis arctis circumſcribere, coactus eſt. Quod detrimentum inde lectores capiant, optime oſtendit *Toepferus Vir Clariſſimus* (g), pluraque hac de re in theſibus huic diſſertationi adiectis expoſui, vnde luce clarius apparet, ſigna monadica *dimenſionalia Fiſcheriana*, longe inferiora effectu eſſe *Hindenburgianis* multiplicibus *combinatoriis*, tantum abeſt (quod inſinuare quidem vult *Fiſcherus*, [*Vorrede* S. V.] lectoribus) vt plus illa valeant, quam haec, efficere.

His vero ſignis alienis *Fiſcherus* non ſolum eodem modo atque *Hindenburgius*, vnde tranſcripſit, vſus eſt, ſed etiam eundem plane ordinem ſecutus, ea primo loco Infinitinomio $Ax^{m} + Bx^{m+r} + Cx^{m+2r} \ldots$ ad dignitatem *exponentis integri poſitiui*, deinde Infinitinomio $1 + Bx^{r} + Cx^{2r} + Dx^{3r} + \ldots$ ac denique Infinitinomio $Ax^{m} + Bx^{m+r} + Cx^{m+2r} \ldots$ ad dignitatem *exponentis indeterminati* euehendo, applicuit, atque ita eandem viam ingreſſus, easdem plane propoſitiones ac formulas elementares invenit, tradidit, problematibus applicuit, et perpauca, quae eſſent alicuius momenti, et quae ipſe inveniſſe iuſto dici poſſit, addi-

(d) *Combinatoriſche Analytik Seite* 16-21.
(e) *Ibid. S.* 61-68.
(f) *Ibid. S.* 21. 22.
(g) *Ibid. vierzehnter Abſchnitt.*

addidit (h). Namque etiam formula, quam non solum libri sui, sed totius fere Analyseos, primariam esse iudicauit, quamque, sicuti reliqua omnia, tanquam inventionem suam venditauit, nulla alia est quam *Eschenbachiana regressoria* (i). Quod ne lectoribus in oculos incurrat statim, aliud ei nomen (*allgemeine Auflösungsreihe*) dedit, signorum autem *Combinatorio - analyticorum* loco, quibus *Eschenbachius* eam exhibuit, sua *dimensionalia*, proque *Coefficientium Binomialium* notis eorum valores posuit. Quam formulam ab *Eschenbachio* Viro Clarissimo sine demonstratione prolatam, etiam *Fischerum* demonstrare non potuisse non miror, cum res multo faciliores demonstrare non potuerit (k). Reli-

(h) *Ibid. zehnter Abschnitt, atque Tabula VII.*

(i) *Ibid. eilfter zwölfter und dreizehnter Abschnitt, atque Tabula VIII.*

(k) Exemplis hoc tantummodo duobus, ex pluribus, probabo:

1) Inuenit *Fischerus* (*Theorie der Dimens. I. Th. §. 122. Seite 101*). $\log. \frac{1-x+x^2}{(1-x)^2} = \frac{1}{1}x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{3}{4}x^4 + \frac{1}{5}x^5 + \frac{0}{6}x^6 + \frac{1}{7}x^7 + \frac{3}{8}x^8 + \frac{4}{9}x^9 + \frac{3}{10}x^{10} + \frac{1}{11}x^{11} + \frac{0}{12}x^{12} + \frac{1}{13}x^{13} + \frac{3}{14}x^{14} + \frac{4}{15}x^{15} + \frac{3}{16}x^{16} + \frac{1}{17}x^{17} + \frac{0}{18}{}^{18} +$ etc. atque legem hic apparentem, sed *a posteriori* ab ipso quaesitam, facillime demonstrare potuisset, si modo non protinus $\frac{x}{(1-x)^2}$ in seriem conuertisset, sed cogitasset, esse $1-x+x^2 = \frac{1+x^3}{1+x}$. Hinc $\log. \frac{1-x+x^2}{(1-x)^2} = \log.(1+x^3) - \log.(1+x) - 2\log.(1-x)$, vnde statim exsurgit *series quaesita* et, vti ipsi visa est praeter necessitatem, *obstinata* cum *termino generali.*

2) Sit $v =$ sin. vers. x, $\frac{1}{2}x = z$, $\sin\frac{1}{2}x = \sin z = y$, atque est

$$\cos.\frac{2z}{p} = 1 - \frac{\frac{4}{p^2}y^2}{1.2} + \frac{\frac{4}{p^2}\left(\frac{4}{p^2}-4\right)y^4}{1.2.3.4} - \text{etc.} = 1 - \frac{\frac{4}{p^2}y^2}{1.2} - \frac{\frac{4}{p^2}\left(4-\frac{4}{p^2}\right)y^4}{1.2.3.4} - \frac{\frac{4}{p^2}\left(4-\frac{4}{p^2}\right)\left(16-\frac{4}{p^2}\right)y^6}{1.2.3.4.5.6} - \frac{\frac{4}{p^2}\left(4-\frac{4}{p^2}\right)\left(16-\frac{4}{p^2}\right)\left(36-\frac{4}{p^2}\right)y^8}{1.2.3.4.5.6.7.8} - \ldots\ldots$$

ergo

$$1 - \cos.\frac{2z}{p} = \text{sin. vers.}\frac{2z}{p} = 2\sin^2\frac{z}{p} = \frac{\frac{4}{p^2}y^2}{1.2} + \frac{\frac{4}{p^2}\left(4-\frac{4}{p^2}\right)y^4}{1.2.3.4} + \frac{\frac{4}{p^2}\left(4-\frac{4}{p^2}\right)\left(16-\frac{4}{p^2}\right)y^6}{1.2.3.4.5.6} + \frac{\frac{4}{p^2}\left(4-\frac{4}{p^2}\right)\left(16-\frac{4}{p^2}\right)\left(36-\frac{4}{p^2}\right)y^8}{1.2.3.4.5.6.7.8} + \ldots$$

atque

$$2\left(p\sin\frac{z}{p}\right)^2 = \frac{4}{1.2}y^2 + \frac{4\left(4-\frac{4}{p^2}\right)y^4}{1.2.3.4} + \frac{4\left(4-\frac{4}{p^2}\right)\left(16-\frac{4}{p^2}\right)y^6}{1.2.3.4.5.6} + \frac{4\left(4-\frac{4}{p^2}\right)\left(16-\frac{4}{p}\right)\left(36-\frac{4}{p^2}\right)y^8}{1.2.3.4.5.6.7.8} + \ldots + \frac{4\left(4-\frac{4}{p^2}\right)\left(16-\frac{4}{p^2}\right)\left(36-\frac{4}{p^2}\right)\ldots\left(4n^2-\frac{4}{p^2}\right)y^{2n+2}}{1.2.3.4.5.6.7.8\ldots(2n+1)(2n+2)}$$

Posito

Reliquum eſt, ut de argumento ipſius diſſertationis breuiter exponam. Signo ([1]) nouo, quo vſus ſum, aliisque, quae *Hindenburgius Vir Celeberrimus* iam introduxit, in §. I. et paſſim in notis breuiſſime explicitis, in §. II-IV relationes quasdam coefficientium in ſerierum dignitatibus per formulas locales propoſui, quibus *prima* in §. V. ſuperſtruitur demonſtratio, quam excipit duplex formulae ipſius regreſſoriae, per ſigna combinatoria et per valores eorum expoſitio, adhibitis ſimul pluribus, ad ipſius formulae vlteriorem explicationem, idoneis exemplis. Aliam §. VI. continet formulam localem longe generaliſſimam, methodo *Bernoulliana* ab *Illuſtri Kaeſtnero* pluribus locis uſurpata, demonſtratam, cum Corollario, in quo potiores, quae inde pendent, formulae ſpecialiores, exhibentur. Huius formulae ſubſidio *altera* Formulae Eſchenbachianae demonſtratio absque omni difficultate in §. VII. abſoluitur. Quibus peractis, in §. VIII. formula proponitur, ſeriei quaeſitae logarithmum naturalem, datis ſolummodo coefficientibus et exponentibus, ex

exprimen

Poſito vero $p = \infty$, mutatur $p \sin \frac{z}{p}$ in z, atque eſt $2z^2 = \frac{4}{1.2}y^2 + \frac{4.4}{1.2.3.4}y^4 + \frac{4.4.16}{1.2.3.4.5.6}y^6$

$+ \frac{4.4.16.36}{1.2.3.4.5.6.7.8}y^8 \ldots\ldots + \frac{4.4.16.36 \ldots\ldots 4n^2}{1.2.3.4.5.6\ldots(2n+1)(2n+2)}y^{2n+2} + \ldots\ldots\ldots = \frac{1}{1.2}2^2y^2$

$+ \frac{1^2}{1.2.3.4}2^4y^4 + \frac{1^2.2^2}{1.2.3.4.5.6}2^6y^6 + \frac{1^2.2^2.3^2}{1.2.3.4.5.6.7.8}2^8y^8 \ldots + \frac{1^2.2^2.3^2 \ldots n^2}{1.2.3.4\ldots(2n+1)(2n+2)}2^{2n+2}y^{2n+2} + \ldots$

$= \frac{1}{1.2}2^2y^2 + \frac{1}{2.3.4}2^4y^4 + \frac{1.2}{3.4.5.6}2^6y^6 + \frac{1.2.3}{4.5.6.7.8}2^8y^8 \ldots + \frac{1.2.3\ldots n}{(n+1)(n+2)\ldots(2n+2)}2^{2n+2}y^{2n+2}$ + etc.

ſiue etiam $\frac{1}{2}z^2 = \frac{1}{1.2}y^2 + \frac{2}{3.4}y^4 + \frac{2.4}{3.5.6}y^6 + \frac{2.4.6}{3.5.7.8}y^8 + \ldots + \frac{2.4.6\ldots 2n}{3.5.7\ldots(2n+1)(2n+2)}y^{2n+2}$ + etc.

Hae aequationes exprimunt quadratum arcus per ſinum. Eſt autem $2z^2 = \frac{1}{2}x^2 = \frac{1}{1.2}2^2y^2$

$+ \frac{1}{2.3.4}2^4y^4 + \frac{1.2}{3.4.5.6}2^6y^6 + \frac{1.2.3}{4.5.6.7.8}2^8y^8 \ldots + \frac{1.2.3\ldots n}{(n+1)(n+2)\ldots(2n+2)}2^{n+2}y^{2n+2} + \ldots$

atque ſin verſ. $x = 2 \sin^2 \frac{1}{2}x$, ſiue $2v = 2^2y^2$, ergo $\frac{1}{2}x^2 = \frac{1}{1.2}.2v + \frac{1}{2.3.4}2^2v^2 + \frac{1.2}{3.4.5.6}2^3v^3$

$+ \frac{1.2.3}{4.5.6.7.8}2^4v^4 + \ldots\ldots\ldots + \frac{1.2.3\ldots n}{(n+1)(n+2)\ldots(2n+2)}2^{n+1}v^{n+1} \ldots\ldots\ldots$

Aequatio haec eſt M (*Theor. der Dimenſionsz. II. Th.* §. 327. *S.* 129.) atque apparet, ad eam demonſtrandam non opus eſſe calculo *implicato*, vti *Fiſcherus* credidit.

Plura huc pertinentia propoſui in theſibus ad finem huius libelli.

(1) *Scalae* ſcilicet p [*a*, *b*, *c*, *d* . . .] vel q [α, β, γ, δ . . .] de qua §. I. expoſui. Series hic p et q *ſolis* cenſentur *coefficientibus* *a*, *b*, *c*, *d*. . . vel α, β, γ, δ. . . hoc eſt, non variabilium, aut ipſarum exponentium (dummodo in arithmetica progreſſione ſequantur) determinatione; quae *ſingula* poſſunt, in una adeo eademque propoſitione, *ſimul diverſa* cogitari.

primens, eaque legitima demonstratione munitur, exemplis illustratur. Formula denique regressoria, quam *Eschenbachius V. Cl.* duabus seriebus sibi aequalibus accomodauit, signorum combinatoriorum et localium compendio in §. IX. simplicius et magis perspicue expressa, agitur vltimo loco, sed breuissime §. X, de *Leibnitiano* ad serierum reversionem pertinente problemate complicatissimo, in quo soluendo coefficientes seriei fictae, tam per terminos reccurrentes, quam per solos datos determinantur. Methodus enim combinatorio-analytica pari vtilitate adhibetur, siue serierum terminos quoslibet extra ordinem (in quo fere dominatur sola) ab antecedentibus independenter, siue sequentes, per regressum ad antecedentes, ex ordine quaeras. Caeterum literae in calculo adhibitae, valores omnes, positiuos et negatiuos, integros et fractos, induere, atque etiam nihilo aequari, possunt, nisi expressa caueatur restrictione (vt §. V.) aut res ipsa loquatur. Sic v. c. litterae, r in §. III et IV, m et 'm in §. VI, et quae serierum terminos saepissime numerat, n, non nisi valores integros positiuos admittere possunt.

Plura quae ad calculum combinatorio-analyticum pertinent, fusius repetere hic non potui, ideoque saepenumero, ad duos *Hindenburgii Celeberrimi* libros provocaui, hac tamen inita ratione, vt etiam ii, qui ducibus istis summis destituti, calculum istum nondum callent, res ipsas hic propositas, earumque demonstrationes intelligere possint. Quodsi mihi contigerit esse tam felici, vt hunc libellum a viris doctis non reprobatum viderim, et quidquam vtilitatis attulerim *Methodo analytico-combinatoriae*, quae maior est, quam vt iustis eam hic praedicare possim laudibus, maximam me felicitatem nactum esse nullus dubito.

Lipsiae d. XXXI. Aug.
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§. I.

§. I.

Si ponitur Series $ax^{s} + bx^{s+d} + cx^{s+2d} \ldots\ldots = p$, erit $(ax^{s} + bx^{s+d} + cx^{s+2d} \ldots)^{m}$ ſiue $p^{m} = p^{m}\daleth_{1} + p^{m}\daleth_{2} + p^{m}\daleth_{3} \ldots\ldots + p^{m}\daleth_{(n+1)} \ldots\ldots$ ſive etiam $p^{m} = p^{m}\varkappa_{1}x^{sm} + p^{m}\varkappa_{2}x^{sm+d} + p^{m}\varkappa_{3}x^{sm+2d} \ldots + p^{m}\varkappa_{(n+1)}\,x^{sm+nd} \ldots\ldots$ (a).

Hae, *Terminorum integrorum*, $p^{m}\daleth_{1}$, $p^{m}\daleth_{2} \ldots p^{m}\daleth_{(n+1)}$, vel *Coefficientium* tantummodo, $p^{m}\varkappa_{1}$, $p^{m}\varkappa_{2}$, $\ldots\; p^{m}\varkappa_{(n+1)}$ ſeriei p^{m} notae, (quarum inter ſe relatio formula generali $p^{m}\daleth_{(n+1)} = p^{m}\varkappa_{(n+1)}x^{sm+nd}$ exprimitur) *Signa Localia* (*Lokalzeichen, Lokalausdrücke*) vocantur, vſusque eorum in Analyſi, inprimis combinatoria, longe maximus eſt.

Signorum Localium $p^{m}\daleth_{1}$, $p^{m}\daleth_{2} \ldots p^{m}\daleth_{(n+1)}$ *valores*, a ſeriei p, non ſolum *Coefficientibus*, ſed etiam *Exponentibus*, ipſiusque *Variabilis* determinatione pendent. Indicandi itaque ſunt, quoties iis vtimur, ſeriei p coefficientes $a, b, c, d, \ldots$ exponentes s, d, litteraque x, variabilem denotans; quod commodius vix effici poterit, quam ſi ſeries ipſa $ax^{s} + bx^{s+d} + cx^{s+2d} \ldots = p$ apponitur.

Valores autem ſignorum localium $p^{m}\varkappa_{1}$, $p^{m}\varkappa_{2}$, $\ldots\; p^{m}\varkappa_{(n+1)}$, de quibus hic potiſſimum agitur, ſeriei p ſolummodo *Coefficientibus* $a, b, c, d, \ldots$ determinantur. Signum itaque hoc: $p\,[a,\ b,\ c,\ d,\ \ldots]$ ſive etiam $p\left[\begin{smallmatrix}1 & 2 & 3 & 4 \ldots \\ a, & b, & c, & d \ldots\end{smallmatrix}\right]$ (praeſertim ubi loco $a, b, c, d \ldots$ coefficientes occurrunt magis compoſiti et complicati) quod *Scalam* appellabo, indicat, Seriei p coefficientes eſſe ex ordine, primum a, ſecundum b, tertium c, quartum $d \ldots$; idque ſemper addam, quotiescunque ſignis $p^{m}\varkappa_{1}$, $p^{m}\varkappa_{2} \ldots$ vtar. Qua re efficitur, vt ſigna haec localia non ſolum ſtatim intelligi, ſed etiam ipſorum valores, ſi opus eſt, ſignis et notis vulgaribus exhiberi poſſint; cui rei hae inſeruiunt *formulae combinatorio - analyticae:*

$$p^{m}\varkappa_{1} = a^{m}$$

$$p^{m}\varkappa_{2} = {}^{m}\mathfrak{A}a^{m-1}\,\mathfrak{a}^{1}A$$

$$p^{m}\varkappa_{3} = {}^{m}\mathfrak{A}a^{m-1}\,\mathfrak{a}^{2}A + {}^{m}\mathfrak{B}a^{m-2}\,\mathfrak{b}^{2}B$$

$$p^{m}\varkappa_{4} = {}^{m}\mathfrak{A}a^{m-1}\,\mathfrak{a}^{3}A + {}^{m}\mathfrak{B}a^{m-2}\,\mathfrak{b}^{3}B + {}^{m}\mathfrak{C}a^{m-3}\,\mathfrak{c}^{3}C$$

$$\text{〃} \qquad \text{〃} \qquad \text{〃} \qquad \text{〃}$$

$$p^{m}\varkappa_{(n+1)} = {}^{m}\mathfrak{A}a^{m-1}\,\mathfrak{a}^{n}A + {}^{m}\mathfrak{B}a^{m-2}\,\mathfrak{b}^{n}B + {}^{m}\mathfrak{C}a^{m-3}\,\mathfrak{c}^{n}C \ldots + {}^{m}\mathfrak{N}a^{m-n}\,\mathfrak{n}^{n}N$$

vbi

(a) *Noui Syſtematis Permutationum, Combinationum ac Variationum primae lineae.* p. XXXIII.

A

vbi *Classibus Combinatoriis* $\mathfrak{a}^n A$, $\mathfrak{b}^n B$, $\mathfrak{c}^n C$ .... respondet *Index* $\begin{pmatrix} b, & c, & d, & e. & .. \\ 1, & 2, & 3, & 4 & ... \end{pmatrix}$(b). Facillimum itaque est, valorem cuiusvis signorum localium $p^m \varkappa 1$, $p^m \varkappa 2$, . . . per scalam appositam, et, qui ab ea pendet, indicem, exhibere. Sic v. c. posito $n = 4$ erit $p^m \varkappa 5 = m a^{m-1} e + \frac{m.\, m-1}{1.\ \ 2.} a^{m-2}(2bd + c^2) + \frac{m.\, m-1.\, m-2}{1.\ \ 2.\ \ 3.} a^{m-3}\, 3b^2 c + \frac{m.\, m-1.\, m-2.\, m-3}{1.\ \ 2.\ \ 3.\ \ 4.} a^{m-4} b^4.$

Alia scala sit: $q\,[\alpha, \beta, \gamma, \delta, \varepsilon \ldots]$ et index $\begin{pmatrix} \beta, & \gamma, & \delta, & \varepsilon. & ... \\ 1, & 2, & 3, & 4. & ... \end{pmatrix}$ atque erit $q^{\frac{f}{g}} \varkappa 4 = \frac{f}{g} \alpha^{\frac{f}{g}-1} \delta + \frac{f.\, f-g}{1.\, 2\ \ g^2} \alpha^{\frac{f}{g}-2}\, 2\beta\gamma + \frac{f.\, f-g.\, f-2g}{1.\ \ 2.\ \ 3.\ \ g^3} \alpha^{\frac{f}{g}-3} \beta^3.$

§. II.

Duae sint scalae: $q\,[\alpha, \beta, \gamma, \delta \ldots]$ et $p\,[q^f \varkappa 1, q^f \varkappa 2, q^f \varkappa 3 \ldots]$, atque erit $p^l \varkappa (n+1) = q^{lf} \varkappa (n+1)$.

Demonstratio.

$(\alpha x + \beta x^2 + \gamma x^3 \ldots)^f = q^f \varkappa 1 x^f + q^f \varkappa 2 x^{f+1} + q^f \varkappa 3 x^{f+2}$ etc. . . . . (§. I.) $= \Sigma$, et $(q^f \varkappa 1 x^f + q^f \varkappa 2 x^{f+1} + q^f \varkappa 3 x^{f+2} \ldots)^l = p^l \varkappa 1 x^{lf} + p^l \varkappa 2 x^{lf+1} + p^l \varkappa 3 x^{lf+2} \ldots + p^l \varkappa (n+1) x^{lf+n} \ldots = \Sigma^l = (\alpha x + \beta x^2 + \gamma x^3 \ldots)^{lf} = q^{lf} \varkappa 1 x^{lf} + q^{lf} \varkappa 2 x^{lf+1} + q^{lf} \varkappa 3 x^{lf+2} \ldots + q^{lf} \varkappa (n+1) x^{lf+n} \ldots$ unde sequitur $p^l \varkappa (n+1) = q^{lf} \varkappa (n+1)$.

§. III.

