A new Duality Based Approach for the Problem of Locating a Semi-Obnoxious Facility

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Notation

$\mathbb{R}^n_+$	$\{x \in \mathbb{R}^n \mid x \geq 0\}$
$\overline{a}^1, \ldots, \overline{a}^{\overline{M}} \in \mathbb{R}^n$	Attracting facilities
$\overline{w}_1, \ldots, \overline{w}_{\overline{M}} > 0$	Weights of attracting facilities
$\overline{B}_1, \ldots, \overline{B}_{\overline{M}} \subseteq \mathbb{R}^n$	Unit balls assigned to the attracting facilities
$\overline{B}^*_1, \ldots, \overline{B}^*_{\overline{M}} \subseteq \mathbb{R}^n$	Dual unit balls of $\overline{B}_1, \ldots, \overline{B}_{\overline{M}}$
$g : \mathbb{R}^n \to \mathbb{R}$	Weighted sum of distances to attracting facilities
$g_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$	Weighted sum of distances to attracting facilities considering a constraints set $\mathcal{H}$
$\underline{a}^1, \ldots, \underline{a}^{\underline{M}} \in \mathbb{R}^n$	Repulsive facilities
$\underline{w}_1, \ldots, \underline{w}_{\underline{M}} > 0$	Weights of repulsive facilities
$\underline{B}_1, \ldots, \underline{B}_{\underline{M}} \subseteq \mathbb{R}^n$	Unit balls assigned to the repulsive facilities
$\underline{B}^*_1, \ldots, \underline{B}^*_{\underline{M}} \subseteq \mathbb{R}^n$	Dual unit balls of $\underline{B}_1, \ldots, \underline{B}_{\underline{M}}$
$h : \mathbb{R}^n \to \mathbb{R}$	Weighted sum of distances to repulsive facilities
(P)	Unconstrained location problem with obnoxious facilities
$\overline{\mathcal{I}}$	Primal grid points w.r.t. attraction
$\underline{\mathcal{I}}$	Primal grid points w.r.t. repulsion
$\overline{G}$	Primal grid w.r.t. attraction

$\underline{G}$	Primal grid w.r.t. repulsion
$\mathcal{X}$	Set of optimal points of (P)
(D)	Toland-Singer-dual problem of (P)
$\overline{\mathcal{I}}_D$	Dual grid points w.r.t. attraction
$\underline{\mathcal{I}}_D$	Dual grid points w.r.t. repulsion
$\overline{G}_D$	Dual grid w.r.t. attraction
$\underline{G}_D$	Dual grid w.r.t. repulsion
$\mathcal{Y}$	Set of optimal points of (D)
$\mathcal{H}$	Convex polyhedral constraints set
$(P_{\mathcal{H}})$	Constrained location problem with obnoxious facilities
$\overline{\mathcal{I}}^{\mathcal{H}}$	Primal constrained grid points w.r.t. attraction
$\overline{G}^{\mathcal{H}}$	Primal constrained grid w.r.t. attraction
$\mathcal{X}_{\mathcal{H}}$	Set of optimal points of $(P_{\mathcal{H}})$
$(D_{\mathcal{H}})$	Toland-Singer-dual problem of $(P_{\mathcal{H}})$
$\overline{\mathcal{I}}_D^{\mathcal{H}}$	Dual constrained grid points w.r.t. attraction
$\overline{G}_D^{\mathcal{H}}$	Dual constrained grid w.r.t. attraction
$\mathcal{Y}_{\mathcal{H}}$	Set of optimal points of $(D_{\mathcal{H}})$
$(LVOP)$	Primal linear vector optimization problem
$(LVOD)$	Dual linear vector optimization problem
$(\overline{LVOP})$	Primal linear vector optimization problem related to attraction
$(\overline{LVOD})$	Dual linear vector optimization problem related to attraction
$(\underline{LVOP})$	Primal linear vector optimization problem related to repulsion
$(\underline{LVOD})$	Dual linear vector optimization problem related to repulsion

(W)	Classical Fermat-Weber Problem
$\operatorname{int} B$	Interior of a set $B \subseteq \mathbb{R}^n$
$\operatorname{ri} B$	Relative interior of a set $B \subseteq \mathbb{R}^n$
$\operatorname{bd} B$	Boundary of a set $B \subseteq \mathbb{R}^n$
$\operatorname{ext}(B)$	Set of extreme points of a set $B \subseteq \mathbb{R}^n$
$\partial f(x)$	Subdifferential of function f at a point x
f^*	Conjugate function of f
σ_B	Support function of a set $B \subseteq \mathbb{R}^n$
$\gamma_B(\cdot)$	Gauge distance associated with unit Ball $B \subseteq \mathbb{R}^n$
$\mathbb{I}_B$	Indicator function of a set $B \subseteq \mathbb{R}^n$
$\operatorname{dom} f$	Effective domain of function f
$\operatorname{epi} f$	Epigraph of function f
$f_1 \,\square\, f_2$	Infimal convolution of the functions f_1 and f_2
$N_B(x)$	Normal cone to a convex set $B \subseteq \mathbb{R}^n$ at $x \in \mathbb{R}^n$
$B_1 + B_2$	Minkowski sum of two sets $B_1, B_2 \subseteq \mathbb{R}^n$
0^+B	Recession cone of the set $B \subseteq \mathbb{R}^n$
$\langle \cdot, \cdot \rangle$	The usual inner product
2^B	Power set of a set $B \subseteq \mathbb{R}^n$
$\|B\|$	Cardinal number of a set $B \subseteq \mathbb{R}^n$

Chapter 1

Introduction

The mathematical field of facility location has gained interest of many researchers who have focused on formulations, geometrical properties and algorithms in a variety of discrete and continuous settings. The goal of facility location problems is to locate a set of new facilities (resources) such that distances, costs or time for satisfying some set of demands (of the customers) with respect to some set of constraints is minimized.

The history of locational analysis can be traced back to the early 17th century, when Fermat proposed the quest to find a point in the plane such that the sum of its distances to three given points with weights equal to one is a minimum. Mathematicians such as Torricelli, Cavalieri, and others took up the challenge to work on this problem.
Later on, in 1909, Weber considered the problem how to locate a single warehouse such that the total distance between the warehouse and several customers is minimized [126]. In 1964, location science attracted researchers interest with a publication by Hakimi (1964) [44], who wanted to locate switching centers in a communications network and police stations in a highway system. Further information about the history of location modeling can be found in the studies by Drezner, Klamroth, Schöbel and Wesolowsky (2002) [31], Eiselt and Marianov (2011) [38] and Wesolowsky (1993) [127].

Meanwhile, the problem of locating desirable facilities such as schools, hospitals, fire stations or post offices has been extensively studied. A large overview on this question can be found amongst others in the books by Drezner (1995) [29], Drezner and Hamacher (2002) [30], Eiselt and Sandbloom (2004) [39], Hamacher (1995) [45], Love, Morris and Wesolowsky (1988) [78], Nickel (2005) [97] and Zanjirani Farahani and Hekmatfa (2009) [130].

In many location models the criteria for finding an optimal location of one or more new facilities have economical issues. The goal is to establish a desirable facility, for instance a new warehouse, service center, post-office, supermarket or fire station, such that travel time or travel costs to a certain amount of given customers are minimized. However, with people getting more and more concerned about their living environment and its impact on health and safety, the undesirable effects of certain types of facilities cannot be left aside. Examples for facilities that provide, to some extent, a disservice to individuals and environment nearby are factories, hazardous facilities, (nuclear) power plants, chemical factories, dump sites, military installations, prisons, airports or train stations, radio or wireless stations, alarm sirens and so on. Although such facilities may obviously be necessary for certain aspects of social life, they may adversely affect the quality of life of people or animals in the surrounding area caused by noise, traffic, stench, pollution or even risk and serious danger. At the same time one cannot afford to select certain sites too far away from the population areas.

In the literature those facilities are called (semi)-obnoxious; (semi)-desirable; push–pull (to use the expressive terminology introduced by Eiselt and Laporte in [37]); or NIMBY (not in my backyard) facilities. Although the history of location theory goes back to the 17th century, the first attempts considering also undesirable facilities in location modeling appeared in the 1970's by Goldman and Dearing (1975) in [43], by Church and Garfinkel (1978) in [27] and by Dasarathy and White (1980) in [28].

A real world example referring to undesirable facilities is cited by Erkut and Neuman (1989) in [42]: "Although cost of power transmission and loss of power during transmission are important issues, Hansen, Peeters and Thisse (1981) [53] point out that the French government chose to locate half of the country's nuclear power plants along the Atlantic coastline, and the Belgian and German borders, at great distances from the large population centers."

The conflicting objectives of locating a facility close to certain demand points and far from others lead to consider models, which combine attracting and repulsive forces. A lot of models have been studied in a great variety of aspects like solution space, distance measures or type of objective. The general goal is to minimize the distances to attracting facilities and to maximize the distances to repulsive ones.

In this thesis we consider a **non-convex single-facility** location problem in the **Euclidean space** ($\mathbb{R}^n$) with a **single push–pull** objective function.

We apply the duality theory by Toland (1978) [121] and Singer (1979) [112] for d.c. optimization problems in order to obtain geometrical properties and duality results for location problems with attraction and repulsion points. Using the special structure of the location problem, we further give statements concerning the existence and attainment of finite optimal solutions as well as a duality based algorithm for determining exact solutions of location problems with obnoxious facilities.

To our best knowledge the duality theory by Toland and Singer seems to be never applied to this kind of non-convex optimization problems in earlier published works.

This study is organized as follows: After proposing a brief overview of literature on location problems with obnoxious facilities in Chapter 2 we introduce the considered location model (P) in detail in Chapter 3. Subsequently, in Chapter 4 we provide basic definitions and properties concerning distance functions (in particular gauge distances (Minkowski, 1911)), as well as foundations from convex analysis [59, 60, 107] and d.c. optimization [2, 55, 57, 61, 80, 81, 82, 83] including the duality theory by Toland (1978) [121] and Singer (1979) [112]. Afterwards, those preliminaries are applied in Chapter 5 in order to formulate a dual problem (D) to the primal problem (P), and to give a necessary and sufficient condition for the existence and attainment of finite optimal solutions as well as geometrical properties and duality statements. As known for the classical location problem [36], we introduce the terms *elementary convex sets and grids with respect to attraction and to repulsion* for both, the primal and the dual problem (P) and (D). In Chapter 6, we apply results from the theory of geometric duality [56] in order to show how primal and dual elementary convex sets are related to each other. Although, we are considering a scalar optimization problem, we show in Chapter 7 that methods from the field of linear vector optimization [5, 76, 51] can be applied in order to determine the primal and dual grid points with respect to attraction and to repulsion. In Chapter 8 we extend our research such that convex polyhedral constraints are considered. Based on the developed duality assertions and the relationship between primal and dual elements we present a primal and a dual algorithm in Chapter 9, which determine exact solutions of the dual pair of optimization problems (P) and (D) by leading back the non-convex problems to a finite number of convex problems. Remarks on the implementation of these algorithms in `MATLAB` are stated in Chapter 10. Finally, in Chapter 11, a conclusion with remarks on possible future researches completes this thesis.

Chapter 2

A Brief Literature Overview and Classification

Facility location models can differ in their objective function, the distance functions, the number and size of facilities which are to be located, and several other decision indices. A great amount of different kinds of location problems has been discussed in the literature during the last decades, which makes it worth to classify them according to their main properties. Such a classification scheme was introduced by Hamacher (1995) in [45] and by Hamacher and Nickel (1998) in [49] (see also Hamacher, Nickel and Schneider (1996) [50]). The main properties can be specified by

pos 1/pos 2/pos 3/pos 4/pos 5

where

pos 1 declares the number of facilities to be located (single or multi facility location problem);

pos 2 describes the solution space, e.g. a continuous space, a discrete space or a network;

pos 3 leaves room for special assumptions and constraints like equal weights, forbidden regions or barriers;

pos 4 defines the type of distance function, for instance Manhattan distances, Euclidean distances, mixed distances, l_p–distances, gauge distances or barrier distances;

pos 5 announces the type of objective function, for instance single- or multi-objective, median or center objective.

Some examples from the literature are given below to indicate the ability of the 5-position classification scheme to describe various kinds of location models:

- The Fermat–Weber problem in the plane with forbidden regions as discussed by Nickel (1995) in [96], Hamacher and Nickel (1995) in [48] can be classified by $1/\mathrm{P}/\mathcal{R}/l_p/\Sigma$.

- The classification schemes $1/\mathbb{R}^2/\mathcal{B}/d_\mathcal{B}/\Sigma$ and $1/\mathbb{R}^2/\mathcal{B}/d_\mathcal{B}/\max$ belong to the Fermat–Weber problem and the center problem with barriers as discussed by Klamroth (2002) in [70].
- The Fermat–Weber problem in the continuous space $\mathbb{R}^n$ with some negative weights, as it is discussed in this study, is classified by $1/\mathbb{R}^n/w_m <> 0/\cdot/\Sigma$.

A collection of efficient algorithms for solving different classes of facility location problems is LoLA (Library of Location Algorithms, see Hamacher, Klamroth, Nickel and Schöbel (1996) [46]). This Software package is based on the classification scheme by Hamacher and Nickel [49].

An overview on classification schemes and facility location software referring to different settings is given in Tafazzoli and Mozafari (2009) [116].

Classification schemes that refer especially to the problem of locating an undesirable facility contain the following aspects, see Erkut and Neuman (1989) [42], Eiselt and Laporte (1995) [37]:

- the number of facilities to be located;
- the solution space;
- the feasible region (discrete, continuous - convex polygon, non-convex polygon, etc.);
- the number of existing facilities (fixed or variable);
- the distance measure;
- the existence of distance constraints (upper bounds to keep undesirable facilities in reach or lower bounds to ensure a minimal distance to the customers);
- the weights (different or equal weights);
- the location of customers (distributed uniformly or located at specific points; in case of multi-facility location problems customers may be assigned to facilities or may be free to choose);
- interactions (only distances between customers and facilities, only distances between facilities or both kinds of distances are to be considered);
- the type of objective (single- or multi–objective; push-, pull- or push–pull objective; median or center objective).

Modeling the optimal location in such situations is surveyed in general by Carrizosa and Plastria in [20] and in a discrete setting by Krarup, Pisinger and Plastria in [71]. Erkut and Neuman (1989) provided in [42] an elaborate classification and a survey with respect to undesirable facilities. Further surveys on location problems with obnoxious facilities are given by Plastria (1996) in [104], by Cappanera (1999) in [16], by Eiselt and Laporte (1995) in [37] and by Moon and Chaudhry (1984) in [94].

In the following we want to give a brief overview on references with respect to some main classification aspects: the **number of facilities to be located**, the **solution space** and the **objective**. We do not make a claim to be complete, instead we focus mainly on frequently cited references.

Concerning the **number of facilities to be located** we distinguish between single- and multi-facility location problems. References referring to **single-facility location problems** are Berman, Drezner and Wesolowsky (1996) [7], Buchanan and Wesolowsky (1993) [15], Carrizosa and Plastria (1998) [19], Church and Garfinkel (1978) [27], Dasarathy and White (1980) [28], Drezner and Wesolowsky (1980, 1983, 1991) [32, 33, 35], Hansen, Peeters, Richard and Thisse (1985) [52], Hansen, Peeters and Thisse (1981) [53], Kaiser and Morin (1993) [64], Karkazis and Karagiorgis (1987) [67], Labbé (1990) [74], Mehrez, Sinunany-Stern and Stulman (1985, 1986) [84, 85], Melachrinoudis (1985, 1988) [86, 87], Melachrinoudis and Cullinane (1985, 1986) [89, 90, 91], Minieka (1983) [93], Nickel and Dudenhöfer (1997) [98], Plastria (1992) [103], Plastria, Gordillo and Carrizosa (2013) [106], Romero-Morales, Carrizosa and Conde (1997) [109] and Ting (1984) [119].

Multi-facility location problems are discussed by Chandrasekaran and Daughety (1981) in [22], by Chaudhry, McCormick and Moon (1986) in [23], by Drezner and Wesolowsky (1985) in [34], by Erkut (1990) in [40], by Erkut, Baptie and v. Hohenbalken (1990) in [41], by Hansen, Peeters and Thisse (1981) in [53], by Kalcsics (2011) in [65], by Karkazis and Papadimitriou (1992) in [68], by Katz, Kedem and Segal (2002) in [69], by Kuby (1987) in [73], by Moon and Chaudhry (1984) in [94], by Shier (1977) in [111], by Suzuki and Drezner (2013) in [115], by Tamir (1991) in [117] and by Ting (1988) in [120].

The continuous space $\mathbb{R}^n$ (frequently the plane $\mathbb{R}^2$) and networks are considered as **solution spaces**. References with regard to the **continuous space** are Buchanan and Wesolowsky (1993) [15], Dasarathy and White (1980) [28], Drezner and Wesolowsky (1983, 1985, 1991, 1980) [33, 34, 35, 32], Hansen, Peeters, Richard and Thisse (1985) [52], Hansen, Peeters, Thisse (1981) [53], Jourani, Michelot and Ndiaye (2009) [63], Kaiser and Morin (1993) [64], Karkazis and Karagiorgis (1987) [67], Krebs and Nickel (2010) [72], Mehrez, Sinunany-Stern and Stulman (1986) [85], Melachrinoudis (1985, 1988) [86, 87], Melachrinoudis and Cullinane (1985, 1986, 1986) [89, 90, 91], Nickel and Dudenhöfer (1997) [98], Plastria (1992) [103], Plastria, Gordillo and Carrizosa (2013) [106] and Romero-Morales, Carrizosa and Conde (1997) [109].

Networks are considered by Berman and Drezner (2000) in [6], by Berman, Drezner and Wesolowsky (1996) in [7], by Berman and Wang (2006, 2008) in [8, 9], by Carrizosa and Conde (2002) in [17], by Carrizosa and Plastria (1998) in [19], by Chandrasekaran and Daughety (1981) in [22], by Chaudhry, McCormick and Moon (1986) in [23], by Church and Garfinkel (1978) in [27], by Erkut (1990) in [40], by Hamacher et al. (2002) in [47], by Kuby (1987) in [73], by Labbé (1990) in [74], by Minieka (1983) in [93], by Moon and Chaudhry (1984) in [94], by Shier (1977) in [111], by Tamir (1991) in [117] and by Ting (1984, 1988) in [119] and [120].

There are various **types of objective functions**. On the one hand we may distinguish the number of objectives. The authors dealt with **single objectives** in the papers by Brimberg and

Juel (1998) [14], by Chen, Hansen, Jaumard and Tuy (1992) [25], by Drezner and Wesolowsky (1991) [35], by Hansen, Peeters and Thisse (1981) [53], by Maranas and Floudas (1994) [79], by Melachrinoudis and Culllinane (1985) [89], by Melachrinoudis and Xanthopulus (2003) [92], by Muños-Pérez and Saameno-Rodríguez (1999) [95], by Nickel and Dudenhöfer (1997) [98], by Plastria and Carrizosa (1999) [105], by Plastria (1991) [102], by Rodríguez, García, Munoz-Pérez and Casermeiro (2006) [108], by Romero-Morales et al. (1997) [109] and by Tamir (2006) [118]; whereas models with **vector-valued objective functions** are considered by Alzorba, Günther and Popovici (2013) in [3], by Blanquero and Carrizosa (2002) in [11], by Brimberg and Juel (1998) in [13], by Carrizosa, Conde and Romero-Morales (1997) in [18], by Carrizosa and Plastria (2000) in [21], by Hamacher, Labbé, Nickel and Skriver (2002) in [47], by Hansen and Thisse (1981) in [54], by Jourani, Michelot, and Ndiaye (2009) in [63], by Karasakal and Nadirler (2008) in [66], by Melachrinoudis (1999) in [88], by Ohsawa (2000) in [99], by Ohsawa, Plastria and Tamura (2006) in [100], by Ohsawa and Tamura (2003) in [101], by Skriver and Andersen (2003) in [114], by Yapicioglu, Dozier and Smith (2004) in [128] and by Yapicioglu, Smith and Dozier (2007) in [129].

On the other hand we may distinguish between push-, pull- and push–pull objectives. Delivering goods or offer services (medical, social, emergency etc.) to customers, make it natural to "pull" a facility as close as possible towards the customer, as known from the classical problem of locating a desirable facility. In case of semi-desirable facilities lower bounds may be used to ensure a certain minimal distance in order to avoid annoying effects of the facility. Hence, the goal of a **pull objective** is to minimize the distances between the new facility and the customers such that the facility is located "as close as possible, but not too close". Vice versa, it seems to be natural to "push" away an undesirable facility as far as possible. In order to avoid pushing the facility towards infinity the facility must be located inside an allowable set, or an upper bound is to be considered as a constraint, e.g. [33]. Hence, the goal of a **push objective** is to maximize the distances between the new facility and the customers such that the facility is located "far away, but within reach".

Drezner and Wesolowsky (1985) consider both in [34], the location problem with pull objective and lower bounds as well as the push objective with upper bounds.

References with push objective in which the authors deal with the maximization of the weighted sum of distances (**maxisum**) are Buchanan and Wesolowsky (1993) [15], Hansen, Peeters, Richard and Thisse (1985) [52], Hansen, Peeters, Thisse (1981) [53], Kaiser and Morin (1993) [64], Plastria (1992) [103], Romero-Morales, Carrizosa and Conde (1997) [109]; whereas the maximization of the shortest distance (**maximin**) is considered in Dasarathy and White (1980) [28], Drezner and Wesolowsky (1980, 1983) [32, 33], Karkazis and Karagiorgis (1987) [67], Mehrez, Sinunany-Stern and Stulman (1986) [85], Melachrinoudis (1985, 1988) [86, 87] and Melachrinoudis and Cullinane (1985, 1986) [89, 90, 91].

Locating facilities, which have some desirable and some undesirable features as well, means to find a compromise solution for instance by aggregation into a single **push–pull objective**: Customers may either try to attract (pull) desirable facilities closer to them, or repel (push) undesirable facilities away from them. Since the need to locate facilities far away from certain points can be quantified through the use of negative weights the objective function constitutes as a **d.c. function** (difference of two convex functions).

Maranas and Floudas (1994) [79] solve this kind of location problem by developing a branch and bound method using rectangular subdivision.

Romero-Morales, Carrizosa and Conde (1997) [109] propose a Big Square Small Square method with a new bounding scheme, which exploits the structure of the problem.

Tuy, Al-Khayyal and Zhou (1995) [124] apply a triangular branch and bound method, where branching follows a normal triangular subdivision scheme. By additionally considering repelling points their paper generalizes the non-convex location problem discussed in Idrissi, Loridan and Michelot (1988) [62], where the goal is to find a location, which maximizes the whole population attracted by the new facility. The further the new facility is established from a customer the less attractive it is (caused by travel time and travel costs). The authors again generalize their study in Al-Khayyal, Tuy, and Zhou (2002) [1].

Tuy (1996) [122] presents a general approach to location problems based on d.c. optimization methods. A branch and bound method is proposed where branching is performed by simplicial subdivision of $\mathbb{R}^n$ and bounds are computed by solving certain linear programs.

Drezner and Wesolowsky (1991) [35] determine exact solutions when distances are rectilinear or squared Euclidean. Further they present heuristic algorithms for the case of squared Euclidean distances. They also formulate conditions for the attainment of finite solutions in case of uniform distance functions.

Nickel and Dudenhöfer (1997) [98] present polynomial algorithms and structural properties based on combinatorial geometrical methods. They use computational geometry and discretization of continuous problems. Their work is heavily based on the structure of level sets.

Chen, Hansen, Jaumard and Tuy (1992) [25] solve the problem by converting the d.c. problem into a concave minimization problem and solve this one by outer approximation. A generalization of this work to multi-source Weber problems, conditional multi-source Weber problems and facilities location problems with limited distances is presented in Chen, Hansen, Jaumard and Tuy (1998) [26].

In this thesis we apply results from the field of d.c. optimization, but in contrast to earlier works we exploit the duality theory by Toland (1978) [121] and Singer (1979) [112] in order to develop duality statements, geometrical properties and an algorithm for determining **exact solutions** of the **non-convex single-facility** location problem with obnoxious facilities in the **Euclidean space** ($\mathbb{R}^n$) with a **single push–pull** objective function.

Chapter 3

Formulation of the Location Problem with Obnoxious Facilities

The goal, when locating a semi-desirable facility, is to minimize the weighted sum of distances to the attracting facilities and to maximize the weighted sum of distances to the repulsive ones. In this study, distances are measured by mixed gauges. Distances induced by gauges, are defined by the well known Minkowski functional[1] with respect to a special set $B \subseteq \mathbb{R}^n$ [36]:

Definition 3.1. (Minkowski 1911) Let B be a closed bounded convex set in $\mathbb{R}^n$ containing the origin in its interior. Then the *gauge* $\gamma_B : \mathbb{R}^n \to \mathbb{R}$ associated with B is a function defined by

$$\gamma_B(x) := \min \{\lambda \geq 0 |\, x \in \lambda B\}.$$

The gauge distance from a point $a \in \mathbb{R}^n$ to $x \in \mathbb{R}^n$ is defined as $\gamma_B(x-a)$. A distance measure function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, specified by a gauge, is given by

$$d(a,x) := \gamma_B(x-a).$$

Vice versa, the set B is called the *unit ball* associated with γ_B and is defined by

$$B := \{x \in \mathbb{R}^n |\, \gamma_B(x) \leq 1\}.$$

We introduce gauge distances more detailed in Section 4.2.

Note that the distance functions, assigned to the repulsion points, may depend on the kind of aversion. For example noises and stench do not need to pass streets. Instead, the application of the Euclidean distance, see (4.4), seems to be reasonable. Meanwhile the distance to other

[1] Note that the Minkowski functional can be defined more general for an arbitrary convex set on a topological space. Concerning the goal of solving location problems in $\mathbb{R}^n$ or especially in the plane ($n = 2$), where gauges are used to measure distances, it is reasonable to define a gauge γ_B on $\mathbb{R}^n$ with the additional demands on the set B as given in Definition 3.1.

obnoxious places may be measured by a distance function given by the course of the roads (e.g., the Manhattan distance, see (4.3)). Hence, we consider a median location problem with mixed distances.

In order to formulate the location problem with obnoxious facilities, we consider $\overline{M} \geq 1$ attraction points $\overline{a}^1, \ldots, \overline{a}^{\overline{M}} \in \mathbb{R}^n$ with weights $\overline{w}_1, \ldots, \overline{w}_{\overline{M}} > 0$ as well as $\underline{M} \geq 1$ repulsion points $\underline{a}^1, \ldots, \underline{a}^{\underline{M}} \in \mathbb{R}^n$, with weights $\underline{w}_1, \ldots, \underline{w}_{\underline{M}} > 0$. The gauge distances, assigned to $\overline{a}^1, \ldots, \overline{a}^{\overline{M}}$ and $\underline{a}^1, \ldots, \underline{a}^{\underline{M}}$, are defined by their associated unit balls $\overline{B}_1, \ldots, \overline{B}_{\overline{M}}$ and $\underline{B}_1, \ldots, \underline{B}_{\underline{M}}$.
In this thesis we focus on polyhedral gauges (see Section 4.2), although several results also hold for gauges in general.

Throughout the entire thesis we study the location problem with obnoxious facilities given by

$$\alpha := \inf_{x \in \mathbb{R}^n} \{g(x) - h(x)\}, \tag{P}$$

with functions $g, h : \mathbb{R}^n \to \mathbb{R}_+$ defined as

$$g(x) := \sum_{m=1}^{\overline{M}} \overline{\phi}_m(x), \qquad \overline{\phi}_m(x) := \overline{w}_m \gamma_{\overline{B}_m}(x - \overline{a}^m), \qquad (m = 1, \ldots, \overline{M}), \tag{3.1}$$

$$h(x) := \sum_{m=1}^{\underline{M}} \underline{\phi}_m(x), \qquad \underline{\phi}_m(x) := \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m), \qquad (m = 1, \ldots, \underline{M}). \tag{3.2}$$

Thus, $g(x)$ is the weighted sum of distances between the new location x and the attraction points $\overline{a}^1, \ldots, \overline{a}^{\overline{M}}$, and $h(x)$ is the weighted sum of distances between x and the repulsion points $\underline{a}^1, \ldots, \underline{a}^{\underline{M}}$. Since the distances to repulsive facilities shall be maximized, the function h obtains a negative sign in (P). One could also consider the repulsion points as facilities with negative weights instead of giving h a negative sign.

Note that the objective function $g - h : \mathbb{R}^n \to \mathbb{R}$ as a difference of two convex functions g and h is a d.c. function (difference of convex functions, see Definition 4.19). In this study we apply the duality theory by Toland [121] and Singer [112] in order to develop geometrical properties, conditions for the existence of a finite optimal solution, duality statements, a description of the relationship between primal and dual elements and a duality based algorithm for determining exact solutions of location problems of type (P) by leading back the non-convex problems to a finite number of convex problems.

Although, in general, the objective function $g - h$ is non-convex, it is possible to exploit the properties of the two convex functions g and h by applying the Toland-Singer-duality.

The notation introduced for formulating the location problem (P), based on the functions g and h as defined in (3.1) and (3.2), are used throughout the entire thesis.

Chapter 4

Preliminaries

In this chapter we provide elementary definitions and properties concerning convex sets and functions in Sections 4.1 and 4.2. We briefly recall the classical Fermat-Weber problem and introduce the concept of elementary convex sets w.r.t. attraction and w.r.t. repulsion in Section 4.3. Finally, in Section 4.4, a short introduction to d.c. optimization problems, including the duality theory by Toland (1978) and Singer (1979), is given.

Instead of providing an extensive overview on these fields, we focus on the main fundamentals, which play a role for this work at hand. Most of the definitions and results presented in Sections 4.1 and 4.2 can be found in the standard literature on convex analysis [59, 60, 107] and references therein. Classical results on gauge distances and locational analysis are presented amongst others in [10, 29, 30, 36, 78, 125, 126], and for frequently cited references with respect to d.c. optimization techniques and the duality theory by Toland and Singer the reader is referred to [2, 55, 57, 61, 80, 81, 82, 83, 112, 121, 123] and references therein.

4.1 Convex Sets and Cones

A set $C \subseteq \mathbb{R}^n$ is called *convex*, if for any pair of distinct points $x^1, x^2 \in C$ the closed line segment

$$\left\{\lambda x^1 + (1-\lambda)x^2 \middle|\, \lambda \in [0,1]\right\}$$

is contained in C. If for any pair of distinct points $x^1, x^2 \in C$ the entire line

$$\left\{\lambda x^1 + (1-\lambda)x^2 \,\middle|\, \lambda \in \mathbb{R}\right\}$$

through x^1 and x^2 is contained in C, then the set C is called *affine*.

Let γ_B be a gauge distance in $\mathbb{R}^n$ associated with a unit ball B, see Definition 3.1. Then the *interior* of a set $C \subseteq \mathbb{R}^n$ is given by

$$\operatorname{int} C := \{x \in \mathbb{R}^n |\; \exists \varepsilon > 0 : \; x + \varepsilon B \subseteq C\},$$

whereas the *relative interior* of a convex set $C \subseteq \mathbb{R}^n$ is the interior of C with respect to the *affine hull* aff C (the smallest affine set containing C), i.e.,

$$\mathrm{ri}\, C = \{x \in \mathrm{aff}\, C |\, \exists \varepsilon > 0 : \, (x + \varepsilon B) \cap (\mathrm{aff}\, C) \subseteq C\} \,.$$

As an example consider the square defined by

$$C := \left\{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 |\, x_1, x_3 \in [0,1],\, x_2 = 0\right\},$$

whose interior is empty, i.e., $\mathrm{int}\, C = \emptyset$, whereas its relative interior is given by

$$\mathrm{ri}\, C = \left\{x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 |\, x_1, x_3 \in (0,1),\, x_2 = 0\right\} \neq \emptyset.$$

A well known relationship between the interior and the relative interior of a convex set is given in the following remark.

Remark 4.1. The interior of a convex set coincides with its relative interior whenever the interior is non-empty.

We call a set $C \subseteq \mathbb{R}^n$ *closed* if $\mathrm{bd}\, C \subseteq C$, where $\mathrm{bd}\, C$ denotes the *boundary* of C and is given by $\mathrm{bd}\, C = \mathbb{R}^n \setminus (\mathrm{int}\, C \cup \mathrm{int}\, \mathbb{R}^n \setminus C)$.

Polyhedral Sets

Let $q \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$. Then the set $\mathcal{H} = \{x \in \mathbb{R}^n |\, \langle q, x\rangle \leq c\}$, where $\langle \cdot, \cdot \rangle$ denotes the standard *inner product* in $\mathbb{R}^n$, is called a closed *half-space* in $\mathbb{R}^n$. The intersection of a finite number of half-spaces is called a *convex polyhedral set* and can be written as

$$S = \{x \in \mathbb{R}^n |\, Ax \leq b\} \,,$$

with suitable $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. If a convex polyhedral set is bounded, then it is called a *convex polytope*.

The following definition plays a role for describing geometric properties of the location problem (P) and its Toland-Singer dual problem (D), which is formulated in Chapter 5.

Definition 4.2. Let $S \neq \emptyset$ be a convex polyhedral set in $\mathbb{R}^n$.

1. A subset $F \subseteq S$ is called an *exposed face* of S if there are $q \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ such that $S \subseteq \{x \in \mathbb{R}^n |\, \langle x, q\rangle \leq c\}$ and $F = \{x \in \mathbb{R}^n |\, \langle x, q\rangle = c\} \cap S$.
2. The exposed face F is called *proper* if $F \neq \emptyset$ and $F \neq S$. Then $\dim(F) < \dim(S)$.
3. A *facet* is an exposed face $\mathcal{F}$ of S with dimension $\dim(\mathcal{F}) = \dim(S) - 1$.
4. An *edge* is an exposed face of dimension one.
5. An exposed face of dimension zero is called an *extreme point*.

Note that $x \in S$ is an extreme point if and only if $S \setminus \{x\}$ is convex. This is the case if x is no relative interior point of any closed line segment in S.

Cones

A set $K \subseteq \mathbb{R}^n$ is called a *cone* if for all $x \in K$ and $\lambda \geq 0$ it holds $\lambda x \in K$. Let $C \subseteq \mathbb{R}^n$ be a convex set. Then the set

$$K := \{\lambda x | \, \lambda \geq 0, \, x \in C\}$$

is called the convex cone *generated* by C. If a polyhedral convex set $S \subseteq \mathbb{R}^n$ contains the origin, then the convex cone generated by S is polyhedral, too.

The *normal cone* to a convex set $B \subseteq \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined as

$$N_B(x) = \begin{cases} \{y \in \mathbb{R}^n | \, \forall z \in B : \, \langle y, z - x \rangle \leq 0\} & \text{if } x \in B, \\ \emptyset & \text{if } x \notin B. \end{cases} \tag{4.1}$$

The elements $y \in N_B(x)$ are said to be normal to the set B at the point x. Note that $N_B(x) = \{0\}$ for all $x \in \operatorname{int} B$ [36].

In Chapter 7 recession cones play a role for applying results from the field of linear vector optimization.

Definition 4.3. [107] The *recession cone* of a convex set $C (\neq \emptyset) \subseteq \mathbb{R}^n$ is defined by the set

$$0^+C := \{y \in \mathbb{R}^n | \, C + \mathbb{R}_+ y \subseteq C\} .$$

The elements of 0^+C are called receding directions or directions of recession of C.

