USUS

LOGARITHMORUM INFINITINOMII

IN

THEORIA AEQUATIONUM

AUCTORE

MAURICIO DE PRASSE

ADIECTA EST TABULA SINGULARIS

LIPSIAE

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§. I.

Institvti Ratio.

Continentur hoc libello Analyseos Combinatoriae *) quaeſtiones tres ex plurimis aliis, quae forma aeque univerſali, ut hic apparent, nusquam, quod equidem ſciam, pertractatae reperiuntur, et quarum inveſtigandarum occaſionem praebuit *mutua relatio exponentium et coefficientium in aequatione:*

$$[1-(ax+bx^2..)]^\alpha [1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^\beta .... [1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^\rho = 1-(Ax+Bx^2+Cx^3...+Nx^n..)$$

Propter *Methodi Combinatoriae* in quaeſtionum analyticarum ſolutione hic adhibitae ipſarumque quaeſtionum indolem viſum eſt mihi, non abs re eſſe, propoſitiones aliquot nondum ſatis notas praeparationis loco praemittere, quibus *formulae dignitatum Polynomii* diverſae et quaedam de *Combinatoriis Involutionibus* hoc in libello obviis, continentur, ut deinde §. XXVII ſeqq. remotis omnibus impedimentis, rei poſſit ſumma afferri ejusque uſus exemplis nonnullis illuſtrari.

Si quid ego effecerim, combinatoriis involutionibus ad Analyſeos theoremata ſummi momenti translatis, eam ob rem ſtudium, quo me disciplinis Matheſeos tradendis dicavi, cuique ut probetur ab illis non alieno, jam in votis habeo: Peritis vero ſatisfacere non, niſi in poſterum, conabor.

§. II.

*) Theoriam artis combinatoriae ejusque ad Analyſin applicationem, Hindenburgius, *Vir Celeberrimus*, in opere, quod inſcribitur: *Novi Syſtematis Permutationum, Combinationum ac Variationum primae lineae* (Lipſiae 1781.), primus ita propoſuit, ut in ea, veluti in fundamento, calculus nitatur omnis. Multa etiam continent ad rem combinatoriam et Analyſin pertinentia, propoſitis ſimul plurimis et exemplis et tabulis combinatoriis, ipſius: *Infinitinomii Dignitatum Hiſtoria, Leges ac Formulae. . .* (Gottingae 1779.). Quibus vero meditationibus peculiaribus, qua via quibusque artificiis ad memorabile illud inventum perductus fuerit, docuit nuper, omnia ſimul enumerans ſcripta eo pertinentia: *Archiv der reinen und angewandten Mathematik.* (Leipzig bey Schäfer 1794.) Heft II. pag. 242.

§. II.

DEFINITIONES.

1) Ars combinatoria res datas ex ordine notat ſive literis *a, b, c, d, e, f, g, h. . . . .*; ſive numeris 1, 2, 3, 4, 5, 6, 7, 8 . . . . .; Signa ſingula *Elementorum*, collecta vero $\begin{pmatrix} a, & b, & c, & d, & e, & f \ldots \\ 1, & 2, & 3, & 4, & 5, & 6 \ldots \end{pmatrix}$ *Indicis* nomen habent.

2) Elementorum conjunctiones Singulae, veluti *abc*; *ddd*; *cecbd*; ſive 123; 444; 15324; *Complexiones* vocantur, et quidem *rite ordinatae*, in quibus (ſi a ſiniſtra ad dextram legas) elementum poſterius nullum antepoſitum eſt priori, e. g. *aab*; *abc*; *abcd*; 112; 123; 1234; Itaque *aba*; *cab*; *abfee*; *aaacb*; 121; 312; 12655; 11132; non ſunt rite ordinatae.

3) Complexiones dividuntur

in *Biniones*, quales ſunt *aa*; *ab*; *bc*; 11; 12; 23;
" *Terniones*, " " *aaa*; *abc*; *cbd*; 111; 123; 324;
" *Quaterniones* " " *abbc*; *adef*; 1223; 1456;
" " " " " " " " "
" *m*tiones " " *abcd .... m*; 1234....m;

prout binis, ternis, quaternis, ... m Elementis conſtant. Ad hujus appellationis analogiam ſingula elementa ipſa dicuntur *Vniones* v. c. *a*; *b*; *c*; 1; 2; 3;

4) Cum reſpicitur ad aliquam conjungendi legem, ſecundum quam ipſae complexiones producuntur,

collectio Vnionum vocatur *prima Claſſis*
" Binionum " *ſecunda* "
" Ternionum " *tertia* "
" " " " "
" *m*tionum " *mta* "

5) *Claſſis rite ordinata* dicitur, cujus complexiones omnes ad inſtar numerorum creſcentium procedunt.

Sic v. c. Complexiones $\begin{pmatrix} 116; & 125; & 134; & 224; & 233 \\ aaf; & abe; & acd; & bbd; & bcc \end{pmatrix}$; non autem $\begin{pmatrix} 224; & 134; & 233; & 116; & 125 \\ bbd; & acd; & bcc; & aaf; & abe \end{pmatrix}$ aut aliter diſpoſitae, ut numeri creſcentes, progrediuntur.

6) *Diſcerptiones numeri n* eae vocantur complexiones numericae, in quibus *elementorum* cujusque complexionis *ſumma* *) aequat numerum *n*.

7) Com-

*) In complexione *ſumma* elementorum cum ipſorum *numero* confundendus non eſt. Illa ab ipſorum elementorum numericorum *magnitudine*, hic a multitudine pendet, et in eadem claſſe conſtanter

7) *Combinationes summae propositae n* eae dicuntur complexiones numericae rite ordinatae (Def. 2), quae simul discerptiones sunt numeri *n* (Def. 6); quo pertinent v. c. summae 5 combinationes a latere collocatae et per classes (Def. 4.) dispositae.

5

14
23

113
122

1112

11111

8) *Permutari* datae combinationis elementa dicuntur, ubi omnibus, quibus possunt, modis sedibus transponuntur suis, et prodeuntes permutando conjunctiones ipsae *Permutationes datae combinationis* vocantur.

§. III.

Problema.

Data Combinatione Summae n (§. II. Def. 7.) *reperire proximè sequentem Classis rite ordinatae* (§. II. Def. 5) et *indicis* (1, 2, 3, 4 ....) (§. Def. 1).

Combinatio data sit v. c. 112445; ubi $n = 17$ et index (1, 2, 3, 4, 5, 6 .....)

Solutio.

I. In data Combinatione quaeratur, a dextra ad sinistram eundo, numerus primus, qui *duabus saltem unitatibus* differat a numero extremo ad dextram et *inventus* (hic 2) *unitate augeatur*. Si talis numerus non reperitur, combinatio data ipsa est classis suae ultima, veluti 233333.

II. Numerus secundum I. auctus (jam 3) in omnibus ad dextram sedibus (quae adsunt) collocetur, excepta tamen extrema.

III. Numeri ad sinistram (si qui adsunt) maneant immutati.

IV. In sede ad dextram extrema ponatur complementum summae *n*. (h. l. complementum est 6)

Vnde

stanter idem est. Sic in complexione 13158 *numerus* elementorum est 5, propter quinque Elementa 1; 3; 1; 5; 8; Elementorum vero *summa* est $1+3+1+5+8=18$. In complexione 25412 numerus elementorum iterum est 5, summa autem 14.

Vnde fit:

		ex	112445
Secundum	I.		..3...
=	I. II.		..333.
=	I. II. III.		11333.
=	I. II. III. IV.		113336

		ex	113336
Secundum	I.		4
=	I. II.		....4.
=	I. II. III.		11334.
=	I. II. III. IV.		113345

		ex	113345
Secundum	I.		...4..
=	I. II.		...44.
=	I. II. III.		113444
=	I. II. III. IV.		113444

Demonstratio.

Paret, illis regulis Discerptionem reperiri. Discerptio vero rite ordinata eſt, quod numerum auctum ſequuntur numeri nulli ipſo minores (reg. III. et IV.). Combinationem denique prodire datae proximam, intelligitur eo, quod ultimam ſedem tenet numerus maximus eorum, quos, ubi regulae I. ſatisfactum eſt, poni licuiſſet.

§. IV.

Problema.

Combinationum ſummae propoſitae n claſſem quamlibet, v. c. mtam (vbi $m < n$) conſtruere, dato Judice (1, 2, 3, 4....).

Solutio.

1) Scribantur $(m - 1)$ unitates, alia juxta aliam, et vltimo loco complementum ad Summam n; i. e. $(n - m + 1)$.

2) Ex prima hac combinatione per regulas §. III. deducatur ſecunda, ex ſecunda tertia atque ita quaelibet poſterior ex proxime priori, donec iſtae regulae amplius adhiberi nequeant, atque omnes Combinationes expreſſae erunt.

Exemplum. Si $n = 10$, $m = 5$, producitur

ſecundum	1) prima combinatio	1 1 1 1 6
=	2) ſucceſſive reliquae omnes	1 1 1 2 5
		1 1 1 3 4
		1 1 2 2 4
		1 1 2 3 3
		1 2 2 2 3
		2 2 2 2 2 *)

§. V.

*) De numerorum diſcerptionibus egerunt Hindenburgius, Vir Celeberrimus, (*Infinit. Dignitatum Hiſt. Leg. ac Form.* p. 73. ſeqq. et p. 129. ſeqq. et *Progr.* quo *Terminorum ab infinitinomii dignitatibus Coefficientes* Moivraeanos *ſequi ordinem lexicographicum oſtenditur* Lipſiae 1795.) et Toepferus, Vir Clariſſimus, (*Combinatoriſche Analytik* etc. p. 68.)

§. V.

Explicatio.

Combinationum summae propositae n classes (§. II. Def. 7.) ex ordine per literas majores Latinas A, B, C, D, ... et numerum n, sequentem in modum, notantur: Sit v. c. summa proposita $n = 8$, notabitur

prima classis combinationum summae 8 i. e.	8	Signo ^{8}A	Index est $(1, 2, 3, 4, 5, 6, 7, 8)$
secunda	17, 26, 35, 44	^{8}B	
tertia	116, 125, 134, 224, 233	^{8}C	
quarta	1115, 1124, 1133, 1223, 2222	^{8}D	
quinta	11114, 11123, 11222	^{8}E	
sexta	111113, 111122	^{8}F	
septima	1111112	^{8}G	
octava	11111111	^{8}H	

Numerus n (hic 8), qui literae majori hic iungitur a laeva, dicitur *Exponens summae* (der Summenexponent). Literae cum numeris *Signa Classium* constituunt.

Scholion I. Index signis combinatoriis semper adiiciendus, quod, illo omisso, combinatoria signa ipsa intelligi nequeunt.

Scholion II. Signa $^{n}M_{(1,2,3\ldots n)}$ et $^{n}\mathcal{M}_{(1,2,3\ldots n)}$ diligenter distinguenda sunt. Illud enim *classem* combinationum summae n *duodecimam*, quia M duodecima est litera alphabeti A, B, C, D, . . ., hoc vero, in quo $\mathcal{M}$ alterius est alphabeti, *mtam* generaliter *classem exprimit*.

§. VI.

§. VI.

Definitio.

Combinationes summae propositae n ex elementis literalibus eae vocantur Complexiones, quae ex numericis ejusdem summae, secundum problema §. IV. productis, oriuntur, si loco numerorum (1, 2, 3, 4 ....) literae his respondentes ponantur, hoc modo:

$$
\begin{array}{lll}
6 = {}^6A & f = {}^6A & g = {}^6A \\[1ex]
\left.\begin{array}{l}15\\24\\33\end{array}\right\} = {}^6B &
\left.\begin{array}{l}ae\\bd\\cc\end{array}\right\} = {}^6B &
\left.\begin{array}{l}bf\\ce\\dd\end{array}\right\} = {}^6B \\[1ex]
\left.\begin{array}{l}114\\123\\222\end{array}\right\} = {}^6C &
\left.\begin{array}{l}aad\\abc\\bbb\end{array}\right\} = {}^6C &
\left.\begin{array}{l}bbe\\bcd\\ccc\end{array}\right\} = {}^6C \\[1ex]
\left.\begin{array}{l}1113\\1122\end{array}\right\} = {}^6D &
\left.\begin{array}{l}aaac\\aabb\end{array}\right\} = {}^6D &
\left.\begin{array}{l}bbbd\\bbcc\end{array}\right\} = {}^6D \\[1ex]
11112 = {}^6E & aaaab = {}^6E & bbbbc = {}^6E \\
111111 = {}^6F & aaaaaa = {}^6F & bbbbbb = {}^6F \\[1ex]
(1, 2, 3, 4, 5, 6) & \binom{a,\, b,\, c,\, d,\, e,\, f}{1,\, 2,\, 3,\, 4,\, 5,\, 6} & \binom{b,\, c,\, d,\, e,\, f,\, g}{1,\, 2,\, 3,\, 4,\, 5,\, 6}
\end{array}
$$

ubi indices inferius hic appositi ostendunt, ad quos numeros literae pertinent. Hinc intelligitur, quid futurum esset, si indices fuissent

$$\binom{c,\, d,\, e,\, f,\, g,\, h}{1,\, 2,\, 3,\, 4,\, 5,\, 6} \text{ vel } \binom{d,\, e,\, f,\, g,\, h,\, i}{1,\, 2,\, 3,\, 4,\, 5,\, 6}$$

§. VII.

Explicatio.

Praefixae signis classium (§. V.) homonymae literae Germanicae minores, veluti

$$\mathfrak{a}^n A;\ \mathfrak{b}^n B;\ \mathfrak{c}^n C;\ \mathfrak{d}^n D;\ \mathfrak{e}^n E;\ \ldots\ldots\ldots\ldots\ \mathfrak{m}^n \mathcal{M}$$

$$\binom{b,\ c,\ d,\ e,\ f,\ g\ \ldots\ldots}{1,\ 2,\ 3,\ 4,\ 5,\ 6\ \ldots\ldots}$$

indicant, singulis Complexionibus literalibus praeponendum esse *Numerum permutationum* *), quo scilicet docetur, quoties complexionis elementa suis transponi queant sedibus (Def. 8.). Sic, posito $n = 8$, est:

$\mathfrak{t}^8 D =$

*) Numerus permutatiorum complexionis *b c d e f g h* .....; ubi elementa numero *m* sunt omnia diversa, est $m.\, m - 1.\, m - 2 \ldots\ldots 3.\, 2.\, 1.$ Quod si autem inter haec aliqua sunt eadem numero β et alia numero γ et alia numero δ etc. fit numerus permutationum

=

$$\mathfrak{d}^8 D \binom{b,\ c,\ d,\ e,\ f \ldots}{1,\ 2,\ 3,\ 4,\ 5 \ldots} = \left\{ \begin{array}{l} 4bbbf \\ 12bbce \\ 6bbdd \\ 12bccd \\ 1cccc \end{array} \right\} \text{ et } \mathfrak{e}^8 E \binom{b,\ c,\ d,\ e \ldots}{1,\ 2,\ 3,\ 4 \ldots} = \left\{ \begin{array}{l} 5bbbbe \\ 20bbbcd \\ 10bbccc \end{array} \right\}.$$

§. VIII.

Theorema.

Si $y = bx + cx^2 + dx^3 + ex^4 + \ldots\ldots\ldots$, potentiae integrae positivae hujus seriei, signis §. VII. explicatis expressae, ex ordine sunt:

$$y^1 = \mathfrak{a}^1 Ax + \mathfrak{a}^2 Ax^2 + \mathfrak{a}^3 Ax^3 + \mathfrak{a}^4 Ax^4 + \mathfrak{a}^5 Ax^5 + \mathfrak{a}^6 Ax^6 + \ldots\ldots + \mathfrak{a}^n Ax^n + \ldots$$

$$y^2 = \mathfrak{b}^2 Bx^2 + \mathfrak{b}^3 Bx^3 + \mathfrak{b}^4 Bx^4 + \mathfrak{b}^5 Bx^5 + \mathfrak{b}^6 Bx^6 + \ldots\ldots + \mathfrak{b}^n Bx^n + \mathfrak{b}^{n+1} Bx^{n+1} + \ldots$$

$$y^3 = \mathfrak{c}^3 Cx^3 + \mathfrak{c}^4 Cx^4 + \mathfrak{c}^5 Cx^5 + \mathfrak{c}^6 Cx^6 + \ldots\ldots\ldots + \mathfrak{c}^n Cx^n + \mathfrak{c}^{n+1} Cx^{n+1} + \mathfrak{c}^{n+2} Cx^{n+2} + \ldots$$

" " " " " " " " "

$$y^m = \mathfrak{m}^m \mathfrak{M} x^m + \mathfrak{m}^{m+1} \mathfrak{M} x^{m+1} + \mathfrak{m}^{m+2} \mathfrak{M} x^{m+2} + \ldots + \mathfrak{m}^n \mathfrak{M} x^n \ldots\ldots + \mathfrak{m}^{n+m-1} \mathfrak{M} x^{n+m-1} \ldots$$

$$\binom{b,\ c,\ d,\ e,\ f,\ g \ldots\ldots}{1,\ 2,\ 3,\ 4,\ 5,\ 6 \ldots\ldots}$$

Termini hic simul apparent generales dignitatum y^1, y^2, y^3, ... y^m.

Hujus Theorematis demonstratio extat in *Infinit. Dign.* §. XXIII. Cf. *Nov. Syst.* p. LIV. 8.

§. IX.

Problema.

Dignitatis y^m §. VIII. terminum construere quemlibet independenter v. c. posito $m = 4$, dignitatis y^4 terminum septimum.

Solu-

$= \frac{m.\ m-1.\ m-2.\ m-3 \ldots\ldots\ldots 3.\ 2.\ 1.}{1.\ 2.\ 3.\ \cdot \beta.\ 1\ 2\ 3 \ldots \gamma.\ 1.\ 2.\ 3 \ldots \delta \text{ etc.}}$ Hinc numerus permutationum combinationis *b c d e f g* est $6.\ 5.\ 4.\ 3.\ 2.\ 1 = 720$ complexionis autem $b^2 c^3 d = \frac{6.\ 5.\ 4.\ 3.\ 2.\ 1}{1.\ 2.\ 1.\ 2.\ 3.\ 1} = 60$. (Cf. Hindenburgii *Infinit. Dign.* §. XIII. p. 32. *Nov. Syst. Perm.* p. 24. *KAESTNERI, VIRI SVMMI*, *Anal. endl. Größen* ed. 1794. §. 34.) Hunc numerum (combinatorium scilicet origine), in complexionibus formularum polynomii occurrentem, *coefficientem polynomialem* dixit Hindenburgius (*Nov. Syst.* p. IX.) ad analogiam Coefficientis binomialis, quocum etiam saepissime conjungitur.

Solutio.

Ex termino *n*to $\mathfrak{m}^{n+m-1}\underset{\left(\begin{smallmatrix} b, c, d \ldots \\ 1, 2, 3 \ldots \end{smallmatrix}\right)}{\mathcal{M}} x^{n+m-1}$ seriei y^m (§. VIII.) sequitur, posito $n = 7$ et $m = 4$, dignitatis y^4 terminus septimus $\underset{\left(\begin{smallmatrix} b, c, d \ldots \\ 1, 2, 3 \ldots \end{smallmatrix}\right)}{\mathfrak{d}^{10}D} x^{10}$. Itaque, ut producatur terminus quaesitus,

I. Discerptionum numeri 10. classis construatur quarta secundum §. IV.	II. Numerorum loco ponantur literae respondentes (§. VI.) secundum Indicem $\left(\begin{smallmatrix} b, c, d, e, f, g \ldots \\ 1, 2, 3, 4, 5, 6 \ldots \end{smallmatrix}\right)$	III. Complexionibus literalibus praefigantur numeri permutationum (Cf. §. VII. *) et singulis potestas x^{10} jungatur.
1117	*bbbh*	4*bbbh* | x^{10}
1126	*bbcg*	12*bbcg*
1135	*bbdf*	12*bbdf*
1144	*bbee*	6*bbee*
1225	*bccf*	12*bccf*
1234	*bcde*	24*bcde*
1333	*bddd*	4*bddd*
2224	*ccce*	4*ccce*
2233	*ccdd*	6*ccdd*

Eodem modo terminus quilibet potestatis y^4 in schemate sequenti reperitur:

	(4)	(5)	(6)	(7)	(8)	(9)
Fit secundum I.	1111	1112	1113	1114	1115	1116
			1122	1123	1124	1125
				1222	1133	1134
					1223	1224
					2222	1233
						2223
secundum II.	*bbbb*	*bbbc*	*bbbd*	*bbbe*	*bbbf*	*bbbg*
			bbcc	*bbcd*	*bbce*	*bbcf*
				bccc	*bbdd*	*bbde*
					bccd	*bcce*
					cccc	*bcdd*
						cccd

Secun-

secundum III. $y^4 = 1bbbb\,x^4 + 4bbbc\,x^5 + \left.\begin{matrix}4bbbd\\ 6bbcc\end{matrix}\right|x^6 + \left.\begin{matrix}4bbbe\\ 12bbcd\\ 4bccc\end{matrix}\right|x^7 + \left.\begin{matrix}4bbbf\\ 12bbce\\ 6bbdd\\ 12bccd\\ 1cccc\end{matrix}\right|x^8 + \left.\begin{matrix}4bbbg\\ 12bbcf\\ 12bbde\\ 12bcce\\ 12bcdd\\ 4cccd\end{matrix}\right|x^9 + \ldots$

Scholion. Terminos quaesitos integros dignitatis y^4 seorsim h. l. scripsimus idque fecimus majoris perspicuitatis gratia, sed non opus est hac distinctione, quia statim potest terminus quilibet per complexiones in II. exhiberi, addendo numeros permutationum complexionibus singulis a laeva, variabilis autem x potestatem respondentem a dextra. Exercitatus paululum poterit etiam, praetereundo I. illico scribere complexiones literales in II. et inde conficere *terminum quaesitum* (III.), qua re totum negotium et contrahitur et sublevatur.