(b) *Nou. Syst. Perm.* §. VI, 7, p. LIV, et *Infin. Dign.* §. XXV, 5, p. 119. $^m\mathfrak{A}$, $^m\mathfrak{B}$, $^m\mathfrak{C}$ ... $^m\mathfrak{N}$, *coefficientes* designant *binomiales* dignitatis mtae, *primum*, *secundum*, *tertium*. . . *ntum*. *Nou. Syst. Perm.* p. XL. Signa $\mathfrak{a}^n A$, $\mathfrak{b}^n B$, $\mathfrak{c}^n C$ . . . . . $\mathfrak{n}^n N$, ad *Indicem* definitum, iis semper adscribendum, qui hic $\begin{pmatrix} b, & c, & d, & e & ... \\ 1, & 2, & 3, & 4 & ... \end{pmatrix}$ est, referenda, indicant, numerum n *discerpendum esse* modis omnibus possibilibus, in *unam*, *duas*, *tres*. . . . *n* partes, sic, vt numeri vel partes minores praecedant, pares vel maiores sequantur; in qualibet autem discerptione ponendas esse, loco numerorum 1, 2, 3, 4 . . . quae numeris istis in Indice respondent, literas *b*, *c*, *d*, *e* . . . et quamlibet denique inde ortam complexionem ducendam esse in *Numerum Permutationum*, qui nimirum indicat, quot variis modis literae, quibus complexio composita est, possint transponi. Sic, pro *complexione generali* $a^\alpha b^\beta c^\gamma d^\delta \ldots r^\varrho$, est $\frac{\nu.\, \nu-1.\, \nu-2.\, \nu-3.\, \nu-4 \ldots 4.\, 3.\, 2.\, 1.}{1.2.3\ldots\alpha.\, 1.2.3\ldots\beta.\, 1.2.3\ldots\gamma.\, 1.2.3\ldots\delta \text{ etc. } 1.2\ldots\varrho}$, vbi $\alpha, \beta, \gamma, \delta \ldots \varrho$. numeri sunt integri positiui, atque $\nu = \alpha + \beta + \gamma + \delta \ldots + \varrho$. *Infin. Dign.* §. XIII. p. 32. *Nov. Syst. Perm.* §. III, 23, p. XXIV. Signum, exempli causa, $\mathfrak{c}^9 C$ hic monet: 1) numerum 9 discerpendum esse in *tres* numeros siue partes omnibus modis possibilibus, sic, vt numeri minores praecedant, pares vero aut maiores sequantur, qui sunt: 1, 1, 7; 1, 2, 6; 1, 3, 5; 1, 4, 4; 2, 2, 5; 2, 3, 4; 3, 3, 3; numero septem 2) in qualibet discerptione

ptione

§. III.

$$s a p^m \varkappa(n+1) + (s+d) b p^m \varkappa n + (s+2d) c p^m \varkappa(n-1) \ldots\ldots + (s+rd)\overset{r}{a} p^m \varkappa(n-r+1) \ldots\ldots$$

$$\ldots + (s+nd)\overset{n}{a} p^m \varkappa 1 = \frac{s(m+1)+nd}{m+1}\, p^{m+1}\varkappa(n+1)$$

$$p\,[a,\ b,\ c,\ d,\ \ldots]$$ (c)

Demonstratio.

$(ax^s + bx^{s+d} + cx^{s+2d} \ldots\ldots + \overset{r}{a}x^{s+rd} + \ldots\ldots + \overset{n}{a}x^{s+nd} \ldots\ldots)^{m+1} = p^{m+1}\varkappa 1 x^{s(m+1)} + p^{m+1}\varkappa 2 x^{s(m+1)+d} + p^{m+1}\varkappa 3 x^{s(m+1)+2d} \ldots\ldots + p^{m+1}\varkappa(n+1) x^{s(m+1)+nd}$, et ſumtis differentialibus:

$$(m+1)(ax^s + bx^{s+d} + cx^{s+2d} \ldots)^m [sax^{s-1} + (s+d)bx^{s+d-1} + (s+2d)cx^{s+2d-1} \ldots + (s+rd)\overset{r}{a}x^{s+rd-1}$$

$$\ldots + (s+nd)\overset{n}{a}x^{s+nd-1} \ldots\ldots]\,dx = [s(m+1)p^{m+1}\varkappa 1 x^{s(m+1)-1} + (s(m+1)+d)p^{m+1}\varkappa 2\, x^{s(m+1)+d-1}$$

$$+ (s(m+1)+2d)p^{m+1}\varkappa 3 x^{s(m+1)+2d-1} \ldots\ldots + (s(m+1)+nd)\, p^{m+1}\varkappa(n+1) x^{s(m+1)+nd-1} + \ldots]\,dx$$

ergo $[p^m\varkappa 1 x^{sm} + p^m\varkappa 2 x^{sm+d} \ldots + p^m\varkappa(n-r+1)\, x^{sm+(n-r)d} + \ldots + p^m\varkappa(n+1) x^{sm+nd} \ldots]$

$$[sax^{s-1} + (s+d)bx^{s+d-1} \ldots\ldots + (s+rd)\overset{r}{a}x^{s+rd-1} \ldots\ldots + (s+nd)\overset{n}{a}x^{s+nd-1} \ldots]$$

$$= \frac{s(m+1)}{m+1}p^{m+1}\varkappa 1\, x^{s(m+1)-1} + \frac{(s(m+1)+d)}{m+1}\, p^{m+1}\varkappa 2 x^{s(m+1)+d-1} + \ldots\ldots\ldots\ldots$$

$\ldots + \frac{(s(m+1)+nd)}{m+1}\, p^{m+1}\varkappa(n+1)\, x^{s(m+1)+nd-1} \ldots$ et facta multiplicatione

ptione loco numerorum 1, 2, 3, 4... ſubſtituendas eſſe reſpectiue literas *b*, *c*, *d*, *e*...., qua ſubſtitutione has nanciſcimur complexiones, *bbb*; *bcg*; *bdf*; *bee*; *ccf*; *cde*; *ddd*; 3) harum denique complexionum quamlibet ducendam eſſe in numerum permutationum, per formulam modo traditam, definitum, vnde fit $\mathfrak{c}^9\mathrm{C} = 3b^2b + 6bcg + 6bdf + 3be^2 + 3c^2f + 6cde + d^3$. Quomodo diſcerptiones omnes ſine errandi periculo poſſint exhiberi, docuit Excell. Hindenburgius, *Infin. Dign.* §. XXII. p. 73 ſeq. et excerpſit Clar. Eſchenbachius in Diſſert. *de Serierum Reuerſione* p. 16 — 20.

In fine huius libelli duas exhibui Tabulas, quarum prior diſcerptiones numerorum 1 — 10, poſterior valores ſignorum $\mathfrak{a}^1\mathrm{A}$, $\mathfrak{a}^2\mathrm{A} \ldots \mathfrak{a}^{10}\mathrm{A}$; $\mathfrak{b}^2\mathrm{B}$, $\mathfrak{b}^3\mathrm{B} \ldots \mathfrak{b}^{10}\mathrm{B}$ etc. ad Indicem $\begin{pmatrix} \beta, & \gamma, & \delta \ldots \\ 1, & 2, & 3 \ldots \end{pmatrix}$ continet. Tabula poſterior, e priori, ſic vt modo memoraui, formata adhiberi etiam poteſt, ſi Index eſt alius, quam $\begin{pmatrix} \beta, & \gamma, & \delta \ldots \\ 1, & 2, & 3 \ldots \end{pmatrix}$ v. c. $\begin{pmatrix} p, & q, & r \ldots \\ 1, & 2, & 3 \ldots \end{pmatrix}$ ſcribendo nimirum loco literarum β, γ, δ ...., literas p, q, r... reſpectiue, eas ſcilicet, quae in Indice numeris 1, 2, 3... ſuo loco ſingulis, reſpondent.

(c) $\overset{r}{a}$, $\overset{n}{a}$, ſignificat, litterarum *a*, *b*, *c*, *d*... (r + 1) tam, (n + 1) tam, quae ſcilicet r aut n locis a prima *a*, remota eſt. *Nov. Syſtem. Perm.* p. XXXVII. Eſt itaque hic littera r ſiue n *diſtantiae exponens*. Conf. *Ibid.* p. LXV, 8, 9. Interdum *diſtantiae exponentes* occurrunt *negatiui*.

$$sap^m\kappa 1\, x^{s(m+1)-1} + sap^m\kappa 2\, x^{s(m+1)+d-1} \ldots\ldots + sap^m\kappa(n+1)\, x^{s(m+1)+nd-1}$$
$$+ (s+d)\, bp^m\kappa 1 x^{s(m+1)+d-1} \ldots\ldots + (s+d)\, bp^m\kappa n x^{s(m+1)+nd-1}$$
$$\vdots$$
$$+ (s+rd)\, \overset{r}{a} p^m\kappa(n-r+1) x^{s(m+1)+nd-1}$$
$$\vdots$$
$$+ (s+nd)\, \overset{n}{a} p^m\kappa 1 x^{s(m+1)+nd-1}$$
$$= \frac{s(m+1)}{m+1} p^{m+1}\kappa 1 x^{s(m+1)-1} + \frac{(s(m+1)+d)}{m+1} p^{m+1}\kappa 2 x^{s(m+1)+d-1} \ldots + \frac{(s(m+1)+nd)}{m+1} p^{m+1}\kappa(n+1) x^{s(m+1)+nd-1}$$

ergo $sap^m\kappa(n+1) + (s+d) bp^m\kappa n + (s+2d) cp^m\kappa(n-1) \ldots + (s+rd)\overset{r}{a} p^m\kappa(n-r+1) \ldots$

$$\ldots + (s+nd)\overset{n}{a} p^m\kappa 1 = \frac{s(m+1)+nd}{m+1} p^{m+1}\kappa(n+1) \quad \text{Q. E. D.}$$

Coroll. I. Sit $s = -n$, et $d = m+1$ atque est

$$-nap^m\kappa(n+1) + (m-n+1) bp^m\kappa n + (2m-n+2) cp^m\kappa(n-1) \ldots + (rm-n+r)\overset{r}{a} p^m\kappa(n-r+1) \ldots$$
$$\ldots + nm\,\overset{n}{a} p^m\kappa 1 = \frac{-n(m+1)+n(m+1)}{m+1} p^{m+1}\kappa(n+1) = 0.$$

ergo

$$\frac{(m-n+1) bp^m\kappa n + (2m-n+2) cp^m\kappa(n-1) \ldots + (rm-n+r)\overset{r}{a} p^m\kappa(n-r+1) \ldots + mn\overset{n}{a} p^m\kappa 1}{na} = p^m\kappa(n+1)$$

Haec formula continet regulam *Illustris Kaestneri*, *Viri Celeberrimi*, pro Infinitinomii Dignitatibus, qua inveniuntur successiue coefficientium $p^m\kappa 2$, $p^m\kappa 3$, $p^m\kappa 4$ etc. sequentes, per omnes antecedentes (d).

Coroll. II. $\dfrac{p^{m+1}\kappa(n+1)}{m+1} = a^m a^n A + \underset{2}{{}^m\mathfrak{A}} a^{m-1} b^n B + \underset{3}{{}^m\mathfrak{B}} a^{m-2} c^n C \ldots + \underset{n}{{}^m\overset{-1}{\mathfrak{N}}} a^{m-n+1} n^n N$ (e)

$p[a, b, c, d, e \ldots]$ $\binom{b,\ c,\ d,\ e \ldots}{1\ \ 2\ \ 3\ \ 4 \ldots}$

Ergo

(d) *Kaestners Analysis des Unendlichen* §. 56. XI. Si enim loco

n, a, b, c, $d \ldots \overset{n}{a}$ $\overset{n+1}{a}$; $p^m\kappa 1$, $p^m\kappa 2$, $p^m\kappa 3 \ldots p^m\kappa(n+1)$, $p^m\kappa(n+2)$ in nostra formula, scribitur $n+1$, 1, $\alpha, \beta, \gamma \ldots \lambda n$, $\lambda(n+1)$, 1, A, B, ... Ln, $L(n+1)$, formula prodit Kaestneriana.

(e) De signorum $p^m\kappa n$ valoribus §. I. Est autem ${}^{m+1}\mathfrak{A} = (m+1)$; ${}^{m+1}\mathfrak{B} = (m+1)\,\underset{2}{{}^m\mathfrak{A}}$; ${}^{m+1}\mathfrak{C} = (m+1)\,\underset{3}{{}^m\mathfrak{B}}$; ... ${}^{m+1}\mathfrak{N} = (m+1)\,\underset{n}{{}^m\overset{-1}{\mathfrak{N}}}$. Scilicet $\overset{-1}{\mathfrak{N}}$, litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ ... designat (n−1)tam. Hinc *Coefficientes binomiales* ${}^m\overset{+r}{\mathfrak{N}}$ et ${}^m\overset{-r}{\mathfrak{N}}$, qui sint et quid significent, facile intelligitur. Nempe *distantiae exponentes* etiam hic, vt alibi, vtiliter adhibentur.

Ergo

$$s a p^m \varkappa(n+1) + {}_{(s+d)} b p^m \varkappa n + {}_{(s+2d)} c p^m \varkappa(n-1) \ldots\ldots + {}_{(s+nd)} \overset{n}{a} p^m \varkappa 1$$

$$= {}_{(s(m+1)+nd)} \left(a^m \mathfrak{a}^n A + {}^m \underset{\overline{2}}{\mathfrak{A}} a^{m-1} \mathfrak{b}^n B + {}^m \underset{\overline{3}}{\mathfrak{B}} a^{m-2} \mathfrak{c}^n C \ldots + {}^m \underset{\overline{n}}{\overset{-1}{\mathfrak{N}}} a^{m-n+1} \mathfrak{n}^n N \right)$$

Hac in formula ponatur $m = -1$, et eſt

$$s a p^{-1} \varkappa(n+1) + {}_{(s+d)} b p^{-1} \varkappa n + {}_{(s+2d)} c p^{-1} \varkappa(n-1) \ldots\ldots + {}_{(s+nd)} \overset{n}{a} p^{-1} \varkappa 1$$

$$= nd \left(\frac{\mathfrak{a}^n A}{a} - \frac{\mathfrak{b}^n B}{2a^2} + \frac{\mathfrak{c}^n C}{3a^3} \ldots\ldots \pm \frac{\mathfrak{n}^n N}{na^n} \right)$$

$$\left(\begin{matrix} b, & c, & d, & e, & \ldots\ldots \\ 1, & 2, & 3, & 4, & \ldots\ldots \end{matrix} \right)$$

Caſum hunc, ſi $m = -1$, ſingulariter ponderandum eſſe credidi, quoniam primo intuitu formula hac §pho propoſita ei applicari non poſſe videatur.

§. IV.

$$s q^f \varkappa 1 \; q^g \varkappa(n+1) + {}_{(s+d)} q^f \varkappa 2 \, q^g \varkappa n + {}_{(s+2d)} q^f \varkappa 3 \; q^g \varkappa(n-1) \ldots + {}_{(s+rd)} q^f \varkappa(r+1) \; q^g \varkappa(n-r+1) \ldots$$

$$\ldots + {}_{(s+nd)} q^f \varkappa(n+1) \; q^g \varkappa 1 = \frac{s(f+g)+ndf}{f+g} . q^{f+g} \varkappa(n+1)$$

$$q \, [\alpha, \; \beta, \; \gamma, \; \delta \ldots]$$

Demonſtratio.

Scala ſit $p\,[a, b, c, d \ldots]$, et eſt (§. III.)

$$s a p^m \varkappa(n+1) + {}_{(s+d)} b p^m \varkappa n + {}_{(s+2d)} c p^m \varkappa(n-1) \ldots + {}_{(s+rd)} \overset{r}{a} p^m \varkappa(n-r+1) + \ldots + {}_{(s+nd)} \overset{n}{a} p^m \varkappa 1$$

$$= \frac{s(m+1)+nd}{m+1} p^{m+1} \varkappa(n+1)$$

Introducatur alia ſcala $q\,[\alpha, \beta, \gamma, \delta, \ldots]$ et ſit $a = q^f \varkappa 1$, $b = q^f \varkappa 2$, $c = q^f \varkappa 3$, etc. atque erit ſcala $p\,[q^f \varkappa 1, \; q^f \varkappa 2, \; q^f \varkappa 3, \ldots]$ ergo (§. II.)

$$s q^f \varkappa 1 \; q^{fm} \varkappa(n+1) + {}_{(s+d)} q^f \varkappa 2 \; q^{fm} \varkappa n + {}_{(s+2d)} q^f \varkappa 3 \; q^{fm} \varkappa(n-1) \ldots + {}_{(s+rd)} q^f \varkappa(r+1) \; q^{fm} \varkappa(n-r+1) \ldots$$

$$\ldots + {}_{(s+nd)} q^f \varkappa(n+1) \, q^{fm} \varkappa 1 = \frac{s(m+1)+nd}{m+1} . q^{fm+f} \varkappa(n+1);$$ et, ſi ponitur $m = \frac{g}{f}$, prodit

$$s q^f \varkappa 1 \; q^g \varkappa(n+1) + {}_{(s+d)} q^f \varkappa 2 \, q^g \varkappa n + {}_{(s+2d)} q^f \varkappa 3 \; q^g \varkappa(n-1) \ldots + {}_{(s+rd)} q^f \varkappa(r+1) \, q^g \varkappa(n-r+1) \ldots$$

$$\ldots + {}_{(s+nd)} q^f \varkappa(n+1) \; q^g \varkappa 1 = \frac{s\left(\frac{g}{f}+1\right)+nd}{\frac{g}{f}+1} . q^{f+g} \varkappa(n+1) = \frac{s(f+g)+ndf}{f+g} q^{f+g} \varkappa(n+1) \quad \text{Q. E. D.}$$

Coroll.

$$\text{Coroll. } \frac{q^{f+g}\varkappa(n+1)}{f+g} = \alpha^{f+g-1}\mathfrak{a}^n A + {}^{f+g-1}\underset{2}{\mathfrak{A}}\alpha^{f+g-2}\mathfrak{b}^n B + {}^{f+g-1}\underset{3}{\mathfrak{B}}\alpha^{f+g-3}\mathfrak{c}^n C$$

$$+ {}^{f+g-1}\underset{4}{\mathfrak{C}}\alpha^{f+g-4}\mathfrak{d}^n D \ldots\ldots {}^{f+g-1}\underset{n}{\overset{n}{\overset{-1}{\mathfrak{N}}}}\alpha^{f+g-n}\mathfrak{n}^n N$$ (vid. §. I. et nota e.)

Ergo

$$s q^f\varkappa 1\, q^g\varkappa(n+1) + (s+d)q^f\varkappa 2\, q^g\varkappa n + (s+2d)q^f\varkappa 3\, q^g\varkappa(n-1) \ldots\ldots + (s+nd)q^f\varkappa(n+1)\, q^g\varkappa 1$$

$$= [s(f+g)+ndf]\left[\frac{\mathfrak{a}^n A}{\alpha} + \frac{{}^{f+g-1}\mathfrak{A}\mathfrak{b}^n B}{2\alpha^2} + \frac{{}^{f+g-1}\mathfrak{B}\mathfrak{c}^n C}{3\alpha^3} \ldots\ldots + \frac{{}^{f+g-1}\overset{-1}{\mathfrak{N}}\mathfrak{n}^n N}{n\alpha^n}\right]\alpha^{f+g}$$

si itaque $g = -f$, vel $f+g = 0$, est

$$s q^f\varkappa 1\, q^{-f}\varkappa(n+1) + (s+d)q^f\varkappa 2\, q^{-f}\varkappa n + (s+2d)q^f\varkappa 3\, q^{-f}\varkappa(n-1) \ldots\ldots + (s+nd)q^f\varkappa(n+1)q^{-f}\varkappa 1$$

$$= ndf\left[\frac{\mathfrak{a}^n A}{\alpha} - \frac{\mathfrak{b}^n B}{2\alpha^2} + \frac{\mathfrak{c}^n C}{3\alpha^3} - \frac{\mathfrak{d}^n D}{4\alpha^4} \ldots\ldots \pm \frac{\mathfrak{n}^n N}{n\alpha^n}\right]$$

$$\left(\begin{matrix}\beta, & \gamma, & \delta, & \varepsilon \ldots\ldots \\ 1, & 2, & 3, & 4 \ldots\ldots\end{matrix}\right)$$

Haec sufficiunt, ad formulae Eschenbachianae demonstrationem primam exhibendam.

§. V.

Problema.

Sit data aequatio: $y^l = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} + \delta x^{r+3d} + \ldots + \overset{n}{\alpha} x^{r+nd} \ldots$ exprimere x^s per seriem, cuius membra secundum dignitates variabilis y progrediuntur, ubi l, r, d, s numeros significare possunt omnes, pro quibus theorema binomiale valet, hac tamen unica restrictione addita, sit neque α neque $r = 0$.

Solutio.