Note that 0^+C is a convex cone containing the origin and it follows directly from the definition that

$$C + 0^+C \subseteq C. \tag{4.2}$$

Obviously, a closed set $C \subseteq \mathbb{R}^n$ is bounded if and only if $0^+C = \{0\}$ [107]. We give some standard examples for convex sets and their recession cones [107]:

$$C_1 := \left\{x \in \mathbb{R}^2 \middle| \, x_2 \geq x_1^2\right\}, \qquad 0^+C_1 = \{0\} \times \mathbb{R}_+,$$
$$C_2 := \left\{x \in \mathbb{R}^2 \middle| \, x_1^2 + x_2^2 \leq 1\right\}, \qquad 0^+C_2 = \left\{(0,0)^T\right\},$$
$$C_3 := \left\{x \in \mathbb{R}^2 \middle| \, x_1 > 0, \, x_2 > 0\right\} \cup \left\{(0,0)^T\right\}, \qquad 0^+C_3 = C_3,$$
$$C_4 := \left\{x \in \mathbb{R}^2 \middle| \, x_1 > 0, \, x_2 \geq \frac{1}{x_1}\right\}, \qquad 0^+C_4 = \mathbb{R}^2_+.$$

Proposition 4.4. [107] The recession cone of a polyhedral set $S = \{x \in \mathbb{R}^n |\, Ax \leq b\}$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, is determined as

$$0^+S = \{x \in \mathbb{R}^n |\, Ax \leq 0\}.$$

4.2 Convex Functions

As usual we define the *effective domain* and the *epigraph* of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned} \operatorname{dom} f &:= \{x \in \mathbb{R}^n |\, f(x) < +\infty\}, \\ \operatorname{epi} f &:= \{(x, r) \in \mathbb{R}^n \times \mathbb{R} |\, r \geq f(x)\}. \end{aligned}$$

Note that $\operatorname{dom} f$ is the projection of $\operatorname{epi} f$ on $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *proper*, if $\operatorname{dom} f \neq \emptyset$. If the epigraph of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is polyhedral then f is called a *polyhedral function* [12]. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *closed* if its epigraph $\operatorname{epi} f$ is closed. Further, we call $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ a *convex function* on $\mathbb{R}^n$ if $\operatorname{epi} f$ is a convex subset of $\mathbb{R}^{n+1}$. It holds that f is convex if and only if for all $x^1, x^2 \in \mathbb{R}^n$ and $\lambda \in [0, 1]$:

$$f(\lambda x^1 + (1 - \lambda)x^2) \leq \lambda f(x^1) + (1 - \lambda)f(x^2).$$

A frequently used example for convex functions is the *indicator function* $\mathbb{I}_B : \mathbb{R}^n \to \{0, +\infty\}$ with respect to a set $B \subseteq \mathbb{R}^n$ given by

$$\mathbb{I}_B(x) := \begin{cases} 0 & \text{if } x \in B, \\ +\infty & \text{otherwise.} \end{cases}$$

The "cross-section" of its epigraph is B. Hence, the indicator function $\mathbb{I}_B$ is convex, if and only if the set B is convex.

Another well known example for convex functions is the Minkowski functional (see Footnote 1 on Page 11).

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *positively homogeneous* if for every $x \in \mathbb{R}^n$ and $\lambda \geq 0$ one has

$$f(\lambda x) = \lambda f(x).$$

Hence, the positive homogeneity of f is equivalent to $\operatorname{epi} f$ being a cone in $\mathbb{R}^{n+1}$. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *subadditive*, if for all $x, y \in \mathbb{R}^n$ it holds

$$f(x + y) \leq f(x) + f(y).$$

If a function is positively homogeneous and subadditive, then the function is called *sublinear*.

Proposition 4.5. [107, Theorem 4.7] A positively homogeneous function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if f is sublinear.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *affine* if both, f and $-f$, are convex.

Remark 4.6. [12] A polyhedral function f is closed and convex and can be decomposed as the pointwise maximum of a finite set of affine functions $f_1, \dots, f_I$ and the indicator function of a non-empty polyhedral set $P \subseteq \mathbb{R}^n$, such that

$$f = \max_{i=1,\dots,I} f_i + \mathbb{I}_P.$$

The *support function* $\sigma_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ of an arbitrary set $A(\neq \emptyset) \subseteq \mathbb{R}^n$ is defined by

$$\sigma_A(y) = \sup_{x \in A} \langle y, x \rangle .$$

Note that the support function is closed and sublinear [59, Prop. 2.1.2]. Moreover, the support function σ_A of a non-empty set $A \subseteq \mathbb{R}^n$ is finite everywhere if and only if A is bounded [59].

Remark 4.7. Let $B(\neq \emptyset) \subseteq \mathbb{R}^n$ be a closed convex set. Then we have

$$N_B(x) = \{y \in \mathbb{R}^n | \, \sigma_B(y) = \langle x, y \rangle\} .$$

Proof. For each $x \in B$ we obtain by (4.1)

$$\begin{aligned} N_B(x) &= \{y \in \mathbb{R}^n | \, \forall z \in B : \, \langle y, z - x \rangle \leq 0\} \\ &= \{y \in \mathbb{R}^n | \, \forall z \in B : \, \langle y, z \rangle \leq \langle y, x \rangle\} \\ &= \{y \in \mathbb{R}^n | \, \sigma_B(y) = \langle y, x \rangle\} . \end{aligned}$$

□

Gauge Distances

In the following we provide some general properties concerning gauges and their corresponding dual gauge distances. An overview on the main properties of gauge distances and their dual functions is given in [110].

Due to the convexity assumption on the set B, some well known properties of the Minkowski functional (see Footnote 1 on Page 11) are non-negativity, subadditivity (or triangle inequality), positive homogeneity, and consequently convexity, see Proposition 4.5. Hence, a gauge distance measure $d(a, x) := \gamma_B(x - a)$, as introduced in Definition 3.1, is also convex and satisfies for all $x, y, z \in \mathbb{R}^n$ and $r \geq 0$:

1. $d(x,y) = \gamma_B(y-x) \geq 0,$ (non-negativity)
2. $d(x,y) \leq d(x,z) + d(z,y),$ (triangle inequality)
3. $d(rx,ry) = rd(x,y).$ (positive homogeneity)

We also find the following properties [59]:

1. *Definiteness:* For all $x \neq 0$ it holds $\gamma_B(x) > 0$ since B is assumed to be closed and bounded by Definition 3.1.

2. *Finiteness*: For all $x \in \mathbb{R}^n$ there exists $\lambda \geq 0$ such that $x \in \lambda B$, since the origin is contained in the interior of B, as assumed in Definition 3.1.

If the set B is strictly convex, i.e.,

$$\left\{\lambda x^1 + (1-\lambda)x^2 \middle|\, \lambda \in (0,1)\right\} \subseteq \operatorname{int} B$$

for any pair of distinct points $x^1, x^2 \in B$, then the associated gauge γ_B is also strictly convex, i.e., $\operatorname{epi} \gamma_B$ is a strictly convex subset of $\mathbb{R}^{n+1}$, and γ_B is said to be a *round gauge*. If the set B is a convex polytope, then the associated gauge γ_B is called a *polyhedral gauge* [36]. In case that the unit ball $B \subseteq \mathbb{R}^n$ is *symmetric* with respect to the origin, i.e., $x \in B$ if and only if $-x \in B$ for all $x \in \mathbb{R}^n$, the gauge γ_B defines a *norm* in $\mathbb{R}^n$ and we have $d(a,x) = d(x,a)$ or, equivalently, $\gamma_B(x-a) = \gamma_B(a-x)$ for all $x, a \in \mathbb{R}^n$ [107]. The most frequently applied distances are the following norms:

- Manhattan or rectilinear distances with polyhedral unit ball

$$B_{\text{Manhattan}} := \left\{ x \in \mathbb{R}^n \middle| \sum_{i=1}^{n} |x_i| \leq 1 \right\}, \tag{4.3}$$

- Euclidean distances with strictly convex unit ball

$$B_{\text{Euclidean}} := \left\{ x \in \mathbb{R}^n \middle| \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \leq 1 \right\}, \tag{4.4}$$

- Tchebychev distances with polyhedral unit ball

$$B_{\text{Tchebychev}} := \left\{ x \in \mathbb{R}^n \middle| \max\left\{|x_1|, \ldots, |x_n|\right\} \leq 1 \right\}. \tag{4.5}$$

As usual, we define the *dual gauge* of γ_B as the gauge associated with the *polar set*

$$B^* := \left\{ y \in \mathbb{R}^n \middle| \forall x \in B : \langle x, y \rangle \leq 1 \right\}, \tag{4.6}$$

i.e., B^* is the *dual unit ball* of B. The dual unit ball B^* is also a closed bounded convex set in $\mathbb{R}^n$, which contains the origin in its interior [107]. It follows directly from Definition 3.1 and

(4.6) that

$$\gamma_{B^*}(y) = \max_{x \in B} \langle x, y \rangle , \tag{4.7}$$

i.e., a gauge γ_{B^*} associated with a dual unit ball B^* coincides with the support function σ_B of the unit ball B, cf. [123, Proposition 1.23].

Gauges and their dual functions are strongly related to each other, as the following properties demonstrate:

1. The polar set $B^{**} := (B^*)^*$ of the polar set B^* is the set B itself [107], i.e., $\gamma_B = \gamma_{B^{**}}$. Hence, by (4.7), a gauge can also be written as

   $$\gamma_B(x) = \max_{y \in B^*} \langle x, y \rangle , \tag{4.8}$$

   and in case of a weighted polyhedral gauge distance γ_B with weight $w > 0$ we have

   $$w\gamma_B(x) = w \max_{y \in B^*} \langle x, y \rangle = \max_{y \in B^*} \langle x, wy \rangle = \max_{y \in wB^*} \langle x, y \rangle = \max_{y \in \mathrm{ext}(wB^*)} \langle x, y \rangle . \tag{4.9}$$

2. If B is polyhedral then its polar B^* is polyhedral, too (and hence if γ_B is a polyhedral gauge then so is the dual gauge γ_{B^*}). In $\mathbb{R}^2$, both polyhedra have the same number of extreme points; in general this property does not hold in $\mathbb{R}^n$ [10, 125]. For instance consider the l_1-norm in $\mathbb{R}^3$ and its dual l_∞-norm.

3. If B is symmetric with respect to the origin then its polar B^* is symmetric, too (and hence if γ_B is a norm then so is the dual gauge γ_{B^*}) [107].

In this thesis we focus on **polyhedral gauges**, although several results also hold for gauges in general.

Subdifferentials

Since polyhedral gauges are not differentiable we need a more general definition of the commonly used derivatives. This leads to the definition of subdifferentials:

Definition 4.8. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function, $x \in \mathrm{dom}\, f$. A vector $y \in \mathbb{R}^n$ is called a *subgradient* of f at x if for each $z \in \mathbb{R}^n$ the inequality

$$f(z) - f(x) \geq \langle y, z - x \rangle$$

is satisfied. The set of all subgradients of f at x is called the *subdifferential* of f at x and is denoted by $\partial f(x)$.

Note that in the case that $\partial f(x)$ is a singleton, the derivative of f exists at x and coincides with $\partial f(x)$.

Theorem 4.9. (Sum Rule for Subdifferentials) [107]
Let $f_1, \dots, f_M : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex functions on $\mathbb{R}^n$. Then, for every $x \in \mathbb{R}^n$ the inclusion

$$\partial \left(\sum_{m=1}^{M} f_m(x) \right) \supseteq \sum_{m=1}^{M} \partial f_m(x)$$

holds. If there exists an element $\widehat{x} \in \cap_{m=1}^{M} \operatorname{dom} f_m$, where every function f_m, except at most one, is continuous, then the above inclusion is in fact an equality for every $x \in \mathbb{R}^n$.

Remark 4.10. (Subdifferentiability of polyhedral functions) [12, Proposition 5.1.1]
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a polyhedral function. Then $\partial f(x) \neq \emptyset$ whenever $x \in \operatorname{dom} f$.

Throughout this work, especially in Chapters 5 and 8, we have to determine several subdifferentials of convex functions. Therefore we briefly provide some basic but important calculus properties below.

Calculus Rules for Subdifferentials [60]

Let $u, v : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper functions, $t \in \mathbb{R}$, $a \in \mathbb{R}^n$. Then we have for all $x \in \mathbb{R}^n$:

(A) Let $v(x) := u(x) + t$, then

$$\partial v(x) = \{y \in \mathbb{R}^n | \forall z \in \mathbb{R}^n : (u(z) + t) - (u(x) + t) \geq \langle y, z - x \rangle\} = \partial u(x).$$

(B) Let $v(x) := tu(x)$, $t > 0$, then

$$\begin{aligned}
\partial v(x) &= \{y \in \mathbb{R}^n | \forall z \in \mathbb{R}^n : tu(z) - tu(x) \geq \langle y, z - x \rangle\} \\
&= \left\{ y \in \mathbb{R}^n \middle| \forall z \in \mathbb{R}^n : u(z) - u(x) \geq \left\langle \frac{y}{t}, z - x \right\rangle \right\} \\
&= \{ty \in \mathbb{R}^n | \forall z \in \mathbb{R}^n : u(z) - u(x) \geq \langle y, z - x \rangle\} = t \cdot \partial u(x).
\end{aligned}$$

(C) Let $v(x) := u(tx)$, $t \neq 0$, then

$$\begin{aligned}
\partial v(x) &= \{y \in \mathbb{R}^n | \forall z \in \mathbb{R}^n : u(tz) - u(tx) \geq \langle y, z - x \rangle\} \\
&= \left\{ y \in \mathbb{R}^n \middle| \forall z \in \mathbb{R}^n : u(tz) - u(tx) \geq \left\langle \frac{y}{t}, tz - tx \right\rangle \right\} \\
&= \{ty \in \mathbb{R}^n | \forall z \in \mathbb{R}^n : u(tz) - u(tx) \geq \langle y, tz - tx \rangle\} = t \cdot \partial u(tx).
\end{aligned}$$

(D) Let $v(x) := u(x - a)$, then $\partial v(x + a) = \partial u(x)$.

(E) Let $v(x) := \langle x, a \rangle$, then

$$\begin{aligned}\partial v(x) &= \{y \in \mathbb{R}^n | \, \forall z \in \mathbb{R}^n : \, \langle z, a \rangle - \langle x, a \rangle \geq \langle y, z - x \rangle\} \\ &= \{y \in \mathbb{R}^n | \, \forall z \in \mathbb{R}^n : \, 0 \geq \langle y - a, z - x \rangle\} = \{a\} \, .\end{aligned}$$

(F) Let $v(x) := u(x) + \langle a, x \rangle$, then by Theorem 4.9 and by (E) we have

$$\partial v(x) = \partial u(x) + \{a\} \, .$$

(G) Let $v_1 \leq v_2, \; v_1(x) = v_2(x)$, then

$$\partial v_1(x) \subseteq \partial v_2(x),$$

since for every $y \in \partial v_1(x)$ and $z \in \mathbb{R}^n$ we have

$$v_2(z) - v_2(x) \geq v_1(z) - v_2(x) = v_1(z) - v_1(x) \geq \langle y, z - x \rangle \, .$$

(H) The subdifferential of the support function σ_B of a set $B \subseteq \mathbb{R}^n$ is

$$\partial \sigma_B(x) = \{y \in B | \, \sigma_B(x) = \langle x, y \rangle\} \, .$$

(I) The subdifferential of a gauge γ_B associated with its unit ball $B \subseteq \mathbb{R}^n$ is given by

$$\partial \gamma_B(x) = \{y \in B^* | \, \gamma_B(x) = \langle x, y \rangle\} \, ,$$

where B^* denotes the dual unit ball of B.

(J) The subdifferential of the indicator function $\mathbb{I}_B$ with respect to a convex set $B(\neq \emptyset) \subseteq \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ coincides with the normal cone $N_B(x)$ to B at x, i.e.,

$$\partial \mathbb{I}_B(x) = N_B(x) = \begin{cases} \{y \in \mathbb{R}^n | \, \forall z \in B : \langle y, z - x \rangle \leq 0\} \, , & \text{if } x \in B, \\ \emptyset, & \text{if } x \notin B. \end{cases}$$

Proof. Let $x \in B$. Then

$$\begin{aligned}\partial \mathbb{I}_B(x) &= \{y \in \mathbb{R}^n | \, \forall z \in \mathbb{R}^n : \, \mathbb{I}_B(z) - \mathbb{I}_B(x) \geq \langle y, z - x \rangle\} \\ &= \{y \in \mathbb{R}^n | \, \forall z \in \mathbb{R}^n : \, \mathbb{I}_B(z) \geq \langle y, z - x \rangle\} \\ &= \{y \in \mathbb{R}^n | \, \forall z \in B : \, 0 \geq \langle y, z - x \rangle\} = N_B(x).\end{aligned}$$

Let otherwise $x \notin B$, i.e., $\mathbb{I}_B(x) = +\infty$. Since $B \neq \emptyset$, for all $y \in \mathbb{R}^n$ there exists $z \in \mathbb{R}^n$ such that $\mathbb{I}_B(z) - \mathbb{I}_B(x) < \langle y, z - x \rangle$. Hence, $\partial \mathbb{I}_B(x) = \emptyset$. □

Conjugate Functions and Infimal Convolution

The duality theory by Toland [121] and Singer [112], which we apply later in Chapters 5 and 8, is based on conjugacy. That is why we next introduce the concept of conjugate functions and some basic but important calculus properties.

Definition 4.11. Given an arbitrary function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, defined by

$$f^*(y) = \sup_{x \in \operatorname{dom} f} \{\langle y, x\rangle - f(x)\}$$

is called the *(Fenchel-) conjugate function* of f.

Note that f^* is a convex and closed function for any proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ [60]. In the case that f is a polyhedral convex function, the conjugate f^* is polyhedral, too [107]. The *biconjugate* of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the function $f^{**} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$f^{**}(x) := (f^*)^*(x) = \sup_{y \in \operatorname{dom} f^*} \{\langle y, x\rangle - f^*(y)\}\,.$$

An immediate consequence of Definition 4.11 is the so called *Fenchel-Young inequality*

$$f^*(y) + f(x) \geq \langle y, x\rangle\,, \tag{4.10}$$

that holds for all $(x, y) \in \operatorname{dom} f \times \mathbb{R}^n$. Obviously it is also true if $x \notin \operatorname{dom} f$.

Fenchel biconjugation [12]: Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. As a direct consequence of the Fenchel-Young inequality (4.10) it holds for all $x \in \mathbb{R}^n$ that

$$f(x) \geq \sup_{y \in \operatorname{dom} f^*} \{\langle x, y\rangle - f^*(y)\} = f^{**}(x). \tag{4.11}$$

Further, the following equivalence holds [12, Theorem 4.2.1]:

$$f^{**} = f \quad \Leftrightarrow \quad f \text{ is convex and closed.} \tag{4.12}$$

If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function, then $f^{**} = cl f$ [12]. Another important property is presented in the following proposition.

Proposition 4.12. [59] Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex closed function, $\partial f(x) \neq \emptyset$. Then

$$y \in \partial f(x) \quad \Leftrightarrow \quad f(x) + f^*(y) = \langle x, y\rangle \quad \Leftrightarrow \quad x \in \partial f^*(y).$$

Proof. By Definition 4.8 of subdifferentials and Definition 4.11 of conjugates it holds for a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ that

$$\begin{aligned}\partial f(x) &= \{y \in \mathbb{R}^n | \, \forall z \in \mathbb{R}^n : \; f(z) - f(x) \geq \langle y, z - x\rangle\} \\ &= \{y \in \mathbb{R}^n | \, \forall z \in \mathbb{R}^n : \; \langle y, x\rangle \geq \langle y, z\rangle - f(z) + f(x)\} \\ &= \left\{ y \in \mathbb{R}^n \middle| \, \langle y, x\rangle \geq \sup_{z \in \mathbb{R}^n} \{\langle y, z\rangle - f(z)\} + f(x) = f^*(y) + f(x) \right\}.\end{aligned}$$

With use of the Fenchel-Young inequality (4.10) we obtain

$$\partial f(x) = \{y \in \mathbb{R}^n | \, \langle y, x\rangle = f^*(y) + f(x)\}. \tag{4.13}$$

Analogously, the subdifferential of the conjugate f^* can be written as

$$\partial f^*(y) = \{x \in \mathbb{R}^n | \, \langle y, x\rangle = f^{**}(x) + f^*(y)\}.$$

Since we assumed the function f to be convex and closed, the assertion follows directly by equivalence (4.12). □

We provide a brief overview on calculus rules for conjugate functions, which are applied in Chapters 5 and 8.

Calculus Rules for Conjugate Functions [60]

Let $u, v : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper functions, $t \in \mathbb{R}$, $a \in \mathbb{R}^n$. Then it holds for all $x, y \in \mathbb{R}^n$:

(a) Let $v(x) := u(x) + t$, then

$$v^*(y) = \sup_{x \in \operatorname{dom} v} \{\langle y, x\rangle - v(x)\} = \sup_{x \in \operatorname{dom} u} \{\langle y, x\rangle - u(x)\} - t = u^*(y) - t.$$

(b) Let $v(x) := tu(x)$, $(t > 0)$, then

$$v^*(y) = \sup_{x \in \operatorname{dom} u} \{\langle x, y\rangle - tu(x)\} = t \sup_{x \in \operatorname{dom} u} \left\{ \left\langle x, \frac{y}{t} \right\rangle - u(x) \right\} = tu^* \left(\frac{y}{t} \right).$$

(c) Let $v(x) := u(tx)$, $(t \neq 0)$, then

$$v^*(y) = \sup_{tx \in \operatorname{dom} u} \{\langle x, y\rangle - u(tx)\} = \sup_{tx \in \operatorname{dom} u} \left\{ \left\langle tx, \frac{y}{t} \right\rangle - u(tx) \right\} = u^* \left(\frac{y}{t} \right).$$

(d) Let $v(x) := u(x - a)$, then

$$\begin{aligned}v^*(y) &= \sup_{x - a \in \operatorname{dom} u} \{\langle x, y\rangle - u(x - a)\} \\ &= \sup_{x - a \in \operatorname{dom} u} \{\langle x - a, y\rangle - u(x - a)\} + \langle a, y\rangle = u^*(y) + \langle a, y\rangle.\end{aligned}$$

(e) Let $v(x) := u(x) + \langle x, a\rangle$, then

$$v^*(y) = \sup_{x\in \operatorname{dom} u} \{\langle x, y\rangle - u(x) - \langle x, a\rangle\} = \sup_{x\in \operatorname{dom} u} \{\langle x, y - a\rangle - u(x)\} = u^*(y - a).$$

(f) Let $v(x) := \langle x, a\rangle$, then

$$v^*(y) = \sup_{x\in\mathbb{R}^n} \{\langle x, y\rangle - \langle x, a\rangle\} = \sup_{x\in\mathbb{R}^n} \langle x, y - a\rangle = \mathbb{I}_{\{a\}}(y).$$

(g) Let $u^1 \leq u^2$, then

$$u_1^*(y) = \sup_{x\in\mathbb{R}^n} \{\langle y, x\rangle - u_1(x)\} \geq \sup_{x\in\mathbb{R}^n} \{\langle y, x\rangle - u_2(x)\} = u_2^*(y).$$

(h) Let $\mathbb{I}_B$ be the indicator function of a non-empty set $B \subseteq \mathbb{R}^n$. Then

$$\mathbb{I}_B^*(y) = \sup_{x\in\mathbb{R}^n} \{\langle x, y\rangle - \mathbb{I}_B(x)\} = \sup_{x\in B}\{\langle x, y\rangle\} = \sigma_B(y)$$

is the support function of B.

(i) Let σ_B be the support function of a non-empty convex set $B \subseteq \mathbb{R}^n$. Then

$$\sigma_B^*(y) = \sup_{x\in\mathbb{R}^n} \{\langle x, y\rangle - \sigma_B(x)\} = \mathbb{I}_{\operatorname{cl} B}(y)$$

is the indicator function of $\operatorname{cl} B$.

Proof. The assertion follows by (h) and (4.12). □

(j) Let γ_B be a gauge associated with the unit ball B. Then $\gamma_B^* = \mathbb{I}_{B^*}$ where B^* is the corresponding dual unit ball.

Proof. The assertion follows by property (i) since $\gamma_B = \sigma_{B^*}$, see (4.8). □

Additionally to the calculus rules above the following Definition 4.13 of an infimal convolution and Theorem 4.15 can be used for determining the conjugate function of a sum of convex functions.

Definition 4.13. [60] Let $f_1, \dots, f_M : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex functions. Their *infimal convolution* is the function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f(x) := (f_1 \,\square\, \cdots \,\square\, f_M)(x) := \inf \left\{ \sum_{m=1}^{M} f_m(x^m) \,\middle|\, \sum_{m=1}^{M} x^m = x \right\}.$$

The infimal convolution is called *exact* at $x = \sum_{m=1}^{M} \widehat{x}^m$ when the infimum is attained at $(\widehat{x}^1, \dots, \widehat{x}^M)$, where $(\widehat{x}^1, \dots, \widehat{x}^M)$ is not necessarily unique with this property.

Within this thesis the following results on infimal convolutions are applied.

Remark 4.14. It is well known that $\operatorname{dom}(f_1 \,\square\, f_2) = \operatorname{dom} f_1 + \operatorname{dom} f_2$ [107] and $\operatorname{epi}(f_1 \,\square\, f_2) = \operatorname{epi} f_1 + \operatorname{epi} f_2$ [58]. Note further that the infimal convolution $f_1 \,\square\, \cdots \,\square\, f_M$ of convex functions $f_1, \ldots, f_M : \mathbb{R}^n \to \mathbb{R}$ is convex, too [12].

Theorem 4.15. [107, Theorem 16.4] Let $f_1, \ldots, f_M : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex functions and assume that $\bigcap_{m=1}^{M} \operatorname{ri}(\operatorname{dom} f_m) \neq \emptyset$. Then

$$(f_1 + \ldots + f_M)^* = f_1^* \,\square\, \ldots \,\square\, f_M^*$$

and, for every $y \in \operatorname{dom}(f_1 + \ldots + f_M)^*$, there exist $y^1, \ldots, y^M$ such that the infimal convolution $f_1^* \,\square\, \ldots \,\square\, f_M^*$ is exact at $y = y^1 + \ldots + y^M$.

Recall that the interior of a convex set coincides with its relative interior whenever the interior is non-empty (see Remark 4.1).

Theorem 4.16. [60, Proposition 3.4.2] Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and let $f := f_1 \,\square\, f_2$ and $y^1 \in \operatorname{dom} f_1$, $y^2 \in \operatorname{dom} f_2$, $y := y^1 + y^2 \in \operatorname{dom} f$. If $\partial f_1(y^1) \cap \partial f_2(y^2) \neq \emptyset$, then f is exact at $y = y^1 + y^2$. If f is exact at $y = y^1 + y^2$, then

$$\partial f_1(y^1) \cap \partial f_2(y^2) = \partial f(y).$$

4.3 Elementary Convex Sets

The general classical Fermat-Weber problem [126] is to minimize the weighted sum of distances to each element of a given set of existing facilities $\mathcal{A} = \{a^1, \ldots, a^M\} \subseteq \mathbb{R}^2$, i.e.,

$$\min_{x \in \mathbb{R}^2} \sum_{m=1}^{M} w_m \gamma_{B_m}(x - a^m), \tag{W}$$

with weights $w_1, \ldots, w_M > 0$ and gauge distances $\gamma_{B_1}, \ldots, \gamma_{B_M}$ defined by the associated unit balls $B_1, \ldots, B_M \subseteq \mathbb{R}^2$.

Consider a unit ball $B \subseteq \mathbb{R}^n$ and a weight $w > 0$. Then the normal cone to the weighted dual ball wB^* at $y \in \operatorname{bd} wB^*$ is the convex cone $N_{wB^*}(y) = \{\lambda x \mid x \in F(y),\ \lambda \geq 0\}$ generated by an exposed face of B, see Definition 4.2, which is defined by $F(y) := \{x \in B \mid \langle y, x \rangle = w\}$, see [36, Section 2].

For the classical Fermat-Weber problem (W) the definition of elementary convex sets was introduced in [36]. In Definition 4.17 we analogously introduce the concept of elementary convex sets w.r.t. attraction and w.r.t. repulsion for the location problem (P) with obnoxious facilities.

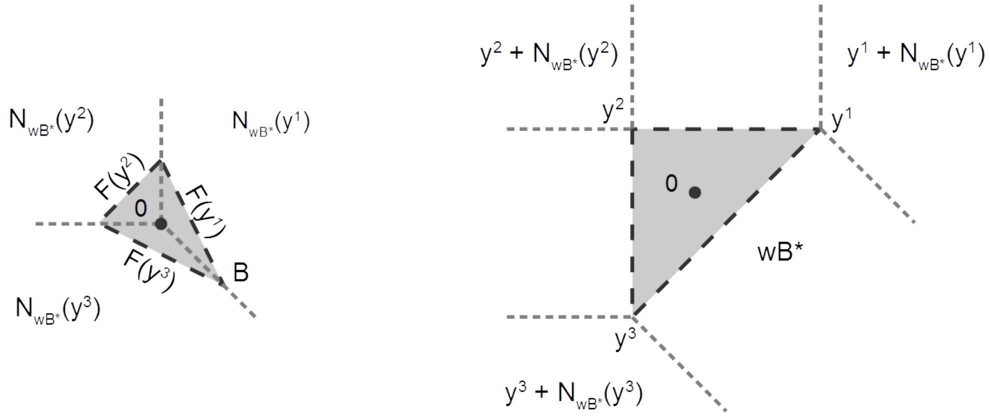


Figure 4.1: A unit ball B and its weighted dual ball wB^*, $(w > 0)$, with extreme points y^m and corresponding normal cones $N_{wB^*}(y^m)$ which coincide with the convex cones generated by the exposed faces $F(y^m)$ of B, for $m = 1, 2, 3$.

Definition 4.17. Consider the location problem (P).

We call a non-empty polyhedral set $\mathcal{C} \subseteq \mathbb{R}^n$ an *elementary convex set w.r.t. attraction*, if there exists a tuple $\overline{\pi} = \left\{\overline{y}^1, \dots, \overline{y}^{\overline{M}}\right\} \in \overline{w}_1 \overline{B}_1^* \times \dots \times \overline{w}_{\overline{M}} \overline{B}_{\overline{M}}^*$, such that

$$\mathcal{C} = \bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m)\right].$$

We call a non-empty polyhedral set $\mathcal{C} \subseteq \mathbb{R}^n$ an *elementary convex set w.r.t. repulsion*, if there exists a tuple $\underline{\pi} = \{\underline{y}^1, \dots, \underline{y}^{\underline{M}}\} \in \underline{w}_1 \underline{B}_1^* \times \dots \times \underline{w}_{\underline{M}} \underline{B}_{\underline{M}}^*$, such that

$$\mathcal{C} = \bigcap_{m=1}^{\underline{M}} \left[\underline{a}^m + N_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m)\right].$$

The half-lines $\overline{a}^m + \mathbb{R}_+ e$, for $e \in \operatorname{ext}(\overline{B}_m)$, $m = 1, \dots, \overline{M}$, and $\underline{a}^m + \mathbb{R}_+ e$, for $e \in \operatorname{ext}(\underline{B}_m)$, $m = 1, \dots, \underline{M}$, are called *fundamental directions*. Moreover, we call the sets

$$\overline{G} := \bigcup_{m=1}^{\overline{M}} \bigcup_{e \in \operatorname{ext}(\overline{B}_m)} \overline{a}^m + \mathbb{R}_+ e \qquad \text{and} \qquad \underline{G} := \bigcup_{m=1}^{\underline{M}} \bigcup_{e \in \operatorname{ext}(\underline{B}_m)} \underline{a}^m + \mathbb{R}_+ e$$

the *(construction) grids* w.r.t. attraction and w.r.t. repulsion, respectively. A point $x \in \mathbb{R}^n$ is called *grid point* (also *intersection point*) w.r.t. attraction (repulsion), if x is an extreme point of an elementary convex set $\mathcal{C}$ w.r.t. attraction (resp. repulsion). We denote by $\overline{\mathcal{I}}$ and $\underline{\mathcal{I}}$ the sets of all grid points w.r.t. attraction and w.r.t. repulsion, respectively.

The following example is considered in different settings within this thesis.

Example 4.18. We consider a location problem with two attracting facilities and one repulsive facility with assigned unit balls, given by their extreme points:

$$\begin{aligned}
\overline{a}^1 &= (2,2)^T, & \text{ext}(\overline{B}_1) &= \{(1,1),(1,-1),(-1,1),(-1,-1)\}\,,\\
\overline{a}^2 &= (9,4)^T, & \text{ext}(\overline{B}_2) &= \{(1,0),(0,1),(-1,0),(0,-1)\}\,,\\
\underline{a}^1 &= (7,1)^T, & \text{ext}(\underline{B}_1) &= \{(0,1),(-1,0),(1,-1)\}\,.
\end{aligned}$$

The corresponding construction grids $\overline{G}$ and $\underline{G}$ are illustrated in Figure 4.2.

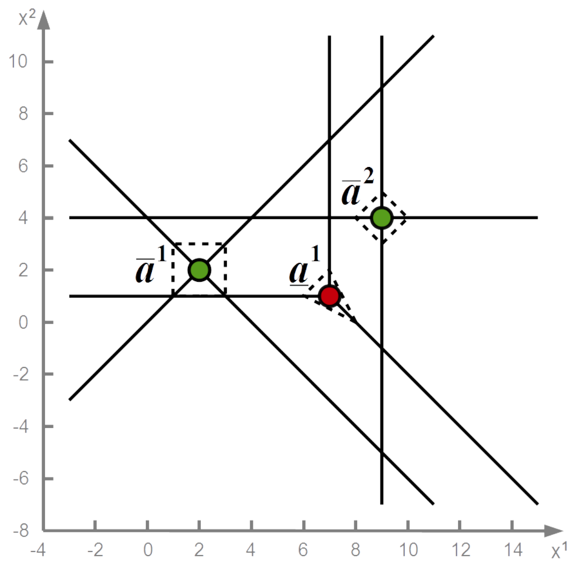


Figure 4.2: Construction grid $\overline{G}$ with respect to attraction points $\overline{a}^1 = (2,2)^T$ and $\overline{a}^2 = (9,4)^T$ and their assigned unit balls $\overline{B}_1, \overline{B}_2$ (green) as well as construction grid $\underline{G}$ with respect to repulsion point $\underline{a}^1 = (7,1)^T$ and its assigned unit ball $\underline{B}_1$ (red).

4.4 D.C. Programming Problems and Toland-Singer Duality

Throughout this entire work we follow the notational convention that

$$(+\infty) - (+\infty) = (+\infty). \tag{4.14}$$

This convention is reasonable as we will see in Section 5.1.

Definition 4.19. The difference $g - h$ of two convex functions $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called *d.c. function* and the minimization problem

$$\inf_{x \in \mathbb{R}^n} \{g(x) - h(x)\},$$

is called *d.c. programming problem.*

Note that, taking into account Definition 4.19, the location problem (P), as defined in Chapter 3, obviously is a d.c. programming problem.

Proposition 4.20. [113, Proposition 8.1] For any pair of functions $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we have

$$\inf_{x\in\mathbb{R}^n} \{g(x) - h^{**}(x)\} = \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*(y)\},$$
$$\inf_{x\in\mathbb{R}^n} \{g(x) - h(x)\} \leq \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*(y)\},$$

where g^* and h^* are the conjugates of g and h.

Proof. Due to the definition of conjugate functions, see Definition (4.11), we have

$$\begin{aligned}\inf_{x\in\mathbb{R}^n} \{g(x) - h^{**}(x)\} &= \inf_{x\in\mathbb{R}^n} \{g(x) - \sup_{y\in\mathbb{R}^n} \{\langle x, y\rangle - h^*(y)\}\} \\ &= \inf_{y\in\mathbb{R}^n} \inf_{x\in\mathbb{R}^n} \{g(x) - \langle x, y\rangle + h^*(y)\} \\ &= \inf_{y\in\mathbb{R}^n} \{h^*(y) - \sup_{x\in\mathbb{R}^n} \{\langle x, y\rangle - g(x)\}\} \\ &= \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*(y)\}.\end{aligned}$$

The second assertion follows since $h^{**} \leq h$, as we know from (4.11). □

From the duality theory by Toland [121] and Singer [112] we obtain the following theorem, which provides a dual d.c. optimization problem for a given primal d.c. program, see also [61, 123].

Theorem 4.21. (**Toland-Singer Duality**, 1978/1979)
[112, 121] Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and closed. Then

$$\alpha := \inf_{x\in\mathbb{R}^n} \{g(x) - h(x)\} = \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*(y)\} =: \beta.$$

Proof. The assertion follows by Proposition 4.20 since h is supposed to be convex and closed, i.e., we can substitute h by h^{**}, see (4.12):

$$\inf_{x\in\mathbb{R}^n} \{g(x) - h(x)\} = \inf_{x\in\mathbb{R}^n} \{g(x) - h^{**}(x)\} = \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*(y)\}.$$

□

Note that in Theorem 4.21 the minimization problem on the left hand side is considered as *primal problem* and the one on the right hand side as the corresponding *dual problem*.

Compared to the classical duality theory of convex analysis, see for instance [131], we do not have a minimization and a maximization problem as well, such that they provide upper and lower bounds for each other. Thus, there does not exist a weak duality result as it is known from the classical Lagrange theory. Nevertheless, this theory by Toland and Singer turns out to be useful in order to solve the given location problem (P). An advantage is that we do not need any special assumptions, such as constraint qualification, on the objective or the constraints of (P), except the convexity of g and h and closedness of h, in order to apply the duality results by Toland and Singer.