§. X.

Explicatio.

Coefficientes binomiales, cujuslibet exponentis ν, sequentem in modum notantur:

$^{\nu}\mathfrak{A}$ exprimit $\frac{\;}{1}$

$^{\nu}\mathfrak{B}$ " $\frac{\nu.(\nu-1)}{1.\;2}$

$^{\nu}\mathfrak{C}$ " $\frac{\nu.(\nu-1).(\nu-2)}{1.\;2.\;3}$

$^{\nu}\mathfrak{D}$ " $\frac{\nu.(\nu-1).(\nu-2).(\nu-3)}{1.\;2.\;3.\;4.}$

" " " "

$^{\nu}\mathfrak{M}$ " $\frac{\nu.(\nu-1).(\nu-2)\ldots(\nu-m+1)}{1.\;2.\;3.\ldots\ldots m}$

Scholion I. Liquet m nullum hic nisi integrum positivum significare posse numerum. Coefficiens binomialis *generalis*, hoc loco $^{\nu}\mathfrak{M}$, litera alterius alphabeti notatur, quo cognoscatur, eum esse generalem.

Scholion II. Jam signa quoque sequentia ex signis §. X et VII. composita facile intelligentur v. c.

$$^{6}\mathfrak{C}\,c\,^{8}C\binom{b,c,d,e,\ldots}{1,2,3,4,\ldots}$$

quod jubet, construi tertiam Classem combinationum summae 8 ex elementis $(b, c, d, e\ldots)$, quae est

$$^{8}C\binom{b,c,d,e,\ldots}{1,2,3,4,\ldots} = \left\{\begin{matrix}b^2g\\ bcf\\ bde\\ c^2e\\ cd^2\end{matrix}\right.$$

dein numerum permutationum cuivis complexioni praeponi, fit

$$\underset{\binom{b,\,c,\,d,\,e\,\ldots}{1,\,2,\,3,\,4\,\ldots}}{c^8C} = \left[\begin{array}{l} 3b^2g \\ 6bcf \\ 6bde \\ 3c^2e \\ 3cd^2 \end{array}\right]$$

complexionum denique summam multiplicari per productum ${}^6\mathfrak{C} = \frac{6.5.4}{1.2.3}$; quo producitur:

$$\underset{\binom{b,\,c,\,d,\,e\,\ldots}{1,\,2,\,3,\,4\,\ldots}}{{}^6\mathfrak{C}\,c^8C} = \frac{6.5.4}{1.2.3}\left[\begin{array}{l} +\,3b^2g \\ +\,6bcf \\ +\,6bde \\ +\,3c^2e \\ +\,3cd^2 \end{array}\right]$$

§. XI.

Theorema.

Sit $az^{\mu} + bz^{\mu+\delta} + cz^{\mu+2\delta}\, dz^{\mu+3\delta} + \ldots\ldots = \mathrm{p}$.

$$\textit{Erit}\ \mathrm{p}^{\nu} = a^{\nu}z^{\nu\mu} + {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^1 A z^{\nu\mu+\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^2A \\ +\,{}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^2B \end{array}\right| z^{\nu\mu+2\delta} + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^3A \\ +\,{}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^3B \\ +\,{}^{\nu}\mathfrak{C}a^{\nu-3}\mathfrak{c}^3C \end{array}\right| z^{\nu\mu+3\delta} \ldots + \left.\begin{array}{l} {}^{\nu}\mathfrak{A}\ a^{\nu-1}\ \mathfrak{a}\ {}^{n}A \\ +\,{}^{\nu}\mathfrak{B}\ a^{\nu-2}\ \mathfrak{b}\ {}^{n}B \\ +\,{}^{\nu}\mathfrak{C}\ a^{\nu-3}\ \mathfrak{c}\ {}^{n}C \\ \quad '' \\ \quad '' \\ +\,{}^{\nu}\mathfrak{M}a^{\nu-m}\mathfrak{m}^{n}\mathcal{M} \\ \quad '' \\ \quad '' \\ +\,{}^{\nu}\mathfrak{N}\ a^{\nu-n}\ \mathfrak{n}^{n}\mathcal{N} \end{array}\right| z^{\nu\mu+n\delta}\ldots$$

$$\binom{b,\ c,\ d,\ e,\ f,\ \ldots\ldots}{1,\ 2,\ 3,\ 4,\ 5,\ \ldots\ldots}$$

ubi μ, δ *et* ν *numeri quicunque positivi, negativi, integri, fracti esse possunt.*

Demonstratio.

Adhibitis signis coefficientium binomialium (§. X.), fit binomium

$$(a+y)^{\nu} = a^{\nu} + {}^{\nu}\mathfrak{A}a^{\nu-1}y^1 + {}^{\nu}\mathfrak{B}a^{\nu-2}y^2 + {}^{\nu}\mathfrak{C}a^{\nu-3}y^3 + {}^{\nu}\mathfrak{D}a^{\nu-4}y^4 + \ldots\ldots$$

Jam, si ponitur

$$y = bx + cx^2 + dx^3 + ex^4 + fx^5 + \ldots\ldots$$

facta substitutione potentiarum hujus seriei, §. VIII. exhibitarum, sequitur:

$(a+bx$

$$(a+bx+cx^2+dx^3+\ldots)^\nu = \left.\begin{array}{l} a^\nu + {}^\nu\mathfrak{A}\, a^{\nu-1} y^1 \\ + {}^\nu\mathfrak{B}\, a^{\nu-2} y^2 \\ + {}^\nu\mathfrak{C}\, a^{\nu-3} y^3 \\ \vdots \\ + {}^\nu\mathfrak{M}\, a^{\nu-m} y^m \\ \vdots \\ + {}^\nu\mathfrak{N}\, a^{\nu-n} y^n \\ \vdots \end{array}\right\} = \begin{array}{l} a^\nu + {}^\nu\mathfrak{A}\, a^{\nu-1}\,[\mathfrak{a}^1 A x + \mathfrak{a}^2 A x^2 + \mathfrak{a}^3 A x^3 + \ldots + \mathfrak{a}^n A x^n + \ldots] \\ + {}^\nu\mathfrak{B}\, a^{\nu-2}\,[\mathfrak{b}^2 B x^2 + \mathfrak{b}^3 B x^3 + \ldots + \mathfrak{b}^n B x^n + \ldots] \\ + {}^\nu\mathfrak{C}\, a^{\nu-3}\,[\mathfrak{c}^3 C x^3 + \ldots + \mathfrak{c}^n C x^n + \ldots] \\ \vdots \\ + {}^\nu\mathfrak{M}\, a^{\nu-m}\,[\mathfrak{m}^m \mathcal{M} x^m + \ldots + \mathfrak{m}^n \mathcal{M} x^n \ldots] \\ \vdots \\ + {}^\nu\mathfrak{N}\, a^{\nu-n}\,[\mathfrak{n}^n \mathcal{N} x^n + \ldots] \\ \vdots \end{array}$$

$$\binom{b,\ c,\ d,\ e,\ f, \ldots}{1,\ 2,\ 3,\ 4,\ 5, \ldots}$$

terminisque ſecundum potentias quantitatis x digeſtis:

$$(a+bx+cx^2+dx^3+\ldots)^\nu =$$

$$a^\nu + {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^1 A x^1 + \left.\begin{array}{l} {}^\nu\mathfrak{A} a^{\nu-2}\mathfrak{a}^2 A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\mathfrak{b}^2 B \end{array}\right| x^2 + \left.\begin{array}{l} {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^3 A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\mathfrak{b}^3 B \\ + {}^\nu\mathfrak{C} a^{\nu-3}\mathfrak{c}^3 C \end{array}\right| x^3 + \ldots\ldots + \left.\begin{array}{l} {}^\nu\mathfrak{A} a^{\nu-1}\ \mathfrak{a}^n A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\ \mathfrak{b}^n B \\ + {}^\nu\mathfrak{C} a^{\nu-3}\ \mathfrak{c}^n C \\ \vdots \\ + {}^\nu\mathfrak{M} a^{\nu-m}\mathfrak{m}^n \mathcal{M} \\ \vdots \\ + {}^\nu\mathfrak{N} a^{\nu-n}\ \mathfrak{n}^n \mathcal{N} \end{array}\right| x^n + \ldots$$

$$\binom{b,\ c,\ d,\ e,\ f, \ldots}{1,\ 2,\ 3,\ 4,\ 5, \ldots}$$

vnde formula prodit Theorematis, poſito utrinque $x = z^\delta$, additoque factore $z^{\nu\mu}$. (Hindenburgius *Nov. Syſt.* p. LIV. 7.)

Scholion. Formulae dignitatum Infinitinomii, his et §. VIII. propoſitae, vocantur *combinatoriae*, quod ſignis utuntur combinatoriis. Pluribus terminis formula theorematis exhibetur in tabula adjecta.

§. XII.

Problema.

Dignitatis $\mathfrak{p}^\nu$ (§.XI.) *conſtruere terminum quemlibet independenter, v. c. undecimum.*

Solutio.

10	$+{}^{\nu}\mathfrak{A}a^{\nu-1}$	$[\,1l\,]$	$={}^{\nu}\mathfrak{A}a^{\nu-1}\,\mathfrak{a}^{10}A$	$z^{\nu\mu+10\delta}$
19		$2bk$		
28		$2ci$		
37	$+{}^{\nu}\mathfrak{B}a^{\nu-2}$	$2dh$	$={}^{\nu}\mathfrak{B}a^{\nu-2}\,\mathfrak{b}^{10}B$	
46		$2eg$		
55		$1ff$		
118		$3bbi$		
127		$6bch$		
136		$6bdg$		
145		$6bef$		
226	$+{}^{\nu}\mathfrak{C}a^{\nu-3}$	$3ccg$	$={}^{\nu}\mathfrak{C}a^{\nu-3}\,\mathfrak{c}^{10}C$	
235		$6cdf$		
244		$3cee$		
334		$3dde$		
1117		$4bbbh$		
1126		$12bbcg$		
1135		$12bbdf$		
1144		$6bbee$		
1225	$+{}^{\nu}\mathfrak{D}a^{\nu-4}$	$12bccf$	$={}^{\nu}\mathfrak{D}a^{\nu-4}\,\mathfrak{d}^{10}D$	
1234		$24bcde$		
1333		$4bddd$		
2224		$4ccce$		
2233		$6ccdd$		
11116		$5bbbbg$		
11125		$20bbbcf$		
11134		$20bbbde$		
11224	$+{}^{\nu}\mathfrak{E}a^{\nu-5}$	$30bbcce$	$={}^{\nu}\mathfrak{E}a^{\nu-5}\,\mathfrak{e}^{10}E$	
11233		$30bbcdd$		
12223		$20bcccd$		
22222		$1ccccc$		
111115		$6bbbbbf$		
111124		$30bbbbce$		
111133	$+{}^{\nu}\mathfrak{F}a^{\nu-6}$	$15bbbbdd$	$={}^{\nu}\mathfrak{F}a^{\nu-6}\,\mathfrak{f}^{10}F$	
111223		$60bbbccd$		
112222		$15bbcccc$		
1111114		$7bbbbbbe$		
1111123	$+{}^{\nu}\mathfrak{G}a_{\nu-7}$	$42bbbbbcd$	$={}^{\nu}\mathfrak{G}a^{\nu-7}\,\mathfrak{g}^{10}G$	
1111222		$35bbbbccc$		
11111113	$+{}^{\nu}\mathfrak{H}a^{\nu-8}$	$8bbbbbbbd$	$={}^{\nu}\mathfrak{H}a^{\nu-8}\,\mathfrak{h}^{10}H$	
11111122		$28bbbbbbcc$		
111111112	$+{}^{\nu}\mathfrak{I}a^{\nu-9}$	$[\,9bbbbbbbbc\,]$	$={}^{\nu}\mathfrak{I}a^{\nu-9}\,\mathfrak{i}^{10}I$	
1111111111	$+{}^{\nu}\mathfrak{K}a^{\nu-10}$	$[\,1bbbbbbbbbb\,]$	$={}^{\nu}\mathfrak{K}a^{\nu-10}\,\mathfrak{k}^{10}K$	

1) Numerus termini quaesiti (h. l. 11) unitate minuatur, discerptionumque numeri residui classes ex ordine omnes construantur (§. IV.)

2) Loco numerorum substituantur (§. VI.) literae respondentes, secundum indicem

$$\begin{pmatrix} b, & c, & d, & e, & f, & g, & h, & i, & k, & l, & \ldots \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & \ldots \end{pmatrix}$$

3) Complexionibus singulis praefigantur numeri permutationum literarum (§. VII.)

4) ${}^{\nu}\mathfrak{A}a^{\nu-1}$ praeponatur *primae* classi
${}^{\nu}\mathfrak{B}a^{\nu-2}$ „ *secundae* „
${}^{\nu}\mathfrak{C}a^{\nu-3}$ „ *tertiae* „
${}^{\nu}\mathfrak{D}a^{\nu-4}$ „ *quartae* „
„ „
„ „
(vide §. X. Scholion *II*)

5) Complexionibus omnibus addatur factor communis $z^{\nu\mu+10\delta}$.

Demonstratio.

Haec terminorum constructio per partes congruit cum termino generali formulae dignitatum infinitinomii combinatoriae §. XI. Q. E. F.

Solutio.

10	$+{}^{\nu}\mathfrak{A}a^{\nu-1}$	[$1l$]	$={}^{\nu}\mathfrak{A}a^{\nu-1}\,\mathfrak{a}\,{}^{10}A$ $\Big\vert\; z^{\nu\mu+10\delta}$
19		$2bk$	
28		$2ci$	
37	$+{}^{\nu}\mathfrak{B}a^{\nu-2}$	$2dh$	$={}^{\nu}\mathfrak{B}a^{\nu-2}\,\mathfrak{b}\,{}^{10}B$
46		$2eg$	
55		$1ff$	
118		$3bbi$	
127		$6bch$	
136		$6bdg$	
145		$6bef$	
226	$+{}^{\nu}\mathfrak{C}a^{\nu-3}$	$3ccg$	$={}^{\nu}\mathfrak{C}a^{\nu-3}\,\mathfrak{c}\,{}^{10}C$
235		$6cdf$	
244		$3cee$	
334		$3dde$	
1117		$4bbbh$	
1126		$12bbcg$	
1135		$12bbdf$	
1144		$6bbee$	
1225	$+{}^{\nu}\mathfrak{D}a^{\nu-4}$	$12bccf$	$={}^{\nu}\mathfrak{D}a^{\nu-4}\,\mathfrak{d}\,{}^{10}D$
1234		$24bcde$	
1333		$4bddd$	
2224		$4ccce$	
2233		$6ccdd$	
11116		$5bbbbg$	
11125		$20bbbcf$	
11134		$20bbbde$	
11224	$+{}^{\nu}\mathfrak{E}a^{\nu-5}$	$30bbcce$	$={}^{\nu}\mathfrak{E}a^{\nu-5}\,\mathfrak{e}\,{}^{10}E$
11233		$30bbcdd$	
12223		$20bcccd$	
22222		$1ccccc$	
111115		$6bbbbbf$	
111124		$30bbbbce$	
111133	$+{}^{\nu}\mathfrak{F}a^{\nu-6}$	$15bbbbdd$	$={}^{\nu}\mathfrak{F}a^{\nu-6}\,\mathfrak{f}\,{}^{10}F$

1) Numerus termini quaesiti (h. l. 11) unitate minuatur, discerptionumque numeri residui classes ex ordine omnes construantur (§. IV.)

2) Loco numerorum substituantur (§. VI.) literae respondentes, secundum indicem

$$\begin{pmatrix} b, & c, & d, & e, & f, & g, & h, & i, & k, & l, & \ldots \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & 10, & \ldots \end{pmatrix}$$

3) Complexionibus singulis praefigantur numeri permutationum literarum (§. VII.)

4) ${}^{\nu}\mathfrak{A}a^{\nu-1}$ praeponatur *primae* classi
${}^{\nu}\mathfrak{B}a^{\nu-2}$ " *secundae* "
${}^{\nu}\mathfrak{C}a^{\nu-3}$ " *tertiae* "
${}^{\nu}\mathfrak{D}a^{\nu-4}$ " *quartae* "
" "
" "
(vide §. X. Scholion *II*)

5) Complexionibus omnibus addatur factor communis $z^{\nu\mu+10\delta}$.

Scholion. Termini si omnes seriei p^{ν} evolvuntur eodem modo, quo hic undecimus, prodit *formula dignitatum infinitinomii algebraice expressa.* Hujus termini primi decem in tabula adjecta conspiciuntur, eamque ob causam hoc loco undecimus evolutus est.

§. XIII.

Explicatio.

Sit $p = az^{\mu} + bz^{\mu+\delta} + cz^{\mu+2\delta} + \ldots$ uti §. XI, *Coefficientes* dignitatis p^{ν} ex ordine his etiam notantur signis:

$p^{\nu}\varkappa 1$ exprimit *coefficientem primum* i.e. a^{ν}

$p^{\nu}\varkappa 2$ = = *secundum* = ${}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^1 A$

$p^{\nu}\varkappa 3$ = = *tertium* = ${}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^2 A + {}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^2 B$

$p^{\nu}\varkappa 4$ = = *quartum* = ${}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^3 A + {}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^3 B + {}^{\nu}\mathfrak{C}a^{\nu-3}\mathfrak{c}^3 C$

= = = = = = =

$p\varkappa(m+1)$ = = (m+1)*tum* = ${}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^m A + {}^{\nu}\mathfrak{B}a^{\nu-2}\mathfrak{b}^m B + {}^{\nu}\mathfrak{C}a^{\nu-3}\mathfrak{c}^m C \ldots + {}^{\nu}\mathfrak{M}a^{\nu-m}\mathfrak{m}^m \mathcal{M}$

$p[a, b, c, d \ldots]$ $\begin{pmatrix} b, & c, & d, & e, & f, & \ldots \\ 1, & 2, & 3, & 4, & 5, & \ldots \end{pmatrix}$

Litera p scilicet seriem indicat, ν dignitatis ipsius exponentem, $\varkappa$ cum numero adposito coefficientem numero ipsi respondentem. Tale signum e. g. $p^{\nu}\varkappa(m+1)$ enunciatur hoc modo: *Seriei* p *ad dignitatem exponentis* ν *evectae coefficiens* $(m+1)$ *tus.* Hinc sequitur:

$$p^{\nu} = p^{\nu}\varkappa 1\, z^{\nu\mu} + p^{\nu}\varkappa 2\, z^{\nu\mu+\delta} + p^{\nu}\varkappa 3\, z^{\nu\mu+2\delta} + p^{\nu}\varkappa 4\, z^{\nu\mu+3\delta} + \ldots\ldots + p^{\nu}\varkappa(m+1)\, z^{\nu\mu+m\delta} + \ldots\ldots$$
$$p[\, a,\ b,\ c,\ d,\ e \ldots\ldots \,]$$

Haec coefficientium notae dignitatis p^{ν} *Signa localia* vocantur, formulaeque ex illis compositae, *formulae locales* *). Signis et formulis localibus adpositum signum $p[a, b, c, d, e\ldots]$ dicitur *Scala* eaque simpliciter seriei datae p coefficientes ex ordine indicat. Scala igitur diligenter ab indice (§. II. Def. 1.) distinguenda est; hoc enim docetur, quos ad coefficientes discerptionum classes referantur **).

Scho-

*) Signa localia *coefficientium* et *terminorum integrorum* (ubi ponitur ⅂ pro $\varkappa$, veluti $p^{\nu}⅂(m+1)$) primus usurpavit Hindenburgius (*Infinit. Dign.* p. 71, 3; p. 93 — 98; 136 — 141). Melior, quae hic adhibetur, notatio *Nov. Syst.* p. XXXIII, 2. De signorum *localium* cum **combinatoriis** comparatione *Ib.* p. LI — LIII. *Id. Paral. ad Ser. Revers.* p. VIII.