Forma seriei quaesitae haec est: $x^s = \dot{A}y^{\frac{ls}{r}} + \dot{B}y^{\frac{l(s+d)}{r}} + \dot{C}y^{\frac{l(s+2d)}{r}} + \dot{D}y^{\frac{l(s+3d)}{r}} + \dot{E}y^{\frac{l(s+4d)}{r}} \ldots$
$\ldots + \overset{n}{\dot{A}}y^{\frac{l(s+nd)}{r}}$ (f)

Scalae

(f) *Coefficientes ficti* punctis notantur supra positis. *Nou. System. Perm.* p. XXXIV. De *forma* seriei fictae cf. *Kaestneri Analysis endlicher Groessen.* §. 690. Si enim $x = gy^\lambda + hy^{\lambda+\delta} \ldots$ aequatio data est, forma seriei quaesitae haec est: $y = \dot{\mathfrak{A}}x^{\frac{1}{\lambda}} + \dot{\mathfrak{B}}x^{\frac{1+\delta}{\lambda}} + \dot{\mathfrak{C}}x^{\frac{1+2\delta}{\lambda}} \ldots$ scribatur loco x, g, y, λ, h, δ respectiue y^l, α, x, r, β, d, et erit $x = \dot{\mathfrak{A}}y^{\frac{l}{r}} + \dot{\mathfrak{B}}y^{\frac{l(1+d)}{r}} + \dot{\mathfrak{C}}y^{\frac{l(1+2d)}{r}} \ldots$ si data aequatio est $y^l = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} \ldots$ unde (sequitur $x^s = \dot{A}y^{\frac{ls}{r}} + \dot{B}y^{\frac{l(s+d)}{r}} + \dot{C}y^{\frac{l(s+2d)}{r}} \ldots\ldots$

Scalae sint

$$q[\alpha, \beta, \gamma, \delta, \ldots\ldots] \text{ et } p = [\dot{A}, \dot{B}, \dot{C}, \dot{D}, \ldots\ldots]$$

Cum sit $y^t = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} + \delta x^{r+3d} \ldots + \overset{n}{\alpha} x^{r+nd}, \ldots\ldots$, erit

$$y^{\frac{ts}{r}} = q^{\frac{s}{r}} \varkappa 1 x^s + q^{\frac{s}{r}} \varkappa 2 x^{s+d} + q^{\frac{s}{r}} \varkappa 3 x^{s+2d} + q^{\frac{s}{r}} \varkappa 4 x^{s+3d} \ldots + q^{\frac{s}{r}} \varkappa(n+1) x^{s+nd} \ldots (\Gamma$$

Et quoniam

$$x^s = \dot{A} y^{\frac{ts}{r}} + \dot{B} y^{\frac{t(s+d)}{r}} + \dot{C} y^{\frac{t(s+2d)}{r}} + \dot{D} y^{\frac{t(s+3d)}{r}} + \ldots + \overset{n}{\dot{A}} y^{\frac{t(s+nd)}{r}} \ldots (\Delta;$$

est etiam

$$x^{s+d} = + p^{\frac{s+d}{s}} \varkappa 1 y^{\frac{t(s+d)}{r}} + p^{\frac{s+d}{s}} \varkappa 2 y^{\frac{t(s+2d)}{r}} + p^{\frac{s+d}{s}} \varkappa 3 y^{\frac{t(s+3d)}{r}} + \ldots + p^{\frac{s+d}{s}} \varkappa n y^{\frac{t(s+nd)}{r}} \ldots\ldots$$

$$x^{s+2d} = + p^{\frac{s+2d}{s}} \varkappa 1 y^{\frac{t(s+2d)}{r}} + p^{\frac{s+2d}{s}} \varkappa 2 y^{\frac{t(s+3d)}{r}} + \ldots + p^{\frac{s+2d}{s}} \varkappa(n-1) y^{\frac{t(s+nd)}{r}} \ldots\ldots$$

$$x^{s+3d} = + p^{\frac{s+3d}{s}} \varkappa 1 y^{\frac{t(s+3d)}{r}} + \ldots + p^{\frac{s+3d}{s}} \varkappa(n-2) y^{\frac{t(s+nd)}{r}} \ldots\ldots$$

$$\vdots$$

$$x^{s+md} = + p^{\frac{s+md}{s}} \varkappa 1 y^{\frac{t(s+md)}{r}} \ldots + p^{\frac{s+md}{s}} \varkappa(n-m+1) y^{\frac{t(s+nd)}{r}} \ldots\ldots$$

$$\vdots$$

$$x^{s+(n-1)d} = + p^{\frac{s+(n-1)d}{s}} \varkappa 1 y^{\frac{t(s+[n-1]d)}{r}} + p^{\frac{s+(n-1)d}{s}} \varkappa 2 y^{\frac{t(s+nd)}{r}} \ldots\ldots$$

$$x^{s+nd} = + p^{\frac{s+nd}{s}} \varkappa 1 y^{\frac{t(s+nd)}{r}} \ldots$$

Subſ

Substitutis his in aequatione Γ valoribus, hanc nanciscimur aequationem:

$$0 = \left.\begin{matrix}\dot{A} q^{\frac{s}{r}} \varkappa_1 \\ -1\end{matrix}\right\} y^{\frac{1s}{r}} + \left.\begin{matrix} \dot{B} q^{\frac{s}{r}} \varkappa_1 \\ + p^{\frac{(s+d)}{s}} \varkappa_1 q^{\frac{s}{r}} \varkappa_2 \end{matrix}\right\} y^{\frac{1(s+d)}{r}} + \left.\begin{matrix} \dot{C} q^{\frac{s}{r}} \varkappa_1 \\ + p^{\frac{s+d}{s}} \varkappa_2 q^{\frac{s}{r}} \varkappa_2 \\ + p^{\frac{s+2d}{s}} \varkappa_1 q^{\frac{s}{r}} \varkappa_3 \end{matrix}\right\} y^{\frac{1(s+2d)}{r}} + \left.\begin{matrix} \dot{D} q^{\frac{s}{r}} \varkappa_1 \\ + p^{\frac{s+d}{s}} \varkappa_3 q^{\frac{s}{r}} \varkappa_2 \\ + p^{\frac{s+2d}{s}} \varkappa_2 q^{\frac{s}{r}} \varkappa_3 \\ + p^{\frac{s+3d}{s}} \varkappa_1 q^{\frac{s}{r}} \varkappa_4 \end{matrix}\right\} y^{\frac{1(s+3d)}{r}} \ldots + \ldots\ldots + \left.\begin{matrix} \overset{n}{\dot{A}} q^{\frac{s}{r}} \varkappa_1 \\ + \ldots + p^{\frac{s+d}{s}} \varkappa_n q^{\frac{s}{r}} \varkappa_2 \\ + \ldots + p^{\frac{s+2d}{s}} \varkappa_{(n-1)} q^{\frac{s}{r}} \varkappa_3 \\ + \ldots + p^{\frac{s+3d}{s}} \varkappa_{(n-2)} q^{\frac{s}{r}} \varkappa_4 \\ + p^{\frac{s+4d}{s}} \varkappa_{(n-3)} q^{\frac{s}{r}} \varkappa_5 \\ \vdots \\ + p^{\frac{s+md}{s}} \varkappa_{(n-m+1)} q^{\frac{s}{r}} \varkappa_{(m+1)} \\ \vdots \\ + p^{\frac{s+(n-1)d}{s}} \varkappa_2 q^{\frac{s}{r}} \varkappa_n \\ + p^{\frac{s+nd}{s}} \varkappa_1 q^{\frac{s}{r}} \varkappa_{(n+1)} \end{matrix}\right\} y^{\frac{1(s+nd)}{r}}$$

Est itaque $\dot{A} q^{\frac{s}{r}} \varkappa_1 - 1 = \dot{A} \alpha^{\frac{s}{r}} - 1 = 0$, et

$$\dot{A} = \alpha^{-\frac{s}{r}} = \frac{s}{s} q^{-\frac{s}{r}} \varkappa_1$$

$$\dot{B} q^{\frac{s}{r}} \varkappa_1 + p^{\frac{s+d}{s}} \varkappa_1 q^{\frac{s}{r}} \varkappa_2 = \dot{B} \alpha^{\frac{s}{r}} + \dot{A}^{\frac{s+d}{s}} \frac{s}{r} \alpha^{\frac{s}{r}-1} \beta = \dot{B} \alpha^{\frac{s}{r}} + \alpha^{-\frac{s+d}{r}} \frac{s}{r} \alpha^{\frac{s}{r}-1} \beta = 0 \text{ et}$$

$$\dot{B} = -\frac{s}{r} \alpha^{-\frac{s+d}{r}-1} \beta = \frac{s}{s+d} \cdot -\frac{s+d}{r} \alpha^{-\frac{s+d}{r}-1} \beta = \frac{s}{s+d} q^{-\frac{s+d}{r}} \varkappa_2$$

Substituantur hi valores in aequatione Δ, et est

$$x^s = \frac{s}{s} q^{-\frac{s}{r}} \varkappa_1 y^{\frac{1s}{r}} + \frac{s}{s+d} q^{-\frac{s+d}{r}} \varkappa_2 y^{\frac{1(s+d)}{r}} \ldots\ldots\ldots (\Delta$$

Valor

Valor coefficientis ficti $\dot{C}$ determinatur hac aequatione:

$\dot{C}q^{\frac{s}{r}}\varkappa 1 + p^{\frac{s+d}{s}}\varkappa 2\, q^{\frac{s}{r}}\varkappa 2 + p^{\frac{s+2d}{s}}\varkappa 1\, q^{\frac{s}{r}}\varkappa 3 = 0$. Eſt autem $p^{\frac{s+d}{s}}\varkappa 2$, ſeriei, x^{s+d} per y exprimentis, coefficiens ſecundus, ergo $= \frac{s+d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 2$; quod obtinemus, ſi in aequationis Δ coefficiente ſecundo, loco s ponatur $s+d$. Continet enim aequatio Δ ſeriem, x^s, per y exprimentem, et loco s, quilibet numerus, ſive integer ſive fractus, ſive poſitiuus ſive negatiuus, ergo etiam $s+d$ ſubſtitui poteſt (g). Eodem modo demonſtratur, eſſe $p^{\frac{s+2d}{s}}\varkappa 1 = \frac{s+2d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 1$; eſt enim $p^{\frac{s+2d}{s}}\varkappa 1$ ſeriei, x^{s+2d}, per y exprimentis, coefficiens primus, cuius valorem invenimus, ſi in aequationis Δ coefficiente primo, loco s ponitur $s+2d$.

Eſt itaque $\dot{C}q^{\frac{s}{r}}\varkappa 1 + \frac{s+d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 2\, q^{\frac{s}{r}}\varkappa 2 + \frac{s+2d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 1\, q^{\frac{s}{r}}\varkappa 3 = 0$, et cum (§. IV.)

$\frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3\, q^{\frac{s}{r}}\varkappa 1 + \frac{s+d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 2\, q^{\frac{s}{r}}\varkappa 2 + \frac{s+2d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 1\, q^{\frac{s}{r}}\varkappa 3 = 0$, etiam

$\dot{C}q^{\frac{s}{r}}\varkappa 1 - \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3\, q^{\frac{s}{r}}\varkappa 1 = 0$, ergo $\dot{C} = \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3$

et $x^s = \frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, y^{\frac{1s}{r}} + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, y^{\frac{1(s+d)}{r}} + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3\, y^{\frac{1(s+2d)}{r}} \ldots\ldots (\Delta$

Coefficiens $\dot{D}$ determinatur hac aequatione: $\dot{D}q^{\frac{s}{r}}\varkappa 1 + p^{\frac{s+d}{s}}\varkappa 3\, q^{\frac{s}{r}}\varkappa 2 + p^{\frac{s+2d}{s}}\varkappa 2\, q^{\frac{s}{r}}\varkappa 3$

$$+ p^{\frac{s+3d}{s}}\varkappa 1\, q^{\frac{s}{r}}\varkappa 4 = 0$$

Eſt

(g) Litteram s *numerum quemlibet* hic poſſe ſignificare, per ſe clarum eſt. Vera itaque eſt aequatio $x^{s+d} = \frac{s+d}{s+d}q^{-\frac{s+d}{r}}\varkappa 1 . y^{\frac{1(s+d)}{r}} + \frac{s+d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 2 . y^{\frac{1(s+2d)}{r}} \ldots$ Euehenda autem aequatione Δ ad dignitatem exponentis $\frac{s+d}{s}$, obtinemus hanc aequationem: $x^{s+d} = p^{\frac{s+d}{s}}\varkappa 1 . y^{\frac{1(s+d)}{r}} + p^{\frac{s+d}{s}}\varkappa 2 . y^{\frac{1(s+2d)}{s}} \ldots$ quae cum *eandem formam* habeat, quam antecedens, idemque exprimat, neceſſario in coefficientibus etiam, cum ea non poteſt non conuenire. Eſt itaque $p^{\frac{s+d}{s}}\varkappa 2 = \frac{s+d}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 2$

Est autem $p^{\frac{s+d}{s}}\varkappa 3 = \frac{s+d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 3$; $p^{\frac{s+2d}{s}}\varkappa 2 = \frac{s+2d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 2$; $p^{\frac{s+3d}{s}}\varkappa 1 = \frac{s+3d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 1$; ergo

$\dot{D} q^{\frac{s}{r}} \varkappa 1 + \frac{s+d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 3\, q^{\frac{s}{r}} \varkappa 2 + \frac{s+2d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 2\, q^{\frac{s}{r}} \varkappa 3 + \frac{s+3d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 1\, q^{\frac{s}{r}} \varkappa 4 = 0$ et cum (§. IV.)

$\frac{s}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 4\, q^{\frac{s}{r}} \varkappa 1 + \frac{s+d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 3\, q^{\frac{s}{r}} \varkappa 2 + \frac{s+2d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 2\, q^{\frac{s}{r}} \varkappa 3 + \frac{s+3d}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 1\, q^{\frac{s}{r}} \varkappa 4 = 0$

etiam $\dot{D} q^{\frac{s}{r}} \varkappa 1 - \frac{s}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 4\, q^{\frac{s}{r}} \varkappa 1 = 0$, unde sequitur $\dot{D} = \frac{s}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 4$, et

$x^s = \frac{s}{s} q^{-\frac{s}{r}} \varkappa 1\, y^{\frac{ls}{r}} + \frac{s}{s+d} q^{-\frac{s+d}{r}} \varkappa 2\, y^{\frac{l(s+d)}{r}} + \frac{s}{s+2d} q^{-\frac{s+2d}{r}} \varkappa 3\, y^{\frac{l(s+2d)}{r}} + \frac{s}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 4\, y^{\frac{l(s+3d)}{r}}, \ldots$ (Δ

Coefficiens $\dot{E}$ determinatur hac aequatione:

$$\dot{E} q^{\frac{s}{r}} \varkappa 1 + p^{\frac{s+d}{s}} \varkappa 4\, q^{\frac{s}{r}} \varkappa 2 + p^{\frac{s+2d}{s}} \varkappa 3\, q^{\frac{s}{r}} \varkappa 3 + p^{\frac{s+3d}{s}} \varkappa 2\, q^{\frac{s}{r}} \varkappa 4 + p^{\frac{s+4d}{s}} \varkappa 1\, q^{\frac{s}{r}} \varkappa 5 = 0$$

Est autem

$p^{\frac{s+d}{s}} \varkappa 4 = \frac{s+d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 4$; $p^{\frac{s+2d}{s}} \varkappa 3 = \frac{s+2d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 3$; $p^{\frac{s+3d}{s}} \varkappa 2 = \frac{s+3d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 2$; $p^{\frac{s+4d}{s}} \varkappa 1 = \frac{s+4d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 1$
ergo

$\dot{E} q^{\frac{s}{r}} \varkappa 1 + \frac{s+d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 4\, q^{\frac{s}{r}} \varkappa 2 + \frac{s+2d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 3\, q^{\frac{s}{r}} \varkappa 3 + \frac{s+3d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 2\, q^{\frac{s}{r}} \varkappa 4 + \frac{s+4d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 1\, q^{\frac{s}{r}} \varkappa 5 = 0$,
et cum (§. IV.)

$$\frac{s}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 5\, q^{\frac{s}{r}} \varkappa 1 + \frac{s+d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 4\, q^{\frac{s}{r}} \varkappa 2 + \frac{s+2d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 3\, q^{\frac{s}{r}} \varkappa 3 + \frac{s+3d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 2\, q^{\frac{s}{r}} \varkappa 4$$
$$+ \frac{s+4d}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 1\, q^{\frac{s}{r}} \varkappa 5 = 0,$$ etiam

$\dot{E} q^{\frac{s}{r}} \varkappa 1 - \frac{s}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 5\, q^{\frac{s}{r}} \varkappa 1 = 0$; unde sequitur $\dot{E} = \frac{s}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 5$, et

$$x^s = \frac{s}{s} q^{-\frac{s}{r}} \varkappa 1\, y^{\frac{ls}{r}} + \frac{s}{s+d} q^{-\frac{s+d}{r}} \varkappa 2\, y^{\frac{l(s+d)}{r}} + \frac{s}{s+2d} q^{-\frac{s+2d}{r}} \varkappa 3\, y^{\frac{l(s+2d)}{r}} + \frac{s}{s+3d} q^{-\frac{s+3d}{r}} \varkappa 4\, y^{\frac{l(s+3d)}{r}}$$
$$+ \frac{s}{s+4d} q^{-\frac{s+4d}{r}} \varkappa 5\, y^{\frac{l(s+4d)}{r}} + \ldots\ldots\ldots\ldots$$ (Δ

Seriei quaesitae coefficiens (n+1)tus $\overset{n}{A}$, determinatur hac aequatione:

$$\overset{n}{A} q^{\frac{s}{r}} \varkappa 1 + p^{\frac{s+d}{s}} \varkappa n\, q^{\frac{s}{r}} \varkappa 2 + p^{\frac{s+2d}{s}} \varkappa (n-1)\, q^{\frac{s}{r}} \varkappa 3 \ldots\ldots + p^{\frac{s+md}{s}} \varkappa (n-m+1)\, q^{\frac{s}{r}} \varkappa (m+1) \ldots\ldots\ldots$$
$$\ldots + p^{\frac{s+(n-1)d}{s}} \varkappa 2\, q^{\frac{s}{r}} \varkappa n + p^{\frac{s+nd}{s}} \varkappa 1\, q^{\frac{s}{r}} \varkappa (n+1) = 0,$$ in qua $m < n$

Ponamus, legem in prioribus quinque coefficientibus obseruatam, locum habere in prioribus n coefficientibus, siue esse

$x^s =$

$$x^s = \frac{s}{s} q^{-\frac{s}{r}} \varkappa 1\, y^{\frac{ls}{r}} + \frac{s}{s+d} q^{-\frac{s+d}{r}} \varkappa 2\, y^{\frac{l(s+d)}{r}} + \frac{s}{s+2d} q^{-\frac{s+2d}{r}} \varkappa 3\, y^{\frac{l(s+2d)}{r}} \ldots\ldots$$

$$\ldots + \frac{s}{s+(n-m)d} q^{-\frac{s+(n-m)d}{r}} \varkappa(n-m+1)\, y^{\frac{l(s+(n-m)d)}{r}} \ldots + \frac{s}{s+(n-3)d} q^{-\frac{s+(n-3)d}{r}} \varkappa(n-2)\, y^{\frac{l(s+(n-3)d)}{r}}$$

$$+ \frac{s}{s+(n-2)d} q^{-\frac{s+(n-2)d}{r}} \varkappa(n-1)\, y^{\frac{l(s+(n-2)d)}{r}} + \frac{s}{s+(n-1)d} q^{-\frac{s+(n-1)d}{r}} \varkappa n\, y^{\frac{l(s+(n-1)d)}{r}} \ldots\ldots (\Delta$$

atque erit $p^{\frac{s+d}{s}} \varkappa n = \frac{s+d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa n$; $p^{\frac{s+2d}{s}} \varkappa(n-1) = \frac{s+2d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n-1)$; $p^{\frac{s+md}{s}} \varkappa(n-m+1) = \frac{s+md}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n-m+1)$; $p^{\frac{s+(n-1)d}{s}} \varkappa 2 = \frac{s+(n-1)d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa 2$; $p^{\frac{s+nd}{s}} \varkappa 1 = \frac{s+nd}{s+nd} q^{-\frac{s+nd}{r}} \varkappa 1$; ergo

$$\overset{n}{\dot{A}} q^{\frac{s}{r}} \varkappa 1 + \frac{s+d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa n\, q^{\frac{s}{r}} \varkappa 2 + \frac{s+2d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n-1)\, q^{\frac{s}{r}} \varkappa 3 \ldots + \frac{s+md}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n-m+1)\, q^{\frac{s}{r}} \varkappa(m+1) \ldots$$

$$\ldots + \frac{s+(n-1)d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa 2\, q^{\frac{s}{r}} \varkappa n + \frac{s+nd}{s+nd} q^{-\frac{s+nd}{r}} \varkappa 1\, q^{\frac{s}{r}} \varkappa(n+1) = 0;$$ et cum $\frac{s}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n+1)\, q^{\frac{s}{r}} \varkappa 1$

$$+ \frac{s+d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa n\, q^{\frac{s}{r}} \varkappa 2 + \frac{s+2d}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n-1)\, q^{\frac{s}{r}} \varkappa 3 \ldots + \frac{s+md}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n-m+1)\, q^{\frac{s}{r}} \varkappa(m+1) \ldots$$

$$+ \frac{s+nd}{s+nd} q^{-\frac{s+nd}{r}} \varkappa 1\, q^{\frac{s}{r}} \varkappa(n+1) = 0;$$ est etiam

$$\overset{n}{\dot{A}} q^{\frac{s}{r}} \varkappa 1 - \frac{s}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n+1)\, q^{\frac{s}{r}} \varkappa 1 = 0, \text{ et } \overset{n}{\dot{A}} = \frac{s}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n+1).$$

Apparet igitur, legem istam si coefficientes n priores sequantur, etiam (n+1)tum sequi. Sequuntur autem eam priores quinque coefficientes, ergo etiam sextus, et cum priores sex, etiam septimus, et omnes, in infinitum

Est itaque $x^s\urcorner(n+1) = \frac{s}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n+1) \cdot y^{\frac{l(s+nd)}{r}}$ *h. e. Coefficiens* (n+1)*tus dignitatis, cuius exponens est* $-\frac{s+nd}{r}$ *seriei datae, ductus in* $\frac{s}{s+nd} y^{\frac{l(s+nd)}{r}}$, *terminum* (n+1)*tum seriei quaesitae constituit.*

Theorema hoc, serierum reuersionis vtique grauissimum, dubito, an alia via, quam arte Combinatorio-Analytica directe possit inveniri.

Breuitatis causa, loco $\frac{s}{r}$, $\frac{s+d}{r}$, $\frac{s+2d}{r}$ .... $\frac{s+nd}{r}$ ponatur respectiue ${}^0m$, ${}^1m$, ${}^2m$... ...${}^nm$, atque, formula iam exposita sic quoque exprimi potest: $x^s\urcorner(n+1) = \frac{{}^0m}{{}^nm} q^{-\overset{n}{m}} \varkappa(n+1)\, y^{\overset{n}{m}l}$.