The following theorem provides necessary optimality conditions for a primal and a dual d.c. programming problem using the subdifferentials of g and h or h^* and g^*, respectively.

Theorem 4.22. [61, 123] (**Necessary Optimality Conditions**)
Let $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and closed functions. If $\widehat{x} \in \operatorname{dom} g \cap \operatorname{dom} h$ is a global minimizer of $g - h$ on $\mathbb{R}^n$, then $\partial h(\widehat{x}) \subseteq \partial g(\widehat{x})$.

Vice versa, if $\widehat{y} \in \operatorname{dom} h^* \cap \operatorname{dom} g^*$ is a global minimizer of $h^* - g^*$ on $\mathbb{R}^n$, then $\partial g^*(\widehat{y}) \subseteq \partial h^*(\widehat{y})$.

Proof. Let $\widehat{x} \in \operatorname{dom} g \cap \operatorname{dom} h$ be a global minimizer of $g - h$, i.e.,

$$g(\widehat{x}) - h(\widehat{x}) \leq g(x) - h(x) \tag{4.15}$$

for all $x \in \mathbb{R}^n$. Let $y \in \partial h(\widehat{x})$. Then we have by (4.15)

$$\langle y, x - \widehat{x} \rangle) \leq h(x) - h(\widehat{x}) \leq g(x) - g(\widehat{x})$$

for all $x \in \mathbb{R}^n$, i.e., $y \in \partial g(\widehat{x})$. The second inclusion follows analogously. □

An important duality based relationship concerning the sets of optimal solutions of dual pairs of d.c. programming problems is the following theorem:

Theorem 4.23. (**Sufficient Optimality Conditions**)
[61, 123] Let $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and closed functions and let $\mathcal{X}$ be the set of (primal) minimizers of $g - h$ on $\mathbb{R}^n$ and $\mathcal{Y}$ the set of (dual) minimizers of $h^* - g^*$ on $\mathbb{R}^n$. Then the following inclusions hold:

$$\mathcal{X} \supseteq \bigcup_{\widehat{y} \in \mathcal{Y}} \partial g^*(\widehat{y}), \qquad \mathcal{Y} \supseteq \bigcup_{\widehat{x} \in \mathcal{X}} \partial h(\widehat{x}),$$

i.e., if $\widehat{x} \in \mathbb{R}^n$ is a (primal) minimizer of $g - h$ on $\mathbb{R}^n$, then any $y \in \partial h(\widehat{x})$ is a minimizer of $h^* - g^*$ on $\mathbb{R}^n$. Vice versa, if $\widehat{y} \in \mathbb{R}^n$ is a minimizer of $h^* - g^*$ on $\mathbb{R}^n$, then any $x \in \partial g^*(\widehat{y})$ is a (primal) minimizer of $g - h$ on $\mathbb{R}^n$.

Proof. Let $\widehat{y} \in \mathcal{Y}$ be an optimal solution of $\inf_{y \in \mathbb{R}^n} \{h^*(y) - g^*(y)\}$. From Theorem 4.21 we have

$$\inf_{x \in \mathbb{R}^n} \{g(x) - h(x)\} = h^*(\widehat{y}) - g^*(\widehat{y}).$$

Let $\widehat{x} \in \partial g^*(\widehat{y})$. Then we know from Theorem 4.22 that $\widehat{x} \in \partial h^*(\widehat{y})$ and by Proposition 4.12

$$g^*(\widehat{y}) + g(\widehat{x}) = \langle \widehat{y}, \widehat{x} \rangle, \qquad h^*(\widehat{y}) + h(\widehat{x}) = \langle \widehat{y}, \widehat{x} \rangle.$$

Hence, we obtain

$$h^*(\widehat{y}) - g^*(\widehat{y}) = -h(\widehat{x}) + g(\widehat{x}),$$

and the first inclusion is true. The second inclusion follows analogously. □

Chapter 5

Duality Assertions for Location Problems with Obnoxious Facilities

In this chapter we formulate a dual problem (D) to the primal problem (P), introduced in Chapter 3, according to the duality theory by Toland [121] and Singer [112], see Theorem 4.21. We give a necessary and sufficient condition for the existence and attainment of finite optimal solutions of the dual pair of optimization problems (P) and (D) in Section 5.1, and geometrical properties and duality statements in Section 5.2. Moreover, we introduce the terms elementary convex sets and grids with respect to attraction and to repulsion for the dual problem (D). Finally, we present discretization results for both problems in Section 5.3.

In order to formulate the Toland-Singer dual problem (D) we determine the conjugate functions g^* and h^* of the functions g and h, as given by (3.1) and (3.2) in the location problem (P):

$$g(x) := \sum_{m=1}^{\overline{M}} \overline{\phi}_m(x), \qquad \overline{\phi}_m(x) := \overline{w}_m \gamma_{\overline{B}_m}(x - \overline{a}^m), \qquad (m = 1, \dots, \overline{M}),$$

$$h(x) := \sum_{m=1}^{\underline{M}} \underline{\phi}_m(x), \qquad \underline{\phi}_m(x) := \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m), \qquad (m = 1, \dots, \underline{M}).$$

By Theorem 4.15 the conjugates g^* and h^* are given by the infimal convolutions

$$g^* = \overline{\phi}_1^* \,\square\, \cdots \,\square\, \overline{\phi}_{\overline{M}}^* \qquad\qquad h^* = \underline{\phi}_1^* \,\square\, \cdots \,\square\, \underline{\phi}_{\underline{M}}^*.$$

Define $\overline{v}_m : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that $\overline{v}_m(x) := \gamma_{\overline{B}_m}(x - \overline{a}^m)$, for all $m = 1, \dots, \overline{M}$. From properties (d) and property (j) in Section 4.2 we have

$$\overline{v}_m^*(\overline{y}^m) = \gamma_{\overline{B}_m^*}(x) + \langle \overline{a}^m, \overline{y}^m \rangle = \mathbb{I}_{\overline{B}_m^*}(\overline{y}^m) + \langle \overline{a}^m, \overline{y}^m \rangle \,.$$

By property (b) we obtain

$$\overline{\phi}_m^*(\overline{y}^m) = \overline{w}_m \overline{v}_m^* \left(\frac{\overline{y}^m}{\overline{w}_m} \right) = \overline{w}_m \left[\left\langle \overline{a}^m, \frac{\overline{y}^m}{\overline{w}_m} \right\rangle + \mathbb{I}_{\overline{B}_m^*} \left(\frac{\overline{y}^m}{\overline{w}_m} \right) \right] = \langle \overline{y}^m, \overline{a}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m).$$

The conjugates $\underline{\phi}_1^*, \ldots, \underline{\phi}_{\underline{M}}^*$ can be determined analogously such that we have

$$\overline{\phi}_m^*(\overline{y}^m) = \langle \overline{y}^m, \overline{a}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m), \qquad (m = 1, \ldots, \overline{M}), \tag{5.1}$$

$$\underline{\phi}_m^*(\underline{y}^m) = \langle \underline{y}^m, \underline{a}^m \rangle + \mathbb{I}_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m), \qquad (m = 1, \ldots, \underline{M}). \tag{5.2}$$

The **Toland-Singer dual problem** of (P), see Theorem 4.21, is the minimization problem:

$$\beta := \inf_{y \in \mathbb{R}^n} \{h^*(y) - g^*(y)\}, \tag{D}$$

with conjugates $g^*, h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ given by

$$\begin{aligned} g^*(y) &= \left(\overline{\phi}_1^* \,\square\, \ldots \,\square\, \overline{\phi}_{\overline{M}}^*\right)(y) \\ &= \inf \left\{ \sum_{m=1}^{\overline{M}} \overline{\phi}_m^*(\overline{y}^m) \,\middle|\, \sum_{m=1}^{\overline{M}} \overline{y}^m = y \right\} \\ &= \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{y}^m, \overline{a}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m)\right] \,\middle|\, \sum_{m=1}^{\overline{M}} \overline{y}^m = y \right\}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} h^*(y) &= \left(\underline{\phi}_1^* \,\square\, \ldots \,\square\, \underline{\phi}_{\underline{M}}^*\right)(y) \\ &= \inf \left\{ \sum_{m=1}^{\underline{M}} \underline{\phi}_m^*(\underline{y}^m) \,\middle|\, \sum_{m=1}^{\underline{M}} \underline{y}^m = y \right\} \\ &= \inf \left\{ \sum_{m=1}^{\underline{M}} \left[\langle \underline{y}^m, \underline{a}^m \rangle + \mathbb{I}_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m)\right] \,\middle|\, \sum_{m=1}^{\underline{M}} \underline{y}^m = y \right\}. \end{aligned} \tag{5.4}$$

Since elements $y \notin \operatorname{dom} h^*$ lead to the objective value $+\infty$, we may minimize over $\operatorname{dom} h^*$ instead of the complete space $\mathbb{R}^n$ in (D).

5.1 Existence of Finite Optimal Solutions

The weights of the repulsive facilities $\underline{a}^1, \ldots, \underline{a}^{\underline{M}}$ in the location problem (P) may have a strong influence on the optimal objective value α. In order to avoid that $\alpha = -\infty$, we give a necessary and sufficient condition for the existence and attainment of finite optimal solutions of the dual pair of optimization problems (P) and (D).

It is reasonable to use the notational convention (4.14), since otherwise, if $(+\infty)-(+\infty) = (-\infty)$, then for each $y \in \mathbb{R}^n \setminus \operatorname{dom} g^*$ the objective value would be $h^*(y) - g^*(y) = -\infty$. Thus, whenever $\operatorname{dom} g^* \neq \mathbb{R}^n$, the optimal objective value of (D) would be $-\infty$.

Proposition 5.1. [113, Remark 8.3] Let $g, h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be two arbitrary functions and g^*, h^* the corresponding conjugates. If $\inf_{x \in \mathbb{R}^n} \{g(x) - h(x)\} > -\infty$ then we have $\operatorname{dom} g \subseteq \operatorname{dom} h$ and $\inf_{y \in \mathbb{R}^n} \{h^*(y) - g^*(y)\} > -\infty$ and hence $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$ follows as well.

By applying Proposition 5.1 we obtain the following necessary and sufficient condition for the finiteness of the optimal objective values of the dual pair of optimization problems (P) and (D).

Theorem 5.2. (**Finiteness Criterion**)
The optimal objective value of the dual pair of optimization problems (P) and (D) is finite, i.e., $\inf_{y\in\mathbb{R}^n}\{h^*(y)-g^*(y)\} = \inf_{x\in\mathbb{R}^n}\{g(x)-h(x)\} \in \mathbb{R}$, if and only if

$$\sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*.$$

Proof. The function g^*, as given in (5.3), attains a finite objective value if and only if there exists a tuple $\left(\overline{y}^1,\dots,\overline{y}^{\overline{M}}\right)$ such that $\sum_{m=1}^{\overline{M}} \overline{y}^m = y$ and $\overline{y}^m \in \overline{w}_m \overline{B}_m^*$ for all $m=1,\dots,\overline{M}$, i.e.,

$$g^*(y)\in\mathbb{R} \qquad \Leftrightarrow \qquad y \in \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*.$$

An analogous statement holds for h^*. Hence, the effective domains of h^* and g^* are given as

$$\operatorname{dom} h^* = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* (\neq \emptyset) \qquad \text{and} \qquad \operatorname{dom} g^* = \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^* (\neq \emptyset). \tag{5.5}$$

From Proposition 5.1 we directly obtain

$$\inf_{y\in\mathbb{R}^n}\{h^*(y)-g^*(y)\}\in\mathbb{R} \qquad \Rightarrow \qquad \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*.$$

Further, the objective value $h^*(y)-g^*(y)$ is finite for all $y\in\operatorname{dom} h^*\cap\operatorname{dom} g^*$. For $y\notin\operatorname{dom} h^*$ we have by (4.14) that $h^*(y)-g^*(y)=+\infty$, i.e., y does not contribute to the infimum of h^*-g^* since $\operatorname{dom} h^*\neq\emptyset$. Hence, we have

$$\inf_{y\in\mathbb{R}^n}\{h^*(y)-g^*(y)\}\in\mathbb{R} \qquad \Leftarrow \qquad \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^* \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*$$

and the assertion holds. □

By Theorem 5.2 we have found a necessary and sufficient condition for the finiteness of the optimal objective values of the dual pair of optimization problems (P) and (D).

The effective domain of h^*, as the weighted sum of a finite number of convex, closed and bounded balls $\underline{B}_1^*,\dots,\underline{B}_{\underline{M}}^*$, is bounded and closed itself.

The function g^*, as the conjugate of the polyhedral function g, is polyhedral, too, and can be decomposed as the sum of a piece-wise affine function $\widehat{g}$ and the indicator function $\mathbb{I}_{\operatorname{dom} g^*}$, such that $g^*=\widehat{g}+\mathbb{I}_{\operatorname{dom} g^*}$, see Remark 4.6. Hence, we know from the Theorem by Weierstrass that

g^* attains a minimum and a maximum in $\operatorname{dom} g^*$. Analogously, h^* attains a minimum and a maximum in $\operatorname{dom} h^*$.
Consequently, the objective function $h^* - g^*$ (and hence the dual optimization problem (D)) attains a minimum in $\operatorname{dom} h^* \cap \operatorname{dom} g^*$ and thus in $\operatorname{dom} h^*$ when $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$. Then also the primal problem (P) does and we can substitute infimum by minimum in both problems.

Remark 5.3. In case that $\overline{B}_1 = \ldots = \overline{B}_{\overline{M}} = \underline{B}_1 = \ldots = \underline{B}_{\underline{M}}$ in Theorem 5.2, the finiteness criterion for the dual pair of optimization problems (P) and (D) simplifies as follows: $\inf_{x\in\mathbb{R}^n} \{g(x) - h(x)\} = \inf_{y\in\mathbb{R}^n}\{h^*(y) - g^*(y)\} \in \mathbb{R}$, if and only if

$$\sum_{m=1}^{\underline{M}} \underline{w}_m \leq \sum_{m=1}^{\overline{M}} \overline{w}_m.$$

Hence, Theorem 5.2 is a generalization of Theorem 1 in [35] (see also [98, Theorem 2.3]).

In case that the optimal objective value is $-\infty$ the problem (P) is already solved - the optimal location of the new facility is infinitely far away. Since this is not applicable for practical issues, one might either change some parameters (for instance the weights) or, alternatively, some constraints might be considered, see Chapter 8. For example it seems to be reasonable to require the new facility to be established within a given city or country.

Remark 5.4. Note that the dual feasible set $\operatorname{dom} h^*$ is bounded, whereas the primal feasible set is unbounded. In Chapter 8 we will see that the boundedness into any direction of the primal feasible set induces unboundedness into this direction within the dual feasible set.

5.2 Geometrical Properties, Duality Statements and Optimality Conditions

In this section we present geometrical properties, duality results and optimality conditions for the dual pair of optimization problems (P) and (D).

Proposition 5.5. For $\overline{\phi}_1^*, \ldots, \overline{\phi}_{\overline{M}}^* : \mathbb{R}^n \to \mathbb{R}$ as given in (5.1), and $\underline{\phi}_1^*, \ldots, \underline{\phi}_{\underline{M}}^* : \mathbb{R}^n \to \mathbb{R}$ as defined in (5.2) we have

$$\partial \overline{\phi}_m^*(\overline{y}^m) = \overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m), \qquad (m = 1, \ldots, \overline{M}),$$
$$\partial \underline{\phi}_m^*(\underline{y}^m) = \underline{a}^m + N_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m), \qquad (m = 1, \ldots, \underline{M}).$$

Proof. From (5.1) we obtain for all $m = 1, \ldots, \overline{M}$ that

$$\partial \overline{\phi}_m^*(\overline{y}^m) = \partial \left[\langle \overline{y}^m, \overline{a}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) \right]. \tag{5.6}$$

Since the inner product $\langle \cdot, \overline{a}^m \rangle$ is continuous and convex on $\mathbb{R}^n$ and the indicator function $\mathbb{I}_{\overline{w}_m \overline{B}_m^*}$ is convex on $\mathbb{R}^n$, we know from Theorem 4.9 that

$$\partial \left[\langle \overline{y}^m, \overline{a}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) \right] = \partial \langle \overline{y}^m, \overline{a}^m \rangle + \partial \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m). \tag{5.7}$$

Further, by properties (E) and (J) in Section 4.2, it holds $\partial \langle \overline{y}^m, \overline{a}^m \rangle = \{\overline{a}^m\}$ and

$$\partial \mathbb{I}_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) = N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m).$$

Thus, by (5.6) and (5.7), we have for all $m = 1, \ldots, \overline{M}$

$$\partial \overline{\phi}_m^*(\overline{y}^m) = \overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m).$$

The second equality follows analogously, by taking into account (5.2). □

Proposition 5.6. The subdifferentials $\partial g^*(y)$ and $\partial h^*(y)$ are non-empty for all $y \in \operatorname{dom} g^*$ and $y \in \operatorname{dom} h^*$, respectively.

Proof. The function g^*, as the conjugate of the polyhedral function g, is polyhedral, too, and can be decomposed as the sum of a piece-wise affine function $\widehat{g}$ and the indicator function $\mathbb{I}_{\operatorname{dom} g^*}$, such that $g^* = \widehat{g} + \mathbb{I}_{\operatorname{dom} g^*}$, see Remark 4.6. By applying the sum rule for subdifferentials, see Theorem 4.9, and property (J) in Section 4.2 we obtain for all $y \in \operatorname{dom} g^*$

$$\partial g^*(y) \supseteq \partial \widehat{g}(y) + \partial \mathbb{I}_{\operatorname{dom} g^*}(y) = \partial \widehat{g}(y) + N_{\operatorname{dom} g^*}(y),$$

which, obviously, is a non-empty set for all $y \in \operatorname{dom} g^*$. Analogously, we obtain $\partial h^*(y) \neq \emptyset$ for all $y \in \operatorname{dom} h^*$. □

Theorem 5.7. For each tuple $\overline{\pi} = (\overline{y}^1, \ldots, \overline{y}^{\overline{M}}) \in \overline{w}_1 \overline{B}_1^* \times \cdots \times \overline{w}_{\overline{M}} \overline{B}_{\overline{M}}^*$ the following statements are equivalent:

1. $\bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m}(\overline{y}^m) \right] \neq \emptyset$,
2. g^* is exact at $y := \sum_{m=1}^{\overline{M}} \overline{y}^m$, i.e., $g^*(y) = \sum_{m=1}^{\overline{M}} \langle \overline{a}^m, \overline{y}^m \rangle$,
3. $\bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m}(\overline{y}^m) \right] = \partial g^* \left(\sum_{m=1}^{\overline{M}} \overline{y}^m \right)$.

Moreover, for each feasible $y \in \operatorname{dom} g^*$ there exists a tuple $\overline{\pi}$ such that these statements hold.

Analogously, for each tuple $\underline{\pi} = (\underline{y}^1, \ldots, \underline{y}^{\underline{M}}) \in \underline{w}_1 \underline{B}_1^* \times \cdots \times \underline{w}_{\underline{M}} \underline{B}_{\underline{M}}^*$ the following statements are equivalent:

1. $\bigcap_{m=1}^{\underline{M}} \left[\underline{a}^m + N_{\underline{w}_m \underline{B}_m}(\underline{y}^m) \right] \neq \emptyset$,
2. h^* is exact at $y := \sum_{m=1}^{\underline{M}} \underline{y}^m$, i.e., $h^*(y) = \sum_{m=1}^{\underline{M}} \langle \underline{a}^m, \underline{y}^m \rangle$,

3. $\bigcap_{m=1}^{\underline{M}} \left[\underline{a}^m + N_{\underline{w}_m \underline{B}_m}(\underline{y}^m)\right] = \partial h^* \left(\sum_{m=1}^{\underline{M}} \underline{y}^m\right)$.

Moreover, for each feasible $y \in \operatorname{dom} h^*$ there exists a tuple $\underline{\pi}$ such that these statements hold.

Proof. We prove the above statements only concerning the function g^*. The second part follows analogously.

1. $\Rightarrow$ 2. $\Rightarrow$ 3.: By Proposition 5.5 we have

$$\bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m)\right] = \bigcap_{m=1}^{\overline{M}} \partial \overline{\phi}_m^*(\overline{y}^m)$$

and by (5.3) we have $g^* = \overline{\phi}_1 \,\square\, \cdots \,\square\, \overline{\phi}_{\overline{M}}$. The assertions follow directly from Theorem 4.16.

3. $\Rightarrow$ 1.: For $(\overline{y}^1, \ldots, \overline{y}^{\overline{M}}) \in \overline{w}_1 \overline{B}_1^* \times \cdots \times \overline{w}_{\overline{M}} \overline{B}_{\overline{M}}^*$ it obviously follows from (5.5) that

$$y := \sum_{m=1}^{\overline{M}} \overline{y}^m \in \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^* = \operatorname{dom} g^*.$$

The assertion follows since $\partial g^*(y) \neq \emptyset$ for all $y \in \operatorname{dom} g^*$, see Proposition 5.6.

The existence of such a tuple $(\overline{y}^1, \ldots, \overline{y}^{\overline{M}}) \in \overline{w}_1 \overline{B}_1^* \times \cdots \times \overline{w}_{\overline{M}} \overline{B}_{\overline{M}}^*$ follows from Theorem 4.15. $\square$

Proposition 5.8. For each $x \in \mathbb{R}^n$ the subdifferentials of g and h are given by

$$\partial g(x) = \sum_{m=1}^{\overline{M}} \operatorname*{argmax}_{y \in \overline{w}_m \overline{B}_m^*} \langle x - \overline{a}^m, y \rangle, \qquad \partial h(x) = \sum_{m=1}^{\underline{M}} \operatorname*{argmax}_{y \in \underline{w}_m \underline{B}_m^*} \langle x - \underline{a}^m, y \rangle.$$

Proof. It holds by (3.1) and the sum rule for subdifferentials, see Theorem 4.9, that

$$\partial g(x) = \partial \sum_{m=1}^{\overline{M}} \overline{\phi}_m(x) = \sum_{m=1}^{\overline{M}} \partial \overline{\phi}_m(x)$$

for all $x \in \mathbb{R}^n$. We show that

$$\partial \overline{\phi}_m(x) = \operatorname*{argmax}_{y \in \overline{w}_m \overline{B}_m^*} \langle x - \overline{a}^m, y \rangle, \qquad (m = 1, \ldots, \overline{M}).$$

Let $x \in \mathbb{R}^n$. Then for all $m = 1, \ldots, \overline{M}$ we obtain the equivalences

$$\begin{aligned}
\overline{y}^m \in \partial\overline{\phi}_m(x) &\Leftrightarrow x \in \partial\overline{\phi}^*_m(\overline{y}^m) && (5.8)\\
&\Leftrightarrow x \in \overline{a}^m + N_{\overline{w}_m\overline{B}^*_m}(\overline{y}^m) && (5.9)\\
&\Leftrightarrow x - \overline{a}^m \in N_{\overline{w}_m\overline{B}^*_m}(\overline{y}^m)\\
&\Leftrightarrow \forall y \in \overline{w}_m\overline{B}^*_m : \langle x - \overline{a}^m, y - \overline{y}^m\rangle \le 0 \quad \text{and} \quad \overline{y}^m \in \overline{w}_m\overline{B}^*_m && (5.10)\\
&\Leftrightarrow \forall y \in \overline{w}_m\overline{B}^*_m : \langle x - \overline{a}^m, y\rangle \le \langle x - \overline{a}^m, \overline{y}^m\rangle \quad \text{and} \quad \overline{y}^m \in \overline{w}_m\overline{B}^*_m\\
&\Leftrightarrow \overline{y}^m \in \underset{y\in\overline{w}_m\overline{B}^*_m}{\operatorname{argmax}} \langle x - \overline{a}^m, y\rangle .
\end{aligned}$$

Equivalence (5.8) holds by Proposition 4.12, and (5.9) holds by Proposition 5.5. Further, equivalence (5.10) holds by definition of the normal cone, see (4.1); $\overline{y}^m \in \overline{w}_m\overline{B}^*_m$ since otherwise $N_{\overline{w}_m\overline{B}^*_m}(\overline{y}^m) = \emptyset$ in contradiction to $x - \overline{a}^m \in N_{\overline{w}_m\overline{B}^*_m}(\overline{y}^m)$. The proof of the second assertion follows analogously. □

Corollary 5.9. For $x \in \mathbb{R}^n$ and $\overline{y}^m \in \overline{w}_m\overline{B}^*_m$, $m \in \{1, \ldots, \overline{M}\}$, the following equivalence holds:

$$x \in \overline{a}^m + N_{\overline{w}_m\overline{B}^*_m}(\overline{y}^m) \qquad \Leftrightarrow \qquad \overline{y}^m \in \underset{y\in\overline{w}_m\overline{B}^*_m}{\operatorname{argmax}} \langle x - \overline{a}^m, y\rangle ,$$

and analogously, for $x \in \mathbb{R}^n$ and $\underline{y}^m \in \underline{w}_m\underline{B}^*_m$, $m \in \{1, \ldots, \underline{M}\}$, we have

$$x \in \underline{a}^m + N_{\underline{w}_m\underline{B}^*_m}(\underline{y}^m) \qquad \Leftrightarrow \qquad \underline{y}^m \in \underset{y\in\underline{w}_m\underline{B}^*_m}{\operatorname{argmax}} \langle x - \underline{a}^m, y\rangle .$$

Proof. From Proposition 5.5 and Proposition 5.8 we have

$$\partial\overline{\phi}^*_m(\overline{y}^m) = \overline{a}^m + N_{\overline{w}_m\overline{B}^*_m}(\overline{y}^m), \qquad \partial\overline{\phi}_m(x) = \underset{y\in\overline{w}_m\overline{B}^*_m}{\operatorname{argmax}} \langle x - \overline{a}^m, y\rangle .$$

The assertion follows by Proposition 4.12. □

From Theorem 5.7 and Definition 4.17 we know that the elementary convex sets for the primal problem (P) coincide with the subdifferentials of the conjugate functions g^* and h^*. Analogously, we define dual elementary convex sets using the subdifferentials of the functions g and h:

Definition 5.10. Consider the dual problem (D). For $x \in \mathbb{R}^n$ we call the sets

$$\partial g(x) = \sum_{m=1}^{\overline{M}} \underset{y\in\overline{w}_m\overline{B}^*_m}{\operatorname{argmax}} \langle x - \overline{a}^m, y\rangle \qquad \text{and} \qquad \partial h(x) = \sum_{m=1}^{\underline{M}} \underset{y\in\underline{w}_m\underline{B}^*_m}{\operatorname{argmax}} \langle x - \underline{a}^m, y\rangle$$

dual elementary convex set w.r.t. attraction and w.r.t. repulsion, respectively.

The corresponding *dual (construction) grids* are given by

$$\overline{G}_D := \bigcup_{\{x\in\mathbb{R}^n \,|\, \dim \partial g(x)=1\}} \partial g(x) \qquad \text{and} \qquad \underline{G}_D := \bigcup_{\{x\in\mathbb{R}^n \,|\, \dim \partial h(x)=1\}} \partial h(x).$$

A point $y \in \mathbb{R}^n$ is called *dual grid point* (also *dual intersection point*) w.r.t. attraction (repulsion), if y is an extreme point of an elementary convex set w.r.t. attraction (resp. repulsion). We denote by $\overline{\mathcal{I}}_D$ and $\underline{\mathcal{I}}_D$ the sets of all grid points w.r.t. attraction and w.r.t. repulsion, respectively.

Remark 5.11. Obviously, the definition of the sets of dual grid points w.r.t. attraction and w.r.t. repulsion, as given in Definition 5.10, is equivalent to the following:

$$\overline{\mathcal{I}}_D := \bigcup_{\{x\in\mathbb{R}^n \,|\, \dim \partial g(x)=0\}} \partial g(x) \; = \{y \in \operatorname{dom} g^* \,|\, \exists x \in \mathbb{R}^n : \{y\} = \partial g(x)\},$$
$$\underline{\mathcal{I}}_D := \bigcup_{\{x\in\mathbb{R}^n \,|\, \dim \partial h(x)=0\}} \partial h(x) \; = \{y \in \operatorname{dom} h^* \,|\, \exists x \in \mathbb{R}^n : \{y\} = \partial h(x)\}.$$

Figure 5.1 illustrates for Example 4.18 the effective domains $\operatorname{dom} g^*$ and $\operatorname{dom} h^*$ as well as the dual construction grids $\overline{G}_D$ with respect to attraction (green grids) and $\underline{G}_D$ with respect to repulsion (red grids) for different choices of attraction and repulsion weights. The shape of the dual elementary convex sets is similar in each picture and differs only depending on the weights.

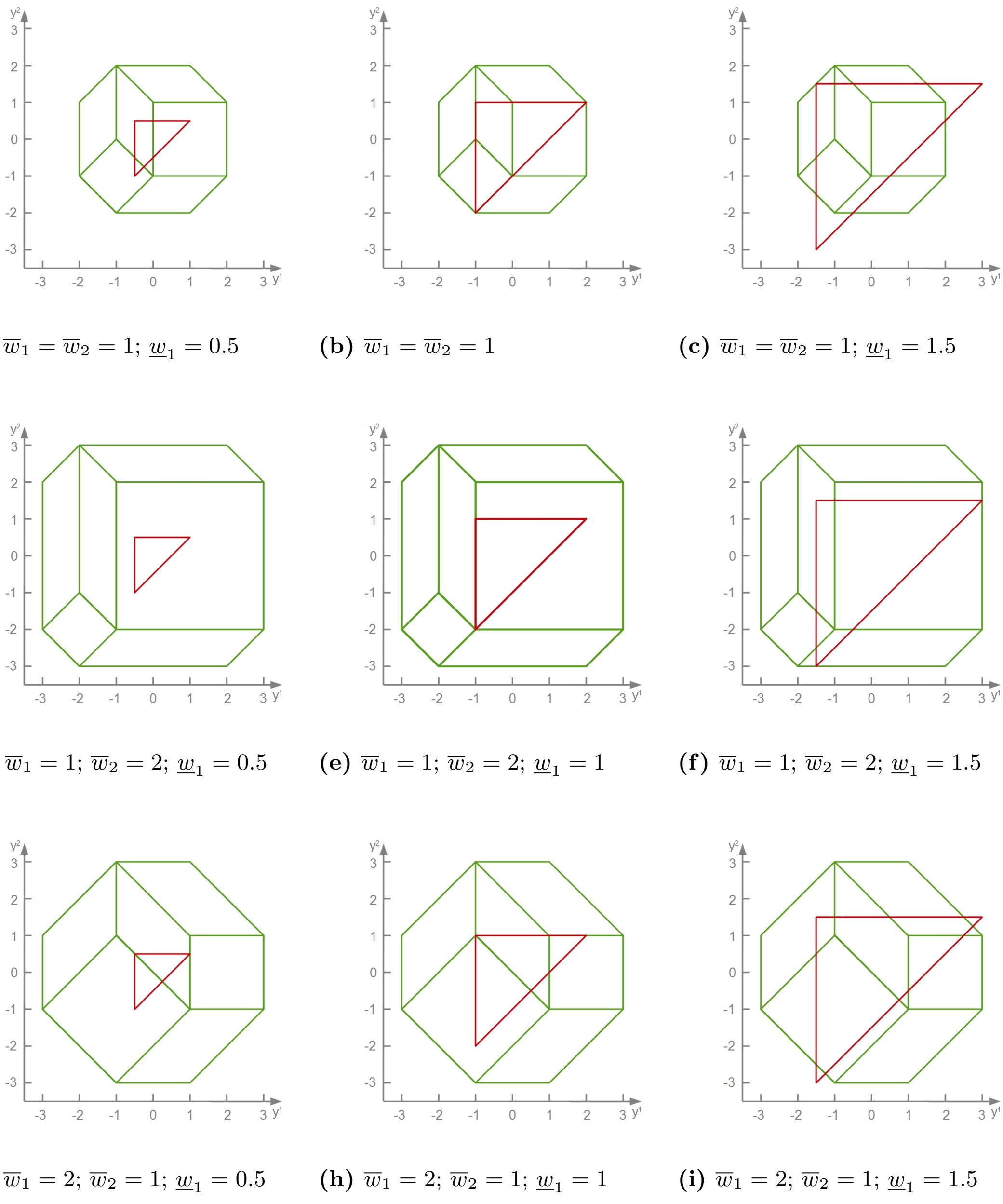
In Figures 5.1c and 5.1i the finiteness criterion, given in Theorem (5.2), is not satisfied since, obviously, $\operatorname{dom} h^* \nsubseteq \operatorname{dom} g^*$.

In Figure 5.1d it holds that $\partial h(x) \subseteq g(x)$ is true, if and only if $x = \overline{a}^2$, where

$$\operatorname{ext}\left(\partial g(\overline{a}^2)\right) = \left\{(-1,-2)^T, (3,-2)^T, (-1,2)^T, (3,2)^T\right\}.$$

Hence, by Theorem 4.22, the only primal element that may be optimal for (P) is $x = \overline{a}^2$.

The relationship between primal and dual elementary convex sets is further described by an inclusion-reversion one-to-one map in Chapter 6.



(a) $\overline{w}_1 = \overline{w}_2 = 1$; $\underline{w}_1 = 0.5$

(b) $\overline{w}_1 = \overline{w}_2 = 1$

(c) $\overline{w}_1 = \overline{w}_2 = 1$; $\underline{w}_1 = 1.5$

(d) $\overline{w}_1 = 1$; $\overline{w}_2 = 2$; $\underline{w}_1 = 0.5$

(e) $\overline{w}_1 = 1$; $\overline{w}_2 = 2$; $\underline{w}_1 = 1$

(f) $\overline{w}_1 = 1$; $\overline{w}_2 = 2$; $\underline{w}_1 = 1.5$

(g) $\overline{w}_1 = 2$; $\overline{w}_2 = 1$; $\underline{w}_1 = 0.5$

(h) $\overline{w}_1 = 2$; $\overline{w}_2 = 1$; $\underline{w}_1 = 1$

(i) $\overline{w}_1 = 2$; $\overline{w}_2 = 1$; $\underline{w}_1 = 1.5$

Figure 5.1: The effective domains of g^* and h^* as well as the dual construction grids $\overline{G}_D$ with respect to attraction (green grids) and $\underline{G}_D$ with respect to repulsion (red grids) for different choices of weights.

Using the following duality statements we are able to determine the set of optimal solutions of (D) out of the set of optimal solutions of (P) and vice versa.

Corollary 5.12. (**Sufficient Optimality Condition for Dual Solutions**)
Let $x \in \bigcap_{m=1}^{\underline{M}} [\underline{a}^m + N_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m)]$ be an optimal solution of the primal problem (P). Then $\sum_{m=1}^{\underline{M}} \underline{y}^m$ is an optimal solution of the dual problem (D).

Proof. By Corollary 5.9 and $x \in \bigcap_{m=1}^{\underline{M}} [\underline{a}^m + N_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m)]$, we have that

$$\underline{y}^m \in \underset{y \in \underline{w}_m \underline{B}_m^*}{\text{argmax}} \langle x - \underline{a}^m, y \rangle$$

for all $m = 1, \ldots, \underline{M}$. By Proposition 5.8 we have $\sum_{m=1}^{\underline{M}} \underline{y}^m \in \partial h(x)$ and thus, by Theorem 4.23, it follows that $\sum_{m=1}^{\underline{M}} \underline{y}^m$ is an optimal solution of the dual problem (D). □

Corollary 5.13. (**Sufficient Optimality Condition for Primal Solutions**)
Let y be an optimal solution of the dual problem (D) such that g^* is exact at $y = \sum_{m=1}^{\overline{M}} \overline{y}^m$ for a tuple $(\overline{y}^1, \ldots, \overline{y}^{\overline{M}}) \in \overline{w}_1 \overline{B}_1^* \times \ldots \times \overline{w}_{\overline{M}} \overline{B}_{\overline{M}}^*$. Then each $x \in \bigcap_{m=1}^{\overline{M}} [\overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m)]$ is optimal for the primal problem (P).

Proof. Let y be an optimal solution of the dual problem (D) and assume that g^* is exact at $y = \sum_{m=1}^{\overline{M}} \overline{y}^m$. Then, by Theorem 5.7, we know that $\bigcap_{m=1}^{\overline{M}} [\overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m)] = \partial g^*(y)$, which is a non-empty set as we know from Proposition 5.6. The assertion follows from Theorem 4.23. □

By applying Corollary 5.12 and Corollary 5.13 we are able to determine the complete set $\mathcal{X}$ of primal optimal solutions when having found the set $\mathcal{Y}$ of dual optimal solutions and vice versa:

Remark 5.14. Let $\mathcal{X}$ be the set of minimizers of (P) and let $\mathcal{Y}$ be the set of minimizers of (D). Then the following equalities hold:

$$\mathcal{X} = \bigcup_{\widehat{y} \in \mathcal{Y}} \partial g^*(\widehat{y}), \qquad \mathcal{Y} = \bigcup_{\widehat{x} \in \mathcal{X}} \partial h(\widehat{x}).$$

Proof. We only prove the first equality, the second one follows analogously.