**) Rothius, *Vir Clarissimus,* in libello cui titulus: *Formulae de serierum reversione demonstratio universalis. Lipsiae* 1793. Scalas introduxit. Signa idem localia in hac dissertatione ingeniose adhibuit, earumque ope formulam de serierum reversione combinatoriam, ab Eschenbachio (1789) propositam, primus et rigorose demonstravit.

Scholion. Formula localis, exempli loco hic propoſita, *dignitatum Infinitinomii formula* appellatur *localis*. Haec quoque in tabula adjecta exprimitur.

* * *

Sed haec de Infinitinomio protuliſſe, ſufficiat. Pergimus jam ad ea explicanda, quae rem noſtram propius attingunt.

---

§. XIV.

Definitio.

Variationes ſummae propoſitae n nominantur diſcerptiones numeri *n* (§. II. Def. 6.) ſine diſcrimine omnes, ſive ſint rite ordinatae, ſive non.

§. XV.

Explicatio.

A

1	1	1	1	1
	2	1	1	1
	1	2	1	1
		3	1	1
	1	1	2	1
		2	2	1
		1	3	1
			4	1
	1	1	1	2
		2	1	2
		1	2	2
			3	2
		1	1	3
			2	3
			1	4
				5

In ſchemate appoſito A, Variationum ſummarum 1, 2, 3, 4, 5 (§. XIV.), Variationes ſummarum minorum a Variationibus ſummarum majorum ita involvuntur, ut illa ex his poſſint exſecari, id quod angulis interjectis docetur, quam ob rem ipſa haec variationum conſtructio *Involutio* vocatur *Combinatoria* *).

§. XVI.

Problema.

Ex involutione Variationum n (§. XV.) *conſtruere Involutionem Variationum ſummae* $(n+1)$.

Solu-

*) De Involutionum et Evolutionum (quae inter reliquas operationes combinatorias facile principatum obtinent) natura, diverſitate et in disquiſitionibus analyticis efficacia, ſumma et utilitate copioſe egit Hindenburgius, *Vir Excellentiſſimus*, (*Archiv der reinen und angewandten Mathematik H.* I. p. 13. ſeqq. *H.* II. III. et IV. *Programma* ſupra (§. IV. *) laudatum.)

Solvtio.

I. Variationi ſummae n cuilibet ad dextram unitas adponatur.

II. Earundem Variationum ſummae n elementa ad dextram extrema unitate augeantur, et, quae ita prodeunt, complexiones complexionibus per I. ortis verticaliter ſubſcribantur.

v. c.

	poſito $n = 1$,	poſito $n = 2$,	poſito $n = 3$,
datum eſt	1	1 1 2	1 1 1 2 1 1 2 3
hinc fit per I.	1 1	1 1 1 2 1	1 1 1 1 2 1 1 1 2 1 3 1
per I. II.	1 1 2	1 1 1 2 1 1 2 3	1 1 1 1 2 1 1 1 2 1 3 1 1 1 2 2 2 1 3 4

Demonstratio.

Ponatur ſolutionem propoſitam problemati ſatisfacere, ſi quaeratur Involutio Variationum ſummarum 1, 2, 3, ... p; Variationes autem ſummae $(p+1)$ regulis traditis non reperiri omnes, ſed unam vel plures deeſſe. Jam quaelibet deficientium Variationum ſummae $(p+1)$ deſineret

vel in ipſam unitatem v. c. 321..... 121

vel in numerum unitate majorem v. c. 213..... 53.

Si *illud*, deficeret 321... 12, ſi *hoc*, deficeret 213.... 52 in Variationibus ſummae praecedentis p; id quod ſuppoſitioni repugnat. Eodem modo oſtenditur, eandem complexionem non poſſe ſaepius occurere. Conſtat autem, ſolutionem ſatisfacere problemati, poſito $p = 1$, et $p = 2$ itaque ſatisfacit etiam pro Variationibus ſummae cujuslibet $(p+1) = 3; 4; 5; \ldots$ quaerendis.

§. XVII.

Explicatio.

B.

1 | 1 | 1 | 1 | 1 | 1 | 1
2 | 1 | 1 | 1 | 1 | 1
3 | 1 | 1 | 1 | 1
2 2 | 1 | 1 | 1
4 | 1 | 1 | 1
3 2 | 1 | 1
5 | 1 | 1
2 2 2 | 1
4 2 | 1
3 3 | 1
6 | 1
3 2 2
5 2
4 3
7

In schemate B Combinationes summarum 1, 2, 3, 4, 5, 6, 7 ita exhibentur, ut elementorum ordo sit inversus ipsaeque Combinationes constituant *Involutionem Combinatoriam* (Cf. §. XV.)

§. XVIII.

Problema.

Ex Involutione Combinationum summae n §. XVII. construere Involutionem Combinationum summae (n + 1).

Solutio.

I. Cuivis Combinationi summae n adponantur a dextra unitas,

II. Vnitate augeantur extrema ad dextram elementa earum complexionum, quae vel duabus terminantur elementis diversis, vel unione (§. II. Def. 3.) constant, et quae ita prodeunt, complexiones complexionibus per I. ortis verticaliter subscribantur,

v. c. posito $n = 1$, datum est 1 |

hinc fit per. I 1 | 1

per I. II. 1 | 1 / 2

posito $n = 2$, datum est 1 | 1 / 2

hinc fit per I. 1 | 1 | 1 / 2 | 1

per I. II. 1 | 1 | 1 / 2 | 1 / 3

posito $n = 3$, datum est 1 | 1 | 1 / 2 | 1 / 3

hinc fit per I. 1 | 1 | 1 | 1 / 2 | 1 | 1 / 3 | 1

per I. II. 1 | 1 | 1 | 1 / 2 | 1 | 1 / 3 | 1 / 2 2 / 4

Solutionis hujus demonstratio eadem est, quae in §. XVI.

Scholion. Involutionum genus ab eo, quod §. XV — XVIII. proposuimus, diversum, hic vero non adhibitum, extat in *Infinit. Dign.* p. 79. 80, et p. 133.

§. XIX.

EXPLICATIO.

Si *Variationes summarum* 1, 2, 3, 4, ... *n* exhibeantur involutorie (§. XVI.), et numerorum loco ponantur literae *a*, *b*, *c*, *d*, ... *n*, haec Involutio Variationum notetur signo:

$$\overset{n}{\mathfrak{J}}\binom{a,\, b,\, c,\, \ldots\, n}{1,\, 2,\, 3,\, \ldots\, n}$$

atque, si cuilibet complexioni numerus primo a laeva suorum elementorum respondens praefigatur, id Variationum genus summae *n* cum suis *Comitibus numericis**) exprimatur signo:

$$\mathfrak{i}\overset{n}{\mathfrak{J}}\binom{a,\, b,\, c,\, \ldots\, n}{1,\, 2,\, 3,\, \ldots\, n}$$

Exempla. Pro summae *n* valoribus 1, 2, 3, 4, 5 .... prodeunt:

$\mathfrak{i}\overset{1}{\mathfrak{J}}\binom{a}{1} = \begin{array}{c|}\underline{1a}\end{array}$ Variationes summae 1 cum suis comitibus,

$\mathfrak{i}\overset{2}{\mathfrak{J}}\binom{a,b}{1,2} = \begin{array}{c|c|}\underline{1a} & a \\ & \underline{2b}\end{array}$ Variationes summae 2 cum suis comitibus,

$\mathfrak{i}\overset{3}{\mathfrak{J}}\binom{a,b,c}{1,2,3} = \begin{array}{c|c|c|}\underline{1a} & a & a \\ & \underline{2b} & a \\ & & 1a\ b \\ & & \underline{3c}\end{array}$ Variationes summae 3 cum suis comitibus,

$\mathfrak{i}\overset{5}{\mathfrak{J}} =$

*) Scilicet HINDENBURGIUS, *V. Cel.* complexionum literalium *Comites numericos* ab ipsarum numeris permutationum sive *coefficientibus polynomialibus* (Cf. §. VII. *) distinguit, et a quacunque polynomialium modificatione, aut alia lege oriundos numeros *Comites complexionum* nominat.

$$\underset{\left(\substack{a,b,c,d\\1,2,3,4}\right)}{i^4\mathfrak{J}} = \begin{array}{r} 1a \mid a \mid a \mid a \\ 2b \mid a \mid a \\ 1a\ b \mid a \\ 3c \mid a \\ 1a\ a\ b \\ 2b\ b \\ 1a\ c \\ 4d \end{array}$$

Variationes summae 4 cum suis comitibus,

$$\underset{\left(\substack{a,b,c,d,e\\1,2,3,4,5}\right)}{i^5\mathfrak{J}} = \begin{array}{r} 1a \mid a \mid a \mid a \mid a \\ 2b \mid a \mid a \mid a \\ 1a\ b \mid a \mid a \\ 3c \mid a \mid a \\ 1a\ a\ b \mid a \\ 2b\ b \mid a \\ 1a\ c \mid a \\ 4d \mid a \\ 1a\ a\ a\ b \\ 2b\ a\ b \\ 1a\ b\ b \\ 3c\ b \\ 1a\ a\ c \\ 2b\ c \\ 1a\ d \\ 5e \end{array}$$

Variationes summae 5 cum suis comitibus,

etc. etc. etc.

§. XX.

Explicatio.

Involutio Combinationum summae n ex elementis $a, b, c, d, \ldots . n$ (§. XVIII et §. VI.) notetur (*Arch. der Math. Heft. IV.* p. 417, 418) signo:

$$\underset{\left(\substack{a,b,c,\ldots n\\1,2,3,\ldots n}\right)}{{}^{n}J} = \underset{\left(\substack{a,b,c,d,\ldots\ldots n\\1,2,3,4,\ldots\ldots n}\right)}{{}^{n}A + {}^{n}B + {}^{n}C + {}^{n}D + \ldots\ldots + {}^{n}\mathcal{N}}$$

et,

et, si singulis complexionibus praefigatur numerus permutationum, signo:

$$\underset{\left(\substack{a,\,b,\,c,\,\ldots n\\ 1,\,2,\,3,\,\ldots\, n}\right)}{j^n J} \quad = \quad \underset{\left(\substack{a,\,b,\,c,\,d,\,\ldots\, n\\ 1,\,2,\,3,\,4,\,\ldots\, n}\right)}{\mathfrak{a}^n A + \mathfrak{b}^n B + \mathfrak{c}^n C + \mathfrak{d}^n D + \ldots\ldots \mathfrak{n}^n \mathcal{N}}$$

Si denique numeri Permutationum per summam n multiplicentur omnes, quilibet autem numero elementorum complexionis, cui adpositus est, dividatur, haec Involutio Combinationum summae n ex elementis $a, b, c, d, \ldots n$ cum *numericis Comitibus* suis (Cf. §. XIX. *) exprimatur signo:

$$\underset{\left(\substack{a,\,b,\,c,\,\ldots n\\ 1,\,2,\,3,\,\ldots\, n}\right)}{\mathfrak{i}^n J} = \underset{\left(\substack{a,\; b,\; c,\; d,\; e,\; \ldots\ldots n\\ 1,\; 2,\; 3,\; 4,\; \ldots\ldots\, n}\right)}{\frac{n}{1}\mathfrak{a}^n A + \frac{n}{2}\mathfrak{b}^n B + \frac{n}{3}\mathfrak{c}^n C + \frac{n}{4}\mathfrak{d}^n D + \ldots + \frac{n}{m}\mathfrak{m}^n \mathcal{M} + \ldots + \frac{n}{n}\mathfrak{n}^n \mathcal{N}}$$

Exempla. Pro summae n valoribus 1, 2, 3, 4, 5, ... prodeunt:

$$\underset{\left(\substack{a\\ 1}\right)}{\mathfrak{i}^1 J} = \begin{array}{l} 1a \end{array} = \underset{\left(\substack{a\\ 1}\right)}{\frac{1}{1}\mathfrak{a}^1 A}$$

$$\underset{\left(\substack{a,\,b\\ 1,\,2}\right)}{\mathfrak{i}^2 J} = \begin{array}{r|l} 1a & a \\ & 2b \end{array} = \underset{\left(\substack{a,\;\; b\\ 1,\;\; 2}\right)}{\frac{2}{1}\mathfrak{a}^2 A + \frac{2}{2}\mathfrak{b}^2 B}$$

$$\underset{\left(\substack{a,\,b,\,c\\ 1,\,2,\,3}\right)}{\mathfrak{i}^3 J} = \begin{array}{r|r|l} 1a & a & a \\ & 3b & a \\ & & 3c \end{array} = \underset{\left(\substack{a,\;\; b,\;\; c\\ 1,\;\; 2,\;\; 3}\right)}{\frac{3}{1}\mathfrak{a}^3 A + \frac{3}{2}\mathfrak{b}^3 B + \frac{3}{3}\mathfrak{c}^3 C}$$

$$\underset{\left(\substack{a,\,b,\,c,\,d\\ 1,\,2,\,3,\,4}\right)}{\mathfrak{i}^4 J} = \begin{array}{r|r|r|l} 1a & a & a & a \\ & 4b & a & a \\ & & 4c & a \\ & & 2b & b \\ & & & 4d \end{array} = \underset{\left(\substack{a,\;\; b,\;\; c,\;\; d\\ 1,\;\; 2,\;\; 3,\;\; 4}\right)}{\frac{4}{1}\mathfrak{a}^4 A + \frac{4}{2}\mathfrak{b}^4 B + \frac{4}{3}\mathfrak{c}^4 C + \frac{4}{4}\mathfrak{d}^4 D}$$

$$\underset{\left(\substack{a,\,b,\,c,\,d,\,e\\ 1,\,2,\,3,\,4,\,5}\right)}{\mathfrak{i}^5 J} = \begin{array}{r|r|r|r|l} 1a & a & a & a & a \\ & 5b & a & a & a \\ & & 5c & a & a \\ & & 5b & b & a \\ & & & 5d & a \\ & & & 5c & b \\ & & & & 5e \end{array} = \underset{\left(\substack{a,\;\; b,\;\; c,\;\; d,\;\; e\\ 1,\;\; 2,\;\; 3,\;\; 4,\;\; 5}\right)}{\frac{5}{1}\mathfrak{a}^5 A + \frac{5}{2}\mathfrak{b}^5 B + \frac{5}{3}\mathfrak{c}^5 C + \frac{5}{4}\mathfrak{d}^5 D + \frac{5}{5}\mathfrak{e}^5 E}$$

Scholion. Signa hic et §. XIX. exhibita sunt *involutoria*, in quibus recta litera J; *Combinationum*, obliqua vero $\mathcal{J}$; *Variationum* involutoriam constructionem (§ XV. et XVII.) secundum indicem $\left(\begin{smallmatrix} a, & b, & c, & d, & e \ldots n \\ 1, & 2, & 3, & 4, & 5 \ldots n \end{smallmatrix}\right)$, litera *n* in fronte a laeva summam et germanicae minores $\mathfrak{i}$ et $\mathfrak{i}$ comites complexionum respondentes significant.

§. XXI.

THEOREMA.

$$Log.nat.[1-(ax+bx^2+cx^3+\ldots)] = -\left(\frac{\mathfrak{i}^1Jx^1}{1}+\frac{\mathfrak{i}^2Jx^2}{2}+\frac{\mathfrak{i}^3Jx^3}{3}+\frac{\mathfrak{i}^4Jx^4}{4}\ldots+\frac{\mathfrak{i}^nJx^n}{n}+\ldots\right)$$
$$\left(\begin{matrix} a, & b, & c, & d, & \ldots & n \\ 1, & 2, & 3, & 4, & \ldots & n \end{matrix}\right)$$

DEMONSTRATIO.

Sit $Log.nat.[1-(ax+bx^2+cx^3\ldots)] = Log.nat.(1-y) = -\left(\frac{y}{1}+\frac{y^2}{2}+\frac{y^3}{3}\ldots+\frac{y^n}{n}\ldots\right)$

unde, dignitatibus seriei y secundum §. VIII. expressis atque ita dispositis, ut coefficientes eiusdem potentiae quantitatis x constituant seriem verticalem, prodit:

$$Log.nat.[1-(ax+bx^2+cx^3+\ldots)] = -\left[\frac{a^1A}{1}x + \left.\begin{matrix}\frac{a^2A}{1}\\ +\frac{b^2B}{2}\end{matrix}\right| x^2 + \left.\begin{matrix}\frac{a^3A}{1}\\ +\frac{b^3B}{2}\\ +\frac{c^3C}{3}\end{matrix}\right| x^3 + \ldots\ldots + \left.\begin{matrix}\frac{a^nA}{1}\\ +\frac{b^nB}{2}\\ +\frac{c^nC}{3}\\ \vdots\\ +\frac{m^n\mathcal{M}}{m}\\ \vdots\\ +\frac{n^n\mathcal{N}}{n}\end{matrix}\right| x^n + \ldots\right]$$
$$\left(\begin{matrix} a, & b, & c, & d, & e \ldots & n, & \ldots \\ 1, & 2, & 3, & 4, & 5 \ldots & n, & \ldots \end{matrix}\right)$$

Coefficiens termini generalis $\left(\frac{a^nA}{1}+\frac{b^nB}{2}+\frac{c^nC}{3}+\ldots+\frac{m^n\mathcal{M}}{m}+\ldots+\frac{n^n\mathcal{N}}{n}\right)x^n$ secundum §. XX. aequalis est $\frac{\mathfrak{i}^nJ}{n}$; itaque, valoribus 1, 2, 3, 4, 5, ... successive positis loco n, aequatio oritur Theorematis.

§. XXII.

§. XXII.

Theorema.

Est
$$\underset{(a)}{i^1\mathfrak{J}} = 1a$$

$$\underset{(a,b)}{i^2\mathfrak{J}} = \underset{(a)}{i^1\mathfrak{J}a} + 2b$$

$$\underset{(a,b,c)}{i^3\mathfrak{J}} = \underset{(a,\ b)}{i^2\mathfrak{J}a + i^1\mathfrak{J}b} + 3c$$

$$\underset{(a,b,c,d)}{i^4\mathfrak{J}} = \underset{(a,\ b,\ c)}{i^3\mathfrak{J}a + i^2\mathfrak{J}b + i^1\mathfrak{J}c} + 4d$$

" "

$$\underset{(a,b,c\ldots n)}{i^n\mathfrak{J}} = \underset{(a,\ b,\ c,\ d,\ e\ \ldots\ \overset{-m}{n},\ \ldots\ \overset{-3}{n},\ \overset{-2}{n},\ \overset{-1}{n},)}{i^{n-1}\mathfrak{J}a + i^{n-2}\mathfrak{J}b + i^{n-3}\mathfrak{J}c + \ldots + i^{n-m}\mathfrak{J}m + \ldots + i^3\mathfrak{J}\overset{-3}{n} + i^2\mathfrak{J}\overset{-2}{n} + i^1\mathfrak{J}\overset{-1}{n}} + nn$$

In his formulis elementa ſingula *a, b, c, d,* . . . ., ſignis involutoriis §. XIX. juncta, ad ſingulas complexiones, quas ſigna exprimunt, referenda eſſe, per ſe clarum eſt. Literae autem cum numeris ſupra poſitis deſignant,

$\overset{-1}{n}$ elementum $(n-1)$ tum,

$\overset{-2}{n}$ " $(n-2)$ tum,

$\overset{-3}{n}$ " $(n-3)$ tum,

" "

$\overset{-m}{n}$ " $(n-m)$ tum,

ubi primum elementum eſt *a*, et *n*tum *n*; nempe numeri -1, -2, -3, . . . $-m$, ſcripti ſupra *n*, *diſtantiam* exponunt *retrorſum* ab *n*to, et proinde *Diſtantiae Exponentes* vocantur. Hindenb. *Nov. Syſt.* p. XXXVII. ſeq. Toepf. *Comb. Anal.* p. 164. ſeq.

Demonstratio.

Theorematis ratio ſtatim ex ipſa conſtructione Variationum §. XV. et explicatione §. XIX. liquet.

§. XXIII.

Theorema.

Elementis datae combinationis permutatis permutationumque alia alii ſubſcripta, uti mos eſt, ſeries elementorum verticalis quaelibet, toties unumquodque continet, quoties prima.

De-

Demonstratio.

Elementum quodvis toties eandem tenet ſedem, quoties reliqua permutari poſſunt, id ex ipſa liquet permutationum definitione (§. II. Def. 8), toties igitur in hac redit ſerie verticali, quoties in illa.

Corollarium I.

Significet $\mathfrak{m}$ numerum permutationum combinationis numericae ſummae n claſſis $m$tae, erit $n.\mathfrak{m}$ ſumma omnium numerorum, quibus permutationes iſtae conſtant, et cum numerus quilibet toties in hac recurrit ſerie verticali, quoties in illa, ſingulae ſeriei verticalis ſumma eſt $\frac{n}{m}.\mathfrak{m}$. Sic:

$$\text{v. c.}\quad \begin{bmatrix} 22233 \\ 22323 \\ 22332 \\ 23223 \\ 23232 \\ 23322 \\ 32223 \\ 32232 \\ 32322 \\ 33222 \end{bmatrix}$$

ſunt permutationes combinationis numericae 22233 (§. II. Def. 8.) in qua $n = 12$, $m = 5$, et $\mathfrak{m} = 10$, unius itaque ſeriei verticalis ſumma eſt: $\frac{12}{5}.10 = 2+2+2+2+2+2+3+3+3+3$

Corollarium II.