Quoniam vero (§. I.) $q^{-\overset{n}{m}} \varkappa(n+1) = \mathfrak{A}\alpha^{-\overset{n}{m}}\; {}^{-\overset{n}{m}-1}a^nA + \mathfrak{B}\alpha^{-\overset{n}{m}}\; {}^{-\overset{n}{m}-2}b^nB + \mathfrak{C}\alpha^{-\overset{n}{m}}\; {}^{-\overset{n}{m}-3}c^nC \ldots$

$$+ \mathfrak{N}\alpha^{-\overset{n}{m}}\; {}^{-\overset{n}{m}-n}n^nN =$$

[$-\overset{n}{m}$ $\mathfrak{A}$

$$\left[\frac{\overset{n}{-m}\mathfrak{A}a^nA}{\alpha}+\frac{\overset{n}{-m}\mathfrak{B}b^nB}{\alpha^2}+\frac{\overset{n}{-m}\mathfrak{C}c^nC}{\alpha^3}\dots+\frac{\overset{n}{-m}\mathfrak{N}n^nN}{\alpha^n}\right]\alpha^{\overset{n}{-m}}\overset{n}{-m}$$, atque $\mathfrak{A}=-{}^nm$;

$\overset{n}{-m}\mathfrak{B}=+{}^nm\frac{\overset{n}{m+1}\mathfrak{A}}{2}$; $\overset{n}{-m}\mathfrak{C}=-{}^nm\frac{\overset{n}{m+2}\mathfrak{B}}{3}$; ... $\overset{n}{-m}\mathfrak{N}=\pm{}^nm\frac{\overset{n}{m+n-1}\overset{-1}{\mathfrak{N}}}{n}$; est etiam

$$x^s 7(n+1)=-{}^0m\left[\frac{a^nA}{\alpha}-\frac{\overset{n}{m+1}\mathfrak{A}b^nB}{2\alpha^2}+\frac{\overset{n}{m+2}\mathfrak{B}c^nC}{3\alpha^3}\dots\mp\frac{{}^n\overset{}{m+n-1}\overset{-1}{\mathfrak{N}}n^nN}{n\alpha^n}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{n}{m}}.$$

Idem hic terminus est, quem Eschenbachius in dissertatione sua p. 24. exhibuit.

Explicitis signis localibus per combinatoria prodit:

$$1)\ x^s=\left(\frac{y^1}{\alpha}\right)^{\overset{0}{m}}+\frac{{}^0m\,\overset{1}{-m}\mathfrak{A}a^1A}{{}^1m\ \alpha}\left(\frac{y^1}{\alpha}\right)^{\overset{1}{m}}+\frac{{}^0m}{{}^2m}\left[\frac{\overset{2}{-m}\mathfrak{A}a^2A}{\alpha}+\frac{\overset{2}{-m}\mathfrak{B}b^2B}{\alpha^2}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{2}{m}}$$

$$+\frac{{}^0m}{{}^3m}\left[\frac{\overset{3}{-m}\mathfrak{A}a^3A}{\alpha}+\frac{\overset{3}{-m}\mathfrak{B}b^3B}{\alpha^2}+\frac{\overset{3}{-m}\mathfrak{C}c^3C}{\alpha^3}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{3}{m}}$$

$$+\frac{{}^0m}{{}^4m}\left[\frac{\overset{4}{-m}\mathfrak{A}a^4A}{\alpha}+\frac{\overset{4}{-m}\mathfrak{B}b^4B}{\alpha^2}+\frac{\overset{4}{-m}\mathfrak{C}c^4C}{\alpha^3}+\frac{\overset{4}{-m}\mathfrak{D}d^4D}{\alpha^4}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{4}{m}}\dots\dots$$

$$\dots+\frac{{}^0m}{{}^nm}\left[\frac{\overset{n}{-m}\mathfrak{A}a^nA}{\alpha}+\frac{\overset{n}{-m}\mathfrak{B}b^nB}{\alpha^2}+\frac{\overset{n}{-m}\mathfrak{C}c^nC}{\alpha^3}\dots+\frac{\overset{n}{-m}\mathfrak{N}n^nN}{\alpha^n}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{n}{m}}\dots\dots$$

$$\begin{pmatrix}\beta, & \gamma, & \delta, & \varepsilon\dots\dots\\ 1, & 2, & 3, & 4\dots\dots\end{pmatrix}$$

atque, substitutis Classium Combinatoriarum valoribus:

$$2)\ x^s=\left(\frac{y^1}{\alpha}\right)^{\overset{0}{m}}+\frac{{}^0m}{{}^1m}\frac{\overset{1}{-m}\mathfrak{A}\beta}{\alpha}\left(\frac{y^1}{\alpha}\right)^{\overset{1}{m}}+\frac{{}^0m}{{}^2m}\left[\frac{\overset{2}{-m}\mathfrak{A}\gamma}{\alpha}+\frac{\overset{2}{-m}\mathfrak{B}\beta^2}{\alpha^2}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{2}{m}}$$

$$+\frac{{}^0m}{{}^3m}\left[\frac{\overset{3}{-m}\mathfrak{A}\delta}{\alpha}+\frac{\overset{3}{-m}\mathfrak{B}2\beta\gamma}{\alpha^2}+\frac{\overset{3}{-m}\mathfrak{C}\beta^3}{\alpha^3}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{3}{m}}$$

$$+\frac{{}^0m}{{}^4m}\left[\frac{\overset{4}{-m}\mathfrak{A}\varepsilon}{\alpha}+\frac{\overset{4}{-m}\mathfrak{B}(2\beta\delta+\gamma^2)}{\alpha^2}+\frac{\overset{4}{-m}\mathfrak{C}3\beta^2\gamma}{\alpha^3}+\frac{\overset{4}{-m}\mathfrak{D}\beta^4}{\alpha^4}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{4}{m}}$$

$$+\frac{{}^0m}{{}^5m}\left[\frac{\overset{5}{-m}\mathfrak{A}\zeta}{\alpha}+\frac{\overset{5}{-m}\mathfrak{B}(2\beta\varepsilon+2\gamma\delta)}{\alpha^2}+\frac{\overset{5}{-m}\mathfrak{C}(3\beta^2\delta+3\beta\gamma^2)}{\alpha^3}+\frac{\overset{5}{-m}\mathfrak{D}4\beta^3\gamma}{\alpha^4}+\frac{\overset{5}{-m}\mathfrak{E}\beta^5}{\alpha^5}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{5}{m}}$$

$+{}^0m$

$$+\frac{{}^{\circ}m}{{}^{6}m}\left[{}^{\overset{6}{-m}}\frac{\mathfrak{A}\eta}{\alpha}+{}^{\overset{6}{-m}}\frac{\mathfrak{B}(2\beta\zeta+2\gamma\varepsilon+\delta^2)}{\alpha^2}+{}^{\overset{6}{-m}}\frac{\mathfrak{C}(3\beta^2\varepsilon+6\beta\gamma\delta+\gamma^3)}{\alpha^3}\right.$$

$$\left.+{}^{\overset{6}{-m}}\frac{\mathfrak{D}(4\beta^3\delta+6\beta^2\gamma^2)}{\alpha^4}+{}^{\overset{6}{-m}}\frac{\mathfrak{E}5\beta^4\gamma}{\alpha^5}+{}^{\overset{6}{-m}}\frac{\mathfrak{F}\beta^6}{\alpha^6}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{6}{m}}+\text{etc.}$$

Exempla I. Sit $y=x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9-\frac{1}{11}x^{11}+\frac{1}{13}x^{13}\ldots$ quaeritur $x^1\urcorner_7$. Eſt igitur $l=s=r=1$, $d=2$, $n=6$, ${}^{\circ}m=1$, $m^n={}^{6}m=13$; $\alpha=1$, $\beta=-\frac{1}{3}$, $\gamma=+\frac{1}{5}$ etc. . . ergo $x^1\urcorner_7=\frac{1}{13}q^{-13}\varkappa 7y^{13}=\frac{1}{13}[{}^{-13}\mathfrak{A}\mathfrak{a}^6A+{}^{-13}\mathfrak{B}\mathfrak{b}^6B+{}^{-13}\mathfrak{C}\mathfrak{c}^6C+{}^{-13}\mathfrak{D}\mathfrak{d}^6D+{}^{-13}\mathfrak{E}\mathfrak{e}^6E+{}^{-13}\mathfrak{F}\mathfrak{f}^6F]y^{13}$; $\mathfrak{a}^6A=\eta=\frac{1}{13}$, $\mathfrak{b}^6B=2\beta\zeta+2\gamma\varepsilon+\delta^2=\frac{2}{3.11}+\frac{2}{5\cdot 9}+\frac{I}{7^2}=\frac{3043}{3^2\cdot 5\cdot 7\cdot^2 11}$; $\mathfrak{c}^6C=3\beta^2\varepsilon+6\beta\gamma\delta+\gamma^3=\frac{3}{9\cdot 9}+\frac{6}{3\cdot 5\cdot 7}+\frac{1}{5^3}=\frac{2414}{3^3\cdot 5^3\cdot 7}$; $\mathfrak{d}^6D=4\beta^3\delta+6\beta^2\gamma^2=\frac{4}{27\cdot 7}+\frac{6}{9.25}=\frac{226}{3^3 5^2 7}$; $\mathfrak{e}^6E=5\beta^4\gamma=\frac{1}{3^4}$; $\mathfrak{f}^6F=\beta^6=\frac{1}{3^6}$; porro ${}^{-13}\mathfrak{A}=-13$, ${}^{-13}\mathfrak{B}=+91$, ${}^{-13}\mathfrak{C}=-455$, ${}^{-13}\mathfrak{D}=+1820$; ${}^{-13}\mathfrak{E}=-6188$, ${}^{-13}\mathfrak{F}=+18564$. Ergo $x^1\urcorner_7=\frac{1}{13}\left[-1+\frac{39559}{3^2.5.7.11}-\frac{31382}{3^3.5^2}+\frac{11752}{3^3.5}-\frac{6188}{3^4}+\frac{6188}{3^5}\right]y^{13}=+\frac{21844}{3^5.5^2 7.11.13}y^{13}$.

Si quaereres $x^{-1}\urcorner_7$, foret, caeteris paribus, $s=-1$, ${}^{\prime}m=-1$, ${}^{6}m=11$, ergo $x^{-1}\urcorner_7=-\frac{1}{11}q^{-11}\varkappa 7y^{11}=-\frac{1}{11}[{}^{-11}\mathfrak{A}\mathfrak{a}^6A+{}^{-11}\mathfrak{B}\mathfrak{b}^6B+{}^{-11}\mathfrak{C}\mathfrak{c}^6C+{}^{-11}\mathfrak{D}\mathfrak{d}^6D+{}^{-11}\mathfrak{E}\mathfrak{e}^6E+{}^{-11}\mathfrak{F}\mathfrak{f}^6F]y^{11}=-\frac{1}{11}\left[-\frac{11}{13}+\frac{6086}{35\cdot 7^2}-\frac{690404}{3^3 5^3 7}+\frac{32318}{3^3 5^2}-\frac{1001}{3^3}+\frac{8008}{3^6}\right]y^{11}=-\frac{1382}{3^6.5^3.7^2.11.13}y^{11}$.

Duo hic termini, quorum alter ad ſeriem x^1, alter ad ſeriem x^{-1} pertinet, extra ordinem, et ab antecedentibus independenter, datis ſolummodo Coefficientibus et Exponentibus, determinati ſunt, quod et *formulae* combinatorio-analyticae *Eſchenbachianae:*

$$x^s\urcorner(n+1)=-{}^{\circ}m\left[\frac{\mathfrak{a}^nA}{\alpha}-{}^{\overset{n}{m+1}}\frac{\mathfrak{A}\mathfrak{b}^nB}{2\alpha^2}+{}^{\overset{n}{m+2}}\frac{\mathfrak{B}\mathfrak{c}^nC}{3\alpha^3}\ldots\ldots\mp{}^{\overset{-1}{{}^{n}m+n-1}}\frac{\mathfrak{N}\mathfrak{n}^nN}{n\alpha^n}\right]\left(\frac{y^1}{\alpha}\right)^{\overset{n}{m}}$$

ope, eodem modo fieri potuiſſet.

Noſtra vero formula, $x^s\urcorner(n+1)=\frac{s}{s+nd}\cdot q^{-\frac{s+nd}{r}}\varkappa(n+1)\cdot y^{\frac{l(s+nd)}{r}}$, qua ſerierum reverſio, ad evectionem ſeriei datae, ad dignitatem exponentis quaeſiti, reducitur, Eſchenbachianae longe praeſtat et grauior eſt. Si enim coefficientes ſeriei datae ſic comparati ſunt, vt digni-

dignitatis eius quaesitae, coefficiens quaesitus, facilius, quam formularum in §. I. propositarum ope, inveniri possit, tunc etiam seriei inversae terminus quilibet eadem facilitate poterit exhiberi.

Ex. II) Sit data aequatio $y = ax + bx^3$, quaeritur 1) x^s per *seriem ascendentem variabilis* y.

Est $l = r = 1$, $d = 2$, ergo $x^s \text{⁊}(n+1) = \frac{s}{s+2n} q^{-(s+2n)} \varkappa(n+1) y^{s+2n}$. Cum autem

$$q^m \varkappa(n+1) = {}^m\mathfrak{N} a^{m-n} b^n = \frac{m \cdot m-1 \cdot m-2 \ldots m-n+1}{1 \cdot 2 \cdot 3 \ldots n} a^{m-n} b^n,$$ est etiam

$$x^s \text{⁊}(n+1) = \pm \frac{s}{s+2n} \cdot \frac{s+2n \cdot s+2n+1 \cdot s+2n+2 \ldots s+3n-1}{1 \cdot 2 \cdot 3 \ldots n} \cdot \frac{b^n y^{s+2n}}{a^{s+3n}}$$

$$= \pm \frac{s \cdot s+2n+1 \cdot s+2n+2 \ldots s+3n-1}{1 \cdot 2 \cdot 3 \ldots n} \frac{b^n y^{s+2n}}{a^{s+3n}}$$ (signum superius valet, si n numerus est par, inferius si impar) atque $x^s = \frac{y^s}{a^s} - \frac{sby^{s+2}}{a^{s+3}} + \frac{s \cdot s+5}{1 \cdot 2} \frac{b^2 y^{s+4}}{a^{s+6}}$

$$- \frac{s \cdot s+7 \cdot s+8}{1 \cdot 2 \cdot 3} \frac{b^3 y^{s+6}}{a^{s+9}} + \frac{s \cdot s+9 \cdot s+10 \cdot s+11}{1 \cdot 2 \cdot 3 \cdot 4} \frac{b^4 y^{s+8}}{a^{s+12}} - \ldots\ldots\ldots$$

$$\ldots \pm \frac{s \cdot s+2n+1 \cdot s+2n+2 \ldots\ldots s+3n-1}{1 \cdot 2 \cdot 3 \ldots\ldots n} \frac{b^n y^{s+2n}}{a^{s+3n}} \ldots\ldots\ldots\ldots$$

2) x^s, per *seriem descendentem variabilis y.*

Quoniam $y = bx^3 + ax$, est $l = 1$, $r = 3$, $d = -2$, ergo

$$x^s \text{⁊}(n+1) = -\frac{s}{2n-s} q^{\frac{2n-s}{3}} \varkappa(n+1) y^{-\frac{(2n-s)}{3}}.$$ Est autem $q^m \varkappa(n+1) = {}^m\mathfrak{N} b^{m-n} a^n$

$$= \frac{m \cdot m-1 \cdot m-2 \ldots m-n+1}{1 \cdot 2 \cdot 3 \ldots n} b^{m-n} a^n;$$ ergo

$$x^s \text{⁊}(n+1) = -\frac{s}{2n-s} \cdot \frac{2n-s \cdot 2n-s-3 \cdot 2n-s-6 \ldots (-n-s+3)}{3 \cdot 6 \cdot 9 \ldots 3n} \frac{a^n y^{-\frac{(2n-s)}{3}}}{b^{\frac{n+s}{3}}}$$

$$= -\frac{s \cdot 2n-s-3 \cdot 2n-s-6 \ldots\ldots\ldots (-n-s+3)}{3 \cdot 6 \cdot 9 \ldots\ldots\ldots 3n} \frac{a^n y^{-\frac{(2n-s)}{3}}}{b^{\frac{n+s}{3}}}$$

$$\text{atque } x^s = \frac{y^{\frac{s}{3}}}{b^{\frac{s}{3}}} - \frac{say^{\frac{s-2}{3}}}{3\, b^{\frac{s+1}{3}}} - \frac{s \cdot 1-s}{3 \cdot 6} \frac{a^2 y^{\frac{s-4}{3}}}{b^{\frac{s+2}{3}}} + \frac{s \cdot 3-s}{3 \cdot 6 \cdot 9} \frac{sa^3 y^{\frac{s-6}{3}}}{b^{\frac{s+3}{3}}}$$

$$+ \frac{s \cdot 5-s \cdot 2-s \cdot s+1}{3 \cdot 6 \cdot 9 \cdot 12} \frac{a^4 y^{\frac{s-8}{3}}}{b^{\frac{s+4}{3}}} + \frac{s \cdot 7-s \cdot 4-s \cdot 1-s \cdot s+2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \frac{a^5 y^{\frac{s-10}{3}}}{b^{\frac{s+5}{3}}}$$

$-\frac{s.9-s.6-s.3-s.s.s+3.a^6 y^{\frac{s-12}{3}}}{3.\ 6.\ 9.\ 12.\ 15.\ 18\ b^{\frac{s+6}{3}}} - \ldots\ldots\ldots\ldots$ Ponatur $s=1$, atque est

$$x = \frac{y^{\frac{1}{3}}}{b^{\frac{1}{3}}} - \frac{ay^{-\frac{1}{3}}}{3b^{\frac{2}{3}}} * + \frac{2.1\,a^3 y^{-\frac{5}{3}}}{3.6.9.\,b^{\frac{4}{3}}} + \frac{4.1.2\,a^4 y^{-\frac{7}{3}}}{3.6.9.12.\,b^{\frac{5}{3}}} * - \frac{8.5.2.1.4\,a^6 y^{-\frac{11}{3}}}{3.6.9.12.15.18.b^{\frac{7}{3}}} - \frac{10.7.4.1.2.5\,a^7 y^{-\frac{13}{3}}}{3.6.9.12.15.18.21 b^{\frac{8}{3}}} \ldots$$

Euanescunt huius seriei terminus tertius, sextus, nonus. . . 3νtus. . . Ponatur enim in formula generali $x^s\,\daleth(n+1) = -\frac{1}{2n-1} q^{\frac{2n-1}{3}} \varkappa(n+1)\, y^{-\frac{(2n-1)}{3}}$; $n = 3\nu - 1$, atque prodit $x\,\daleth 3\nu = -\frac{1}{6\nu-3} q^{2\nu-1} \varkappa 3\nu\, y^{-2\nu+1}$. Constat vero $q^{2\nu-1} = (bx^3 + a\,x)^{2\nu-1}$ terminis tantummodo 2ν, ergo $q^{2\nu-1}\varkappa 3\nu = 0$, atque $x\,\daleth 3\nu = 0$.

Ex. III. Sit data aequatio $y^1 = x^r + x^{r+d} + \frac{x^{r+2d}}{1.2} + \frac{x^{r+3d}}{1.2.3} + \frac{x^{r+4d}}{1.2.3.4} \ldots$ quaeritur x^s. Est hic $q^m \varkappa(n+1) = \frac{m^n}{1.2.3\ldots n}$, ergo $x^s\,\daleth(n+1) = \frac{\frac{s}{s+nd}\cdot\left(-\frac{s+nd}{r}\right)^n}{1\,2\,3\,4\ldots\ldots n}\, y^{\frac{1(s+nd)}{r}}$

$= \pm \frac{s(s+nd)^{n-1}}{1.2.3\ldots n.r^n} y^{\frac{1(s+nd)}{r}}$, atque $x^s = y^{\frac{1s}{r}} - \frac{s}{r} y^{\frac{1(s+d)}{r}} + \frac{s(s+2d)^1}{1.\ 2.\ r^2} y^{\frac{1(s+2d)}{r}} - \frac{s(s+3d)^2}{1.2.3.r^3} y^{\frac{1(s+3d)}{r}}$

$+ \frac{s(s+4d)^3}{1.2.3.4\ r^4} y^{\frac{1(s+4d)}{r}} - \ldots$

Apparebit inde, quanto cum fructu haec de serierum reversione formula $x^s\,\daleth(n+1) = \frac{s}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n+1)\, y^{\frac{1(s+nd)}{r}}$ adhiberi possit, neque dubito plura adhuc, de quibus nondum cogitaui, ei inesse.

Sed *formulam* nunc *localem* proponam longe complicatissimam, eamque generalissimam, cuius ope alia adhuc, formulae iam expositae regressoriae, ideoque etiam Eschenbachianae, demonstratio facillime poterit exhiberi.

§. VI.

$$\frac{s}{f.\ g} q^f\varkappa 1\, q^g \varkappa(n+1) + \frac{(s+c)}{(f+d)(g-d)} q^{f+d}\varkappa 2\, q^{g-d}\varkappa n \ldots + \frac{(s+mc)}{(f+md)(g-md)} q^{f+md}\varkappa(m+1)\, q^{g-md}\varkappa(n-m+1) \ldots$$

$$\ldots + \frac{(s+nc)}{(f+nd)(g-nd)} q^{f+nd}\varkappa(n+1)\, q^{g-nd}\varkappa 1 = \frac{s(f+g)+n(cf-ds)}{f(f+g)(g-nd)} q^{f+g}\varkappa(n+1)$$

Demon-

Demonstratio.