By Theorem 4.23 we already have the inclusions

$$\mathcal{X} \supseteq \bigcup_{\widehat{y} \in \mathcal{Y}} \partial g^*(\widehat{y}) \qquad \mathcal{Y} \supseteq \bigcup_{\widehat{x} \in \mathcal{X}} \partial h(\widehat{x}).$$

Let $\widehat{x} \in \mathcal{X}$. Then it holds that $\partial h(\widehat{x}) \subseteq \mathcal{Y}$ and by Theorem 4.22 we have $\partial h(\widehat{x}) \subseteq \partial g(\widehat{x})$. Due to the fact that h is a polyhedral function we have $\partial h(\widehat{x}) \neq \emptyset$, see Remark 4.10. For $\widehat{y} \in \partial h(\widehat{x})$ we have by Proposition 4.12 that $\widehat{x} \in \partial g^*(\widehat{y})$ and the opposite inclusion

$$\mathcal{X} \subseteq \bigcup_{\widehat{y} \in \mathcal{Y}} \partial g^*(\widehat{y})$$

is satisfied as well. □

5.3 Discretization Results

In preparation for an algorithm for solving the location problem (P) with obnoxious facilities, the following discretization results play a mayor role.

Corollary 5.15. **(Discretization Result for the Primal Problem (P))**
The set $\overline{\mathcal{I}}$ of primal grid points in $\overline{G}$ w.r.t. attraction is a finite dominating set for the optimal points of the location problem (P), i.e., $\overline{\mathcal{I}} \cap \mathcal{X} \neq \emptyset$, where $\mathcal{X}$ is the set of minimizers of (P).

Proof. From Remark 5.14 we know that $\mathcal{X} = \bigcup_{y \in \mathcal{Y}} \partial g^*(y)$, where the subdifferentials $\partial g^*(y)$ are closed polyhedral elementary convex sets. Hence, we have $\operatorname{ext}(\partial g^*(y)) \subseteq \overline{\mathcal{I}} \cap \mathcal{X}$ for all $y \in \mathcal{Y}$. □

Note, that the discretization result in Corollary 5.15 is a generalization of the discretization result for the classical Fermat-Weber problem (W), see [36].

Corollary 5.16. **(Discretization Result for the Dual Problem (D))**
The set $\underline{\mathcal{I}}_D$ of dual grid points in $\underline{G}_D$ w.r.t. repulsion is a finite dominating set for the optimal points of the dual problem (D), i.e., $\underline{\mathcal{I}}_D \cap \mathcal{Y} \neq \emptyset$, where $\mathcal{Y}$ is the set of minimizers of (D).

Proof. From Remark 5.14 we know that $\mathcal{Y} = \bigcup_{x \in \mathcal{X}} \partial h(x)$, where the subdifferentials $\partial h(x)$ are closed polyhedral elementary convex sets. Hence, we have $\operatorname{ext}(\partial h(x)) \subseteq \underline{\mathcal{I}}_D \cap \mathcal{Y}$ for all $x \in \mathcal{X}$. □

Chapter 6

An Assignment between Primal and Dual Elements Based on Geometric Duality

In this chapter, we apply results from the theory of geometric duality [56] in order to describe the assignment between primal and dual elementary convex sets by an inclusion reversing one-to-one mapping.

Consider the cone

$$K := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} |\, x = 0,\, r \geq 0\}\,, \tag{6.1}$$

and the epigraph of a proper closed convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. A proper face F, see Definition 4.2, of $\operatorname{epi} f$ is called *K-minimal* if all of its points are minimal with respect to K, i.e.,

$$\forall y \in F: \; (y - K \backslash (-K)) \cap \operatorname{epi} f = \emptyset. \tag{6.2}$$

Proposition 6.1. [56, Proposition 3.2] A subset $F \subseteq \operatorname{epi} f$ is a K-minimal exposed face, see Definition 4.2, of $\operatorname{epi} f$ if and only if there exists $\widehat{y} \in \operatorname{dom} f^*$ with $\partial f^*(\widehat{y}) \neq \emptyset$ such that

$$F = \left\{(x, f(x)) \in \mathbb{R}^{n+1} |\; \widehat{y} \in \partial f(x)\right\}.$$

Further, a subset $F^* \subseteq \operatorname{epi} f^*$ is a K-minimal exposed face of $\operatorname{epi} f^*$ if and only if there exists $\widehat{x} \in \operatorname{dom} f$ with $\partial f(\widehat{x}) \neq \emptyset$ such that

$$F^* = \left\{(y, f^*(y)) \in \mathbb{R}^{n+1} |\; y \in \partial f(\widehat{x})\right\}.$$

The following theorem describes an assignment between K-minimal exposed faces F of $\operatorname{epi} f$ and K-minimal exposed faces F^* of $\operatorname{epi} f^*$.

Theorem 6.2. (**Geometric Duality Map for Epigraphs**)
[56, Theorem 3.3] The mapping $\psi :\ 2^{\mathbb{R}^{n+1}} \to 2^{\mathbb{R}^{n+1}}$ defined by

$$\psi(F^*) := \bigcap_{(y,f^*(y))\in F^*} \left\{(x, f(x)) \in \mathbb{R}^{n+1} |\ y \in \partial f(x)\right\}$$

is an inclusion reversing one-to-one mapping ($F_1^* \subseteq F_2^*$ if and only if $\psi(F_1^*) \supseteq \psi(F_2^*)$) between K-minimal exposed faces of epi f^* and K-minimal exposed faces of epi f. Its inverse mapping is given by

$$\psi^*(F) := \bigcap_{(x,f(x))\in F} \left\{(y, f^*(y)) \in \mathbb{R}^{n+1} |\ y \in \partial f(x)\right\}.$$

Subsection 4.2 in [56] also provides the result

$$\dim F^* + \dim \psi(F^*) = n \tag{6.3}$$

concerning the mapping ψ in Theorem 6.2.

Geometric Duality Map for Elementary Convex Sets

Since the function g, as given by (3.1) in the location problem (P), and its conjugate g^*, see (5.3) in the dual problem (D), are polyhedral and convex, the epigraphs epi $g \subseteq \mathbb{R}^{n+1}$ and epi $g^* \subseteq \mathbb{R}^{n+1}$ are polyhedral and convex sets.

According to Proposition 6.1 each exposed K-minimal face $\overline{F}$ of epi g is given by a suitable $y \in \operatorname{dom} g^*$ as

$$\overline{F} = \{(x, g(x))|\ x \in \partial g^*(y)\} =: \overline{F}_y$$

and each exposed K-minimal face $\overline{F}^*$ of epi g^* is given by a suitable $x \in \operatorname{dom} g$ as

$$\overline{F}^* = \{(y, g^*(y))|\ y \in \partial g(x)\} =: \overline{F}^*_x.$$

According to Theorem 6.2 and Equation (6.3) the map $\overline{\psi}^* :\ 2^{\mathbb{R}^{n+1}} \to 2^{\mathbb{R}^{n+1}}$ defined by

$$\overline{\psi}^*(\overline{F}) = \bigcap_{(x,g(x))\in\overline{F}} \overline{F}^*_x$$

provides an inclusion-reversing one-to-one map between the K-minimal faces of epi g and epi g^* such that

$$\dim \overline{F} + \dim \overline{\psi}^*(\overline{F}) = n.$$

Remark 6.3. The projection of an exposed face $\overline{F}_y = \{(x, g(x))|\, x \in \partial g^*(y)\} \subseteq \operatorname{epi} g$ onto $\mathbb{R}^n$ obviously coincides with the elementary convex set $\partial g^*(y)$, and the projection of an exposed face $\overline{F}^*_x = \{(y, g^*(y))|\, y \in \partial g(x)\} \subseteq \operatorname{epi} g^*$ onto $\mathbb{R}^n$ coincides with the elementary convex set $\partial g(x)$. On the other hand, the elementary convex sets $\partial g^*(y)$ and $\partial g(x)$ generate the exposed faces

$$\overline{F}_y = \{(x, g(x))|\, x \in \partial g^*(y)\} \qquad \text{and} \qquad \overline{F}^*_x = \{(y, g^*(y))|\, y \in \partial g(x)\}\,,$$

i.e., there is a one-to-one relationship between the K-minimal faces $\overline{F}_y$ of $\operatorname{epi} g$ and $\overline{F}^*_x$ of $\operatorname{epi} g^*$ and the elementary convex sets $\partial g^*(y)$ and $\partial g(x)$, respectively, such that

$$\begin{aligned} (x, g(x)) \in \overline{F}_y \Leftrightarrow x \in \partial g^*(y), \qquad & \dim \overline{F}_y = \dim \partial g^*(y),\\ (y, g^*(y)) \in \overline{F}^*_x \Leftrightarrow y \in \partial g(x), \qquad & \dim \overline{F}^*_x = \dim \partial g(x). \end{aligned}$$

Consequently the following duality result holds:

Theorem 6.4. (**Assignment between Primal and Dual Elementary Convex Sets**) Let $\overline{\mathfrak{C}} := \{\partial g^*(y)|\, y \in \operatorname{dom} g^*\} \subseteq 2^{\mathbb{R}^n}$ and $\overline{\mathfrak{D}} := \{\partial g(x)|\, x \in \mathbb{R}^n\} \subseteq 2^{\mathbb{R}^n}$. The function $\overline{\Psi}^* : \overline{\mathfrak{C}} \to \overline{\mathfrak{D}}$, defined by

$$\overline{\Psi}^*(\partial g^*(y)) := \bigcap_{x \in \partial g^*(y)} \partial g(x),$$

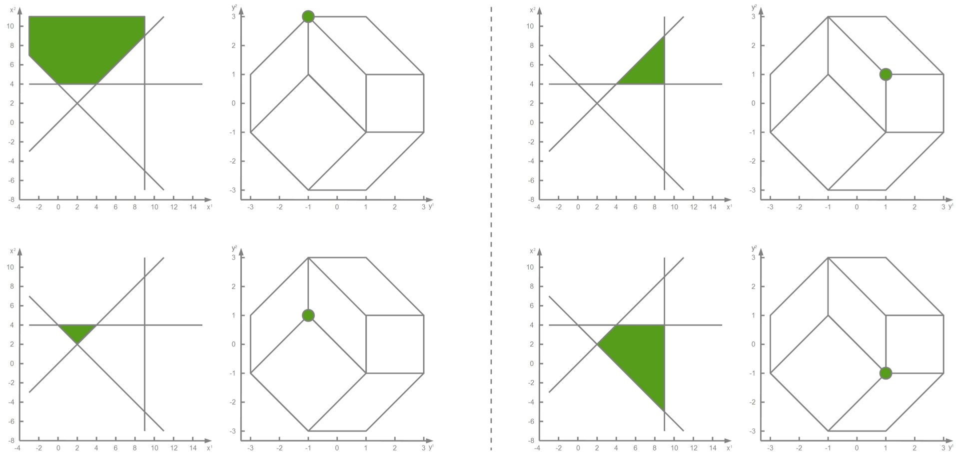
provides an inclusion-reversing one-to-one map between the primal and dual elementary convex sets, such that

$$\dim \partial g^*(y) + \dim \overline{\Psi}^*(\partial g^*(y)) = n.$$

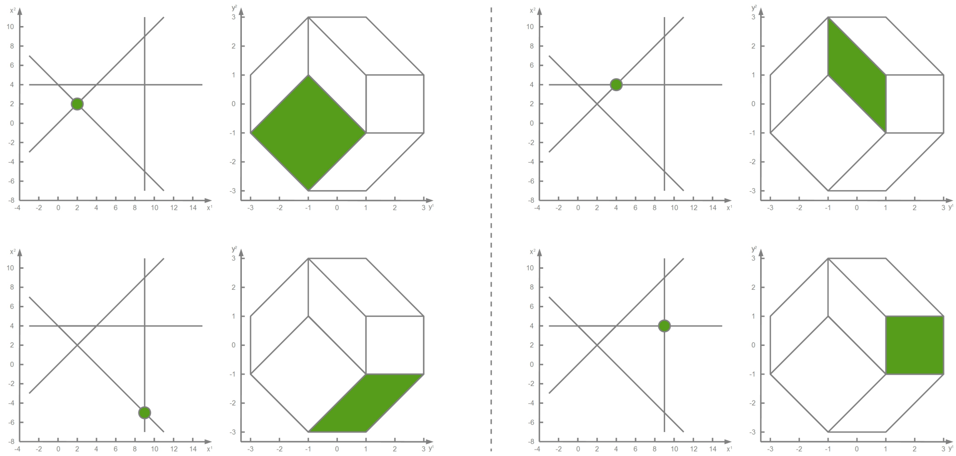
Note that the results of this chapter hold analogously for the epigraphs of h and h^* and the corresponding primal and dual elementary convex sets w.r.t. repulsion.

Example

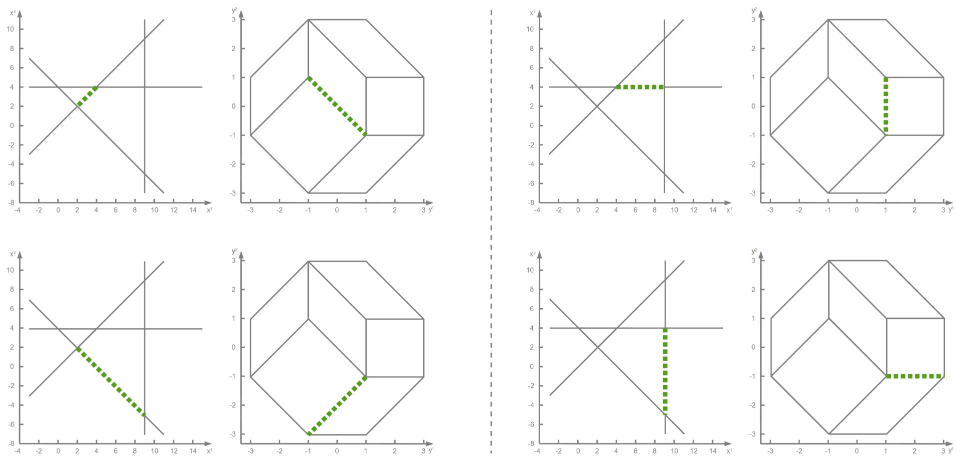
The assignment between primal and dual elementary convex sets in case of the 2-dimensional space $\mathbb{R}^2$ is illustrated in Figure 6.1. The figure refers to the attracting points of Example 4.18 with weights $\overline{w}_1 = 2$ and $\overline{w}_2 = 1$ and illustrates the primal and the dual construction grids $\overline{G}$ and $\overline{G}_D$ w.r.t. attraction. The sum of the dimensions of each dual pair of assigned elementary convex sets is $\dim \partial g^*(y) + \dim \overline{\Psi}(\partial g^*(y)) = 2$.



(a) Assignment between primal elementary convex cells and corresponding dual grid points.



(b) Assignment between primal grid points and corresponding dual elementary convex cells.



(c) Assignment between edges (see Definition 4.2) of primal elementary convex sets and corresponding dual edges.

Figure 6.1: Primal construction grid $\overline{G}$ as well as the corresponding dual construction grid $\overline{G}_D$ for the two attracting facilities $\overline{a}^1$ and $\overline{a}^2$ of Example 4.18 and weights $\overline{w}_1 = 2$ and $\overline{w}_2 = 1$.

Chapter 7

An Application of Benson's Algorithm for Solving the Scalar Location Problem (P)

Although, we are considering a scalar optimization problem, we show in this chapter that methods from the field of linear vector optimization, in particular the well known Benson algorithm [5, 76, 51], can be applied in order to determine the primal and dual grid points with respect to attraction and to repulsion, as introduced in Definitions 4.17 and 5.10. Based on the discretization results presented in Section 5.3, those grid points play a role in Chapter 9 for deriving an algorithm for solving the location problem (P) with obnoxious facilities.

Let $C \subseteq \mathbb{R}^q$ be a polyhedral cone with non-empty interior, which does not contain any line. Consider the *linear vector optimization problem*

$$C - \min Px, \qquad P \in \mathbb{R}^{q\times n}, \tag{LVOP}$$

subject to the set $S := \{x \in \mathbb{R}^n |\, Bx \geq b\}$, $B \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. The set $\mathcal{P} := P[S] + C$ is called *upper image* of (LVOP). A point $\widehat{x} \in S$ is called a *minimizer* of (LVOP), if

$$P[S] \cap (P\widehat{x} - C\setminus \{0\}) = \emptyset.$$

The *geometric dual problem* [51] is

$$K - \max D^*(u, \eta), \tag{LVOD}$$

with respect to the cone K, given in (6.1), the linear objective function $D^* : \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^q$ defined by

$$D^*(u, \eta) := (\eta_1, \ldots, \eta_{q-1}, \langle b, u\rangle),$$

and the feasible set

$$T := \left\{(u, \eta) \in \mathbb{R}^m_+ \times \mathbb{R}^q \middle|\, B^T u = P^T\eta,\ \langle c, \eta\rangle = 1,\ Y^T\eta \geq 0\right\}, \tag{7.1}$$

where Y is a matrix whose columns are generators of the ordering cone C and $c \in \mathbb{R}^q$ is a vector contained in the interior of C with positive last entry. The set $\mathcal{D}^* := D^*[T] - K$ is called *lower image* of (LVOD). A pair $(\widehat{u}, \widehat{\eta}) \in T$ is called a *maximizer* of (LVOD), if

$$D^*[T] \cap (D^*(\widehat{u}, \widehat{\eta}) + K \setminus \{0\}) = \emptyset.$$

To solve such a linear vector optimization problem (LVOP) one can use the well known Benson algorithm [5]. The extended version of Benson's algorithm, presented in [51], solves the linear vector optimization problem (LVOP) and its geometric dual problem (LVOD) by determining an inequality representation as well as the extreme points and extreme directions of the polyhedral sets $P[S] + C$ and $D^*[T] - K$ with P, S, C and D^*, T, K as given above.

A Primal Linear Vector Optimization Problem Related to Repulsion

In order to apply Benson's method, we show in the following, that the epigraph of h, where $h(x) = \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m)$, as defined by (3.2) in the location problem (P), coincides with the upper image of a linear vector optimization problem related to repulsion.

We use a "primal" representation of the epigraph of h, "primal" in the sense that this representation is based on the primal unit balls $\underline{B}_1, \dots \underline{B}_{\underline{M}}$:

$$\begin{aligned}
&\operatorname{epi} h \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \,\middle|\, t \geq h(x) = \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m) \right\} \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \,\middle|\, t \geq \sum_{m=1}^{\underline{M}} \underline{w}_m \min\{\lambda \geq 0 |\ (x - \underline{a}^m) \in \lambda \underline{B}_m\} \right\} \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \,\middle|\, \exists \lambda_1, \dots, \lambda_{\underline{M}} \geq 0 :\ \forall m = 1, \dots, \underline{M} :\ (x - \underline{a}^m) \in \lambda_m \underline{B}_m,\ t \geq \sum_{m=1}^{\underline{M}} \underline{w}_m \lambda_m, \right\}.
\end{aligned}$$

Further, we define the feasible set

$$\begin{aligned}
\underline{S} := \Big\{ \big(x, (\lambda_1, \dots, \lambda_{\underline{M}}), t\big) \in \mathbb{R}^n \times \mathbb{R}^{\underline{M}} \times \mathbb{R} \Big| \\
\lambda_1, \dots, \lambda_{\underline{M}} \geq 0,\ \forall m = 1, \dots, \underline{M} :\ (x - \underline{a}^m) \in \lambda_m \underline{B}_m,\ t \geq \sum_{m=1}^{\underline{M}} \underline{w}_m \lambda_m, \Bigg\}.
\end{aligned} \tag{7.2}$$

In the following we denote by E_n the identity matrix in $\mathbb{R}^n$ and by $0_{n,\underline{M}}$, $0_{n,1}$, $0_{1,\underline{M}}$, $0_{1,n}$ the zero matrices in the dimensions defined by their indices. Writing the unit balls $\underline{B}_1, \dots, \underline{B}_{\underline{M}}$ as

$$\underline{B}_m = \{x \in \mathbb{R}^n |\ \underline{A}_m x \leq \underline{b}_m\}, \qquad \underline{A}_m \in \mathbb{R}^{\underline{r}_m \times n},\ \underline{b}^m \in \mathbb{R}^{\underline{r}_m}, \qquad (m = 1, \dots, \underline{M}), \tag{7.3}$$

and the constraints of the feasible set $\underline{S}$ in (7.2) into a common system of inequalities, we obtain

$$\underline{S} = \left\{ z = \big(x, (\lambda_1, \dots, \lambda_{\underline{M}}), t\big) \in \mathbb{R}^n \times \mathbb{R}^{\underline{M}} \times \mathbb{R} \,\middle|\, \underline{B} z \geq \underline{b} \right\}$$

with

$$\underline{B} := \left(\begin{array}{c|ccc|c} 0_{1,n} & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0_{1,n} & 0 & & 1 & 0 \\ \hline 0_{1,n} & -\underline{w}_1 & \cdots & -\underline{w}_{\underline{M}} & 1 \\ \hline -\underline{A}_1 & \underline{b}^1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -\underline{A}_{\underline{M}} & 0 & & \underline{b}^{\underline{M}} & 0 \end{array} \right), \qquad \underline{b} := \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \hline -\underline{A}_1 \cdot \underline{a}^1 \\ \vdots \\ -\underline{A}_{\underline{M}} \cdot \underline{a}^{\underline{M}} \end{array} \right). \tag{7.4}$$

The **linear vector optimization problem related to repulsion** can be formulated as

$$\underline{C} - \min_{x \in \underline{S}} \underline{P} x, \tag{$\underline{\text{LVOP}}$}$$

with suitable matrix $\underline{P}\colon\ \mathbb{R}^{n+\underline{M}+1} \to \mathbb{R}^{n+1}$ defined as

$$\underline{P} := \begin{pmatrix} E_n & 0_{n,\underline{M}} & 0_{n,1} \\ 0_{1,n} & 0_{1,\underline{M}} & 1 \end{pmatrix},$$

and ordering cone $\underline{C}$, which is chosen as the recession cone of the epigraph of h, i.e., $\underline{C} := 0^+ \operatorname{epi} h$. Hence, by (4.2) and due to $0 \in 0^+ \operatorname{epi} h$, we obtain for the upper image of ($\underline{\text{LVOP}}$)

$$\underline{P}[\underline{S}] + \underline{C} = \operatorname{epi} h + 0^+ \operatorname{epi} h = \operatorname{epi} h.$$

The recession cone $\underline{C} = 0^+ \operatorname{epi} h$ is determined below.

Analogously, for the function g, as defined by (3.1) in the location problem (P), we can define a **linear vector optimization problem related to attraction**

$$\overline{C} - \min_{x \in \overline{S}} \overline{P} x, \tag{$\overline{LVOP}$}$$

with suitable matrix $\overline{P} \in \mathbb{R}^{(n+1)\times(n+\overline{M}+1)}$, feasible set $\overline{S} \subseteq \mathbb{R}^{n+\overline{M}+1}$ and ordering cone $\overline{C} := 0^+ \operatorname{epi} g$, such that $\overline{P}[\overline{S}] + \overline{C} = \operatorname{epi} g$.

Determination of the Cones $\underline{C}$ and $\overline{C}$

In order to determine the recession cone $\underline{C} = 0^+ \operatorname{epi} h$, we use a "dual" representation of the epigraph, "dual" in the sense that this representation is based on the dual unit balls $\underline{B}_1^*, \dots, \underline{B}_{\underline{M}}^*$.

By (4.9) it holds

$$\begin{aligned}
&\operatorname{epi} h \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \;\middle|\; t \geq h(x) = \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m) = \sum_{m=1}^{\underline{M}} \max_{\underline{y}^m \in \operatorname{ext}(\underline{w}_m \underline{B}_m^*)} \left\langle x - \underline{a}^m, \underline{y}^m \right\rangle \right\} \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \;\middle|\; \forall (\underline{y}^1, \dots, \underline{y}^{\underline{M}}) \in \operatorname{ext}(\underline{w}_1 \underline{B}_1^*) \times \cdots \times \operatorname{ext}(\underline{w}_{\underline{M}} \underline{B}_{\underline{M}}^*) : \sum_{m=1}^{\underline{M}} \left\langle \underline{a}^m, \underline{y}^m \right\rangle \geq \sum_{m=1}^{\underline{M}} \left\langle x, \underline{y}^m \right\rangle - t \right\}.
\end{aligned}$$

By applying Proposition 4.4 we obtain for the recession cone

$$\begin{aligned}
\underline{C} &= 0^+ \operatorname{epi} h \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \;\middle|\; \forall (\underline{y}^1, \dots, \underline{y}^{\underline{M}}) \in \operatorname{ext}(\underline{w}_1 \underline{B}_1^*) \times \cdots \times \operatorname{ext}(\underline{w}_{\underline{M}} \underline{B}_{\underline{M}}^*) : \; 0 \geq \sum_{m=1}^{\underline{M}} \left\langle x, \underline{y}^m \right\rangle - t \right\} \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \;\middle|\; t \geq \sum_{m=1}^{\underline{M}} \max_{\underline{y}^m \in \underline{w}_m \underline{B}_m^*} \left\langle x, \underline{y}^m \right\rangle \right\} \\
&= \left\{ (x,t) \in \mathbb{R}^{n+1} \;\middle|\; t \geq \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(x) \right\}. \qquad (7.5)
\end{aligned}$$

The matrix $\underline{Y}$, whose columns are the generators of the ordering cone $\underline{C}$, contains as columns the vectors

$$\left(d, \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(d) \right) \in \mathbb{R}^{n+1},$$

for which d is a fundamental direction of at least one of the unit balls $\underline{B}_1, \dots, \underline{B}_{\underline{M}}$.

Analogously, the recession cone $\overline{C} = O^+ \operatorname{epi} g$ can be written as

$$\overline{C} = 0^+ \operatorname{epi} g = \left\{ (x,t) \in \mathbb{R}^{n+1} \;\middle|\; t \geq \sum_{m=1}^{\overline{M}} \overline{w}_m \gamma_{\overline{B}_m}(x) \right\},$$

and the matrix $\overline{Y}$, whose columns are the generators of $\overline{C}$, contains as columns the vectors

$$\left(d, \sum_{m=1}^{\overline{M}} \overline{w}_m \gamma_{\overline{B}_m}(d) \right) \in \mathbb{R}^{n+1},$$

for which d is a fundamental direction of at least one of the unit balls $\overline{B}_1, \dots, \overline{B}_{\overline{M}}$.

A Dual Linear Vector Optimization Problem Related to Repulsion

In the following we show that the epigraph of the function h^* corresponds to the lower image of the dual linear vector optimization problem related to repulsion.

Taking into account $\underline{B}, \underline{b}, \underline{P}, \underline{C}, \underline{S}$, as defined in the primal linear vector optimization problem (LVOP), as well as the cone K, as defined in (6.1), a **geometric dual vector optimization problem related to repulsion** is given by

$$K - \max_{(u,\eta)\in \underline{T}} \underline{D}^*(u,\eta), \tag{LVOD}$$

with objective function

$$\underline{D}^*(u,\eta) : = (\eta_1, \dots, \eta_n, \langle \underline{b}, u\rangle) \tag{7.6}$$

and feasible set

$$\underline{T} := \{(u,\eta) |\, \text{(i)-(vi) are satisfied}\}\,, \tag{7.7}$$

where

(i) $u = (u_1, \dots, u_{\underline{M}}, v, \underline{z}^1, \dots, \underline{z}^{\underline{M}})$ with $(u_1, \dots, u_{\underline{M}}) \in \mathbb{R}^{\underline{M}}_+$, $v \in \mathbb{R}_+$ and $\underline{z}^m \in \mathbb{R}^{\underline{r}_m}_+$ for $m = 1, \dots, \underline{M}$, where $\underline{r}_m$ is the number of constraints, which define the unit ball $\underline{B}_m$, see (7.3),

$\eta \in \mathbb{R}^{n+1}$, $y := -(\eta_1, \dots, \eta_n)$,

(ii) $\eta_{n+1} = 1$,

(iii) $v = 1$,

(iv) $y = \sum_{m=1}^{\underline{M}} \underline{A}_m^T \underline{z}^m$,

(v) $u_m = \underline{w}_m - \langle \underline{b}^m, \underline{z}^m\rangle$ for all $m = 1, \dots, \underline{M}$,

(vi) For all $(x,t) \in \underline{C}$ it holds $\langle x, y\rangle \leq t$.

Proposition 7.1. Set $c := (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Then the set $\underline{T}$ has the structure as given in (7.1), i.e., the following equivalences hold:

1. (ii) is equivalent to $\langle c, \eta\rangle = 1$.
2. (iii), (iv) and (v) are equivalent to $B^T u = P^T \eta$.
3. (vi) holds if and only if $Y^T \eta \geq 0$, where $\underline{Y}$ is a matrix, whose columns generate the ordering cone $\underline{C}$.

Proof.

1. For $c := (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ we obtain $\langle c, \eta \rangle = 1$ if and only if $\eta_{n+1} = 1$, which coincides with property (ii).

2. We further have equivalence between properties (iii), (iv) and (v) and $\underline{B}^T u = \underline{P}^T \eta$:

$$\underline{B}^T u = \left(\begin{array}{c} -\sum_{m=1}^{M} \underline{A}_m^T \underline{z}^m \\ \hline u_1 - \underline{w}_1 + \langle b^1, \underline{z}^1 \rangle \\ \vdots \\ u_{\underline{M}} - \underline{w}_{\underline{M}} + \langle b^{\underline{M}}, \underline{z}^{\underline{M}} \rangle \\ \hline 1 \end{array} \right) = \left(\begin{array}{c} -y \\ \hline 0 \\ \vdots \\ 0 \\ \hline 1 \end{array} \right) = \underline{P}^T \eta.$$

3. Let $\underline{Y}$ be a matrix, whose columns generate the ordering cone $\underline{C}$. Then it holds $\underline{Y}^T \eta \geq 0$ if and only if $\left\langle \binom{x}{t}, \eta \right\rangle \geq 0$ for all $(x, t) \in \underline{C}$, where

$$\left\langle \begin{pmatrix} x \\ t \end{pmatrix}, \eta \right\rangle = \left\langle \begin{pmatrix} x \\ t \end{pmatrix}, \begin{pmatrix} -y \\ 1 \end{pmatrix} \right\rangle = -\langle x, y \rangle + t \geq 0 \qquad \Leftrightarrow \qquad \langle x, y \rangle \leq t,$$

such that property (vi) is equivalent to $\underline{Y}^T \eta \geq 0$.

□

Proposition 7.2. Let the properties (i) and (v) be satisfied, i.e., $\langle \underline{b}^m, \underline{z}^m \rangle \leq \underline{w}_m$ and $\underline{z}_m \geq 0$ for all $m = 1, \dots, \underline{M}$. Then it holds $\underline{A}_m^T \underline{z}^m \in \underline{w}_m \underline{B}_m^*$ for all $m = 1, \dots, \underline{M}$.

Proof. For all $\widehat{x} \in \underline{B}_m = \{x \in \mathbb{R}^n | \, \underline{A}_m x \leq \underline{b}^m\}$ we obtain

$$\langle \widehat{x}, \underline{A}_m^T \underline{z}^m \rangle = \langle \underline{A}_m \widehat{x}, \underline{z}^m \rangle \leq \langle \underline{b}^m, \underline{z}^m \rangle \leq \underline{w}_m.$$

The assertion follows since $\underline{w}_m \underline{B}_m^* = \{y \in \mathbb{R}^n | \, \forall x \in \underline{B}_m : \langle x, y \rangle \leq \underline{w}_m\}$, see (4.6). □

Proposition 7.3. The properties (i), (iv) and (v) imply the property (vi).

Proof. Let $(x, t) \in \underline{C}$ and assume that $\underline{z}^1, \dots, \underline{z}^{\underline{M}}$ and y are given such that they satisfy the properties (i), (iv) and (v). Then we obtain by (7.5), (4.9) and Proposition 7.2

$$t \geq \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(x) = \sum_{m=1}^{\underline{M}} \max_{\underline{y}^m \in \underline{w}_m \underline{B}_m^*} \langle x, \underline{y}^m \rangle \geq \sum_{m=1}^{\underline{M}} \langle x, \underline{A}_m^T \underline{z}^m \rangle = \left\langle x, \sum_{m=1}^{\underline{M}} \underline{A}_m^T \underline{z}^m \right\rangle = \langle x, y \rangle .$$

□

Proposition 7.4. For the lower image of the dual problem (LVOD) and the epigraph of h^* it holds

$$-\left(\underline{D}^*[\underline{T}] - K\right) = \operatorname{epi} h^*.$$

Proof. For the dual objective function, as given in (7.6), we have by property (i) and by (7.4)

$$\underline{D}^*(u,\eta) = (\eta_1,\dots,\eta_n,\langle \underline{b},u\rangle) = -\left(y, \sum_{m=1}^{M} \langle \underline{A}_m \underline{a}^m, \underline{z}^m\rangle\right) = -\left(y, \sum_{m=1}^{M} \langle \underline{a}^m, \underline{A}_m^T \underline{z}^m\rangle\right).$$

Taking into account the definition of the set $\underline{T}$, see (7.7), and Proposition 7.3 we obtain

$$\begin{aligned} -\underline{D}^*[\underline{T}] = \Big\{(y,t) \in \mathbb{R}^{n+1} \,\Big|\, \exists \underline{z}^1,\dots,\underline{z}^{\underline{M}} \geq 0 :& \\ \forall m = 1,\dots,\underline{M} : \langle \underline{b}^m, \underline{z}^m\rangle \leq \underline{w}_m,\ y = \sum_{m=1}^{M} \underline{A}_m^T \underline{z}^m,\ t = \sum_{m=1}^{M} \langle \underline{a}^m, \underline{A}_m^T \underline{z}^m\rangle \Big\}&. \end{aligned}$$

By definition of the ordering cone K, in (6.1), and Proposition 7.2 it follows

$$\begin{aligned} &-\left(\underline{D}^*[\underline{T}] - K\right) \\ &= \left\{ (y,t) \in \mathbb{R}^{n+1} \,\middle|\, \exists \left(\underline{y}^1,\dots,\underline{y}^{\underline{M}}\right) \in \underline{w}_1 \underline{B}_1^* \times \dots \times \underline{w}_{\underline{M}} \underline{B}_{\underline{M}}^* : y = \sum_{m=1}^{M} \underline{y}^m,\ t \geq \sum_{m=1}^{M} \langle \underline{a}^m, \underline{y}^m\rangle \right\} \\ &= \left\{ (y,t) \in \mathbb{R}^{n+1} \,\middle|\, t \geq \inf \left\{ \sum_{m=1}^{M} \langle \underline{a}^m, \underline{y}^m\rangle + \mathbb{I}_{\underline{w}_m \underline{B}_m^*}(\underline{y}^m) \,\middle|\, y = \sum_{m=1}^{M} \underline{y}^m \right\} = h^*(y) \right\} = \operatorname{epi} h^*, \end{aligned}$$

and the assertion holds. ☐

Analogously, we can define a **geometric dual vector optimization problem related to attraction**

$$K - \max_{(u,\eta)\in\overline{T}} \overline{D}^*(u,\eta), \qquad (\overline{LVOD})$$

with objective function $\overline{D}^*$, dual feasible set $\overline{T}$ and ordering cone K, as given in (6.1), such that $\operatorname{epi} g^* = -\left(\overline{D}^*[\overline{T}] - K\right)$.

Conclusion

We constructed a dual pair of linear vector optimization problems (LVOP) and (LVOD) such that we obtain for the upper image $\underline{P}[\underline{S}] + \underline{C} = \operatorname{epi} h$ and for the lower image $\underline{D}^*[\underline{T}] - K = -\operatorname{epi} h^*$. Finding the exposed faces, see Definition 4.2, of $\operatorname{epi} h$ and $\operatorname{epi} h^*$ means to minimize the linear vector optimization problem (LVOP) and to maximize its geometric dual problem (LVOD). This can be done by applying Benson's algorithm. A numerical implementation of the primal and dual variant of Benson's algorithm, as presented in [51], is `bensolve-1.2`.[1]

The projections of the extreme points of $\operatorname{epi} h$ and $\operatorname{epi} h^*$ onto $\mathbb{R}^n$ coincide with the primal and dual grid points $x \in \underline{\mathcal{I}}$ and $y \in \underline{\mathcal{I}}_D$, w.r.t. repulsion, see Remark 6.3.