Elementis datae combinationis Summae n, claſſis $m$tae et indicis $\binom{a,b,c,d,e,..n}{1,2,3,4,5,..n}$ permutatis, et praepoſito cuique permutationi numero, qui primo ad laevam ipſius elemento reſpondet, ea re productum data combinatione expreſſum multiplicatur per $\frac{n}{m}.\mathfrak{m}$ ſecundum Corollarium I. Sic data combinatione $bbbcc$ eſt $n = 12$, $m = 5$, $\mathfrak{m} = 10$.

$$\text{atque} \left\{ \begin{matrix} 2bbbcc \\ 2bbcbc \\ 2bbccb \\ 2bcbbc \\ 2bcbcb \\ 2bccbb \\ 3cbbbc \\ 3cbbcb \\ 3cbcbb \\ 3ccbbb \end{matrix} \right\} = \frac{12}{5}.10.bbbcc = 24\,bbbcc.$$

§. XXIV.

Theorema.

$$n\left[\tfrac{1}{1}a^n A + \tfrac{1}{2}b^n B + \tfrac{1}{3}c^n C + \tfrac{1}{4}d^n D + \ldots + \tfrac{1}{m}m^n \mathcal{M} + \ldots \tfrac{1}{n}n^n \mathcal{N}\right]_{\binom{a,\,b,\,c,\,d,\,\ldots\ldots\,n}{1,\,2,\,3,\,4,\,\ldots\ldots\,n}} = \overset{n}{\mathfrak{i}}J_{\binom{a,b,c,..n}{1,2,3,..n}} = \overset{n}{\mathfrak{i}}\mathcal{J}_{\binom{a,b,c,..n}{1,2,3,..n}}$$

Demonstratio.

I. Ipſa liquet explicatione §. XX. eſſe

$$n\left(\frac{1}{1}\mathfrak{a}^n A + \frac{1}{2}\mathfrak{b}^n B + \frac{1}{3}\mathfrak{c}^n C + \frac{1}{4}\mathfrak{d}^n D + \ldots\ldots + \frac{1}{m}\mathfrak{m}^n \mathcal{M} + \ldots \frac{1}{n}\mathfrak{n}^n \mathcal{N}\right) = \quad \mathfrak{i}^n \mathfrak{J}$$

$$\begin{pmatrix} a, & b, & c, & d, & \ldots\ldots & n \\ 1, & 2, & 3, & 4, & \ldots\ldots & n \end{pmatrix} \qquad \begin{pmatrix} a, & b, & c, & \ldots & n \\ 1, & 2, & 3, & \ldots & n \end{pmatrix}$$

II. Si pro Combinationibus ſummae n, Claſſis $m$tae cum numeris permutationum ſuarum i.e. pro:

$$\mathfrak{m}^n \mathcal{M}$$
$$\begin{pmatrix} a, & b, & c, & \ldots & n \\ 1, & 2, & 3, & \ldots & n \end{pmatrix}$$

ipſae harum Combinationum ſubſtituantur permutationes (i.e. Variationum ſummae n Claſſis $m$ta Def. §. XIV.) et cuique permutationi praefigatur numerus primo elementorum ſuorum reſpondens; eo producta ſigno:

$$^{n}\mathcal{M}$$
$$\begin{pmatrix} a, & b, & c, & \ldots & n \\ 1, & 2, & 3, & \ldots & n \end{pmatrix}$$

expreſſa ſecundum Corollarium II. §. XXIII. multiplicantur per $\frac{n}{m}.\mathfrak{m}$ (valor ſcilicet numeri permutationum $\mathfrak{m}$ pendet a diverſitate factorum cujusvis illorum productorum v. §. VII.*)

Itaque $\quad \frac{n}{1}\mathfrak{a}^n A + \frac{n}{2}\mathfrak{b}^n B + \frac{n}{3}\mathfrak{c}^n C + \frac{n}{4}\mathfrak{d}^n D + \ldots + \frac{n}{m}\mathfrak{m}^n \mathcal{M} + \ldots + \frac{n}{n}\mathfrak{n}^n \mathcal{N}$

$$\begin{pmatrix} a, & b, & c, & d, & e, & \ldots\ldots & n \\ 1, & 2, & 3, & 4, & 5, & \ldots\ldots & n \end{pmatrix}$$

Variationibus exprimitur, ſi prima, ſecunda, tertia, verbo Claſſes omnes Variationum ſummae n conſtruuntur et cuique variationi proponitur numerus primo ſuorum elementorum reſpondens. His variationibus autem involutorie diſpoſitis oritur (§. XIX.):

$$\mathfrak{i}^n \mathfrak{J} = \frac{n}{1}\mathfrak{a}^n A + \frac{n}{2}\mathfrak{b}^n B + \frac{n}{3}\mathfrak{c}^n C + \frac{n}{4}\mathfrak{d}^n D + \ldots + \frac{n}{m}\mathfrak{m}^n \mathcal{M} + \ldots + \frac{n}{n}\mathfrak{n}^n \mathcal{N}$$

$$\begin{pmatrix} a, & b, & c, & \ldots & n \\ 1, & 2, & 3, & \ldots & n \end{pmatrix} \qquad \begin{pmatrix} a, & b, & c, & d, & \ldots\ldots & n \\ 1, & 2, & 3, & 4, & \ldots\ldots & n \end{pmatrix}$$

Supereſt ut exempla afferantur:

Poſito $n = 1$, eſt $\underline{1a}| = \underline{1a}| = 1\left(\frac{1}{1}\mathfrak{a}^1 A\right)$

$$\begin{pmatrix} a \\ 1 \end{pmatrix}$$

$n = 2$, $\quad \begin{array}{l|l|} \underline{1a} & a \\ \;\underline{2b} & \end{array} = \begin{array}{l|l|} \underline{1a} & a \\ \;\underline{2b} & \end{array} = 2\left(\frac{1}{1}\mathfrak{a}^2 A + \frac{1}{1}\mathfrak{b}^2 B\right)$

$$\begin{pmatrix} a, & b, \\ 1, & 2, \end{pmatrix}$$

$n = 3$, $\quad \begin{array}{l|l|l|} \underline{1a} & a & a \\ 2b & a & \\ 1a & b & \\ \;\underline{3c} & & \end{array} = \begin{array}{l|l|l|} \underline{1a} & a & a \\ \underline{3b} & a & \\ \;\underline{3c} & & \end{array} = 3\left(\frac{1}{1}\mathfrak{a}^3 A + \frac{1}{2}\mathfrak{b}^3 B + \frac{1}{3}\mathfrak{c}^3 C\right)$

$$\begin{pmatrix} a, & b, & c, \\ 1, & 2, & 3, \end{pmatrix}$$

Po-

Posito $n = 4$ est
$$\begin{array}{llll} 1a & a & a & a \\ 2b & a & a & \\ 1a\;b & a & & \\ 3c & a & & \\ 1a\;a\;b & & & \\ 2b\;b & & & \\ 1a\;c & & & \\ 4d & & & \end{array} = \begin{array}{llll} 1a & a & a & a \\ 4b & a & a & \\ 4c & a & & \\ 2b\;b & & & \\ 4d & & & \end{array} = 4\left(\frac{1}{1}\mathfrak{a}^4 A + \frac{1}{2}\mathfrak{b}^4 B + \frac{1}{3}\mathfrak{c}^4 C + \frac{1}{4}\mathfrak{d}^4 D\right)$$
$$\left(\begin{matrix} a, & b, & c, & d \\ {\scriptstyle 1,} & {\scriptstyle 2,} & {\scriptstyle 3,} & {\scriptstyle 4} \end{matrix}\right)$$

§. XXV

Explicatio.

Ponatur:

$$ {}^{1}\Sigma = \underset{(a)}{i^1 J}\,\alpha + \underset{(\mathfrak{a})}{i^1 J}\,\beta + \ldots + \underset{(\mathfrak{A})}{i^1 J}\,\varrho = 1a|\alpha + 1\mathfrak{a}|\beta + \ldots + 1\mathfrak{A}|\varrho $$

$$ {}^{2}\Sigma = \underset{(a,b)}{i^2 J}\,\alpha + \underset{(\mathfrak{a},\mathfrak{b})}{i^2 J}\,\beta + \ldots + \underset{(\mathfrak{A},\mathfrak{B})}{i^2 J}\,\varrho = \begin{array}{ll} 1a & a \\ 2b & \end{array}\Big|\alpha + \begin{array}{ll} 1\mathfrak{a} & \mathfrak{a} \\ 2\mathfrak{b} & \end{array}\Big|\beta + \ldots + \begin{array}{ll} 1\mathfrak{A} & \mathfrak{A} \\ 2\mathfrak{B} & \end{array}\Big|\varrho $$

$$ {}^{3}\Sigma = \underset{(a,b,c)}{i^3 J}\,\alpha + \underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}{i^3 J}\,\beta + \ldots + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{i^3 J}\,\varrho = \begin{array}{lll} 1a & a & a \\ 3b & a & \\ 3c & & \end{array}\Big|\alpha + \begin{array}{lll} 1\mathfrak{a} & \mathfrak{a} & \mathfrak{a} \\ 3\mathfrak{b} & \mathfrak{a} & \\ 3\mathfrak{c} & & \end{array}\Big|\beta + \ldots + \begin{array}{lll} 1\mathfrak{A} & \mathfrak{A} & \mathfrak{A} \\ 3\mathfrak{B} & \mathfrak{A} & \\ 3\mathfrak{C} & & \end{array}\Big|\varrho $$

$$ {}^{n}\Sigma = \underset{(a,b,..n)}{i^n J}\,\alpha + \underset{(\mathfrak{a},\mathfrak{b},..\mathfrak{n})}{i^n J}\,\beta + \ldots + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^n J}\,\varrho $$

(Conf. Explicationem §. XX.)

§. XXVI.

Theorema.

$$ Log.\,nat.[1-(ax+bx^2..)]^\alpha\,[1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^\beta..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^\varrho = \left(\frac{{}^{1}\Sigma}{1}x + \frac{{}^{2}\Sigma}{2}x^2 + .. + \frac{{}^{n}\Sigma}{n}x^n.. \right. $$

DEMONSTRATIO.

Est enim (§. XXI.), si cuique termino index adponatur:

$$\alpha\, Log.\, nat. [1-(ax+bx^2+cx^3+..)] = -\left(\underset{(a)}{i^1 J\alpha}\frac{x}{1} + \underset{(a,b)}{i^2 J\alpha}\frac{x^2}{2} + \underset{(a,b,c)}{i^3 J\alpha}\frac{x^3}{3} + \ldots + \underset{(a,b,..n)}{i^n J\alpha}\frac{x^n}{n} + \ldots\right)$$

$$\beta\, Log.\, nat. [1-(\mathfrak{a}x+\mathfrak{b}x^2+\mathfrak{c}x^3+..)] = -\left(\underset{(\mathfrak{a})}{i^1 J\beta}\frac{x}{1} + \underset{(\mathfrak{a},\mathfrak{b})}{i^2 J\beta}\frac{x^2}{2} + \underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}{i^3 J\beta}\frac{x^3}{3} + \ldots + \underset{(\mathfrak{a},\mathfrak{b},..\mathfrak{n})}{i^n J\beta}\frac{x^n}{n} + \ldots\right)$$

$$\varrho\, Log.\, nat. [1-(\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3+..)] = -\left(\underset{(\mathfrak{A})}{i^1 J\varrho}\frac{x}{1} + \underset{(\mathfrak{A},\mathfrak{B})}{i^2 J\varrho}\frac{x^2}{2} + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{i^3 J\varrho}\frac{x^3}{3} + \ldots + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^n J\varrho}\frac{x^n}{n} + \ldots\right)$$

unde, si coefficientes earundem potentiarum quantitatis x per omnes series verticales colligantur et signis §. XXV. exprimantur, formula prodit proposita.

§. XXVII.

Haec ubi praemissa sunt, ad ipsius libelli summam convertimur. In aequatione scilicet supra (§. I.) proposita:

$$[1-(ax+bx^2\ldots)]^\alpha [1-(\mathfrak{a}x+\mathfrak{b}x^2..)]^\beta \ldots [1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^\varrho = 1-(\text{A}x+\text{B}x^2+\text{C}x^3..+\text{N}x^n\ldots)$$

ubi numerus factorum indeterminatus est atque Coefficientium et Exponentium valores quicunque esse possunt,

I. Summa Involutionum $\underset{(a,b,c..n)}{i^n J\alpha} + \underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c},..\mathfrak{n})}{i^n J\beta} \ldots + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C}..\mathfrak{N})}{i^n J\varrho}$; per Coefficientes A, B, C, .. N;

II. Coefficientes A, B, C, D, E, F, ... N, per Exponentes $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \ldots\ldots \varrho$; et Coefficientes $(a, b, c, d,..)$; $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \ldots)$; ... $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \ldots)$;

III. Exponentes $\alpha, \beta, \gamma, \delta, ..\varrho$, per Coefficientes $(a, b, c, d, \ldots)$; $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d},..)$; .. $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \ldots)$; et A, B, C, D, ... exprimentur.

Harum relationum prima et secunda, quibus multa communia sunt, proximis duobus paragraphis afferuntur, de tertia vero ab illis longe diversa, problemate de Incognitarum Eliminatione interjecto, actum est §. XXXI.

§. XXVIII.

Theorema.

Data aequatione §. XXVII. exposita:

$$[1-(ax+bx^2..)]^\alpha[1-(ax+bx^2..)]^\beta..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^\varrho = 1-(Ax+Bx^2+Cx^3...+Nx^n...)$$

sequitur:

$$I.\ {}^n\Sigma = \underset{(A,B,C,..N)}{i^nJ} = \underset{(a,b,..n)}{i^nJ\alpha} + \underset{(a,b,..n)}{i^nJ\beta} + ... + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^nJ\varrho}.$$

$$II.\ {}^n\Sigma = {}^{n-1}\Sigma A + {}^{n-2}\Sigma B + {}^{n-3}\Sigma C + ... + {}^{n-m}\Sigma M + ... + {}^1\Sigma \overset{-1}{N} + n N.$$

$$= \begin{cases} +\alpha\,[\underset{(a,\ b,\ c,\ .....\ n)}{i^{n-1}\mathfrak{J}a + i^{n-2}\mathfrak{J}b + i^{n-3}\mathfrak{J}c + ... + i^{n-m}\mathfrak{J}m + ... + i^1\mathfrak{J}\overset{-1}{n}} + n n] \\ +\beta\,[\underset{(a,\ b,\ c,\ .....\ n)}{i^{n-1}\mathfrak{J}a + i^{n-2}\mathfrak{J}b + i^{n-3}\mathfrak{J}c + ... + i^{n-m}\mathfrak{J}m + ... + i^1\mathfrak{J}\overset{-1}{n}} + n n] \\ \vdots \\ +\varrho\,[\underset{(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ .....\ \mathfrak{N})}{i^{n-1}\mathfrak{J}\mathfrak{A} + i^{n-2}\mathfrak{J}\mathfrak{B} + i^{n-3}\mathfrak{J}\mathfrak{C} + ... + i^{n-m}\mathfrak{J}\mathfrak{M} + ... + i^1\mathfrak{J}\overset{-1}{\mathfrak{N}}} + n\mathfrak{N}] \end{cases}$$

Demonstratio.

Secundum §. XXI. est:

$$Log.nat.\,[1-(Ax+Bx^2+Cx^3+Dx^4..+Nx^n..)] = -\underset{\left(\substack{A,\ B,\ C,\ D,\ ...N \\ 1,\ 2,\ 3,\ 4,\ ...\ n}\right)}{\left(\frac{i^1Jx^1}{1}+\frac{i^2Jx^2}{2}+\frac{i^3Jx^3}{3}....+\frac{i^nJx^n}{n}..\right)}$$

et secundum §. XXVI.

$$Log.nat.\,[1-(ax+bx^2..)]^\alpha[1-(ax+bx^2..)]^\beta..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^\varrho = -\left[\frac{{}^1\Sigma x}{1}+\frac{{}^2\Sigma x^2}{2}...+\frac{{}^n\Sigma x^n}{n}..\right]$$

Itaque, utriusque seriei coefficientibus comparatis, sequitur:

$$I.\ {}^n\Sigma = \underset{(A,B,C,..N)}{i^nJ} = \underset{(a,b,..n)}{i^nJ\alpha} + \underset{(a,b,..n)}{i^nJ\beta} + ... + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^nJ\varrho}$$

Jam si adhibentur Combinationum loco Variationes, uti traditum est §. XXIV. fit:

$${}^n\Sigma = \underset{(A,B,C,.N)}{i^n\mathfrak{J}} = \underset{(a,b,..n)}{i^n\mathfrak{J}\alpha} + \underset{(a,b,..n)}{i^n\mathfrak{J}\beta} + ... + \underset{(\mathfrak{A},\mathfrak{B},..\mathfrak{N})}{i^n\mathfrak{J}\varrho}$$

five, fignis involutoriis fecundum §. XXII. evolutis:

$${}^{n}\Sigma = i^{n-1}\mathfrak{J}A + i^{n-2}\mathfrak{J}B + i^{n-3}\mathfrak{J}C + \ldots + i^{n-m}\mathfrak{J}M + \ldots + i^{1}\mathfrak{J}\overset{-1}{N} + nN$$
$$(A,\ B,\ C,\ \ldots\ \overset{-1}{N})$$

cumque ex formula *I* et §. XXIV. pofito (n-m) loco (n), fequatur:

$$\underset{(A,B,C..)}{{}^{n-m}\Sigma = i^{n-m}J} = \underset{(A,B,C..)}{i^{n-m}\mathfrak{J}.}$$

Formula prodit univerfalis, terminis recurrentibus compofita:

$$II.\ {}^{n}\Sigma = {}^{n-1}\Sigma A + {}^{n-2}\Sigma B + {}^{n-3}\Sigma C + \ldots + {}^{n-m}\Sigma M + \ldots + {}^{1}\Sigma \overset{-1}{N} + nN$$

$$= \begin{cases} + \alpha\,[i^{n-1}\mathfrak{J}a + i^{n-2}\mathfrak{J}b + i^{n-3}\mathfrak{J}c + \ldots + i^{n-m}\mathfrak{J}m + \ldots + i^{1}\mathfrak{J}\overset{-1}{n} + nn] \\ \qquad (a,\ b,\ c,\ \ldots\ \overset{-1}{n}) \\ + \beta\,[i^{n-1}\mathfrak{J}\mathfrak{a} + i^{n-2}\mathfrak{J}\mathfrak{b} + i^{n-3}\mathfrak{J}\mathfrak{c} + \ldots + i^{n-m}\mathfrak{J}\mathfrak{m} + \ldots + i^{1}\mathfrak{J}\overset{-1}{\mathfrak{n}} + n\mathfrak{n}] \\ \quad : \qquad (\mathfrak{a},\ \mathfrak{b},\ \mathfrak{c},\ \ldots\ \overset{-1}{\mathfrak{n}}) \\ \quad : \\ + \varrho\,[i^{n-1}\mathfrak{J}\mathfrak{A} + i^{n-2}\mathfrak{J}\mathfrak{B} + i^{n-3}\mathfrak{J}\mathfrak{C} + \ldots + i^{n-m}\mathfrak{J}\mathfrak{M} + \ldots + i^{1}\mathfrak{J}\overset{-1}{\mathfrak{N}} + n\mathfrak{N}] \\ \qquad (\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \ldots\ \overset{-1}{\mathfrak{M}}) \end{cases}$$