Si verae sunt priores n formulae, quae oriuntur, tribuendo in formula proposita generali litterae n successiue valores 0, 1, 2, ... m, ... 'm, . . . n — 1, vera etiam erit formula (n + 1)ta. Posito enim,

1); $\frac{s}{fg}q^{f}\varkappa 1\, q^{g}\varkappa 1 = \frac{s(f+g)}{f(f+g)g}q^{f+g}\varkappa 1$

2); $\frac{s}{fg}q^{f}\varkappa 1\, q^{g}\varkappa 2 + \frac{(s+c)}{(f+d)(g-d)}q^{f+d}\varkappa 2\, q^{g-d}\varkappa 1 = \frac{s(f+g)+cf-ds}{f(f+g)(g-d)}q^{f+g}\varkappa 2$

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m + 1); $\frac{s}{fg}q^{f}\varkappa 1\, q^{g}\varkappa(m+1) + \frac{(s+c)}{(f+d)(g-d)}q^{f+d}\varkappa 2\, q^{g-d}\varkappa m \ldots + \frac{(s+mc)}{(f+md)(g-md)}q^{f+md}\varkappa(m+1)\, q^{g-md}\varkappa 1$

$= \frac{s(f+g)+m(cf-ds)}{f(f+g)(g-md)}q^{f+g}\varkappa(m+1)$

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'm + 1); $\frac{s}{fg}q^{f}\varkappa 1\, q^{g}\varkappa('m+1) + \frac{(s+c)}{(f+d)(g-d)}q^{f+d}\varkappa 2\, q^{g-d}\varkappa 'm \ldots + \frac{(s+mc)}{(f+md)(g-md)}q^{f+md}\varkappa(m+1)\, q^{g-md}\varkappa('m-m+1)$

$\ldots + \frac{(s+'mc)}{(f+'md)(g-'md)}q^{f+'md}\varkappa('m+1)\, q^{g-'md}\varkappa 1 = \frac{s(f+g)+'m(cf-ds)}{f(f+g)(g-'md)}q^{f+g}\varkappa('m+1)$

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n); $\frac{s}{fg}q^{f}\varkappa 1\, q^{g}\varkappa n + \frac{(s+c)}{(f+d)(g-d)}q^{f+d}\varkappa 2\, q^{g-d}\varkappa(n-1) \ldots + \frac{(s+mc)}{(f+md)(g-md)}q^{f+md}\varkappa(m+1)\, q^{g-md}\varkappa(n-m) \ldots$

$\ldots + \frac{(s+(n-1)c)}{(f+(n-1)d)(g-(n-1)d)}q^{f+(n-1)d}\varkappa n\, q^{g-(n-1)d}\varkappa 1 = \frac{s(f+g)+(n-1)(cf-ds)}{f(f+g)(g-(n-1)d)}q^{f+g}\varkappa n$

atque vltimo loco

n + 1); $\frac{s}{fg}q^{f}\varkappa 1\, q^{g}\varkappa(n+1) + \frac{(s+c)}{(f+d)(g-d)}q^{f+d}\varkappa 2\, q^{g-d}\varkappa n \ldots + \frac{(s+mc)}{(f+md)(g-md)}q^{f+md}\varkappa(m+1)\, q^{g-md}\varkappa(n-m+1)$

$\ldots + \frac{s+(n-1)c}{(f+(n-1)d)(g-(n-1)d)}q^{f+(n-1)d}\varkappa n\, q^{g-(n-1)d}\varkappa 2 + \frac{(s+nc)}{(f+nd)(g-nd)}q^{f+nd}\varkappa(n+1)\, q^{g-nd}\varkappa 1 = S$

ductisque hisce (n + 1) aequationibus: (n + 1)ta, nta, ... ('m + 1)ta, ... (m + 1)ta, ... 2da, 1ma, *singulis respective* in $(g-nd)q^{nd-g}\varkappa 1$; $(g-[n-1]d)q^{nd-g}\varkappa 2$; . . . $(g-'md)q^{nd-g}\varkappa(n-'m+1)$; . . . $(g-md)q^{nd-g}\varkappa(n-m+1)$; . . . . $(g-d)q^{nd-g}\varkappa n$; $gq^{nd-g}\varkappa(n+1)$; Erit:

	1	2	m + 1	n	n + 1	⊙

n+1); $\frac{s(g-nd)}{f.\,g} q^{f}_{\varkappa1} q^{g}_{\varkappa(n+1)} q^{nd-g}_{\varkappa1} + \frac{(s+c)(g-nd)}{(f+d)(g-d)} q^{f+d}_{\varkappa2} q^{g-d}_{\varkappa n} q^{nd-g}_{\varkappa1} \dots\dots + \frac{(s+mc)(g-nd)}{(f+md)(g-md)} q^{f+md}_{\varkappa(m+1)} q^{g-md}_{\varkappa(n-m+1)} q^{nd-g}_{\varkappa1} \dots\dots + \frac{(s+[n-1]c)(g-nd)}{(f+[n-1]d)(g-[n-1]d)} q^{f+[n-1]d}_{\varkappa n} q^{g-[n-1]d}_{\varkappa2} q^{nd-g}_{\varkappa1} + \frac{(s+nc)(g-nd)}{(f+nd)(g-nd)} q^{f+nd}_{\varkappa(n+1)} q^{g-nd}_{\varkappa1} q^{nd-g}_{\varkappa1} = (g-nd) q^{nd-g}_{\varkappa1} S$

n); $\frac{s(g-[n-1]d)}{f.\,g} q^{f}_{\varkappa1} q^{g}_{\varkappa n} q^{nd-g}_{\varkappa2} + \frac{(s+c)(g-[n-1]d)}{(f+d)(g-d)} q^{f+d}_{\varkappa2} q^{g-d}_{\varkappa(n-1)} q^{nd-g}_{\varkappa2} \dots\dots + \frac{(s+mc)(g-[n-1]d)}{(f+md)(g-md)} q^{f+md}_{\varkappa(m+1)} q^{g-md}_{\varkappa(n-m)} q^{nd-g}_{\varkappa2} \dots\dots + \frac{(s+[n-1]c)(g-[n-1]d)}{(f+[n-1]d)(g-[n-1]d)} q^{f+(n-1)d}_{\varkappa n} q^{g-[n-1]d}_{\varkappa1} q^{nd-g}_{\varkappa2} = \frac{s(f+g)+[n-1](cf-ds)}{f(f+g)} q^{f+g}_{\varkappa n} q^{nd-g}_{\varkappa2}$

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'm+1); $\frac{s(g-'md)}{f.\,g} q^{f}_{\varkappa1} q^{g}_{\varkappa('m+1)} q^{nd-g}_{\varkappa(n-'m+1)} + \frac{(s+c)(g-'md)}{(f+d)(g-d)} q^{f+d}_{\varkappa2} q^{g-d}_{\varkappa'm} q^{nd-g}_{\varkappa(n-'m+1)} \dots + \frac{(s+mc)(g-'md)}{(f+md)(g-md)} q^{f+md}_{\varkappa(m+1)} q^{g-md}_{\varkappa('m-m+1)} q^{nd-g}_{\varkappa(n-'m+1)} \dots\dots = \frac{s(f+g)+'m(cf-ds)}{f(f+g)} q^{f+g}_{\varkappa('m+1)} q^{nd-g}_{\varkappa(n-'m+1)}$

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m+1); $\frac{s(g-md)}{f.\,g} q^{f}_{\varkappa1} q^{g}_{\varkappa(m+1)} q^{nd-g}_{\varkappa(n-m+1)} + \frac{(s+c)(g-md)}{(f+d)(g-d)} q^{f+d}_{\varkappa2} q^{g-d}_{\varkappa m} q^{nd-g}_{\varkappa(n-m+1)} \dots + \frac{(s+mc)(g-md)}{(f+md)(g-md)} q^{f+md}_{\varkappa(m+1)} q^{g-md}_{\varkappa1} q^{nd-g}_{\varkappa(n-m+1)} = \frac{s(f+g)+m(cf-ds)}{f(f+g)} q^{f+g}_{\varkappa(m+1)} q^{nd-g}_{\varkappa(n-m+1)}$

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2); $\frac{s(g-d)}{f.\,g} q^{f}_{\varkappa1} q^{g}_{\varkappa2} q^{nd-g}_{\varkappa n} + \frac{(s+c)(g-d)}{(f+d)(g-d)} q^{f+d}_{\varkappa2} q^{g-d}_{\varkappa1} q^{nd-g}_{\varkappa n} = \frac{s(f+g)+(cf-ds)}{f(f+g)} q^{f+g}_{\varkappa2} q^{nd-g}_{\varkappa n}$

1); $\frac{s\,g}{f.g} q^{f}_{\varkappa1} q^{g}_{\varkappa1} q^{nd-g}_{\varkappa(n+1)} = \frac{s(f+g)}{f(f+g)} q^{f+g}_{\varkappa1} q^{nd-g}_{\varkappa(n+1)}$

Quas (n+1) aequationes, si addendo coniungimus, *summam* invenimus, (§. IV.) seriei *verticalis primae* $= 0$, *secundae* $= 0$ etc. $(m+1)$tae $= 0$, etc. ntae $= 0$; seriei autem $\odot = \frac{s+nc}{f+nd} q^{f+nd}_{\varkappa(n+1)} - \frac{s(f+g)+n(cf-ds)}{f(f+g)} q^{f+g}_{\varkappa(n+1)} q^{nd-g}_{\varkappa1} + (g-nd) q^{nd-g}_{\varkappa1} S$. Est itaque $\frac{(s+nc)(g-nd)}{(f+nd)(g-nd)} q^{f+nd}_{\varkappa(n+1)} q^{g-nd}_{\varkappa1} q^{nd-g}_{\varkappa1} = \frac{s+nc}{f+nd} q^{f+nd}_{\varkappa(n+1)} = \frac{s+nc}{f+nd} q^{f+nd}_{\varkappa(n+1)} - \frac{s(f+g)+n(cf-ds)}{f(f+g)} q^{f+g}_{\varkappa(n+1)} q^{nd-g}_{\varkappa1} + (g-nd) q^{nd-g}_{\varkappa1} S$, atque $\frac{s(f+g)+n(cf-ds)}{f(f+g)} q^{f+g}_{\varkappa(n+1)} q^{nd-g}_{\varkappa1} = (g-nd) q^{nd-g}_{\varkappa1}.S$; unde sequitur $S = \frac{s(f+g)+n(cf-ds)}{f(f+g)(g-nd)} q^{f+g}_{\varkappa(n+1)}$. Si itaque verae sunt priores n formulae, vera etiam est (n+1)ta.

Est autem, pro scala q $[\alpha, \beta, \gamma, \delta \dots]$,

1); $\frac{s}{fg} q^{f}_{\varkappa1} q^{g}_{\varkappa1} = \frac{s}{fg} \alpha^{f}.\,\alpha^{g} = \frac{s}{f.g} \alpha^{f+g} = \frac{s(f+g)}{f(f+g)g} q^{f+g}_{\varkappa1}$

2); $\frac{s}{fg} q^{f}_{\varkappa1} q^{g}_{\varkappa2} + \frac{(s+c)}{(f+d)(g-d)} q^{f+d}_{\varkappa2} q^{g-d}_{\varkappa1} = \frac{s}{f}\alpha^{f}.\,\alpha^{g-1}\beta + \frac{s+c}{g-d}\alpha^{f+d-1}\beta\,\alpha^{g-d} = \left(\frac{s}{f} + \frac{s+c}{g-d}\right)\alpha^{f+g-1}\beta = \frac{s(f+g)+cf-ds}{f(f+g)(g-d)}(f+g)\alpha^{f+g-1}\beta = \frac{s(f+g)+cf-ds}{f(f+g)(g-d)} q^{f+g}_{\varkappa2}$

Siue, vera est formula *prima* atque *secunda*, ergo etiam *tertia* vera est, cumque igitur *priores tres* formulae verae sint, vera etiam est *quarta*, cumque sic argumentando *continuare* possimus quovsque velimus, verae etiam sunt *reliquae* quae sequuntur formulae in infinitum. Vera igitur etiam *formula* est initio paragraphi *proposita*. Q. E. D.

Coroll. E formula ista generali, plures speciales possunt deduci. Posito enim $g = h+nd$, prodit

A) $\frac{s}{f(h+nd)} q^{f}_{\varkappa1} q^{h+nd}_{\varkappa(n+1)} + \frac{(s+c)}{(f+d)(h+[n-1]d)} q^{f+d}_{\varkappa2} q^{h+[n-1]d}_{\varkappa n} \dots + \frac{(s+mc)}{(f+md)(h+[n-m]d)} q^{f+md}_{\varkappa(m+1)} q^{h+(n-m)d}_{\varkappa(n-m+1)} \dots$

$\dots\dots + \frac{s+nc}{(f+nd)h} q^{f+nd}_{\varkappa(n+1)} q^{h}_{\varkappa1} = \frac{s(f+h+nd)+n(cf-ds)}{f(f+h+nd)h} q^{f+h+nd}_{\varkappa(n+1)} = \frac{s(f+h)+ncf}{fh(f+h+nd)} q^{f+h+nd}_{\varkappa(n+1)}$

Positis vero $s = fg$, $c = -fd$, nanciscimur

B) $\frac{f}{f} q^{f}_{\varkappa1} q^{g}_{\varkappa(n+1)} + \frac{f}{f+d} q^{f+d}_{\varkappa2} q^{g-d}_{\varkappa n} \dots + \frac{f}{f+md} q^{f+md}_{\varkappa(m+1)} q^{g-md}_{\varkappa(n-m+1)} \dots + \frac{f}{f+nd} q^{f+nd}_{\varkappa(n+1)} q^{g-nd}_{\varkappa1}$

$= \frac{fg(f+g)-n(df^2+dfg)}{f(f+g)(g-nd)} q^{f+g}_{\varkappa(n+1)} = q^{f+g}_{\varkappa(n+1)}$

Formula A, positis $s = fh$, $c = 0$, mutatur in hanc

C) $\frac{fh}{f(h+nd)} q^{f}_{\varkappa1} q^{h+nd}_{\varkappa(n+1)} + \frac{fh}{(f+d)(h+[n-1]d)} q^{f+d}_{\varkappa2} q^{h+(n-1)d}_{\varkappa n} \dots + \frac{fh}{(f+md)(h+[n-m]d)} q^{f+md}_{\varkappa(m+1)} q^{h+(n-m)d}_{\varkappa(n-m+1)} \dots$

$\dots\dots + \frac{fh}{(f+nd)h} q^{f+nd}_{\varkappa(n+1)} q^{h}_{\varkappa1} = \frac{fh(f+h)}{fh(f+h+nd)} q^{f+h+nd}_{\varkappa(n+1)} = \frac{f+h}{f+h+nd} q^{f+h+nd}_{\varkappa(n+1)}$

Ponantur in formula B loco f, d, g, *respectiue* $-\frac{s}{r}$, $-\frac{d}{r}$, $+\frac{s}{r}$, atque erit

D, 1) $\frac{s}{s} q^{-\frac{s}{r}}_{\varkappa1} q^{\frac{s}{r}}_{\varkappa(n+1)} + \frac{s}{s+d} q^{-\frac{s+d}{r}}_{\varkappa2} q^{\frac{s+d}{r}}_{\varkappa n} \dots + \frac{s}{s+md} q^{-\frac{s+md}{r}}_{\varkappa(m+1)} q^{\frac{s+md}{r}}_{\varkappa(n-m+1)} \dots + \frac{s}{s+nd} q^{-\frac{s+nd}{r}}_{\varkappa(n+1)} q^{\frac{s+nd}{r}}_{\varkappa1}$

$= q^{0}_{\varkappa(n+1)} = 0$ (si $n > 0$). Ergo, pro literae n valoribus 1, 2, 3, 4 . . . quibuscunque,

$$D,\ 2)\ \frac{s}{s+nd}q^{-\frac{s+nd}{r}}\varkappa(n+1) = -\Big\{\frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, q^{\frac{s}{r}}\varkappa(n+1) + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, q^{\frac{s+d}{r}}\varkappa n \,\ldots\ldots$$

$$\ldots + \frac{s}{s+md}q^{-\frac{s+md}{r}}\varkappa(m+1)q^{\frac{s+md}{r}}\varkappa(n-m+1) \ldots + \frac{s}{s+(n-1)d}q^{-\frac{s+(n-1)d}{r}}\varkappa n\, q^{\frac{s+(n-1)d}{r}}\varkappa 2\Big\} : q^{\frac{s+nd}{r}}\varkappa 1$$

Hac formula vtemur in altera, quae nunc ſequitur, formulae Eſchenbachianae demonſtratione.

§. VII.

Si data eſt aequatio $y^l = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} + \delta x^{r+3d} \ldots\ldots$ erit

$$x^s = \frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, y^{\frac{ls}{r}} + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, y^{\frac{l(s+d)}{r}} + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3\, y^{\frac{l(s+2d)}{r}} + \ldots\ldots$$

$$\ldots + \frac{s}{s+nd}q^{-\frac{s+nd}{r}}\varkappa(n+1)y^{\frac{l(s+nd)}{r}}\ ; \qquad q\,[\alpha,\ \beta,\ \gamma,\ \delta \ldots]$$

Demonſtratio.

Sit $x^s = \dot{A}y^{\frac{ls}{r}} + \dot{B}y^{\frac{l(s+d)}{r}} + \dot{C}y^{\frac{l(s+2d)}{r}} + \dot{D}y^{\frac{l(s+3d)}{r}} \ldots\ldots\ldots (\Gamma$

Quoniam

$$y^l = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} + \delta x^{r+3d} \ldots\ldots \text{ eſt } (\S.\ I.)$$

$$y^{\frac{ls}{r}} = q^{\frac{s}{r}}\varkappa 1\, x^s + q^{\frac{s}{r}}\varkappa 2\, x^{s+d} + q^{\frac{s}{r}}\varkappa 3\, x^{s+2d} + q^{\frac{s}{r}}\varkappa 4\, x^{s+3d} + \ldots\ldots$$

$$y^{\frac{l(s+d)}{r}} = q^{\frac{s+d}{r}}\varkappa 1\, x^{s+d} + q^{\frac{s+d}{r}}\varkappa 2\, x^{s+2d} + q^{\frac{s+d}{r}}\varkappa 3\, x^{s+3d} + \ldots\ldots$$

$$y^{\frac{l(s+2d)}{r}} = q^{\frac{s+2d}{r}}\varkappa 1\, x^{s+2d} + q^{\frac{s+2d}{r}}\varkappa 2\, x^{s+3d} + \ldots\ldots$$

$$y^{\frac{l(s+3d)}{r}} = q^{\frac{s+3d}{r}}\varkappa 1\, x^{s+3d} + \ldots\ldots$$

atque, ſubſtitutis iſtis valoribus in aequatione Γ, invenimus

$$0 = \left.\begin{matrix} +\dot{A}q^{\frac{s}{r}}\varkappa 1 \\ -1 \end{matrix}\right\} x^s \left.\begin{matrix} +\dot{A}q^{\frac{s}{r}}\varkappa 2 \\ +\dot{B}q^{\frac{s+d}{r}}\varkappa 1 \end{matrix}\right\} x^{s+d} \left.\begin{matrix} +\dot{A}q^{\frac{s}{r}}\varkappa 3 \\ +\dot{B}q^{\frac{s+d}{r}}\varkappa 2 \\ +\dot{C}q^{\frac{s+2d}{r}}\varkappa 1 \end{matrix}\right\} x^{s+2d} \left.\begin{matrix} +\dot{A}q^{\frac{s}{r}}\varkappa 4 \\ +\dot{B}q^{\frac{s+d}{r}}\varkappa 3 \\ +\dot{C}q^{\frac{s+2d}{r}}\varkappa 2 \\ +\dot{D}q^{\frac{s+3d}{r}}\varkappa 1 \end{matrix}\right\} x^{s+3d} \ldots\ldots$$

Est itaque

1) $\dot{A} = \frac{1}{q^{\frac{s}{r}}\kappa_1} = \frac{1}{\alpha^{\frac{s}{r}}} = \alpha^{-\frac{s}{r}} = \frac{s}{s}q^{-\frac{s}{r}}\kappa_1$

2) $\dot{B} = -\dot{A}q^{\frac{s}{r}}\kappa_2 : q^{\frac{s+d}{r}}\kappa_1 = -\frac{s}{s}q^{-\frac{s}{r}}\kappa_1 q^{\frac{s}{r}}\kappa_2 : q^{\frac{s+d}{r}}\kappa_1 = \frac{s}{s+d}q^{-\frac{s+d}{r}}\kappa_2$ (§. VI. Coroll. D, 2 pro $n = 1$)

3) $\dot{C} = -\left(\dot{A}q^{\frac{s}{r}}\kappa_3 + \dot{B}q^{\frac{s+d}{r}}\kappa_2\right) : q^{\frac{s+2d}{r}}\kappa_1 = -\left(\frac{s}{s}q^{-\frac{s}{r}}\kappa_1 q^{\frac{s}{r}}\kappa_3 + \frac{s}{s+d}q^{-\frac{s+d}{r}}\kappa_2 q^{\frac{s+d}{r}}\kappa_2\right) : q^{\frac{s+2d}{r}}\kappa_1$

$= \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\kappa_3$ (Ibid. pro $n = 2$)

4) $\dot{D} = -\left(\dot{A}q^{\frac{s}{r}}\kappa_4 + \dot{B}q^{\frac{s+d}{r}}\kappa_3 + \dot{C}q^{\frac{s+2d}{r}}\kappa_2\right) : q^{\frac{s+3d}{r}}\kappa_1 = -\left(\frac{s}{s}q^{-\frac{s}{r}}\kappa_1 q^{\frac{s}{r}}\kappa_4 + \frac{s}{s+d}q^{-\frac{s+d}{r}}\kappa_2 q^{\frac{s+d}{r}}\kappa_3\right.$

$\left. + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\kappa_3 q^{\frac{s+2d}{r}}\kappa_2\right) : q^{\frac{s\ 3d}{r}}\kappa_1 = \frac{s}{s+3d}q^{-\frac{s+3d}{r}}\kappa_4$ (Ibid. pro $n = 3$)

In oculos hic statim incurrit, legem istam, quam coefficientium fictorum $\dot{A}$, $\dot{B}$, $\dot{C}$, $\dot{D}$ valores hic sequuntur, in omnibus sequentibus etiam esse valituram.

Est itaque $x^s = \frac{s}{s}q^{-\frac{s}{r}}\kappa_1 y^{\frac{ls}{r}} + \frac{s}{s+d}q^{-\frac{s+d}{r}}\kappa_2 y^{\frac{l(s+d)}{r}} \dots + \frac{s}{s+nd}q^{-\frac{s+nd}{r}}\kappa_{(n+1)} y^{\frac{l(s+nd)}{r}} \dots$

Q. E. D.

§. VIII.

Problema.

Data aequatione $y^l = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} \dots$ quaerere logarithmum naturalem variabilis x per seriem, secundum dignitates variabilis y progredientem.