Based on the discretization results, presented in Section 5.3, those grid points play a role in Chapter 9 for deriving an algorithm for solving the location problem (P) with obnoxious facilities.

Analogous results hold for the function g and its conjugate g^*.

[1] http://ito.mathematik.uni-halle.de/~loehne

Chapter 8

Locating an Obnoxious Facility under Consideration of Polyhedral Constraints

In practical applications it is reasonable to establish the new facility within a given area, a city or a country for instance. Throughout this chapter we extend our research such that convex polyhedral constraints are considered. We formulate a dual problem to the constrained location problem, based on the duality theory by Toland [121] and Singer [112], see Theorem 4.21. In Section 8.1 we provide some geometrical properties and duality statements. In Section 8.2 we state a criterion for the finiteness of the objective value of such a constrained location problem. In Section 8.3 we give optimality conditions and discretization results for both problems. Finally, we illustrate the results of this chapter by some examples in Section 8.4.

Let $\mathcal{H} \subseteq \mathbb{R}^n$ be given by the intersection of I closed half-spaces such that

$$\mathcal{H} := \bigcap_{i=1}^{I} \mathcal{H}_i, \qquad \mathcal{H}_i := \left\{ x \in \mathbb{R}^n | \ \left\langle x, q^i \right\rangle \leq c_i \right\}, \qquad (i = 1, \ldots, I), \tag{8.1}$$

with $q^i \in \mathbb{R}^n \setminus \{0\}$ and $c_i \in \mathbb{R}$, for $i = 1, \ldots, I$. We assume that the half-spaces $\mathcal{H}_1, \ldots, \mathcal{H}_I$ are given such that their intersection $\mathcal{H}$ has non-empty interior and none of them is redundant, i.e., none of them can be omitted without changing the structure of $\mathcal{H}$.

We consider the location problem with obnoxious facilities and convex constraints

$$\min_{x \in \mathcal{H}} \{ g(x) - h(x) \}, \tag{8.2}$$

with

$$g(x) = \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{\gamma}_m (x - \overline{a}^m), \qquad h(x) = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{\gamma}_m (x - \underline{a}^m).$$

as given by (3.1) and (3.2) in the location problem (P). Formulating (8.2) as an unconstrained *d.c. problem* by using the indicator function, we obtain

$$\alpha_{\mathcal{H}} := \min_{x\in\mathbb{R}^n} \{g_{\mathcal{H}}(x) - h(x)\}, \tag{$P_{\mathcal{H}}$}$$

where $g_{\mathcal{H}}(x) := g(x) + \mathbb{I}_{\mathcal{H}}(x)$. Obviously, the constraints set $\mathcal{H}$ does not affect the function h.

Formulation of the Dual Problem

In order to formulate a Toland-Singer dual problem ($D_{\mathcal{H}}$) we determine the conjugate function $g^*_{\mathcal{H}}$ of the function $g_{\mathcal{H}}$.

Proposition 8.1. Let $\mathcal{H}(\neq \emptyset) \subseteq \mathbb{R}^n$ be a convex set. Then the conjugate $g^*_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\begin{aligned} g^*_{\mathcal{H}}(y) &= \inf_{\overline{y}^0\in\mathbb{R}^n} \{g^*(y - \overline{y}^0) + \sigma_{\mathcal{H}}(\overline{y}^0)\} \\ &= \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m\rangle + \mathbb{I}_{\overline{w}_m B^*_m}(\overline{y}^m)\right] + \sigma_{\mathcal{H}}(\overline{y}^0) \,\middle|\, \sum_{m=0}^{\overline{M}} \overline{y}^m = y \right\}. \end{aligned}$$

with $g^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, as given by (5.3) in (D).

Proof. The intersection $\operatorname{int}\operatorname{dom} g \cap \operatorname{int}\operatorname{dom} \mathbb{I}_{\mathcal{H}}$ is non-empty since $\operatorname{dom} g = \mathbb{R}^n$ and $\operatorname{int} \mathcal{H} \subseteq \mathbb{R}^n$ is supposed to be non-empty. Further, the functions g and $\mathbb{I}_{\mathcal{H}}$ are convex. Hence, the assertion follows by Theorem 4.15 and property (h) in Section 4.2:

$$g^*_{\mathcal{H}}(y) = (g + \mathbb{I}_{\mathcal{H}})^*(y) = (g^* \,\square\, \mathbb{I}^*_{\mathcal{H}})(y) = (g^* \,\square\, \sigma_{\mathcal{H}})(y) = \inf_{\overline{y}^0\in\mathbb{R}^n} \left\{g^*(y - \overline{y}^0) + \sigma_{\mathcal{H}}(\overline{y}^0)\right\}.$$

Taking into account the structure of g^*, as given by (5.3) in (D), we obtain the more specific formulation

$$\begin{aligned} g^*_{\mathcal{H}}(y) &= (g + \mathbb{I}_{\mathcal{H}})^*(y) \\ &= \left(\overline{\phi}_1 + \ldots + \overline{\phi}_{\overline{M}} + \mathbb{I}_{\mathcal{H}}\right)^*(y) \\ &= \left(\overline{\phi}^*_1 \,\square\, \ldots \,\square\, \overline{\phi}^*_{\overline{M}} \,\square\, \sigma_{\mathcal{H}}\right)(y) \\ &= \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m\rangle + \mathbb{I}_{\overline{w}_m B^*_m}(\overline{y}^m)\right] + \sigma_{\mathcal{H}}(\overline{y}^0) \,\middle|\, \sum_{m=0}^{\overline{M}} \overline{y}^m = y \right\}, \end{aligned}$$

with $\overline{\phi}_1, \ldots, \overline{\phi}_{\overline{M}}$ as given by (3.1) in the location problem (P) and their conjugates $\overline{\phi}^*_1, \ldots, \overline{\phi}^*_{\overline{M}}$ as given by (5.1) in the dual problem (D). □

The **Toland-Singer dual problem** of ($P_{\mathcal{H}}$), see Theorem 4.21, is defined by

$$\beta_{\mathcal{H}} := \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*_{\mathcal{H}}(y)\} \tag{$D_{\mathcal{H}}$}$$

with conjugates $h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, as given by (5.4) in (D), and $g^*_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, as given by Proposition 8.1.

8.1 Geometrical Properties

In this section we provide some geometrical properties and duality statements of the dual pair of optimization problems ($P_{\mathcal{H}}$) and ($D_{\mathcal{H}}$).

Proposition 8.2. Let $\mathcal{H} = \bigcap_{i=1}^{I} \mathcal{H}_i$, $\mathcal{H}_i = \left\{x \in \mathbb{R}^n \middle| \left\langle x, q^i \right\rangle \leq c_i\right\}$ with q^i, c_i, for $i = 1, \ldots, I$, as defined in (8.1). Moreover, define for $x \in \mathcal{H}$ the index set $\mathfrak{I}_x := \left\{i \in \{1, \ldots, I\} \middle| \left\langle x, q^i \right\rangle = c_i\right\}$. Then we have

$$\partial g_{\mathcal{H}}(x) = \partial g(x) + N_{\mathcal{H}}(x), \tag{8.3}$$

where

$$N_{\mathcal{H}}(x) = \begin{cases} \sum\limits_{i \in \mathfrak{I}_x} \mathbb{R}_+ q^i, & \text{if } x \in \operatorname{bd} \mathcal{H}, \\ \{0\}, & \text{if } x \in \operatorname{int} \mathcal{H}, \\ \emptyset, & \text{if } x \notin \mathcal{H}, \end{cases}$$

and $\partial g(x)$ is given in Proposition 5.8.

Proof. Equation (8.3) follows from Theorem 4.9 and property (J) in Section 4.2:

$$\partial g_{\mathcal{H}}(x) = \partial (g + \mathbb{I}_{\mathcal{H}})(x) = \partial g(x) + \partial \mathbb{I}_{\mathcal{H}}(x) = \partial g(x) + N_{\mathcal{H}}(x).$$

From Theorem 4.9 and property (J) in Section 4.2 we obtain for $x \in \mathcal{H}$

$$N_{\mathcal{H}}(x) = \partial \mathbb{I}_{\mathcal{H}}(x) = \partial \sum_{i=1}^{I} \mathbb{I}_{\mathcal{H}_i}(x) = \sum_{i=1}^{I} \partial \mathbb{I}_{\mathcal{H}_i}(x) = \sum_{i=1}^{I} N_{\mathcal{H}_i}(x) = \begin{cases} \sum_{i \in \mathfrak{I}_x} \mathbb{R}_+ q^i, & x \in \operatorname{bd} \mathcal{H}, \\ \{0\}, & x \in \operatorname{int} \mathcal{H}. \end{cases}$$

The last equation holds since $N_{\mathcal{H}_i}(x) = \{0\}$ for $x \in \operatorname{int} \mathcal{H}_i$, i.e., whenever $i \notin \mathfrak{I}_x$, and $N_{\mathcal{H}_i}(x) = \mathbb{R}_+ q^i$ for $x \in \operatorname{bd} \mathcal{H}_i$, i.e., whenever $i \in \mathfrak{I}_x$. □

Proposition 8.3. The subdifferentials $\partial g^*_{\mathcal{H}}(y)$, $\partial g_{\mathcal{H}}(x)$ and $\partial \sigma_{\mathcal{H}}(y)$ are non-empty for all $y \in \operatorname{dom} g^*_{\mathcal{H}}$, $x \in \operatorname{dom} g_{\mathcal{H}} = \mathcal{H}$ and $y \in \operatorname{dom} \sigma_{\mathcal{H}}$, respectively.

Proof. The function $\mathbb{I}_{\mathcal{H}}$ is polyhedral, whenever $\mathcal{H}$ is a polyhedral set. Hence, the function $g_{\mathcal{H}} = g + \mathbb{I}_{\mathcal{H}}$ and the conjugates $\mathbb{I}^*_{\mathcal{H}} = \sigma_{\mathcal{H}}$ and $g^*_{\mathcal{H}}$ are polyhedral, too. The assertions follow by Remark 4.10. □

Proposition 8.4. For $\mathcal{H} = \bigcap_{i=1}^{I} \mathcal{H}_i$, $\mathcal{H}_i = \left\{x \in \mathbb{R}^n \middle| \left\langle x, q^i \right\rangle = c_i\right\}$ with q^i, c_i $(i = 1, \ldots, I)$ as defined in (8.1) the effective domain of $\sigma_{\mathcal{H}}$ is

$$\operatorname{dom} \sigma_{\mathcal{H}} = \sum_{i=1}^{I} \mathbb{R}_+ q^i.$$

Proof. Let $y \in \operatorname{dom} \sigma_{\mathcal{H}}$. Then by Proposition 8.3 and property (h) in Section 4.2 it holds

$$\partial \sigma_{\mathcal{H}}(y) = \{x \in \mathcal{H} | \, \sigma_{\mathcal{H}}(y) = \langle x, y \rangle\} \neq \emptyset.$$

By Proposition 4.12, properties (i) and (J) in Section 4.2 and Proposition 8.2 we have

$$x \in \partial \sigma_{\mathcal{H}}(y) \qquad \Leftrightarrow \qquad y \in \partial \sigma^*_{\mathcal{H}}(x) = \partial \mathbb{I}_{\mathcal{H}}(x) = N_{\mathcal{H}}(x) \subseteq \sum_{i=1}^{I} \mathbb{R}_+ q^i.$$

Let $y \in \sum_{i=1}^{I} \mathbb{R}_+ q^i$. Then there exist $\lambda_1, \ldots, \lambda_I \geq 0$ such that $y = \sum_{i=1}^{I} \lambda_i q^i$. Hence, we obtain

$$\sigma_{\mathcal{H}}(y) = \sup_{x \in \mathcal{H}} \langle x, y \rangle = \sup_{x \in \mathcal{H}} \left\langle x, \sum_{i=1}^{I} \lambda_i q^i \right\rangle = \sup_{x \in \mathcal{H}} \sum_{i=1}^{I} \lambda_i \left\langle x, q^i \right\rangle \leq \sum_{i=1}^{I} \lambda_i c_i < +\infty,$$

i.e., $y \in \operatorname{dom} \sigma_{\mathcal{H}}$. □

Proposition 8.5. Let $\mathcal{H} = \bigcap_{i=1}^{I} \mathcal{H}_i$, $\mathcal{H}_i = \left\{x \in \mathbb{R}^n \middle| \left\langle x, q^i \right\rangle \leq c_i\right\}$ with q^i, c_i, for $i = 1, \ldots, I$, as defined in (8.1). Moreover, define for $x \in \mathcal{H}$ the index set $\mathfrak{I}_x := \left\{i \in \{1, \ldots, I\} \middle| \left\langle x, q^i \right\rangle = c_i\right\}$. Then for $(y, \overline{y}^0) \in \operatorname{dom} g^*_{\mathcal{H}} \times \operatorname{dom} \sigma_{\mathcal{H}}$ with $y - \overline{y}^0 \in \operatorname{dom} g^*$ we have

$$\partial g^*_{\mathcal{H}}(y) \supseteq \partial g^*(y - \overline{y}^0) \cap \partial \sigma_{\mathcal{H}}(\overline{y}^0), \tag{8.4}$$

where

$$\partial \sigma_{\mathcal{H}}(\overline{y}^0) = \begin{cases} \left\{ x \in \operatorname{bd} \mathcal{H} \middle| \overline{y}^0 \in \displaystyle\sum_{i \in \mathfrak{I}_x} \mathbb{R}_+ q^i \right\}, & \text{if } \overline{y}^0 \in \operatorname{dom} \sigma_{\mathcal{H}} \backslash \{0\}, \\ \mathcal{H}, & \text{if } \overline{y}^0 = 0, \\ \emptyset, & \text{if } \overline{y}^0 \notin \operatorname{dom} \sigma_{\mathcal{H}}, \end{cases}$$

and $\partial g^*(y - \overline{y}^0)$ as given in Section 5.2.

Proof. Let $(y, \overline{y}^0) \in \operatorname{dom} g^*_{\mathcal{H}} \times \operatorname{dom} \sigma_{\mathcal{H}}$ such that $y - \overline{y}^0 \in \operatorname{dom} g^*$ and $\partial g^*(y - \overline{y}^0) \cap \partial \sigma_{\mathcal{H}}(\overline{y}^0) \neq \emptyset$. Then inclusion (8.4) holds by Theorem 4.16 and Proposition 8.1.

Let $\overline{y}^0 = 0$. Then, by property (H) in Section 4.2, it holds

$$\partial \sigma_{\mathcal{H}}(\overline{y}^0) = \left\{x \in \mathcal{H} \middle| \, \sigma_{\mathcal{H}}(\overline{y}^0) = \left\langle x, \overline{y}^0 \right\rangle\right\} = \mathcal{H}.$$

Let $\overline{y}^0 \in \operatorname{dom} \sigma_{\mathcal{H}} \setminus \{0\}$. By Remark 4.7, property (H) in Section 4.2 and Proposition 8.2 it follows

$$\partial\sigma_{\mathcal{H}}(\overline{y}^0) = \left\{x \in \mathcal{H} \middle| \, \sigma_{\mathcal{H}}(\overline{y}^0) = \langle x, \overline{y}^0\rangle\right\} = \left\{x \in \mathcal{H} \middle| \, \overline{y}^0 \in N_{\mathcal{H}}(x)\right\} = \left\{x \in \operatorname{bd}\mathcal{H} \middle| \, \overline{y}^0 \in \sum_{i\in\mathfrak{I}_x} \mathbb{R}_+ q^i\right\}.$$

Let $\overline{y}^0 \notin \operatorname{dom} \sigma_{\mathcal{H}}$, then $\partial\sigma_{\mathcal{H}}(\overline{y}^0) = \emptyset$. □

Theorem 8.6. For each pair $(y, \overline{y}^0) \in \operatorname{dom} g^*_{\mathcal{H}} \times \operatorname{dom} \sigma_{\mathcal{H}}$ with $y - \overline{y}^0 \in \operatorname{dom} g^*$ the following statements are equivalent:

1. $\partial g^*(y - \overline{y}^0) \cap \partial\sigma_{\mathcal{H}}(\overline{y}^0) \neq \emptyset$,
2. $g^*_{\mathcal{H}}$ is exact at $y := (y - \overline{y}^0) + \overline{y}^0$,
3. $\partial g^*(y - \overline{y}^0) \cap \partial\sigma_{\mathcal{H}}(\overline{y}^0) = \partial g^*_{\mathcal{H}}(y)$.

Moreover, for each feasible $y \in \operatorname{dom} g^*_{\mathcal{H}}$ there exists $\overline{y}^0 \in \operatorname{dom} \sigma_{\mathcal{H}}$ such that these statements hold.

Proof. 1. ⇒ 2. ⇒ 3. Those implications follow directly by Theorem 4.16.

3. ⇒ 1. The assertion follows since $\partial g^*_{\mathcal{H}}(y) \neq \emptyset$ for all $y \in \operatorname{dom} g^*_{\mathcal{H}}$, see Proposition 8.3.

The existence of such a pair $(y, \overline{y}^0) \in \operatorname{dom} g^*_{\mathcal{H}} \times \operatorname{dom} \sigma_{\mathcal{H}}$ follows from Theorem 4.15. □

Corollary 8.7. If $\partial g^*(y) \cap \mathcal{H} \neq \emptyset$, then it holds $\partial g^*_{\mathcal{H}}(y) = \partial g^*(y) \cap \mathcal{H}$.

8.2 Existence of Finite Optimal Solutions

In general, constraints influence the finiteness of the optimal objective value of the location problem. A necessary and sufficient criterion for the finiteness of the optimal objective value of the dual pair of optimization problems $(P_{\mathcal{H}})$ and $(D_{\mathcal{H}})$, under consideration of the polyhedral constraints as given in (8.1), is the following:

Theorem 8.8. **(Finiteness Criterion)**
The optimal objective value of the dual pair of optimization problems $(P_{\mathcal{H}})$ and $(D_{\mathcal{H}})$ is finite, i.e., $\inf_{x\in\mathbb{R}^n} \{g_{\mathcal{H}}(x) - h(x)\} = \inf_{y\in\mathbb{R}^n} \{h^*(y) - g^*_{\mathcal{H}}(y)\} \in \mathbb{R}$, if and only if

$$\sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}^*_m \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}^*_m + \sum_{i=1}^{I} \mathbb{R}_+ q^i.$$

Proof. By Remark 4.14 and Proposition 8.1 we have $\operatorname{dom} g^*_{\mathcal{H}} = \operatorname{dom} g^* + \operatorname{dom} \sigma_{\mathcal{H}}$ $(\neq \emptyset)$. Taking into account the domains of g^*, h^* and $\sigma_{\mathcal{H}}$, as given in (5.5) and Proposition 8.4, the assertion follows from Proposition 5.1 analogously as in Theorem 5.2. □

Remark 8.9. In case of one half-space constraint, i.e., for $\mathcal{H} = \{x \in \mathbb{R}^n | \ \langle x, q^1 \rangle = c_1\}$ with q^1, c_1 as defined in (8.1) it holds: If $\operatorname{dom} h^*$ and $\operatorname{dom} g^*$ are symmetric with respect to the origin then

$$\operatorname{dom} h^* \subseteq \operatorname{dom} g^*_{\mathcal{H}} \Leftrightarrow \operatorname{dom} h^* \subseteq \operatorname{dom} g^*.$$

If the number I of half-space constraints is greater than one, then the assertion may be wrong.

Proof. Let $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*_{\mathcal{H}}$ and $y \in \operatorname{dom} h^*$. Since $\operatorname{dom} h^*$ is symmetric we also have $-y \in \operatorname{dom} h^*$ and $y, -y \in \operatorname{dom} g^*_{\mathcal{H}}$.

Since $\operatorname{dom} g^*_{\mathcal{H}} = \operatorname{dom} g^* + \operatorname{dom} \mathbb{R}_+ q^1$, there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$y - \lambda_1 q^1 \in \operatorname{dom} g^*, \qquad -y - \lambda_2 q^1 \in \operatorname{dom} g^*.$$

Due to the symmetry of $\operatorname{dom} g^*$ we have

$$-y + \lambda_1 q^1 \in \operatorname{dom} g^*, \qquad y + \lambda_2 q^1 \in \operatorname{dom} g^*.$$

Hence, the line segment $[y - \lambda_1 q^1, y + \lambda_2 q^1]$ is a subset of the convex set $\operatorname{dom} g^*$, and $y \in \operatorname{dom} g^*$.

Let $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$. By $0 \in \mathbb{R}_+ q^1$ it follows that $\operatorname{dom} g^* \subseteq \operatorname{dom} g^*_{\mathcal{H}}$ and the assertion holds. □

If $\mathcal{H}$ is a bounded set the finiteness is even assured since then $\operatorname{dom} \sigma_{\mathcal{H}} = \mathbb{R}^n$ and hence $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*_{\mathcal{H}} = \operatorname{dom} g^* + \operatorname{dom} \sigma_{\mathcal{H}} = \mathbb{R}^n$. An example is illustrated in Figure 8.1d in Section 8.4.

Moreover, the constraints $\mathcal{H}_i = \{x \in \mathbb{R}^n | \ \langle q^i, x \rangle \leq c_i\}$, for $i = 1, \dots, I$, in the constrained location problem $(P_{\mathcal{H}})$, provoke into each direction $q \in \sum_{i=1}^{I} \mathbb{R}_+ q^i$ the boundedness of $\operatorname{dom} g_{\mathcal{H}}$ and the unboundedness of $\operatorname{dom} g^*_{\mathcal{H}}$.

8.3 Optimality Conditions and Discretization Results

Using the following duality statements we are able to determine the set of optimal solutions of $(D_{\mathcal{H}})$ out of the set of optimal solutions of $(P_{\mathcal{H}})$ and vice versa.

Corollary 8.10. (**Sufficient Optimality Condition for Dual Solutions**)
Let $x \in \bigcap_{m=1}^{M} [\underline{a}^m + N_{\underline{w}_m \underline{B}^*_m}(\underline{y}^m)]$ be an optimal solution of the primal problem $(P_{\mathcal{H}})$. Then $\sum_{m=1}^{M} \underline{y}^m$ is an optimal solution of the dual problem $(D_{\mathcal{H}})$.

Proof. This result follows analogously to Corollary 5.12, since the function h and its conjugate h^* are not influenced by the constraints $\mathcal{H}$. □

Corollary 8.11. (**Sufficient Optimality Condition for Primal Solutions**)
Let y be an optimal solution of the dual problem $(D_{\mathcal{H}})$ such that $g^*_{\mathcal{H}}$ is exact at $y = (y - \overline{y}^0) + \overline{y}^0$, $(y - \overline{y}^0, \overline{y}^0) \in \operatorname{dom} g^* \times \operatorname{dom} \sigma_{\mathcal{H}}$. Then each $x \in \partial g^*(y - \overline{y}^0) \cap \partial \sigma_{\mathcal{H}}(\overline{y}^0)$ is optimal for the primal problem $(P_{\mathcal{H}})$.

Proof. Let y be an optimal solution of the dual problem ($D_{\mathcal{H}}$) and assume that g^* is exact at $y = (y - \overline{y}^0) + \overline{y}^0$. Then, by Theorem 8.6, we know that

$$\partial g^*(y - \overline{y}^0) \,\cap\, \partial \sigma_{\mathcal{H}}(\overline{y}^0) = \partial g^*_{\mathcal{H}}(y),$$

which is a non-empty set by Proposition 8.3. The assertion follows by Remark 5.14. □

By applying Remark 5.14, Corollary 8.10 and Corollary 8.11 we are able to determine the complete set $\mathcal{X}_{\mathcal{H}}$ of primal optimal solutions when having found the set $\mathcal{Y}_{\mathcal{H}}$ of dual optimal solutions and vice versa.

Definition 8.12. For $x \in \mathcal{H}$ and $y \in \operatorname{dom} g^*_{\mathcal{H}}$ we call the subdifferentials $\partial g_{\mathcal{H}}(x)$ and $\partial g^*_{\mathcal{H}}(y)$ *primal and dual constrained elementary convex sets w.r.t. attraction*. The *primal and dual constrained (construction) grids* are defined by

$$\overline{G}^{\mathcal{H}} := \bigcup_{\left\{y \in \operatorname{dom} g^*_{\mathcal{H}} \,\middle|\, \dim \partial g^*_{\mathcal{H}}(y) = 1\right\}} \partial g^*_{\mathcal{H}}(y), \qquad \overline{G}^{\mathcal{H}}_D := \bigcup_{\{x \in \mathcal{H} \mid \dim \partial g_{\mathcal{H}}(x) = 1\}} \partial g_{\mathcal{H}}(x).$$

A point $x \in \mathbb{R}^n$ is called *primal constrained grid point* w.r.t. attraction, if x is an extreme point of a primal constrained elementary convex set w.r.t. attraction. We denote by $\overline{\mathcal{I}}^{\mathcal{H}}$ the set of all primal constrained grid points w.r.t. attraction.

A point $y \in \mathbb{R}^n$ is called *dual constrained grid point* w.r.t. attraction, if y is an extreme point of a dual constrained elementary convex set w.r.t. attraction. We denote by $\overline{\mathcal{I}}^{\mathcal{H}}_D$ the set of all dual constrained grid points w.r.t. attraction.

Note that primal and dual elementary convex sets w.r.t. repulsion in case of convex polyhedral constraints coincide with the subdifferentials $\partial h(x)$ and $\partial h^*(y)$, as given in Section 5.2, since the functions h and h^* are not influenced by the constraints.

Theorem 8.13. (Geometric Duality Map for the Constrained Case))
Let $\overline{\mathfrak{C}}_{\mathcal{H}} := \{\partial g^*_{\mathcal{H}}(y) \mid y \in \operatorname{dom} g^*_{\mathcal{H}}\} \subseteq 2^{\mathbb{R}^n}$ and $\overline{\mathfrak{D}}_{\mathcal{H}} := \{\partial g_{\mathcal{H}}(x) \mid x \in \operatorname{dom} g_{\mathcal{H}}\} \subseteq 2^{\mathbb{R}^n}$. The function $\overline{\Phi}^*_{\mathcal{H}} : \overline{\mathfrak{C}}_{\mathcal{H}} \to \overline{\mathfrak{D}}_{\mathcal{H}}$ defined by

$$\overline{\Phi}^*_{\mathcal{H}}(\partial g^*_{\mathcal{H}}(y)) := \bigcap_{x \in \partial g^*_{\mathcal{H}}(y)} \partial g_{\mathcal{H}}(x)$$

provides an inclusion-reversing one-to-one map between the primal and dual elementary convex sets such that

$$\dim \partial g^*_{\mathcal{H}}(y) + \dim \overline{\Phi}^*_{\mathcal{H}}(\partial g^*_{\mathcal{H}}(y)) = n.$$

Proof. The assertion follows analogously as for Theorem 6.4 since $g_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R}$ is a polyhedral convex function. □

An example in $\mathbb{R}^2$ is illustrated in Figure 8.1, where $\dim \partial g_{\mathcal{H}}(x) + \dim \partial g^*_{\mathcal{H}}(y) = 2$ for each pair of assigned elementary convex sets.

The discretization results for an unconstrained location problem (P) with obnoxious facilities, given in Corollaries 5.15 and 5.16, can analogously be formulated for the more general case of a constrained location problem $(P_{\mathcal{H}})$ with obnoxious facilities.

Corollary 8.14. **(Discretization Result for the Primal Problem $(P_{\mathcal{H}})$)**
The set $\overline{\mathcal{I}}^{\mathcal{H}}$ of primal grid points in $\overline{G}^{\mathcal{H}}$ w.r.t. attraction is a finite dominating set for the optimal points of the location problem $(P_{\mathcal{H}})$, i.e., $\overline{\mathcal{I}}^{\mathcal{H}} \cap \mathcal{X}_{\mathcal{H}} \neq \emptyset$, where $\mathcal{X}_{\mathcal{H}}$ is the set of minimizers of $(P_{\mathcal{H}})$.

Proof. By Remark 5.14 we have $\mathcal{X}_{\mathcal{H}} = \bigcup_{y \in \mathcal{Y}_{\mathcal{H}}} \partial g^*_{\mathcal{H}}(y)$, where the subdifferentials $\partial g^*_{\mathcal{H}}(y)$ are closed polyhedral elementary convex sets. Hence, we have $\operatorname{ext}(\partial g^*_{\mathcal{H}}(y)) \subseteq \overline{\mathcal{I}}^{\mathcal{H}} \cap \mathcal{X}_{\mathcal{H}}$ for all $y \in \mathcal{Y}_{\mathcal{H}}$. □

Corollary 8.15. **(Discretization Result for the Dual Problem $(D_{\mathcal{H}})$)**
The set $\underline{\mathcal{I}}_D$ of all dual grid points in $\underline{G}_D$ w.r.t. repulsion is a finite dominating set for the optimal points of the dual problem $(D_{\mathcal{H}})$, i.e., $\underline{\mathcal{I}}_D \cap \mathcal{Y}_{\mathcal{H}} \neq \emptyset$, where $\mathcal{Y}_{\mathcal{H}}$ is the set of minimizers of (D).

Proof. By Remark 5.14 we have $\mathcal{Y}_{\mathcal{H}} = \bigcup_{x \in \mathcal{X}_{\mathcal{H}}} \partial h(x)$, where the subdifferentials $\partial h(x)$ are closed polyhedral elementary convex sets. Hence, we have $\operatorname{ext}(\partial h(x)) \subseteq \underline{\mathcal{I}} \cap \mathcal{Y}_{\mathcal{H}}$ for all $x \in \mathcal{X}_{\mathcal{H}}$. □

8.4 Example

In order to illustrate the influence of linear constraints we give some examples in $\mathbb{R}^2$. In Figure 8.1 we marked for Example 4.18 the different kinds of primal and dual elementary convex sets $\partial g^*_{\mathcal{H}}(y)$ and $\partial g_{\mathcal{H}}(x)$ by different colors and line styles. We distinguish the following categories of elementary convex sets:

1. $\partial g^*(y) \cap \mathcal{H} \neq \emptyset$, i.e., $\partial g^*_{\mathcal{H}}(y) = \partial g^*(y) \cap \mathcal{H}$
   a) If $\partial g^*(y) \subseteq \mathcal{H}$, then the primal elementary convex set $\partial g^*_{\mathcal{H}}(y)$ is not influenced by the constraints set $\mathcal{H}$, i.e., $\partial g^*_{\mathcal{H}}(y) = \partial g^*(y)$.
      i. $\dim \partial g^*(y) = 0$ (green)
      ii. $\dim \partial g^*(y) = 1$ (some continuous lines)
      iii. $\dim \partial g^*(y) = 2$ (red).
   b) If $\partial g^*(y) \nsubseteq \mathcal{H}$, then the primal elementary convex set $\partial g^*_{\mathcal{H}}(y)$ is influenced by the constraints set $\mathcal{H}$, i.e., $\partial g^*_{\mathcal{H}}(y) = \partial g^*(y) \cap \mathcal{H} \neq \partial g^*(y)$.
      i. $\dim \partial g^*(y) = 1$ (some continuous lines)
      ii. $\dim \partial g^*(y) = 2$ (blue).

2. $\partial g^*(y) \cap \mathcal{H} = \emptyset$
   a) $\dim \partial g^*_{\mathcal{H}}(y) = 0$
      i. $\partial g^*_{\mathcal{H}}(y) \subseteq \operatorname{bd} \mathcal{H} \cap \overline{G}$ (purple)
      ii. $\partial g^*_{\mathcal{H}}(y) \subseteq \operatorname{ext}(\mathcal{H})$ (orange).
   b) $\dim \partial g^*_{\mathcal{H}}(y) = 1$

      $\partial g^*_{\mathcal{H}}(y) \subseteq \operatorname{bd} \mathcal{H}$ (dashed lines).

The assigned dual elementary convex sets, see Theorem 8.13, are identified in the same manner as their respective primal elementary convex sets.

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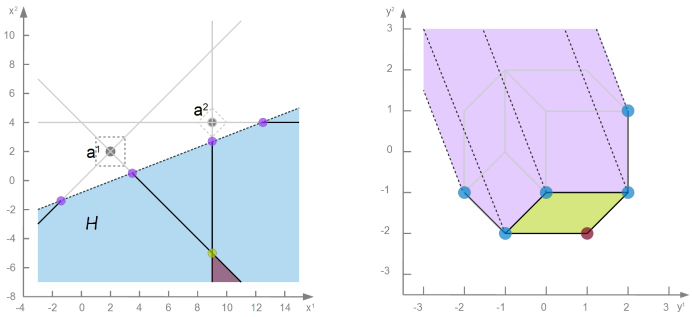
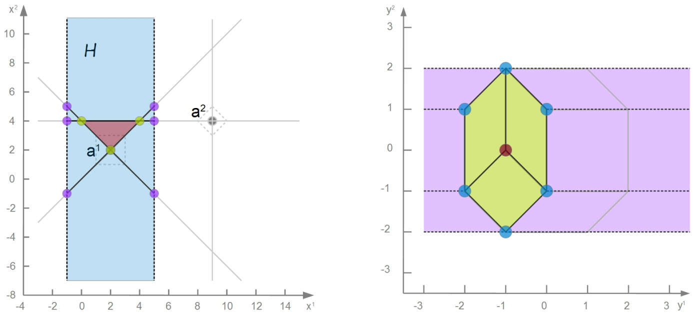


$$\mathcal{H}_1 = \left\{ x \in \mathbb{R}^n \,\middle|\, \left\langle (-2,5)^T, x \right\rangle \le -4 \right\},$$

$$\operatorname{dom} \sigma_{\mathcal{H}} = \mathbb{R}_+ \left\{ (-2,5)^T \right\}.$$

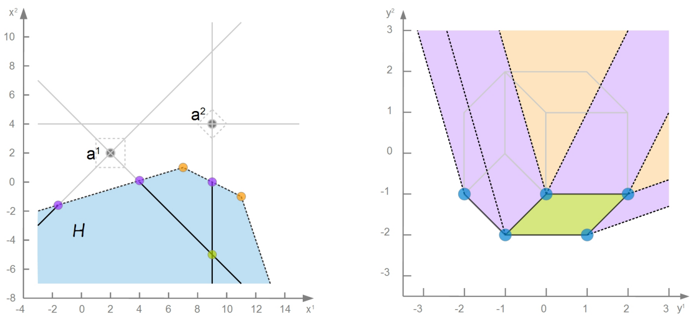
(a) One single half-space constraint.



$$\mathcal{H}_1 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (1,0)^T, x \right\rangle \le 5 \right\},$$
$$\mathcal{H}_2 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (-1,0)^T, x \right\rangle \le 1 \right\},$$

$$\operatorname{dom} \sigma_{\mathcal{H}} = \mathbb{R} \times \{0\}.$$

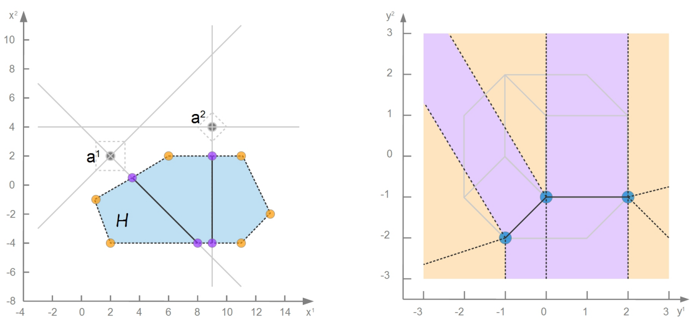
(b) Two parallel half-space constraints.



$$\mathcal{H}_1 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (-3,10)^T, x \right\rangle \le -11 \right\},$$
$$\mathcal{H}_2 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (2,4)^T, x \right\rangle \le 18 \right\},$$
$$\mathcal{H}_3 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (3,1)^T, x \right\rangle \le 32 \right\},$$

$$\operatorname{dom} \sigma_{\mathcal{H}} = \mathbb{R}_+ \left\{ (-3,10)^T \right\} + \mathbb{R}_+ \left\{ (2,4)^T \right\} + \mathbb{R}_+ \left\{ (3,1)^T \right\}.$$

(c) Three half-space constraints.



$$\mathcal{H}_1 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (-4,7)^T, x \right\rangle \le -10 \right\},$$
$$\mathcal{H}_2 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (0,1)^T, x \right\rangle \le 2 \right\},$$
$$\mathcal{H}_3 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (2,1)^T, x \right\rangle \le 24 \right\},$$
$$\mathcal{H}_4 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (1,-1)^T, x \right\rangle \le 15 \right\},$$
$$\mathcal{H}_5 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (0,-1)^T, x \right\rangle \le 4 \right\},$$
$$\mathcal{H}_6 = \left\{ x \in \mathbb{R}^2 \,\middle|\, \left\langle (-3,-1)^T, x \right\rangle \le -2 \right\},$$

$$\operatorname{dom} \sigma_{\mathcal{H}} = \mathbb{R}^2.$$

(d) Bounded closed polyhedral constraints.