Ex formula *I.* fequitur, pofito $n = 1;\ 2;\ 3;\ 4;$

$${}^{1}\Sigma = \underline{1A}| = \underline{1a}|\,\alpha + \underline{1\mathfrak{a}}|\,\beta + \ldots\ldots\ldots + \underline{1\mathfrak{A}}|\,\varrho$$

$${}^{2}\Sigma = \begin{array}{c|c|} \underline{1A} & A \\ \underline{2B} & \end{array} = \begin{array}{c|c|} \underline{1a} & a \\ \underline{2b} & \end{array}\alpha + \begin{array}{c|c|} \underline{1\mathfrak{a}} & \mathfrak{a} \\ \underline{2\mathfrak{b}} & \end{array}\beta + \ldots\ldots\ldots + \begin{array}{c|c|} \underline{1\mathfrak{A}} & \mathfrak{A} \\ \underline{2\mathfrak{B}} & \end{array}\varrho$$

$${}^{3}\Sigma = \begin{array}{c|c|c|} \underline{1A} & A & A \\ & \underline{3B} & A \\ & & \underline{3C} \end{array} = \begin{array}{c|c|c|} \underline{1a} & a & a \\ & \underline{3b} & a \\ & & \underline{3c} \end{array}\alpha + \begin{array}{c|c|c|} \underline{1\mathfrak{a}} & \mathfrak{a} & \mathfrak{a} \\ & \underline{3\mathfrak{b}} & \mathfrak{a} \\ & & \underline{3\mathfrak{c}} \end{array}\beta + \ldots\ldots + \begin{array}{c|c|c|} \underline{1\mathfrak{A}} & \mathfrak{A} & \mathfrak{A} \\ & \underline{3\mathfrak{B}} & \mathfrak{A} \\ & & \underline{3\mathfrak{C}} \end{array}\varrho$$

$${}^{4}\Sigma = \begin{array}{c|c|c|c|} \underline{1A} & A & A & A \\ & \underline{4B} & A & A \\ & & \underline{4C} & A \\ & & 2B & B \\ & & & \underline{4D} \end{array} = \begin{array}{c|c|c|c|} \underline{1a} & a & a & a \\ & \underline{4b} & a & a \\ & & \underline{4c} & a \\ & & 2b & b \\ & & & \underline{4d} \end{array}\alpha + \begin{array}{c|c|c|c|} \underline{1\mathfrak{a}} & \mathfrak{a} & \mathfrak{a} & \mathfrak{a} \\ & \underline{4\mathfrak{b}} & \mathfrak{a} & \mathfrak{a} \\ & & \underline{4\mathfrak{c}} & \mathfrak{a} \\ & & 2\mathfrak{b} & \mathfrak{b} \\ & & & \underline{4\mathfrak{d}} \end{array}\beta + \ldots + \begin{array}{c|c|c|c|} \underline{1\mathfrak{A}} & \mathfrak{A} & \mathfrak{A} & \mathfrak{A} \\ & \underline{4\mathfrak{B}} & \mathfrak{A} & \mathfrak{A} \\ & & \underline{4\mathfrak{C}} & \mathfrak{A} \\ & & 2\mathfrak{B} & \mathfrak{B} \\ & & & \underline{4\mathfrak{D}} \end{array}\varrho$$

Ex-

Ex formula *II.* vero sequitur, posito $n = 1;\ 2;\ 3;\ 4;$

$$^{1}\Sigma = 1\,\mathrm{A} = (1\,a)\,\alpha + (1\,\mathfrak{a})\,\beta + \ldots\ldots + (1\,\mathfrak{A})\,\varrho$$

$$^{2}\Sigma = {}^{1}\Sigma\mathrm{A} + 2\,\mathrm{B} = \underset{(a)}{(i^{1}\mathfrak{J}a} + 2b)\,\alpha + \underset{(\mathfrak{a})}{(i^{1}\mathfrak{J}\mathfrak{a}} + 2\mathfrak{b})\,\beta + \ldots + \underset{(\mathfrak{A})}{(i^{1}\mathfrak{J}\mathfrak{A}} + 2\mathfrak{B})\,\varrho$$

$$^{3}\Sigma = {}^{2}\Sigma\mathrm{A} + {}^{1}\Sigma\mathrm{B} + 3\mathrm{C} = \underset{(a,\ b,)}{(i^{2}\mathfrak{J}a + i^{1}\mathfrak{J}b} + 3c)\,\alpha + \underset{(\mathfrak{a},\ \mathfrak{b})}{(i^{2}\mathfrak{J}\mathfrak{a} + i^{1}\mathfrak{J}\mathfrak{b}} + 3\mathfrak{c})\,\beta + \ldots + \underset{(\mathfrak{A},\ \mathfrak{B})}{(i^{2}\mathfrak{J}\mathfrak{A} + i^{1}\mathfrak{J}\mathfrak{B}} + 3\mathfrak{C})\,\varrho$$

$$^{4}\Sigma = {}^{3}\Sigma\mathrm{A} + {}^{2}\Sigma\mathrm{B} + {}^{1}\Sigma\mathrm{C} + 4\mathrm{D} = \underset{(a,\ b,\ c)}{(i^{3}\mathfrak{J}a + i^{2}\mathfrak{J}b + i^{1}\mathfrak{J}c} + 4d)\,\alpha + \ldots + \underset{(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C})}{(i^{3}\mathfrak{J}\mathfrak{A} + i^{2}\mathfrak{J}\mathfrak{B} + i^{1}\mathfrak{J}\mathfrak{C}} + 4\mathfrak{D})\,\varrho$$

§. XXIX.

Theorema.

Ex aequatione §. XXVII. *proposita*

$$[1-(ax+bx^{2}\ldots)]^{\alpha}.[1-(\mathfrak{a}x+\mathfrak{b}x^{2}\ldots)]^{\beta}\ldots[1-(\mathfrak{A}x+\mathfrak{B}x^{2}\ldots)]^{\varrho} = 1-(\mathrm{A}x+\mathrm{B}x^{2}+\mathrm{C}x^{3}\ldots+\mathrm{N}x^{n}\ldots)$$

sequitur:

$$\mathrm{N} = \frac{1}{1}\mathfrak{a}^{n}\mathrm{A} - \frac{1}{1.2}\mathfrak{b}^{n}\mathrm{B} + \frac{1}{1.2.3}\mathfrak{c}^{n}\mathrm{C} - \ldots + \frac{1}{1.2.3\ldots m}\mathfrak{m}^{n}\mathcal{M} + \ldots + \frac{1}{1.2\ldots n}\mathfrak{n}^{n}\mathcal{N}$$

$$\left(\begin{matrix} \frac{^{1}\Sigma}{1}, & \frac{^{2}\Sigma}{2}, & \frac{^{3}\Sigma}{3}, & \frac{^{4}\Sigma}{4}, & \ldots\ldots & \frac{^{n}\Sigma}{n} \\ 1, & 2, & 3, & 4, & \ldots\ldots & n \end{matrix}\right)$$

Demonstratio.

Logarithmus naturalis factorum ad sinistram in aequatione est (§. XXVI.):

$$-\left(\frac{^{1}\Sigma}{1}x^{1} + \frac{^{2}\Sigma}{2}x^{2} + \frac{^{3}\Sigma}{3}x^{3} + \frac{^{4}\Sigma}{4}x^{4} + \ldots + \frac{^{n}\Sigma}{n}x^{n} + \ldots\right)$$

cui aequale sit $-\lambda$. Jam si h basin logarithmorum naturalium denotat, constat esse:

$$h^{-\lambda} = 1 - \frac{1}{1}\lambda + \frac{1}{1,2}\lambda^{2} - \frac{1}{1.2.3}\lambda^{3} + \ldots \mp \frac{1}{1.2.3\ldots n}\lambda^{n} \pm \ldots$$

ex qua aequatione, valore quantitatis $-\lambda$ restituto et dignitatibus eiusdem secundum §. VIII. expressis, sequitur:

$$h^{-[\frac{{}^1\Sigma}{1}x+\frac{{}^2\Sigma}{2}x^2\ldots]} = 1-\mathfrak{a}^1Ax - \left.\begin{array}{l}\frac{1}{1}\mathfrak{a}^2A\\ +\frac{1}{1.2}\mathfrak{b}^2B\end{array}\right| x^2 - \left.\begin{array}{l}\frac{1}{1}\mathfrak{a}^3A\\ +\frac{1}{1.2}\mathfrak{b}^3B\\ -\frac{1}{1.2.3}\mathfrak{c}^3C\end{array}\right| x^3 - \begin{array}{l}\ldots\\ +\\ -\end{array} - \left.\begin{array}{l}\frac{1}{1}\mathfrak{a}^nA\\ +\frac{1}{1.2}\mathfrak{b}^nB\\ -\frac{1}{1.2.3.}\mathfrak{c}^nC\\ \ldots\\ \mp\frac{1}{1.2..n}\mathfrak{n}^n\mathcal{N}\end{array}\right| x^n\ldots \quad [Q$$

$$\left(\frac{{}^1\Sigma}{1},\frac{{}^2\Sigma}{2},\frac{{}^3\Sigma}{3},\ldots\frac{{}^n\Sigma}{n},\ldots\right)_{1,\,2,\,3,\,\ldots\,n}$$

Log.nat. $[1-(ax+bx^2..)]^{\alpha}..[1-(\mathfrak{A}x+\mathfrak{B}x^2..)]^{\varrho}$

Est vero $1-[Ax+Bx^2+Cx^3+..+Nx^n..]=h$

$$-[\frac{{}^1\Sigma x}{1}+\frac{{}^2\Sigma x^2}{2}+\frac{{}^3\Sigma x^3}{3}\ldots+\frac{{}^n\Sigma x^n}{n}\ldots]$$
$$=h$$

Itaque, serierum $1-[Ax+Bx^2+Cx^3+\ldots+Nx^n\ldots]$ et Q; comparatione instituta, reperitur:

$$N=\frac{1}{1}\mathfrak{a}^nA-\frac{1}{1.2}\mathfrak{b}^nB+\frac{1}{1.2.3}\mathfrak{c}^nC-\frac{1}{1.2.3.4.}\mathfrak{d}^nD+\ldots\pm\frac{1}{1.2\ldots m}\mathfrak{m}^n\mathcal{M}\mp\ldots\pm\frac{1}{1.2\ldots n}\mathfrak{n}^n\mathcal{N}$$
$$\left(\frac{{}^1\Sigma}{1},\frac{{}^2\Sigma}{2},\frac{{}^3\Sigma}{3},\ldots\frac{{}^n\Sigma}{n}\right)_{1,\,2,\,3,\,\ldots\,n,}$$

Sic est, posito $n=1;\ 2;\ 3;\ 4;$

$$A=\frac{1}{1}\frac{{}^1\Sigma}{1}$$

$$B=\frac{1}{1}\frac{{}^2\Sigma}{2}-\frac{1}{1.2}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}$$

$$C=\frac{1}{1}\frac{{}^3\Sigma}{3}-\frac{1}{1.2}\,2\frac{{}^1\Sigma}{1}\frac{{}^2\Sigma}{2}+\frac{1}{1.2.3.}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}$$

$$D=\frac{1}{1}\frac{{}^4\Sigma}{4}-\frac{1}{1.2}\left[\begin{array}{l}2\frac{{}^1\Sigma}{1}\frac{{}^3\Sigma}{3}\\ \frac{{}^2\Sigma}{2}\frac{{}^2\Sigma}{2}\end{array}\right]+\frac{1}{1.2.3.}\,3\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^2\Sigma}{2}-\frac{1}{1.2.3.4}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}\frac{{}^1\Sigma}{1}.$$

§. XXX.

§. XXX.

Problema.

Datis aequationibus numero r

$$
\begin{array}{l}
{}^1s = {}^1a\alpha + {}^1b\beta + {}^1c\gamma + \ldots\ldots + {}^1r\varrho \\
{}^2s = {}^2a\alpha + {}^2b\beta + {}^2c\gamma + \ldots\ldots + {}^2r\varrho \\
{}^3s = {}^3a\alpha + {}^3b\beta + {}^3c\gamma + \ldots\ldots + {}^3r\varrho \\
\vdots \\
{}^rs = {}^ra\alpha + {}^rb\beta + {}^rc\gamma + \ldots\ldots + {}^rr\varrho \;{}^{*)}
\end{array}
$$

valores quantitatum incognitarum $\alpha, \beta, \gamma, \ldots\ldots \varrho$ *eliminando quaerere.*

Solutio.

Pars prior. Reperitur α regulis, quae sequuntur:

1) Complexio $abcd \ldots\ldots r$ toties in columna verticali scribatur, quoties elementa, $1, 2, 3, 4 \ldots\ldots r$ permutari possunt (Cf. §. VII*).

2) In hac Columna (1) literis singulis Complexionis primae adscribantur, veluti exponentes, a laeva elementa singula primae permutationum a numeris $1, 2, 3 \ldots\ldots r$ oriundarum (prodit ${}^1a\,{}^2b\,{}^3c \ldots\ldots {}^rr$), et literis complexionis secundae elementa secundae permutationis, literis complexionis tertiae elementa tertiae permutationis, etc.

3) I. Complexionibus columnae (2), quae locis imparibus constitutae sunt, praefigantur vicissim signa + et — donec ${}^1a$ mutetur in ${}^2a$, deinde signa — et + donec ${}^2a$ mutetur in ${}^3a$, deinde iterum signa + et — donec ${}^3a$ mutetur in ${}^4a$, etc.

II. Complexionum paribus locis constitutarum cuique praefigatur contrarium signum complexionis proxime praecedentis.

4) Repetantur ea quae supra (1, 2, 3) praecepta fuerunt, sed tamen ita, ut in hac altera, quae prodit, columna ubique ponatur s loco a.

5) *Summa* complexionum cum suis signis columnae *posterioris*, per *summam* prioris divisa, valorem aequabit quantitatis α.

Pars

*) Hic literae $a, b, c, \ldots\ldots$ significant *coefficientes*, primum, secundum, tertium etc. literarum autem exponentes a laeva $1, 2, 3 \ldots\ldots$ *aequationes*, primam, secundam, tertiam etc. Sic verbi causa ${}^2c$ secundae aequationis tertium coefficientem denotat, et ${}^3a$ aequationis tertiae coefficientem primum.

Pars posterior. Eodem modo inveniuntur $\beta, \gamma, \delta \ldots \varrho$, nisi quod s loco $b, c, d \ldots r$, respective ponatur (ut in 4. *Partis prioris* s pro a.).

Exempla.

1) Sit $r = 1$ et $\varrho = \alpha$. Ergo ${}^{1}s = {}^{1}a\alpha$ et $\alpha = {}^{1}s : {}^{1}a$

2) Sit $r = 2$ et $\varrho = \beta$. Ergo

$$\begin{aligned} {}^{1}s &= {}^{1}a\alpha + {}^{1}b\beta \\ {}^{2}s &= {}^{2}a\alpha + {}^{2}b\beta \end{aligned} \quad \text{et} \quad \begin{aligned} \alpha &= \begin{bmatrix} +{}^{1}s\,{}^{2}b \\ -{}^{2}s\,{}^{1}b \end{bmatrix} : \begin{bmatrix} +{}^{1}a\,{}^{2}b \\ -{}^{2}a\,{}^{1}b \end{bmatrix} \\ \beta &= \begin{bmatrix} +{}^{1}a\,{}^{2}s \\ -{}^{2}a\,{}^{1}s \end{bmatrix} : \begin{bmatrix} +{}^{1}a\,{}^{2}b \\ -{}^{2}a\,{}^{1}b \end{bmatrix} \end{aligned}$$

3) Sit $r = 3$ et $\varrho = \gamma$. Ergo

$$\begin{aligned} {}^{1}s &= {}^{1}a\alpha + {}^{1}b\beta + {}^{1}c\gamma \\ {}^{2}s &= {}^{2}a\alpha + {}^{2}b\beta + {}^{2}c\gamma \\ {}^{3}s &= {}^{3}a\alpha + {}^{3}b\beta + {}^{3}c\gamma \end{aligned} \quad \text{et}$$

$$\alpha = \begin{Bmatrix} +{}^{1}s\,{}^{2}b\,{}^{3}c \\ -{}^{1}s\,{}^{3}b\,{}^{2}c \\ -{}^{2}s\,{}^{1}b\,{}^{3}c \\ +{}^{2}s\,{}^{3}b\,{}^{1}c \\ +{}^{3}s\,{}^{1}b\,{}^{2}c \\ -{}^{3}s\,{}^{2}b\,{}^{1}c \end{Bmatrix} : \begin{Bmatrix} +{}^{1}a\,{}^{2}b\,{}^{3}c \\ -{}^{1}a\,{}^{3}b\,{}^{2}c \\ -{}^{2}a\,{}^{1}b\,{}^{3}c \\ +{}^{2}a\,{}^{3}b\,{}^{1}c \\ +{}^{3}a\,{}^{1}b\,{}^{2}c \\ -{}^{3}a\,{}^{2}b\,{}^{1}c \end{Bmatrix} \quad \beta = \begin{Bmatrix} +{}^{1}a\,{}^{2}s\,{}^{3}c \\ -{}^{1}a\,{}^{3}s\,{}^{2}c \\ -{}^{2}a\,{}^{1}s\,{}^{3}c \\ +{}^{2}a\,{}^{3}s\,{}^{1}c \\ +{}^{3}a\,{}^{1}s\,{}^{2}c \\ -{}^{3}a\,{}^{2}s\,{}^{1}c \end{Bmatrix} : \begin{Bmatrix} +{}^{1}a\,{}^{2}b\,{}^{3}c \\ -{}^{1}a\,{}^{3}b\,{}^{2}c \\ -{}^{2}a\,{}^{1}b\,{}^{3}c \\ +{}^{2}a\,{}^{3}b\,{}^{1}c \\ +{}^{3}a\,{}^{1}b\,{}^{2}c \\ -{}^{3}a\,{}^{2}b\,{}^{1}c \end{Bmatrix} \quad \gamma = \begin{Bmatrix} +{}^{1}a\,{}^{2}b\,{}^{3}s \\ -{}^{1}a\,{}^{3}b\,{}^{2}s \\ -{}^{2}a\,{}^{1}b\,{}^{3}s \\ +{}^{2}a\,{}^{3}b\,{}^{1}s \\ +{}^{3}a\,{}^{1}b\,{}^{2}s \\ -{}^{3}a\,{}^{2}b\,{}^{1}s \end{Bmatrix} : \begin{Bmatrix} +{}^{1}a\,{}^{2}b\,{}^{3}c \\ -{}^{1}a\,{}^{3}b\,{}^{2}c \\ -{}^{2}a\,{}^{1}b\,{}^{3}c \\ +{}^{2}a\,{}^{3}b\,{}^{1}c \\ +{}^{3}a\,{}^{1}b\,{}^{2}c \\ -{}^{3}a\,{}^{2}b\,{}^{1}c \end{Bmatrix}$$

Severa solutionis hujus demonstratio nititur in theoria *Variationum, omissis quidem repetitionibus*, id quod, alia scribendi occasione data, ostendam.

Scholion. Problematis solutio convenit cum regulis *Crameri* (*Introduction à l' Analyse des Courbes* p. 656. seqq. *Num. I. et II. De l' Evanouissement des Inconnues*). Easdem Hindenburgius, simul cum regulis de eadem re a Bezolto traditis (Bezout *Théorie générale des Equations algébriques*), amplissime pertractavit in praefatione libelli *Rüdigeriani*, cui titulus est: *Specimen analyticum de lineis curvis secundi ordinis*, ubi etiam p. XLVI — XLVIII. regula traditur datae complexionis permutationes reperiundi, qua h. l. (Sol. Pars prior 1, 2.) opus est.

§. XXXI.

Problema.

Propositae aequationis:

$$[1-(ax+bx^2..)]^{\alpha}[1-(\mathfrak{a}x+\mathfrak{b}x^2...)]^{\beta}..[1-(\mathfrak{A}x+\mathfrak{B}x^2)]^{\varrho} = 1-(\text{A}x+\text{B}x^2+\text{C}x^3..+\text{N}x^n..)$$

exponentes incognitos $\alpha, \beta, .. \varrho$ *per coefficientes datos*, $a, b, c\ldots$; $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}..$; $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\ldots$; et A, B, C... exprimere.

Solu-

Solutio.

Secundum §. XXVIII. posito $n = 1, 2, 3, \ldots r$, est:

$$\underset{(A)}{i^1 J} = \underset{(a)}{i^1 J \alpha} + \underset{(\mathfrak{a})}{i^1 J \beta} + \ldots\ldots + \underset{(\mathfrak{A})}{i^1 J \varrho}$$

$$\underset{(A,B)}{i^2 J} = \underset{(a,b)}{i^2 J \alpha} + \underset{(\mathfrak{a},\mathfrak{b})}{i^2 J \beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B})}{i^2 J \varrho}$$

$$\underset{(A,B,C)}{i^3 J} = \underset{(a,b,c)}{i^3 J \alpha} + \underset{(\mathfrak{a},\mathfrak{b},\mathfrak{c})}{i^3 J \beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},\mathfrak{C})}{i^1 J \varrho}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\underset{(A,B,\ldots R)}{i^r J} = \underset{(a,b,\ldots r)}{i^r J \alpha} + \underset{(\mathfrak{a},\mathfrak{b},\ldots \mathfrak{r})}{i^r J \beta} + \ldots\ldots + \underset{(\mathfrak{A},\mathfrak{B},\ldots \mathfrak{R})}{i^r J \varrho}$$

Ex his aequationibus eruuntur (§. XXX.) valores literarum $\alpha, \beta \ldots \varrho$ quaesiti. Nam,

1) posito $r = 1$ et $\varrho = \alpha$, fit $\underset{(A)}{i^1 J} = \underset{(a)}{i^1 J \alpha}$. Ergo: $\alpha = + \underset{(A)}{i^1 J} : \underset{(a)}{i^1 J}$

2) posito $r = 2$ et $\varrho = \beta$, fit

$$\underset{(A)}{i^1 J} = \underset{(a)}{i^1 J \alpha} + \underset{(\mathfrak{a})}{i^1 J \beta}$$
$$\underset{(A,B)}{i^2 J} = \underset{(a,b)}{i^2 J \alpha} + \underset{(\mathfrak{a},\mathfrak{b})}{i^2 J \beta}$$

Ergo:

$$\alpha = \underset{(A,B)\,(\mathfrak{a},\mathfrak{b})}{\begin{Bmatrix} + i^1 J & i^2 J \\ - i^2 J & i^1 J \end{Bmatrix}} : \underset{(a,b)\,(\mathfrak{a},\mathfrak{b})}{\begin{Bmatrix} + i^1 J & i^2 J \\ - i^2 J & i^1 J \end{Bmatrix}} \qquad \beta = \underset{(a,b)\,(A,B)}{\begin{Bmatrix} + i^1 J & i^2 J \\ - i^2 J & i^1 J \end{Bmatrix}} : \underset{(a,b)\,(\mathfrak{a},\mathfrak{b})}{\begin{Bmatrix} + i^1 J & i^2 J \\ - i^2 J & i^1 J \end{Bmatrix}}$$

ubi quilibet index pertinet ad signa involutoria seriei verticalis super ipsum positae.