Solutio.

Si $y^l = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} \dots$ est (§. V. et §. VII.) $x^s = \frac{s}{s}q^{-\frac{s}{r}}\kappa_1 y^{\frac{ls}{r}} + \frac{s}{s+d}q^{-\frac{s+d}{r}}\kappa_2 y^{\frac{l(s+d)}{r}} + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\kappa_3 y^{\frac{l(s+2d)}{r}} \dots\dots = \alpha^{-\frac{s}{r}} y^{\frac{ls}{r}} + \frac{s}{s+d}q^{-\frac{s+d}{r}}\kappa_2 y^{\frac{l(s+d)}{r}} + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\kappa_3 y^{\frac{l(s+2d)}{r}} \dots + \frac{s}{s+nd}q^{-\frac{s+nd}{r}}\kappa_{(n+1)} y^{\frac{l(s+nd)}{r}} \dots$ ergo $\frac{x^s - 1}{s} = \frac{\left(\alpha^{-\frac{1}{r}} y^{\frac{l}{r}}\right)^s - 1}{s} + \frac{1}{s+d}q^{-\frac{s+d}{r}}\kappa_2 y^{\frac{l(s+d)}{r}} + \frac{1}{s+2d}q^{-\frac{s+2d}{r}}\kappa_3 y^{\frac{l(s+2d)}{r}} \dots + \frac{1}{s+nd}q^{-\frac{s+nd}{r}}\kappa_{(n+1)} y^{\frac{l(s+nd)}{r}} + \dots$

Posito

Posito iam $s = \frac{1}{\infty}$ mutatur $\frac{x^s - 1}{s}$ in log. x, atque $\frac{\left(\alpha^{-\frac{1}{r}} y^{\frac{1}{r}}\right)^s - 1}{s}$ in log. $\alpha^{-\frac{1}{r}} y^{\frac{1}{r}}$; Est itaque log. $x = \log. \alpha^{-\frac{1}{r}} y^{\frac{1}{r}} + \frac{1}{d} q^{-\frac{d}{r}} \varkappa 2\, y^{\frac{1d}{r}} + \frac{1}{2d} q^{-\frac{2d}{r}} \varkappa 3\, y^{\frac{2ld}{r}} + \ldots + \frac{1}{nd} q^{-\frac{nd}{r}} \varkappa(n+1) y^{\frac{nld}{r}} + \ldots$ Q. E. I.

Exempla. Sit $y = \alpha x + \beta x^3 + \gamma x^5 \ldots$ erit log. $x = \log. \frac{y}{\alpha} + \frac{1}{2} q^{-2} \varkappa 2\, y^2 + \frac{1}{4} q^{-4} \varkappa 3\, y^4 + \frac{1}{6} q^{-6} \varkappa 4\, y^6 + \frac{1}{8} q^{-8} \varkappa 5\, y^8 + \frac{1}{10} q^{-10} \varkappa 6\, y^{10} + \ldots = \log. \frac{y}{\alpha} - \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^2 + \left[-\frac{\gamma}{\alpha} + \frac{5\beta^2}{2\alpha^2}\right]\left(\frac{y}{\alpha}\right)^4 + \left[-\frac{\delta}{\alpha} + \frac{7\beta\gamma}{\alpha^2} - \frac{28\beta^3}{3\alpha^3}\right]\left(\frac{y}{\alpha}\right)^6 + \left[-\frac{\varepsilon}{\alpha} + \frac{9\beta\delta}{\alpha^2} + \frac{9\gamma^2}{2\alpha^2} - \frac{45\beta^2\gamma}{\alpha^3} + \frac{165\beta^4}{4\alpha^4}\right]\left(\frac{y}{\alpha}\right)^8 + \left[-\frac{\zeta}{\alpha} + \frac{11\beta\varepsilon}{\alpha^2} + \frac{11\gamma\delta}{\alpha^2} - \frac{66\beta^2\delta}{\alpha^3} - \frac{66\beta\gamma^2}{\alpha^3} + \frac{286\beta^3\gamma}{\alpha^4} - \frac{1001\beta^5}{5\alpha^5}\right]\left(\frac{y}{\alpha}\right)^{10} +$ etc.

1) Sit $x = \text{tang } y$, siue $y = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \frac{1}{11}x^{11} \ldots$ atque erit log. $x = \log. \text{tang } y = \log. y + \frac{1}{3}y^2 + \left[-\frac{1}{5} + \frac{5}{2.9}\right]y^4 + \left[\frac{1}{7} - \frac{7}{3.5} + \frac{28}{3.3^3}\right]y^6 + \left[-\frac{1}{9} + \frac{9}{3.7} + \frac{9}{2.5^2} - \frac{45}{3^2.5} + \frac{165}{4.3^4}\right]y^8 + \left[\frac{1}{11} - \frac{11}{3.9} - \frac{11}{5.7} + \frac{66}{3^2.7} + \frac{66}{3.5^2} - \frac{286}{3^2.5} + \frac{1001}{5.3^5}\right]y^{10}$ etc. $= \log. y + \frac{1}{3}y^2 + \frac{7}{90}y^4 + \frac{62}{2835}y^6 + \frac{127}{18900}y^8 + \frac{146}{66825}y^{10} + \ldots$ (h).

2) Sit $x = \text{sin } y$, siue $y = x + \frac{1}{2.3}x^3 + \frac{1.3}{2.4.5}x^5 + \frac{1.3.5}{2.4.6.7}x^7 + \frac{1.3.5.7}{2.4.6.8.9.}x^9 + \frac{1.3.5.7.9}{2.4.6.8.10.11}x^{11} + \ldots = x + \frac{1}{2.3}x^3 + \frac{3}{2^3.5}x^5 + \frac{5}{2^4.7}x^7 + \frac{5.7}{2^7.3^2}x^9 + \frac{7.9}{2^8.11}x^{11} + \ldots$ atque est log. $x = \log. \text{sin } y = \log. y - \frac{1}{2.3}y^2 + \left[-\frac{3}{2^3.5} + \frac{5}{2.2^2.3^2}\right]y^4 + \left[-\frac{5}{2^4.7} + \frac{7.3}{2.3.2^3.5} - \frac{28}{3.2^3.3^3}\right]y^6 + \left[-\frac{5.7}{2^7.3^2} + \frac{9.5}{2.3.2^4.7} + \frac{9.3^2}{2.2^6\,5^2} - \frac{45.3}{2^2.3^2.2^3.5} + \frac{165}{4.2^4.3^4}\right]y^8 + \left[-\frac{7.9}{2^8.11} + \frac{11.5.7.}{2.3.2^7.3^2} + \frac{11.3.5}{2^3.5\,2^4.7} - \frac{66.5}{2^2.3^2.2^4.7} - \frac{66.3^2}{2.3.2^6.5^2} + \frac{286.3}{2^3.3^3.2^3.5} - \frac{1001}{5.2^5.3^5}\right]y^{10} + \ldots$
$\ldots = \log y - \frac{1}{6}y^2 - \frac{1}{180}y^4 - \frac{1}{2835}y^6 - \frac{1}{37800}y^8 - \frac{1}{467775}y^{10} - \ldots$ (i).

3)

(h) Si $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, *numeros* significant *Bernoullianos*, est log. tang $y = \log. y + \frac{2^3(2^1-1)}{1.2.2}\mathfrak{A}y^2 + \frac{2^5(2^3-1)}{1.2.3.4.4}\mathfrak{B}y^4 + \frac{2^7(2^5-1)}{1.2.3.4.5.6.6}\mathfrak{C}y^6 \ldots$

(i) log. sin $y = \log y - \frac{2^2\mathfrak{A}}{1.2.2}y^2 - \frac{2^4\mathfrak{B}y^4}{1.2.3.4.4} - \frac{2^6\mathfrak{C}\,y^6}{1.2.3.4.5.6.6} \ldots$ Haec formula aeque ac antecedens (nota h)

3) Sit $y = ax + bx^3$. Quoniam hic $q^m \varkappa(n+1) = {}^m\mathfrak{N}a^{m-n}b^n$, atque, $\log x\mathfrak{Z}(n+1)$

$$= \frac{1}{2n} q^{-2n}\varkappa(n+1) y^{2n} = \frac{1}{2n}{}^{-2n}\mathfrak{N}a^{-3n}b^n y^{2n} = \pm \frac{2n+1.\,2n+2.\,2n+3.\,..\,3n-1}{1.\quad 2.\quad 3.\quad 4\,...\,n} \cdot \frac{b^n y^{2n}}{a^{3n}}$$

erit $\log x = \log \frac{y}{a} - \frac{by^2}{a^3} + \frac{5}{1.2}\frac{b^2y^4}{a^6} - \frac{7.8}{1.2.3}\frac{b^3y^6}{a^9} + \frac{9.10.11.}{1.2.3.4}\frac{b^4y^8}{a^{12}} - \dots\dots$

§. IX.

Problema.

Data aequatione $az^{fr} + bz^{f(r+d)} + cz^{f(r+2d)} \dots = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} \dots$ quaerere x^s, per seriem secundum dignitates variabilis z progredientem.

Solutio.

Pro *Scalis* p $[a, b, c, d \dots]$ et q $[\alpha, \beta, \gamma, \delta \dots]$, erit, introducta variabili nova y, vtrique seriei aequali (§. V. et VII.).

$$x^s = \frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, y^{\frac{s}{r}} + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, y^{\frac{s+d}{r}} + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3\, y^{\frac{s+2d}{r}} + \frac{s}{s+3d}q^{-\frac{s+3d}{r}}\varkappa 4\, y^{\frac{s+3d}{r}} + \dots$$

Substituta vero loco y ubique serie $az^{fr} + bz^{f(r+d)} + cz^{f(r+2d)} + \dots$ per *Methodum*, ab *Hindenburgio* V. C. sic dictam, *potentiarum*, fit:

$$x^s = \frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, p^{\frac{s}{r}}\varkappa 1 \Big] z^{fs} + \left[\begin{array}{l} \frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, p^{\frac{s}{r}}\varkappa 2 \\ + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, p^{\frac{s+d}{r}}\varkappa 1 \end{array}\right] z^{f(s+d)} + \left[\begin{array}{l} \frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, p^{\frac{s}{r}}\varkappa 3 \\ + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, p^{\frac{s+d}{r}}\varkappa 2 \\ + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3\, p^{\frac{s+2d}{r}}\varkappa 1 \end{array}\right] z^{f(s+2d)} + \left[\begin{array}{l} \frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, p^{\frac{s}{r}}\varkappa 4 \\ + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, p^{\frac{s+d}{r}}\varkappa 3 \\ + \frac{s}{s+2d}q^{-\frac{s+2d}{r}}\varkappa 3\, p^{\frac{s+2d}{r}}\varkappa 2 \\ + \frac{s}{s+3d}q^{-\frac{s+3d}{r}}\varkappa 4\, p^{\frac{s+3d}{r}}\varkappa 1 \end{array}\right] z^{f(s+3d)}$$

+ etc. *Terminus* huius seriei *generalis* est

$$x^s\mathfrak{Z}(n+1) = \Big[\frac{s}{s}q^{-\frac{s}{r}}\varkappa 1\, p^{\frac{s}{r}}\varkappa(n+1) + \frac{s}{s+d}q^{-\frac{s+d}{r}}\varkappa 2\, p^{\frac{s+d}{r}}\varkappa n \dots + \frac{s}{s+\nu d}q^{-\frac{s+\nu d}{r}}\varkappa(\nu+1)\, p^{\frac{s+\nu d}{r}}\varkappa(n-\nu+1) \dots$$
$$\dots + \frac{s}{s+nd}q^{-\frac{s+nd}{r}}\varkappa(n+1)\, p^{\frac{s+nd}{r}}\varkappa 1\Big] z^{f(s+nd)}$$

Ex-

(nota h) facillime ope calculi differentialis demonstratur, adhibitis aequationibus: tang y = cotang. y − 2cotang. 2y, et tang $y = \frac{2^2(2^2-1)}{1\ 2}\mathfrak{A}y + \frac{2^4(2^4-1)}{1.\ 2.\ 3.\ 4}\mathfrak{B}y^3 + \frac{2^6(2^6-1)}{1.2.3.4.5.6}\mathfrak{C}y^5 \dots$ *Leipziger Magazin für reine und angewandte Mathematick; Zweites Stück* 1786. p. 269. Coefficientes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \dots$ etiam hic (vt in nota h) *numeros Bernoullianos* significant.

Exemplum. Sit $z + {}^{-3}\mathfrak{A}z^2 + {}^{-3}\mathfrak{B}z^3 + {}^{-3}\mathfrak{C}z^4 \ldots = x^2 + {}^{\frac{1}{2}}\mathfrak{A}x^4 + {}^{\frac{1}{2}}\mathfrak{B}x^6 + {}^{\frac{1}{2}}\mathfrak{C}x^8 \ldots$ (ſiue $z\,(1+z)^{-3} = x^2\,(1+x^2)^{\frac{1}{2}}$) et quaeratur $x^{\frac{5}{2}}$. Quoniam hic $s = \frac{5}{2}$, $f = \frac{1}{2}$, $r = d = 2$,

$$q^m\varkappa(n+1) = {}^{\frac{m}{2}}\mathfrak{N},\; p^m\varkappa(n+1) = {}^{-3m}\mathfrak{N},\text{ erit } x^{\frac{5}{2}} = z^{\frac{5}{4}} + \left[{}^{-\frac{15}{4}}\mathfrak{A} + \tfrac{5}{9}\,{}^{-\frac{9}{8}}\mathfrak{A}\right] z^{-\frac{9}{4}}$$

$$+ \left[{}^{-\frac{15}{4}}\mathfrak{B} + \tfrac{5}{9}\,{}^{-\frac{9}{8}}\mathfrak{A}\,{}^{-\frac{27}{4}}\mathfrak{A} + \tfrac{5}{13}\,{}^{-\frac{13}{8}}\mathfrak{B}\right] z^{\frac{13}{4}} + \left[{}^{-\frac{15}{4}}\mathfrak{C} + \tfrac{5}{9}\,{}^{-\frac{9}{8}}\mathfrak{A}\,{}^{-\frac{27}{4}}\mathfrak{B} + \tfrac{5}{13}\,{}^{-\frac{13}{8}}\mathfrak{B}\,{}^{-\frac{39}{4}}\mathfrak{A} + \tfrac{5}{17}\,{}^{-\frac{17}{8}}\mathfrak{C}\right] z^{\frac{17}{4}}$$

$$\text{etc.} \ldots = z^{\frac{5}{4}} - \tfrac{35}{8} z^{\frac{9}{4}} + \tfrac{1785}{128} z^{\frac{13}{4}} - \tfrac{43785}{1024} z^{\frac{17}{4}} + \ldots$$

Coroll. *Formula generalior* $az^f + bz^{f+g} + cz^{f+2g} \ldots = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} \ldots$ ad ſpecialiorem hac §pho propoſitam reduci poteſt, ſi $\frac{rg}{fd}$ numerus eſt *rationalis poſitiuus.* Sic v. c. aequatio $az^3 + bz^5 + cz^7 \ldots = \alpha x + \beta x^2 + \gamma x^3 \ldots$ ita exprimi poteſt: $az^3 + 0.\,z^4 + bz^5 + 0\,z^6 \ldots = \alpha x^{\frac{3}{3}} + 0.\,x^{\frac{4}{3}} + 0.\,x^{\frac{5}{3}} + \beta x^{\frac{6}{3}} + 0.\,x^{\frac{7}{3}} \ldots$ Idem valet de *formula generaliſſima:*

$(az^f + bz^{f+g} + cz^{f+2g} \ldots)^{\varkappa} = \alpha x^r + \beta x^{r+d} + \gamma x^{r+2d} \ldots$ ſi $\frac{rg}{\varkappa fd}$ numerus eſt *rationalis poſitiuus.*

Sed haec ſufficiant. Agam adhuc breuiter de *Leibnitiano* ad *ſerierum reverſionem* pertinente *problemate*, quod ipſis *Leibnitii* verbis *Hindenburgius Vir Excellentiſſimus* praefationi operis de *Infinitinominii Dignitatibus* p. XV. ſeqq. inſeruit.

§. X.

P r o b l e m a.

Sit data aequatio $0 = 01y + 02y^2 + 03y^3 + 04y^4 + 05y^5 \ldots + (-10 + 11y + 12y^2 + 13y^3 + 14y^4 + 15y^5 \ldots)z + (20 + 21y + 22y^2 + 23y^3 \ldots)z^2 + (30 + 31y + 32y^2 + 33y^3 \ldots)z^3 + (40 + 41y \ldots)z^4 \ldots$ etc. (k) exprimere z per ſeriem, ſecundum dignitates variabilis y progredientem.

S o l u t i o.

Nihil aliud neceſſarium eſt, quam vt in aequatione ficta $z = \dot{A}y + \dot{B}y^2 + \dot{C}y^3$ $\dot{D}y^4 + \dot{E}y^5 + \ldots$ determinentur valores coefficientium $\dot{A}$, $\dot{B}$, $\dot{C}$, $\dot{D}$, $\dot{E} \ldots$ quod ſic fiat:

Cum	$z =$	$\dot{A}y$	$+\ \dot{B}y^2$	$+\ \dot{C}y^3$	$+\ \dot{D}y^4$	$+\ \dot{E}y^5 \ldots$
eſt	$z^2 =$		$\mathfrak{b}^2By^2$	$+\ \mathfrak{b}^3By^3$	$+\ \mathfrak{b}^4By^4$	$+\ \mathfrak{b}^5By^5 \ldots$
„	$z^3 =$			$+\ \mathfrak{c}^3Cy^3$	$+\ \mathfrak{c}^4Cy^4$	$+\ \mathfrak{c}^5Cy^5 \ldots$
„	$z^4 =$				$+\ \mathfrak{d}^4Dy^4$	$+\ \mathfrak{d}^5Dy^5 \ldots$
„	$z^5 =$					$+\ \mathfrak{e}^5Ey^5 \ldots$

Sub-

(k) Aequationis coefficientes loco litterarum *numeris* hic deſignati ſunt, qui itaque tantum fictitii ſeu ſuppoſititii ſunt, quorumque nota prior potentiam litterae z, poſterior, potentiae litterae y aequalis eſt, ad quam coefficiens pertinet. De Coefficientium per numeros notatione Leibnitiana egit quoque *Hindenburgius* V. C. *Nov. Syſt. Comb.* p. XXXV.

Substituantur hic valores in aequatione data, atque haec prodit formula:

$$0 = (01y + 02y^2 + 03y^3 + 04y^4 + 05y^5 \ldots)$$
$$+ (-10 + 11y + 12y^2 + 13y^3 + 14y^4 \ldots)(\dot{A}y + \dot{B}y^2 + \dot{C}y^3 + \dot{D}y^4 + \dot{E}y^5 \ldots)$$
$$+ (20 + 21y + 22y^2 + 23y^3 \ldots)(b^2By^2 + b^3By^3 + b^4By^4 + b^5By^5 \ldots)$$
$$+ (30 + 31y + 32y^2 \ldots)(c^3Cy^3 + c^4Cy^4 + c^5Cy^5 \ldots)$$
$$+ (40 + 41y \ldots)(d^4Dy^4 + d^5Dy^5 \ldots) + (50 \ldots)(e^5Ey^5 \ldots) + \ldots$$ siue

$$0 = \left.\begin{array}{l} 01 \\ -10\,\dot{A} \end{array}\right] y + \left.\begin{array}{l} 02 \\ -10\,\dot{B} \\ +11\,\dot{A} \\ \\ \\ \\ +20\,b^2B \end{array}\right] y^2 + \left.\begin{array}{l} 03 \\ -10\,\dot{C} \\ +11\,\dot{B} \\ +12\,\dot{A} \\ \\ \\ +20\,b^3B \\ +21\,b^2B \\ \\ \\ +30\,c^3C \end{array}\right] y^3 + \left.\begin{array}{l} 04 \\ -10\,\dot{D} \\ +11\,\dot{C} \\ +12\,\dot{B} \\ +13\,\dot{A} \\ \\ +20\,b^4B \\ +21\,b^3B \\ +22\,b^2B \\ \\ +30\,c^4C \\ +31\,c^3C \\ \\ +40\,d^4D \end{array}\right] y^4 + \left.\begin{array}{l} 05 \\ -10\,\dot{E} \\ +11\,\dot{D} \\ +12\,\dot{C} \\ +13\,\dot{B} \\ +14\,\dot{A} \\ +20\,b^5B \\ +21\,b^4B \\ +22\,b^3B \\ +23\,b^2B \\ +30\,c^5C \\ +31\,c^4C \\ +32\,c^3C \\ +40\,d^5D \\ +41\,d^4D \\ +50\,e^5E \end{array}\right] y^5$$

Est itaque

$$\dot{A} = 01 : 10$$
$$\dot{B} = [02 + 11\,\dot{A} + 20\,b^2B] : 10$$
$$\dot{C} = \left[\begin{array}{l} 03 + 11\,\dot{B} + 20\,b^3B + 30\,c^3C \\ \quad + 12\,\dot{A} + 21\,b^2B \end{array}\right] : 10$$
$$\dot{D} = \left[\begin{array}{l} 04 + 11\,\dot{C} + 20\,b^4B + 30\,c^4C + 40\,d^4D \\ \quad + 12\,\dot{B} + 21\,b^3B + 31\,c^3C \\ \quad + 13\,\dot{A} + 22\,b^2B \end{array}\right] : 10$$
$$\dot{E} = \left[\begin{array}{l} 05 + 11\,\dot{D} + 20\,b^5B + 30\,c^5C + 40\,d^5D + 50\,e^5E \\ \quad + 12\,\dot{C} + 21\,b^4B + 31\,c^4C + 41\,d^4D \\ \quad + 13\,\dot{B} + 22\,b^3B + 32\,c^3C \\ \quad + 14\,\dot{A} + 23\,b^2B \end{array}\right] : 10$$

Index, ad claſſes combinatorias pertinens, eſt $\begin{bmatrix}\dot{A}, & \dot{B}, & \dot{C}, & \dot{D}\ldots\ldots\\ 1, & 2, & 3, & 4\ldots\ldots\end{bmatrix}$ (1)

Lex combinationis complicatiſſima, quam hic coefficientes ſeruant a *Leibnitio* verbis perquam obſcuris pronunciata, ſignis hic combinatorio-analyticis clariſſima reddita eſt. Iudicari itaque poteſt, etiam ex hoc exemplo, qua vtilitate, quoque commodo, ſigna analytico-combinatoria in Analyſi poſſint adhiberi.