Figure 8.1: Primal and dual grids w.r.t. attraction for Example 4.18 under consideration of different polyhedral constraints.

Chapter 9

An Algorithm for Solving Location Problems with Obnoxious Facilities

In this chapter we formulate a primal and a dual algorithm for solving the problem of locating an obnoxious facility.

In Sections 9.1 and 9.2 we present different methods for determining the sets $\overline{\mathcal{I}}$ and $\underline{\mathcal{I}}_D$ of primal grid points w.r.t. attraction and dual grid points w.r.t. repulsion. Based on the Corollaries 5.15 and 5.16, we use those grid points in Section 9.3 in order to formulate a primal and a dual algorithm for solving the dual pair of optimization problems (P) and (D). The algorithms are illustrated by an example in Section 9.4. Moreover, the special case of no repulsion is shortly discussed in Section 9.5. Finally, algorithms for the more general case of the constrained location problem $(P_{\mathcal{H}})$ and its dual problem $(D_{\mathcal{H}})$ are presented in Section 9.6.

9.1 Determination of Grid Points w.r.t. Attraction

In this section we present different methods for determining the set $\overline{\mathcal{I}}$ of primal grid points w.r.t. attraction. These grid points are determined in step 2 and used in in step 3 of the primal Algorithm 9.2 for solving the optimization problem (P).

Application of Benson's Algorithm for Determining $\overline{\mathcal{I}}$ and $\overline{\mathcal{I}}_D$

Although we are considering a scalar optimization problem it is possible to apply methods from the field of linear vector optimization in order to determine the primal and dual grid points w.r.t. attraction. In particular, we apply the well known Benson algorithm [5, 51, 76]. We consider the objectives $\overline{P}, \overline{D}^*$, feasible sets $\overline{S}, \overline{T}$ and ordering cones $\overline{C}, K$ as introduced in Chapter 7. Then the epigraphs of the functions g and g^*, as given by (3.1) and (5.3) in the location problem (P) and its dual problem (D), can be written as

$$\operatorname{epi} g = \overline{P}[\overline{S}] + \overline{C} \qquad \text{and} \qquad \operatorname{epi} g^* = -\left(\overline{D}^*[\overline{T}] - K\right),$$

i.e., the upper and the lower image of the dual pair of linear vector optimization problems

$$\overline{C}-\min \overline{P}[\overline{S}], \tag{$\overline{LVOP}$}$$

$$K-\max \overline{D}^*[\overline{T}]. \tag{$\overline{LVOD}$}$$

Minimizing the linear vector optimization problem ($\overline{LVOP}$) and maximizing its geometric dual problem ($\overline{LVOD}$) means to find the exposed faces, see Definition 4.2, and the extreme points of $\operatorname{epi} g$ and $\operatorname{epi} g^*$. The grid points $x \in \overline{\mathcal{I}}$ and $y \in \overline{\mathcal{I}}_D$ then are given as the projections of those extreme points onto $\mathbb{R}^n$, see Remark 6.3, such that

$$\overline{\mathcal{I}} := \left\{x \in \mathbb{R}^n \,\middle|\, (x, g(x)) \in \operatorname{ext}\left(\overline{P}[\overline{S}] + \overline{C}\right)\right\},$$
$$\overline{\mathcal{I}}_D := \left\{y \in \operatorname{dom} g^* \,\middle|\, (y, g^*(y)) \in \operatorname{ext}\left(-(\overline{D}^*[\overline{T}] - K)\right)\right\}.$$

Note that the set of primal and dual grid points w.r.t. repulsion may be determined completely analogously.

Determining $\overline{\mathcal{I}}$ by Intersection of Fundamental Directions

The half-lines $\overline{a}^m + \mathbb{R}_+ e$ for all $e \in \operatorname{ext}(\overline{B}_m)$, $m \in \{1, \ldots, \overline{M}\}$, are called fundamental directions and generate the primal grid w.r.t. attraction. The set $\overline{\mathcal{I}}$ of grid points coincides with the set of intersection points of all fundamental directions, such that

$$\overline{\mathcal{I}} := \bigcup_{(m_1,m_2)\in\{1,\ldots,\overline{M}\}^2} \Big\{ \left(\overline{a}^{m_1} + \mathbb{R}_+ e^1\right) \cap \left(\overline{a}^{m_2} + \mathbb{R}_+ e^2\right) \Big| \; (e^1, e^2) \in \operatorname{ext}(\overline{B}_{m_1}) \times \operatorname{ext}(\overline{B}_{m_2}),\, e^2 \notin \mathbb{R}_+ e^1 \Big\}.$$

Determining $\overline{\mathcal{I}}$ in the Special Case of Manhattan Distances

In the special case that the assigned distance function for each attracting facility in the location problem (P) is the Manhattan norm (4.3), i.e.,

$$\overline{B}_1 = \ldots = \overline{B}_{\overline{M}} = B_{\text{Manhattan}} = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n |x_i| \leq 1 \right\},$$

we directly obtain the set $\overline{\mathcal{I}}$ of primal grid points w.r.t. attraction as

$$\overline{\mathcal{I}} := \left\{\overline{a}_1^1, \ldots, \overline{a}_1^{\overline{M}}\right\} \times \cdots \times \left\{\overline{a}_n^1, \ldots, \overline{a}_n^{\overline{M}}\right\}.$$

9.2 Determination of Grid Points w.r.t. Repulsion

In this section we present different methods for determining the dual grid points w.r.t. repulsion. These grid points are determined in step 2 and used in in step 3 of the dual Algorithm 9.3 for solving the dual pair of optimization problems (P) and (D).

Application of Benson's Algorithm for Determining $\underline{\mathcal{I}}$ and $\underline{\mathcal{I}}_D$

Consider the objectives $\underline{P}, \underline{D}^*$, feasible sets $\underline{S}, \underline{T}$ and ordering cones $\underline{C}, K$ as introduced in Chapter 7. The epigraphs of the functions h and h^*, as given by (3.2) and (5.4) in the location problem (P) and its dual problem (D), can be written as

$$\operatorname{epi} h = \underline{P}[\underline{S}] + \underline{C} \qquad \text{and} \qquad \operatorname{epi} h^* = -\left(\underline{D}^*[\underline{T}] - K\right),$$

i.e., the upper and the lower image of a dual pair of linear vector optimization problems

$$\underline{C}-\min \underline{P}[\underline{S}], \tag{LVOP}$$

$$K-\max \underline{D}^*[\underline{T}]. \tag{LVOD}$$

Minimizing the linear vector optimization problem (LVOP) and maximizing its geometric dual problem LVOD means to find the exposed faces and the extreme points of $\operatorname{epi} h$ and $\operatorname{epi} h^*$. The grid points $x \in \underline{\mathcal{I}}$ and $y \in \underline{\mathcal{I}}_D$ then are given as the projections of those extreme points onto $\mathbb{R}^n$, see Remark 6.3, such that

$$\underline{\mathcal{I}} := \{x \in \mathbb{R}^n |\, (x, h(x)) \in \operatorname{ext}(\underline{P}[\underline{S}] + \underline{C})\},$$
$$\underline{\mathcal{I}}_D := \{y \in \operatorname{dom} h^* |\, (y, h^*(y)) \in \operatorname{ext}(-(\underline{D}^*[\underline{T}] - K))\}.$$

Determining $\underline{\mathcal{I}}_D$ in the Special Case of Manhattan Distances

Consider the special case that the corresponding distance function for each repulsive facility in the location problem (P) is the Manhattan norm (4.3), i.e.,

$$\underline{B}_1 = \ldots = \underline{B}_M = B_{\text{Manhattan}} = \left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} |x_i| \leq 1 \right\}.$$

By Remark 5.11 the set $\underline{\mathcal{I}}_D$ of dual grid points w.r.t. repulsion is given by

$$\underline{\mathcal{I}}_D = \{y \in \operatorname{dom} h^* |\, \exists x \in \mathbb{R}^n : \{y\} = \partial h(x)\}.$$

Those dual elementary convex sets are defined (see Definition 5.10) by the subdifferentials

$$\partial h(x) = \sum_{m=1}^{M} \underset{\underline{y}^m \in \underline{w}_m \underline{B}_m^*}{\operatorname{argmax}} \left\langle x - \underline{a}^m, \underline{y}^m \right\rangle.$$

In case of Manhattan distances the weighted dual unit balls $\underline{w}_m \underline{B}_m^*$ can be written as

$$\underline{w}_m \underline{B}_m^* = [-\underline{w}_m, \underline{w}_m] \times \ldots \times [-\underline{w}_m, \underline{w}_m] \subseteq \mathbb{R}^n,$$

for all $m = 1, \ldots, \underline{M}$, such that we obtain

$$\partial h(x) = \sum_{m=1}^{\underline{M}} \sum_{i=1}^{n} \underset{\underline{y}_i^m \in [-\underline{w}_m, \underline{w}_m]}{\operatorname{argmax}} (x_i - \underline{a}_i^m) \cdot \underline{y}_i^m.$$

Further, for $i = 1, 2, \ldots, n$ and $m = 1, \ldots, \underline{M}$ it holds

$$\underset{\underline{y}_i^m \in [-\underline{w}_m, \underline{w}_m]}{\operatorname{argmax}} (x_i - \underline{a}_i^m) \cdot \underline{y}_i^m = \begin{cases} \{-\underline{w}_m\}, & \text{if } x_i < \underline{a}_i^m, \\ [-\underline{w}_m, \underline{w}_m], & \text{if } x_i = \underline{a}_i^m, \\ \{\underline{w}_m\}, & \text{if } x_i > \underline{a}_i^m. \end{cases}$$

Remark 9.1. Note that we do not consider the case that $x_i = \underline{a}_i^m$ since otherwise $\dim \partial h(x) > 0$ in contradiction to $\{y\} = \partial h(x)$.

Let $x \in \mathbb{R}^n$ with $\partial h(x) = \{y\}$. We define for $i = 1, 2, \ldots, n$

$$D_i(x) := \{m \in \{1, \ldots, \underline{M}\} \,|\, \underline{a}_i^m < x_i\}, \qquad D_i^c(x) := \{1, \ldots, \underline{M}\} \setminus D_i(x).$$

If $D_i(x), D_i^c(x) \neq \emptyset$ then the i-th component of y is given by

$$y_i = \sum_{m \in D_i(x)} \underline{w}_m - \sum_{m \in D_i^c(x)} \underline{w}_m,$$
$$h_i^*(y_i) := \sum_{m \in D_i(x)} \underline{a}_i^m \underline{w}_m - \sum_{m \in D_i^c(x)} \underline{a}_i^m \underline{w}_m,$$

otherwise if $x_i < \min\left\{\underline{a}_i^1, \ldots, \underline{a}_i^{\underline{M}}\right\}$ then

$$y_i = -\sum_{m=1}^{\underline{M}} \underline{w}_m, \qquad h_i^*(y_i) := -\sum_{m=1}^{\underline{M}} \underline{a}_i^m \underline{w}_m,$$

and if $x_i > \max\left\{\underline{a}_i^1, \ldots, \underline{a}_i^{\underline{M}}\right\}$ then we have

$$y_i = \sum_{m=1}^{\underline{M}} \underline{w}_m, \qquad h_i^*(y_i) := \sum_{m=1}^{\underline{M}} \underline{a}_i^m \underline{w}_m.$$

For $y = (y_1, \ldots, y_n)^T \in \underline{\mathcal{I}}_D$ we obtain

$$h^*(y) = \sum_{i=1}^{n} h_i^*(y_i).$$

In Algorithm 9.1 these results are used for determining the set $\underline{\mathcal{I}}_D$ of dual grid points w.r.t. repulsion in the case of Manhattan distances for all repulsive facilities $\underline{a}^1, \ldots, \underline{a}^{\underline{M}}$.

Algorithm 9.1 (Determination of dual set $\underline{\mathcal{I}}_D$ of grid points w.r.t. repulsion in Case of Manhattan Norm)**.**

Input: $\underline{a}^m, \underline{w}_m$, $(m = 1, \ldots, \underline{M})$.

Output: $\underline{\mathcal{I}}_D$, $h^*[\underline{\mathcal{I}}_D]$.

1. For $i = 1, 2, \ldots, n$ and $\widehat{m} = 1, \ldots, \underline{M}$ determine

$$D_i(\widehat{m}) : = \left\{ m \in \{1, \ldots, \underline{M}\} \,\middle|\, \underline{a}_i^m \leq \underline{a}_i^{\widehat{m}} \right\}, \qquad D_i^c(\widehat{m}) := \{1, \ldots, \underline{M}\} \setminus D_i(\widehat{m}),$$

$$y_i(0) : = -\sum_{m=1}^{\underline{M}} \underline{w}_m,$$

$$y_i(\widehat{m}) : = \begin{cases} \displaystyle\sum_{m \in D_i(\widehat{m})} \underline{w}_m - \sum_{m \in D_i^c(\widehat{m})} \underline{w}_m, & \text{if } D_i^c(\widehat{m}) \neq \emptyset, \\ \displaystyle\sum_{m=1}^{\underline{M}} \underline{w}_m, & \text{if } D_i^c(\widehat{m}) = \emptyset, \end{cases}$$

$$h_i^*(0) : = -\sum_{m=1}^{\underline{M}} \underline{a}_i^m \underline{w}_m,$$

$$h_i^*(\widehat{m}) : = \begin{cases} \displaystyle\sum_{m \in D_i(\widehat{m})} \underline{a}_i^m \underline{w}_m - \sum_{m \in D_i^c(\widehat{m})} \underline{a}_i^m \underline{w}_m, & \text{if } D_i(\widehat{m}), D_i^c(\widehat{m}) \neq \emptyset, \\ \displaystyle\sum_{m=1}^{\underline{M}} \underline{a}_i^m \underline{w}_m, & \text{if } D_i^c(\widehat{m}) = \emptyset. \end{cases}$$

2. Determine

$$\underline{\mathcal{I}}_D = \bigcup_{\widehat{m}=0}^{\underline{M}} \{y_1(\widehat{m})\} \times \ldots \times \bigcup_{\widehat{m}=0}^{\underline{M}} \{y_n(\widehat{m})\} \subseteq \mathbb{R}^n.$$

3. For each $y := (y_1(\widehat{m}_1), \ldots, y_n(\widehat{m}_n)) \in \underline{\mathcal{I}}_D$, with $\widehat{m}_1, \ldots, \widehat{m}_n \in \{0, \ldots, \underline{M}\}$, set

$$h^*(y) = \sum_{i=1}^{n} h_i^*(\widehat{m}_i).$$

Example 9.2. Suppose that we have given the repulsion points $\underline{a}^1 = (1,1)$, $\underline{a}^2 = (2,3)$, $\underline{a}^3 = (3,2)$ and $\underline{a}^4 = (2,0)$ with corresponding weights $\underline{w}_1 = 2$, $\underline{w}_2 = 1$, $\underline{w}_3 = 2$ and $\underline{w}_4 = 3$. We obtain

$$\begin{aligned} D_1(1) &= \{1\}\,, & D_2(1) &= \{1,4\}\,, \\ D_1(2) &= \{1,2,4\}\,, & D_2(2) &= \{1,2,3,4\}\,, \\ D_1(3) &= \{1,2,3,4\}\,, & D_2(3) &= \{1,3,4\}\,, \\ D_1(4) &= \{1,2,4\}\,, & D_2(4) &= \{4\}\,, \end{aligned}$$

$$\begin{aligned} y_1(0) &= -(\underline{w}_1 + \underline{w}_2 + \underline{w}_3 + \underline{w}_4) = -8, & y_2(0) &= -(\underline{w}_1 + \underline{w}_2 + \underline{w}_3 + \underline{w}_4) = -8, \\ y_1(1) &= \underline{w}_1 - (\underline{w}_2 + \underline{w}_3 + \underline{w}_4) = -4, & y_2(1) &= (\underline{w}_1 + \underline{w}_4) - (\underline{w}_2 + \underline{w}_3) = 2, \\ y_1(2) &= (\underline{w}_1 + \underline{w}_2 + \underline{w}_4) - \underline{w}_3 = 4, & y_2(2) &= \underline{w}_1 + \underline{w}_2 + \underline{w}_3 + \underline{w}_4 = 8, \\ y_1(3) &= \underline{w}_1 + \underline{w}_2 + \underline{w}_3 + \underline{w}_4 = 8, & y_2(3) &= (\underline{w}_1 + \underline{w}_3 + \underline{w}_4) - \underline{w}_2 = 6, \\ y_1(4) &= (\underline{w}_1 + \underline{w}_2 + \underline{w}_4) - \underline{w}_3 = 4, & y_2(4) &= \underline{w}_4 - (\underline{w}_1 + \underline{w}_2 + \underline{w}_3) = -2, \end{aligned}$$

and hence $\underline{\mathcal{I}}_D = \{-8,-4,4,8\} \times \{-8,-2,2,6,8\}$.

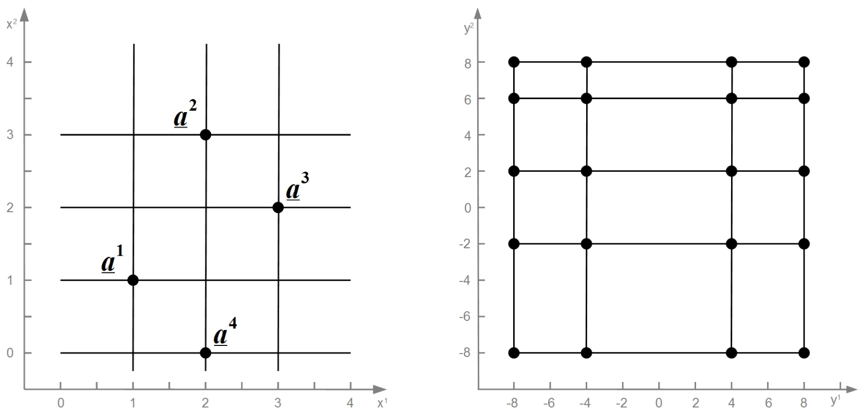


Figure 9.1: Primal and dual grid w.r.t. repulsion in case of Manhattan distances.

9.3 Solving the Location Problem with Obnoxious Facilities

Before starting the solving procedure, we apply Theorem 5.2 in step 1 of both, the primal Algorithm 9.2 and the dual Algorithm 9.3, in order to check the finiteness of the optimal objective value of the dual pair of optimization problems (P) and (D). In case of an infinite optimal objective value, i.e., $\alpha = \beta := -\infty$, the algorithm quits. The primal set of optimal solutions is $\mathcal{X} = \emptyset$ and the dual set of optimal solutions is $\mathcal{Y} := \operatorname{dom} h^* \setminus \operatorname{dom} g^*$.

The Primal Algorithm

In case of a finite optimal objective value, one of the methods, presented in Section 9.1, is applied in step 2 of Algorithm 9.2 for determining the set $\overline{\mathcal{I}}$ of primal grid points w.r.t. attraction as well as $g[\overline{\mathcal{I}}]$. In step 3, for each $x \in \overline{\mathcal{I}}$, the objective value $h(x)$ is determined. Finally, in step 4, the primal minimal objective value

$$\alpha := \min_{x \in \overline{\mathcal{I}}} \{g(x - h(x)\},$$

the primal set of optimal grid points

$$\mathcal{X} := \operatorname*{argmin}_{x \in \overline{\mathcal{I}}} \{g(x) - h(x)\},$$

and the dual set of optimal points

$$\mathcal{Y} := \bigcup_{x \in \mathcal{X}} \partial h(x) = \bigcup_{x \in \mathcal{X}} \sum_{m=1}^{M} \operatorname*{argmax}_{\underline{y}^m \in \operatorname{ext}(\underline{w}_m \underline{B}^*_m)} \langle x - \underline{a}^m, \underline{y}^m \rangle$$

are determined, based on Corollary 5.12 and Proposition 5.8. By Theorem 4.21 the dual optimal objective value β coincides with the primal one α.

The Dual Algorithm

In case of a finite optimal objective value, one of the methods presented in Section 9.2, is applied in step 2 of Algorithm 9.3 for determining the set $\underline{\mathcal{I}}_D$ of dual grid points w.r.t. repulsion. In step 3, for each $y \in \underline{\mathcal{I}}_D$ the objective value $g^*(y)$ is determined by solving the linear optimization problem

$$g^*(y) = \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}^*_M}(\overline{y}^m) \right] \middle| \sum_{m=1}^{\overline{M}} \overline{y}^m = y \right\}.$$

The tuples $\left(\overline{y}^1, \ldots, \overline{y}^{\overline{M}}\right)$, for which the infimum is attained, are used in step 4, where the dual minimal objective value

$$\beta := \min_{y \in \underline{\mathcal{I}}_D} \{h^*(y) - g^*(y)\},$$

the dual set of optimal grid points

$$\mathcal{Y} := \operatorname*{argmin}_{y \in \mathcal{I}_D} \{h^*(y) - g^*(y)\},$$

and the primal set of optimal points

$$\mathcal{X} := \bigcup_{y \in \mathcal{Y}} \partial g^*(y) = \bigcup_{y \in \mathcal{Y}} \bigcap_{m=1,\ldots,\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m)\right],$$

are determined, based on Corollary 5.13 and Theorem 5.7. By Theorem 4.21 the primal optimal objective value α coincides with the dual one β.

Algorithm 9.2 (Primal Algorithm for Solving the Location Problem (P))**.**

Input: $\overline{a}^m, \overline{w}_m, \overline{B}^*_m$, $(m = 1, \ldots, \overline{M})$; $\underline{a}^m, \underline{w}_m, \underline{B}^*_m$, $(m = 1, \ldots, \underline{M})$.

Output: The sets $\mathcal{X}$ and $\mathcal{Y}$ of optimal grid points of (P) and (D); optimal objective value α.

1. Check finiteness of the optimal objective value using the condition

$$\operatorname{dom} h^* = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}^*_m \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}^*_m = \operatorname{dom} g^*$$

   given in Theorem 5.2. If the condition is satisfied go on with 2. Otherwise set $\alpha := -\infty$, $\mathcal{Y} := \operatorname{dom} h^* \setminus \operatorname{dom} g^*$ and $\mathcal{X} := \emptyset$ and `STOP`.

2. Determine the set $\overline{\mathcal{I}}$ as well as $g[\overline{\mathcal{I}}]$ by using one of the methods presented in Section 9.1.

3. For all $x \in \overline{\mathcal{I}}$ determine $h(x) := \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m)$.

4. Determine the optimal objective value

$$\alpha := \min_{x \in \overline{\mathcal{I}}} \{g(x) - h(x)\},$$

   the primal set of optimal grid points

$$\mathcal{X} := \operatorname*{argmin}_{x \in \overline{\mathcal{I}}} \{g(x) - h(x)\},$$

   and the dual set of optimal points

$$\mathcal{Y} := \bigcup_{x \in \mathcal{X}} \partial h(x) = \bigcup_{x \in \mathcal{X}} \sum_{m=1}^{\underline{M}} \operatorname*{argmax}_{\underline{y}^m \in \operatorname{ext}(\underline{w}_m \underline{B}^*_m)} \left\langle x - \underline{a}^m, \underline{y}^m \right\rangle.$$

Algorithm 9.3 (Dual Algorithm for Solving the Location Problem (P))**.**

Input: $\overline{a}^m, \overline{w}_m, \overline{B}^*_m$, $(m = 1, \dots, \overline{M})$; $\underline{a}^m, \underline{w}_m, \underline{B}^*_m$, $(m = 1, \dots, \underline{M})$.

Output: The sets $\mathcal{X}$ and $\mathcal{Y}$ of optimal grid points of (P) and (D); optimal objective value β.

1. Check finiteness of the optimal objective value using the condition

$$\operatorname{dom} h^* = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}^*_m \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}^*_m = \operatorname{dom} g^*$$

given in Theorem 5.2. If the condition is satisfied go on with 2. Otherwise set $\beta := -\infty$, $\mathcal{Y} := \operatorname{dom} h^* \setminus \operatorname{dom} g^*$ and $\mathcal{X} := \emptyset$ and `STOP`.

2. Determine the set $\underline{\mathcal{I}}_D$ and $h^*[\underline{\mathcal{I}}_D]$ by using one of the methods presented in Section 9.2.

3. For all $y \in \underline{\mathcal{I}}_D$ determine the objective value $g^*(y)$ by solving the linear optimization problem

$$g^*(y) = \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}^*_M}(\overline{y}^m) \right] \middle| \sum_{m=1}^{\overline{M}} \overline{y}^m = y \right\}$$

and determine the set $\widehat{\mathcal{Y}}$ of tuple $(\overline{y}^1, \dots, \overline{y}^{\overline{M}})$ for which the infimum is attained.

4. Determine the optimal objective value

$$\beta := \min_{y \in \underline{\mathcal{I}}_D} \{ h^*(y) - g^*(y) \},$$

the dual set of optimal grid points

$$\mathcal{Y} := \operatorname*{argmin}_{y \in \underline{\mathcal{I}}_D} \{ h^*(y) - g^*(y) \},$$

and the primal set of optimal points

$$\mathcal{X} := \bigcup_{y \in \mathcal{Y}} \partial g^*(y) = \bigcup_{y \in \mathcal{Y}} \bigcap_{m=1,\dots,\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m) \right].$$

Obviously, the time complexity of Algorithm 9.2 strongly depends on the number of primal grid points w.r.t. attraction. The time complexity of Algorithm 9.3 strongly depends on the number of dual grid points w.r.t. repulsion. For each grid point, a linear optimization problem is to be solved in Algorithm 9.3. The complexity of each sub-problem depends on the number of attraction points. In Section 10.4 we present some examples in order to demonstrate the influence of the number of facilities (and hence grid points) on the running times.
Further considerations on duality based algorithms for practical usage, including results concerning complexity and efficiency, will be presented in forthcoming studies.

9.4 Example

We consider again the location problem introduced in Example 4.18, i.e., we have two attraction points $\overline{a}^1 = (2,2)^T$ and $\overline{a}^2 = (9,4)^T$ and one repulsion point $\underline{a}^1 = (7,1)^T$ with assigned primal and dual unit balls given by

$$\begin{aligned}
&\text{ext}(\overline{B}_1) = \left\{\pm(1,1)^T, \pm(1,-1)^T\right\}, &&\text{ext}(\overline{B}_1^*) = \left\{\pm(1,0)^T, \pm(0,1)^T\right\},\\
&\text{ext}(\overline{B}_2) = \left\{\pm(1,0)^T, \pm(0,1)^T\right\}, &&\text{ext}(\overline{B}_2^*) = \left\{\pm(1,1)^T, \pm(1,-1)^T\right\},\\
&\text{ext}(\underline{B}_1) = \left\{(-1,0)^T, (0,1)^T, (1,-1)^T\right\}, &&\text{ext}(\underline{B}_1^*) = \left\{(-1,1)^T, (-1,-2)^T, (2,1)^T\right\}.
\end{aligned}$$

Moreover, we choose the weights such that $\overline{w}^1 = \overline{w}^2 = \underline{w}^1 = 1$.

Application of the Primal Algorithm

1. The finiteness criteria $\text{dom}\, h^* \subseteq \text{dom}\, g^*$ is satisfied since the effective domains of h^* and g^* are given by

$$\begin{aligned}
\text{ext}(\text{dom}\, h^*) &= \text{ext}\left(\sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^*\right) = \left\{(-1,1)^T, (-1,-2)^T, (2,1)^T\right\},\\
\text{ext}(\text{dom}\, g^*) &= \text{ext}\left(\sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*\right) = [\{-1,1\} \times \{-2,2\}] \cup [\{-2,2\} \times \{-1,1\}].
\end{aligned}$$

2. The primal grid points $x \in \overline{\mathcal{I}}$ w.r.t. attraction and the corresponding objective values $g(x)$ are given by

$x \in \overline{\mathcal{I}}$	$(0,4)^T$	$(2,2)^T$	$(4,4)^T$	$(9,-5)^T$	$(9,4)^T$	$(9,9)^T$
$g(x)$	11	9	7	16	7	12

where by (4.3) and (4.5)

$$g(x) = \max\left\{\left|x_1 - \overline{a}_1^1\right|, \left|x_2 - \overline{a}_2^1\right|\right\} + \left|x_1 - \overline{a}_1^2\right| + \left|x_2 - \overline{a}_2^2\right|.$$

3. For each grid point $x \in \overline{\mathcal{I}}$ we determine the corresponding objective value $h(x)$ such that

$x \in \overline{\mathcal{I}}$	$(0,4)^T$	$(2,2)^T$	$(4,4)^T$	$(9,-5)^T$	$(9,4)^T$	$(9,9)^T$
$h(x)$	10	6	6	10	7	12

where by (4.9)

$$h(x) = \max_{y \in \underline{B}_1^*} \left\langle x - \underline{a}^1, y \right\rangle.$$

The corresponding primal elementary convex sets $\partial h(x)$ are illustrated in Figure 9.2.

4. Obviously, the optimal objective value is

$$\alpha = \min_{x \in \overline{\mathcal{I}}} \{g(x) - h(x)\} = \min\{1, 3, 1, 6, 0, 0\} = 0,$$

and the set of optimal primal grid points is

$$\mathcal{X} = \operatorname*{argmin}_{x \in \overline{\mathcal{I}}} \{g(x) - h(x)\} = \left\{(9,4)^T, (9,9)^T\right\}.$$

The set of optimal dual solutions results as

$$\mathcal{Y} = \bigcup_{x \in \mathcal{X}} \partial h(x) = \operatorname*{argmax}_{y \in \underline{B}_1^*} \left\langle (9,4)^T - \underline{a}^1, y \right\rangle \cup \operatorname*{argmax}_{y \in \underline{B}_1^*} \left\langle (9,9)^T - \underline{a}^1, y \right\rangle = \left\{(2,1)^T\right\}.$$

Application of the Dual Algorithm

1. The finiteness criteria $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$ is satisfied since the effective domains of h^* and g^* are given by

$$\operatorname{ext}(\operatorname{dom} h^*) = \operatorname{ext}\left(\sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}_m^*\right) = \left\{(-1,1)^T, (-1,-2)^T, (2,1)^T\right\},$$

$$\operatorname{ext}(\operatorname{dom} g^*) = \operatorname{ext}\left(\sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}_m^*\right) = [\{-1,1\} \times \{-2,2\}] \cup [\{-2,2\} \times \{-1,1\}].$$

2. Obviously, the set of primal grid points $\underline{\mathcal{I}}$ w.r.t. repulsion contains only the element $x = \underline{a}^1$ and we obtain

$$\partial h(x) = \operatorname*{argmax}_{y \in \underline{B}_1^*} \left\langle \underline{a}^1 - \underline{a}^1, y \right\rangle = \underline{B}_1^*.$$

The dual grid points $y \in \underline{\mathcal{I}}_D$ w.r.t. repulsion and the corresponding objective values $h^*(y)$ are given by

$y \in \underline{\mathcal{I}}_D$	$h^*(y)$
$(-1,1)^T$	-6
$(-1,-2)^T$	-9
$(2,1)^T$	15

with $h^*(y) = \left\langle \underline{a}^1, y \right\rangle$.

3. For each grid point $y \in \underline{\mathcal{I}}_D$ we determine the corresponding objective value $g^*(y)$ such that

$y \in \underline{\mathcal{I}}_D$	$\{\overline{y}^1, \overline{y}^2\}$	$g^*(y)$
$(-1,1)^T$	$\left\{(0,1)^T, (-1,0)^T\right\}$	-7
$(-1,-2)^T$	$\left\{(0,-1)^T, (-1,-1)^T\right\}$	-15
$(2,1)^T$	$\left\{(1,0)^T, (1,1)^T\right\}$	15

The corresponding primal elementary convex sets $\partial g^*(y)$ are illustrated in Figure 9.3.

4. Obviously, the optimal objective value is

$$\alpha = \min_{y \in \mathcal{I}_D} \{h^*(y) - g^*(y)\} = \min\{1, 6, 0\} = 0,$$

and the set of optimal dual grid points is

$$\mathcal{Y} = \operatorname*{argmin}_{y \in \mathcal{I}_D} \{h^*(y) - g^*(y)\} = \left\{(2,1)^T\right\}.$$

The set of optimal primal solutions results as

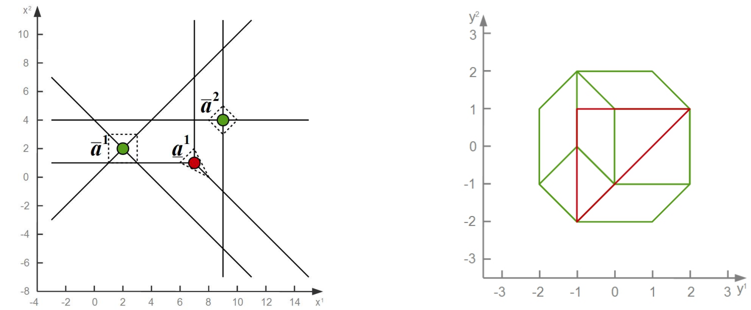
$$\begin{aligned} \mathcal{X} = \bigcup_{y \in \mathcal{Y}} \partial g^*(y) = \partial g^*((2,1)^T) &= [\overline{a}^1 + N_{\overline{B}_1^*}((1,0)^T)] \cap [\overline{a}^2 + N_{\overline{B}_2^*}((1,1)^T)] \\ &= [\overline{a}^1 + \mathbb{R}_+(1,1)^T + \mathbb{R}_+(1,-1)^T] \cap [\overline{a}^2 + \mathbb{R}_+^2]. \end{aligned}$$

Remark 9.3. Without solving the location problem, it is easy to see in Figure 9.2c that the dual grid point $y = (-1,-2)^T$ can not be optimal for the dual problem (D), since the necessary optimality condition $\partial g^*(y) \subseteq \partial h^*(y)$, in Theorem 4.22, is not satisfied:

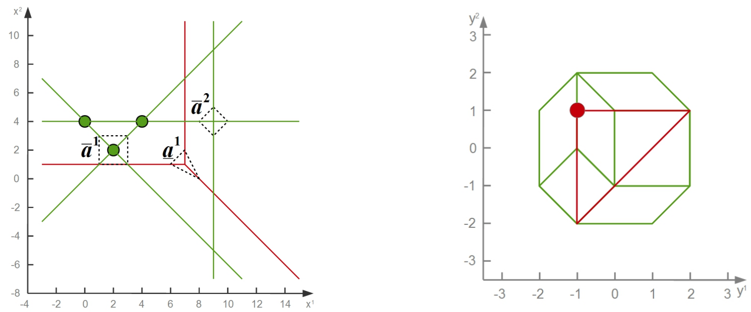
$$\begin{aligned} \partial g^*((-1,-2)^T) &= [\overline{a}^1 + \mathbb{R}_+(-1,-1)^T + \mathbb{R}_+(1,-1)^T] \cap [\overline{a}^2 - \mathbb{R}_+^2], \\ \partial h^*((-1,-2)^T) &= [\underline{a}^1 + \mathbb{R}_+(-1,0)^T + \mathbb{R}_+(1,-1)^T]. \end{aligned}$$

Analogously, it is easy to see in Figure 9.2 that the primal grid point $y = (2,2)^T$ can not be optimal for the primal problem (P), since the necessary optimality condition $\partial h(x) \subseteq \partial g(x)$, in Theorem 4.22, is not satisfied:

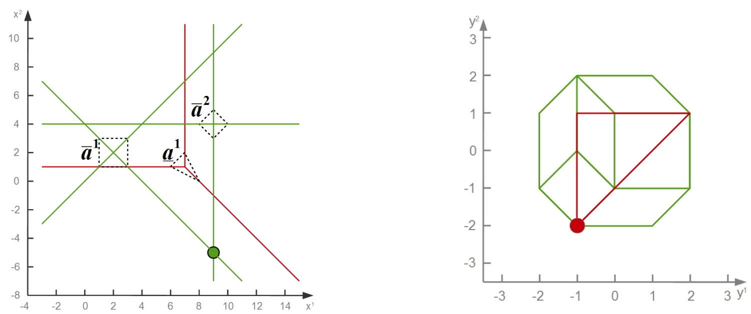
$$\begin{aligned} \operatorname{ext}(\partial g((2,2)^T)) &= \left\{(-2,-1)^T; (-1,-2)^T; (-1,0)^T; (0,-1)^T\right\}, \\ \partial h((2,2)^T) &= \left\{(-1,1)^T\right\}. \end{aligned}$$



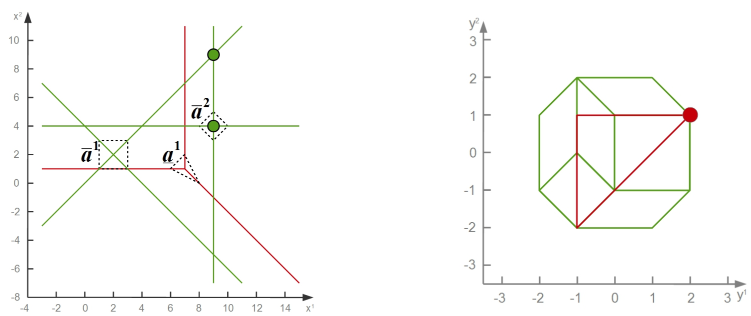
(a) Primal and dual construction grids.