Scholion. Dignum notari hoc problema, quippe methodorum vulgarium nulla, quod equidem sciam, ad illud solvendum sufficiat, ejusdem potius solutio videatur impossibilis, cum incognitae *plures* $\alpha, \beta, \gamma, \ldots \varrho$, *una* ex aequatione sint determinandae. *Analysis* igitur *Combinatoria* h. l. uti saepius, aliis methodis efficit majora.

Sed haec jam sufficiant, plura in additamentis.

ADDI-

ADDITAMENTA.

§ 1.

Summam libelli hactenus propoſita complectuntur. Varia Corollaria et Exempla, quae adhuc reliqua ſunt, nunc demum ſequuntur, quod relationibus §. XXVIII. – XXXI. exhibitis interjecta, ipſarum ordinem turbaſſent.

§ 2.

Si in aequatione §. XXVII. expoſita factorum multiplicandorum omnes ſunt binomiorum potentiae *) veluti:

$$(1-ax)^{\alpha}(1-bx)^{\beta}(1-cx)^{\gamma}(1-dx)^{\delta}\ldots(1-rx)^{\varrho}=1-(\mathrm{A}x+\mathrm{B}x^2+\mathrm{C}x^3+\mathrm{D}x^4+\ldots+\mathrm{N}x^n+\ldots)$$

tunc eſt (§. XXV.):

$$^{n}\Sigma = a^{n}\alpha + b^{n}\beta + c^{n}\gamma + d^{n}\delta \ldots + r^{n}\varrho$$

et ſequitur ex §. XXVIII. (mutato ſcilicet valore ſigni $^{n}\Sigma$)

$$I.\quad {}^{n}\Sigma = \underset{(\mathrm{A},\mathrm{B},\mathrm{C},\ldots\mathrm{N})}{\mathfrak{i}^{n}\mathfrak{J}} = a^{n}\alpha + b^{n}\beta + c^{n}\gamma + d^{n}\delta + \ldots + r^{n}\varrho$$

$$II.\quad {}^{n}\Sigma = {}^{n-1}\Sigma\mathrm{A} + {}^{n-2}\Sigma\mathrm{B} + {}^{n-3}\Sigma\mathrm{C} + \ldots + {}^{n-m}\Sigma\mathrm{M} \ldots + {}^{1}\Sigma\overset{\cdot 1}{\mathrm{N}} + n\mathrm{N} = a^{n}\alpha + b^{n}\beta + c^{n}\gamma \ldots + r^{n}\varrho$$

atque ex §. XXIX.

$$III.\quad \mathrm{N} = \frac{1}{1}\mathfrak{a}^{n}\mathfrak{A} - \frac{1}{1.2}\mathfrak{b}^{n}\mathfrak{B} + \frac{1}{1.2.3}\mathfrak{c}^{n}\mathfrak{C} - \frac{1}{1.2.3.4}\mathfrak{d}^{n}\mathfrak{D} + \ldots \pm \frac{1}{1.2..m}\mathfrak{m}^{n}\mathfrak{M} \mp \ldots \pm \frac{1}{1.2..n}\mathfrak{n}^{n}\mathfrak{N}$$

$$\underset{1,\quad 2,\quad 3,\quad 4,\ \ldots\ n}{\left(\frac{^{1}\Sigma}{1}, \frac{^{2}\Sigma}{2}, \frac{^{3}\Sigma}{3}, \frac{^{4}\Sigma}{4}, \ldots \frac{^{n}\Sigma}{n}\right)}$$

*) Hanc aequationem tractarunt T. Simpson (*Philoſophical Tranſactions Vol. XLVII.* 1751. p. 20.) et G. F. Tempelhofius, *Vir Strenuiſſimus*, (*Anfangsgründe der Analyſis des Unendlichen* p. 301.)

§. 3.

Problema.

Summam reperire ſeriei infinitae:

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \frac{1}{7^{2n}} + \ldots\ldots,$$

in qua n eſſe poteſt numerus quilibet integer poſitivus.

Solutio.

$$\frac{Sin z}{z} = \left(1 - \frac{1.z^2}{1^2\pi^2}\right)\left(1 - \frac{1.z^2}{2^2\pi^2}\right)\left(1 - \frac{1.z^2}{3^2\pi^2}\right)\cdots = 1 - \left(\frac{z^2}{1.2.3} - \frac{z^4}{1.2.3.4.5} + \cdots \pm \frac{z^{2n}}{1.2\ldots(2n+1)} \mp \ldots\right)$$

Itaque ſecundum §. 2. formulam *I* *)

$$^{n}\Sigma = \frac{i^{n}J}{\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \ldots \frac{\mp 1}{1.2..(2n+1)}\right)} = \left(\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \cdots\right)\frac{1}{\pi^{2n}}$$

et ſecundum formulam *II.*

$$^{n}\Sigma = \frac{^{n-1}\Sigma}{1.2.3} - \frac{^{n-2}\Sigma}{1.2.3.4.5} + \frac{^{n-3}\Sigma}{1.2.3.4.5.6.7} - \cdots \pm \frac{^{n-m}\Sigma}{1.2\ldots(2m-1)} \mp \cdots \mp \frac{^{1}\Sigma}{1.2..(2n-1)} \pm \frac{n}{1.2..(2n+1)}$$

$$= \left(\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \cdots\right)\frac{1}{\pi^{2n}}$$

Corollarium.

Multiplicetur per $\frac{2}{2^{2n}}$ aequatio ſequens:

*) Valores literarum in §. 2. hoc loco ſunt:

$x = z^2$; $\alpha = 1$, $\beta = 1$; $\gamma = 1$; $\delta = 1$; ....

$a = \frac{1}{1^2\pi^2}$; $b = \frac{1}{2^2\pi^2}$; $c = \frac{1}{3^2\pi^2}$; $d = \frac{1}{4^2\pi^2}$; ....

$A = \frac{1}{1.2.3}$; $B = \frac{-1}{1.2.3.4.5}$; $C = \frac{1}{1.2.3.4.5.6.7}$; $D = \frac{-1}{1.2.3.4.5.6.7.8.9}$; .....

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\ldots\ldots=\frac{i^{n}J\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\ldots\frac{\pm 1}{1.2\ldots(2n+1)}\right)}$$

hinc oritur:

$$\frac{2}{2^{2n}}+\frac{2}{4^{2n}}+\frac{2}{6^{2n}}+\frac{2}{8^{2n}}+\ldots\ldots=\frac{\frac{2}{2^{2n}}i^{n}J\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\ldots\frac{\pm 1}{1.2..(2n+1)}\right)}$$

qua serie de priori detracta, provenit:

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}-\ldots\ldots=\frac{\left(1-\frac{1}{2^{2n-1}}\right)i^{n}J\pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5}\ldots\frac{\pm 1}{1.2..(2n+1)}\right)}$$

§. 4

PROBLEMA.

Summam reperire seriei infinitae:

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\frac{1}{9^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}+\ldots\ldots$$

in qua n numerus esse potest quilibet integer positivus.

SOLUTIO.

$$Cos\,z=\left(1-\frac{2^2z^2}{1^2\pi^2}\right)\left(1-\frac{2^2z^2}{3^2\pi^2}\right)\left(1-\frac{2^2z^2}{5^2\pi^2}\right)\ldots=1-\left(\frac{z^2}{1.2}-\frac{z^4}{1.2.3.4}+\frac{z^6}{1.2.3.4.5.6}\ldots\right)$$

Itaque secundum §. 2. formulam *I.* *)

 $^{n}\Sigma =$

*) Valores literarum in §. 2. hoc loco sunt: $x = z^2$; porro $\alpha = 1$

$$^{n}\Sigma = \frac{[^{n}]}{\left(\frac{1}{1.2}, \frac{-1}{1.2.3.4}, \ldots \frac{\pm 1}{1.2\ldots 2n}\right)} = \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \ldots\right)\frac{2^{2n}}{\pi^{2n}}$$

et secundum formulam II.

$$^{n}\Sigma = \frac{^{n-1}\Sigma}{1.2} - \frac{^{n-2}\Sigma}{1.2.3.4} + \frac{^{n-3}\Sigma}{1.2.3.4.5.6} - \ldots \mp \frac{^{1}\Sigma}{1.2\ldots(2n-2)} \pm \frac{n}{1.2\ldots 2n} = \left(1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} \ldots\right)\frac{2^{2n}}{\pi^{2n}}$$

§. 5.

Problema.

Invenire summam seriei infinitae

$$\frac{1}{1^{\varkappa}} \mp \frac{1}{3^{\varkappa}} + \frac{1}{5^{\varkappa}} \mp \frac{1}{7^{\varkappa}} + \frac{1}{9^{\varkappa}} \mp \frac{1}{11^{\varkappa}} + \frac{1}{13^{\varkappa}} \mp \ldots,$$

ubi superiora signa valent, si $\varkappa$ *est numerus integer positivus impar, inferiora, si est par.*

Solutio.

Secundum Eulerum (*Introductio in Analysin infinitorum Lib. I.* §. 171. et 175) est, posito $\frac{\pi x}{4} = z$:

$$Cos\, z + Sin\, z \begin{cases} = \left(1 + \frac{4z}{1\pi}\right)\left(1 - \frac{4z}{3\pi}\right)\left(1 + \frac{4z}{5\pi}\right)\left(1 - \frac{4z}{7\pi}\right)\left(1 + \frac{4z}{9\pi}\right)\left(1 - \frac{4z}{11\pi}\right)\ldots \\ = 1 - \left(-\frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3}\, \frac{z^4}{1.2.3.4} - \ldots \mp \frac{z^{2n-1}}{1.2(2n-1)} \pm \frac{z^{2n}}{1.2\ldots 2n} \pm \ldots\right) \end{cases}$$

Itaque

$$\alpha = 1;\quad \beta = 1;\quad \gamma = 1;\quad \delta = 1;\ldots$$

$$a = \frac{2^2}{2^2\pi^2};\ b = \frac{2^2}{3^2\pi^2};\ c = \frac{2^2}{5^2\pi^2}\ d = \frac{2^2}{7^2\pi^2};\ldots$$

$$A = \frac{1}{1.2};\ B = \frac{-1}{1.2.3.4};\ C = \frac{1}{1.2.3.4.5.6};\ D = \frac{1-}{1.2.3.4.5.6.7.8};\ldots$$

Itaque secundum §. 2. formulam I. *)

$$-{}^{2n-1}\Sigma = \frac{-\,\{^{n-1}J}{\left(\frac{-1}{1},\frac{1}{1.2},\frac{1}{1.2.3},\frac{-1}{1.2.3.4},\ldots\frac{\mp 1}{1.2..(2n-1)}\right)} = \left(\frac{1}{1^{2n-1}}-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\frac{1}{7^{2n-1}}+\ldots\right)\frac{4^{2n-1}}{\pi^{2n-1}}$$

$$+{}^{2n}\Sigma = \frac{+\,\{^{2n}J}{\left(\frac{-1}{1},\frac{1}{1.2},\frac{1}{1.2.3},\frac{-1}{1.2.3.4},\ldots\frac{\pm 1}{1.2..2n}\right)} = \left(\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\ldots\right)\frac{4^{2n}}{\pi^{2n}}$$

et secundum formulam II.

$$-{}^{2n-1}\Sigma = -\frac{{}^{2n-2}\Sigma}{1}+\frac{{}^{2n-3}\Sigma}{1.2}+\frac{{}^{2n-4}\Sigma}{1.2.3}-\ldots\pm\frac{{}^{2}\Sigma}{1.1..(2n-3)} = \left(\frac{1}{1^{2n-1}}-\frac{1}{3^{2n-1}}+\frac{1}{5^{2n-1}}-\ldots\right)\frac{4^{2n-1}}{\pi^{2n-1}}$$

$$+{}^{2n}\Sigma = -\frac{{}^{2n-1}\Sigma}{1}+\frac{{}^{2n-2}\Sigma}{1.2}+\ldots\pm\frac{{}^{3}\Sigma}{1.2..(2n-3)}\mp\frac{{}^{2}\Sigma}{1.2..(2n-2)}\mp\frac{{}^{1}\Sigma}{1.2..(2n-1)} = \left(\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}\ldots\right)\frac{4^{2n}}{\pi^{2n}}$$

in quibus formulis signa superiora valent, si *n* est numerus impar, si par, inferiora.

COROLLARIUM.

Quia (§. 3.)

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\ldots = \frac{\{^{n}J\ \pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2.3.4.5.6},\ldots\right)}$$

et

$$\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{6^{2n}}+\frac{1}{8^{2n}}+\ldots = \frac{\frac{1}{2^{2n}}\{^{n}J\ \pi^{2n}}{\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2.3.4.5.6},\ldots\right)}$$

prodit,

*) Pro hac scilicet aequatione in §. 2.

$x = z;\ \alpha = 1;\ \beta = 1;\ \gamma = 1;\ \delta = 1;\ldots$

$a = -\frac{4}{1\pi};\ b = \frac{4}{3\pi};\ c = -\frac{4}{5\pi};\ d = \frac{4}{7\pi};\ldots$

$A = \frac{-1}{1};\ B = \frac{1}{1.2};\ C = \frac{1}{1.2.3};\ D = \frac{-1}{1.2.3.4};\ E = \frac{-1}{1.2.3.4.5};\ F = \frac{1}{1.2.3.4.5.6};\ldots$

prodit, posteriore serie de priore detracta,

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\dots = \left(1-\frac{1}{2^{2n}}\right)[^{n}].\pi^{2n}$$
$$\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2.3.4.5.6.7},\dots\right)$$

Ejusdem seriei summa inventa est §. 4 et 5, unde sequitur:

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}\dots = [^{n}]\left(\frac{\pi}{2}\right)^{2n} = (2^{2n}-1)[^{n}]\left(\frac{\pi}{2}\right)^{2n} = \frac{1}{2^{2n}}[^{n}]\left(\frac{\pi}{2}\right)^{2n}$$
$$\left(\frac{1}{1.2},\frac{-1}{1.2.3.4},\frac{1}{1.2.3\,4.5.6},\dots\right)\quad\left(\frac{1}{1.2.3},\frac{-1}{1.2.3.4.5},\frac{1}{1.2..7},\dots\right)\quad\left(\frac{-1}{1},\frac{+1}{1.2},\frac{+1}{1.2.3},\frac{-1}{1.2.3.4},\dots\right);$$

§. 6.

SERIERUM QUARUNDAM SUMMAE PROPONUNTUR INDEPENDENTES, QUAE TERMINORUM RECURRENTIUM AUXILIO HUCUSQUE EXHIBITA SUNT.

In egregio opere (*Versuch einer neuen Summationsmethode. Berlin* 1788.) PFAFFIUS, *Vir Celeberrimus*, cum aliis, quas instituit, quaestionibus gravissimis, magnum formularum recurrentium numerum tradidit, quibus multarum serierum summae ex circuli rectificatione pendentes exprimuntur. Ex iisdem nonnullae h. l. deliguntur, quarum summae arctissimo vinculo cum propositis (§. 3. 4. 5) conjunctae, earundem auxilio in formulas independentes transformantur.

Summam seriei infinitae, ubi m numerus integer positivus est,

$$1+\frac{1}{2^{2m}}+\frac{1}{3^{2m}}+\frac{1}{4^{2m}}+\frac{1}{5^{2m}}+\dots+\frac{1}{n^{2m}}+\dots$$

PFAFFIUS in opere laudato ita exhibet:

$$\frac{\pi^2}{1.2.3}\Sigma\frac{1}{n^{2m-2}}-\frac{\pi^4}{1.2.3.4.5}\Sigma\frac{1}{n^{2m-4}}+\dots\mp\frac{\pi^{2m-2}}{1.2..(2m-1)}\Sigma\frac{1}{n^2}\pm\frac{\pi^{2m}m}{1.2.3..(2m+1)}\ ^{*})$$

ean-

*) Nempe signo Σ PFAFFIUS exprimit summam seriei infinitae, cujus terminus generalis ea est functio numeri n, quae huic signo adjicitur, sic Σ N summa est omnium valorum functionis N, qui proveniunt, posito $n = 1, 2, 3, 4 \dots$ Signo Σ ± N IDEM notat seriei summam, in qua terminorum signa (+ —) alternantur (PFAFF. ibid. p. 4. **).

eandem (§. 3) reperimus esse

$$i^m J \pi^{2m} \left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2..7}, \ldots\right)$$

Pfaffiana formula recurrit, nostra vero est independens. Haec praemittenda erant, ut sequentia intelligantur, ubi Pf. cum numero adposito *Pfaffiani* operis paginam notat.

I. Summa seriei infinitae $Sin\ \varphi + \frac{Sin\ 2\varphi}{2^{2m-1}} + \frac{Sin\ 3\varphi}{3^{2m-1}} + \frac{Sin\ 4\varphi}{4^{2m-1}} + \ldots\ldots$

$$= \varphi \Sigma \frac{1}{2^{2m-2}} - \frac{\varphi^3}{1.2.3} \Sigma \frac{1}{2^{2m-4}} + \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)} \Sigma \frac{1}{n^2} \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)} \frac{\pi}{2} \mp \frac{\varphi^{2m-1}}{1.2..(2m-1)} \frac{1}{2} \qquad \text{(Pf. 10.}$$

$$= \varphi i^{m-1} J \pi^{2m-2} - \frac{\varphi^3}{1.2.3} i^{m-2} J \pi^{2m-4} + \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)} i^1 J \pi^2 \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)} \frac{\pi}{2} + \frac{\varphi^{2m-1}}{1.2..(2m-1)} \frac{1}{2} \qquad \text{(§. 3.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1,2,3,4,5}, \frac{1}{1.2.3.4.5.6.7}, \frac{-1}{1.2,3\ldots9}, \ldots\right)$$

II. Summa seriei infinitae $Sin\ \varphi - \frac{Sin\ 2\varphi}{2^{2m-1}} + \frac{Sin\ 3\varphi}{3^{2m-1}} - \frac{Sin\ 4\varphi}{4^{2m-1}} + \ldots$

$$= \varphi \Sigma \pm \frac{1}{n^{2m-2}} - \frac{\varphi^3}{1.2.3} \Sigma \pm \frac{1}{n^{2m-4}} + \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)} \Sigma \pm \frac{1}{n^2} \pm \frac{\varphi^{2m-1}}{1.2..(2m-1)} \frac{1}{2} \qquad \text{(Pf. 8.}$$

$$= \varphi\left(1 - \frac{1}{2^{2m-3}}\right) i^{m-1} J \pi^{2m-2} - \frac{\varphi^3}{1.2.3}\left(1 - \frac{1}{2^{2m-5}}\right) i^{m-2} J \pi^{2n-4} \ldots \mp \frac{\varphi^{2m-3}}{1.2..(2m-3)}\left(1 - \frac{1}{2}\right) i^1 J \pi^2 \pm \frac{\varphi^{2m-1}}{1.2..(2m-1)} \frac{1}{2} \qquad \text{(§.3.C.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\ldots\right)$$

III. Summa seriei infinitae $Cos\ \varphi + \frac{Cos\ 2\varphi}{2^{2n}} + \frac{Cos\ 3\varphi}{3^{2m}} + \frac{Cos\ 4\varphi}{4^{2m}} + \ldots\ldots$

$$= \Sigma \frac{1}{n^{2m}} - \frac{\varphi^2}{1.2} \Sigma \frac{1}{n^{2m-2}} + \ldots \pm \frac{\varphi^{2m-2}}{1.2.(2m-2)} \Sigma \frac{1}{n^2} \mp \frac{\varphi^{2m-1}}{1.2..(2m-1)} \frac{1}{2} \pm \frac{1}{2} \frac{\varphi^{2m}}{1,2..2m} \qquad \text{(Pf. 12.}$$

$$= i^m J \pi^{2m} - \frac{\varphi^2}{1.2} i^{m-1} J \pi^{2m-2} + \ldots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)} i^1 J \pi^2 \mp \frac{\varphi^{2m-1}}{1.2..(2m-1)} \cdot \frac{1}{2} \pm \frac{1}{2} \frac{\varphi^{2m}}{1,2,,2m} \qquad \text{(§. 3.}$$

$$\left(\frac{1}{1,2,3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\right)$$

IV. Sum-

IV. Summa ſeriei infinitae $Cos\,\varphi - \frac{Cos\,2\,\varphi}{2^{2m}} + \frac{Cos\,3\,\varphi}{3^{2m}} - \frac{Cos\,4\,\varphi}{4^{2m}} + \ldots$

$$= \Sigma \pm \frac{1}{n^{2m}} - \frac{\varphi}{1.2} \Sigma \pm \frac{1}{n^{2m-2}} + \ldots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)} \Sigma \pm \frac{1}{n^2} \mp \frac{1}{2} \frac{\varphi^{2m}}{1.2..2m} \quad \text{(Pf. 12.}$$

$$= \left(1 - \frac{1}{2^{2m-1}}\right) {}^{(m)}J \pi^{2m} - \frac{\varphi^2}{1.2} \left(1 - \frac{1}{2^{2m-3}}\right) {}^{(m-1)}J \pi^{2m-2} \ldots \pm \frac{\varphi^{2m-2}}{1.2..(2m-2)} \left(1 - \frac{1}{2}\right) {}^{(1)}J \pi^2 \mp \frac{1}{2} \frac{\varphi^2}{1.2..2m} \quad \text{(§. 3 C.}$$

$$\left(\frac{1}{1.2.3}, \frac{-1}{1.2.3.4.5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\right)$$

§. 7.