Caeterum apparet, coefficientium $\dot{B}$, $\dot{C}$, $\dot{D}$, $\dot{E}$ etc. nullum extra ordinem his formulis exhiberi; quoniam aequationi, qua determinatur, omnes inſunt antecedentes. Si itaque deſiderentur formulae, quibus valores coefficientium iſtorum, *datis* ſolummodo *coefficientibus*, et ab *antecedentibus independenter*, exhiberentur, opus eſt, vt in qualibet aequatione, valores antecedentium ſubſtituantur. Fiat hoc, et has nanciſcimur aequationes:

$$\dot{A} = 01.10^{-1}$$

$$\dot{B} = 02.10^{-1} + 01.11.10^{-2} + 20.01^2.10^{-3}$$

$$\dot{C} = 03.10^{-1} + \left.\begin{matrix}01.12\\ 02.11\end{matrix}\right]10^{-2} + \left.\begin{matrix}(2)20.01.02\\ +\quad 01^2.21\\ +\quad 01.11^2\end{matrix}\right]10^{-3} + \left.\begin{matrix}(3)20.01^2.11.\\ 30.01^3\end{matrix}\right]10^{-4} + (2)20^2.01^3\,]10^{-5}$$

$$\dot{D} = 04.10^{-1} + \left.\begin{matrix}01.13\\ +\ 02.12\\ +\ 03.11\end{matrix}\right]10^{-2} + \left.\begin{matrix}(2)20.01.03\\ +\quad 20.02^2\\ +\quad 01^2.22\\ +\ (2)01.11.12\\ +\ (2)01.21.02\\ +\quad 11^2.02\end{matrix}\right]10^{-3} + \left.\begin{matrix}(3)\ 20.01^2.12\\ +\ (6)20.01.11.02\\ +\ (3)\ 30.01^2.02\\ +\quad 01^3.31\\ +\ (3)\ 01^2.11.21\\ +\quad 01.11^3\end{matrix}\right]10^{-4} + \left.\begin{matrix}(6)20^2.01^2.02\\ +\ (4)\ 20.01^3.21\\ +\ (6)20.01^2.11^2\\ +\ (4)\ 30\ 01^3.11\\ +\quad 40.01^4\end{matrix}\right]10^{-5}$$

$$+ \left.\begin{matrix}(10)20^2.01^3.11\\ +\ (5)\ 20.30.01^4\end{matrix}\right]10^{-6} + (5)20^3.01^4\,]10^{-7}$$

ubi numeri, parentheſibus incluſi, *veros* ſignificant *numeros*.

Quos valores, cum paulo accuratius contemplatus eſſem, legem in iis inveni, quae antea nondum eſt obſeruata, quaeque haec eſt:

Continet coefficiens quilibet:

1) Vnionem, cuius nota prior, 0, poſterior, numero coefficientis quaerendi aequalis eſt, ductum in 10^{-1}.

2) Se-

---

(1) Valores claſſium combinatoriarum hic occurrentium, depromi poſſunt ex Tab. II. huic libello adiuncta, ſcribendo ubique loco β, γ, δ, ε, ζ . . . reſpectiue $\dot{A}$, $\dot{B}$, $\dot{C}$, $\dot{D}$, $\dot{E}$......

2) Seriem binionum, ductam in 10^{-2}, ternionum, ductam in 10^{-3}, quaternionum, ductam in 10^{-4} etc., earumque omnium possibilium, quae ex datis coefficientibus (excepto unico 10) formari possunt, legemque sequuntur hanc:

3) Summa notarum *posteriorum* aequalis est numero coefficientis quaerendi, cuius complexio ipsa membrum est, *priorum* vero exponenti classis, ad quam complexio pertinet, demta unitate.

d) Coefficiens numericus, in quamlibet complexionem ductus, est quotiens, ortus, ex diuisione numeri permutationum, ad complexionem pertinentis, per exponentem classis, ad quam complexio pertinet.

Complexio, v. c. (2) 20.01.03 occurrit in valore coefficientis $\dot{D}$; est itaque summa notarum posteriorum $0+1+3=4$, quoniam $\dot{D}$ quartus est coefficiens, priorum vero, $2+0+0=3-1$, quoniam ternio est. Coefficiens numericus 2, ductus in complexionem, est quotiens, ortus, si numerus permutationum, ad complexionem istam pertinens, 6, diuiditur per exponentem classis, 3, ad quam complexio, quoniam ternio est, pertinent. In alia complexione (3) $30.01^2.02$, quae etiam valorem coefficientis $\dot{D}$ ingreditur, est summa notarum posteriorum $0+1+1+2=4$, priorum $3+0+0+0=4-1$, et coefficiens numericus $3=\frac{12}{4}$.

Diximus supra, complexiones omnes possibiles, quae ex datis coefficientibus formatae, legem sequuntur praescriptam, exhibendas esse; notandum vero est, nullam complexionem plus quam vice simplici occurrere posse, siue nullam complexionem admittendam esse, quae, mutato partium, h. e. coefficientium ex quibus composita est, situ, iam adfuit.

Methodum, cuius ope omnes complexiones possibiles, sine magno erui possunt labore, inueni quidem, afferre autem nolui, quoniam, praeterea quod ipsa facillima sit, [consistit enim in numerorum discerptionibus (m) atque variationibus (n)] eius inuestigatio etiam, non difficilis est. Methodi istius ope, coefficiens quidem quilibet extra ordinem, ab antecedentibus independenter, datis solummodo coefficientibus, potest determinari, sed commodum inde proueniens, vt opus non sit coefficientibus anterioribus, alia ex parte perit hic fere totum, et multitudine membrorum tollitur, quae iam in coefficiente quinto permagna est. Verum tamen si quis terminum aliquem extra ordinem desideret, methodus alia quaerendi eum, ab antecedentibus independenter,

 nulla

(m) *Infinitinomii Dignit* § XXII. p. 73.

(n) *Ibid.* §. XXVII, 4. pag. 129.

nulla ſuppetit. Patet autem, quaerendorum, ſiue independenter, ſiue per antecedentes, terminorum in hoc aliisque id genus problematibus, *leges*, nulla alia ratione poſſe, tam breuiter, tamque perſpicue, tamque etiam ad vſum accommodate, quam *formularum combinatorio-analyticarum* ope, exhiberi.

Res ipſa tam memorabilis mihi viſa fuit, vt plane ſilentio eam praeterire noluerim, praeſertim cum coefficientes etiam ſeriei, quamcunque dignitatem exponentis integri poſitiui litterae z, per y experimentis, ſimilem legem obſeruent. Conſtat enim $n^m N$, ad indicem $\begin{bmatrix} \dot{A}, & \dot{B}, & \dot{C}, & \dot{D} \ldots \\ 1, & 2, & 3, & 4 \ldots \end{bmatrix}$ pertinens, ex ſerie con(n)nationum, ducta in 10^{-n}, ex ſerie con($n+1$)nationum ducta in $10^{-(n+1)}$ etc. usque tandem ad con($2m-n$)nationem unam peruenerimus, quae ducta eſt in $10^{-(2m-n)}$. Serierum iſtarum complexio quaevis multiplicata eſt in coefficientem numericum, qui eſt quotiens, ortus, ex diuiſione, producti e numero permutationum, in n, per exponentem claſſis; Summa notarum poſteriorum aequalis eſt m, priorum exponenti claſſis, demtis n unitatibus, habetque quaelibet ſeries complexiones omnes, quae ex datis coefficientibus (excepto, ut antea, unico 10) ſic formari poſſunt, vt legem iſtam ſequantur.

THESES.

THESES.

I.

Ad iuuenum ingenium acuendum et formandum, aptiſſima Geometria eſt.

II.

Errant, qui exiſtimant, lucis et ſoni reflexiones iisdem plane legibus contineri.

III.

Varia argumenta, quae contra motus exiſtentiam in medium attulerunt veteres philoſophi, merae ſunt argutiae, mera ſophiſmata. Dari veras, easdemque grauiſſimas, difficultates in motibus corporum definiendis, Optica oſtendit cum Aſtronomia.

IIII.

Nullius corporis poſſumus in Phyſica aut Aſtronomia motum definire abſolutum. Motus, quos nouimus et definimus, omnes ſunt relatiui.

V.

Formula *Cl. Fiſcheri* (a)

$$\text{A)}\ x = -\frac{a}{b} + \frac{ca^3}{b^4} - \frac{6}{2}\frac{c^2a^5}{b^7} + \frac{8.9}{2.3}\frac{c^3a^7}{b^{10}} - \text{etc.}$$

qua radicem aequationis $0 = a + bx + cx^3$ exhibuit, a *formula Cardani* tota pendet, et inde poteſt derivari, tantum abeſt, vt illa ad formulam a Cardanica diuerſam et ſimpliciorem viam monſtrare poſſit; id quod *Fiſcherus* non vidit, operamque adeo in collatione formulae ſuae cum Cardanica inutiliter collocauit. Ponatur enim in formula

$$\text{Cardanica}\ x = \sqrt[3]{\left\{-\frac{1}{2}\frac{a}{c} + \sqrt{\left(\frac{1}{4}\frac{a^2}{c^2} + \frac{1}{27}\frac{b^3}{c^3}\right)}\right\}} + \sqrt[3]{\left\{-\frac{1}{2}\frac{a}{c} - \sqrt{\left(\frac{1}{4}\frac{a^2}{c^2} + \frac{1}{27}\frac{b^3}{c^3}\right)}\right\}}$$

breuitatis cauſa $\frac{1}{2}\frac{a}{c} = A$, $\sqrt{\left(\frac{1}{4}\frac{a^2}{c^2} + \frac{1}{27}\frac{b^3}{c^3}\right)} = B$, atque eſt

$$x = \sqrt[3]{-A+B} + \sqrt[3]{-A-B} = \sqrt[3]{B-A} - \sqrt[3]{B+A}$$

Eſt autem:

$$\sqrt[3]{B-A} = B^{\frac{1}{3}} - {}^{\frac{1}{3}}\mathfrak{A}B^{-\frac{2}{3}}A + {}^{\frac{1}{3}}\mathfrak{B}B^{-\frac{5}{3}}A^2 - {}^{\frac{1}{3}}\mathfrak{C}B^{-\frac{8}{3}}A^3 + \text{etc.}$$

$$\sqrt[3]{B+A} = B^{\frac{1}{3}} + {}^{\frac{1}{3}}\mathfrak{A}B^{-\frac{2}{3}}A + {}^{\frac{1}{3}}\mathfrak{B}B^{-\frac{5}{3}}A^2 + {}^{\frac{1}{3}}\mathfrak{C}B^{-\frac{8}{3}}A^3 + \text{etc.}$$

$$\text{Ergo}\ x = -2\left\{{}^{\frac{1}{3}}\mathfrak{A}B^{-\frac{2}{3}}A + {}^{\frac{1}{3}}\mathfrak{C}B^{-\frac{8}{3}}A^3 + {}^{\frac{1}{3}}\mathfrak{E}B^{-\frac{14}{3}}A^5 + \ldots\ldots\right\}$$

Hinc

(a) *Theorie der Dimenſionszeichen Th. II. §. 211, 212. S. 26-29.*

Hinc, ſi loco A et B, eorum valores iterum ſubſtituuntur

$$x = -\frac{2.\,\frac{1}{2}\,{}^{\frac{1}{3}}\mathfrak{A}\,\frac{a}{c}}{\left(\frac{1}{27}\frac{b^3}{c^3}+\frac{1}{4}\frac{a^2}{c^2}\right)^{\frac{1}{3}}} - \frac{2.\,\frac{1}{2^3}\,{}^{\frac{1}{3}}\mathfrak{C}\,\frac{a^3}{c^3}}{\left(\frac{1}{27}\frac{b^3}{c^3}+\frac{1}{4}\frac{a^2}{c^2}\right)^{\frac{4}{3}}} - \frac{2.\,\frac{1}{2^5}\,{}^{\frac{1}{3}}\mathfrak{E}\,\frac{a^5}{c^5}}{\left(\frac{1}{27}\frac{b^3}{c^3}+\frac{1}{4}\frac{a^2}{c^2}\right)^{\frac{7}{3}}} - \text{etc.}$$

$$= -\frac{3\,{}^{\frac{1}{3}}\mathfrak{A}a}{\left(b^3+\frac{3^3}{2^2}a^2c\right)^{\frac{1}{3}}} - \frac{3^4.\,\frac{1}{2^2}\,{}^{\frac{1}{3}}\mathfrak{C}a^3c}{\left(b^3+\frac{3^3}{2^2}a^2c\right)^{\frac{4}{3}}} - \frac{3^7.\,\frac{1}{2^4}\,{}^{\frac{1}{3}}\mathfrak{E}a^5c^2}{\left(b^3+\frac{3^3}{2^2}a^2c\right)^{\frac{7}{3}}} - \text{etc.}$$

cuius aequationis ſi terminus quilibet in ſeriem infinitam expanditur, prodit

$$x = -3\,{}^{\frac{1}{3}}\mathfrak{A}\frac{a}{b} - \frac{3^4}{2^2}\left[{}^{-\frac{1}{3}}\mathfrak{A}\,{}^{\frac{1}{3}}\mathfrak{A} + {}^{\frac{1}{3}}\mathfrak{C}\right]\frac{ca^3}{b^4} - \frac{3^7}{2^4}\left[{}^{-\frac{1}{3}}\mathfrak{B}\,{}^{\frac{1}{3}}\mathfrak{A} + {}^{-\frac{4}{3}}\mathfrak{A}\,{}^{\frac{1}{3}}\mathfrak{C} + {}^{\frac{1}{3}}\mathfrak{E}\right]\frac{c^2a^5}{b^7} - \text{etc.}$$

quae formula, facta Coefficientium Binomialium numerica reductione, ipſiſſimam *Fiſcheri* exhibet *aequationem* A.

VI.

Falſo *Fiſcherus* exiſtimat, (b) *Dignitatum huiusmodi:*

$$y^n = (a+b+c+d+e+f+\text{etc.})^n$$

Euolutionem complicatiſſimam, taediique pleniſſimam eſſe, et quidem *per ſe* ac *ſua natura*, remedium autem tollendis difficultatibus afferri nullum poſſe. Scilicet *Methodus*, pro huiusmodi dignitatibus *longe facillima*, *Hindenburgiana* vires *theoriae Fiſcherianae* tranſcendit, quae *Complexionibus* tantum, ſiue *ſectionibus* vtitur *numeri definiti.* Adhibitis autem *Complexionibus ſimpliciter*, *admiſſis* quidem *repetitionibus*, quarum vſum *Fiſcherus* neglexit, formula prodit *combinatorio-analytica*,

Pro numero n, et *qualitate*, et *quantitate*, quocunque:

$$y^n = a^n + {}^n\mathfrak{A}a^{n-1}\,a'A + {}^n\mathfrak{B}a^{n-2}\,b'B.\;.\;.\;+ {}^n\mathfrak{M}a^{n-m}\,m'M.\;.\;.\;.\;.$$
$$(b,\ c,\ d,\ e,\ f,\ g,\ .\;.\;.\;.\;.)$$

Pro numero autem n *integro poſitiuo:*

$$\underset{(a,b,c,d\ldots)}{y^n = n'N} = a^n + {}^n\mathfrak{A}a^{n-1}\,a'A + {}^n\mathfrak{B}a^{n-2}\,b'B + {}^n\mathfrak{C}a^{n-3}\,c'C.\;.\;.\;.\;+ a^0 n'N \quad \text{(c)}$$
$$(b,\ c,\ d,\ e.\;.\;.\;.\;.)$$

Complexiones autem *ſimplices* rerum quotlibet datarum *a*, *b*, *c*, *d*, . . . ſecundum *Indicem* Claſſis vel *definitum* vel *indefinitum* quemcunque; h. e. *Claſſes* 'A, 'B, 'C, 'D...'N, per regulas ac formulas ab *Exc. Hindenburgio* (*Nov. Syſt. Perm.* p. XIX. atque *Infin. Dign.* p. 17, 18, et 157, 158.) traditas, pari facilitate per elementa propoſita *a*, *b*, *c*, *d*. . . . exhiberi, qua numeri poſſunt ex notis vulgaribus 0, 1, 2, 3. . . ſcribi ex ordine; quo fit, vt dignitatum hic propoſitarum y^n reddantur *longe facillimae* Euolutiones. Ex-

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(b) *Theorie der Dimenſionszeichen Th. I.* §. 50. *S.* 32.

(c) Hanc formulam, pro numero *integro poſitivo* n, excitauit quoque *Clariſſimus Toepferus* in libelli: *Combinatoriſche Analytik und Theorie der Dimenſionszeichen, in Parallele geſtellt*, pag. 154. et Tab. VII, II, a. Sed ibi, typothetarum vitio, poſitum $n'\mathfrak{N}$ pro $n'\mathfrak{N}$ vel $n'\mathcal{N}$, atque etiam *Coefficientes* exciderunt *binomiales* ${}^n\mathfrak{A}$, ${}^n\mathfrak{B}$, ${}^n\mathfrak{C}$. . . ., quod, vt monerem lectores, harum rerum non ſatis peritos, me rogauit.

VII.

Expoſitio *Legis*, qua *coefficientes numerici* dignitatum a ſerie

$$z = y + \frac{1}{2.3}y^3 + \frac{1.3}{2.4.5}y^5 + \frac{1.3.5}{2.4.6.7}y^7 + \text{etc.}$$

(quae *arcum* z exprimit per *ſinum* ipſius y) continentur, vires *Theoriae Fiſcherianae* ſuperat. Ad *Complexiones* enim illa *ſimplices, non admiſſis repetitionibus*, deducit. Hinc formulae prodeunt *combinatorio-analyticae*,

pro dignitatibus, *prima* et *ſecunda*

$$\frac{z^1}{1} = y + \frac{1}{2.3}y^3 + \frac{1.3}{2.4.5}y^5 + \frac{1.3.5}{2.4.6.7}y^7 + \frac{1.3.5.7}{2.4.6.8.9}y^9 + \text{etc.}$$

$$\frac{z^2}{1.2} = \frac{1}{2}y^2 + \frac{2}{3.4}y^4 + \frac{2.4}{3.5.6}y^6 + \frac{2.4.6}{3.5.7.8}y^8 + \frac{2.4.6.8.}{3.5.7.9.10}y^{10} + \text{etc.} \quad \text{(d)}$$

pro dignitatibus, *tertia* et *quarta*

$$\frac{z^3}{1.2.3} = \frac{1}{1.2.3}\left[\frac{1}{1^2}\right]y^3 + \frac{1.3}{2.4.5}\left[\frac{1}{1^2} + \frac{1}{3^2}\right]y^5 + \frac{1.3.5}{2.4.6.7}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right]y^7$$
$$+ \frac{1.3.5.7}{2.4.6.8.9}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}\right]y^9 + \text{etc.} \quad \text{(e)}$$

$$\frac{z^4}{1.2.3.4} = \frac{2}{3.4}\left[\frac{1}{2^2}\right]y^4 + \frac{2.4}{3.5.6}\left[\frac{1}{2^2} + \frac{1}{4^2}\right]y^6 + \frac{2.4.6}{3.5.7.8}\left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2}\right]y^8 + \frac{2.4.6.8}{3.5.7.9.10}\left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2}\right]y^{10} + \text{etc.}$$

pro dignitatibus, *quinta* et *ſexta*

$$\frac{z^5}{1.2.3.4.5} = \frac{1.3}{2.4.5}\overset{B^{\backprime}}{\left(\frac{1}{1^2}, \frac{1}{3^2}\right)}y^5 + \frac{1.3.5}{2.4.6.7}\overset{B^{\backprime}}{\left(\frac{1}{1^2}, \frac{1}{3^2}, \frac{1}{5^2}\right)}y^7 + \frac{1.3.5.7}{2.4.6.8.9}\overset{B^{\backprime}}{\left(\frac{1}{1^2}, \frac{1}{3^2}, \frac{1}{5^2}, \frac{1}{7^2}\right)}y^9 + \text{etc.}$$

$$\frac{z^6}{1.2.3.4.5.6} = \frac{2.4}{3.5.6}\overset{B^{\backprime}}{\left(\frac{1}{2^2}, \frac{1}{4^2}\right)}y^6 + \frac{2.4.6}{3.5.7.8}\overset{B^{\backprime}}{\left(\frac{1}{2^2}, \frac{1}{4^2}, \frac{1}{6^2}\right)}y^8 + \frac{2.4.6.8}{3.5.7.9.10}\overset{B^{\backprime}}{\left(\frac{1}{2^2}, \frac{1}{4^2}, \frac{1}{6^2}, \frac{1}{8^2}\right)}y^{10} \text{ etc.} \quad \text{(f)}$$

ſimiliterque in *reliquis* dignitatibus, ſic vt

in *ſeptima* et *octava* dignitate *Terniones*

C\`

(d) Cf. Praefatio huius libelli.