(b) $x = (0,4)^T, x = (2,2)^T, x = (4,4)^T$; corresponding dual elementary convex set $\partial h((0,4)^T) = \partial h((2,2)^T) = \partial h((4,4)^T) = \{(-1,1)^T\}$.

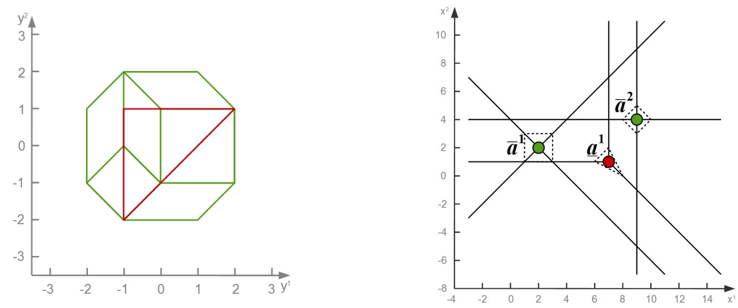


(c) $x = (9,-5)^T$; corresponding dual elementary convex set $\partial h((9,-5)^T) = \{(-1,-2)^T\}$.

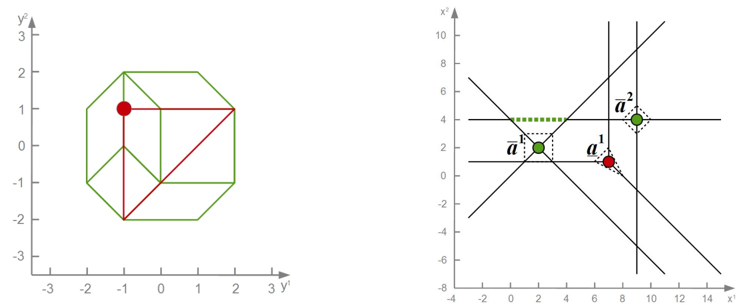


(d) $x = (9,4)^T, x = (9,9)^T$; corresponding dual elementary convex set $\partial h((9,4)^T) = \partial h((9,9)^T) = \{(2,1)^T\}$.

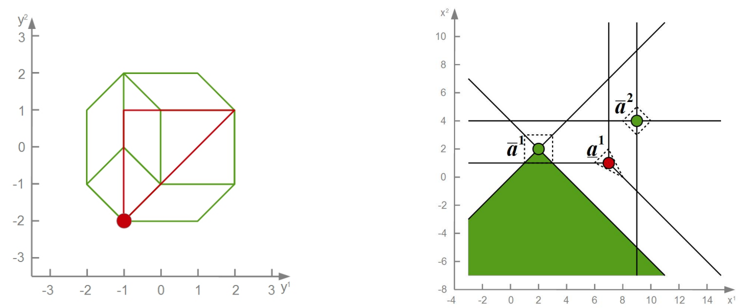
Figure 9.2: Primal grid points $x \in \overline{\mathcal{I}}$ (green) and the assigned dual elementary convex sets $\partial h(x)$ (red) for the example in Section 9.4.



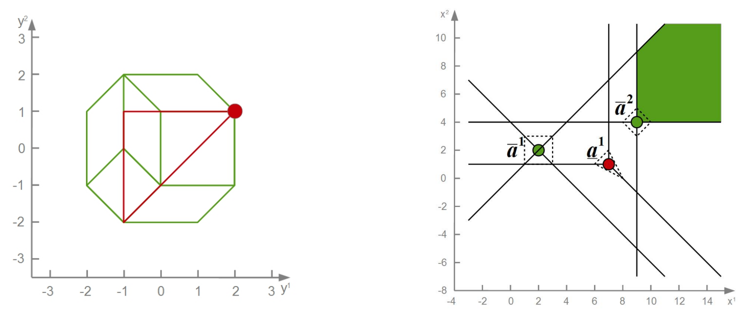
(a) Dual and primal construction grids.



(b) $y = (-1,1)^T$; corresponding primal elementary convex set $\partial g^*((-1,1)^T)$.



(c) $y = (-1,-2)^T$; corresponding primal elementary convex set $\partial g^*((-1,-2)^T)$.



(d) $y = (2,1)^T$; corresponding primal elementary convex set $\partial g^*((2,1)^T)$.

Figure 9.3: Dual grid points $y \in \mathcal{I}_D$ (red) and the assigned primal elementary convex sets $\partial g^*(y)$ (green) for the example in Section 9.4.

9.5 The Special Case of no Repulsion

For the classical Fermat-Weber problem (W), as given in Section 4.3, the number of repulsion points is equal to zero, i.e., $\underline{M} = 0$. In this case the function $h(x)$ does not contribute to the objective function $g(x) - h(x)$ in (P), in particular $h \equiv 0$. The conjugate h^* then is given by

$$h^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y \rangle - 0\} = \mathbb{I}_{\{0\}}(y)$$

and thus the effective domain results as $\operatorname{dom} h^* = \{0\} = \underline{\mathcal{I}}_D = \mathcal{Y}$. Hence, the dual set of optimal solutions is $\mathcal{Y} = \{0\}$. The primal set of optimal points is given by $\mathcal{X} := \partial g^*(0)$, and the optimal objective value α is finite. The dual problem (D) simplifies to

$$-g^*(0) = -\min \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m) \right] \middle| \sum_{m=1}^{\overline{M}} \overline{y}^m = 0 \right\},$$

and the dual algorithm reduces as follows:

Algorithm 9.4 (Dual algorithm in case of $\underline{M} = 0$)**.**

Input: $\overline{a}^m, \overline{w}_m, \overline{B}^*_m$, $(m = 1, \dots, \overline{M})$.

Output: The set $\mathcal{X}$ of optimal grid points of of ($P_{\mathcal{H}}$); optimal objective value α.

Determine

$$\alpha := -\min \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m) \right] \middle| \sum_{m=1}^{\overline{M}} \overline{y}^m = 0 \right\},$$

$$\mathcal{X} := \bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m}(\overline{y}^m) \right].$$

In [36, Lemma 3.1] the authors give a statement on the existence of such a tuple of elements $y^1, \dots, y^M$ and a sufficient condition for a tuple to provide optimal solutions for this special case of no repulsion. Nevertheless, we do not know from Lemma 3.1 in [36] how to determine such a tuple. Using the duality based results of our study, we are able to determine such a tuple on the one hand and also to generalize the assertion of Lemma 3.1 in [36] such, that an arbitrary number of repulsion points may be considered, on the other hand.

9.6 An Algorithm for Solving the Constrained Problem with Obnoxious Facilities

The Algorithms 9.2 and 9.3 can be generalized to the constrained case by substituting the unconstrained function g and its conjugate g^* by the constrained function $g_{\mathcal{H}}$ and its conjugate

$g^*_{\mathcal{H}}$; and by taking into account the primal constrained set of grid points $\overline{\mathcal{I}}^{\mathcal{H}}$, see Definition 8.12. Note that the generalized Benson algorithm, as presented in [51], is not applicable for determining the grid points $x \in \overline{\mathcal{I}}^{\mathcal{H}}$ and $y \in \overline{\mathcal{I}}^{\mathcal{H}}_D$, as described in Chapter 7 and Section 9.1. In case of constraints the recession cone $C := 0^+ \operatorname{epi} g_{\mathcal{H}}$ may have empty interior such that the assumption $c \in \operatorname{int} C$, see (7.1) in Chapter 7, is not satisfiable. A further generalization of the algorithms is necessary to be able to deal with epigraphs of arbitrary polyhedral convex functions, the development is in progress, see [77].

Algorithm 9.5 (Primal Algorithm for Solving the Constrained Location Problem ($P_{\mathcal{H}}$))**.**

Input: $\overline{a}^m, \overline{w}_m, \overline{B}^*_m$, $(m = 1, \ldots, \overline{M})$; $\underline{a}^m, \underline{w}_m, \underline{B}^*_m$, $(m = 1, \ldots, \underline{M})$; $\mathcal{H}$.

Output: The sets $\mathcal{X}_{\mathcal{H}}$ and $\mathcal{Y}_{\mathcal{H}}$ of optimal grid points of ($P_{\mathcal{H}}$) and ($D_{\mathcal{H}}$), the optimal objective value $\alpha_{\mathcal{H}}$.

1. Check finiteness of the optimal objective value using the condition
$$\operatorname{dom} h^* = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}^*_m \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}^*_m + \sum_{i=1}^{I} \mathbb{R}_+ q^i = \operatorname{dom} g^*_{\mathcal{H}}$$
given in Theorem 8.8. If the condition is satisfied go on with 2. Otherwise set $\alpha_{\mathcal{H}} := -\infty$, $\mathcal{Y}_{\mathcal{H}} := \operatorname{dom} h^* \setminus \operatorname{dom} g^*_{\mathcal{H}}$ and $\mathcal{X}_{\mathcal{H}} := \emptyset$ and `STOP`.
2. Determine the set $\overline{\mathcal{I}}^{\mathcal{H}}$ as well as $g[\overline{\mathcal{I}}^{\mathcal{H}}]$.
3. For all $x \in \overline{\mathcal{I}}^{\mathcal{H}}$ determine $h(x) := \sum_{m=1}^{\underline{M}} \underline{w}_m \gamma_{\underline{B}_m}(x - \underline{a}^m)$.
4. Determine the optimal objective value
$$\alpha_{\mathcal{H}} := \min_{x \in \overline{\mathcal{I}}^{\mathcal{H}}} \{g_{\mathcal{H}}(x) - h(x)\},$$
the primal set of optimal grid points
$$\mathcal{X}_{\mathcal{H}} := \operatorname*{argmin}_{x \in \overline{\mathcal{I}}^{\mathcal{H}}} \{g_{\mathcal{H}}(x) - h(x)\},$$
and the dual set of optimal grid points
$$\mathcal{Y}_{\mathcal{H}} := \bigcup_{x \in \mathcal{X}} \partial h(x) = \bigcup_{x \in \mathcal{X}_{\mathcal{H}}} \sum_{m=1}^{\underline{M}} \operatorname*{argmax}_{\underline{y}^m \in \operatorname{ext}(\underline{w}_m \underline{B}^*_m)} \langle x - \underline{a}^m, \underline{y}^m \rangle .$$

Algorithm 9.6 (Dual Algorithm for Solving the Constrained Location Problem ($P_{\mathcal{H}}$)).

Input: $\overline{a}^m, \overline{w}_m, \overline{B}^*_m$, $(m = 1, \ldots, \overline{M})$; $\underline{a}^m, \underline{w}_m, \underline{B}^*_m$, $(m = 1, \ldots, \underline{M})$; $\mathcal{H}$.

Output: The sets $\mathcal{X}_{\mathcal{H}}$ and $\mathcal{Y}_{\mathcal{H}}$ of optimal grid points of ($P_{\mathcal{H}}$) and ($D_{\mathcal{H}}$), the optimal objective value $\beta_{\mathcal{H}}$.

1. Check finiteness of the optimal objective value using the condition

$$\operatorname{dom} h^* = \sum_{m=1}^{\underline{M}} \underline{w}_m \underline{B}^*_m \subseteq \sum_{m=1}^{\overline{M}} \overline{w}_m \overline{B}^*_m + \sum_{i=1}^{I} \mathbb{R}_+ q^i = \operatorname{dom} g^*_{\mathcal{H}}$$

   given in Theorem 8.8. If the condition is satisfied go on with 2. Otherwise set $\beta_{\mathcal{H}} := -\infty$, $\mathcal{Y}_{\mathcal{H}} := \operatorname{dom} h^* \setminus \operatorname{dom} g^*_{\mathcal{H}}$ and $\mathcal{X}_{\mathcal{H}} := \emptyset$ and `STOP`.

2. Determine the set $\underline{\mathcal{I}}_D$ and $h^*[\underline{\mathcal{I}}_D]$ by using one of the methods presented in Section 9.2.

3. For all $y \in \underline{\mathcal{I}}_D$ determine the objective value $g^*_{\mathcal{H}}(y)$ by solving the linear optimization problem

$$\begin{aligned} g^*_{\mathcal{H}}(y) &= \inf_{\overline{y}^0 \in \mathbb{R}^n} \left\{ g^*(y - \overline{y}^0) + \sigma_{\mathcal{H}}(\overline{y}^0) \right\} \\ &= \inf \left\{ \sum_{m=1}^{\overline{M}} \left[\langle \overline{a}^m, \overline{y}^m \rangle + \mathbb{I}_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m) \right] + \sigma_{\mathcal{H}}(\overline{y}^0) \,\middle|\, \sum_{m=0}^{\overline{M}} \overline{y}^m = y \right\}. \end{aligned}$$

   and determine the set $\widehat{\mathcal{Y}}$ of tuple $(\overline{y}^0, \ldots, \overline{y}^{\overline{M}})$ for which the infimum is attained.

4. Determine the optimal objective value

$$\beta_{\mathcal{H}} := \min_{y \in \underline{\mathcal{I}}_D} \{ h^*(y) - g^*_{\mathcal{H}}(y) \},$$

   the dual set of optimal grid points

$$\mathcal{Y}_{\mathcal{H}} := \operatorname*{argmin}_{y \in \underline{\mathcal{I}}_D} \{ h^*(y) - g^*_{\mathcal{H}}(y) \},$$

   and the primal set of optimal grid points

$$\mathcal{X}_{\mathcal{H}} := \bigcup_{y \in \mathcal{Y}_{\mathcal{H}}} \partial g^*_{\mathcal{H}}(y) = \bigcup_{y \in \mathcal{Y}_{\mathcal{H}}} \left[\partial \sigma_{\mathcal{H}}(\overline{y}^0) \cap \bigcap_{m=1,\ldots,\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}^*_m}(\overline{y}^m) \right] \right].$$

Chapter 10

A Matlab Implementation for the 2-Dimensional Case

Based on the duality and discretization results given in Chapter 5 we implemented in `Matlab` the Algorithms 9.2 and 9.3 for solving the location problem (P) with obnoxious facilities in the **plane**. The aim of this chapter is to give some advice on the implementation. In Sections 10.1 and 10.2 we give some remarks on the required input and the resulting output arguments. A choice of subroutines is described in detail and an overview on the program structure is given in Section 10.3. Finally, in Section 10.4, we demonstrate some examples solved by the implementation.

Throughout this chapter we use the following notation: Consider all (pairwise different) unit balls $B_1, \ldots, B_J \subseteq \mathbb{R}^2$, $(J \leq \overline{M} + \underline{M})$, that are assigned to an attracting or a repulsive facility in the location problem (P), defined in Chapter 3. We define for $j = 1, \ldots, J$ the sets

$$\overline{T}_j := \left\{m \in \left\{1, \ldots, \overline{M}\right\} \middle| \overline{B}_m = B_j\right\}, \qquad \underline{T}_j := \left\{m \in \left\{1, \ldots, \underline{M}\right\} \middle| \underline{B}_m = B_j\right\}, \tag{10.1}$$

and the index sets

$$\overline{\mathfrak{J}} := \left\{j \in \{1, \ldots, J\} \middle| \overline{T}_j \neq \emptyset\right\}, \qquad \underline{\mathfrak{J}} := \left\{j \in \{1, \ldots, J\} \middle| \underline{T}_j \neq \emptyset\right\},$$

Further, we define vectors $\overline{u} \in \mathbb{R}^{\overline{M}}$ and $\underline{u} \in \mathbb{R}^{\underline{M}}$, such that

$$\overline{u}(m) = j \quad \Leftrightarrow \quad m \in \overline{T}_j \quad (m = 1, \ldots, \overline{M}), \tag{10.2}$$

$$\underline{u}(m) = j \quad \Leftrightarrow \quad m \in \underline{T}_j \quad (m = 1, \ldots, \underline{M}), \tag{10.3}$$

which is justified since by (10.1) there exists for all $m = 1, \ldots, \overline{M}$ exactly one $j \in \{1, \ldots, J\}$ such that $m \in \overline{T}_j$ and analogously there exists for all $m = 1, \ldots, \underline{M}$ exactly one $j \in \{1, \ldots, J\}$ such that $m \in \underline{T}_j$.

Remark 10.1. Let $B = \left\{x \in \mathbb{R}^n \middle| \widehat{A}x \leq \widehat{b}\right\}$, $\widehat{A} \in \mathbb{R}^{r\times n}, b \in \mathbb{R}^r$, be a constraint representation of the unit ball $B \subseteq \mathbb{R}^n$. Since by Definition 3.1 the origin is assumed to belong to the interior of B, it holds: $\widehat{b} > 0$. Hence, there exists a representation $B = \{x \in \mathbb{R}^n | \, Ax \leq \mathbb{1}\}$, $A \in \mathbb{R}^{r\times n}$.

This remark leads to the following definition:

Definition 10.2. A matrix $A \in \mathbb{R}^{r\times n}$ is called associated with the unit ball $B \subseteq \mathbb{R}^n$ if

$$B = \{x \in \mathbb{R}^n | \, Ax \leq \mathbb{1}\}.$$

10.1 Input Arguments

To start the calculations one has to call the function

`ObnoxiousSolve(ap,ar,up,ur,B,Options,BenOptions)`,

where `ap,ar,up,ur,B` are required input arguments and `Options` and `BenOptions` are optional arguments.

Required Input Arguments

We explain the mandatory arguments with help of the following example:

Example 10.3. Consider the location problem (P) with six attracting and three repulsive facilities:

$$\begin{array}{llll}
\overline{a}^1 = (1,10)^T, & \overline{w}_1 = 12, & \underline{a}^1 = (6,7)^T, & \underline{w}_1 = 6, \\
\overline{a}^2 = (5,5)^T, & \overline{w}_2 = 7, & \underline{a}^2 = (3,4)^T, & \underline{w}_2 = 5, \\
\overline{a}^3 = (4,6)^T, & \overline{w}_3 = 8, & \underline{a}^3 = (3,9)^T, & \underline{w}_3 = 2. \\
\overline{a}^4 = (9,8)^T, & \overline{w}_4 = 10, & & \\
\overline{a}^5 = (4,9)^T, & \overline{w}_5 = 8, & & \\
\overline{a}^6 = (4,10)^T, & \overline{w}_6 = 14, & &
\end{array}$$

We assign the following unit balls to these facilities:

$$B_j = \{x \in \mathbb{R}^n | \, A_j x \leq \mathbb{1}\}, \qquad (j = 1, \ldots, 4),$$

where the associated matrices are given as

$$A_1 := \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ -1 & -1 \\ 1 & -1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_3 := \begin{pmatrix} -1 & 1 \\ 2 & 1 \\ -1 & -2 \end{pmatrix}, \quad A_4 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

The unit balls are assigned to the facilities by the vectors

$$\overline{u} = [1, 3, 1, 2, 3, 4], \qquad \underline{u} = [3, 2, 2],$$

such that

B_1 is assigned to $\overline{a}^1$ and $\overline{a}^3$, B_3 is assigned to $\overline{a}^2$, $\overline{a}^5$ and $\underline{a}^1$,

B_2 is assigned to $\overline{a}^4$ and $\underline{a}^2$, $\underline{a}^3$, B_4 is assigned to $\overline{a}^6$.

We obtain by application of (10.2) and (10.3) that $\overline{\mathfrak{J}} = \{1, 2, 3, 4\}$, $\underline{\mathfrak{J}} = \{2, 3\}$ and

$$\overline{T}_1 = \{1, 3\}, \quad \overline{T}_2 = \{4\}, \quad \overline{T}_3 = \{2, 5\}, \quad \overline{T}_4 = \{6\},$$
$$\underline{T}_1 = \emptyset, \quad \underline{T}_2 = \{2, 3\}, \quad \underline{T}_3 = \{1\}, \quad \underline{T}_4 = \emptyset.$$

In order to realize the input of those arguments in `Matlab`, we define a structure `B` containing the matrices A_1, A_2, A_3, A_4 as well as structures `ap` and `ar` containing the coordinates and weights of the attraction and repulsion points, respectively. Finally, we call the function `ObnoxiousSolve` in order to start the computations.

```
B(1)=struct('ball',[1 1;-1 1;-1 -1; 1 -1]);
B(2)=struct('ball',[1 0;0 1;-1 0; 0 -1]);
B(3)=struct('ball',[-1 1;2 1;-1 -2]);
B(4)=struct('ball',[0 1;-1 0;1 -1]);

up=[1 3 1 2 3 4];
ur=[3 2 2];

ap(1)=struct('loc',[1 10],'weight',12);
ap(2)=struct('loc',[5  5],'weight',7);
ap(3)=struct('loc',[4  6],'weight',8);
ap(4)=struct('loc',[9  8],'weight',10);
ap(5)=struct('loc',[4  9],'weight',8);
ap(6)=struct('loc',[4 10],'weight',14);

ar(1)=struct('loc',[6  7],'weight',6);
ar(2)=struct('loc',[3  4],'weight',5);
ar(3)=struct('loc',[3  9],'weight',2);

[solD,solP,grid,time]=ObnoxiousSolve(ap,ar,up,ur,B)
```

Optional Input Arguments

The argument `Options` may be set in order to predefine special details as listed below. The default setting is

```
Options=struct('info',1,'out',2,'solve',d,'plot',5);
```

- `Options.info`: display or suppress status information
  - 0: suppress status information
  - 1: display status information (default).
- `Options.out`: select output data
  - 0: suppress output data
  - 1: display optimal solutions
  - 2: display optimal solutions as well as `time` and `grid` (default).
- `Options.solve`: choose algorithm to be applied
  - 'd': use dual algorithm (see Algorithm 9.3) (default)
  - 'p': use primal algorithm (see Algorithm 9.2)
  - 'b': use dual and primal algorithm as well (see Algorithms 9.2 and 9.3).
- `Options.plot`: select elements to be plotted
  - 0: plot facilities, dual domains
  - 1: plot facilities, dual domains, solutions
  - 2: plot facilities, dual domains, primal grids
  - 3: plot facilities, dual domains, primal grids, solutions
  - 4: plot facilities, dual domains, primal grids, dual grid points
  - 5: plot facilities, dual domains, primal grids, dual grid points, solutions (default)
  - 6: suppress plotting.

The argument `BenOptions` may be set in order to predefine special details referring to the implementation `bensolve-1.2.`[1], which is a generalized version of Benson's algorithm [5, 51] and is applied in our implementation in order to determine primal and dual grid points. The default setting (by `ObnoxiousSolve`) is

```
BenOptions=struct('info',0,'lp_solver',3);
```

For more details call `help bensolve`.

[1] http://ito.mathematik.uni-halle.de/~loehne/

- `BenOptions.info`: display information for `bensolve`
  - 0: suppress display information for `bensolve` (default)
  - 1: display information for `bensolve`
  - 2: display more information for `bensolve`.
- `BenOptions.lp_solver`: select LP solver for `bensolve`
  - 0: `MATLAB linprog`
  - 1: `cdd` with criss-cross-method
  - 2: `cdd` with simplex
  - 3: `glpk` (revised simplex) (default).

10.2 Output Arguments

Depending on the chosen output option, the function `ObnoxiousSolve` returns the following arguments (as far as they were determined):

1. When `Options.out>0`, then the calculated primal and dual solutions are displayed.

   The structures `solP` and `solD` contain

   - the optimal primal grid points $x \in \overline{\mathcal{I}} \cap \mathcal{X}$,
   - the optimal dual grid points $y \in \underline{\mathcal{I}}_D \cap \mathcal{Y}$,
   - the optimal objective value α,

   obtained by executing the primal Algorithm 9.2 and the dual Algorithm 9.3, respectively.

2. If `Options.out=2` then, additionally, the output arguments `grid` and `time` are displayed.

   The fields of the structure `grid` contain

   - the primal attraction grid points $x \in \overline{\mathcal{I}}$,
   - the primal repulsion grid points $x \in \underline{\mathcal{I}}$,
   - the dual attraction grid points $y \in \overline{\mathcal{I}}_D$,
   - the dual repulsion grid points $y \in \underline{\mathcal{I}}_D$.

   The fields of the structure `time` contain the running times for

   - determining the primal and dual grid w.r.t. attraction,
   - determining the primal and dual grid w.r.t. repulsion,
   - solving the location problem (P) by executing the primal Algorithm 9.2,
   - solving the location problem (P) by executing the dual Algorithm 9.3.

10.3 Subroutines of the Matlab Implementation

In this section we present some of the subroutines of the function `ObnoxiousSolve`.

1. For a matrix A associated with the unit ball $B = \left\{x \in \mathbb{R}^2 \middle| Ax \leq \mathbb{1}\right\}$, the subroutine `DualBall` determines a matrix D, whose rows contain the extreme points of B. The matrix D represents equivalently the matrix associated with the dual unit ball $B^* = \left\{x \in \mathbb{R}^2 \middle| Dx \leq \mathbb{1}\right\}$.

```
function [D]=DualBall(A)
A=unique(A,'rows');
k = convhull(A(:,1),A(:,2));
Q=A(k,1:2);
D=zeros(size(Q,1)-1,2);
for i=1:size(Q,1)-1
    x=linsolve([Q(i,:);Q(i+1,:)],[1;1]);
    D(i,:)=x';
end
```

   Amongst others, this subroutine is used for determining the matrices associated with the dual unit balls $B_1^*, \dots, B_J^*$, which then are stored in the field `B(j).dualBall` for all $j = 1, \dots, J$.

2. The subroutine `Domain` determines the extreme points of $\operatorname{dom} g^*$ and $\operatorname{dom} h^*$.

   By defining vectors $\overline{w} \in \mathbb{R}_+^{|\overline{\mathfrak{J}}|}$ and $\underline{w} \in \mathbb{R}_+^{|\underline{\mathfrak{J}}|}$ such that

$$\overline{w}(j) := \sum_{m \in \overline{T}_j} \overline{w}_m, \qquad \forall j \in \overline{\mathfrak{J}},$$

$$\underline{w}(j) := \sum_{m \in \underline{T}_j} \underline{w}_m, \qquad \forall j \in \underline{\mathfrak{J}},$$

   the domains of g^* and h^* can be determined by

$$\operatorname{dom} g^* = \sum_{j \in \overline{\mathfrak{J}}} B_j^* \cdot \overline{w}(j), \qquad \operatorname{dom} h^* = \sum_{j \in \underline{\mathfrak{J}}} B_j^* \cdot \underline{w}(j).$$

   Compared to (5.5) the number of unit balls, which are to be summed up, reduces from $\overline{M}$ to $|\overline{\mathfrak{J}}|$ and from $\underline{M}$ to $|\underline{\mathfrak{J}}|$, since each ball is taken into account only once. Especially in the case that $J = 1$, see Remark 5.3, we have

$$\operatorname{dom} g^* = B^* \cdot \sum_{m=1}^{\overline{M}} \overline{w}_m, \qquad \operatorname{dom} h^* = B^* \cdot \sum_{m=1}^{\underline{M}} \underline{w}_m.$$

```
function [Bp,Br,B]=Domain(ap,ar,B,up,ur)
Bp=[0 0]; Br=[0 0];
Mp=length(up); Mr=length(ur);
for m=1:Mp
    for i=1:size(B,2)
        if up(m)==i
            B(i).pweight=B(i).pweight+ap(m).weight;
        end
    end
end
for i=1:size(B,2)
    Bp=minksum(Bp,B(i).pweight*B(i).ball);
end
for m=1:Mr
    for i=1:size(B,2)
        if ur(m)==i
            B(i).rweight=B(i).rweight+ar(m).weight;
        end
    end
end
for i=1:size(B,2)
    Br=minksum(Br,B(i).rweight*B(i).ball);
end
```

3. Taking into account the matrices `Bp` and `Br`, whose rows contain the extreme points of $\operatorname{dom} g^*$ and $\operatorname{dom} h^*$, respectively, the subroutine `Finite` evaluates the condition for the existence of a finite optimal solution of the location problem (P), based on Theorem 5.2.

   Let $\overline{B}_{con}$ be the matrix associated with $\operatorname{dom} g^*$, i.e., $\operatorname{dom} g^* = \left\{y \in \mathbb{R}^2 \middle| \overline{B}_{con} y \leq \mathbb{1}\right\}$. Then it holds

$$\operatorname{dom} h^* \subseteq \operatorname{dom} g^* \qquad \Leftrightarrow \qquad \forall y \in \operatorname{ext}(\operatorname{dom} h^*) : \ \overline{B}_{con} y \leq \mathbb{1}.$$

   The subroutine `Finite` returns a decision variable L which is defined as

$$L := \begin{cases} 1, & \text{if } \operatorname{dom} h^* \subseteq \operatorname{dom} g^*, \\ 0, & \text{if otherwise.} \end{cases}$$

```
function[L]=Finite(Bp,Br)
k = convhull(Bp(:,1),Bp(:,2));
Bp=[Bp(k,1),Bp(k,2)];
Bp_con=dual_ball(Bp);
if Bp_con*Br'<=ones(size(Bp_con,1),size(Br,1))
    L=0;
else
    L=1;
end
```

4. Based on equation (4.9), the subroutine `Gauge` determines the gauge distance, associated with the unit ball $B = \{x \in \mathbb{R}^2 \,|\, Ax \leq \mathbb{1}\}$, $A \in \mathbb{R}^{r\times 2}$, between a point x and the origin:

$$\gamma_B(x) = \max_{y\in B^*} \langle x, y\rangle = \max_{y\in \mathrm{ext}(B^*)} \langle x, y\rangle = \max_{i=1,\dots,r} [Ax]_i.$$

```
function [gaugedist]=Gauge(A,x)
gaugedist=max(A*x);
```

5. Based on Corollary 5.13, the function `GetPrimal` determines for each optimal dual grid point $y \in \underline{\mathcal{I}}_D$ the corresponding primal solutions, given by the elementary convex set $\partial g^*(y)$. We obtain

$$x \in \partial g^*(y) = \bigcap_{m=1}^{\overline{M}} \left[\overline{a}^m + N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m)\right],$$

where for all $m = 1, \dots, \overline{M}$

$$\begin{aligned}(x - \overline{a}^m) \in N_{\overline{w}_m \overline{B}_m^*}(\overline{y}^m) &\Leftrightarrow \forall y \in \overline{w}_m \overline{B}_m^* : & \langle x - \overline{a}^m, y - \overline{y}^m\rangle \leq 0\\ &\Leftrightarrow \forall y \in \mathrm{ext}(\overline{w}_m \overline{B}_m^*) : & \langle x, y - \overline{y}^m\rangle \leq \langle \overline{a}^m, y - \overline{y}^m\rangle .\end{aligned}$$

Since the rows of the matrix $\overline{A}_m$, associated with the unit ball $\overline{B}_m$, contain the extreme points of the dual unit ball $\overline{B}_m^*$, we obtain

$$\begin{aligned}\partial g^*(y) = \Big\{x \in \mathbb{R}^2 \,\Big|\, &\forall m = 1, \dots, \overline{M};\ \forall i = 1, \dots, \overline{r}_m :\\ &\big\langle x, \overline{w}_m[\overline{A}_m]_i - \overline{y}^m\big\rangle \leq \big\langle \overline{a}^m, \overline{w}_m[\overline{A}_m]_i - \overline{y}^m\big\rangle\Big\}.\end{aligned}$$

```
function [B,b]=GetPrimal(yp,ap,Mp)
B=[];b=[];
for m=1:Mp
    Bm=ap(m).weight*ap(m).ball;
    for i=1:size(Bm,1)
        By=Bm(i,:)-yp(m,:);
        B=[B;By];
        b=[b;By*(ap(m).loc)'];
    end
end
```

The determination of the corresponding extreme points can be realized by vertex enumeration [4, 24, 51].

6. Based on Proposition 5.8 and Corollary 5.12, the function `GetDual` determines for each optimal primal grid point $x \in \overline{\mathcal{I}}$ the corresponding dual solutions, given by the elementary convex set $\partial h(x)$. Since the extreme points of the dual balls $\underline{B}_1^*, \dots, \underline{B}_{\underline{M}}^*$ coincide with the

rows of the matrices $\underline{A}_1, \ldots, \underline{A}_M$ associated with the respective primal balls $\underline{B}_1, \ldots, \underline{B}_M$, we obtain

$$\partial h(x) = \sum_{m=1}^{M} \underset{\underline{y}^m \in \underline{w}_m \underline{B}^*_m}{\operatorname{argmax}} \left\langle x - \underline{a}^m, \underline{y}^m \right\rangle$$

where

$$\operatorname{ext}(\underline{w}_m \underline{B}^*_m) = \left\{ \underline{w}_m [\underline{A}_m]_1, \ldots, \underline{w}_m [\underline{A}_m]_{\underline{r}_m} \right\}.$$

```
function Y=GetDual(x,ar,Mr)
Y=[0 0];
for m=1:Mr
    Ym=[];
    u=ar(m).weight*ar(m).ball;
    d=u*(x-ar(m).loc)';
    z=max(d);
    ind=find(d>z-0.0001);
    for i=1:length(ind)
        Ym=[Ym;u(ind(i),:)];
    end
    Y=minksum(Y,Ym);
end
```

Overview on the Program Structure

The function `ObnoxiousSolve` is organized as follows:

0. The subroutine `SetDefault` sets the default options for the function `ObnoxiousSolve`.

1. For a matrix A, associated with the unit ball $B = \left\{x \in \mathbb{R}^2 \middle| Ax \leq \mathbb{1}\right\}$, the subroutine `DualBall` determines a matrix D, whose rows contain the extreme points of B. The matrix D represents equivalently the matrix associated with the dual ball $B^* = \left\{x \in \mathbb{R}^2 \middle| Dx \leq \mathbb{1}\right\}$.

2. The function `Domain` determines the extreme points of $\operatorname{dom} g^*$ and $\operatorname{dom} h^*$.

3. The function `Finite` evaluates the finiteness condition $\operatorname{dom} h^* \subseteq \operatorname{dom} g^*$ (see Theorem 5.2 and step 1 in the Algorithms 9.2 and 9.3). If the condition is not satisfied, the program displays "`Solution is INFINITE.`" before quitting. Otherwise, if the finiteness condition is true, the program displays "`Solution is finite.`" and starts the determination of optimal solutions.

4. `SolvePrimal` executes the steps 2, 3 and 4 in the primal Algorithm 9.2 in order to determine the optimal primal and dual grid points for the dual pair of optimization problems (P) and (D).

a) The subroutine `Gauge` is applied in order to calculate the objective values $h(x)$ for all $x \in \overline{\mathcal{I}}$ (step 3 in Algorithm 9.2) and for determining the matrix Y, whose columns generate the recession cone $0^+ \operatorname{epi} g$, see Section 7.

b) The function `bensolve` determines the primal and the dual grid points $x \in \overline{\mathcal{I}}$ and $y \in \overline{\mathcal{I}}_D$ as well as the objective values $g(x)$ and $g^*(y)$, see step 2 in Algorithm 9.2. The subroutine is called by `bensolve(P,B,b,Y,[],c,ben_opt)`. The required input arguments `P,B,b,Y,c` can be chosen as they are presented in Chapter 7.

c) Based on Proposition 5.8 and Corollary 5.12, the function `GetDual` determines for each optimal primal grid point $x \in \overline{\mathcal{I}}$ the corresponding dual solutions, given by the elementary convex set $\partial h(x)$, see step 4 in Algorithm 9.2.