Problema.

Summam reperire ſeriei infinitae

$$Cos\,\varphi - \frac{Cos\,3\,\varphi}{3^{2m-1}} + \frac{Cos\,5\,\varphi}{5^{2m-1}} - \frac{Cos\,7\,\varphi}{7^{2m-1}} + \ldots\ldots$$

in qua m numerus eſt integer poſitivus.

Solutio.

Pfaffiana methodo Coſinus arcuum φ, $3\,\varphi$, $5\,\varphi$, ... ſolvantur in ſeries infinitas, eae multiplicentur reſpective per 1; $-\frac{1}{3^{2m-1}}$; $+\frac{1}{5^{2m-1}}$; ... deinde factores numeratori et denominatori ſingulorum terminorum communes extinguantur, termini ipſi ſecundum potentias arcus φ diſponantur, et coefficientes dignitatum arcus φ in ſummas colligantur prodibit auxilio §. 5. ſumma ſeriei propoſitae

$$= -{}^{(2m-1)}J \frac{\pi^{2m-1}}{4^{2m-1}} + {}^{(2m-3)}J \frac{\pi^{2m-3}}{4^{2m-3}} \frac{\varphi^2}{1.2} - \ldots \mp {}^{(3)}J \frac{\pi^3}{4^3} \frac{\varphi^{2m-4}}{1.2..(2m-4)} \pm {}^{(1)}J \frac{\pi^1}{4^1} \frac{\varphi^{2m-2}}{1.2..(2m-2)}$$

$$\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3}, \frac{-1}{1.2.3.4}, \frac{-1}{1.2.3.4.5}, \ldots\right)$$

(Conferatur Pfaffius l. c. X. 3. p. 39. ſeq.)

§. 8.

P r o b l e m a.

Summam reperire seriei infinitae, in qua m numerus est integer positivus:

$$Sin\,\varphi - \frac{Sin\,3\,\varphi}{3^{2m}} + \frac{Sin\,5\,\varphi}{5^{2m}} - \frac{Sin\,7\,\varphi}{7^{2m}} + \ldots .$$

S o l u t i o.

Sinus solvantur in series infinitas, caeteraque agantur uti §. 7. prodit summa quaesita:

$$= -\mathrm{i}^{2m-1}]\frac{\pi^{2m-1}}{4^{2m-1}}\varphi + \mathrm{i}^{2m-3}]\frac{\pi^{2m-3}}{4^{2m-3}}\frac{\varphi^3}{1.2.3} - \ldots \pm \mathrm{i}^{3}]\frac{\pi^3}{4^3}\frac{\varphi^{2m-3}}{1.2..(2m-3)} \mp \mathrm{i}^{1}]\frac{\pi^1}{4^1}\frac{\varphi^{2m-1}}{1.2..(2m-1)}$$

$$\left(\frac{-1}{1}, \frac{1}{1.2}, \frac{1}{1.2.3.}, \frac{-1}{1.2.3.4.}, \ldots\right)$$

(Conferatur Pfaffius l. c. X. 5. p. 41.)

§. 9.

Eadem methodo, quae §. 7. et 8 adhibita est, sed auxilio Corollarii §. 3. eruuntur summae serierum sequentium, in quibus *m* numerus est integer positivus:

$$1)\ \frac{Sin\,2\,\varphi}{2^{2m+1}} - \frac{Sin\,4\,\varphi}{4^{2m+1}} + \frac{Sin\,6\,\varphi}{6^{2m+1}} - \ldots = \left(1 - \frac{1}{2^{2m-1}}\right)\mathrm{i}^{m}]\frac{\pi^{2m}}{2^{2m}}\varphi - \left(1 - \frac{1}{2^{2m-3}}\right)\mathrm{i}^{m-1}]\frac{\pi^{2m-2}}{2^{2m-2}}\frac{\varphi^3}{1.2.3} + \ldots$$

$$\ldots \pm \left(1 - \frac{1}{2}\right)\mathrm{i}^{1}]\frac{\pi^2}{2^2}\frac{\varphi^{2m-1}}{1.2..(2m-1)} \mp \frac{1}{2}\frac{\varphi^{2m+1}}{1.2..(2m+1)}$$

$$2)\ \frac{Cos\,2\,\varphi}{2^{2m}} - \frac{Cos\,4\,\varphi}{4^{2m}} + \frac{Cos\,6\,\varphi}{6^{2m}} - \ldots = \left(1 - \frac{1}{2^{2m-1}}\right)\mathrm{i}^{m}]\frac{\pi^{2m}}{2^{2m}} - \left(1 - \frac{1}{2^{2m-3}}\right)\mathrm{i}^{m-1}]\frac{\pi^{2m-2}}{2^{2m-2}}\frac{\varphi^2}{1.2} + \ldots .$$

$$\ldots \pm \left(1 - \frac{1}{2}\right)\mathrm{i}^{1}]\frac{\pi^2}{2^2}\frac{\varphi^{2m-2}}{1.2..(2m-2)} \mp \frac{\varphi^{2m}}{1.2..2m}$$

ubi ad indicem $\left(\frac{1}{1.2.3}, \frac{-1}{1,2,3,4,5}, \frac{1}{1.2.3.4.5.6.7}, \ldots\right)$ signa involutoria referuntur. Est enim secundum Eulerum (*Institt. Calc. Diff. P. II.* §. 185.)

$$1 - 2^{2m} + 3^{2m} - 4^{2m} + 5^{2m} - \ldots = 0$$

Hoc itaque loco serierum trigonometricarum summis, sublato omni terminorum recursu, campus patet latissimus.

§. 10.

PROBLEMA.

Producti ex factoribus numero infinitis, secundum potentias variabilis x ordinati,

$$\left(1-\frac{x}{b}\right)^{a}\left(1-\frac{x}{b^{2}}\right)^{a}\left(1-\frac{x}{b^{3}}\right)^{a}\ldots\ldots=1-(\mathrm{A}x+\mathrm{B}x^{2}+\mathrm{C}x^{3}+\ldots\ldots+\mathrm{N}x^{n}+\ldots)$$

invenire coefficientem quemlibet a prioribus independenter.

SOLUTIO.

Est hoc loco secundum §. 2. et formulam III.

$$\frac{{}^{n}\Sigma}{n}=\frac{1}{n}\left(\frac{a}{b^{n}}+\frac{a}{b^{2n}}+\frac{a}{b^{3n}}+\frac{a}{b^{4n}}+\ldots\right)=\frac{a}{n(b^{n}-1)}$$

atque

$$\mathrm{N}=\frac{1}{1}\mathfrak{a}^{n}\mathrm{A}-\frac{1}{1.2}\mathfrak{b}^{n}\mathrm{B}+\frac{1}{1.2.3}\mathfrak{c}^{n}\mathrm{C}-\ldots\pm\frac{1}{1.2\ldots m}\mathfrak{m}^{n}\mathcal{M}\ldots\pm\frac{1}{1.2..n}\mathfrak{n}^{n}\mathcal{N}$$

$$\left(\begin{matrix}\frac{a}{1(b^{1}-1)}, & \frac{a}{2(b^{2}-1)}, & \frac{a}{3(b^{3}-1)} & \ldots\ldots & \frac{a}{n(b^{n}-1)}\\ 1, & 2, & 3, & \ldots & n\end{matrix}\right)$$

unde prodit, posito n = 1, 2, 3, ...

$$\mathrm{A}=\frac{a}{b-1};\ \mathrm{B}=\frac{a}{2(b^{2}-1)}-\frac{a^{2}}{2(b-1)^{2}};\ \mathrm{C}=\frac{a}{3(b^{3}-1)}-\frac{a^{2}}{2(b-1)(b^{2}-1)}+\frac{a^{3}}{2.3.(b-1)^{3}}$$

§. 11.

Quodsi in aequatione §. 2. substituatur

$$\left\{\begin{matrix}\text{pro} & \alpha, & \beta, & \gamma, & \delta, & \varepsilon, & \zeta,\ldots\\ \text{respective} & +\alpha, & -\alpha, & +\beta, & -\beta, & +\gamma, & -\gamma,\ldots\end{matrix}\right\}\text{ et }\left\{\begin{matrix}\text{pro} & a, & b, & c, & d, & e, & f,\ldots\\ \text{respective} & +a, & -a, & -b, & +b, & +c, & -c,\ldots\end{matrix}\right\}\text{ oritur}$$

$$(1-ax)^{\alpha}(1+ax)^{-\alpha}(1+bx)^{+\beta}(-bx)^{-\beta}(1-cx)^{+\gamma}(1+cx)^{-\gamma}\ldots=1-(\mathrm{A}x+\mathrm{B}x^{2}+\mathrm{C}x^{3}\ldots+\mathrm{N}x^{n}..)$$

ubi numerus factorum semper est par, sive sit infinitus, sive finitus, atque fit:

$${}^{2m-1}\Sigma=2\left[a^{2m-1}\alpha-b^{2m-1}\beta+c^{2m-1}\gamma-d^{2m-1}\delta+\ldots\right]$$
$${}^{2m}\Sigma=0$$

Hinc

Hinc provenit:

I. $^{2m-1}\Sigma = {}^{2m-3}\Sigma B + {}^{2m-5}\Sigma D + \ldots + {}^{1}\overset{m-2}{\Sigma M} + (2m-1)\overset{m-1}{M} = 2\,[a^{2m-1}\alpha - b^{2m-1}\beta + c^{2m-1}\gamma - \ldots]$

II. $^{2m}\Sigma = {}^{2m-1}\Sigma A + {}^{2m-3}\Sigma C + \ldots + {}^{3}\overset{m-3}{\Sigma M} + {}^{1}\overset{m-1}{\Sigma M} + 2m\overset{m}{M} = 0$

III. $$N = \pm\left(\frac{1}{1.2.n}\mathfrak{n}^n\mathcal{N} + \frac{1}{1.2..(n-2)}\mathfrak{n}^n\overset{-2}{\mathcal{N}} + \frac{1}{1.2..(n-4)}\mathfrak{n}^n\overset{-4}{\mathcal{N}} + \ldots\ldots + \begin{Bmatrix}\frac{1}{1}\,\mathfrak{a}^n A \\ \frac{1}{1.2}\,\mathfrak{b}^n B\end{Bmatrix}\right) \text{*)}$$

$$\begin{pmatrix}\frac{^1\Sigma}{1}, & 0, & \frac{^3\Sigma}{3}, & 0, & \frac{^5\Sigma}{5}, & 0, & \frac{^7\Sigma}{7}, & \ldots\ldots \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots\ldots\end{pmatrix}$$

in qua tertia formula $\begin{cases}\text{superiora signa } + \text{ et } \frac{1}{1}\mathfrak{a}^n A \text{ valent, si } n \text{ numerus est impar,} \\ \text{inferiora } \; - \text{ et } \frac{1}{1.2}\mathfrak{b}^n B, \text{ si } n \text{ est par.}\end{cases}$

§. 12.

Problema.

Invenire valores secundum potentias quantitatis x digestos productorum, quae sequuntur:

1) $$\left(\frac{1-x}{1+x}\right)^{Sin\,\phi}\left(\frac{1+\frac{1}{2}x}{1-\frac{1}{2}x}\right)^{Sin\,2\phi}\left(\frac{1-\frac{1}{3}x}{+\frac{1}{3}x}\right)^{Sin\,3\phi}\left(\frac{1+\frac{1}{4}x}{1-\frac{1}{4}x}\right)^{Sin\,4\phi}\ldots = 1-(Ax+Bx^2+Cx^3\ldots+Nx^n\ldots)$$

2) $$\left(\frac{1-x}{1+x}\right)^{Cos\,\phi}\left(\frac{1+\frac{1}{3}x}{1-\frac{1}{3}x}\right)^{Cos\,3\phi}\left(\frac{1-\frac{1}{5}x}{1+\frac{1}{5}x}\right)^{Cos\,5\phi}\left(\frac{1+\frac{1}{7}x}{1-\frac{1}{7}x}\right)^{Cos\,7\phi}\ldots = 1-(\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3\ldots+\mathfrak{N}x^n\ldots)$$

Solutio.

Secundum §. 11. **) est in producto 1)

$$^{2m-1}\Sigma = 2\left[Sin\,\phi - \frac{Sin\,2\,\phi}{2^{2m-1}} + \frac{Sin\,3\,\phi}{3^{2m-1}} - \frac{Sin\,4\,\phi}{4^{2m-1}} + \ldots\ldots\right]$$

in

*) Scripsi h. l. $\mathfrak{n}^n\overset{-2}{\mathcal{N}}$ pro $\overset{-2}{\mathfrak{n}^n}\overset{-2}{\mathcal{N}}$ etc. quod hoc compendio nulla obscuritas nascitur.

**) Est scilicet in §. 11. intuitu prioris producti

$\alpha = Sin\,\phi;\ \beta = Sin\,2\,\phi;\ \gamma = Sin\,3\,\phi;\ldots\ a = 1;\ b = \frac{1}{2};\ c = \frac{1}{3};\ d = \frac{1}{4};\ldots$

intuitu posterioris

$\alpha = Cos\,\phi;\ \beta = Cos\,3\,\phi;\ \gamma = Cos\,5\,\phi;\ldots\ a = 1;\ b = \frac{1}{3};\ c = \frac{1}{5};\ d = \frac{1}{7};\ldots$

in producto 2)

$$ {}^{2m-1}\Sigma = 2\left[Cos\,\varphi - \frac{Cos\,3\varphi}{3^{2m-1}} + \frac{Cos\,5\varphi}{5^{2m-1}} - \frac{Cos\,7\varphi}{7^{2m-1}} + \ldots\ldots \right] $$

cum vero harum ſerierum ſummae cognitae ſint (§. 6. II. et §. 7.), ſimul utriusque ſeriei coefficiens quilibet independenter erui poteſt. Invenitur:

$$ A = \varphi; \quad B = -\frac{\varphi^2}{1.2}; \quad C = \frac{\varphi^3}{1.2.3}\left(\frac{2\varphi^2 + \pi^2}{3}\right); \ldots $$

$$ \mathfrak{A} = \frac{\pi}{2}; \quad \mathfrak{B} = -\frac{\pi^2}{8}; \quad \mathfrak{C} = \frac{\pi}{4}\left(\frac{\pi^2}{4^2} + \frac{\varphi^2}{3}\right); \ldots $$

§. 13.

Ex aequatione

$$ (1-ax)(1-bx)(1-cx)\ldots(1-rx) = 1 - (Ax + Bx^2 + Cx^3 + \ldots + Nx^n \ldots + Rx^r) $$

ſequitur ſecundum §. 2.

$$ \text{I.} \quad {}^n\Sigma = \underset{(A,\,B,\,C,\,\ldots\,N)}{i^n J} = a^n + b^n + c^n + \ldots + r^n $$

$$ \text{II.} \quad {}^n\Sigma = {}^{n-1}\Sigma A + {}^{n-2}\Sigma B + \ldots\ldots + {}^{1}\Sigma \overset{-1}{N} + nN = a^n + b^n + c^n + \ldots + r^n $$

$$ \text{III.} \quad N = \frac{1}{1}\mathfrak{a}^n\mathfrak{A} - \frac{1}{1.2}\mathfrak{b}^n\mathfrak{B} + \frac{1}{1.2.3}\mathfrak{c}^n\mathfrak{C} - \ldots \pm \frac{1}{1.2..m}\mathfrak{m}^n\mathcal{M} \mp \ldots\ldots \pm \frac{1}{1.2..n}\mathfrak{n}^n\mathcal{N} $$

$$ \left(\frac{{}^1\Sigma}{1}, \frac{{}^2\Sigma}{2}, \frac{{}^3\Sigma}{3}, \ldots \frac{{}^n\Sigma}{n}\right) $$

Quodſi ponatur $1 - (Ax + Bx^2 + \ldots + Rx^r) = 0$, ſimul $a, b, c, \ldots r$; ſunt radices aequationis

$$ y^r - (Ay^{r-1} + By^{r-2} + Cy^{r-3} + \ldots + Ny^{r-n} + \ldots + R) = 0 $$

in qua $y = \frac{1}{x}$ (*) et formularum propoſitarum prima et ſecunda exhibent ſummam radicum aequa-

*) Ex aequatione $(1-ax)(1-bx)\ldots(1-rx) = 1-(Ax + Bx^2 \ldots + Rx^r) = 0$ ſequitur,

poſito $x = \frac{1}{y}$; $\left(1-\frac{a}{y}\right)\left(1-\frac{b}{y}\right)\ldots\left(1-\frac{r}{y}\right) = 1-\left(\frac{A}{y} + \frac{B}{y^2} + \ldots + \frac{R}{y^r}\right) = 0$

ſive $\left(\frac{y-a}{y}\right)\left(\frac{y-b}{y}\right)\ldots\left(\frac{y-r}{y}\right) = 1-\left(\frac{A}{y} + \frac{B}{y^2} + \ldots + \frac{R}{y^r}\right) = 0$

et ſi haec aequatio multiplicetur per y^r

$$ (y-a)(y-b)\ldots(y-r) = y^r - [Ay^{r-1} + By^{r-2} \ldots + Ry^0] = 0 $$

aequationis hujus ad potentiam *n*tam elevatarum; altera quidem independenter, altera vero, Newtoni, Baermanni, Kaestneri, Euleri, Tempelhofii, aliorumque Analystarum exemplo *), insertis praecedentibus radicum potentiis.

De tertia formula unum moneam: Exhibet ea valorem coefficientis N, sed constat, eundem aequare *n*tam classem omnium complexionum rite ordinatarum indicis, $(a, b, c, \ldots r)$, in quibus singulis nullum elementum bis vel saepius occurrit, (Kaestneri *Analysis endl. Grössen* §. 224.) i. e. *n*tam classem *Combinationum simpliciter, indicis* $(a, b, c, d, \ldots r)$ *et omissis quidem repetitionibus* **), quam Hindenburgius, *Vir Celeberrimus*, hoc notat signo:

$$\underset{(a,\ b,\ c,\ \ldots r)}{\mathcal{N}'}$$

Itaque, duplici valore coefficientis N invento, haec prodit relatio:

$$\mathrm{N} = \underset{(a,b,c,.r)}{\mathcal{N}'} = \frac{1}{1}\mathfrak{a}^n A - \frac{1}{1{,}2}\mathfrak{b}^n B + \ldots \pm \frac{1}{1.2..m}\mathfrak{m}^n \mathcal{M} \mp \ldots \pm \frac{1}{1.2..n}\mathfrak{n}^n \mathcal{N}$$

$$\left(\frac{{}^1\Sigma}{1}, \frac{{}^2\Sigma}{2}, \frac{{}^3\Sigma}{3}, \ldots \frac{{}^n\Sigma}{n}\right)$$

§. 14.

*) Theorema, quod formula ${}^n\Sigma = {}^{n-1}\Sigma A + {}^{n-2}\Sigma B + \ldots + {}^1\Sigma \overset{-1}{N} + nN$ exprimitur, Newtonus (*Arithmetica Universalis in fine capitis de transmutationibus aequationum* p. 192. *Editionis s'Gravesandinae*) proposuit, sed nullam ejus demonstrationem. Kaestnerus, *Vir Illustris*, illud demonstravit, afferens simul alia, quae pertinent ad hoc theorema (*Analysis endlicher Grössen* §. 751). *Eulerianas* hujus theorematis demonstrationes, Michelsen, *Vir Celeberrimus*, in additamentis suis ad Euleri *Introductionem in Analysin infinitorum* collegit (*Zusätze zum 10ten Capitel des 1sten Buchs*).