(e) Hinc ſi, 2z = x, adeoque ſin. verſ. 2z = ſin. verſ. x = v, atque $v = 2y^2$, ſiue $\left(\frac{v}{2}\right)^{\frac{1}{2}} = y$, erit

$$\frac{1}{1.2.3} \cdot \frac{1}{8}x^3 = \frac{1}{1.2.3}\left[\frac{1}{1^2}\right]\left(\frac{v}{2}\right)^{\frac{3}{2}} + \frac{1.3}{2.4.5}\left[\frac{1}{1^2} + \frac{1}{3^2}\right]\left(\frac{v}{2}\right)^{\frac{5}{2}} + \frac{1.3.5}{2.4.6.7}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2}\right]\left(\frac{v}{2}\right)^{\frac{7}{2}} ..$$

Series haec eſt *Fiſcheriana* K, diuiſa per 2. 3. 8, cuius legem exponere non potuit. *Theorie der Dimenſ. II. Th. §. 327. S. 128.*

(f) Nempe *Hindenburgius*, *Vir Celeberr. Claſſes Combinationum ſimpliciter*, *admiſſis quidem repetitionibus*, ſignis 'A, 'B, 'C, 'D...., ſed, *non admiſſis repetitionibus*, ſignis A\`, B\`, C\`, D\`. . . . . notat, ſubſcripto ſimul *ſingulis Claſſibus* Indice, ſi non idem eſt in omnibus Claſſibus, ſed diverſus, ſiue *Coefficientibus ipſis* $\underset{(a,\, b,\, c\, \ldots)}{B^{\backprime}}$ $\underset{(\alpha,\, \beta,\, \gamma\, \ldots)}{C^{\backprime}}$ etc. ſiue, vt hic eſt *Coefficientium* tantum *numero*: $\underset{(a,\, b)}{B^{\backprime}}$ $\underset{(a,\, b,\, c)}{B^{\backprime}}$ etc. vel $\underset{(a,\, b,\, c)}{C^{\backprime}}$ $\underset{(a,\, b,\, c,\, d)}{C^{\backprime}}$ etc. Quod ſi vero *idem* manet per omnes Claſſes *Index*, formulae *ille* combinatorio-analyticae, more conſueto, non ſingulis Claſſibus, apponitur.

$$\overset{C'}{\left(\frac{1}{1^2},\frac{1}{3^2},\frac{1}{5^2}\right)}\overset{C'}{\left(\frac{1}{1^2},\frac{1}{3^2},\frac{1}{5^2},\frac{1}{7^2}\right)}\overset{C'}{\left(\frac{1}{1^2},\frac{1}{3^2},\frac{1}{5^2},\frac{1}{7^2},\frac{1}{9^2}\right)}\ldots \text{et} \overset{C'}{\left(\frac{1}{2^2},\frac{1}{4^2},\frac{1}{6^2}\right)}\overset{C''}{\left(\frac{1}{2^2},\frac{1}{4^2},\frac{1}{6^2},\frac{1}{8^2}\right)}\ldots$$

et in $(2n+1)$*ta* et $(2n+2)$*ta* dignitate, N*tiones* occurrant; ad indices nimirum $\left(\frac{1}{1^2},\frac{1}{3^2},\frac{1}{5^2}\ldots\right)$ pro $(2n+1)$*ta*, et $\left(\frac{1}{2^2},\frac{1}{4^2},\frac{1}{6^2},\frac{1}{8^2}\ldots\right)$ pro $(2n+2)$*ta* dignitate.

VIII.

Ex VI et VII liquido apparet, pessime et sibi et lectoribus consuluisse *Fischerum*, quod *Complexionum* tantummodo *numeri definiti*, et *admissis quidem repetitionibus*, rationem habuerit, *reliquas* autem *Complexiones* omnes, *admissis* et *non admissis repetitionibus*, simul cum *Permutationibus*, neglexerit. Scilicet *Fischerus* V. Cl. hausit sua quae habet et fundamenti loco posuit, omnia, *signa* nempe *dimensionalia*, quae vocat, eorumque *per numeros euolutionem*, *propositiones* etiam de *Infinitinomii dignitatibus elementares*, quae *Methodi potentiarum* inprimis ope problematibus aliis quam plurimis accommodantur; haec omnia hausit *Fischerus* ex *Hindenburgii Viri Celeberrimi* libro, cuius ne titulum quidem nominauit, de *Infinitinomii Dignitatibus*, in quo *Methodus Hindenburgiana* solis fere complexionibus numeri definiti admissis repetitionibus continetur, neglectis reliquis Complexionibus, *Doctrinae* etiam *Combinatoriae* in vniuersum vsus in Analysi, praeter duo illa Problemata de *sectionibus numeri definiti*, non tam clare atque perspicue, vt factum est postea, nec verbis nec exemplis expositus, *Characteristica* denique *combinatorio-analytica* non adeo perfecta et numeris omnibus absoluta est, vti ea in altero *Hindenburgii* libro, biennio post edito: *Novi Systematis Permutationum Combinationum* ac *Variationum primae lineae*, apparet. Atque hunc quidem librum *Hindenburgianum* expressis verbis in Praefatione libri sui excitauit *Fischerus*, sed tamen, cum combinatoriis, quae in eo tractantur, problematibus, susque deque habuit, haud dubie propterea, quod horum problematum in Analysi vsum longe maximum non perspiceret, existimans, se posse, alia prorsus, et longe faciliori ratione, quam si lectores per combinatorios, qui *ipsi videbantur scilicet*, anfractus deduceret, *theoriae* nimirum, quam vocat, *signorum dimensionalium* ope, ea omnia, quae *Hindenburgius* praestitit et praestare potest, atque etiam, si Diis placet, maiora efficere (g). Quam vana autem haec fuerit, quamque misere eum fefellerit spes, euentus docuit, et amicissimus *Toepferus*, edito peculiari, bonaeque frugis plenissimo libello (*Combinatorische Analytick* etc.), inprimis autem huius libelli sectione vltima, docte atque peregregie demonstrauit.

Quodsi autem *Fischerus* in propositionibus atque problematibus facilioribus sibi consulere non potuit, in grauioribus eum ac longe difficilioribus *combinatorio-analyticis* hoc potuisse verisimile non est; quodquidem argumentum, per se grauissimum, plenam demonstrandi vim accipit, si cum iis, quae hic atque in praefatione proposui, comparatur.

(g) Expressis verbis contendit *Fischerus* (in Praefat. p. V.) *unicam* et *simplicem* et *facilem* characteristicam, *a se* adhibitam, ad problematum ab *Hindenburgio* propositorum (*Nov. Syst Comb.* p. XXVI-XXXII.) ducere posse solutionem. *Signa* nimirum intelligit *dimensionalia*, ab *Hindenburgii* characteristica *combinatoria* desumta, sed *vitiose* expressa et cum *detrimento Harmoniae*, quae in concursu signorum *Hindenburgianorum* conspicua est, et occasionem adeo, propositiones nouas reperiundi, praebere potest. *Toepf. Combin. Analyt.* S. 54—61 .S. 148. Nota ζ und S. 170—173.

TABU-

TABULAE COMBINATORIAE

1) Sectionum siue Discerptionum Numerorum definitorum inde ab 1-10.

2) Valorum huiusmodi Sectionum numericarum, positis literis loco numerorum et additis Permutationum numeris.

Tabula. I.

Complexionum Classes a numeris 1 —— 10. Classes Complexionum, numeri propositi n, prima, secunda, tertia, etc. designantur per ^{n}A, ^{n}B, ^{n}C, . . . etc. *Infinitinomii Dignitat.* §. XXII. p. 85, 7 et p. 166. *Discerptionum* problema traditur p. 73. seq.

^{1}A	1
^{2}A	2
^{2}B	1,1
^{3}A	3
^{3}B	1,2
^{3}C	1,1,1
^{4}A	4
^{4}B	1,3
	2,2
^{4}C	1,1,2
^{4}D	1,1,1,1
^{5}A	5
^{5}B	1,4
	2,3
^{5}C	1,1,3
	1,2,2
^{5}D	1,1,1,2
^{5}E	1,1,1,1,1
^{6}A	6
^{6}B	1,5
	2,4
	3,3
^{6}C	1,1,4
	1,2,3
	2,2,2
^{6}D	1,1,1,3
	1,1,2,2
^{6}E	1,1,1,1,2
^{6}F	1,1,1,1,1,1
^{7}A	7
^{7}B	1,6
	2,5
	3,4
^{7}C	1,1,5
	1,2,4
	1,3,3
	2,2,3
^{7}D	1,1,1,4
	1,1,2,3
	1,2,2,2
^{7}E	1,1,1,1,3
	1,1,1,2,2
^{7}F	1,1,1,1,1,2
^{7}G	1,1,1,1,1,1,1
^{8}A	8
^{8}B	1,7
	2,6
	3,5
	4,4
^{8}C	1,1,6
	1,2,5
	1,3,4
	2,2,4
	2,3,3
^{8}D	1,1,1,5
	1,1,2,4
	1,1,3,3
	1,2,2,3
	2,2,2,2
^{8}E	1,1,1,1,4
	1,1,1,2,3
	1,1,2,2,2
^{8}F	1,1,1,1,1,3
	1,1,1,1,2,2
^{8}G	1,1,1,1,1,1,2
^{8}H	1,1,1,1,1,1,1,1
^{9}A	9
^{9}B	1,8
	2,7
	3,6
	4,5
^{9}C	1,1,7
	1,2,6
	1,3,5
	1,4,4
	2,2,5
	2,3,4
	3,3,3
^{9}D	1,1,1,6
	1,1,2,5
	1,1,3,4
	1,2,2,4
	1,2,3,3
	2,2,2,3
^{9}E	1,1,1,1,5
	1,1,1,2,4
	1,1,1,3,3
	1,1,2,2,3
	1,2,2,2,2
^{9}F	1,1,1,1,1,4
	1,1,1,1,2,3
	1,1,1,2,2,2
^{9}G	1,1,1,1,1,1,3
	1,1,1,1,1,2,2
^{9}H	1,1,1,1,1,1,1,2
^{9}I	1,1,1,1,1,1,1,1,1
^{10}A	10
^{10}B	1,9
	2,8
	3,7
	4,6
	5,5
^{10}C	1,1,8
	1,2,7
	1,3,6
	1,4,5
	2,2,6
	2,3,5
	2,4,4
	3,3,4
^{10}D	1,1,1,7
	1,1,2,6
	1,1,3,5
	1,1,4,4
	1,2,2,5
	1,2,3,4
	1,3,3,3
	2,2,2,4
	2,2,3,3
^{10}E	1,1,1,1,6
	1,1,1,2,5
	1,1,1,3,4
	1,1,2,2,4
	1,1,2,3,3
	1,2,2,2,3
	2,2,2,2,2
^{10}F	1,1,1,1,1,5
	1,1,1,1,2,4
	1,1,1,1,3,3
	1,1,1,2,2,3
	1,1,2,2,2,2
^{10}G	1,1,1,1,1,1,4
	1,1,1,1,1,2,3
	1,1,1,1,2,2,2
^{10}H	1,1,1,1,1,1,1,3
	1,1,1,1,1,1,2,2
^{10}I	1,1,1,1,1,1,1,1,2
^{10}K	1,1,1,1,1,1,1,1,1,1

Regulam, quamcunque Classem a praecedentibus independenter reperiundi, tradidit *Toepferus Cl. Combin. Anal. S.* 80 — 83.

Tabula. II.

Complexionum Classes, cum Coefficientibus polynomialibus numericis, siue numeris permutationum, positis pro

Numeris 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 Tabulae I

Coefficientibus $\beta, \gamma, \delta, \varepsilon, \zeta, \eta, \vartheta, \iota, \varkappa, \lambda$. Conf. pag. 2. nota b

$a^1A =$	β
$a^2A =$	γ
$b^2B =$	β^2
$a^3A =$	δ
$b^3B =$	$2\beta\gamma$
$c^3C =$	β^3
$a^4A =$	ε
$b^4B =$	$2\beta\delta$, γ^2
$c^4C =$	$3\beta^2\gamma$
$d^4D =$	β^4
$a^5A =$	ζ
$b^5B =$	$2\beta\varepsilon$, $2\gamma\delta$
$c^5C =$	$3\beta^2\delta$, $3\beta\gamma^2$
$d^5D =$	$4\beta^3\gamma$
$e^5E =$	β^5
$a^6A =$	η
$b^6B =$	$2\beta\zeta$, $2\gamma\varepsilon$, δ^2
$c^6C =$	$3\beta^2\varepsilon$, $6\beta\gamma\delta$, γ^3
$d^6D =$	$4\beta^3\delta$, $6\beta^2\gamma^2$
$e^6E =$	$5\beta^4\gamma$
$f^6F =$	β^6
$a^7A =$	ϑ
$b^7B =$	$2\beta\eta$, $2\gamma\zeta$, $2\delta\varepsilon$
$c^7C =$	$3\beta^2\zeta$, $6\beta\gamma\varepsilon$, $3\beta\delta^2$, $3\gamma^2\delta$
$d^7D =$	$4\beta^3\varepsilon$, $12\beta^2\gamma\delta$, $4\beta\gamma^3$
$e^7E =$	$5\beta^4\delta$, $10\beta^3\gamma^2$
$f^7F =$	$6\beta^5\gamma$
$g^7G =$	β^7
$a^8A =$	ι
$b^8B =$	$2\beta\vartheta$, $2\gamma\eta$, $2\delta\zeta$, ε^2
$c^8C =$	$3\beta^2\eta$, $6\beta\gamma\zeta$, $6\beta\delta\varepsilon$, $3\gamma^2\varepsilon$, $3\gamma\delta^2$
$d^8D =$	$4\beta^3\zeta$, $12\beta^2\gamma\varepsilon$, $6\beta^2\delta^2$, $12\beta\gamma^2\delta$, γ^4
$e^8E =$	$5\beta^4\varepsilon$, $20\beta^3\gamma\delta$, $10\beta^2\gamma^3$
$f^8F =$	$6\beta^5\delta$, $15\beta^4\gamma^2$
$g^8G =$	$7\beta^6\gamma$
$h^8H =$	β^8
$a^9A =$	$\varkappa$
$b^9B =$	$2\beta\iota$, $2\gamma\vartheta$, $2\delta\eta$, $2\varepsilon\zeta$
$c^9C =$	$3\beta^2\vartheta$, $6\beta\gamma\eta$, $6\beta\delta\zeta$, $3\beta\varepsilon^2$, $3\gamma^2\zeta$, $6\gamma\delta\varepsilon$, δ^3
$d^9D =$	$4\beta^3\eta$, $12\beta^2\gamma\zeta$, $12\beta^2\delta\varepsilon$, $12\beta\gamma^2\varepsilon$, $12\beta\gamma\delta^2$, $4\gamma^3\delta$
$e^9E =$	$5\beta^4\zeta$, $20\beta^3\gamma\varepsilon$, $10\beta^3\delta^2$, $30\beta^2\gamma^2\delta$, $5\beta\gamma^4$
$f^9F =$	$6\beta^5\varepsilon$, $30\beta^4\gamma\delta$, $20\beta^3\gamma^3$
$g^9G =$	$7\beta^6\delta$, $21\beta^5\gamma^2$
$h^9H =$	$8\beta^7\gamma$
$i^9I =$	β^9
$a^{10}A =$	λ
$b^{10}B =$	$2\beta\varkappa$, $2\gamma\iota$, $2\delta\vartheta$, $2\varepsilon\eta$, ζ^2
$c^{10}C =$	$3\beta^2\iota$, $6\beta\gamma\vartheta$, $6\beta\delta\eta$, $6\beta\varepsilon\zeta$, $3\gamma^2\eta$, $6\gamma\delta\zeta$, $3\gamma\varepsilon^2$, $3\delta^2\varepsilon$
$d^{10}D =$	$4\beta^3\vartheta$, $12\beta^2\gamma\eta$, $12\beta^2\delta\zeta$, $6\beta^2\varepsilon^2$, $12\beta\gamma^2\zeta$, $24\beta\gamma\delta\varepsilon$, $4\beta\delta^3$, $4\gamma^3\varepsilon$, $6\gamma^2\delta^2$
$e^{10}E =$	$5\beta^4\eta$, $20\beta^3\gamma\zeta$, $20\beta^3\delta\varepsilon$, $30\beta^2\gamma^2\varepsilon$, $30\beta^2\gamma\delta^2$, $20\beta\gamma^3\delta$, γ^5
$f^{10}F =$	$6\beta^5\zeta$, $30\beta^4\gamma\varepsilon$, $15\beta^4\delta^2$, $60\beta^3\gamma^2\delta$, $15\beta^2\gamma^4$
$g^{10}G =$	$7\beta^6\varepsilon$, $42\beta^5\gamma\delta$, $35\beta^4\gamma^3$
$h^{10}H =$	$8\beta^7\delta$, $28\beta^6\gamma^2$
$i^{10}I =$	$9\beta^8\gamma$
$k^{10}K =$	β^{10}

CORRIGENDA ET ADDENDA.

Praef. p. IV. lin. 3. *lege* ſpecimen *p. VI. lin.* 7. *l.* earum valores *ib. not.* (k) *lin. vlt. deſunt puncta*.... *p. VII. not.* (k) *lin.* 7. *l.* $2^{2n+2}\, y^{2n+2}$ *ib. lin.* 9. *l.* $2^4 v^4$ *p. VIII. lin.* 3. *l.* accommodauit *Diſſert. p.* 2. *not.* (*b*) *lin.* 10. *l.* Hinc, pro *p.* 3. *lin.* 6. *l.* $p^{m+1} \varkappa(n+1)\, x^{s(m+1)+nd}$.... *p.* 4. *lin.* 1, 2, 3, 4 et 5. *in fine deſunt puncta*.... *ibid. not.* (*e*) *lin* 1. *l.* $p^m \varkappa(n+1)$ *p.* 6. *lin.* 2. *deeſt ſignum* + *ib. lin. vlt. deſunt puncta*... *p.* 7. *lin.* 2. *l.* $p[\dot{A}, \dot{B}, \dot{C}, \dot{D}, \ldots]$ *p.* 8. *lin.* 2. *l.* $y^{\frac{1(s+nd)}{r}}$... *p.* 9. *not.* (*g*) *lin.* 4. *l.* $p^{\frac{s+d}{s}} \varkappa^2 y^{\frac{1(s+2d)}{r}}$.... *p.* 11. *lin.* 7. *l.* et cum (§. IV.) *p.* 13. *lin.* 4. *l.* ${}^{n}m = {}^{6}m$ *ibid. lin.* 11. *l.* ${}^{0}m = -1$ *p.* 22. *lin.* 11. *l.* $-\frac{286}{3^3 \cdot 5}$ *ibid. not.* (*h*) *lin.* 1. *l.* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$... *p.* 24. *not.* (*k*) *lin.* 2. *l.* nota prior potentiae *p.* 30. *not.* (*c*) *lin.* 3. *l.* poſitum $n'\mathfrak{N}$ pro $n'\mathcal{N}$ vel $n'N$

Praeterea monere conuenit,

1) *Cl. Toepferum* Claſſium combinatoriarum ${}^{n}A$, ${}^{n}B$, ${}^{n}C$.... *indefinitas mtas, ntas*.... exhibuiſſe per ${}^{n}\mathcal{M}$, ${}^{n}\mathcal{N}$....(Combin. Anal. p. 161, 162. et Tab. I, II.) pro quibus ego litteras formae maioris, ſed pariter latinas ${}^{n}\mathrm{M}$, ${}^{n}\mathrm{N}$.... ſubſtitui. Scilicet *Exc. Hindenburgius*, pro huiusmodi Claſſibus *indefinitis*, vſus eſt, quod optimum eſt haud dubie, literis, non maioris ſed plane alius alphabeti, quod *ſcriptas* refert *litteras latinas* (Infin. Dign. §. XXI, 7. p. 85; conf. p. 93. etc.). Sed huiusmodi litterarum in typographia, in qua Toepferiana illa et haec mea diſſertatio, formulis exſcriptae ſunt, non ſuppetebant exemplaria.

2) Pro ſignis

Eſchenbachianis m, ${}^{I}m$, ${}^{II}m$, ${}^{III}m$, ${}^{IV}m$, ${}^{V}m$, ${}^{VI}m$... ${}^{n}m$ (*de Ser. Reuerſ.* p. 23-25; *Toepf.* Tab. VIII, A, B)

ego poſui ſigna ${}^{0}m$, ${}^{1}m$, ${}^{2}m$, ${}^{3}m$, ${}^{4}m$, ${}^{5}m$, ${}^{6}m$... ${}^{n}m$ (in hac mea diſſert. p. 11 — 13.)

ne quis ſigna haec paululum inter ſe diuerſa, diuerſum etiam ſenſum habere, et inter ſe diſcrepare, exiſtimet.

3) Si eueniret, vt in formula, ſeriei inverſae terminum generalem exhibente $x^s \daleth(n+1) = \frac{s}{s+nd} q^{-\frac{s+nd}{r}} \varkappa(n+1)\, y^{\frac{1(s+nd)}{r}}$ (pag. 11. ſeqq.) fieret $s + nd = 0$, adeoque etiam ${}^{n}m = 0$, fore, quia ${}^{0}m = \frac{s}{r}$ et ${}^{1}\mathfrak{A} = {}^{2}\mathfrak{B} = {}^{3}\mathfrak{C} \ldots = {}^{n}\mathfrak{N} = 1 = \left(\frac{y^1}{\alpha}\right)^0$

$$x^s \daleth(n+1) = -\frac{s}{r}\left[\frac{a\,{}^{n}A}{\alpha} - \frac{b\,{}^{n}B}{2\alpha^2} + \frac{c\,{}^{n}C}{3\alpha^3} \cdots\cdots \mp \frac{n\,{}^{n}N}{n\alpha^n}\right] = x^s\, \varkappa(n+1)$$

$$\left(\begin{matrix}\beta, & \gamma, & \delta, & \varepsilon, & \zeta \ldots\ldots \\ 1, & 2, & 3, & 4, & 5 \ldots\ldots\end{matrix}\right)$$

ſiue etiam $x^s \daleth(n+1) = -\frac{s}{r} \log. q\, \varkappa(n+1) = x^s \varkappa(n+1)$

Erinnerung an den Buchbinder.

Die Bogen müſſen, wegen S. 18, 19. nicht zu tief gefalzt, und nicht zu ſtark beſchnitten werden.

FORMULAE

DE SERIERUM REVERSIONE

DEMONSTRATIO VNIVERSALIS

SIGNIS LOCALIBUS

COMBINATORIO-ANALYTICORUM

VICARIIS

EXHIBITA

DISSERTATIO ACADEMICA

AUCTORE

M. HENRICO AUGUSTO ROTHE

DRESDANO.

LIPSIAE

LITTERIS SOMMERIIS.