5. `SolveDual` executes the steps 2, 3 and 4 in the dual Algorithm 9.3, in order to determine the optimal primal and dual grid points for the dual pair of optimization problems (P) and (D).

   a) The subroutine `Gauge` is applied in order to determine the matrix Y, whose columns generate the recession cone $0^+ \operatorname{epi} h$, see Section 7.

   b) The function `bensolve` determines the primal and dual grid points $x \in \underline{\mathcal{I}}$ and $y \in \underline{\mathcal{I}}_D$ as well as the objective values values $h(x)$ and $h^*(y)$, see step 2 in Algorithm 9.3.

   c) Based on Corollary 5.13, the function `GetPrimal` determines for the optimal dual grid points $y \in \underline{\mathcal{I}}_D$ the corresponding primal solutions given by the elementary convex set $\partial g^*(y)$, see step 4 in Algorithm 9.3.

6. `ObnoxiousPlot` illustrates (depending on the predefined plot options) the existing facilities, the primal grids w.r.t. attraction and repulsion, the dual domains w.r.t. attraction and repulsion and the optimal primal and dual grid points.

Matlab Code of the Main Routine

```
function [solD,solP,grid,time]=ObnoxiousSolve(ap,ar,up,ur,B,Options,BenOptions)

display('********************************')
display('*       ObnoxiousSolve         *')
display('********************************')

solP=[];
solD=[];
grid=[];
time=[];

Mp=length(up);
Mr=length(ur);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 0. set (default) options
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
[Options,BenOptions]=SetDefault(Options,BenOptions);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 1. determine dual balls and initialize
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if Options.info>0
    display('DETERMINING DUAL BALLS')
end
for i=1:size(B,2)
    B(i).pweight=0;
    B(i).rweight=0;
    B(i).dualBall=DualBall(B(i).ball);
end
for m=1:Mp
    ap(m).ball=B(up(m)).ball;
    ap(m).dualBall=B(up(m)).dualBall;
end
for m=1:Mr
    ar(m).ball=B(ur(m)).ball;
    ar(m).dualBall=B(ur(m)).dualBall;
end
if Options.info>0
    display('done.')
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 2. determine domain g* and domain h*
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if Options.info>0
    display('DETERMINING DOMAINS OF g* AND h*...')
end
```

```
[Bp,Br,B]=Domain(ap,ar,B,up,ur);
if Options.info>0
    display('done.')
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 3. check if dom h* subset of dom g*
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if Options.info>0
    display('CHECKING FINITENESS...')
end
L=Finite(Bp,Br);
if L>0
    display('Solution is INFINITE.')
else
    if Options.info>0
        display('Solution is finite.')
    end

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % 4./5. solve primal and/or dual problem
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    if options.solve=='p'
        [solP,grid.primal_att,grid.dual_att,time.att_grid,...
                    time.primal_solve]=SolvePrimal(ap,ar,Options,BenOptions);
    elseif options.solve=='d'
        [solD,grid.primal_rep,grid.dual_rep,time.rep_grid,...
                    time.dual_solve]=SolveDual(ap,ar,Mp,Mr,Options,BenOptions);
    elseif options.solve=='b'
        [solP,grid.primal_att,grid.dual_att,time.att_grid,...
                    time.primal_solve]=SolvePrimal(ap,ar,Options,BenOptions);
        [solD,grid.primal_rep,grid.dual_rep,time.rep_grid,...
                    time.dual_solve]=SolveDual(ap,ar,Mp,Mr,Options,BenOptions);
    else
        error('Invalid entry in Option.solve chosen.')
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% 6. plot
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ObnoxiousPlot(ap,ar,solP,solD,grid,Bp,Br,Options)
```

10.4 Examples

In this section we illustrate some examples solved by `ObnoxiousSolve`. The first Example 10.4 serves as a general explanation concerning the interpretation of the output arguments. The Examples 10.5, 10.6, 10.7 and 10.8 demonstrate the difference between the primal Algorithm 9.2 and the dual Algorithm 9.3 concerning time complexity. The case that no finite solution exists is illustrated in Example 10.9.

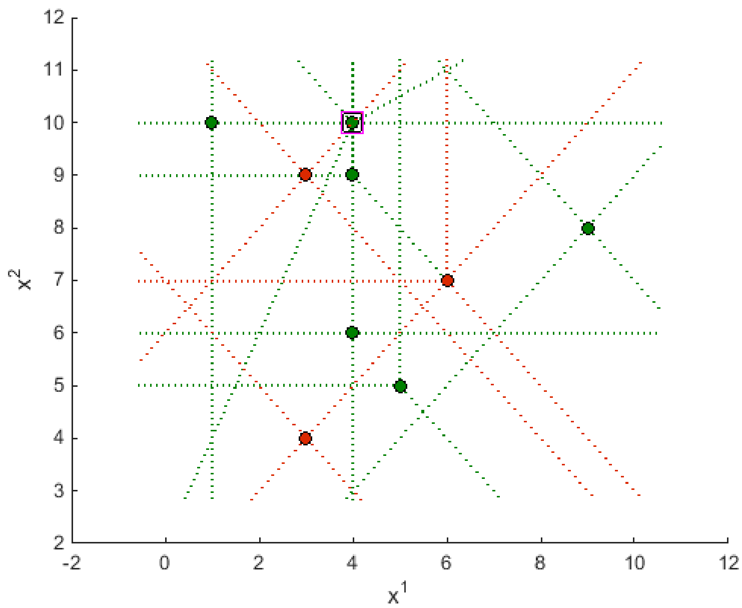
Example 10.4. For the data of Example 10.3 the implementation `ObnoxiousSolve` returns the following output:

```
time =
        att_grid: 1.4920
    primal_solve: 0.0112
        rep_grid: 0.1548
      dual_solve: 0.1650

grid =
    primal_att: [3x33 double]
      dual_att: [3x45 double]
    primal_rep: [3x9 double]
      dual_rep: [3x16 double]

solP.primal =
    4.0000      10.0000

solP.dual =
    −16     43
    −4      31

solP.obj_val =
    −14.0000

solD.primal =
    4       10

solD.dual =
    −16.0000    43.0000
    −4          31

solD.obj_val =
    −14
```

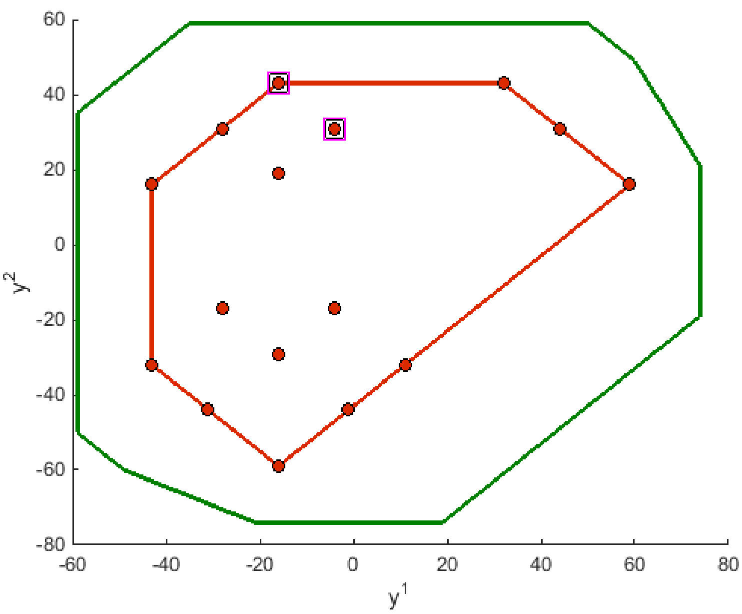
Both, the primal Algorithm 9.2 as well as the dual Algorithm 9.3, lead to the optimal primal grid point $x = (4, 10)^T$, the optimal dual grid points $y = (-16, 43)^T$ and $y = (-4, 31)^T$ and the optimal objective value $\alpha = -14$.

Moreover, the primal Algorithm 9.2 returns the sets $\overline{\mathcal{I}}$ and $\overline{\mathcal{I}}_D$ of 33 primal and 45 dual grid points w.r.t. attraction, where each $x \in \overline{\mathcal{I}}$ and $y \in \overline{\mathcal{I}}_D$ has three entries: its coordinates in

$\mathbb{R}^2$ and the objective values $g(x)$ and $g^*(y)$, respectively. Analogously, the dual Algorithm 9.3 returns the sets $\underline{\mathcal{I}}$ and $\underline{\mathcal{I}}_D$ of 9 primal and 16 dual grid points w.r.t. repulsion, where each $x \in \underline{\mathcal{I}}$ and $y \in \underline{\mathcal{I}}_D$ has three entries: its coordinates in $\mathbb{R}^2$ and the objective values $h(x)$ and $h^*(y)$, respectively.



(a) Attracting points and resulting primal grid w.r.t. attraction (green); repulsive facilities and resulting grid w.r.t. repulsion (red); optimal primal grid point $x = (4, 10)^T$ (square).



(b) Domain of g^* (green); domain of h^* and dual grid points $y \in \underline{\mathcal{I}}_D$ w.r.t. repulsion (red); optimal dual grid points $y = (-16, 43)$ and $y = (-4, 31)$ (squares).

Figure 10.1: Primal and dual plot for Example 10.4.

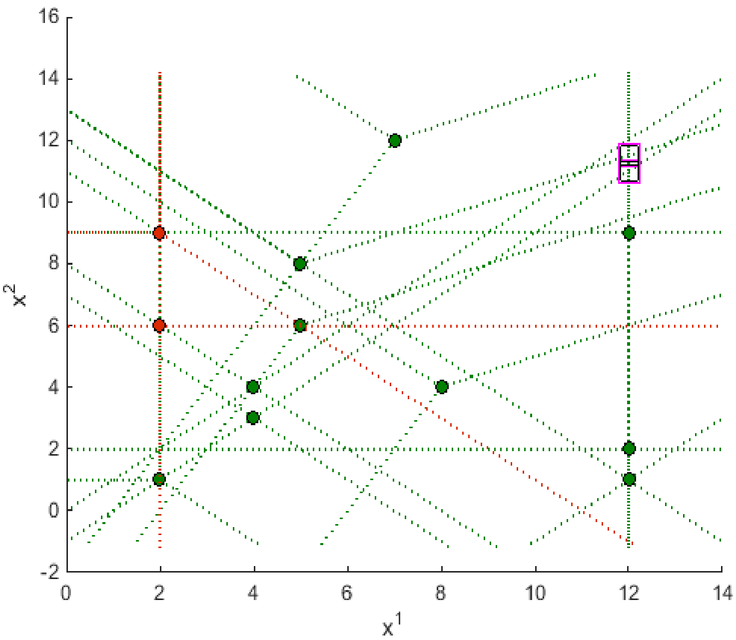
Example 10.5. Consider the location problem (P) with ten attracting and two repulsive facilities as follows:

```
B(1)=struct('ball',[1 1;-1 1;-1 -1; 1 -1]);
B(2)=struct('ball',[1 0;0 1;-1 0; 0 -1]);
B(3)=struct('ball',[-1 1;2 1;-1 -2]);
B(4)=struct('ball',[0 1;-1 0;1 -1]);

up=[4 2 4 3 1 1 4 2 2 4];
ur=[3 1];

ap(1) =struct('loc',[8  4],'weight',2);
ap(2) =struct('loc',[4  3],'weight',2);
ap(3) =struct('loc',[7 12],'weight',3);
ap(4) =struct('loc',[2  1],'weight',1);
ap(5) =struct('loc',[12 9],'weight',1);
ap(6) =struct('loc',[12 2],'weight',2);
ap(7) =struct('loc',[5  6],'weight',1);
ap(8) =struct('loc',[4  4],'weight',1);
ap(9) =struct('loc',[12 1],'weight',3);
ap(10)=struct('loc',[5  8],'weight',3);

ar(1) =struct('loc',[2  9],'weight',3);
ar(2) =struct('loc',[2  6],'weight',3);

[solD,solP,grid,time]=ObnoxiousSolve(ap,ar,up,ur,B,struct('solve','b'),[]);
```

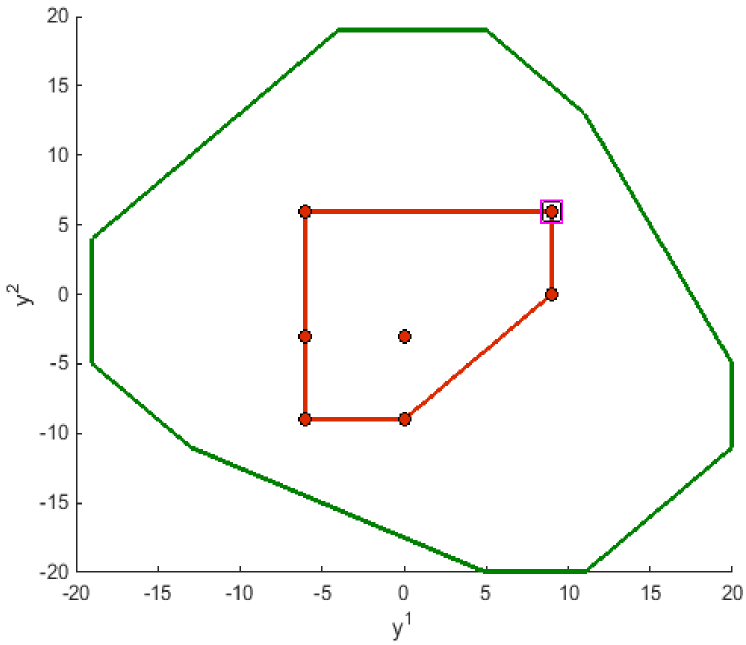
We obtain the following results:

```
time =
        att_grid: 5.3228
    primal_solve: 0.0051
        rep_grid: 0.0619
      dual_solve: 0.0257

grid =
    primal_att: [3x80 double]
      dual_att: [3x102 double]
    primal_rep: [3x3 double]
      dual_rep: [3x7 double]
```

Obviously, the time complexity is mainly influenced by the number of grid points w.r.t. attraction. We have 80 primal and 102 dual grid points w.r.t. attraction (computed by executing the primal Algorithm 9.2) and only 3 primal and 7 dual grid points w.r.t. repulsion (computed by executing the dual Algorithm 9.3). Hence, for this example it seems to be advantageous to apply the dual algorithm.



(a) Attracting points and resulting primal grid w.r.t. attraction (green); repulsive facilities and resulting grid w.r.t. repulsion (red); optimal primal grid points $x = (12, 11)^T$ and $x = (12, 11.5)^T$ (squares).



(b) Domain of g^* (green); domain of h^* and dual grid points $y \in \underline{\mathcal{I}}_D$ w.r.t. repulsion (red); optimal dual grid point $y = (9, 6)^T$ (square).

Figure 10.2: Primal and dual plot for Example 10.5.

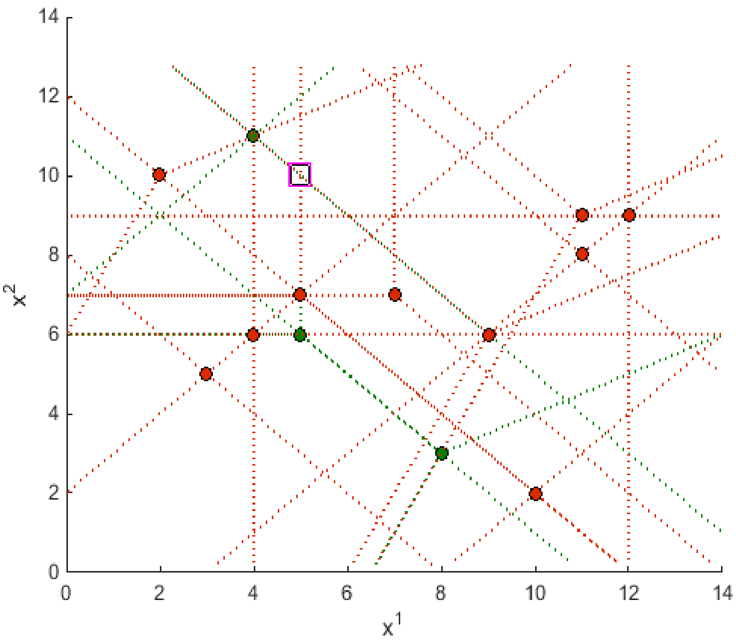
Example 10.6. Consider the location problem (P) with three attracting and ten repulsive facilities as follows:

```
B(1)=struct('ball',[1 1;-1 1;-1 -1; 1 -1]);
B(2)=struct('ball',[1 0;0 1;-1 0; 0 -1]);
B(3)=struct('ball',[-1 1;2 1;-1 -2]);
B(4)=struct('ball',[0 1;-1 0;1 -1]);

up=[2 3 4];
ur=[2 1 4 4 4 2 3 3 1 2];

ap(1) =struct('loc',[4 11],'weight',10);
ap(2) =struct('loc',[5  6],'weight',8);
ap(3) =struct('loc',[8  3],'weight',4);

ar(1) =struct('loc',[11 8],'weight',1);
ar(2) =struct('loc',[4  6],'weight',2);
ar(3) =struct('loc',[11 9],'weight',1);
ar(4) =struct('loc',[9  6],'weight',1);
ar(5) =struct('loc',[2 10],'weight',1);
ar(6) =struct('loc',[3  5],'weight',1);
ar(7) =struct('loc',[7  7],'weight',1);
ar(8) =struct('loc',[5  7],'weight',1);
ar(9) =struct('loc',[12 9],'weight',1);
ar(10)=struct('loc',[10 2],'weight',1);

[solD,solP,grid,time]=ObnoxiousSolve(ap,ar,up,ur,B,struct('solve','b'),[]);
```

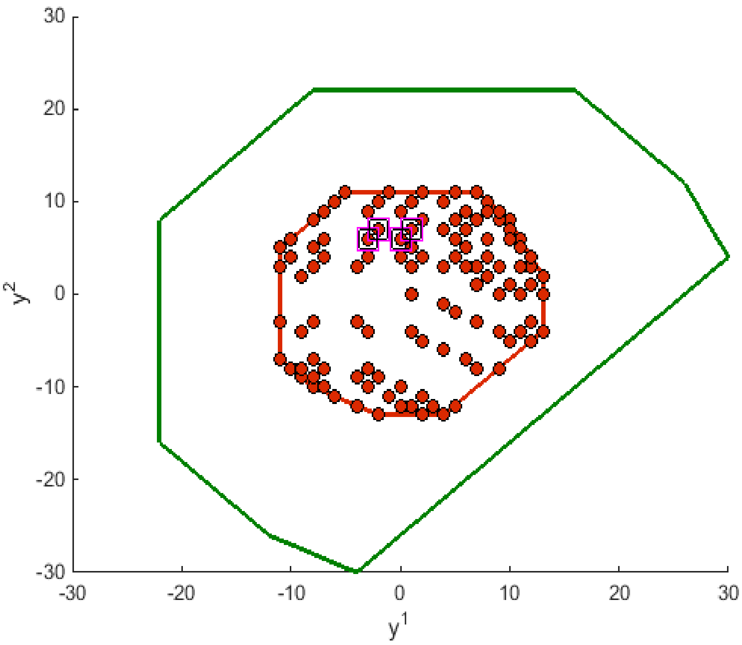
We obtain the following results:

```
time =
        att_grid: 0.0489
    primal_solve: 0.0013
        rep_grid: 2.2252
      dual_solve: 0.0424

grid =
    primal_att: [3x8 double]
      dual_att: [3x14 double]
    primal_rep: [3x88 double]
      dual_rep: [3x113 double]
```

Obviously, the time complexity is mainly influenced by the number of grid points w.r.t. repulsion. We have 8 primal and 14 dual grid points w.r.t. attraction (computed by executing the primal Algorithm 9.2) but 88 primal and 113 dual grid points w.r.t. repulsion (computed by executing the dual Algorithm 9.3). Hence, for this example it seems to be advantageous to apply the primal algorithm.



(a) Attracting points and resulting primal grid w.r.t. attraction (green); repulsive facilities and resulting grid w.r.t. repulsion (red); optimal primal grid point $x = (5, 10)^T$ (square).



(b) Domain of g^* (green); domain of h^* and dual grid points $y \in \mathcal{I}_D$ w.r.t. repulsion (red); optimal dual grid points $y = (-3, 6)^T$, $y = (-2, 7)^T$, $y = (0, 6)^T$, $y = (1, 7)^T$ (squares).

Figure 10.3: Primal and dual plot for Example 10.6.

Example 10.7. Consider the location problem (P) with ten attracting and ten repulsive facilities as follows:

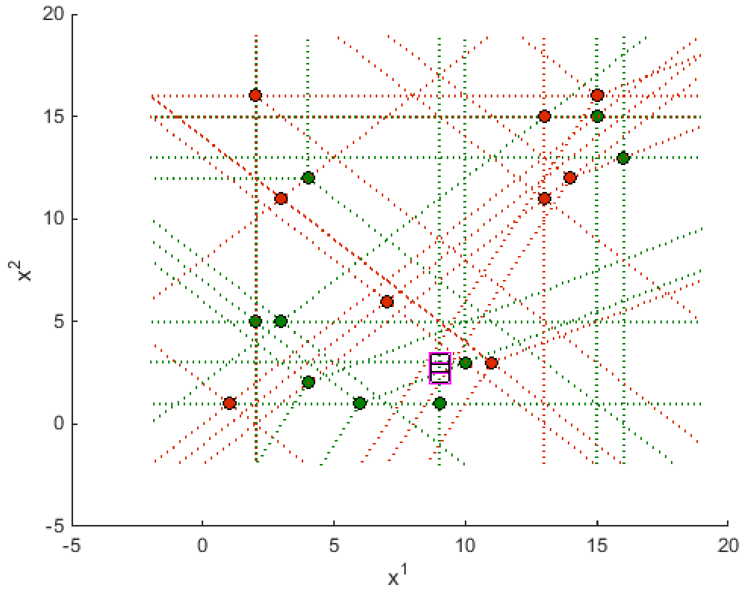
```
B(1)=struct('ball',[1 1;-1 1;-1 -1; 1 -1]);
B(2)=struct('ball',[1 0;0 1;-1 0; 0 -1]);
B(3)=struct('ball',[-1 1;2 1;-1 -2]);
B(4)=struct('ball',[0 1;-1 0;1 -1]);

up=[1 3 2 1 1 4 3 2 4 1];
ur=[3 4 2 2 2 4 4 1 1 2];

ap(1) =struct('loc',[15 15],'weight',7);
ap(2) =struct('loc',[4  12],'weight',4);
ap(3) =struct('loc',[3   5],'weight',10);
ap(4) =struct('loc',[2   5],'weight',1);
ap(5) =struct('loc',[9   1],'weight',10);
ap(6) =struct('loc',[4   2],'weight',4);
ap(7) =struct('loc',[10  3],'weight',11);
ap(8) =struct('loc',[7   6],'weight',1);
ap(9) =struct('loc',[6   1],'weight',6);
ap(10)=struct('loc',[16 13],'weight',1);

ar(1) =struct('loc',[2  16],'weight',5);
ar(2) =struct('loc',[11  3],'weight',5);
ar(3) =struct('loc',[7   6],'weight',5);
ar(4) =struct('loc',[1   1],'weight',4);
ar(5) =struct('loc',[13 11],'weight',5);
ar(6) =struct('loc',[15 16],'weight',2);
ar(7) =struct('loc',[14 12],'weight',3);
ar(8) =struct('loc',[2  16],'weight',2);
ar(9) =struct('loc',[13 15],'weight',1);
ar(10)=struct('loc',[3  11],'weight',3);

[solD,solP,grid,time]=ObnoxiousSolve(ap,ar,up,ur,B,struct('solve','b'),[]);
```

We obtain the following results:

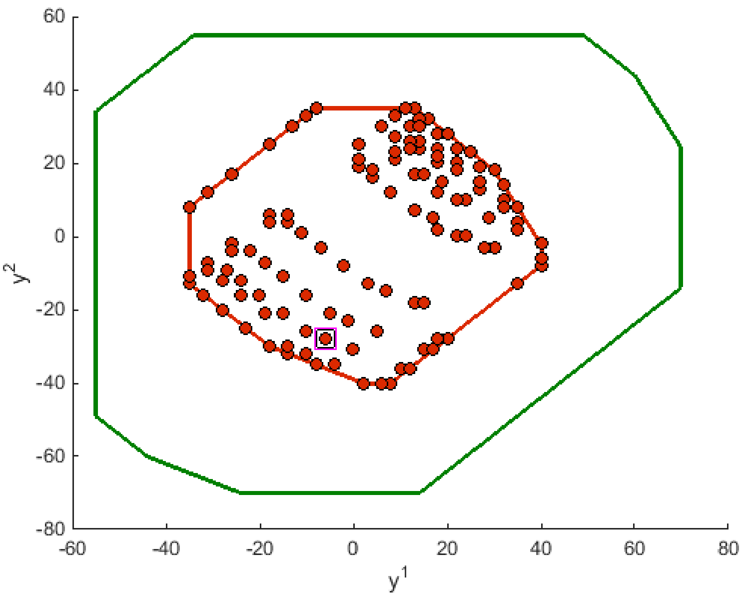
```
time =
        att_grid: 6.5522
    primal_solve: 0.0165
        rep_grid: 4.1830
      dual_solve: 0.0607

grid =
    primal_att: [3x106 double]
      dual_att: [3x129 double]
    primal_rep: [3x94 double]
      dual_rep: [3x121 double]
```

Obviously, the time complexity is mainly influenced by both, the number of grid points w.r.t.

attraction and the number of grid points w.r.t. repulsion. We have 106 primal and 129 dual grid points w.r.t. attraction (computed by executing the primal Algorithm 9.2) and 94 primal and 121 dual grid points w.r.t. repulsion (computed by executing the dual Algorithm 9.3). Hence, both algorithms seem to be comparable concerning running times for this example.



(a) Attracting points and resulting primal grid w.r.t. attraction (green); repulsive facilities and resulting grid w.r.t. repulsion (red); optimal primal grid points $x = (9, 3)^T$ and $x = (9, 2.5)^T$ (squares).



(b) Domain of g^* (green); domain of h^* and dual grid points $y \in \mathcal{I}_D$ w.r.t. repulsion (red); optimal dual grid point $y = (-6, -28)^T$ (square).

Figure 10.4: Primal and dual plot for Example 10.7.

Example 10.8. Consider the location problem (P) with randomly chosen coordinates and weights for $\overline{M}$ attracting and $\underline{M} = 1$ repulsive facilities:

```
B(1)=struct('ball',[1 1;-1 1;-1 -1; 1 -1]);
B(2)=struct('ball',[1 0;0 1;-1 0; 0 -1]);
B(3)=struct('ball',[-1 1;2 1;-1 -2]);
B(4)=struct('ball',[0 1;-1 0;1 -1]);

ur=ceil((size(B,2))*rand(1,Mr));
up=ceil((size(B,2))*rand(1,Mp));

for m=1:Mp
    ap(m).loc=ceil(rand(1,2)*(Mp+Mr));
    ap(m).weight=ceil(rand(1,1)*(Mr+1));
end
for m=1:Mr
    ar(m).loc=ceil(rand(1,2)*(Mp+Mr));
    ar(m).weight=ceil(rand(1,1)*0.5*(Mp+1));
end
```

We first solve the location problem with $\overline{M} = 20$ attraction points by executing the primal Algorithm 9.2, i.e.,

```
Mp=20; Mr=1;
[sol,solP,grid,time]=Obnoxious_Solve(ap,ar,up,ur,B,struct('solve','p'),[]);
```

and we obtain

```
time =
        att_grid: 44.3437
    primal_solve: 0.0067

grid =
    primal_att: [3x244 double]
      dual_att: [3x317 double]
```

Further, we solve the location problem with $\overline{M} = 500$ attraction points by executing the dual Algorithm 9.3, i.e.,

```
Mp=500; Mr=1;
[sol,solP,grid,time]=Obnoxious_Solve(ap,ar,up,ur,B,struct('solve','d'),[]);
```

and we obtain

```
time =
      rep_grid: 0.0153
    dual_solve: 17.3229

grid =
    primal_rep: [3x1 double]
      dual_rep: [3x3 double]
```

When solving the location problem (P) with 20 attraction points by executing the primal Algorithm 9.2, the time complexity consists mainly of determining the 244 primal and 317 dual grid points w.r.t. attraction. When solving the location problem (P) with a much greater amount of 500 attraction points by executing the dual Algorithm 9.3, the time complexity consists mainly of determining the optimal solutions. There are only one primal and three dual grid points. For each grid point, a linear optimization problem is to be solved in step 3 of Algorithm 9.3, where the running time for each sub-problem depends on the number of attraction points, see also Section 9.3.

As we can see, solving the problem with $\overline{M} = 20$ attraction points by executing the primal algorithm needs more than twice the time for solving the problem with $\overline{M} = 500$ attraction points by executing the dual algorithm. Hence, when there is a large number of attracting facilities and a small number of repulsive ones, then it is beneficial to apply the dual Algorithm instead of the primal one.

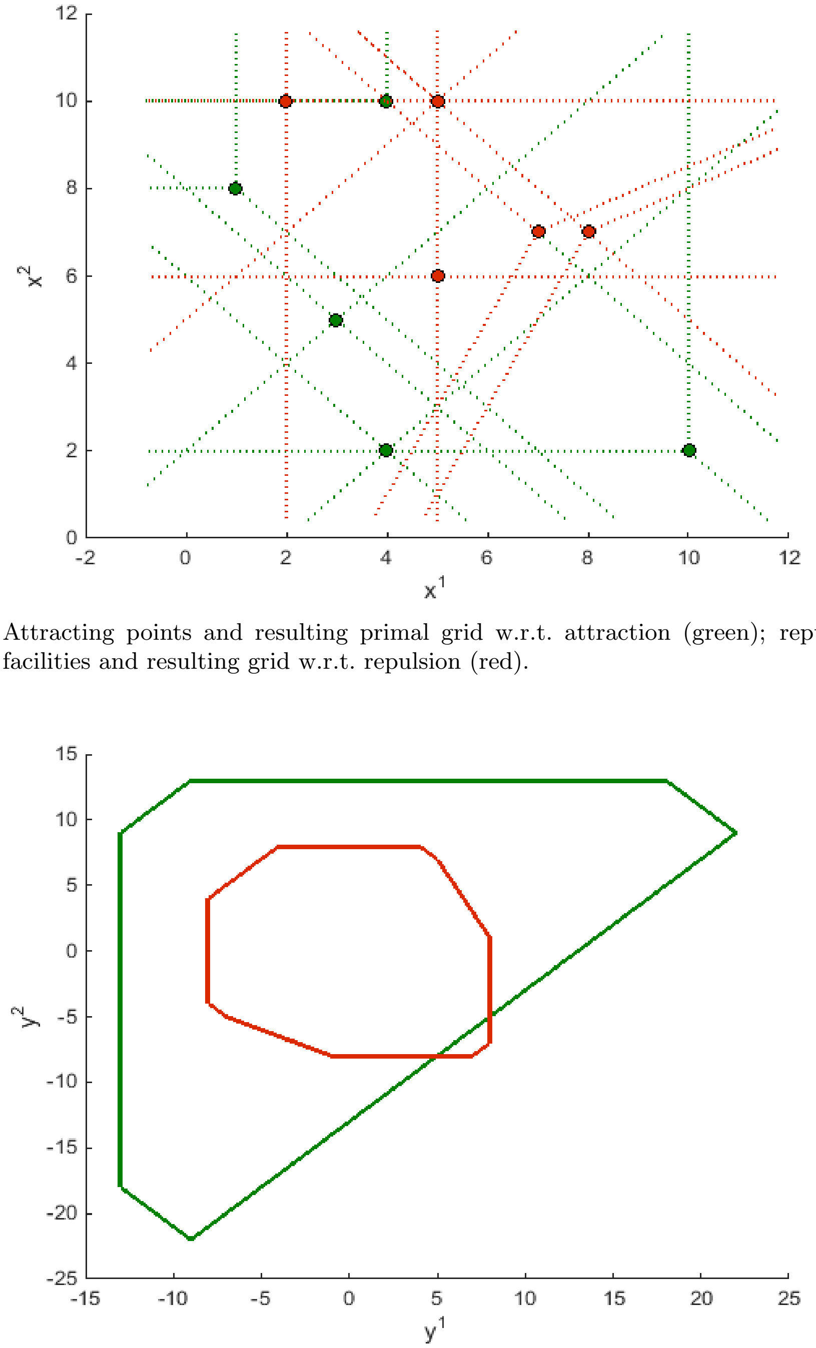
Example 10.9. Consider the location problem (P) with five attracting and five repulsive facilities as follows:

```
B(1)=struct('ball',[1 1;-1 1;-1 -1; 1 -1]);
B(2)=struct('ball',[1 0;0 1;-1 0; 0 -1]);
B(3)=struct('ball',[-1 1;2 1;-1 -2]);
B(4)=struct('ball',[0 1;-1 0;1 -1]);

up=[2 3 2 3 3];
ur=[1 2 4 4 1];

ap(1)=struct('loc',[3  5],'weight',1);
ap(2)=struct('loc',[10 2],'weight',1);
ap(3)=struct('loc',[4  2],'weight',3);
ap(4)=struct('loc',[4 10],'weight',6);
ap(5)=struct('loc',[1  8],'weight',2);

ar(1)=struct('loc',[5  6],'weight',3);
ar(2)=struct('loc',[5 10],'weight',1);
ar(3)=struct('loc',[8  7],'weight',2);
ar(4)=struct('loc',[7  7],'weight',1);
ar(5)=struct('loc',[2 10],'weight',1);

[sol,solP,grid,time]=ObnoxiousSolve(ap,ar,up,ur,B,struct('solve','b'),[]);
```

For this example the finiteness criterion is not satisfied (cf. subroutine `Finite` in Section 10.3), such that the implementation `ObnoxiousSolve` returns:

```
Solution is INFINITE.
```

The existing facilities and the corresponding primal grids as well as the domains of g^* and h^* are illustrated in Figure 10.5.



(a) Attracting points and resulting primal grid w.r.t. attraction (green); repulsive facilities and resulting grid w.r.t. repulsion (red).

(b) Domain of g^* (green); domain of h^* (red).

Figure 10.5: Primal and dual plot for Example 10.9.

Chapter 11

Conclusion

In this thesis we present a new approach for solving the non-convex location problem (P) with obnoxious facilities. This approach is based on the special structure of (P) and the fact that the objective function can be written as a d.c. function. By applying the duality theory by Toland [121] and Singer [112] we obtain a suitable dual problem (D).

We introduce the concepts of primal and dual elementary convex sets with respect to attraction and to repulsion, as well as corresponding grids and grid points.

Although, we consider a scalar optimization problem, we show the remarkable fact, that methods from the field of linear vector optimization can be applied in order to determine those grid points with respect to attraction and to repulsion.

The relationship between primal and dual elementary convex sets is described, based on results from the field of geometric duality [56], by an inclusion reversing one-to-one mapping.

Moreover, we present properties of primal and dual elementary convex sets and state duality assertions and discretization results based on the duality theory by Toland and Singer.

The obtained results are applied in order to formulate a dual and a primal algorithm, which determine exact solutions by leading back the non-convex optimization problems to a finite number of convex problems.

The developed algorithms are implemented as `Matlab` functions. It turns out that, in case of few attraction points with a small amount of fundamental directions, the primal algorithm is beneficial. Whereas, in case of few repulsive facilities with a small amount of fundamental directions the dual algorithm is advantageous.

In this thesis we also consider the more general case of a constrained location problem $(P_{\mathcal{H}})$ by introducing an indicator function. Analogously to the unconstrained case, we define a dual problem $(D_{\mathcal{H}})$ as well as primal and dual elementary convex sets with respect to attraction and to repulsion, taking into account the constraints. We show that most of the results obtained for the unconstrained location problem can be generalized for the constrained case.

For future research, further improvements of the presented algorithms might be possible. Furthermore, the case of round gauges may be studied in more detail. Many results obtained in this research can in fact also be formulated for this case, but are not applicable for an algorithm from the present point of view. Moreover, it seems to be suitable to consider further constraints, such as barriers, which may influence travel time and travel costs, or forbidden regions for the obnoxious facility. Also the problem of locating multiple obnoxious facilities may be discussed. Another interesting future research may focus on the question, how the duality theory by Toland and Singer may also be applied for different kinds of location problems with obnoxious facilities, such as network problems or problems with center objective function.

Index

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit

A new Duality Based Approach for the Problem of Locating a Semi-Obnoxious Facility

selbständig und ohne fremde Hilfe verfasst, keine anderen als die von mir angegebenen Quellen und Hilfsmittel benutzt und die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.
Die Arbeit wurde in gleicher oder ähnlicher Form weder einer anderen Fakultät vorgelegt noch veröffentlicht.

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Vorträge und Publikationen

Vorträge

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