**) In tabula adposita Combinationum simpliciter, indicis (a, b, c, d, e) et omissis quidem repetitionibus, duarum vicinarum classium posterior ex priori oritur, si quaelibet prioris classis complexio ante indicis elementa, novissimum ipsius elementum insequentia, successive ponitur, atque ita complexiones ordinantur, ut quae in idem desinunt elementum, eadem in serie verticali collocentur.

a	*b*	*c*	*d*	*e*)	= A'
	ab	*ac*	*ad*	*ae*	= B'
		bc	*bd*	*be*	
			cd	*ce*	
				de	
		abc	*abd*	*abe*	= C'
			acd	*ace*	
				ade	
			bcd	*bce*	
				bde	
				cde	
			abcd	*abce*	= D'
				abde	
				acde	
				bcde	
				abcde	= E'

(a, b, c, d, e)

§. 14.

Problema.

Data aequatione

$$[1-(1x+1x^2)]^\alpha[1-(2x+2x^2)]^\beta=1-(3x+4x^2+Cx^3+Dx^4+\ldots)$$

reperire exponentes α, β *et coefficientes* C, D, E, ...

Solutio.

Pars prior. Est

$\overset{1}{i}J=1$	$\overset{1}{i}J=2$	$\overset{1}{i}J=3$
$\overset{2}{i}J=3$	$\overset{2}{i}J=8$	$\overset{2}{i}J=17$
$\overset{3}{i}J=4$	$\overset{3}{i}J=20$	$\overset{3}{i}J=63+3C$
$\overset{4}{i}J=7$	$\overset{4}{i}J=56$	$\overset{4}{i}J=257+12C+4D$
(1,1)	(2,2)	(3,4,C,D)

Secundum §. XXXI. reperitur:

$$\alpha=\left\{\begin{matrix}+\overset{1}{i}J & \overset{2}{i}J\\ -\overset{2}{i}J & \overset{1}{i}J\end{matrix}\right\}_{(3,4)(2,2)}:\left\{\begin{matrix}+\overset{1}{i}J & \overset{2}{i}J\\ -\overset{2}{i}J & \overset{1}{i}J\end{matrix}\right\}_{(1,1)(2,2)}=-5;\quad \beta=\left\{\begin{matrix}+\overset{1}{i}J & \overset{2}{i}J\\ -\overset{2}{i}J & \overset{1}{i}J\end{matrix}\right\}_{(1,1)(3,4)}:\left\{\begin{matrix}+\overset{1}{i}J & \overset{2}{i}J\\ -\overset{2}{i}J & \overset{1}{i}J\end{matrix}\right\}_{(1,1)(2,2)}=4$$

Pars posterior. Secundum §. XXVIII.

$$\underset{(3,4,C)}{\overset{3}{i}J}=-\underset{(1,1)}{\overset{3}{i}J}5+\underset{(2,2)}{\overset{3}{i}J}4;\quad \text{hoc est: } 63+3C=-4.5+20.4$$

ex qua aequatione eruitur $C=-1$. Secundum eundem §.

$$\underset{(3,4,C,D)}{\overset{4}{i}J}=-\underset{(1,1)}{\overset{4}{i}J}5+\underset{(2,2)}{\overset{4}{i}J}4\quad \text{hoc est: } 257+12C+4D=-7.5+56.4$$

unde sequitur: $D=-14$.

Eadem ratione valor reliquorum coefficientium E, F, G... successive invenitur.

His licet paucis exemplis relationum supra §. XXVIII. seqq. propositarum usus apparebit uberrimus.

EX OFFICINA KRAMERIANA

Pag.	*Lin.*	*Errata.*	*Corrige*
1	4	adhibitae	adhibitae novitatem
3	12	(§. Def. 1)	(§. II. Def. 1)
4	4	113444	11344.
—	8	III.	II.
7	6 a fine	$123\ldots\gamma$	$1.2.3\ldots\gamma$
8	19	$5ccdd$	$6ccdd$
9	16	$\frac{}{1}$	$\frac{\nu}{1}$
—	ult.	fit	unde fit
11	9	${}^{\nu}\mathfrak{A}a^{\nu-2}\mathfrak{a}^2A$	${}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^2A$
—	6 a fine	his	hic
13	9	${}^{\nu}\mathfrak{A}a\ \mathfrak{a}^1A$	${}^{\nu}\mathfrak{A}a^{\nu-1}\mathfrak{a}^1A$
—	12	$p\varkappa(m+1)$	$p^{\nu}\varkappa(m+1)$
—	13 a fine	Haec	Hae
14	13	illa	illae
—	6 a fine	*Variationum n*	*Variationum ſummae n*
16	11	adponantur	adponatur
—	12	duabus	duobus
20	8	$\binom{a,\ b,\ c,\ d,\ldots\ n}{1,\ 2,\ 3,\ 4,\ldots\ n}$	$\binom{a,\ b,\ c,\ d,\ldots\ n,\ldots}{1,\ 2,\ 3,\ 4,\ldots\ n,\ldots}$
21	6 a fine	XV	XVI
23	6 a fine	$\frac{1}{1}\mathfrak{b}^2B$	$\frac{1}{2}\mathfrak{b}^2B$
24	ult.	$\left(\frac{{}^1\Sigma}{1}x+\frac{{}^2\Sigma}{2}x+\ldots+\frac{\Sigma}{n}x^n\ldots\right)$	$-\left(\frac{{}^1\Sigma}{1}x+\frac{{}^2\Sigma}{2}x^2+\ldots+\frac{{}^n\Sigma}{n}x^n\ldots\right)$
26	9 a fine	$\binom{A,\ B,\ C,\ D,\ldots\ N}{1,\ 2,\ 3,\ 4,\ldots\ n}$	$\binom{A,\ B,\ C,\ D,\ldots\ N,\ldots}{1,\ 2,\ 3,\ 4,\ldots\ n,\ldots}$
27	11	nn	nn
28	14	$+\ +\ +$	$\pm\ \mp\ \pm$ (uti pag. 29. lin. 8.)
31	*Ex.* no. 3	$r = s$	$r = 3$
36	2 a fine	$\frac{2^2}{5\pi^2}\,d$	$\frac{2^2}{5^2\pi^2};d$

Pag.	*Lin.*	*Errata.*	*Corrige.*
37	5 et 7 a fine	$\frac{1}{1.2.3.4.5.6}$	$\frac{1}{1.2.3.4.5.6.7}$
38	5	nventa	inventa
—	10	EXHIBITA	EXHIBITAE
39	9	$\frac{Sin\,2\varphi}{2^{1m-1}}$	$\frac{Sin\,2\varphi}{2^{2m-1}}$
—	6	$\varphi\Sigma\frac{1}{2^{2m-2}}-\frac{\varphi^3}{1.2.3}\Sigma\frac{1}{2^{2m-4}}$	$\varphi\Sigma\frac{1}{n^{2m-2}}-\frac{\varphi^3}{1.2.3}\Sigma\frac{1}{n^{2m-4}}$
—	7	$+\frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2}$	$\mp\frac{\varphi^{2m-1}}{1.2..(2m-1)}\frac{1}{2}$
40	5 a fine	colligantur	colligantur:
42	4 a fine	$(-bx)^{-\beta}$	$(1-bx)^{-\beta}$
43	9 a fine	$\left(\frac{1-\frac{1}{3}x}{+\frac{1}{3}x}\right)^{Sin\,3\varphi}$	$\left(\frac{1-\frac{1}{3}x}{1+\frac{1}{3}x}\right)^{Sin\,3\varphi}$

$$p^{\nu} = (az^{\mu} + bz^{\mu+\delta} + cz^{\mu+2\delta} + dz^{\mu+3\delta} +$$

$$p^{\nu} = p^{\nu}\kappa 1\, z^{\nu\mu} + p^{\nu}\kappa 2\, z^{\nu\mu+\delta} \cdots\cdots + p^{\nu}\kappa(n+1)\, z^{\nu\mu+n\delta} + \ldots\ldots$$

$$p^{\nu} = a^{\nu}z^{\nu\mu} + {}^{\nu}\mathfrak{A}a^{\nu-1}a^{1}Az^{\nu\mu+\delta} + {}^{\nu}\mathfrak{A}\ldots + {}^{\nu}\mathfrak{A}\ldots \cdots\cdots \left.\begin{array}{l} + {}^{\nu}\mathfrak{A}\; a^{\nu-1}\; \mathfrak{a}\; {}^{n}A \\ + {}^{\nu}\mathfrak{B}\; a^{\nu-2}\; \mathfrak{b}\; {}^{n}B \\ + {}^{\nu}\mathfrak{C}\; a^{\nu-3}\; \mathfrak{c}\; {}^{n}C \\ + {}^{\nu}\mathfrak{D}\; a^{\nu-4}\; \mathfrak{d}\; {}^{n}D \\ + {}^{\nu}\mathfrak{E}\; a^{\nu-5}\; \mathfrak{e}\; {}^{n}E \\ + {}^{\nu}\mathfrak{F}\; a^{\nu-6}\; \mathfrak{f}\; {}^{n}F \\ \quad\;\; '' \\ \quad\;\; '' \\ + {}^{\nu}\mathfrak{M}a^{\nu-m}\mathfrak{m}^{n}\mathcal{M} \\ \quad\;\; '' \\ \quad\;\; '' \\ + {}^{\nu}\mathfrak{N}\; a^{\nu-n}\; \mathfrak{n}^{n}\mathcal{N} \end{array}\right| z^{\nu\mu+n\delta} + \ldots\ldots$$

$$p^{\nu} = a^{\nu}z^{\nu\mu} + {}^{\nu}\mathfrak{A}a^{\nu-1}bz^{\nu\mu+\delta} + \ldots + \left.\begin{array}{l} + {}^{\nu}\mathfrak{A}a^{\nu-1}\; i \\ + {}^{\nu}\mathfrak{B}a^{\nu-2}\begin{cases} 2bh \\ 2cg \\ 2df \\ e^{2} \end{cases} \\ + {}^{\nu}\mathfrak{C}a^{\nu-3}\begin{cases} 3b^{2}g \\ 6bcf \\ 6bde \\ 3c^{2}e \end{cases} \end{array}\right| z^{\nu\mu+8\delta} \left.\begin{array}{l} + {}^{\nu}\mathfrak{A}a^{\nu-1}\; k \\ + {}^{\nu}\mathfrak{B}a^{\nu-2}\begin{cases} 2bi \\ 2ch \\ 2dg \\ 2ef \end{cases} \\ + {}^{\nu}\mathfrak{C}a^{\nu-3}\begin{cases} 3b^{2}h \\ 6bcg \\ 6bdf \\ 3be^{2} \\ 3c^{2}f \end{cases} \end{array}\right| z^{\nu\mu+9\delta} + \ldots\ldots$$

$$p^\nu = (az^\mu + bz^{\mu+\delta} + cz^{\mu+2\delta} + dz^{\mu+3\delta} + \ldots\ldots)^\nu$$

$$p^\nu = p^\nu\varkappa 1\, z^{\nu\mu} + p^\nu\varkappa 2\, z^{\nu\mu+\delta} + p^\nu\varkappa 3\, z^{\nu\mu+2\delta} + p^\nu\varkappa 4\, z^{\nu\mu+3\delta} + p^\nu\varkappa 5\, z^{\nu\mu+4\delta} + p^\nu\varkappa 6\, z^{\nu\mu+5\delta} + p^\nu\varkappa 7\, z^{\nu\mu+6\delta} + \ldots\ldots\ldots\ldots + p^\nu\varkappa(n+1)\, z^{\nu\mu+n\delta} + \ldots\ldots$$

$$p\,[\,a, b, c, d, e, f, g, h, i, k, \ldots\,]$$

$$p^\nu = a^\nu z^{\nu\mu} + {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^1 A z^{\nu\mu+\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^2 A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\mathfrak{b}^2 B \end{array}\right| z^{\nu\mu+2\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^3 A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\mathfrak{b}^3 B \\ + {}^\nu\mathfrak{C} a^{\nu-3}\mathfrak{c}^3 C \end{array}\right| z^{\nu\mu+3\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^4 A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\mathfrak{b}^4 B \\ + {}^\nu\mathfrak{C} a^{\nu-3}\mathfrak{c}^4 C \\ + {}^\nu\mathfrak{D} a^{\nu-4}\mathfrak{d}^4 D \end{array}\right| z^{\nu\mu+4\delta}$$

$$+ \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^5 A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\mathfrak{b}^5 B \\ + {}^\nu\mathfrak{C} a^{\nu-3}\mathfrak{c}^5 C \\ + {}^\nu\mathfrak{D} a^{\nu-4}\mathfrak{d}^5 D \\ + {}^\nu\mathfrak{E} a^{\nu-5}\mathfrak{e}^5 E \end{array}\right| z^{\nu\mu+5\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\mathfrak{a}^6 A \\ + {}^\nu\mathfrak{B} a^{\nu-2}\mathfrak{b}^6 B \\ + {}^\nu\mathfrak{C} a^{\nu-3}\mathfrak{c}^6 C \\ + {}^\nu\mathfrak{D} a^{\nu-4}\mathfrak{d}^6 D \\ + {}^\nu\mathfrak{E} a^{\nu-5}\mathfrak{e}^6 E \\ + {}^\nu\mathfrak{F} a^{\nu-6}\mathfrak{f}^6 F \end{array}\right| z^{\nu\mu+6\delta} + \ldots\ldots\ldots\ldots + \left.\begin{array}{l} + {}^\nu\mathfrak{A}\, a^{\nu-1}\, \mathfrak{a}^n A \\ + {}^\nu\mathfrak{B}\, a^{\nu-2}\, \mathfrak{b}^n B \\ + {}^\nu\mathfrak{C}\, a^{\nu-3}\, \mathfrak{c}^n C \\ + {}^\nu\mathfrak{D}\, a^{\nu-4}\, \mathfrak{d}^n D \\ + {}^\nu\mathfrak{E}\, a^{\nu-5}\, \mathfrak{e}^n E \\ + {}^\nu\mathfrak{F}\, a^{\nu-6}\, \mathfrak{f}^n F \\ \quad '' \\ \quad '' \\ + {}^\nu\mathfrak{M}\, a^{\nu-m}\, \mathfrak{m}^n \mathcal{M} \\ \quad '' \\ \quad '' \\ + {}^\nu\mathfrak{N}\, a^{\nu-n}\, \mathfrak{n}^n \mathcal{N} \end{array}\right| z^{\nu\mu+n\delta} + \ldots\ldots$$

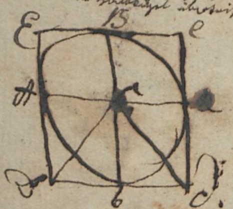
$$\left(\begin{array}{cccccccccc} b, & c, & d, & e, & f, & g, & h, & i, & k, & \ldots\ldots \\ 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \ldots\ldots \end{array}\right)$$

$$p^\nu = a^\nu z^{\nu\mu} + {}^\nu\mathfrak{A} a^{\nu-1} b z^{\nu\mu+\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; c \\ + {}^\nu\mathfrak{B} a^{\nu-2}\; b^2 \end{array}\right| z^{\nu\mu+2\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; d \\ + {}^\nu\mathfrak{B} a^{\nu-2}\; 2bc \\ + {}^\nu\mathfrak{C} a^{\nu-3}\; b^3 \end{array}\right| z^{\nu\mu+3\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; e \\ + {}^\nu\mathfrak{B} a^{\nu-2} \begin{cases} 2bd \\ c^2 \end{cases} \\ + {}^\nu\mathfrak{C} a^{\nu-3}\; 3b^2c \\ + {}^\nu\mathfrak{D} a^{\nu-4}\; b^4 \end{array}\right| z^{\nu\mu+4\delta}$$

$$+ \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; f \\ + {}^\nu\mathfrak{B} a^{\nu-2} \begin{cases} 2be \\ 2cd \end{cases} \\ + {}^\nu\mathfrak{C} a^{\nu-3} \begin{cases} 3b^2d \\ 3bc^2 \end{cases} \\ + {}^\nu\mathfrak{D} a^{\nu-4}\; 4b^3c \\ + {}^\nu\mathfrak{E} a^{\nu-5}\; b^5 \end{array}\right| z^{\nu\mu+5\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; g \\ + {}^\nu\mathfrak{B} a^{\nu-2} \begin{cases} 2bf \\ 2ce \\ d^2 \end{cases} \\ + {}^\nu\mathfrak{C} a^{\nu-3} \begin{cases} 3b^2e \\ 6bcd \\ c^3 \end{cases} \\ + {}^\nu\mathfrak{D} a^{\nu-4} \begin{cases} 4b^3d \\ 6b^2c^2 \end{cases} \\ + {}^\nu\mathfrak{E} a^{\nu-5}\; 5b^4c \\ + {}^\nu\mathfrak{F} a^{\nu-6}\; b^6 \end{array}\right| z^{\nu\mu+6\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; h \\ + {}^\nu\mathfrak{B} a^{\nu-2} \begin{cases} 2bg \\ 2cf \\ 2de \end{cases} \\ + {}^\nu\mathfrak{C} a^{\nu-3} \begin{cases} 3b^2f \\ 6bce \\ 3bd^2 \\ 3c^2d \end{cases} \\ + {}^\nu\mathfrak{D} a^{\nu-4} \begin{cases} 4b^3e \\ 12b^2cd \\ 4bc^3 \end{cases} \\ + {}^\nu\mathfrak{E} a^{\nu-5} \begin{cases} 5b^4d \\ 10b^3c^2 \end{cases} \\ + {}^\nu\mathfrak{F} a^{\nu-6}\; 6b^5c \\ + {}^\nu\mathfrak{G} a^{\nu-7}\; b^7 \end{array}\right| z^{\nu\mu+7\delta}$$

$$+ \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; i \\ + {}^\nu\mathfrak{B} a^{\nu-2} \begin{cases} 2bh \\ 2cg \\ 2df \\ e^2 \end{cases} \\ + {}^\nu\mathfrak{C} a^{\nu-3} \begin{cases} 3b^2g \\ 6bcf \\ 6bde \\ 3c^2e \\ 3cd^2 \end{cases} \\ + {}^\nu\mathfrak{D} a^{\nu-4} \begin{cases} 4b^3f \\ 12b^2ce \\ 6b^2d^2 \\ 12bc^2d \\ c^4 \end{cases} \\ + {}^\nu\mathfrak{E} a^{\nu-5} \begin{cases} 5b^4e \\ 20b^3cd \\ 10b^2c^3 \end{cases} \\ + {}^\nu\mathfrak{F} a^{\nu-6} \begin{cases} 6b^5d \\ 15b^4c^2 \end{cases} \\ + {}^\nu\mathfrak{G} a^{\nu-7}\; 7b^6c \\ + {}^\nu\mathfrak{H} a^{\nu-8}\; b^8 \end{array}\right| z^{\nu\mu+8\delta} + \left.\begin{array}{l} + {}^\nu\mathfrak{A} a^{\nu-1}\; k \\ + {}^\nu\mathfrak{B} a^{\nu-2} \begin{cases} 2bi \\ 2ch \\ 2dg \\ 2ef \end{cases} \\ + {}^\nu\mathfrak{C} a^{\nu-3} \begin{cases} 3b^2h \\ 6bcg \\ 6bdf \\ 3be^2 \\ 3c^2f \\ 6cde \\ d^3 \end{cases} \\ + {}^\nu\mathfrak{D} a^{\nu-4} \begin{cases} 4b^3g \\ 12b^2cf \\ 12b^2de \\ 12bc^2e \\ 12bcd^2 \\ 4c^3d \end{cases} \\ + {}^\nu\mathfrak{E} a^{\nu-5} \begin{cases} 5b^4f \\ 20b^3ce \\ 10b^3d^2 \\ 30b^2c^2d \\ 5bc^4 \end{cases} \\ + {}^\nu\mathfrak{F} a^{\nu-6} \begin{cases} 6b^5e \\ 30b^4cd \\ 20b^3c^3 \end{cases} \\ + {}^\nu\mathfrak{G} a^{\nu-7} \begin{cases} 7b^6d \\ 21b^5c^2 \end{cases} \\ + {}^\nu\mathfrak{H} a^{\nu-8}\; 8b^7c \\ + {}^\nu\mathfrak{I} a^{\nu-9}\; b^9 \end{array}\right| z^{\nu\mu+9\delta} + \ldots\ldots$$

Die Summe der äußern Cilinder läßt sich kleiner machen als jeder Körper welcher die Halbkugel übertrifft.
Die Summe der innern Cilinder kann größer gemacht werden als jeder Körper den die Halbkugel übertrifft.



$b\cdot f(z)+b\cdot f(x)$
$x\cdot f(z)+x\cdot f(x)$

$x - a - b\sqrt{-1}$

$x - a + b\sqrt{-1}$

USUS

LOGARITHMORUM INFINITINOMII

IN

THEORIA AEQUATIONUM

AUCTORE

MAURICIO DE PRASSE

ADIECTA EST TABULA SINGULARIS

LIPSIAE

APUD CHRIST. THEOPH. RABENHORST

1796.